Solving Nonconvex QCQP: Convex-Relaxation then Local Refinement

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HQCQP: Homogeneous Case

Homogeneous Quadratically Constrained Quadratic Programming: Given symmetric matrices $Q,\ A_i's$, find global optimal solution of the following optimization problem

$$\text{(HQCQP)}\quad \begin{array}{rll} z^* := & \text{Maximize} & \mathbf{x}^T Q\mathbf{x}\ (= Q \bullet (\mathbf{x}\mathbf{x}^T)) \\ & \text{s.t.} & \mathbf{x}^T A_i \mathbf{x}\ (\leq, =, \geq)\ b_i,\ \forall i = 1, ..., m. \end{array}$$

Note that if $\mathbf{x}^*$ is a (global) optimal solution, so is $-\mathbf{x}^*$.

Rank-Constrained SDP Formulation:

$$\begin{array}{rll} z^* := & \text{Maximize}_{X \in \mathcal{S}^n} & Q \bullet X \\ & \text{s.t.} & A_i \bullet X\ (\leq, =, \geq)\ b_i,\ \forall i = 1, ..., m, \\ & & X \succeq \mathbf{0},\ \text{rank}(X) = 1. \end{array}$$

SDP Relaxation: remove the rank constraint.

QCQP: General Case

Quadratically Constrained Quadratic Programming:

$$\begin{array}{lll} & z^* := \quad \text{Maximize} & \mathbf{x}^T Q\mathbf{x} + 2\mathbf{q}^T\mathbf{x} \\ \text{(QCQP)} & & \\ & \text{s.t.} & \mathbf{x}^T A_i\mathbf{x} + 2\mathbf{a}_i^T\mathbf{x}\ (\leq, =, \geq)\ b_i,\ \forall i = 1, ..., m. \end{array}$$

that can be reformulated:

$$\begin{array}{lll} z^* := & \text{Maximize} & Q \bullet Y + 2x_{n+1}\mathbf{q}^T\mathbf{x} \\ & \text{s.t.} & A_i \bullet Y + 2x_{n+1}\mathbf{a}_i^T\mathbf{x}\ (\leq, =, \geq)\ b_i,\ \forall i = 1, ..., m, \\ & & Y - \mathbf{x}\mathbf{x}^T = \mathbf{0}. \end{array}$$

SDP Relaxation: $Y - \mathbf{x}\mathbf{x}^T \succeq \mathbf{0}$ or $\begin{pmatrix} 1 & \mathbf{x}^T \\ \mathbf{x} & Y \end{pmatrix} \succeq \mathbf{0}.$

Alternative Relaxation

For each quadratic function $\mathbf{x}^T A\mathbf{x}$, split A as

$$A = A_+ - A_-$$

where both A_+ and A_- are PSD. Factorize both of them such that, e.g., $A_+ = R_+^T R_+$, and introduce the linear constraints $R_+\mathbf{x} = \mathbf{z}_+$ so that $\mathbf{x}^T A_+\mathbf{x} = \|\mathbf{z}_+\|^2$.

Many-Small-SDP-Cone Relaxation:

- For each diagonally quadratic variable z, adding another auxiliary variable y such that $y = z^2$.
- Then relax it to $y - z^2 \geq 0$ or

$$\begin{pmatrix} 1 & z \\ z & y \end{pmatrix} \succeq \mathbf{0}.$$

QCQP Applications

- GUROBI "solves" nonconvex QCQP by a "branching and cut" algorithm; see several Youtube videos.
- Hub-Location, Quadratic Assignment, ...
- Nash Equilibrium, Bimatrix-Game, Linear Complementarity Problem, ...
- Communication, Data Science, ...
- Sensor Network Localization, Ball Packing, Kissing Number,...

Example: SNL

Given a graph $G = (V, E)$ and sets of non–negative weights, say $\{d_{ij} : (i, j) \in E\}$, the goal is to compute a realization of G in the Euclidean space $\mathbf{R}^d$ for a given low dimension d, where the distance information is preserved.

More precisely: given anchors $\mathbf{a}_k \in \mathbf{R}^d$, $d_{ij} \in N_x$, and $\hat{d}_{kj} \in N_a$, find $\mathbf{x}_i \in \mathbf{R}^d$ such that

$$\|\mathbf{x}_i - \mathbf{x}_j\|^2 = d_{ij}^2, \ \forall\, (i, j) \in N_x, \ i < j,$$
$$\|\mathbf{a}_k - \mathbf{x}_j\|^2 = \hat{d}_{kj}^2, \ \forall\, (k, j) \in N_a,$$

This is a QCQP feasibility problem.

Matrix Representation and SDP Relaxation

Let $X = [\mathbf{x}_1 \; \mathbf{x}_2 \; ... \; \mathbf{x}_n]$ be the $d \times n$ matrix that needs to be determined and $\mathbf{e}_j$ be the vector of all zero except 1 at the jth position. The SDP relaxation is also an SDP feasibility problem:

$$
\begin{aligned}
&(\mathbf{e}_i - \mathbf{e}_j)^T(\mathbf{e}_i - \mathbf{e}_j) \bullet Y = d_{ij}^2, \; \forall\, i, j \in N_x, \; i < j, \\
&(\mathbf{a}_k; -\mathbf{e}_j)^T(\mathbf{a}_k; -\mathbf{e}_j) \bullet \begin{pmatrix} I & X \\ X^T & Y \end{pmatrix} = \hat{d}_{kj}^2, \; \forall\, k, j \in N_a, \\
&Y \succeq X^T X.
\end{aligned}
$$

First-Step: Solve the SDP feasibility problem and let X^{SDP} be the position solution of the optimal matrix solution. If the problem is "anchor-free", let X^{SDP} be the eigenvectors of Y^{SDP} corresponding to the d largest eigenvalues.

Second-Step: Using X^{SDP} as the initial solution and apply SDM in minimizing the square residual function:

$$
\min_X \sum_{(i<j,j)\in N_x} (\|\mathbf{x}_i - \mathbf{x}_j\|^2 - d_{ij}^2)^2 + \sum_{(k,j)\in N_a} (\|\mathbf{a}_k - \mathbf{x}_j\|^2 - \hat{d}_{kj}^2)^2.
$$

Example: The Kissing Number Problem

Given a unit ball centered at the origin in dimension d, the maximum number of other unit balls can touch or kiss the centered unit-ball?

The Kissing Problem as a QCQP: Given n unit-balls, could we find their center locations for all of them such that the following problem is feasible:

$$\begin{aligned}\|\mathbf{x}_i - \mathbf{x}_j\|^2 &\geq 4, \ \forall 1 \leq i \neq j \leq n,\\ \|\mathbf{x}_i\|^2 &= 4, \ \forall i\\ \mathbf{x}_i &\in R^d\end{aligned}$$

Solving SDP relaxation (may add anchors and an objective leading to a low-rank solution):

$$\begin{aligned}(\mathbf{e}_i - \mathbf{e}_j)^T Y (\mathbf{e}_i - \mathbf{e}_j) &\geq 4, \ \forall i \neq j,\\ \mathbf{e}_i^T Y \mathbf{e}_i &= 4, \ \forall i\\ Y &\succeq \mathbf{0}; \quad (\text{Rank}(Y) = d)\end{aligned}$$

Then minimizing a nonconvex residual function.

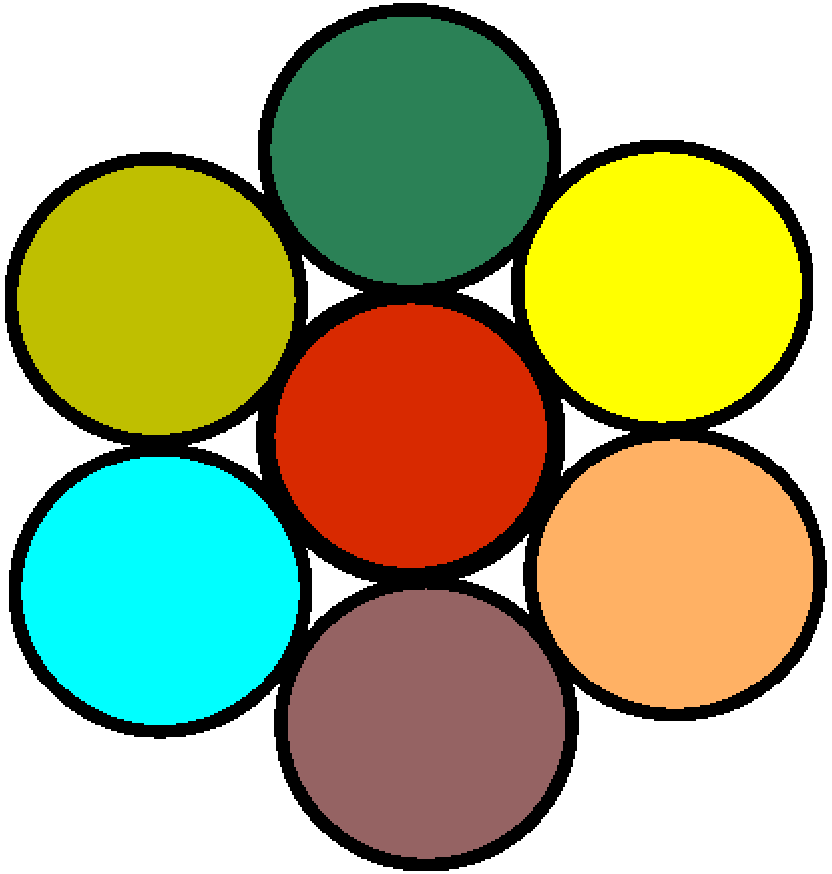

Figure 1: Kissing Localization in 2D

Project Proposal

- COPT-DSDP Solver:
  - initialization, homogeneous model, detect infeasibility
  - new cholesky factorization, numerical improvement
- Global QCQP solver (GUROBI 9.1):
  - Add SDP relaxation and then use GUROBI9.1
  - Develop own COPT-QCQP solver: finite branching, smart cut, etc.
  - Test on Kissing Number problem for $d = 4, 5, 6$ some of which are open
- Distributionally Robust Optimization Solver (SOCP and SDP based):
  - SOCP and small SDP cone relaxation
  - COPT-DRO financial product

Reading List

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