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Relaxations for QCQP

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1 Relaxations for General QCQP

As a convention, we assume data matrices are symmetric, i.e., $Q, A_i \in \mathcal{S}^n$

Recall homogeneous QCQP for $x \in \mathbb{R}^n$:

$$
\begin{array}{lll}
(\textbf{HQCQP}) & \text{Maximize} & x^T Q x \\
& \text{s.t.} & x^T A_i x \ (\leq, =, \geq)\ b_i, \forall i = 1, \dots, m \\
& & 0 \leq x \leq 1
\end{array} \tag{1}
$$

And inhomogeneous QCQP,

$$
\begin{array}{lll}
(\textbf{QCQP}) & \text{Maximize} & x^T Q x + 2q^T x \\
& \text{s.t.} & x^T A_i x + 2a_i^T x \ (\leq, =, \geq)\ b_i \\
& & 0 \leq x \leq 1
\end{array} \tag{2}
$$

We mark some of the trivial techniques below.

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- One can always reformulate (2) into a homogeneous problem by increasing the dimension of variables by 1.

$$\begin{aligned} \text{Maximize} \quad & x^T Q x + 2q^T x \\ & = \begin{bmatrix} x^T & t \end{bmatrix} \begin{bmatrix} Q & q \\ q^T & o \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} \\ \text{s.t.} \quad & -1 \le t \le 1 \end{aligned} \tag{3}$$

- Also, a symmetrized version can be achieved by fact that $x^T A x = x^T A^T x$, let

$$\tilde{A} := \frac{A + A^T}{2} \tag{4}$$

1.1 SDP Relaxation

For $x \in \mathbb{R}^n$, we have: $x^T A_i x = A_i \bullet (xx^T)$ and $xx^T \in \mathcal{S}_+^n$, which results in following relaxation using semidefinite cones, also called *lifting* method or Shor relaxation,

$$\begin{aligned} (\textbf{Shor-Basic}) \quad \text{Maximize} \quad & Q \bullet Y + 2q^T x \\ \text{s.t.} \quad & Y - xx^T \succeq 0 \text{ or } \begin{bmatrix} 1 & x^T \\ x & Y \end{bmatrix} \succeq 0 \\ & A_i \bullet Y + 2a_i^T x \ (\le, =, \ge) \ b_i, \forall i \\ & 0 \le x \le 1 \end{aligned} \tag{5}$$

Notice QCQP with matrix variables can also be reformulated into a SDP based problem, let $X \in \mathbb{R}^{n \times d}$, then $X^T A_i X = A_i \bullet (XX^T)$, similarly,

$$\begin{aligned} \text{Maximize} \quad & Q \bullet Y \\ \text{s.t.} \quad & Y - XX^T \succeq 0 \text{ or } \begin{bmatrix} I_d & X^T \\ X & Y \end{bmatrix} \succeq 0 \\ & A_i \bullet Y \ (\le, =, \ge) \ b_i, \forall i \end{aligned} \tag{6}$$

SDP relaxations (5) can be unbounded in some case. A simple improvement to (5) is to add bounds for the diagonal entries.

$$\begin{aligned} (\textbf{Shor}) \quad \text{Maximize} \quad & Q \bullet Y + 2q^T x \\ \text{s.t.} \quad & Y - xx^T \succeq 0 \text{ or } \begin{bmatrix} 1 & x^T \\ x & Y \end{bmatrix} \succeq 0 \\ & A_i \bullet Y + 2a_i^T x \ (\le, =, \ge) \ b_i, \forall i \\ & 0 \le x \le 1 \\ & \operatorname{diag}(Y) \le x \end{aligned} \tag{7}$$

1.1.1 RLT

There are many further enhancements to (7), see [**?**] for discussion on the strength of different relaxations. Here we discuss a few widely used methods using copositive cones and reformulation-linearization-techniques (RLT) cuts.

1.1.2 Copositive

1.1.3 Dual

Consider the dual of primal SDP relaxation.

$$\begin{aligned} L &= \max_{x,X} \; Q \bullet X + q^T x + \sum_i \left(\lambda_i A_i \bullet X + \lambda_i a_i^T x - \lambda_i b_i \right) + \mu^T x - \mu^T e \\ &\quad - \begin{bmatrix} X & x \\ x^T & 1 \end{bmatrix} \bullet \begin{bmatrix} Y & y \\ y^T & 1 \end{bmatrix} \\ &= \max_{x,X} \; -\lambda^T b - \mu^T e - 1 + X \bullet (Q + \sum_i \lambda_i A_i - Y) + x^T (q + \sum_i \lambda_i a_i + \mu - 2y) \end{aligned} \tag{8}$$

And thus the dual,

$$\begin{aligned} \min_{\lambda,\mu,y,Y} \quad & \lambda^T b + \mu^T e + 1 \\ \text{s.t.} \quad & Y = Q + \sum_i \lambda_i A_i \\ & q + \sum_i \lambda_i a_i + \mu - 2y \leq 0 \\ & Y \succeq yy^T \\ & \lambda, \mu \geq 0 \end{aligned} \tag{9}$$

1.2 SOCP Relaxation

We now consider another way of relaxing the original QCQP problem. Consider symmetric indefinite matrix $Q \in S^n$ and its spectral decomposition.

$$\begin{aligned} Q &= V \Lambda V^T = \sum_j^n \lambda_j v_j v_j^T \\ \Lambda &= \mathrm{diag}(\lambda) \end{aligned} \tag{10}$$

Without loss of generality, we assume first r eigenvalues are positive, $\lambda_1, ..., \lambda_r \geq 0, r \leq n$. The quadratic form $x^T Q x$ can also be partitioned into positive and negative parts:

$$x^T Q x = \sum_{j=1}^r \lambda_j x^T v_j v_j^T x + \sum_{j=r+1}^n \lambda_j x^T v_j v_j^T x \tag{11}$$

By letting $s_j \geq z_j^2, z_j = v_j^T x, j = 1, \cdots, n$, then (11) can be expressed by introducing n (2-d) quadratic cones, literally:

$$x^T Q x \leq \sum_j s_j \cdot \lambda_j, \ (s_j, v_j^T x) \in \mathcal{Q}^2 \tag{12}$$

This substitution uses a set of small quadratic cones instead of one semidefinite matrix of size n^2. With some abuse of notation, suppose $\lambda_i, V_i, i = 0, \cdots, m$ are eigenvalues and vectors for Q and $A_i, i = 1, \cdots, m$, respectively. Following the same routine for each constraint, we describe the Many-Small-Cone (MSC) relaxation to QCQP, namely,

$$\begin{array}{llll}
(\mathbf{MSC}) & \text{Maximize}: & y_0^T \lambda_0 + q^T x & \\
& \text{s.t.} & V_i z_i = x & i = 0, ..., m \\
& & y_i^T \lambda_i + a_i^T x \leq b_i & i = 1, ..., m \\
& & y_i \geq z_i \circ z_i & i = 0, ..., m \\
& & y_i^T e \leq x^T e & i = 0, \cdots, m
\end{array} \tag{13}$$

The last set of constraints are placed to resolve unboundedness for the fact that the similarity transformation by any orthogonal basis $V_i, \forall i$ preserves the value of $\mathbf{trace}$ operator, namely:

$$y_i^T e = \mathbf{trace}(V_i^T x x^T V_i) = \mathbf{trace}(x x^T) \leq x^T e \tag{14}$$

This method is closely related to D.C. and Convex SOCP relaxations to QCQP, see [**?**], [**?**], [**?**], [**?**]. Recently, [**?**] mention a similar formulation, by defining $C_i = V_i \mathbf{diag}(\sqrt{|\lambda_i|})$. We list it below for convenience.

$$\begin{array}{llll}
(\mathbf{MSC\text{-}Luo}) & \text{Maximize}: & y_0^T e + q^T x & \\
& \text{s.t.} & C_i z_i = x & i = 0, ..., m \\
& & y_i^T \lambda_i + a_i^T x \leq b_i & i = 1, ..., m \\
& & y_i \geq z_i \circ z_i & i = 0, ..., m \\
& & y^T \dfrac{1}{|\lambda|} \leq x^T e & i = 0, \cdots, m
\end{array} \tag{15}$$

In [**?**], box constraints for z_i can be calculated by its definition. For the case where $x \in [0, 1]$, we show bounds are redundant and the two formulations are equivalent.

Theorem 1 *The relaxations* (13), (15) *are equivalent.*

Proof We show the solution for any one of the relaxations can be derived from another.
Suppose (x_0, z_0, y_0) is a feasible solution to (13), then $(x_0, \sqrt{\mathbf{diag}(|\lambda|)} z_0, \mathbf{diag}(|\lambda|) \cdot y_0)$ is feasible to (15). Conversely, if (x_0, z_0, y_0) is feasible to (15), then we can construct $(x_0, \frac{1}{\sqrt{\mathbf{diag}(|\lambda|)}} z_0, \frac{1}{\mathbf{diag}(|\lambda|)} \cdot y_0)$ that is also feasible to (13). □