Assignment 4

Please submit it electronically to ELMS. This assignment is 6% in your total points. For the simplicity of the grading, the total points for the assignment are 60. Note that we will reward the use of Latex for typesetting with bonus points (an extra 5% of your points).

Problem 1. *The Fourier transform and translation invariance.* The quantum Fourier transform on n qubits is defined as the transformation

$$|x\rangle \mapsto \frac{1}{\sqrt{2^n}} \sum_{y=0}^{2^n-1} e^{2\pi ixy/2^n} |y\rangle$$

where we identify n-bit strings and the integers they represent in binary. More generally, for any nonnegative integer N, we can define the quantum Fourier transform modulo N as

$$|x\rangle \overset{F_N}{\mapsto} \frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} e^{2\pi ixy/N} |y\rangle$$

where the state space is $\mathbb{C}^N$, with orthonormal basis $\{|0\rangle, |1\rangle, \ldots, |N-1\rangle\}$. Let P denote the unitary operation that adds 1 modulo N: for any $x \in \{0, 1, \ldots, N-1\}$, $P|x\rangle = |x+1 \bmod N\rangle$.

1. *(2 points)* Show that F_N is a unitary transformation.

   Answer:
   Let $w = e^{\frac{2\pi i}{n}}, a, b \in \{0, 1, ..., n-1\}$

   Then $|a\rangle \overset{F_N}{\mapsto} \frac{1}{n} \sum_{y=0}^{n-1} e^{\frac{2\pi iay}{n}} |y\rangle$ and $|b\rangle \overset{F_N}{\mapsto} \frac{1}{n} \sum_{y=0}^{n-1} e^{\frac{2\pi iby}{n}} |y\rangle$

   Let result of a,b $= |\phi_a\rangle, |\phi_b\rangle$
   Therefore the inner product of $|\phi_a\rangle, |\phi_b\rangle = \langle\phi_a|\phi_b\rangle = \frac{1}{n} \sum_{y=0}^{n-1} (w^{b-a})^y$

   Using the formula $\sum_{k=0}^{n-1} x^k = \frac{x^n-1}{x-1} for x \neq 1$, the fact that $\sum_{k=0}^{n-1} 1^k = n$, and the fact that $w^{nl} = 1$ for any integer l because $w = e^{\frac{2\pi i}{n}}$ we know that $\langle\phi_a|\phi_b\rangle = 1$ if a = b and 0 otherwise.

   Therefore $\{|\phi_a\rangle, ..., |\phi_{n-1}\rangle\}$ is orthonormal. therefore F_n is a unitary transformation

2. *(5 points)* Show that the Fourier basis states are eigenvectors of P. What are their eigenvalues? (Equivalently, show that $F_N^{-1} P F_N$ is diagonal, and find its diagonal entries.)

   Answer:
   $PF_n|x\rangle = \frac{1}{\sqrt{n}} \sum_{y=0}^{n-1} e^{\frac{2\pi ixy}{n}} |y + 1 mod N\rangle$

$= \frac{1}{\sqrt{n}} \sum_{y=0}^{n-1} e^{\frac{2\pi ix(y+1}{n}} * e^{\frac{2\pi i - x}{n}} |y + 1modN\rangle$
$= e^{\frac{2\pi i-x}{n}} \frac{1}{\sqrt{n}} \sum_{y=0}^{n-1} e^{\frac{2\pi ix(y+1)}{n}} |y+1modN\rangle$

Because at y = n-1, the term of the summation is equal to 0, the summation is the same as the original with all the other sum terms at indexes in front of the original sum.

Therefore we get $e^{\frac{2\pi ix(y+1)}{n}} |y + 1modN\rangle = e^{\frac{2\pi i-x}{n}} F_n|x\rangle$
Therefore the Fourier basis states are eigenvectors of P and the eig vals are $e^{\frac{2\pi i-x}{n}}$ for each basis state $|\psi_i\rangle$

3. *(3 points)* Let $|\psi\rangle$ be a state of n qubits. Show that if $P|\psi\rangle$ is measured in the Fourier basis (or equivalently, if we apply the inverse Fourier transform and then measure in the computational basis), the probabilities of all measurement outcomes are the same as if the state had been $|\psi\rangle$.

Answer:

Let $|\psi\rangle = \sum_{i=0}^{n-1} \alpha_i|\widehat{i}$ where $\widehat{i} = F_n|\widehat{i}\rangle$ and α_i is the amplitude for each basis.
So $P|\psi\rangle = \sum_{i=0}^{n-1} \alpha_i P(|\widehat{i}\rangle) = \sum_{i=0}^{n-1} \alpha_i e^{2\pi i - x/n}|\widehat{i}\rangle$ by 1.2 result.
Now we have the probability $= \sum_{i=0}^{n-1} \alpha_i e^{(2\pi i - x)/n} \sum_{i=0}^{n-1} \alpha_i' e^{2\pi ix/n} = \alpha_i \alpha_i'$
This is the same probability as if $|\psi\rangle$ was measured in $\sum_{i=0}^{n-1} \alpha_i|\widehat{i}\rangle$

Problem 2. *Factoring 21.*

1. *(3 points)* Suppose that, when running Shor's algorithm to factor the number 21, you choose the value $a = 2$. What is the order r of a mod 21?

   Answer: $order = 6$ when a = 2. 2^6 mod 21 = 1

2. *(3 points)* Give an expression for the probabilities of the possible measurement outcomes when performing phase estimation with n bits of precision in Shor's algorithm.

   Answer:
   $p(y, f(x_0)) = \frac{1}{2^n} \sum_{y}^{2^{n-1}} \sum_{b=0}^{m-1} ||e^{\frac{2\pi ixy}{2^n}}||^2$

3. *(3 points)* In the execution of Shor's algorithm considered in part (a), suppose you perform phase estimation with $n = 7$ bits of precision. Plot the probabilities of the possible measurement outcomes obtained by the algorithm. You are encouraged to use software to produce your plot.

4. *(3 points)* Compute $\gcd(21, a^{r/2} - 1)$ and $\gcd(21, a^{r/2} + 1)$. How do they relate to the prime factors of 21?

5. *(3 points)* How would your above answers change if instead of taking $a = 2$, you had taken $a = 5$?

Problem 3. *Density matrices.* Consider the ensemble in which the state $|0\rangle$ occurs with probability 3/5 and the state $(|0\rangle + |1\rangle)/\sqrt{2}$ occurs with probability 2/5.

1. *(2 points)* What is the density matrix ρ of this ensemble?

   Answer:

   $\frac{3}{5}(|0\rangle\langle 0|) + \frac{2}{5}(\frac{1}{2}(|0\rangle + |1\rangle)(\langle 0| + \langle 1|))$

   $\frac{3}{5}|0\rangle\langle 0| + \frac{1}{5}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|)$

   $\rho = \frac{4}{5}|0\rangle\langle 0| + \frac{1}{5}|0\rangle\langle 1| + \frac{1}{5}|1\rangle\langle 0| + \frac{1}{5}|1\rangle\langle 1|)$

2. *(3 points)* Write ρ in the form $\frac{1}{2}(I + r_x X + r_y Y + r_z Z)$, and plot ρ as a point in the Bloch sphere.

   Answer:
   $\rho = \frac{4}{5}|0\rangle\langle 0| + \frac{1}{5}|0\rangle\langle 1| + \frac{1}{5}|1\rangle\langle 0| + \frac{1}{5}|1\rangle\langle 1|) = \frac{4}{5}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + \frac{1}{5}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} + \frac{1}{5}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} + \frac{1}{5}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$

   $\rho = \frac{1}{5}\begin{pmatrix}4 & 1\\ 1 & 1\end{pmatrix}$
   $= \frac{1}{5}\begin{pmatrix}4 & 0\\ 0 & 1\end{pmatrix} + \frac{1}{5}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$
   $= \frac{1}{2}(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \frac{2}{5}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + \frac{3}{5}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix})$
   $\rho = \frac{1}{2}[I + \frac{2}{5}X + \frac{3}{5}Z]$

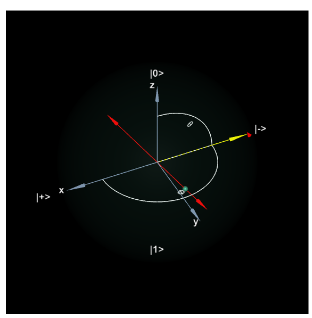

Figure 1: Bloch Sphere Plot

3. *(3 points)* Suppose we measure the state in the computational basis. What is the probability of getting the outcome 0? Compute this both by averaging over the ensemble of pure states and by computing $(\rho|0\rangle\langle 0|)$, and show that the results are consistent.

   Answer:

   Averaging over the ensemble:
   $p(|0\rangle) = \langle 0|\rho|0\rangle$
   $= \frac{4}{5}$
   *Terms from [$\frac{4}{5}|0\rangle\langle 0| + \frac{1}{5}|0\rangle\langle 1| + \frac{1}{5}|1\rangle\langle 0| + \frac{1}{5}|1\rangle\langle 1|)$]
   with a 1 will zero out leaving only the first 00 term, hence 4/5

$tr(p|0\rangle\langle 0|)$:

$tr(\frac{1}{5}\begin{pmatrix}4 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}) = tr(\frac{1}{5}\begin{pmatrix}4 & 0\\ 1 & 0\end{pmatrix}) = \frac{4}{5}$

Answer is consistent with both methods!

4. *(3 points)* How does the density matrix change if we apply the Hadamard gate? Compute this both by applying the Hadamard gate to each pure state in the ensemble and finding the corresponding density matrix, and by computing $H\rho H^{\dagger}$.

Answer:

It flips the probability

$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = p\frac{3}{5}$
$H\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |0\rangle = p\frac{2}{5}$

$p' = \frac{1}{10}(7|0\rangle\langle 0| + 3(|0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|)) = \frac{1}{10}\begin{pmatrix}7 & 3\\ 3 & 3\end{pmatrix}$

With HpH† method:

$H\rho H^{\dagger} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\frac{1}{5}\begin{pmatrix}4 & 1\\ 1 & 1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} = \frac{1}{10}\begin{pmatrix}7 & 3\\ 3 & 3\end{pmatrix}$

Same results

Problem 4. *Local operations and the partial trace.*

1. *(3 points)* Let $|\psi\rangle = \frac{\sqrt{3}}{2}|00\rangle + \frac{1}{2}|11\rangle$. Let ρ denote the density matrix of $|\psi\rangle$ and let ρ' denote the density matrix of $(I \otimes H)|\psi\rangle$. Compute ρ and ρ'.

$\rho = |\psi\rangle\langle\psi|$

$\rho = (\frac{\sqrt{3}}{2}|00\rangle + \frac{1}{2}|11\rangle)(\frac{\sqrt{3}}{2}\langle 00| + \frac{1}{2}\langle 11|)$

$$\rho = \begin{pmatrix}\frac{3}{4} & 0 & 0 & \frac{\sqrt{3}}{4}\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ \frac{\sqrt{3}}{4} & 0 & 0 & \frac{1}{4}\end{pmatrix}$$

$\rho' = ((I \otimes H)|\psi\rangle)((I \otimes H)|\psi\rangle)^\dagger$

$$\rho' = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{3}}{2} \\ 0 \\ 0 \\ 1/2 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{3}}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{-1}{2\sqrt{2}} \end{pmatrix} \rho' = \begin{pmatrix} \frac{3}{8} & \frac{3}{8} & \frac{\sqrt{3}}{8} & -\frac{\sqrt{3}}{8} \\ \frac{3}{8} & \frac{3}{8} & \frac{\sqrt{3}}{8} & -\frac{\sqrt{3}}{8} \\ \frac{\sqrt{3}}{8} & \frac{\sqrt{3}}{8} & \frac{1}{8} & -\frac{1}{8} \\ -\frac{\sqrt{3}}{8} & -\frac{\sqrt{3}}{8} & -\frac{1}{8} & \frac{1}{8} \end{pmatrix}$$

2. *(3 points)* Compute $_B(\rho)$ and $_B(\rho')$, where B refers to the second qubit.

   Answer:
   $Tr_b(\rho) = \begin{pmatrix} 3/4 & 0 \\ 0 & 1/4 \end{pmatrix}$
   $Tr_b(\rho') = \begin{pmatrix} 3/4 & 0 \\ 0 & 1/4 \end{pmatrix}$

3. *(4 points)* Let ρ be a density matrix for a quantum system with a bipartite state space $A \otimes B$. Let I denote the identity operation on system A, and let U be a unitary operation on system B. Prove that $_B(\rho) =_B ((I \otimes U)\rho(I \otimes U^\dagger))$.

   Answer:
   $*UU^\dagger = I$

   $Tr_b((I \otimes U)\rho(I \otimes U^\dagger))$
   $= Tr_b(\rho(I \otimes U)(I \otimes U^\dagger))$
   $= Tr_b(\rho(I \otimes I))$
   $= Tr_b(\rho)$
   done

4. *(3 points)* Show that the converse of part (c) holds for pure states. In other words, show that if $|\psi\rangle$ and $|\phi\rangle$ are bipartite pure states, and $_B(|\psi\rangle\langle\psi|) =_B (|\phi\rangle\langle\phi|)$, then there is a unitary operation U acting on system B such that $|\phi\rangle = (I \otimes U)|\psi\rangle$.

5. *(2 points)* Does the converse of part (c) hold for general density matrices? Prove or disprove it.

   Answer:

   $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} B = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$
   if the converse holds for general density matrices then, $A = (I \bigotimes U)B$
   $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix} \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$

   But, no such U exists (U would have to be 2 and 0 at the same time) therefore (c) does not hold for general density matrices

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Problem 5. *Product and entangled states.* Determine which of the following states are entangled. If the state is not entangled, show how to write it as a tensor product; if it is entangled, prove this.

1. *(3 points)* $\frac{2}{3}|00\rangle + \frac{1}{3}|01\rangle - \frac{2}{3}|11\rangle$

   Answer:

   $(\frac{2}{3}|00\rangle + \frac{1}{3}|01\rangle - \frac{2}{3}|11\rangle)(\frac{2}{3}\langle 00| + \frac{1}{3}\langle 01| - \frac{2}{3}\langle 11|)$

   Reduce to density matrix:

   $$\begin{pmatrix} \frac{4}{9} & \frac{2}{9} & 0 & -\frac{4}{9} \\ \frac{2}{9} & \frac{1}{9} & 0 & -\frac{2}{9} \\ 0 & 0 & 0 & 1 \\ -\frac{4}{9} & -\frac{2}{9} & 0 & \frac{4}{9} \end{pmatrix}$$

   Partial Trace:

   $$\begin{pmatrix} \frac{5}{9} & \frac{-2}{9} \\ \frac{-2}{9} & \frac{4}{9} \end{pmatrix}$$

   Because the rank of the partial trace is greater than one, we know the state is entangled

2. *(3 points)* $\frac{1}{2}(|00\rangle - i|01\rangle + i|10\rangle + |11\rangle)$

   Answer:

   $\frac{1}{4}((|00\rangle - i|01\rangle + i|10\rangle + |11\rangle)((\langle 00| + i\langle 01| - i\langle 10| + \langle 11|)$

   Reduce to density matrix:

   $$\frac{1}{4}\begin{pmatrix} 1 & -i & i & 1 \\ i & 1 & -1 & i \\ -i & -1 & 1 & -i \\ 1 & -i & i & 1 \end{pmatrix}$$

   Partial Trace:

   $$\begin{pmatrix} 2 & 2i \\ -2i & 2 \end{pmatrix}$$

   Because the rank of the partial trace is equal to 1 we know it is not entangled

3. *(3 points)* $\frac{1}{2}(|00\rangle - |01\rangle + |10\rangle + |11\rangle)$

   Answer:

   $\frac{1}{4}((|00\rangle - |01\rangle + |10\rangle + |11\rangle)((\langle 00| - \langle 01| + \langle 10| + \langle 11|)$

   Reduce to density matrix:

$$\frac{1}{4}\begin{pmatrix} 1 & -1 & 1 & 1 \\ -1 & 1 & -1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & 1 \end{pmatrix}$$

Partial Trace:

$$\begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

Because the rank of the partial trace is greater than one, we know the state is entangled