Assignment 1

Computational Intelligence, SS2020

Team Members		
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1 Maximum Likelihood Estimation of Model Parameters

1.1 Which measurement in sc.2 is exponentially distributed?

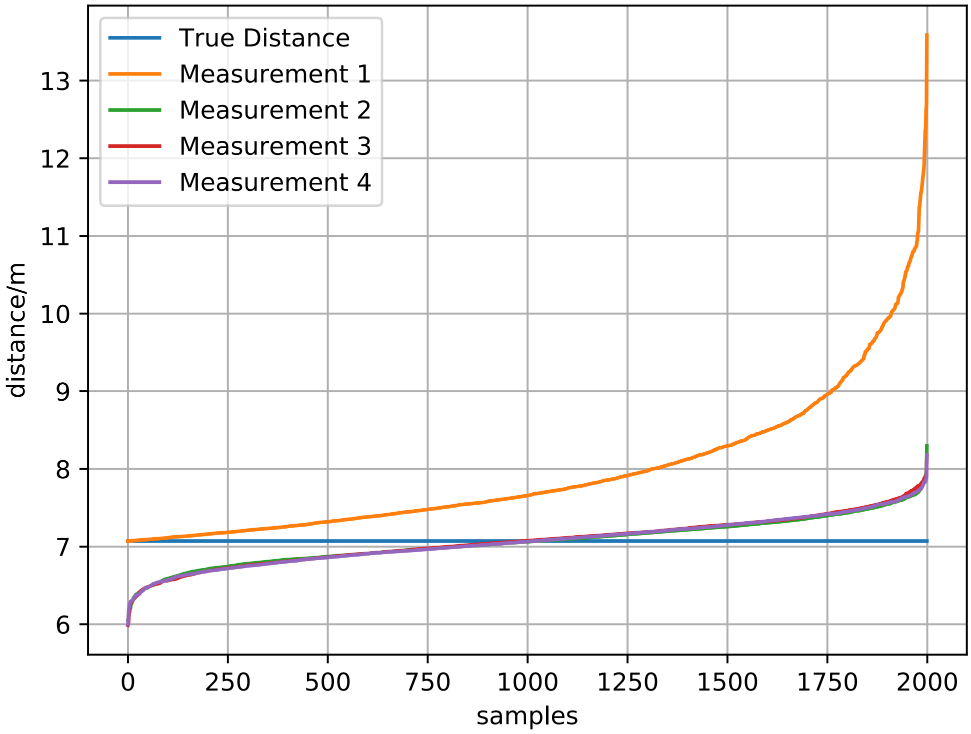

Figure 1: Scenario 2 with mixed measurement models for the anchors

- In figure 1 we can see that measurement 1 in scenario 2 is exponentially distributed. It is the only distribution which is not negative over all samples and its slope is rising exponentially for rising x-values.

1.2 Derivation for the maximum likelihood solution

- Analytical derivation for the Gaussian distribution:

$$p(\tilde{d}_n(a_i, \mathbf{p}) \mid \mathbf{p}) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{[\tilde{d}_n(a_i,\mathbf{p}) - d(a_i,\mathbf{p})]^2}{2\sigma^2}}$$

The data is independent identically distributed (iid), therefore the likelihood function is the product of all individual likelihoods

$$P(\tilde{d}_n(a_i, \mathbf{p}) \mid \mathbf{p}) = \prod_{n=0}^{N-1} \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{[\tilde{d}_n(a_i,\mathbf{p}) - d(a_i,\mathbf{p})]^2}{2\sigma^2}}$$

To convert the product to a sum we apply the natural logarithm.

$$
\begin{aligned}
L(\tilde{d}_n(a_i,\mathbf{p}) \mid \mathbf{p}) &= ln\left[\prod_{n=0}^{N-1} \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{[\tilde{d}_n(a_i,\mathbf{p})-d(a_i,\mathbf{p})]^2}{2\sigma^2}}\right] \\
L(\tilde{d}_n(a_i,\mathbf{p}) \mid \mathbf{p}) &= \sum_{n=0}^{N-1}\left[ln(\frac{1}{\sqrt{2\pi\sigma^2}}) - \frac{[\tilde{d}_n(a_i,\mathbf{p})-d(a_i,\mathbf{p})]^2}{2\sigma^2}\right]
\end{aligned}
$$

Since we want to find the parameter σ^2, which maximizes the probability of the distance, we derive $L(\tilde{d}_n(a_i,\mathbf{p}) \mid \mathbf{p})$ and set it to zero.

$$
\begin{aligned}
L(\tilde{d}_n(a_i,\mathbf{p}) \mid \mathbf{p}) &= \sum_{n=0}^{N-1}\left[ln(1) - \frac{1}{2}ln(2\pi\sigma^2) - \frac{[\tilde{d}_n(a_i,\mathbf{p})-d(a_i,\mathbf{p})]^2}{2\sigma^2}\right] \quad \mid \tfrac{\partial}{\partial\sigma^2} \\
\tfrac{\partial}{\partial\sigma^2} L(\tilde{d}_n(a_i,\mathbf{p}) \mid \mathbf{p}) &= \sum_{n=0}^{N-1}\left[-\frac{1}{\sigma^2} + \frac{[\tilde{d}_n(a_i,\mathbf{p})-d(a_i,\mathbf{p})]^2}{\sigma^4}\right] \quad \stackrel{!}{=} 0 \\
0 &= \sum_{n=0}^{N-1}\left[-\frac{1}{\sigma^2} + \frac{[\tilde{d}_n(a_i,\mathbf{p})-d(a_i,\mathbf{p})]^2}{\sigma^4}\right] \\
\tfrac{N}{\sigma^2} &= \sum_{n=0}^{N-1} \frac{[\tilde{d}_n(a_i,\mathbf{p})-d(a_i,\mathbf{p})]^2}{\sigma^4} \\
\sigma^2 &= \frac{\sum_{n=0}^{N-1}[\tilde{d}_n(a_i,\mathbf{p})-d(a_i,\mathbf{p})]^2}{N}
\end{aligned}
$$

- Analytical derivation for the Exponential distribution:

$$
p(\tilde{d}_n(a_i,\mathbf{p}) \mid \mathbf{p}) = \begin{cases} \lambda_i e^{-\lambda_i[\tilde{d}_n(a_i,\mathbf{p})-d(a_i,p)]} & , \tilde{d}_n(a_i,\mathbf{p}) \geq d(a_i,\mathbf{p}) \\ 0 & , \text{else} \end{cases}
$$

$$
\begin{aligned}
L(\tilde{d}_n(a_i,\mathbf{p}) \mid \mathbf{p}) &= \sum_{n=0}^{N-1} ln(\lambda_i) - \lambda_i[\tilde{d}_n(a_i,\mathbf{p})-d(a_i,\mathbf{p})] \quad \mid \tfrac{\partial}{\partial\lambda_i} \\
\tfrac{\partial}{\partial\lambda_i} L(\tilde{d}_n(a_i,\mathbf{p}) \mid \mathbf{p}) &= \tfrac{N}{\lambda_i} - \sum_{n=0}^{N-1}[\tilde{d}_n(a_i,\mathbf{p})-d(a_i,\mathbf{p})] \quad \stackrel{!}{=} 0 \\
\tfrac{N}{\lambda_i} &= \sum_{n=0}^{N-1}[\tilde{d}_n(a_i,\mathbf{p})-d(a_i,\mathbf{p})] \\
\lambda_i &= \frac{N}{\sum_{n=0}^{N-1}[\tilde{d}_n(a_i,\mathbf{p})-d(a_i,\mathbf{p})]} \quad , \tilde{d}_n(a_i,\mathbf{p}) \geq d(a_i,\mathbf{p})
\end{aligned}
$$

2 Estimation of the Position

2.1 Least-Squares Estimation of the Position

- Analytical conversion of the ML estimation equation:

$$\hat{\mathbf{p}}_{ML}(n) = \underset{\mathbf{p}}{\text{argmax}} \prod_{i=0}^{N_A-1} p(\tilde{d}_n(a_i, \mathbf{p}) \mid \mathbf{p})$$
$$\hat{\mathbf{p}}_{ML}(n) = \underset{\mathbf{p}}{\text{argmax}}\, ln \left[\prod_{i=0}^{N_A-1} p(\tilde{d}_n(a_i, \mathbf{p}) \mid \mathbf{p}) \right]$$
$$\hat{\mathbf{p}}_{ML}(n) = \underset{\mathbf{p}}{\text{argmax}} \sum_{i=0}^{N_A-1} ln \left(\frac{1}{\sqrt{2\pi\sigma_i^2}} \right) - \frac{[\tilde{d}_n(a_i, \mathbf{p}) - d(a_i, \mathbf{p})]^2}{2\sigma_i^2}$$

Because in scenario 1 we only use Gaussian models for all anchors that were calibrated with the same distance to the reference position, we can assume that $\sigma_i^2 = \sigma^2\ \forall i$. That means the ln-term can be neglected since it only shifts the value of the maximum by a constant but does not affect its position. Similarly $\frac{1}{2\sigma^2}$ can be omitted, as it is also just a scaling factor. Furthermore $\underset{\mathbf{p}}{\text{argmax}}\,(-\dots)$ is equivalent to $\underset{\mathbf{p}}{\text{argmin}}\,(\dots)$. Thus:

$$\hat{\mathbf{p}}_{ML}(n) = \underset{\mathbf{p}}{\text{argmin}} \sum_{i=0}^{N_A-1} [\tilde{d}_n(a_i, \mathbf{p}) - d(a_i, \mathbf{p})]^2 = \hat{\mathbf{p}}_{LS}(n)$$

2.2 Gauss-Newton Algorithm for Position Estimation

- Analytical solution for the Jacobian matrix.

$$[J(p)]_{i,1} = \frac{\partial}{\partial x} \left[\tilde{d}_n(a_i, \mathbf{p}) - \sqrt{(x_i - x)^2 + (y_i - y)^2} \right]$$
$$[J(p)]_{i,1} = -\frac{[(-)2(x_i - x)]}{2 \cdot \sqrt{(x_i - x)^2 + (y_i - y)^2}}$$
$$[J(p)]_{i,1} = \frac{(x_i - x)}{\sqrt{(x_i - x)^2 + (y_i - y)^2}}$$
$$[J(p)]_{i,2} = \frac{(y_i - y)}{\sqrt{(x_i - x)^2 + (y_i - y)^2}}$$

	Scenario 1	Scenario 2		Scenario 3
		with exponential anchor	without exponential anchor	
Error mean μ_e	0.278	0.640	0.399	1.265
Error variance σ_e^2	0.022	0.275	0.054	0.939

Table 1: .

2.3 Numerical Maximum Likelihood Estimation of the Position

2.3.1 Single Measurement

- The numerical maximum likelihood estimate is computed by finding the maximum of the joint likelihood of all anchors evaluated within a 2D-grid enclosed by the anchors. Because of the i.i.d.-assumption the joint likelihood can be calculated as the product of all individual exponential likelihoods:

$$p(\tilde{\mathbf{d}}_n(\mathbf{p})|\mathbf{p}) = \begin{cases} \prod_{i=0}^{N_A-1} p(\tilde{d}_n(a_i, \mathbf{p})|\mathbf{p}), & \text{if } \tilde{d}_n(a_i, \mathbf{p}) \geq d(a_i, \mathbf{p})\ \forall i \\ 0 & \text{else} \end{cases}$$

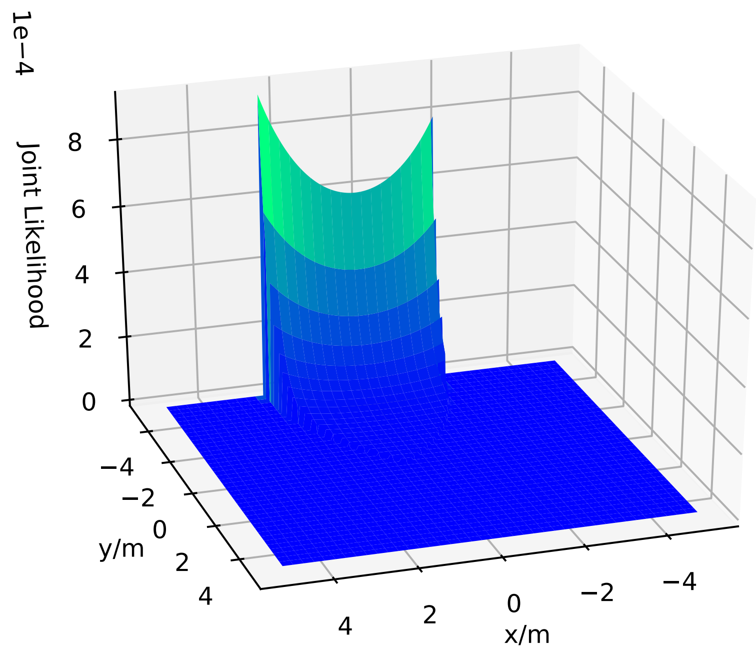

Figure 2: $p(\tilde{\mathbf{d}}_0(\mathbf{p})|\mathbf{p})$ evaluated within a 2D-grid enclosed by the anchors.

- Because of its nonlinearity the joint likelihood function of the first sample $n = 0$ has two local maxima (s. figure 2). If we used a gradient ascent algorithm with a random starting position, it would stop once it reaches any of them, possibly leading to a false estimation of the position, if the found maximum is not global.

- The found maximum at $\mathbf{p}_{0,NML} = \begin{bmatrix} 2.5 \\ -5 \end{bmatrix}$ is not at the true position. **Of Course That's Because...???**

2.3.2 Multiple Measurements

	Least-Squares	Numerical Maximum Likelihood	Bayes
Error mean μ_e	1.265	0.915	0.680
Error variance σ_e^2	0.939	0.489	0.095

Table 2: Error-mean and -variance of Least-Squares-, Numerical Maximum Likelihood- and Bayes-Estimation.

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