UNC GILLINGS SCHOOL OF GLOBAL PUBLIC HEALTH

Robust and efficient estimation in the presence of a randomly censored covariate

Brian Richardson

Acknowledgements

Seong-Ho Lee, PhD

Tanya Garcia, PhD

Yanyuan Ma, PhD

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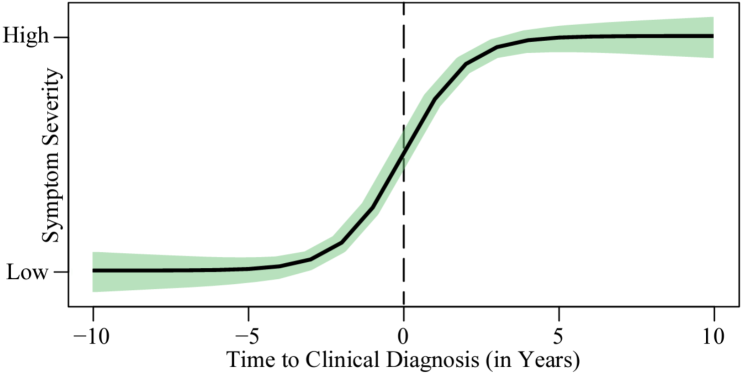
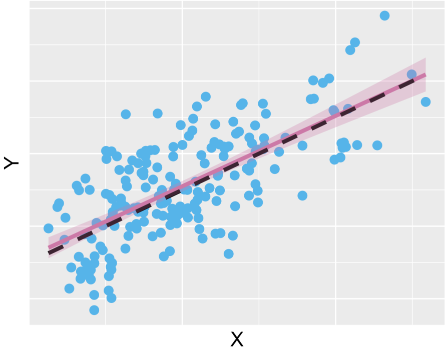
Huntington's Disease and Censored Covariates



Sarah C Lotspeich et al. "Making Sense of Censored Covariates: Statistical Methods for Studies of Huntington's Disease". In: *Annual Review of Statistics and Its Application* 11 (2024)

Censored Covariates: a Simple Example



- Regression model:
  $E(Y) = \beta_0 + \beta_1 X$

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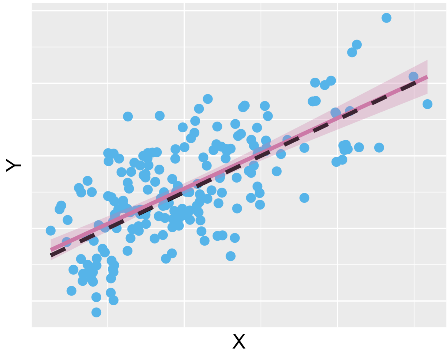

- Regression model:
  $\mathrm{E}(Y) = \beta_0 + \beta_1 X$
- Estimate $\boldsymbol{\beta} = (\beta_0, \beta_1)^T$ with least squares/maximum likelihood

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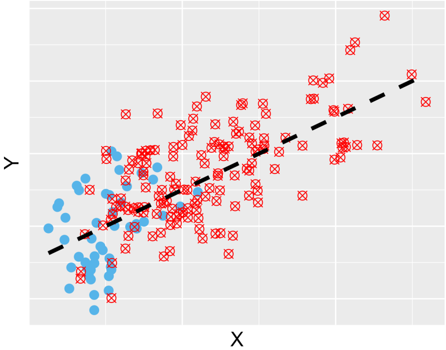

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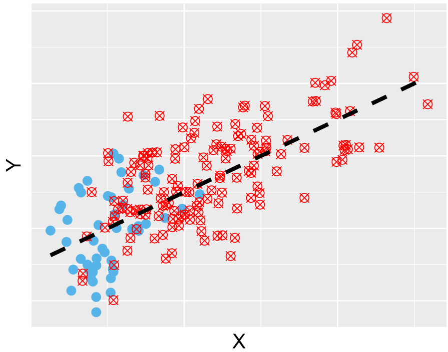

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- $W = \min(X, C)$
- $\Delta = I(X \leq C)$

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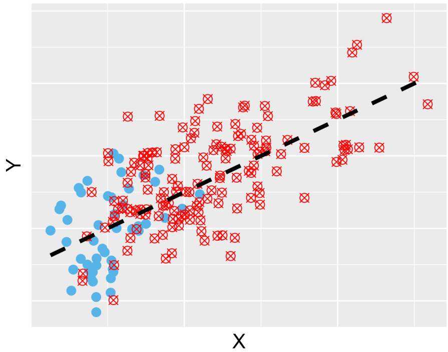
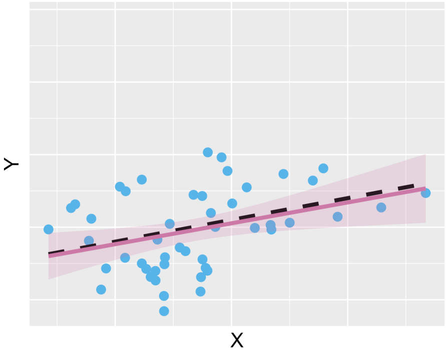


Problem: X is censored by a **random censoring time** C

- $W = \min(X, C)$
- $\Delta = I(X \leq C)$
- assume: $C \perp\!\!\!\perp (X, Y)$

Complete Case Analysis



Only use observations where X is *uncensored*

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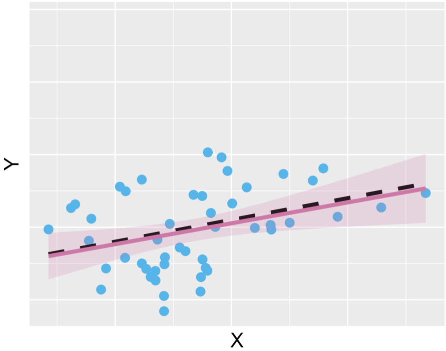
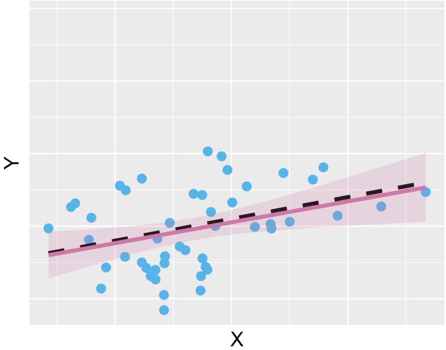


Only use observations where X is *uncensored*

✓ Consistent

Complete Case Analysis



Only use observations where X is *uncensored*

- ✓ Consistent
- ✗ Inefficient

Maximum Likelihood Estimation (MLE)

$$f_{Y,W,\Delta}(y, w, \delta, \boldsymbol{\beta}, \boxed{\boldsymbol{\alpha}}) \propto \underbrace{\{f_{Y|X}(y, w, \boldsymbol{\beta})\}^{\delta}}_{\text{uncensored}} \underbrace{\left\{\int_{w}^{\infty} f_{Y|X}(y, x, \boldsymbol{\beta}) \boxed{f_X(x, \boldsymbol{\alpha})} dx\right\}^{1-\delta}}_{\text{censored}}$$

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$$\mathbf{S}_{\boldsymbol{\beta}}(y, w, \delta, \boldsymbol{\beta}) \equiv \frac{\partial}{\partial \boldsymbol{\beta}} \log f_{Y,W,\Delta}(y, w, \delta, \boldsymbol{\beta}, \boldsymbol{\alpha}), \qquad \sum_{i=1}^{n} \mathbf{S}_{\boldsymbol{\beta}}(Y_i, W_i, \Delta_i, \boldsymbol{\beta}) = \mathbf{0}$$

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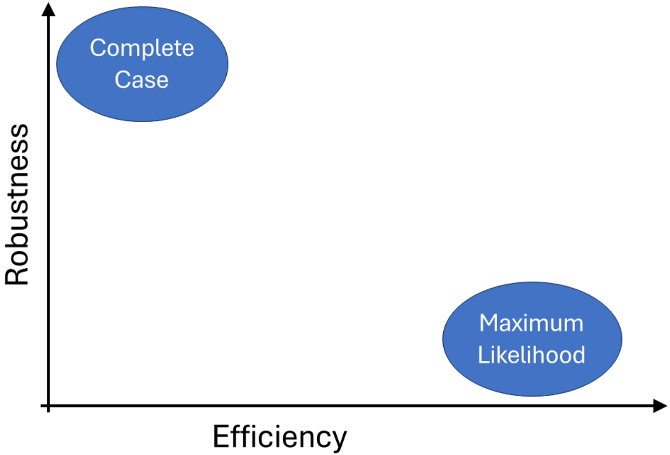
✓ consistent

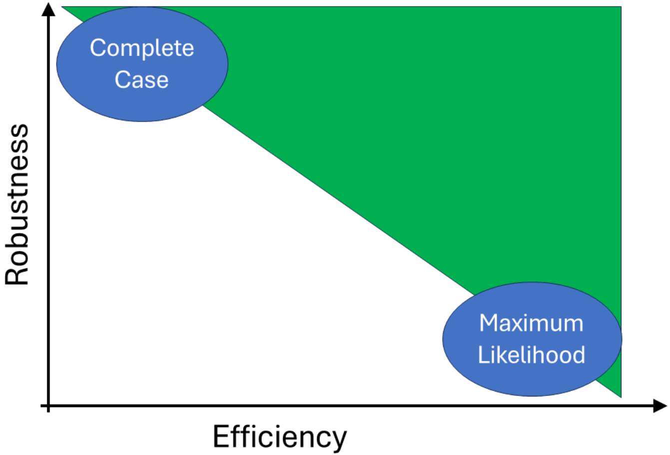
✓ fully efficient

✗ inconsistent when model for **nuisance parameter** f_X is incorrect

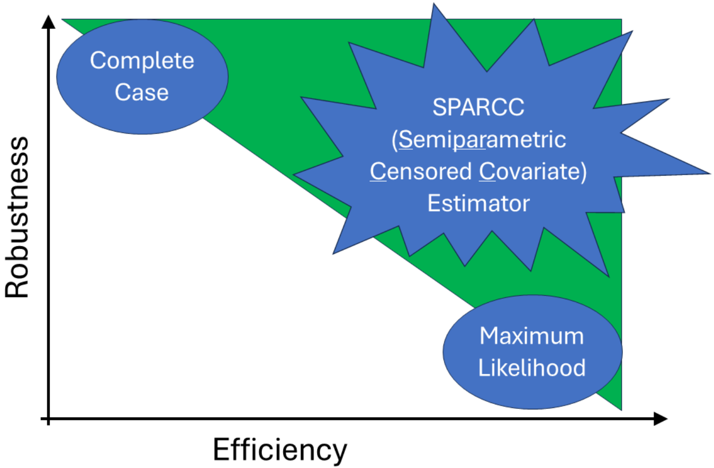
Existing Methods



Existing Opportunity



A New Approach



The Semiparametric Recipe

- **of interest**: parameter $\boldsymbol{\beta}$ characterizing $Y|X$

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- **of interest**: parameter $\boldsymbol{\beta}$ characterizing $Y|X$
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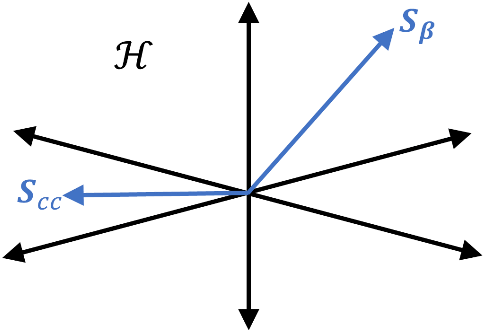
- **semiparametric**: avoid parametric assumptions for $\boldsymbol{\eta}$
- **goal:** find semiparametric efficient estimating function

The Semiparametric Recipe



- **Hilbert space** of estimating functions

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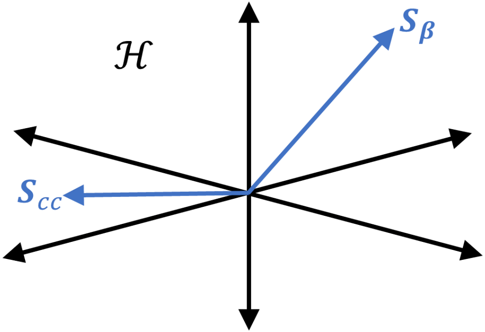

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- **covariance inner product** $\langle \boldsymbol{h}, \boldsymbol{g} \rangle \equiv \mathrm{E}(\boldsymbol{h}^T \boldsymbol{g})$

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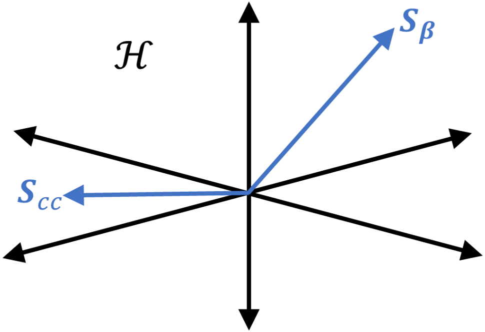
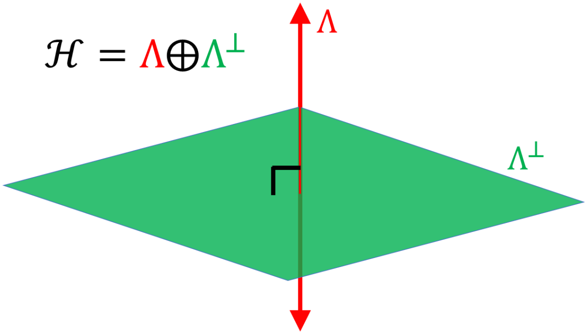


- **Hilbert space** of estimating functions
- **covariance inner product** $\langle \boldsymbol{h}, \boldsymbol{g} \rangle \equiv \mathrm{E}(\boldsymbol{h}^T \boldsymbol{g})$
- **orthogonal** $\Leftrightarrow$ **uncorrelated**

$$\boldsymbol{h} \perp \boldsymbol{g} \iff \langle \boldsymbol{h}, \boldsymbol{g} \rangle = 0$$

The Semiparametric Recipe



- construct Λ using **nuisance scores**

$$\partial \log f_{Y,W,\Delta}(y, w, \delta, \boldsymbol{\beta}, \boldsymbol{\eta}) / \partial \boldsymbol{\eta}$$

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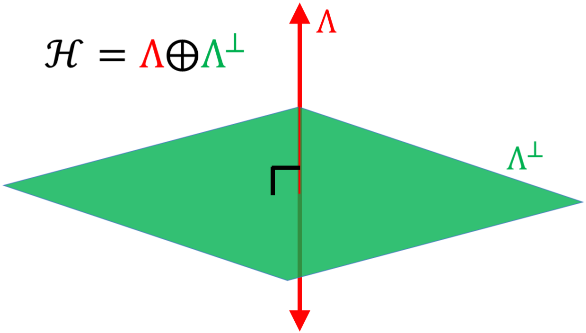

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- orthogonal complement $\Lambda^{\perp}$

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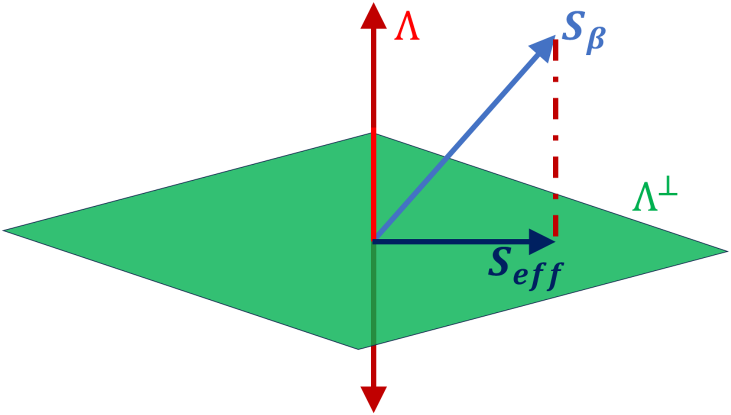

Implementing the SPARCC Estimator

The **SPARCC Estimator** $\widehat{\boldsymbol{\beta}}$ is the solution to

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Computing $\mathbf{S}_{\text{eff}}$ requires $\boldsymbol{\eta} = (f_X, f_C)$. These distributions can be modeled either **parametrically** or **nonparametrically**

Properties of the SPARCC Estimator

With **parametric** f_X, f_C, $\widehat{\boldsymbol{\beta}}$ is:

- ✓ **doubly robust**: consistent if one of f_X, f_C is correctly specified,
- ✓ **semiparametric efficient** if f_X, f_C are *both* correctly specified

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With **nonparametric** f_X, f_C, $\widehat{\beta}$ is:

- ✓ **consistent**
- ✓ **semiparametric efficient**

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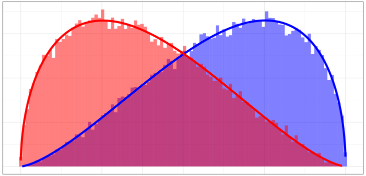
$(Z = 0, Z = 1)$

f_X correct:



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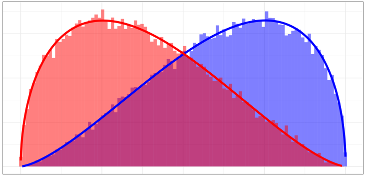

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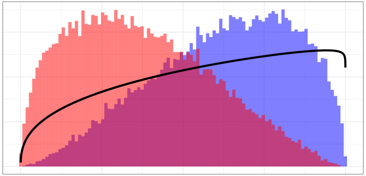
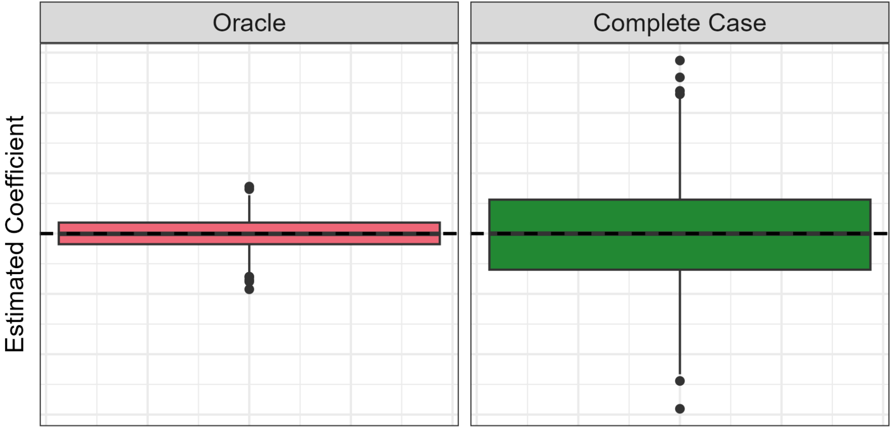


Simulation Results: Robustness



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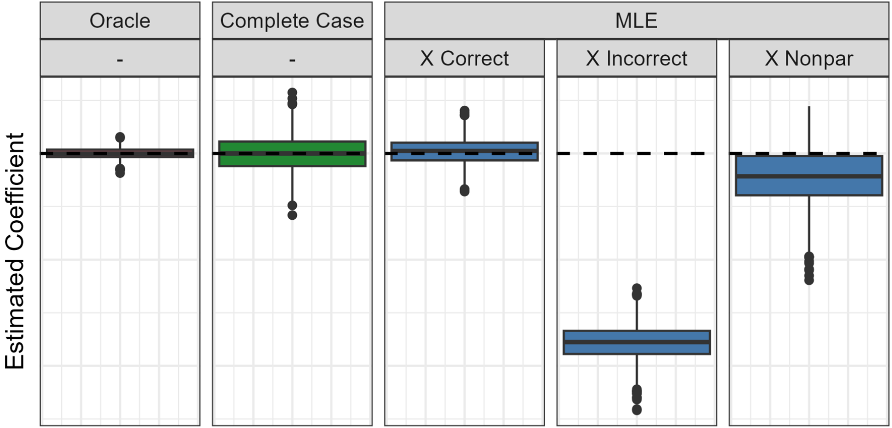

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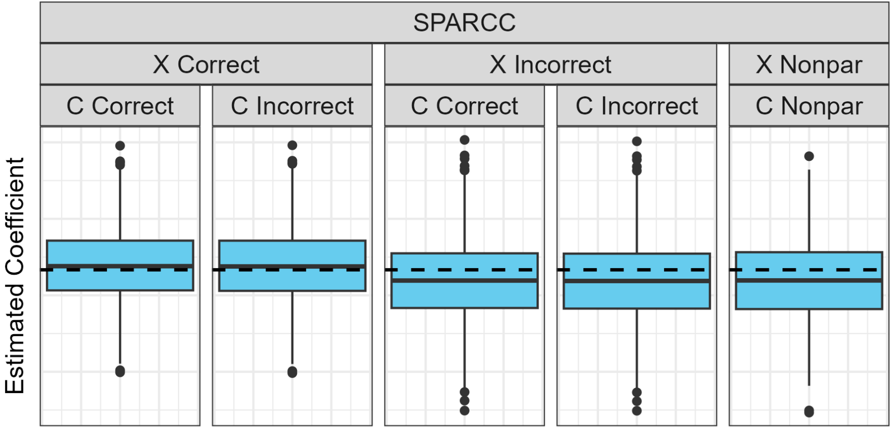
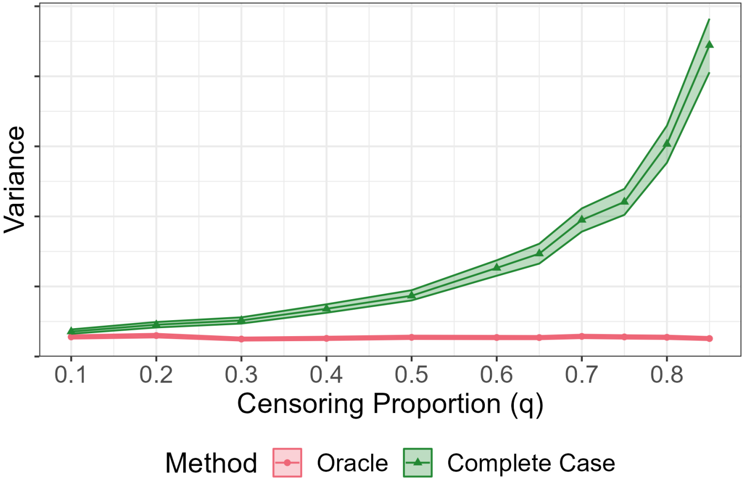


Simulation Results: Efficiency



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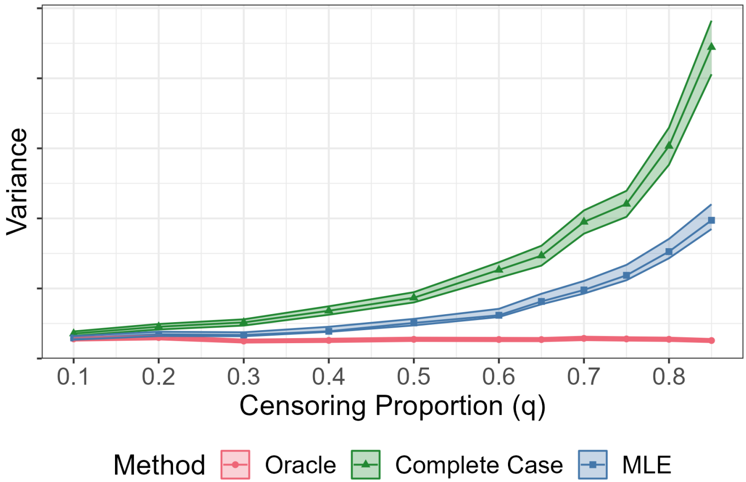

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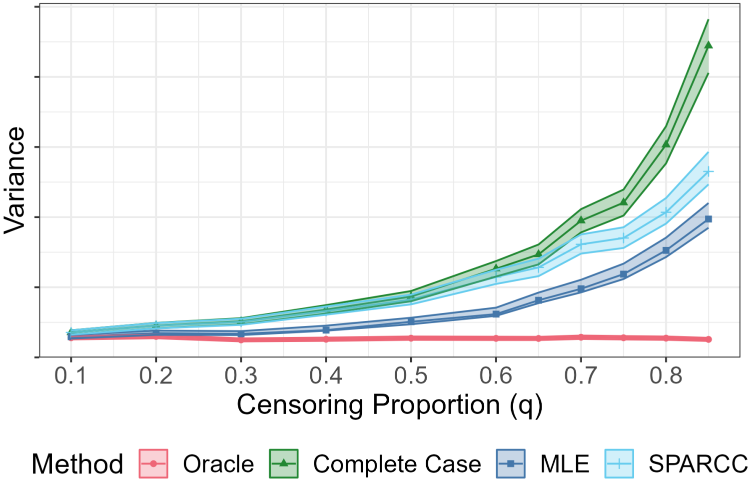

SPARCC: Semiparametric Censored Covariate Estimation

R package available at https://github.com/brian-d-richardson/sparcc

Appendix I: MLE Score Function

$$\mathbf{S}_{\boldsymbol{\beta}}(y, w, \delta, \mathbf{z}, \boldsymbol{\beta}) = \underbrace{\delta \mathbf{S}^{\mathrm{F}}_{\boldsymbol{\beta}}(y, w, \mathbf{z}, \boldsymbol{\beta})}_{\text{uncensored}} + \underbrace{(1-\delta)\frac{\mathrm{E}\{\mathrm{I}(X > w)\mathbf{S}^{\mathrm{F}}_{\boldsymbol{\beta}}(y, X, \mathbf{z}, \boldsymbol{\beta}) \mid y, \mathbf{z}\}}{\mathrm{E}\{\mathrm{I}(X > w) \mid y, \mathbf{z}\}}}_{\text{censored}}$$

Appendix II: Efficient Score Function

$$\mathbf{S}_{\text{eff}}(y, w, \delta, \mathbf{z}, \boldsymbol{\beta}) \equiv \delta\{\mathbf{S}^{\mathrm{F}}_{\boldsymbol{\beta}}(y, w, \mathbf{z}, \boldsymbol{\beta}) - \boxed{\mathbf{a}(w, z, \boldsymbol{\beta})}\}$$
$$+ (1-\delta)\frac{\mathrm{E}[\mathrm{I}(X > w)\{\mathbf{S}^{\mathrm{F}}_{\boldsymbol{\beta}}(y, X, \mathbf{z}, \boldsymbol{\beta}) - \boxed{\mathbf{a}(X, \mathbf{z}, \boldsymbol{\beta})}\} \mid y, \mathbf{z}]}{\mathrm{E}\{\mathrm{I}(X > w) \mid y, \mathbf{z}\}},$$

Appendix II: Efficient Score Function

$$
\begin{aligned}
\mathbf{S}_{\text{eff}}(y, w, \delta, \mathbf{z}, \boldsymbol{\beta}) &\equiv \delta\{\mathbf{S}^{\mathrm{F}}_{\boldsymbol{\beta}}(y, w, \mathbf{z}, \boldsymbol{\beta}) - \boxed{\mathbf{a}(w, z, \boldsymbol{\beta})}\} \\
&+ (1-\delta)\frac{\mathrm{E}[\mathrm{I}(X > w)\{\mathbf{S}^{\mathrm{F}}_{\boldsymbol{\beta}}(y, X, \mathbf{z}, \boldsymbol{\beta}) - \boxed{\mathbf{a}(X, \mathbf{z}, \boldsymbol{\beta})}\} \mid y, \mathbf{z}]}{\mathrm{E}\{\mathrm{I}(X > w) \mid y, \mathbf{z}\}},
\end{aligned}
$$

where $\mathbf{a}(x, \mathbf{z}, \boldsymbol{\beta})$ satisfies

$$
\begin{aligned}
&\mathrm{E}\{\mathrm{I}(x \le C) \mid \mathbf{z}\}\mathbf{a}(x, \mathbf{z}, \boldsymbol{\beta}) + \mathrm{E}\left[\mathrm{I}(x > C)\frac{\mathrm{E}\{\mathrm{I}(X > C)\mathbf{a}(X, \mathbf{z}, \boldsymbol{\beta}) \mid Y, C, \mathbf{z}\}}{\mathrm{E}\{\mathrm{I}(X > C) \mid Y, C, \mathbf{z}\}} \mid x, \mathbf{z}\right] \\
&= \mathrm{E}\left[\mathrm{I}(x > C)\frac{\mathrm{E}\{\mathrm{I}(X > C)\mathbf{S}^{\mathrm{F}}_{\boldsymbol{\beta}}(Y, X, \mathbf{z}, \boldsymbol{\beta}) \mid Y, C, \mathbf{z}\}}{\mathrm{E}\{\mathrm{I}(X > C) \mid Y, C, \mathbf{z}\}} \mid x, \mathbf{z}\right]
\end{aligned}
$$

Thank you! Any questions?

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✉: brichson@ad.unc.edu