Stochastic Processes in Cell Biology II: Supplementary material

Paul C. Bressloff

August 17, 2024

Chapter 13
Self-organization and assembly of cellular structures

A. Mean field theory for interacting particle systems

Large systems of interacting particles arise in a wide range of applications in the natural and social sciences. For example, in physics the particles could represent electrons or ions in a plasma, molecules in passive or active fluids, or galaxies in a cosmological model. On the other hand, particles in biological applications tend to be micro-organisms such as cells or bacteria that can exhibit non-trivial aggregation phenomena such as motility-based phase separation (see Sect. 15.7). Finally, in economics or social sciences, particles typically represent individual "agents". A major challenge is how to reduce the complexity of such systems. A classical approach is to derive a macroscopic model that provides a continuous description of the dynamics in terms of global densities evolving according to non-linear partial differential equations. Such kinetic formulations date back to the foundations of statistical mechanics and the Boltzmann equation of dilute gases interacting via direct collisions. In recent years, however, much of the focus has been on the mean field limit of particles with long range or collisionless interactions. Two paradigmatic examples are interacting Brownian particles in the overdamped regime and the Kuramoto model of coupled phase oscillators (see Sect. 15.5).

The classical Dean-Kawasaki (DK) equation is a stochastic partial differential equation (SPDE) that describes fluctuations in the global density of N over-damped Brownian particles with positions $\mathbf{X}_j(t) \in \mathbb{R}^d$ at time t [1, 2]. Within the context of non-equilibrium statistical physics, the DK equation is commonly combined with dynamical density functional theory (DDFT) in order to derive hydrodynamical models of interacting particle systems [5, 3, 4, 6]. It is an exact equation for the global density (or empirical measure) in the distributional sense, and plays an important role in the stochastic and numerical analysis of interacting particle systems [7, 8, 9, 10, 11, 12, 13]. There is also considerable mathematical interest in the rigorous stochastic analysis of the mean field limit $N \to \infty$ for overdamped Brownian particles with weak interactions, see for example Refs. [14, 15, 16, 17, 18]. In particular, if the initial positions of the N particles are independent and identically distributed, then for a wide range of systems it can be proven that ρ converges in distribution to the solution of the McKean-Vlasov (MV) equation [19]; the latter is a nonlocal nonlinear Fokker-Planck (FP) equation for the mean field density. The interacting particle system is said to satisfy the propagation of chaos property. The MV equation can also be derived directly from the DK equation by taking expectations with respect to the independent white noise processes and imposing a mean field ansatz. The MV equation is of interest in its own right, since it can support multiple stationary solutions and associated phase transitions [20, 21]. This has been explored in various configurations, including double-well confinement and Curie-Weiss (quadratic) pairwise interactions on $\mathbb{R}$ [22, 23, 24], and interacting particles on a torus [25, 26]. A well-known example of the latter is the stochastic Kuramoto model of interacting phase oscillators with sinusoidal coupling and quenched disorder due to the random distribution of natural frequencies [27, 28, 29]. The well-known continuum model for the density of phase oscillators [30, 31, 32, 33] is precisely the MV equation for the global density in the mean field limit $N \to \infty$, whose existence can be proven rigorously using propagation of chaos [34].

A.1 Weakly interacting Brownian particles and the McKean-Vlasov equation

Consider N overdamped Brownian particles in $\mathbb{R}^d$. Let $\mathbf{X}_j(t) \in \mathbb{R}^d$ denote the position of the jth particle at time t, $j = 1,\ldots,N$. We assume that the particles are subject to a common external conservative force $\mathbf{F}(\mathbf{x}) = -\boldsymbol{\nabla} V(\mathbf{x})$ and interact via a pairwise potential K. That is, the force on a particle at $\mathbf{x}$ due to a particle at $\mathbf{y}$ is $-\boldsymbol{\nabla} K(\mathbf{x}-\mathbf{y})$, where differentiation is with respect to $\mathbf{x}$. The particle positions $\mathbf{X}_j(t)$ evolve according to the SDE[1]

$$d\mathbf{X}_j(t) = -\frac{1}{\gamma}\left[\boldsymbol{\nabla} V(\mathbf{X}_j(t)) + \frac{1}{N}\sum_{k=1}^N \boldsymbol{\nabla} K(\mathbf{X}_j(t)-\mathbf{X}_k(t))\right]dt + \sqrt{2D}d\mathbf{W}_j(t) \quad \text{(A.1)}$$

where $\mathbf{W}_j$ is a vector of independent Wiener processes. Following the "hydrodynamic" formulation of Ref. [1], we define the global density (or empirical measure)

$$\rho(x,t) = \frac{1}{N}\sum_{j=1}^N \delta(\mathbf{x}-\mathbf{X}_j(t)), \quad \text{(A.2)}$$

and introduce an arbitrary smooth test function $f(\mathbf{x})$ of compact support. It follows that

---

[1] In the original formulation of Ref. [1] the interaction potential is not scaled by a factor $1/N$ and the global density is taken to be $\rho(x,t) = \sum_{j=1}^N \delta(\mathbf{x}-\mathbf{X}_j(t))$. It is necessary to include the factor $1/N$ in order to apply a mean field ansatz.

$$\frac{1}{N}\sum_{j=1}^{N} f(\mathbf{X}_j(t)) = \int_{\mathbb{R}^d} \rho(\mathbf{x},t) f(\mathbf{x}) d\mathbf{x}, \tag{A.3}$$

and

$$\left[\int_{\mathbb{R}^d} d\mathbf{x}\, f(\mathbf{x}) \frac{\partial \rho(\mathbf{x},t)}{\partial t}\right] dt = \frac{1}{N}\sum_{j=1}^{N} df(X_j(t)).$$

Using Itô's lemma (see Sect. 2.2), we find that

$$\begin{aligned} &\int_{\mathbb{R}^d} d\mathbf{x}\, f(\mathbf{x}) \frac{\partial \rho(\mathbf{x},t)}{\partial t} \qquad \text{(A.4)} \\ &= \int_{\mathbb{R}^d} d\mathbf{x} \left[\frac{\sqrt{2D}\boldsymbol{\nabla} f(\mathbf{x})}{N} \cdot \sum_{j=1}^{N} \rho_j(x,t) \boldsymbol{\xi}_j(t) + \rho(\mathbf{x},t) \left(D\boldsymbol{\nabla}^2 f(\mathbf{x}) - \boldsymbol{\nabla} f(\mathbf{x}) \cdot \mathcal{V}[\mathbf{x},t,\rho] \right) \right], \end{aligned}$$

where $\rho_j(\mathbf{x},t) = \delta(\mathbf{x} - \mathbf{X}_j(t))$, $d\mathbf{W}_j(t) = \boldsymbol{\xi}_j(t)dt$, and

$$\mathcal{V}[\mathbf{x},t,\rho] = \frac{1}{\gamma}\left[\boldsymbol{\nabla} V(\mathbf{x}) + \int_{\mathbb{R}^d} d\mathbf{y}\, \rho(\mathbf{y},t) \boldsymbol{\nabla} K(\mathbf{x}-\mathbf{y})\right]. \tag{A.5}$$

Integrating by parts the various terms involving derivatives of f and using the fact that f is arbitrary yields the following SPDE for ρ:

$$\frac{\partial \rho(\mathbf{x},t)}{\partial t} = -\sqrt{\frac{2D}{N^2}} \sum_{j=1}^{N} \boldsymbol{\nabla} \cdot \left[\rho_j(\mathbf{x},t) \boldsymbol{\xi}_j(t)\right] + D\boldsymbol{\nabla}^2 \rho(\mathbf{x},t) + \boldsymbol{\nabla} \cdot \left(\rho(\mathbf{x},t) \mathcal{V}[\mathbf{x},t,\rho]\right). \tag{A.6}$$

As it stands, equation (A.6) is not a closed equation for ρ due to the noise terms. Following Ref. [1], we introduce the space-dependent Gaussian noise term

$$\xi(\mathbf{x},t) = -\frac{1}{N}\sum_{j=1}^{N} \boldsymbol{\nabla} \cdot \left[\rho_j(\mathbf{x},t) \boldsymbol{\xi}_j(t)\right], \tag{A.7}$$

with zero mean and the correlation function

$$\langle \xi(\mathbf{x},t) \xi(\mathbf{y},t') \rangle = \frac{\delta(t-t')}{N^2} \sum_{j=1}^{N} \boldsymbol{\nabla}_{\mathbf{x}} \cdot \boldsymbol{\nabla}_{\mathbf{y}} \left(\rho_j(\mathbf{x},t) \rho_j(\mathbf{y},t) \right).$$

Since $\rho_j(\mathbf{x},t)\rho_j(\mathbf{y},t) = \delta(\mathbf{x}-\mathbf{y})\rho_j(\mathbf{x},t)$, it follows that

$$\langle \xi(\mathbf{x},t) \xi(\mathbf{y},t') \rangle = \frac{1}{N}\delta(t-t') \boldsymbol{\nabla}_{\mathbf{x}} \cdot \boldsymbol{\nabla}_{\mathbf{y}} \left(\delta(\mathbf{x}-\mathbf{y}) \rho(\mathbf{x},t) \right).$$

Finally, we introduce the global density-dependent noise field

$$\widehat{\xi}(\mathbf{x},t) = \frac{1}{\sqrt{N}}\boldsymbol{\nabla}\cdot\left(\eta(\mathbf{x},t)\sqrt{\rho}(\mathbf{x},t)\right), \tag{A.8}$$

where $\eta(\mathbf{x},t)$ is a global white noise field whose components satisfy

$$\langle\eta^{\sigma}(\mathbf{x},t)\eta^{\sigma'}(\mathbf{y},t')\rangle = \delta(t-t')\delta(\mathbf{x}-\mathbf{y})\delta_{\sigma,\sigma'}. \tag{A.9}$$

It can be checked that the Gaussian noises ξ and $\widehat{\xi}$ have the same correlation functions and are thus statistically identical. We thus obtain the classical DK equation [1, 2]

$$\frac{\partial\rho(\mathbf{x},t)}{\partial t} = \sqrt{\frac{2D}{N}}\boldsymbol{\nabla}\cdot\left[\sqrt{\rho(\mathbf{x},t)}\boldsymbol{\eta}(\mathbf{x},t)\right] + D\boldsymbol{\nabla}^2\rho(\mathbf{x},t) + \boldsymbol{\nabla}\cdot\left(\rho(\mathbf{x},t)\mathcal{V}[\mathbf{x},t,\rho]\right) \tag{A.10}$$

Although equation (A.10) is exact in the weak sense, it is highly singular. Moreover, averaging with respect to the Gaussian white noise processes results in a moment closure problem for the one-particle density $\langle\rho\rangle$. That is, setting $p(\mathbf{x},t) = \left\langle \rho(\mathbf{x},t)\right\rangle$, we have

$$\begin{aligned}\frac{\partial p(\mathbf{x},t)}{\partial t} &= D\boldsymbol{\nabla}^2 p(\mathbf{x},t) + \frac{1}{\gamma}\boldsymbol{\nabla}\cdot\left(p(\mathbf{x},t)\boldsymbol{\nabla}V(\mathbf{x})\right)\\ &\quad + \frac{1}{\gamma}\boldsymbol{\nabla}\cdot\left(\int_{\mathbb{R}^d}\boldsymbol{\nabla}K(\mathbf{x}-\mathbf{y})\left\langle\rho(\mathbf{x},t)\rho(\mathbf{y},t)\right\rangle\right).\end{aligned} \tag{A.11}$$

As it stands, $p(\mathbf{x},t)$ couples to the two-point correlation function, which in turn couples to the three-point correlation function etc. Therefore, we now take the thermodynamic limit $N\to\infty$ under the mean field ansatz

$$\left\langle\rho(\mathbf{x},t)\rho(\mathbf{y},t)\right\rangle = \left\langle\rho(\mathbf{x},t)\right\rangle\left\langle\rho(\mathbf{y},t)\right\rangle = p(\mathbf{x},t)p(\mathbf{y},t). \tag{A.12}$$

This yields the deterministic MV equation

$$\frac{\partial p(\mathbf{x},t)}{\partial t} = D\boldsymbol{\nabla}^2\phi(\mathbf{x},t) + \boldsymbol{\nabla}\cdot\left(p(\mathbf{x},t)\mathcal{V}[\mathbf{x},t,p]\right). \tag{A.13a}$$

with

$$\mathcal{V}[\mathbf{x},t,p] = \frac{1}{\gamma}\left[\boldsymbol{\nabla}V(\mathbf{x}) + \int_{\mathbb{R}^d} d\mathbf{y}\, p(\mathbf{y},t)\boldsymbol{\nabla}K(\mathbf{x}-\mathbf{y})\right]. \tag{A.13b}$$

The derivation of classical MV equation [19], and the validity of the mean field ansatz can be proven using propagation of chaos [14, 15, 16, 17, 18]. The latter is essentially a version of the law of large numbers, so that simulations for large but finite N generate macroscopic quantities that are consistent with solutions to the deterministic MV equation up to $O(1/\sqrt{N})$ errors.

A.2 Stationary solutions of the 1D McKean-Vlasov equation

A classical result in statistical physics is that for a finite system of overdamped Brownian particles subject to conservative forces, the corresponding linear Fokker-Planck (FP) equation has a unique stationary solution given by the Boltzmann distribution. More specifically, the joint probability density $p(\mathbf{x}_1,\ldots,\mathbf{x}_N,t)$ evolves according to the multivariate FP equation

$$\frac{\partial p}{\partial t} = D\sum_{j=1}^{N} \boldsymbol{\nabla}_j^2 p + \frac{1}{\gamma}\sum_{j=1}^{N} \boldsymbol{\nabla}_j \cdot (\boldsymbol{\nabla}_j U(\mathbf{x}_1,\ldots,\mathbf{x}_N)p), \tag{A.14}$$

where $\boldsymbol{\nabla}_j$ indicates differentiation with respect to $\mathbf{x}_j$, and

$$U(\mathbf{x}_1,\ldots,\mathbf{x}_N) = \sum_{j=1}^{N} V(\mathbf{x}_j) + \frac{1}{2N}\sum_{j,k=1}^{N} K(\mathbf{x}_j - \mathbf{x}_k). \tag{A.15}$$

equation (A.14) has the unique stationary solution

$$p = Z^{-1}\mathrm{e}^{-\beta U}, \quad Z = \int \left[\prod_{j=1}^{N} d\mathbf{x}_j\right] \mathrm{e}^{-\beta U(\mathbf{x}_1,\ldots,\mathbf{x}_N)}, \tag{A.16}$$

with $\beta = 1/(k_B T)$. (We are assuming that $Z < \infty$ for the given choice of potentials K and V.) The existence of a unique stationary density for the finite system reflects the fact that the dynamics is ergodic. However, ergodicity may break down in the thermodynamic limit $N \to \infty$, resulting in the coexistence of multiple stationary states and their associated phase transitions. This has been explored for an infinite system of interacting Brownian particles using the MV equation. (A.13) [20]. Examples include double-well confinement and Curie-Weiss (quadratic) interactions on $\mathbb{R}$ [22, 23, 24], and interacting particles on a torus [25, 26]. In the specific case of the Curie-Weiss potential $K(\mathbf{x}-\mathbf{y}) = \lambda(\mathbf{x}-\mathbf{y})^2/2$, the coupling term in the SDE (A.1) becomes $-\lambda(\mathbf{X}_j(t) - \overline{\mathbf{X}}(t))$ where $\overline{\mathbf{X}}(t) = N^{-1}\sum_{k=1}^{N}\mathbf{X}_k(t)$. It is an example of a cooperative coupling that tends to make the system relax towards the "center of gravity" of the multi-particle ensemble. If $V(\mathbf{x})$ is given by a multi-well potential then there is competition between the cooperative interactions and the tendency of particles to be distributed across the different potential wells according to the classical Boltzmann distribution. Here we explore stationary solutions by considering a 1D version of the MV equation (A.13) with Curie-Weiss coupling:

$$\frac{\partial p(x,t)}{\partial t} = D\frac{\partial^2 p(x,t)}{\partial x^2} + \frac{\partial}{\partial x}\bigg(p(x,t)\mathcal{V}[x,t,p]\bigg), \tag{A.17}$$

with

$$\mathcal{V}[x,t,p] = \frac{1}{\gamma}\left[V'(x) + \lambda\int_{-\infty}^{\infty}(x-y)p(y,t)dy\right]. \tag{A.18}$$

The steady state version of equation (A.17) is $J_0'(x) = 0$,where

$$J_0(x) := -D\frac{\partial p_0(x)}{\partial x} - \beta D p_0(x)\left(V'(x) + \lambda \int_{-\infty}^{\infty} (x-y)p_0(y)dy\right).$$

(We use the subscript 0 to indicate no resetting.) The integral term reduces to $\lambda(x - \langle y \rangle)$ with $\langle y \rangle = \int_0^\infty y p_0(y)dy$. Suppose, for the moment, that $\langle y \rangle = \ell$ for some fixed ℓ, which then acts as a parameter of the density p_0. The normalizability of $p_0(x)$ implies that $J_0(\pm\infty) = 0$ and so $J_0(x) = 0$ for all x. It follows that, for fixed ℓ, the stationary density is given by a Boltzmann distribution:

$$p_0 = p_0(x;\ell) = Z(\ell)^{-1}\exp\left(-\beta[V(x) + \lambda x^2/2 - \ell\lambda x]\right). \tag{A.19}$$

The factor $Z(\ell)$ ensures the normalization $\int_0^\infty p_0(x;\ell)dx = 1$. The unknown parameter ℓ is determined by imposing the self-consistency condition

$$\ell = m_0(\ell) \equiv \int_{-\infty}^{\infty} x p_0(x;\ell)dx. \tag{A.20}$$

The number of equilibrium solutions is then equal to the number of solutions of equation (A.20). First, consider the quadratic confining potential $V(x) = \nu x^2/2$, $\nu > 0$. We have

$$Z(\ell) = \int_{-\infty}^{\infty} \mathrm{e}^{-\beta[(\nu+\lambda)x^2/2 - \ell\lambda x]}dx = \sqrt{\frac{2\pi}{\beta[\nu+\lambda]}}\mathrm{e}^{\beta\ell^2\lambda^2/2[\nu+\lambda]}, \tag{A.21}$$

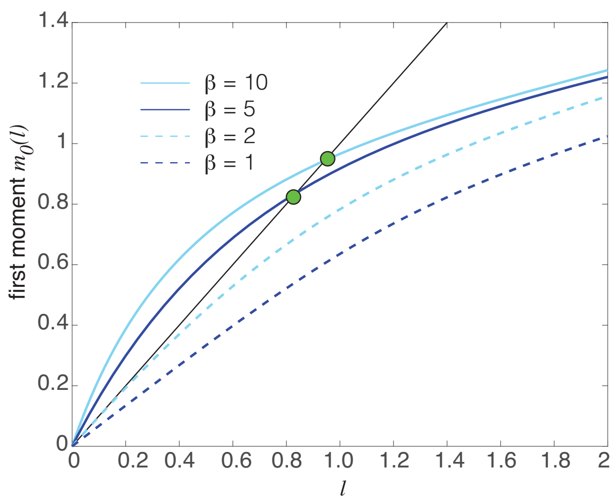

Fig. 13.1: Stationary solution of the 1D McKean-Vlasov equation (A.17) for $V(x) = x^4/4 - x^2/2$. Plot of the first moment $m_0(\ell)$ as a function of ℓ and various inverse temperatures β. The nonzero intercepts with the diagonal determine the positive definite solution ℓ_0. We also take $\lambda = 1$.

and equation (A.20) becomes

$$\ell = Z(\ell)^{-1}\int_0^\infty x\mathrm{e}^{-\beta[(\nu+\lambda)x^2/2-\ell\lambda x]}dx = \frac{1}{\lambda\beta}\frac{\partial \log Z(\ell)}{\partial \ell} = \frac{\ell\lambda}{\nu+\lambda}. \tag{A.22}$$

It follows that $\ell = 0$ and

$$p_0(x;0) = \sqrt{\frac{\beta[\nu+\lambda]}{2\pi}}\exp\left(-\beta(\nu+\lambda)x^2/2\right). \tag{A.23}$$

Hence, the interactions simply modify the effective strength of the quadratic potential.

The situation is more complicated when $V(x)$ has at least two minima, because the tendency of the Boltzmann distribution to localize around both minima competes with the cooperative effects of the Curie-Weiss potential. As an example, consider the double-well potential $V(x) = x^4/4 - x^2/2$. Although it is no longer possible to analytically solve the corresponding self-consistency equation (A.20), one can prove that there exists a phase transition at a critical temperature T_c such that $\ell = 0$ for $T > T_c$ and $\ell = \pm\ell_0 \neq 0$ for $T < T_c$ [22, 23, 24]. This is illustrated in Fig. 13.1 by plotting the first moment function $m(\ell)$ for different values of β. We find numerically that $\beta_c \approx 2$ when $\lambda = 1$, which is consistent with the critical point obtained in Refs. [22, 23].

A.3 Dynamical density functional theory (DDFT)

One of the crucial assumptions of mean field theory is that the particles are weakly interacting, In particular, the pairwise interaction potential K in equation (A.1) is scaled by the factor $1/N$. In the absence of this scaling, equation (A.11) becomes

$$\begin{aligned}\frac{\partial u(\mathbf{x},t)}{\partial t} &= D\boldsymbol{\nabla}^2 u(\mathbf{x},t) + \frac{1}{\gamma}\boldsymbol{\nabla}\cdot\Big(u(\mathbf{x},t)\boldsymbol{\nabla}V(\mathbf{x})\Big)\\ &\quad + \frac{1}{\gamma}\boldsymbol{\nabla}\cdot\left(\int_{\mathbb{R}^d}\boldsymbol{\nabla}K(\mathbf{x}-\mathbf{y})\Big\langle\rho(\mathbf{x},t)\rho(\mathbf{y},t)\Big\rangle\right),\end{aligned} \tag{A.24}$$

where

$$u(\mathbf{x},t) = \langle\rho(\mathbf{x},t)\rangle = \left\langle\sum_{j=1}^N \delta(\mathbf{x}-\mathbf{X}_j(t))\right\rangle. \tag{A.25}$$

The exact mean field limit no longer exists. However, one can use an alternative method to achieve moment closure of equation (A.11) for the one-body density, which is known as dynamical density functional theory (DDFT) [5, 3, 4, 6]. A crucial assumption of DDFT is that the relaxation of the system is sufficiently slow such that the pair correlation function can be equated with that of a corresponding equilibrium system at each point in time [6]. This allows one to approximate equation

(A.24) by the closed equation

$$\frac{\partial u(\mathbf{x},t)}{\partial t} = -\boldsymbol{\nabla}\cdot\mathbf{J}(\mathbf{x},t), \tag{A.26}$$

where

$$\mathbf{J}(\mathbf{x},t) = -D\left\{\boldsymbol{\nabla}u(\mathbf{x},t) + \beta u(\mathbf{x},t)\boldsymbol{\nabla}[V(\mathbf{x}) + \mu^{\text{ex}}(\mathbf{x},t)]\right\}, \tag{A.27}$$

Here

$$\mu^{\text{ex}}(\mathbf{x},t) = \frac{\delta F^{\text{ex}}[u(\mathbf{x},t)]}{\delta u(\mathbf{x},t)}, \tag{A.28}$$

and $F^{\text{ex}}[u]$ is the equilibrium excess free energy functional with the equilibrium density profiles replaced by non-equilibrium ones. One of the features of DDFT is that $F^{\text{ex}}[u]$ is independent of the actual external potential. Note that equation (A.26) is a version of the generalized Fick's law derived in Box 13A using linear response theory.

A.4 McKean-Vlasov equation for the classical Kuramoto model

One of the most studied interacting particle systems is the Kuramoto model of weakly-coupled, near identical limit-cycle oscillators with a sinusoidal phase interaction function [27, 28, 29]. The deterministic version of the model takes the form of a system of nonlinear phase equations

$$\frac{d\theta_j}{dt} = \omega_j + \frac{\lambda}{N}\sum_{k=1}^{N}\sin(\theta_k - \theta_j), \tag{A.29}$$

where $\theta_j(t) \in [0,2\pi]$ is the phase of the jth oscillator with natural frequency ω_j, and $\lambda \geq 0$ is the coupling strength. The frequencies ω_j are typically assumed to be distributed according to a probability density $g(\omega)$ with (i) $g(-\omega) = g(\omega)$ and (ii) $g(0) \geq g(\omega)$ for all $w \in [0,\infty)$. Without loss of generality, one can always take $g(\omega)$ to have zero mean by going to a rotating frame if necessary. One method for investigating the collective behavior of the Kuramoto model is to assume that it has a well-defined mean field limit $N \to \infty$ involving a continuum of oscillators distributed on the circle [31, 32, 33]. Let $\sigma_0(\theta,t,\omega)$ denote the fraction of oscillators with natural frequency ω that lie between θ and $\theta + d\theta$ at time t with

$$\int_0^{2\pi}\sigma_0(\theta,t,\omega)d\theta = 1. \tag{A.30}$$

More precisely, $\sigma_0(\theta,t,\omega)$ is a population density that is conditioned on the natural frequency of the oscillators, see below. Since the total number of oscillators is fixed,

σ evolves according to the continuity or Liouville equation

$$\frac{\partial \sigma_0}{\partial t} = -\frac{\partial}{\partial \theta}(\sigma_0 v_0), \tag{A.31}$$

where

$$v_0(\theta,t,\omega) = \omega + \lambda \int_0^{2\pi} d\theta' \int_{-\infty}^{\infty} d\omega' \sin(\theta'-\theta)\sigma_0(\theta',t,\omega')g(\omega'). \tag{A.32}$$

It is also possible to consider a stochastic version of the Kuramoto model [30]. If $\Theta_j(t) \in [0,2\pi)$ denotes the stochastic phase of the jth oscillator at time t, then the corresponding SDE is

$$d\Theta_j(t) = \left[\omega_j + \frac{\lambda}{N}\sum_{k=1}^{N} \sin(\Theta_k(t)-\Theta_j(t))\right]dt + \sqrt{2D}dW_j(t) \tag{A.33}$$

for $j = 1,\ldots,N$, where $W_j(t)$ is an independent Wiener process. The corresponding continuum model now takes the form of a nonlinear FP equation on the circle:

$$\frac{\partial \sigma_0}{\partial t} = -\frac{\partial}{\partial \theta}(\sigma_0 v_0) + D\frac{\partial^2 \sigma_0}{\partial \theta^2}. \tag{A.34}$$

An alternative interpretation of the SDE (A.33) is a system of Brownian particles on an N-torus with pairwise coupling and quenched disorder due to the random distribution of natural frequencies. It follows that Eq. (A.34) is equivalent to the corresponding MV equation for the global density in the mean field limit. The existence of the latter has been proven rigorously using propagation of chaos [34], and has been extended to a wider class of interacting particle systems on the torus [26, 16]. The mean field limit also applies to the deterministic Kuramoto model in the case of an ensemble of initial conditions. The next step is to introduce the global density or empirical measure

$$\rho(\theta,t,\omega) = \frac{1}{N}\sum_{j=1}^{N} \delta(\theta-\Theta_j(t))\delta(\omega-\omega_j). \tag{A.35}$$

It is important to note that the stochastic density ρ is distinct from the deterministic density σ_0 for the noiseless Kuramoto model. Moreover, the former has the normalization

$$\int_0^{2\pi} \rho(\theta,t,\omega)d\theta = \frac{1}{N}\sum_{j=1}^{N} \delta(\omega-\omega_j). \tag{A.36}$$

Taking expectations with respect to the quenched random frequencies, we have

$$\mathbb{E}[\rho(\theta,t,\omega)] = \frac{1}{N}\sum_{j=1}^{N}\delta(\theta-\Theta_j(t))\mathbb{E}[\delta(\omega-\omega_j)]$$
$$= \frac{g(\omega)}{N}\sum_{j=1}^{N}\delta(\theta-\Theta_j(t)), \tag{A.37}$$

which implies that $\int_0^{2\pi}\mathbb{E}[\rho(\theta,t,\omega)]d\theta = g(\omega)$. Consider an arbitrary smooth test function $f:[0,2\pi]\times\mathbb{R}\to\mathbb{R}$ with

$$f(\Theta_j(t),\omega_j) = \int_0^{2\pi}d\theta\int_{\mathbb{R}}d\omega\rho_j(\theta,t,\omega)f(\theta,\omega), \tag{A.38}$$

and

$$\left[\int_0^{2\pi}d\theta\int_{\mathbb{R}}d\omega f(\theta,\omega)\frac{\partial\rho_j(\theta,t,\omega)}{\partial t}\right]dt = df(\Theta_j,\omega_j).$$

Using Itô's lemma along analogous lines to the case of a Brownian gas, we find that

$$\int_0^{2\pi}d\theta\int_{\mathbb{R}}d\omega f(\theta,\omega)\frac{\partial\rho(\theta,t,\omega)}{\partial t}$$
$$= \int_0^{2\pi}d\theta\int_{\mathbb{R}}d\omega\left[\partial_\theta f(\theta,\omega)\frac{\sqrt{2D}}{N}\sum_{j=1}^{N}\rho_j(\theta,t,\omega)\xi_j(t)\right.$$
$$\left.+\rho(\theta,t,\omega)\left(D\partial_\theta^2 f(\theta,\omega)+\partial_\theta f(\theta,\omega)\mathcal{V}[\theta,t,\omega,\rho]\right)\right],$$

where

$$\mathcal{V}[\theta,t,\omega,\rho] = \omega+\lambda\int_0^{2\pi}d\theta'\int_{\mathbb{R}}d\omega'\rho(\theta',t,\omega')\sin(\theta'-\theta). \tag{A.39}$$

Integrating by parts the various terms involving derivatives of f and using the fact that f is arbitrary yields the following SPDE for ρ:

$$\frac{\partial\rho(\theta,t,\omega)}{\partial t} = -\sqrt{\frac{2D}{N^2}}\sum_{j=1}^{N}\frac{\partial}{\partial\theta}\left[\rho_j(\theta,t,\omega)\xi_j(t)\right]+D\frac{\partial^2}{\partial\theta^2}\rho(\theta,t,\omega)$$
$$-\frac{\partial}{\partial\theta}\left(\rho(\theta,t,\omega)\mathcal{V}[\theta,t,\omega,\rho]\right) \tag{A.40}$$

Following along similar lines to the derivation of equation (A.10), we introduce the white noise term

$$\xi(\theta,t,\omega) = -\frac{1}{N}\sum_{j=1}^{N}\partial_\theta\left[\rho_j(\theta,t,\omega)\xi_j(t)\right], \tag{A.41}$$

which has zero mean and correlation function

$$\langle \xi(\theta,t,\omega)\xi(\theta',t',\omega')\rangle = \frac{1}{N^2}\delta(t-t')\sum_{j=1}^{N}\partial_\theta\partial_{\theta'}\bigg(\rho_j(\theta,t,\omega)\rho_j(\theta',t,\omega')\bigg).$$

Since $\rho_j(\theta,t,\omega)\rho_j(\theta',t,\omega') = \delta(\theta-\theta')\delta(\omega-\omega')\rho_j(\theta,t,\omega)$, it follows that

$$\langle \xi(\theta,t,\omega)\xi(\theta',t',\omega')\rangle = \frac{1}{N}\delta(t-t')\partial_\theta\partial_{\theta'}\bigg(\delta(\theta-\theta')\delta(\omega-\omega')\rho(\theta,t,\omega)\bigg).$$

Finally, we introduce the global density-dependent noise field

$$\widehat{\xi}(\theta,t,\omega) = \frac{1}{\sqrt{N}}\partial_\theta\bigg(\eta(\theta,t,\omega)\sqrt{\rho(\theta,t,\omega)}\bigg), \tag{A.42}$$

where $\eta(\theta,t,\omega)$ is a global white noise field whose components satisfy

$$\langle \eta(\theta,t,\omega)\eta(\theta',t',\omega')\rangle = \delta(t-t')\delta(\theta-\theta)\delta(\omega-\omega'). \tag{A.43}$$

It can be checked that the Gaussian noises ξ and $\widehat{\xi}$ have the same correlation functions and are thus statistically identical. We thus obtain the generalized DK equation for the stochastic Kuramoto model with resetting:

$$\begin{aligned}\frac{\partial\rho(\theta,t,\omega)}{\partial t} &= \sqrt{\frac{2D}{N}}\frac{\partial}{\partial\theta}\left[\sqrt{\rho(\theta,t,\omega)}\eta(\theta,t,\omega)\right] + D\frac{\partial^2}{\partial\theta^2}\rho(\theta,t,\omega)\\ &\quad - \frac{\partial}{\partial\theta}\bigg(\rho(\theta,t,\omega)\mathcal{V}[\theta,t,\omega,\rho]\bigg).\end{aligned} \tag{A.44}$$

As in the case of the DK equation (A.10), taking expectations with respect to the white noise processes results in a moment closure problem. However, assuming a mean field ansatz in the thermodynamic limit leads to the following deterministic MV equation for the mean field $\phi(\theta,t,\omega) = \bigg\langle\rho(\theta,t,\omega)\bigg\rangle$:

$$\frac{\partial\phi(\theta,t,\omega)}{\partial t} = D\frac{\partial^2}{\partial\theta^2}\phi(\theta,t,\omega) - \frac{\partial}{\partial\theta}\bigg(\phi(\theta,t,\omega)\mathcal{V}[\theta,t,\omega,\phi]\bigg) \tag{A.45}$$

As in the case of interacting Brownian particles, a stationary solution of the MV equation (A.45) has to be determined self-consistently. Now, however, the self-consistency condition involves the first circular moment

$$Z_1(t) = R(t)\mathrm{e}^{i\psi(t)} := \int_0^{2\pi}d\theta\int_{-\infty}^{\infty}d\omega\,\mathrm{e}^{i\theta}\phi(\theta,t,\omega), \tag{A.46}$$

rather than the first moment $\ell = \langle x\rangle$ for the Curie-Weiss potential, say. Substituting equation (A.46) into the MV equation (A.45) gives

$$\frac{\partial \phi(\theta,t,\omega)}{\partial t} = D\frac{\partial^2 \phi(\theta,t,\omega)}{\partial \theta^2} - \frac{\partial}{\partial \theta}\left[\Big(\omega + \lambda R(t)\sin(\psi(t)-\theta(t))\Big)\phi(\theta,t,\omega)\right]. \tag{A.47}$$

The amplitude $R(t)$ is a measure of the degree of synchronization with $R = 1$ signifying complete synchrony and $R = 0$ corresponding to the incoherent state $\phi(\theta,t,\omega) = g(\omega)/2\pi$, which is a solution of equation (A.45). In principle, one could now proceed by solving the time-independent version of (A.47) for fixed Z_1 and then substituting the resulting stationary solution $\phi_{Z_1}(\theta,\omega)$ into equation (A.46) to determine Z_1. However, the calculation of ϕ_{Z_1} is nontrivial since we no longer have a stationary Boltzmann distribution on the circle.

An alternative representation of the MV equation can be obtained by considering the Fourier series expansion

$$\phi(\theta,t,\omega) = \frac{g(\omega)}{2\pi}\left(1+\sum_{m=1}^{\infty}\left[\phi_m(\omega,t)\mathrm{e}^{im\theta} + \text{c. c.}\right]\right), \tag{A.48}$$

with

$$\phi_m(\omega,t) = \langle \mathrm{e}^{-im\theta}\rangle := \int_0^{2\pi} \mathrm{e}^{-im\theta}\phi(\theta,t,\omega)\frac{d\theta}{2\pi}. \tag{A.49}$$

Solving the initial value problem for ϕ is then equivalent to solving an infinite hierarchy of equations for the coefficients ϕ_m:

$$\frac{\partial \phi_m}{\partial t} + im\omega\phi_m + \frac{\lambda m}{2}\left[\phi_{m+1}Z_1 - \phi_{m-1}Z_1^*\right], \tag{A.50}$$

with

$$Z_1(t) = \int_{-\infty}^{\infty} g(\omega)\phi_1^*(\omega,t)d\omega. \tag{A.51}$$

References

1. D. S. Dean. "Langevin equation for the density of a system of interacting Langevin processes." *J. Phys. A* **29** L613-L617 (1996)
2. K. Kawasaki. "Microscopic analyses of the dynamical density functional equation of dense fluids." *J. Stat. Phys.* **93** 527-546 (1998).
3. A. J. Archer and R. Evans. "Dynamical density functional theory and its application to spinodal decomposition." *J. Chem. Phys.* **121** 4246-4254 (2004)
4. A. J. Archer and M. Rauscher. "Dynamical density functional theory for interacting Brownian particles: stochastic or deterministic?" *J. Phys. A: Math. Gen.* **37** 9325 (2004)
5. U. M. B. Marconi and P. Tarazona. "Dynamic density functional theory of fluids." *J. Chem. Phys.* **110** 8032-8044 (1999)
6. M. te Vrugt, H. Lowen and R. Wittkowski. "Classical dynamical density functional theory: from fundamentals to applications." *Adv. Phys.* **69** 121 (2021).

7. N. Dirr, M. Stamatakis and J. Zimmer. ”Entropic and gradient flow formulations for nonlinear diffusion.” *J. Math. Phys.* **57** 081505 (2016).
8. V. Konarovskyi, T. Lehmann and M.-K. von Renesse. “Dean-Kawasaki dynamics: ill-posedness vs. triviality.” *Electron. Commun. Probab.* **24** 1-9 (2019).
9. V. Konarovskyi, T. Lehmann and M.-K. von Renesse. “On Dean-Kawasaki dynamics with smooth drift potential.” *J. Stat. Phys.* **178**:666-681 (2020).
10. C. Djurdjevac, J. Koppl and A. Djurdjevac. “Feedback loops in opinion dynamics of agent-based models with multiplicative noise.” *Entropy* **24** 1352 (2022).
11. A. Djurdjevac, H. Kremp and N. Perkowski. “Weak error analysis for a nonlinear SPDE approximation of the Dean-Kawasaki equation.” arXiv preprint https://arxiv.org/abs/2212.11714 (2022).
12. B. Fehrman and B. Gess. “Non-equilibrium large deviations and parabolic-hyperbolic PDE with irregular drift.” *Invent. Math.* **234** 573-636 (2023).
13. F. Cornalba and J. Fischer. “The Dean-Kawasaki equation and the structure of density fluctuations in systems of diffusing particles.” *Arch. Ration. Mech. Anal.,* **247** 59 (2023).
14. K. Oelschlager. “A martingale approach to the law of large numbers for weakly interacting stochastic processes.” *Ann. Probab.* **12** 458-479 (1984).
15. P. E. Jabin and Z. Wang. “Mean field limit for stochastic particle systems.” In *Active Particles* Volume 1. Ed. by N. Bellomo, P. Degond, and E. Tadmor. Modelling and Simulation in Science, Engineering and Technology. Cham: Springer International Publishing, pp. 379-402 (2017).
16. M. G. Delgadino, R. S. Gvalani and G. A. Pavliotis. “On the diffusive-mean field limit for weakly interacting diffusions exhibiting phase transitions.” *Arch. Rational Mech. Anal.* **241** 91-148 (2021).
17. L.-P. Chaintron and A. Diez. “Propagation of chaos: A review of models, methods and applications. I. Models and methods.” *Kinetic and related models* **15** 895-1015 (2022).
18. L.-P. Chaintron and A. Diez. “Propagation of chaos: A review of models, methods and applications. II. Applications. *Kinetic and related models*” **15** 1017-1173 (2022).
19. H. P. McKean. “A class of Markov processes associated with nonlinear parabolic equations.” *Proc. Natl. Acad. Sci. USA* **56**1907-1911 (1996).
20. Y. Tamura. “On asymptotic behaviors of the solution of a nonlinear diffusion equation.” J. Fac. Sci. Univ. Tokyo Sect. IA Math. 31 195-221 (1984).
21. J. Tugaut, Phase transitions of McKean-Vlasov processes in double-wells landscape, Stochastics **86**, 257 (2014).
22. R. C. Desai and R. Zwanzig. “Statistical mechanics of a nonlinear stochastic model.” *J. Stat. Phys.* **19** 1-24 (1978).
23. D. A. Dawson. “Critical dynamics and fluctuations for a mean-field model of cooperative behavior.” *J. Stat. Phys.* **31** 2-85 (1983).
24. S. N. Gomez, S. Kalliadasis, G. A. Pavliotis and P. Yatsyshin. “Dynamics of the Desai-Zwanzig model in a multiwell and random energy landscapes.” *Phys. Rev. E* **99** 032109 (2019).
25. L. Chayes and V. Panferov. “The McKean-Vlasov equation in finite volume.” *J. Stat. Phys.* **138** 351-380 (2010).
26. J. A. Carrillo, R. S. Gvalani, G. A. Pavliotis and A. Schlichting. “Long-time behaviour and phase transitions for the McKean-Vlasov equation on the torus.” *Archive for Rational Mechanics and Analysis* **235** 635-690 (2020).
27. Y. Kuramoto *Chemical Oscillations, Waves, and Turbulence* Springer (1984).
28. S. H. Strogatz. “From Kuramoto to Crawford: Exploring the onset of synchronization in populations of coupled oscillators.” *Physica D* **143** 1-20 (2000).
29. J. A. Acebron, L. L. Bonilla, C. L. Perez Vicente, P. Ritort and R. Spigler. “The Kuramoto model: A simple paradigm for synchronization phenomena.” *Rev. Mod. Phys.* **77** 137-185 (2005).
30. H. Sakaguchi. “Cooperative phenomena in coupled oscillator systems under external fields.” *Prog. Theor. Phys.***79** 39-46 (1988).
31. S. H. Strogatz and R. E. Mirollo. “Stability of incoherence of a population of coupled oscillators.” *J. Stat. Phys.* **63** 613-635 (1991)

32. J. D. Crawford. “Amplitude expansions for instabilities in populations of globally-coupled oscillators.” *J. Stat. Phys.* **74** 1047-1084 (1994).
33. J. D. Crawford and K. T. R. Davies. “Synchronization of globally coupled phase oscillators: Singularities and scaling for general couplings.” *Physica D* **125** 1-46 (1999).
34. P. Dai Pra and F. den Hollander. “McKean-Vlasov limit for interacting random processes in random media.” *J. Stat. Phys.* **84** 735-772 (1996).