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Relaxations for QCQP

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Contents

1 Relaxations for QCQP

As a convention, we assume data matrices are symmetric, i.e., $Q, A_i \in \mathcal{S}^n$

Recall homogeneous QCQP for $x \in \mathbb{R}^n$:

$$\begin{aligned} \text{Maximize} \quad & x^T Q x \\ \text{s.t.} \quad & x^T A_i x \ (\leq, =, \geq) \ b_i, \forall i = 1, \dots, m \\ & 0 \leq x \leq 1 \end{aligned} \tag{1}$$

And inhomogeneous QCQP,

$$\begin{aligned} \text{Maximize} \quad & x^T Q x + 2q^T x \\ \text{s.t.} \quad & x^T A_i x + 2a_i^T x \ (\leq, =, \geq) \ b_i \\ & 0 \leq x \leq 1 \end{aligned} \tag{2}$$

We mark some of the trivial techniques below.

One can always reformulate (2) into a homogeneous problem by increasing the dimension of variables by 1.

$$\begin{aligned} \text{Maximize} \quad & x^T Q x + 2q^T x \\ & = \begin{bmatrix} x^T & t \end{bmatrix} \begin{bmatrix} Q & q \\ q^T & o \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} \\ \text{s.t.} \quad & -1 \leq t \leq 1 \end{aligned} \tag{3}$$

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Also, one can solve a symmetric version since $x^T A x = x^T A^T x$ by letting,

$$\tilde{A} := \frac{A + A^T}{2} \tag{4}$$

1.1 SDP Relaxation

For $x \in \mathbb{R}^n$, we have: $x^T A_i x = A_i \bullet (xx^T)$ and $xx^T \in \mathcal{S}_+^n$, which results in following relaxation using semidefinite cones, also called *lifting* method or Shor relaxation,

$$\begin{aligned} \text{Maximize} \quad & Q \bullet Y + 2q^T x \\ \text{s.t.} \quad & Y - xx^T \succeq 0 \text{ or } \begin{bmatrix} 1 & x^T \\ x & Y \end{bmatrix} \succeq 0 \\ & A_i \bullet Y + 2a_i^T x \; (\le, =, \ge) \; b_i, \forall i \\ & 0 \le x \le 1 \end{aligned} \tag{5}$$

Notice QCQP with matrix variables can also be reformulated into a SDP based problem, let $X \in \mathbb{R}^{n \times d}$, then $X^T A_i X = A_i \bullet (XX^T)$

$$\begin{aligned} \text{Maximize} \quad & Q \bullet Y \\ \text{s.t.} \quad & Y - XX^T \succeq 0 \text{ or } \begin{bmatrix} I_d & X^T \\ X & Y \end{bmatrix} \succeq 0 \\ & A_i \bullet Y \; (\le, =, \ge) \; b_i, \forall i \end{aligned} \tag{6}$$

SDP relaxation (5) can be unbounded in some case. A simple improvement to (5) is to add bounds for the diagonal entries.

$$\begin{aligned} \text{Maximize} \quad & Q \bullet Y + 2q^T x \\ \text{s.t.} \quad & Y - xx^T \succeq 0 \text{ or } \begin{bmatrix} 1 & x^T \\ x & Y \end{bmatrix} \succeq 0 \\ & A_i \bullet Y + 2a_i^T x \; (\le, =, \ge) \; b_i, \forall i \\ & 0 \le x \le 1 \\ & \operatorname{diag}(Y) \le x \end{aligned} \tag{7}$$

There are many further enhancements to (7), see [**?**] for discussion on the strength of different relaxations. Here we discuss a few widely used methods using copositive cones and reformulation-linearization-techniques (RLT) cuts.

Consider the dual of primal SDP relaxation.

$$\begin{aligned} L = & \max_{x,X} \; Q \bullet X + q^T x + \sum_i \left(\lambda_i A_i \bullet X + \lambda_i a_i^T x - \lambda_i b_i \right) + \mu^T x - \mu^T e \\ & - \begin{bmatrix} X & x \\ x^T & 1 \end{bmatrix} \bullet \begin{bmatrix} Y & y \\ y^T & 1 \end{bmatrix} \\ = & \max_{x,X} \; -\lambda^T b - \mu^T e - 1 + X \bullet (Q + \sum_i \lambda_i A_i - Y) + x^T (q + \sum_i \lambda_i a_i + \mu - 2y) \end{aligned} \tag{8}$$

And thus the dual,

$$
\begin{aligned}
\min_{\lambda,\mu,y,Y} \quad & \lambda^T b + \mu^T e + 1 \\
\text{s.t.} \quad & Y = Q + \sum_i \lambda_i A_i \\
& q + \sum_i \lambda_i a_i + \mu - 2y \leq 0 \\
& Y \succeq y y^T \\
& \lambda, \mu \geq 0
\end{aligned}
\tag{9}
$$

1.2 SOCP Relaxation

We now consider another way of relaxing the original QCQP problem. Consider symmetric indefinite matrix $Q \in \mathcal{S}^n$ and its spectral decomposition.

$$
\begin{aligned}
& Q = V \Lambda V^T = \sum_j^n \lambda_j v_j v_j^T \\
& \Lambda = \operatorname{diag}(\lambda)
\end{aligned}
\tag{10}
$$

Without loss of generality, we assume first r eigenvalues are positive, $\lambda_1, ..., \lambda_r \geq 0, r \leq n$. The quadratic form $x^T Q x$ can also be partitioned into positive and negative parts:

$$
x^T Q x = \sum_{j=1}^r \lambda_j x^T v_j v_j^T x + \sum_{j=r+1}^n \lambda_j x^T v_j v_j^T x \tag{11}
$$

By letting $y_j \geq z_j^2, z_j = v_j^T x, j = 1, \cdots, n$, we introduce n (small) quadratic cones, then (11) can be rewritten as:

$$
x^T Q x \leq \lambda^T y, \ (y_j, v_j^T x) \in \mathcal{Q}^2 \tag{12}
$$

A natural Many-Small-Cone (MSC) relaxation to QCQP can be written as: