Helical Auxetic Yarn Modelling

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Helical Auxetic Sensors

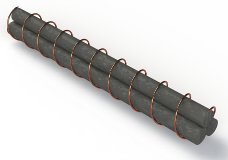


Figure: Unstretched state

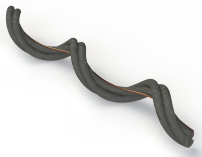


Figure: Stretched state

- Want a model to determine exact geometry at arbitrary strain.
- Parameters:
  - Stretchable elastic fibre (initially core) with circular cross-section, initial diameter $d_e(\varepsilon = 0)$, length l_0, and Poisson's ratio ν.
  - Perfectly inelastic wire (initially winding) with circular cross-section, diameter d_w, n turns, and pitch $p_0 = \frac{l_0}{n}$.
- Use model to calculate capacitance?

Approach to Modelling

Approach:

1. Obtain equation for a helical coil in space.
2. Determine the constraints that link the geometry of the elastic and inelastic fibres w.r.t. overall sensor strain.
3. Slice the geometry with a plane along the sensor axis to get the cross-section.
4. Compute capacitance computationally from the 2-D cross-section geometry.

Assumptions:

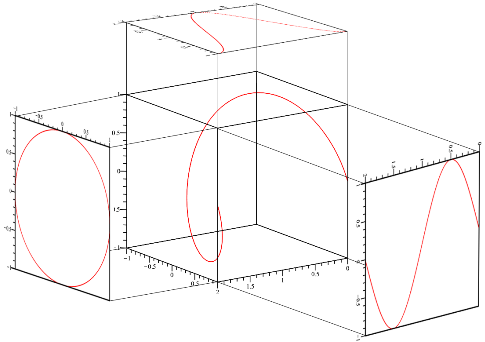
- Elastic and stiff fibres are concentric.
- The wire helix geometry is independent of the elastic element.
- The elastic fibre is stiff and it does not comply to the wire (maintains a circular cross-section along the helical path, with point contact).
- Capactiance may be calculated from the cross-section, integrated along the sensor (i.e., no self-capacitance between coils).

Helical Curve

- Begin with the equation for a helical path in space parameterized by t.
- Helix will run along the x-axis with radius r_M and pitch p.

Equation for a helical path

$$\vec{h}(t) = \begin{cases} x(t) &= \frac{p}{2\pi} t \\ y(t) &= r_M \cos(t) \\ z(t) &= r_M \sin(t) \end{cases} \tag{1}$$



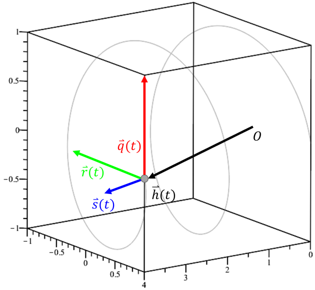
Helical Coil with Volume: Coordinate Set-Up[1]

- Need to add thickness to the helix by tracing its path with a circular cross-section.
- We require an orthonormal basis coordinate system $\{\vec{p}(t), \vec{q}(t), \vec{r}(t)\}$ for this circle, where:
  - $\vec{p}(t)$ is tangent to the helical path at point $\vec{h}(t)$
  - $\vec{q}(t)$ is normal to the path (pointing toward the centre)
  - $\vec{r}(t)$ is orthogonal to both $\vec{p}$ and $\vec{q}$

$$\vec{p}(t) = \frac{\frac{d\vec{h}(t)}{dt}}{\left|\left|\frac{d\vec{h}(t)}{dt}\right|\right|} = \frac{1}{\sqrt{(\frac{p}{2\pi})^2 + r_M^2}} \begin{pmatrix} \frac{p}{2\pi} & -r_M \sin(t) & r_M \cos(t) \end{pmatrix} \quad (2)$$

$$\vec{q}(t) = \frac{\frac{d\vec{p}(t)}{dt}}{\left|\left|\frac{d\vec{p}(t)}{dt}\right|\right|} = \begin{pmatrix} 0 & -\cos(t) & -\sin(t) \end{pmatrix} \quad (3)$$

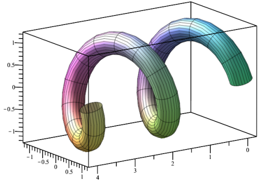
$$\vec{r}(t) = \vec{p}(t) \times \vec{q}(t) = \frac{1}{(\frac{p}{2\pi})^2 + r_M^2} \begin{pmatrix} r_M & \frac{p}{2\pi} \sin(t) & -\frac{p}{2\pi} \cos(t) \end{pmatrix} \quad (4)$$



[1] what's the equation of helix surface?

Helical Coil with Volume: Revolving the Circle

- Draw a circle of radius r_m along a plane spanned by $\vec{r}(t)$ and $\vec{s}(t)$
- Introduce parameterization of the circle with variable u
- Convert into (x, y, z) coordinates of the helical reference frame
- Add in parameter θ to control the starting rotational position



$$\vec{H}(t,u) = \overbrace{\vec{h}(t)}^{\text{helix}} + \overbrace{r_m\vec{r}(t)\cos(u) + r_m\vec{s}(t)\sin(u)}^{\text{circle}} \quad t = 0,\ldots,2\pi\cdot n \quad u = 0,\ldots,2\pi \tag{5}$$

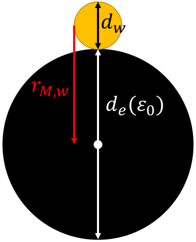
Parameterized helical coil

$$\vec{H}(t,u) = \begin{cases} x(t,u) = \frac{p}{2\pi}t + \frac{r_M r_m \sin(u)}{\sqrt{r_M^2+\left(\frac{p}{2\pi}\right)^2}} \\ y(t,u) = r_M\cos(t+\theta) - r_m\cos(t+\theta)\cos(u) + \frac{p\cdot r_m \sin(t+\theta)\sin(u)}{2\pi\sqrt{r_M^2+\left(\frac{p}{2\pi}\right)^2}} \\ z(t,u) = r_M\sin(t+\theta) - r_m\sin(t+\theta)\cos(u) - \frac{p\cdot r_m \cos(t+\theta)\sin(u)}{2\pi\sqrt{r_M^2+\left(\frac{p}{2\pi}\right)^2}} \end{cases} \tag{6}$$

Behaviour at Endpoints

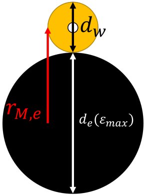
- At zero strain ($\varepsilon = 0$) the following holds:

	Wire	Elastic
D	d_w	$d_e(0)$
l	$n\sqrt{(\pi(d_w + d_e(0)))^2 + p^2}$	l
r_m	$\frac{1}{2}d_w$	$\frac{1}{2}d_e(0)$
r_M	$\frac{1}{2}(d_w + d_e(0))$	0



- At the maximum strain (ε_{max}), the following holds:

	Wire	Elastic
d	d_w	$d_e(\varepsilon_{max})$
l	$n\sqrt{(\pi(d_w + d_e(0)))^2 + p^2}$	$\star$
r_m	$\frac{1}{2}d_w$	$\frac{1}{2}d_e(\varepsilon_{max})$
r_M	0	$\frac{1}{2}(d_w + d_e(\varepsilon_{max}))$



Geometry of the Wire Coil

- The wire helix minor radius is constant and its major radius is only dependent on initial conditions and strain ε.
- The wire does not stretch, so we can produce an equality to solve for its r_M as a function of ε.
- Use the fact that changes linearly with strain $p(\varepsilon) = p(0)(1+\varepsilon)$.

$$\begin{aligned} l_w(0) &= l_w(\varepsilon) \\ n\sqrt{(\pi(d_w + d_e(0)))^2 + p^2} &= n\sqrt{(\pi(2r_{M,w}(\varepsilon)))^2 + (p(1+\varepsilon))^2} \\ r_{M,w}(\varepsilon) &= \frac{1}{2\pi}\sqrt{\pi^2(d_e(0) + d_w(0))^2 - p^2\varepsilon(\varepsilon+2)} \end{aligned} \tag{7}$$

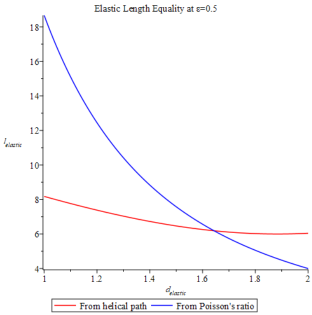
- From these, the geometry of the wire coil (r_M, r_m, p) at any strain is exactly defined.
- What about the elastic coil?

Constraints between Elastic and Wire

- The axis of the helices shifts from the centre of the elastic at $\varepsilon = 0$ to the centre of the wire at ε_{max} following:

$$r_{M,e}(\varepsilon) = \frac{1}{2}\left(d_e(\varepsilon) + d_w\right) - r_{M,w}(\varepsilon) \tag{8}$$

- We now know expressions for (r_M, r_m, p) of the elastic coil at arbitrary strain, but they are all dependent on the elastic diameter $d_e(\varepsilon)$.
- Two equations can be written for the elastic length $l_e(\varepsilon)$:
  1. Length from helical path: $l_{e,h}(\varepsilon) = f(d_e(\varepsilon), \varepsilon)$
  2. Length from Poisson's ratio: $l_{e,\nu}(\varepsilon) = f(d_e(\varepsilon))$
- Can equate $l_{e,h}(\varepsilon) = l_{e,\nu}(\varepsilon)$ and solve for $d_e(\varepsilon)$ numerically.



From Helical Path

$$l_{e,h}(\varepsilon) = n\sqrt{4\pi^2\left(\frac{d_e(\varepsilon)+d_w}{2} - \frac{1}{2\pi}\sqrt{(d_e(0)+d_w)^2\pi^2 - p^2\varepsilon(\varepsilon+2)}\right)^2 + p^2(1+\varepsilon)^2} \tag{9}$$

From Poisson's Ratio

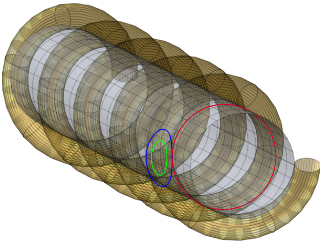
$$l_{e,\nu}(\varepsilon) = \exp\left(-\frac{\ln\left(\frac{d_e(\varepsilon)}{d_e(0)}\right)}{\nu}\right) l_0 \tag{10}$$

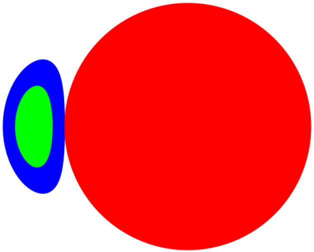
Final Geometrical Model

- An example for a helical auxetic sensor with $d_e(0) = 1$, $d_w = 0.1$, $p(0) = 2$, $\nu = 0.45$.

Slicing the Helices

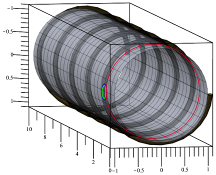
- Conveniently, we want cross-sections along the x-axis of the helix (in the zy-plane).
- If we get the x-component of $\vec{H}(t, u)$ and solve it for t, we can substitute it back into the other components and find the equation for the slice of $\vec{H}(t, u)$ with a plane at $x = 0$.
- Repeat for both wire and elastic helices, and optionally for the wire core to delineate the dielectric layer.

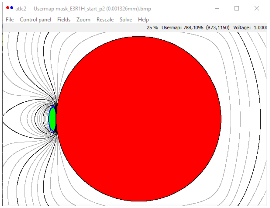




Calculating Capacitance

- ATLC2[2]: calculates transmission line parameters from cross-section.
- Multiply by length of strained sensor to estimate capacitance.





C and Gp results :

C = 569.366 pF
Gp = 0.000000077604
Zo = 5.92 Ohms if vf=0.989
L = 0.0200 uH if vf=0.989
Vo = 0.0000 V Qdif = -3.85 %
Run time = 1.7 minutes, 41 MB

[2] Arbitrary Transmission Line Calculator