Laboratory Manual

PHSC 12720 Exoplanets

The University of Chicago

Spring 2019

Labs

LAB 1

Scale of the solar system and local stellar environment

> *Space is big. You just won't believe how vastly, hugely, mind-bogglingly big it is. I mean, you may think it's a long way down the road to the chemist's, but that's just peanuts to space.* Douglas Adams, The Hitchhiker's Guide to the Galaxy

Why can't we just see planets orbiting other stars with normal observational techniques like looking through larger and larger telescopes? Through this lab, we hope you gain a felt sense appreciation for the scale of star systems and the distance between the Sun and our closest stellar neighbor, and see why we need to use special techniques to detect exoplanets.

1.1 Your current intuition

In order to gauge your current sense of where things are in relation to each other, each member of your group will first make three different small, qualitative scale models. **For this section, do not use a book, or the Internet, other people, or any other resource, to guide your efforts.** This will help you see where your current intuition lies. And when you compare to others, do not change your guess — it is expected that you might not have an accurate conception yet.

1. Individually, without looking at your groupmates' work, take a blank sheet of paper and draw a horizontal line across the page. On the left end of it, place a dot and mark it "Sun". On the right end, place a dot and mark it "Neptune". Now place and label a dot for your best guess orbital distance for each of the seven other planets orbiting the Sun.

2. Next, make a similar scale model, this time placing the following objects on it: the Sun, Neptune, our nearest star Proxima Centauri, and its planet, Proxima Centauri b.

3. Finally, draw the Sun and all 8 planets with your best guess of their relative sizes — you should end up with 9 circles. The distances do not need to be scaled as well for this estimate.

4. Compare your drawings with your groupmates. Write down any surprises or big differences that you had for your report.

1.2 Making an accurate scale model

Now that you have your current sense of it, you will make an accurate scale model of the three scales you made above. Bigger is better, so you will use the hallway that extends on the second floor from the south end of Kersten, across the skywalk, through Eckhardt, and into the Accelerator Building.

Available equipment: measuring wheel, paper, masking or label tape, scissors, markers

Your group will be assigned one of the 3 scale models described in the previous section to create in the hallway. After you create it, you will give a short (~ 5 min) presentation to the class, describing what you found and what your impression is, and walking the class through the scale model down the hallway.

Tips

- To make a scale model, you will need to gather the size or distance information for the objects in your model using any resource you'd like, then divide each of those by the same number to create your scale. You can use a spreadsheet to make this task easier.
- For the two distance scale models, you will need to measure the distance you have to work with in the hallway. Ensure that you can see from one end to the other. Prop open the skywalk doors if needed. You should place some kind of upright sign, perhaps taped down, at each spot where an object is in your model.
- For the size scale model, since all the objects are mostly spherical, you can create your model by cutting out circles of paper. For the larger objects, feel free to tape sheets of paper together to make a bigger circle.

1.3 Appreciating distances

Answer the following questions:

1. Using a nominal speed of a car on Earth, how long would it take to drive:
   a) once around the equator?
   b) from the Earth to the Sun?
   c) from the Sun to Neptune?
   d) from the Sun to Proxima Centauri?
2. If you wanted to travel each of these distances in 1 year, how fast would you need to go in each case?
3. For the longest distance, how does that speed compare to the speed of light, which is the fastest anything can go?

1.4 Report checklist

Include the following in your lab report. See Appendix A for formatting details.

- Your intuition estimates from Section 1.1 and your reflection of how they compare to your group mates'.
- A table of the distances/sizes and scaled version that you created in Section 1.2.
- Worked solutions to the questions in Section 1.3.
- A 100–200 word reflection on the assignment — was there anything that surprised you? How do you see your place in the universe, given the scale of the solar system and how far away even the nearest star is?

LAB 2

Detecting exoplanets with the radial velocity method

2.1 Introduction

In the Fall of 1995, two Swiss astronomers announced evidence for a planet orbiting the star 51 Pegasi, a groundbreaking discovery since the star, 51 Pegasi, is very similar to our own Sun. Since 1995, thousands of potential exoplanets have been detected and the field of exoplanet science has become a pillar of modern astronomy.

In this lab, we will explore the radial velocity technique, the method used to detect the first exoplanets and a powerful technique for studying exosolar planetary systems.

Grading

Each bolded instruction is worth 4 points, and each numbered question is worth 2 points. We include 10 points for attendance and participation to arrive at 60 points for this lab.

2.2 Building Intuition

The radial velocity method works because in a planetary system, both the host star and its planet orbit the Center of Mass for the system. Since the host star is also moving, the star has a non-zero velocity, which we can study through the Doppler effect on light.

Open the link for the Center of Mass webpage (`http://astro.unl.edu/naap/esp/centerofmass.html`). The animation in the upper left illustrates the orbit of two bodies about their Center of Mass. Read through the webpage and use the sliders to explore how varying separation and relative mass changes the Center of Mass.

For this lab, the following values will be useful:

- 1 AU $= 1.496 \times 10^{11}$ m (approximate distance between the Earth and the Sun)
- 1 $\mathrm{M_J} = 1.898 \times 10^{27}$ kg (mass of Jupiter)
- 1 $\mathrm{M_{Sun}} = 1.989 \times 10^{30}$ kg (mass of the Sun)
- $G = 6.674 \times 10^{-11}\ \mathrm{N\,m^2/kg^2}$ (Newtonian constant of gravitation)

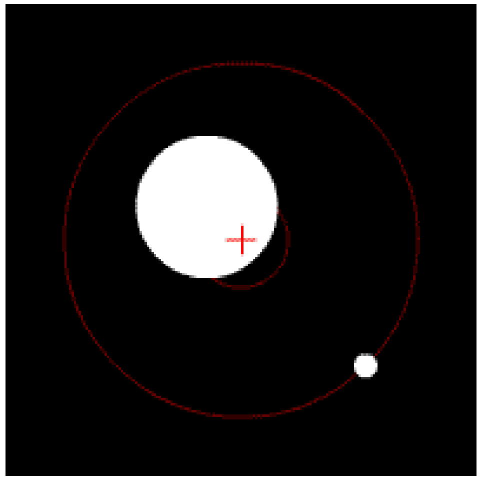

Figure 2.1: Schematic of a star (upper-left) and planet (lower-right) orbiting a common center-of-mass.

Consider the picture in Figure 2.1 and define $M_{\mathrm{p}} = 0.5M_{\mathrm{J}}$ as the mass of the planet (the smaller object), $M_{\mathrm{S}} = 1.0M_{\mathrm{Sun}}$ as the mass of the star (the larger object), and $d = 0.0527\,\mathrm{AU}$ be the separation between the two. Assume the system is in a circular orbit.

1. How far is the Center of Mass from the center of the star (in AU)?
2. What is the star's orbital radius (in AU)?

Since the star is moving around the Center of Mass, it has a non-zero velocity. Let the orbital period of the system be $T = 4.23$ days.

3. What is the orbital velocity of the host star?
4. Sketch the figure, then on that sketch, draw arrows showing the velocity at different parts of the orbit. What happens to the direction of the star's velocity as the planet traces out its orbit?

Astronomers can use spectroscopic techniques for measuring the velocity of stars along the line of sight between the earth and the star. In the illustration in Figure 2.1, suppose the earth is to the right of the system so that we are observing the system edge on from the right side of the page. **Draw a graph** similar to Figure 2.2 and then draw on it the "line of sight" velocity for the star as a function of time (hint: it will be a trigonometric function).

5. How are the period and amplitude related to the orbital period and the star's velocity?

Consider a similar system only with a planet that is twice the mass ($M_{\mathrm{p}} = M_{\mathrm{J}}, M_{\mathrm{S}} = M_{\mathrm{Sun}}$, $d = 0.0527$ AU, and $T = 4.23$ days).

6. What is the system's host star orbital radius (in AU) and orbital velocity (in m/s)?
7. How do these compare with the values for the previous system with a less massive planet?

Sketch the line of sight velocity for this more massive system on the same plot that you already drew.

8. How does the line of sight velocity depend on the planet's mass?

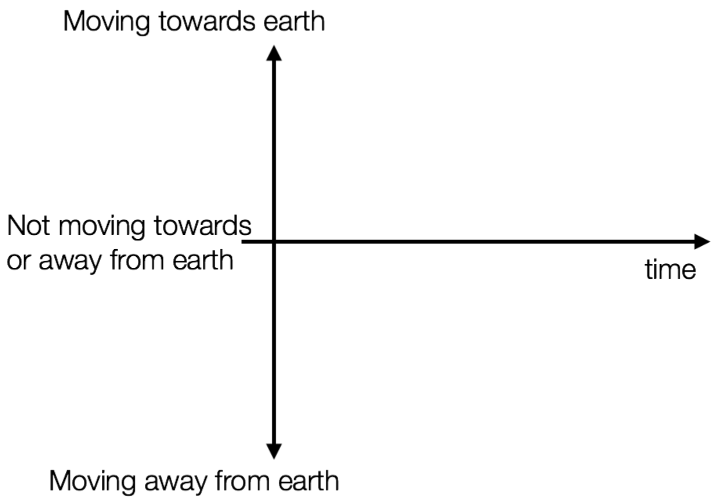

Figure 2.2: Graph to draw for radial velocity estimation.

Open the link for the radial velocity simulator (`http://astro.unl.edu/naap/esp/animations/radialVelocitySimulator.html`). Under “Visualization Controls” click “show multiple views.” Under “Planet Properties” set the eccentricity to zero. Under “Animation Controls,” press “Start Animation” to set things in motion. Vary the mass of the planet, the mass of the star and the semimajor axis and note how the radial velocity graph changes. Vary the other parameters (eccentricity, longitude, inclination) and see how the radial velocity changes. **Include your findings in your report.**

2.3 Radial velocity of 51 Peg

Table 2.1 lists actual line of sight velocities for 51 Peg measured by Marcy & Buttler in 1995 at the Lick Observatory in California.

Use the free software SciDAVis to plot this data. You can download and install it on your computer, or you can use the lab computers, which have it installed already.

Then, fit an equation to the data. To do so in SciDAVis, first save the project, so you won't accidentally lose your work. Then, select Analysis ▸ Fit Wizard... from the drop-down menu. Type your desired equation into the large box. You can use any of the functions you can find in the lists at the top of the window and combine them how you would like. For fit parameters, use the letters a, b, c, and so on. For each fit parameter you use, include it in the list of parameters. For example, if you think the data are best represented by a tangent function, you could put “`b*tan(c*x+d)`” in the box. Then click the “Fit >>” button, then the “Fit” button to fit the data with this equation. This finds

Day	V (m/s)	Day	V (m/s)	Day	V (m/s)	Day	V (m/s)	Day	V (m/s)	Day	V (m/s)	Day	V (m/s)
0.6	-20.2	3.6	-35.1	5.6	45.3	7.7	-22.6	9.6	25.1	10.8	51	13.6	2.7
0.7	-8.1	3.7	-42.6	5.7	47.6	7.8	-31.7	9.7	35.7	11.7	-2.5	13.7	17.6
0.8	5.6	4.6	-33.5	5.8	56.2	8.6	-44.1	9.8	41.2	11.8	-4.6		
1.6	56.4	4.7	-27.5	6.6	65.3	8.7	-37.1	10.6	61.3	12.6	-38.5		
1.7	66.8	4.8	-22.7	6.7	62.5	8.8	-35.3	10.7	56.9	12.7	-48.7		

Table 2.1: Line-of-sight velocities for 51 Pegasi measured over time.

the parameters for the equation that make it best fit the data. Experiment to find an equation that seems to fit well.

You can retrieve the fit parameters and their uncertainties from the Results Log on the main window. **Include the graph, the fit parameters, and their uncertainties in your report.**

9. From your fit to the data, what is the orbital period (days) and host star orbital velocity (m/s)?
10. What is the orbital radius (in AU) for the host star?

Kepler's third law relates the orbital period to the system's semimajor axis. In the case where the planet's mass is much smaller than the star's mass, Kepler's third law is

$$P^2 = \frac{4\pi^2}{GM} a^3 \,, \tag{2.1}$$

where P is the orbital period, G is Newton's constant, M is the mass of the star, and a is the semimajor axis. The mass of 51Peg is $1.0 M_{\rm Sun}$.

11. According to your fitted orbital period and Kepler's third law, what is the system's semimajor axis?

The host star orbital radius and the semimajor axis are related by the Center of Mass.

12. Using the system's semimajor axis, the mass of 51Peg, and the host star's orbital period, determine the planet's mass (in units of $M_{\rm J}$).

2.4 Radial velocities for other systems

Go to the webpage for SystemicLive (http://www.stefanom.org/systemic-live/). Note that while the graphics within the tutorial are currently not visible on the website, the data visualization and analysis software are still functional). Click "Open Systemic" to start the program. Then on that first page, scroll down to "Tutorials and Resources" and click on the link "51 Pegged: Rediscovering the first exoplanet with Systemic Live" and follow that tutorial. You will use the Systemic software to analyze radial velocity data for 51Peg.

Note that one key analysis tool you will use in the remainder of the lab is the power spectrum of the radial velocity data. We know that we can create any periodic function from the sum of sinusoids of different frequencies and phases. The power spectrum tells us which frequencies dominate that decomposition. Therefore, peaks in the power spectrum correspond to periodicities in the data, and hence point to possible planetary periods in the radial velocity measurements.

13. How do the mass and separation results from Systemic compare with your earlier 51Peg numbers? Print out a plot of the "Phased Radial Velocity" and write on it the Period and Mass for the system.
14. Use Systemic to determine orbit parameters for the following systems: 47Uma, 70Vir. Print out plots for your "Phased Radial Velocity" for each system. Write on your plots the Period and Mass for each system orbit.
15. Use Systemic to analyze the data for system: upsand. Hint, there are multiple planets for this system. How many planets do you find? What are their masses and orbital periods?

2.5 Reflection

Do some research on the history and background of the systems you analyzed (51Peg, 47Uma, 70Vir, and upsand). Write a one paragraph summary of this lab and discuss the following:

16. What was the historical context and significance of these measurements?
17. How do they relate to what you did in this lab?

LAB 3

Detecting exoplanets with the transit method

3.1 Part 1

See other handout on Canvas for Part 1.

3.2 Part 2

Throughout the upcoming week, you will use one of the MicroObservatory telescopes, built and maintained by the Harvard-Smithsonian Center for Astrophysics and located at the Whipple Observatory in Amado, Arizona to take a series of images of a “target” star in order to calculate a light curve for that star, which could be used to learn about the planet(s) orbiting them. These images will form the basis of your subsequent investigation in the Image Lab on the LSE website.

During this class, you will determine which stars you are imaging and decide among your group who will schedule the observations. After you do this, you will use the LSE website to analyze an example image, so you can analyze your own images later at home for the report.

The report will be due two weeks from today’s lab, rather than one week, to account for the extra work at home.

Scheduling observations

Throughout the upcoming week, you will use one of the MicroObservatory telescopes, built and maintained by the Harvard-Smithsonian Center for Astrophysics and located at the Whipple Observatory in Amado, Arizona to take a series of images of a “target” star. These images will form the basis of your subsequent investigation in the Image Lab on the LSE website.

Log on to LSE using the credentials emailed to you and click “Go to the Lab.”

Click on the “Telescope” tab and go through the sections on the right side of the page to learn about the remote observatory we are using and how to schedule observations.

Next, look at the transit calendar file posted on the canvas course website, “2019_April_May_transit_calendar.pdf,” which lists observable transits by the remote observatory each night. Note, times are for Tucson, AZ, not Chicago. Work out with your TA and rest of the section a plan for observing a few transits over the course of the week. Each transit will require observing for $\sim$ 1 hour before and $\sim$ 1 hour after the transit for a total observation time of 4–5 hours. This corresponds to 80– 100 exposures. One possible arrangement would be to have smaller groups responsible for scheduling different segments of the total observation, e.g.: group 1 can schedule observing for the 1st hour, group 2 would schedule the second hour, etc... Make plans for observing a number of systems throughout the week. If possible, conduct multiple observations of the same system. *Note: you can only schedule observations for the current night, so you will need to login and schedule your observations on the appropriate day of the week.*

Once your observations have been completed, you can analyze them on your own, but in lab we will practice this analysis with example data.

Click on the "Image Lab" and go through the 9 page tutorial. Practice your analysis on the "Demo_Images" for TRES-3. Make sure you subtract and appropriate Dark Image (note that the image filenames include the date and time of the exposure).

Once it's been taken, analyze the data from your scheduled observations. Be sure to press the "Calculate & Record" button to save your results.

Combine your data with the rest of the sections (you might have to re- do the previous week's activities) by looking at the "class graph." **Save this light curve and turn it in with your write up.**

3.3 Report checklist and grading

Each item below is worth 10 points, and there is an additional 10 points for attendance and participation.

- Data table from Part 1.
- Plots of your light curves for Planet 1, Planet 2, and all of the planets you observed with LSE.
- Discuss how different features of the light curve connected to physical properties of the orbital system.
- What is your interpretation of the temperatures for the simulated planets (Planet 1 & 2)?
- List which scheduled observations your group performed. Were you able to see a light curve from your scheduled observations? Why or why not?

LAB 4

Using Kepler's Laws to find the density of Jupiter

4.1 Introduction

So far, you have been using several techniques to learn about the properties of planets orbiting other stars based one what we can observe from Earth, including using orbital properties to discover the mass of planets. The same physical laws can be used to study planets and their moons.

To study the composition of a planet, it is useful to know its density — then one can learn more about whether it is rocky or gaseous. In this lab, you will use Kepler's Laws to find the density of Jupiter, given orbital properties of its moon. In this case, we can use actual images of the these using Stone Edge Observatory, a remotely operated telescope in California.

Under nominal circumstances, we would have you schedule the observations yourself, so you could analyze the data that you, yourself, took. However, the observatory is not operating well enough right now to let that happen. Fortunately, we have archival images that were taken in 2017 by this observatory that you can analyze.

4.2 Using Kepler's laws

Keplers third law relates the orbital period to the systems semi-major axis. In the case where the planets mass is much smaller than the stars mass, Keplers third law is:

$$P^2 = \frac{4\pi^2}{GM}a^2\,, \tag{4.1}$$

where P is the orbital period, G is Newton's constant (6.674×10^{11} N m^2/kg^2), M is the mass of Jupiter, and a is the semi-major axis.

We can rewrite the third law as:

$$\frac{a^3}{P^2} = \frac{G}{4\pi^2}\left(\frac{4}{3}\pi R^3 \rho\right) = \rho\frac{GR^3}{3\pi} \tag{4.2}$$

where R is the radius of Jupiter and ρ is the density of Jupiter. Solving for ρ gives:

$$\rho = \left(\frac{a}{R}\right)^3 \frac{3\pi}{GP^2} \tag{4.3}$$

The goal for this lab is to use our data to determine the ratio (a/R) for each moon. We can then combine that with the moons period to estimate the density of Jupiter.

To determine the semi-major axis, a, for a moon's orbit, we can assume that the orbit is actually a circular orbit, and we know that we are viewing the orbits edge-on. So what we are seeing is a projection of a circular motion onto one dimension, which results in a sinusoidal motion, with the amplitude of that motion being the radius of the orbit (and thus the semi-major axis). So here we will analyze images taken at different times, plot the angular position of the moon over time, and fit those data points to a sine (or, equivalently, a cosine) function, and the amplitude will be the semi-major axis, a.

4.3 The set of observations

The images can be found on Canvas, in the Files ¿ Lab folder, in the compressed file "jupiter_data.zip".

Observations were taken on four separate dates between April 14 and 23, 2017, as stated in the names of the files. Images were taken in each of the g, r, i, and *clear* bands. These refer approximately to the color filters used in each case: green, red, infrared, and clear (no filter). See Figure 4.1 for the frequency response of such a filter set. All images were taken with a 0.05 second exposure time.

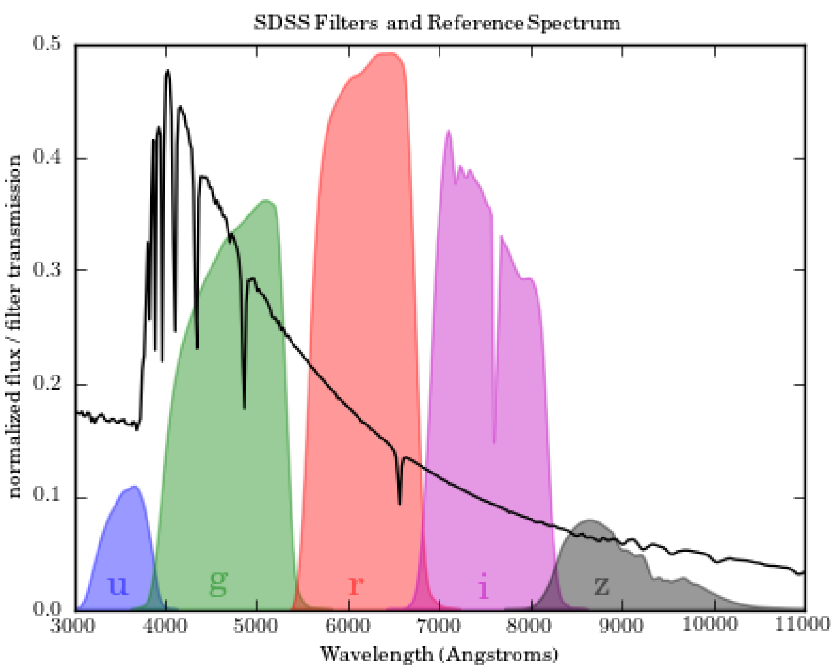

Figure 4.1: Typical transmission rates of various filters used in astronomy. This is from the Sloan Digital Sky Survey.

4.4 Make color images

All CCDs which are used in astronomical images are monochrome — they do not have different pixels for different wavelengths of light, which allows them to achieve higher resolution. To create a color image, we must combine the images taken with different color filters and add color ourselves.

One can easily make RGB images in DS9 by selecting *Frame > New Frame RGB*, which allows one to upload different .fits files for each color with *File > Open* depending on the color selected in the pop-up window. You can then scale each image as needed to make 3-color images. **Make an RGB image for Jupiter.**

4.5 Analysis

You will use the *clear* images to measure the position of the moons relative to Jupiter and *gri* images to measure the radius of Jupiter. Since we will be using the ratio of moon position to Jupiter radius, we can use whatever units are convenient to measure the distance, as long as we are consistent, as the unit divides by itself in the calculation.

1. **Use** DS9 to load and analyze each of the FITS files. To get a better view, select "Scale ¿¿ Log" from the drop-down menus. Using the Ruler function (described in the following paragraph), **for each timestamp, estimate the radius** of Jupiter from the *i*-filter file (where available) **and the position** of each moon relative to Jupiter from the *clear*-filter file. Keep track of whether the moon is on the East or West side of Jupiter by making all locations on the East side positive and locations on the West side negative.

You can use the DS9 *Ruler* function to measure the distance (in pixels) between two points in the image. To use the *Ruler*, first go to the Region menu from the toolbar at the top. Select the *Shape* sub-menu and click *Ruler*. Then, click the *Edit* menu button and select *Region*. You can then use the mouse to draw a line between two points and DS9 will calculate the distance for you. If you have trouble using the *Ruler* feature, you can also read off the (x, y) coordinates for each point and calculate the distance yourself. **Record your values** in a data table in a spreadsheet.

2. **Record the time** the image was taken by going to the *File* menu and then *Display Fits Header*. Towards the bottom of the header will be listed the *DATE-OBS* which will be in UTC time. To identify each moon at the time of the picture, use the following site: `http://www.shallowsky.com/jupiter/`, be sure to enter the correct date into the site.

3. For each moon, find the semi-major axis of its orbit by fitting a sine curve to the data, using the following procedure:

   a) Copy your data into a new table in SciDAVis. It should have the time in days (include fraction of days) in the x column and the x/R ratio in the y column. For your report, format the table and graph according to Appendix A.

   b) Use SciDAVis to fit the following function to the data:

   $$X_i = A \cos\left(\frac{2\pi T_i}{P} + \phi\right) . \tag{4.4}$$

   Here, X_i is the position of the moon scaled to the radius of Jupiter, A the amplitude of the sinusoid, which corresponds to the corresponds to the orbit semi-major axis divided by R_{Jupiter}, T_i the time of the observation, P the period of the moon's orbit, and ϕ the phase of the orbit. X_i and T_i are the data you obtain with your observations, and A and ϕ are the parameters you will fit. Enter the period for the relevant moon from Table 4.1.

The amplitude from your fit corresponds to the ratio, (a/R) which we will use in Kepler's Third law.

Moon	Period (days)	ρ_{Jupiter} (kg/m^3)
Ganymede	7.1546	
Io	1.7691	
Europa	3.5512	
Calisto	16.689	

Table 4.1: Four moons of Jupiter and their periods.

4. For each moon, use the amplitude of the curve together with Keplers Third law to estimate the density of Jupiter. Make sure you use the right units for your values. *Hint: look at the units for the Gravitational constant. What units do you need for the moons orbital period?*
5. Take the average of the densities, and use the standard deviation of them for the uncertainty. Report this value, with uncertainty, in your report.
6. Compare your result to a value of the density you find online, using the uncertainties and the procedure in Appendix B.3. How close are they? What are possible sources of uncertainty of your measurement?
7. Rocky planets in our solar system have densities ranging from 3000–5000 kg/m^3. Given this and your result, would you conclude that Jupiter is rocky or gaseous, and why?

4.6 Report checklist and grading

Each item below is worth 10 points, and there is an additional 10 points for attendance and participation.

- Data table including times and positions for each moon, as well as the radius of Jupiter.
- Graphs of position (x/R) vs. time for each moon, with the fitted curve plotted as well.
- Answers or evidence of completion for Steps 5–7.

APPENDIX A

Lab Report Format

A.1 General

- The report should be typed for ease of reading. Text should be double-spaced, and the page margins (including headers and footers) should be approximately 2.5 cm, for ease of marking by the grader. Each page should be numbered.
- The first page should include the title of the lab; lab section day, time, and number; and the names of the members of your lab team.

A.2 Organizing the report

The report should follow the sequence of the lab manual. Answers to questions and inclusion of tables and figures should appear in the order they are referenced in the manual. In general, include the following:

- Any procedure that you performed that is different from what is described in the lab manual.
- Any data that you've collected: tables, figures, measured values, sketches. Whenever possible, include an estimate of the uncertainty of measured values.
- Any calculations that you perform using your data, and the final results of your calculation. Note that you must show your work in order to demonstrate to the grader that you have actually done it. Even if you're just plugging numbers into an equation, you should write down the equation and all the values that go into it. This includes calculating uncertainty and propagation of uncertainty.
- If you are using software to perform a calculation, you should explicitly record what you've done. For example, "Using Excel we fit a straight line to the velocity vs. time graph. The resulting equation is $v = (0.92\,\mathrm{m/s^2})t + 0.2\,\mathrm{m/s}$.
- Answers to any questions that appear in the lab handout. Each answer requires providing justification for your answer.
- At the end of each experiment, you should discuss the findings and reflect deeply on the quality and importance of the findings. This can be both in the frame of a scientist conducting the experiment ("What did the experiment tell us about the world?") and in the frame of a student ("What skills or mindsets did I learn?").

A.3 Graphs, Tables, and Figures

Any graph, table, or figure (a figure is any graphic, for example a sketch) should include a caption describing what it is about and what features are important, or any helpful orientation to it. The reader should be able to understand the basics of what a graph, table, or figure is saying and why it is important without referring to the text. For more examples, see any such element in this lab manual.

Each of these elements has some particular conventions.

Tables

A table is a way to represent tabular data in a quantitative, precise form. Each column in the table should have a heading that describes the quantity name and the unit abbreviation in parentheses. For example, if you are reporting distance in parsecs, then the column heading should be something like "distance (pc)". This way, when reporting the distance itself in the column, you do not need to list the unit with every number.

Graphs

A graph is a visual way of representing data. It is helpful for communicating a visual summary of the data and any patterns that are found.

The following are necessary elements of a graph of two-dimensional data (for example, distance vs. time, or current vs. voltage) presented in a scatter plot.

– **Proper axes.** The conventional way of reading a graph is to see how the variable on the vertical axis changes when the variable on the horizontal axis changes. If there are independent and dependent variables, then the independent variable should be along the horizontal axis.

– **Axis labels.** The axes should each be labeled with the quantity name and the unit abbreviation in parentheses. For example, if you are plotting distance in parsecs, then the axis label should be something like "distance (pc)".

– **Uncertainty bars.** If any quantities have an uncertainty, then these should be represented with so-called "error bars", along both axes if present. If the uncertainties are smaller than the symbol used for the data points, then this should be explained in the caption.

APPENDIX B

Analysis of Uncertainty

A physical quantity consists of a value, unit, and uncertainty. For example, "$5 \pm 1\,\mathrm{m}$" means that the writer believes the true value of the quantity to most likely lie within 4 and 6 meters[1]. Without knowing the uncertainty of a value, the quantity is next to useless. For example, in our daily lives, we use an implied uncertainty. If I say that we should meet at around 5:00 pm, and I arrive at 5:05 pm, you will probably consider that within the range that you would expect. Perhaps your implied uncertainty is plus or minus 15 minutes. On the other hand, if I said that we would meet at 5:07 pm, then if I arrive at 5:10 pm, you might be confused, since the implied uncertainty of that time value is more like 1 minute.

Scientists use the mathematics of probability and statistics, along with some intuition, to be precise and clear when talking about uncertainty, and it is vital to understand and report the uncertainty of quantitative results that we present.

B.1 Types of measurement uncertainty

For simplicity, we limit ourselves to the consideration of two types of uncertainty in this lab course, instrumental and random uncertainty.

Instrumental uncertainties

Every measuring instrument has an inherent uncertainty that is determined by the precision of the instrument. Usually this value is taken as a half of the smallest increment of the instrument's scale. For example, 0.5 mm is the precision of a standard metric ruler; 0.5 s is the precision of a watch, etc. For electronic digital displays, the equipment's manual often gives the instrument's resolution, which may be larger than that given by the rule above.

Instrumental uncertainties are the easiest ones to estimate, but they are not the only source of the uncertainty in your measured value. You must be a skillful experimentalist to get rid of all other sources of uncertainty so that all that is left is instrumental uncertainty.

[1] The phrase "most likely" can mean different things depending on who is writing. If a physicist gives the value and does not given a further explanation, we can assume that they mean that the measurements are randomly distributed according to a normal distribution around the value given, with a standard deviation of the uncertainty given. So if one were to make the same measurement again, the author believes it has a 68% chance of falling within the range given. Disciplines other than physics may intend the uncertainty to be 2 standard deviations.

Random uncertainties

Very often when you measure the same physical quantity multiple times, you can get different results each time you measure it. That happens because different uncontrollable factors affect your results randomly. This type of uncertainty, random uncertainty, can be estimated only by repeating the same measurement several times. For example if you measure the distance from a cannon to the place where the fired cannonball hits the ground, you could get different distances every time you repeat the same experiment.

For example, say you took three measurements and obtained 55.7, 49.0, 52.5, 42.4, and 60.2 meters. We can quantify the variation in these measurements by finding their standard deviation using a calculator, spreadsheet, or the formula (assuming the data distributed according to a normal distribution)

$$\sigma = \sqrt{\sum_{i=1}^{N} \frac{(x_i - \bar{x})^2}{N-1}}\,, \tag{B.1}$$

where $\{x_1, x_2, \ldots, x_N\}$ are the measured values, $\bar{x}$ is the mean of those values, and N is the number of measurements. For our example, the resulting standard deviation is 6.8 meters. Generally we are interested not in the variation of the measurements themselves, but how uncertain we are of the average of the measurements. The uncertainty of this mean value is given, for a normal distribution, by the so-called "standard deviation of the mean", which can be found by dividing the standard deviation by the square root of the number of measurements,

$$\sigma_{\text{mean}} = \frac{\sigma}{\sqrt{N}}\,. \tag{B.2}$$

So, in this example, the uncertainty of the mean is 3.0 meters. We can thus report the length as $52 \pm 3\,\text{m}$.

Note that if we take more measurements, the standard deviation of those measurements will not generally change, since the variability of our measurements shouldn't change over time. However, the standard deviation of the mean, and thus the uncertainty, will decrease.

B.2 Propagation of uncertainty

When we use an uncertain quantity in a calculation, the result is also uncertain. To determine by how much, we give some simple rules for basic calculations, and then a more general rule for use with any calculation which requires knowledge of calculus. Note that these rules are strictly valid only for values that are normally distributed, though for the purpose of this course, we will use these formulas regardless of the underlying distributions, unless otherwise stated, for simplicity.

If the measurements are completely independent of each other, then for quantities $a \pm \delta a$ and $b \pm \delta b$, we can use the following formulas:

$$\text{For } c = a + b \text{ (or for subtraction), } \delta c = \sqrt{(\delta a)^2 + (\delta b)^2} \tag{B.3}$$

$$\text{For } c = ab \text{ (or for division), } \frac{\delta c}{c} = \sqrt{\left(\frac{\delta a}{a}\right)^2 + \left(\frac{\delta b}{b}\right)^2} \tag{B.4}$$

$$\text{For } c = a^n,\ \frac{\delta c}{c} = n\frac{\delta a}{a} \tag{B.5}$$

For other calculations, there is a more general formula not discussed here.

Expression	Implied uncertainty
12	0.5
12.0	0.05
120	5
120.	0.5

Table B.1: Expression of numbers and their implied uncertainty.

What if there is no reported uncertainty?

Sometimes you'll be calculating with numbers that have no uncertainty given. In some cases, the number is exact. For example, the circumference C of a circle is given by $C = 2\pi r$. Here, the coefficient, 2π, is an exact quantity and you can treat its uncertainty as zero. If you find a value that you think is uncertain, but the uncertainty is not given, a good rule of thumb is to assume that the uncertainty is half the right-most significant digit. So if you are given a measured length of 1400 m, then you might assume that the uncertainty is 50 m. This is an assumption, however, and should be described as such in your lab report. For more examples, see Table B.1.

How many digits to report?

After even a single calculation, a calculator will often give ten or more digits in an answer. For example, if I travel 11.3 ± 0.1km in 350 ± 10s, then my average speed will be the distance divided by the duration. Entering this into my calculator, I get the resulting value "`0.0322857142857143`". Perhaps it is obvious that my distance and duration measurements were not precise enough for all of those digits to be useful information. We can use the propagated uncertainty to decide how many decimals to include. Using the formulas above, I find that the uncertainty in the speed is given by my calculator as "`9.65683578099600e-04`", where the '`e`' stands for "times ten to the". I definitely do not know my uncertainty to 14 decimal places. For reporting uncertainties, it general suffices to use just the 1 or 2 left-most significant digits, unless you have a more sophisticated method of quantifying your uncertainties. So here, I would round this to 1 significant digit, resulting in an uncertainty of 0.001 km/s. Now I have a guide for how many digits to report in my value. Any decimal places to the right of the one given in the uncertainty are distinctly unhelpful, so I report my average speed as "0.032 ± 0.001 km/s". You may also see the equivalent, more succinct notation "0.032(1) km/s".

B.3 Comparing two values

If we compare two quantities and want to find out how different they are from each other, we can use a measure we call a t' value (pronounced "tee prime"). This measure is not a standard statistical measure, but it is simple and its meaning is clear for us.

Operationally, for two quantities having the same unit, $a \pm \delta a$ and $b \pm \delta b$, the measure is defined as[2]

$$t' = \frac{|a-b|}{\sqrt{(\delta a)^2 + (\delta b)^2}} \tag{B.6}$$

If $t' \lesssim 1$, then the values are so close to each other that they are indistinguishable. It is either that they represent the same true value, or that the measurement should be improved to reduce the uncertainty.

[2] Statistically, if δa and δb are uncorrelated, random uncertainties, then t' represents how many standard deviations the difference $a - b$ is away from zero.

If $1 \lesssim t' \lesssim 3$, then the result is inconclusive. One should improve the experiment to reduce the uncertainty.

If $t' \gtrsim 3$, then the true values are very probably different from each other.