# Symbol error probability of M-ary PSK

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## 1 Introduction

In the $M$-ary phase shift keying scheme (M-PSK), one of $M$ possible signals is transmitted by modulating the phase of the carrier waveform as shown in the expression below

$$s_m(t) = Re\left[g(t)e^{j\frac{2\pi(m-1)}{M}}e^{j2\pi f_c t}\right]$$

where $m = 1, 2, ..., M,\ f_c$ is the carrier frequency, and $g(t)$ is a pulse (Proakis 5th). At a given bit energy, a M-PSK system's error rate will increase with $M$. Therefore, understanding the relationship between the probability of error and $M$ is essential for understanding M-PSK performance and for meeting design specifications. This report explores the relationship between error probability ($P_e$) and $M$ by investigating theoretical $P_e$ values and Monte Carlo simulation results.

## 2 Methods

At a given $M$ value, the exact formula for a M-PSK system's error probability is

$$P_M = 1 - 2\int_0^\infty \frac{e^{-(r_0-\sqrt{E_s})^2/N_0}}{\sqrt{\pi}N_0}\left(\int_0^{r_0\tan(\pi/M)} \frac{e^{-r_1^2/N_0}}{\sqrt{\pi}N_0}\,dr_1\right)dr_0$$

Where $E_s = E_b \cdot \log_2(M)$ is the energy per symbol and $E_b$ is the energy per bit.

The upper and lower bounds on the error probability are given by the following expression.

$$Q\left(\sqrt{\frac{2E_b}{N_0}(\log_2 M)\sin^2(\pi/M)}\right) < P_M < 2Q\left(\sqrt{\frac{2E_b}{N_0}(\log_2 M)\sin^2(\pi/M)}\right)$$

Theoretical $P_M$ values and upper/lower bounds were calculated in MATLAB. Calculations were performed for M = 2, 4, 8 and 16. Additionally, $P_M$ values were estimated with a Monte Carlo simulation.

The Monte Carlo simluation was carried out for each of the above $M$ values. A $m$ value is randomly generated and from it the received (noisy) signal is calculated according to

$$r_1 = \sqrt{E_s}\cos(2\pi(m-1)/M) + n_1$$

$$r_2 = \sqrt{E_s}\sin(2\pi(m-1)/M) + n_2$$

where the $n_i$ terms are independent Gaussian random variables with zero mean and variance $N_0/2$. The received signal is compared to the ground-truth $m$ value to determine whether a bit error has occurred. That is, we identify errors by checking whether the received bit lies in the decision region corresponding to $m$. For each bit energy, $P_M$ is computed after a total of 100 bit errors have accumulated.

# 3 Results

Figure 1 shows the theoretical results, simulated data and bounds, which become less tight as $M$ increases. At $P_e = 10^{-2}$, the bounds on $M = 2$, $M = 4$ and $M = 8$ have a width of about 1 dB (note that because $\log_2(2)\sin^2(\pi/2) = \log_2(4)sin^2(\pi/4)$, the bounds for $M = 2$ and $M = 4$ are the same). However, it is clear from this figure that the $M = 16$ bounds are far less strict. In fact, the width of the bounds for $M = 16$ is approximately 4 dB at $P_e = 10^{-1}$.

If a bit error rate of $10^{-2}$ were required, a 2-PSK or 4-PSK system is preferrable. As expected, $M = 2$ has the most favorable SNR. The gap between $M = 2$ and $M = 4$ at $P_e = 10^{-2}$ is about 1 dB. However, increasing from $M = 1$ to $M = 8$ will cost more than 4 dB at this $P_e$ value.

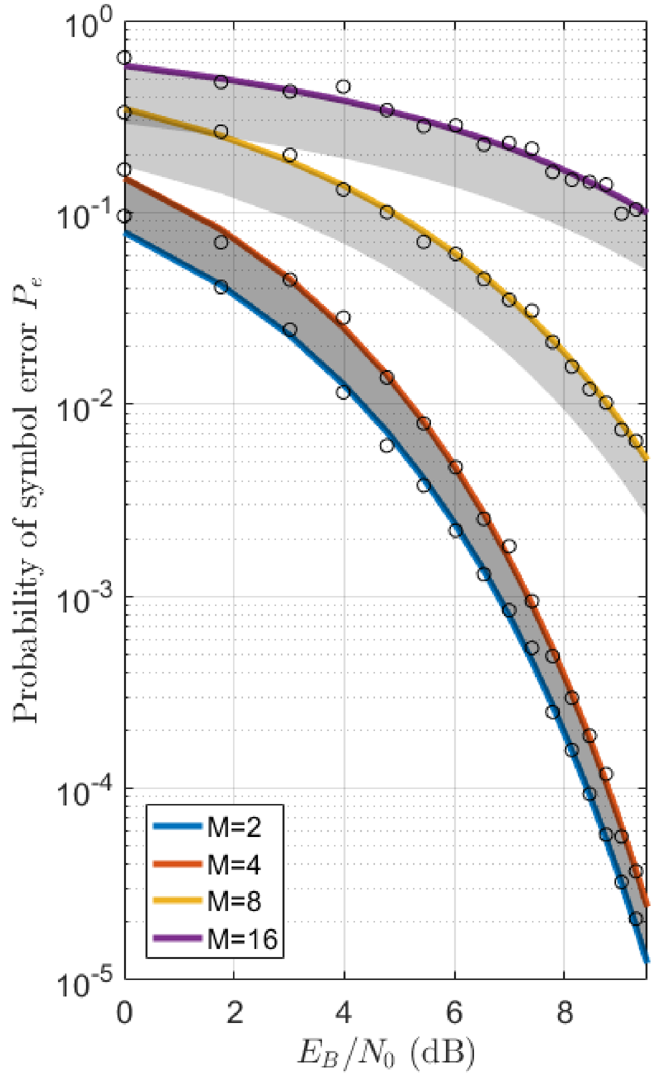


Figure 1: M-PSK error probabiblity vs. $E_b/N_0$ for M = 2, 4, 8 and 16. Upper and lower bounds are shown by the gray patches. The exact expression is shown by the solid lines. Monte Carlo simulation results are denoted by the 'o' markers.