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HW #7
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Prob. #2

$$\min \frac{1}{2}\cdot\|X-Y\|_F^2$$

subject to $g(z)\leq\tau$, $L(x)=z$

Write Lagrangian...

$$\mathcal{L}(v) = \frac{1}{2}\|X-Y\|_F^2 + v^T(L(x)-z)$$

$$h(\lambda,v) = \inf_{\substack{x,z \\ \text{s.t. } g(z)\leq\tau}} \mathcal{L}(x,v,z)$$

$$= \inf_{\substack{x,z \\ \text{s.t. } g(z)\leq\tau}} \left(\frac{1}{2}\|X-Y\|_F^2 + v^T\cdot L(x) - v^T\cdot z\right)$$

$$= \underbrace{\inf_x\left(\frac{1}{2}\|X-Y\|_F^2 + v^T\cdot L(x)\right)}_{①} + \underbrace{\inf_{\substack{z \\ \text{s.t. } g(z)\leq\tau}}\left(-v^T z\right)}_{②} \longrightarrow \text{decouple...}$$

① $\inf_x\left(\frac{1}{2}\|X-Y\|_F^2 + v^T\cdot L(x)\right)$

$$\Rightarrow \nabla_x\left(\frac{1}{2}\|X-Y\|_F^2 + v^T\cdot L(x)\right) = 0$$

$$\frac{1}{2}\cdot\nabla_x\|X-Y\|_F^2 + \nabla_x\langle v^T, L(x)\rangle = 0$$

$$\nabla_x\langle v^T, L(x)\rangle = \nabla_x\langle L^* v, X\rangle = L^* v$$

$$\frac{1}{2}\nabla_x\left(tr\left((X-Y)^T(X-Y)\right)\right) + L^* v = 0$$

$$\frac{1}{2}\nabla_x\left(tr\left(X^TX - YX - XY + Y^TY\right)\right) + L^* v = 0$$

$$\frac{1}{2}\left(2\cdot X - 2\cdot Y\right) + L^* v = 0$$

$$\frac{1}{\cancel{2}}\cdot\cancel{2}(X-Y) + L^* v = 0$$

$$\Rightarrow \underline{\underline{X_{opt}}} = Y - L^* v$$

② $\inf_{z} \left(-v^T \cdot z\right)$

s.t. $g(z) \le \tau$

$= \inf_{z} \left(-v^T \cdot z + I_{\{g(z) \le \tau\}}\right)$ → Set C...

$C = \{\|g(z)\|_1 \le \tau\}$

$= -\sup_{z} \left(v^T \cdot z - I_C\right)$

$= -\sup_{z} \langle v^T, z\rangle - I_C(z)$

$= -\sup_{z \in C} \langle v^T, z\rangle$

$= -\sup_{\|z\|_1 \le \tau} \langle v^T, z\rangle$

$\underline{\underline{z_{opt}}} = -\tau \cdot \|v^T\|_\infty$

☆ conjugate of indicator function of a norm ball is the dual norm

- Since $p=1$, by Hölder's inequality, it is the $L-\infty$ norm.

☆ Hölders ineq,

$\langle x, y\rangle \le \|x\|_p \cdot \|y\|_q$

$\Rightarrow \langle v^T, z\rangle \le \tau \cdot \|v^T\|_\infty$

$\rightarrow p = 1, q = \infty$

$$\Rightarrow h(v) = \inf_{x}\left(\tfrac{1}{2}\|X - Y\|_F^2 + v^T \cdot L(x)\right) + \inf_{\substack{z \\ \text{s.t. } g(z) \le \tau}} \left(-v^T z\right)$$

$$= \tfrac{1}{2}\|Y - L^* v - \cancel{Y}\|_F^2 + L^* v\left(Y - L^* v\right) - \tau \cdot \|v^T\|_\infty$$

$$= \boxed{\tfrac{1}{2}\|L^* v\|_F^2 + L^* v\left(Y - L^* v\right) - \tau \cdot \|v^T\|_\infty}$$