A Review on Recent Developments in Quadratic Optimization and SOCP Based Method for QCQP

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The QCQP

We consider the QCQP,

$$
\begin{aligned}
(\textbf{HQCQP}) \quad \max \quad & x^\top Q x \\
\text{s.t.} \quad & x^\top A_i x \; (\leq, =, \geq) \; b_i, \forall i = 1, \ldots, m \\
& 0 \leq x \leq 1
\end{aligned}
\tag{1}
$$

and inhomogeneous QCQP,

$$
\begin{aligned}
(\textbf{QCQP}) \quad \max \quad & x^\top Q x + q^\top x \\
\text{s.t.} \quad & x^\top A_i x + a_i^\top x \; (\leq, =, \geq) \; b_i \\
& 0 \leq x \leq 1
\end{aligned}
\tag{2}
$$

SDR to QCQP I

$$
\begin{aligned}
(\textbf{Shor}) \quad \max \quad & Q \bullet Y + q^\top x \\
\text{s.t.} \quad & Y - xx^\top \succeq 0 \text{ or } \begin{bmatrix} 1 & x^\top \\ x & Y \end{bmatrix} \succeq 0 \\
& A_i \bullet Y + a_i^\top x \ \leq \ b_i, \forall i \\
& 0 \leq x \leq 1
\end{aligned} \tag{3}
$$

The "homogenization" trick can also be used, by adding an auxillary variable $t \in \{-1, 1\}$,

$$
\begin{aligned}
(\textbf{HShor}) \quad \max \quad & \begin{bmatrix} Q & q/2 \\ q^\top/2 & o \end{bmatrix} \tilde{X} \\
\text{s.t.} \quad & \tilde{X} = \begin{bmatrix} x \\ t \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix}^\top
\end{aligned} \tag{4}
$$

- Provides good relaxation bound, some cases are exact. [YZ03], [SZ03], [BY19], [WJ21]

SDR to QCQP II

- Randomized/deterministic method to project back $X^* \to x^*$.
- Fruitful results using simultaneous diagonalizations, [Jia16], [WJ21]

Lift-and-Project: LP based Method I

A typical BB framework using lifted matrix X...[AHJS00], [Lin05]

1. Solve the lifted formulation without $X \succeq 0$
2. Spatial branching such as $x \leq l \vee x \geq u$
3. Cutting planes
   - 1 minimize $\|X^n - xx^\top\|$ (rank-one approximation)
   - 2 ensure $X \succeq 0$ SDP cuts, for example, ensure $2 \times 2, 3 \times 3$ matrix minors are PSD.
   - 3 and so on..

Another set of important cutting planes for Lift-and-Project is the McCormick envelop (or RLT), for example, if $x \in [l, u]$

$$(x - l)(x - u)^\top \leq 0 \tag{5}$$

Lift-and-Project: LP based Method II

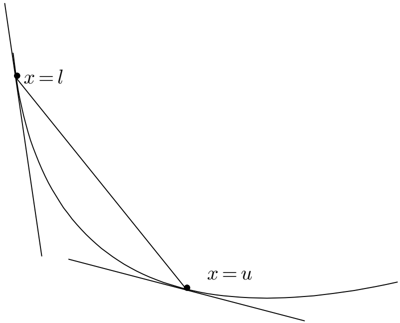

Figure: Illustration of RLT

Lift-and-Project: SDP based Method I

Similar to LP framework except ...

1. $X \succeq 0$ is ensured.
2. Special structure such as chordal graph can be captalized
   - 1 [NFF$^+$03] and implementation in SDPA.
   - 2 [BYZ00] rank-one coeffient matrices and dual scaling. (DSDP)

Problems in Lift-and-Project

- Scales badly using lifted $X = xx^\top$ if $x \in \mathbb{R}^n$ as n increases...
- LP relaxation produces mediocre bounds (without PSD)
- SDP is hard to warm-start (important for BB to reuse the nodes)

Motivation: SOCP I

SOCP relaxation can be seen as a middle ground.

- SOCP ($n + c$) is expected to be faster than L-and-P ($n \times n$)
- Recent warmstart schemes for homogeneous self-dual conic program, good news for BB, [SAY13]

Key problems to solve

- **(upper bound)** Convexification techniques for an indefinite homogeneous quadratic function by Q

$$S = \{(z, x) \mid x^\top Qx \leq z\} \tag{6}$$

  Many alternatives but still a question to find "good" convexifications.

- **(upper bound)** Cutting planes other than RLT
- **(lower bound)** Efficient local method to find primal feasible solutions

Recent results related to QP

- Second-order Cone Representable (SOCPr) sets and hidden convexity. [BTDH14].
- Aggregation based convexification. For two quadratic functions, [BKK17], [MV17]. For three quadratic functions [DMS21]
- Intersection of a quadratic set and polytope. [SD20]
- Maximal S-free sets, [CCD+15], [Mic13]. Note [Mic13] also discuss how to lift first-order cut planes for general qudratic function over nonconvex sets.
- †Simultaneous Diagonalization ..., [Jia16], [WJ21], [WKK20]
- †Disjunctive programming

† not yet covered.

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The intersection of two quadratic sets, examples I

A convex example,

$$
\begin{aligned}
&x^2 + y^2 \leq z \\
&(x-1)^2 + (y-1)^2 + z^2 \leq 5
\end{aligned}
$$

The intersection of two quadratic sets, examples II

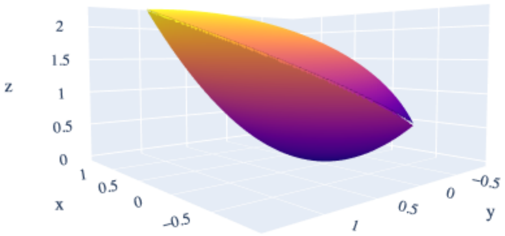

Figure: Example of a paraboloid vs ellipsoid (convex)

The intersection of two quadratic sets, examples III

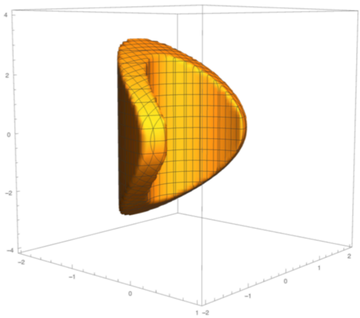

Figure: Example of three quadratic functions by [DMS21] and constructed convex hull

Definitions I

Definition

(Aggregation of inequalities) Given a set of inequalities

$$X = \{x : f_i(x) \leq 0, i = 1, \ldots, m\}$$

An aggregation by $\lambda \in \mathbb{R}_+^m$ is defined as,

$$X(\lambda) = \left\{ x : \sum_i \lambda_i \cdot f_i(x) \leq 0 \right\} \tag{7}$$

Definition

(SOCPr sets) A cone F_+ is SOCr if it can be expressed as $F^+ = \left\{x : \left\|B^T x\right\| \leq b^T x\right\}$, where nonzeros columns in B are independent and $b \notin range(B)$

Aggregation for two inequalities

Consider intersection of two quadratic function.

$$F_0 = \{x : x^\top A_0 x \le 0\}, F_1 = \{x : x^\top A_1 x \le 0\}$$
$$F_s = \{x : x^\top A_s x \le 0\}, A_s = sA_1 + (1-s)A_0$$

- Suppose F_0 is SOCr, F_1 is a general quadratic set.
- [BKK17] shows a unified approach to convexify intersection. Similar results in [MV17]. Useful in Trust-region subproblems.

Aggregation for two inequalities I

(Burer), cf. [BKK17]. Consider two quadratic inequalities and convex aggregation

$$F_0 = \{x : x^T A_0 x \leq 0\}, F_1 = \{x : x^T A_1 x \leq 0\}$$
$$F_s = \{x : x^T A_s x \leq 0\}, A_s = sA_1 + (1-s)A_0$$

Assume,

- A_0 has at least one positive eigenvalue and exactly one negative eigenvalue, and can be decomposed into,
$$A_0 = BB^T - bb^T \tag{8}$$
where nonzero columns of B are independent, $b \notin (B)$, such that we define,
$$F_0^+ = \{x : x^T BB^T x \leq b^T x\} \tag{9}$$
- $F_0 \cap F_1$ has nonempty interior.
$$\textbf{int}(F_0 \cap F_1) \neq \emptyset \tag{10}$$

Aggregation for two inequalities II

- A_0 is nonsingular, or either A_1 is positive or negative definite on null space of A_0; and s is computed in the following way,

$$T = \{s : A_s \text{has single negative eigenvalue}\}$$

$$s = \begin{cases} 0 & A_1 \text{ n.d in null space of } A_0 \\ \min(T \cap (0,1]) & \text{else.} \end{cases}$$

Then $\textbf{cl.conic.hull}(F_0^+ \cap F_1) \subseteq F_0^+ \cap F_s^+$

- General Rule: Find a good F_0 then attach to the F_1 of interest.
- Application, (Trust Region Subproblem) $F_0 = \{x : x^T x \leq \delta\}$

Aggregation for other cases

- [SD20] consider one Q-constraint over a polytope, applied in a bilinear bipartite graph problem.
- [DMS21] shows conditions for existence of aggregation for three quadratic constraints.
- Awkward situation for arbitrary number of Q-constraints

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Maximal S-Free and Intersection cut I

- Recently, use reverse convex approach is a hot topic in IP.

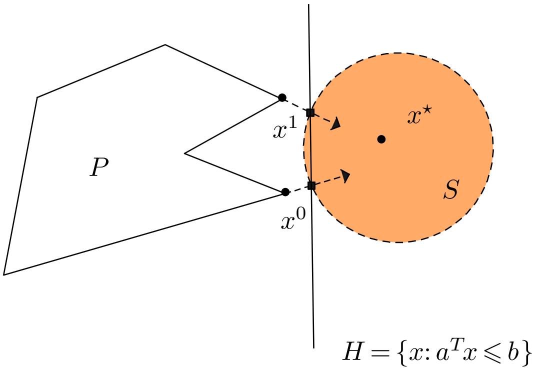

Figure: An illustration of intersection cut using linear relaxation and quadratic free sets

Maximal S-Free and Intersection cut II

- In LP relaxation, use basic solution (extreme points), and rays to find the intersection point with a convex set S
- Then any point of $\mathbb{R}^n - \mathbf{int}(S)$ can be cut off.
- Easy to construct by solving an 1-d. equation. [CMS21]
- Can also be strengthened by lifing.

Lifing for Mixed-Integer Programs

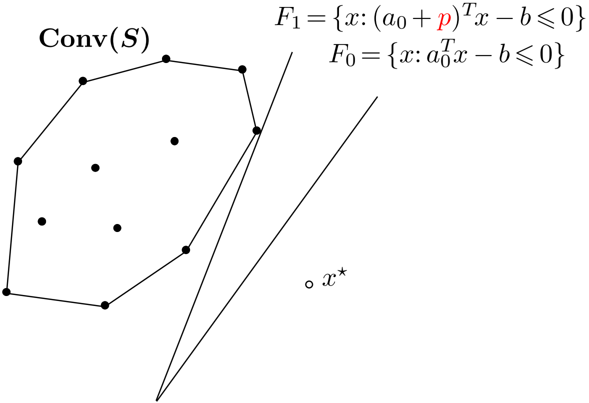

Figure: An illustration for lifing in MILP

Maximal S-Free and Lifting I

What if no LP (no rays and extreme points)?

- Find piecewise linear understimations and lifting.
- e.g., Consider $z = x^T Qx$ convex and minimize over $R^n - \textbf{int}(P)$, P is a good convex set like polytope.

$$\begin{aligned} z &\geq x^T Qx \\ &\geq 2y^T Q(x-y) + y^T Qy, \forall y \\ \textbf{By: } \nabla z_x &= 2Qx, \forall x \end{aligned}$$

- Too weak in most cases.

Maximal S-Free and Lifting II

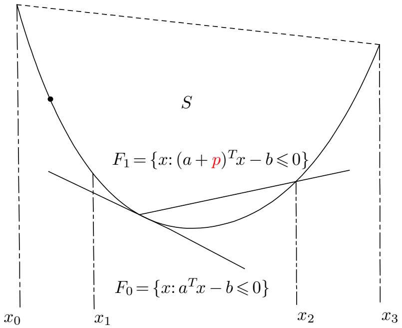

Figure: Lifting a piecewise linear model. Suppose S is the epigraph. $S = \{z \geq x^T Qx\}, x \in [x_0, x_1] \cup [x_2, x_3]\}$ Then the original first-order cut F_0 can be lifted to pass the border $(x = x_2)$. F_1 shows the lifted version.

Maximal S-Free and Lifting III

- Lifting inequality,

$$\begin{aligned} z &\geq x^T Qx \\ &\geq 2y^T Q(x-y) + y^T Qy, \forall y \\ \textbf{Lifting:} \quad \dagger\, \alpha, p \quad &\geq (2y^T Q + \alpha \cdot p^T)(x-y) + y^T Qy, \forall y \end{aligned}$$

- Get a lifting inquality is equivalent to solve an optimization problem ... where † is feasible for all $y \notin P$,

$$\begin{aligned} \hat{\alpha}(p) &\doteq \max_{\alpha}\{\alpha : \dagger \text{ is valid for } R^n - \textbf{int}(P)\} \\ &\Rightarrow \min_{\alpha}\{\alpha : \dagger \text{ is invalid for } \textbf{int}(P)\} \end{aligned}$$

Sounds terrifying.

Maximal S-Free and Lifting, (cont.)

Good news, :), $\hat{\alpha}(p)$ is tractable for some special P !

- [Mic13], [BM14] shows $\hat{\alpha}(p)$ is polynomially solvable for P is an ellipsoid, bounded polyhedron, etc.; p is required to "point into" $\mathbf{int}(P)$
- Missing closed-form solutions, may have to solve optimization problem when needed.

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Application: construct stronger convex relaxations I

For unconstrained QCQP, max $x^T Qx$ can be seen the intersection of convex and nonconvex sets,

$$F_1 = \{(x, z) : z - x^\top Qx - q^\top x \leq 0\}$$
$$F_0 = \{(x, s) : \|x\|^2 \leq s\}$$

Scaling, choose some $t > 0$ such that $tI_n - Q \succeq 0$

$$\begin{aligned} &\max z \\ &\text{s.t. } z + x^\top (tI - Q)x - q^\top x - t \cdot s \leq 0 \\ &\qquad \|x\|^2 \leq s \end{aligned}$$

- One can verify $t = \lambda_{\max}(Q)$ is the best choice, creating the convex hull of $F_0 \cap F_1$, [BKK17], [MV17]
- Easy to construct similar sets for quadratic constraints

Application: construct stronger convex relaxations II

For the case where s is unknown, but if we know $x \in [l, u]$
Use any $\Sigma \succeq 0$, and replace $\|x\|^2 \leq s$
Consider the intersection of the following,

$$F_1 = \{(x, z) : z - x^\top Qx - q^\top x \leq 0\}$$
$$F_0 = \{x : (x - l)^\top \Sigma(x - u) \leq 0\}$$

Similar procedure can be used to construct **conv**$(F_0 \cap F_1)$

- Limited improvement, need combinations of 2 or 3 functions.
- How to construct F_0 in a smart way is still a question.

Q&A

END

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