A QCQP Solver

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QCQP

Recall QCQP:

$$
\begin{aligned}
\text{Maximize} \quad & x^T Q x + q^T x \\
\text{s.t.} \quad & x^T A_i x + a_i^T x \; (\leq, =, \geq) \; b_i \\
& 0 \leq x \leq e
\end{aligned}
\tag{1}
$$

MSC

Consider MSC relaxation with spectral decomposition, $Q = V_0 \mathbf{diag}(\lambda_0) V_0^T, A_i = V_i \mathbf{diag}(\lambda_i) V_i^T$

$$
\begin{aligned}
(\mathbf{MSC}) \quad \text{Maximize :} \quad & y_0^T \lambda_0 \\
\text{s.t.} \quad & V_i^T x = z_i, \cancel{V_i z_i = x} && i = 0, ..., m \\
& y_i^T \lambda_i + a_i^T x \leq b_i && i = 1, ..., m \\
& y_i \geq z_i \circ z_i && i = 0, ..., m \\
& y_i^T e \leq x^T e && i = 0, \cdots, m
\end{aligned} \tag{2}
$$

Let Ω defines the set of second-order cones for $(x, \boldsymbol{y}, \boldsymbol{z})$

$$
\Omega = \left\{ (x, z, y) : \begin{array}{ll} V_i^T x = z_i & i = 0, ..., m \\ y_i^T \lambda_i + a_i^T x \leq b_i & i = 1, ..., m \\ y_i \geq z_i \circ z_i & i = 0, ..., m \end{array} \right\} \tag{3}
$$

MSC on the surface of norm balls

Optimal conditions for MSC,

- if $(y^*)^T e = \|x^*\|^2$, then (x^*, y^*) is the solution
- we notice $y_i^T e \geq \|z_i\|^2 = \|x\|^2$ is guaranteed.
- we want to have,

$$y_i^T e \leq \|x\|^2$$

  which is clearly nonconvex

Norm-constrained MSC (NMSC)

Notice,

$$\|x\|^2 = \max_{\|\xi\| \le \sqrt{s}} \xi^T x \tag{4}$$

So we add slack variable s, t, ξ and bilinear constraint.

$$\begin{aligned}
(\textbf{MSC}) \quad \text{Maximize}: \quad & y_0^T \lambda_0 && && (5)\\
\text{s.t.} \quad & (y, z, x) \in \Omega && && (6)\\
& y_i^T e \le t && i = 0, \cdots, m && (7)\\
(\kappa) \quad & t = s && i = 0, \cdots, m && (8)\\
(\mu) \quad & \xi^T x = t && && (9)\\
& \xi^T \xi \le s && && (10)
\end{aligned}$$

If s, t, ξ, y, z, x is the solution, then y, z, x is the optimal solution for MSC.

NMSC: the ADMM approach

This allows the augmented Lagrangian function,

$$\mathscr{L}(x, y, z, \xi, s, \kappa, \mu) = -y_0^T \lambda_0 + \kappa(t - s) + \mu(\xi^T x - t) + \frac{\rho}{2}(t - s)^2 + \frac{\rho}{2}(\xi^T x - s)^2$$

The ADMM iteration,

$$\begin{aligned}
(x, y, z, t)^{k+1} &= \arg\min_{(x,y,z)\in\Omega, t\geq 0} L\left(x, y, z, \xi^k, s^k, \kappa^k, \mu^k\right) \\
(s, \xi)^{k+1} &= \arg\min_{(s,\xi)\in\mathscr{Q}} L\left((x, y, z, t)^{k+1}, \xi, s, \kappa^k, \mu^k\right) \\
\kappa^{k+1} &= \kappa^k + \rho\left(t^{k+1} - s^{k+1}\right) \\
\mu^{k+1} &= \mu^k + \rho\left(\langle \xi^{k+1}, x^{k+1}\rangle - s^{k+1}\right)
\end{aligned}$$

where $\mathscr{Q}(\cdot)$ forms a simple SOCP for s, ξ,

$$\mathscr{Q}(x) = \left\{(s, \xi) : \|\xi\|^2 \leq s\right\} \tag{11}$$

Simple test on ADMM

	n:m:id	t	best_bound	best_obj	relax_obj	nodes	method
0	5:5:0	0.03	5.56	5.56	5.56	29.0	grb
1	5:5:0	8.44	5.56	5.56	5.56	171.0	admm_msc
0	50:20:0	200.00	189.34	87.69	189.34	839.0	grb
1	50:20:0	200.04	123.06	122.99	123.00	248.0	admm_msc
0	50:50:0	200.00	197.20	68.50	197.20	395.0	grb
1	50:50:0	200.39	159.97	157.23	157.36	86.0	admm_msc
0	100:20:0	400.00	777.92	90.51	777.92	65.0	grb
1	100:20:0	402.83	385.68	383.19	383.28	130.0	admm_msc
0	100:50:0	400.01	817.60	115.00	817.60	12.0	grb
1	100:50:0	406.29	367.47	358.75	359.59	61.0	admm_msc
0	200:5:0	1000.00	3205.11	111.11	3205.11	2.0	grb
1	200:5:0	1002.45	519.80	519.37	519.38	375.0	admm_msc
0	200:20:0	1000.01	4050.97	135.87	4050.97	1.0	grb
1	200:20:0	1006.92	528.21	519.58	519.88	74.0	admm_msc
0	QPLIB_1055	200.00	33.28	33.03	33.28	911.0	grb
1	QPLIB_1055	200.58	33.05	33.04	33.04	231.0	admm_msc