A Review on SOCP Relaxations for QCQP

Chuwen Zhang

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The QCQP

We consider the QCQP,

$$\begin{aligned} (\textbf{HQCQP}) \quad \max \quad & x^T Q x \\ \text{s.t.} \quad & x^T A_i x \,(\leq, =, \geq)\, b_i, \forall i = 1, \ldots, m \\ & 0 \leq x \leq 1 \end{aligned} \tag{1}$$

and inhomogeneous QCQP,

$$\begin{aligned} (\textbf{QCQP}) \quad \max \quad & x^T Q x + q^T x \\ \text{s.t.} \quad & x^T A_i x + a_i^T x \,(\leq, =, \geq)\, b_i \\ & 0 \leq x \leq 1 \end{aligned} \tag{2}$$

Dilemma of SOCP relaxations

Scale every indefinite inequalities.

$$
\begin{aligned}
&\max z \\
&\text{s.t. } z - x^T Q x \leq 0 \\
&\qquad x^T(A_i)x + a_i^T x \leqslant b_i \cdot s, i = 1, \ldots, m
\end{aligned}
\tag{3}
$$

In the above case, let $t_i = \lambda_{\max}(Q_i)$, convexify by maximum eigenvalue. Call this **SCALE**

$$
\begin{aligned}
z + x^T(t_i I_n - Q_i x) &\leq t_i s \\
\|x\|^2 &= s
\end{aligned}
$$

1. Relaxation bounds are worse than SDR
2. Cannot produce feasible solutions along the way
3. Need more cutting planes

Recent results related to QP

1. **Aggregation** Second-order Cone Representable (SOCPr) sets and hidden convexity. [BTDH14].
2. **Aggregation** Aggregation based convexification. For two quadratic functions, [BKK17], [MV17]. For three quadratic functions [DMS21]
3. **Aggregation** Intersection of a quadratic set and polytope. [SD20]
4. **Maximal-S-Free** Maximal S-free sets, [CCD+15], [Mic13]. Note [Mic13] also discuss how to lift first-order cut planes for general qudratic function over nonconvex sets.
5. †Simultaneous Diagonalization ..., [Jia16], [WJ21], [WKK20]
6. †Disjunctive programming

† 5, 6 not yet covered. 1, 2 can be used to tighten convex relaxation. 4. use to generate cuts. 3. to be investigated.

Aggregation: idea I

Definition

(Aggregation of inequalities) Given a set of inequalities

$$X = \{x : f_i(x) \leq 0, i = 1, \ldots, m\}$$

An aggregation by $\lambda \in \mathbb{R}^m_+$ is defined as,

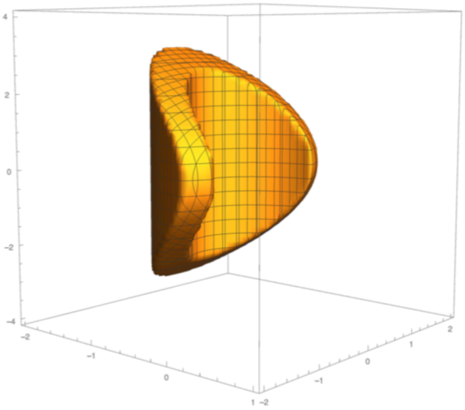
$$X(\lambda) = \left\{ x : \sum_i \lambda_i \cdot f_i(x) \leq 0 \right\} \tag{4}$$

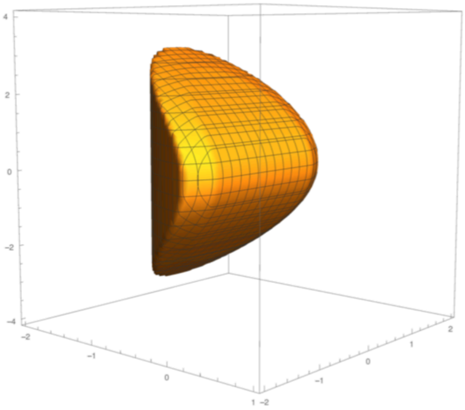
Definition

(SOCPr sets) A cone F_+ is SOCr if it can be expressed as $F^+ = \left\{x : \left\|B^T x\right\| \leq b^T x\right\}$, where nonzeros columns in B are independent and $b \notin range(B)$

Aggregation: idea II

Some Illustration (Three quadratic functions) Example by [DMS21] and constructed convex hull





Aggregation for two inequalities

Consider intersection of two quadratic function.

$$F_0 = \{x : x^T A_0 x \leq 0\}, F_1 = \{x : x^T A_1 x \leq 0\}$$
$$F_s = \{x : x^T A_s x \leq 0\}, A_s = sA_1 + (1-s)A_0$$

- Suppose F_0 is SOCr, F_1 is a general quadratic set.
- (2017) [BKK17] shows a unified approach to convexify intersection, similar results in [MV17], useful in Trust-region subproblems. problems like $\|x\|^2 \leq s$
- (2020) [SD20] consider one Q-constraint over a polytope, applied in a bilinear bipartite graph problem.
- (2021) [DMS21] shows conditions for existence of aggregation for three quadratic constraints.
- No easy answer to intersection of arbitrary number of Q-constraints

Some potential improvements to SOCP relaxations I

Problem: **SCALE** method is actually producing convex hull of,

$$\begin{aligned} x^T Q_i x + q_i x + b_i &\leq 0 \\ \|x\|^2 &\leq s \end{aligned}$$

verified by [BKK17], but we have to estimate s.
Workaround: use box $x \in [l, u]$ to replace $\|x\|^2 \leq s$, for any $\Sigma \geq 0$

$$(x - l)^T \Sigma (x - u) \leq 0 \tag{5}$$

When $\Sigma \succeq 0$, it is an ellipsoid covering the box.

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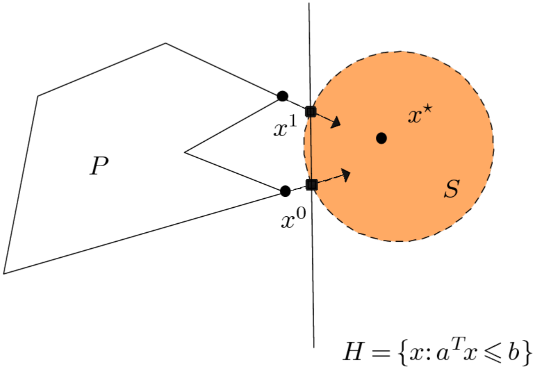
Maximal S-Free and intersection cuts

Linear relaxation and quadratic free, [CMS21], illustration



Maximal S-Free and lifting

Lift a first order underestimation,
Consider $x^T Qx$ convex and minimize over $R^n - \mathbf{int}(P)$, P is a good convex set like polytope.

$$\begin{aligned} z &\geq x^T Qx \\ &\geq 2y^T Q(x-y) + y^T Qy, \forall y \\ \textbf{Lifting:} \quad \dagger\ \alpha, p \quad &\geq (2y^T Q + \alpha \cdot p^T)(x-y) + y^T Qy, \forall y \end{aligned}$$

Get α, p by solving an optimization problem. $\dagger$ is feasible for all $y \notin P$ **LFO**

Future Work

1. ~~Implement the workaround for $\|x\|^2 \leq s$~~
2. In **LFO** and **convexification**, both has to figure out a proper convex set P.
3. Summarize the readings in disjunctive and simultaneous diagonalization.

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