Global QCQP Solver

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SDP relaxations: Many-small-cone

Recall relaxation: Let $A = A^+ + A^-$ be symmetric where $A^-, A^+ \succeq 0$, which allows Cholesky decomposition,

$$U^+(U^+)^T = A^+$$

since U^+ may be low-rank, we can define z^+ accordingly,

$$\begin{aligned} (U^+)^T x &= z^+ \\ x^T A^+ x &= ||z^+||^2 = \sum_i (z^+)_i^2 \\ y_i &= (z_i^+)^2, \begin{bmatrix} 1 & z_i^+ \\ z_i^+ & y_i \end{bmatrix} \succeq 0, \forall i \end{aligned} \tag{1}$$

Decompose "Matrices"

One way to do this is:

- Symmetrize: re-define $Q := \frac{Q+Q^T}{2}$, by the fact that $x^T Q x = x^T (\frac{Q+Q^T}{2}) x$
- Compute eigenvalue decomposition:

$$Q = U \Gamma U^T$$

- Partition columns of U by $I^+ \in \mathbb{R}^{n\times n}, I^+_{ii} = 1$ if $\Gamma_i > 0$
- We define $U^+ = U \cdot \sqrt{(I^+) \cdot \Gamma}$, $U^- = U \cdot \sqrt{-(I^-) \cdot \Gamma}$
- We have:

$$Q = U^+ (U^+)^T - U^- (U^-)^T$$

Many-Small-Cone: MSC

$$
\begin{array}{lll}
\text{Maximize} & (y^+)^T e - (y^-)^T e + q^T x & \\
\text{s.t.} & \begin{bmatrix} y_i^+ & z_i^+ \\ z_i^+ & 1 \end{bmatrix} \succeq 0, \begin{bmatrix} y_i^- & z_i^- \\ z_i^- & 1 \end{bmatrix} \succeq 0 & \forall i \\
& (U^+)^T x = z^+, (U^-)^T x = z^- & \\
& \begin{bmatrix} Y_{j,i}^+ & Z_{j,i}^+ \\ Z_{j,i}^+ & 1 \end{bmatrix} \succeq 0, \begin{bmatrix} Y_{j,i}^- & Z_{j,i}^- \\ Z_{j,i}^- & 1 \end{bmatrix} \succeq 0 & \forall j, \forall i \\
& (U_j^+)^T x = Z_j^+, (U_j^-)^T x = Z_j^- & \forall j \\
& (Y_j^+)^T e - (Y_j^-)^T e + a_j^T x (\leq, =, \geq)\ b_j & \forall j
\end{array}
\tag{2}
$$

where
$Q = U^+(U^+)^T - U^-(U^-)^T, A_j = U_j^+(U_j^+)^T - U_j^-(U_j^-)^T, j = 1, \ldots, m$

MSC: Implementation

- Use sparsity, we need at most n nonzero variable z for the objective and each one of quadratic constraints.
- How to bound $y, Y_j, j = 1, ..., m$?
- Or bound $z, Z_j, j = 1, ..., m$ then use linearization (RLT) for cutting planes