A QCQP Solver

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QCQP

Recall QCQP:

$$
\begin{aligned}
\text{Maximize} \quad & x^T Q x + q^T x \\
\text{s.t.} \quad & x^T A_i x + a_i^T x \, (\leq, =, \geq) \; b_i \\
& 0 \leq x \leq e
\end{aligned}
\tag{1}
$$

- Q, A_i maybe indefinite

Convex Relaxations with SOCP

Now assume we are using a suitable relaxation, we choose second-order cones. No lifted matrix $X \in \mathscr{S}^n$ is allowed.

- Method 1, (MSC) Many-small-cones, based on spectral decomposition, then we approximate on 1-D quadratic functions that form the many but smaller (actually 2D) smaller cones. Use Spectral decomposition, if $Q = V\Gamma V^T$, then assume V is full rank, the approximation actually reduces to the following hard constraints,

$$\begin{aligned} V_i^T x &= z_i, i = 1, \ldots, m \\ y_i &= z_i \circ z_i \Rightarrow y^T e = \|x\|^2 \end{aligned} \tag{2}$$

- Method 2, scale Q, A_i to be positive/negative semi-definite, then do convex relaxation based on Cholesky $Q + \lambda I_n = RR^T$, method 2 is simply,

$$\begin{aligned} &x^T(Q_i + t_i I_n)x + q_i^T x \leqslant b_i + t_i \cdot s, i = 1, \ldots, m \\ &\|x\|^2 = s \end{aligned} \tag{3}$$

- DC, use some $Q = Q_+ - Q_-$ where two parts a negative semi-definite (for max problem) or psd (for min problem).

Current progress

We see these methods are, not too surprisingly, the same, and they suffer the identical difficulties. In essence, we wish to solve a convex optimization problem subject to the norm sphere: n-dimensional $\|x\|^2 = s$ or equivalently $x_i^2 = \rho_i$

- if we are have,
$$x^T x \leq s, \text{ i.e.}, \begin{bmatrix} 1/2 \\ s \\ x \end{bmatrix} \in \mathscr{Q}$$
we only need the reverse inequality,
$$x^T x \geq s$$
- (B-C) intuitively solved by RLT like inequalities. readily plugged into current B-C framework. Need research on new cutting planes.
$$x^T x = \rho^T e, \rho_i = (x^2)_i \tag{4}$$
$$(\mathcal{B}) \quad \rho \leq (u + l) \circ x_i - l \circ u \tag{5}$$
- (ADMM) or use a local method, maybe useful to feasibility problem, like kissing #

ADMM, the ALM

Notice,

$$\|x\|^2 = \max_{\|\xi\| \le \sqrt{s}} \xi^T x \tag{6}$$

So we add slack variable s, t, ξ and bilinear constraint.

$$\begin{aligned}
(\mathbf{MSC}) \quad \text{Maximize}: \quad & y_0^T \lambda_0 && && (7)\\
\text{s.t.} \quad & (y, z, x) \in \Omega && && (8)\\
& y_i^T e \le t && i = 0, \cdots, m && (9)\\
(\kappa) \quad & t = s && i = 0, \cdots, m && (10)\\
(\mu) \quad & \xi^T x = t && && (11)\\
& \xi^T \xi \le s && && (12)
\end{aligned}$$

If s, t, ξ, y, z, x is the solution, then y, z, x is the solution for MSC. This allows the augmented Lagrangian function,

$$\mathscr{L}(x, y, z, \xi, s, \kappa, \mu) = -y_0^T \lambda_0 + \kappa(t - s) + \mu(\xi^T x - t) + \frac{\rho}{2}(t - s)^2 + \frac{\rho}{2}(\xi^T x - s)^2$$

ADMM, iteration

The ADMM iteration,

$$
\begin{aligned}
(x, y, z, t)^{k+1} &= \arg\min_{(x,y,z)\in\Omega, t\geq 0} L\left(x, y, z, \xi^k, s^k, \kappa^k, \mu^k\right) \\
(s, \xi)^{k+1} &= \arg\min_{(s,\xi)\in\mathscr{Q}} L\left((x, y, z, t)^{k+1}, \xi, s, \kappa^k, \mu^k\right) \\
\kappa^{k+1} &= \kappa^k + \rho\left(t^{k+1} - s^{k+1}\right) \\
\mu^{k+1} &= \mu^k + \rho\left(\langle \xi^{k+1}, x^{k+1}\rangle - s^{k+1}\right)
\end{aligned}
$$

where $\mathscr{Q}(\cdot)$ forms a simple SOCP for s, ξ,

$$
\mathscr{Q}(x) = \left\{(s, \xi) : \|\xi\|^2 \leq s\right\} \tag{13}
$$

ADMM, kissing

for balls in R^d, let $X \in R^{n \times d}$, use the following formulation,

$$
\begin{aligned}
\|x_i - x_j\| \geq 4 \Rightarrow \|x_i + x_j\|^2 \leq s \leq 12 \\
\|x_i\| \leq 2 \\
\|\xi_i\| \leq 2 \\
(\mu)\ x_i^T \xi_i = 4 \\
\mathcal{L} = \sum_i \mu_i (x_i^T \xi_i - 4) + \rho \sum_i (x_i^T \xi_i - 4)^2
\end{aligned}
$$

This is not convergent! Refine this into a local method.

Cutting planes for QCQP

S-free sets

- Gonzalo Muñoz, '20, '21, IPCO, Maximal quadratic free sets and intersection cuts. keywords: S-free sets, the reverse of nonconvex S is convex, find maximal $C \in R^n - S$, in QCQP, use LP relaxation, extreme rays to create intersection cuts.
- Bienstock, outer-product-free sets. '20

Convex hulls for some special structure, SOCPr (SOCP-representable)

- Convexify some nonconvex quadratic regions. e.g., Burer '17, quadratic function with one negative eigenvalue (norm balls) intersection with almost any quadratic function (ellipsoid, paraboloids, hyperbolic paraboloid, hyperboloid)
- Modaresi, Vielma' 17, Convex hull of two quadratic or a conic quadratic and a quadratic inequality
- Santana, Dey' 20, The Convex Hull of a Quadratic Constraint over a Polytope
- Dey et al' 19, New SOCP relaxation and branching rule for bipartite bilinear programs

Linear disjunction

Burer '17

For intersection with a paraboloid and quadratic function (indefinite)

$$\mathcal{Y} = \left\{ y = \begin{pmatrix} \tilde{y} \\ y_n \end{pmatrix} \in R^n : \begin{array}{c} \tilde{y}^T\tilde{y} \leq y_n \\ \tilde{y}^T\tilde{Q}\tilde{y} + 2g^T y + f \leq 0 \end{array} \right\}$$

Homogenize: $x = [\tilde{y}, y_n, x_{n+1}]$

$$A_0 := \begin{bmatrix} I & 0 & 0 \\ 0^T & 0 & -\frac{1}{2} \\ 0^T & -\frac{1}{2} & 0 \end{bmatrix}, \quad A_1 := \begin{bmatrix} \tilde{Q} & 0 & \tilde{g} \\ 0^T & 0 & g_n \\ \tilde{g}^T & g_n & f \end{bmatrix}, \quad H^1 := \{x : x_{n+1} = 1\}$$

Def $$A_t := \begin{bmatrix} (1-t)I + t\tilde{Q} & 0 & t\tilde{g} \\ 0^T & 0 & (1-t)\left(-\frac{1}{2}\right) + tg_n \\ t\tilde{g}^T & (1-t)\left(-\frac{1}{2}\right) + tg_n & tf \end{bmatrix}$$

$$t = 1/(1-\lambda), \lambda = \min(\lambda(Q)) < 0$$

$$\mathcal{Y}_t = \left\{ y : \tilde{y}^T\tilde{y} \leq y_n, x^T A_t x \leq 0 \right\}$$

$$\Rightarrow \quad \mathcal{Y}_t = \mathrm{cl.convex.hull}(\mathcal{Y})$$

Cutting planes for QCQP

- use SOCPr to improve QCQP relaxations
- consider cases for box constraint (so as to incorporate Branch and Cut)