SOCP Relaxations for QCQP

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The QCQP

We consider the QCQP,

$$\begin{array}{llll} (\textbf{QCQP}) & \max & x^T Qx + q^T x & \\ & \text{s.t.} & x^T A_i x + a_i^T x \, (\leq, =, \geq) \; b_i & \\ & & 0 \leq x \leq 1 & \end{array} \tag{1}$$

For simplicity we first consider $m = 0$, i.e.,

$$\begin{array}{lll} (\textbf{QCQP}) & \max & x^T Qx + q^T x \\ & \text{s.t.} & 0 \leq x \leq 1 \end{array} \tag{2}$$

Convex Relaxations with SOCP I

Now assume we use second-order cones. No lifted matrix $X \succeq 0$ is allowed.
Method 1, (MSC) Many-small-cones, based on diagonalization, then we approximate on 1-D quadratic functions that form the many but smaller (actually 2D) smaller cones. Use Spectral decomposition, if $Q = V \Gamma V^T$, then assume V is full rank, the nonconvexity moves to the norm constraints.

$$\begin{aligned} V_i^T x = z_i, i = 1, \ldots, m \\ y_i = z_i \circ z_i \Rightarrow y^T e = \|x\|^2 \end{aligned} \tag{3}$$

Method 2, (Scaling) For any indefinite quadratic matrix, scale by identity matrices.

$$\begin{aligned} & x^T (Q_i + t_i I_n) x + q_i^T x \leq b_i + t_i \cdot s, i = 1, \ldots, m \\ & \|x\|^2 = s \end{aligned} \tag{4}$$

Then bound s as whole or let $\sum_i \rho_i = s$ and bound ρ separately.

Convex Relaxations with SOCP II

More Complex... Use D.C., i.e., $Q = Q_+ - Q_-$ where two parts a negative semi-definite (for max problem) or psd (for min problem). Since it is a max problem, let,

$$
\begin{aligned}
&\dagger\ z \leq \rho - x^T Q_- x + q^T x \\
&\ddagger\ \rho \leq x^T Q_+ x
\end{aligned}
$$

Then naturally we can at least come up with two convexification,

1. As before, let $(x-l)^T(x-u) \leq 0$ to convexify $\ddagger$, equivalent to compute $\lambda_{\max}(Q_+)$, this would be equivalent to Scaling and MSC
2. Use $\dagger$ to convexify $\ddagger$. i.e., find α, such that,

$$
\alpha \cdot Q_- - Q_+ \succeq 0
$$

Convex Relaxations with SOCP III

Summary

1. All mentioned above can be viewed as construction of convex hull of two quadratic sets. And can be covered by the scheme in [BKK17]
2. Can also be interpreted from the cutting plane scheme by [BM14]. For example, in the D.C.,

$$
\begin{aligned}
&\dagger \; z \leq \rho - x^T Q_- x + q^T x \\
&\ddagger \; \rho \leq x^T Q_+ x
\end{aligned}
$$

it views $\ddagger$ as reverse convex set of the paraboloid.

$$\ddagger \;\Leftrightarrow \{x : x \in \mathbb{R}^n - P\}, P = \{(x, \rho) : x^T Q_+ x \leq \rho\} \tag{5}$$

Convex Relaxations with SOCP IV

At each solution it cuts off $(\bar{x}, \bar{\rho}) \in P$ by a linear inequality. Lift a first order underestimation,

$$(\dagger)\ \rho - z \geq (2Q_{-}\bar{x} - q)^T (x - \bar{x}) + \bar{x}^T Q_{-}\bar{x} - q^T x, \forall \bar{x}$$

Lifting: $\dagger$

$$\begin{aligned} \rho - z \geq\ & \bar{x}^T Q_{-}\bar{x} - q^T \bar{x} \\ & + \left(\begin{bmatrix} 2Q_{-}\bar{x} - q \\ 0 \end{bmatrix} - \alpha \begin{bmatrix} 2Q_{+}\bar{x} \\ -1 \end{bmatrix} \right)^T \begin{bmatrix} x - \bar{x} \\ \bar{\rho} - \bar{x}^T Q_{+}\bar{x} \end{bmatrix} \end{aligned}$$

Convex Relaxations with SOCP V

Theorem, [BM14]. $\alpha^\star = 1/\lambda_{\max}(Q_+)$, then lifting inequality is the tightest.

This can be seen as supporting hyperplanes for the convex hull of $\dagger, \ddagger$. If we already convexify and deploy the SOCP to it, the cutting planes will be useless.

In conclusion, the different schemes are actually equivalent, in different views of 1. convex hull by aggregation, 2. cutting plane to reverse convex set.

Convexification with the Box Constraint I

In a B-C framework (simple case when x is restricted to a small box $x \in [l, u]$),
we use n-dimensional RLT cut by a doubly nonnegative matrix $B, B \succeq 0, B \geq 0, \text{ s.t. }$,

$$(x - l)^T B(x - u) \leq 0$$

1. to convexify a $A \rightarrow$ by B solve α

$$\begin{aligned} & x^T Ax + a^T x + b \leq 0 \\ \Leftrightarrow \quad & \{\alpha > 0 : \alpha B + A \succeq 0\} \end{aligned}$$

interpretation: S-lemma.

Convexification with the Box Constraint II

2. if $B = I_n$, it is equivalent to scaling, solve $\lambda_{\max}(A)$
3. if do D.C., adopt a DNN decomposition,

$$B = Q_+ - Q_-, \ , \text{s.t. } , Q_+ \succeq 0, Q_- \succeq 0, Q_- \geq 0 \tag{6}$$

   do not have much improvements.
4. in MSC, we do not need n-d RLTs, since we have $(x_i - l_i)(x_i - u_i) \leq 0$ for every i, this includes all n-d cuts.

Up to now, we explore,

- convexification by two quadratic sets
- possible implementation in B-C, previously, use $B = I$ to convexify all...
- test on unconstrained case (with box only), $B = I$ is already the best in most test case, need more analysis.

Questions & Future Work I

Focus on box constrained case,

1. for a general optimization model,

$$\begin{aligned} & f(x), \text{ s.t. } x \in S, S \text{ compact} \\ \Rightarrow? \quad & \max z, \text{ s.t. } (x, z) \in \mathbf{conv}(S \cap \{(x, z) : z \le f(x)\}) \end{aligned}$$

2. give extreme points of a A and the box
3. incorporate nonlinear intersection cut, [MV17]

Consider mixed cones, use X in some ways,

1. RLT on off-diagonal entries, $(x_i - l_i)(x_j - u_j) \le 0$
2. Strengthen by [YZ03] ?

$$\begin{aligned} & a^T x \le b, \|x\| \le \sqrt{\delta}, \\ \Rightarrow & \|(b - a^T x)x\| = \|bx - Xa\| \le \sqrt{\delta}(b - a^T x) \end{aligned}$$

[GDF+14], handle the case where δ is a variable.

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