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Doubly robust estimation under a randomly censored covariate

Brian Richardson

Slides Available Online

Acknowledgements

Seong-Ho Lee, PhD

Tanya Garcia, PhD

Yanyuan Ma, PhD

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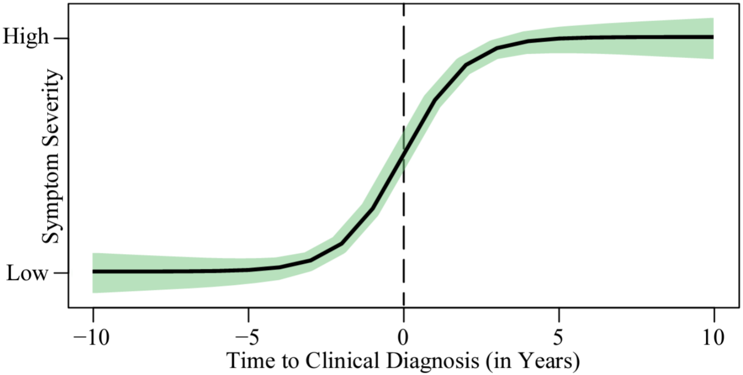
Huntington's Disease and Censored Covariates



Lotspeich et. al. (2024)

Censored Covariates: a Simple Example

- Regression model: $\mathrm{E}(Y) = \beta_0 + \beta_1 X$

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- Regression model:
  $\mathrm{E}(Y) = \beta_0 + \beta_1 X$
- Estimate $\boldsymbol{\beta} = (\beta_0, \beta_1)^T$ by solving:

$$\sum_{i=1}^{n} \underbrace{(Y_i - \beta_0 - \beta_1 X_i)(1, X_i)^T}_{\text{score function: } \mathbf{S}^F_{\boldsymbol{\beta}}} = \mathbf{0}$$

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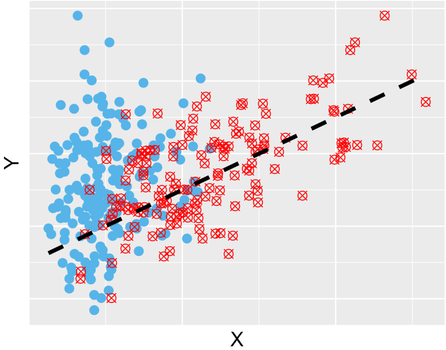

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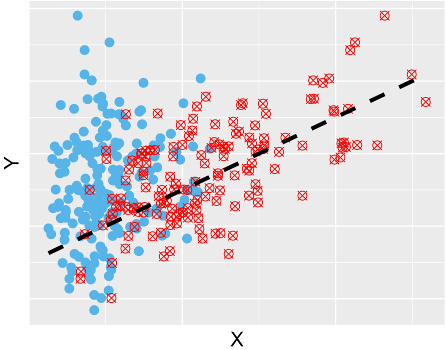

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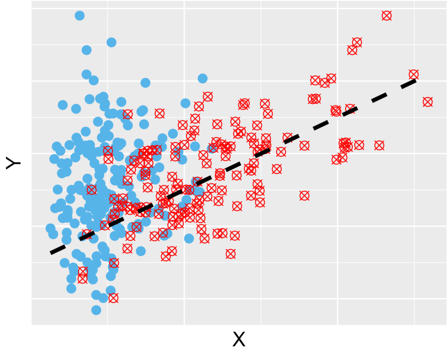
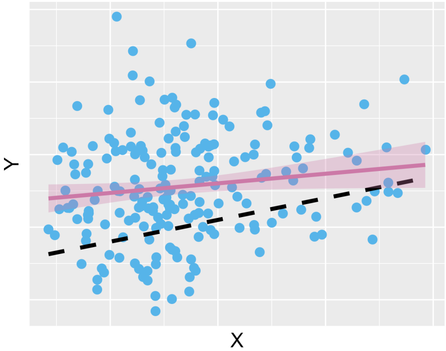


Problem: X is censored by a **random censoring time** C

- $W = \min(X, C)$
- $\Delta = I(X \leq C)$
- assume: $C \perp\!\!\!\perp (X, Y)$

A Naive Approach



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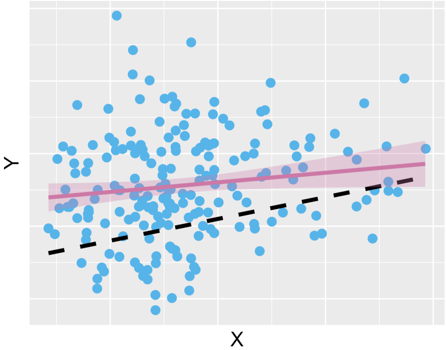

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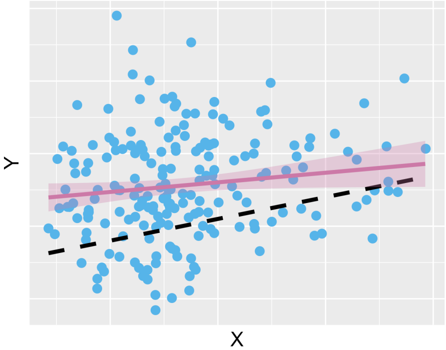

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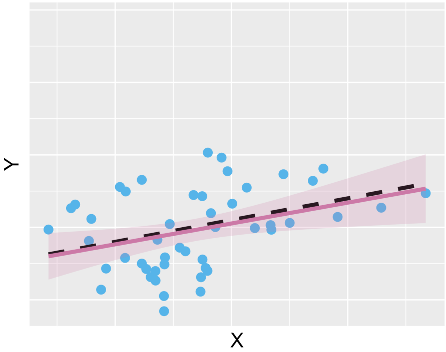
✗ Inconsistent estimator of $\boldsymbol{\beta}$

Complete Case Analysis



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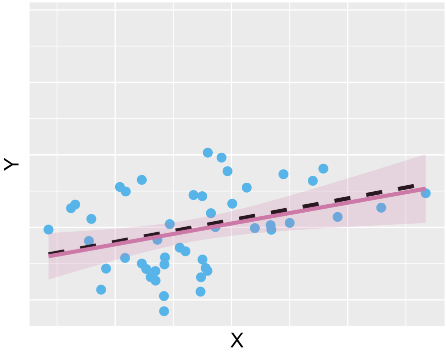

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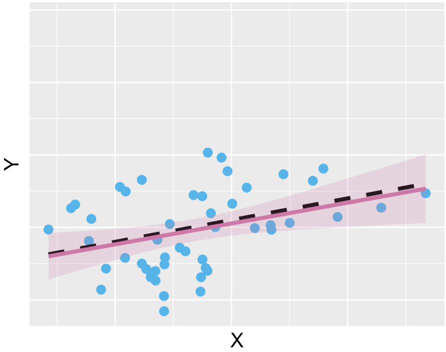

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✓ Consistent

✗ Inefficient

Maximum Likelihood Estimation (MLE)

$$f_{Y,W,\Delta}(y, w, \delta, \boldsymbol{\beta}, \boxed{\boldsymbol{\alpha}}) \propto \underbrace{\{f_{Y|X}(y, w, \boldsymbol{\beta})\}^{\delta}}_{\text{uncensored}} \underbrace{\left\{\int_{w}^{\infty} f_{Y|X}(y, x, \boldsymbol{\beta}) \boxed{f_X(x, \boldsymbol{\alpha})} dx\right\}^{1-\delta}}_{\text{censored}}$$

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Maximum Likelihood Estimation (MLE)

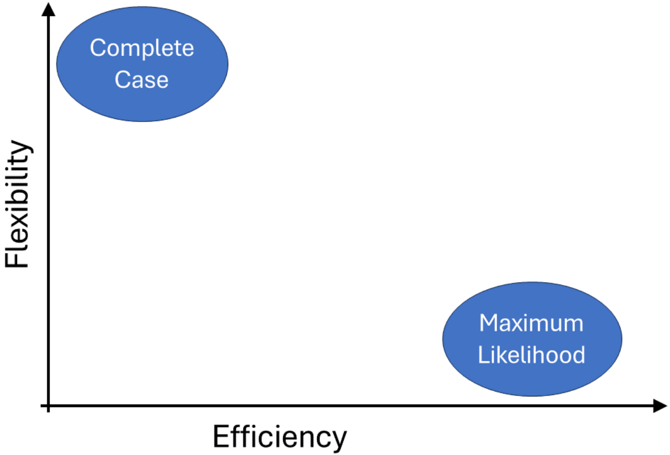
Pros

- ✓ consistent estimator of β
- ✓ fully efficient

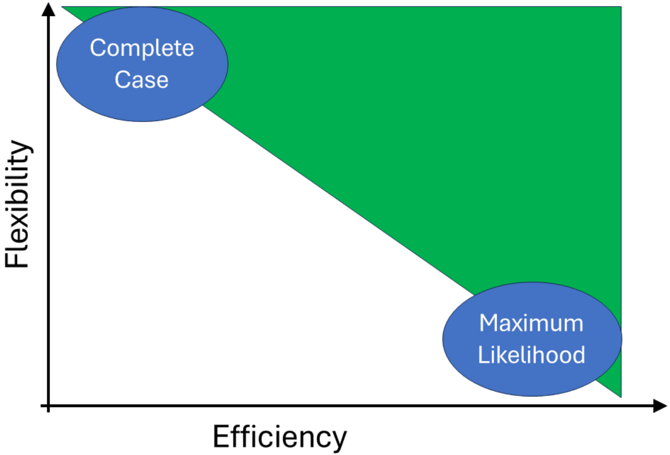
Cons

- ✗ requires model for **nuisance parameter** f_X
- ✗ inconsistent estimator when model for f_X is incorrect

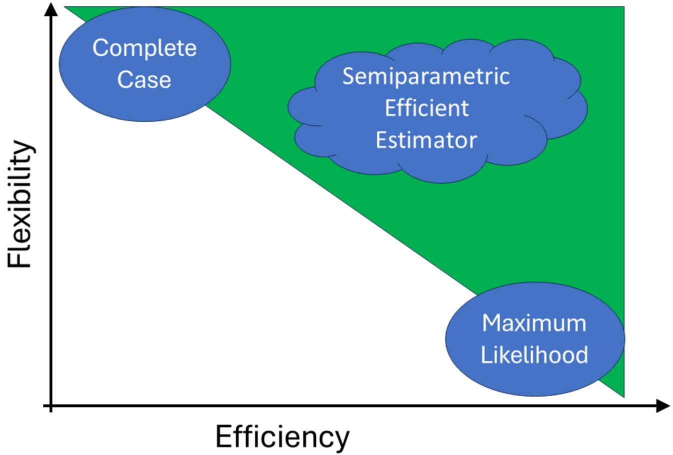
Existing Methods



Existing Opportunity



A New Approach



The Semiparametric Recipe

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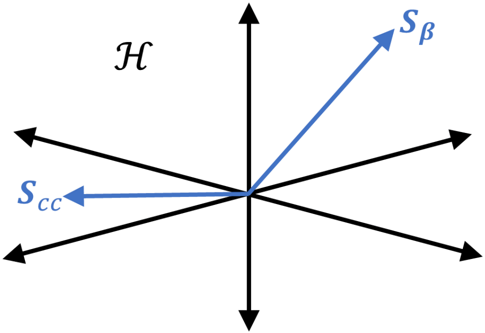
- **goal**: to find the estimating function resulting in a semiparametric efficient estimator
- **semiparametric**: infinite dimensional nuisance parameters f_X, f_C
- Geometric approach from Tsiatis (2006)

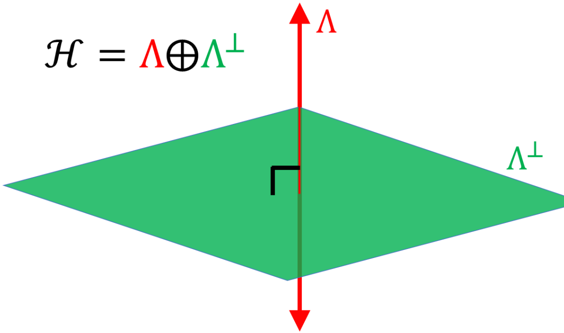
The Semiparametric Recipe

- **Hilbert space** of estimating functions
- **covariance inner product** $\langle \boldsymbol{h}, \boldsymbol{g} \rangle = \mathrm{E}(\boldsymbol{h}^T \boldsymbol{g})$



The Semiparametric Recipe

- construct Λ using **nuisance scores**
- orthogonal complement $\Lambda^\perp$



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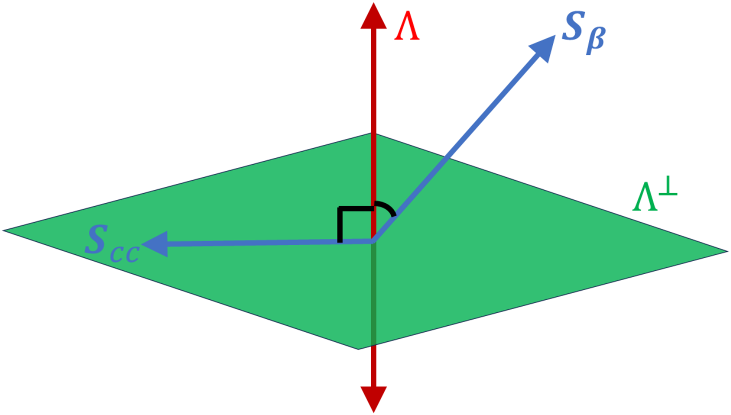

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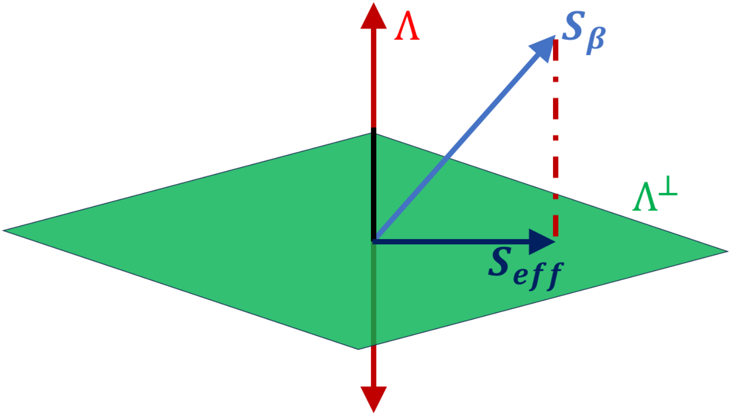

Properties of the Proposed Estimator

The **semiparametric efficient estimator** $\widehat{\boldsymbol{\beta}}_{\text{eff}}$ is the solution to

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- ✓ consistent and asymptotically normal
- ✓ **Doubly Robust**: $\widehat{\boldsymbol{\beta}}_{\text{eff}}$ is consistent if at least one of f_X, f_C is correctly specified
- ✓ **Locally Efficiency**: If f_X, f_C are *both* correctly specified, then $\widehat{\boldsymbol{\beta}}_{\text{eff}}$ achieves the **semiparametric efficiency bound**

Simulation Setup

- $Y|X \sim N(\beta_0 + \beta_1 X, \sigma^2)$

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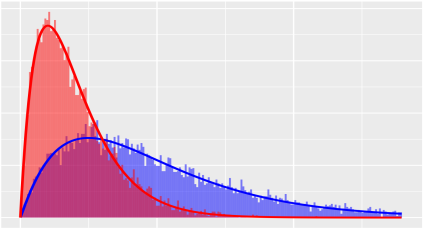
- X, C possibly **misspecified** as exponential

X, C correct



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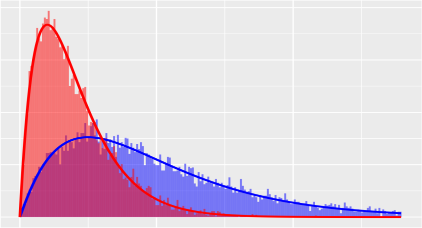
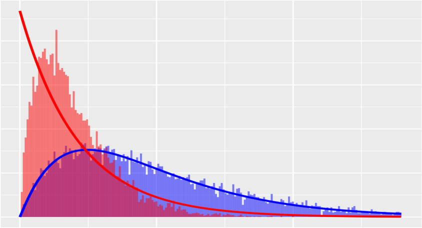


X correct, C incorrect



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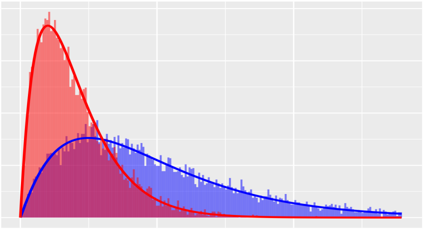

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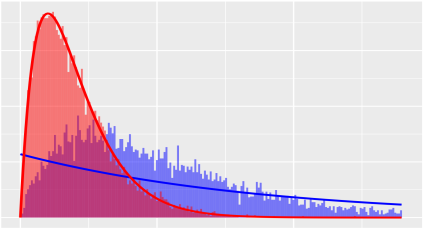

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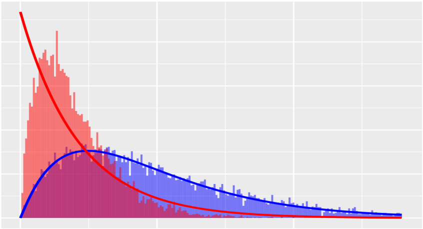

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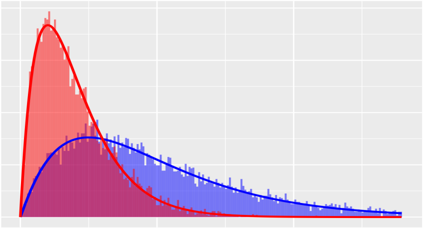

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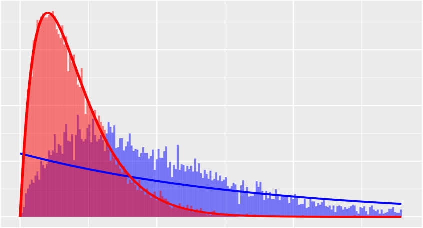

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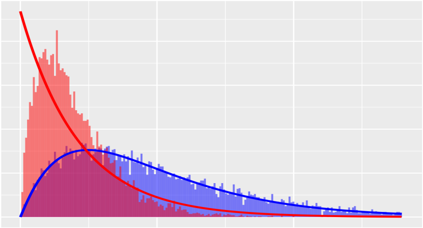

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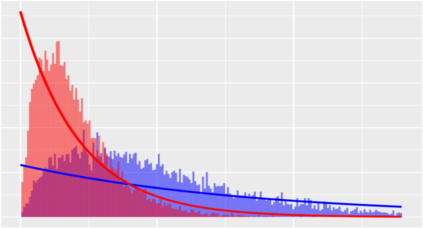
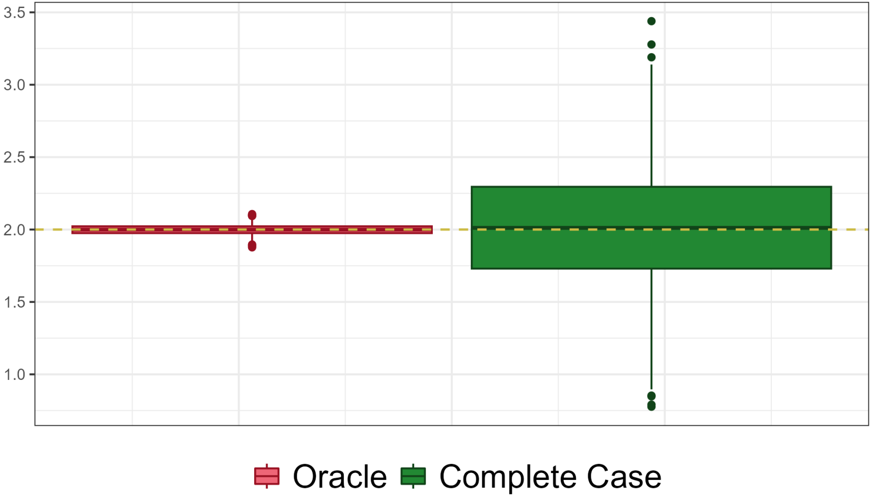


Simulation Results



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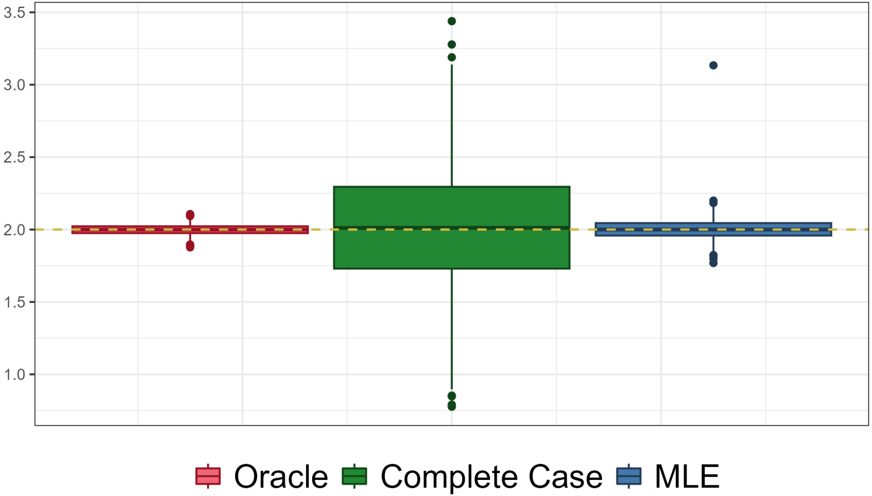

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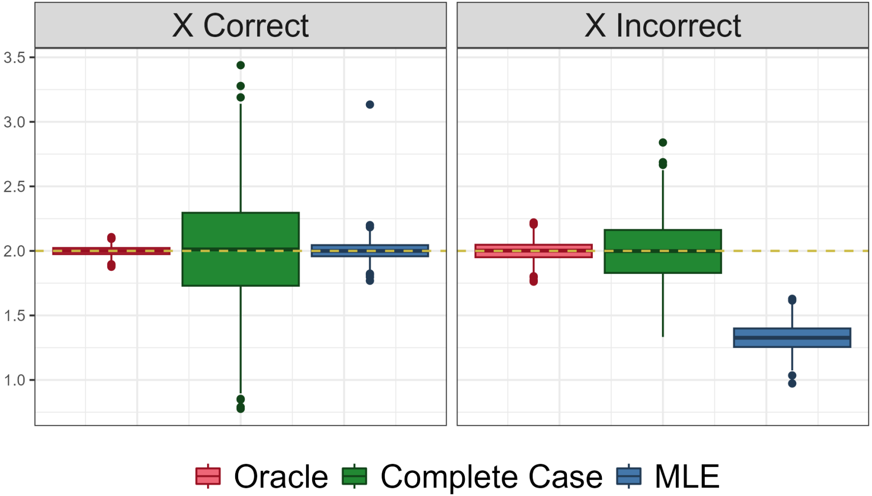

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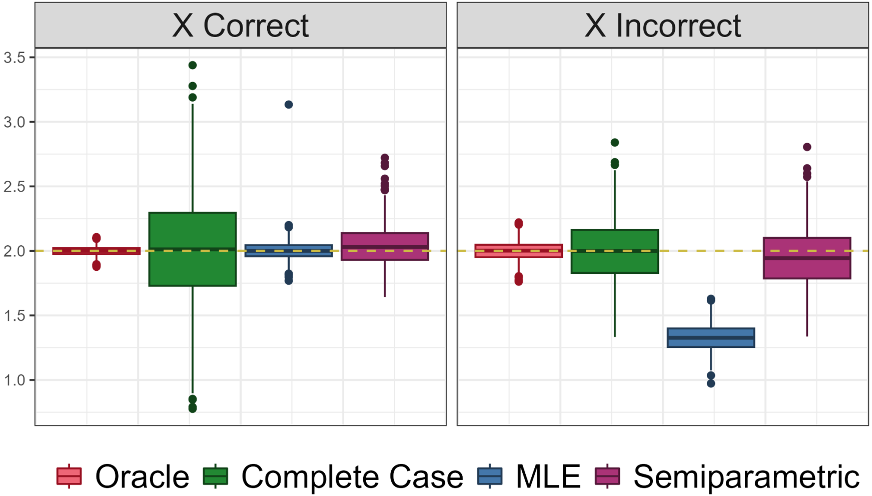

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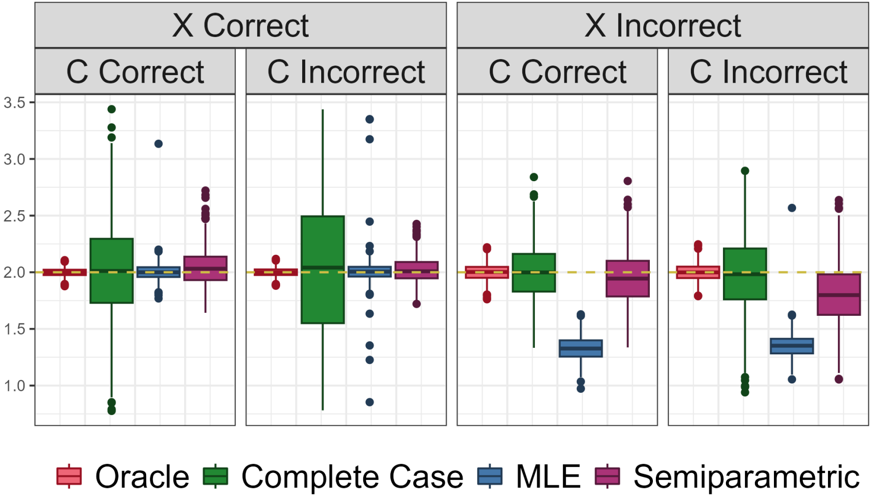

Generalizations

The methods presented here extend to:

- Nonlinear $\mathrm{E}(Y|X) = m(X, \boldsymbol{\beta})$

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- Additional uncensored covariates $\mathbf{Z}$
  - $\mathrm{E}(Y|X, \mathbf{Z}) = m(X, \mathbf{Z}, \boldsymbol{\beta})$
  - Nuisance distributions become $f_{X|\mathbf{Z}}, f_{C|\mathbf{Z}}, f_{\mathbf{Z}}$

SPARCC: Semiparametric Censored Covariate Estimation

R package available at https://github.com/brian-d-richardson/sparcc

Thank you! Any questions?

Brian Richardson
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