Due date: October 19, 2020.

Type your solution using Word or LaTeX,
submit a PDF document on canvas
Handwritten solutions will not be graded.

Problem 1 [70 = 5 + 30 + 5 + 10 + 15 + 5]

The goal of this problem is to compute an upper bound on the area of a spherical cap (see Fig. 1). We have seen in class an upper bound in the context of Lévy's lemma. The bound we will derive will be almost as tight as that of Lévy's lemma.

We consider the unit sphere $S^{n-1}(1)$ in $\mathbb{R}^n$. Let $B^n(0,1)$ be the unit ball centered at the origin. Let $\varepsilon \in [0, 1/\sqrt{2}]$, we now proceed to define the spherical cap, $\kappa(x_0, \varepsilon)$, about a point x_0 $S^{n-1}(1)$, denote by

$$\kappa(x_0, \varepsilon) = \left\{ x \in S^{n-1}(1), \langle x, x_0 \rangle \geq \varepsilon \right\}, \quad \varepsilon \in [0, 1/\sqrt{2}]. \tag{1}$$

Please note that ε does not refer to the radius, but rather to the height at which we cut the sphere $S^{n-1}(1)$ perpendicular to the axis defined by x_0 (see Fig. 1). Formally, ε is the cosine of the solid angle centered around x_0. For any point $x \in \kappa$, the projection of x on the axis defined by x_0, $\langle x, x_0 \rangle$, cannot be lower than ε (see Fig. 1).

In the following we keep both x_0 and ε constant; to alleviate notations we will simply write κ to refer to $\kappa(x_0, \varepsilon)$. We define x_1 to be the point in the unit ball, $B^n(0,1)$, such that (see Figs. 1 and 3),

$$x_1 = \varepsilon x_0. \tag{2}$$

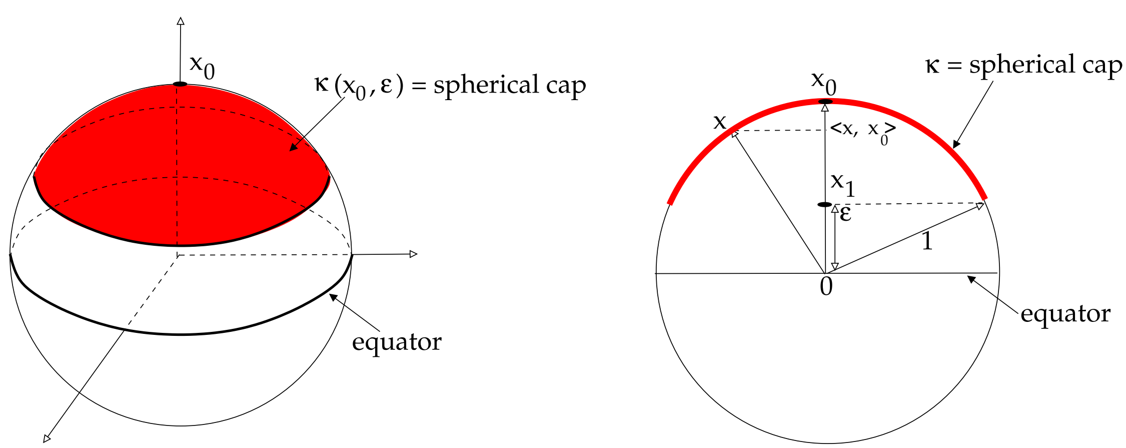


Figure 1: Three-dimensional (left) and two-dimensional (right) views of the the ε-*spherical cap* $\kappa(x_0, \varepsilon)$.

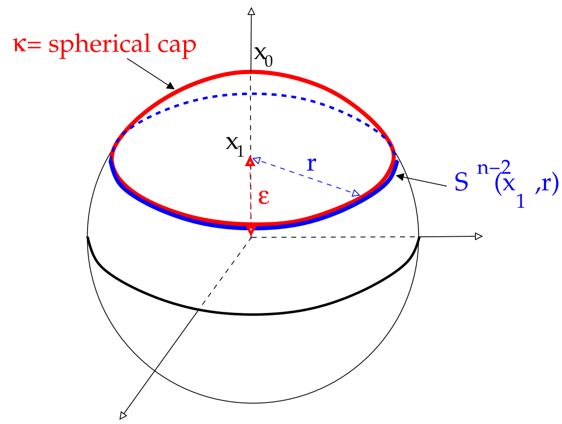


Figure 2: The sphere $S^{n-2}(x_1, r)$ is the blue circle at the opening of the spherical cap κ.

Finally, let $S^{n-2}(x_1, r)$ be the $n-2$-dimensional sphere centered around x_1 defined by

$$S^{n-2}(x_1, r) = \left\{ x \in \mathrm{S}^{\text{n-1}}(1) : \langle x, x_0 \rangle = \varepsilon \right\}. \tag{3}$$

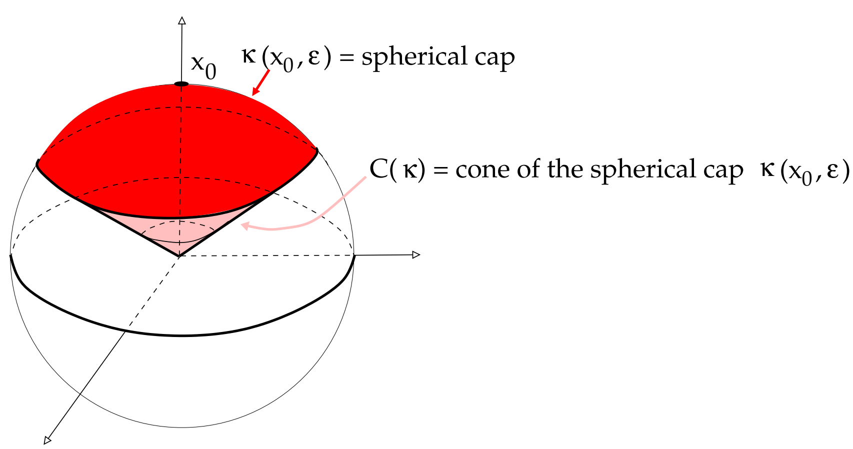
$S^{n-2}(x_1, r)$ is the border of the opening of the cap, shown as a blue circle in Fig. 2. While you should think of $S^{n-2}(x_1, r)$ as a circle when you compare it to $S^{n-1}(0, 1)$, we prefer to call it a sphere because circles are really 2-dimensional objects.

Assignment [20 = 10 + 10]

1. Compute the radius r of the sphere $S^{n-2}(x_1, r)$ (see Fig. 2).
2. Compute the volume of $B^n(x_1, r)$ as a function of the volume of the unit ball, $B^n(0, 1)$ (see Fig. 4).

We define the cone $C(\kappa)$ of the spherical cap κ (see Fig 3),

$$C(\kappa) = \{x \in \mathrm{B}^{\mathrm{n}}(0, 1); \exists x_0 \in \kappa, \exists 0 \le \alpha \le 1, x = \alpha x_0\}. \tag{4}$$



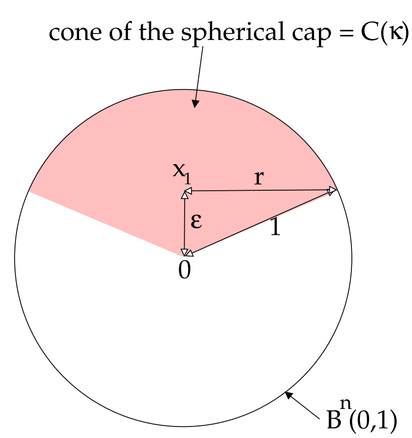


Figure 3: Three-dimensional (left) and two-dimensional (right) views of the cone $C(\kappa)$ of the spherical cap κ.

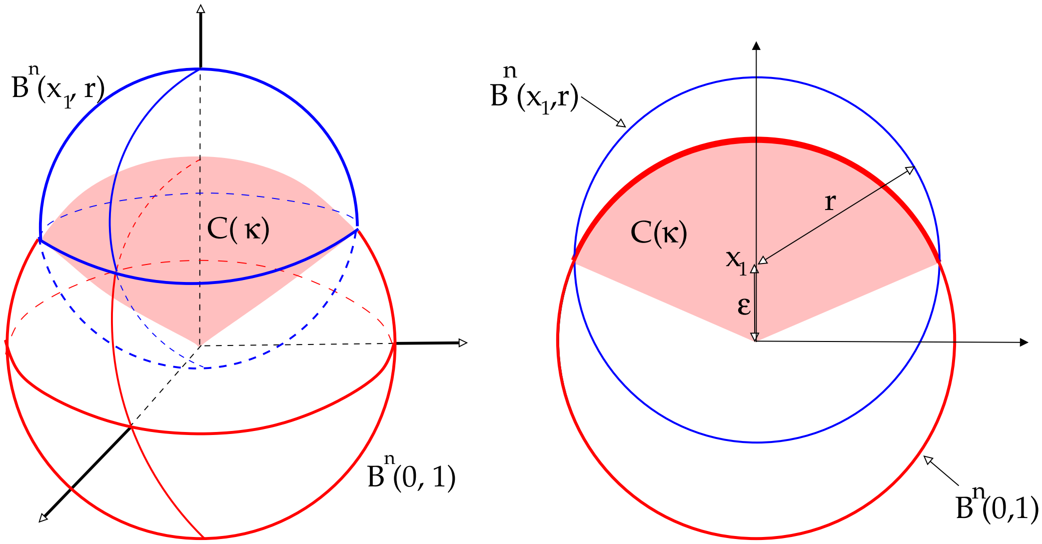


Figure 4: Three-dimensional (left) and two-dimensional (right) views of the balls $B^n(x_1, r)$, and $\mathrm{B}^n(0, 1)$. The cone $C(\kappa)$ (pink region) is inside $B^n(x_1, r)$.

Assignment [80 = 50 + 10 + 15 + 5]

3. Prove that
$$C(\kappa) \subset B^n(x_1, r), \tag{5}$$
where $B^n(x_1, r)$ is the ball of radius r centered at x_1 (see Fig. 4).

4. Using the inequality $1 + x \leq e^x$ conclude that
$$\mathrm{vol}(C(\kappa)) \leq e^{-n\varepsilon^2/2}\, \mathrm{vol}\,(\mathrm{B}^n(0, 1))\,. \tag{6}$$

5. Using the relationship between the surface of κ, and the volume of $C(\kappa)$, prove that
$$\mu^{n-1}(\kappa(x_0, \varepsilon)) \leq e^{-n\varepsilon^2/2}, \tag{7}$$
where μ^{n-1} is the uniform measure on the sphere $\mathrm{S}^{n-1}(1)$.

6. Compare this estimate with that of Paul Lévy's theorem.