

# Relaxations for QCQP

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## 1 Relaxations for General QCQP

As a convention, we assume data matrices are symmetric, i.e.,  $Q, A_i \in S^n$

Recall homogeneous QCQP for  $x \in \mathbb{R}^n$ :

$$\begin{aligned} \text{(HQCQP)} \quad & \text{Maximize} \quad x^T Q x \\ & \text{s.t.} \quad x^T A_i x (\leq, =, \geq) b_i, \forall i = 1, \dots, m \\ & \quad 0 \leq x \leq 1 \end{aligned} \tag{1}$$

And inhomogeneous QCQP,

$$\begin{aligned} \text{(QCQP)} \quad & \text{Maximize} \quad x^T Q x + q^T x \\ & \text{s.t.} \quad x^T A_i x + a_i^T x (\leq, =, \geq) b_i \\ & \quad 0 \leq x \leq 1 \end{aligned} \tag{2}$$

We now assume a standard form with less-than-or-equal-to constraints. We mark some of the trivial techniques below.

- One can always reformulate (2) into a homogeneous problem by increasing the dimension of variables by 1.

$$\begin{aligned}
\text{Maximize} \quad & x^T Q x + q^T x \\
& = [x^T \ t] \begin{bmatrix} Q & q/2 \\ q^T/2 & o \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} \\
\text{s.t.} \quad & -1 \leq t \leq 1
\end{aligned} \tag{3}$$

- Also, a symmetrized version can be achieved by fact that  $x^T A x = x^T A^T x$ , let

$$\tilde{A} := \frac{A + A^T}{2} \tag{4}$$

### 1.1 SDP Relaxation

For  $x \in \mathbb{R}^n$ , we have:  $x^T A_i x = A_i \bullet (xx^T)$  and  $xx^T \in S_+^n$ , which results in following relaxation using semidefinite cones, also called *lifting* method or Shor relaxation,

$$\begin{aligned}
\text{(Shor-Basic)} \quad & \text{Maximize} \quad Q \bullet Y + q^T x \\
\text{s.t.} \quad & Y - xx^T \succeq 0 \text{ or } \begin{bmatrix} 1 & x^T \\ x & Y \end{bmatrix} \succeq 0 \\
& A_i \bullet Y + a_i^T x \leq b_i, \forall i \\
& 0 \leq x \leq 1
\end{aligned} \tag{5}$$

Notice QCQP with matrix variables can also be reformulated into a SDP based problem, let  $X \in \mathbb{R}^{n \times d}$ , then  $X^T A_i X = A_i \bullet (XX^T)$ , similarly,

$$\begin{aligned}
& \text{Maximize} \quad Q \bullet Y \\
\text{s.t.} \quad & Y - XX^T \succeq 0 \text{ or } \begin{bmatrix} I_d & X^T \\ X & Y \end{bmatrix} \succeq 0 \\
& A_i \bullet Y \leq b_i, \forall i
\end{aligned} \tag{6}$$

SDP relaxations (5) can be unbounded in some case. A simple improvement to (5) is to add bounds for the diagonal entries.

$$\begin{aligned}
\text{(Shor)} \quad & \text{Maximize} \quad Q \bullet Y + q^T x \\
\text{s.t.} \quad & Y - xx^T \succeq 0 \text{ or } \begin{bmatrix} 1 & x^T \\ x & Y \end{bmatrix} \succeq 0 \\
& A_i \bullet Y + a_i^T x \leq b_i, \forall i \\
& 0 \leq x \leq 1 \\
& \text{diag}(Y) \leq x
\end{aligned} \tag{7}$$

### 1.1.1 Dual

Consider the dual of primal SDP relaxation with diagonal bound, cf. (7)

$$\begin{aligned}
L &= \min_{x,Y} -Q \bullet Y - q^T x + \sum_i (\lambda_i A_i \bullet Y + \lambda_i a_i^T x - \lambda_i b_i) \\
&\quad \mathbf{diag}(v) \bullet Y - v^T x + \mu^T x - \mu^T e \\
&\quad - \begin{bmatrix} Y & x \\ x^T & 1 \end{bmatrix} \bullet \begin{bmatrix} Z & y \\ y^T & \alpha \end{bmatrix} \\
&= \min_{x,Y} -\lambda^T b - \mu^T e - \alpha \\
&\quad + Y \bullet \left( -Q + \sum_i \lambda_i A_i - Z + \mathbf{diag}(v) \right) \\
&\quad + x^T \left( -q + \sum_i \lambda_i a_i + \mu - 2y - v \right)
\end{aligned} \tag{8}$$

And thus the dual  $\phi(\cdot) = \max_{(\cdot)} L$ :

$$\begin{aligned}
(\text{Dual-Shor}) \quad &\text{Minimize: } \lambda^T b + \mu^T e + \alpha \\
&\text{s.t. } Q = \sum_i \lambda_i A_i + \mathbf{diag}(v) - Z \\
&\quad \sum_i \lambda_i a_i + \mu - 2y - v - q \geq 0 \\
&\quad \begin{bmatrix} Z & y \\ y^T & \alpha \end{bmatrix} \geq 0 \\
&\quad \lambda, \mu, v \geq 0
\end{aligned} \tag{9}$$

### 1.1.2 RLT

There are many further enhancements to (7), see [?] for discussion on the strength of different relaxations. Here we discuss a few widely used methods using copositive cones and reformulation-linearization-techniques (RLT) cuts.

### 1.1.3 Copositive

## 1.2 SOCP Relaxation

We now consider another way of relaxing the original QCQP problem. Consider symmetric indefinite matrix  $Q \in S^n$  and its spectral decomposition.

$$\begin{aligned}
Q &= V \Lambda V^T = \sum_j^n \lambda_j v_j v_j^T \\
\Lambda &= \text{diag}(\lambda)
\end{aligned} \tag{10}$$

Without loss of generality, we assume first  $r$  eigenvalues are positive,  $\lambda_1, \dots, \lambda_r \geq 0, r \leq n$ . The quadratic form  $x^T Q x$  can also be partitioned into positive and negative parts:

$$x^T Q x = \sum_{j=1}^r \lambda_j x^T v_j v_j^T x + \sum_{j=r+1}^n \lambda_j x^T v_j v_j^T x \quad (11)$$

By letting  $y_j \geq z_j^2, z_j = v_j^T x, j = 1, \dots, n$ , then (11) can be expressed by introducing  $n$  (2- $d$ ) quadratic cones, literally,

$$x^T Q x \leq \sum_j y_j \cdot \lambda_j, (y_j, v_j^T x) \in \mathcal{Q}^2 \quad (12)$$

This substitution uses a set of small quadratic cones instead of one semidefinite matrix of size  $n^2$ . With some abuse of notation, suppose  $\lambda_i, V_i, i = 0, \dots, m$  are eigenvalues and vectors for  $Q$  and  $A_i, i = 1, \dots, m$ , respectively. Following the same routine for each constraint, we describe the Many-Small-Cone (MSC) relaxation to QCQP, namely,

$$\begin{aligned} \text{(MSC)} \quad & \text{Maximize : } y_0^T \lambda_0 + q^T x \\ \text{s.t.} \quad & V_i z_i = x \quad i = 0, \dots, m \\ & y_i^T \lambda_i + a_i^T x \leq b_i \quad i = 1, \dots, m \\ & y_i \geq z_i \circ z_i \quad i = 0, \dots, m \\ & y_i^T e \leq x^T e \quad i = 0, \dots, m \end{aligned} \quad (13)$$

The last set of constraints are placed to resolve unboundedness for the fact that the similarity transformation by any orthogonal basis  $V_i, \forall i$  preserves the value of **trace** operator, namely:

$$y_i^T e = \text{trace}(V_i^T x x^T V_i) = \text{trace}(x x^T) \leq x^T e \quad (14)$$

This method is closely related to D.C. and Convex SOCP relaxations to QCQP, see [?], [?], [?], [?]. Recently, [?] mention a similar formulation, by defining  $C_i = V_i \text{diag}(\sqrt{|\lambda_i|})$ . We list it below for convenience.

$$\begin{aligned} \text{(MSC-Luo)} \quad & \text{Maximize : } y_0^T e + q^T x \\ \text{s.t.} \quad & C_i z_i = x \quad i = 0, \dots, m \\ & y_i^T \lambda_i + a_i^T x \leq b_i \quad i = 1, \dots, m \\ & y_i \geq z_i \circ z_i \quad i = 0, \dots, m \\ & y_i^T \frac{1}{|\lambda_i|} \leq x^T e \quad i = 0, \dots, m \end{aligned} \quad (15)$$

In [?], box constraints for  $z_i$  can be calculated by its definition. For the case where  $x \in [0, 1]$ , we show bounds are redundant and the two formulations are equivalent.

**Theorem 1** *The relaxations (13), (15) are equivalent.*

*Proof* We show the solution for any one of the relaxations can be derived from another.

Suppose  $(x_0, z_0, y_0)$  is a feasible solution to (13), then  $(x_0, \sqrt{\mathbf{diag}(|\lambda|)}z_0, \mathbf{diag}(|\lambda|) \cdot y_0)$  is feasible to (15). Conversely, if  $(x_0, z_0, y_0)$  is feasible to (15), then we can construct  $(x_0, \frac{1}{\sqrt{\mathbf{diag}(|\lambda|)}}z_0, \frac{1}{\mathbf{diag}(|\lambda|)} \cdot y_0)$  that is also feasible to (13).  $\square$