On the Design of Stable, High Performance Sigma Delta Modulators

M.A.Sc. Thesis Defence

Brett Hannigan
`bch@alumni.ubc.ca`

2018-12-04

School of Biomedical Engineering
University of British Columbia

Table of Contents

Introduction

Project Objectives

Primary Objective

To develop a systematic method of design for sigma delta A/D converters for the recording of bio-signals.

Project Objectives

Primary Objective

To develop a systematic method of design for sigma delta A/D converters for the recording of bio-signals.

Ideally, the goals of the method are to:

- Model the nonlinear system accurately in a way that allows analysis of existing designs.

Primary Objective

To develop a systematic method of design for sigma delta A/D converters for the recording of bio-signals.

Ideally, the goals of the method are to:

- Model the nonlinear system accurately in a way that allows analysis of existing designs.
- Reduce dependence on simulation.

Primary Objective

To develop a systematic method of design for sigma delta A/D converters for the recording of bio-signals.

Ideally, the goals of the method are to:

- Model the nonlinear system accurately in a way that allows analysis of existing designs.
- Reduce dependence on simulation.
- Provide a way to design guaranteed stable sigma delta modulators in a way that minimizes conservatism.

Oversampling

Sampling a signal at a rate higher than what the Nyquist-Shannon sampling theorem would dictate.

Oversampling

Sampling a signal at a rate higher than what the Nyquist-Shannon sampling theorem would dictate.

Noise Shaping

The use of a filter to push quantization noise out of the signal band by wrapping the quantizer in a feedback loop.

Principles of Sigma Delta Modulation

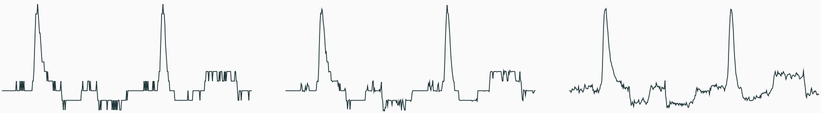

Figure 1: An example EEG signal [1] digitized to 5 bits with naïve quantization (left), 10 times oversampled quantization (middle), and first-order sigma delta modulation (right).

Oversampling

Sampling a signal at a rate higher than what the Nyquist-Shannon sampling theorem would dictate.

Noise Shaping

The use of a filter to push quantization noise out of the signal band by wrapping the quantizer in a feedback loop.

Basic Structure of a Sigma Delta Modulator

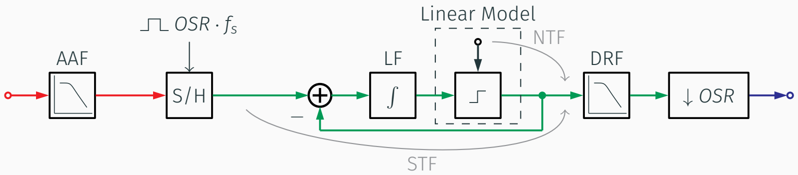

Figure 2: A simplified block diagram of a discrete-time sigma delta A/D converter.

Basic Structure of a Sigma Delta Modulator

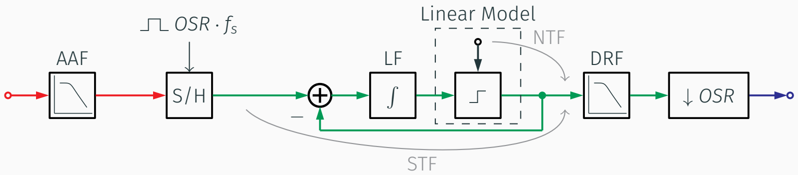

Figure 2: A simplified block diagram of a discrete-time sigma delta A/D converter.

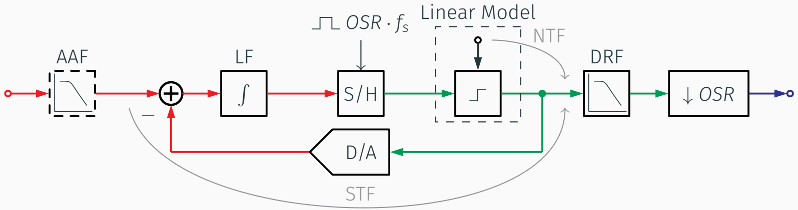

Figure 3: A simplified block diagram of a continuous-time sigma delta A/D converter.

THE UNIVERSITY OF BRITISH COLUMBIA

The nonlinear quantizer in the forward path makes stability analysis difficult. Loop filter design is commonly done in one of several ways:

Design of the Loop Filter

The nonlinear quantizer in the forward path makes stability analysis difficult. Loop filter design is commonly done in one of several ways:

- Pure integrator – DC-stable for low order loops [2].

Design of the Loop Filter

THE UNIVERSITY OF BRITISH COLUMBIA

The nonlinear quantizer in the forward path makes stability analysis difficult. Loop filter design is commonly done in one of several ways:

- Pure integrator – DC-stable for low order loops [2].
- Prototype NTF – noise rejection of linear model chosen from a family of filters [3, Appx. B].

Design of the Loop Filter

The nonlinear quantizer in the forward path makes stability analysis difficult. Loop filter design is commonly done in one of several ways:

- Pure integrator – DC-stable for low order loops [2].
- Prototype NTF – noise rejection of linear model chosen from a family of filters [3, Appx. B].
- Optimization-based approaches – wide range of techniques.

Design of the Loop Filter

The nonlinear quantizer in the forward path makes stability analysis difficult. Loop filter design is commonly done in one of several ways:

- Pure integrator – DC-stable for low order loops [2].
- Prototype NTF – noise rejection of linear model chosen from a family of filters [3, Appx. B].
- Optimization-based approaches – wide range of techniques.

The design process often relies on extensive simulation to confirm that stability is likely during normal operation and the circuit may include complicated instability detection and recovery mechanisms [4, 5, 6, 7].

Stability and Performance

$\mathcal{H}_\infty$ Stability Criterion

Lee's rule states that a modulator is probably stable if the peak of the NTF frequency response is less than a heuristic value.

$$||NTF(z)||_\infty \lessapprox 2$$

$\mathcal{H}_\infty$ Stability Criterion

Lee's rule states that a modulator is probably stable if the peak of the NTF frequency response is less than a heuristic value.

$$||NTF(z)||_\infty \lessapprox 2$$

- Rule-of-thumb developed empirically from observations on a 4th order DT modulator [8].

$\mathcal{H}_\infty$ Stability Criterion

Lee's rule states that a modulator is probably stable if the peak of the NTF frequency response is less than a heuristic value.

$$||NTF(z)||_\infty \lessapprox 2$$

- Rule-of-thumb developed empirically from observations on a 4th order DT modulator [8].
- Generally conservative for 2nd order loops, approximately correct for 3rd order, and inadequate for higher order [9].

THE UNIVERSITY OF BRITISH COLUMBIA

$\mathcal{H}_\infty$ Stability Criterion

Lee's rule states that a modulator is probably stable if the peak of the NTF frequency response is less than a heuristic value.

$$||NTF(z)||_\infty \lessapprox 2$$

- Rule-of-thumb developed empirically from observations on a 4th order DT modulator [8].
- Generally conservative for 2nd order loops, approximately correct for 3rd order, and inadequate for higher order [9].
- Common in existing design tools [3, Appx. B].

$\mathcal{H}_\infty$ Stability Criterion

Lee's rule states that a modulator is probably stable if the peak of the NTF frequency response is less than a heuristic value.

$$||NTF(z)||_\infty \lessapprox 2$$

- Rule-of-thumb developed empirically from observations on a 4th order DT modulator [8].
- Generally conservative for 2nd order loops, approximately correct for 3rd order, and inadequate for higher order [9].
- Common in existing design tools [3, Appx. B].
- Easy to apply as a control optimization problem.

$\mathcal{H}_\infty$ Stability Criterion

Lee's rule states that a modulator is probably stable if the peak of the NTF frequency response is less than a heuristic value.

$$||NTF(z)||_\infty \lessapprox 2$$

- Rule-of-thumb developed empirically from observations on a 4th order DT modulator [8].
- Generally conservative for 2nd order loops, approximately correct for 3rd order, and inadequate for higher order [9].
- Common in existing design tools [3, Appx. B].
- Easy to apply as a control optimization problem.
- No straightforward relationship between maximum stable input and the Lee criterion value.

Root Locus Stability Criterion

The describing function of the nonlinear quantizer is a variable gain dependent on quantizer input amplitude. A modulator loop is stable if the root locus remains in the stable region of the complex plane when sweeping through valid quantizer gains.

$$|NTF(z, K)| \leq 1 \quad \forall K \in [k_l, k_h]$$

Root Locus Stability Criterion

The describing function of the nonlinear quantizer is a variable gain dependent on quantizer input amplitude. A modulator loop is stable if the root locus remains in the stable region of the complex plane when sweeping through valid quantizer gains.

$$|NTF(z,K)| \leq 1 \quad \forall K \in [k_l, k_h]$$

- Has been used to produce designs by examining pole and zero departure angles [10, 11, 12].

Root Locus Stability Criterion

The describing function of the nonlinear quantizer is a variable gain dependent on quantizer input amplitude. A modulator loop is stable if the root locus remains in the stable region of the complex plane when sweeping through valid quantizer gains.

$$|NTF(z, K)| \leq 1 \quad \forall K \in [k_l, k_h]$$

- Has been used to produce designs by examining pole and zero departure angles [10, 11, 12].
- Can predict the maximum stable input amplitude.

THE UNIVERSITY OF BRITISH COLUMBIA

$\mathcal{H}_2$ Stability Criterion

The $\mathcal{H}_2$ stability criterion predicts stability for a class of norm-bounded input signals r if the squared 2-norm (power gain) of the NTF is less than a value calculated by placing assumptions on the statistical distribution of the quantizer input signal u.

$$||NTF(z)||_2^2 \leq f(r, u)$$

$\mathcal{H}_2$ Stability Criterion

The $\mathcal{H}_2$ stability criterion predicts stability for a class of norm-bounded input signals r if the squared 2-norm (power gain) of the NTF is less than a value calculated by placing assumptions on the statistical distribution of the quantizer input signal u.

$$||NTF(z)||_2^2 \leq f(r, u)$$

- Splits signals into DC baseline with superposition of the AC component, uses additional degrees of freedom to enforce that the quantization error is uncorrelated with quantizer input [13].

$\mathcal{H}_2$ Stability Criterion

The $\mathcal{H}_2$ stability criterion predicts stability for a class of norm-bounded input signals r if the squared 2-norm (power gain) of the NTF is less than a value calculated by placing assumptions on the statistical distribution of the quantizer input signal u.

$$||NTF(z)||_2^2 \leq f(r, u)$$

- Splits signals into DC baseline with superposition of the AC component, uses additional degrees of freedom to enforce that the quantization error is uncorrelated with quantizer input [13].
- A good approximation as long as the chosen quantizer input PDF is accurate.

$\mathcal{H}_2$ Stability Criterion

The $\mathcal{H}_2$ stability criterion predicts stability for a class of norm-bounded input signals r if the squared 2-norm (power gain) of the NTF is less than a value calculated by placing assumptions on the statistical distribution of the quantizer input signal u.

$$||NTF(z)||_2^2 \leq f(r, u)$$

- Splits signals into DC baseline with superposition of the AC component, uses additional degrees of freedom to enforce that the quantization error is uncorrelated with quantizer input [13].
- A good approximation as long as the chosen quantizer input PDF is accurate.
- Can predict the maximum stable input amplitude.

ℓ_1 Stability Criterion

The input to the quantizer can be bounded for a class of norm-bounded input signals r if the absolute value of the sum of loop filter impulse response coefficients (maximum peak-to-peak gain) is bounded .

$$\min_{K} ||ntf_K[n]||_1 \leq 3 - ||r||_\infty$$

THE UNIVERSITY OF BRITISH COLUMBIA

ℓ_1 Stability Criterion

The input to the quantizer can be bounded for a class of norm-bounded input signals r if the absolute value of the sum of loop filter impulse response coefficients (maximum peak-to-peak gain) is bounded .

$$\min_{K} ||ntf_K[n]||_1 \leq 3 - ||r||_\infty$$

- Sufficient criterion for BIBO stability [14].

THE UNIVERSITY OF BRITISH COLUMBIA

ℓ_1 Stability Criterion

The input to the quantizer can be bounded for a class of norm-bounded input signals *r* if the absolute value of the sum of loop filter impulse response coefficients (maximum peak-to-peak gain) is bounded .

$$\min_K ||ntf_K[n]||_1 \leq 3 - ||r||_\infty$$

- Sufficient criterion for BIBO stability [14].
- Uses the worst-case gain of the filter, usually resulting in very conservative designs [13].

ℓ_1 Stability Criterion

The input to the quantizer can be bounded for a class of norm-bounded input signals r if the absolute value of the sum of loop filter impulse response coefficients (maximum peak-to-peak gain) is bounded .

$$\min_{K} ||ntf_K[n]||_1 \leq 3 - ||r||_\infty$$

- Sufficient criterion for BIBO stability [14].
- Uses the worst-case gain of the filter, usually resulting in very conservative designs [13].
- Determines the maximum stable input amplitude.

Performance Goal

$$\min ||NTF(z)||_\infty \quad z \in [j\omega_l, j\omega_h]$$

Performance Goal

$$\min ||NTF(z)||_\infty \quad z \in [j\omega_l, j\omega_h]$$

- Want to maximize the quantization noise rejection in the signal band.

Performance

Performance Goal

$$\min ||NTF(z)||_\infty \quad z \in [j\omega_l, j\omega_h]$$

- Want to maximize the quantization noise rejection in the signal band.
- Two main options:
  - Addition of weighting filters to the NTF channel, then use $\mathcal{H}_\infty$ control techniques.

Performance Goal

$$\min ||NTF(z)||_\infty \quad z \in [j\omega_l, j\omega_h]$$

- Want to maximize the quantization noise rejection in the signal band.
- Two main options:
  - Addition of weighting filters to the NTF channel, then use $\mathcal{H}_\infty$ control techniques.
  - Apply the Generalized KYP lemma which provides a link between a finite-frequency inequality and a set of LMIs.

Similar Work

Similar Work

Table 1: A comparison of some recent work on sigma delta modulator design as a control optimization problem.

Ref.	Optimized norms	NTF Type	Performance goal	Stability criteria
[15]	$\mathcal{H}_\infty$, $\mathcal{H}_2$, ℓ_1	IIR	Weighting filters	$\mathcal{H}_\infty$, $\mathcal{H}_2$

Similar Work

Table 1: A comparison of some recent work on sigma delta modulator design as a control optimization problem.

Ref.	Optimized norms	NTF Type	Performance goal	Stability criteria
[15]	$\mathcal{H}_\infty$, $\mathcal{H}_2$, ℓ_1	IIR	Weighting filters	$\mathcal{H}_\infty$, $\mathcal{H}_2$
[16]	$\mathcal{H}_\infty$	IIR[1]	GKYP	Not reported

Table 1: A comparison of some recent work on sigma delta modulator design as a control optimization problem.

Ref.	Optimized norms	NTF Type	Performance goal	Stability criteria
[15]	$\mathcal{H}_\infty$, $\mathcal{H}_2$, ℓ_1	IIR	Weighting filters	$\mathcal{H}_\infty$, $\mathcal{H}_2$
[16]	$\mathcal{H}_\infty$	IIR[1]	GKYP	Not reported
[17]	$\mathcal{H}_\infty$	FIR	GKYP	ℓ_1 mentioned, $\mathcal{H}_\infty$ used

Similar Work

Table 1: A comparison of some recent work on sigma delta modulator design as a control optimization problem.

Ref.	Optimized norms	NTF Type	Performance goal	Stability criteria
[15]	$\mathcal{H}_\infty$, $\mathcal{H}_2$, ℓ_1	IIR	Weighting filters	$\mathcal{H}_\infty$, $\mathcal{H}_2$
[16]	$\mathcal{H}_\infty$	IIR[1]	GKYP	Not reported
[17]	$\mathcal{H}_\infty$	FIR	GKYP	ℓ_1 mentioned, $\mathcal{H}_\infty$ used
[18]	$\mathcal{H}_\infty$	IIR	GKYP	$\mathcal{H}_\infty$

Similar Work

Table 1: A comparison of some recent work on sigma delta modulator design as a control optimization problem.

Ref.	Optimized norms	NTF Type	Performance goal	Stability criteria
[15]	$\mathcal{H}_\infty$, $\mathcal{H}_2$, ℓ_1	IIR	Weighting filters	$\mathcal{H}_\infty$, $\mathcal{H}_2$
[16]	$\mathcal{H}_\infty$	IIR[1]	GKYP	Not reported
[17]	$\mathcal{H}_\infty$	FIR	GKYP	ℓ_1 mentioned, $\mathcal{H}_\infty$ used
[18]	$\mathcal{H}_\infty$	IIR	GKYP	$\mathcal{H}_\infty$
[19]	$\mathcal{H}_\infty$, $\mathcal{H}_2$, ℓ_1	FIR	Weighting filters	ℓ_1 mentioned, $\mathcal{H}_\infty$ used

Table 1: A comparison of some recent work on sigma delta modulator design as a control optimization problem.

Ref.	Optimized norms	NTF Type	Performance goal	Stability criteria
[15]	$\mathcal{H}_\infty$, $\mathcal{H}_2$, ℓ_1	IIR	Weighting filters	$\mathcal{H}_\infty$, $\mathcal{H}_2$
[16]	$\mathcal{H}_\infty$	IIR[1]	GKYP	Not reported
[17]	$\mathcal{H}_\infty$	FIR	GKYP	ℓ_1 mentioned, $\mathcal{H}_\infty$ used
[18]	$\mathcal{H}_\infty$	IIR	GKYP	$\mathcal{H}_\infty$
[19]	$\mathcal{H}_\infty$, $\mathcal{H}_2$, ℓ_1	FIR	Weighting filters	ℓ_1 mentioned, $\mathcal{H}_\infty$ used
This	$\mathcal{H}_2$, ℓ_1	IIR	GKYP	$\mathcal{H}_\infty$, root locus, $\mathcal{H}_2$, ℓ_1

[1] Only the zeros of the IIR filter are optimized.

Optimization

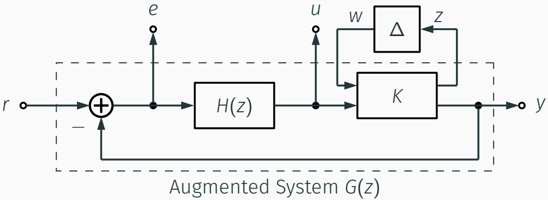

Figure 4: The augmented system model with channels of interest extracted.

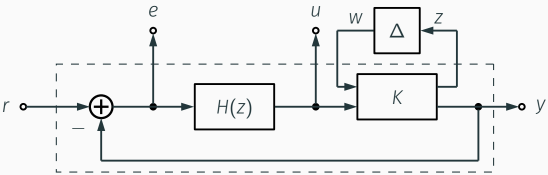

Figure 4: The augmented system model with channels of interest extracted.

$$G : \begin{bmatrix} \dot{x} \\ z \\ e \\ u \\ y \end{bmatrix} = \left[\begin{array}{c|cc} A_H - k_{22}B_HC_H & -k_{21}B_H & B_H \\ \hline k_{12}C_H & k_{11} & 0 \\ -k_{22}C_H & -k_{21} & 1 \\ C_H & 0 & 0 \\ k_{22}C_H & k_{21} & 0 \end{array}\right] \begin{bmatrix} x \\ w \\ r \end{bmatrix}$$

LMIs exist for the GKYP Lemma [20], $\mathcal{H}_2$ norm [21], and $\star$-norm [22, 23], an upper bound of the ℓ_1 norm, but there are some caveats:

LMIs exist for the GKYP Lemma [20], $\mathcal{H}_2$ norm [21], and $\star$-norm [22, 23], an upper bound of the ℓ_1 norm, but there are some caveats:

- LMIs are non-convex for IIR filters because there is a product term of the pole coefficients.

LMIs exist for the GKYP Lemma [20], $\mathcal{H}_2$ norm [21], and $\star$-norm [22, 23], an upper bound of the ℓ_1 norm, but there are some caveats:

- LMIs are non-convex for IIR filters because there is a product term of the pole coefficients.
- The $\star$-norm LMI has a non-convex scalar term.

Convexification

THE UNIVERSITY OF BRITISH COLUMBIA

The authors in [18] showed how to manipulate the GKYP LMI by assuming the augmented system is in CCF so that there is only one occurrence of the non-convex term in a form like:

$$\cdots + \begin{bmatrix} aa^T & a \\ a^T & 1 \end{bmatrix} \geq 0$$

The authors in [18] showed how to manipulate the GKYP LMI by assuming the augmented system is in CCF so that there is only one occurrence of the non-convex term in a form like:

$$\cdots + \begin{bmatrix} aa^T & a \\ a^T & 1 \end{bmatrix} \geq 0$$

With this reduction, attempted the following:

- Use of general non-convex solver.

Convexification

The authors in [18] showed how to manipulate the GKYP LMI by assuming the augmented system is in CCF so that there is only one occurrence of the non-convex term in a form like:

$$\cdots + \begin{bmatrix} aa^T & a \\ a^T & 1 \end{bmatrix} \geq 0$$

With this reduction, attempted the following:

- Use of general non-convex solver.
- Use of rank-constrained LMI solver.

The authors in [18] showed how to manipulate the GKYP LMI by assuming the augmented system is in CCF so that there is only one occurrence of the non-convex term in a form like:

$$\cdots + \begin{bmatrix} aa^T & a \\ a^T & 1 \end{bmatrix} \geq 0$$

With this reduction, attempted the following:

- Use of general non-convex solver.
- Use of rank-constrained LMI solver.
- Applying Shor's relaxation to linearize the problem.

The authors in [18] showed how to manipulate the GKYP LMI by assuming the augmented system is in CCF so that there is only one occurrence of the non-convex term in a form like:

$$\cdots + \begin{bmatrix} aa^T & a \\ a^T & 1 \end{bmatrix} \geq 0$$

With this reduction, attempted the following:

- Use of general non-convex solver.
- Use of rank-constrained LMI solver.
- Applying Shor's relaxation to linearize the problem.
- Performing an iterative method [24].

Examples

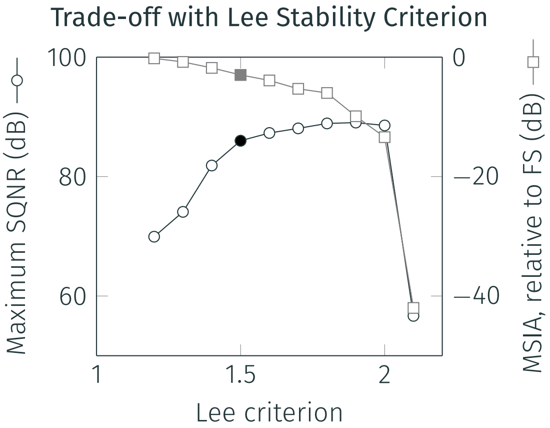

Figure 5: The performance (maximum simulated SQNR) and stability (simulated MSIA) achieved with the $\mathcal{H}_\infty$ modulator design for different Lee criterion goals.

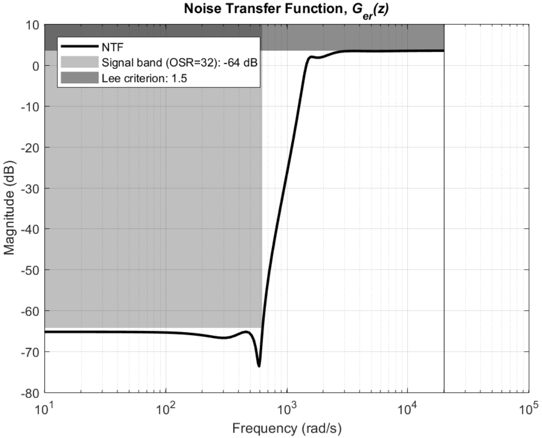

Figure 6: The noise transfer function generated with the $\mathcal{H}_\infty$ stability criterion and associated optimization targets.

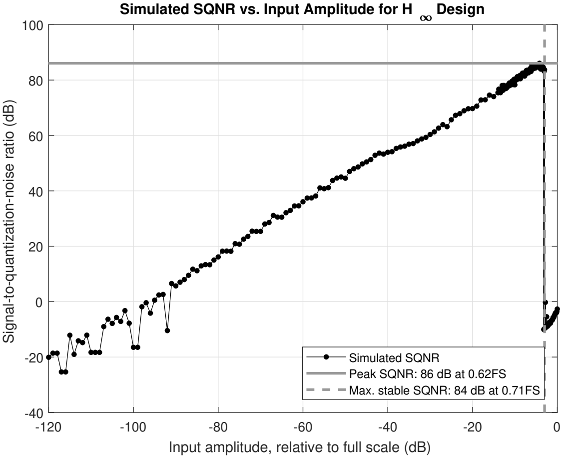

Figure 7: Simulation data for the modulator generated with the $\mathcal{H}_\infty$ stability critierion.

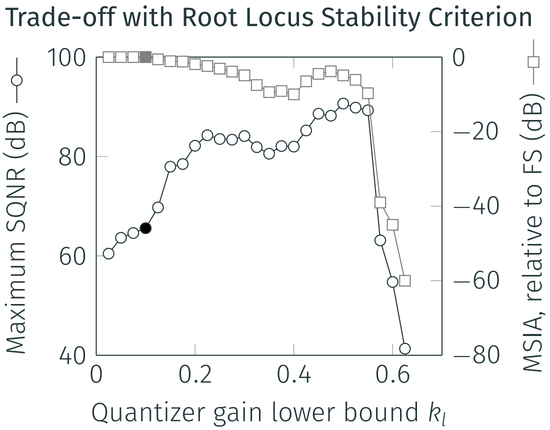

Figure 8: The performance (maximum simulated SQNR) and stability (simulated MSIA) achieved with the root locus modulator design for different quantizer gain robustness goals.

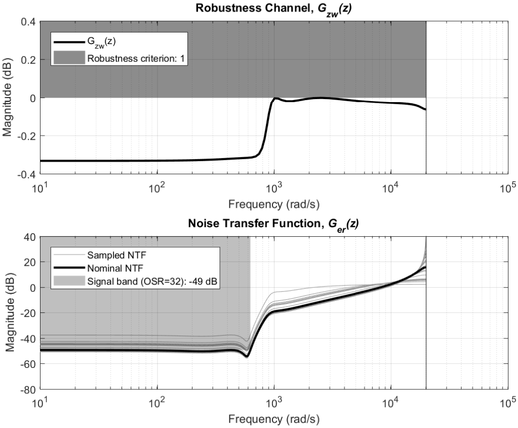

Figure 9: The noise transfer function generated with the root locus stability criterion and associated optimization targets.

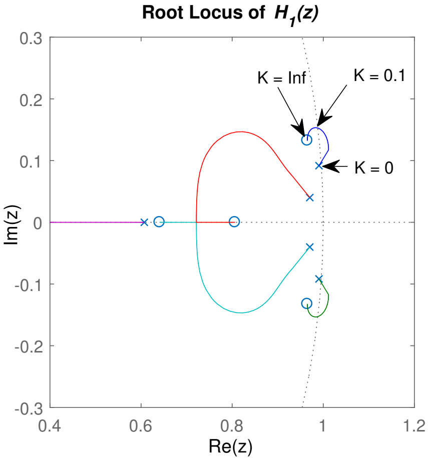

Figure 10: The root locus for the design produced when $[k_l, k_h] = [0.1, \infty)$.

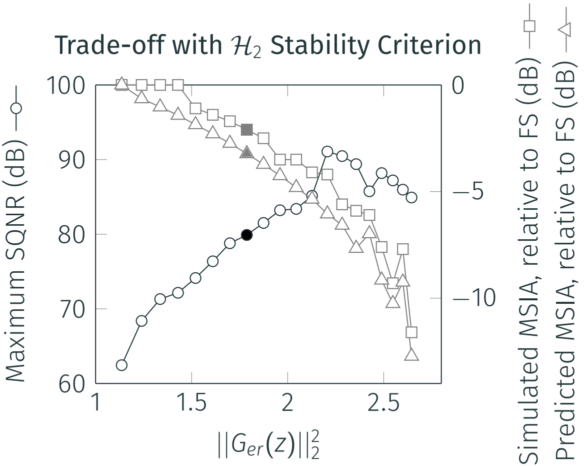

Figure 11: The performance (maximum simulated SQNR) and stability achieved with the modulator design for $\mathcal{H}_2$ norm goals.

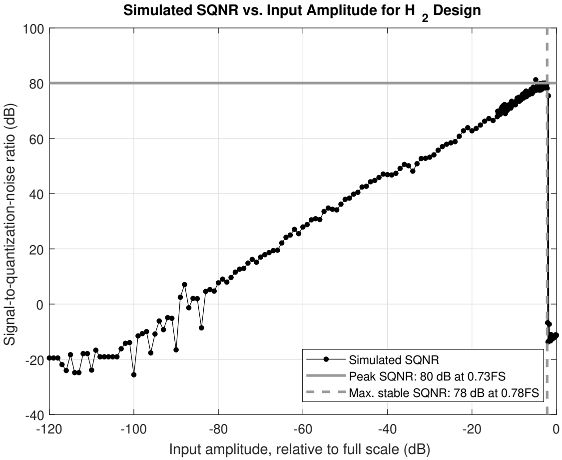

Figure 12: Simulation data for the modulator generated with the $\mathcal{H}_2$ stability criterion.

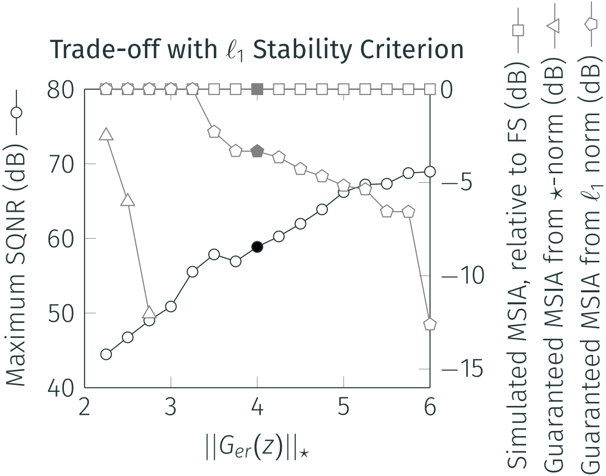

Figure 13: The performance (maximum simulated SQNR) and stability achieved with the modulator design for $\star$-norm goals.

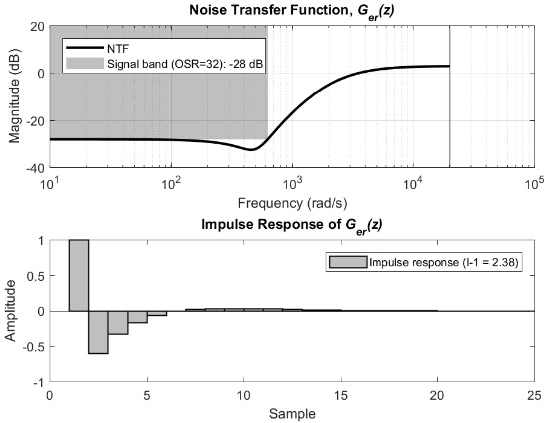

Figure 14: The noise transfer function generated with the ℓ_1 stability criterion and associated optimization targets.

Table 2: The performance and stability observations from simulations done on modulators designed using each stability criterion.

Type	Peak SQNR	MSIA (pred.)	MSIA (sim.)	LRIA
`DSToolbox`	87 dB	N/A	0.76 FS	−96 dB
$\mathcal{H}_\infty$	86 dB at 0.62 FS	N/A	0.71 FS	−91 dB
Root locus	66 dB	N/A	1 FS	−52 dB
$\mathcal{H}_2$	78 dB at 0.73 FS	0.69 FS	0.78 FS	−84 dB
ℓ_1	59 dB	0.68 FS	1 FS	−41 dB

Conclusion

Contributions

- Development of an algorithm that unites sigma delta modulator design using $\mathcal{H}_\infty$, $\mathcal{H}_2$, and ℓ_1 stability criteria with the GKYP performance goal supporting both FIR and IIR filters.

Contributions

- Development of an algorithm that unites sigma delta modulator design using $\mathcal{H}_\infty$, $\mathcal{H}_2$, and ℓ_1 stability criteria with the GKYP performance goal supporting both FIR and IIR filters.
- Extending the LMI system from [18] to be compatible with other channels of the augmented system.

Contributions

- Development of an algorithm that unites sigma delta modulator design using $\mathcal{H}_\infty$, $\mathcal{H}_2$, and ℓ_1 stability criteria with the GKYP performance goal supporting both FIR and IIR filters.
- Extending the LMI system from [18] to be compatible with other channels of the augmented system.
- Modelling the quantizer gain as an uncertainty and using optimization to enforce stability for a range of quantizer gains.

Contributions

- Development of an algorithm that unites sigma delta modulator design using $\mathcal{H}_\infty$, $\mathcal{H}_2$, and ℓ_1 stability criteria with the GKYP performance goal supporting both FIR and IIR filters.
- Extending the LMI system from [18] to be compatible with other channels of the augmented system.
- Modelling the quantizer gain as an uncertainty and using optimization to enforce stability for a range of quantizer gains.
- Presenting a proof-of-concept of using this work to directly design continuous-time loop filters.

Contributions

- Development of an algorithm that unites sigma delta modulator design using $\mathcal{H}_\infty$, $\mathcal{H}_2$, and ℓ_1 stability criteria with the GKYP performance goal supporting both FIR and IIR filters.
- Extending the LMI system from [18] to be compatible with other channels of the augmented system.
- Modelling the quantizer gain as an uncertainty and using optimization to enforce stability for a range of quantizer gains.
- Presenting a proof-of-concept of using this work to directly design continuous-time loop filters.

Questions?

- B. Blankertz, G. Dornhege, M. Krauledat, K. R. Müller, and G. Curio, "The non-invasive Berlin Brain-Computer Interface: Fast acquisition of effective performance in untrained subjects," *NeuroImage*, vol. 37, no. 2, pp. 539–550, 2007.
- S. Hein and A. Zakhor, "On the Stability of Sigma Delta Modulators," *IEEE Transactions on Signal Processing*, vol. 41, no. 7, pp. 2322–2348, 1993.
- R. Schreier and G. C. Temes, *Understanding Delta-Sigma Data Converters*, vol. 53.
  Wiley, 1997.

THE UNIVERSITY OF BRITISH COLUMBIA

- N. Wong and T.-s. Ng, “Fast detection of instability in sigma-delta modulators based on unstable embedded limit cycles,” *IEEE Transactions on Circuits and Systems II*, vol. 51, no. 8, pp. 442–449, 2004.
- N. S. Sooch, “Gain Scaling of Oversampled Analog-to-Digital Converters,” 1989.
- S. M. Moussavi and B. H. Leung, “High-Order Single-Stage Single-Bit Oversampling A/D Converter Stabilized with Local Feedback Loops,” *IEEE Transactions on Circuits and Systems*, vol. 41, no. 1, pp. 19–25, 1994.
- F. O. Eynde, G. M. Yin, and W. Sansen, “A CMOS Fourth-order 14b 500k-sample/s Sigma-delta ADC Converter,” 1991.

- K. C. H. Chao, S. Nadeem, W. L. Lee, and C. G. Sodini, "A Higher Order Topology for Interpolative Modulators for Oversampling A/D Converters," *IEEE Transactions on Circuits and Systems*, vol. 37, no. 3, pp. 309–318, 1990.
- R. Schreier, "An Empirical Study of High-Order Single-Bit Delta-Sigma Modulators," *IEEE Transactions on Circuits and Systems II: Analog and Digital Signal Processing*, vol. 40, no. 8, pp. 461–466, 1993.
- C.-c. Yang, K.-d. Chen, W.-C. Wang, and T.-h. Kuo, "Transfer function design of stable high-order sigma-delta modulators with root locus inside unit circle," in *Proceedings. IEEE Asia-Pacific Conference on ASIC*, pp. 5–8, 2002.

THE UNIVERSITY OF BRITISH COLUMBIA

- T.-H. Kuo, C.-C. Yang, K.-D. Chen, and W. C. Wang, “Design Method for High-Order Sigma Delta Modulator Stabilized by Departure Angles Designed to Keep Root Loci in Unit Circle,” *IEEE Transactions on Circuits and Systems II: Express Briefs*, vol. 53, no. 10, pp. 1083–1087, 2006.
- K. Kang, *Simulation, and Overload and Stability Analysis of Continuous Time Sigma Delta Modulator.*
  PhD thesis, University of Nevada, 2014.
- L. Risbo, *Sigma Delta Modulators - Stability Analysis and Optimization.*
  Doctor of philosophy, Technical University of Denmark, 1994.

D. Anastassiou, “Error Diffusion Coding for A/D Conversion,” *IEEE Transactions on Circuits and Systems*, vol. 36, no. 9, pp. 1175–1186, 1989.

A. Oberoi, *A Convex Optimization Approach to the Design of Multiobjective Discrete Time Systems*.
Master of science, Rochester Institute of Technology, 2004.

M. M. Osqui and A. Megretski, “Semidefinite Programming in Analysis and Optimization of Performance of Sigma-Delta Modulators for Low Frequencies,” in *Proceedings of the 2007 American Control Conference*, no. 6, pp. 3582–3587, 2007.

M. Nagahara and Y. Yamamoto, “Frequency Domain Min-Max Optimization of Noise-Shaping Delta-Sigma Modulators,” *IEEE Transactions on Signal Processing*, vol. 60, no. 6, pp. 1–12, 2012.

- X. Li, C. Yu, and H. Gao, “Design of delta-sigma modulators via generalized Kalman-Yakubovich-Popov lemma,” *Automatica*, vol. 50, no. 10, pp. 2700–2708, 2014.
- M. R. Tariq and S. Ohno, “Unified LMI-based design of $\Delta\Sigma$ modulators,” *EURASIP Journal on Advances in Signal Processing*, vol. 2016, no. 1, p. 29, 2016.
- T. Iwasaki and S. Hara, “Generalized KYP Lemma: Unified Frequency Domain Inequalities with Design Applications,” *IEEE Trans. Autom. Control*, vol. 50, no. 1, pp. 41–59, 2005.
- I. Masubuchi, A. Ohara, and N. Suda, “LMI-based controller synthesis: a unified formulation and solution,” *Robust and Nonlinear Control*, vol. 8, no. 9, pp. 669–686, 1998.

- J. Bu and M. Sznaier, "Linear matrix inequality approach to synthesizing low-order suboptimal mixed l1/Hp controllers," *Automatica*, vol. 36, no. 7, pp. 957–963, 2000.
- A. Oberoi and J. C. Cockburn, "A simplified LMI approach to l1 Controller Design," in *Proceedings of the 2005 American Control Conference*, (Portland), pp. 1788–1792, 2005.
- S. L. Shishkin, "Optimization under non-convex Quadratic Matrix Inequality constraints with application to design of optimal sparse controller," *IFAC-PapersOnLine*, vol. 50, no. 1, pp. 10754–10759, 2017.

Modelling Uncertain Quantizer Gain

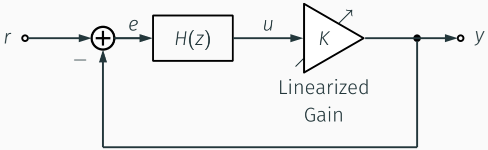

Figure 15: The sigma delta loop with the quantizer represented as an uncertain gain.

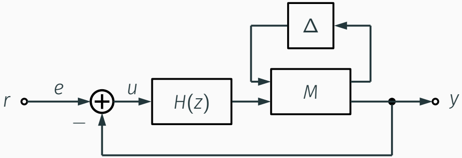

Figure 16: The linearized block diagram with the quantizer replaced by a multiplicative uncertainty extracted via LFT.

Iterative Convexification Procedure

- Find feasible initial condition a_0.
- Separate out non-convex term:

$$\begin{bmatrix} aa^T & a \\ a^T & 0 \end{bmatrix} = \begin{bmatrix} (a_0 + a_1)(a_0 + a_1)^T - a_1 a_1^T & a_0 + a_1 \\ (a_0 + a_1)^T & 0 \end{bmatrix} =$$

$$\begin{bmatrix} (a_0 a_0^T + a_0 a_1^T + a_1 a_0^T & a_0 + a_1 \\ (a_0 + a_1)^T & 0 \end{bmatrix}$$

- Using known a_0, solve the non-convex problem in a_1.
- Repeat process until termination criteria met.

$$a = a_0 + a_1$$

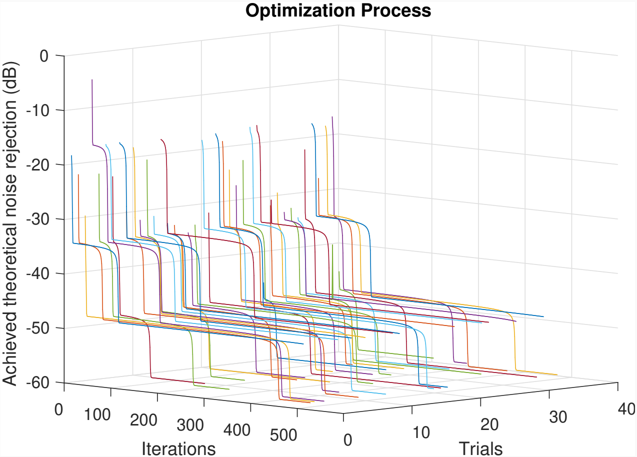

Figure 17: An example of the dependence of the iterative optimization scheme on initial conditions.

Limitations

- The influence of integrator state saturation is not captured by the model.
- LMI expressions are not feasible for loop filters with poles or zeros exactly on the unit circle.
- Convergence is dependent on choice of optimization parameters and initial conditions.
- It can be difficult to avoid pole-zero cancellations with high order designs.

Future Work

- Incorporate constraints on transfer function coefficients for ease of implementation.
- Explore implementation of a sigma delta DAC designed using this method on an FPGA.
- Find better optimization targets for continuous-time modulator design.