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Relaxations for QCQP

Chuwen Zhang

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1 Relaxations for General QCQP

As a convention, we assume data matrices are symmetric, i.e., $Q, A_i \in \mathcal{S}^n$

Recall homogeneous QCQP for $x \in \mathbb{R}^n$:

$$\begin{aligned} (\textbf{HQCQP}) \quad \text{Maximize} \quad & x^T Q x \\ \text{s.t.} \quad & x^T A_i x \ (\leq, =, \geq)\ b_i, \forall i = 1, \dots, m \\ & 0 \leq x \leq 1 \end{aligned} \tag{1}$$

And inhomogeneous QCQP,

$$\begin{aligned} (\textbf{QCQP}) \quad \text{Maximize} \quad & x^T Q x + q^T x \\ \text{s.t.} \quad & x^T A_i x + a_i^T x \ (\leq, =, \geq)\ b_i \\ & 0 \leq x \leq 1 \end{aligned} \tag{2}$$

We now assume a standard form with less-than-or-equal-to constraints. We mark some of the trivial techniques below.

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- One can always reformulate (2) into a homogeneous problem by increasing the dimension of variables by 1.

$$\begin{aligned} \text{Maximize} \quad & x^T Q x + q^T x \\ & = \begin{bmatrix} x^T & t \end{bmatrix} \begin{bmatrix} Q & q/2 \\ q^T/2 & o \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} \\ \text{s.t.} \quad & -1 \le t \le 1 \end{aligned} \tag{3}$$

- Also, a symmetrized version can be achieved by fact that $x^T A x = x^T A^T x$, let

$$\tilde{A} := \frac{A + A^T}{2} \tag{4}$$

1.1 SDP Relaxation

For $x \in \mathbb{R}^n$, we have: $x^T A_i x = A_i \bullet (xx^T)$ and $xx^T \in \mathcal{S}^n_+$, which results in following relaxation using semidefinite cones, also called *lifting* method or Shor relaxation,

$$\begin{aligned} (\textbf{Shor-Basic}) \quad \text{Maximize} \quad & Q \bullet Y + q^T x \\ \text{s.t.} \quad & Y - xx^T \succeq 0 \text{ or } \begin{bmatrix} 1 & x^T \\ x & Y \end{bmatrix} \succeq 0 \\ & A_i \bullet Y + a_i^T x \ \le \ b_i, \forall i \\ & 0 \le x \le 1 \end{aligned} \tag{5}$$

Notice QCQP with matrix variables can also be reformulated into a SDP based problem, let $X \in \mathbb{R}^{n \times d}$, then $X^T A_i X = A_i \bullet (XX^T)$, similarly,

$$\begin{aligned} \text{Maximize} \quad & Q \bullet Y \\ \text{s.t.} \quad & Y - XX^T \succeq 0 \text{ or } \begin{bmatrix} I_d & X^T \\ X & Y \end{bmatrix} \succeq 0 \\ & A_i \bullet Y \ \le \ b_i, \forall i \end{aligned} \tag{6}$$

SDP relaxations (5) can be unbounded in some case. A simple improvement to (5) is to add bounds for the diagonal entries.

$$\begin{aligned} (\textbf{Shor}) \quad \text{Maximize} \quad & Q \bullet Y + q^T x \\ \text{s.t.} \quad & Y - xx^T \succeq 0 \text{ or } \begin{bmatrix} 1 & x^T \\ x & Y \end{bmatrix} \succeq 0 \\ & A_i \bullet Y + a_i^T x \ \le \ b_i, \forall i \\ & 0 \le x \le 1 \\ & \operatorname{diag}(Y) \le x \end{aligned} \tag{7}$$

1.1.1 Dual

Consider the dual of primal SDP relaxation with diagonal bound, cf. (7)

$$
\begin{aligned}
L = \min_{x,Y} & \; - Q \bullet Y - q^T x + \sum_i \left(\lambda_i A_i \bullet Y + \lambda_i a_i^T x - \lambda_i b_i \right) \\
& \mathbf{diag}(v) \bullet Y - v^T x + \mu^T x - \mu^T e \\
& - \begin{bmatrix} Y & x \\ x^T & 1 \end{bmatrix} \bullet \begin{bmatrix} Z & y \\ y^T & \alpha \end{bmatrix} \\
= \min_{x,Y} & \; -\lambda^T b - \mu^T e - \alpha \\
& + Y \bullet \left(-Q + \sum_i \lambda_i A_i - Z + \mathbf{diag}(v) \right) \\
& + x^T \left(-q + \sum_i \lambda_i a_i + \mu - 2y - v \right)
\end{aligned} \tag{8}
$$

And thus the dual $\phi(\cdot) = \max_{(\cdot)} L$:

$$
\begin{aligned}
(\mathbf{Dual\text{-}Shor}) \quad \text{Minimize:} \quad & \lambda^T b + \mu^T e + \alpha \\
\text{s.t.} \quad & Q = \sum_i \lambda_i A_i + \mathbf{diag}(v) - Z \\
& \sum_i \lambda_i a_i + \mu - 2y - v - q \geq 0 \\
& \begin{bmatrix} Z & y \\ y^T & \alpha \end{bmatrix} \succeq 0 \\
& \lambda, \mu, v \geq 0
\end{aligned} \tag{9}
$$

1.1.2 RLT

There are many further enhancements to (7), see [**?**] for discussion on the strength of different relaxations. Here we discuss a few widely used methods using copositive cones and reformulation-linearization-techniques (RLT) cuts.

1.1.3 Copositive

1.2 SOCP Relaxation

We now consider another way of relaxing the original QCQP problem. Consider symmetric indefinite matrix $Q \in S^n$ and its spectral decomposition.

$$
\begin{aligned}
Q &= V \Lambda V^T = \sum_j^n \lambda_j v_j v_j^T \\
\Lambda &= \operatorname{diag}(\lambda)
\end{aligned} \tag{10}
$$

Without loss of generality, we assume first r eigenvalues are positive, $\lambda_1, ..., \lambda_r \geq 0, r \leq n$. The quadratic form $x^T Qx$ can also be partitioned into positive and negative parts:

$$x^T Qx = \sum_{j=1}^{r} \lambda_j x^T v_j v_j^T x + \sum_{j=r+1}^{n} \lambda_j x^T v_j v_j^T x \tag{11}$$

By letting $s_j \geq z_j^2, z_j = v_j^T x, j = 1, \cdots, n$, then (11) can be expressed by introducing n (2-d) quadratic cones, literally,

$$x^T Qx \leq \sum_j s_j \cdot \lambda_j,\ (s_j, v_j^T x) \in Q^2 \tag{12}$$

This substitution uses a set of small quadratic cones instead of one semidefinite matrix of size n^2. With some abuse of notation, suppose $\lambda_i, V_i, i = 0, \cdots, m$ are eigenvalues and vectors for Q and $A_i, i = 1, \cdots, m$, respectively. Following the same routine for each constraint, we describe the Many-Small-Cone (MSC) relaxation to QCQP, namely,

$$\begin{array}{lrll} (\mathbf{MSC}) & \text{Maximize}: & y_0^T \lambda_0 + q^T x & \\ & \text{s.t.} & V_i z_i = x & i = 0, ..., m \\ & & y_i^T \lambda_i + a_i^T x \leq b_i & i = 1, ..., m \\ & & y_i \geq z_i \circ z_i & i = 0, ..., m \\ & & y_i^T e \leq x^T e & i = 0, \cdots, m \end{array} \tag{13}$$

The last set of constraints are placed to resolve unboundedness for the fact that the similarity transformation by any orthogonal basis $V_i, \forall i$ preserves the value of **trace** operator, namely:

$$y_i^T e = \mathbf{trace}(V_i^T x x^T V_i) = \mathbf{trace}(x x^T) \leq x^T e \tag{14}$$

This method is closely related to D.C. and Convex SOCP relaxations to QCQP, see [**?**], [**?**], [**?**], [**?**]. Recently, [**?**] mention a similar formulation, by defining $C_i = V_i \mathbf{diag}(\sqrt{|\lambda_i|})$. We list it below for convenience.

$$\begin{array}{lrll} (\mathbf{MSC\text{-}Luo}) & \text{Maximize}: & y_0^T e + q^T x & \\ & \text{s.t.} & C_i z_i = x & i = 0, ..., m \\ & & y_i^T \lambda_i + a_i^T x \leq b_i & i = 1, ..., m \\ & & y_i \geq z_i \circ z_i & i = 0, ..., m \\ & & y^T \dfrac{1}{|\lambda|} \leq x^T e & i = 0, \cdots, m \end{array} \tag{15}$$

In [**?**], box constraints for z_i can be calculated by its definition. For the case where $x \in [0, 1]$, we show bounds are redundant and the two formulations are equivalent.

Theorem 1 *The relaxations* (13)*,* (15) *are equivalent.*

Proof We show the solution for any one of the relaxations can be derived from another.
Suppose (x_0, z_0, y_0) is a feasible solution to (13), then $(x_0, \sqrt{\mathbf{diag}(|\lambda|)}z_0, \mathbf{diag}(|\lambda|) \cdot y_0)$ is feasible to (15). Conversely, if (x_0, z_0, y_0) is feasible to (15), then we can construct $(x_0, \frac{1}{\sqrt{\mathbf{diag}(|\lambda|)}}z_0, \frac{1}{\mathbf{diag}(|\lambda|)} \cdot y_0)$ that is also feasible to (13). $\square$