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The Generalized Missile Equations of Motion

2.1 Coordinate Systems

2.1.1 Transformation Properties of Vectors

In a rectangular system of coordinates, a vector can be completely specified by its components. These components depend, of course, upon the orientation of the coordinate system, and the same vector may be described by many different triplets of components, each of which refers to a particular system of axes. The three components that represent a vector in one set of axes, will be related to the components along another set of axes, as are the coordinates of a point in the two systems. In fact, the components of a vector may be regarded as the coordinates of the end of the vector drawn from the origin. This fact is expressed by saying that the scalar components of a vector transform as do the coordinates of a point. It is possible to concentrate attention entirely on the three components of a vector and to ignore its geometrical aspect. A vector would then be defined as a set of three numbers that transform as do the coordinates of a point when the system of axes is rotated. It is often convenient to designate the coordinate axes by numbers instead of letters x, y, z so that the components of a vector will be a_1, a_2, and a_3. The designation for the whole vector is a_i, where it is understood that the subscript i can take on the value 1, 2, or 3. A vector equation is then written in the form

$$a_i = b_i . \tag{2.1}$$

This represents three equations, one for each value of the subscript i. The rotation of a system of coordinates about the origin may be represented by nine quantities $\gamma_{ij'}$, where $\gamma_{ij'}$ is the cosine of the angle between the i-axis in one position of the coordinates and the j-axis in the other position. These nine quantities give the angles made by each of the axes in one position with each of the axes in the other. They are also the coefficients in the expression for the transformation of the coordinates of a point. The cosines can be conveniently kept in order by writing them in the form of a matrix:

$$\begin{bmatrix} \gamma_{11j'} & \gamma_{12'} & \gamma_{13'} \\ \gamma_{21'} & \gamma_{22'} & \gamma_{23'} \\ \gamma_{31'} & \gamma_{32'} & \gamma_{33'} \end{bmatrix}. \tag{2.2}$$

Of the nine quantities, only three are independent, since there are six independent relations between them. Since $\gamma_{ij'}$ can be considered as the component along the j'-axis in one coordinate system of a unit vector along the i-axis in the other, then

$$\gamma_{i1'}^2 + \gamma_{i2'}^2 + \gamma_{i3'}^2 = \sum_{j'} \gamma_{ij'}^2 = 1. \tag{2.3a}$$

This will be true for every value of i. Similarly,

$$\sum_{i'} \gamma_{ij'}^2 = 1. \tag{2.3b}$$

The components of a vector, or the coordinates of a point, can be transformed from one system of coordinates to the other by

$$a_i = \gamma_{i1'}a_{1'} + \gamma_{i2'}a_{2'} + \gamma_{i3'}a_{3'} = \gamma_{ij'}a_{j'}. \tag{2.4}$$

Here $a_{j'}$ represents the components of the vector a in one system of coordinates, and a_i the components in the other. The summation sign is omitted in the last term, since it is to be understood that a sum is to be carried out over all three values of any index that is repeated.

2.1.2 Linear Vector Functions

If a vector is a function of a single scalar variable, such as time, each component of the vector is independently a function of this variable. If the vector is a linear function of time, then each component is proportional to the time. A vector may also be a function of another vector. In general, this implies that each component of the function depends on each component of the independent vector. Moreover, a vector is a linear function of another vector if each component of the first is a linear function of the three components of the second. This requires nine independent coefficients of proportionality. The statement that a is a linear function of b means that

$$\begin{aligned} a_1 &= C_{11}b_1 + C_{12}b_2 + C_{13}b_3, \\ a_2 &= C_{21}b_1 + C_{22}b_2 + C_{23}b_3, \\ a_3 &= C_{31}b_1 + C_{32}b_2 + C_{33}b_3. \end{aligned} \tag{2.5}$$

Using the summation convention as in (2.4), this becomes

$$a_i = C_{ij}b_j. \tag{2.6}$$

A relationship such as that in (2.6) must be independent of the coordinate system in spite of the fact that the notation is clearly based on specific coordinates. The components a_i and b_i are with reference to a particular coordinate system. The constants C_{ij} also have reference to specific axes, but they must so transform with a rotation of axes that a given vector b always leads to the same vector a.

If the coordinate system is rotated about the origin, the vector components will change so that

$$a_i = \gamma_{ij'} a_{j'} = C_{ij} \gamma_{jk'} b_{k'}. \tag{2.7}$$

If both sides of this equation are multiplied by $\gamma_{l'i}$ and the equations for the three values of i are added, the result is

$$\gamma_{l'i} \gamma_{ij'} a_{j'} = a_{l'} = (\gamma_{l'i} C_{ij} \gamma_{jk'}) b_{k'}. \tag{2.8}$$

If the quantity $\gamma_{l'i} C_{ij} \gamma_{jk'}$ is called $C_{l'k'}$, then

$$a_{i'} = C_{l'k'} b_{k'}. \tag{2.9}$$

This relationship between the components in this system of coordinates is the same vector relationship as was expressed by the C_{ik} in the original system of coordinates.

2.1.3 Tensors

Tensor is a general name given to quantities that transform in prescribed ways when the coordinate system is rotated. A *scalar* is a tensor of rank 0, for it is independent of the coordinate system. A *vector* is a tensor of rank 1. Its components transform as do the coordinates of a point. A *tensor* of rank 2 has components that transform as do the quantities C_{ij}. Put another way, a scalar is a quantity whose specification (in any coordinate system) requires just one number. On the other hand, a vector (originally defined as a directed line segment) is a quantity whose specification requires three numbers, namely, its components with respect to some basis. In essence, scalars and vectors are both special cases of a more general object called a *tensor of order n*, whose specification in any given coordinate system require 3^n numbers, again called the *components* of the tensor. In fact,

scalars are tensors of order 0, with $3^0 = 1$ components,
vectors are tensors of order 1, with $3^1 = 3$ components.

Tensors can be added or subtracted by adding or subtracting their corresponding components. They can also be multiplied in various ways by multiplying components in various combinations. These and other possible operations with tensors will not be described here.

A tensor of the second rank is said to be symmetric if $C_{ij} = C_{ji}$ and to be antisymmetric if $C_{ij} = -C_{ji}$. An antisymmetric tensor has its diagonal components equal to zero. Any tensor may be regarded as the sum of a symmetric and an antisymmetric part for

$$C_{ij} = \tfrac{1}{2}[C_{ij} + C_{ji}] + \tfrac{1}{2}[C_{ij} - C_{ji}] \tag{2.10a}$$

and

$$\tfrac{1}{2}[C_{ij} + C_{ji}] = S_{ij} \quad \tfrac{1}{2}[C_{ij} - C_{ji}] = A_{ij}, \tag{2.10b}$$

where S_{ij} is symmetric and A_{ij} is antisymmetric. Numerous physical quantities have the properties of tensors of the second rank, so that the inertial properties of a rigid body can be described by the symmetric tensor of inertia. By way of illustration, consider that we are given two vectors A and B. There are nine products of a component of A with a component of B. Thus,

$$A_i B_i (i, k = 1, 2, 3).$$

Suppose we transform to a new coordinate system K', in which A and B have components A_i' and B_k'. Then the transformation of a coordinate system can be expressed as

$$A_i = \alpha_{i'} A_k,$$

where A_k, A_i' are the components of the vector in the old and new coordinate systems K and K', respectively, and $\alpha_{i'k}$ is the cosine of the angle between the ith axis of K' and the kth axis of K. Thus,

$$A_i' = \alpha_{i'k} A_i,\ B_k' = \alpha_{k'm} B_m,$$

and hence

$$A_i' B_k' = \alpha_{i'l} \alpha_{k'm} A_l B_m.$$

Therefore, $A_i B_k$ is a second-order tensor.

2.1.4 Coordinate Transformations

There are three commonly used methods of expressing the orientation of one three-axis coordinate system with respect to another. The three methods are (1) *Euler* angles, (2) *direction cosines*, and (3) *quaternions*. The Euler angle method, which is the conventional designation relating a moving-axis system to a fixed-axis system, is used frequently in missile and aircraft mechanizations and/or simulations. The common designations of the Euler angles are roll (ϕ), pitch (θ), and yaw (ψ). Its strengths lie in a relatively simple mechanization in digital computer simulation of vehicle (i.e., missile or aircraft) dynamics. Another beneficial aspect of this technique is that the Euler angle rates and the Euler angles have an easily interpreted physical significance. The negative attribute to the Euler angle coordinate transformation method is the mathematical singularity that exists when the pitch angle θ approaches 90°. The direction cosine method yields the direction cosine matrix (*DCM*), which defines the transformation between a fixed frame, say frame a, and a rotating frame, say frame b, such as the vehicle body axes. Specifically, the *DCM* is an array of direction cosines expressed in the form

$$C_a^b = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix},$$

where c_{jk} is the direction cosine between the jth axis in the a frame and the kth axis in the b frame. Since each axis system has three unit vectors, there are nine direction cosines. Direction cosines have the advantage of being free of any singularities such as arise in the Euler angle formulation at 90° pitch angle. The main disadvantage of this method is the number of equations that must be solved due to the constraint equations. (Note that by constraint equations we mean $c_{11} = c_{22}c_{33} - c_{23}c_{32}$, $c_{21} = c_{13}c_{32} - c_{12}c_{33}$, etc.)

In order to resolve the ambiguity resulting from the singularity in the Euler angle representation of rotations about the three axes, a four-parameter system was first developed by Euler in 1776. Subsequently, Hamilton modified it in 1843, and he named this system the quaternion system. Therefore, a quaternion [Q] is a quadruple of real numbers, which can be written as a three-dimensional vector. Hamilton adopted a vector notation in the form

$$[Q] = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3 = (q_0, q_1, q_2, q_3) = (q_0, \mathbf{q}), \tag{2.11}$$

where q_0, q_1, q_2, q_3 are real numbers and the set $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ forms a basis for a quaternion vector space. From the orthogonality property of quaternions, we have

$$q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1. \tag{2.12}$$

In terms of the Euler angles ϕ, θ, ψ, we have

$$\begin{aligned} q_0 &= \cos(\psi/2)\cos(\theta/2)\cos(\phi/2) - \sin(\psi/2)\sin(\theta/2)\sin(\phi/2), \\ q_1 &= \sin(\theta/2)\sin(\phi/2)\cos(\psi/2) + \sin(\psi/2)\cos(\theta/2)\cos(\phi/2), \\ q_2 &= \sin(\theta/2)\cos(\psi/2)\cos(\phi/2) - \sin(\psi/2)\sin(\phi/2)\cos(\theta/2), \\ q_3 &= \sin(\phi/2)\cos(\psi/2)\cos(\theta/2) + \sin(\psi/2)\sin(\theta/2)\cos(\phi/2). \end{aligned}$$

Suppose now that we wish to transform any vector, say $\mathbf{V}$, from body coordinates $\mathbf{V}^b$ into the navigational coordinates $\mathbf{V}^n$. This transformation can be expressed as follows:

$$\mathbf{V}^n = C_b^n \mathbf{V}^b,$$

where C_b^n is the direction cosine matrix, or equivalently, using quaternions,

$$\mathbf{V}^n = q\mathbf{V}^b q^*,$$

where q^* is the conjugate of q. Then [7]

$$C_b^n = \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1q_2 - q_0q_3) & 2(q_1q_3 + q_0q_2) \\ 2(q_1q_2 + q_0q_3) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2q_3 - q_0q_1) \\ 2(q_1q_3 - q_0q_2) & 2(q_2q_3 + q_0q_1) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}.$$

For more details on the quaternion and its properties, the reader is referred to [7].

The coordinate system that will be adopted in the present discussion is a right-handed system with the positive x-axis along the missile's longitudinal axis, the y-axis positive to the right (or aircraft right wing), and the z-axis positive down (i.e., the z-axis is defined by the cross product of the x- and y-axis). This coordinate system is also known as *north-east-down* (*NED*) in reference to the inertial north-east-down sign convention [5], [7]. It should be noted here that the coordinate system used in the present development is the same one used in aircraft. Four orthogonal-axes systems are usually defined to develop the appropriate equations of vehicle (aircraft or missile) motion. They are as follows:

1. The *inertial* frame, which is fixed in space, and for which Newton's Laws of Motion are valid.
2. An *Earth-centered* frame that rotates with the Earth.
3. An *Earth-surface* frame that is parallel to the Earth's surface, and whose origin is at the vehicle's center of gravity (cg) defined in north, east, and down directions.
4. The conventional *body* axes are selected to represent the vehicle. The center of this frame is at the cg of the vehicle, and its components are forward, out of the right wing, and down.

In ballistic missiles, two other common coordinate systems are used. These coordinate systems are

1. *Launch Centered Inertial:* This system is inertially fixed and is centered at launch site at the instant of launch. In this system, the x-axis is commonly taken to be in the horizontal plane and in the direction of launch, the positive z-axis vertical, and the y-axis completing the right-handed coordinate system.
2. *Launch Centered Earth-Fixed:* This is an Earth-fixed coordinate system, having the same orientation as the inertial coordinate system (1). This system is advantageous in gimbaled inertial platforms in that it is not necessary to remove the Earth rate torquing signal from the gyroscopes at launch.

Figure 2.1 illustrates two posible methods for defining the missile body axes with respect to the Earth and/or inertial reference axes. These coordinate frames will be used to define the missile's position and angular orientation in space.

Referring to Figure 2.1, we will denote the Earth-fixed coordinate system by (X_e, Y_e, Z_e). In this right-handed coordinate system, the $X_e - Y_e$ lie in the horizontal plane, and the Z_e-axis points down vertically in the direction of gravity. (Note that the position of the missile's center of gravity at any instant of time is given in this coordinate system). The second coordinate system, the body axis system, denoted by (X_b, Y_b, Z_b), is fixed with respect to the missile, and thus moves with the missile. This is the missile body coordinate system. The positive X_b-axis coincides with the missile's center line (or longitudinal axis) or forward direction. The positive Y_b-axis is to the right of the X_b-axis in the horizontal plane and is designated as the pitch axis. The positive Z_b-axis is the yaw axis and points down. This coordinate system is similar to the *NED* system. The Euler angles (ψ, θ, ϕ) are commonly used to define the missile's attitude

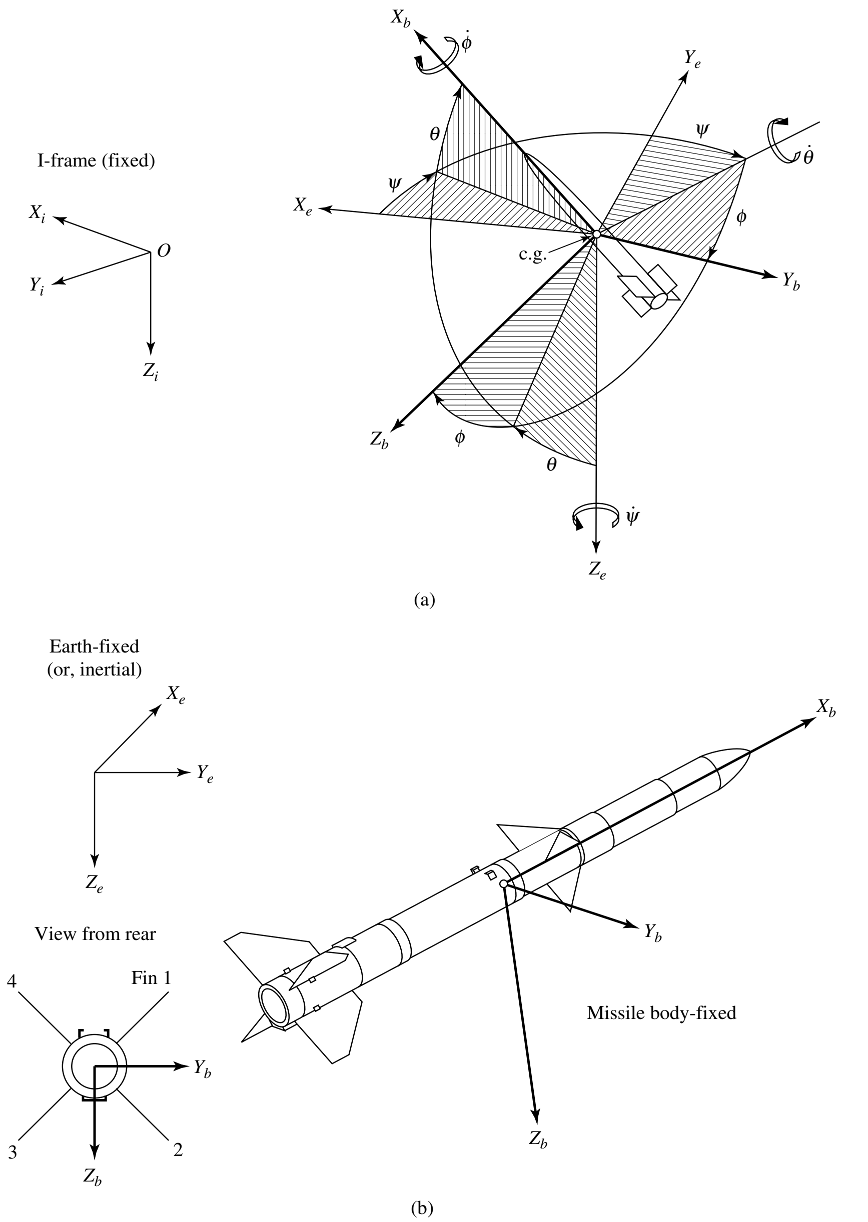

Fig. 2.1. Orientation of the missile axes with respect to the Earth-fixed axes.

with respect to the Earth-fixed axes. These Euler angles are illustrated in Figure 2.1, whereby the order of rotation of the missile axes is *yaw*, *pitch*, and *roll*. This figure also illustrates the angular rates of the Euler angles. The transformation C_e^b from the Earth-fixed axes coordinate system to the missile body-axes frame is achieved by a

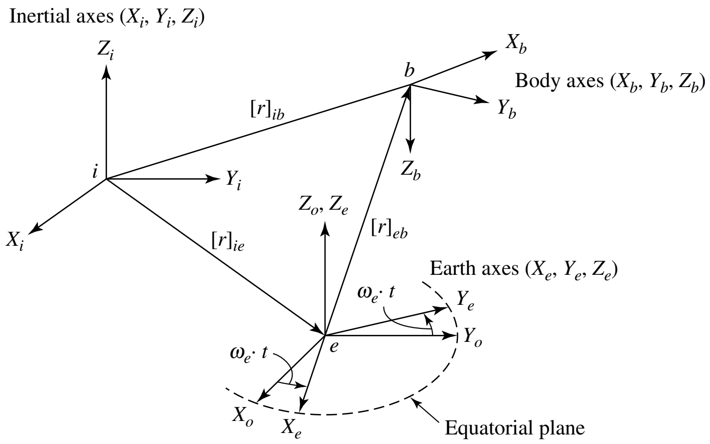

Fig. 2.2. Representation of the inertial coordinate system (inertial, Earth, and body coordinate systems).

yaw, pitch, and roll rotation about the longitudinal, lateral, and normal (i.e., vertical) axes, respectively. The resultant transformation matrix C_e^b is [2], [7]

$$
\begin{aligned}
C_e^b &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \sin\phi\cos\theta \\ \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi & \cos\phi\cos\theta \end{bmatrix}.
\end{aligned}
$$

It should be noted here that ambiguities (or singularities) can result from using the above transformation (i.e., as θ, ϕ, $\psi \to 90°$). Therefore, in order to avoid these ambiguities, the ranges of the Euler angles (ϕ, θ, ψ) are limited as follows:

$$
\begin{array}{lcl}
-\pi \le \phi < \pi & \text{or} & 0 \le \phi < 2\pi, \\
-\pi \le \psi < \pi, & & \\
-\pi/2 \le \theta \le \pi/2 & \text{or} & 0 \le \psi < 2\pi.
\end{array}
$$

The inertial coordinate system described above is shown in Figure 2.2.

2.2 Rigid-Body Equations of Motion

In this section we will consider a typical missile and derive the equations of motion according to Newton's laws. In deriving the rigid-body equations of motion, the following assumptions will be made:

1. **Rigid Body:** A rigid body is an idealized system of particles. Furthermore, it will be assumed that the body does not undergo any change in size or shape.

Translation of the body results in that every line in the body remains parallel to its original position at all times. Consequently, the rigid body can be treated as a particle whose mass is that of the body and is concentrated at the center of mass. In assuming a rigid body, the aeroelastic effects are not included in the equations. With this assumption, the forces acting between individual elements of mass are eliminated. Furthermore, it allows the airframe motion to be described completely by a translation of the center of gravity and by a rotation about this point. In addition, the airframe is assumed to have a plane of symmetry coinciding with the vertical plane of reference. The vertical plane of reference is the plane defined by the missile X_b- and Z_b-axes as shown in Figure 2.1. The Y_b-axis, which is perpendicular to this plane of symmetry, is the principal axis, and the products of inertia I_{XY} and I_{YZ} vanish.

2. **Aerodynamic Symmetry in Roll:** The aerodynamic forces and moments acting on the vehicle are assumed to be invariant with the roll position of the missile relative to the free-stream velocity vector. Consequently, this assumption greatly simplifies the equations of motion by eliminating the aerodynamic cross-coupling terms between the roll motion and the pitch and yaw motions. In addition, a different set of aerodynamic characteristics for the pitch and yaw is not required.
3. **Mass:** A constant mass will be assumed, that is, $dm/dt \cong 0$.

In addition, the following assumptions are commonly made:

4. The missile equations of motion are written in the body-axes coordinate frame.
5. A spherical Earth rotating at a constant angular velocity is assumed.
6. The vehicle aerodynamics are nonlinear.
7. The undisturbed atmosphere rotates with the Earth.
8. The winds are defined with respect to the Earth.
9. An inverse-square gravitational law is used for the spherical Earth model.
10. The gradients of the low-frequency winds are small enough to be neglected.

Furthermore, in the present development, it will be assumed that the missile has *six degrees of freedom* (6-DOF). The six degrees of freedom consist of (1) three translations, and (2) three rotations, along and about the missile (X_b, Y_b, Z_b) axes. These motions are illustrated in Figure 2.3, the translations being (u, v, w) and the rotations (P, Q, R). In compact form, the traslation and rotation of a rigid body may be expressed mathematically by the following equations:

$$\mathit{Translation}: \sum \mathbf{F} = m\mathbf{a}, \tag{2.13}$$

$$\mathit{Rotation}: \sum \tau = \frac{d}{dt}(\boldsymbol{r} \times m\mathbf{V}) \tag{2.14}$$

where $\sum \tau$ is the net torque on the system.

Aerodynamic forces and moments are assumed to be functions of the *Mach** number (M) and nonlinear with flow incidence angle. Furthermore, the introduction

*The Mach number is expressed as $M = V_M/V_s$, where V_M is the velocity of the missile and V_s is the local velocity of sound, a piecewise linear function of the missile's altitude.

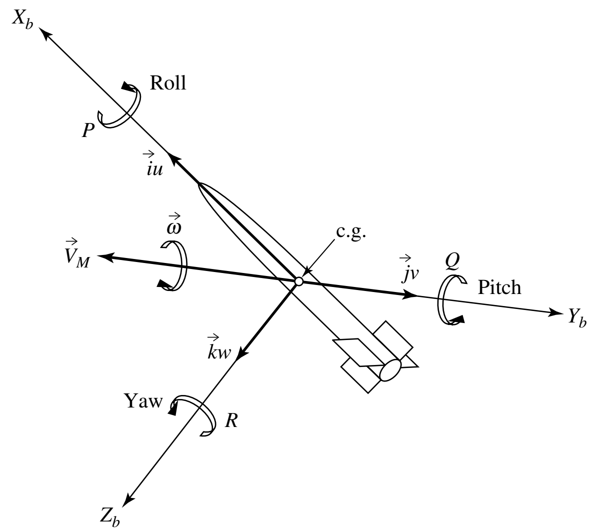

Fig. 2.3. Representation of the missile's six degrees of freedom.

of surface winds in a trajectory during launch can create flow incidence angles that are very large, on the order of 90°. Nonlinear aerodynamic characteristics with respect to flow incidence angle must be assumed to simulate the launch motion under the effect of wind. Since Mach number varies considerably in a missile trajectory, it is necessary to assume that the aerodynamic characteristics vary with Mach number.

The linear velocity of the missile **V** can be broken up into components u, v, and w along the missile (X_b, Y_b, Z_b) body axes, respectively. Mathematically, we can write the missile vector velocity, V_M, in terms of the components as

$$\mathbf{V}_M = u\mathbf{i} + v\mathbf{j} + w\mathbf{k},$$

where (**i**, **j**, **k**) are the unit vectors along the respective missile body axes. The magnitude of the missile velocity is given by

$$|\mathbf{V}_M| = V_M = (u^2 + v^2 + w^2)^{1/2}.$$

These components are illustrated in Figure 2.3.

In a similar manner, the missile's angular velocity vector $\boldsymbol{\omega}$ can be broken up into the components P, Q, and R about the (X_b, Y_b , Z_b) axes, respectively, as follows:

$$\boldsymbol{\omega} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k},$$

where P is the roll rate, Q is the pitch rate, and R is the yaw rate. Note that some authors use lowercase letters for roll, pitch, and yaw rates instead of uppercase letters. Therefore, these linear and rotational velocity components constitute the 6-*DOF* of the missile. As stated in the beginning of this section, the rigid-body equations of motion are obtained from Newton's second law, which states that the summation of all external *forces* acting on a body is equal to the time rate of the momentum of the body, and the summation of the external *moments* acting on the body is equal to the time rate of change of *moment of momentum* (angular momentum). Specifically, Newton's laws of motion were formulated for a single particle. Assuming that the mass m of the particle is multiplied by its velocity $\mathbf{V}$, then the product

$$\mathbf{p} = m\mathbf{V} \tag{2.15}$$

is called the *linear momentum*. Thus, the linear momentum is a vector quantity having the same direction and sense as $\mathbf{V}$. For a system of n particles, the linear momentum is the summation of the linear momenta of all particles in the system. Thus [8],

$$\mathbf{p} = \sum_{i=1}^{n} (m_i \mathbf{V}_i) = m_1\mathbf{V}_1 + m_2\mathbf{V}_2 + \cdots + m_n\mathbf{V}_n, \tag{2.16}$$

where i denotes the ith particle, and n denotes the number of particles in the system. Note that the time rates of change of linear and angular momentum are referred to an absolute or inertial reference frame. For many problems of interest in airplane and missile dynamics, an axis system fixed to the Earth can be used as an inertial reference frame (see Figure 2.1). Mathematically, Newton's second law can be expressed in terms of conservation of both linear and angular momentum by the following vector equations [1], [8], [11]:

$$\sum \mathbf{F} = \left.\frac{d(m\mathbf{V}_M)}{dt}\right]_I, \tag{2.17a}$$

$$\sum \mathbf{M} = \left.\frac{d\mathbf{H}}{dt}\right]_I, \tag{2.17b}$$

where m is the mass, $\mathbf{H}$ the angular momentum, and the symbol $]_I$ indicates the time rate of change of the vector with respect to inertial space. Note that (2.17a) is simply

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}, \tag{2.18a}$$

or

$$\mathbf{F} = m\left(\frac{d\mathbf{V}}{dt}\right) = m\mathbf{a}. \tag{2.18b}$$

Equations (2.17a) and (2.17b) can be rewritten in scalar form, consisting of three force equations and three moment equations as follows:

$$F_x = \frac{d(mu)}{dt},\ F_y = \frac{d(mv)}{dt},\ F_z = \frac{d(mw)}{dt}, \tag{2.19}$$

where F_x, F_y, F_z and u, v, w are the components of the force and velocity along the missile's X_b, Y_b, and Z_b axes, respectively. Normally, these force components are composed of contributions due to (1) aerodynamic, (2) propulsive, and (3) gravitational forces acting on the missile. In a similar manner, the moment equations can be expressed as follows [6]:

$$L = \frac{dH_x}{dt}, \quad M = \frac{dH_y}{dt}, \quad N = \frac{dH_z}{dt}, \tag{2.20}$$

where L, M, N are the roll moment, pitch moment, and yaw moment, respectively, and H_x, H_y, H_z are the components of the moment of momentum along the body X, Y, and Z axes, respectively.

At this point, let us summarize the various forces, moments, and axes used in developing the missile 6-*DOF* equations of motion.

Force:

$$\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k},$$

where F_x, F_y, F_z are the (x, y, z) components of the force.

Velocity:

$$\mathbf{V} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k},$$

where u, v, w are the velocity components along the (x, y, z) axes, respectively.

Moment of External Forces:

$$\sum \Delta\mathbf{M} = L\mathbf{i} + M\mathbf{j} + N\mathbf{k},$$

where L is the rolling moment, M is the pitching moment, and N is the yawing moment.

Angular Momentum:

$$\mathbf{H} = H_x\mathbf{i} + H_y\mathbf{j} + H_z\mathbf{k},$$

where H_x, H_y, H_z are the components of the angular momentum along the x, y, z axes, respectively.

Angular Velocity:

$$\boldsymbol{\omega} = \omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k},$$

where P is the roll rate, Q is the pitch rate, and R is the yaw rate. ($\mathbf{i}$ = unit vector along the x-axis, $\mathbf{j}$ = unit vector along the y-axis, and $\mathbf{k}$ = unit vector along the z-axis).

We now wish to develop an expression for the time rate of change of the velocity vector with respect to the Earth. Before we do this, we note that in general, a vector $\mathbf{A}$ can be transformed from a fixed (e.g., inertial) to a rotating coordinate system by the relation [6], [7]

$$\left(\frac{d\mathbf{A}}{dt}\right)_{\text{fixed}(X',Y',Z')} = \left[\frac{d\mathbf{A}}{dt}\right]_{\text{rot.}(X,Y,Z)} + \boldsymbol{\omega} \times \mathbf{A}, \tag{2.21a}$$

Table 2.1. Axis and Moment Nomenclature

(a) Axis Definition

Axis	Direction	Name	Linear Velocity	Angular Displacement	Angular Rates
OX	Forward	Roll	u	ϕ	P
OY	Right Wing	Pitch	v	θ	Q
OZ	Downward	Yaw	w	ψ	R

(b) Moment Designation

Axis	Moment of Inertia	Product of Inertia	Force	Moment
OX	I_x	$I_{xy} = 0$	F_x	L
OY	I_y	$I_{yx} = 0$	F_y	M
OZ	I_z	$I_{zx} \neq 0$	F_z	N

or

$$\left(\frac{d\mathbf{A}}{dt}\right)_{\text{inertial}} = \left[\frac{d\mathbf{A}}{dt}\right]_{\text{body}} + \boldsymbol{\omega} \times \mathbf{V}_M, \tag{2.21b}$$

where $\boldsymbol{\omega}$ is the angular velocity of the missile body coordinate system (X, Y, Z) relative to the fixed (inertial) system (X', Y', Z'), and $\times$ denotes the vector cross product. Normally, the missile's linear velocity V_M is expressed in the Earth-fixed axis system, so that (2.21a) can be written in the form

$$\left(\frac{d\mathbf{V}_M}{dt}\right)_E = \left(\frac{d\mathbf{V}_M}{dt}\right)_{\text{rot.coord.}} + \boldsymbol{\omega} \times \mathbf{V}_M, \tag{2.22}$$

where ω is the total angular velocity vector of the missile with respect to the Earth. In terms of the body axes, we can write the force equation in the form

$$\mathbf{F} = m\left[\frac{d\mathbf{V}_M}{dt}\right]_{\text{body}} + m(\boldsymbol{\omega} \times \mathbf{V}_M). \tag{2.23}$$

The first part on the right-hand side of (2.22) can be written as

$$\left(\frac{d\mathbf{V}_M}{dt}\right)_{\text{rot.coord.}} = \left(\frac{du}{dt}\right)\mathbf{i} + \left(\frac{dv}{dt}\right)\mathbf{j} + \left(\frac{dw}{dt}\right)\mathbf{k}, \tag{2.24}$$

where

$(du/dt) =$ forward (or longitudinal) acceleration,
$(dv/dt) =$ right wing (or lateral) acceleration,
$(dw/dt) =$ downward (or vertical) acceleration,

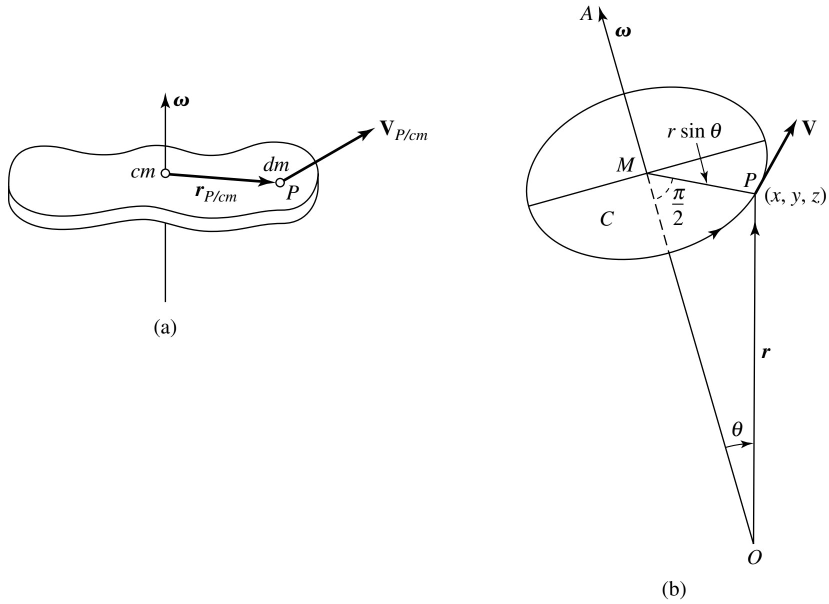

Fig. 2.4. General rigid body with angular velocity vector $\boldsymbol{\omega}$ about its center of mass.

and the vector cross product as

$$\boldsymbol{\omega} \times \mathbf{V}_M = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ P & Q & R \\ u & v & w \end{bmatrix} = (wQ - vR)\mathbf{i} + (uR - wP)\mathbf{j} + (vP - uQ)\mathbf{k}. \tag{2.25}$$

Next, from (2.17a) we can write the sum of the forces as

$$\sum \Delta \mathbf{F} = \sum \Delta F_x \mathbf{i} + \sum \Delta F_y \mathbf{j} + \sum \Delta F_z \mathbf{k}. \tag{2.26}$$

Equating the components of (2.24), (2.25), and (2.26) yields the missile's *linear equations of motion*. Thus, for a missile with an $X_b - Z_b$ plane of symmetry (see rigid-body assumption #1) we have

$$\sum \Delta F_x = m(\dot{u} + wQ - vR), \tag{2.27a}$$

$$\sum \Delta F_y = m(\dot{v} + uR - wP), \tag{2.27b}$$

$$\sum \Delta F_z = m(\dot{w} + vP - uQ). \tag{2.27c}$$

From (2.17b) we can obtain in a similar manner the equations of angular motion. However, before we develop these equations, an expression for $\mathbf{H}$ is needed. To this end, consider Figure 2.4.

Now let dm be an element of mass of the missile, $\mathbf{V}$ the velocity of the elemental mass relative to the inertial frame, and $\delta\mathbf{F}$ the resulting force acting on the elemental mass.

First of all, and with reference to Figure 2.4(a), the position vector of any particle of the rigid body in a Newtonian frame of reference is the vector sum of the position vector of the center of the mass and the position vector of the particle with respect to the center of mass. Mathematically,

$$\boldsymbol{r}_p = \boldsymbol{r}_{cm} + \boldsymbol{r}_{p/cm},$$

where

$$\begin{aligned}
\boldsymbol{r}_p &= \text{the position vector of the particle,}\\
\boldsymbol{r}_{cm} &= \text{position vector of the center of mass of the particle,}\\
\boldsymbol{r}_{p/cm} &= \text{position vector of the particle with respect to the}\\
&\quad\ \text{center of mass.}
\end{aligned}$$

Note that if this equation is differentiated, we obtain

$$\frac{d\boldsymbol{r}_p}{dt} = \frac{d\boldsymbol{r}_{cm}}{dt} + \frac{d\boldsymbol{r}_{p/cm}}{dt}.$$

Also, from Figure 2.4(a) we can write the velocity of the point p in the form

$$\mathbf{V}_p = \frac{d(\boldsymbol{r}_{cm})}{dt} + \boldsymbol{\omega} \times \boldsymbol{r}_{p/cm},$$

or

$$\mathbf{V}_p = \mathbf{V}_{cm} + \mathbf{V}_{p/cm}.$$

Then, from Newton's second law we have

$$\delta\mathbf{F} = dm\left(\frac{d\mathbf{V}}{dt}\right). \tag{2.28}$$

The total external force acting on the missile is found by summing all the elements of the missile. Therefore,

$$\sum \delta\mathbf{F} = \mathbf{F}. \tag{2.29}$$

The velocity of the differential mass dm is

$$\mathbf{V} = \mathbf{V}_{cm} + \left(\frac{d\mathbf{r}}{dt}\right), \tag{2.30}$$

where $\mathbf{V}_{cm}$ is the velocity of the center of mass (cm) of the missile, and $d\mathbf{r}/dt$ is the velocity of the element relative to the center of mass. Substituting (2.30) for the velocity into (2.29) results in

$$\sum \delta\mathbf{F} = \mathbf{F} = \left(\frac{d}{dt}\right)\sum\left[\mathbf{V}_{cm} + \left(\frac{d\mathbf{r}}{dt}\right)\right] dm. \tag{2.31}$$

Assuming that the mass of the missile is constant, (2.31) can be written in the form

$$\mathbf{F} = m\left(\frac{d\mathbf{V}_{cm}}{dt}\right) + \left(\frac{d}{dt}\right)\sum\left(\frac{d\mathbf{r}}{dt}\right)dm, \tag{2.32a}$$

or

$$\mathbf{F} = m\left(\frac{d\mathbf{V}_{cm}}{dt}\right) + \left(\frac{d^2}{dt^2}\right)\sum\mathbf{r}dm. \tag{2.32b}$$

Since $\mathbf{r}$ is measured from the center of the mass, the summation $\sum\mathbf{r}dm$ is equal to 0. Thus, the force equation becomes simply

$$\mathbf{F} = m\left(\frac{d\mathbf{V}_{cm}}{dt}\right), \tag{2.33}$$

which relates the external force on the missile to the motion of the vehicle's center of mass. Similarly, we can develop the moment equation referred to a moving center of mass. For the differential element of mass, dm, the moment equation can then be written as

$$\delta\mathbf{M} = d\left(\frac{\delta\mathbf{H}}{dt}\right) = \left(\frac{d}{dt}\right)(\mathbf{r}\times\mathbf{V})dm. \tag{2.34}$$

The velocity of the mass element can be expressed in terms of the velocity of the center of mass and the relative velocity of the mass element to the center of mass. Therefore,

$$\mathbf{V}_p = \mathbf{V}_{cm} + \left(\frac{d\mathbf{r}_{p/cm}}{dt}\right) = \mathbf{V}_{cm} + \boldsymbol{\omega}\times\mathbf{r}, \tag{2.35}$$

where $\boldsymbol{\omega}$ is the angular velocity vector of the vehicle and $\mathbf{r}$ is the position of the mass element measured from the center of mass (see Figure (2.4a)). In relation to (2.35) and Figure 2.4(a), we can write the equation

$$\left(\frac{d\mathbf{r}}{dt}\right)_{\text{inertial}} = \left[\frac{d\mathbf{r}}{dt}\right]_{\text{rel. to coord.}} + \boldsymbol{\omega}\times\mathbf{r}.$$

The reader will note that this is the well-known *Coriolis* equation, which is important in dynamics where body axes are used. Furthermore, it will be noted that the term $\boldsymbol{\omega}\times\mathbf{r}$ occurs in addition to the vector change relative to the coordinate system, so that the total derivative relative to the inertial axes is expressed by this equation. The rigid-body assumption implies that $d\mathbf{r}_{p/cm}/dt = 0$. Therefore, we can write the linear velocity of the point p in the simple form

$$\mathbf{V}_p = \boldsymbol{\omega}\times\mathbf{r}_{p/cm.}$$

In general, the moment about an arbitrary point O of the momentum $\mathbf{p} = m\mathbf{V}$ (2.15) of a particle is

$$\mathbf{H} = \mathbf{r}\times m\mathbf{V} = m\mathbf{r}\times(\boldsymbol{\omega}\times\mathbf{r}).$$

Referring to Figure (2.4(b)), it will be observed that this vector is perpendicular to both $\mathbf{r}$ and $\mathbf{V}$. Furthermore, this vector lies along MP and is directed toward M. The *moment of momentum* (or *angular momentum*) of the entire body about O is therefore

$$\mathbf{H} = \sum \mathbf{r} \times m\mathbf{V} = \sum m\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r}) = \sum m[\boldsymbol{\omega}(\mathbf{r} \cdot \mathbf{r}) - \mathbf{r}(\mathbf{r} \cdot \boldsymbol{\omega})]$$

(note that this result was obtained using the formula $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$). This equation can also be written as

$$\mathbf{H} = \left(\sum mr^2\right) \boldsymbol{\omega} - \sum m\mathbf{r}(\mathbf{r} \cdot \boldsymbol{\omega}).$$

From Figure (2.4(a)), the total moment of momentum can be written as

$$\mathbf{H} = \sum \delta\mathbf{H} = \sum (\mathbf{r} \times \mathbf{V}_{cm})dm + \sum [\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})]dm. \tag{2.36}$$

Note that the velocity $\mathbf{V}_{cm}$ is constant with respect to the summation and can be taken outside the summation sign. Thus [1], [3]

$$\mathbf{H} = \sum \mathbf{r}dm \times \mathbf{V}_{cm} + \sum [\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})]dm. \tag{2.37a}$$

As stated above, the first term on the right-hand side of (2.37a) is zero. Therefore, we have simply

$$\delta\mathbf{H} = \sum [\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})]dm \tag{2.37b}$$

and

$$\mathbf{H} = \int \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})dm. \tag{2.37c}$$

Performing the vector operations in (2.37c) and noting that

$$\begin{aligned} \boldsymbol{\omega} &= \omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}, \\ \mathbf{r} &= x\mathbf{i} + y\mathbf{j} + z\mathbf{k}, \end{aligned}$$

we have

$$\begin{aligned} \boldsymbol{\omega} \times \mathbf{r} &= \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ P & Q & R \\ x & y & z \end{bmatrix} \\ &= (zQ - yR)\mathbf{i} + (xR - zP)\mathbf{j} + (yP - xQ)\mathbf{k}. \end{aligned} \tag{2.38a}$$

Finally,

$$\begin{aligned} \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r}) &= \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ (zQ - yR) & (xR - zP) & (yP - xQ) \end{bmatrix} \\ &= \mathbf{i}[(y^2 + z^2)P - xyQ - xzR] + \mathbf{j}[(z^2 + x^2)Q - yzR - xyP] \\ &\quad + \mathbf{k}[(x^2 + y^2)R - xzP - yzQ]. \end{aligned} \tag{2.38b}$$

Substituting (2.38b) into (2.37c), we have

$$\mathbf{H} = \int \mathbf{i}[(y^2+z^2)P - xyQ - xzR]dm + \int \mathbf{j}[(z^2+x^2)Q - yzR - xyP]dm$$
$$+ \int \mathbf{k}[(x^2+y^2)R - xzP - yzQ]dm, \quad (2.38c)$$

where the $\int (y^2+z^2)$ is defined as the moment of inertia, I_x, and $\int xydm$ is defined as the product of inertia, I_{xy}. The remaining integrals in (2.38c) are similarly defined. By proper positioning of the body axis system, one can make the products of inertia $I_{xy} = I_{yz}$ equal to 0. This will be true if we can assume that the x-y plane is a plane of symmetry of the missile. Consequently, (2.38c) can be rewritten in component form as follows:

$$H_x = P\int (y^2+z^2)dm - R\int xzdm = PI_x - RI_{xz}, \quad (2.39a)$$
$$H_y = Q\int (x^2+z^2)dm = QI_y, \quad (2.39b)$$
$$H_z = R\int (x^2+y^2)dm - P\int xzdm = RI_z - PI_{xz}. \quad (2.39c)$$

From (2.17b), we note that the time rate of $\mathbf{H}$ is required. Now, since $\mathbf{H}$ can change in magnitude and direction, (2.17b) can be written as [1]

$$\sum \Delta\mathbf{M} = 1_H\left(\frac{d\mathbf{H}}{dt}\right) + \boldsymbol{\omega}\times\mathbf{H}. \quad (2.40)$$

Next, the components of $1_H(d\mathbf{H}/dt)$ assume the form

$$\frac{dH_x}{dt} = \left(\frac{dP}{dt}\right)I_x - \left(\frac{dR}{dt}\right)I_{xz}, \quad (2.41a)$$
$$\frac{dH_y}{dt} = \left(\frac{dQ}{dt}\right)I_y, \quad (2.41b)$$
$$\frac{dH_z}{dt} = \left(\frac{dR}{dt}\right)I_z - \left(\frac{dP}{dt}\right)I_{xz}. \quad (2.41c)$$

Since initially we assumed a rigid body with constant mass, the time rates of change of the moments and products of inertia are zero. The vector cross product in (2.40) is

$$\boldsymbol{\omega}\times\mathbf{H} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ P & Q & R \\ H_x & H_y & H_z \end{bmatrix}$$
$$= (QH_z - RH_y)\mathbf{i} + (RH_x - PH_z)\mathbf{j} + (PH_y - QH_x)\mathbf{k}. \quad (2.42)$$

Similar to (2.26), we can write an equation for the summation of all moments in the form

$$\sum \Delta \mathbf{M} = \mathbf{i} \sum \Delta L + \mathbf{j} \sum \Delta M + \mathbf{k} \sum \Delta N. \tag{2.43}$$

Equating the components of (2.41), (2.42), and (2.43) and substituting for H_x, H_y, and H_z from (2.39) yields the *angular momentum equations*. Thus [1], [5],

$$\sum \Delta L = \dot{P} I_x - \dot{R} I_{xz} + QR(I_z - I_y) - PQI_{xz}, \tag{2.44a}$$

$$\sum \Delta M = \dot{Q} I_y + PR(I_x - I_z) + (P^2 - R^2) I_{xz}, \tag{2.44b}$$

$$\sum \Delta N = \dot{R} I_z - \dot{P} I_{xz} + PQ(I_y - I_x) + QRI_{xz}, \tag{2.44c}$$

or

$$\sum \Delta L = \dot{P} I_x + (I_z - I_y)QR - (\dot{R} + PQ) I_{xz}, \tag{2.44d}$$

$$\sum \Delta M = \dot{Q} I_y + (I_x - I_z)PR + (P^2 - R^2) I_{xz}, \tag{2.44e}$$

$$\sum \Delta N = \dot{R} I_z + (I_y - I_x)PQ - (\dot{P} - QR) I_{xz}, \tag{2.44f}$$

where dP/dt is the roll acceleration, dQ/dt is the pitch acceleration, and dR/dt is the yaw acceleration. The set of equations (2.27a)–(2.27c) and (2.44d)–(2.44f) or (2.44a)–(2.44c) represents the complete 6-*DOF* missile equations of motion. Specifically, equations (2.27) describe the translation, and equations (2.44) describe the rotation of a body. The set of equations (2.27) and (2.44) are six simultaneous nonlinear equations of motion, with six variables u, v, w, P, Q, and R, which completely describe the behavior of a rigid body. Moreover, these equations can be solved with a digital computer using numerical integration techniques. An analytical solution of sufficient accuracy can be obtained by linearizing these equations. These equations are also known as *Euler's equations*. Note that I_x, I_y, I_{xz} are constant for a given rigid body because of our choice of coordinate axes. Due to the usual symmetry of the aircraft (or missile) about the x-y plane, the products of inertia that involve y are usually omitted, and the moment equations may be rewritten as follows (note that for cruciform missiles with rotational symmetry, $I_y = I_z$ and $I_{xz} = 0$):

$$\Delta L = \dot{P} I_x + QR(I_z - I_y), \tag{2.45a}$$

$$\Delta M = \dot{Q} I_y + (I_x - I_z)PR, \tag{2.45b}$$

$$\Delta N = \dot{R} I_z + (I_y - I_x)PQ. \tag{2.45c}$$

It should be noted that the L and N equations indicate that a rolling or yawing moment excites angular velocities about all three axes. Therefore, except for certain cases,

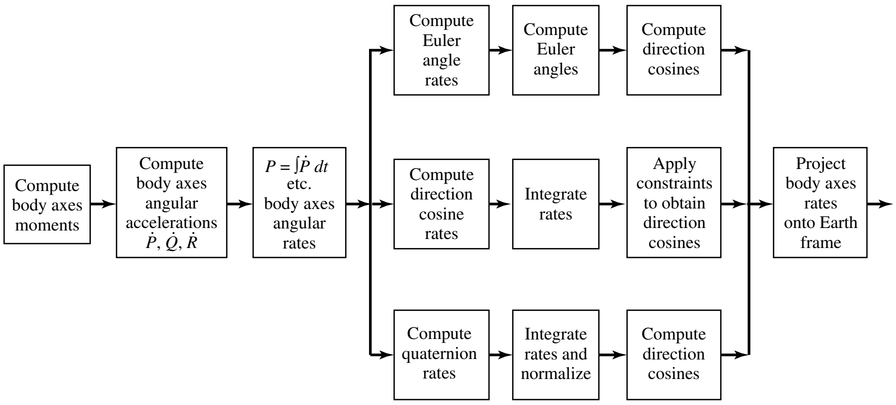

Fig. 2.5. Rotational dynamics of a rigid body.

these equations cannot be decoupled. Solving (2.45a)–(2.45c) for $dP/dt, dQ/dt$, and dR/dt, we obtain the *rotation* accelerations as follows:

$$\frac{dP}{dt} = QR[(I_y - I_z)/I_x] + (L/I_x), \tag{2.46a}$$

$$\frac{dQ}{dt} = PR[(I_z - I_x)/I_y] + (M/I_y), \tag{2.46b}$$

$$\frac{dR}{dt} = PQ[(I_x - I_y)/I_z] + (N/I_z). \tag{2.46c}$$

The relationship of the three coordinate systems discussed in Section 2.1 can be described in terms of the body dynamics. Figure 2.5 illustrates the manner in which these three methods are integrated into computational sequence of representing the vehicle dynamics.

The equations for the angular velocities $(d\psi/dt, d\phi/dt, d\theta/dt)$ in terms of the Euler angles (ψ, ϕ, θ) and the rates (P, Q, R) can be written from Figure 2.1 as follows [1]:

$$\frac{d\psi}{dt} = (Q \sin\phi + R\cos\phi)/\cos\theta, \tag{2.47a}$$

$$\frac{d\phi}{dt} = P + \left(\frac{d\psi}{dt}\right)\sin\theta, \tag{2.47b}$$

$$\frac{d\theta}{dt} = Q\cos\phi - R\sin\phi, \tag{2.47c}$$

where P is the roll rate, Q is the pitch rate, and R is the yaw rate. The values of (ψ, ϕ, θ) can be obtained by integrating (2.47a)–(2.47c). Thus,

$$\psi = \psi_0 + \int_0^t \left(\frac{d\psi}{dt}\right) dt, \tag{2.48a}$$

$$\phi = \phi_0 + \int_0^t \left(\frac{d\phi}{dt}\right) dt, \tag{2.48b}$$

$$\theta = \theta_0 + \int_0^t \left(\frac{d\theta}{dt}\right) dt. \tag{2.48c}$$

From the transformation matrix C_e^b of Section 2.1, the components of the missile velocity dX_e/dt, dY_e/dt, dZ_e/dt in the Earth-fixed coordinate system (X_e, Y_e, Z_e) in terms of (u, v, w) and (ψ, ϕ, θ) are given as follows:

$$\begin{aligned}
\frac{dX_e}{dt} &= (\cos\theta\cos\psi)u + (\cos\psi\sin\phi\sin\theta - \sin\psi\cos\phi)v \\
&\quad + (\cos\psi\cos\phi\sin\theta + \sin\psi\sin\phi)w, \\
\frac{dY_e}{dt} &= (\cos\theta\sin\psi)u + (\sin\psi\sin\phi\sin\theta + \cos\psi\cos\phi)v \\
&\quad + (\sin\psi\cos\phi\sin\theta - \cos\psi\sin\phi)w, \\
\frac{dZ_e}{dt} &= -(\sin\theta)u + (\sin\phi\cos\theta)v + (\cos\theta\cos\phi)w,
\end{aligned}$$

or in matrix form,

$$\frac{d}{dt}\begin{bmatrix} X_e \\ Y_e \\ Z_e \end{bmatrix} = C_e^b \begin{bmatrix} u \\ v \\ w \end{bmatrix}. \tag{2.49}$$

From (2.49) we can obtain the equations for (X_e, Y_e, Z_e) in the form

$$X_e = X_{e,0} + \int_0^t \left(\frac{dX_e}{dt}\right) dt, \tag{2.50a}$$

$$Y_e = Y_{e,0} + \int_0^t \left(\frac{dY_e}{dt}\right) dt, \tag{2.50b}$$

$$Z_e = Z_{e,0} + \int_0^t \left(\frac{dZ_e}{dt}\right) dt, \tag{2.50c}$$

and the altitude is

$$h = -Z_e. \tag{2.50d}$$

In the foregoing discussion, only the missile velocities relative to the ground or inertial velocities have been mentioned. If wind is being considered, the missile velocities relative to the wind must be computed, since these velocities are needed in computing the aerodynamic forces and moments (see Chapter 3).

It should be noted here that stability and control for fixed-wing aircraft are assessed through six rigid-body degrees of freedom models. Rotorcraft models provide three more degrees of freedom for the main flapping plus a rotational degree of freedom. Additional degrees of freedom for structural modes and other dynamic components

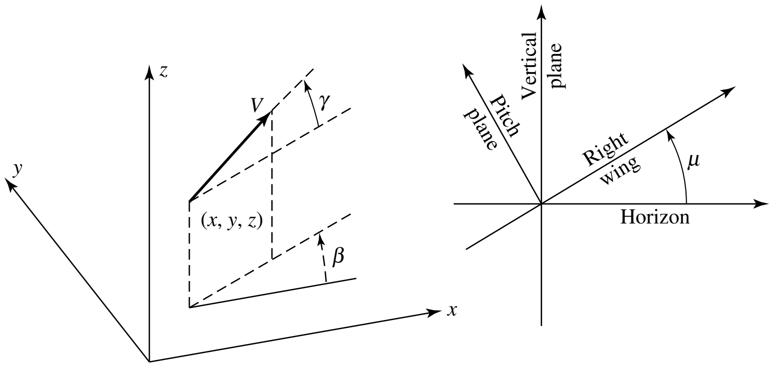

Fig. 2.6. Coordinate system

(e.g., transmissions) can be added as necessary. The aircraft models are based on aerodynamic coefficient representations of the major aircraft components, including the wing, fuselage, vertical tail, and horizontal tail. Mass distribution is represented by the center-of-gravity location and mass moments of inertia for the aircraft. A stability analysis is performed by trimming the forces and moments on the aircraft model for each flight condition. Force and moment derivatives are obtained through perturbations from trim in the state and control variables. These derivatives are used to represent the rigid-body motion of the aircraft as a set of linear first-order differential equations. The matrix representation of the aircraft motion is then used in the linear analysis package MATLAB* to assess stability and to investigate feedback control design. Aircraft dynamics and control system conceptual designs are typically analyzed with respect to dynamic performance, stability, and pilot/vehicle interface.

Example 1. In this example, we will consider an aircraft whose equations of motion can be represented as a point mass, based on five variables (i.e., 5-*DOF*). The coordinate system for this example is illustrated in Figure 2.6.

Furthermore, the variables are defined as follows:

$$
\begin{aligned}
\text{Let}\, x, y, z &= \text{position variables,} \\
\mathbf{V} &= \text{velocity vector,} \\
\alpha &= \text{angle-of-attack } (AOA), \\
\beta &= \text{velocity heading angle,} \\
\gamma &= \text{velocity elevation angle,} \\
\mu &= \text{orientation angle of the aircraft body axes relative} \\
&\ \text{to the velocity vector.}
\end{aligned}
$$

From the definition of the above variables, the orientation of the velocity vector **V** is through the angles β and γ, while the orientation of the aircraft body axes relative

* MATLAB is a commercially available software package for use on a personal computer.

to the velocity vector is through the angle μ and the *AOA* α in the pitch plane. The yaw of the aircraft about the velocity vector, **V**, is assumed to be zero (i.e., sufficient control power exists that all maneuvers are *coordinated*). For this 5-*DOF* (x, y, z, α, μ) point mass model, the equations of motion are as follows:

$$\begin{aligned}
x &= V\cos\gamma\cos\beta,\\
y &= V\cos\gamma\sin\beta,\\
z &= V\sin\gamma,\\
\dot{V} &= \frac{1}{m}[T\cos\alpha - D] - g\sin\gamma,\\
\dot{\beta} &= \frac{1}{mV}[T\sin\alpha + L](\sin\mu/\cos\gamma),\\
\dot{\gamma} &= \frac{1}{mV}[T\sin\alpha + L]\cos\mu - (g/V)\cos\gamma,\\
m &= m(M, z, n),\\
\dot{n} &= \dot{n}(\alpha, V_{IAS}),\\
\dot{\alpha} &= \dot{\alpha}(\alpha, V_{IAS}),\\
\dot{\mu} &= \dot{\mu}(\alpha, V_{IAS}),
\end{aligned}$$

where

$$\begin{aligned}
M &= \text{Mach number},\\
g &= \text{acceleration of gravity},\\
m &= \text{mass},\\
n &= \text{throttle setting},\\
V_{IAS} &= \text{indicated airspeed},\\
T &= \text{Thrust} = T(M, z, n),\\
D &= \text{drag} = \tfrac{1}{2}\rho V^2 S C_D \text{ (see Section 3.1)},\\
L &= \text{lift} = \tfrac{1}{2}\rho V^2 S C_L \text{ (see Section 3.1)},\\
\rho &= \text{atmospheric density} = \rho(z)\\
&\quad \text{(i.e., a function of altitude)},\\
S &= \text{aerodynamic reference area},\\
C_D &= \text{coefficient of drag},\\
C_L &= \text{coefficient of lift}.
\end{aligned}$$

Several approximations can be made in the above model. These are:

1. The $d\beta/dt$ equation of motion becomes undefined for vertical (i.e., $\gamma = \pm 90°$) flight.
2. The dn/dt, $d\alpha/dt$, $d\mu/dt$ equations are at best first-order approximations to actual aircraft control response.

From the above results and discussion, a 6-*DOF* model can be implemented that approximates an actual 6-*DOF* control response with a standard transfer function/filter whose input constants can be selected by the designer to more accurately match actual aircraft/missile control response. The roll, pitch, and yaw transfer functions are then as follows:

Roll: $\dfrac{P_{\text{stab}}(s)}{P_{\text{stab}_{\text{cmd}}}(s)} = \dfrac{1}{\tau s + 1}$

Pitch: $\dfrac{n_z(s)}{n_{z_{\text{cmd}}}(s)} = \dfrac{\omega^2(\tau s + 1)}{s^2 + 2\zeta\omega s + \omega^2}$

Yaw: $\dfrac{n_y(s)}{n_{y_{\text{cmd}}}(s)} = \dfrac{\omega^2}{s^2 + 2\zeta\omega s + \omega^2}$

where τ is the time constant, s is the Laplace operator, and ω is the frequency.

Under 6-*DOF* modeling, the $d\mu/dt$, $d\gamma/dt$, and $d\beta/dt$ kinematic relationships are

$$\begin{aligned}
\frac{d\mu}{dt} &= P + \tan\gamma(Q\sin\mu + R\cos\mu),\\
\frac{d\gamma}{dt} &= Q\cos\mu - R\sin\mu,\\
\frac{d\beta}{dt} &= \sec\gamma(Q\sin\mu + R\cos\mu),
\end{aligned}$$

where

$$\begin{aligned}
P &= \text{body axes roll rate},\\
Q &= \text{body axes pitch rate},\\
R &= \text{body axes yaw rate}.
\end{aligned}$$

Next, in order to eliminate the $d\beta/dt$ equations anomaly at $\gamma = \pm 90°$, the quaternion system of coordinates will be used; the kinematic rate equations are [7]

$$\begin{aligned}
\frac{de_1}{dt} &= (-e_4P - e_3Q - e_2R)/2,\\
\frac{de_2}{dt} &= (-e_3P - e_4Q - e_1R)/2,\\
\frac{de_3}{dt} &= (-e_2P + e_1Q - e_4R)/2,\\
\frac{de_4}{dt} &= (-e_1P - e_2Q + e_3R)/2,
\end{aligned}$$

and the Earth-to-body direction cosine matrix is, as before,

$$C_e^b = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix},$$

where

$$
\begin{aligned}
C_{11} &= e_1^2 - e_2^2 - e_3^2 + e_4^2, \\
C_{12} &= 2(e_3 e_4 + e_1 e_2), \\
C_{13} &= 2(e_2 e_4 - e_1 e_3), \\
C_{21} &= 2(e_3 e_4 - e_1 e_2), \\
C_{22} &= e_1^2 - e_2^2 + e_3^2 - e_4^2, \\
C_{23} &= 2(e_2 e_3 + e_1 e_4), \\
C_{31} &= 2(e_1 e_3 + e_2 e_4), \\
C_{32} &= 2(e_2 e_3 - e_1 e_4), \\
C_{33} &= e_1^2 + e_2^2 - e_3^2 - e_4^2,
\end{aligned}
$$

and

$$
\begin{aligned}
\beta &= \tan^{-1}(C_{12}/C_{11}), \\
\gamma &= -\sin^{-1}(C_{13}), \\
\mu &= \tan^{-1}(C_{23}/C_{33}).
\end{aligned}
$$

Finally, we note that the same 6-*DOF* equations of motion can be used to model both aircraft and missiles.

Example 2. Based on the discussion thus far, let us now consider in this example a 6-*DOF* aerodynamic model. Furthermore, let us assume an *NED* coordinate system, in which all units are metric. This model is designed for a generic aircraft. A quaternion fast-processing technique will be employed to simulate aircraft navigation. This technique avoids not only time-consuming trigonometric computations in the fast-rate direction cosine updating, but also singularities in aircraft attitude determination [7].

6-*DOF* Initialization

Before processing begins, these initialization functions must be performed. Compute the initialized Earth reference velocity:

$$
\begin{aligned}
U_e &= V * \cos(\theta) * \cos(\psi), \\
V_e &= V * \cos(\phi) * \sin(\psi), \\
W_e &= -V * \sin(\theta),
\end{aligned}
$$

where

$$
\begin{aligned}
U_e &= \text{Earth } X\text{-velocity}, \\
V_e &= \text{Earth } Y\text{-velocity}, \\
W_e &= \text{Earth } Z\text{-velocity}, \\
\theta &= \text{pitch angle}, \\
\phi &= \text{roll angle}, \\
\psi &= \text{yaw angle}, \\
V &= \text{airspeed}.
\end{aligned}
$$

Compute Initialized Quaternions

$$
\begin{aligned}
A &= [\sin(\psi/2) * \sin(\theta/2) * \cos(\phi/2)] - [\cos(\psi/2) * \cos(\theta/2) * \sin(\phi/2)],\\
B &= -1 * [\cos(\psi/2) * \sin(\theta/2) * \cos(\phi/2)] - [\sin(\psi/2) * \cos(\theta/2) * \sin(\phi/2)],\\
C &= -1 * [\sin(\psi/2) * \cos(\theta/2) * \cos(\phi/2)] + [\cos(\psi/2) * \sin(\theta/2) * \sin(\phi/2)],\\
D &= -1 * [\cos(\psi/2) * \cos(\theta/2) * \cos(\phi/2)] - [\sin(\psi/2) * \sin(\theta/2) * \sin(\phi/2)],
\end{aligned}
$$

where $A, B, C, D =$ quaternion parameters of the direction cosine matrix.

Now compute the initialized direction cosine matrix:

$$
\begin{aligned}
Cm(1,1) &= A^2 - B^2 - C^2 + D^2,\\
Cm(1,2) &= 2 * (A * B - C * D),\\
Cm(1,3) &= 2 * (A * C + B * D),\\
Cm(2,1) &= 2 * (A * B + C * D),\\
Cm(2,2) &= -1 * A^2 + B^2 - C^2 + D^2,\\
Cm(2,3) &= 2 * (B * C - A * D),\\
Cm(3,1) &= 2 * (A * C - B * D),\\
Cm(3,2) &= 2 * (B * C + A * D),\\
Cm(3,3) &= -1 * A^2 - B^2 + C^2 + D^2,
\end{aligned}
$$

where Cm direction cosine matrix.

Compute the initial body velocity

$$
\begin{aligned}
U_b &= Cm(1,1) * U_e + Cm(2,1) * V_e + Cm(3,1) * W_e,\\
V_b &= Cm(1,2) * U_e + Cm(2,2) * V_e + Cm(3,2) * W_e,\\
W_b &= Cm(1,3) * U_e + Cm(2,3) * V_e + Cm(3,3) * W_e,
\end{aligned}
$$

where

$$
\begin{aligned}
U_b &= \text{body } X\text{-velocity},\\
V_b &= \text{body } Y\text{-velocity},\\
W_b &= \text{body } Z\text{-velocity}.
\end{aligned}
$$

6-*DOF* Processing

The following computations are performed at every simulation cycle.

Compute the dynamic pressure

$$q = \tfrac{1}{2}\rho V^2,$$

where

$$
\begin{aligned}
q &= \text{dynamic pressure},\\
\rho &= \text{pressure in standard atmosphere},\\
V &= \text{airspeed}.
\end{aligned}
$$

Compute the wing lift

$$L = C_L * q * S,$$

where

$$\begin{aligned} L &= \text{lift}, \\ C_L &= \text{coefficient of lift}, \\ S &= \text{surface area of the wing}. \end{aligned}$$

Compute the wing drag

$$D = C_D * q * S,$$

where C_D = coefficient of drag.

Compute the lift acceleration

$$L_a = L/w,$$

where w = weight of the airplane.

Compute the drag acceleration

$$D_a = D/w.$$

Compute the thrust acceleration

$$T_a = T/m,$$

where

$$\begin{aligned} m &= \text{mass of the airplane}, \\ T &= \text{thrust}. \end{aligned}$$

Compute the body accelerations

$$\begin{aligned} X_{ba} &= T_a * L_a * \sin(\alpha) - D_a * \cos(\alpha) + Cm(3,1) * g + R * V_b - Q * W_b, \\ Y_{ba} &= Cm(3,2) * g - R * U_b + P * W_b, \\ Z_{ba} &= -1 * L_a * \cos(\alpha) - D_a * \sin(\alpha) + Cm(3,3) * g + Q * X_b - P * V_b, \end{aligned}$$

where

$$\begin{aligned} X_{ba} &= X\text{-axis body acceleration}, \\ Y_{ba} &= Y\text{-axis body acceleration}, \\ Z_{ba} &= Z\text{-axis body acceleration}, \\ P &= \text{roll rate}, \\ Q &= \text{pitch rate}, \\ R &= \text{yaw rate}, \\ g &= \text{acceleration due to gravity}, \\ \alpha &= \text{angle of attack}. \end{aligned}$$

Compute the Earth accelerations

$$
\begin{aligned}
X_{ea} &= Cm(1,1) * X_{ba} + Cm(1,2) * Y_{ba} + Cm(1,3) * Z_{ba},\\
Y_{ea} &= Cm(2,1) * X_{ba} + Cm(2,2) * Y_{ba} + Cm(2,3) * Z_{ba},\\
Z_{ea} &= Cm(3,1) * X_{ba} + Cm(3,2) * Y_{ba} + Cm(3,3) * Z_{ba},
\end{aligned}
$$

where

$$
\begin{aligned}
X_{ea} &= X\text{-axis Earth acceleration},\\
Y_{ea} &= Y\text{-axis Earth acceleration},\\
Z_{ea} &= Z\text{-axis Earth acceleration}.
\end{aligned}
$$

Compute the angular deltas

$$
\begin{aligned}
\Delta\theta &= Q * t_i,\\
\Delta\phi &= P * t_i,\\
\Delta\psi &= R * t_i,
\end{aligned}
$$

where

$$
\begin{aligned}
\Delta\theta &= \text{pitch delta},\\
\Delta\phi &= \text{roll delta},\\
\Delta\psi &= \text{yaw delta},\\
t_i &= \text{simulation cycle time}.
\end{aligned}
$$

Compute Cn and Sn

$$
\begin{aligned}
Cn &= 1.0 - (\Delta\theta^2 + \Delta\phi^2 + \Delta\psi^2)/8 + (\Delta\theta^4 + \Delta\phi^4 + \Delta\psi^4)/384,\\
Sn &= 0.5 - (\Delta\theta^2 + \Delta\phi^2 + \Delta\psi^2)/48,
\end{aligned}
$$

where

$$
\begin{aligned}
Cn &= n\text{th-order Maclaurin Series of } \cos(\Delta\theta/2),\\
Sn &= n\text{th-order Maclaurin Series of } \sin(\Delta\theta/2),\\
\Delta\theta &= \text{total body angle increment in } t_i.
\end{aligned}
$$

Compute the Quaternions

$$
\begin{aligned}
A' &= A * Cn + B * Sn * \Delta\psi + C * -1 * Sn * \Delta\theta + D * Sn * \Delta\phi,\\
B' &= A * -1 * Sn * \Delta\psi + B * Cn + C * Sn * \Delta\phi + D * Sn * \Delta\theta,\\
C' &= A * Sn * \Delta\theta + B * -1 * Sn * \Delta\phi + C * Cn + D * Sn * \Delta\psi,\\
D' &= A * -1 * Sn * \Delta\phi + B * -1 * Sn * \Delta\theta + C * -1 * Sn * \Delta\psi,\\
A &= A',\\
B &= B',\\
C &= C',\\
D &= D'.
\end{aligned}
$$

Renormalize the Quaternions

$$
\begin{aligned}
\text{Normalizer} &= 0.5 * (3.0 - A^2 - B^2 - C^2 - D^2),\\
A &= A * \text{Normalizer},\\
B &= B * \text{Normalizer},\\
C &= C * \text{Normalizer},\\
D &= D * \text{Normalizer}.
\end{aligned}
$$

Compute the direction cosine matrix

$$
\begin{aligned}
Cm(1,1) &= A^2 - B^2 - C^2 + D^2,\\
Cm(1,2) &= 2 * (A * B - C * D),\\
Cm(1,3) &= 2 * (A * C + B * D),\\
Cm(2,1) &= 2 * (A * B + C * D),\\
Cm(2,2) &= -1 * A^2 + B^2 - C^2 + D^2,\\
Cm(2,3) &= 2 * (B * C - A * D),\\
Cm(3,1) &= 2 * (A * C - B * D),\\
Cm(3,2) &= 2 * (B * C + A * D),\\
Cm(3,3) &= -1 * A^2 - B^2 + C^2 + D^2.
\end{aligned}
$$

Compute the Earth velocities

$$
\begin{aligned}
U_e &= U_e + X_{ea} * t_i,\\
V_e &= V_e + Y_{ea} * t_i,\\
W_e &= W_e + Z_{ea} * t_i.
\end{aligned}
$$

Compute the body velocities

$$
\begin{aligned}
U_b &= Cm(1,1) * U_e + Cm(2,1) * V_e + Cm(3,1) * W_e,\\
V_b &= Cm(1,2) * U_e + Cm(2,2) * V_e + Cm(3,2) * W_e,\\
W_e &= Cm(1,3) * U_e + Cm(2,3) * V_e + Cm(3,3) * W_e.
\end{aligned}
$$

Compute the airspeed

$$
V = (U_e^2 + V_e^2 + W_e^2)^{1/2}.
$$

Compute the Earth referenced position

$$
\begin{aligned}
U_e &= U_e * t_i,\\
V_e &= V_e * t_i,\\
W_e &= W_e * t_i.
\end{aligned}
$$

Compute the angle of attack

$$\alpha = A\tan 2(W_b/U_b).$$

Compute the attitudes

$$\begin{aligned}\theta &= A\sin(-1 * Cm(3,1)),\\ \phi &= A\tan 2(Cm(3,2)/Cm(3,3)),\\ \psi &= A\tan 2(Cm(2,1)/Cm(1,1)).\end{aligned}$$

Compute the sideslip angle

$$\beta = A\tan 2(V_b/U_b),$$

where β is the sideslip angle.

Compute the flightpath angle

$$\gamma = A\tan 2((-1 * W_e)/(U_e^2 + V_e^2)^{1/2}),$$

where γ is the flightpath angle.

Earlier in this section, the equations of motion for a missile were discussed assuming the missile to be a rigid body. However, all materials exhibit deformation under the action of forces: *elasticity* when a given force produces a definite deformation, which vanishes if the force is removed; *plasticity* if the removal of the force leaves permanent deformation; *flow* if the deformation continually increases without limit under the action of forces, however small.

A "fluid" is material that flows. Actual fluids fall into two categories, namely, gases and liquids. A "gas" will ultimately fill any closed space to which it has access and is therefore classified as a (highly) *compressive fluid*. A "liquid" at constant temperature and pressure has a definite volume and when placed in an open vessel will take under the action of gravity the form of the lower part of the vessel and will be bounded above by a horizontal free surface. All known liquids are to some extent compressible. For most purposes it is, however, sufficient to regard liquids as *incompressible fluids*. It should be pointed out that for speeds that are not comparable with that of sound, the effect of compressibility on atmospheric air can be neglected, and in many experiments that are carried out in wind tunnels the air is considered to be a liquid, in the above sense, which may conveniently be called *incompressible air*.

All liquids (and gases) in common with solids exhibit *viscosity* arising from internal friction in the substance. For those readers interested in pursuing more thoroughly the area of incompressible air and/or fluid flow, the *Navier–Stokes* equation is a good start. The *Euler* and *Navier–Stokes* equations describe the motion of a fluid in $\mathbf{R}^n$ ($n = 2$ or 3). These equations are to be solved for an unknown velocity vector $\mathbf{u}(\boldsymbol{x}, t) = (\mathbf{u}_i(\boldsymbol{x}, t))_{1\le i\le n} \in \mathbf{R}^n$ and pressure $p(\boldsymbol{x}, t) \in \mathbf{R}$, defined for position $\boldsymbol{x} \in \mathbf{R}^n$ and time $t \ge 0$. We restrict attention here to incompressible fluids filling all of $\mathbf{R}^n$. The *Navier–Stokes* equations are then given by

$$(\partial/\partial t)u_i + \sum_{j=1}^{n} u_j(\partial u_i/\partial x_j) = \nu\Delta u_i - (\partial p/\partial x_i) + f_i(x,t) \quad (\boldsymbol{x} \in \mathbf{R}^n, t \ge 0), \tag{2.51}$$

$$\operatorname{div} u = \sum_{i=1}^{n} (\partial u_i / \partial x_i) = 0 \quad (\boldsymbol{x} \in \mathbf{R}^n, t \geq 0), \tag{2.52}$$

with initial conditions

$$\mathbf{u}(\boldsymbol{x}, 0) = \mathbf{u}^0(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbf{R}^n, \tag{2.53}$$

where

$$\Delta = \sum_{i=1}^{n} (\partial^2 / \partial x_i^2)$$

is the Laplacian in the space variables.

Here, $\boldsymbol{u}^0(\boldsymbol{x})$ is a given $\boldsymbol{C}^\infty$ divergence-free vector field on $\mathbf{R}^n$, $f_i(x, t)$ are the components of a given externally applied force (e.g., gravity), and ν is a positive coefficient (the viscosity). Equation (2.51) is just Newton's law $\boldsymbol{f} = m\boldsymbol{a}$ for a fluid element subject to the external force $\boldsymbol{f} = (f_i(x, t))_{1 \leq i \leq n}$ and to the force arising from pressure and friction. Equation (2.52) just says that the fluid is *incompressible*. For physically reasonable solutions, we want to make sure that $\boldsymbol{u}(\boldsymbol{x}, t)$ does not grow large as $|\boldsymbol{x}| \to \infty$. Moreover, we accept a solution of equations (2.51)–(2.53) as physically reasonable only if it satisfies

$$p, \boldsymbol{u} \in \boldsymbol{C}^\infty(\mathbf{R}^n \times [0, \infty))$$

and

$$\int_{\mathbf{R}^n} |\boldsymbol{u}(\mathbf{x}, t)|^2 dx < \boldsymbol{C} \quad \forall t \geq 0 \,(\text{bounded energy}).$$

2.3 D'Alembert's Principle

In Section 2.2 we discussed the rigid-body equations of motion. Specifically, we discussed Newton's second law as given by (2.17b) and (2.18b). In the fundamental equation (2.18b), $\mathbf{F} = m\mathbf{a}$, the quantity $m(-\mathbf{a})$ is called the *reversed effective force* or *inertia force*. D'Alembert's principle is based on Newton's second and third laws of motion and states that 'the inertia force is in equilibrium with the external applied force,' or

$$\mathbf{F} + m(-\mathbf{a}) = 0. \tag{2.54}$$

This principle has the effect of reducing a dynamical problem to a problem in statics and may thus make it easier to solve. Based on the principle of *virtual work,* * which was established for the case of static equilibrium, we can proceed as follows: Let $\mathbf{p}$ be the momentum of a particle in the system, and separate the forces acting on it into an

* Consider a particle acted upon by several forces. If the particle is in equilibrium, the resultant $\mathbf{R}$ of the forces must vanish, and the work done by the forces is a virtual displacement $\delta\mathbf{r}$ is zero. Thus, $\mathbf{R} \cdot \delta\mathbf{r} = 0$.

applied force **F** and a constraint force **f**. Then the equation of motion of the particle can be written as [9]

$$\mathbf{F}+\mathbf{f}-\left(\frac{d\mathbf{p}}{dt}\right)=0.$$

The quantity $(d\mathbf{p}/dt)$ is usually referred to as the *reverse effective force* discussed above. Note that the virtual work of the constraint force is zero, since **f** and $\delta\mathbf{r}$ are mutually perpendicular. The virtual work of the forces acting on the particle is

$$\left[\mathbf{F}_i-\left(\frac{d\mathbf{p}_i}{dt}\right)\right]\cdot\delta\mathbf{r}_i=0 \quad (i=1,2,\ldots,N),$$

and for a system of N particles,

$$\sum_{i=1}^{N}\left[\mathbf{F}_i-\left(\frac{d\mathbf{p}_i}{dt}\right)\right]\cdot\delta\mathbf{r}_i=0 \quad (i=1,2,\ldots,N).$$

Another way of writing this equation is

$$\sum_{i=1}^{N}\left[\mathbf{F}_i-m_i\left(\frac{d^2\mathbf{r}_i}{dt^2}\right)\right]\cdot\delta\mathbf{r}_i=0 \quad (i=1,2,\ldots,N),$$

where $\mathbf{r}_i$ is the position vector of the particle. The term $-m_i(d^2\mathbf{r}_i/dt^2)$ has the dimensions of force and is known as the *inertia force* acting on the ith particle (see also discussion above). This is the *Lagrangian* form of d'Alembert's principle and is one of the most important equations of classical dynamics.

2.4 Lagrange's Equations for Rotating Coordinate Systems

The missiles considered thus far were assumed to obey the laws of rigid bodies. However, in analyzing the dynamics of flexible missiles, such as intermediate-range ballistic missiles or intercontinental ballistic missiles, it is convenient to use a set of coordinates moving with the missile. In this case, the missile can be considered as a system of particles whose position relative to the moving axes can be defined by generalized coordinates q_i. Specifically, we will consider the motion of a *holonomic** system with n degrees of freedom. Let $(q_1, q_2, \ldots, q_n)$ be the coordinates that specify the configuration of the system at time t. Furthermore, we will consider a mechanical system of n particles whose coordinates are $(x_1, y_1, z_1, x_2, y_2, z_2, \ldots, x_n, y_n, z_n)$. The motion of the system is known when the value of every coordinate is known as a function of time. Suppose now that the system moves from a certain configuration given by $(x'_1, \ldots, z'_n)$ at time t_1 to another configuration given by $(x''_1, y''_1, \ldots, z''_n)$ at

*A dynamical system for which a displacement represented by arbitrary infinitesimal changes in the coordinates is, in general, a possible displacement is said to be *holonomic*. When this condition is not satisfied, the system is said to be *nonholonomic*.

time t_2. During all of the motion between these two configurations, the Newtonian equations of motion will be followed, and the acceleration of each particle will be given by the total force acting on it. Moreover, this motion can be described by expressing each coordinate as a function of time.

There are then $3n$ dependent variables depending on the one independent variable t. These functions can be written in the form

$$x_1 = x_1(t),\ y_1 = y_1(t), \ldots, z_n = z_n(t). \tag{2.55}$$

In deriving the equations of motion, it is common practice to start the derivation using the concepts of kinetic and potential energies of the system using Lagrange's equation. As will be noted, however, these equations differ from the usual Lagrange equations for fixed coordinates. Now consider some other way in which the system might have moved from the initial configuration to the final configuration in the same amount of time, $t_2 - t_1$. This new motion is to be one that satisfies the geometric conditions, or the constraints of the problem. If this new motion is just slightly different from the original motion, the coordinates, as functions of time, can be written as follows:

$$x_1(t) + \delta x_1(t),\ y_1(t) + \delta y_1(t), \ldots, z_n(t) + \delta z_n(t).$$

The variation of a coordinate x is a function of time and is the difference between the x coordinate of the comparison path and that of the true path. It is also assumed that the true path is a continuous function with continuous first derivatives satisfying Newton's equations. The same ideas apply to the comparison path. Therefore,

$$\delta x_1(t_1) = \delta x_1(t_2) = \delta y_1(t_1) = \delta y_1(t_2) = \cdots = \delta z_n(t_2) = 0. \tag{2.56}$$

The true path was originally defined in terms of the Newtonian equations of motion. For the true path there are $3n$ equations of the form

$$m_i \left(\frac{d^2 x_i}{dt^2} \right) = X_i, \tag{2.57}$$

where m_i typifies the mass of one of the particles of the system. The quantity X_i may be a function of the coordinates, of the time explicitly, or of both. It may be considered, however, as a function of time only, since the dependence on the coordinates is a dependence upon the positions of the particles, and these are uniquely determined by the time along any path that may be considered. In general, the coordinates of the individual particles (referenced to some fixed set of rectangular coordinates) are known functions of the coordinates $(q_1, q_2, \ldots, q_n)$ of the system, and possibly of t also. Let this dependence be expressed by the equations [11]

$$\begin{aligned} x_i &= f_i(q_1, q_2, \ldots, q_n, t), \\ y_i &= g_i(q_1, q_2, \ldots, q_n, t), \\ z_i &= h_i(q_1, q_2, \ldots, q_n, t). \end{aligned} \tag{2.58}$$

Furthermore, let (X_i, Y_i, Z_i) be the components of the total force (external) acting on the particle m_i. Then, the equations of motion of this particle are

$$m_j\left(\frac{d^2x_j}{dt^2}\right) = X_i, \quad m_j\left(\frac{d^2y_j}{dt^2}\right) = Y_i, \quad m_j\left(\frac{d^2z_j}{dt^2}\right) = Z_i. \tag{2.59}$$

If each component of the force X_i is now multiplied by the variation of path in the direction of the force and all the resulting equations are added together, the result is [4], [11]

$$\begin{aligned}\delta U &= \sum_i (X_i\delta x_i + Y_i\delta y_i + Z_i\delta z_i),\\ &= \sum_i m_i\left(\frac{d^2x_i}{dt^2}\delta x_i + \frac{d^2y_i}{dt^2}\delta y_i + \frac{d^2z_i}{dt^2}\delta z_i\right)\\ &= \sum_i m_i\left[\frac{d}{dt}(\dot{x}_i\delta x_i + \dot{y}_i\delta y_i + \dot{z}_i\delta z_i) - \dot{x}_i\delta\dot{x}_i - \dot{y}_i\delta\dot{y}_i - \dot{z}_i\delta\dot{z}_i\right]\end{aligned} \tag{2.60}$$

where the symbol $\sum$ denotes summation over all the particles of the system; this can be either an integration (if the particles are united into rigid bodies) or a summation over a discrete aggregate of particles. The quantity δU is defined by the first equality in (2.60). It is the work done by the forces of the system during the infinitesimal displacement $(\delta x_i, \ldots, \delta z_n)$ and is a function of the time and the independent coordinates of the system. If the forces do not depend explicitly on the time, δU can be expressed as a function of the coordinates only. The last part of (2.60) represents the variation of the kinetic energy δT. Hence the equation can be written as [3]

$$\delta T + \delta U = \sum\nolimits_i m_i\frac{d}{dt}(x_i\delta x_i + y_i\delta y_i + z_i\delta z_i). \tag{2.61}$$

It should be noted that in the above expressions t is the independent variable. Now, if both sides of (2.61) are integrated with respect to this independent variable between the limits t_1 and t_2, the result is

$$\int_{t_1}^{t_2}(\delta T + \delta U)dt = \delta\int_{t_1}^{t_2} T\,dt + \int_{t_1}^{t_2}\delta U\,dt = 0. \tag{2.62}$$

In this equation, we note that the right-hand side is zero because all of the variations are zero at both limits. Therefore, (2.62) is a property of the path that satisfies the equations of motion, and this property furnishes a way of defining the true path of the system. In the special case in which the forces are conservative, that is, when they can be derived from the potential energy, δU is the negative of the variation of the potential energy. Consequently, we have

$$\delta\int_{t_1}^{t_2}(T - U)dt = \delta\int_{t_1}^{t_2} L\,dt = 0, \tag{2.63}$$

where T is the kinetic energy and U is the potential energy.

Commonly, the quantity $(T-U)$ is denoted by L and is called the *Lagrangian function* or the kinetic potential of the system, or $L=T-U$. The function L, defined as the excess of kinetic energy over potential energy, is the most fundamental quantity in the mathematical analysis of mechanical problems. The Lagrangian function can be expressed in any convenient coordinate system, and the variation principle will still apply. Thus, if we introduce a new function L of the variables $(q_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n, t)$, defined by the equation

$$L=T-U,$$

then Lagrange's equations can be written in the form [11]

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)-\frac{\partial L}{\partial q_i}=0 \qquad \begin{array}{l} i=1,2,\ldots,n, \\ t_1 \le t \le t_2. \end{array} \tag{2.64}$$

Hamilton's principle for the motion of a mechanical system states that

$$\delta \int_{t_1}^{t_2} L(q_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n, t)dt. \tag{2.65}$$

In (2.65) the q's represent the coordinates necessary to specify the configuration of the system. Note that the time appears explicitly in the Lagrangian function only in case the forces are explicit functions of time, or the coordinates used are in motion. In the simple conservative cases the Lagrangian function depends upon the coordinates and their first derivatives only. If, as has been assumed, the coordinates are all independent, then the path can be described by the set of differential equations (2.64).

The Euler–Lagrange equations for Hamilton's principle (2.64) are usually called simply Lagrange's equations. They contain nothing more than was contained in the Newtonian equations, but they have the decided advantage that the coordinates may be of any kind whatever. It is necessary only to write the potential and kinetic energies in the desired coordinates to obtain the equations of motion by simple differentiation. This is usually much simpler than transforming the differential equations themselves. Finally, we note here that Lagrange's equations and Newton's equations are entirely equivalent.

Now, if the $(x\text{-}y)$ plane is rotated by an angle θ, the coordinate axes in plane motion will have three degrees of freedom, namely, x_0, y_0, and θ, which can be varied independently. In this case, the Lagrange equations can be written in the form [4], [9]

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{x}_0}-\dot{\theta}\frac{\partial T}{\partial \dot{y}_0}=\sum F_x \tag{2.66a}$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{y}_0}+\dot{\theta}\frac{\partial T}{\partial \dot{x}_0}=\sum F_y \tag{2.66b}$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}}+\dot{x}_0\frac{\partial T}{\partial \dot{y}_0}-\dot{y}_0\frac{\partial T}{\partial \dot{x}_0}=\sum M_0 \tag{2.66c}$$

where the equation for the q_i remains unaltered. In accounting for the terms in these equations, the partials $\partial T/\partial \dot{x}_0$ and $\partial T/\partial \dot{y}_0$ will be recognized as the general momenta, and $\partial T/\partial \dot{\theta}$ as the generalized angular momentum. The linear momentum can then be represented by the expression

$$\boldsymbol{p} = (\partial T/\partial \dot{x}_0)\mathbf{i} + (\partial T/\partial \dot{y}_0)\mathbf{j}. \tag{2.67}$$

Since the force equation is the rate of change of the linear momentum, we have

$$\boldsymbol{F} = [d\boldsymbol{p}/dt] + \boldsymbol{\omega} \times \boldsymbol{p}. \tag{2.68}$$

The terms of (2.66a) and (2.66b) are immediately accounted for. Moreover, the terms of (2.66c) can be identified from the expression

$$\mathbf{M}_0 = \left(\frac{d\boldsymbol{h}_0}{dt}\right) + \left(\frac{d\boldsymbol{R}_0}{dt}\right) \times \sum m_i \left(\frac{d\boldsymbol{r}_i}{dt}\right). \tag{2.69}$$

The term $\partial T/\partial \dot{\theta}$ is the angular momentum $\boldsymbol{h}_0$, and the remaining two terms are equal to $(d\boldsymbol{R}_0/dt) \times \sum m_i (d\boldsymbol{r}_i/dt)$, where $d\boldsymbol{R}_0/dt = (dx_0/dt)\mathbf{i} + (dy_0/dt)\mathbf{j}$.

Example 3. A typical example illustrating the above principles will now be given. Specifically, we will work out Problem 2, p. 118, of reference [4]. Consider a particle moving in a plane attracted toward the origin of coordinates with a force inversely proportional to the square of the distance from it. In plane polar coordinates (r, θ) one has

$$U = -\left(\frac{k}{r}\right) \quad \text{and} \quad T = \left(\frac{m}{2}\right)\left[\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2\right].$$

From these expressions, we form the *Lagrangian function* as follows:

$$L = T - U = \left(\frac{m}{2}\right)\left[\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2\right] + \left(\frac{k}{r}\right).$$

Furthermore, using (2.64), we obtain the derivatives as

$$\frac{\partial L}{\partial \dot{r}} = m\left(\frac{dr}{dt}\right) \quad \text{and} \quad \frac{\partial L}{\partial r} = mr\left(\frac{d\theta}{dt}\right)^2 - \left(\frac{k}{r^2}\right),$$

which give for this equation of motion

$$m\left(\frac{d^2r}{dt^2}\right) - mr\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{k}{r^2}\right) = 0.$$

For the other equation in the variable θ, we have

$$\frac{\partial L}{\partial \dot{\theta}} = mr^2\left(\frac{d\theta}{dt}\right) \quad \text{and} \quad \frac{\partial L}{\partial \theta} = 0,$$

so that this equation of motion is

$$m\left(\frac{d}{dt}\right)\left[r^2\left(\frac{d\theta}{dt}\right)\right]=0.$$

Note that since θ is not explicitly present in L, the derivative of L with respect to $d\theta/dt$ is a constant.

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Missile Guidance and Control Systems
Siouris, G.M.
2004, XIV, 666 p., Hardcover
ISBN: 978-0-387-00726-7