Global QCQP Solver

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1 Semidefinite Relaxation

We consider two types of SDP relaxation for canonical QCQP. We first consider for the case where x is a vector, i.e., $x \in \mathbb{R}^n$.

$$\begin{aligned} \text{Maximize} \quad & x^T Q x \\ \text{s.t.} \quad & x^T A_i x \; (\leq, =, \geq) \; b_i, \forall i = 1, \ldots, m \end{aligned} \tag{1.1}$$

And for inhomogeneous QCQP,

$$\begin{aligned} \text{Maximize} \quad & x^T Q x + 2q^T x \\ \text{s.t.} \quad & x^T A_i x + 2a_i^T x \ (\leq, =, \geq) \ b_i \end{aligned} \tag{1.2}$$

for inhomogeneous case, we notice:

$$\begin{aligned} & x^T Q x + 2q^T x \\ & = \begin{bmatrix} x^T & t \end{bmatrix} \begin{bmatrix} Q & q \\ q^T & o \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} \\ \text{s.t.} \quad & 1 \leq tle1 \end{aligned}$$

we can use a homogeneous reformulation where the size of problem by 1.

1.1 Method I

1.1.1 Vector case

for $X \in \mathbb{R}^n$, we have: $x^T A_i x = A_i \bullet (xx^T)$

$$\begin{aligned} \text{Maximize} \quad & Q \bullet Y \\ \text{s.t.} \quad & Y - xx^T \succeq 0 \text{ or } \begin{bmatrix} 1 & x^T \\ x & Y \end{bmatrix} \succeq 0 \\ & A_i \bullet Y \ (\leq, =, \geq) \ b_i, \forall i \end{aligned} \tag{1.3}$$

1.1.2 Matrix case

for $X \in \mathbb{R}^{n \times d}$, we have: $X^T A_i X = A_i \bullet (XX^T)$

$$\begin{aligned} \text{Maximize} \quad & Q \bullet Y \\ \text{s.t.} \quad & Y - XX^T \succeq 0 \text{ or } \begin{bmatrix} I_d & X^T \\ X & Y \end{bmatrix} \succeq 0 \\ & A_i \bullet Y \ (\leq, =, \geq) \ b_i, \forall i \end{aligned} \tag{1.4}$$

1.1.3 Inhomogeneous

SDP relaxation,

$$\begin{aligned} \text{Maximize} \quad & Q \bullet Y + 2q^T x \\ \text{s.t.} \quad & Y - xx^T \succeq 0 \text{ or } \begin{bmatrix} 1 & x^T \\ x & Y \end{bmatrix} \succeq 0 \\ & A_i \bullet Y + 2a_i^T x \; (\leq, =, \geq) \; b_i, \forall i \end{aligned} \tag{1.5}$$

Alternative

The above formulation could be unbounded. We homogenize by letting $y = (x; t)$,

$$\begin{aligned} \text{Maximize} \quad & y^T \begin{bmatrix} Q & q \\ q^T & 0 \end{bmatrix} y \\ \text{s.t.} \quad & y^T \begin{bmatrix} A_i & a_i \\ a_i^T & 0 \end{bmatrix} y \; (\leq, =, \geq) \; b_i, \forall i \end{aligned} \tag{1.6}$$

1.2 Method II

Let $A = A_+ + A_-$ where $A_-, A_+ \succeq 0$, so we do Cholesky $R_+^T R_+ = A_+$, since R_+ may be low-rank, then we can define z_+ according the rank.

$$\begin{aligned} & R_+ x = z_+, x^T A_+ x = ||z_+||^2 = \sum_i (z_+)_i^2 \\ & y_i = (z_+)_i^2, \begin{bmatrix} 1 & (z_+)_i \\ (z_+)_i & y_i \end{bmatrix} \succeq 0, \forall i \end{aligned} \tag{1.7}$$

This is the so-called “many-small-cone” method.

1.3 Remark

1.3.1 Extending to matrix and tensor case

We first develop for the vector case: $x \in \mathbb{R}^n$, whereas QCQP is not limited to vector case:

- vectors, $x \in \mathbb{R}^n$, max-cut, quadratic knapsack problem
- matrices, $x \in \mathbb{R}^{n \times d}$, quadratic assignment problem, SNL, kissing number.

For example, SNL uses $X \in \mathbb{R}^{n\times d}$ for d-dimensional coordinates. For higher dimensional case, followings can be done:

- One may however using the vectorized method, i.e., $x = \mathtt{vec}(X)$ to reformulate the matrix-based optimization problem, given the SDP bounds by original and vectorized relaxations are equivalent with mile assumptions. (see Ding et al. (2011))
- the above method may create a matrix of very large dimension resulted from Kronecker product.
- ultimately, the solver should provide an option to use user specified relaxations.

1.4 Tests

We test on specific applications:

- vectors, $x \in \mathbb{R}^n$, max-cut, quadratic knapsack problem
- matrices, $x \in \mathbb{R}^{n\times d}$, QAP, SNL, kissing number.

and a recent general new library as present in `qplib`, `http://qplib.zib.de/instances.html`, see Furini et al. (2019).

		solve_time	best_bound	best_obj
prob_num	method			
0	gurobi	0.008010	1609.0	1609.000000
	gurobi_rel	0.006286	1781.581401	1781.581125
	sdp_helberg	0.006992	-	1865.679912
	sdp_qcqp1	0.006028	-	2526.824086
	sdp_qcqp1_no_x	0.005026	-	2018.968153

2 Branch and Cut Algorithm

For $x \in \mathbb{R}^n$, we have: $x^T A_i x = A_i \bullet (xx^T)$

$$\begin{aligned} \text{Maximize} \quad & Q \bullet Y \\ \text{s.t.} \quad & Y - xx^T \succeq 0 \text{ or } \begin{bmatrix} 1 & x^T \\ x & Y \end{bmatrix} \succeq 0 \\ & A_i \bullet Y \ (\leq, =, \geq) \ b_i, \forall i \end{aligned} \tag{2.1}$$

For MINLP and more specifically for QP, see Belotti et al. (2013), Misener and Floudas (2013). More recently, spacial branch-and-cut method Chen et al. (2017) for QP with complex variables.

Branching node and value selection, namely, select which $x_i, i = 1, ..., N$ and value α, for left and right children, for example:

$$x_i \leq \alpha, x_i \geq \alpha$$

- Audet et al. (2000) using RLT relaxation and LP, literally, $W_{ij} \approx x_i x_j, v_i \approx x_i^2$. The branching is essentially based on $\|W_{ij} - x_i x_j\|, \|v_i - x_i^2\|$, at each node we solve a linear programming relaxation.
- Linderoth (2005) a B-B based on subdividing the feasible region into the Cartesian product of triangles and rectangles.

Branching value:

Some source code to look at:

- Couenne: https://www.coin-or.org/Couenne/
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2.1 Root

The root of the problem can selected from different SDPs as we discussed before. It is a question to answer whether we should use the SDP with the tightest bound.

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Appendix