DUE DATES: PART I: FRIDAY, AUGUST 28, 2020; PART II: FRIDAY, SEPTEMBER 4, 2020.

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In this problem, you will construct a sequence of random variables that obeys the weak law of large numbers ($n^{-1}\sum_1^n X_i$ converges to 0 in probability), but not the strong law ($n^{-1}\sum_1^n X_i$ almost surely does not converge).

PART I

Let $\Omega_1, \Omega_2, \ldots$ be an infinite sequence of independent events. Let $p_i = \mathbb{P}(\Omega_i)$.

1. Prove that the probability that none of the Ω_i, with $i > i_0$, occur is given by

$$\prod_{i=i_0+1}^{\infty} (1 - p_i) \tag{1}$$

2. Prove that

$$\mathbb{P}(\text{no } \Omega_i \text{ with } i > i_0 \text{ occur}) \leq \exp\left\{-\sum_{i=i_0+1}^{\infty} p_i\right\} \tag{2}$$

   Hint: use the fact that $1 + x \leq e^x$.

3. Prove that if

$$\sum_{i=1}^{\infty} \mathbb{P}(\Omega_i) \quad \text{diverges,} \tag{3}$$

   then

$$\mathbb{P}(\text{infinitely many } \Omega_i \text{ occur}) = 1. \tag{4}$$

We now consider a sequence $\{X_i\}$, $i \geq 1$ of independent random variables such that for each positive integer $i \geq 2$,

$$\mathbb{P}(X_i = -i) = \mathbb{P}(X_i = i) = \frac{1}{2i\log i}, \qquad \text{and} \qquad \mathbb{P}(X_i = 0) = 1 - \frac{1}{i\log i}, \tag{5}$$

with

$$\mathbb{P}(X_1 = 0) = 1. \tag{6}$$

Let

$$S_n = \sum_{i=1}^{n} X_i, \qquad \text{and} \qquad V_n = \mathbb{E}\left[S_n^2\right] - \{\mathbb{E}[S_n]\}^2. \tag{7}$$

4. Compute $\mathbb{E}[S_n]$ and V_n.

5. Prove that

$$\forall \varepsilon > 0, \lim_{n\to\infty} \mathbb{P}\left(\left|\frac{S_n}{n}\right| \geq \varepsilon\right) = 0, \tag{8}$$

to wit the X_i obey the weak law of large numbers.
Hint: to prove (8), it is sufficient to prove that $n^{-1}S_n$ converges to zero in mean square, that is

$$\lim_{n\to\infty} \mathbb{E}\left[\left|\frac{S_n}{n}\right|^2\right] = 0. \tag{9}$$

Part II

In the rest of the problem, we will prove that the X_i do not obey the strong law of large numbers. By contradiction, assume that $n^{-1}\sum_{i=1}^{n} X_i \to 0$ almost surely.

6. Prove that

$$\frac{X_n}{n} \text{ converges to } 0 \text{ almost surely.} \tag{10}$$

7. Deduce from the previous question that

$$\mathbb{P}\left(\left|\frac{X_i}{i}\right| \geq 1 \quad \text{occurs for infinitely many } i\right) = 0. \tag{11}$$

8. Let Ω_i be the event $\left|\frac{X_i}{i}\right| \geq 1$. Compute $\mathbb{P}(\Omega_i)$.

9. Prove that

$$\lim_{n\to\infty} \sum_{i=1}^{n} P(\Omega_i) = \infty \tag{12}$$

Hint: use the integral test.

10. Prove that

$$\mathbb{P}\left(\frac{S_n}{n} \to 0\right) = 0. \tag{13}$$

11. Conclude.