Geometric Model of 4-Component Helical Auxetic Capacitor Sensors

The 4-ply helical auxetic capacitive sensor is composed of two twisted elastic fibre and two conductive fibres wire. The elastic fibres have accurate geometry but the conductive fibres are approximated with circular cross-sections.

Input parameters:

```
> wire_diameter := 0.35;
  initial_elastic_diameter := 1;
  initial_pitch := 2.8;
  elastic_poissons_ratio := 0.4;
  final_strain := 0.4;
  number_of_turns := 2;
  run_name := "E3R1-02";
  with(plots) :
  with(ColorTools) :
  plotsetup(default) :
```

$$wire\_diameter := 0.35$$
$$initial\_elastic\_diameter := 1$$
$$initial\_pitch := 2.8$$
$$elastic\_poissons\_ratio := 0.4$$
$$final\_strain := 0.4$$
$$number\_of\_turns := 2$$
$$run\_name := \text{"E3R1-02"} \quad \textbf{(1)}$$

First, define the equation for a helix in Cartesian (x, y, z) coordinates, parameterized by (t, u). The parameter t ranges from 0 to 2πn, where n is the number of total turns. u ranges from 0 to 2π. Plot an example of a single-turn helix with major radius r_M, minor radius r_m, and pitch p.

$$> helix := (t, u, \theta, r_M, r_m, p) \rightarrow \left[\frac{1}{2\cdot\pi} p\cdot t + \frac{r_M\cdot r_m\cdot \sin(u)}{\mathrm{sqrt}\left(r_M^{\,2} + \left(\frac{p}{2\cdot\pi}\right)^2\right)}, \right.$$
$$r_M\cdot\cos(t+\theta) - r_m\cdot\cos(t+\theta)\cdot\cos(u) + \frac{p\cdot r_m\cdot\sin(t+\theta)\cdot\sin(u)}{2\cdot\pi\cdot\,\mathrm{sqrt}\left(r_M^{\,2} + \left(\frac{p}{2\cdot\pi}\right)^2\right)},$$
$$\left. r_M\cdot\sin(t+\theta) - r_m\cdot\sin(t+\theta)\cdot\cos(u) - \frac{p\cdot r_m\cdot\cos(t+\theta)\cdot\sin(u)}{2\cdot\pi\cdot\,\mathrm{sqrt}\left(r_M^{\,2} + \left(\frac{p}{2\cdot\pi}\right)^2\right)} \right];$$

$$helix := \left(t, u, \theta, r_M, r_m, p\right) \mapsto \left[\frac{p \cdot t}{2 \cdot \pi} + \frac{r_M \cdot r_m \cdot \sin(u)}{\sqrt{r_M^2 + \frac{p^2}{4 \cdot \pi^2}}}, r_M \cdot \cos(t + \theta) - r_m \cdot \cos(t + \theta) \cdot \cos(u) + \frac{p \cdot r_m \cdot \sin(t + \theta) \cdot \sin(u)}{2 \cdot \pi \cdot \sqrt{r_M^2 + \frac{p^2}{4 \cdot \pi^2}}}, r_M \cdot \sin(t + \theta) - r_m \cdot \sin(t + \theta) \cdot \cos(u) - \frac{p \cdot r_m \cdot \cos(t + \theta) \cdot \sin(u)}{2 \cdot \pi \cdot \sqrt{r_M^2 + \frac{p^2}{4 \cdot \pi^2}}}\right] \quad (2)$$

The function for the length of a helical path with radius r, ptich p, and n turns.

> $helical\_length := (r, p, n) \rightarrow n \cdot \mathrm{sqrt}\left((2 \cdot \pi \cdot r)^2 + p^2\right)$

$$helical\_length := (r, p, n) \mapsto n \cdot \sqrt{4 \cdot \pi^2 \cdot r^2 + p^2} \quad (3)$$

Strain and raidal strain of an isotropic rod from axial strain and its Poisson's ratio.

> $strain := \left(l, l_0\right) \rightarrow \frac{\left(l - l_0\right)}{l_0};$

$poisson\_strain\_y := \left(\varepsilon_x, \nu\right) \rightarrow -\left(1 - \left(1 + \varepsilon_x\right)^{-\nu}\right);$

$$strain := \left(l, l_0\right) \mapsto \frac{l - l_0}{l_0}$$

$$poisson\_strain\_y := \left(\varepsilon_x, \nu\right) \mapsto -1 + \left(\varepsilon_x + 1\right)^{-\nu} \quad (4)$$

The following strain expressions are for the elastic element in transverse and axial directions.

> $elastic\_radial\_strain := strain\left(d_{elastic}, d_{elastic,\, i}\right);$

$elastic\_axial\_strain := strain\left(l_{elastic}, l_{elastic,\, i}\right)$

$$elastic\_radial\_strain := \frac{d_{elastic} - d_{elastic,i}}{d_{elastic,i}}$$

$$elastic\_axial\_strain := \frac{l_{elastic} - l_{elastic,i}}{l_{elastic,i}} \quad (5)$$

We equate the above expressions using the Poisson's ratio function defined in ??.

> $elastic\_strain := elastic\_radial\_strain = poisson\_strain\_y\left(elastic\_axial\_strain, \nu_{elastic}\right):$

$elastic\_diameter := solve\left(elastic\_strain, d_{elastic}\right)$

$$elastic\_diameter := \left(\frac{l_{elastic} - l_{elastic,i}}{l_{elastic,i}} + 1\right)^{-\nu_{elastic}} d_{elastic,i} \quad (6)$$

> $pitch\_from\_strain := p_i \cdot (1+\varepsilon_x);$

$elastic\_length\_from\_helical\_path := l_{elastic} = helical\_length\left(\frac{elastic\_diameter}{2}, pitch\_from\_strain, number\_of\_turns\right)$

$$pitch\_from\_strain := p_i\,(1+\varepsilon_x)$$

$$elastic\_length\_from\_helical\_path := l_{elastic} = 2\sqrt{\pi^2\left(\left(\frac{l_{elastic}-l_{elastic,i}}{l_{elastic,i}}+1\right)^{-\nu_{elastic}}\right)^2 d_{elastic,i}{}^2 + p_i^2\,(1+\varepsilon_x)^2} \quad \textbf{(7)}$$

> $elastic\_length\_from\_helical\_path := solve(elastic\_length\_from\_helical\_path, l_{elastic})$

$$elastic\_length\_from\_helical\_path := 2\,RootOf\left(-\pi^2\left(\left(\frac{2\_Z}{l_{elastic,i}}\right)^{-\nu_{elastic}}\right)^2 d_{elastic,i}{}^2 - p_i^2\,\varepsilon_x^2 - 2\,p_i^2\,\varepsilon_x - p_i^2 + \_Z^2\right) \quad \textbf{(8)}$$

> $initial\_elastic\_length := evalf\left(helical\_length\left(\frac{initial\_elastic\_diameter}{2}, initial\_pitch, number\_of\_turns\right)\right);$

$elastic\_length\_by\_strain := unapply\left(subs\left(\{l_{elastic,\,i} = initial\_elastic\_length, \nu_{elastic} = elastic\_poissons\_ratio, d_{elastic,\,i} = initial\_elastic\_diameter, p_i = initial\_pitch\}, elastic\_length\_from\_helical\_path\right), \varepsilon_x\right)$

$$initial\_elastic\_length := 8.416556160$$

$$elastic\_length\_by\_strain := \varepsilon_x \mapsto 2\cdot RootOf\left(-\frac{31.15898409}{\_Z^{0.8}} - 7.84\cdot\varepsilon_x^2 - 15.68\cdot\varepsilon_x - 7.84 + \_Z^2\right) \quad \textbf{(9)}$$

Take the real root.

> $initial\_elastic\_length;$
$final\_elastic\_length := allvalues(elastic\_length\_by\_strain(final\_strain))[1];$
$final\_elastic\_diameter := evalf\left(subs\left(\{l_{elastic} = final\_elastic\_length, l_{elastic,\,i} = initial\_elastic\_length, \nu_{elastic} = elastic\_poissons\_ratio, d_{elastic,\,i} = initial\_elastic\_diameter\}, elastic\_diameter\right)\right);$
$final\_pitch := subs\left(\{p_i = initial\_pitch, \varepsilon_x = final\_strain\}, pitch\_from\_strain\right)$

$$8.416556160$$

$$final\_elastic\_length := 9.816859328$$

$$final\_elastic\_diameter := 0.9402963346$$
$$final\_pitch := 3.92 \tag{10}$$

We solve for the equations of the helix sliced by a plane in the x axis, which is will be needed later to generate the cross-section images. This is done by solving the helix x equation for t, and substituting it into the other helix equations to eliminate t. Now, new parameter x defines the location of the plane, and the equations trace out a helical cross-section.

> $t\_cross\_section := solve\left(x = helix\left(t, u, \theta, r_M, r_m, p\right)[1], t\right);$
> $helix\_cross\_section := unapply\left(simplify\left(subs\left(\{t = t\_cross\_section\}, helix\left(t, u, \theta, r_M, r_m, p\right)\right)\right), x, u, \theta, r_M, r_m, p\right):$

$$t\_cross\_section := -\frac{2\left(2 r_M r_m \sin(u)\pi - x\sqrt{4\pi^2 r_M^2 + p^2}\right)\pi}{p\sqrt{4\pi^2 r_M^2 + p^2}} \tag{11}$$

Initial Geometry

> $core\_cs\_initial := helix\_cross\_section\left(0, u, \pi, \frac{initial\_elastic\_diameter}{2}, \frac{initial\_elastic\_diameter}{2}, initial\_pitch\right):$
> $Q := core\_cs\_initial[2..3]:$
> $Q_\perp := [-simplify(diff(Q[2], u)), simplify(diff(Q[1], u))]:$

> $u\_pos\_initial := fsolve\left(\frac{-Q[1]}{\cos\left(\tan^{-1}\left(\frac{Q_\perp[2]}{Q_\perp[1]}\right)\right)} - \frac{wire\_diameter}{2} = 0, u, 0..\pi\right)$

$$u\_pos\_initial := 1.647633449 \tag{12}$$

> $evalf(subs(u = u\_pos\_initial, Q[2]))$

$$0.6217417664 \tag{13}$$

> $major\_radius\_wire\_initial := evalf\left(subs\left(u = u\_pos\_initial, -\frac{Q_\perp[2]}{Q_\perp[1]} \cdot Q[1] + Q[2]\right)\right)$

$$major\_radius\_wire\_initial := 0.7533156755 \tag{14}$$

> $cartesian\_to\_polar := (x, y) \rightarrow \left(simplify\left(\text{sqrt}\left(\left(x + \frac{initial\_elastic\_diameter}{2}\right)^2 + y^2\right)\right), simplify\left(\tan^{-1}\left(\frac{y}{\left(x + \frac{initial\_elastic\_diameter}{2}\right)}\right)\right)\right):$

> $circular\_cross\_section := \left(u, \theta, r_M, r_m\right) \rightarrow \left[0, r_M\cos(\theta) + r_m\cos(u), r_M\sin(\theta) + r_m\sin(u)\right]$

$$circular\_cross\_section := \left(u, \theta, r_M, r_m\right) \mapsto \left[0, r_M \cdot \cos(\theta) + r_m \cdot \cos(u), r_M \cdot \sin(\theta) + r_m \cdot \sin(u)\right] \tag{15}$$

> $core1cs\_initial := helix\_cross\_section\left(0, u, 0, \frac{initial\_elastic\_diameter}{2},\right.$

$\frac{initial\_elastic\_diameter}{2}, initial\_pitch\Big)$:

$core2cs\_initial := helix\_cross\_section\Big(0, u, \pi, \frac{initial\_elastic\_diameter}{2},$
$\frac{initial\_elastic\_diameter}{2}, initial\_pitch\Big)$:

$winding1cs\_initial := circular\_cross\_section\Big(u, \frac{\pi}{2}, major\_radius\_wire\_initial,$
$\frac{wire\_diameter}{2}\Big)$:

$winding2cs\_initial := circular\_cross\_section\Big(u, \frac{3 \cdot \pi}{2}, major\_radius\_wire\_initial,$
$\frac{wire\_diameter}{2}\Big)$:

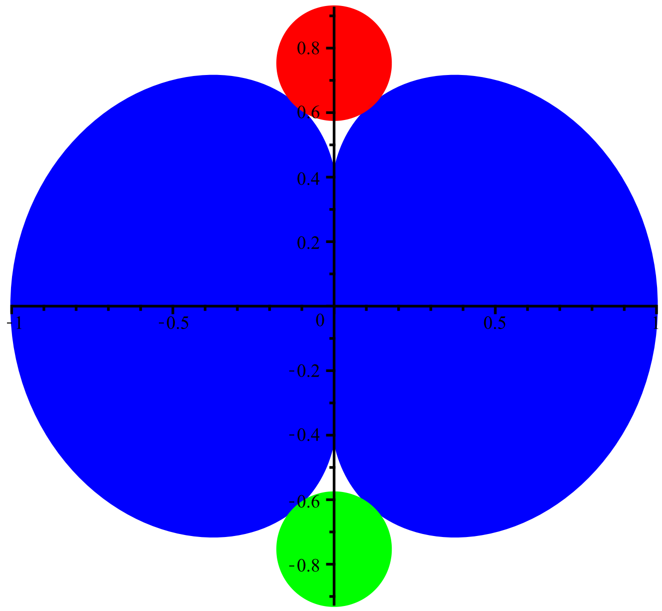
$plot\_core1cs\_initial := plot\big([core1cs\_initial[2], core1cs\_initial[3], u = 0..2 \cdot \pi], filled = [color = Color([0, 0, 1]), transparency = 0], color = Color([0, 0, 1]), thickness = 0, scaling = constrained\big)$:

$plot\_core2cs\_initial := plot\big([core2cs\_initial[2], core2cs\_initial[3], u = 0..2 \cdot \pi], filled = [color = Color([0, 0, 1]), transparency = 0], color = Color([0, 0, 1]), thickness = 0, scaling = constrained\big)$:

$plot\_winding1cs\_initial := plot\big([winding1cs\_initial[2], winding1cs\_initial[3], u = 0..2 \cdot \pi], filled = [color = Color([1, 0, 0]), transparency = 0], color = Color([1, 0, 0]), thickness = 0, scaling = constrained\big)$:

$plot\_winding2cs\_initial := plot\big([winding2cs\_initial[2], winding2cs\_initial[3], u = 0..2 \cdot \pi], filled = [color = Color([0, 1, 0]), transparency = 0], color = Color([0, 1, 0]), thickness = 0, scaling = constrained\big)$:

$display([plot\_core1cs\_initial, plot\_core2cs\_initial, plot\_winding1cs\_initial, plot\_winding2cs\_initial])$



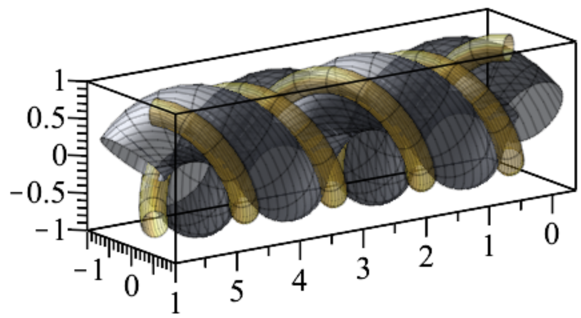
> $core1\_initial := plot3d\left(helix\left(t, u, 0, \frac{initial\_elastic\_diameter}{2}, \frac{initial\_elastic\_diameter}{2}, initial\_pitch\right), t = 0..2 \cdot number\_of\_turns \cdot \pi, u = 0..2 \cdot \pi, color = Color\left(\left[\frac{84}{255}, \frac{88}{255}, \frac{99}{255}\right]\right), transparency = 0.35, scaling = constrained\right):$

$core2\_initial := plot3d\left(helix\left(t, u, \pi, \frac{initial\_elastic\_diameter}{2}, \frac{initial\_elastic\_diameter}{2}, initial\_pitch\right), t = 0..2 \cdot number\_of\_turns \cdot \pi, u = 0..2 \cdot \pi, color = Color\left(\left[\frac{84}{255}, \frac{88}{255}, \frac{99}{255}\right]\right), transparency = 0.35, scaling = constrained\right):$

$winding1\_initial := plot3d\left(helix\left(t, u, \frac{\pi}{2}, major\_radius\_wire\_initial, \frac{wire\_diameter}{2}, initial\_pitch\right), t = 0..2 \cdot number\_of\_turns \cdot \pi, u = 0..2 \cdot \pi, color = Color\left(\left[\frac{249}{255}, \frac{200}{255}, \frac{70}{255}\right]\right), transparency = 0.45, scaling = constrained\right):$

$winding2\_initial := plot3d\left(helix\left(t, u, \frac{3 \cdot \pi}{2}, major\_radius\_wire\_initial, \frac{wire\_diameter}{2}, initial\_pitch\right), t = 0..2 \cdot number\_of\_turns \cdot \pi, u = 0..2 \cdot \pi, color = Color\left(\left[\frac{249}{255}, \frac{200}{255}, \frac{70}{255}\right]\right), transparency = 0.45, scaling = constrained\right):$

$display([core1\_initial, core2\_initial, winding1\_initial, winding2\_initial])$



Final Geometry

> $l_{w,i} := helical\_length(major\_radius\_wire\_initial, initial\_pitch, number\_of\_turns);$
> $r_{M,wire} := solve\left(l_{w,i} = helical\_length(r, p, n), r\right)[1];$
> $r_{M,wire} := simplify\left(subs\left(\{p = pitch\_from\_strain\}, r_{M,wire}\right)\right);$
> $r_{M,wire} := unapply\left(simplify\left(subs\left(\{p_i = initial\_pitch, n = number\_of\_turns\}, r_{M,wire}\right)\right), \varepsilon_x\right);$

$$l_{w,i} := 10.99879818$$

$$r_{M,wire} := \frac{3.183098862 \times 10^{-9} \sqrt{-2.500000000 \times 10^{15} p^2 n^2 + 3.024339035 \times 10^{17}}}{n}$$

$$r_{M,wire} := \frac{3.183098862 \times 10^{-9} \sqrt{3.024339035 \times 10^{17} - 2.500000000 \times 10^{15} \left(\varepsilon_x + 1.\right)^2 n^2 p_i^2}}{n}$$

$$r_{M,wire} := \varepsilon_x \mapsto 1.591549431 \times 10^{-9} \cdot \sqrt{2.240339035 \times 10^{17} - 7.840000000 \times 10^{16} \cdot \varepsilon_x^2 - 1.568000000 \times 10^{17} \cdot \varepsilon_x} \quad (16)$$

> $major\_radius\_wire\_final := r_{M,\,wire}(final\_strain)$

$$major\_radius\_wire\_final := 0.6138717846 \quad (17)$$

> $core1cs\_final := helix\_cross\_section\left(0, u, 0, \frac{final\_elastic\_diameter}{2}, \frac{final\_elastic\_diameter}{2}, final\_pitch\right):$

$core2cs\_final := helix\_cross\_section\left(0, u, \pi, \frac{final\_elastic\_diameter}{2}, \frac{final\_elastic\_diameter}{2}, final\_pitch\right):$

$winding1cs\_final := circular\_cross\_section\left(u, \frac{\pi}{2}, major\_radius\_wire\_final, \frac{wire\_diameter}{2}\right):$

$winding2cs\_final := circular\_cross\_section\left(u, \frac{3 \cdot \pi}{2}, major\_radius\_wire\_final, \frac{wire\_diameter}{2}\right):$

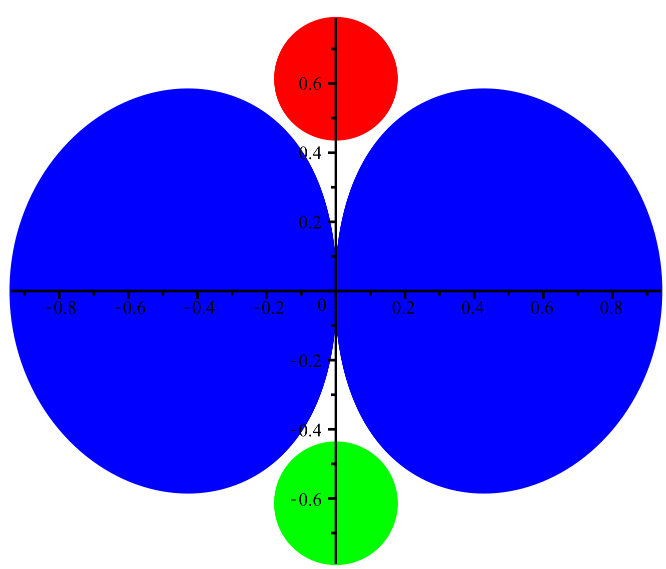
$plot\_core1cs\_final := plot([core1cs\_final[2], core1cs\_final[3], u = 0..2 \cdot \pi], filled = [color = Color([0, 0, 1]), transparency = 0], color = Color([0, 0, 1]), thickness = 0, scaling = constrained):$

$plot\_core2cs\_final := plot([core2cs\_final[2], core2cs\_final[3], u = 0..2 \cdot \pi], filled = [color = Color([0, 0, 1]), transparency = 0], color = Color([0, 0, 1]), thickness = 0, scaling = constrained):$

$plot\_winding1cs\_final := plot([winding1cs\_final[2], winding1cs\_final[3], u = 0..2 \cdot \pi], filled = [color = Color([1, 0, 0]), transparency = 0], color = Color([1, 0, 0]), thickness = 0, scaling = constrained):$

$plot\_winding2cs\_final := plot([winding2cs\_final[2], winding2cs\_final[3], u = 0..2 \cdot \pi], filled = [color = Color([0, 1, 0]), transparency = 0], color = Color([0, 1, 0]), thickness = 0, scaling = constrained):$

$display([plot\_core1cs\_final, plot\_core2cs\_final, plot\_winding1cs\_final, plot\_winding2cs\_final])$



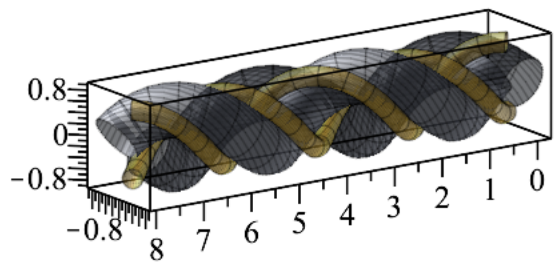
$$core1\_final := plot3d\left(helix\left(t, u, 0, \frac{final\_elastic\_diameter}{2}, \frac{final\_elastic\_diameter}{2}, final\_pitch\right), t = 0..2\cdot number\_of\_turns\cdot\pi, u = 0..2\cdot\pi, color = Color\left(\left[\frac{84}{255}, \frac{88}{255}, \frac{99}{255}\right]\right), transparency = 0.35, scaling = constrained\right):$$

$$core2\_final := plot3d\left(helix\left(t, u, \pi, \frac{final\_elastic\_diameter}{2}, \frac{final\_elastic\_diameter}{2}, final\_pitch\right), t = 0..2\cdot number\_of\_turns\cdot\pi, u = 0..2\cdot\pi, color = Color\left(\left[\frac{84}{255}, \frac{88}{255}, \frac{99}{255}\right]\right), transparency = 0.35, scaling = constrained\right):$$

$$winding1\_final := plot3d\left(helix\left(t, u, \frac{\pi}{2}, major\_radius\_wire\_final, \frac{wire\_diameter}{2}, final\_pitch\right), t = 0..2\cdot number\_of\_turns\cdot\pi, u = 0..2\cdot\pi, color = Color\left(\left[\frac{249}{255}, \frac{200}{255}, \frac{70}{255}\right]\right), transparency = 0.45, scaling = constrained\right):$$

$$winding2\_final := plot3d\left(helix\left(t, u, \frac{3\cdot\pi}{2}, major\_radius\_wire\_final, \frac{wire\_diameter}{2}, final\_pitch\right), t = 0..2\cdot number\_of\_turns\cdot\pi, u = 0..2\cdot\pi, color = Color\left(\left[\frac{249}{255}, \frac{200}{255}, \frac{70}{255}\right]\right), transparency = 0.45, scaling = constrained\right):$$

$$display([core1\_final, core2\_final, winding1\_final, winding2\_final])$$



Calculation of Capacitance versus Strain

> $$parallel\_wire\_capacitance := (d, a, l, \varepsilon) \rightarrow \frac{\pi \cdot \varepsilon \cdot l}{\ln\left(\frac{d}{2 \cdot a} + \mathrm{sqrt}\left(\frac{d^2}{2 \cdot a} - 1\right)\right)};$$

$$parallel\_wire\_capacitance := (d, a, l, \varepsilon) \mapsto \frac{\pi \cdot \varepsilon \cdot l}{\ln\left(\frac{d}{2 \cdot a} + \sqrt{\frac{d^2}{2 \cdot a} - 1}\right)} \quad \textbf{(18)}$$

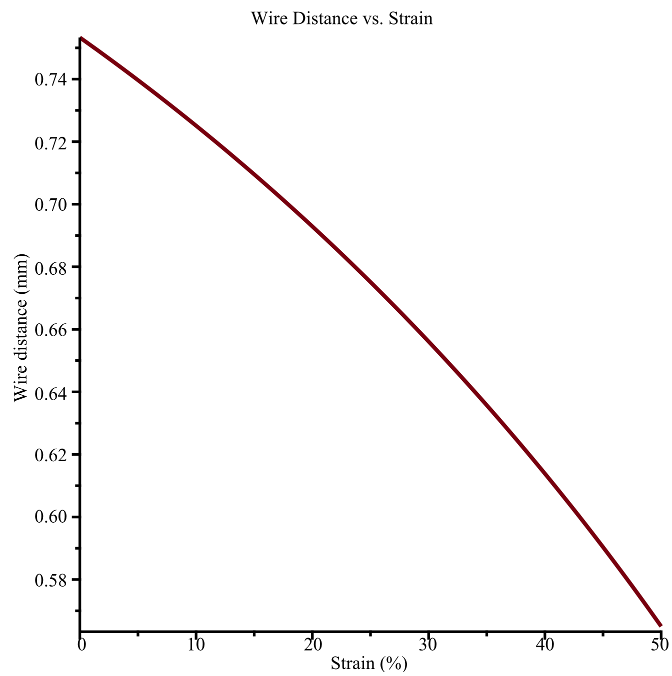
> $strain\_values := Vector([seq(0..final\_strain, numelems = 100)]) :$
$wire\_distance := Vector(numelems(strain\_values)) :$
$capacitance := Vector(numelems(strain\_values)) :$
for i **from** 1 **to** $numelems(strain\_values)$ **do**
$wire\_distance[i] := r_{M,wire}(strain\_values[i]) :$
$capacitance[i] := parallel\_wire\_capacitance\left(wire\_distance[i], \frac{wire\_diameter}{2}, subs\left(\left\{p_i = initial\_pitch, \varepsilon_x = strain\_values[i]\right\}, pitch\_from\_strain\right), 8.84\mathrm{e}{-12}\right) :$
end do:
$with(ExcelTools) :$

```
Export(strain_values, "Strain.xls");
Export(capacitance, "Capacitance.xls");
```

```
Error, (in ExcelTools:-Export) Could not open the file. T:\Auxetic Capacitive Sensors\01 Scripts\Capacitance.xls (The process cannot access the file because it is being used by another process)
```

Plot the distance between wires versus strain.

```
> plot(strain_values·100, wire_distance, title = "Wire Distance vs. Strain", labels = ["Strain (%)", "Wire distance (mm)"], labeldirections = ["horizontal", "vertical"])
```



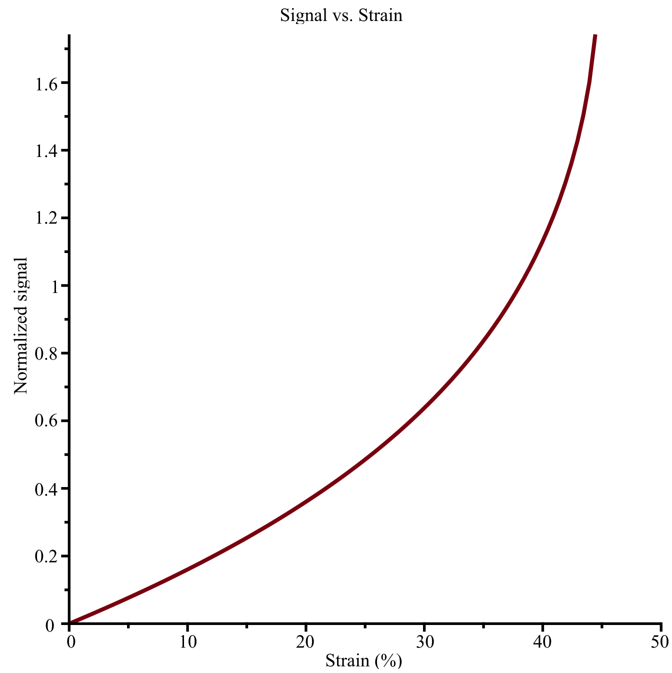
Plot the estimated capacitance between the wires versus strain.

```
> plot(strain_values·100, capacitance·1e12, title = "Capacitance vs. Strain", labels = ["Strain (%)", "Capacitance (pF)"], labeldirections = ["horizontal", "vertical"])
```

```
Error, (in plot) two lists or Vectors of numerical values expected
```

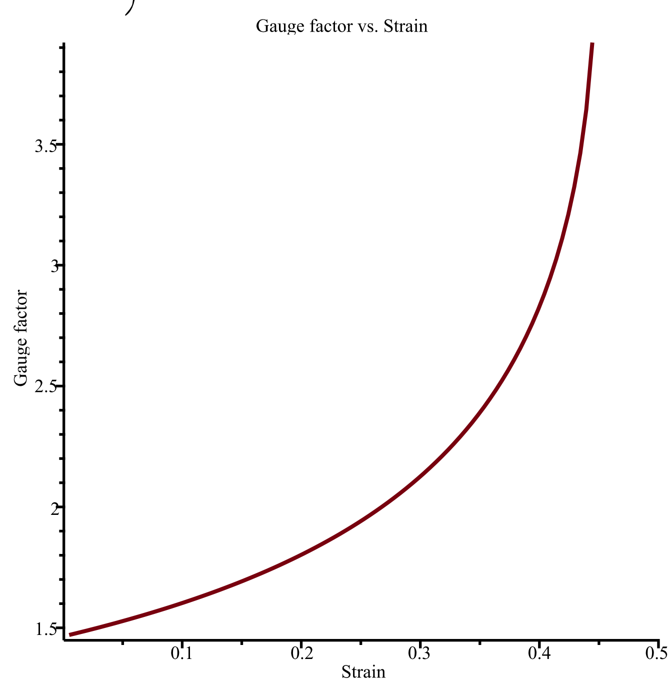
Plot the normalized signal (ΔC/C0) versus strain.

> $plot\left(strain\_values \cdot 100, \dfrac{(capacitance - \sim capacitance[1])}{\sim capacitance[1]}, title = \text{"Signal vs. Strain"}, labels = [\text{"Strain (\%)"}, \text{"Normalized signal"}], labeldirections = [\text{"horizontal"}, \text{"vertical"}] \right)$



Plot the sensitivity (gauge factor) versus strain.

$$> \ plot\left(strain\_values[2..100], \frac{\left(\frac{(capacitance[2..100] - \sim capacitance[1])}{\sim capacitance[1]}\right)}{\sim strain\_values[2..100]}, title = \text{"Gauge factor vs. Strain"}, labels = [\text{"Strain"}, \text{"Gauge factor"}], labeldirections = [\text{"horizontal"}, \text{"vertical"}]\right)$$



>