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Doubly robust estimation under a randomly censored covariate

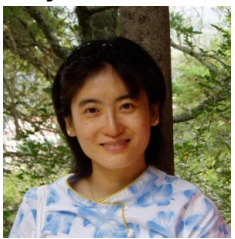
Brian Richardson

Acknowledgements

Seong-Ho Lee, PhD

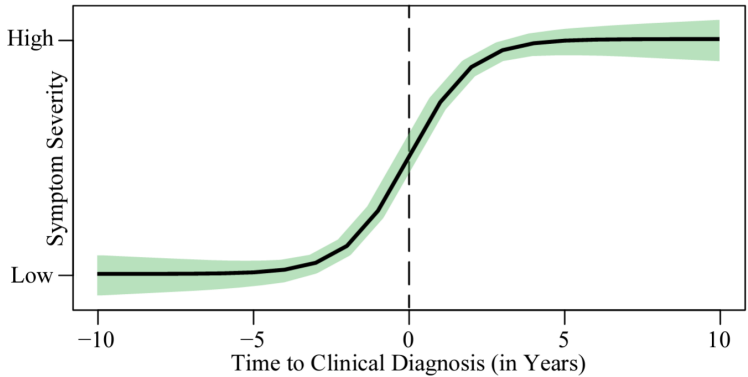
Tanya Garcia, PhD

Yanyuan Ma, PhD



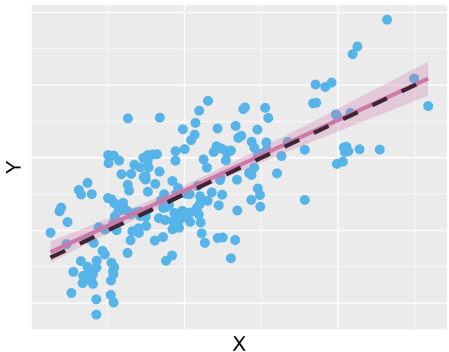
This research was supported by the National Institute of Environmental Health Sciences grant T32ES007018.

Huntington's Disease and Censored Covariates



Sarah C Lotspeich et al. "Making Sense of Censored Covariates: Statistical Methods for Studies of Huntington's Disease". In: *Annual Review of Statistics and Its Application* 11 (2024)

Censored Covariates: a Simple Example



- Regression model:
  $\mathrm{E}(Y) = \beta_0 + \beta_1 X$

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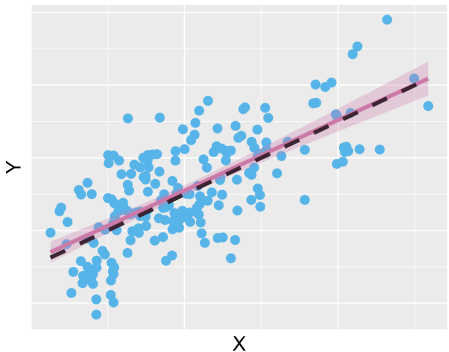
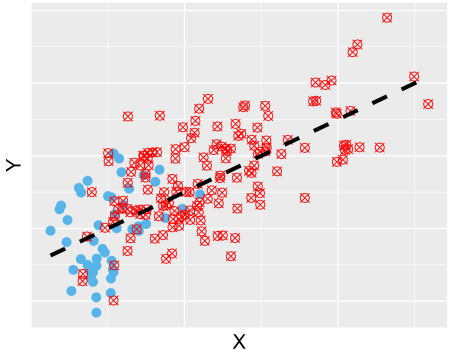


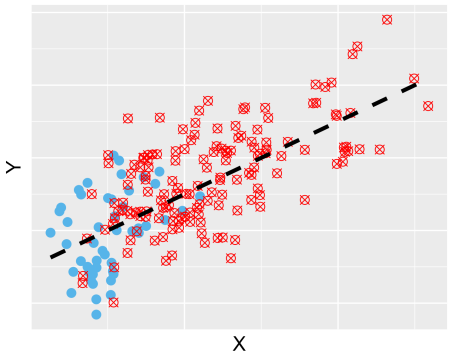
- Regression model: $\mathrm{E}(Y) = \beta_0 + \beta_1 X$
- Estimate $\boldsymbol{\beta} = (\beta_0, \beta_1)^T$ with least squares/maximum likelihood

Censored Covariates: a Simple Example



Problem: X is censored by a **random censoring time** C

Censored Covariates: a Simple Example



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- $W = \min(X, C)$
- $\Delta = I(X \leq C)$

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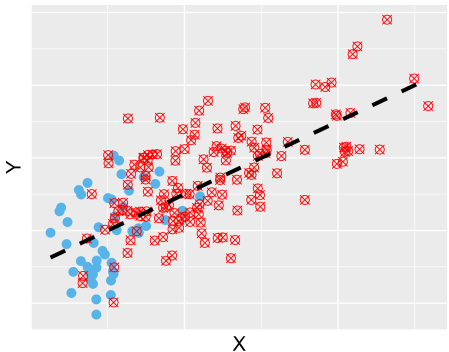
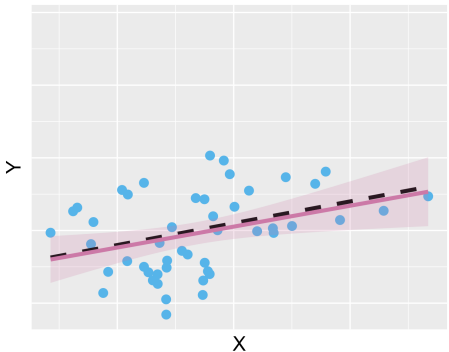


Problem: X is censored by a **random censoring time** C

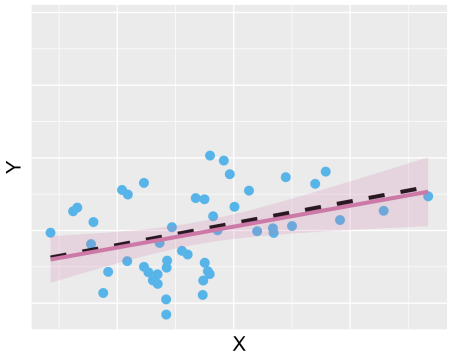
- $W = \min(X, C)$
- $\Delta = I(X \leq C)$
- assume: $C \perp\!\!\!\perp (X, Y)$

Complete Case Analysis



Only use observations where X is *uncensored*

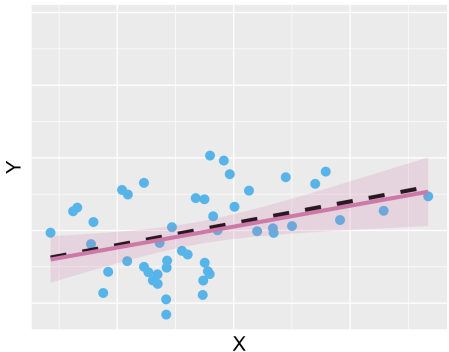
Complete Case Analysis



Only use observations where X is *uncensored*

✓ Consistent

Complete Case Analysis



Only use observations where X is *uncensored*

✓ Consistent

✗ Inefficient

Maximum Likelihood Estimation (MLE)

$$f_{Y,W,\Delta}(y, w, \delta, \boldsymbol{\beta}, \boxed{\boldsymbol{\alpha}}) \propto \underbrace{\{f_{Y|X}(y, w, \boldsymbol{\beta})\}^{\delta}}_{\text{uncensored}} \underbrace{\left\{ \int_{w}^{\infty} f_{Y|X}(y, x, \boldsymbol{\beta}) \boxed{f_X(x, \boldsymbol{\alpha})} \, dx \right\}^{1-\delta}}_{\text{censored}}$$

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$$\mathbf{S}_{\boldsymbol{\beta}}(y, w, \delta, \boldsymbol{\beta}) \equiv \frac{\partial}{\partial \boldsymbol{\beta}} \log f_{Y,W,\Delta}(y, w, \delta, \boldsymbol{\beta}, \boldsymbol{\alpha}), \qquad \sum_{i=1}^{n} \mathbf{S}_{\boldsymbol{\beta}}(Y_i, W_i, \Delta_i, \boldsymbol{\beta}) = \mathbf{0}$$

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- ✓ fully efficient

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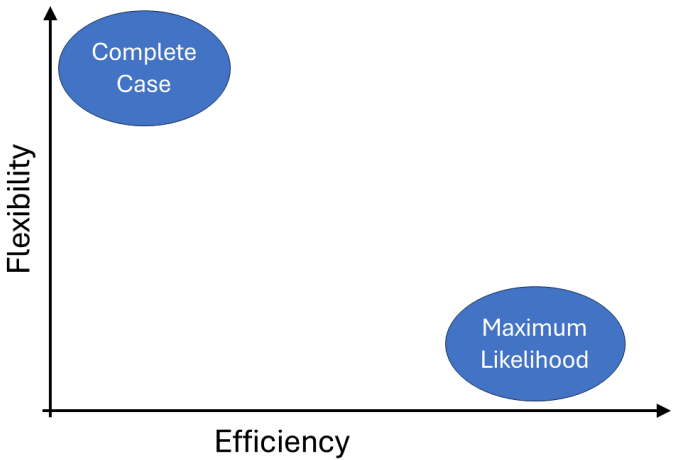
✓ consistent

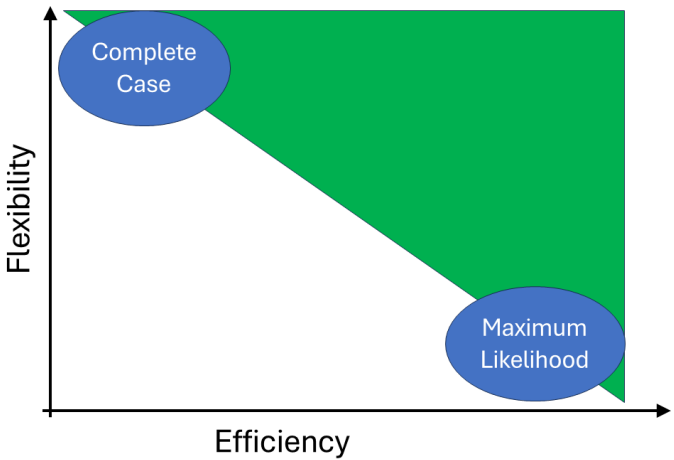
✓ fully efficient

✗ inconsistent when model for **nuisance parameter** f_X is incorrect

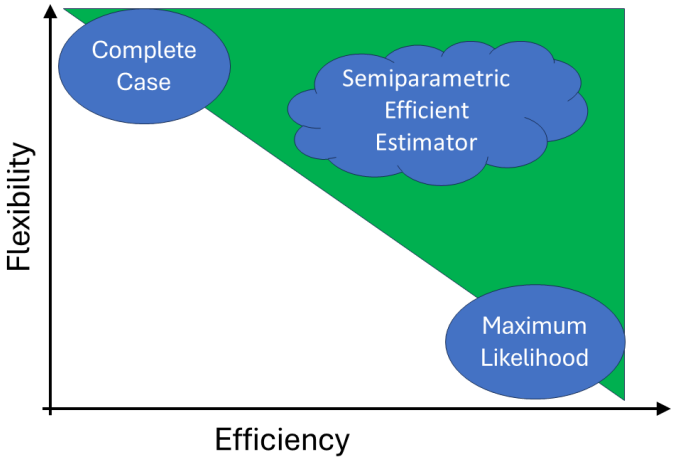
Existing Methods



Existing Opportunity



A New Approach



The Semiparametric Recipe

- **goal**: to find the estimating function resulting in a semiparametric efficient estimator

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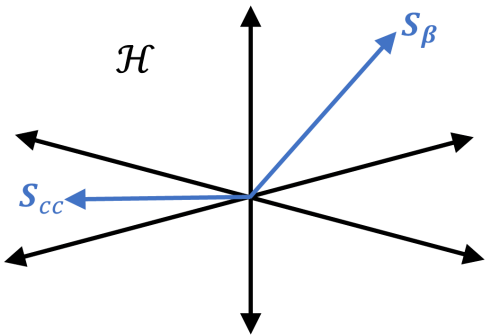
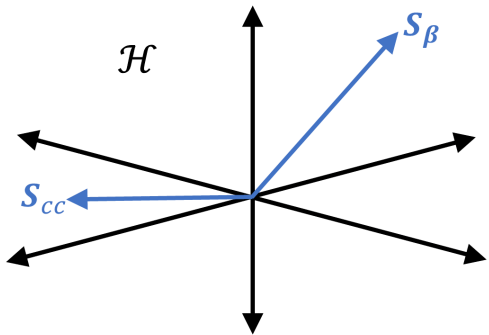
- Geometric approach from Tsiatis, *Semiparametric theory and missing data*

The Semiparametric Recipe



- **Hilbert space** of estimating functions

The Semiparametric Recipe



- **Hilbert space** of estimating functions
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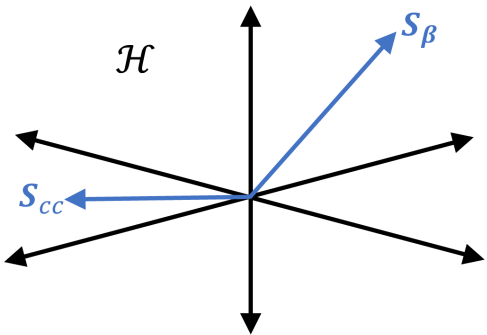
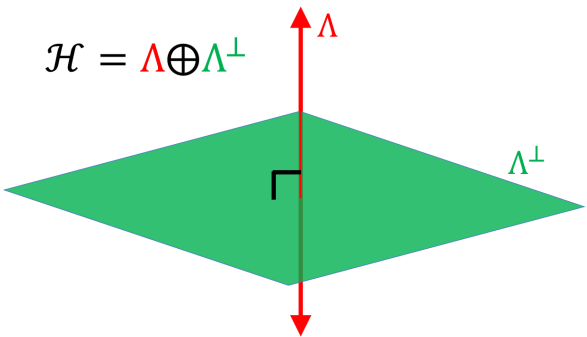
The Semiparametric Recipe



- **Hilbert space** of estimating functions
- **covariance inner product** $\langle \boldsymbol{h}, \boldsymbol{g} \rangle \equiv \mathrm{E}(\boldsymbol{h}^T \boldsymbol{g})$
- **orthogonal** $\Leftrightarrow$ **uncorrelated**

$$\boldsymbol{h} \perp \boldsymbol{g} \iff \langle \boldsymbol{h}, \boldsymbol{g} \rangle = 0$$

The Semiparametric Recipe



- construct Λ using **nuisance scores**

$$\partial \log f_{Y,W,\Delta}(y, w, \delta, \boldsymbol{\beta}, \boldsymbol{\eta}) / \partial \boldsymbol{\eta}$$

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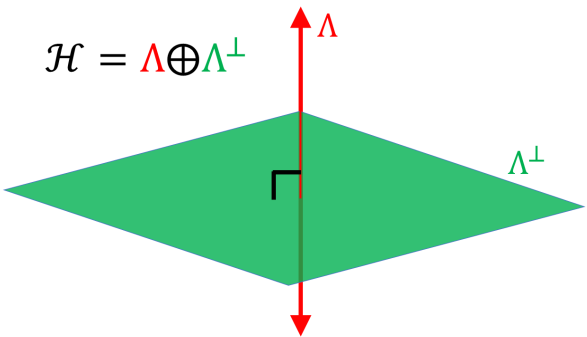

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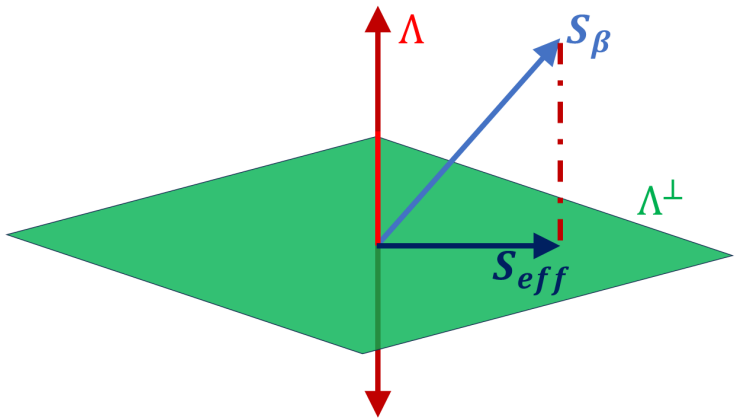
- orthogonal complement $\Lambda^{\perp}$

The Semiparametric Recipe



Properties of the Proposed Estimator

The **semiparametric efficient estimator** $\widehat{\boldsymbol{\beta}}_{\text{eff}}$ is the solution to

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- ✓ **Locally Efficiency**: If f_X, f_C are *both* correctly specified, then $\widehat{\boldsymbol{\beta}}_{\text{eff}}$ achieves the **semiparametric efficiency bound**

Simulation Setup

- $Y|X \sim N(\beta_0 + \beta_1 X, \sigma^2)$

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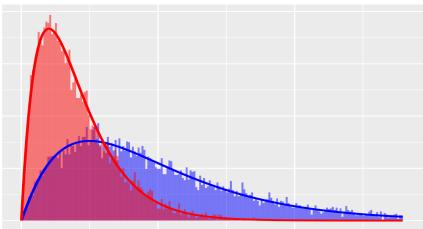
- X, C possibly **misspecified** as exponential

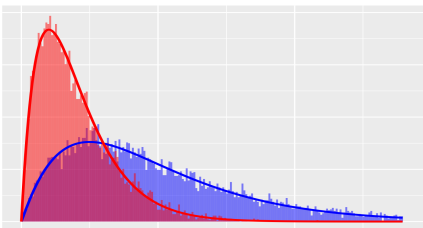
Simulating Model Misspecification

X, C correct

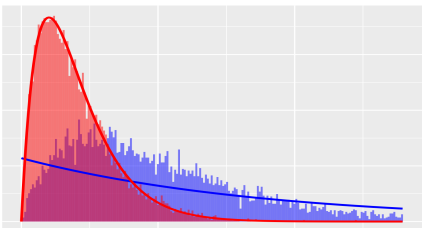


Simulating Model Misspecification

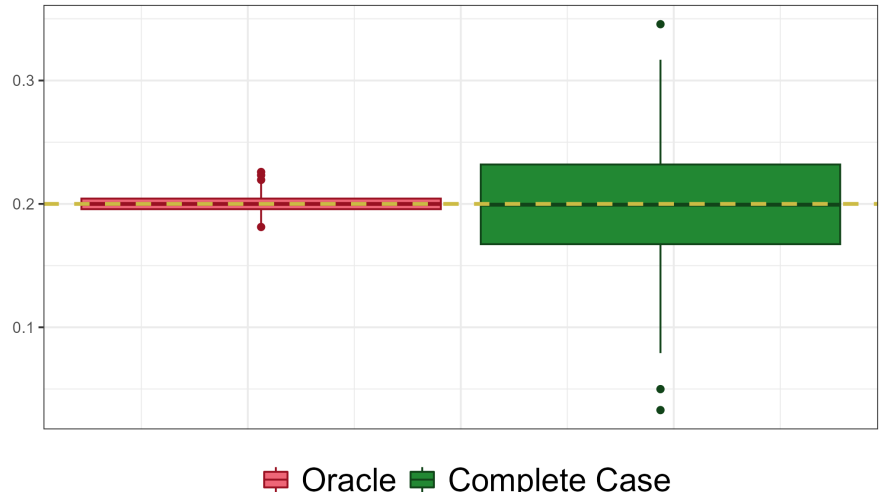
X, C correct



X incorrect, C correct



Simulation Results



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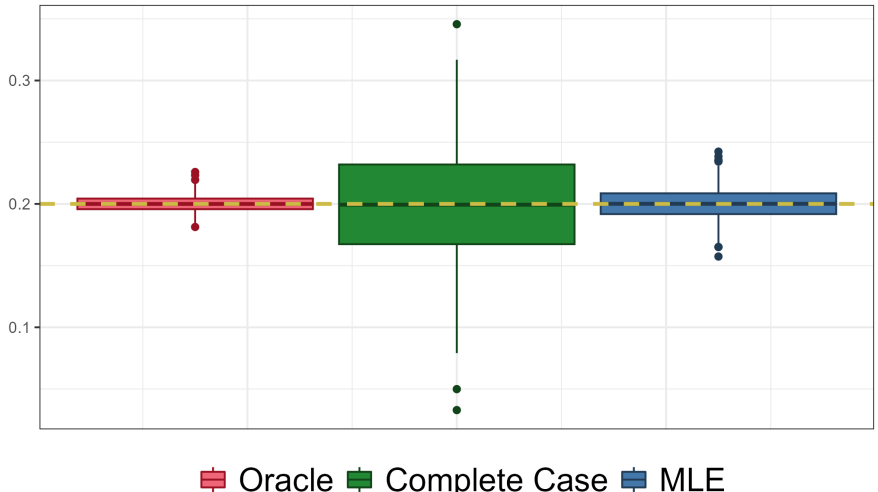

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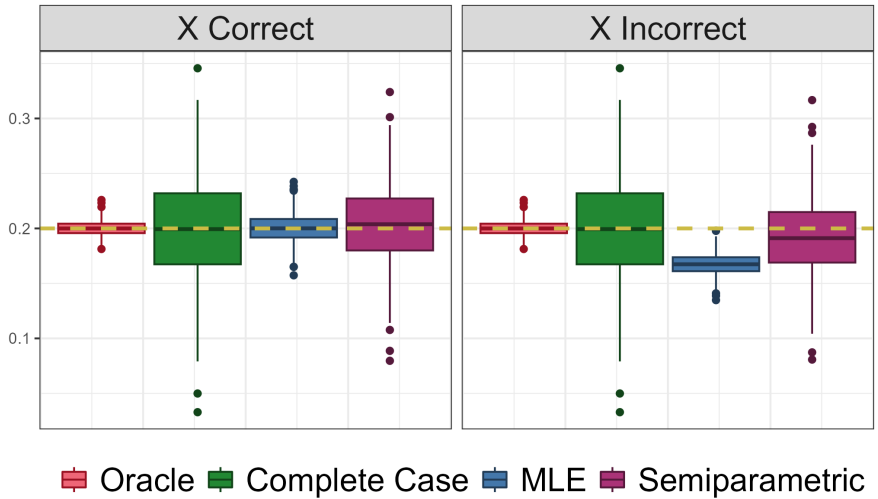

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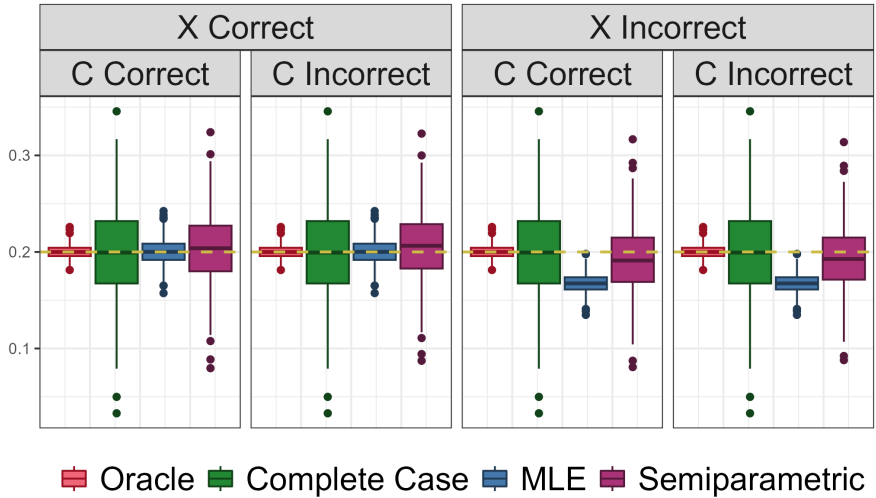

Generalizations

The methods presented here extend to:

- Nonlinear $\mathrm{E}(Y|X) = m(X, \boldsymbol{\beta})$

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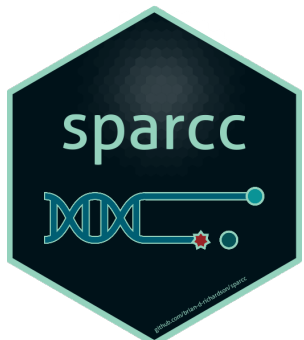
- Additional uncensored covariates $\mathbf{Z}$
  - $\mathrm{E}(Y|X, \mathbf{Z}) = m(X, \mathbf{Z}, \boldsymbol{\beta})$
  - Nuisance distributions become $f_{X|\mathbf{Z}}, f_{C|\mathbf{Z}}, f_{\mathbf{Z}}$

SPARCC: Semiparametric Censored Covariate Estimation



R package available at https://github.com/brian-d-richardson/sparcc

Appendix I: MLE Score Function

$$\mathbf{S}_{\boldsymbol{\beta}}(y, w, \delta, \mathbf{z}, \boldsymbol{\beta}) = \underbrace{\delta \mathbf{S}^{\mathrm{F}}_{\boldsymbol{\beta}}(y, w, \mathbf{z}, \boldsymbol{\beta})}_{\text{uncensored}} + \underbrace{(1-\delta)\frac{\mathrm{E}\{\mathrm{I}(X > w)\mathbf{S}^{\mathrm{F}}_{\boldsymbol{\beta}}(y, X, \mathbf{z}, \boldsymbol{\beta}) \mid y, \mathbf{z}\}}{\mathrm{E}\{\mathrm{I}(X > w) \mid y, \mathbf{z}\}}}_{\text{censored}}$$

Appendix II: Efficient Score Function

$$\mathbf{S}_{\text{eff}}(y, w, \delta, \mathbf{z}, \boldsymbol{\beta}) \equiv \delta\{\mathbf{S}^{\text{F}}_{\boldsymbol{\beta}}(y, w, \mathbf{z}, \boldsymbol{\beta}) - \boxed{\mathbf{a}(w, z, \boldsymbol{\beta})}\}$$
$$+ (1-\delta)\frac{\text{E}[\text{I}(X > w)\{\mathbf{S}^{\text{F}}_{\boldsymbol{\beta}}(y, X, \mathbf{z}, \boldsymbol{\beta}) - \boxed{\mathbf{a}(X, \mathbf{z}, \boldsymbol{\beta})}\} \mid y, \mathbf{z}]}{\text{E}\{\text{I}(X > w) \mid y, \mathbf{z}\}},$$

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where $\mathbf{a}(x, \mathbf{z}, \boldsymbol{\beta})$ satisfies

$$\text{E}\{\text{I}(x \leq C) \mid \mathbf{z}\}\mathbf{a}(x, \mathbf{z}, \boldsymbol{\beta}) + \text{E}\left[\text{I}(x > C)\frac{\text{E}\{\text{I}(X > C)\mathbf{a}(X, \mathbf{z}, \boldsymbol{\beta}) \mid Y, C, \mathbf{z}\}}{\text{E}\{\text{I}(X > C) \mid Y, C, \mathbf{z}\}} \mid x, \mathbf{z}\right]$$
$$= \text{E}\left[\text{I}(x > C)\frac{\text{E}\{\text{I}(X > C)\mathbf{S}^{\text{F}}_{\boldsymbol{\beta}}(Y, X, \mathbf{z}, \boldsymbol{\beta}) \mid Y, C, \mathbf{z}\}}{\text{E}\{\text{I}(X > C) \mid Y, C, \mathbf{z}\}} \mid x, \mathbf{z}\right]$$

Thank you! Any questions?

Brian Richardson
✉: brichson@ad.unc.edu