CMSC/PHYS 457, Introduction to Quantum Computing
Spring 2023, University of Maryland

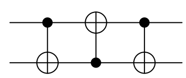
Due: March 2nd, 2023
Date: March 3, 2023

Assignment 2

Please submit it electronically to ELMS. This assignment is 6% in your final grade. For the simplicity of the grading, the total number of points for the assignment is 60.

Problem 1. *Circuit identities.*

1. *(5 points)* Show that the following circuit swaps two qubits:



Answer:

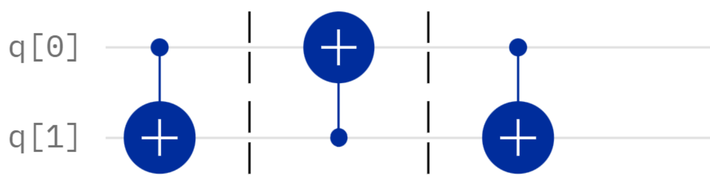
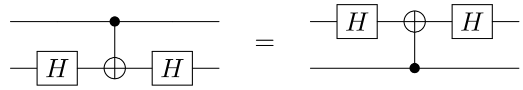


Figure 1: Swap Circuit:
T_1: 1st dotted line, T_2: 2nd dotted line

Qubit	Input	T_1	T_2	Output
q[0]	0	0	0	0
q[1]	0	0	0	0
q[0]	0	0	1	1
q[1]	1	1	1	0
q[0]	1	1	0	0
q[1]	0	1	1	1
q[0]	1	1	1	1
q[1]	1	0	0	1

Table containing the 4 cases for inputs and the value at each signifigant time stamp as well as input showing that input swapped

2. *(5 points)* Verify the following circuit identity:



Answer:

Left-Hand Side:

$$LHS = I \bigotimes H * CNOT * I \bigotimes H$$

$$LHS = 1/\sqrt{2}\begin{pmatrix}1 & 1 & 0 & 0\\ 1 & -1 & 0 & 0\\ 0 & 0 & 1 & 1\\ 0 & 0 & 1 & -1\end{pmatrix} * \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{pmatrix} * 1/\sqrt{2}\begin{pmatrix}1 & 1 & 0 & 0\\ 1 & -1 & 0 & 0\\ 0 & 0 & 1 & 1\\ 0 & 0 & 1 & -1\end{pmatrix}$$

$$LHS = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}$$

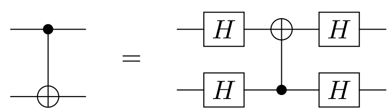
$$RHS = H \bigotimes I * R - CNOT * H \bigotimes I$$

$$\text{RHS} = 1/\sqrt{2}\begin{pmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\\ 1 & 0 & -1 & 0\\ 0 & 1 & 0 & -1\end{pmatrix} * \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\end{pmatrix} * 1/\sqrt{2}\begin{pmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\\ 1 & 0 & -1 & 0\\ 0 & 1 & 0 & -1\end{pmatrix}$$

$$RHS = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}$$

Therefore, LHS == RHS

3. *(5 points)* Verify the following circuit identity:



Answer:

LHS = CNOT

$$LHS = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{pmatrix}$$

$$\text{RHS} = \text{H} \bigotimes H * R - CNOT * H \bigotimes H$$

$$\text{RHS} = 1/2\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & -1 & 1 & -1\\ 1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1\end{pmatrix} * \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\end{pmatrix} * 1/2\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & -1 & 1 & -1\\ 1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1\end{pmatrix}$$

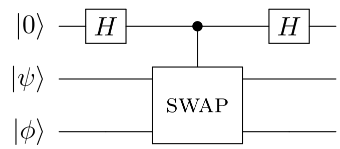
$$RHS = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{pmatrix}$$

Therefore, LHS == RHS

Give an interpretation of this identity: The identity shows us that if we apply the H gate to both qubits on either side of our reversed CNOT gate we can flip it back to a CNOT gate, allowing us to switch which qubits are the control and target and vice versa

Problem 2. *Swap test.*

1. *(5 points)* Let $|\psi\rangle$ and $|\phi\rangle$ be arbitrary single-qubit states (not necessarily computational basis states), and let SWAP denote the 2-qubit gate that swaps its input qubits (i.e., $\text{SWAP}|x\rangle|y\rangle = |y\rangle|x\rangle$ for any $x, y \in \{0, 1\}$). Compute the output of the following quantum circuit:



Answer: $\frac{1}{2}(|0\rangle|\psi\rangle|\phi\rangle + |1\rangle|\psi\rangle|\phi\rangle + |0\rangle|\phi\rangle|\psi\rangle - |1\rangle|\phi\rangle|\psi\rangle)$

2. *(5 points)* Suppose the top qubit in the above circuit is measured in the computational basis. What is the probability that the measurement result is 0?

Answer: $\frac{1}{2}(1 + \langle\phi||\psi\rangle^2)$

3. *(3 points)* If the result of measuring the top qubit in the computational basis is 0, what is the (normalized) post-measurement state of the remaining two qubits?

Answer: $\frac{M_i|\phi\rangle}{\sqrt{p(i)}}$ $M_i == I$

$\frac{1}{\sqrt{2}}(|\phi\rangle|\psi\rangle + |\psi\rangle|\phi\rangle$

4. *(2 points)* How do the results of the previous parts change if $|\psi\rangle$ and $|\phi\rangle$ are n-qubit states, and SWAP denotes the $2n$-qubit gate that swaps the first n qubits with the last n qubits?

Answer: The only thing that changes is adding the qubits to the state, the probability of the system will stay the same.

Problem 3. *The Hadamard gate and qubit rotations*

1. *(5 points)* Suppose that $(n_x, n_y, n_z) \in^3$ is a unit vector and $\theta \in$. Show that

$$e^{-i\frac{\theta}{2}(n_x X + n_y Y + n_z Z)} = \cos(\tfrac{\theta}{2})\, I - i \sin(\tfrac{\theta}{2})\,(n_x X + n_y Y + n_z Z).$$

Answer:
If we set A = $(n_x X + n_y Y + n_z Z)$
We know $A^2 == I$ because $X^2, Y^2, Z^2 == I$ and $(n_x, n_y, n_z) \in^3$is a unit vector
Then by using the Taylor series of e we have
$\sum_{k=0}^{\infty} \frac{1}{k!}(-i\frac{\theta}{2}A)^k = e^{\frac{-i\theta}{2}A}$
Even Odd Term Split: $\sum_{k=0}^{\infty} \frac{1}{2k!}(-i\frac{\theta}{2})^{2k} + \sum_{k=0}^{\infty} \frac{1}{(2k+1)!}(-i\frac{\theta}{2})^{2k+1}A$
Then we apply the Taylor series of cos(x) and sin(x) to get $\cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}A = \cos\frac{\theta}{2}I - \sin\frac{\theta}{2}(n_x X + n_y Y + n_z Z)$

2. *(5 points)* Find a unit vector $(n_x, n_y, n_z) \in^3$ and numbers $\phi, \theta \in$ so that

$$H = e^{i\phi} e^{-i\frac{\theta}{2}(n_x X + n_y Y + n_z Z)},$$

where H denotes the Hadamard gate. What does this mean in terms of the Bloch sphere?

Answer:

$e^{i\phi}e^{-i\frac{\theta}{2}(n_xX+n_yY+n_zZ)} = e^{i\phi}\begin{pmatrix} \cos\frac{\theta}{2} - i\sin\frac{\theta}{2}n_z & -in_x\sin\frac{\theta}{2} - n_y\sin\frac{\theta}{2} \\ -in_x\sin\frac{\theta}{2} - n_y\sin\frac{\theta}{2} & \cos\frac{\theta}{2} - i\sin\frac{\theta}{2}n_z \end{pmatrix}$
if $\theta = \pi$, then $e^{i\phi}\begin{pmatrix} -in_z & -in_x - n_y \\ -in_x + n_y & in_z \end{pmatrix}$
if $(n_x, n_y, n_z) = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$, then$\begin{pmatrix} -i\frac{1}{\sqrt{2}} & -i\frac{1}{\sqrt{2}} \\ -i\frac{1}{\sqrt{2}} & i\frac{1}{\sqrt{2}} \end{pmatrix}$
If $\phi = \frac{\pi}{2}, e^{i\frac{\pi}{2}} = i$
So we get $i\begin{pmatrix} -i\frac{1}{\sqrt{2}} & -i\frac{1}{\sqrt{2}} \\ -i\frac{1}{\sqrt{2}} & i\frac{1}{\sqrt{2}} \end{pmatrix}$ -¿ $\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ -¿ H, therefore $u = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}), \phi = \frac{\pi}{2}, \theta = \pi$
In the Bloch sphere we can conclude H $= \cos\frac{\pi}{2}|0\rangle + (\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})\sin\frac{\pi}{2}|1\rangle = i|1\rangle$

3. *(5 points)* Write the Hadamard gate as a product of rotations about the x and y axes. In particular, find $\alpha, \beta, \gamma, \phi \in$ such that $H = e^{i\phi}R_y(\gamma)R_x(\beta)R_y(\alpha)$.
   Answer: $e^{i\phi}R_y(\gamma)R_x(\beta)R_y(\alpha) = e^{i\phi}\begin{pmatrix} \cos\frac{\gamma}{2} & -\sin\frac{\gamma}{2} \\ \sin\frac{\gamma}{2} & \cos\frac{\gamma}{2} \end{pmatrix}\begin{pmatrix} \cos\frac{\beta}{2} & -i\sin\frac{\beta}{2} \\ -i\sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} \cos\frac{\alpha}{2} & -\sin\frac{\alpha}{2} \\ \sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix}$
   If you plug in $\gamma = -\frac{\pi}{4}, \beta = \pi, \alpha = \frac{\pi}{4}$ and simplify, you get:
   $i\begin{pmatrix} -i\frac{1}{\sqrt{2}} & -i\frac{1}{\sqrt{2}} \\ -i\frac{1}{\sqrt{2}} & i\frac{1}{\sqrt{2}} \end{pmatrix}$
   Which simplifies to H (same as in the earlier part)

Problem 4. *Universality of gate sets.* Prove that each of the following gate sets either is or is not universal. You may use the fact that the set $\{\textsc{cnot}, H, T\}$ is universal.

1. *(5 points)* $\{H, T\}$
   Answer: This set isn't universal (for non 1-bit) because there is no entangling gate within the set

2. *(5 points)* $\{\textsc{cnot}, T\}$
   Answer: This set isn't universal because we do not have two rotational gate's over two non-parallel axes (T rotates but C-NOT does not)

3. *(5 points)* $\{\textsc{cnot}, H\}$
   Answer: This set isn't universal because we do not have two rotational gate's over two non-parallel axes (H rotates but C-NOT does not)

4. *(Bonus: 10 points)* $\{\textsc{cnot}, H, T^2\}$
   Answer: This set isn't universal because while the set does rotate around two non-parallel axes and have an entangling gate, T^2 simplifies to the S gate which rotates the Z-Axis but does not cover the irrational numbers so the set isn't universal

---