QCQP: Progress Report

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The QCQP

We consider the QCQP

$$
\begin{aligned}
(\textbf{QCQP}) \quad \max \quad & x^T Q x + q^T x \\
\text{s.t.} \quad & x^T A_i x + a_i^T x \,(\leq, =, \geq)\, b_i
\end{aligned} \tag{1}
$$

Rank-r Indefiniteness I

Formally, a quadratic inequality induced by a symmetric matrix A can be expressed as the following,

$$x^T \left(\sum_{j \in J_+} \lambda_j v_j v_j^T \right) x + a^T x \leq b + \left(\sum_{j \in J_-} \lambda_j v_j v_j^T \right) \tag{2}$$

- For clarity, we say a quadratic inequality$x^T A x + a^T x \leq b$ is *rank-r indefinite* if first r eigenvalues are negative.
- In comparison, a maximization problem with matrix Q is rank-r indefinite if last r eigenvalues are nonnegative.

Rank-1 Indefinite Quadratic Inequality I

Suppose $Q = RR^T - aa^T$
For quadratic inquality, it can be written as,

$$x^T RR^T x \leq (a^T x)^2$$

Then we natually solve the problem in two disjoint subregions, by $a^T x \geq 0 \vee a^T x \leq 0$,

$$\begin{aligned} (P1) \quad & \|R^T x\| \leq -a^T x \\ (P2) \quad & \|R^T x\| \leq a^T x \end{aligned}$$

Implementation,

- Create offline subregions at depth 0 (multiple root nodes)
- Seems impossible to do explicit disjunctions for inhomogeneous case.
- Apply branch and bound.

Rank-1 Indefinite Maximization I

Consider maximization

$$
\begin{aligned}
\max \quad & x^T(aa^T - RR^T)x + q^T x \\
\Rightarrow \max \quad & z \\
\text{s.t.} \quad & z + x^T RR^T x - q^T x \leq (a^T x)^2
\end{aligned}
$$

We analyze,

$$S = \{(z, x) : x^T RR^T x + z - q^T x \leq (a^T x)^2\}$$

1. Simply branch on $sign(a^T x)$ will prune one region very fast by bound.
2. Seems impossible to do explicit disjunctions as homogeneous inequality.
3. If using best bound rule in node selection, it has no obvious advantage.
4. Apply branch and bound.

Extension to rank-r I

Consider quadratic maximization,

$$\begin{aligned} \max \quad & z \\ \text{s.t.} \quad & \sum_{j \in J_-} \lambda_j z_j - q^T x + z \le \sum_{j \in J_+} \lambda_j z_j \\ & y_j \ge (x_j^T v_j)^2, j = 1, ..., n \end{aligned} \tag{3}$$

With box or ball as regularity (else (3) will be unbounded.)

- ball: $x \in B(0, \delta)$, $x \in [0, 1]^n$
- ellipsoid: $x \in \{x : x^T A x + a^T x \le \delta^2\}$
- regularity implies, (explains why ball constraint is easier.)

$$y^T e = \|x\|^2 \tag{4}$$

Computational Results I

- We test on "unconstrained" (with only a box or ball on x)
- MSC is very effective in the test problems. as $r \nearrow$, the problem becomes hard.
- MSC is tight with only ball constraints? (TRS)

Future Work I

- Extension to an indefinite quadratic inequalities
- Plug in ADMM as primal feasible solution.

Future Work: Multiple Quadratic Constraints I

Our old MSC.

$$
\begin{aligned}
\text{Maximize} \quad & y_0^T \lambda_0 + q^T x & & \\
\text{s.t.} \quad & V_i z_i = x & i = 0, ..., m & \quad (5) \\
& y_i^T \lambda_i + a_i^T x \leq b_i & i = 1, ..., m & \quad (6) \\
& y_i = z_i \circ z_i & i = 0, ..., m & \quad (7)
\end{aligned}
$$

- In this formulation, we need $m \times n$ auxillary pairs (z, y) by allowing different bases V_i
- Actually this may not be necessary...

Future Work: Multiple Quadratic Constraints II

Consider one indefinite A with a rank-r indefinite Q, we can convexify A by...

$$Q = \sum_{J_+} \lambda_j v_j v_j^T - \sum_{J_-} \lambda_j v_j v_j^T$$

$$x^T A_+ x - x^T A - x + a^T x \leq b \tag{8}$$

$$\Rightarrow \quad x^T \left(A + V_- \Gamma V_-^T\right) x + a^T x \leq b + \underbrace{x^T (V_- \Gamma V_-^T) x}_{\textbf{diag}(\Gamma)^T y} \tag{9}$$

- Then we do not need to add more $\{y\}$ for this constraint.
- The only question left is whether we can find Γ such that,

$$V_- \Gamma V_-^T - A_- \succeq 0 \tag{10}$$

- This is weaker than simultaneous diagonalization via congruence.

Future Work: ADMM in the original x I

Notice,

$$\|x\|^2 = \max_{\|\xi\| \leq \sqrt{s}} \xi^T x \tag{11}$$

So we add slack variable s, t, ξ and bilinear constraint.

$$
\begin{aligned}
(\textbf{MSC}) \quad \text{Maximize}: \quad & y_0^T \lambda_0 && && (12)\\
\text{s.t.} \quad & (y, z, x) \in \Omega && && (13)\\
& y_i^T e \leq t && i = 0, \cdots, m && (14)\\
(\kappa) \quad & t = s && i = 0, \cdots, m && (15)\\
(\mu) \quad & \xi^T x = t && && (16)\\
& \xi^T \xi \leq s && && (17)
\end{aligned}
$$

If s, t, ξ, y, z, x is the solution, then y, z, x is the optimal solution for MSC.
This allows the augmented Lagrangian function,

$$\mathscr{L}(x, y, z, \xi, s, \kappa, \mu) = -y_0^T \lambda_0 + \kappa(t - s) + \mu(\xi^T x - t) + \frac{\rho}{2}(t - s)^2 + \frac{\rho}{2}(\xi^T x - s)^2$$

Future Work: ADMM in the original x II

The ADMM iteration,

$$\begin{aligned}
(x, y, z, t)^{k+1} &= \arg\min_{(x,y,z)\in\Omega, t\geq 0} L\left(x, y, z, \xi^k, s^k, \kappa^k, \mu^k\right) \\
(s, \xi)^{k+1} &= \arg\min_{(s,\xi)\in\mathscr{Q}} L\left((x, y, z, t)^{k+1}, \xi, s, \kappa^k, \mu^k\right) \\
\kappa^{k+1} &= \kappa^k + \rho\left(t^{k+1} - s^{k+1}\right) \\
\mu^{k+1} &= \mu^k + \rho\left(\langle \xi^{k+1}, x^{k+1}\rangle - s^{k+1}\right)
\end{aligned}$$

where $\mathscr{Q}(\cdot)$ forms a simple SOCP for s, ξ,

$$\mathscr{Q}(x) = \left\{(s, \xi) : \|\xi\|^2 \leq s\right\} \tag{18}$$

Future Work: ADMM in the original x III

The size of auxillary variable ξ equals to n, if r is small,
We can actually shrink the size of above problem.

$$
\begin{aligned}
(\textbf{MSC}) \quad \text{Maximize}: \quad & y_0^T \lambda_0 && && (19)\\
\text{s.t.} \quad & (y, z, x) \in \Omega && && (20)\\
& y_i^T e = s && i = 0, \cdots, m && (21)\\
(\mu_i, i \in I_+) \quad & \xi_i \cdot (v_i^T x) = y_i && i \in I_+ && (22)\\
& \xi_i^2 \leq y_i, (v_i^T x)^2 \leq y_i && && (23)
\end{aligned}
$$

The ADMM,

$$
\mathcal{L} = y^T \lambda + \sum_{i \in I_+} \mu_i \left[\xi_i \cdot (v_i^T x) - y_i \right] + \frac{\rho}{2} \sum_{i \in I_+} (\xi_i \cdot (v_i^T x) - y_i)^2 \tag{24}
$$

Which is easier than old ADMM for general case.