Notes on Stochastic Programming

Jingyuan Yang, Chuwen Zhang
Stochastic Programming Reading Group

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1 Mathematical Background

1.1 Functions

Definition 1.1. *An extended real valued function $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ is proper if $f(x) > -\infty$ for all $x \in \mathbb{R}^n$ and its domain, dom f, is nonempty.*

Definition 1.2. *An extended real valued function $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ is lower semicontinuous if at every point $x_0 \in \mathbb{R}^n$, $f(x_0) \le \liminf_{x \to x_0} f(x)$.*

f is lower semicontinuous iff its epigraph is a closed subset of $\mathbb{R}^{n+1}$.

Definition 1.3. *An extended real valued function $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ is polyhedral if it is proper convex and lower semicontinuous, its domain is a convex closed polyhedron, and $f(\cdot)$ is piecewise linear on its domain.*

Definition 1.4. *We refer $\mathscr{G}$ which is a mapping from Ω into the set of subsets of $\mathbb{R}^n$ as a multifunction.*

Definition 1.5. *The function $F(x, \omega)$ is random lower semicontinuous if the associated epigraphical multifunction $\omega \mapsto$ epi $F(\cdot, \omega)$ is closed valued and measurable.*

Theorem 1.1. *Let $F : \mathbb{R}^n \times \Omega \to \bar{\mathbb{R}}$ be a random lower semicontinuous function. Then the optimal value function $v(\omega)$ and the optimal solution multifunction $X^*(\omega)$ are both measurable.*

2 Linear Two-Stage Problems

2.1 Basic Properties

We discuss two-stage stochastic linear programming problems of the form

$$\begin{array}{ll} \min_x & c^T x + \mathbb{E}[Q(x, \xi)] \\ \text{s.t.} & Ax = b, x \ge 0, \end{array} \tag{1}$$

where $Q(x, \xi)$ is the optimal value of the second-stage problem

$$\begin{array}{ll} \min_y & q^T y \\ \text{s.t.} & Tx + Wy = hy, y \geq 0. \end{array} \tag{2}$$

The dual problem of problem (2) can be written in the form

$$\begin{array}{ll} \max_\pi & \pi^T (h - Tx) \\ \text{s.t.} & W^T \pi \leq q. \end{array} \tag{3}$$

We define

$$s_q(\chi) \triangleq \inf\{q^T y : Wy = \chi, y \geq 0\}.$$

Clearly, $Q(x, \xi) = s_q(h - Tx)$. In addition, by duality theory,

$$s_q(\chi) = \sup_{\pi \in \Pi(q)} \pi^T \chi,$$

where $\Pi(q) \triangleq \{\pi : W^T \pi \leq q\}$.

Proposition 2.1. *For any given ξ, $Q(\cdot, \xi)$ is convex. Moreover, if $\{\pi : W^T \pi \leq q\}$ is nonempty and problem (2) is feasible for at least one x, then the function $Q(\cdot, \xi)$ is polyhedral.*

Proposition 2.2. *Suppose that for given $x = x_0$ and $\xi \in \Xi$, the value $Q(x_0, \xi)$ is finite. Then $Q(\cdot, \xi)$ is subdifferentiable at x_0 and*

$$\partial Q(x_0, \xi) = -T^T \mathfrak{D}(x_0, \xi),$$

where $\mathfrak{D}(x_0, \xi) \triangleq \arg\max_{\pi \in \Pi(q)} \pi^T (h - Tx)$ is the set of optimal solutions of the dual problem (3).

2.1.1 The domain of $s_q(\chi)$

The *positive hull* of a matrix W is defined as

$$\text{pos } W \triangleq \{\chi : \chi = Wy, y \geq 0\}.$$

The recession cone of $\Pi(q)$ is equal to

$$\Pi_0 \triangleq \{\pi : W^T \pi \leq 0\}.$$

Denote the polar cone to Π_0 as Π_0^*. We have

$$\text{dom } s_q = \text{pos } W = \Pi_0^*.$$

2.2 The Expected Recourse Cost

$$\phi(x) \triangleq \mathbb{E}[Q(x, \xi)].$$

With discrete distribution of ξ, if for at least one scenario, the corresponding second-stage problem is infeasible, $\phi(x)$ is $+\infty$.

To ensure that $\phi(x)$ is well defined, we have to verify two conditions:

1. $Q(x, \cdot)$ is measurable [followed by Theorem 1.1];

2. either $\mathbb{E}[Q(x, \xi)_+]$ or $\mathbb{E}[(-Q(x, \xi))_+]$ is finite.

Proposition 2.3. *Suppose that the recourse is fixed and*

$$\mathbb{E}[||q||||h||] < +\infty \textit{ and } \mathbb{E}[||q||||T||] < +\infty.$$

Consider a point $x \in \mathbb{R}^n$. Then $\mathbb{E}[Q(x, \xi)_+]$ is finite iff $h - Tx \in pos\ W$ holds w.p. 1.

Proof. By Hoffman's lemma.

Moreover, if the recourse is complete, $\phi(\cdot)$ is well defined and is less than $+\infty$. Since the function $\phi(\cdot)$ is convex, we have that if $\phi(\cdot)$ is finite valued in at least one point, then $\phi(\cdot)$ is finite valued on the entire space $\mathbb{R}^n$.

Proposition 2.4. *Suppose that the probability distribution of ξ has finite support $\Xi = \{\xi_1, \ldots, \xi_K\}$ and $\phi(\cdot)$ has a finite value in at least one point $\bar{\xi} \in \mathbb{R}^n$. Then the function $\phi(\cdot)$ is polyhedral, and for any $\xi_0 \in dom\ \phi$,*

$$\partial \phi(x_0) = \sum_k p_k \partial Q(x_0, \xi_K).$$

Proof. By Proposition 2.1.

Remark. ϕ is differentiable at x_0 iff for every ξ_k, the corresponding second-stage dual problem has a unique optimal solution.

Proposition 2.5. *Suppose that the expectation function $\phi(\cdot)$ is proper and its domain has a nonempty interior. Then for any $x_0 \in dom\ phi$,,*

$$\partial \phi(x_0) = -\mathbb{E}[T^T \mathfrak{D}(x_0, \xi)] + \mathcal{N}_{dom\ \phi}(x_0),$$

where

$$\mathfrak{D}(x_0, \xi) \triangleq arg \max_{\pi \in \Pi(q)} \pi^T (h - Tx).$$

Moreover, ϕ is differentiable at x_0 iff x_0 belongs to the interior of dom ϕ and the set $\mathfrak{D}(x_0, \xi)$ is a singleton w.p. 1.

Proposition 2.6. *Suppose that (i) the recourse is fixed, (ii)for a.e. q the set $\Pi(q)$ is nonempty, (iii) condition $\mathbb{E}[||q||||h||] < +\infty$ and $\mathbb{E}[||q||||T||] < +\infty$ holds, (iv) the conditional distribution of h, given (T, q) is absolutely continuous for almost all (T, q). Then ϕ is continuously differentiable on the interior of its domain.*

In the case of a continuous distribution of ξ, the expectation operator smoothes the piecewise linear function $Q(\cdot, \xi)$.

Theorem 2.7. *Let $\bar{x}$ be a feasible solution of problem* (1)*. Then $\bar{x}$ is an optimal solution iff there exist $\pi_k \in \mathfrak{D}(\bar{x}, \xi_k), k = 1, \ldots, K$, and $\mu \in \mathbb{R}^m$ such that*

$$\begin{gathered} \sum_k p_k T_k^T \pi_k + A^T \mu \leq c \\ \bar{x}^T (c - \sum_k p_k T_k^T \pi_k - A^T \mu) = 0. \end{gathered} \tag{4}$$

If we deal with general distributions of the problem's data, additional conditions are needed to ensure the subdifferentiability of the expected recourse cost and the existence of Lagrange multipliers.

Theorem 2.8. *Let $\bar{x}$ be a feasible solution of problem* (1)*. Suppose that the expected recourse cost function $\phi(\cdot)$ is proper, int(dom ϕ)$\cap X$ is nonempty, and $\mathcal{N}_{dom\ \phi}(\bar{x}) \subset \mathcal{N}_X(\bar{x})$. Then $\bar{x}$ is an optimal solution iff there exist a measurable function $\pi(\omega) \in (x, \xi(\omega))$, and a vector $\mu \in \mathbb{R}^m$ such that*

$$\begin{gathered} \mathbb{E}[T^T \pi] + A^T \mu \leq c \\ \bar{x}^T (c - \mathbb{E}[T^T \pi] - A^T \mu) = 0. \end{gathered} \tag{5}$$

The assumptions can be substituted by i) the recourse is fixed, (ii)for a.e. q the set $\Pi(q)$ is nonempty, (iii) condition $\mathbb{E}[||q||||h||] < +\infty$ and $\mathbb{E}[||q||||T||] < +\infty$ holds and T is deterministic.