APPM 5515: High Dimensional Probability– Fall 2020 — Homework 1 pt. 2

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6. Prove that

$$\frac{X_n}{n} \quad \text{converges to 0 almost surely.} \tag{1}$$

Proof. From the question prompt, we know that $n^{-1}S_n = n^{-1}\sum_{i=1}^{n} X_i \to 0$ almost surely. So,

$$n^{-1}\sum_{i=1}^{n} X_i = \frac{1}{n}\left[X_1 + X_2 + X_3 + \cdots + X_n\right]$$

$$= \frac{X_1}{n} + \frac{X_2}{n} + \frac{X_3}{n} + \cdots + \frac{X_n}{n}$$

and we know that $n^{-1}S_n = n^{-1}\sum_{i=1}^{n} X_i$ converges to 0 almost surely. The only way this is possible is if each term also converges to 0 almost surely. Therefore, we know the final term, $\frac{X_n}{n}$ converges to 0 almost surely. ■

7. Deduce from the previous equation that

$$P\left(\left|\frac{X_i}{i}\right| \geq 1 \text{ occurs for infinitely many i}\right) = 0 \tag{2}$$

Proof. That is, $\left|\frac{X_1}{1}\right| \geq 1 \cap \left|\frac{X_2}{2}\right| \geq 1 \cap \left|\frac{X_3}{3}\right| \geq 1 \cap \cdots \cap \left|\frac{X_n}{n}\right| \geq 1$
From the previous homework, we know that $X_i = i, -i$, or 0. Using these values, we get the expression for

$$\left|\frac{X_i}{i}\right| = \left|\frac{i}{i}\right| \text{ or } \left|\frac{-i}{i}\right| \text{ or } \left|\frac{0}{i}\right|$$

$$= |1| \text{ or } |-1| \text{ or } |0|$$

$$= 1 \text{ or } 1 \text{ or } 0$$

but we know that $P(X_i = 0) = 1 - \frac{1}{i\log i} \neq 1, \forall i \in Z^+ \setminus 1$. However, we can take the limit of i at infinity and see that the probability that $X_i = 0$ approaches 1.

$$\lim_{i \to \infty}\left[1 - \frac{1}{i\log i}\right] \to \left[1 - \frac{1}{\infty}\right] = 1.$$

This implies that as i approaches infinity, that X_i begins to take 0 as its value. Therefore, the magnitude $\left|\frac{X_i}{i}\right|$ is no longer greater or equal to 1 for the rest of the i's and that the corresponding probability is 0. ■

8. Let Ω_i be the event $\left|\frac{X_i}{i}\right| \geq 1$. Compute $P(\Omega_i)$

$$P(\Omega_i) = P\left(\left|\frac{X_i}{i}\right| \geq 1\right)$$
$$= \frac{1}{2i \log i} + \frac{1}{2i \log i} + 0$$
$$= 2\left(\frac{1}{2i \log i}\right)$$
$$= \frac{1}{i \log i}$$

9. Prove that

$$\lim_{n\to\infty} \sum_{i=1}^{n} P(\Omega_i) = \infty \tag{3}$$

Proof. We will prove this via the integral test. Namely, the integral test holds for $m > N = 1$ and that $f(m)$ for $m > 1$, $f(m)$ is a positive, non-increasing function (monotonically decreasing). This allows us to apply the integral test and to conclude that the convergence behavior of the series is the same as the integral. The following test is applied:

$$\int_{1}^{\infty} \frac{1}{m \log m} \, dm = \lim_{a\to\infty} \int_{1}^{a} \frac{1}{m \log m} \, dm$$
$$= \lim_{a\to\infty} \left[\int_{0}^{\log(a)} \frac{1}{u} \, du\right] \qquad (\text{let } u = \log m)$$
$$= \lim_{a\to\infty} \left[\log|u| \, \Big|_{0}^{\log(a)}\right]$$
$$= \lim_{a\to\infty} \left[\log|\log(a)| - \log|0|\right]$$
$$= \infty - (-\infty)$$
$$= \infty$$
$$\implies \text{ this series is divergent.}$$

■

10. Prove that

$$P\left(\frac{S_n}{n} \to 0\right) = 0 \tag{4}$$

Proof. Assume that $\exists \, \epsilon > 0$. Then, using the values from above, we can test the absolute convergence(assuming that is the meaning behind the arrow)

$$\frac{S_n}{n} = \frac{1}{n} \cdot \sum_{i=1}^{n} X_i$$
$$\leq \frac{1}{n} \cdot \sum_{i=1}^{n} |X_i|$$
$$= \frac{1}{n} \cdot [|X_1| + |X_2| + |X_3| + \cdots + |X_n|]$$

We know that $X_i = -i, 0,$ or i, but due to the magnitude, we know it takes either i or 0. Continuing the proof,

$$\frac{S_n}{n} = \sum_{i=0}^{n} \frac{i}{n}$$

$$= 0 + \frac{1}{n} + \frac{2}{n} + \frac{3}{n} + \cdots + \frac{n}{n}$$

$$> 1 \text{ (assuming } n > 0)$$

Since the sum is always greater than 1 regardless of the positive n you choose, there is no possible way for the sum to converge to 0. Therefore, $P\left(\frac{S_n}{n} \to 0\right) = 0$. Assuming the arrow is any type of convergence, the series also fails the divergence test and so diverges regardless of the convergence method. This also corroborates the conclusion reached. ∎

11. Conclude.
Since we started with the assumption that $n^{-1}S_n = n^{-1}\sum_{i=1}^{n} X_i \to 0$ almost surely, and in 10), we showed that $P(\frac{S_n}{n} \to 0) = 0$, we can say that by contradition, $n^{-1}\sum_{i=1}^{n} X_i$ does not converge to 0 almost surely. And by extension, this shows that the X_i's do not obey the strong law of large numbers.