APPM 5515: High Dimensional Probability– Fall 2020 — Homework 6

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1. Prove $M = E[A]$

Proof. We know $A = TBT^T$, but since we stipulated $T = I$, we can simply use $A = B$ to find M.
Note: I simplified the formation of M for notational convenience. Recall that M is a block matrix formation and so the different values occur at the half way point in each direction.
Thus,

$$\begin{aligned} E[A] = E[B] &= E \begin{bmatrix} b_{1,1}(p) & \cdots & b_{1,n}(q) \\ \vdots & & \vdots \\ b_{n,1}(p) & \cdots & b_{n,n}(p) \end{bmatrix} \\ &= \begin{bmatrix} E[b_{1,1}(p)] & \cdots & E[b_{1,n}(q)] \\ \vdots & & \vdots \\ E[b_{n,1}(p)] & \cdots & E[b_{n,n}(p)] \end{bmatrix} \\ &= \begin{bmatrix} p & \cdots & q \\ \vdots & & \vdots \\ q & \cdots & p \end{bmatrix} \\ &= M \end{aligned}$$

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2. Find $E[D]$.
Since it is a diagonal matrix, and we know how to compute d_i, we can say

$$E[D_{i,j}] = E[0] = 0 \text{ when } i \neq j \tag{1a}$$

$$E[D_{i,j}] = E[d_i] = 0 \text{ when } i = j \tag{1b}$$

Thus, we have

$$\begin{aligned} E[d_i] &= E\left[\sum_{j=1}^{n} A_{i,j}\right] \\ &= \sum_{j=1}^{n} E\left[A_{i,j}\right] \end{aligned}$$

$$= \frac{(p+q)n}{2}$$

And so, we have $E[D]$ be defined as

$$E[D] = \begin{bmatrix} \frac{(p+q)n}{2} & 0 & \cdots & 0 \\ 0 & \frac{(p+q)n}{2} & 0 & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \frac{(p+q)n}{2} \end{bmatrix} \tag{2}$$

3. To prove, w_1 is an eigenvector of M, we simply see if it satisfies the definition of an eigenvector. Thus We show $Mw_1 = \mu_1 w_1$.

$$Mw_1 = \mu_1 \frac{1}{\sqrt{n}}\mathbf{1}$$
$$= \frac{1}{\sqrt{n}} \begin{bmatrix} \frac{(p+q)n}{2} \\ \frac{(p+q)n}{2} \\ \vdots \\ \frac{(p+q)n}{2} \end{bmatrix}$$
$$= \frac{(p+q)n}{2\sqrt{n}}\mathbf{1}$$

Thus, w_1 is an eigenvector and $\boxed{\mu_1 = \frac{(p+q)n}{2}}$.

4. Using the same process, we can show $Mw_2 = \mu_2 w_2$.

$$Mw_2 = \frac{1}{\sqrt{n}} \begin{bmatrix} \frac{(p-q)n}{2} \\ \frac{(p-q)n}{2} \\ \vdots \\ \frac{(p-q)n}{2} \end{bmatrix}$$
$$= \frac{(p-q)n}{2\sqrt{n}} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ -1 \\ -1 \\ \vdots \\ -1 \end{bmatrix}$$
$$= \frac{(p-q)n}{2} w_2$$

Thus, w_2 is an eigenvector and its eigenvalue is $\boxed{\mu_2 = \frac{(p-q)n}{2}}$.

5. From the given defined functions of w_3 and w_4, we can graph them together:

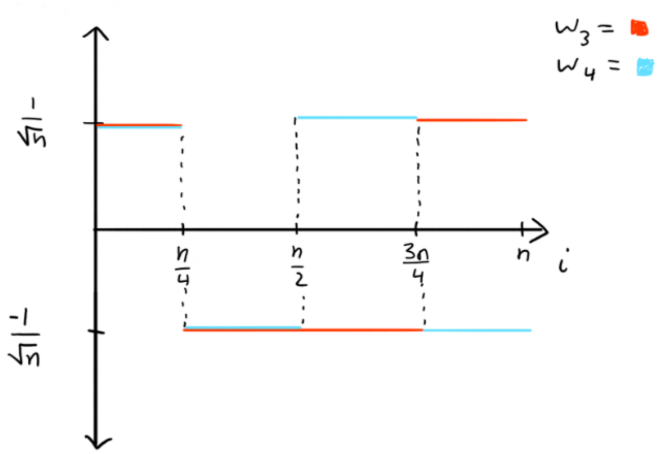

Figure 1: w_3 and w_4

6. *Proof.* To prove w_3 and w_4 are in the null space of $M = E[A]$, we show that $Mw_n = 0, n = 3, 4$. First, w_3.

$$Mw_3 = \begin{bmatrix} p \cdot \frac{n}{4} - (p \cdot \frac{n}{4} + q \cdot \frac{n}{4}) + q \cdot \frac{n}{4} \\ \vdots \\ q \cdot \frac{n}{4} - (q \cdot \frac{n}{4} + p \cdot \frac{n}{4}) + p \cdot \frac{n}{4} \\ \vdots \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Second, w_4,

$$Mw_3 = \begin{bmatrix} p \cdot \frac{n}{4} - p \cdot \frac{n}{4} + q \cdot \frac{n}{4} - q \cdot \frac{n}{4} \\ \vdots \\ q \cdot \frac{n}{4} - q \cdot \frac{n}{4} + p \cdot \frac{n}{4} - p \cdot \frac{n}{4} \\ \vdots \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Thus, both w_3, w_4 are in the null space of M. ∎

7. *Proof.* To prove $M = \mu_1 w_1 w_1^T + \mu_2 w_2 w_2^T$, we simply plug in our values.
Note: I simplified the formation of M for notational convenience. Recall that M is a block matrix formation and so the different values occur at the half way point in each direction.

Doing this, we get

$$
\begin{aligned}
M &= \mu_1 \left(\frac{1}{n} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 & \cdots \end{bmatrix} \right) + \mu_2 \left(\frac{1}{n} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ -1 \\ -1 \\ \vdots \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & \cdots & -1 & -1 & \cdots \end{bmatrix} \right) \\
&= \frac{\mu_1}{n} \left(\begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix} \right) + \frac{\mu_2}{n} \left(\begin{bmatrix} 1 & \cdots & -1 & \cdots \\ \vdots & & \vdots & \\ 1 & \cdots & -1 & \cdots \\ -1 & \cdots & 1 & \cdots \\ \vdots & & \vdots & \\ -1 & \cdots & 1 & \cdots \end{bmatrix} \right) \\
&= \frac{1}{2} \left(\begin{bmatrix} (p+q)+(p-q) & \cdots & (p+q)+(q-p) \\ \vdots & & \vdots \\ (p+q)+(q-p) & \cdots & (p+q)+(p-q) \end{bmatrix} \right) \\
&= \frac{1}{2} \left(\begin{bmatrix} 2p & \cdots & 2q \\ \vdots & & \vdots \\ 2q & \cdots & 2p \end{bmatrix} \right) \\
&= M
\end{aligned}
$$

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8. In general, we can compute a number (up to the number of communities - given a trivial eigenvector) of eigenvalues and eigenvectors. Then, after inspecting its eigenvector(s), we can separate our communities by looking at the sign of the eigenvector(s). This will allow us to cluster certain communities together given that we started with an adjacency matrix.

9. To find $E[\beta]$ and $Var[\beta]$, we simply calculate directly:
   First, $E[\beta]$:

$$
\begin{aligned}
E[\beta] &= (1-p)p + (-p)(1-p) \\
&= \boxed{0}
\end{aligned}
$$

   Second, $Var[\beta]$:

$$
\begin{aligned}
Var[\beta] &= E[\beta^2] - E^2[\beta] \\
&= \boxed{(1-p)^2 p + p^2(1-p)}
\end{aligned}
$$

10. We can show that β is sub-gaussian in a number of ways. One way is that we know bernoulli is sub-gaussian and so translating its values, and so its mean, will not affect its tail distributions as it only affects where they are displaced. Thus, it is sub-gaussian. Another way is to use

the bound shown here and to see that it satisfies the definition of a sub-gaussian norm, that is

$$\|Q\|_{\Psi_2} = \inf\{\lambda > 0, E\left[e^{Q^2/\lambda^2}\right] \leq 2\}$$

Here, we say

$$e^{\frac{\beta^2}{\|\beta\|^2_{\Psi_2}}} = 2 \tag{3}$$

which implies (via the hint) $1 + (e-1)\frac{\beta^2}{\|\beta\|^2_{\Psi_2}} = 2$. And so, solving for $\|\beta\|^2_{\Psi_2}$, we get $\|\beta\|^2_{\Psi_2} = (e-1)\beta^2 \leq (e-1)$. Finally, this implies that $\|\beta\|_{\Psi_2} \leq \sqrt{e-1}$. Unfortunately, I could not get the bound to be divided in half.

11. *Proof.* Due to how X is constructed, we get

$$X = \left[\begin{array}{c|c} \beta(p) & \beta(q) \\ \hline \beta(q) & \beta(p) \end{array}\right]$$

Thus,

$$\left[\begin{array}{c|c} E[\beta(p)] & E[\beta(q)] \\ \hline E[\beta(q)] & E[\beta(p)] \end{array}\right] = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & 0 \end{array}\right] = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}_{n\times n}$$

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12. *Proof.* Recall that if our matrix, A, is square, that is $n = m$, then we can say $\|A\| = |\lambda_1|$. Thus, We have $E[\|X\|] = E[|\lambda_1|]$. And so, from the textbook, we can use corollary 4.4.8 to say

$$\|X\| \leq c_1 K(\sqrt{n} + t) \tag{4}$$

And so,

$$E[\|X\|] \leq c_1\sqrt{n} \tag{5}$$

since $Var[X] \leq 1$ and t goes away on expectation. Thus, we have

$$E[\|X\|] = E[|\lambda_1|] \leq c_1\sqrt{n} \tag{6}$$

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13. *Proof.* Using Weyl-Lidskii Theorem, and letting $A = M, E = X$, we can say for $\lambda_i, i = 1, 2,$

$$|\lambda_1(M) - \lambda_1(M + X)| \leq \|X\| \tag{7}$$

Using our previous answers, we can use the fact that $\|X\| \sim \sqrt{n}$ and $\lambda_i \sim n$, as well as $A = M + X$, to deduce

$$\begin{aligned} &|\lambda_i(M) - \lambda_i(M + X)| \leq \|X\| \\ &\implies -c\sqrt{n} + \lambda_i(M) \leq \lambda_i(A) \leq \lambda_i(M) + c\sqrt{n} \\ &\qquad \implies \lambda_i(M) \leq \lambda_i(A) \leq \lambda_i(M) \text{ (for large } n \text{, since } n > \sqrt{n}) \end{aligned}$$

Thus, by a similar proof of squeeze theorem, the eigenvalues of A approach the eigenvalues of $E[A] = M$. Note, we are considering non-trivial eigenvalues as our high-dimensional matrix is low rank ($r = 2$). Additionally, the bounds on p and q allow for our eigenvalue to not overlap between our first and second eigenvalues on their edge of bounding. This then satisfies the premise of the theorem that they are ordered for any value of n and so can be considered in full. ■

$\boxed{\textbf{— END —}}$