Global QCQP Solver

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SDP relaxations: Method-I

Recall relaxation Method-I:
for $X \in \mathbb{R}^n$, homogeneous case:

$$
\begin{aligned}
\text{Maximize} \quad & Q \bullet Y \\
\text{s.t.} \quad & Y - xx^T \succeq 0 \text{ or } \begin{bmatrix} 1 & x^T \\ x & Y \end{bmatrix} \succeq 0 \\
& A_i \bullet Y (\leq, =, \geq) b_i, \forall i
\end{aligned}
\tag{1}
$$

Method-I: Inhomogeneous Case

for inhomogeneous case:

$$
\begin{aligned}
\text{Maximize} \quad & Q \bullet Y + 2q^T x \\
\text{s.t.} \quad & Y - xx^T \succeq 0 \text{ or } \begin{bmatrix} 1 & x^T \\ x & Y \end{bmatrix} \succeq 0 \\
& A_i \bullet Y + 2a_i^T x (\leq, =, \geq) b_i, \forall i
\end{aligned} \tag{2}
$$

Method-I: Inhomogeneous Case, Compact

We homogenize by letting $y = (x; t)$,

$$\begin{array}{rl} \text{Maximize} & y^T \begin{bmatrix} Q & q \\ q^T & 0 \end{bmatrix} y \\ \text{s.t.} & y^T \begin{bmatrix} A_i & a_i \\ a_i^T & 0 \end{bmatrix} y \, (\leq, =, \geq) \; b_i, \forall i \end{array} \tag{3}$$

and SDP accordingly,

$$\begin{array}{rl} \text{Maximize} & \begin{bmatrix} Q & q \\ q^T & 0 \end{bmatrix} \bullet Y \\ \text{s.t.} & \begin{bmatrix} A_i & a_i \\ a_i^T & 0 \end{bmatrix} \bullet Y \, (\leq, =, \geq) \; b_i, \forall i \\ & \begin{bmatrix} 1 & y^T \\ y & Y \end{bmatrix} \succeq 0 \end{array} \tag{4}$$

Modifications

Unboundedness:

- We found this unbounded on QKP (quadratic knapsack problem). May also be the case for QP with linear constraints only: $A_i = 0, \forall i$
- Solution: add $\mathsf{diag}(Y) = x$

Box Constraints:

- For $0 \leq x \leq e$, we can use $\mathsf{diag}(Y) = x$
- For $l \leq x \leq u$, we can use $\mathsf{diag}(Y) = x$ + linear transformation P such that: $Pz = x, 0 \leq z \leq e$

Example: QKP

The quadratic knapsack problem, we add $\mathsf{diag}(Y) = x$

$$\begin{aligned} \text{Maximize} \quad & x^T Q x + 2 q^T x \\ \text{s.t.} \quad & a_i^T x \leq b_i, \forall i \\ & 0 \leq x \leq e \end{aligned} \tag{5}$$

Adopt SDP: Method-I (2)

Example: Results $n = 100$

prob_num	method	solve_time	best_bound	best_obj
jeu_100_25_1	gurobi	0.33	18751.41	18528.00
	gurobi_rel	0.42	18906.74	18604.47
	sdp_qcqp1	19.11	-	23140.56
jeu_100_25_2	gurobi	0.04	56576.0	56121.00
	gurobi_rel	0.07	56576.0	56561.75
	sdp_qcqp1	22.80	-	57450.54
jeu_3_n_1	gurobi	0.00	20.0	20.00
	gurobi_rel	0.00	22.31	21.75
	sdp_qcqp1	0.00	-	24.55
jeu_10_100_13	gurobi	0.01	1612.03	1541.00
	gurobi_rel	0.00	1782.36	1735.25
	sdp_qcqp1	0.01	-	1865.68
jeu_2_100_1	gurobi	0.00	140.0	140.00
	gurobi_rel	0.00	140.0	140.00
	sdp_qcqp1	0.00	-	140.00

for full results refer to the excel.

Future Work

- Compare Method-I (normal (2) and compact (3))
- Test on box constraints, extend to d-dimensional x, i.e., matrices
- Test on SNL and other "real" QCQPs, use Gurobi as benchmark.
- Can we improve basic relaxation Method-I?[1]
- Implement Method-II

[1] Xiaowei Bao, Nikolaos V. Sahinidis, and Mohit Tawarmalani. "Semidefinite relaxations for quadratically constrained quadratic programming: A review and comparisons". In: *Mathematical programming* 129.1 (2011). Publisher: Springer, pp. 129–157.