Global QCQP Solver

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QCQP SDP relaxations

Consider (non-)convex QCQP:

$$\begin{aligned} f := \text{Maximize} \quad & x^T Q x + q^T x \\ \text{s.t.} \quad & x^T A_i x + a_i^T x (\leq, =, \geq) b_i, \forall i \\ & 0 \leq x \leq 1 \end{aligned} \tag{1}$$

And Shor relaxation with Y

$$\begin{aligned} z := \text{Maximize} \quad & Q \bullet Y + q^T x \\ \text{s.t.} \quad & Y - xx^T \succeq 0 \text{ or } \begin{bmatrix} 1 & x^T \\ x & Y \end{bmatrix} \succeq 0 \\ & A_i \bullet Y + a_i^T x (\leq, =, \geq) b_i, \forall i \end{aligned} \tag{2}$$

A Simple Spatial Branch-and-Cut Algorithm

$\epsilon^{\text{feas}}, \epsilon^{\text{opt}}$ are feasibility and optimality tolerance, respectively.
Define residual function: $\varepsilon(x, Y) = |Y - xx^T|$
At current solution x^k, Y^k

- Compute $\varepsilon^k = \varepsilon(x^k, Y^k)$ as the residual
- If $\max_{ij} \varepsilon_{ij}^k \leq \epsilon^{\text{feas}}$, stop
- Choose $i := \arg\max_i \sum_j \varepsilon_{ij}^k$ to branch, then choose secondary axis $j = \arg\max_j \varepsilon_{ij}^k$
- Branch as two, left: $x_i \in [l_i, u_i := x_i^k]$; right: $x_i \in [l_i := x_i^k, u_i]$;
- For each child, add RLT cuts to cut off y_{ij}^k

$$
\begin{aligned}
y_{ij} - x_i \cdot u_j - x_j \cdot l_i + l_i \cdot u_j &\leq 0 \\
y_{ij} - x_i \cdot l_j - x_j \cdot u_i + l_j \cdot u_i &\leq 0 \\
y_{ij} - x_i \cdot l_j - x_j \cdot l_i + l_i \cdot l_j &\geq 0 \\
y_{ij} - x_i \cdot u_j - x_j \cdot u_i + u_i \cdot u_j &\geq 0
\end{aligned}
$$

- Pop next problem with the best parent bound.

Literature Review

- Spatial branching, [Belotti et al., 2013], [Chen et al., 2017]
- Branching by KKT, [Burer and Vandenbussche, 2008], [Chen and Burer, 2011]
- Eigenvalue decomposition and branch, [Lu et al., 2017]

Questions & Future Work

Coding:

- Avoid create QP repetitively
- Use warm-start (a)
- d-dimensional extension

Methodology:

- Improve branching rules
- Improve initial relaxation using a tighter formulation
- Improve RLT cuts, add more cuts: SDP, BQP, ... see Gurobi 9.0, 9.1 docs
- Early pruning: use bound estimate?

Methodology?:

- Warm-start (b) at the parent solution to avoid solving child node from scratch? (like dual simplex)

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