ROBOT DYNAMICS

1 POSITION

NOMENCLATURE

$_{\mathcal{A}}\mathbf{r}_{BC}$	Vector from point B to C (w.r.t. frame $\mathcal{A}$)
$\mathcal{X}_{P,A}$	Position parameterization of point A

CARTESIAN SYSTEM

- Orthonormal basis $\mathbf{e}_x^{\mathcal{A}}$, $\mathbf{e}_y^{\mathcal{A}}$, $\mathbf{e}_z^{\mathcal{A}}$
- Cartesian coordinates $\mathcal{X}_{Pc} = [x, y, z]^\top$
- Position vector $_{\mathcal{A}}\mathbf{r} = x\mathbf{e}_x^{\mathcal{A}} + y\mathbf{e}_y^{\mathcal{A}} + z\mathbf{e}_z^{\mathcal{A}} = [x, y, z]^\top$
- Vector addition $_{\mathcal{A}}\mathbf{r}_{AC} + {_{\mathcal{A}}\mathbf{r}_{AB}} + {_{\mathcal{A}}\mathbf{r}_{BC}}$ applies
  ⚠ only in Cartesian coordinates

CYLINDRICAL SYSTEM

- Cylindrical coordinates $\mathcal{X}_{Pz} = [\rho, \theta, z]^\top$
- Position vector $_{\mathcal{A}}\mathbf{r} = [\rho\cos\theta, \rho\sin\theta, z]^\top$

SPHERICAL SYSTEM

- Spherical coordinates $\mathcal{X}_{Ps} = [r, \theta, \phi]^\top$
- Position vector $_{\mathcal{A}}\mathbf{r} = [r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi]^\top$

2 LINEAR VELOCITY

NOMENCLATURE

$_{\mathcal{A}}\dot{\mathbf{r}}_{BC}$	Linear velocity of point C relative to point B (w.r.t. frame $\mathcal{A}$)
$\mathbf{v}_C = {_{\mathcal{I}}\dot{\mathbf{r}}_{IC}}$	Absolute linear velocity of a point C (only w.r.t. any fixed frame $\mathcal{I}$ with origin I)
$\mathbf{a}_C = \dot{\mathbf{v}}_C$	Absolute acceleration of a point C
$\mathbf{\Omega}_{\mathcal{B}} = {_{\mathcal{I}}\boldsymbol{\omega}_{\mathcal{I}\mathcal{B}}}$	Absolute angular velocity of a body $\mathcal{B}$ (only w.r.t. a fixed frame $\mathcal{I}$)
$\mathbf{\Psi}_{\mathcal{B}} = \dot{\mathbf{\Omega}}_{\mathcal{B}}$	Absolute angular acceleration of a body $\mathcal{B}$

RELATION TO $\mathcal{X}_P$

- $\mathbf{E}_P(\mathcal{X}_P)$ relates linear velocity $\dot{\mathbf{r}}$ to the time derivative of the (position) representation $\mathcal{X}_P$

$$\dot{\mathbf{r}} = \underbrace{\frac{\partial \mathbf{r}}{\partial \mathcal{X}_P}}_{=:E_P(\mathcal{X}_P)} \dot{\mathcal{X}}_P$$

$$\boxed{\begin{aligned}\dot{\mathbf{r}} &= \mathbf{E}_P(\mathcal{X}_P)\dot{\mathcal{X}}_P \\ \Longleftrightarrow \dot{\mathcal{X}}_P &= \mathbf{E}_P^{-1}(\mathcal{X}_P)\dot{\mathbf{r}}\end{aligned}}$$

  - Cartesian system

$$\mathbf{E}_{Pc}(\mathcal{X}_{Pc}) = \mathbf{E}_{Pc}^{-1}(\mathcal{X}_{Pc}) = \mathbb{I}$$

  - Cylindrical system

$$\mathbf{E}_{Pz}(\mathcal{X}_{Pz}) = \begin{bmatrix} \cos\theta & -\rho\sin\theta & 0 \\ \sin\theta & \rho\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{E}_{Pz}^{-1}(\mathcal{X}_{Pz}) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\frac{\sin\theta}{\rho} & \frac{\cos\theta}{\rho} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

VELOCITY IN RIGID MOVING BODIES

- Rigid body formulation

$$_{\mathcal{I}}\mathbf{r}_{IC} = {_{\mathcal{I}}\mathbf{r}_{IB}} + {_{\mathcal{I}}\mathbf{r}_{BC}} = {_{\mathcal{I}}\mathbf{r}_{IB}} + \mathbf{C}_{\mathcal{I}\mathcal{B}}\,{_{\mathcal{B}}\mathbf{r}_{BC}}$$

where B, C are points on a rigid body $\mathcal{B}$

$$\begin{aligned}\Longrightarrow {_{\mathcal{I}}\dot{\mathbf{r}}_{IC}} &= {_{\mathcal{I}}\dot{\mathbf{r}}_{IC}} + \mathbf{C}_{\mathcal{I}\mathcal{C}}\,{_{\mathcal{B}}\dot{\mathbf{r}}_{BC}}^{\,0} + \dot{\mathbf{C}}_{\mathcal{I}\mathcal{B}}\,{_{\mathcal{B}}\mathbf{r}_{BC}} \\ &= {_{\mathcal{I}}\dot{\mathbf{r}}_{IB}} + \left[{_{\mathcal{I}}\boldsymbol{\omega}_{\mathcal{I}\mathcal{B}}}\right]_\times \mathbf{C}_{\mathcal{I}\mathcal{B}}\,{_{\mathcal{B}}\mathbf{r}_{BC}} \\ &= {_{\mathcal{I}}\dot{\mathbf{r}}_{IB}} + {_{\mathcal{I}}\boldsymbol{\omega}_{\mathcal{I}\mathcal{B}}} \times {_{\mathcal{I}}\mathbf{r}_{BC}}\end{aligned}$$

using $\dot{\mathbf{C}}_{\mathcal{I}\mathcal{B}} = \left[{_{\mathcal{I}}\boldsymbol{\omega}_{\mathcal{I}\mathcal{B}}}\right]_\times \mathbf{C}_{\mathcal{I}\mathcal{B}}$

$$\boxed{\mathbf{v}_C = \mathbf{v}_B + \mathbf{\Omega}_{\mathcal{B}} \times {_{\mathcal{I}}\mathbf{r}_{BC}}}$$

3 ROTATION MATRICES

NOMENCLATURE

$\mathbf{C}_{\mathcal{A}\mathcal{B}}$	Passive rotation matrix (from frame $\mathcal{B}$ to $\mathcal{A}$) aka direction cosine matrix (DCM)
$\mathbf{R}$	Active rotation matrix

FRAME TRANSFORMATION

- Rotation matrix $\mathbf{C}_{\mathcal{A}\mathcal{B}}$ transforms vectors w.r.t. frame $\mathcal{B}$ to vectors w.r.t. frame $\mathcal{A}$

$$\boxed{_{\mathcal{A}}\mathbf{r}_{AC} = \mathbf{C}_{\mathcal{A}\mathcal{B}} \cdot {_{\mathcal{B}}\mathbf{r}_{AC}}}$$

- Columns of $\mathbf{C}_{\mathcal{A}\mathcal{B}}$ are unit vectors of $\mathcal{B}$ w.r.t. $\mathcal{A}$

$$\boxed{\mathbf{C}_{\mathcal{A}\mathcal{B}} = \begin{bmatrix} _{\mathcal{A}}\mathbf{e}_x^{\mathcal{B}} & _{\mathcal{A}}\mathbf{e}_y^{\mathcal{B}} & _{\mathcal{A}}\mathbf{e}_z^{\mathcal{B}} \end{bmatrix}}$$

PROPERTIES

- Orthogonality

$$\mathbf{C}_{\mathcal{B}\mathcal{A}} = \mathbf{C}_{\mathcal{A}\mathcal{B}}^{-1} = \mathbf{C}_{\mathcal{A}\mathcal{B}}^\top$$

  - Imposes 6 constraints on the 9 parameters
- No singularity problem
- Special orthonormal group

$$\mathbf{C} \in \mathrm{SO}(3) := \{\mathbf{C} \in \mathbb{R}^{3\times 3} \mid \mathbf{C}^\top\mathbf{C} = \mathbb{I}, \det(\mathbf{C}) = \pm 1\}$$

- Concatenation of rotations

$$\boxed{\mathbf{C}_{\mathcal{A}\mathcal{C}} = \mathbf{C}_{\mathcal{A}\mathcal{B}}\mathbf{C}_{\mathcal{B}\mathcal{C}}}$$

ACTIVE/PASSIVE ROTATIONS

- Passive rotations map the same vector from frame $\mathcal{B}$ to $\mathcal{A}$

$$_{\mathcal{A}}\mathbf{u} = \mathbf{C}_{\mathcal{A}\mathcal{B}} \cdot {_{\mathcal{B}}\mathbf{u}}$$

- Active rotations rotate a vector in thee same frame

$$_{\mathcal{A}}\mathbf{v} = \mathbf{R} \cdot {_{\mathcal{A}}\mathbf{u}}$$

ELEMENTARY ROTATIONS

- Passive rotation about the x-axis:

$$\mathbf{C}_x(\varphi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{bmatrix}, \quad \mathbf{R}_x(\varphi) = \mathbf{C}_x^\top(\varphi)$$

- Passive rotation about the y-axis:

$$\mathbf{C}_y(\varphi) = \begin{bmatrix} \cos\varphi & 0 & \sin\varphi \\ 0 & 1 & 0 \\ -\sin\varphi & 0 & \cos\varphi \end{bmatrix}, \quad \mathbf{R}_y(\varphi) = \mathbf{C}_y^\top(\varphi)$$

- Passive rotation about the z-axis:

$$\mathbf{C}_z(\varphi) = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{R}_z(\varphi) = \mathbf{C}_z^\top(\varphi)$$

ROTATION COMPOSITIONS

- Concatenate a rotation $\mathbf{C}_1$ with a successive rotation...
  - $\mathbf{C}_2$ defined in moving (intrinsic) axes: $\mathbf{C} = \mathbf{C}_1\mathbf{C}_2$ (postmultiply)
  - $\mathbf{C}_2$ defined in fixed (extrinsic) axes: $\mathbf{C} = \mathbf{C}_2\mathbf{C}_1$ (premultiply)
- Example: two interpretations for $\mathbf{C}_z(\psi)\mathbf{C}_y(\theta)\mathbf{C}_x(\phi)$
  - Intrinsic convention (intrinsic ZYX Euler angles)
    1. Rotate around z
    2. Rotate around rotated y
    3. Rotate around rotated x
  - Extrinsic convention (extrinsic XYZ Euler angles)
    1. Rotate around x
    2. Rotate around original y
    3. Rotate around original z

Note: intrinsic ZYX = extrinsic XYZ
⚠ The intrinsic convention will be used hereinafter

4 EULER ANGLES

PROPERTIES

- 3 parameters (rotation angles)
- Singularity problem: there exists a certain configuration in which there is no longer a conversion between ω and $\dot{\mathcal{X}}_R$
- Different conventions
  - Proper Euler angles: ZXZ, XYX, YZY, ZYZ, XZX, YXY
    * Share an axis for the first and last rotation
  - Tait-Bryan angles: XYZ, YZX, ZXY, XZY, ZYX, YXZ
    * Rotations about three distinct axes
    * Example: ZYX Tait-Bryan angles $\boldsymbol{\chi}_{R,\mathrm{Euler},ZYX} = \begin{bmatrix} z \\ y \\ x \end{bmatrix}$

CONVERSION OF ZYX TAIT-BRYAN ANGLES

- ZYX Tait-Bryan angles → rotation matrix

$$\begin{aligned}\mathbf{C}_{\mathcal{A}\mathcal{D}} &= \mathbf{C}_{\mathcal{A}\mathcal{B}}(z)\mathbf{C}_{\mathcal{B}\mathcal{C}}(y)\mathbf{C}_{\mathcal{C}\mathcal{D}}(x) \\ &= \mathbf{C}_z(z)\mathbf{C}_y(y)\mathbf{C}_x(x) \\ &= \begin{bmatrix} c_y c_z & c_z s_x s_y - c_x s_z & s_x s_z + c_x c_z s_y \\ c_y s_z & c_x c_z + s_x s_y s_z & c_x s_y s_z - c_z s_x \\ -s_y & c_y s_x & c_x c_y \end{bmatrix}\end{aligned}$$

- Rotation matrix → ZYX Tait-Bryan angles

$$\mathbf{C}_{\mathcal{A}\mathcal{D}} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

$$\boldsymbol{\chi}_{R,\mathrm{Euler},ZYX} = \begin{bmatrix} z \\ y \\ x \end{bmatrix} = \begin{bmatrix} \mathrm{atan2}(c_{21}, c_{11}) \\ \mathrm{atan2}\left(-c_{31}, \sqrt{c_{32}^2 + c_{33}^2}\right) \\ \mathrm{atan2}(c_{32}, c_{33}) \end{bmatrix}$$

5 ANGLE AXIS

PROPERTIES

- 4 parameters
  - Rotation angle $\theta \in \mathbb{R}$
  - Rotation axis $\mathbf{n} \in \mathbb{R}^3$

$$\boldsymbol{\chi}_{R,\mathrm{AngleAxis}} = \begin{bmatrix} \theta \\ \mathbf{n} \end{bmatrix}$$

- Unitary constraint: $\|\mathbf{n}\| = 1$
- Singularity problem (at $\theta = 0$)
- Convertible to rotation/Euler vector (3 parameters)

$$\boldsymbol{\chi}_{R,\mathrm{RotVec}} = \theta\mathbf{n}$$

  - In general, rotational vectors are not proper vectors

CONVERSIONS

- Angle axis → rotation matrix

$$\mathbf{C}_{\mathcal{A}\mathcal{B}}(\theta, \mathbf{n}) = \cos(\theta)\mathbb{I}_{3\times 3} - \sin(\theta)\left[\mathbf{n}\right]_\times + (1 - \cos(\theta))\mathbf{n}\mathbf{n}^\top$$

$$= \begin{bmatrix} n_x^2(1 - c_\theta) + c_\theta & n_x n_y(1 - c_\theta) - n_z s_\theta & n_x n_z(1 - c_\theta) + n_y s_\theta \\ n_x n_y(1 - c_\theta) + n_z s_\theta & n_y^2(1 - c_\theta) + c_\theta & n_y n_z(1 - c_\theta) - n_x s_\theta \\ n_x n_z(1 - c_\theta) - n_y s_\theta & n_y n_z(1 - c_\theta) + n_x s_\theta & n_z^2(1 - c_\theta) + c_\theta \end{bmatrix}$$

- Rotation matrix → angle axis

$$\boldsymbol{\chi}_{R,\mathrm{AngleAxis}} = \begin{bmatrix} \theta \\ \mathbf{n} \end{bmatrix} = \begin{bmatrix} \arccos\left(\frac{c_{11} + c_{22} + c_{33} - 1}{2}\right) \\ \mathbf{n} = \frac{1}{2\sin\theta} \begin{bmatrix} c_{32} - c_{23} \\ c_{13} - c_{31} \\ c_{21} - c_{12} \end{bmatrix} \end{bmatrix}$$

6 UNIT QUATERNIONS

PROPERTIES

- 4 parameters ξ_{0-3}
- Vector representation:
$$\boldsymbol{\chi}_{R,\text{Quat}} = \boldsymbol{\xi} = \begin{bmatrix} \xi_0 \\ \check{\boldsymbol{\xi}} \end{bmatrix}$$
- Unitary constraint: $\xi_0^2 + \xi_1^2 + \xi_2^2 + \xi_3^2 = 1$
- No singularity problem
- 4D complex number representation:
$$\boldsymbol{\xi} = \xi_0 + \xi_1 i + \xi_2 j + \xi_3 k$$
  - Real part $\xi_0 = \cos\left(\frac{\theta}{2}\right)$
  - Imaginary part $\check{\boldsymbol{\xi}} = \sin\left(\frac{\theta}{2}\right)\mathbf{n} = \begin{bmatrix} \xi_1 & \xi_2 & \xi_3 \end{bmatrix}^\top$
- Inverse $\boldsymbol{\xi}^{-1} \equiv \boldsymbol{\xi}^*$ (conjugate) $\equiv \bar{\boldsymbol{\xi}} \equiv \boldsymbol{\xi}^\top = \begin{bmatrix} \xi_0 \\ -\check{\boldsymbol{\xi}} \end{bmatrix}$
- Identity: $\boldsymbol{\xi} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^\top$
- Double cover: $\boldsymbol{\xi}$ and $-\boldsymbol{\xi}$ represent the same rotation

HAMILTONIAN CONVENTION

$$\boldsymbol{\xi} = \xi_0 + \xi_1 i + \xi_2 j + \xi_3 k$$
$$i^2 = j^2 = k^2 = ijk = -1$$
$$ij = -ji = -ijk^2 = k$$
$$jk = -kj = i$$
$$ki = -ik = j$$

CONVERSIONS

- Rotation matrix $\to$ unit quaternion:
$$\boldsymbol{\chi}_{R,\text{Quat}} = \boldsymbol{\xi}_{\mathcal{AD}} = \begin{bmatrix} \sqrt{c_{11} + c_{22} + c_{33} + 1} \\ \text{sgn}(c_{32} - c_{23})\sqrt{c_{11} - c_{22} - c_{33} + 1} \\ \text{sgn}(c_{13} - c_{31})\sqrt{c_{22} - c_{33} - c_{11} + 1} \\ \text{sgn}(c_{21} - c_{12})\sqrt{c_{33} - c_{11} - c_{22} + 1} \end{bmatrix}$$
- Unit quaternion $\to$ rotation matrix:
$$\mathbf{C}_{\mathcal{AB}} = \mathbb{I}_{3\times3} + 2\xi_0 \left[\check{\boldsymbol{\xi}}\right]_\times + 2\left[\check{\boldsymbol{\xi}}\right]_\times^2 = (2\xi_0^2 - 1)\mathbb{I}_{3\times3} + 2\xi_0\left[\check{\boldsymbol{\xi}}\right]_\times + 2\check{\boldsymbol{\xi}}\check{\boldsymbol{\xi}}^\top$$
$$= \begin{bmatrix} \xi_0^2 + \xi_1^2 - \xi_2^2 - \xi_3^2 & 2\xi_1\xi_2 - 2\xi_0\xi_3 & 2\xi_0\xi_2 + 2\xi_1\xi_3 \\ 2\xi_0\xi_3 + 2\xi_1\xi_2 & \xi_0^2 - \xi_1^2 + \xi_2^2 - \xi_3^2 & 2\xi_2\xi_3 - 2\xi_0\xi_1 \\ 2\xi_1\xi_3 - 2\xi_0\xi_2 & 2\xi_0\xi_1 + 2\xi_2\xi_3 & \xi_0^2 - \xi_1^2 - \xi_2^2 + \xi_3^2 \end{bmatrix}$$

PRODUCT OF TWO QUATERNIONS

$$\mathbf{q} \otimes \mathbf{p} = (q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k})(p_0 + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k})$$
$$= \cdots = q_0p_0 - q_1p_1 - q_2p_2 - q_3p_3 + \ldots$$
$$\cdots + (q_0p_1 + q_1p_0 + q_2p_3 - q_3p_2)\mathbf{i} + \ldots$$
$$\cdots + (q_0p_2 - q_1p_3 + q_2p_0 + q_3p_1)\mathbf{j} + \ldots$$
$$\cdots + (q_0p_3 + q_1p_2 - q_2p_1 + q_3p_0)\mathbf{k}$$
$$= \begin{bmatrix} q_0p_0 - \check{\mathbf{q}}^\top\check{\mathbf{p}} \\ q_0\check{\mathbf{p}} + p_0\check{\mathbf{q}} + [\check{\mathbf{q}}]_\times\check{\mathbf{p}} \end{bmatrix}$$
$$= \begin{bmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{bmatrix}\begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}$$
$$= \underbrace{\begin{bmatrix} q_0 & -\check{\mathbf{q}}^\top \\ \check{\mathbf{q}} & q_0\mathbb{I}_{3\times3} + [\check{\mathbf{q}}]_\times \end{bmatrix}}_{=:M_l(q) \text{ (left matrix of q)}}\mathbf{p} = \mathbf{M}_l(\mathbf{q})\mathbf{p}$$
$$= \begin{bmatrix} p_0 & -p_1 & -p_2 & -p_3 \\ p_1 & p_0 & p_3 & -p_2 \\ p_2 & -p_3 & p_0 & p_1 \\ p_3 & p_2 & -p_1 & p_0 \end{bmatrix}\begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}$$
$$= \underbrace{\begin{bmatrix} p_0 & -\check{\mathbf{p}}^\top \\ \check{\mathbf{p}} & p_0\mathbb{I}_{3\times3} - [\check{\mathbf{p}}]_\times \end{bmatrix}}_{=:M_r(p) \text{ (right matrix of p)}}\mathbf{q} = \mathbf{M}_r(\mathbf{p})\mathbf{q}$$

ROTATING A VECTOR

- Pure (imaginary) quaternion $\mathbf{p}$ of a coordinate vector $_{\mathcal{A}}\mathbf{r}$
$$\mathbf{p}(_{\mathcal{A}}\mathbf{r}) = \begin{bmatrix} 0 \\ _{\mathcal{A}}\mathbf{r} \end{bmatrix}$$
- Transform $_{\mathcal{A}}\mathbf{r}$ from frame $\mathcal{A}$ to $\mathcal{B}$ with the quaternion $\boldsymbol{\xi}_{\mathcal{BA}}$
$$\mathbf{p}(_{\mathcal{B}}\mathbf{r}) = \boldsymbol{\xi}_{\mathcal{BA}} \otimes \mathbf{p}(_{\mathcal{A}}\mathbf{r}) \otimes \boldsymbol{\xi}_{\mathcal{BA}}^\top$$
$$= \mathbf{M}_l(\boldsymbol{\xi}_{\mathcal{BA}})\mathbf{M}_r(\boldsymbol{\xi}_{\mathcal{BA}}^\top)\,\mathbf{p}(_{\mathcal{A}}\mathbf{r})$$

Note: analogous to $_{\mathcal{B}}\mathbf{r} = \mathbf{C}_{\mathcal{BA}}\,_{\mathcal{A}}\mathbf{r}$

7 HOMOGENEOUS TRANSFORMATION

NOMENCLATURE

HOMOGENEOUS TRANSFORMATION

- Transform a position vector $_{\mathcal{B}}\mathbf{r}_{BC}$ to $_{\mathcal{A}}\mathbf{r}_{AC}$ using a rotation and translation
$$_{\mathcal{A}}\mathbf{r}_{AC} = {}_{\mathcal{A}}\mathbf{r}_{AB} + {}_{\mathcal{A}}\mathbf{r}_{BC}$$
$$= {}_{\mathcal{A}}\mathbf{r}_{AB} + \mathbf{C}_{\mathcal{AB}} \cdot {}_{\mathcal{B}}\mathbf{r}_{BC}$$
- Combine into a single transformation $\mathbf{T}_{\mathcal{AB}}$
$$\begin{bmatrix} _{\mathcal{A}}\mathbf{r}_{AC} \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{C}_{\mathcal{AB}} & _{\mathcal{A}}\mathbf{r}_{AB} \\ \mathcal{O}_{1\times3} & 1 \end{bmatrix}}_{\mathbf{T}_{\mathcal{AB}}}\begin{bmatrix} _{\mathcal{B}}\mathbf{r}_{BC} \\ 1 \end{bmatrix}$$

PROPERTIES

- Inverse
$$\mathbf{T}_{\mathcal{AB}}^{-1} = \begin{bmatrix} \mathbf{C}_{\mathcal{AB}}^\top & \overbrace{-\mathbf{C}_{\mathcal{AB}}^\top\,_{\mathcal{A}}\mathbf{r}_{AB}}^{_{\mathcal{B}}\mathbf{r}_{BA}} \\ \mathcal{O}_{1\times3} & 1 \end{bmatrix}$$
- Concatenation of transformations
$$\mathbf{T}_{\mathcal{AC}} = \mathbf{T}_{\mathcal{AB}}\mathbf{T}_{\mathcal{BC}}$$

8 ANGULAR VELOCITY

NOMENCLATURE

PROPERTIES

- Angular velocity $_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{AB}}$ describes the relative rotational velocity of frame $\mathcal{B}$ w.r.t. frame $\mathcal{A}$ (expressed in frame $\mathcal{A}$)
$$_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{AB}} = \lim_{\epsilon \to 0}\frac{_{\mathcal{A}}\boldsymbol{\varphi}_{\mathcal{B}(t)\mathcal{B}(t+\epsilon)}}{\epsilon}$$
where $\boldsymbol{\varphi}$ is a rotation vector
  - For $\epsilon \to 0$, the angular velocity is defined as the ratio of a proper vector and a scalar
- Opposite direction
$$\boldsymbol{\omega}_{\mathcal{AB}} = -\boldsymbol{\omega}_{\mathcal{BA}}$$
- Transformation of angular velocity
$$_{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{AB}} = \mathbf{C}_{\mathcal{BA}} \cdot {}_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{AB}}$$
- Addition of relative velocities
$$_{\mathcal{D}}\boldsymbol{\omega}_{\mathcal{AC}} = {}_{\mathcal{D}}\boldsymbol{\omega}_{\mathcal{AB}} + {}_{\mathcal{D}}\boldsymbol{\omega}_{\mathcal{BC}}$$

RELATION TO $\mathcal{X}_R$

- $\mathbf{E}_R(\boldsymbol{\mathcal{X}}_R) \in \mathbb{R}^{3\times\dim(\boldsymbol{\mathcal{X}}_R)}$ relates angular velocity ω to the time derivative of the rotation representation $\dot{\boldsymbol{\mathcal{X}}}_R$
$$_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{AB}} = \mathbf{E}_R(\boldsymbol{\mathcal{X}}_R)\dot{\boldsymbol{\mathcal{X}}}_R$$
$$\iff \dot{\boldsymbol{\mathcal{X}}}_R = \mathbf{E}_R^{-1}(\boldsymbol{\mathcal{X}}_R)_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{AB}}$$

DERIVING $\mathbf{E}_{R,\text{Euler},ZYX}$

- ZYX Tait-Bryan angles $\boldsymbol{\mathcal{X}}_R = \begin{bmatrix} \psi & \theta & \phi \end{bmatrix}^\top$ defining
$$\mathbf{C}_{\mathcal{AD}} = \mathbf{C}_{\mathcal{AB}}(\psi)\mathbf{C}_{\mathcal{BC}}(\theta)\mathbf{C}_{\mathcal{CD}}(\phi)$$
  1. $\mathbf{C}_{\mathcal{AB}}$ is rotation (yaw ψ) around $\mathbf{e}_z^{\mathcal{I}}$ i.e. $\mathbf{C}_z(\psi)$
  2. $\mathbf{C}_{\mathcal{BC}}$ is rotation (pitch θ) around $\mathbf{e}_y^1$ i.e. $\mathbf{C}_y(\theta)$
  3. $\mathbf{C}_{\mathcal{CD}}$ is rotation (roll ϕ) around $\mathbf{e}_x^2$ i.e. $\mathbf{C}_x(\phi)$
$$_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{AD}} = {}_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{AB}} + {}_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{BC}} + {}_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{CD}}$$
$$= {}_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{AB}} + \mathbf{C}_{\mathcal{AB}}\,_{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{BC}} + \mathbf{C}_{\mathcal{AB}}\mathbf{C}_{\mathcal{BC}}\,_{\mathcal{C}}\boldsymbol{\omega}_{\mathcal{CD}}$$
$$= {}_{\mathcal{A}}\mathbf{e}_z^{\mathcal{A}}\dot{\psi} + \mathbf{C}_{\mathcal{AB}}\,_{\mathcal{B}}\mathbf{e}_y^{\mathcal{B}}\dot{\theta} + \mathbf{C}_{\mathcal{AB}}\mathbf{C}_{\mathcal{BC}}\,_{\mathcal{C}}\mathbf{e}_x^{\mathcal{C}}\dot{\phi}$$
$$= \begin{bmatrix} _{\mathcal{A}}\mathbf{e}_z^{\mathcal{A}} & \mathbf{C}_{\mathcal{AB}}\,_{\mathcal{B}}\mathbf{e}_y^{\mathcal{B}} & \mathbf{C}_{\mathcal{AB}}\mathbf{C}_{\mathcal{BC}}\,_{\mathcal{C}}\mathbf{e}_x^{\mathcal{C}} \end{bmatrix}\dot{\boldsymbol{\mathcal{X}}}_R$$
$$= \begin{bmatrix} _{\mathcal{A}}\mathbf{e}_z^{\mathcal{A}} & \mathbf{C}_z(\psi)\,_{\mathcal{B}}\mathbf{e}_y^{\mathcal{B}} & \mathbf{C}_z(\psi)\mathbf{C}_y(\theta)\,_{\mathcal{C}}\mathbf{e}_x^{\mathcal{C}} \end{bmatrix}\dot{\boldsymbol{\mathcal{X}}}_R$$

$$\mathbf{E}_{R,\text{Euler},ZYX} = \begin{bmatrix} 0 & -\sin\psi & \cos\psi\cos\theta \\ 0 & \cos\phi & \sin\phi\cos\theta \\ 1 & 0 & -\sin\theta \end{bmatrix}$$

$$\mathbf{E}_{R,\text{Euler},ZYX}^{-1} = \begin{bmatrix} \frac{\cos\psi\sin\theta}{\cos\theta} & \frac{\sin\theta\sin\psi}{\cos\theta} & 1 \\ -\sin\psi & \cos\psi & 0 \\ \frac{\cos\psi}{\cos\theta} & \frac{\sin\psi}{\cos\theta} & 0 \end{bmatrix}$$

MORE $\mathbf{E}_R$

- Angle axis
$$\mathbf{E}_{R,\text{AngleAxis}} = \begin{bmatrix} \mathbf{n} & \sin(\theta)\mathbb{I}_{3\times3} + (1 - \cos\theta)\left[\mathbf{n}\right]_\times \end{bmatrix}$$
$$\mathbf{E}_{R,\text{AngleAxis}}^{-1} = \begin{bmatrix} \mathbf{n}^\top \\ -\frac{1}{2}\frac{\sin\theta}{1-\cos\theta}\left[\mathbf{n}\right]_\times^2 - \frac{1}{2}\left[\mathbf{n}\right]_\times \end{bmatrix}$$
- Rotation vector
$$\mathbf{E}_{R,\text{RotVec}} = \left[\mathbb{I}_{3\times3} + \left[\boldsymbol{\varphi}\right]_\times\left(\frac{1-\cos\|\boldsymbol{\varphi}\|}{\|\boldsymbol{\varphi}\|^2}\right) + \left[\boldsymbol{\varphi}\right]_\times^2\left(\frac{\|\boldsymbol{\varphi}\|-\sin\|\boldsymbol{\varphi}\|}{\|\boldsymbol{\varphi}\|^3}\right)\right]$$
$$\mathbf{E}_{R,\text{RotVec}}^{-1} = \left[\mathbb{I}_{3\times3} - \frac{1}{2}\left[\boldsymbol{\varphi}\right]_\times + \left[\boldsymbol{\varphi}\right]_\times^2\frac{1}{\|\boldsymbol{\varphi}\|^2}\left(1 - \frac{\|\boldsymbol{\varphi}\|}{2}\frac{\sin\|\boldsymbol{\varphi}\|}{1-\cos\|\boldsymbol{\varphi}\|}\right)\right]$$
- Quaternion
$$\mathbf{E}_{R,\text{Quat}} = 2\mathbf{H}(\boldsymbol{\xi}) \iff \mathbf{E}_{R,\text{Quat}}^{-1} = \frac{1}{2}\mathbf{H}(\boldsymbol{\xi})^\top$$
$$\mathbf{H}(\boldsymbol{\xi}) = \begin{bmatrix} -\check{\boldsymbol{\xi}} & [\check{\boldsymbol{\xi}}]_\times + \xi_0\mathbb{I}_{3\times3} \end{bmatrix} \in \mathbb{R}^{3\times4} = \begin{bmatrix} -\xi_1 & \xi_0 & -\xi_3 & \xi_2 \\ -\xi_2 & \xi_3 & \xi_0 & -\xi_1 \\ -\xi_3 & -\xi_2 & \xi_1 & \xi_0 \end{bmatrix}$$

ANGULAR VELOCITY FROM PASSIVE ROTATION

$$[_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{AB}}]_\times = \dot{\mathbf{C}}_{\mathcal{AB}}\mathbf{C}_{\mathcal{AB}}^\top$$
$$[_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{AB}}]_\times = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}, \quad _{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{AB}} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

9 KINEMATICS OF SYSTEMS OF BODIES

NOMENCLATURE

GENERALIZED ROBOT ARM

- n_j joints
  - Revolute (1 DOF)
  - Prismatic (1 DOF)
- $n_l = n_j + 1$ links (n_j moving links, 1 fixed link)
- $6n_j - 5n_j = n_j = \dim(\mathbf{q})$ DOFs
  - Moving links (n_j) need 6 parameters
  - 1 DOF joints (n_j) impose 5 constraints
  - Number of DOFs = number of minimal coordinates

SERIAL KINEMATIC LINKAGES

- Configuration parameters
  - Generalized coordinates: a set of scalar parameters $\mathbf{q}$ that fully specify the robot's configuration (complete, independent, not unique)
$$\mathbf{q} = \begin{bmatrix} q_1 \\ \vdots \\ q_{n_j} \end{bmatrix} \in \mathbb{R}^{n_j}$$
  - End-effector configuration parameters: a set of m parameters that fully specify the end-effector position and orientation w.r.t. the inertial frame
$$\boldsymbol{\mathcal{X}}_e = \begin{bmatrix} \boldsymbol{\mathcal{X}}_{P,e} \\ \boldsymbol{\mathcal{X}}_{R,e} \end{bmatrix} = \begin{bmatrix} \mathcal{X}_1 \\ \vdots \\ \mathcal{X}_{m_e} \end{bmatrix} \in \mathbb{R}^{m_e}$$
$$m_e := \begin{cases} 3 & \text{robot arm in 2D} \\ 6 & \text{robot arm in 3D} \end{cases}$$
  - Task space: independent of the robot
  - Operational space coordinates: m_{e0} DOFs of end-effector (independent)
$$\boldsymbol{\mathcal{X}}_{e0} = \begin{bmatrix} \boldsymbol{\mathcal{X}}_{P,e0} \\ \boldsymbol{\mathcal{X}}_{R,e0} \end{bmatrix} = \begin{bmatrix} \mathcal{X}_1 \\ \vdots \\ \mathcal{X}_{m_{e0}} \end{bmatrix} \in \mathbb{R}^{m_{e0}}$$

FORWARD KINEMATICS

- End-effector configuration $\boldsymbol{\mathcal{X}}_e$ as a function of generalized coordinates $\mathbf{q}$
$$\boldsymbol{\mathcal{X}}_e = \boldsymbol{\mathcal{X}}_e(\mathbf{q}) \in \mathbb{R}^{m_e}$$
  - Homogeneous transformation from inertial frame origin to end-effector
$$\mathbf{T}_{\mathcal{IE}}(\mathbf{q}) = \mathbf{T}_{\mathcal{I}0} \cdot \left(\prod_{k=1}^{n_j}\mathbf{T}_{k-1,k}(q_k)\right) \cdot \mathbf{T}_{n_j\mathcal{E}}$$
$$= \begin{bmatrix} \mathbf{C}_{\mathcal{IE}}(\mathbf{q}) & _{\mathcal{I}}\mathbf{r}_{\mathcal{IE}}(\mathbf{q}) \\ \mathbf{0}_{1\times3} & 1 \end{bmatrix} \in \mathbb{R}^{4\times4}$$
- Example: planar (2D) robot arm (in xz plane) with $n_j = 3$ joints (angles q_{1-3}) and $n_l = 4$ links (lengths l_{0-3})
$$\mathbf{T}_{\mathcal{IE}} = \mathbf{T}_{\mathcal{I}0}\mathbf{T}_{01}(\mathbf{q}_1, l_0)\mathbf{T}_{12}(\mathbf{q}_2, l_1)\mathbf{T}_{23}(\mathbf{q}_3, l_2)\mathbf{T}_{3\mathcal{E}}(l_3)$$
$$\boldsymbol{\mathcal{X}}_{P,e}(\mathbf{q}) = \begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} l_1\sin q_1 + l_2\sin(q_1 + q_2) + l_3\sin(q_1 + q_2 + q_3) \\ l_0 + l_1\cos q_1 + l_2\cos(q_1 + q_2) + l_3\cos(q_1 + q_2 + q_3) \end{bmatrix}$$
$$\boldsymbol{\mathcal{X}}_{R,e}(\mathbf{q}) = q_1 + q_2 + q_3$$

ANALYTICAL JACOBIAN

- Relates time derivative of configuration parameters $\mathcal{X}_e$ to generalized velocities $\dot{\mathbf{q}}$
- Depends on parameterization of a point (example: end-effector e)

$$\mathcal{X}_e + \delta\mathcal{X}_e = \mathcal{X}(\mathbf{q} + \delta\mathbf{q}) = \mathcal{X}_e(\mathbf{q}) + \frac{\partial\mathcal{X}_e}{\partial\mathbf{q}}\delta\mathbf{q} + \mathcal{O}(\delta\mathbf{q}^2)$$

$$\implies \delta\mathcal{X}_e \approx \underbrace{\frac{\partial\mathcal{X}_e}{\partial\mathbf{q}}}_{J_{A,e}(\mathbf{q})}\delta\mathbf{q}$$

$$\boxed{\begin{aligned}\dot{\mathcal{X}}_e &= \mathbf{J}_{A,e}(\mathbf{q})\dot{\mathbf{q}} \\ \mathbf{J}_{A,e}(\mathbf{q}) &= \begin{bmatrix} \frac{\partial\mathcal{X}_{e,1}}{\partial q_1} & \cdots & \frac{\partial\mathcal{X}_{e,1}}{\partial q_{n_j}} \\ \vdots & \ddots & \vdots \\ \frac{\partial\mathcal{X}_{e,m}}{\partial q_1} & \cdots & \frac{\partial\mathcal{X}_{e,m}}{\partial q_{n_j}} \end{bmatrix} \in \mathbb{R}^{\dim(\mathcal{X}_e)\times n_q}\end{aligned}}$$

- Mainly used for numeric algorithms
- Decomposition into position and orientation part

$$\mathbf{J}_{A,e} = \begin{bmatrix} \mathbf{J}_{A_P,e} \\ \mathbf{J}_{A_R,e} \end{bmatrix} = \begin{bmatrix} \frac{\partial\mathcal{X}_{P,e}}{\partial\mathbf{q}} \\ \frac{\partial\mathcal{X}_{R,\mathcal{E}}}{\partial\mathbf{q}} \end{bmatrix}$$

- Example: planar (2D) robot arm from earlier

$$\mathbf{J}_{A_P,e}(\mathbf{q}) = \begin{bmatrix} l_1\cos_1 + l_2\cos_{12} + l_3\cos_{123} & l_2\cos_{12} + l_3\cos_{123} & l_3\cos_{123} \\ -l_1\sin_1 - l_2\sin_{12} - l_3\sin_{123} & -l_2\sin_{12} - l_3\sin_{123} & -l_3\sin_{123} \end{bmatrix}$$

$$\mathbf{J}_{A_R,e}(\mathbf{q}) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

GEOMETRIC JACOBIAN

- Relates end-effector twist $_\mathcal{A}\mathbf{w}_e$ to generalized velocities $\dot{\mathbf{q}}$
- Independent of parameterization

$$\boxed{\begin{aligned} _\mathcal{A}\mathbf{w}_e &= {_\mathcal{A}}\mathbf{J}_{0,e}(\mathbf{q})\dot{\mathbf{q}} \\ _\mathcal{A}\mathbf{w}_e &= \begin{bmatrix} _\mathcal{A}\dot{\mathbf{r}}_{AE} \\ _\mathcal{A}\boldsymbol{\omega}_{\mathcal{AE}} \end{bmatrix} \in \mathbb{R}^6 \\ _\mathcal{A}\mathbf{J}_{0,e}(\mathbf{q}) &\in \mathbb{R}^{6\times n_q} \end{aligned}}$$

- Unique for every robot
- More common than analytical Jacobian
- Algebra:

$$_\mathcal{A}\mathbf{w}_C = {_\mathcal{A}}\mathbf{w}_B + {_\mathcal{A}}\mathbf{w}_{BC}$$
$$_\mathcal{A}\mathbf{J}_{0,C} = {_\mathcal{A}}\mathbf{J}_{0,B} + {_\mathcal{A}}\mathbf{J}_{0,BC}$$

where $_\mathcal{A}\mathbf{J}_{0,BC}$ is the relative Jacobian from point B to C

- Conversion between geometric and analytical Jacobian

$$_\mathcal{A}\mathbf{w}_e = \mathbf{E}_e(\mathcal{X}_e)\dot{\mathcal{X}}_e$$
$$\mathbf{E}_e(\mathcal{X}_e) = \begin{bmatrix} \mathbf{E}_{P,e}(\mathcal{X}_{P,E}) \\ \mathbf{E}_{R,e}(\mathcal{X}_{R,\mathcal{E}}) \end{bmatrix}$$

$$\boxed{_\mathcal{A}\mathbf{J}_{0,e} = \mathbf{E}_e(\mathcal{X}_e)\mathbf{J}_{A,e}(\mathbf{q})}$$

- Change of base

$$_\mathcal{I}\mathbf{J}_0 = \mathbf{C}_{\mathcal{IA}} \, {_\mathcal{A}}\mathbf{J}_0 \, \mathbf{C}_{\mathcal{IA}}^\top$$

GEOMETRIC JACOBIAN 2D EXAMPLE

- 3-link planar arm, in xy plane, end-effector parameterized with Cartesian coordinates

$$\boxed{_\mathcal{I}\mathbf{J}_{0,e} = \mathbf{J}_{A,e}, \quad \mathbf{E}_e = \mathbb{I}, \quad \dot{\mathcal{X}}_e = {_\mathcal{I}}\mathbf{w}_e}$$

1. Introduce coordinate frames $0 - 3$ (inertial frame $0 \equiv \mathcal{I}$)
2. Introduce generalized coordinates q_{1-3}
3. Determine end-effector position

$$_\mathcal{I}\mathbf{r}_{\mathcal{IE}}(\mathbf{q}) = \begin{bmatrix} l_0 + l_1 c_1 + l_2 c_{12} + l_3 c_{123} \\ l_1 s_1 + l_2 s_{12} + l_3 s_{123} \\ 0 \end{bmatrix}$$

4. Compute Jacobian

$$_\mathcal{I}\mathbf{J}_{0_P,e} = \mathbf{J}_{A_P,e} = \frac{\partial}{\partial\mathbf{q}}{_\mathcal{I}}\mathbf{r}_{\mathcal{IE}}(\mathbf{q}) = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} - l_3 s_{123} & -l_2 s_{12} - l_3 s_{123} & -l_3 s_{123} \\ l_1 c_1 + l_2 c_{12} + l_3 c_{123} & l_2 c_{12} + l_3 c_{123} & l_3 c_{123} \\ 0 & 0 & 0 \end{bmatrix}$$

GEOMETRIC JACOBIAN DERIVATION

- Rigid body formulation at a link (body) k (frame also denoted using index k)

$$_\mathcal{I}\dot{\mathbf{r}}_{Ik} = {_\mathcal{I}}\dot{\mathbf{r}}_{I(k-1)} + {_\mathcal{I}}\boldsymbol{\omega}_{\mathcal{I}(k-1)} \times {_\mathcal{I}}\mathbf{r}_{(k-1)k}$$
$$\iff \mathbf{v}_k = \mathbf{v}_{(k-1)} + \boldsymbol{\Omega}_{(k-1)} \times {_\mathcal{I}}\mathbf{r}_{(k-1)k}$$

- Apply this to all links up to end-effector body $\mathcal{E}$ at index $k = n + 1$ with origin E, using the property that the base link at index $k = 0$ is fixed i.e. $\mathbf{v}_0 = \mathbf{0}$

$$_\mathcal{I}\dot{\mathbf{r}}_{IE} = \sum_{k=1}^n {_\mathcal{I}}\boldsymbol{\omega}_{\mathcal{I}k} \times {_\mathcal{I}}\mathbf{r}_{k(k+1)}$$
$$\iff \mathbf{v}_E = \sum_{k=1}^n \boldsymbol{\Omega}_k \times {_\mathcal{I}}\mathbf{r}_{k(k+1)}$$

- Angular velocity propagation

$$_\mathcal{I}\boldsymbol{\omega}_{\mathcal{I}k} = {_\mathcal{I}}\boldsymbol{\omega}_{\mathcal{I}(k-1)} + {_\mathcal{I}}\boldsymbol{\omega}_{(k-1)k}$$

with $_\mathcal{I}\boldsymbol{\omega}_{(k-1)k} = {_\mathcal{I}}\mathbf{n}_k\dot{q}_k$, where q_k is the (1 DOF) joint angle (normal direction $_\mathcal{I}\mathbf{n}_k$) of link k w.r.t. link $k-1$

$$\implies {_\mathcal{I}}\boldsymbol{\omega}_{\mathcal{I}k} = \sum_{i=1}^k {_\mathcal{I}}\mathbf{n}_i\dot{q}_i$$

- Write in matrix form to get geometric rotation Jacobian

$$_\mathcal{I}\boldsymbol{\omega}_{\mathcal{IE}} = \sum_{i=1}^n {_\mathcal{I}}\mathbf{n}_i\dot{q}_i = \underbrace{\begin{bmatrix} _\mathcal{I}\mathbf{n}_1 & _\mathcal{I}\mathbf{n}_2 & \cdots & _\mathcal{I}\mathbf{n}_n \end{bmatrix}}_{_\mathcal{I}J_{0R,e}} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

- End-effector velocity

$$\begin{aligned} _\mathcal{I}\dot{\mathbf{r}}_{IE} &= \sum_{k=1}^n \left(\sum_{i=1}^k ({_\mathcal{I}}\mathbf{n}_i\dot{q}_i) \times {_\mathcal{I}}\mathbf{r}_{k(k+1)} \right) \\ &= \sum_{k=1}^n {_\mathcal{I}}\mathbf{n}_k\dot{q}_k \times \sum_{i=k}^n {_\mathcal{I}}\mathbf{r}_{i(i+1)} \\ &= \sum_{k=1}^n {_\mathcal{I}}\mathbf{n}_k\dot{q}_k \times {_\mathcal{I}}\mathbf{r}_{k(n+1)} \end{aligned}$$

- Write in matrix form to get geometric position Jacobian

$$_\mathcal{I}\dot{\mathbf{r}}_{IE} = \sum_{k=1}^n {_\mathcal{I}}\mathbf{n}_k\dot{q}_k \times {_\mathcal{I}}\mathbf{r}_{k(n+1)} = \underbrace{\begin{bmatrix} _\mathcal{I}\mathbf{n}_1 \times {_\mathcal{I}}\mathbf{r}_{1(n+1)} & \cdots & _\mathcal{I}\mathbf{n}_n \times {_\mathcal{I}}\mathbf{r}_{n(n+1)} \end{bmatrix}}_{_\mathcal{I}J_{0_P,e}} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

Note: $_\mathcal{I}\mathbf{n}_k = \mathbf{C}_{\mathcal{I}(k-1)\,(k-1)}\mathbf{n}_k$

- Concatenate into the full geometric Jacobian

$$_\mathcal{I}\mathbf{J}_{0,e} = \begin{bmatrix} _\mathcal{I}\mathbf{J}_{0_P,e} \\ _\mathcal{I}\mathbf{J}_{0_R,e} \end{bmatrix}$$

STEPS TO GET GEOMETRIC JACOBIAN

1. Determine the passive rotation matrices $\mathbf{C}_{\mathcal{I}k}$ between all links ($k = 1, \ldots, n$) using concatenations of elementary rotations, recalling that link/frame $k = 0 \equiv \mathcal{I}$

$$\mathbf{C}_{\mathcal{I}k} = \prod_{i=1}^k \mathbf{C}_{(i-1)i}, \; k = 0, \ldots, n$$

2. Determine the rotation axes $_\mathcal{I}\mathbf{n}_k$ ($k = 1, \ldots, n$)
   (a) In local frame i.e. $_{(k-1)}\mathbf{n}_k$ ($k = 1, \ldots, n$)
   (b) In inertial frame using transformations from step 1
   $$_\mathcal{I}\mathbf{n}_k = \mathbf{C}_{\mathcal{I}(k-1)\,(k-1)}\mathbf{n}_k, \; k = 1, \ldots, n$$
3. Determine the end-effector position vectors $_\mathcal{I}\mathbf{r}_{k(n+1)}$
   (a) Determine the position vectors between adjacent frames $_k\mathbf{r}_{k(k+1)}$ ($k = 1, \ldots, n$)
   (b) Transform the vectors from (a) into the inertial frame
   $$_\mathcal{I}\mathbf{r}_{k(k+1)} = \mathbf{C}_{\mathcal{I}k}\,{_k}\mathbf{r}_{k(k+1)}, \; k = 1, \ldots, n$$
   (c) Add the vectors from (b) to get
   $$_\mathcal{I}\mathbf{r}_{k(n+1)} = \sum_{i=k}^n {_\mathcal{I}}\mathbf{r}_{i(i+1)}, \; k = 1, \ldots, n$$
4. Determine $_\mathcal{I}\mathbf{J}_{0_P,e}$ and $_\mathcal{I}\mathbf{J}_{0_R,e}$ with the matrix definitions

10 KINEMATIC CONTROL

INVERSE KINEMATICS

- Inverse kinematics: generalized coordinates $\mathbf{q}$ as a function of end-effector configuration $\mathcal{X}_e$

$$\mathbf{q} = \mathbf{q}(\mathcal{X}_e)$$

- Inverse differential kinematics
  - Jacobians map velocities from joint- to task-space
    * Geometric ($\dot{\mathbf{q}} \mapsto {_\mathcal{I}}\mathbf{w}_e$): $_\mathcal{I}\mathbf{J}_{0,e} \in \mathbb{R}^{\dim(_\mathcal{I}\mathbf{w}_e)\times\dim(\mathbf{q})}$
    * Analytical ($\dot{\mathbf{q}} \mapsto \dot{\mathcal{X}}_e$): $\mathbf{J}_{A,e} \in \mathbb{R}^{\dim(\mathcal{X}_e)\times\dim(\mathbf{q})}$
  - (Moore-Penrose) pseudo-inverse of Jacobian for inverse mapping

$$\boxed{\dot{\mathbf{q}} = {_\mathcal{I}}\mathbf{J}_{0,e}^\dagger \, {_\mathcal{I}}\mathbf{w}_e^*}$$

Note: $\mathbf{J}^\dagger = \mathbf{J}^\top(\mathbf{J}\mathbf{J}^\top)^{-1} = (\mathbf{J}^\top\mathbf{J})^{-1}\mathbf{J}$

MATRIX INVERSION

- **Square** matrices
  - $n \times n$ matrix $\mathbf{A}$ is invertible/nonsingular if $\exists \, n \times n$ matrix $\mathbf{B}$ s.t. $\mathbf{AB} = \mathbf{BA} = \mathbb{I}_{n\times n}$ or simply $\det(\mathbf{A}) \neq 0$
- **Non-square** matrices (generalization of the inverse): Moore-Penrose pseudo inverse (most common)
  - Generalized inverse must fulfill condition $ABA = \mathbf{A}$
    * In general, $\mathbf{B}$ is not unique
    * Moore-Penrose addresses this non-uniqueness by enforcing three additional conditions ($\mathbf{A} \in \mathbb{R}^{m\times n}$)
    $$\mathbf{A}^\dagger\mathbf{A}\mathbf{A}^\dagger = \mathbf{A}^\dagger$$
    $$(\mathbf{A}^\dagger\mathbf{A})^\top = \mathbf{A}^\dagger\mathbf{A}$$
    $$(\mathbf{A}\mathbf{A}^\dagger)^\top = \mathbf{A}\mathbf{A}^\dagger$$
  - **Tall** matrices: $m \geq n$
    * Full column rank; row rank deficient
    * Inverse of $\mathbf{A}^\top\mathbf{A}$ is defined
    $$\mathbf{A}^\dagger = (\mathbf{A}^\top\mathbf{A})^{-1}\mathbf{A}^\top$$
    where $\mathbf{A}^\dagger$ is "left inverse" of $\mathbf{A}$ i.e. $\mathbf{A}^\dagger\mathbf{A} = \mathbb{I}$
    * For linear set of equations $\mathbf{Ax} = \mathbf{b}$ there are more equations than variables, therefore no exact solution
      · $\mathbf{A}^\dagger\mathbf{b}$ minimizes the least squares error $||\mathbf{Ax} - \mathbf{b}||_2^2$
  - **Wide** matrices: $m \leq n$
    * Full row rank; column rank deficient
    * Inverse of $\mathbf{A}\mathbf{A}^\top$ is defined
    $$\mathbf{A}^\dagger = \mathbf{A}^\top(\mathbf{A}\mathbf{A}^\top)^{-1}$$
    where $\mathbf{A}^\dagger$ is "right inverse" of $\mathbf{A}$ i.e. $\mathbf{A}\mathbf{A}^\dagger = \mathbb{I}$
    * For linear set of equations $\mathbf{Ax} = \mathbf{b}$ there are more variables than equations, therefore multiple solutions
    $$\mathbf{x} = \mathbf{A}^\dagger\mathbf{b} + \mathcal{N}(\mathbf{A})\mathbf{x}_0, \; \mathcal{N}(\mathbf{A}) = \mathbf{N}_A = \mathbb{I} - \mathbf{A}^\dagger\mathbf{A}, \; \mathbf{A}\mathbf{N}_A = 0$$
      · $\mathbf{x} = \mathbf{A}^\dagger\mathbf{b}$ minimizes $||\mathbf{x}||_2$ while ensuring $\mathbf{Ax} = \mathbf{b}$
- Rank of a matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$
  - $rank(\mathbf{A})$ is largest number of columns of $\mathbf{A}$ that constitute a linearly independent set (this set is not necessarily unique but the cardinality i.e. rank is)
  - Row rank equals column rank: $rank(\mathbf{A}) = rank(\mathbf{A}^\top)$
  - Full rank: $rank(\mathbf{A}) = \min(m, n)$
  - Column-/row rank deficient: rows/columns are not linearly independent

SINGULARITIES AND REDUNDANCY

- Singularity: joint-space configuration $\mathbf{q}_s$ such that $_\mathcal{I}\mathbf{J}_{0,e}(\mathbf{q}_s) =: \mathbf{J}$ is column-rank deficient
  - Boundary singularity: e.g. manipulator stretched out; prevented in motion planning
  - Internal singularity
  - Jacobian is poorly conditioned
  - Small desired twist $_\mathcal{I}\mathbf{w}_e^*$ produce high $\dot{\mathbf{q}}$
- Damped version of Moore-Penrose pseudo inverse
$$\dot{\mathbf{q}} = \mathbf{J}^\top(\mathbf{J}\mathbf{J}^\top + \lambda^2\mathbb{I})^{-1}\,{_\mathcal{I}}\mathbf{w}_e^*, \quad \lambda \in \mathbb{R} > 0$$
$$\iff \arg\min_{\dot{\mathbf{q}}} ||{_\mathcal{I}}\mathbf{w}_e^* - \mathbf{J}\dot{\mathbf{q}}||^2 + \lambda^2||\dot{\mathbf{q}}||^2$$

- Redundancy: task-space has lower dimension than the joint-space i.e. $\dim(_\mathcal{I}\mathbf{w}_e) < \dim(\mathbf{q})$
  - Redundancy implies infinite solutions
  $$\dot{\mathbf{q}} = \mathbf{J}^\dagger \, {_\mathcal{I}}\mathbf{w}_e^* + \mathbf{N}\dot{\mathbf{q}}_0$$
  where $\mathbf{N} = \mathcal{N}(\mathbf{J})$ is the null-space ($\mathcal{N}$) projection for which it holds that
  $$\mathbf{JN} = \mathbf{0}$$
- Null-space allows joint-space motion without changing task-space motion:
$$\mathbf{J}\dot{\mathbf{q}} = \mathbf{J}\left(\mathbf{J}^\dagger \, {_\mathcal{I}}\mathbf{w}_e^* + \mathbf{N}\dot{\mathbf{q}}_0\right) = {_\mathcal{I}}\mathbf{w}_e^*$$
  - Null-space projection matrix computation
  $$\mathbf{N} = \mathbb{I} - \mathbf{J}^\dagger\mathbf{J}$$

MULTI-TASK CONTROL

- Manipulation/locomotion (of redundant systems) as combination of high level tasks
$$\text{task}_i := \{\mathbf{J}_i, \mathbf{w}_i^*\}$$

- Multi-task with equal priority
  - Stack n_t tasks and use pseudo-inverse
$$\dot{\mathbf{q}} = \underbrace{\begin{bmatrix} \mathbf{J}_1 \\ \vdots \\ \mathbf{J}_{n_t} \end{bmatrix}^\dagger}_{\bar{\mathbf{J}}} \underbrace{\begin{bmatrix} \mathbf{w}_1^* \\ \vdots \\ \mathbf{w}_{n_t}^* \end{bmatrix}}_{\bar{\mathbf{w}}}$$
$$= \arg\min_q ||\bar{\mathbf{J}}\mathbf{q} - \bar{\mathbf{w}}||_2 = \arg\min_q ||\dot{\mathbf{q}}||_2 \text{ subject to } \bar{\mathbf{J}}\dot{\mathbf{q}} = \bar{\mathbf{w}}$$
  - Weigh some tasks higher than others with weight matrix $\mathbf{W} = \text{diag}(w_1, \dots, w_m)$ changing $\bar{\mathbf{J}}$ to $\bar{\mathbf{J}}^W$
$$\bar{\mathbf{J}}^{W\dagger} = \left(\bar{\mathbf{J}}^\top \mathbf{W} \bar{\mathbf{J}} \right)^{-1} \bar{\mathbf{J}}^\top \mathbf{W}$$

- Multi-task with prioritization (ensuring strict priority i.e. preceding task satisfied before attempting to satisfy next)
  - Task 1 (highest priority)
$$\dot{\mathbf{q}} = \mathbf{J}_1^\dagger \mathbf{w}_1^* + \mathbf{N}_1 \dot{\mathbf{q}}_0$$
  - Task 2 (solution mustn't violate task 1)
$$\mathbf{w}_2 = \mathbf{J}_2 \dot{\mathbf{q}} = \mathbf{J}_2 (\mathbf{J}_1^\dagger \mathbf{w}_1^* + \mathbf{N}_1 \dot{\mathbf{q}}_0)$$
    * Solve for $\dot{\mathbf{q}}_0$
$$\dot{\mathbf{q}}_0 = (\mathbf{J}_2 \mathbf{N}_1)^\dagger (\mathbf{w}_2^* - \mathbf{J}_2 \mathbf{J}_1^\dagger \mathbf{w}_1^*)$$
    * New solution for task 1 (without violating it)
$$\dot{\mathbf{q}} = \mathbf{J}_1^\dagger \mathbf{w}_1^* + \mathbf{N}_1 (\mathbf{J}_2 \mathbf{N}_1)^\dagger (\mathbf{w}_2^* - \mathbf{J}_2 \mathbf{J}_1^\dagger \mathbf{w}_1^*)$$
  - Generalization for n_t tasks
$$\dot{\mathbf{q}} = \sum_{i=1}^{n_t} \bar{\mathbf{N}}_{i-1} \dot{\mathbf{q}}_i, \quad \dot{\mathbf{q}}_i = (\mathbf{J}_i \bar{\mathbf{N}}_{i-1})^\dagger \left(\mathbf{w}_i^* - \mathbf{J}_i \sum_{k=1}^{i-1} \bar{\mathbf{N}}_{k-1} \dot{\mathbf{q}}_k \right)$$
  where $\bar{\mathbf{N}}_i$ is the null-space projection of the stacked Jacobian $\bar{\mathbf{J}}_i = \begin{bmatrix} \mathbf{J}_1^\top & \dots & \mathbf{J}_{i-1}^\top \end{bmatrix}^\top$

ITERATIVE INVERSE DIFFERENTIAL KINEMATICS

- Iterative solution to the inverse kinematics problem with target configuration $\mathcal{X}_e^*$ and initial joint space guess $\mathbf{q}^0$

  $q \leftarrow q^0$ ▷ Start configuration
  while $||\mathcal{X}_e^* - \mathcal{X}_e(q)|| \geq \text{tol}$ **do** ▷ While solution is not reached
     $J_{A,e} \leftarrow J_{A,e}(q) = \frac{\partial \mathcal{X}_e}{\partial q}(q)$ ▷ Evaluate Jacobian
     $J_{A,e}^\dagger \leftarrow (J_{A,e}(q))^\dagger$ ▷ Compute pseudo-inverse
     $\Delta \mathcal{X}_e \leftarrow \mathcal{X}_e^* - \mathcal{X}_e(q)$ ▷ Compute configuration error
     $q \leftarrow q + J_{A,e}^\dagger \Delta \mathcal{X}_e$ ▷ Update generalized coordinates
  end while

  - Use scaling factor $k \in (0,1)$ to prevent overshoot (slows convergence)
$$\mathbf{q} \leftarrow \mathbf{q} + k \mathbf{J}_{A,e}^\dagger \Delta \mathcal{X}_e$$
  - Use damped pseudo-inverse (Levenberg-Marquardt) or transpose to prevent problems due to ill-conditioned inversion problem in singular configurations
$$\mathbf{q} \leftarrow \mathbf{q} + \mathbf{J}_{A,e}^\top (\mathbf{J}_{A,e} \mathbf{J}_{A,e}^\top + \lambda^2 \mathbb{I})^{-1} \Delta \mathcal{X}_e$$
$$\text{or} \quad \mathbf{q} \leftarrow \mathbf{q} + \alpha \mathbf{J}_{A,e}^\top \Delta \mathcal{X}_e$$

ROTATION ERROR

- $\mathbf{J}_{A,e}$ depends on parameterization and thus affects convergence from start to target configuration
- Rotate along shortest path in SO(3) using rotation vector
$$\Delta \mathcal{X}_e = \mathcal{X}_{R,\text{RotVec}} = \theta \mathbf{n} =: \Delta \boldsymbol{\varphi}$$
- Extract rotation vector $_{\mathcal{I}}\Delta\boldsymbol{\varphi}$ from relative rotation between the current orientation $\boldsymbol{\varphi}_k$ (frame $\mathcal{K}$) and target orientation $\boldsymbol{\varphi}^*$ (frame $\mathcal{T}$)
$$\mathbf{C}_{\mathcal{KT}}(\Delta\boldsymbol{\varphi}) = \mathbf{C}_{\mathcal{IK}}^\top(\boldsymbol{\varphi}_k) \mathbf{C}_{\mathcal{IT}}(\boldsymbol{\varphi}^*)$$
$$_{\mathcal{K}}\Delta\boldsymbol{\varphi} = {_{\mathcal{T}}}\Delta\boldsymbol{\varphi} = \text{RotVec}(\mathbf{C}_{\mathcal{KT}})$$
$$_{\mathcal{I}}\Delta\boldsymbol{\varphi} = \mathbf{C}_{\mathcal{IK}} {_{\mathcal{K}}}\Delta\boldsymbol{\varphi} = \mathbf{C}_{\mathcal{IK}} \text{RotVec}(\mathbf{C}_{\mathcal{KT}})$$
$$= \text{RotVec}(\mathbf{C}_{\mathcal{IK}} \mathbf{C}_{\mathcal{KT}} \mathbf{C}_{\mathcal{IK}}^\top)$$
$$= \text{RotVec}(\mathbf{C}_{\mathcal{IT}} \mathbf{C}_{\mathcal{IK}}^\top)$$
- Given small rotation offsets, rotating along the shortest path allows using the geometric Jacobian
$$\mathbf{q} \leftarrow \mathbf{q} + k_{p_R} {_{\mathcal{I}}}\mathbf{J}_{0_R,e}^\dagger {_{\mathcal{I}}}\Delta\boldsymbol{\varphi}$$

KINEMATIC TRAJECTORY CONTROL

- Position
  - End-effector desired motion $_{\mathcal{I}}\mathbf{r}_{IE}^*(t)$ and $_{\mathcal{I}}\dot{\mathbf{r}}_{IE}^*(t)$
  - Tracking error $\Delta_{\mathcal{I}}\mathbf{r}_{IE}(t) = {_{\mathcal{I}}}\mathbf{r}_{IE}^*(t) - {_{\mathcal{I}}}\mathbf{r}_{IE}(t)$
  - Nonlinear stabilizing control law
$$\dot{\mathbf{q}}^* = {_{\mathcal{I}}}\mathbf{J}_{0_P,e}^\dagger ({_{\mathcal{I}}}\dot{\mathbf{r}}_{IE}^*(t) + k_{p_P}\Delta_{\mathcal{I}}\mathbf{r}_{IE}(t))$$
    * Desired velocity used as feedforward
    * P controller for feedback
- Orientation
  - Nonlinear stabilizing control law
$$\dot{\mathbf{q}}^* = {_{\mathcal{I}}}\mathbf{J}_{0_R,e}^\dagger ({_{\mathcal{I}}}\boldsymbol{\omega}_{\mathcal{IE}}^*(t) + k_{p_R} {_{\mathcal{I}}}\Delta\boldsymbol{\varphi})$$

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11 DYNAMICS

DYNAMICS

- Description of the cause of motion
- Input: torque/force τ acting on system
- Output: motion $\ddot{\mathbf{q}}$ of system
- Methods deriving equations of motion (EOM) based on principle of virtual work
  - Newton-Euler
  - Projected Newton-Euler
  - Lagrange II

NOMENCLATURE

dm	Infinitesimal mass element on body $\mathcal{B}$
M	Point of particle dm
S	Center of gravity (COG) and origin of $\mathcal{B}$
$\mathcal{B}$	Body containing particles dm
$d\mathbf{F}_{\text{ext}}$	Resultant external force acting on dm
δW	Virtual work
$\delta(\cdot)$	Virtual quantity
$\mathbf{v}_M := {_{\mathcal{I}}}\dot{\mathbf{r}}_{IM}$	Absolute velocity of particle dm
$\mathbf{a}_M := {_{\mathcal{I}}}\ddot{\mathbf{r}}_{IM}$	Absolute acceleration of particle dm
$\boldsymbol{\rho} := {_{\mathcal{I}}}\mathbf{r}_{SM}$	Vector from S to a particle dm
$\mathbf{F}_{\text{ext}}$	Resultant external force on S
$\mathbf{T}_{\text{ext}}$	Resultant external torque on $\mathcal{B}$ w.r.t S
$\mathbf{p}_S$	Linear momentum/impulse of S
$\mathbf{N}_S$	Angular momentum of S
$\mathbf{v}_S := {_{\mathcal{I}}}\dot{\mathbf{r}}_{IS}$	Absolute velocity of S
$\mathbf{a}_S := {_{\mathcal{I}}}\ddot{\mathbf{r}}_{IS}$	Absolute acceleration of S
$\mathbf{I}_S$	Inertial tensor of $\mathcal{B}$ w.r.t. S
$\boldsymbol{\Omega}_{\mathcal{B}} := {_{\mathcal{I}}}\ddot{\mathbf{r}}_{IS}$	Absolute angular velocity of $\mathcal{B}$
$\boldsymbol{\Psi}_{\mathcal{B}} := \dot{\boldsymbol{\Omega}}_{\mathcal{B}}$	Absolute angular acceleration of $\mathcal{B}$
n_q	Number of generalized coordinates
n_c	Number of contact forces
n_b	Number of bodies in multi-body system
$\mathbf{M}(\mathbf{q}) \in \mathbb{R}^{n_q \times n_q}$	Generalized mass matrix
$\mathbf{b}(\mathbf{q}, \dot{\mathbf{q}}) \in \mathbb{R}^{n_q}$	Coriolis and centrifugal terms
$\mathbf{g}(\mathbf{q}) \in \mathbb{R}^{n_q}$	Gravitational terms
$\boldsymbol{\tau} \in \mathbb{R}^{n_q}$	Forces/torques acting in direction of generalized coordinates (all $\mathbf{q}$ assumed actuated)
$\mathbf{F}_{\text{ext}} \in \mathbb{R}^{3n_c}$	External forces acting on system (e.g. contacts)
$\mathbf{J}_{\text{ext}}(\mathbf{q}) \in \mathbb{R}^{3n_c \times n_q}$	Position Jacobian of external forces
$\mathcal{L}$	Lagrangian
$\mathcal{T}$	Kinetic energy
$\mathcal{U}$	Potential energy

BODY PROPERTIES

- Body mass
$$m := \int_{\mathcal{B}} dm$$
- Center of mass/gravity
$$\mathbf{0} := \int_{\mathcal{B}} \boldsymbol{\rho} \, dm$$
- Inertia matrix/inertial tensor around COG
$$\mathbf{I}_S := \int_{\mathcal{B}} -[\boldsymbol{\rho}]_\times^2 \, dm = \int_{\mathcal{B}} [\boldsymbol{\rho}]_\times [\boldsymbol{\rho}]_\times^\top \, dm$$

PRINCIPLE OF VIRTUAL WORK (D'ALEMBERT)

- Dynamic equilibrium imposes zero virtual work
$$\delta W = \int_{\mathcal{B}} \delta_{\mathcal{I}}\mathbf{r}_{IM}^\top \cdot (_{\mathcal{I}}\ddot{\mathbf{r}}_{IM} dm - d\mathbf{F}_{\text{ext}}) = 0$$

⚠ Looking at a single rigid body
- Variational notation with δ describes, for a fixed instance in time, all possible directions the quantity may move while satisfying applicable constraints
- Quantities
  - Absolute velocity
$$_{\mathcal{I}}\mathbf{r}_{IM} = {_{\mathcal{I}}}\mathbf{r}_{IS} + {_{\mathcal{I}}}\mathbf{r}_{SM}$$
$$_{\mathcal{I}}\dot{\mathbf{r}}_{IM} = {_{\mathcal{I}}}\dot{\mathbf{r}}_{IS} + {_{\mathcal{I}}}\boldsymbol{\omega}_{\mathcal{IB}} \times {_{\mathcal{I}}}\mathbf{r}_{SM}$$
$$\iff \mathbf{v}_M = \mathbf{v}_S + \boldsymbol{\Omega}_{\mathcal{B}} \times \boldsymbol{\rho}$$
$$= \begin{bmatrix} \mathbb{I}_{3\times3} & -[\boldsymbol{\rho}]_\times \end{bmatrix} \begin{bmatrix} \mathbf{v}_S \\ \boldsymbol{\Omega}_{\mathcal{B}} \end{bmatrix}$$
  - Absolute acceleration
$$_{\mathcal{I}}\ddot{\mathbf{r}}_{IM} = \mathbf{a}_M = \mathbf{a}_S + \boldsymbol{\Psi}_{\mathcal{B}} \times \boldsymbol{\rho} + \boldsymbol{\Omega}_{\mathcal{B}} \times (\boldsymbol{\Omega}_{\mathcal{B}} \times \boldsymbol{\rho})$$
$$= \begin{bmatrix} \mathbb{I}_{3\times3} & -[\boldsymbol{\rho}]_\times \end{bmatrix} \begin{bmatrix} \mathbf{a}_S \\ \boldsymbol{\Psi}_{\mathcal{B}} \end{bmatrix} + [\boldsymbol{\Omega}_{\mathcal{B}}]_\times [\boldsymbol{\Omega}_{\mathcal{B}}]_\times \boldsymbol{\rho}$$
  - Virtual displacement
$$\delta_{\mathcal{I}}\mathbf{r}_{IM} = \delta_{\mathcal{I}}\mathbf{r}_{IS} + \delta_{\mathcal{I}}\boldsymbol{\omega}_{\mathcal{IB}} \times \boldsymbol{\rho} = \begin{bmatrix} \mathbb{I}_{3\times3} & -[\boldsymbol{\rho}]_\times \end{bmatrix} \begin{bmatrix} \delta_{\mathcal{I}}\mathbf{r}_{IS} \\ \delta_{\mathcal{I}}\boldsymbol{\omega}_{\mathcal{IB}} \end{bmatrix}$$

- Plug expressions above into D'Alembert's Principle and use body properties as well as $\int_{\mathcal{B}} [\boldsymbol{\rho}]_\times d\mathbf{F}_{\text{ext}} = \mathbf{T}_{\text{ext}}$, etc.
$$\delta W = \begin{bmatrix} \delta_{\mathcal{I}}\mathbf{r}_{IS} \\ \delta_{\mathcal{I}}\boldsymbol{\omega}_{\mathcal{IB}} \end{bmatrix}^\top \left(\begin{bmatrix} \mathbb{I}_{3\times3}m & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_S \end{bmatrix} \begin{bmatrix} \mathbf{a}_S \\ \boldsymbol{\Psi}_{\mathcal{B}} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ [\boldsymbol{\Omega}_{\mathcal{B}}]_\times \mathbf{I}_S\boldsymbol{\Omega}_{\mathcal{B}} \end{bmatrix} - \begin{bmatrix} \mathbf{F}_{\text{ext}} \\ \mathbf{T}_{\text{ext}} \end{bmatrix} \right) = 0$$

- Use the definitions
$$\mathbf{p}_S = m\mathbf{v}_S$$
$$\mathbf{N}_S = \mathbf{I}_S\boldsymbol{\Omega}_{\mathcal{B}}$$
$$\dot{\mathbf{p}}_S = m\mathbf{a}_S$$
$$\dot{\mathbf{N}}_S = \mathbf{I}_S\boldsymbol{\Psi}_{\mathcal{B}} + \boldsymbol{\Omega}_{\mathcal{B}} \times \mathbf{I}_S\boldsymbol{\Omega}_{\mathcal{B}}$$

$$\begin{bmatrix} \delta_{\mathcal{I}}\mathbf{r}_{IS} \\ \delta_{\mathcal{I}}\boldsymbol{\omega}_{\mathcal{IB}} \end{bmatrix}^\top \left(\begin{bmatrix} \dot{\mathbf{p}}_S \\ \dot{\mathbf{N}}_S \end{bmatrix} - \begin{bmatrix} \mathbf{F}_{\text{ext}} \\ \mathbf{T}_{\text{ext}} \end{bmatrix} \right) = 0 \quad \forall \begin{bmatrix} \delta_{\mathcal{I}}\mathbf{r}_{IS} \\ \delta_{\mathcal{I}}\boldsymbol{\omega}_{\mathcal{IB}} \end{bmatrix}_{\text{consistent}}$$

- For a single, free, rigid rigid body (can move in all directions)
$$\boxed{\begin{aligned} \dot{\mathbf{p}}_S &= \mathbf{F}_{\text{ext}} \\ \dot{\mathbf{N}}_S &= \mathbf{T}_{\text{ext}} \end{aligned}}$$

EULER'S LAWS OF MOTION

- Conservation of linear and angular momentum
$$\boxed{\begin{aligned} \dot{\mathbf{p}}_S &= {}^d/_{dt}(m\mathbf{v}_S) = m\mathbf{a}_S = \mathbf{F}_{\text{ext}} \\ \dot{\mathbf{N}}_S &= {}^d/_{dt}(\mathbf{I}_S\boldsymbol{\Omega}_{\mathcal{B}}) = \mathbf{I}_S\boldsymbol{\Psi}_{\mathcal{B}} + \boldsymbol{\Omega}_{\mathcal{B}} \times \mathbf{I}_S\boldsymbol{\Omega}_{\mathcal{B}} = \mathbf{T}_{\text{ext}} \end{aligned}}$$

⚠ Only valid for free rigid bodies (and w.r.t. inertial frame)

FIXED BASE DYNAMICS

$$\boxed{\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{b}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{g}(\mathbf{q}) = \boldsymbol{\tau} + \mathbf{J}_{\text{ext}}(\mathbf{q})^\top \mathbf{F}_{\text{ext}}}$$

NEWTON-EULER METHOD

- Pros and cons
  ⊕ Intuitively clear and direct access to constraining forces
  ⊖ Huge combinatorial problem with many bodies
- Idea
  1. Free body diagrams of all bodies
  2. Introduce constraining force at body interfaces
  3. Apply Euler's laws of motion to individual bodies
  4. Eliminate the constraining forces
- Example: 3D manipulator with n_j 1-DOF joints
  - n_j moving links with 1 DOF $\implies$ n_j generalized coordinates and $5n_j$ constraining forces/torques

PROJECTED NEWTON-EULER METHOD

- Apply principle of virtual work to a multi-body system (body frames $\mathcal{B}_i$ with COGs S_i, $i = 1, \dots, n_b$)

$$\sum_{i=1}^{n_b} \begin{bmatrix} \delta_{\mathcal{I}} \mathbf{r}_{IS_i} \\ \delta_{\mathcal{I}} \boldsymbol{\omega}_{\mathcal{IB}_i} \end{bmatrix}^\top \left(\begin{bmatrix} \dot{\mathbf{p}}_{S_i} \\ \mathbf{N}_{S_i} \end{bmatrix} - \begin{bmatrix} \mathbf{F}_{\text{ext},i} \\ \mathbf{T}_{\text{ext},i} \end{bmatrix} \right) = 0 \quad \forall \begin{bmatrix} \delta_{\mathcal{I}} \mathbf{r}_{IS_i} \\ \delta_{\mathcal{I}} \boldsymbol{\omega}_{\mathcal{IB}_i} \end{bmatrix}_{\text{consistent}}$$

- Express change of impulse and angular momentum in generalized coordinates
  - Twist and its time derivative

$$_{\mathcal{I}}\mathbf{w}_{S_i} = \begin{bmatrix} \mathbf{v}_{S_i} \\ \boldsymbol{\Omega}_{\mathcal{B}} \end{bmatrix} = \begin{bmatrix} _{\mathcal{I}}\mathbf{J}_{0_P,S_i} \\ _{\mathcal{I}}\mathbf{J}_{0_R,S_i} \end{bmatrix} \dot{\mathbf{q}}$$

$$_{\mathcal{I}}\dot{\mathbf{w}}_{S_i} = \begin{bmatrix} \mathbf{a}_{S_i} \\ \boldsymbol{\Psi}_{\mathcal{B}} \end{bmatrix} = \begin{bmatrix} _{\mathcal{I}}\mathbf{J}_{0_P,S_i} \\ _{\mathcal{I}}\mathbf{J}_{0_R,S_i} \end{bmatrix} \ddot{\mathbf{q}} + \begin{bmatrix} _{\mathcal{I}}\dot{\mathbf{J}}_{0_P,S_i} \\ _{\mathcal{I}}\dot{\mathbf{J}}_{0_R,S_i} \end{bmatrix} \dot{\mathbf{q}}$$

  where $_{\mathcal{I}}\mathbf{J}_{0,S_i}$ is the geometric Jacobian of S_i
  - Impulse and angular momentum time derivatives

$$\begin{bmatrix} \dot{\mathbf{p}}_{S_i} \\ \mathbf{N}_{S_i} \end{bmatrix} = \begin{bmatrix} m_i \mathbf{a}_{S_i} \\ \mathbf{I}_{S_i} \boldsymbol{\Psi}_{\mathcal{B}_i} + \boldsymbol{\Omega}_{\mathcal{B}_i} \times \mathbf{I}_{S_i} \boldsymbol{\Omega}_{\mathcal{B}_i} \end{bmatrix}$$

$$= \begin{bmatrix} m_i \, _{\mathcal{I}}\mathbf{J}_{0_P,S_i} \\ \mathbf{I}_{S_i} \, _{\mathcal{I}}\mathbf{J}_{0_R,S_i} \end{bmatrix} \ddot{\mathbf{q}} + \begin{bmatrix} m_i \, _{\mathcal{I}}\mathbf{J}_{0_P,S_i}\dot{\mathbf{q}} \\ \mathbf{I}_{S_i} \, _{\mathcal{I}}\dot{\mathbf{J}}_{0_R,S_i}\dot{\mathbf{q}} + \, _{\mathcal{I}}\mathbf{J}_{0_R,S_i}\dot{\mathbf{q}} \times \mathbf{I}_{S_i} \, _{\mathcal{I}}\mathbf{J}_{0_R,S_i}\dot{\mathbf{q}} \end{bmatrix}$$

- Express virtual displacements in generalized coordinates

$$\begin{bmatrix} \delta_{\mathcal{I}}\mathbf{r}_{IS_i} \\ \delta_{\mathcal{I}}\boldsymbol{\omega}_{\mathcal{IB}_i} \end{bmatrix} = \begin{bmatrix} _{\mathcal{I}}\mathbf{J}_{0_P,S_i} \\ _{\mathcal{I}}\mathbf{J}_{0_R,S_i} \end{bmatrix} \delta\mathbf{q}$$

- Plug above expressions into principle of virtual work for a multi-body system

$$\delta\mathbf{q}^\top \Bigg(\sum_{i=1}^{n_b} \underbrace{\begin{bmatrix} _{\mathcal{I}}\mathbf{J}_{0_P,S_i} \\ _{\mathcal{I}}\mathbf{J}_{0_R,S_i} \end{bmatrix}^\top \begin{bmatrix} m_i \, _{\mathcal{I}}\mathbf{J}_{0_P,S_i} \\ \mathbf{I}_{S_i} \, _{\mathcal{I}}\mathbf{J}_{0_R,S_i} \end{bmatrix}}_{\mathsf{M(q)}} \ddot{\mathbf{q}} + \dots$$

$$\dots + \underbrace{\begin{bmatrix} _{\mathcal{I}}\mathbf{J}_{0_P,S_i} \\ _{\mathcal{I}}\mathbf{J}_{0_R,S_i} \end{bmatrix}^\top \begin{bmatrix} m_i \, _{\mathcal{I}}\mathbf{J}_{0_P,S_i}\dot{\mathbf{q}} \\ \mathbf{I}_{S_i} \, _{\mathcal{I}}\dot{\mathbf{J}}_{0_R,S_i}\dot{\mathbf{q}} + \, _{\mathcal{I}}\mathbf{J}_{0_R,S_i}\dot{\mathbf{q}} \times \mathbf{I}_{S_i} \, _{\mathcal{I}}\mathbf{J}_{0_R,S_i}\dot{\mathbf{q}} \end{bmatrix}}_{\mathsf{b(q,\dot{q})}} - \dots$$

$$\dots - \underbrace{\begin{bmatrix} _{\mathcal{I}}\mathbf{J}_{0_P,S_i} \\ _{\mathcal{I}}\mathbf{J}_{0_R,S_i} \end{bmatrix}^\top \begin{bmatrix} \mathbf{F}_{\text{ext},i} \\ \mathbf{T}_{\text{ext},i} \end{bmatrix}}_{\mathsf{g(q)}} \Bigg) = 0 \quad \forall \delta\mathbf{q}_{\text{consistent}}$$

- Extract matrices for the equations of motion
  - Mass matrix

$$\mathbf{M}(\mathbf{q}) = \sum_{i=1}^{n_b} \left(_{\mathcal{A}}\mathbf{J}_{0_P,S_i}^\top m_i \, _{\mathcal{A}}\mathbf{J}_{0_P,S_i} + \, _{\mathcal{B}}\mathbf{J}_{0_R,S_i}^\top \, _{\mathcal{B}}\mathbf{I}_{S_i} \, _{\mathcal{B}}\mathbf{J}_{0_R,S_i} \right)$$

  - Coriolis and centrifugal terms

$$\mathbf{b}(\mathbf{q},\dot{\mathbf{q}}) = \sum_{i=1}^{n_b} \Big(_{\mathcal{A}}\mathbf{J}_{0_P,S_i}^\top m_i \, _{\mathcal{A}}\dot{\mathbf{J}}_{0_P,S_i}\dot{\mathbf{q}} + \, _{\mathcal{B}}\mathbf{J}_{0_R,S_i}^\top \cdot \dots$$

$$\dots \cdot \left(_{\mathcal{B}}\mathbf{I}_{S_i} \, _{\mathcal{B}}\dot{\mathbf{J}}_{0_R,S_i}\dot{\mathbf{q}} + \, _{\mathcal{B}}\boldsymbol{\Omega}_{\mathcal{B}_i} \times \, _{\mathcal{B}}\mathbf{I}_{S_i} \, _{\mathcal{B}}\boldsymbol{\Omega}_{\mathcal{B}_i} \right) \Big)$$

  - Gravitational terms

$$\mathbf{g}(\mathbf{q}) = \sum_{i=1}^{n_b} \left(- _{\mathcal{A}}\mathbf{J}_{0_P,S_i}^\top \, _{\mathcal{A}}\mathbf{F}_{g_i} \right)$$

  ⚠ $\mathbf{M}$, $\mathbf{b}$, $\mathbf{g}$ are in "generalized space" (multiplied with $\mathbf{q}$) so summation terms can be in different frames $\mathcal{A}$, $\mathcal{B}$)

LAGRANGE II METHOD

- Lagrangian $\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}) = \mathcal{T}(\mathbf{q}, \dot{\mathbf{q}}) - \mathcal{U}(\mathbf{q})$
- Lagrangian equation

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}}\right)^\top - \left(\frac{\partial \mathcal{L}}{\partial \mathbf{q}}\right)^\top = \boldsymbol{\tau}$$

$$\Longleftrightarrow \frac{d}{dt}\left(\frac{\partial \mathcal{T}}{\partial \dot{\mathbf{q}}}\right)^\top - \left(\frac{\partial \mathcal{T}}{\partial \mathbf{q}}\right)^\top + \left(\frac{\partial \mathcal{U}}{\partial \mathbf{q}}\right)^\top = \boldsymbol{\tau}$$

- Kinetic energy

$$T = \sum_{i=1}^{n_b} \left(\frac{1}{2}m_i \, _{\mathcal{A}}\dot{\mathbf{r}}_{AS_i}^\top \, _{\mathcal{A}}\dot{\mathbf{r}}_{AS_i} + \frac{1}{2} \, _{\mathcal{B}}\boldsymbol{\Omega}_{\mathcal{B}_i}^\top \, _{\mathcal{B}}\mathbf{I}_{S_i} \, _{\mathcal{B}}\boldsymbol{\Omega}_{\mathcal{B}_i} \right)$$

$$= \frac{1}{2}\dot{\mathbf{q}}^\top \sum_{i=1}^{n_b} \left(_{\mathcal{A}}\mathbf{J}_{0_P,S_i}^\top m_i \, _{\mathcal{A}}\mathbf{J}_{0_P,S_i} + \, _{\mathcal{B}}\mathbf{J}_{0_R,S_i}^\top \, _{\mathcal{B}}\mathbf{I}_{S_i} \, _{\mathcal{B}}\mathbf{J}_{0_R,S_i} \right) \dot{\mathbf{q}}$$

$$= \frac{1}{2}\dot{\mathbf{q}}^\top \mathbf{M}(\mathbf{q})\dot{\mathbf{q}}$$
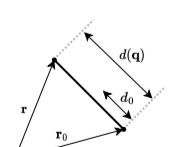
- Potential energy
  - Gravitational forces

$$_{\mathcal{I}}\mathbf{F}_{g_i} = m_i g \, _{\mathcal{I}}\mathbf{e}_g$$

$$\mathcal{U}_g = -\sum_{i=1}^{n_b} \, _{\mathcal{I}}\mathbf{r}_{IS_i}^\top \, _{\mathcal{I}}\mathbf{F}_{g_i}$$

  - Spring forces

$$\mathbf{F}_E = k_j \left(||\mathbf{r} - \mathbf{r}_0|| - d_0 \right) \frac{\mathbf{r} - \mathbf{r}_0}{||\mathbf{r} - \mathbf{r}_0||}$$

$$\mathcal{U}_{E_j} = \frac{1}{2}k_j \left(d(\mathbf{q}) - d_0 \right)^2$$

EXTERNAL FORCES AND TORQUES

- Generalized force $\boldsymbol{\tau}$ (represented in the space of generalized coordinates) can have contributions from:
  - External forces $\mathbf{F}_{\text{ext},j}$ acting in points P_j

$$\boldsymbol{\tau}_{F,\text{ext}} = \sum_{j=1}^{n_{f,\text{ext}}} \, _{\mathcal{A}}\mathbf{J}_{0_P,P_j}^\top \, _{\mathcal{A}}\mathbf{F}_{\text{ext},j}$$

  - External torques $\mathbf{T}_{\text{ext},k}$ acting on bodies $\mathcal{B}_k$

$$\boldsymbol{\tau}_{T,\text{ext}} = \sum_{k=1}^{n_{m,\text{ext}}} \, _{\mathcal{A}}\mathbf{J}_{0_R,\mathcal{B}_k}^\top \, _{\mathcal{A}}\mathbf{T}_{\text{ext},k}$$

  - Combined

$$\boldsymbol{\tau}_{\text{ext}} = \boldsymbol{\tau}_{F,\text{ext}} + \boldsymbol{\tau}_{T,\text{ext}}$$

  - Actuators
    * Actuator acting between body links $\mathcal{B}_{k-1}$ and $\mathcal{B}_k$ (in points B_{k-1} and B_k, respectively) imposes a force $\mathbf{F}_{a,k}$ and/or torque $\mathbf{T}_{a,k}$ on both links equally and in opposite directions
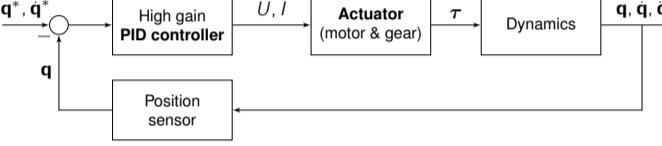
$$\boldsymbol{\tau}_{a,k} = \left(_{\mathcal{A}}\mathbf{J}_{0_P,B_k} - \, _{\mathcal{A}}\mathbf{J}_{0_P,B_{k-1}} \right)^\top \, _{\mathcal{A}}\mathbf{F}_{a,k} + \dots$$

$$\dots + \left(_{\mathcal{A}}\mathbf{J}_{0_R,\mathcal{B}_k} - \, _{\mathcal{A}}\mathbf{J}_{0_R,\mathcal{B}_{k-1}} \right)^\top \, _{\mathcal{A}}\mathbf{T}_{a,k}$$

  - Total

$$\boldsymbol{\tau} = \boldsymbol{\tau}_{\text{ext}} + \sum_{k=1}^{n_A} \boldsymbol{\tau}_{a,k}$$
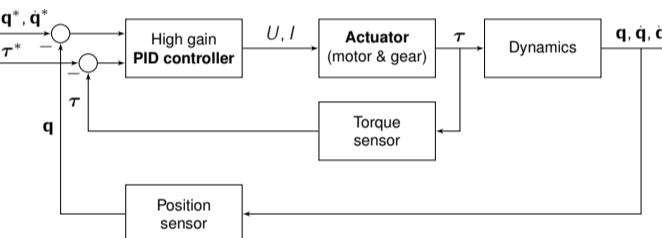
12   DYNAMICS

CLASSICAL POSITION CONTROL

- Joint level position feedback
- High PID gains guarantee good joint level tracking
- Disturbances (load, etc.) are compensated by PID

JOINT TORQUE CONTROL

- Active regulation of system dynamics
- Model-based load compensation
- Interaction force control

JOINT IMPEDANCE CONTROL

- Impedance control (PD law for torque)

$$\boldsymbol{\tau}^* = \mathbf{k}_p(\mathbf{q}^* - \mathbf{q}) + \mathbf{k}_d(\dot{\mathbf{q}}^* - \dot{\mathbf{q}})$$

  - Static conditions: $\dot{\mathbf{q}} = \ddot{\mathbf{q}} = 0$

$$\Longrightarrow \mathbf{g}(\mathbf{q}) = \boldsymbol{\tau} = \mathbf{k}_p(\mathbf{q}^* - \mathbf{q})$$

  - Adding integrator may introduce additional problems
- Impedance control with gravity compensation

$$\boldsymbol{\tau}^* = \mathbf{k}_p(\mathbf{q}^* - \mathbf{q}) + \mathbf{k}_d(\dot{\mathbf{q}}^* - \dot{\mathbf{q}}) + \hat{\mathbf{g}}(\mathbf{q})$$

  - Static conditions: $\dot{\mathbf{q}} = \ddot{\mathbf{q}} = 0$

$$\Longrightarrow \mathbf{g}(\mathbf{q}) = \boldsymbol{\tau} = \mathbf{k}_p(\mathbf{q}^* - \mathbf{q}) + \hat{\mathbf{g}}(\mathbf{q})$$

- Configuration dependent load causes control performance reduction (the inertia seen at each joint varies with the robot configuration, so PD gains are selected for some average configuration)

JOINT-SPACE INVERSE DYNAMICS CONTROL

- Inverse dynamics control (PD law for acceleration)

$$\ddot{\mathbf{q}}^* = \mathbf{k}_p(\mathbf{q}^* - \mathbf{q}) + \mathbf{k}_d(\dot{\mathbf{q}}^* - \dot{\mathbf{q}})$$

$$\boldsymbol{\tau}^* = \hat{\mathbf{M}}(\mathbf{q})\ddot{\mathbf{q}}^* + \hat{\mathbf{b}}(\mathbf{q}, \dot{\mathbf{q}}) + \hat{\mathbf{g}}(\mathbf{q})$$

  - Can achieve great performance
  - Requires accurate modeling
- Perfect model case: $\hat{\mathbf{M}} = \mathbf{M}$, $\hat{\mathbf{b}} = \mathbf{b}$, $\hat{\mathbf{g}} = \mathbf{g}$

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{b}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{g}(\mathbf{q}) = \boldsymbol{\tau}^* = \hat{\mathbf{M}}(\mathbf{q})\ddot{\mathbf{q}}^* + \hat{\mathbf{b}}(\mathbf{q}, \dot{\mathbf{q}}) + \hat{\mathbf{g}}(\mathbf{q})$$

$$\Longrightarrow \ddot{\mathbf{q}} = \ddot{\mathbf{q}}^* = \mathbf{k}_p(\mathbf{q}^* - \mathbf{q}) + \mathbf{k}_d(\dot{\mathbf{q}}^* - \dot{\mathbf{q}})$$

  - Every joint behaves like a decoupled mass spring damper with unitary mass
    * Eigenfrequency (natural frequency) $\omega = \sqrt{k_p}$
    * Dimensionless damping $D = \frac{k_d}{2\sqrt{k_p}}$

TASK-SPACE DYNAMICS CONTROL

- Motivation: motion in joint space often hard to describe
- Single task
  - Inverse dynamics formulation

$$\dot{\mathbf{w}}_1^* = \begin{bmatrix} \ddot{\mathbf{r}}_1^* \\ \dot{\boldsymbol{\omega}}_1^* \end{bmatrix} = \frac{d}{dt}(\mathbf{J}_{0,1}\dot{\mathbf{q}}^*) = \mathbf{J}_{0,1}\ddot{\mathbf{q}}^* + \dot{\mathbf{J}}_1\dot{\mathbf{q}}^*$$

$$\Longrightarrow \ddot{\mathbf{q}}^* = \mathbf{J}_{0,1}^\dagger \left(\dot{\mathbf{w}}_1^* - \dot{\mathbf{J}}_1\dot{\mathbf{q}}^* \right)$$

  - Quadratic optimization (QP) formulation
    * Tasks to fulfill

$$\boldsymbol{\tau} = \mathbf{M}\ddot{\mathbf{q}} + \mathbf{b} + \mathbf{g}$$

$$\dot{\mathbf{w}}_e^* = \mathbf{J}_{0,e}\ddot{\mathbf{q}} + \dot{\mathbf{J}}_{0,e}\dot{\mathbf{q}}$$

    * Minimize for $\ddot{\mathbf{q}}$ and $\boldsymbol{\tau}$

$$\arg\min_{\ddot{\mathbf{q}}, \boldsymbol{\tau}} \left\| \begin{bmatrix} \mathbf{J}_{0,e} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ \boldsymbol{\tau} \end{bmatrix} - (\dot{\mathbf{w}}_e^* - \dot{\mathbf{J}}_{0,e}\dot{\mathbf{q}}) \right\|_2$$

$$\text{subject to } \begin{bmatrix} \mathbf{M} & -\mathbb{I} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ \boldsymbol{\tau} \end{bmatrix} + \mathbf{b} + \mathbf{g} = \mathbf{0}$$

- Multiple tasks
  - Equal priorities

$$\ddot{\mathbf{q}}^* = \begin{bmatrix} \mathbf{J}_1 \\ \vdots \\ \mathbf{J}_{n_t} \end{bmatrix}^\dagger \left(\begin{bmatrix} \dot{\mathbf{w}}_1^* \\ \vdots \\ \dot{\mathbf{w}}_{n_t}^* \end{bmatrix} - \begin{bmatrix} \dot{\mathbf{J}}_1 \\ \vdots \\ \dot{\mathbf{J}}_{n_t} \end{bmatrix} \dot{\mathbf{q}} \right)$$

  - Prioritization (hierarchical: attempt to fulfill a task as best as possible only if the next higher priority task is fulfilled)

$$\ddot{\mathbf{q}}^* = \sum_{i=1}^{n_t} \mathbf{N}_i\ddot{\mathbf{q}}_i, \quad \ddot{\mathbf{q}}_i = (\mathbf{J}_i\mathbf{N}_i)^\dagger \left(\dot{\mathbf{w}}_i^* - \dot{\mathbf{J}}_i\dot{\mathbf{q}} - \mathbf{J}\sum_{k=1}^{i-1}\mathbf{N}_k\ddot{\mathbf{q}}_k \right)$$

END-EFFECTOR DYNAMICS

- End-effector dynamics

$$\ddot{\mathbf{q}} = \mathbf{M}^{-1}(\boldsymbol{\tau} - \mathbf{b} - \mathbf{g})$$

$$\dot{\mathbf{w}}_e^* = \mathbf{J}_{0,e}\ddot{\mathbf{q}} + \dot{\mathbf{J}}_{0,e}\dot{\mathbf{q}}$$

$$\Longrightarrow \dot{\mathbf{w}}_e = \mathbf{J}_{0,e}\mathbf{M}^{-1}(\boldsymbol{\tau} - \mathbf{b} - \mathbf{g}) + \dot{\mathbf{J}}_{0,e}\dot{\mathbf{q}}$$

$$\Longrightarrow \dot{\mathbf{w}}_e = \mathbf{J}_{0,e}\mathbf{M}^{-1}(\mathbf{J}_{0,e}^\top\mathbf{F}_e - \mathbf{b} - \mathbf{g}) + \dot{\mathbf{J}}_{0,e}\dot{\mathbf{q}}$$

$$\Longrightarrow \dot{\mathbf{w}}_e - \dot{\mathbf{J}}_{0,e}\dot{\mathbf{q}} + \mathbf{J}_{0,e}\mathbf{M}^{-1}\mathbf{b} + \mathbf{J}_{0,e}\mathbf{M}^{-1}\mathbf{g} = \mathbf{J}_{0,e}\mathbf{M}^{-1}\mathbf{J}_{0,e}^\top\mathbf{F}_e$$

$$\Longrightarrow \left(\mathbf{J}_{0,e}\mathbf{M}^{-1}\mathbf{J}_{0,e}^\top \right)^{-1} \left(\dot{\mathbf{w}}_e - \dot{\mathbf{J}}_{0,e}\dot{\mathbf{q}} + \mathbf{J}_{0,e}\mathbf{M}^{-1}\mathbf{b} + \mathbf{J}_{0,e}\mathbf{M}^{-1}\mathbf{g} \right) = \mathbf{F}_e$$

$$\boxed{\boldsymbol{\Lambda}\dot{\mathbf{w}}_e + \boldsymbol{\mu} + \mathbf{p} = \mathbf{F}_e}$$

  where

$$\boxed{\begin{aligned} \boldsymbol{\Lambda} &= \left(\mathbf{J}_{0,e}\mathbf{M}^{-1}\mathbf{J}_{0,e}^\top \right)^{-1} \\ \boldsymbol{\mu} &= \boldsymbol{\Lambda}\mathbf{J}_{0,e}\mathbf{M}^{-1}\mathbf{b} - \boldsymbol{\Lambda}\dot{\mathbf{J}}_{0,e}\dot{\mathbf{q}} \\ \mathbf{p} &= \boldsymbol{\Lambda}\mathbf{J}_{0,e}\mathbf{M}^{-1}\mathbf{g} \end{aligned}}$$

  ⚠ $\mathbf{F}_e$ acts on system

END-EFFECTOR MOTION CONTROL

- Desired end-effector acceleration

$$\dot{\mathbf{w}}_e^* = \mathbf{k}_p \begin{bmatrix} \mathbf{r}_e^* - \mathbf{r}_e \\ \Delta\boldsymbol{\varphi}_e \end{bmatrix} + \mathbf{k}_d(\mathbf{w}_e^* - \mathbf{w}_e) + \dot{\mathbf{w}}_{e,\text{ff}}(t)$$

  where $\Delta\boldsymbol{\varphi}_e \approx \mathbf{E}_R(\boldsymbol{\mathcal{X}}_{R,e}^* - \boldsymbol{\mathcal{X}}_{R,e})$ (for small errors) is the end-effector rotation error and $\dot{\mathbf{w}}_{e,\text{ff}}(t)$ is trajectory control feedforward term
- End-effector dynamics give corresponding joint torque

$$\boldsymbol{\tau}^* = \hat{\mathbf{J}}_e^\top\mathbf{F}_e = \hat{\mathbf{J}}_e^\top \left(\hat{\boldsymbol{\Lambda}}_e\dot{\mathbf{w}}_e^* + \hat{\boldsymbol{\mu}} + \hat{\mathbf{p}} \right) + \mathbf{N}(\mathbf{J}_{0,e}^\top)\boldsymbol{\tau}_0$$

  where

$$\mathbf{N}(\mathbf{J}_{0,e}^\top) = \left(\mathbb{I} - \mathbf{J}_{0,e}^\top(\mathbf{J}_{0,e}\mathbf{M}^{-1}\mathbf{J}_{0,e}^\top)^{-1}\mathbf{J}_{0,e}\mathbf{M}^{-1} \right)$$

  is the null space projection of the transposed geometric end-effector Jacobian (if the torque is applied in the nullspace, the end-effector acceleration does not change)
- Substitute $\boldsymbol{\tau}^*$ back into dynamics equation, assume $\boldsymbol{\tau}_0 = 0$ and use the definition of $\boldsymbol{\Lambda}$

$$\ddot{\mathbf{q}} = \mathbf{M}^{-1}(\boldsymbol{\tau}^* - \mathbf{b} - \mathbf{g})$$

$$= \mathbf{M}^{-1} \left(\hat{\mathbf{J}}_e^\top \left(\hat{\boldsymbol{\Lambda}}_e\dot{\mathbf{w}}_e^* + \hat{\boldsymbol{\mu}} + \hat{\mathbf{p}} \right) - \mathbf{b} - \mathbf{g} \right)$$

$$= \mathbf{M}^{-1} \left(\hat{\mathbf{J}}_e^\top \left((\mathbf{J}_{0,e}\mathbf{M}^{-1}\mathbf{J}_{0,e}^\top)^{-1}\dot{\mathbf{w}}_e^* + \hat{\boldsymbol{\mu}} + \hat{\mathbf{p}} \right) - \mathbf{b} - \mathbf{g} \right)$$

OPERATIONAL SPACE CONTROL

- Extend end-effector dynamics with contact force

$$\boldsymbol{\Lambda}\dot{\mathbf{w}}_e + \boldsymbol{\mu} + \mathbf{p} + \mathbf{F}_c = \mathbf{F}_e$$

  - $\mathbf{F}_c$ acts on surface i.e. is executed by robot
- Introduce selection matrices $\mathbf{S}_{M,F}$ (in inertial frame) to separate motion and force directions

$$\boldsymbol{\tau}^* = \hat{\mathbf{J}}_e^\top \left(\hat{\boldsymbol{\Lambda}}_e\mathbf{S}_M\dot{\mathbf{w}}_e^* + \mathbf{S}_F\mathbf{F}_c^* + \hat{\boldsymbol{\mu}} + \hat{\mathbf{p}} \right)$$

SELECTION MATRICES

- Selection matrices ensure that incompatible force and motion components are removed (if motions already compatible then they have no effect)
- Selection matrices $\mathbf{\Sigma}_{M,F}$ in end-effector frame

$$\mathbf{\Sigma}_{M,F} = \begin{bmatrix} \mathbf{\Sigma}_{(M,F)P} & 0 \\ 0 & \mathbf{\Sigma}_{(M,F)R} \end{bmatrix} \in \mathbb{R}^{6\times 6}$$

  – The 3 diagonal entries of $\mathbf{\Sigma}_{MP}$ are
  $$\begin{cases} 1 & \text{EE can translate} \\ 0 & \text{EE can apply force} \end{cases}$$

  – The 3 diagonal entries of $\mathbf{\Sigma}_{MR}$ are
  $$\begin{cases} 1 & \text{EE can rotate} \\ 0 & \text{EE can apply torque} \end{cases}$$

- $\mathbf{\Sigma}_M$ (for motion control) and $\mathbf{\Sigma}_F$ (for force/torque control) are related through
$$\mathbf{\Sigma}_{F(P,R)} = \mathbb{I} - \mathbf{\Sigma}_{M(P,R)}$$

- Transform to inertial frame using $\mathbf{C}$ if $\mathbf{F}_c^*$ and/or $\mathbf{\dot{w}}_e^*$ are w.r.t. the inertial frame
$$\mathbf{S}_{M,F} = \begin{bmatrix} \mathbf{C}^\top \mathbf{\Sigma}_{(M,F)P} \mathbf{C} & 0 \\ 0 & \mathbf{C}^\top \mathbf{\Sigma}_{(M,F)R} \mathbf{C} \end{bmatrix}$$

GENERAL LEAST SQUARE OPTIMIZATION

$$\boxed{\mathbf{Ax} - \mathbf{b} = 0}$$

$$\Longleftrightarrow \mathbf{x} = \mathbf{A}^\dagger \mathbf{b}$$
$$\Longleftrightarrow \arg\min_{\mathbf{x}} ||\mathbf{Ax} - \mathbf{b}||_2$$
$$\Longleftrightarrow \min ||\mathbf{x}||_2 \text{ subject to } \mathbf{Ax} - \mathbf{b} = 0$$

$$\boxed{\mathbf{A}_1\mathbf{x}_1 + \mathbf{A}_2\mathbf{x}_2 - \mathbf{b} = 0}$$

$$\Longleftrightarrow \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \end{bmatrix}^\dagger \mathbf{b}$$

$$\Longleftrightarrow \arg\min_{\mathbf{x}_1,\mathbf{x}_2} \left|\left| \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} - \mathbf{b} \right|\right|_2$$

$$\Longleftrightarrow \min \left|\left| \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} \right|\right|_2 \text{ subject to } \mathbf{A}_1\mathbf{x}_1 + \mathbf{A}_2\mathbf{x}_2 - \mathbf{b} = 0$$

$$\boxed{\begin{array}{c} \mathbf{A}_1\mathbf{x} - \mathbf{b}_1 = 0 \\ \mathbf{A}_2\mathbf{x} - \mathbf{b}_2 = 0 \end{array}}$$

- Equal priority
$$\mathbf{x} = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{bmatrix}^\dagger \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix}$$

$$\Longleftrightarrow \arg\min_{\mathbf{x}} \left|\left| \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{bmatrix} \mathbf{x} - \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix} \right|\right|_2$$

$$\Longleftrightarrow \min ||\mathbf{x}||_2 \text{ subject to } \mathbf{A}_1\mathbf{x} - \mathbf{b}_1 = 0, \ \mathbf{A}_2\mathbf{x} - \mathbf{b}_2 = 0$$

- Prioritization
$$\mathbf{x} = \mathbf{A}_1^\dagger \mathbf{b}_1 + \mathcal{N}(\mathbf{A}_1)\mathbf{x}_0$$
$$\mathbf{A}_2\mathbf{x} - \mathbf{b}_2 = \mathbf{A}_2\left(\mathbf{A}_1^\dagger \mathbf{b}_1 + \mathcal{N}(\mathbf{A}_1)\mathbf{x}_0\right) - \mathbf{b}_2 = 0$$
$$\mathbf{x}_0 = (\mathbf{A}_2 \mathcal{N}(\mathbf{A}_1))^\dagger (\mathbf{b}_2 - \mathbf{A}_2\mathbf{A}_1^\dagger \mathbf{b}_1)$$
$$\Longleftrightarrow \arg\min_{\mathbf{x}} ||\mathbf{A}_2\mathbf{x} - \mathbf{b}_2||_2 \text{ subject to } ||\mathbf{A}_1\mathbf{x} - \mathbf{b}_1|| = \mathbf{c}_1$$

13 FLOATING BASE DYNAMICS

FLOATING BASE KINEMATICS

- Unactuated base in point B (can only move through contacts with environment) denoted with index b
- Generalized coordinates
$$\mathbf{q} = \begin{bmatrix} \mathbf{q}_b \\ \mathbf{q}_j \end{bmatrix}$$

where
$$\mathbf{q}_b = \begin{bmatrix} \mathbf{q}_{b_P} = {}_\mathcal{I}\mathbf{r}_{IB} \\ \mathbf{q}_{b_R} = \mathbf{\mathcal{X}}_{R,b} \end{bmatrix} \in \mathbb{R}^3 \times SO(3)$$

  – Base orientation $\mathbf{q}_{b_R}$ can be parameterized using any rotation representation $\mathbf{\mathcal{X}}_{R,b}$
  – At least $n_{b0} = 6$ base generalized coordinates in 3D

- Generalized velocities and accelerations
  – Common choice
$$\mathbf{u} = \begin{bmatrix} {}_\mathcal{I}\mathbf{v}_B \\ {}_\mathcal{B}\boldsymbol{\omega}_{IB} \\ \dot{\varphi}_1 \\ \vdots \\ \dot{\varphi}_{n_j} \end{bmatrix}, \quad \dot{\mathbf{u}} = \begin{bmatrix} {}_\mathcal{I}\mathbf{a}_B \\ {}_\mathcal{B}\boldsymbol{\psi}_{IB} \\ \ddot{\varphi}_1 \\ \vdots \\ \ddot{\varphi}_{n_j} \end{bmatrix}$$

  – Linear mapping between $\dot{\mathbf{q}}$ and $\mathbf{u}$
$$\mathbf{u} = \mathbf{E}_{fb}\dot{\mathbf{q}}, \quad \mathbf{E}_{fb} = \begin{bmatrix} \mathbb{I}_{3\times 3} & 0 & 0 \\ 0 & \mathbf{E}_{\mathcal{X}_{R,b}} & 0 \\ 0 & 0 & \mathbb{I}_{n_j \times n_j} \end{bmatrix}$$

- ⚠ In literature, $\dot{\mathbf{q}}$ is commonly used instead of $\mathbf{u}$
  * Note: $\mathbf{w} = \mathbf{J}_0\dot{\mathbf{q}} \Longleftrightarrow \mathbf{w} = \mathbf{J}_0\mathbf{E}_{fb}^{-1}\mathbf{u}$
  * For notation accuracy and brevity, define
  $$\bar{\mathbf{J}}_0 := \mathbf{J}_0\mathbf{E}_{fb}^{-1} \implies \mathbf{w} = \bar{\mathbf{J}}_0\mathbf{u}$$

FLOATING BASE DIFFERENTIAL KINEMATICS

- Position of an arbitrary point Q on the robot
$${}_\mathcal{I}\mathbf{r}_{IQ}(\mathbf{q}) = {}_\mathcal{I}\mathbf{r}_{IB}(\mathbf{q}) + \mathbf{C}_{\mathcal{IB}}(\mathbf{q})\, {}_\mathcal{B}\mathbf{r}_{BQ}(\mathbf{q})$$

- Velocity of Q
$${}_\mathcal{I}\mathbf{v}_Q = {}_\mathcal{I}\mathbf{v}_B + \dot{\mathbf{C}}_{\mathcal{IB}}\, {}_\mathcal{B}\mathbf{r}_{BQ} + \mathbf{C}_{\mathcal{IB}}\, {}_\mathcal{B}\dot{\mathbf{r}}_{BQ}$$
$$= {}_\mathcal{I}\mathbf{v}_B + \mathbf{C}_{\mathcal{IB}}\left[{}_\mathcal{B}\boldsymbol{\omega}_{IB}\right]_\times\, {}_\mathcal{B}\mathbf{r}_{BQ} + \mathbf{C}_{\mathcal{IB}}\, {}_\mathcal{B}\dot{\mathbf{r}}_{BQ}$$
$$= {}_\mathcal{I}\mathbf{v}_B - \mathbf{C}_{\mathcal{IB}}\left[{}_\mathcal{B}\mathbf{r}_{BQ}\right]_\times\, {}_\mathcal{B}\boldsymbol{\omega}_{IB} + \mathbf{C}_{\mathcal{IB}}\, {}_\mathcal{B}\dot{\mathbf{r}}_{BQ}$$
$$= {}_\mathcal{I}\mathbf{v}_B - \mathbf{C}_{\mathcal{IB}}\left[{}_\mathcal{B}\mathbf{r}_{BQ}\right]_\times\, {}_\mathcal{B}\boldsymbol{\omega}_{IB} + \mathbf{C}_{\mathcal{IB}}\, {}_\mathcal{B}\mathbf{J}_{0_{P_{q_j}},Q}(\mathbf{q}_j)(\mathbf{q}_j)\dot{\mathbf{q}}_j$$
$$= \underbrace{\begin{bmatrix} \mathbb{I}_{3\times 3} & -\mathbf{C}_{\mathcal{IB}}\left[{}_\mathcal{B}\mathbf{r}_{BQ}\right]_\times & \mathbf{C}_{\mathcal{IB}}\, {}_\mathcal{B}\mathbf{J}_{0_{P_{q_j}},Q}(\mathbf{q}_j) \end{bmatrix}}_{{}_\mathcal{J}\mathbf{J}_Q(\mathbf{q})} \mathbf{u}$$

CONTACT CONSTRAINTS

- A contact point C_i is not allowed to move
$${}_\mathcal{I}\mathbf{r}_{IC_i} = \text{const}$$
$$\implies {}_\mathcal{I}\dot{\mathbf{r}}_{IC_i} = \mathbf{0} \Longleftrightarrow {}_\mathcal{I}\bar{\mathbf{J}}_{0_P,C_i}\mathbf{u} = \mathbf{0}$$
$$\implies {}_\mathcal{I}\ddot{\mathbf{r}}_{IC_i} = \mathbf{0} \Longleftrightarrow {}_\mathcal{I}\bar{\mathbf{J}}_{0_P,C_i}\dot{\mathbf{u}} + {}_\mathcal{I}\dot{\bar{\mathbf{J}}}_{0_P,C_i}\mathbf{u} = \mathbf{0}$$

- Stack of constraints (contact Jacobians)
$${}_\mathcal{I}\bar{\mathbf{J}}_{0_P,c} := \begin{bmatrix} {}_\mathcal{I}\bar{\mathbf{J}}_{0_P,C_1} \\ \vdots \\ {}_\mathcal{I}\bar{\mathbf{J}}_{0_P,C_{n_c}} \end{bmatrix}$$

  ⚠ Hereinafter, let ${}_\mathcal{I}\bar{\mathbf{J}}_{0_P,c} \equiv {}_\mathcal{I}\bar{\mathbf{J}}_c$

- Stack of n_c contact points
$$\mathbf{r}_c := \begin{bmatrix} {}_\mathcal{I}\mathbf{r}_{IC_1} \\ \vdots \\ {}_\mathcal{I}\mathbf{r}_{IC_{n_c}} \end{bmatrix}$$

- System motion $\mathbf{u}, \dot{\mathbf{u}}$ that doesn't violate contact constraints
$$\dot{\mathbf{r}}_c = \bar{\mathbf{J}}_c\mathbf{u} = \mathbf{0} \Longleftrightarrow \mathbf{u} = \bar{\mathbf{J}}_c^\dagger \mathbf{0} + \mathbf{N}_c\mathbf{u}_0 = \mathbf{N}_c\mathbf{u}_0$$
$$\ddot{\mathbf{r}}_c = \bar{\mathbf{J}}_c\dot{\mathbf{u}} + \dot{\bar{\mathbf{J}}}_c\mathbf{u} = \mathbf{0} \Longleftrightarrow \dot{\mathbf{u}} = \bar{\mathbf{J}}_c^\dagger(-\dot{\bar{\mathbf{J}}}_c\mathbf{u}) + \mathbf{N}_c\dot{\mathbf{u}}_0$$
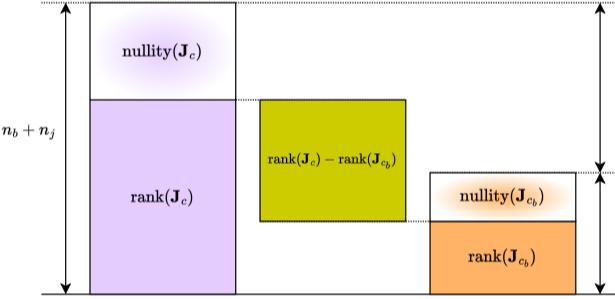
  – Can be used e.g. as an additional task
  ⚠ Hereinafter, assume $\bar{\mathbf{J}}_c = \mathbf{J}_c$

CONTACT JACOBIAN PROPERTIES

- Separation into base and joints part
$$\mathbf{J}_c = \begin{bmatrix} \mathbf{J}_{c_b} & \mathbf{J}_{c_j} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{\partial\mathbf{r}_c}{\partial\mathbf{q}_b} & \frac{\partial\mathbf{r}_c}{\partial\mathbf{q}_j} \end{bmatrix} \in \begin{cases} \mathbb{R}^{3n_c\times(n_b+n_j)} & \text{3D, point contacts} \\ \mathbb{R}^{2n_c\times(n_b+n_j)} & \text{2D, point contacts} \end{cases}$$

- Physical interpretation

n_b	Base DOFs
n_j	Joint DOFs
$n_b + n_j$	Total DOFs
$\text{rank}(\mathbf{J}_c)$	Number of independent constraints (locks on the DOFs)
$\text{nullity}(\mathbf{J}_c)$	Number of DOFs for which motion is admissible
$\text{rank}(\mathbf{J}_{c_b})$	Number of constrained base DOFs (if all joints locked)
$\text{nullity}(\mathbf{J}_{c_b})$	Number of unconstrained (free) base DOFs (if all joints locked)
$\text{rank}(\mathbf{J}_c) - \text{rank}(\mathbf{J}_{c_b})$	Number of kinematic joint constraints (internal constraints) that must be satisfied by joints

  – Note: nullity($\mathbf{A}$) is the dimension of the nullspace of $\mathbf{A}$
  – $6 - \text{rank}(\mathbf{J}_{c_b})$ unconstrained base DOFs

FLOATING BASE DYNAMICS

$$\boxed{\mathbf{M}(\mathbf{q})\dot{\mathbf{u}} + \mathbf{b}(\mathbf{q},\mathbf{u}) + \mathbf{g}(\mathbf{q}) = \mathbf{S}^\top\boldsymbol{\tau} + \mathbf{J}_{\text{ext}}^\top\mathbf{F}_{\text{ext}}}$$
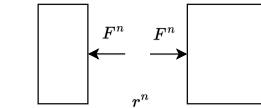
		Dimensions
$\mathbf{q}$	Generalized coordinates	$n_q := n_b + n_j$
$\mathbf{u}$	Generalized velocities	$n_u := 6 + n_j$
$\dot{\mathbf{u}}$	Generalized accelerations	n_u
$\mathbf{M}(\mathbf{q})$	Mass matrix	$n_u \times n_u$
$\mathbf{b}(\mathbf{q},\mathbf{u})$	Centrifugal and Coriolis forces	n_u
$\mathbf{g}(\mathbf{q})$	Gravity forces	n_u
$\boldsymbol{\tau}$	Forces/torques acting in direction of generalized coordinates (n_τ is number of actuated DOFs)	n_τ
$\mathbf{S}$	Selection matrix of actuated joints (corrects dimension)	$n_\tau \times n_u$
$\mathbf{F}_{\text{ext}}$	External forces acting on system (e.g. contacts)	$3n_c$
$\mathbf{J}_{\text{ext}}$	Pos. Jacobian of external forces	$3n_c \times n_u$

Note: the EoM can't be used directly for control due to the occurence of (unknown) contact forces

HARD CONTACT MODEL

- Closed contact
$$r^n = 0, \ \dot{r}^n = 0$$

- Linear complementarity constraint
$$\ddot{r}^n \geq 0, \ F^n \geq 0, \ \ddot{r}^n F^n = 0$$

SOFT CONTACT MODEL

- Spring-damper environment
$$\mathbf{F}_c = k_p(\mathbf{r}_c - \mathbf{r}_{c0}) + k_d\dot{\mathbf{r}}_c$$
where $\mathbf{r}_{c0}$ is the point of first contact with the environment, and $\mathbf{F}_c = 0$ if not in contact i.e. never negative

- Contact parameters selected as numerical (not physical) parameters to control tradeoff between numerical stability ("stiff" EoM needs small timesteps and can lead to instability) and physical accuracy

CONSTRAINT/SUPPORT CONSISTENT DYNAMICS

- Derivation (goal: eliminate $\mathbf{F}_c$ from the EoM)
  – Hard contact constraint
$$\dot{\mathbf{r}}_c = \mathbf{J}_c\mathbf{u} = \mathbf{0} \implies \ddot{\mathbf{r}}_c = \mathbf{J}_c\dot{\mathbf{u}} + \dot{\mathbf{J}}_c\mathbf{u} = \mathbf{0}$$
$$\implies \dot{\mathbf{J}}_c\mathbf{u} = -\mathbf{J}_c\dot{\mathbf{u}}$$

  – Floating base dynamics
$$\mathbf{M}\dot{\mathbf{u}} + \mathbf{b} + \mathbf{g} + \mathbf{J}_c^\top\mathbf{F}_c = \mathbf{S}^\top\boldsymbol{\tau} \implies \dot{\mathbf{u}} = \mathbf{M}^{-1}\left(\mathbf{S}^\top\boldsymbol{\tau} - (\mathbf{b} + \mathbf{g} + \mathbf{J}_c^\top\mathbf{F}_c)\right)$$
$$\ddot{\mathbf{r}}_c = \mathbf{J}_c\mathbf{M}^{-1}\left(\mathbf{S}^\top\boldsymbol{\tau} - (\mathbf{b} + \mathbf{g} + \mathbf{J}_c^\top\mathbf{F}_c)\right) + \dot{\mathbf{J}}_c\mathbf{u} = \mathbf{0}$$
$$\implies \mathbf{F}_c = \left(\mathbf{J}_c\mathbf{M}^{-1}\mathbf{J}_c^\top\right)^{-1}\left(\mathbf{J}_c\mathbf{M}^{-1}\left(\mathbf{S}^\top\boldsymbol{\tau} - (\mathbf{b} + \mathbf{g})\right) + \dot{\mathbf{J}}_c\mathbf{u}\right)$$

  – Insert $\mathbf{F}_c$ into floating base dynamics
$$\mathbf{M}\dot{\mathbf{u}} + \mathbf{b} + \mathbf{g} + \mathbf{J}_c^\top\left(\mathbf{J}_c\mathbf{M}^{-1}\mathbf{J}_c^\top\right)^{-1}\left(\mathbf{J}_c\mathbf{M}^{-1}\left(\mathbf{S}^\top\boldsymbol{\tau} - (\mathbf{b} + \mathbf{g})\right) + \dot{\mathbf{J}}_c\mathbf{u}\right) = \mathbf{S}^\top\boldsymbol{\tau}$$

  – Use hard contact constraint $\dot{\mathbf{J}}_c\mathbf{u} = -\mathbf{J}_c\dot{\mathbf{u}}$
$$\mathbf{M}\dot{\mathbf{u}} + \mathbf{b} + \mathbf{g} + \mathbf{J}_c^\top\left(\mathbf{J}_c\mathbf{M}^{-1}\mathbf{J}_c^\top\right)^{-1}\left(\mathbf{J}_c\mathbf{M}^{-1}\left(\mathbf{S}^\top\boldsymbol{\tau} - (\mathbf{b} + \mathbf{g})\right) - \mathbf{J}_c\dot{\mathbf{u}}\right) = \mathbf{S}^\top\boldsymbol{\tau}$$

  – Multiply by $\mathbf{N}_c^\top$
  where $\mathbf{N}_c := \mathbb{I} - \mathbf{M}^{-1}\mathbf{J}_c^\top(\mathbf{J}_c\mathbf{M}^{-1}\mathbf{J}_c^\top)^{-1}\mathbf{J}_c$
  and use $\mathbf{J}_c\mathbf{N}_c = 0$ i.e. $\mathbf{N}_c^\top\mathbf{J}_c^\top = 0$
$$\mathbf{N}_c^\top\left(\mathbf{M}\dot{\mathbf{u}} + \mathbf{b} + \mathbf{g}\right) = \mathbf{N}_c^\top\mathbf{S}^\top\boldsymbol{\tau}$$

CONTACT DYNAMICS

- Assume perfect inelastic collision (Newtonian law)
- Impulse transfer at contact
- Integration of a single point in time
$$\int_{\{t_0\}}\left(\mathbf{M}\dot{\mathbf{u}} + \mathbf{b} + \mathbf{g} + \mathbf{J}_c^\top\mathbf{F}_c - \mathbf{S}^\top\boldsymbol{\tau}\right)dt = \mathbf{M}(\mathbf{u}^+ - \mathbf{u}^-) + \mathbf{J}_c^\top\mathcal{F}_c = 0$$
where $\mathcal{F}_c$ is an impulsive force and $\mathbf{u}^-$, $\mathbf{u}^+$ are pre- and post-impact generalized velocities, respectively

  – Solve for $\mathcal{F}_c$ and use the condition $\dot{\mathbf{r}}_c^+ = \mathbf{J}_c\mathbf{u}^+ = 0$
$$\mathcal{F}_c = \left(\mathbf{J}_c\mathbf{M}^{-1}\mathbf{J}_c^\top\right)^{-1}\mathbf{J}_c\mathbf{u}^- = \mathbf{\Lambda}_c\dot{\mathbf{r}}_c^-$$

- Change in generalized velocity
$$\Delta\mathbf{u} = \mathbf{u}^+ - \mathbf{u}^- = -\mathbf{M}^{-1}\mathbf{J}_c^\top\left(\mathbf{J}_c\mathbf{M}^{-1}\mathbf{J}_c^\top\right)^{-1}\mathbf{J}_c\mathbf{u}^-$$

  – Post-impact generalized velocity
$$\mathbf{u}^+ = \left(\mathbb{I} - \mathbf{M}^{-1}\mathbf{J}_c^\top(\mathbf{J}_c\mathbf{M}^{-1}\mathbf{J}_c^\top)^{-1}\mathbf{J}_c\right)\mathbf{u}^- = \mathbf{N}_c\mathbf{u}^-$$

- Energy loss
$$E_{\text{loss}} = \Delta E_K = -\frac{1}{2}\Delta\mathbf{u}^\top\mathbf{M}\Delta\mathbf{u} = -\frac{1}{2}\Delta\dot{\mathbf{r}}_c^\top\mathbf{\Lambda}_c\Delta\dot{\mathbf{r}}_c = -\frac{1}{2}(\dot{\mathbf{r}}_c^-)^\top\mathbf{\Lambda}_c\dot{\mathbf{r}}_c^-$$

SUPPORT CONSISTENT INVERSE DYNAMICS

$$\mathbf{N}_c^\top\left(\mathbf{M}\dot{\mathbf{u}} + \mathbf{b} + \mathbf{g}\right) = \mathbf{N}_c^\top\mathbf{S}^\top\boldsymbol{\tau}$$
$$\implies \boldsymbol{\tau}^* = \left(\mathbf{N}_c^\top\mathbf{S}^\top\right)^\dagger\mathbf{N}_c^\top\left(\mathbf{M}\dot{\mathbf{u}} + \mathbf{b} + \mathbf{g}\right) + \mathcal{N}\left(\mathbf{N}_c^\top\mathbf{S}^\top\right)\boldsymbol{\tau}_0^*$$

- Internal force directions allow changing the force with the environment without changing the motion of the system (nullspace torque term)

TASK SPACE CONTROL AS QUADRATIC PROGRAM

- General form: $\arg\min_{\mathbf{x}} ||\mathbf{A}_i\mathbf{x} - \mathbf{b}_i||_2, \quad \mathbf{x} = \begin{bmatrix} \dot{\mathbf{u}} & \mathbf{F}_c & \boldsymbol{\tau} \end{bmatrix}^\top$

|	
To fulfill EoM $\mathbf{M}\dot{\mathbf{u}} + \mathbf{b} + \mathbf{g} + \mathbf{J}_c^\top\mathbf{F}_c = \mathbf{S}^\top\boldsymbol{\tau}$	$\mathbf{A} = \begin{bmatrix} \hat{\mathbf{M}} & \hat{\mathbf{J}}_c^\top & -\mathbf{S}^\top \end{bmatrix}$ $\mathbf{b} = -\hat{\mathbf{b}} - \hat{\mathbf{g}}$
Motion tasks $\mathbf{J}_i\dot{\mathbf{u}} + \dot{\mathbf{J}}_i\mathbf{u} = \dot{\mathbf{w}}^*$	$\mathbf{A} = \begin{bmatrix} \hat{\mathbf{J}}_i & 0 & 0 \end{bmatrix}$ $\mathbf{b} = \dot{\mathbf{w}}^* - \hat{\dot{\mathbf{J}}}_i\mathbf{u}$
Force tasks $\mathbf{F}_c = \mathbf{F}_i^*$	$\mathbf{A} = \begin{bmatrix} 0 & \mathbb{I} & 0 \end{bmatrix}$ $\mathbf{b} = \mathbf{F}_i^*$
Torque minimization	$\mathbf{A} = \begin{bmatrix} 0 & 0 & \mathbb{I} \end{bmatrix}$ $\mathbf{b} = 0$

14 LEGGED ROBOTICS

IDEAL ACTUATOR FOR LEGGED ROBOTICS

- Ideal torque source (high bandwidth, high accuracy)
- Energy efficiency
- High maximum joint torque
- High maximum joint velocity
- Small size and weight
- Robustness to impacts, etc.
- Large range of motion
- Low price
- User-friendly

ACTUATION PRINCIPLES

- High-geared motor with torque sensor
  - ⊕ Very compact
  - ⊕ Motor can be operated at high speed
  - ⊕ High reflected inertia
  - ⊖ Low gearbox efficiency
  - ⊖ Impact loads can destroy the gear
- High-geared motor with serial spring (series elastic actuator)
  - ⊕ Very compact
  - ⊕ Precise torque regulation
  - ⊕ Spring decouples actuator and link inertia (robustness as motor inertia is not seen during impact)
  - ⊕ Additional spring dynamics (temporary energy storage and power/speed amplification)
  - ⊖ Low control bandwidth

- Low-geared high-torque motor (pseudo direct drive)
  - ⊕ Low reflected inertia due to low gear ratio (impact robust, high speed and power)
  - ⊕ High bandwidth current control (force control)
  - ⊖ Relatively large (hard to integrate)
- Hydraulic actuation
  - ⊕ High force at small size/weight
  - ⊕ Very rugged
  - ⊕ Pressure sensor provides direct force feedback
  - ⊖ Onboard pump required
  - ⊖ Hard to downscale
  - ⊖ Energetically inefficient
  - ⊖ Can leak
- Pneumatic muscle actuators
  - ⊕ Lightweight
  - ⊕ High maximum contraction force
  - ⊖ Often with off-board pump
  - ⊖ Works only in contraction
  - ⊖ Nonlinear contraction-force-pressure characteristics
  - ⊖ Difficult to control
  - ⊖ Can be quite loud

OTHER ACTUATION TYPES
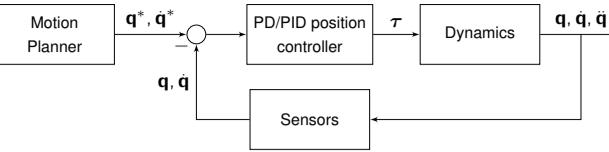
- New, unconventional actuator types
  - Shape memory alloy (SMA)
  - Electro-Active Polymer (EAP)
  - Piezo-electric
- Open issues
  - Low output force levels
  - Low displacement (strain)
  - Need kV power supplies
  - Low control bandwidth

STATIC VS. DYNAMICS STABILITY

- Statically stable
  - Bodyweight supported by at least three legs
  - Robot will not fall if all joints stop instantaneously
  - Safe, slow and inefficient
- Dynamic walking
  - Robot will fall if not continuously moving
  - Less than three legs can be in ground contact
  - Fast, efficient and demanding for actuation/control

KINEMATIC CONTROL
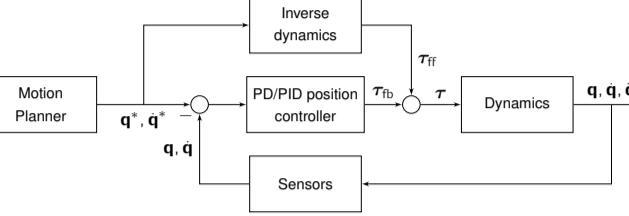
- High-gain joint position trajectory tracking

- ⊖ Performs poorly on unknown terrain

JOINT SPACE INVERSE DYNAMICS

- Low-gain joint control with model compensation

SUPPORT CONSISTENT INVERSE DYNAMICS

- Joint acceleration from multiple objectives
  - Track the swing leg (point F on foot)
    $$_\mathcal{I}\ddot{\mathbf{r}}_{IF} = \mathbf{J}_{F_P}\ddot{\mathbf{q}} + \dot{\mathbf{J}}_{F_P}\dot{\mathbf{q}}$$
    $$_\mathcal{I}\ddot{\mathbf{r}}^*_{IF} = \mathbf{k}_p(_\mathcal{I}\mathbf{r}^*_{IF} - _\mathcal{I}\mathbf{r}_{IF}) + \mathbf{k}_d(\dot{\mathbf{r}}^*_{IF} - \dot{\mathbf{r}}_{IF})$$
  - Move the base
    $$\dot{\mathbf{w}}_B = \mathbf{J}_B\ddot{\mathbf{q}} + \dot{\mathbf{J}}_B\dot{\mathbf{q}}$$
    $$\dot{\mathbf{w}}^*_B = \mathbf{k}_p\left(\begin{bmatrix}\mathbf{r}^*\\\varphi^*\end{bmatrix} - \begin{bmatrix}\mathbf{r}\\\varphi\end{bmatrix}\right) + k_d(\mathbf{w}^* - \mathbf{w})$$
  - Ensure contact constraint
    $$\ddot{\mathbf{r}}_c = \mathbf{J}_{c_P}\ddot{\mathbf{q}} + \dot{\mathbf{J}}_{c_P}\dot{\mathbf{q}} = \mathbf{0}$$
- Above tasks impose $3 + 6 + 9 = 18$ constraints $\implies$ fully define (18 DOF) system motion
- Inverse dynamics control
  $$\boldsymbol{\tau}^* = \left(\hat{\mathbf{N}}_c^\top \mathbf{S}^\top\right)^\dagger \hat{\mathbf{N}}_c^\top \left(\hat{\mathbf{M}}\dot{\mathbf{u}}^* + \hat{\mathbf{b}} + \hat{\mathbf{g}}\right)$$
  $$\ddot{\mathbf{q}}^* = \begin{bmatrix}\mathbf{J}_{F_P}\\\mathbf{J}_B\\\mathbf{J}_{c_P}\end{bmatrix}^{-1}\begin{bmatrix}_\mathcal{I}\ddot{\mathbf{r}}^*_{IF}\\\dot{\mathbf{w}}^*_B\\\mathbf{0}\end{bmatrix} - \begin{bmatrix}\dot{\mathbf{J}}_{F_P}\\\dot{\mathbf{J}}_B\\\dot{\mathbf{J}}_{c_P}\end{bmatrix}\dot{\mathbf{q}} \rightarrow \dot{\mathbf{u}}^*$$
- Alternative: task-space inverse dynamics control (direcly regulating in task space as sequential QP)

LOCOMOTION AS OPTIMIZATION PROBLEM

- Inverse dynamics as constrained, prioritized optimization
  - Step 1: move base
    $$\arg\min_{\ddot{q}} ||\dot{\mathbf{w}}^*_B(t) - \mathbf{J}_B\ddot{\mathbf{q}} - \dot{\mathbf{J}}_B\dot{\mathbf{q}}|| \quad \text{subject to}$$

$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{b} + \mathbf{g} + \mathbf{J}_c^\top \mathbf{F}_c = \mathbf{S}^\top \boldsymbol{\tau}$	EoM
$\mathbf{J}_{c_P}\ddot{\mathbf{q}} + \dot{\mathbf{J}}_{c_P}\dot{\mathbf{q}} = \mathbf{0}$	Contact constraint
$\mathbf{F}_{c,n_i} > F_{n,\min}$	Normal contact force
$\mu\mathbf{F}_{c,n_i} > ||\mathbf{F}_{c,t_i}||_2$	Friction cone

  - Step 2: move swing leg
    $$\arg\min_{\ddot{q}} ||_\mathcal{I}\ddot{\mathbf{r}}^*_{IF}(t) - \mathbf{J}_{F_P}\ddot{\mathbf{q}} - \dot{\mathbf{J}}_{F_P}\dot{\mathbf{q}}|| \quad \text{subject to}$$

$\dot{\mathbf{w}}^*_B(t) - \mathbf{J}_B\ddot{\mathbf{q}} - \dot{\mathbf{J}}_B\dot{\mathbf{q}} = c_1$	Higher prio unaffected
$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{b} + \mathbf{g} + \mathbf{J}_c^\top \mathbf{F}_c = \mathbf{S}^\top \boldsymbol{\tau}$	EoM
$\mathbf{F}_{c,n_i} > F_{n,\min}$	Normal contact force
$\mu\mathbf{F}_{c,n_i} > ||\mathbf{F}_{c,t_i}||_2$	Friction cone
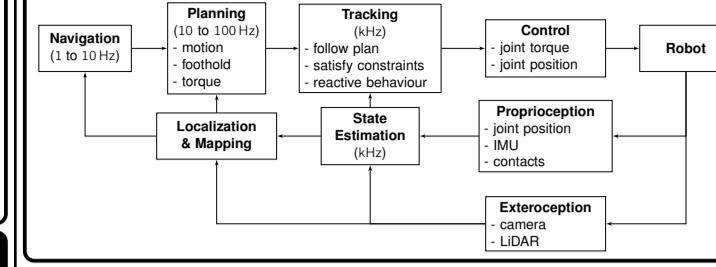
  - Final step: minimize e.g. torque $\boldsymbol{\tau}$ or tangential contact forces $\mathbf{F}_{c,t_i}$ such that all other tasks are still fulfilled

STABILITY ANALYSIS THROUGH LIMIT CYCLES

- Poincaré map $\mathbf{x}_{k+1} = P(\mathbf{x}_k)$
- Fix point characterized by $\mathbf{x}^* = P(\mathbf{x}^*)$
- Linearization of mapping $\Delta\mathbf{x}_{k+1} = \frac{\partial P}{\partial \mathbf{x}}\Delta\mathbf{x}_k = \boldsymbol{\Phi}\Delta\mathbf{x}_k$
- The system is stable iff all eigenvalues $\lambda_i(\boldsymbol{\Phi}) < 1$

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15 ANYmal

LOCOMOTION CONTROL PIPELINE

FLOATING BASE MODEL

$$\mathbf{M}(\mathbf{q})\dot{\mathbf{u}} + \mathbf{h}(\mathbf{q}, \mathbf{u}) = \mathbf{S}^\top \boldsymbol{\tau} + \mathbf{J}_S^\top \boldsymbol{\lambda}$$

- Support/contact Jacobian $\mathbf{J}_S$ consists of n_c stacked (geometric) end-effector Jacobians $\mathbf{J}_{0,E_{n_c}}$
  $$\mathbf{J}_S = \begin{bmatrix}_\mathcal{I}\mathbf{J}_{0,E_1}\\\vdots\\_\mathcal{I}\mathbf{J}_{0,E_{n_c}}\end{bmatrix}$$
  where
  $$_\mathcal{I}\mathbf{J}_{0,E_k} = \begin{bmatrix}\mathbb{I} & -\mathbf{C}_{\mathcal{IB}}\left[_\mathcal{B}\mathbf{r}_{BE_k}\right]_\times & \mathbf{0} & \dots & _\mathcal{I}\mathbf{J}_{0,BE_k} & \dots & \mathbf{0}\end{bmatrix}$$
  where $_\mathcal{I}\mathbf{J}_{0,BE_k}$ is the relative (geometric) Jacobian from the base B to end effector E_k
- Stacked external (ground contact) forces
  $$\boldsymbol{\lambda} = \begin{bmatrix}\boldsymbol{\lambda}_1\\\vdots\\\boldsymbol{\lambda}_{n_c}\end{bmatrix}$$

WHOLE BODY CONTROL IDEA

1. Separate the EoM into base and joint dynamics
   $$\begin{bmatrix}\textcolor{red}{\mathbf{M}_b}\\\mathbf{M}_j\end{bmatrix}\dot{\mathbf{u}} + \begin{bmatrix}\textcolor{red}{\mathbf{h}_b}\\\mathbf{h}_j\end{bmatrix} = \begin{bmatrix}\textcolor{red}{\mathbf{0}}\\\boldsymbol{\tau}\end{bmatrix} + \begin{bmatrix}\textcolor{red}{\mathbf{J}_{S_b}^\top}\\\mathbf{J}_{S_j}^\top\end{bmatrix}\boldsymbol{\lambda}$$
2. Define a vector of quantities used for control
   $$\boldsymbol{\xi} = \begin{bmatrix}\dot{\mathbf{u}}\\\boldsymbol{\lambda}\end{bmatrix}$$
   Note: $\boldsymbol{\tau}$ follows directly from this
3. Break down locomotion problem into simpler tasks of the form
   $$\mathbf{T}_p = \begin{cases}\mathbf{W}_{eq,p}(\mathbf{A}_p\boldsymbol{\xi} - \mathbf{b}_p) = \mathbf{0}\\\mathbf{W}_{ineq,p}(\mathbf{D}_p\boldsymbol{\xi} - \mathbf{f}_p) \le \mathbf{0}\end{cases}$$
   where the $\mathbf{W}$ are weighting factors
4. Solve the tasks hierarchically i.e. project the constraints into the nullspace of higher priority tasks
5. Given $\dot{\mathbf{u}}^*, \boldsymbol{\tau}^*$ from the optimization, obtain $\boldsymbol{\tau}_{des}$ from the joint part of the EoM
   $$\boldsymbol{\tau}_{des} = \mathbf{M}_j(\mathbf{q})\dot{\mathbf{u}}^* + \mathbf{h}_j(\mathbf{q}, \mathbf{u}) - \mathbf{J}_{S_j}^\top\boldsymbol{\lambda}^*$$
6. Use $\boldsymbol{\tau}_{des}$ as a feedforward term
   $$\boldsymbol{\tau}^* = \boldsymbol{\tau}_{des} + \mathbf{k}_p\tilde{\mathbf{q}} + \mathbf{k}_d\check{\mathbf{q}}$$

WHOLE BODY CONTROL TASKS

Base equations of motion	$\begin{bmatrix}\mathbf{M}_b & -\mathbf{J}_{S_b}^\top\end{bmatrix}\boldsymbol{\xi} = -\mathbf{h}$
Force limits	$(_\mathcal{I}\mathbf{h} - _\mathcal{I}\mathbf{n}\mu)^\top {}_\mathcal{I}\boldsymbol{\lambda}_k \le 0$
	$-(_\mathcal{I}\mathbf{h} + _\mathcal{I}\mathbf{n}\mu)^\top {}_\mathcal{I}\boldsymbol{\lambda}_k \le 0$
	$(_\mathcal{I}\mathbf{l} - _\mathcal{I}\mathbf{n}\mu)^\top {}_\mathcal{I}\boldsymbol{\lambda}_k \le 0$
	$-(_\mathcal{I}\mathbf{l} + _\mathcal{I}\mathbf{n}\mu)^\top {}_\mathcal{I}\boldsymbol{\lambda}_k \le 0$
Torque limits	$\boldsymbol{\tau}_{\min} \le \boldsymbol{\tau} \le \boldsymbol{\tau}_{\max}$
No motion at contact points	$\begin{bmatrix}\mathbf{J}_S & \mathbf{0}\end{bmatrix}\boldsymbol{\xi} = -\dot{\mathbf{J}}_S\mathbf{u}$
CoM motion tracking	$\ddot{\mathbf{r}}_B^* = \ddot{\mathbf{r}}_{B,des} + \mathbf{k}_p\Delta\mathbf{r}_B + \mathbf{k}_d\Delta\dot{\mathbf{r}}_B$
	$\begin{bmatrix}\mathbf{J}_{0,B} & \mathbf{0}\end{bmatrix}\boldsymbol{\xi} = \ddot{\mathbf{r}}_B^* - \dot{\mathbf{J}}_{0,B}\mathbf{u}$
Torso angular motion tracking	
Foot motion tracking	
End-effector motion tracking	
End-effector force tracking	$\begin{bmatrix}\mathbf{0} & \mathbb{I}\end{bmatrix}\boldsymbol{\xi} = \begin{bmatrix}\mathbf{0} & \dots & \boldsymbol{\lambda}^*\end{bmatrix}^\top$
Torso orientation adaptation (depending on arm config)	
Contact force minimization	$\begin{bmatrix}\mathbf{0} & \mathbb{I}\end{bmatrix}\boldsymbol{\xi} = \mathbf{0}$

⚠ Tasks in descending priority order

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16 ROTORCRAFTS

OVERVIEW

- Rotorcrafts: aircraft which produces lift from a rotary wing turning in a plane close to horizontal
- Pros and cons
  - ⊕ Agility
  - ⊕ Ability to hover
  - ⊕ Ability to vertically take-off and land
  - ⊖ Complexity
  - ⊖ High maintenance costs
  - ⊖ Poor efficiency, especially in forward flight

LARGE SCALE ROTORCRAFTS

- Helicopter
  - Power driven main rotor
  - Main rotor tilted to fly forward
  - Air flow from top to bottom
- Autogyro
  - Un-driven main rotor, tilted away
  - Forward propeller for propulsion
  - Rotor generates uplift like a wing
  - Air flow from bottom to top
  - Not capable of hovering
- Gyrodyne
  - Power driven main rotor
  - Main rotor can't tilt
  - Additional propeller for propulsion
  - Air flows from top to bottom

HELICOPTER TYPES

- Single rotor
- ➕ Most efficient
- ➖ Limited payload
- ➖ Need to balance counter-torque with tail rotor
- Multi rotor
- ➖ Reduced efficiency due to multiple rotors and down-wash interference
- ➕ Able to lift more payload
- ➕ Even number of rotors balance counter torque
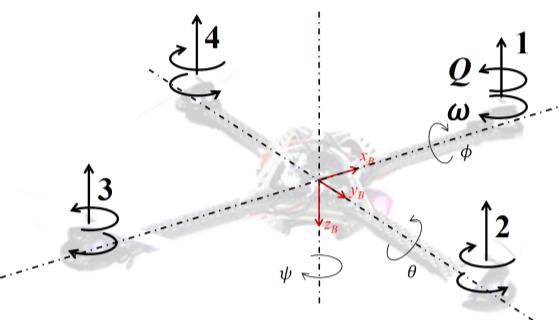
UAS/MAV ROTORCRAFTS

- Quadrotor/multicopter
  - 4+ propellers in cross configuration
  - Direct drive (no gearbox)
  - Very good torque compensation
  - High maneuverability/agility
  - High dynamics (dynamic control required)
- Standard helicopter
  - Most efficient design
  - Swashplate mechanism for controlling cyclic pitch
  - Tail rotor required
  - Low dynamics
  - Low agility
  - Difficult to scale down in size
- Ducted fan
  - Fix propeller
  - Torques produced by control surfaces
  - Heavy
  - Compact, protection of propellers
- Coaxial
  - Complex mechanisms
  - Passively stable
  - Compact
  - Suitable for miniaturization
- Omnidirectional multicopter
  - Propellers rotatable around connecting axes
  - Can fly in all directions at any attitude of the main body
  - Generates forces in all directions
  - Complex control and aerodynamic coupling

NOMENCLATURE

$\mathcal{E}$	Inertial (Earth-fixed) frame with origin E
B	Center of gravity (COG) of $\mathcal{B}$
$\mathcal{B}$	Body frame with origin B
ϕ, θ, ψ	Roll, pitch and yaw angles
$_\mathcal{B}\omega_{\mathcal{EB}}$	Angular velocity of $\mathcal{B}$ (body angular rates) w.r.t. $\mathcal{E}$ (expressed in $\mathcal{B}$) with components p, q, r
$_\mathcal{B}\dot{\mathbf{r}}_{EB}$	Body velocity (expressed in $\mathcal{B}$) with components u, v, w
T_i	Propeller thrust forces
Q_i	Propeller drag torques
$\omega_{p,i}$	Propeller angular rates
l	Arm length
h	Propeller height (from COG to propeller plane)
m	Total mass
$\mathbf{I}$	Inertia tensor of $\mathcal{B}$ w.r.t. B
$\mathbf{F}_{\text{Aero}}$	Resultant aerodynamic force on B
$\mathbf{F}$	Resultant force on B
$\mathbf{M}_{\text{Aero}}$	Resultant aerodynamic torque on B
$\mathbf{M}$	Resultant moment/torque on $\mathcal{B}$ w.r.t. B
b	Propeller thrust constant
d	Propeller drag constant
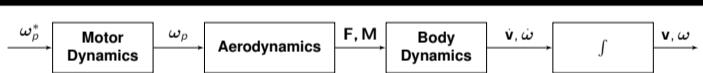
MODELING PURPOSES

- System analysis: model allows evaluating characteristics (stability, controllability, power consumption, etc.) of future aircraft in flight or its behavior in various conditions
- Control law design and simulation: model allows comparing various control techniques and tune their parameters

MODELING ASSUMPTIONS

- CoG and body frame origin coincide
- No interaction with ground or other surfaces
- Rigid and symmetric structure
- Rigid propellers
- No fuselage drag
- Frame (body) is symmetric in xz- and yz-plane (diagonal inertia tensor)

MODEL COMPONENTS

- System input: desired rotor speeds ω_r^* (or the corresponding motor voltage inputs)
- Motor dynamics are very fast and can therefore be neglected for control design

ATTITUDE

- Representation using yaw, pitch, roll Tait-Bryan angles

$$\mathbf{C}_{\mathcal{EB}} = \mathbf{C}_{\mathcal{E}1}(\psi)\mathbf{C}_{12}(\theta)\mathbf{C}_{2\mathcal{B}}(\phi)$$
$$= \mathbf{C}_z(\psi)\mathbf{C}_y(\theta)\mathbf{C}_x(\phi)$$
$$= \begin{bmatrix} c_\theta c_\psi & c_\psi s_\phi s_\theta - c_\phi s_\psi & s_\phi s_\psi + c_\phi c_\psi s_\theta \\ c_\theta s_\psi & c_\phi c_\psi + s_\phi s_\theta s_\psi & c_\phi s_\theta s_\psi - c_\psi s_\phi \\ -s_\theta & c_\theta s_\phi & c_\phi c_\theta \end{bmatrix}$$

$$\mathbf{C}_{\mathcal{BE}} = \mathbf{C}_{\mathcal{EB}}^\top = \begin{bmatrix} c_\theta c_\psi & c_\theta s_\psi & -s_\theta \\ c_\psi s_\phi s_\theta - c_\phi s_\psi & c_\phi c_\psi + s_\phi s_\theta s_\psi & c_\theta s_\phi \\ s_\phi s_\psi + c_\phi c_\psi s_\theta & c_\phi s_\theta s_\psi - c_\psi s_\phi & c_\phi c_\theta \end{bmatrix}$$

- Limits

$$\phi \in (-\pi, \pi)$$
$$\theta \in (-\pi/2, \pi/2)$$
$$\psi \in (-\pi, \pi)$$

ANGULAR VELOCITY

- Rotation matrix $\mathbf{C}_{\mathcal{EB}}$ from $\mathcal{B}$ to $\mathcal{E}$

$$\mathbf{C}_{\mathcal{EB}} = \mathbf{C}_{\mathcal{E}1}(\psi)\mathbf{C}_{12}(\theta)\mathbf{C}_{2\mathcal{B}}(\phi)$$

1. $\mathbf{C}_{\mathcal{E}1}$ is rotation (yaw ψ) around $\mathbf{e}_z^\mathcal{E}$ ($\mathbf{C}_z(\psi)$)
2. $\mathbf{C}_{12}$ is rotation (pitch θ) around $\mathbf{e}_y^1$ ($\mathbf{C}_y(\theta)$)
3. $\mathbf{C}_{2\mathcal{B}}$ is rotation (roll ϕ) around $\mathbf{e}_x^2$ ($\mathbf{C}_x(\phi)$)

- ZYX Tait-Bryan angles (3-2-1 intrinsic Euler angles)

$$\boldsymbol{\Theta} = \begin{bmatrix} \phi & \theta & \psi \end{bmatrix}^\top$$

- Relation to angular rates $\dot{\boldsymbol{\Theta}}$

$$\boldsymbol{\Theta} = \begin{bmatrix} \dot\phi & \dot\theta & \dot\psi \end{bmatrix}^\top \neq {_\mathcal{B}}\omega_{\mathcal{EB}} = \begin{bmatrix} p & q & r \end{bmatrix}^\top$$

$$\omega_{\mathcal{EB}} = \omega_{\mathcal{E}1} + \omega_{12} + \omega_{2\mathcal{B}} = \dot\psi\mathbf{e}_z^\mathcal{E} + \dot\theta\mathbf{e}_y^1 + \dot\phi\mathbf{e}_x^2$$
$$= \dot\psi\mathbf{e}_z^1 + \dot\theta\mathbf{e}_y^2 + \dot\phi\mathbf{e}_x^\mathcal{B}$$

$$_\mathcal{B}\omega_{\mathcal{EB}} = \dot\psi\mathbf{C}_{\mathcal{B}1}\mathbf{e}_z^1 + \dot\theta\mathbf{C}_{\mathcal{B}2}\mathbf{e}_y^2 + \dot\phi\mathbf{e}_x^\mathcal{B}$$
$$= \begin{bmatrix} \mathbf{e}_x^\mathcal{B} & \mathbf{C}_{\mathcal{B}2}\mathbf{e}_y^2 & \mathbf{C}_{\mathcal{B}1}\mathbf{e}_z^1 \end{bmatrix} \boldsymbol{\Theta}$$
$$= \begin{bmatrix} \mathbf{e}_x^\mathcal{B} & \mathbf{C}_x^\top(\phi)\mathbf{e}_y^2 & \mathbf{C}_{\mathcal{B}2}\mathbf{C}_{21}\mathbf{e}_z^1 \end{bmatrix} \dot{\boldsymbol{\Theta}}$$
$$= \begin{bmatrix} \mathbf{e}_x^\mathcal{B} & \mathbf{C}_x^\top(\phi)\mathbf{e}_y^2 & \mathbf{C}_x^\top(\phi)\mathbf{C}_y^\top(\theta)\mathbf{e}_z^1 \end{bmatrix} \dot{\boldsymbol{\Theta}}$$

$$_\mathcal{B}\omega_{\mathcal{EB}} = \underbrace{\begin{bmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\phi & \sin\phi\cos\theta \\ 0 & -\sin\phi & \cos\phi\cos\theta \end{bmatrix}}_{\mathbf{E}_{R,\text{Euler},ZYX}} \dot{\boldsymbol{\Theta}}$$

- Linearization around hover

$$\mathbf{E}_{R,\text{Euler},ZYX}\Big|_{\phi=\theta=0} = \mathbb{I}_{3\times3} \iff \dot{\boldsymbol{\Theta}} = {_\mathcal{B}}\omega_{\mathcal{EB}}$$

- Singularity: $\det(\mathbf{E}_{R,\text{Euler},ZYX}) = -\cos\theta$
  $\implies$ gimbal lock at $\theta = \pm\pi/2$

UAV DYNAMICS

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{b}(\mathbf{q},\dot{\mathbf{q}}) + \mathbf{g}(\mathbf{q}) + \mathbf{J}_{\text{ext}}^\top\mathbf{F}_{\text{ext}} = \mathbf{S}^\top\boldsymbol{\tau}_{\text{act}}$$

$\mathbf{q}$	Generalized coordinates
$\mathbf{M}(\mathbf{q})$	Mass matrix
$\mathbf{b}(\mathbf{q},\dot{\mathbf{q}})$	Centrifugal and Coriolis forces
$\mathbf{g}(\mathbf{q})$	Gravity forces
$\boldsymbol{\tau}_{\text{act}}$	Actuation torques
$\mathbf{S}$	Selection matrix of actuated joints
$\mathbf{F}_{\text{ext}}$	External forces (exerted by system)
$\mathbf{J}_{\text{ext}}$	(Geometric) Jacobian of external forces

EQUATION OF FORCES

1. Conservation of linear momentum
$$\dot{\mathbf{p}}_S = {}^d/_{dt}(m\mathbf{v}_S) = m\mathbf{a}_S = \mathbf{F}_{\text{ext}}$$
2. Modified notation
$$\dot{\mathbf{p}}_B = {}^d/_{dt}(m\dot{\mathbf{r}}_{EB}) = m\ddot{\mathbf{r}}_{EB} = \mathbf{F}$$
3. Alternative formulation
$$\frac{d}{dt}\bigg|_\mathcal{E}(m\dot{\mathbf{r}}_{EB}) = \sum_j \mathbf{F}_j$$

   where subscript $\mathcal{E}$ denotes time differentiation w.r.t. the inertial frame $\mathcal{E}$ and $\dot{\mathbf{r}}_{EB}$ is the velocity of the COG B
4. Change of frame gives
$$\sum_j \mathbf{F}_j = \frac{d}{dt}\bigg|_\mathcal{B}(m\dot{\mathbf{r}}_{EB}) + \omega_{\mathcal{EB}} \times (m\dot{\mathbf{r}}_{EB})$$
5. Assume constant mass and express vectors in $\mathcal{B}$
$$\frac{1}{m}{_\mathcal{B}}\mathbf{F} = {_\mathcal{B}}\ddot{\mathbf{r}}_{EB} + {_\mathcal{B}}\omega_{\mathcal{EB}} \times {_\mathcal{B}}\dot{\mathbf{r}}_{EB}$$
6. The resultant force $\mathbf{F}$ is composed of aerodynamic forces and gravity
$$\frac{1}{m}{_\mathcal{B}}\mathbf{F}_{\text{Aero}} + \mathbf{C}_{\mathcal{BE}}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}g = \begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} + \begin{bmatrix} p \\ q \\ r \end{bmatrix} \times \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$
$$\implies \frac{1}{m}{_\mathcal{B}}\mathbf{F}_{\text{Aero}} + \begin{bmatrix} -\sin\theta \\ \sin\phi\cos\theta \\ \cos\phi\cos\theta \end{bmatrix}g = \begin{bmatrix} qw - rv \\ ru - pw \\ pv - qu \end{bmatrix} = \begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix}$$

EQUATION OF MOMENTS

1. Conservation of angular momentum
$$\dot{\mathbf{N}}_S = {}^d/_{dt}(\mathbf{I}_S\boldsymbol{\Omega}_\mathcal{B}) = \mathbf{I}_S\boldsymbol{\Psi}_\mathcal{B} + \boldsymbol{\Omega}_\mathcal{B} \times \mathbf{I}_S\boldsymbol{\Omega}_\mathcal{B} = \mathbf{T}$$
2. Modified notation
$$\dot{\mathbf{N}}_B = {}^d/_{dt}(\mathbf{I}_B\,\omega_{\mathcal{EB}}) = \mathbf{I}_B\,\dot{\omega}_{\mathcal{EB}} + \omega_{\mathcal{EB}} \times \mathbf{I}_B\,\omega_{\mathcal{EB}} = \mathbf{M}$$
3. Express vectors in $\mathcal{B}$
$$\mathbf{I}_B\,{_\mathcal{B}}\dot{\omega}_{\mathcal{EB}} + {_\mathcal{B}}\omega_{\mathcal{EB}} \times \mathbf{I}_B\,{_\mathcal{B}}\omega_{\mathcal{EB}} = {_\mathcal{B}}\mathbf{M}$$
4. The resultant moment M is composed of aerodynamic moments

$$\begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}\begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} + \begin{bmatrix} p \\ q \\ r \end{bmatrix} \times \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}\begin{bmatrix} p \\ q \\ r \end{bmatrix} = {_\mathcal{B}}\mathbf{M}_{\text{Aero}}$$

$$\implies \begin{bmatrix} I_{xx}\dot{p} + qr(I_{zz} - I_{yy}) \\ I_{yy}\dot{q} + pr(I_{xx} - I_{zz}) \\ I_{zz}\dot{r} \end{bmatrix} = {_\mathcal{B}}\mathbf{M}_{\text{Aero}}$$

using $I_{yy} = I_{zz}$ (assumption)

BODY DYNAMICS

- Equation of forces and moments in matrix form

$$\begin{bmatrix} m\mathbb{I}_{3\times3} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_B \end{bmatrix} \begin{bmatrix} {}_{\mathcal{B}}\ddot{\mathbf{r}}_{EB} \\ {}_{\mathcal{B}}\dot{\boldsymbol{\omega}}_{\mathcal{E}\mathcal{B}} \end{bmatrix} + \begin{bmatrix} {}_{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{E}\mathcal{B}} \times m\, {}_{\mathcal{B}}\dot{\mathbf{r}}_{EB} \\ {}_{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{E}\mathcal{B}} \times \mathbf{I}_B\, {}_{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{E}\mathcal{B}} \end{bmatrix} = \begin{bmatrix} {}_{\mathcal{B}}\mathbf{F} \\ {}_{\mathcal{B}}\mathbf{M} \end{bmatrix}$$

AERODYNAMIC FORCES

$${}_{\mathcal{B}}\mathbf{F}_{\text{Aero}} = {}_{\mathcal{B}}\mathbf{F}_{\text{Thrust}} = \begin{bmatrix} 0 \\ 0 \\ -b(\omega_{p_1}^2 + \omega_{p_2}^2 + \omega_{p_3}^2 + \omega_{p_4}^2) \end{bmatrix}$$

- Thrust forces in the shaft direction

$${}_{\mathcal{B}}\mathbf{F}_{\text{Thrust}} = \sum_{i=1}^{4} \begin{bmatrix} 0 \\ 0 \\ -T_i \end{bmatrix}, \quad T_i = b_i \omega_{p,i}^2$$

- Additional forces (neglected during hover)
  - Hub forces along the horizontal speed

$${}_{\mathcal{B}}\mathbf{H} = \sum_{i=1}^{4} H_i \frac{{}_{\mathcal{B}}\dot{\mathbf{r}}_{EB_h}}{||{}_{\mathcal{B}}\dot{\mathbf{r}}_{EB_h}||}, \quad {}_{\mathcal{B}}\dot{\mathbf{r}}_{EB_h} = \begin{bmatrix} u & v & 0 \end{bmatrix}^\top$$

AERODYNAMIC MOMENTS

$${}_{\mathcal{B}}\mathbf{M}_{\text{Aero}} = {}_{\mathcal{B}}\mathbf{M}_{\text{Thrust}} + {}_{\mathcal{B}}\mathbf{M}_{\text{Drag}} = \begin{bmatrix} lb(\omega_{p_4}^2 - \omega_{p_2}^2) \\ lb(\omega_{p_1}^2 - \omega_{p_3}^2) \\ d(-\omega_{p_1}^2 + \omega_{p_2}^2 - \omega_{p_3}^2 + \omega_{p_4}^2) \end{bmatrix}$$

- Thrust induced moment from propeller rotations

$${}_{\mathcal{B}}\mathbf{M}_{\text{Thrust}} = \begin{bmatrix} l(T_4 - T_2) \\ l(T_1 - T_3) \\ 0 \end{bmatrix}$$

- Drag torques (define required motor power)

$${}_{\mathcal{B}}\mathbf{M}_{\text{Drag}} = \begin{bmatrix} 0 \\ 0 \\ \sum_{i=1}^{4} Q_i(-1)^i \end{bmatrix}, \quad Q_i = d_i \omega_{p,i}^2$$

- Additional moments (neglected during hover)
  - Inertial counter torques

$${}_{\mathcal{B}}\mathbf{M}_{\text{CT}_i} = \mathbf{I}_{\text{Prop}_i} \begin{bmatrix} 0 \\ 0 \\ \omega_{p,i} \end{bmatrix}$$

  - Propeller gyro effect

$${}_{\mathcal{B}}\mathbf{M}_{\text{Gyro}_i} = \begin{bmatrix} \mathbf{I}_{\text{Prop}_i} \omega_{p,i} \end{bmatrix} \times {}_{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{E}\mathcal{B}}$$

  - Rolling moments

$${}_{\mathcal{B}}\mathbf{R} = \sum_{i=1}^{4} R_i(-1)^{i-1} \frac{{}_{\mathcal{B}}\dot{\mathbf{r}}_{EB_h}}{||{}_{\mathcal{B}}\dot{\mathbf{r}}_{EB_h}||}$$

  - Hub moments

$${}_{\mathcal{B}}\mathbf{M}_{\text{Hub}} = \sum_{i=1}^{4} H_i\, {}_{\mathcal{B}}\mathbf{p}_{,i} \times \frac{{}_{\mathcal{B}}\dot{\mathbf{r}}_{EB_h}}{||{}_{\mathcal{B}}\dot{\mathbf{r}}_{EB_h}||}$$

HOVER MODEL APPROXIMATION

- Equations of motion in body frame
  - Translational dynamics

$$\boxed{\begin{aligned} m\dot{u} &= m(rv - qw) - mg\sin\theta \\ m\dot{v} &= m(pw - ru) + mg\sin\phi\cos\theta \\ m\dot{w} &= m(qu - pv) + mg\cos\phi\cos\theta - \dots \\ &\quad \dots - b(\omega_{p_1}^2 + \omega_{p_2}^2 + \omega_{p_3}^2 + \omega_{p_4}^2) \end{aligned}}$$

  - Rotational dynamics

$$\boxed{\begin{aligned} I_{xx}\dot{p} &= qr(I_{yy} - I_{zz}) + lb(\omega_{p_4}^2 - \omega_{p_2}^2) \\ I_{yy}\dot{q} &= pr(I_{zz} - I_{xx}) + lb(\omega_{p_1}^2 - \omega_{p_3}^2) \\ I_{zz}\dot{r} &= d(-\omega_{p_1}^2 + \omega_{p_2}^2 - \omega_{p_3}^2 + \omega_{p_4}^2) \end{aligned}}$$

PROPELLER AERODYNAMICS

- Propeller in hover
  - Thrust force T (perpendicular to propeller plane)

$$T = \frac{1}{2}\rho A_p C_T (\omega_p R_p)^2$$

  - Drag torque Q (torque around propeller plane; opposite to prop spin direction)

$$Q = \frac{1}{2}\rho A_p C_Q (\omega_p R_p)^2 R_p$$

  - C_T, C_Q depend on blade pitch angle (propeller geometry), Reynolds number, etc.
- Propeller in forward flight
  - Hub force H (perpendicular to T; opposes horizontal flight direction)

$$H = \frac{1}{2}\rho A_p C_H (\omega_p R_p)^2$$

  - Rolling moment R (around flight direction)

$$R = \frac{1}{2}\rho A_p C_R (\omega_p R_p)^2 R_p$$

  - C_H, C_R depend on the advance ratio $\mu = \frac{||{}_{\mathcal{B}}\dot{\mathbf{r}}_{EB_h}||}{\omega_p R_p}$

MOMENTUM THEORY

- Idea: by actio-reactio, the power put into the fluid to change its momentum downwards is the thrust force at the propeller

MOMENTUM THEORY ASSUMPTIONS

- Infinitely thin propeller disc area A_p
- Uniform thrust and velocity distribution over disc area (allows 1D flow analysis)
- Quasi-static airflow (constant flow properties)
- No viscous effects (no profile drag)
- Incompressible flow
- Stationary control volume A

FLUID DYNAMICS PRINCIPLES

- **Conservation of fluid mass**: mass flow in and out of control volume is equal in a quasi-static flow

$$\int_A \rho(\mathbf{u}\cdot\mathbf{n})\,dA = 0$$

  - Fluid density ρ
  - Flow speed $\mathbf{u}$ at surface element dA
  - Surface element normal unit vector $\mathbf{n}$

- **Conservation of fluid momentum**: net force on fluid is the change in momentum of fluid

$$\int_A \rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n})\,dA = -\int_A p\mathbf{n}\,dA + \mathbf{F}$$

assuming unconstrained flow (net pressure force is 0)

  - Pressure p at surface element dA
  - Net force $\mathbf{F}$ on fluid

- **Conservation of energy**: work done on fluid results in gain of kinetic energy

$$\frac{1}{2}\int_A \rho u^2(\mathbf{u}\cdot\mathbf{n})\,dA = \frac{dE}{dt} = P$$

  - Energy E
  - Power P

MOMENTUM THEORY RESULTS

- Speed constant across propeller

$$u_1 = u_2$$

- Pressure change across propeller
- Far wake slipstream velocity is twice the induced velocity

$$u_3 = 2u_1$$

- Thrust force $T = 2\rho A_p(V + u_1)u_1$
- In hover ($V = 0$)
  - Thrust force $T = 2\rho A_p u_1^2$
  - Slipstream tube

$$A_0 = \infty \implies A_3 = A_p/2$$

HOVER POWER CONSUMPTION

- Ideal power to produce thrust

$$P = T(V + u_1)$$

  - In hover

$$P = \frac{T^{\frac{3}{2}}}{\sqrt{2\rho A_p}}, \quad T = mg$$

  * Reducing power by decreasing disc loading $\frac{T}{A_p}$ i.e. increasing A_p
    · Mechanical constraint: tip Mach number
    · More profile/structural drag
    · Longer tail, etc.

PROPELLER EFFICIENCY

- Figure of merit

$$FM = \frac{\text{Ideal power required to hover}}{\text{Actual power required to hover}} < 1$$

  - Ideal power given by momentum theory
  - Actual power includes profile drag, blade-tip vortex, etc.
  - FM used to compare different propellers with the same disc loading

BLADE ELEMENT MOMENTUM THEORY (BEMT)

- Divide propeller into blade elements dr
- Calculate forces for each (2D) airfoil element and sum them up
  - Angle of attack and Reynolds number from relative airflow V consisting of tangential velocity $\omega_p r$ and axial velocity $V_p + u_{\text{ind}}$
    * Main component V_p of axial velocity calculated using Momentum Theory
    * u_{ind} is the induced velocity
  - Lift and drag polar provide lift and drag coefficient based on angle of attack

DC MOTOR MODEL

- Mechanical system
  - Change in rotational speed depends on generated motor torque T_m and drag torque Q

$$I_m \frac{\omega_m}{dt} = T_m(t) - Q(t)$$

  - Electromagnetic field in coil generates $T_m(t)$

$$T_m(t) = k_T i(t)$$

  with torque constant k_T and current $i(t)$

- Electrical system
  - Voltage balance

$$L\frac{d}{dt}i(t) = U(t) - Ri(t) - U_{\text{Ind}}(t)$$

  with coil inductance L, resistance R, input voltage $U(t)$ and induced voltage U_{Ind} (back EMF from rotating coil inducing opposing current)

- Electrical dynamics usually much faster than mechanical, therefore approximate the full (second order) system as a as first order system

18 QUADCOPTER CONTROL

OVERVIEW

- Goal: move quadcopter in space
- 6 DOF, underactuated system (not all states are independently controllable)
- System input: 4 motor speeds (propellers in cross configuration, with two pairs $(1, 3)$ and $(2, 4)$ turning in opposite directions; motor 1 points forward)
- Full state feedback control (assume all states are known)

ASSUMPTIONS

- Motor dynamics negligible
- Flight at low speeds
  - Only dominant aerodynamic forces considered
  - Linearization about small roll and pitch angles

INTUITION

- Vertical control (along z-axis): simultaneous change in all rotor speeds
- Directional control (around z-axis): rotor speed imbalance between the two rotor pairs
- Longitudinal control (forward): converse change of rotor speeds 1 and 3
- Lateral control (sideways): converse change of rotor speeds 2 and 4

VIRTUAL CONTROL INPUTS

- Defining (4) virtual control inputs simplifies the equations of motion (yielding decoupled and linear inputs)
  - Moments along each axis

$$U_2 := lb(\omega_{p,4}^2 - \omega_{p,2}^2)$$
$$\implies I_{xx}\dot{p} = qr(I_{yy} - I_{zz}) + U_2$$
$$U_3 := lb(\omega_{p,1}^2 - \omega_{p,3}^2)$$
$$\implies I_{yy}\dot{q} = pr(I_{zz} - I_{xx}) + U_3$$
$$U_4 := d(-\omega_{p,1}^2 + \omega_{p,2}^2 - \omega_{p,3}^2 + \omega_{p,3}^2)$$
$$\implies I_{zz}\dot{r} = U_4$$

  - Total thrust

$$U_1 = b(\omega_{p,1}^2 + \omega_{p,2}^2 + \omega_{p,3}^2 + \omega_{p,4}^2)$$
$$\implies m\dot{u} = m(rv - qw) - mg\sin\theta$$
$$m\dot{v} = m(pw - ru) + mg\sin\phi\cos\theta$$
$$m\dot{w} = m(qu - pv) + mg\cos\phi\cos\theta - U_1$$

HIERARCHICAL (CASCADED) CONTROL

- Rotational dynamics independent of translational dynamics
- Cascaded control structure
  - Inner loop: attitude control
    * Inputs: U_1, U_2, U_3
  - Outer loop: position control
    * Inputs: U_1, ϕ, θ

LINEARIZED DYNAMICS AROUND HOVER

- Equilibrium (hover conditions)
  - $\phi = \theta = 0 \implies \mathbf{E}_{R,\text{Euler},ZYX} = \mathbb{I}_{3\times3} \implies {}_{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{E}\mathcal{B}} = \Theta$
  - $p = q = r = 0$
  - $U_2 = U_3 = U_4 = 0$
  - $U_1 = mg$
- Linearized rotational dynamics (no couplings)

$$\ddot{\phi} = \frac{1}{I_{xx}}U_2, \quad \ddot{\theta} = \frac{1}{I_{yy}}U_3, \quad \ddot{\psi} = \frac{1}{I_{zz}}U_4$$

PD ATTITUDE CONTROL

- P controller not sufficient
  - Example: roll subsystem

  $$U_2 = k_p(\phi^* - \phi) \implies \ddot{\phi} = \frac{1}{I_{xx}}k_p(\phi^* - \phi)$$

  (Undamped harmonic oscillator)

- PD controller for the 3 separate attitude subsystems

  $$U_2 = k_{p,\text{Roll}}(\phi^* - \phi) - \dot{\phi}k_{d,\text{Roll}}$$
  $$U_3 = k_{p,\text{Pitch}}(\theta^* - \theta) - \dot{\theta}k_{d,\text{Pitch}}$$
  $$U_4 = k_{p,\text{Yaw}}(\psi^* - \psi) - \dot{\psi}k_{d,\text{Yaw}}$$

- Angular rates from transformation of body angular rates
- Avoid integral elements in controller
- Parameter tuning
  - k_p chosen to get desired convergence time
  - k_d chosen to get desired damping

  ⚠ Actuator dynamics limit the control signal bandwidth and actuator signals mustn't saturate motors

- Thrust input $U_1 = T^*$

CONTROL ALLOCATION

- Transform virtual control inputs to rotor speeds using the allocation matrix

$$\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} b & b & b & b \\ 0 & -lb & 0 & lb \\ lb & 0 & -lb & 0 \\ -d & d & -d & d \end{bmatrix} \begin{bmatrix} \omega_{p,1}^2 \\ \omega_{p,2}^2 \\ \omega_{p,3}^2 \\ \omega_{p,4}^2 \end{bmatrix}$$

$$\implies \begin{bmatrix} \omega_{p,1}^2 \\ \omega_{p,2}^2 \\ \omega_{p,3}^2 \\ \omega_{p,4}^2 \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{4b} & 0 & \frac{1}{2lb} & -\frac{1}{4d} \\ \frac{1}{4b} & -\frac{1}{2lb} & 0 & \frac{1}{4d} \\ \frac{1}{4b} & 0 & -\frac{1}{2lb} & -\frac{1}{4d} \\ \frac{1}{4b} & \frac{1}{2lb} & 0 & \frac{1}{4d} \end{bmatrix}}_{\text{Allocation matrix}} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix}$$

- Limit motor speeds to feasible (positive) values

TRANSLATIONAL DYNAMICS W.R.T. INERTIAL FRAME

1. Conservation of linear momentum
   $$\dot{\mathbf{p}}_S = {}^d/_{dt}(m\mathbf{v}_S) = m\mathbf{a}_S = \mathbf{F}_{\text{ext}}$$
2. Modified notation
   $$\dot{\mathbf{p}}_B = {}^d/_{dt}(m\dot{r}_{EB}) = m\ddot{r}_{EB} = \mathbf{F}$$
3. Express vectors in $\mathcal{E}$
   $$m\,{}_{\mathcal{E}}\ddot{r}_{EB} = {}_{\mathcal{E}}\mathbf{F}$$
4. The resultant force is composed of the thrust (other aerodynamic forces neglected) and gravity

   $${}_{\mathcal{E}}\ddot{r}_{EB} = \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix} + \frac{1}{m}\mathbf{C}_{\mathcal{EB}}\,{}_{\mathcal{B}}\mathbf{F}_{\text{Thrust}} = \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix} + \frac{1}{m}\mathbf{C}_{\mathcal{EB}}\begin{bmatrix} 0 \\ 0 \\ -U_1 \end{bmatrix}$$

5. Define the components of the thrust force vector w.r.t $\mathcal{E}$

   $${}_{\mathcal{E}}\mathbf{F}_{\text{Thrust}} = \begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix} = U_1 \begin{bmatrix} s_\phi s_\psi + c_\phi c_\psi s_\theta \\ c_\phi s_\theta s_\psi - c_\psi s_\phi \\ c_\phi c_\theta \end{bmatrix}$$

   $$\implies {}_{\mathcal{E}}\ddot{r}_{EB} = \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix} - \frac{1}{m}{}_{\mathcal{E}}\mathbf{F}_{\text{Thrust}}$$

ALTITUDE CONTROL

- Vertical equation of translation dynamics w.r.t. $\mathcal{E}$
  $$\ddot{z} = g - \frac{1}{m}U_1 \cos\phi \cos\theta$$
- Define vertical thrust force w.r.t. $\mathcal{E}$ as virtual control
  $$T_z := U_1 \cos\phi \cos\theta \implies \ddot{z} = g - \frac{1}{m}T_z$$
- PD altitude control
  $$T_z = -k_{p,z}(z^* - z) + k_{d,z}\dot{z} - mg$$
- Transformation back to $\mathcal{B}$
  $$U_1 = \frac{T_z}{\cos\phi \cos\theta}$$

FULL POSITION CONTROL

1. 3 separate PD controllers for directions of thrust force vector w.r.t. $\mathcal{E}$
   $$\begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix} = \begin{bmatrix} -k_{p,x}(x^* - x) + k_{d,x}\dot{x} \\ -k_{p,y}(y^* - y) + k_{d,y}\dot{y} \\ -k_{p,z}(z^* - z) + k_{d,z}\dot{z} - mg \end{bmatrix}$$

2. Calculate the corresponding total thrust
   $$U_1 = \sqrt{T_x^2 + T_y^2 + T_z^2}$$

3. Calculate the corresponding roll and pitch angles after aligning the thrust force vector with the yaw angle

   $$\frac{1}{U_1}\begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix} = \begin{bmatrix} s_\phi s_\psi + c_\phi c_\psi s_\theta \\ c_\phi s_\theta s_\psi - c_\psi s_\phi \\ c_\phi c_\theta \end{bmatrix}$$

   $$\implies \frac{1}{U_1}\mathbf{C}_{\mathcal{E}1}(z,\psi)^\top \begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix} = \mathbf{C}_{\mathcal{E}1}(z,\psi)^\top \begin{bmatrix} s_\phi s_\psi + c_\phi c_\psi s_\theta \\ c_\phi s_\theta s_\psi - c_\psi s_\phi \\ c_\phi c_\theta \end{bmatrix}$$

   $$\implies \frac{1}{U_1}\mathbf{C}_{1\mathcal{E}}(z,\psi)\begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix} = \begin{bmatrix} c_\psi & s_\psi & 0 \\ -s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} s_\phi s_\psi + c_\phi c_\psi s_\theta \\ c_\phi s_\theta s_\psi - c_\psi s_\phi \\ c_\phi c_\theta \end{bmatrix}$$

   $$= \begin{bmatrix} \sin\theta \cos\phi \\ -\sin\phi \\ \cos\theta \cos\phi \end{bmatrix}$$

4. PD attitude control
5. Control allocation

19 FIXED WING UAV MODELING

NOMENCLATURE

$\mathcal{I}$	Inertial frame with origin I
${}_{\mathcal{B}}\mathbf{v}_a := {}_{\mathcal{B}}\dot{r}_{IB}$	Body velocity with components u, v, w
$\mathbf{V}_\infty$	Free-stream velocity
V	Airspeed
$\mathcal{W}$	Wind frame
α	Angle of attack
β	Sideslip angle
ϕ, θ, ψ	Roll, pitch and yaw angles
γ	Flight path angle (horizontal to ${}_{\mathcal{B}}\mathbf{v}_a$)
ξ	Heading angle (North to ${}_{\mathcal{B}}\mathbf{v}_a$)
$\mathcal{X}$	Course angle (North to ${}_{\mathcal{I}}\mathbf{v}$)
$\mathbf{v}_w$	Wind velocity
${}_{\mathcal{I}}\mathbf{v} := {}_{\mathcal{I}}\dot{r}_{IB}$	Ground-based inertial velocity (ground speed)
Θ	Attitude with components ϕ, θ, ψ
$\dot{\Theta}$	Angular rates
$\mathbf{W}$	Weight force in the COG
T	Thrust force
ϵ_T	Thrust offset angle
SF	Side force
L	Aircraft lift
D	Aircraft drag
ρ	Air density
S	Surface/planform area
c_L	Aircraft lift coefficient
c_D	Aircraft drag coefficient
L_m, M_m, N_m	Rolling, pitching, yawing moment
$L_{m_T}, M_{m_T}, N_{m_T}$	Rolling, pitching, yawing moment due to propulsion
c_l, c_m, c_n	Rolling, pitching, yawing moment coefficient
b	Wing span
$\bar{c}$	Mean geometric chord
$\mathbf{F}_T$	Resultant thrust force
SF	Side force from side-slip

MODELING ASSUMPTIONS

- UAV is rigid, symmetric structure (constant, diagonal $\mathbf{I}_B$)
- UAV is electric $\implies$ constant mass
- Motor without dynamics of gyroscopic effects
- Simplified aerodynamics
  - Operation in linear lift domain (no stall)
  - Fuselage lift/drag lumped
  - No side-slip
  - No interference effects (prop wash, etc.)
- Constant wind $\implies {}^d/_{dt}\,{}_{\mathcal{I}}\mathbf{v}_w = 0$
- Earth is flat and no effects from Coriolis acceleration i.e. Earth is an inertial (or Galilean) frame

FICTITIOUS (INERTIAL) FORCES

- Forces that appear to act on a mass whose motion is described using a non-inertial (rotating/linearly accelerating) frame
- Consequence of the acceleration of the physical object the non-inertial frame is connected to
- Examples: Coriolis and centrifugal force

BODY FRAME $\mathcal{B}$

- Origin in COG B; x forward, y right, z down
- Free-stream velocity $\mathbf{V}_\infty = -{}_{\mathcal{B}}\mathbf{v}_a$
- Air-mass relative speed (airspeed) $V = \sqrt{u^2 + v^2 + w^2}$

INERTIAL FRAME $\mathcal{I}$

- North, East, Down (NED)
  - Northing $N \equiv \mathbf{e}_x^{\mathcal{I}}$, Easting $E \equiv \mathbf{e}_y^{\mathcal{I}}$, Down $D \equiv \mathbf{e}_z^{\mathcal{I}}$
- Flat Earth assumption
- Origin in start position

WIND FRAME $\mathcal{W}$

- Origin in COG B; $\mathbf{e}_x^{\mathcal{W}}$ along ${}_{\mathcal{B}}\mathbf{v}_a$ i.e. opposite to $\mathbf{V}_\infty$
- Angle of attack $\alpha = \arctan {}^w/_u$
- Sideslip angle $\beta = \arcsin {}^v/_V$

GROUND SPEED

$${}_{\mathcal{I}}\mathbf{v} = \mathbf{C}_{\mathcal{IB}}\,{}_{\mathcal{B}}\mathbf{v}_a + {}_{\mathcal{I}}\mathbf{v}_w = \begin{bmatrix} V\cos\gamma\cos\xi + {}_{\mathcal{I}}\mathbf{v}_{w,N} \\ V\cos\gamma\sin\xi + {}_{\mathcal{I}}\mathbf{v}_{w,E} \\ -V\sin\gamma + {}_{\mathcal{I}}\mathbf{v}_{w,D} \end{bmatrix}$$

⚠ Measured by GNSS

NON-AERODYNAMIC FORCES

- Weight
  $${}_{\mathcal{I}}\mathbf{W} = mg\,\mathbf{e}_z^{\mathcal{I}}$$
- Thrust of the propeller T offset from $\mathbf{e}_x^{\mathcal{B}}$ by angle ϵ_T
  $${}_{\mathcal{B}}\mathbf{F}_{\text{Thrust}} = \begin{bmatrix} T\cos\epsilon_T \\ 0 \\ -T\sin\epsilon_T \end{bmatrix}$$

AERODYNAMIC FORCES

- Lift (wing, tail, fuselage contributions lumped)
  $$L = \frac{1}{2}\rho V^2 S c_L, \quad {}_{\mathcal{B}}\mathbf{F}_{\text{Lift}} = \begin{bmatrix} L\sin\alpha \\ 0 \\ -L\cos\alpha \end{bmatrix} \perp {}_{\mathcal{B}}\mathbf{v}_a$$
- Drag (lumped)
  $$D = \frac{1}{2}\rho V^2 S c_D, \quad {}_{\mathcal{B}}\mathbf{F}_{\text{Drag}} = \begin{bmatrix} -D\cos\alpha \\ 0 \\ -D\sin\alpha \end{bmatrix} \parallel {}_{\mathcal{B}}\mathbf{v}_a$$

  assuming no sideslip
- Side force (assumed zero)
  $${}_{\mathcal{B}}\mathbf{F}_{\text{Side}} = SF\,\mathbf{e}_y^{\mathcal{B}}$$

AERODYNAMIC MOMENTS

- Rolling moment
  $$L_m = \frac{1}{2}\rho V^2 S b c_l$$
  - Wing span b
- Pitching moment
  $$M_m = \frac{1}{2}\rho V^2 S \bar{c} c_m$$
  - Mean geometric chord $\bar{c}$
- Yawing moment
  $$N_m = \frac{1}{2}\rho V^2 S b c_n$$

COMPONENT BUILD-UP APPROACH

- Practical model structure for nominal flight regimes
- Aerodynamic forces and moments built up from both static and dynamic components, summed from each part of the aircraft
- Example: aircraft lift coefficient as 2${}^{\text{nd}}$ order expansion
  $$c_L \approx f(\alpha, q, \delta_e)$$
  $$= c_{L_0} + c_{L_\alpha}\alpha + c_{L_{\alpha^2}}\alpha^2 + c_{L_q}\hat{q} + c_{L_{\delta_e}}\delta_e + c_{L_{\alpha\delta_e}}\alpha\delta_e$$
  where e.g. $c_{L_\alpha} = {}^{\partial c_L}/_{\partial\alpha}$

BODY DYNAMICS

- Translation

  $$\boxed{\frac{1}{m}{}_{\mathcal{B}}\mathbf{F} = {}_{\mathcal{B}}\dot{\mathbf{v}}_a + {}_{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{IB}} \times {}_{\mathcal{B}}\mathbf{v}_a}$$

  $${}_{\mathcal{B}}\mathbf{F} = {}_{\mathcal{B}}\mathbf{F}_{\text{Lift}} + {}_{\mathcal{B}}\mathbf{F}_{\text{Drag}} + {}_{\mathcal{B}}\mathbf{F}_{\text{Side}} + {}_{\mathcal{B}}\mathbf{F}_{\text{Thrust}} + \mathbf{C}_{\mathcal{BI}}\,{}_{\mathcal{I}}\mathbf{W}$$
  $$= \begin{bmatrix} T\cos\epsilon_T - D\cos\alpha + L\sin\alpha - mg\sin\theta \\ SF + mg\sin\phi\cos\theta \\ -T\sin\epsilon_T - D\sin\alpha - L\cos\alpha + mg\cos\phi\cos\theta \end{bmatrix}$$

- Rotation

  $$\boxed{\mathbf{I}_B\,{}_{\mathcal{B}}\dot{\boldsymbol{\omega}}_{\mathcal{IB}} + {}_{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{IB}} \times \mathbf{I}_B\,{}_{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{IB}} = {}_{\mathcal{B}}\mathbf{M}}$$

  $${}_{\mathcal{B}}\mathbf{M} = \begin{bmatrix} L_m \\ M_m \\ N_m \end{bmatrix} + \begin{bmatrix} L_{m_T} \\ M_{m_T} \\ N_{m_T} \end{bmatrix}$$

MOMENT OF INERTIA

$$_\mathcal{B}\mathbf{I} = \begin{bmatrix} I_{xx} & 0 & I_{xz} \\ 0 & I_{yy} & 0 \\ I_{xz} & 0 & I_{zz} \end{bmatrix}$$

- Symmetric about y
- I_{xz} typically small
- Can be determined in swing tests

2D BODY DYNAMICS

- General

$$m\dot{V} = T\cos(\alpha + \epsilon_T) - D - mg\sin\gamma \quad (x)$$
$$mV\dot{\gamma} = L + T\sin(\alpha + \epsilon_T) - mg\cos\gamma \quad (y)$$

- Stationary level flight: $T = D$, $L = mg$
- Stationary gliding flight: $D = mg\sin\gamma$, $L = mg\cos\gamma$

  – Max. range ("best glide"): $\tan\gamma_{\min} = \left((c_L/c_D)_{\max}\right)^{-1}$
  – Max. endurance i.e. $(V_{\text{sink}})_{\min}$ at $(c_L^3/c_D^2)_{\max}$

20 FIXED WING UAV CONTROL

NOMENCLATURE

$\delta_T \in [0, 1]$	Throttle control input (normalized)
$\delta_E \in [-1, 1]$	Elevator control input (normalized)
$\delta_A \in [-1, 1]$	Aileron control input (normalized)
$\delta_R \in [-1, 1]$	Rudder control input (normalized)

REMARKS ON AIRCRAFT CONTROL

- Inherently nonlinear (especially longitudinal axis)
- Low control authority
- Actuator saturation
- Double integrator characteristics
- Underactuated MIMO system (4 inputs, 6 outputs/DOFs)

CONTROL TECHNIQUES
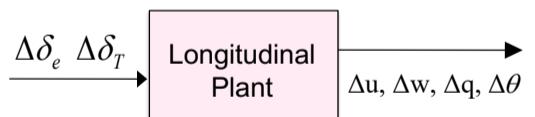
- Cascaded PID loops
- Optimal control (LQR)
- Robust control (H_∞, H_2 loop-sharing)
- Adaptive control
- Linear/nonlinear model predictive control
- (Nonlinear) dynamic inversion

THE PLANT
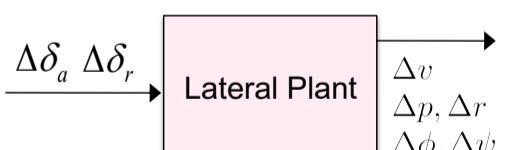
- Control inputs $\delta_T, \delta_E, \delta_A, \delta_R$

  – Positive deflections cause positive moments
  – Ailerons may have differential (e.g. more "up" for same "down") to combat adverse yawing

- States

  – Body velocities (u, v, w) estimated using Pitot-static tube (V), airflow vanes/multi-hole probe (β, α) and GNSS ($_\mathcal{I}\mathbf{v}$)
  – Body rates (p, q, r) estimated using IMU gyroscope
  – Euler angles ϕ, θ, ψ estimated using IMU accelerometer ($_\mathcal{B}\mathbf{a}$) and magnetometer (ψ)
  – Inertial position r_N, r_E, r_D estimated using GNSS

PLANT LINEARIZATION

- Longitudinal plant

$\Delta\delta_e$ $\Delta\delta_T$ → Longitudinal Plant → $\Delta u, \Delta w, \Delta q, \Delta\theta$

- Lateral plant
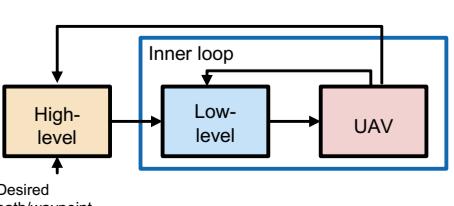
$\Delta\delta_a$ $\Delta\delta_r$ → Lateral Plant → Δv, $\Delta p, \Delta r$, $\Delta\phi, \Delta\psi$
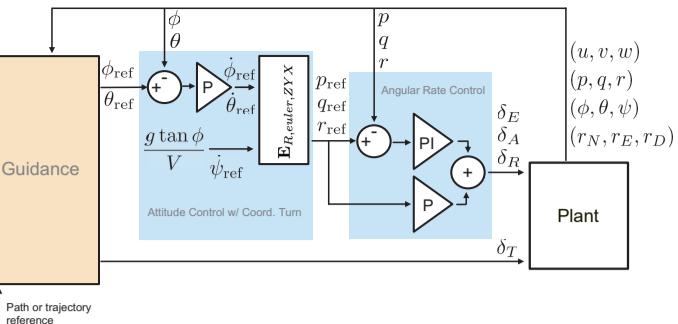
### CASCADED CONTROL LOOPS

- Control (low level part)

  – Goal: Stabilize attitude
  – Dynamics (actuator control inputs to states) challenging to globally identify in a nonlinear, high-fidelity form (thus linearizations common)

- Guidance (high level part)

  – Goals:
    * Follow a reference trajectory (position control)
    * Reject constant low-frequency disturbances e.g. constant wind
  – Dynamics typically only consist of kinematics (thus no system identification needed)

SIMPLE CASCADED CONTROL

- Need integrator wind-up protection
- Dynamic pressure scaling (scale actuator output inversely with airspeed i.e. $1/v^2$)
- Bandwidth (rate) of inner (low-level) loop should be sufficiently higher than outer (guidance) loop
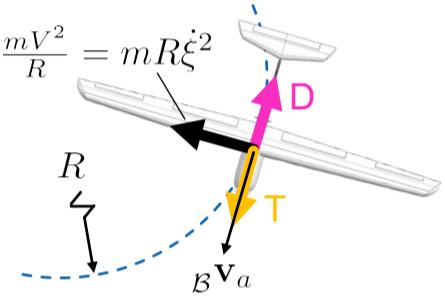
STATIONARY LEVEL COORDINATED TURN

- Stationarity: $_\mathcal{B}\dot{\mathbf{v}}_a = \mathbf{0}, {}_\mathcal{B}\dot{\boldsymbol{\omega}} = \mathbf{0}$
- Turning: $\phi = \text{const.} \neq 0$
- Level: $\alpha = \theta \implies \gamma = 0 \implies h = \text{const.}$
- Coordinated: $SF = 0$ i.e. no sideslip $\beta = 0 \implies \xi = \psi$ (centripetal force only from lift component)
- Force balances in front view ($\perp \mathbf{v}_a$)
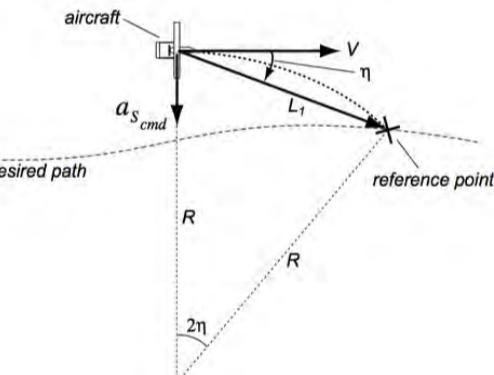
$$L\cos\phi = mg$$
$$\frac{mV^2}{R} = L\sin\phi + T\underbrace{\sin\epsilon_T}_{\approx 0}\sin\phi$$

- Minimum speed $V_{\min} \propto \sqrt{1/\cos\phi}$
- Heading/yaw rate $\dot{\xi} = \dot{\psi} = V/R = \frac{g\tan\phi}{V}$
- Additionally assume $D = T$ (thrust acts along drag axis)

$$\frac{mV^2}{R} = mR\dot{\xi}^2$$
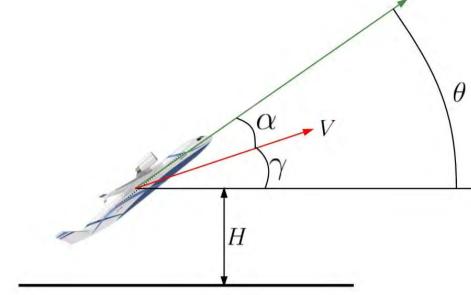
$\mathcal{L}_1$ GUIDANCE

- Lateral-directional path following guidance (with stationary level coordinated turns)

  – Lookahead vector of length L_1
 – Error angle η
  – Roll reference ϕ_{ref}

$$\sin\eta = \frac{L_1}{2R} \implies R = \frac{L_1}{2\sin\eta}$$
$$a_{s_{\text{cmd}}} = \frac{V^2}{R} = \frac{2V^2\sin\eta}{L_1} \implies \dot{\xi}_{\text{ref}} = \frac{a_{s_{\text{cmd}}}}{V}$$
$$\dot{\xi}_{\text{ref}} = \frac{g\tan\phi_{\text{ref}}}{V} \implies \phi_{\text{ref}} = \arctan\left(\frac{a_{s_{\text{cmd}}}}{g}\right)$$

TOTAL ENERGY CONTROL SYSTEM (TECS)

- Control altitude and airspeed

  – Total energy E (rate $\dot{E}$)

$$E = E_K + E_P = \frac{1}{2}mV^2 + mgH$$
$$\frac{\dot{E}}{mg} = \frac{V\dot{V}}{g} + \dot{H}$$

  – Total energy E_{spec} (rate $\dot{E}_{\text{spec}}$)

$$\dot{E}_{\text{spec}} = \frac{\dot{E}}{mgV} = \frac{\dot{V}}{g} + \frac{\dot{H}}{V} = \frac{\dot{V}}{g} + \sin\gamma \approx \frac{\dot{V}}{g} + \gamma$$

  – Energy distribution E_{dist} (rate $\dot{E}_{\text{dist}}$)

$$\dot{E}_{\text{dist}} = \gamma - \frac{\dot{V}}{g}$$