Assignment 1

Computational Intelligence, SS2020

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1 Maximum Likelihood Estimation of Model Parameters

- Analytical derivation for the Gaussian distribution:

$$p(\tilde{d}_n(a_i, \mathbf{p}) \mid \mathbf{p}) \quad = \quad \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{[\tilde{d}_n(a_i,\mathbf{p})-d(a_i,\mathbf{p})]^2}{2\sigma^2}}$$

The data is independent identically distributed (iid), therefore the likelihood function is the sum of all individual likelihoods

$$P(\tilde{d}_n(a_i, \mathbf{p}) \mid \mathbf{p}) \quad = \quad \prod_{n=0}^{N-1} \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{[\tilde{d}_n(a_i,\mathbf{p})-d(a_i,\mathbf{p})]^2}{2\sigma^2}}$$

To convert the product to sum we use the logarithm.

$$L(\tilde{d}_n(a_i, \mathbf{p}) \mid \mathbf{p}) \quad = \quad ln\left[\prod_{n=0}^{N-1} \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{[\tilde{d}_n(a_i,\mathbf{p})-d(a_i,\mathbf{p})]^2}{2\sigma^2}}\right]$$

$$L(\tilde{d}_n(a_i, \mathbf{p}) \mid \mathbf{p}) \quad = \quad \sum_{n=0}^{N-1}\left[ln(\frac{1}{\sqrt{2\pi\sigma^2}}) - \frac{[\tilde{d}_n(a_i,\mathbf{p})-d(a_i,\mathbf{p})]^2}{2\sigma^2}\right]$$

Since we want to find the parameter σ^2, which maximizes the probability of the distance, we derive $L(\tilde{d}_n(a_i, \mathbf{p}) \mid \mathbf{p})$ and set it to zero.

$$L(\tilde{d}_n(a_i, \mathbf{p}) \mid \mathbf{p}) \quad = \quad \sum_{n=0}^{N-1}\left[ln(1) - ln(\sqrt{2\pi\sigma^2} - \frac{[\tilde{d}_n(a_i,\mathbf{p})-d(a_i,\mathbf{p})]^2}{2\sigma^2}\right] \quad \mid \frac{\partial}{\partial\sigma^2}$$

$$\frac{\partial}{\partial\sigma^2} L(\tilde{d}_n(a_i, \mathbf{p}) \mid \mathbf{p}) \quad = \quad \sum_{n=0}^{N-1}\left[-\frac{1}{\sigma^2} + \frac{[\tilde{d}_n(a_i,\mathbf{p})-d(a_i,\mathbf{p})]^2}{\sigma^4}\right] \quad \stackrel{!}{=} 0$$

$$0 \quad = \quad \sum_{n=0}^{N-1}\left[-\frac{1}{\sigma^2} + \frac{[\tilde{d}_n(a_i,\mathbf{p})-d(a_i,\mathbf{p})]^2}{\sigma^4}\right]$$

$$\frac{N}{\sigma^2} \quad = \quad \sum_{n=0}^{N-1} \frac{[\tilde{d}_n(a_i,\mathbf{p})-d(a_i,\mathbf{p})]^2}{\sigma^4}$$

$$\sigma^2 \quad = \quad \sum_{n=0}^{N-1} \frac{[\tilde{d}_n(a_i,\mathbf{p})-d(a_i,\mathbf{p})]^2}{N}$$

- Analytical derivation for the Exponential distribution:

$$p(\tilde{d}_n(a_i, \mathbf{p}) \mid \mathbf{p}) = \begin{cases} \lambda_i e^{-\lambda_i[\tilde{d}_n(a_i,\mathbf{p})-d(a_i,p)]} & , \tilde{d}_n(a_i,\mathbf{p}) \geq d(a_i,\mathbf{p}) \\ 0 & , \text{else} \end{cases}$$

$$
\begin{aligned}
L(\tilde{d}_n(a_i,\mathbf{p}) \mid \mathbf{p}) &= \sum_{n=0}^{N-1} ln(\lambda_i) - \lambda_i[\tilde{d}_n(a_i,\mathbf{p}) - d(a_i,\mathbf{p})] \quad \mid \tfrac{\partial}{\partial \lambda_i} \\
\tfrac{\partial}{\partial \lambda_i} L(\tilde{d}_n(a_i,\mathbf{p}) \mid \mathbf{p}) &= \tfrac{N}{\lambda_i} - \sum_{n=0}^{N-1} [\tilde{d}_n(a_i,\mathbf{p}) - d(a_i,\mathbf{p})] \quad \stackrel{!}{=} 0 \\
\tfrac{N}{\lambda_i} &= \sum_{n=0}^{N-1} [\tilde{d}_n(a_i,\mathbf{p}) - d(a_i,\mathbf{p})] \\
\lambda_i &= \frac{N}{\sum_{n=0}^{N-1} [\tilde{d}_n(a_i,\mathbf{p}) - d(a_i,p)]} \quad , \tilde{d}_n(a_i,\mathbf{p}) \geq d(a_i,\mathbf{p})
\end{aligned}
$$

2 Estimation of the Position

2.1 Least-Squares Estimation of the Position

- Analytical conversion of the ML estimation equation:

$$
\begin{aligned}
\hat{\mathbf{p}}_{ML}(n) &= \underset{\mathbf{p}}{\operatorname{argmax}} \prod_{n=0}^{N-1} p(\tilde{d}_n(a_i,\mathbf{p}) \mid \mathbf{p}) \\
\hat{\mathbf{p}}_{ML}(n) &= \underset{\mathbf{p}}{\operatorname{argmax}}\, ln \left[\prod_{n=0}^{N-1} p(\tilde{d}_n(a_i,\mathbf{p}) \mid \mathbf{p}) \right] \\
\hat{\mathbf{p}}_{ML}(n) &= \underset{\mathbf{p}}{\operatorname{argmax}} \sum_{i=0}^{N_A-1} ln \left(\frac{1}{\sqrt{2\pi\sigma^2}} - \frac{[\tilde{d}_n(a_i,\mathbf{p}) - d(a_i,\mathbf{p})]^2}{2\sigma^2} \right)
\end{aligned}
$$

The ln-term can be negclected since it's just a scaling factor. Furthermore $-\underset{\mathbf{p}}{\operatorname{argmax}}$ is equal to $\underset{\mathbf{p}}{\operatorname{argmin}}$.

$$
\hat{\mathbf{p}}_{ML}(n) = \underset{\mathbf{p}}{\operatorname{argmin}} \sum_{i=0}^{N_A-1} \frac{[\tilde{d}_n(a_i,\mathbf{p}) - d(a_i,\mathbf{p})]^2}{2\sigma^2} \approx \hat{\mathbf{p}}_{LS}(n)
$$

2.2 Gauss-Newton Algorithm for Position Estimation

- Analytical solution for the Jacobian matrix.

$$
\begin{aligned}
[J(p)]_{i,1} &= \frac{\partial}{\partial x} \left[\tilde{d}_n(a_i,\mathbf{p}) - \sqrt{(x_i-x)^2 + (y_i-y)^2} \right] \\
[J(p)]_{i,1} &= -\frac{[(-)2(x_i-x)]}{2 \cdot \sqrt{(x_i-x)^2+(y_i-y)^2}} \\
[J(p)]_{i,1} &= \frac{(x_i-x)}{\sqrt{(x_i-x)^2+(y_i-y)^2}} \\
[J(p)]_{i,2} &= \frac{(y_i-y)}{\sqrt{(x_i-x)^2+(y_i-y)^2}}
\end{aligned}
$$

2.3 Numerical Maximum Likelihood Estimation of the Position

	Scenario 1	Scenario 2		Scenario 3
		with exponential anchor	without exponential anchor	
Error mean μ_e	0.2779	0.6402	0.3988	1.2697
Error variance σ_e^2	0.0216	0.2745	0.0541	0.9622

Table 1: .

	Least-Squares	Numerical Maximum Likelihood	Bayes
Error mean μ_e	0.9621	0.9154	0.3080
Error variance σ_e^2	1.2697	0.4889	0.0135

Table 2: .