APPM 5515: High Dimensional Probability– Fall 2020 — Homework 1 pt. 1

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1. Prove that the probability that none of the Ω_i, with $i > i_0$, occur is given by

$$\prod_{i=i_0+1}^{\infty} (1 - p_i)$$

Proof. First note that the events are *independent*, and that the probability that each occur is p_i. Then the probability that event Ω_i does not occur is $1 - p_i$.

Now, since the events are independent, the joint probability that both occur are the product of the probabilities, that is $p_1 p_2 \cdots p_i$. The same follows for the probability that none occur. And when you constrict $i > i_0$, we get the following that none of the events occur:

$$\boxed{\prod_{i=i_0+1}^{\infty} (1 - p_i)}$$

□

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2. Prove the following inequality

$$\prod_{i=i_0+1}^{\infty} (1 - p_i) \leq \exp\left\{ - \sum_{i=i_0+1}^{\infty} p_i \right\}$$

Proof. First, assume $x = -p_i$. Then our equation becomes

$$\prod_{i=i_0+1}^{\infty} (1 + x)$$

Now take the log and simplify

$$\begin{aligned} \prod_{i=i_0+1}^{\infty} (1 + x) = & \\ &= \ln\left(\prod_{i=i_0+1}^{\infty} (1 + x) \right) \\ &= \sum_{i=i_0+1}^{\infty} \ln(1 + x) \end{aligned}$$

$$\implies \leq \sum_{i=i_0+1}^{\infty} x$$

$$\therefore \boxed{\prod_{i=i_0+1}^{\infty}(1-p_i) \leq \exp\left\{-\sum_{i=i_0+1}^{\infty} p_i\right\}}$$

□

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3. Prove that if

$$\sum_{i=1}^{\infty} P(\Omega_i) \qquad \textbf{diverges,}$$

then

$$P(\textbf{infinitely many } \Omega_i \textbf{ occur}) = 1.$$

Proof. From question 1, we can set a bound for the above series. That is

$$\prod_{i=i_0+1}^{\infty}(1-p_i) \leq \exp\left\{-\sum_{i=i_0+1}^{\infty} p_i\right\}$$

Since $p_i > 0$,

$$\exp\left\{-\sum_{i=i_0+1}^{\infty} p_i\right\} \to 0$$

$$\implies \prod_{i=i_0+1}^{\infty}(1-p_i) \to 0$$

This then implies that the probability that none of them occur is 0, which is the complement that the probability that infinite of them occur is 1. □

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4. Compute $E[S_n]$ and V_n
First let us compute $E[S_n]$.

$$\begin{aligned} E[S_n] &= \\ &= E\left[\sum_{i=1}^{n} E(X_i)\right] \\ &= \sum_{i=1}^{n} E(X_i) \\ &= \sum_{i=1}^{n} X_i \cdot P(X_i) \end{aligned}$$

Now, we note that X_i is a sequence of events that takes discreet values, that is $X_i = -i, i, \text{or } 0$. Also note that when $i = 1$, its probability is 0. Summing all possible events within the sample space, we get the following for the expectation value:

$$
\begin{aligned}
E(S_n) &= \\
&= \overbrace{1 \cdot 0}^{X_1=0} + \underbrace{0 \cdot \left(1 - \frac{1}{i \log i}\right)}_{X_i \text{ when } X_i=0 \text{ for } i \geq 2} + \underbrace{(i - i) \cdot \frac{1}{2i \log i}}_{X_i \text{ when } X_i=i,-i \text{ for } i\geq 2} \\
&= 0 + 0 + 0 \\
&= 0
\end{aligned}
$$

Similarly, we know the definition of the variance is the following

$$V_n = E[S_n^2] - E^2[S_n]$$

We also know that X_i is independent from eachother. This leads to the following identity

$$V_n = V(S_n) = n \cdot V[X_i]$$

Formally, we can calculate...

$$
\begin{aligned}
V(S_n) &= \\
&= n \cdot V[X_i] \\
&= n \cdot \left[E[X_i^2] - E^2[X_i]\right] \\
&= n \cdot \left[\frac{i}{\log i} - 0\right] \\
&= \frac{n \cdot i}{\log i}
\end{aligned}
$$

$$\therefore \boxed{E(S_n) = 0 \text{ and } V(S_n) = \frac{n \cdot i}{\log i}}$$

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5. Prove

$$\forall \epsilon > 0, \lim_{n\to\infty} P\left(\left|\frac{S_n}{n}\right| \geq \epsilon\right) = 0,$$

Proof. It is sufficient to prove

$$\lim_{n\to\infty} E\left[\left|\frac{S_n}{n}\right|^2\right] = 0.$$

Now, expanding on this from our calculations in (4), we can obtain the following process

$$\lim_{n\to\infty} E\left[\left|\frac{S_n}{n}\right|^2\right] =$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \frac{1}{n^2} E\left[|S_n|^2\right] \\
&= \lim_{n\to\infty} \frac{1}{n^2} E\left[\left(\sum_{i=1}^{n} X_i\right)^2\right] \\
&= \lim_{n\to\infty} \frac{1}{n^2} \frac{n \cdot i}{\log i} \\
&= \lim_{n\to\infty} \frac{i}{n \log i} \\
&\to 0 \text{ as } n \to \infty
\end{aligned}
$$

$\square$