APPM 5515: High Dimensional Probability– Fall 2020 — Homework 4

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1. Prove that $\mathbf{Q}$ is an isometry "in expectation",

$$E\left[\|\mathbf{Q}x\|^2\right] = E\left[\left\|\frac{1}{\sqrt{d}}y\right\|^2\right] = ||x||^2 \tag{1}$$

Solution:

Proof. Note that

$$\frac{1}{\sqrt{d}}y = \left[\tfrac{1}{\sqrt{d}}y_1, \cdots, \tfrac{1}{\sqrt{d}}y_d\right]^T$$

Thus,

$$\begin{aligned}\left\|\frac{1}{\sqrt{d}}y\right\|^2 &= \left(\sqrt{\left(\frac{1}{\sqrt{d}}y_1\right)^2 + \cdots + \left(\frac{1}{\sqrt{d}}y_d\right)^2}\right)^2 \\ &= \frac{1}{d}y_1^2 + \cdots + \frac{1}{d}y_d^2 \\ &= \frac{1}{d}\sum_{i=1}^{d} y_i^2\end{aligned}$$

So,

$$\begin{aligned}E\left[\left\|\frac{1}{\sqrt{d}}y\right\|^2\right] &= E\left[\frac{1}{d}\sum_{i=1}^{d} y_i^2\right] \\ &= \frac{1}{d}E\left[\sum_{i=1}^{d} y_i^2\right] \\ &= \frac{1}{d}\sum_{i=1}^{d} E\left[y_i^2\right]\end{aligned}$$

But, y_i are from $\mathbf{G}$ whose entries are independent, zero-mean, unit variance. Thus,

$$\frac{1}{d}\sum_{i=1}^{d} E\left[y_i^2\right] = \frac{1}{d}\sum_{i=1}^{d} x_i^2$$

$$= \frac{1}{d} \cdot d \cdot \|x\|_2^2$$
$$= \|x\|_2^2$$

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2. Prove that the y_i are independent sub-Guassian random variables that that there exists a $c_1 > 0$ s.t.,
$$\|y_i\|_{\Psi_2} \leq \|x\|_2 \frac{v}{\sqrt{c_1}}$$

Solution:

Proof. Notice that $x_i \in \mathbb{R}$ and that $g_{i,j}$ are independent sub-Guassian random variables. Then note that $y = \mathbf{G}x$, so this implies that $\mathbf{G}x = \begin{bmatrix} \overbrace{g_{1,1}x_1 + g_{1,2}x_2 + \cdots + g_{1,n}x_n}^{y_1} \\ \vdots \\ \underbrace{g_{d,1}x_1 + g_{d,2}x_2 + \cdots + g_{d,n}x_n}_{y_d} \end{bmatrix}$ Now, define $z_j = g_{i,j}x_j$, and so, by a proposition made in class, we know if $z_1, \cdots, z_n$ are independent sub-Guassian random variables then,

$$\sum_{j=1}^{n} z_j = y_i$$

is also sub-Guassian. Now to prove the inequality. We know from the proposition there exists a constant $c_1 > 0, c_1 \in \mathbb{R}$ s.t.

$$\left\| \sum_{j=1}^{n} z_j \right\|_{\Psi_2} \leq c_1 \cdot \sum_{j=1}^{n} \|z_j\|_{\Psi_2}$$

which in our case
$$\|y_i\|_{\Psi_2} \leq c_1 \cdot \sum_{j=1}^{n} \|g_{i,j}x_j\|_{\Psi_2}$$

Thus, we can do the following

$$\begin{aligned} c_1 \sum_{j=1}^{n} \|g_{i,j}x_j\|_{\Psi_2} &= \sum_{j=1}^{n} \|g_{i,j}\|_{\Psi_2} \cdot |x_j| \\ &\leq c_1 \sum_{j=1}^{n} max\|g_{i,j}\|_{\Psi_2} \cdot |x_j| \\ &= c_1 \sum_{j=1}^{n} v \cdot |x_j| \\ &= c_1 \cdot v \cdot \|x\|_1 \end{aligned}$$

$$\leq c_1 \cdot v \cdot \|x\|_2 \qquad \text{(where } c_1 = \frac{1}{\sqrt{c_2}}\text{)}$$

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3. Compute $\mathbb{E}[y_i]$ and $Var[y_i]$ as a function of $\|x\|$

   Solution:

   Proof. First we show $E[y_i]$.

$$\begin{aligned} E[y_i] &= E\left[\sum_{j=1}^{n} g_{i,j} x_j\right] = \sum_{j=1}^{n} E\left[g_{i,j} x_j\right] \\ &= \sum_{j=1}^{n} x_j E\left[g_{i,j}\right] \\ &= \sum_{j=1}^{n} x_j 0 && \text{(since } g_{i,j} \text{ is sub-Guassian with mean 0)} \\ &= \sum_{j=1}^{n} 0 \\ &= 0 \end{aligned}$$

   Now, we show $Var[y_i]$.

$$\begin{aligned} Var[y_i] &= E[y_i^2] - E^2[y_i] \\ &= E[y_i^2] \\ &= E\left[\left(\underbrace{\sum_{j=1}^{n} g_{i,j} x_j}_{N\left(0, \sum_{j=1}^{n} x_j^2\right)}\right)^2\right] \\ &= \sum_{j=1}^{n} x_j^2 && \text{(since it is just Guassian, we can take the variance out)} \\ &= \|x\|_2^2 \end{aligned}$$

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4. Prove that y_i^2 is sub-exponential and that

$$\|y_i^2\|_{\Psi_1} = \|y_i\|_{\Psi_2}^2$$

   Solution:

Proof. From a lemma in class, we know that if X and Y are sub-Guassian, then their product, XY is subexponential. It also follows that if we define $X = y_i$ and $Y = y_i$ then we can say $XY = y_i \cdot y_i = y_i^2$ is sub-exponential. Finally, the lemma also says

$$\|XY\|_{\Psi_1} \leq \|X\|_{\Psi_2}\|Y\|_{\Psi_2}$$

which implies

$$\|y_i \cdot y_i\|_{\Psi_1} = \|y_i^2\|_{\Psi_1} \leq \|y_i\|_{\Psi_2}\|y_i\|_{\Psi_2} = \|y_i\|_{\Psi_2}^2$$

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5. Use the corollary of the general Bernstein inequality which was presented in class, to show that there exists an $\alpha > 0$ and $\beta > 0$ s.t.

$$P\left(\left|\frac{1}{d}\sum_{i=1}^{d} z_i\right| \geq \epsilon\right) \leq \exp 2\left(-\alpha \cdot \min\left(\frac{\epsilon^2}{\beta^2}, \frac{\epsilon}{\beta}\right) d\right)$$

Solution:

Proof. Since the above equation is the corollary we are supposed to use, we can simply justify our assumptions and then it follows that the above formula works. For that, we need to make sure z_i is an independent, mean-zero, sub-exponential RV. To show mean-zero

$$\begin{aligned} E[z_i] &= E\left[y_i^2 - E[y_i^2]\right] \\ &= E\left[y_i^2\right] - E[y_i^2] \\ &= 0 \end{aligned}$$

Therefore it is mean zero. Now, by the problem statement on the homework, it says we can show z_i is sub-exponential with the inequality. Therefore, by definition we showed it is subexponential.

Lastly, we show it is independent. For this, we know y_i is independent and that any function on an independent random varible remains independent. Thus, y_i^2 is independent. Now note that $E[y_i^2]$ is a constant and so redefining our random variable to account for the constant does not affect independent. Thus, z_i is independent.

Because we showed that z_i is an independent, mean-zero, subexponential random variable, we can conclude that the general Bernstein inequality will hold. ■

6. If we choose

$$d = \frac{2\beta^2}{\alpha\epsilon^2}\log\left(N/\sqrt{\delta}\right)$$

then

$$P\left(\left|\frac{1}{d}\sum_{i=1}^{d} z_i\right| \geq \epsilon\right) \leq \frac{2\delta}{N^2}$$

Solution:

Proof. Plugging in d to the formula in problem 5, we get

$$P\left(\left|\frac{1}{d}\sum_{i=1}^{d} z_i\right| \geq \epsilon\right) \leq \exp 2\left(-\alpha \cdot \min\left(\frac{\epsilon^2}{\beta^2}, \frac{\epsilon}{\beta}\right)\frac{2\beta^2}{\alpha\epsilon^2}\log\left(N/\sqrt{\delta}\right)\right)$$

where the alphas cancel. Also note that in small deviation regime, $\frac{\epsilon}{\beta} < 1$ which implies that if you square it, it will become even smaller. Thus, $\frac{\epsilon^2}{\beta^2}$ is the minimum. And so,

$$\begin{aligned}
P\left(\left|\frac{1}{d}\sum_{i=1}^{d} z_i\right| \geq \epsilon\right) &\leq \exp 2\left(-1 \cdot \frac{\epsilon^2}{\beta^2}\frac{2\beta^2}{\epsilon^2}\log\left(N/\sqrt{\delta}\right)\right) \\
&= 2\exp\left(-2\log\left(N/\sqrt{\delta}\right)\right) \\
&= 2\exp\left[\log\left(N/\sqrt{\delta}\right)^{-2}\right] \\
&= 2\left(\frac{N}{\sqrt{\delta}}\right)^{-2} \\
&= \frac{2\delta}{N^2}
\end{aligned}$$

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7. Prove the following:

$$P\left(\mathbf{G} = (g_{i,j}); \forall 1 \leq k < l \leq N, \left|\frac{\|\mathbf{Q}(x_k - x_l)\|^2}{\|x_k - x_l\|^2} - 1\right| \geq \epsilon\right) \leq \delta$$

Solution:

Proof. Let $x = x_k - x_l$. Then,

$$P\left(\left|\frac{\|\mathbf{Q}x\|^2}{\|x\|^2} - 1\right| \geq \epsilon\right) \leq \delta$$

but $\mathbf{Q}x = \frac{1}{\sqrt{d}}\mathbf{G}x = \frac{1}{\sqrt{d}}y$. So,

$$\begin{aligned}
P\left(\left|\frac{\|\frac{1}{\sqrt{d}}y\|^2}{\|x\|^2} - 1\right| \geq \epsilon\right) &\leq \delta \\
P\left(\left|\frac{\frac{1}{d}\cdot\sum_{i=1}^{d} y_i^2}{\|x\|^2} - 1\right| \geq \epsilon\right) &\leq \delta \\
P\left(\left|\frac{\frac{1}{d}\cdot\sum_{i=1}^{d} y_i^2 - \|x\|^2}{\|x\|^2}\right| \geq \epsilon\right) &\leq \delta
\end{aligned}$$

And now recall that $\|x\|^2 = Var[y_i] = E[y_i^2]$. Thus,

$$P\left(\left|\frac{\frac{1}{d}\cdot\sum_{i=1}^{d} y_i^2 - E[y_i^2]}{\|x\|^2}\right| \geq \epsilon\right) \leq \delta$$
$$P\left(\left|\frac{\frac{1}{d}\sum_{i=1}^{d}\left(y_i^2 - E\left[y_i^2\right]\right)}{\|x\|^2}\right| \geq \epsilon\right) \leq \delta$$

And if we let $t = \epsilon \cdot \|x\|^2$ and $z_i = y_i^2 - E\left[y_i^2\right]$, then using the equation we proved in the above question (and in the small deviation regime), we get

$$P\left(\left|\frac{1}{d}\sum_{i=1}^{d} z_i\right| \geq t\right) \leq \frac{2\delta}{N^2}$$

Now recall that $\forall 1 \leq k < l \leq N$. This implies that if $k, l \in \mathbb{N}$, then $N \geq 2$, meaning that $N^2 \geq 4 > 2$ and so

$$P\left(\left|\frac{1}{d}\sum_{i=1}^{d} z_i\right| \geq t\right) < \delta$$

$\blacksquare$

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