Spring Semester 2023

AIRCRAFT AERODYNAMICS & FLIGHT MECHANICS

27.04.2023

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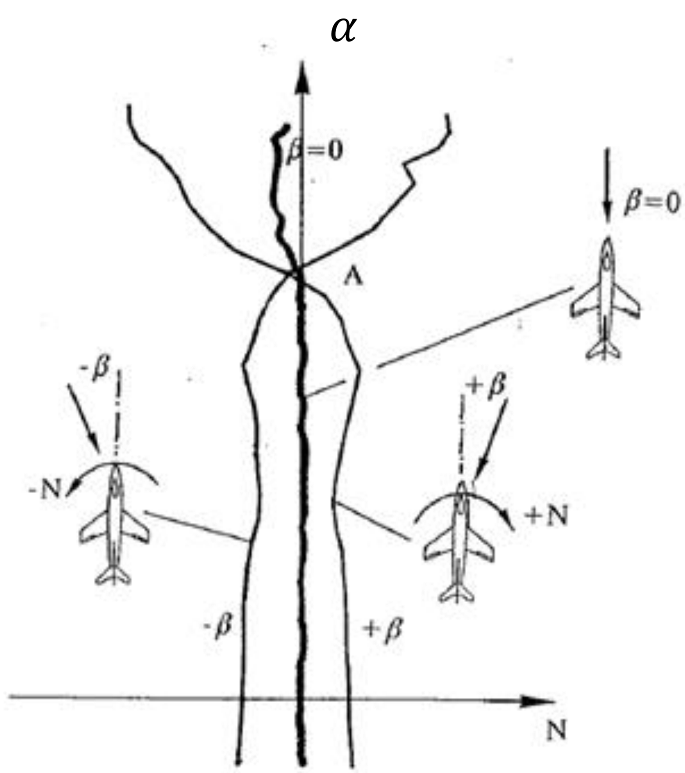
This lecture is adapted with permission from the lecture “Ausgewählte Kapitel der Flugtechnik” by Dr. Jürg Wildi

Recap

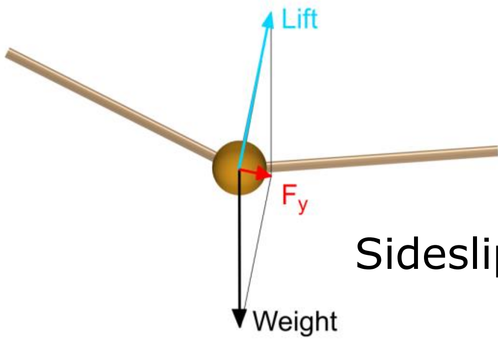
Directional static stability



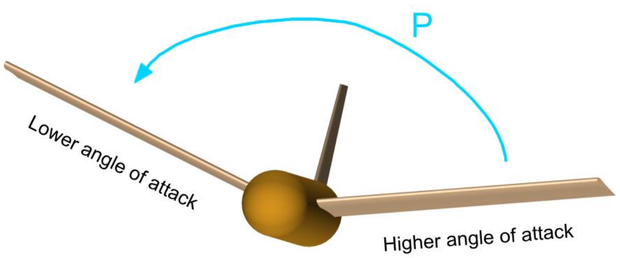
Restoring yawing moment N

$c_{n_\beta} > 0$

Lateral static stability



Sideslip due to roll



Restoring rolling moment L

$c_{l_\beta} < 0$

Dihedral effect

Flight Dynamics

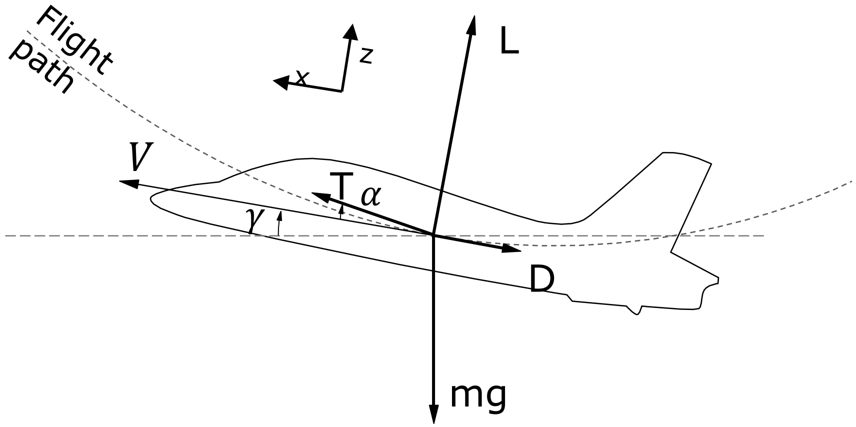
Equations of Motion – Point Mass

Velocity vector coordinate frame

Forces

$$m\dot{V} = T cos(\alpha + \sigma) - D - mg\ sin(\gamma)$$

$$mV\dot{\gamma} = L + T sin(\alpha + \sigma) - mg\ cos(\gamma)$$

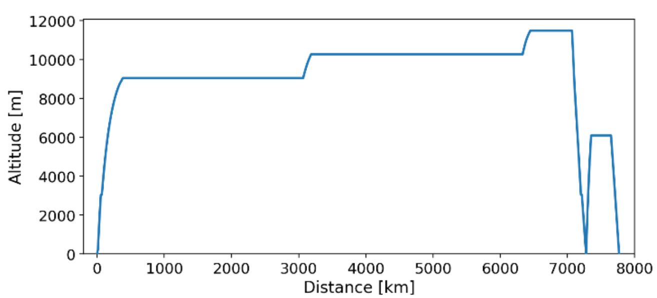


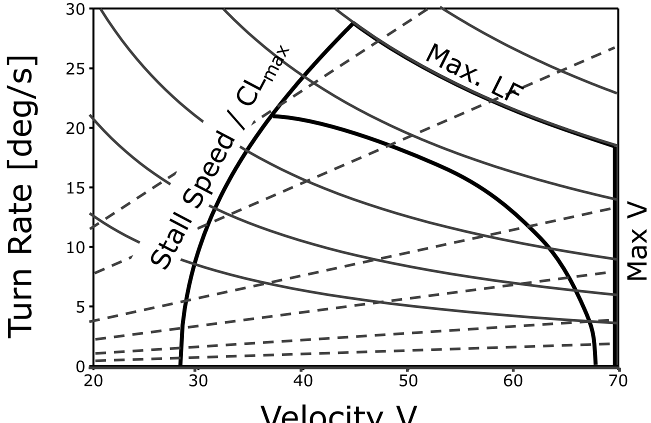
Kinematics

$$\dot{h} = V sin(\gamma)$$

$$\dot{x} = V cos(\gamma)$$ where x is the horizontal distance

Sufficient to answer most aircraft performance questions, including full mission simulations

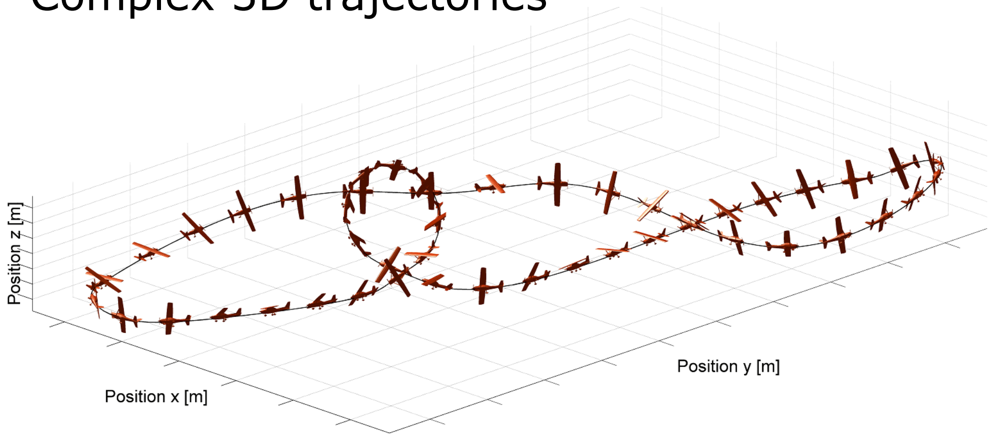




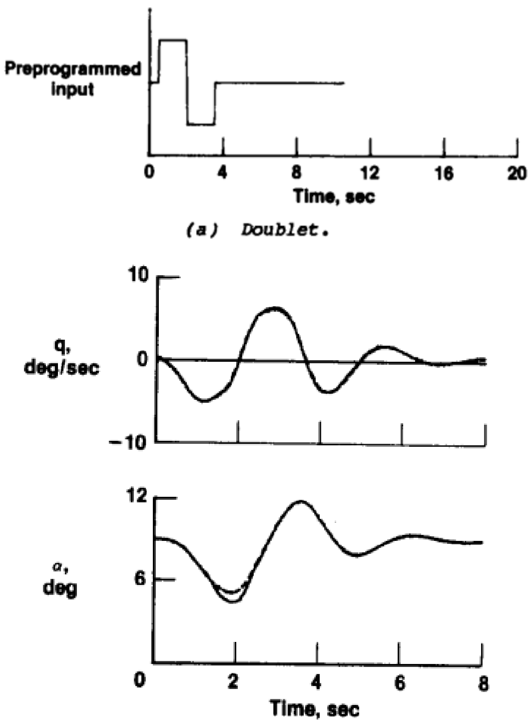
Dynamics of Flight

Interested in time dependency of e.g. attitude
time evolution → dynamics

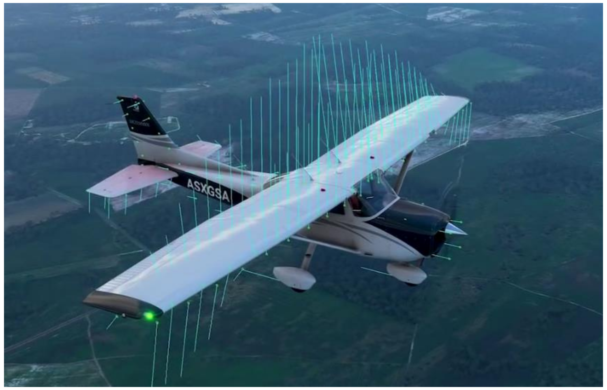
Complex 3D trajectories



Time-response of attitude dynamics



Flight simulation



6-DoF Equations of Motion

Forces

$$m(\dot{U} - VR + WQ) = -mg\sin\theta + F_{A_x} + F_{T_x}$$

$$m(\dot{V} + UR - WP) = mg\sin\phi\cos\theta + F_{A_y} + F_{T_y}$$

$$m(\dot{W} - UQ + VP) = mg\cos\phi\cos\theta + F_{A_z} + F_{T_z}$$

Moments

$$I_{xx}\dot{P} - I_{xz}\dot{R} - I_{xz}PQ + (I_{zz} - I_{yy})RQ = L_A + L_T$$

$$I_{yy}\dot{Q} + (I_{xx} - I_{zz})PR + I_{xz}(P^2 - R^2) = M_A + M_T$$

$$I_{zz}\dot{R} - I_{xz}\dot{P} + (I_{yy} - I_{xx})PQ + I_{xz}QR = N_A + N_T$$

Kinematics

$$P = \dot{\phi} - \dot{\psi}\sin\theta$$

$$Q = \dot{\theta}\cos\phi + \dot{\psi}\cos\theta\sin\phi$$

$$R = \dot{\psi}\cos\theta\cos\phi - \dot{\theta}\sin\phi$$

$\Rightarrow$ Derivation from first principles

Reference Frames and Axes

Inertial System: A reference frame that is not accelerating (e.g. non-rotating)
Newtons laws are valid in an inertial system

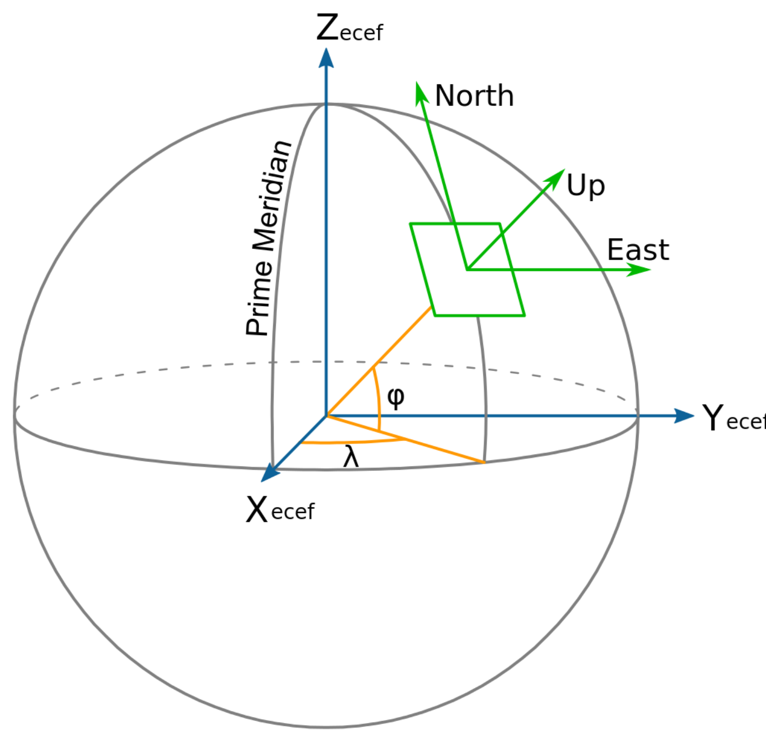
GPS coords in this frame

ECI: Earth centered inertial (e.g. J2000)
ECEF: Earth centered earth fixed – rotates with the earth (e.g. WGS 84)
Local Tangent Frame – tangent plane with origin on the earth surface

rotates w/ Earth

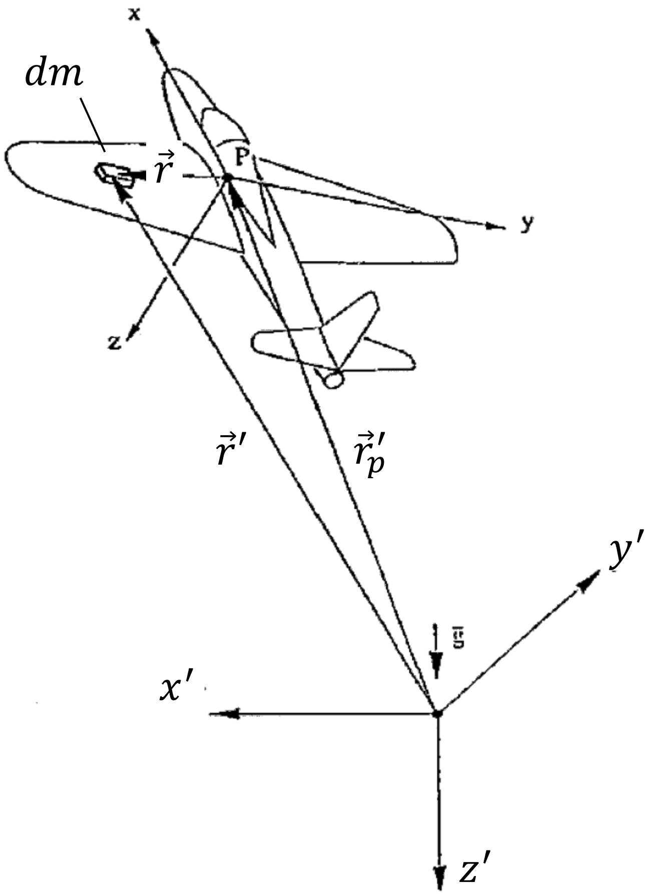
Local Tangent Frames:
ENU (x=East, y=North, z=Up)
NED (x= North, y=East, z=Down)

we will use NED to derive EOM



For investigating airplane dynamics, a local tangent plane (*flat earth*) system is sufficient and can be considered *inertial*

Rigid-Body Dynamics



- Inertial System (x', y', z')

  The NED coordinate system (x', y', z') is earth fixed and assumed to be non-rotating. Therefore, we can use Newtons law

$$\frac{d\vec{p}}{dt} = m\frac{d\vec{v}}{dt} = \vec{F}$$

$\vec{p}$: linear momentum

- Body Axis System (x, y, z)

  The aircraft is a collection of elements with mass dm. The location of a mass element can be described in the body axis system (x, y, z) with origin P with the position vector $\bar{r}$ or in the earth fixed system with the position vector $\bar{r}'$

  Assumption: the airplane is a rigid body

External Forces

The external forces are due to gravity, aerodynamics and propulsion

Gravity

$\vec{R} = \rho_A \vec{g}$

ρ_A: Density of the aircraft

$\vec{g}$: Gravitational acceleration vector

Aerodynamics and Thrust

$\vec{F} =$ Aerodynamic force + thrust force

Note: forces are per unit surface area

Derivation of the Equations of Motion

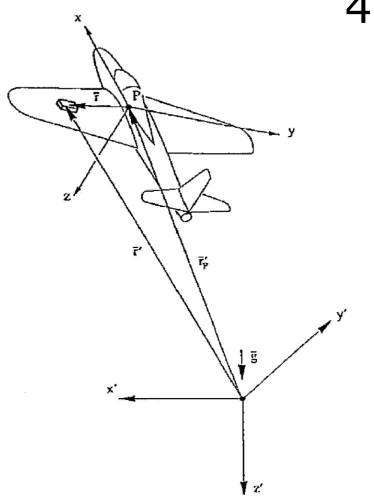
Procedure

1) Formulation of the equations for linear and angular momentum in the inertial system (x', y', z') in vector notation

2) Linear momentum and angular momentum w.r.t the aircraft's center of mass (or C.G.)

3) Transformation of the momentum equations into the body-axis system (x, y, z)

4) Change from vector notation to component form



Inertial Frame

1) Formulation of the equations for linear and angular momentum in the inertial system (x', y', z') in vector notation

$\frac{d}{dt}(mv) \rightarrow \frac{d}{dt}\left(\rho V \frac{dr}{dt}\right)$

RIGID BODIES

Linear momentum

$$\frac{d}{dt}\int \rho_A \frac{d\vec{r}'}{dt} dV = \int_V \rho_A \vec{g} dV + \int_S \vec{F} dS \qquad \left\{ m\frac{d\vec{v}}{dt} = \sum Forces \right\}$$

time derivative of position vector wrt inertial frame

volume integral (total gravitational force)

aerodynamic (surface) forces (drag, thrust, etc.)

Angular momentum

$$\frac{d}{dt}\int_V \vec{r}' \times \rho_A \frac{d\vec{r}'}{dt} dV = \int_V \vec{r}' \times \rho_A \vec{g} dV + \int_S \vec{r}' \times \vec{F} dS$$

rate of change of angular momentum

$$\left\{ \frac{d}{dt}(\vec{r} \times m\vec{v}) = \sum Moments \right\}$$

Mass

$$m = \int_V \rho_A dV$$

Center of Mass / Gravity

2) Linear momentum and angular momentum w.r.t. the aircraft's center of mass

Coordinate system (x, y, z)

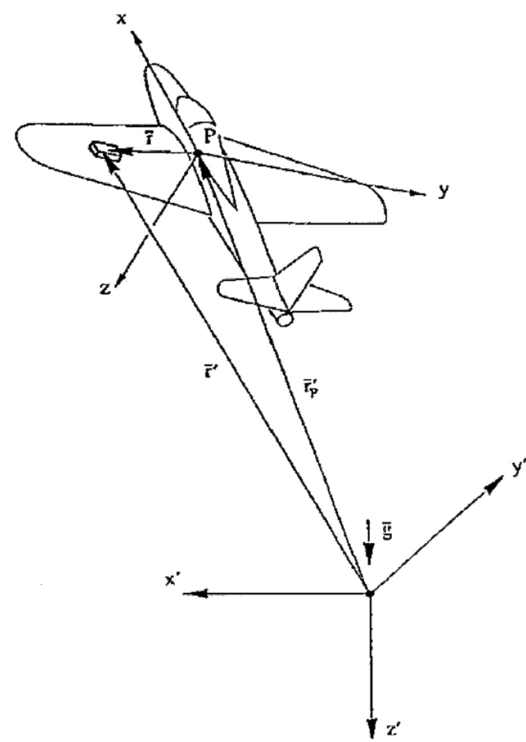
Origin: P

Location of the C.G. in the earth-fixed system: $\vec{r}_p'$

Location of a mass element, relative to P: $\vec{r}$

Position vector $\vec{r}' = \vec{r}_p' + \vec{r}$

inertial → ref. pt   ref pt → mass element



Reference point P = Center of Mass (C.G.) def.

$$\int_V \vec{r} \rho_A dV = 0 \qquad \left\{ \int_V \vec{r} dm = 0 \right\}$$

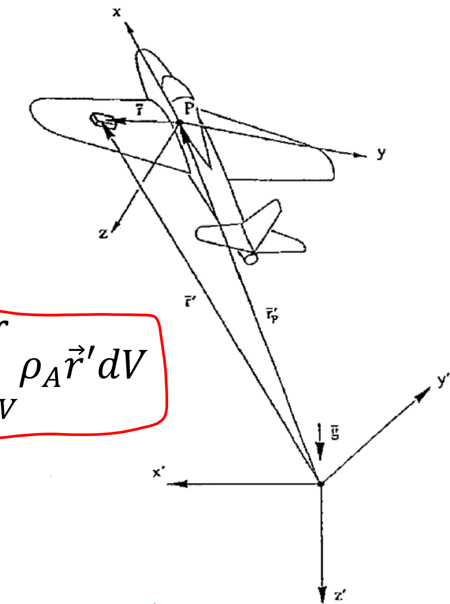
(to simplify eqns)

Position vector of the C.G. in the earth-fixed system (x', y', z')

$$\vec{r}_p' = \frac{1}{m} \int_V \rho_A \vec{r}' dV$$

Linear Momentum

Linear momentum $\quad \frac{d}{dt}\int \rho_A \frac{d\vec{r}'}{dt} dV = \int_V \rho_A \vec{g} dV + \int_S \vec{F} dS$



System (x', y', z')

~ aerodyn. forces

~ thrust

rhs: $\quad \int_V \rho_A \vec{g} dV + \int_S \vec{F} dS = \underline{m\vec{g} + \vec{F}_A + \vec{F}_T}$

$$\vec{r}_p' = \frac{1}{m}\boxed{\int_V \rho_A \vec{r}' dV}$$

lhs: $\quad \frac{d}{dt}\int_V \rho_A \frac{d\vec{r}'}{dt} dV = \frac{d}{dt}\left(\frac{d}{dt}\boxed{\int_V \rho_A \vec{r}' dV}\right) = m\frac{d}{dt}\frac{d\vec{r}_p'}{dt} = m\frac{d\vec{v}_p}{dt}$

collapsing 3D object → point mass

$\vec{v}_p = \frac{d\vec{r}_p'}{dt}$: Velocity of the aircraft's C.G. (Point P)

$$\boxed{m\frac{d\vec{v}_p}{dt} = m\vec{g} + \vec{F}_A + \vec{F}_T}$$

Rate of change of linear momentum = sum of external forces (Euler's first law)

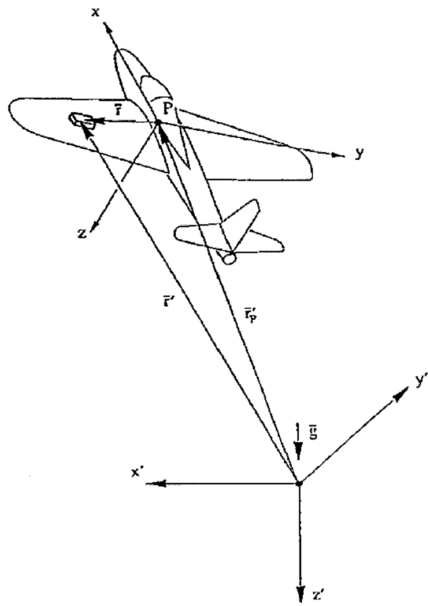
Angular Momentum

Angular momentum $\int_S \vec{r} \times \vec{F} dS = \vec{M}_A + \vec{M}_T$

System (x', y', z')

$\vec{M}_A$: Aerodynamic Moments

$\vec{M}_T$: Thrust Moments



with $\vec{r}' = \vec{r}_p' + \vec{r}$

$$\boxed{\frac{d}{dt}\int_V \vec{r} \times \frac{d\vec{r}}{dt}\rho_A dV = \vec{M}_A + \vec{M}_T}$$

Rate of change of angular momentum = sum of external moments (Euler's second law)

Issue: time dependent function inside the volume integral

EoM in the Body Axis System

Angular momentum $$\frac{d}{dt}\int_V \vec{r}\times\frac{d\vec{r}}{dt}\rho_A dV = \vec{M}_A + \vec{M}_T$$

time dependent function

3) Transformation of the momentum equations into the body-axis system (x, y, z)

Transport Theorem: → basic kinematic equation (relates the time derivative of an Euclidean vector as evaluated in a non-rotating (inertial) frame to its time derivative in a non-inertial frame (body frame here).

$$\frac{d\vec{A}}{dt} = \frac{\delta\vec{A}}{\delta t} + \vec{\omega}\times\vec{A}$$

where

$\frac{\delta\vec{A}}{\delta t}$: Time derivative in the rotating frame

$\vec{\omega}$: Angular velocity of the rotating system

Derivative in the earth fixed frame (x', y', z') (inertial)

Body-fixed frame (x, y, z)

analogous to:

$$\boxed{\underline{v}_A = \underline{v}_B + \underline{\omega}\times\underline{r}_{BA}}$$

ABBA

but general formulation: $\frac{d}{dt}\underline{f} = \left[\left(\frac{d}{dt}\right)_r + \underbrace{\Omega}_{\text{angular velocity of rotating coord. sys.}} \times\right]\underline{f}$

(can be used for any Euclidean vector $\underline{f}$) here: ~~speed~~ velocity

Linear Momentum

$$m\boxed{\frac{d\vec{v}_p}{dt}} = m\vec{g} + \vec{F}_A + \vec{F}_T$$

Linear momentum in the inertial frame

$A = v_p$ here

using $\boxed{\dfrac{d\vec{A}}{dt}} = \dfrac{\delta\vec{A}}{\delta t} + \vec{\omega} \times \vec{A}$

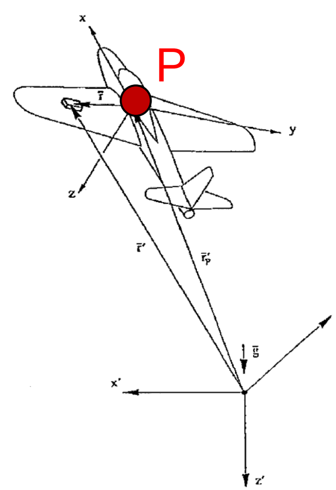
$$m\frac{d\vec{v}_p}{dt} = m\left(\frac{\delta\vec{v}_p}{\delta t} + \vec{\omega} \times \vec{v}_p\right) = m(\dot{\vec{v}}_p + \vec{\omega} \times \vec{v}_p)$$

$\dot{\underline{v}}_p = \dfrac{\delta \underline{v}_p}{\delta t}$

Note: time derivative in rotating frame (body frame)

$$\boxed{m(\dot{\vec{v}}_p + \vec{\omega} \times \vec{v}_p) = m\vec{g} + \vec{F}_A + \vec{F}_T}$$

Linear momentum in the body frame



Angular Momentum

$$\frac{d}{dt}\int_V \vec{r}\times\frac{d\vec{r}}{dt}\rho_A dV = \vec{M}_A + \vec{M}_T$$ Angular momentum in the inertial frame

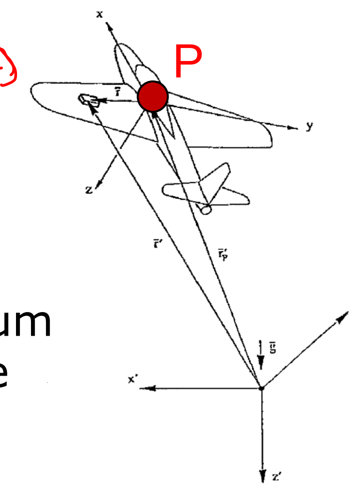
using $$\frac{d\vec{A}}{dt} = \frac{\delta\vec{A}}{\delta t} + \vec{\omega}\times\vec{A}$$

apply transport theorem to $\underline{r}$ $(\underline{A} = \underline{r})$

$$\frac{d}{dt}\int_V \vec{r}\times\frac{d\vec{r}}{dt}\rho_A dV = \int_V \vec{r}\times\frac{d}{dt}\boxed{\frac{d\vec{r}}{dt}}\rho_A dV = \int_V \vec{r}\times\frac{d}{dt}\boxed{\left(\dot{\vec{r}} + \vec{\omega}\times\vec{r}\right)}\rho_A dV \qquad \dot{\vec{r}} = \frac{\delta\vec{r}}{\delta t}$$

$$= \int \vec{r}\times\underbrace{\left(\ddot{\vec{r}} + \dot{\vec{\omega}}\times\vec{r} + 2\vec{\omega}\times\dot{\vec{r}} + \vec{\omega}\times(\vec{\omega}\times\vec{r})\right)}_{\frac{d}{dt}\left[\dot{\underline{r}} + \underline{\omega}\times\underline{r}\right]}\rho_A dV$$

Note: $\frac{d}{dt} \neq \frac{\delta}{\delta t}$ (so we can't just use product rule)



Aircraft = rigid body $\qquad \dot{\vec{r}} = 0$ and $\ddot{\vec{r}} = 0$

position of mass elements does not change wrt. P (CG)

$$\boxed{\int_V \vec{r}\times\left(\dot{\vec{\omega}}\times\vec{r} + \vec{\omega}\times(\vec{\omega}\times\vec{r})\right)\rho_A dV = \vec{M}_A + \vec{M}_T}$$

Angular momentum in the body frame

Angular Momentum

$$\int_V \vec{r} \times \left(\vec{\dot{\omega}} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r})\right) \rho_A dV = \vec{M}_A + \vec{M}_T$$

Angular momentum in the body frame
- Vector notation

ω (the rotational velocity of the aircraft) is the only time dependent variable. $\vec{\omega}$ is independent of the volume integral, since $\vec{\omega}$ (and $\vec{\dot{\omega}}$) is a property of the coordinate system (x, y, z).

The time dependent integral on the lhs was therefore eliminated by the transformation from the earth-fixed into the body-fixed, rotating coordinate system.

EoM in Component Form

Vector Form of the EoM is useful to formulate & describe the physics

To model time-depend behavior, the component form is more suitable.

Unit vectors in x, y and z direction (body axis): (i, j, k)

Example:

$\vec{\mathrm{r}} = ix + jy + kz$

Location of a mass element dm
In the body-axis coordinate system

Alternative Notation:

$$\vec{\mathrm{r}} = x \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{i} + y \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{j} + \mathrm{z} \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{k} \Rightarrow \vec{\mathrm{r}} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Components

Location $\vec{\mathrm{r}} = ix + jy + kz$

Forces

$$\vec{F}_A = iF_{A_x} + jF_{A_y} + kF_{A_z}$$

$$\vec{F}_T = iF_{T_x} + jF_{T_y} + kF_{T_z}$$

$$\vec{g} = ig_x + jg_y + kg_z$$

A: Aerodynamics
T: Thrust

Moments

$$\vec{M}_A = iL_A + jM_A + kN_A$$

$$\vec{M}_T = iL_T + jM_T + kN_T$$

L: Rolling moment
M: Pitching moment
N: Yawing moment

Velocity $\vec{v}_P = iU + jV + kW$

Angular velocity $\vec{\omega} = iP + jQ + kR$

P: Roll rate
Q: Pitch rate
R: Yaw rate

Components

$$\vec{F}_A = iF_{A_x} + jF_{A_y} + kF_{A_z}$$

$$\vec{F}_T = iF_{T_x} + jF_{T_y} + kF_{T_z}$$

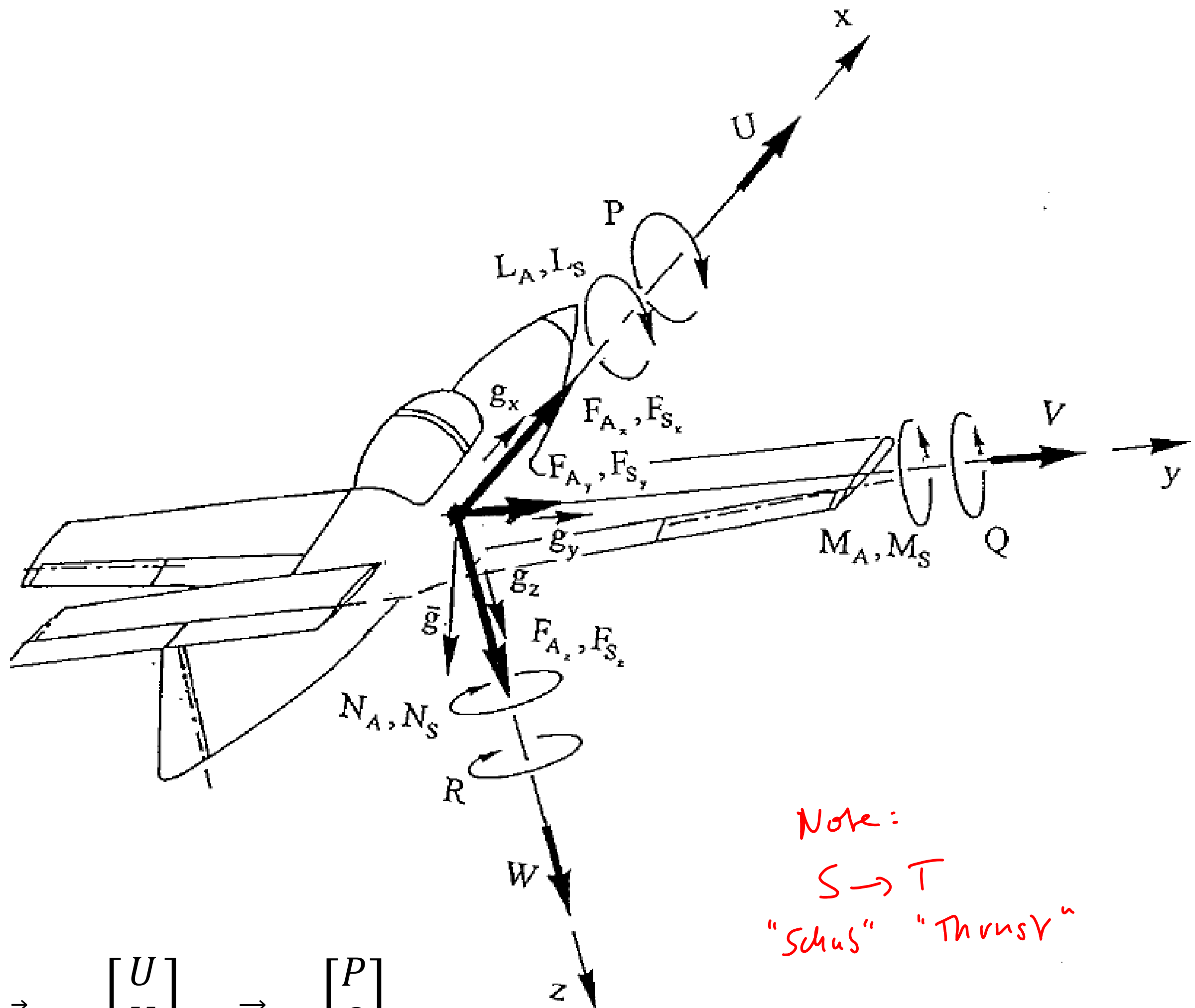
$$\vec{g} = ig_x + jg_y + kg_z$$

$$\vec{M}_A = iL_A + jM_A + kN_A$$

$$\vec{M}_T = iL_T + jM_T + kN_T$$

$$\vec{v}_P = iU + jV + kW$$

$$\vec{\omega} = iP + jQ + kR$$



Note:
S → T
"Schub" "Thrust"

$$\vec{v}_P = \begin{bmatrix} U \\ V \\ W \end{bmatrix} \quad \vec{\omega} = \begin{bmatrix} P \\ Q \\ R \end{bmatrix}$$

Linear Momentum in Component Form

$$m\left(\dot{\vec{v}}_p + \vec{\omega} \times \vec{v}_p\right) = m\vec{g} + \vec{F}_A + \vec{F}_T$$

Linear momentum in the body frame in vector notation

Use the definitions of the components

$$\vec{v}_P = \begin{bmatrix} U \\ V \\ W \end{bmatrix} \quad \dot{\vec{v}}_p = \begin{bmatrix} \dot{U} \\ \dot{V} \\ \dot{W} \end{bmatrix} \quad \vec{\omega} = \begin{bmatrix} P \\ Q \\ R \end{bmatrix}$$

$$\underline{\omega} \times \underline{v}_p = \begin{bmatrix} P \\ Q \\ R \end{bmatrix} \times \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} QW - RV \\ Ru - PW \\ PV - Qu \end{bmatrix}$$

and expanding the cross product results in

$$\boxed{\begin{aligned} m\left(\dot{U} - VR + WQ\right) &= mg_x + F_{A_x} + F_{T_x} \\ m\left(\dot{V} + UR - WP\right) &= mg_y + F_{A_y} + F_{T_y} \\ m\left(\dot{W} - UQ + VP\right) &= mg_z + F_{A_z} + F_{T_z} \end{aligned}}$$

Linear momentum in the body frame in component form

Angular Momentum in Component Form

Angular momentum in the body frame in vector notation

$$\int_V \vec{r} \times \left(\underbrace{\dot{\vec{\omega}} \times \vec{r}}_{1} + \underbrace{\vec{\omega} \times (\vec{\omega} \times \vec{r})}_{2} \right) \rho_A dV = \vec{M}_A + \vec{M}_T$$

Expanding the lhs results in two terms:

$$\underbrace{\int_V \vec{r} \times (\dot{\vec{\omega}} \times \vec{r}) \rho_A dV}_{1} + \underbrace{\int_V \vec{r} \times \left(\vec{\omega} \times (\vec{\omega} \times \vec{r})\right) \rho_A dV}_{2} = \vec{M}_A + \vec{M}_T$$

These two terms can be formulated in the component form and simplified

Angular Momentum in Component Form

LOOK AT THIS

$$\underbrace{\int_V \vec{r} \times (\vec{\omega} \times \vec{r})\rho_A dV}_{1} + \int_V \vec{r} \times (\vec{\omega} \times (\vec{\omega} \times \vec{r}))\rho_A dV = \vec{M}_A + \vec{M}_T$$

using $\quad \vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})$

replace two cross product w/ difference of two dot products

$$\int_V \vec{r} \times (\vec{\omega} \times \vec{r})\rho_A dV = \underbrace{\int_V \vec{\omega}\cdot(\vec{r}\cdot\vec{r})\rho_A dV}_{1.1} - \underbrace{\int_V \vec{r}\cdot(\vec{r}\cdot\vec{\omega})\rho_A dV}_{1.2}$$

Write terms 1.1 and 1.2 in component form:

1.1: $$\int_V \vec{\omega}\cdot(\vec{r}\cdot\vec{r})\rho_A\, dV = \left(i\dot{P} + j\dot{Q} + k\dot{R}\right)\int_V (x^2+y^2+z^2)\rho_A dV$$

1.2: $$\int_V \vec{r}\cdot(\vec{r}\cdot\vec{\omega})\rho_A\, dV = \int_V (ix+jy+kz)\left(x\dot{P}+y\dot{Q}+z\dot{R}\right)\rho_A dV$$

Angular Momentum in Component Form

$$\int_V \vec{r} \times (\vec{\omega} \times \vec{r})\rho_A dV = \int_V \vec{\omega} \cdot (\vec{r} \cdot \vec{r})\rho_A dV - \int_V \vec{r} \cdot (\vec{r} \cdot \vec{\omega})\rho_A dV$$

$$= \underbrace{(i\dot{P} + j\dot{Q} + k\dot{R})}_{\text{vector}} \int_V (x^2 + y^2 + z^2)\rho_A dV - \int_V \underbrace{(ix + jy + kz)}_{\text{vector}}(x\dot{P} + y\dot{Q} + z\dot{R})\rho_A dV$$

Re-ordering by i, j, k

only geometry; nothing to do w/ dynamics/kinematics anymore

$$= i\left\{\dot{P}\int_V (y^2 + z^2)\rho_A dV - \dot{Q}\int_V xy\rho_A dV - \dot{R}\int_V xz\rho_A dV\right\} +$$

$$j\left\{\dot{Q}\int_V (x^2 + z^2)\rho_A dV - \dot{P}\int_V yx\rho_A dV - \dot{R}\int_V yz\rho_A dV\right\} +$$

$$k\left\{\dot{R}\int_V (x^2 + y^2)\rho_A dV - \dot{P}\int_V zx\rho_A dV - \dot{Q}\int_V zy\rho_A dV\right\}$$

Moment of inertia I of a rigid body

Moment of Inertia

how our aircraft reacts to rotations

Moment of Inertia w.r.t. the x-Axis

$$I_{xx} = \int_V (y^2 + z^2)\rho_A dV$$

$$I_{xy} = \int_V xy\rho_A dV$$

$$I_{xz} = \int_V xz\rho_A dV$$

Moment of Inertia w.r.t. the y-Axis

$$I_{yy} = \int_V (x^2 + z^2)\rho_A dV$$

$$I_{yx} = I_{xy} = \int_V xy\rho_A dV$$

$$I_{yz} = \int_V yz\rho_A dV$$

Moment of Inertia w.r.t. the z-Axis

$$I_{zz} = \int_V (x^2 + y^2)\rho_A dV$$

$$I_{zx} = I_{xz} = \int_V xz\rho_A dV$$

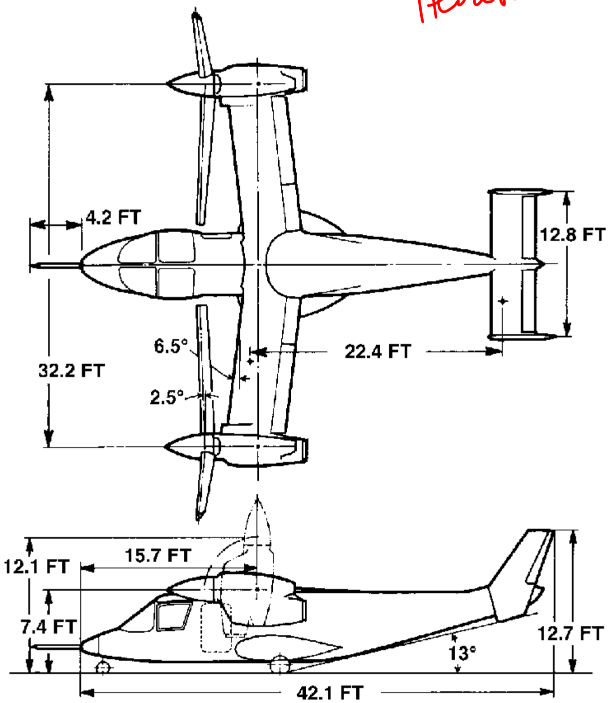
$$I_{zy} = I_{yz} = \int_V yz\rho_A dV$$

The moments of inertia are important properties that define the dynamic stability, the control characteristics and the handling of an aircraft.

The moments of inertia can be computed if the aircraft geometry is known and the mass distribution is given

Geometry / Layout / Inboard drawing

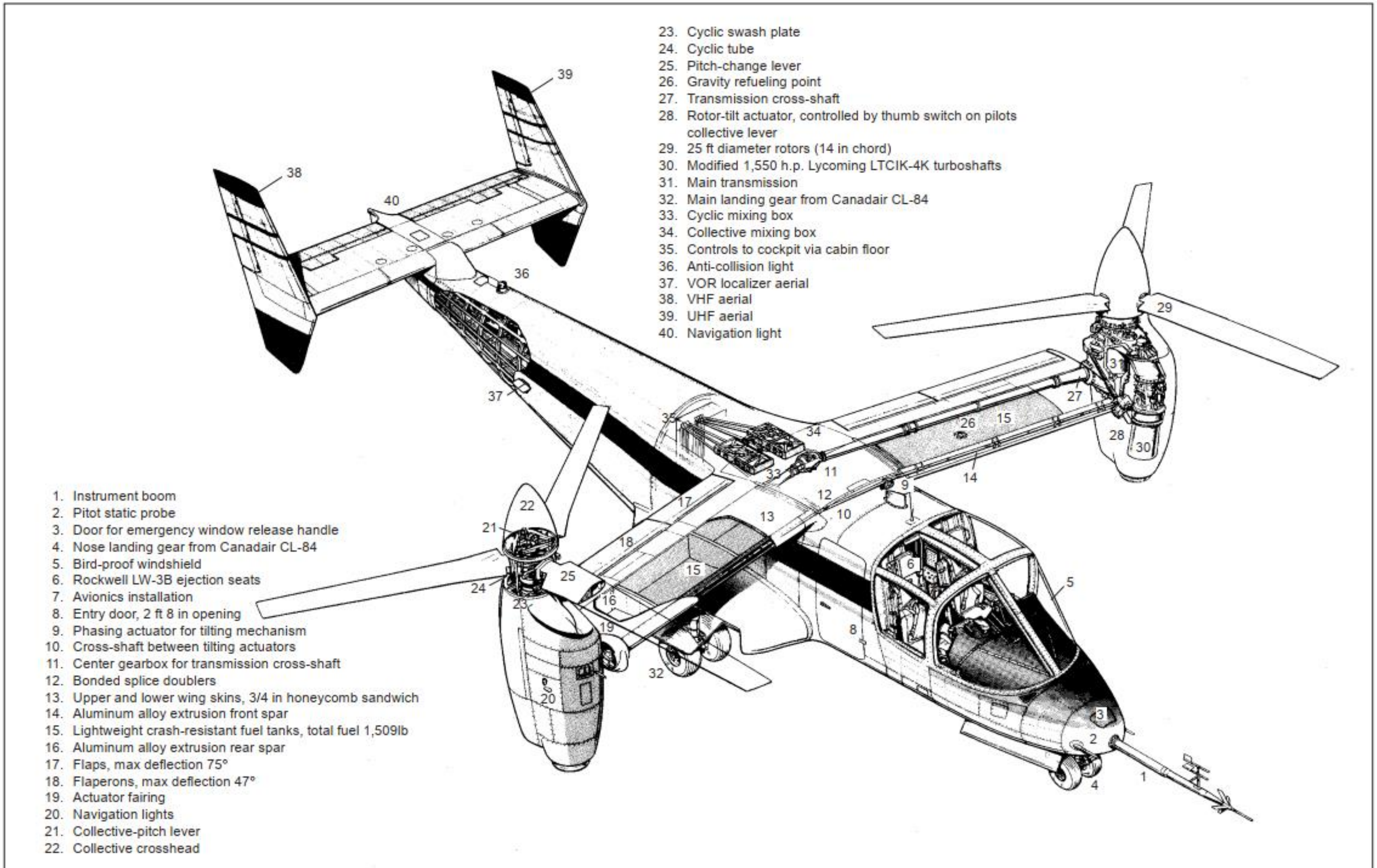
Iteration 1



Bell XV-15 – experimental tiltrotor (1977)

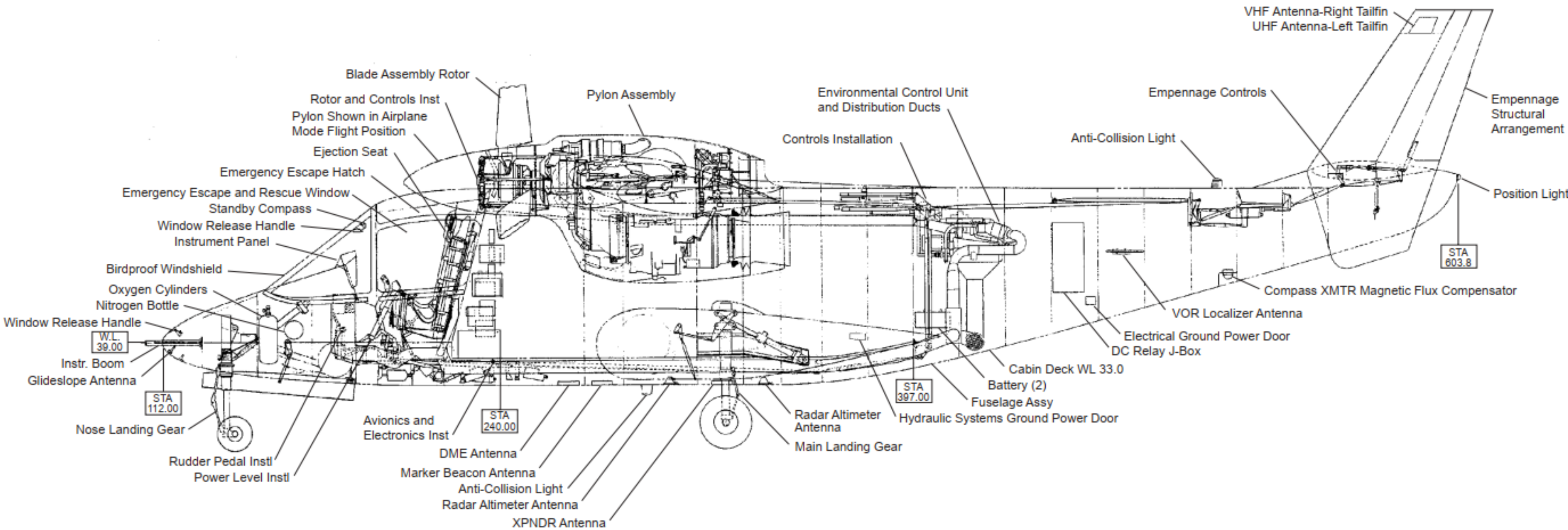
More detailed iteration

Geometry / Layout / Inboard drawing



Source: NASA SP-2000-4517

Geometry / Layout / Inboard drawing



dynamics eqns are detailed (MOI) => save until late design stage when pos & masses of major component known

Source: NASA SP-2000-4517

Angular Momentum in Component Form

Review of the steps

$$\int_V \vec{r} \times (\vec{\dot{\omega}} \times \vec{r})\rho_A dV = \int_V \vec{\dot{\omega}} \cdot (\vec{r} \cdot \vec{r})\rho_A dV - \int_V \vec{r} \cdot (\vec{r} \cdot \vec{\dot{\omega}})\rho_A dV$$

$$= \left(i\dot{P} + j\dot{Q} + k\dot{R}\right) \int_V (x^2 + y^2 + z^2)\rho_A dV + \int_V (ix + jy + kz)\left(x\dot{P} + y\dot{Q} + z\dot{R}\right)\rho_A dV$$

$$= i\left\{\dot{P} I_{xx} - \dot{Q} I_{xy} - \dot{R} I_{xz}\right\} +$$

$$j\left\{\dot{Q} I_{yy} - \dot{P} I_{xy} - \dot{R} I_{yz}\right\} +$$

$$k\left\{\dot{R} I_{zz} - \dot{P} I_{xz} - \dot{Q} I_{yz}\right\} +$$

Angular Momentum in Component Form

$$\underbrace{\int_V \vec{r} \times (\dot{\vec{\omega}} \times \vec{r})\rho_A dV}_{1} + \underbrace{\int_V \vec{r} \times \big(\vec{\omega} \times (\vec{\omega} \times \vec{r})\big)\rho_A dV}_{2} = \vec{M}_A + \vec{M}_T$$

Expand the second term:

$$\underbrace{\int_V \vec{r} \times \vec{\omega}(\vec{r} \cdot \vec{\omega})\rho_A dV}_{\Downarrow} - \underbrace{\int \vec{r} \times \vec{r}\,(\vec{\omega} \cdot \vec{\omega})\rho_A dV}$$ = 0 (cross product = 0)

b/w two same vectors ($\underline{v} \times \underline{v} = 0$)

⇓ Use the component form and expand

$$\int_V (ix + jy + kz) \times (iP + jQ + kR)(Px + Qy + Rz)\rho_A dV =$$

$$i\{I_{xy}PR + I_{yz}(R^2 - Q^2) - I_{xz}PQ + RQ\big(I_{zz} - I_{yy}\big)\} +$$

$$j\{(I_{xx} - I_{zz})PR + I_{xz}(P^2 - R^2) - I_{xy}QR + I_{yz}PQ\} +$$

$$k\{\big(I_{yy} - I_{xx}\big)PQ + I_{xy}(Q^2 - P^2) + I_{xz}QR - I_{yz}PR\}$$

Angular Momentum in Component Form

$$\int_V \vec{r} \times \left(\dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r})\right) \rho_A dV = \vec{M}_A + \vec{M}_T$$

Angular momentum in vector form in the body axis system

Combing the results from the first and second term and using symmetry:

For symmetric aircraft (in the x-y plane): $I_{xy} = I_{yz} = 0$

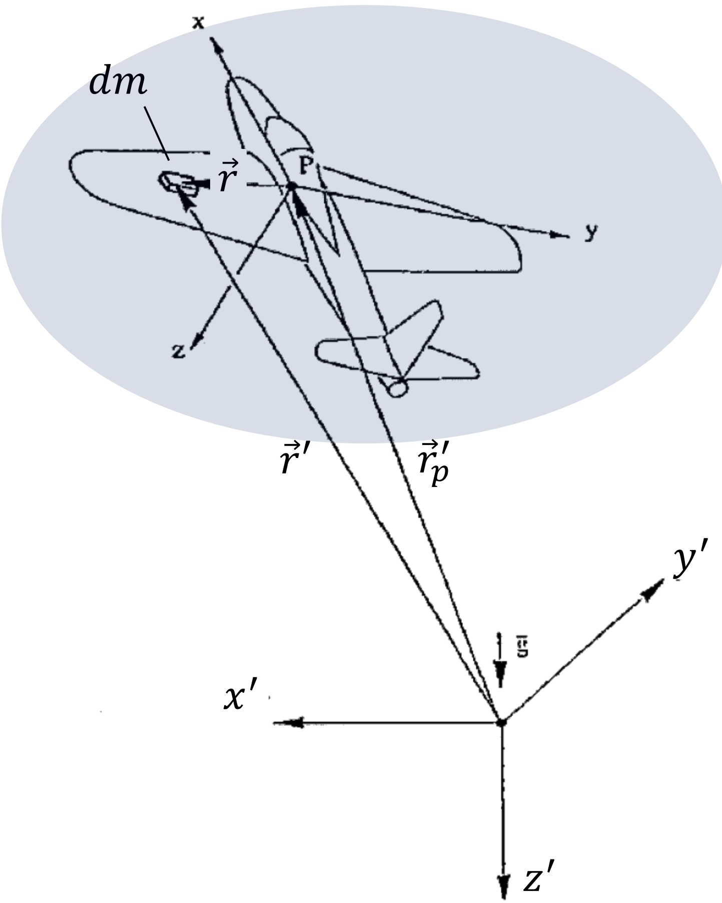
$$I_{xx}\dot{P} - I_{xz}\dot{R} - I_{xz}PQ + \left(I_{zz} - I_{yy}\right)RQ = L_A + L_T$$

$$I_{yy}\dot{Q} + (I_{xx} - I_{zz})PR + I_{xz}(P^2 - R^2) = M_A + M_T$$

$$I_{zz}\dot{R} - I_{xz}\dot{P} + \left(I_{yy} - I_{xx}\right)PQ + I_{xz}QR = N_A + N_T$$

Angular momentum in component form in the body axis system

Coordinate Systems



- Inertial System (x', y', z')
- Body Axis System (x, y, z)

Coordinate Transformation

≠ reference from transf.

$(i, j, k)_{earth}$ $(i, j, k)_{body}$

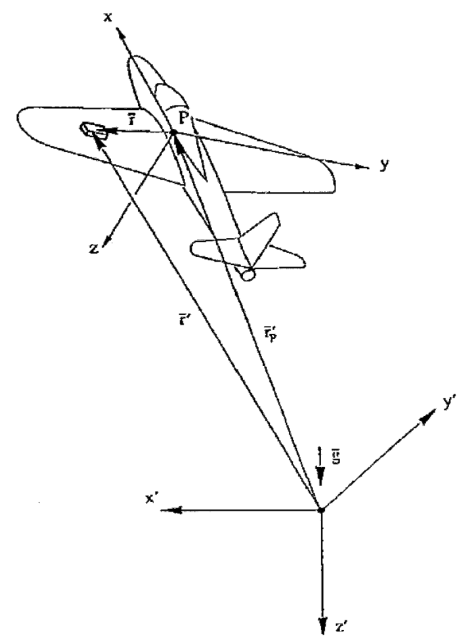
Position and orientation of the aircraft in the earth-fixed coordinate system (x', y', z')	⟺	Position and orientation of the body-fixed system (x, y, z) in the earth-fixed system

Step 1

Translation of the earth-fixed system (x', y', z') into the reference point of the aircraft (Point P, C.G.). The new system is denoted (x_1, y_1, z_1)

Step 2

The orientation of the system (x_1, y_1, z_1) relative to the body-fixed system (x, y, z) can be described through three successive rotations



Step 2
The orientation of the system (x_1, y_1, z_1) relative to the body-fixed system (x, y, z) can be described through three successive rotations

1) Rotation of (x_1, y_1, z_1) around the z_1-Axis by the heading angle ψ
-> (x_2, y_2, z_2) (note: $z_1 = z_2$)

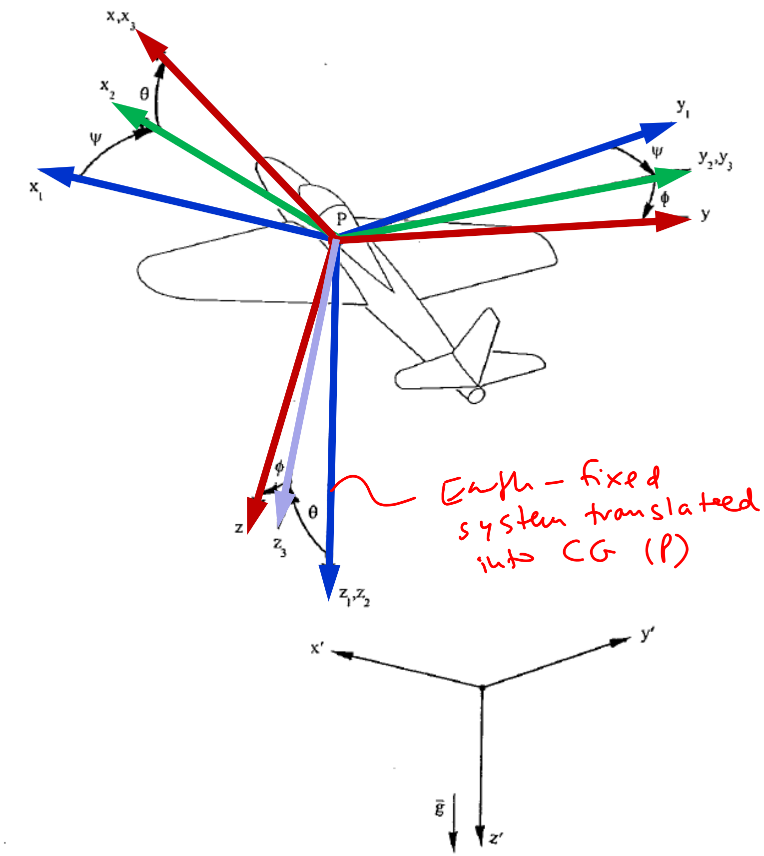
align heading

2) Rotation of (x_2, y_2, z_2) around the y_2-Axis by the pitch angle θ
-> (x_3, y_3, z_3) (note: $y_2 = y_3$)

align pitch

3) Rotation of (x_3, y_3, z_3) around the x_3-Axis by the roll angle ϕ
-> (x, y, z) (note: $x_3 = x$)

align roll



Coordinate Transformation

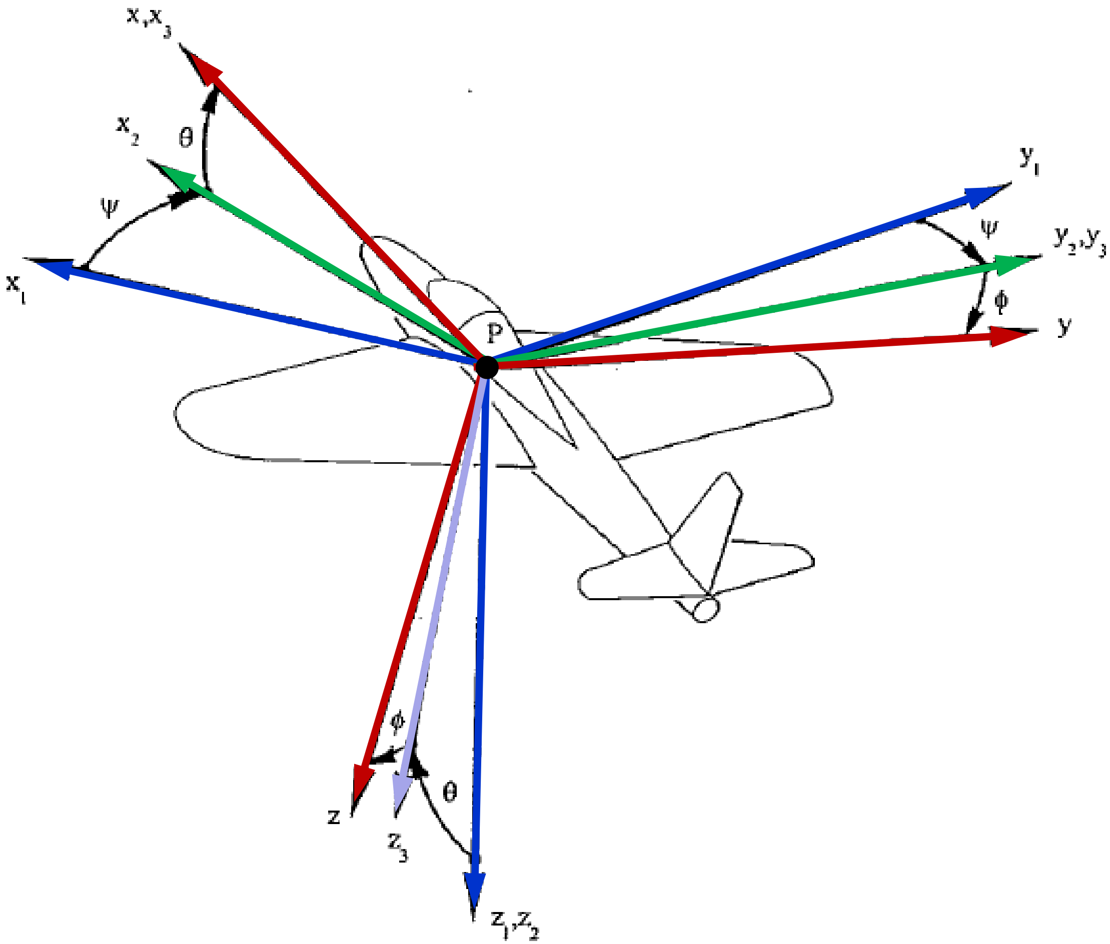
Euler Angles:

ψ, θ, ϕ

YAW PITCH ROLL

The order of the rotations are important

1. Yaw (ψ)
2. Pitch (θ)
3. Roll (ϕ)



Trajectory (Flight Path)

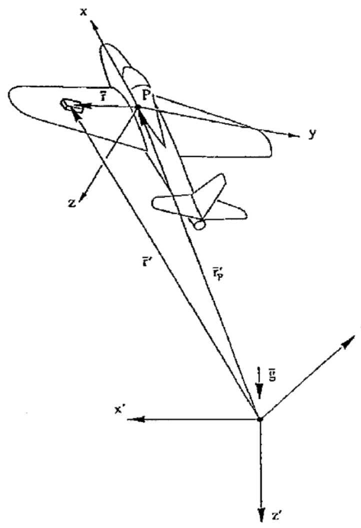
Body-fixed system (x, y, z) $\Longleftrightarrow$ Earth-fixed system (x', y', z')

$\vec{v}_P$ with the components U,V,W

$\vec{v}_P$ with the components $(\dot{x}', \dot{y}', \dot{z}')$

Step 1

Translation of the earth-fixed system (x', y', z') into the reference point of the aircraft (Point P, C.G.). The new system is denoted (x_1, y_1, z_1)



$$u_1 = \dot{x}'$$

$$v_1 = \dot{y}'$$

$$w_1 = \dot{z}'$$

velocities do not change after translating frame

Step 2

Rotation of the (x_1, y_1, z_1) system around the three Euler angles

Step 2

translated sys

Rotation of the (x_1, y_1, z_1) system around the three Euler angles

$$\begin{bmatrix} u_1 \\ v_1 \\ w_1 \end{bmatrix} = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_2 \\ v_2 \\ w_2 \end{bmatrix}$$

before after

1) Rotation of (x_1, y_1, z_1) around the z_1-Axis by the heading angle ψ
(x_1, y_1, z_1) -> (x_2, y_2, z_2)

$$\begin{bmatrix} u_2 \\ v_2 \\ w_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} u_3 \\ v_3 \\ w_3 \end{bmatrix}$$

2) Rotation of (x_2, y_2, z_2) around the y_2-Axis by the pitch angle θ
(x_2, y_2, z_2) -> (x_3, y_3, z_3)

$$\begin{bmatrix} u_3 \\ v_3 \\ w_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} U \\ V \\ W \end{bmatrix}$$

3) Rotation of around the x_3-Axis by the roll angle ϕ
(x_3, y_3, z_3) -> (x, y, z)

Trajectory (Flight Path)

$$\begin{bmatrix} u_1 \\ v_1 \\ w_1 \end{bmatrix} = \underbrace{\begin{bmatrix} \dot{x}' \\ \dot{y}' \\ \dot{z}' \end{bmatrix}}_{\text{time derivatives of CG position wrt. inertial frame}} = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} U \\ V \\ W \end{bmatrix}$$

Relation between the velocity in the body-fixed system and the velocity in the earth fixed system.

The trajectory (position of the aircraft) in the earth-fixed system is obtained by numerical integration, e.g.

$$\int \dot{x}' dt = x'(t)$$

$$= \begin{bmatrix} \cos^2\psi & -\sin\psi & \sin\theta\cos\psi \\ \sin\psi\cos\theta & \cos\psi & \sin\psi\sin\theta \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\psi & -\sin\psi\cos\phi + \sin\theta\sin\phi\cos\psi & \sin\psi\sin\phi + \sin\theta\cos\psi\cos\phi \\ \sin\psi\cos\theta & \cos\psi\cos\phi + \sin\psi\sin\theta\sin\phi & -\sin\phi\cos\psi + \sin\psi\sin\theta\cos\phi \\ -\sin\theta & \sin\phi\cos\theta & \cos\theta\cos\phi \end{bmatrix}$$

Euler-Angle Rates and Body-Axis Rates

$$\vec{\omega} = iP + jQ + kR = \begin{bmatrix} P \\ Q \\ R \end{bmatrix}_{\text{inertial (Earth)}}$$

$$\vec{\omega} = \vec{\dot{\psi}} + \vec{\dot{\theta}} + \vec{\dot{\phi}} = \begin{bmatrix} \dot{\psi}_1 \\ \dot{\psi}_2 \\ \dot{\psi}_3 \end{bmatrix}_{\text{frame 1 (Earth)}} + \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}_{\text{frame 2}} + \begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \\ \dot{\phi}_3 \end{bmatrix}_{\text{frame 3}}$$

Note: !

$$\boxed{\begin{matrix} P \\ Q \\ R \end{matrix}} \begin{matrix} \neq \\ \neq \\ \neq \end{matrix} \boxed{\begin{matrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{matrix}}$$

yaw, pitch, roll rates (body-axis rates)

Euler angle rates

$\dot{\psi}$ Is measured in the inertial frame (x_1, y_1, z_1)

$\dot{\theta}$ Is measured in the intermediate frame (x_2, y_2, z_2)

$\dot{\phi}$ Is measured in the intermediate frame (x_3, y_3, z_3)

Using successive Euler rotations:

$$\begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\phi & \cos\theta\sin\phi \\ 0 & -\sin\phi & \cos\theta\cos\phi \end{bmatrix}}_{T} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}$$

$$P = \dot{\phi} - \dot{\psi}\sin\theta$$

$$Q = \dot{\theta}\cos\phi + \dot{\psi}\cos\theta\sin\phi$$

$$R = \dot{\psi}\cos\theta\cos\phi - \dot{\theta}\sin\phi$$

w/ T^{-1}:

$$\dot{\phi} = P + Q\sin\phi\tan\theta + R\cos\phi\tan\theta$$

$$\dot{\theta} = Q\cos\phi - R\sin\phi$$

$$\dot{\psi} = (Q\sin\phi + R\cos\phi)\frac{1}{\cos\theta}$$

pitch 90° gives singularity (gimbal lock)

Linear Momentum (Forces)

$$m(\dot{U} - VR + WQ) = mg_x + F_{A_x} + F_{T_x}$$

$$m(\dot{V} + UR - WP) = mg_y + F_{A_y} + F_{T_y}$$

$$m(\dot{W} - UQ + VP) = mg_z + F_{A_z} + F_{T_z}$$

Linear momentum in the body frame component form

$\Downarrow$ Coordinate transformation of the gravity vector

$$m(\dot{U} - VR + WQ) = -mg\sin\theta + F_{A_x} + F_{T_x}$$

$$m(\dot{V} + UR - WP) = mg\sin\phi\cos\theta + F_{A_y} + F_{T_y}$$

$$m(\dot{W} - UQ + VP) = mg\cos\phi\cos\theta + F_{A_z} + F_{T_z}$$

Summary

Balance of Forces

$$m(\dot{U} - VR + WQ) = -mg\sin\theta + F_{A_x} + F_{T_x}$$

$$m(\dot{V} + UR - WP) = mg\sin\phi\cos\theta + F_{A_y} + F_{T_y}$$

$$m(\dot{W} - UQ + VP) = mg\cos\phi\cos\theta + F_{A_z} + F_{T_z}$$

Balance of Moments

$$I_{xx}\dot{P} - I_{xz}\dot{R} - I_{xz}PQ + (I_{zz} - I_{yy})RQ = L_A + L_T$$

$$I_{yy}\dot{Q} + (I_{xx} - I_{zz})PR + I_{xz}(P^2 - R^2) = M_A + M_T$$

$$I_{zz}\dot{R} - I_{xz}\dot{P} + (I_{yy} - I_{xx})PQ + I_{xz}QR = N_A + N_T$$

Kinematic Equations

$$P = \dot{\phi} - \dot{\psi}\sin\theta$$

$$Q = \dot{\theta}\cos\phi + \dot{\psi}\cos\theta\sin\phi$$

$$R = \dot{\psi}\cos\theta\cos\phi - \dot{\theta}\sin\phi$$

Solution to the EoM

Time-domain solutions

Six non-linear differential equations
Six variables (U,V,W,P,Q,R)
In addition the pitch angle θ and roll angle ϕ via the kinematic relations from the body-rates (P,Q,R)

$\Rightarrow$ Numerical Integration

Dynamic Stability Investigation

$\Rightarrow$ Simplification

Small-perturbation equations – superposition of a stationary flight state with small disturbances

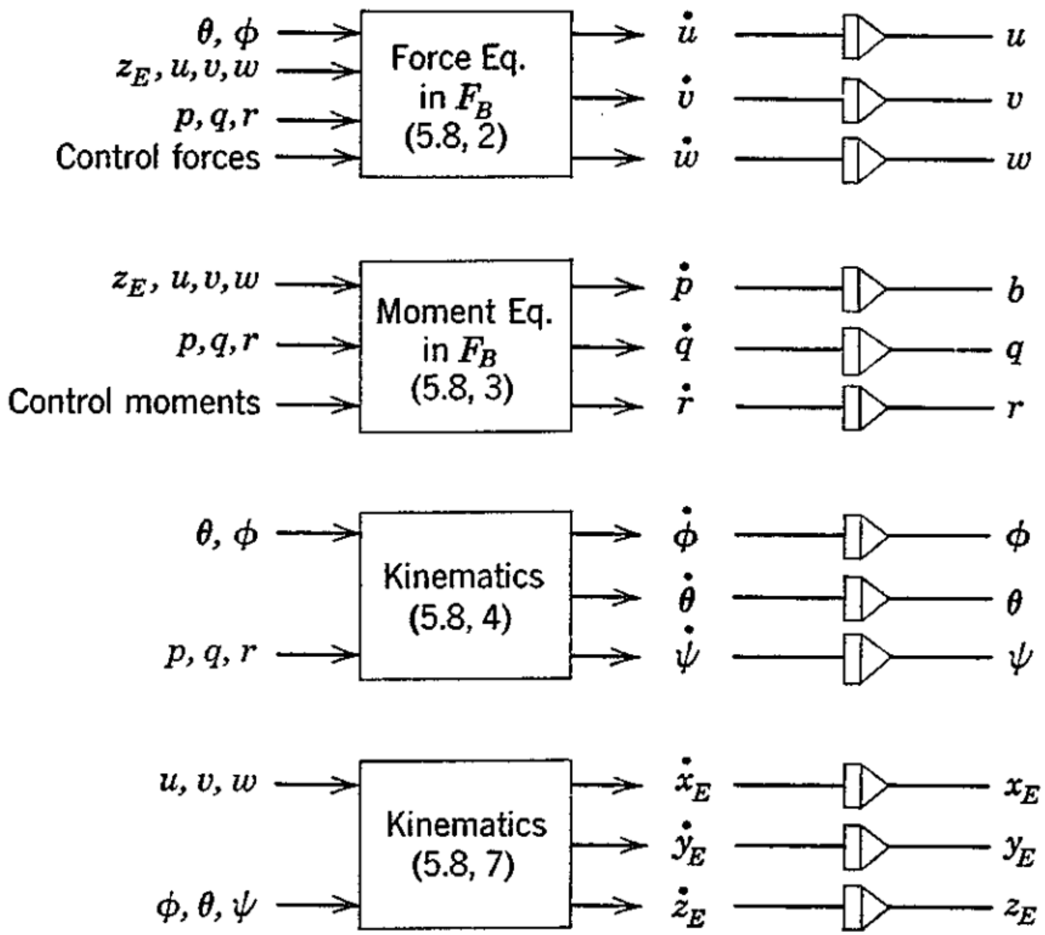
$$U = U_1 + u(t)$$

$$P = P_1 + p(t)$$

$$Y = Y_1 + y(t)$$

stationary components

Numerical Integration



Linear Momentum

Angular Momentum

Euler-Angle Rates

Trajectory

Block diagram of equations for vehicle with plane of symmetry. Body axes. Flat-Earth approximation.

Source: Etkin, B. *Dynamics of atmospheric flight*

Stationary Flight

Definition A stationary flight state is a flight condition where all variables in the body-fixed coordinate system are constant, i.e. are not time dependent.

$$\dot{\vec{v}}_P = 0 \qquad \vec{v}_P = const$$

$$\dot{\vec{\omega}} = 0 \qquad \vec{\omega} = const$$

Examples

- Straight flight (level cruise, climb or descent)
- Horizontal turn with constant turn-rate and radius
- Pull-up (wings level)

EoM for Stationary Flight

Stationary flight:

$$\dot{\vec{v}}_P = 0 \qquad \dot{\vec{\omega}} = 0$$

No linear and angular acceleration

Balance of Forces

$$m(-VR + WQ) = -mg\sin\theta + F_{A_x} + F_{T_x}$$

$$m(UR - WP) = mg\sin\phi\cos\theta + F_{A_y} + F_{T_y}$$

$$m(-UQ + VP) = mg\cos\phi\cos\theta + F_{A_z} + F_{T_z}$$

Balance of Moments

$$-I_{xz}PQ + \left(I_{zz} - I_{yy}\right)RQ = L_A + L_T$$

$$(I_{xx} - I_{zz})PR + I_{xz}(P^2 - R^2) = M_A + M_T$$

$$\left(I_{yy} - I_{xx}\right)PQ + I_{xz}QR = N_A + N_T$$

Kinematic Equations

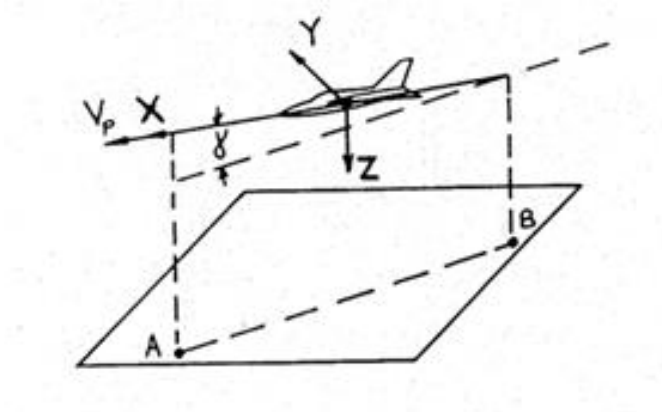
$$P = \dot{\phi} - \dot{\psi}\sin\theta$$

$$Q = \dot{\theta}\cos\phi + \dot{\psi}\cos\theta\sin\phi$$

$$R = \dot{\psi}\cos\theta\cos\phi - \dot{\theta}\sin\phi$$

Stationary Straight Flight

$\vec{\omega} = 0 \qquad P = Q = R = 0$



Balance of Forces

$$0 = -mg\,sin\,\theta + F_{A_x} + F_{T_x}$$

$$0 = mg\,sin\,\phi\,cos\,\theta + F_{A_y} + F_{T_y}$$

$$0 = mg\,cos\,\phi\,cos\,\theta + F_{A_z} + F_{T_z}$$

Balance of Moments

$$0 = L_A + L_T$$

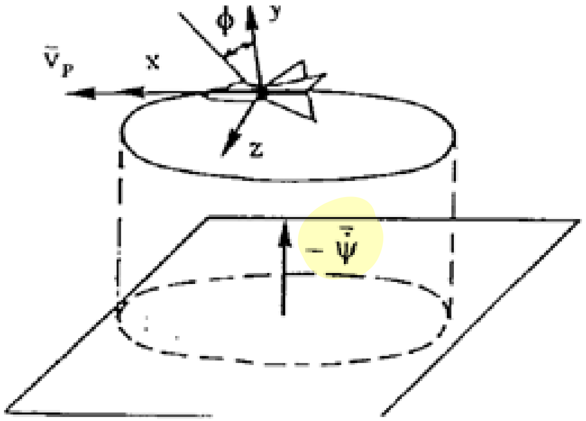
$$0 = M_A + M_T$$

$$0 = N_A + N_T$$

The kinematic equations are not needed

Stationary Horizontal Turn

$$\vec{\omega} = \vec{k}'\dot{\psi}' = \vec{k}\dot{\psi}$$



The angular velocity vector $\vec{\omega}$ is oriented vertically in the earth-fixed coordinate system (x', y', z')

During the turn only the yaw angle ψ changes, the pitch and roll angles stay constant

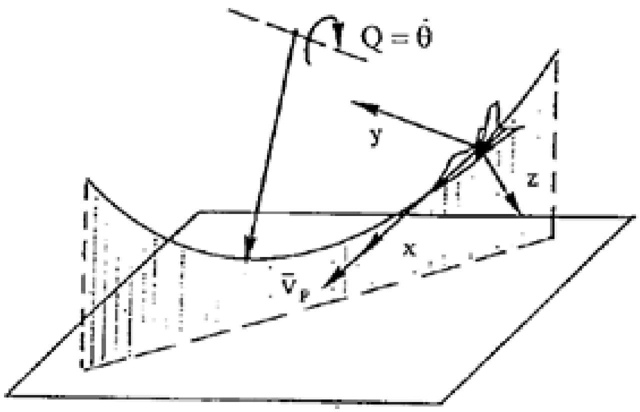
The kinematic equations simplify to

$$P = -\dot{\psi}\sin\theta$$

$$Q = \dot{\psi}\cos\theta\sin\phi$$

$$R = \dot{\psi}\cos\theta\cos\phi$$

Pull-Up



$V = 0$ $\quad P = 0$ $\quad \phi = 0$

$R = 0$

Balance of Forces

$$m(WQ) = -mg\sin\theta + F_{A_x} + F_{T_x}$$

$$0 = F_{A_y} + F_{T_y}$$

$$m(-UQ) = mg\cos\phi + F_{A_z} + F_{T_z}$$

Balance of Moments

$$0 = L_A + L_T$$

$$0 = M_A + M_T$$

$$0 = N_A + N_T$$

Kinematic Equations

$$Q = \dot{\theta}$$

6-DoF Equations of Motion – What is Missing

Forces

$$m(\dot{U} - VR + WQ) = -mg\sin\theta + F_{A_x} + F_{T_x}$$

$$m(\dot{V} + UR - WP) = mg\sin\phi\cos\theta + F_{A_y} + F_{T_y}$$

$$m(\dot{W} - UQ + VP) = mg\cos\phi\cos\theta + F_{A_z} + F_{T_z}$$

Moments

$$I_{xx}\dot{P} - I_{xz}\dot{R} - I_{xz}PQ + (I_{zz} - I_{yy})RQ = L_A + L_T$$

$$I_{yy}\dot{Q} + (I_{xx} - I_{zz})PR + I_{xz}(P^2 - R^2) = M_A + M_T$$

$$I_{zz}\dot{R} - I_{xz}\dot{P} + (I_{yy} - I_{xx})PQ + I_{xz}QR = N_A + N_T$$

Kinematics

$$P = \dot{\phi} - \dot{\psi}\sin\theta$$

$$Q = \dot{\theta}\cos\phi + \dot{\psi}\cos\theta\sin\phi$$

$$R = \dot{\psi}\cos\theta\cos\phi - \dot{\theta}\sin\phi$$

Aerodynamics

- Forces
- Moments

Propulsion

- Forces
- Moments

6-DoF Equations of Motion - Summary

- Formulation of the linear and angular momentum equations for a mass element, in an inertial system
- Definition of the center of gravity of mass elements in a rigid body
- Change of reference frame from the inertial system into the body-fixed system
- Changed notation from vector to components

$$\vec{v}_P = \begin{bmatrix} U \\ V \\ W \end{bmatrix} \quad \dot{\vec{v}}_p = \begin{bmatrix} \dot{U} \\ \dot{V} \\ \dot{W} \end{bmatrix} \quad \vec{\omega} = \begin{bmatrix} P \\ Q \\ R \end{bmatrix} \qquad I = \begin{bmatrix} I_{xx} & 0 & I_{xz} \\ 0 & I_{yy} & 0 \\ I_{zx} & 0 & I_{zz} \end{bmatrix}$$

- Described the orientation of the body-axis system with Euler-angles
- Defined special cases of the 6-DoF equations for stationary flight: straight-line, horizontal turn and pull-up

Equations of Motion – Point Mass

Linear momentum in the inertial frame

$$\frac{d}{dt}\int \rho_A \frac{d\vec{r}'}{dt} dV = \int_V \rho_A \vec{g} dV + \int_S \vec{F} dS$$

⇨ Linear momentum in the velocity vector coordinate frame, 2D

Forces

$$m\dot{V} = T cos(\alpha + \sigma) - D - mg\, sin(\gamma)$$

$$mV\dot{\gamma} = L + T sin(\alpha + \sigma) - mg\, cos(\gamma)$$

Kinematics

$$\dot{h} = V sin(\gamma)$$

$$\dot{x} = V cos(\gamma)$$