Notes on Stochastic Programming

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[1]Stochastic Programming Reading Group

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1 Preface

This monograph records the reading notes for the subject “stochastic programming” starting from Spring 2022. The goal is to increase the authors’ familiarity in this fascinating field.

2 Mathematical Background

The part is based on [Shapiro et al., 2014], [Birge and Louveaux, 2011] and so forth.

2.1 Basic Convex Analysis

Firstly, we introduce some definitions.

K_C	recession cone to set C
C^*	polar of cone to set C
$\mathcal{N}_C$	normal cone to set C
$\mathcal{R}_C$	radial cone to set C
$\mathcal{T}_C$	tangent cone to set C

Table 1: A summary of convex sets

Definition 1. The basic convex sets.

i The recession cone K_C of the set C is formed by vectors h such that for any $x \in C$ and any $t > 0$ it follows that $x + t \cdot h \in C$.

Theorem 2. *The theorems for the recession cone.*

i The K_C is convex if C is convex,

ii and is closed if the set C is closed.

iii The convex set C is bounded iff. $K_C = \{0\}$

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3 Two-Stage Problems

Introduction

To understand the difficulties of multistage (integer) programs, we first look at the structural properties of the value functions.

Consider the two-stage problem,

$$\begin{aligned}\min_{x\in\mathcal{X}} \ & c^T x + \mathcal{Q}(x) \\ \text{s.t. } & Ax \le b, x \in \mathcal{X}\end{aligned}$$

where the second stage recourse, the value function $\mathcal{Q}(x)$ is defined as the expectation with a recourse variable y.

$$\begin{aligned}\mathcal{Q}(x) &= \mathbb{E}Q(x, \boldsymbol{\xi}) \\ &\boldsymbol{\xi}(\omega) = (T, W, h)(\omega) \\ &T(\boldsymbol{\omega})x + W(\boldsymbol{\omega})y = h(\boldsymbol{\omega}), y \ge 0, \forall \boldsymbol{\omega} \\ Q(x, \boldsymbol{\xi}) &= \min_{y\in\mathcal{Y}} q^T y\end{aligned}$$

Remark

- Assume random variable $\boldsymbol{\omega}$ living on some probability space, $(\Omega, \mathcal{F}, P)$
- The first stage variable x is made before a realization of $\boldsymbol{\xi}$, i.e., the “here-and-now” ones.
- The second stage variable y, to be precise, should be $\boldsymbol{y}(\omega)$ that actually depends on $\boldsymbol{\omega}$

Analysis on the value function

We first assume (x, y) are continuous, for example, $\mathcal{X}, \mathcal{Y}$ are convex. Define the function the support function $s_q(\chi)$ of set $\Pi(q)$, we notice,

$$\begin{aligned}\chi &= h - Tx \\ \Pi(q) &\doteq \left\{\pi : W^\top \pi \le q\right\} \\ s_q(\chi) &\doteq \inf\left\{q^\top y : Wy = \chi, \quad y \ge 0\right\} \\ \Rightarrow s_q(\chi) &= \sup_{\pi\in\Pi(q)} \pi^\top \chi\end{aligned}$$

Obviously,

Theorem 4. *The value function $Q(x, \boldsymbol{\xi}) = s_q(\chi)$, furthermore*

1. *Q is a homogeneous polyhedron function supporting $\Pi(q)$*
2. *Also, the subdifferential of Q could also be defined.*

$$\partial Q\left(x_0, \boldsymbol{\xi}\right) = -T^\top \mathcal{D}\left(x_0, \boldsymbol{\xi}\right)$$

where

$$\mathcal{D}(x, \boldsymbol{\xi}) = \pi^* \doteq \arg\max_{\pi\in\Pi(q)} \pi^\top (h - Tx)$$

4 Analysis on Value Function

Now we inspect the existence of Q. Firstly, we introduce the definition,

Definition 5. Classification of recourse. The recourse problem is,

i (Fixed) if the matrix $W(\boldsymbol{\omega}) = W$ is fixed (not random).

ii (Complete) if the system $\{y : Wy = \chi, y \geq 0\}$ has a solution for every χ

iii (Simple) if both complete and fixed

iv (Relatively complete) relatively complete if for every feasible x, the feasible set of the second-stage problem is nonempty for almost everywhere (a.e.) $\boldsymbol{\omega} \in \Omega$.

For a recourse to be complete, $\Pi(q)$ must be bounded, i.e., the recession cone $\Pi(0) = \{0\}$.

Two-stage integer program

For the integer second stage case, first look at the example in [Schultz, 2003]

$$\min_{(y,y')} \left\{ q^T y + q'^T y' : Tx + Wy + W'y' = h(\omega), y \in Z_+^{\bar{n}_2}, y' \in R_+^{n_2'} \right\}$$

References

[Birge and Louveaux, 2011] Birge, J. R. and Louveaux, F. (2011). *Introduction to Stochastic Programming*. Springer Series in Operations Research and Financial Engineering. Springer-Verlag New York, 2 edition.

[Schultz, 2003] Schultz, R. (2003). Stochastic programming with integer variables. *Mathematical Programming*, 97(1):285–309.

[Shapiro et al., 2014] Shapiro, A., Dentcheva, D., and Ruszczyński, A. (2014). *Lectures on stochastic programming: modeling and theory*. SIAM.