GILLINGS SCHOOL OF
GLOBAL PUBLIC HEALTH

Doubly robust estimation under a randomly censored covariate

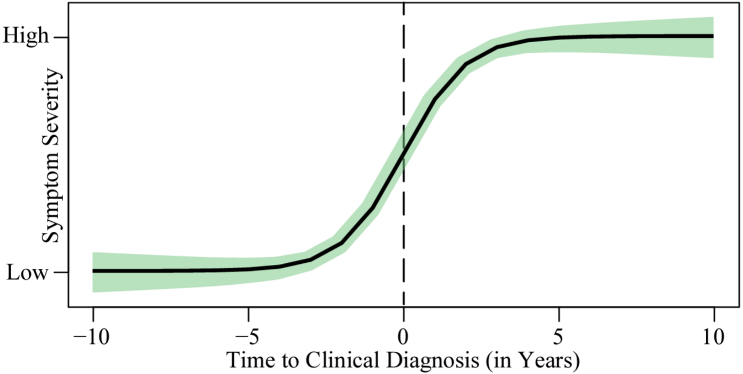
Brian Richardson

Acknowledgements

Seong-Ho Lee, PhD Tanya Garcia, PhD Yanyuan Ma, PhD

This research was supported by the National Institute of Environmental Health Sciences grant T32ES007018.

Huntington's Disease and Censored Covariates



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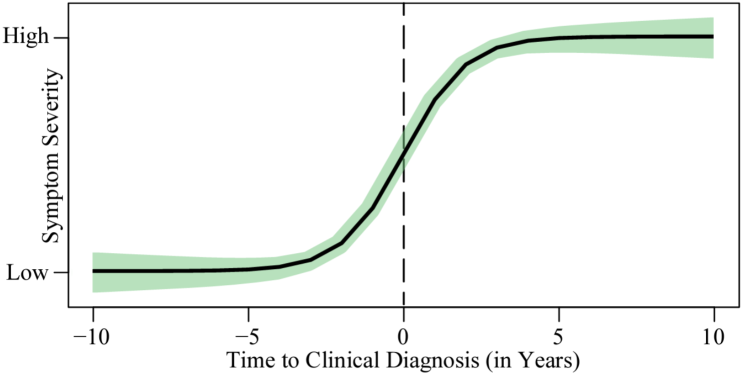
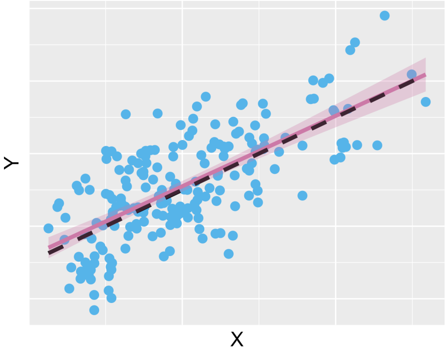


Lotspeich et al., "Making Sense of Censored Covariates: Statistical Methods for Studies of Huntington's Disease"

Censored Covariates: a Simple Example



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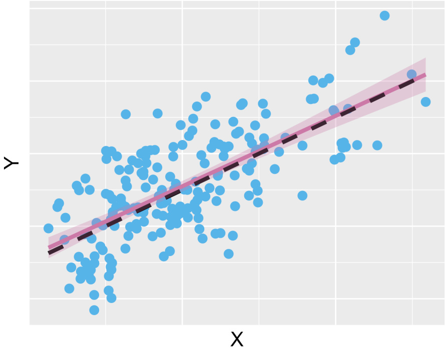

- Regression model:
  $\mathrm{E}(Y) = \beta_0 + \beta_1 X$
- Estimate $\boldsymbol{\beta} = (\beta_0, \beta_1)^T$ with least squares/maximum likelihood

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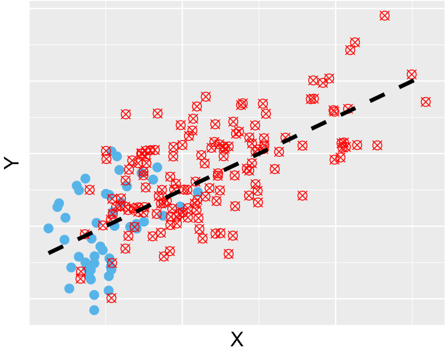

Problem: X is censored by a **random censoring time** C

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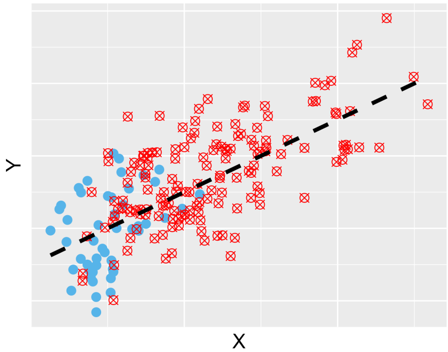

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- $\Delta = I(X \leq C)$

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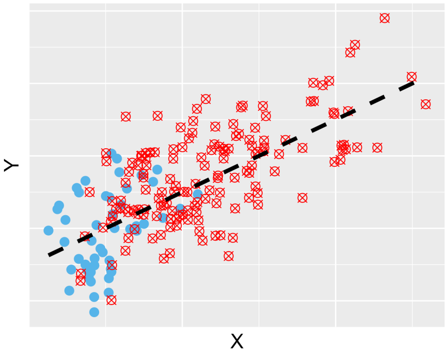
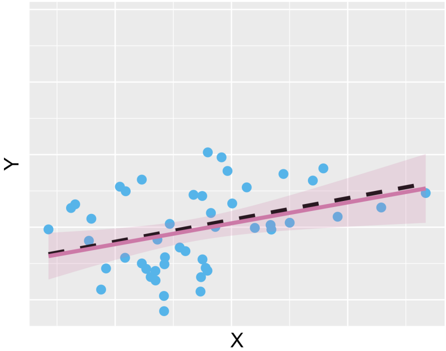


Problem: X is censored by a **random censoring time** C

- $W = \min(X, C)$
- $\Delta = I(X \leq C)$
- assume: $C \perp\!\!\!\perp (X, Y)$

Complete Case Analysis



Only use observations where X is *uncensored*

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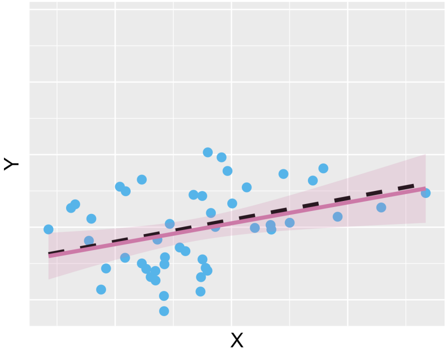

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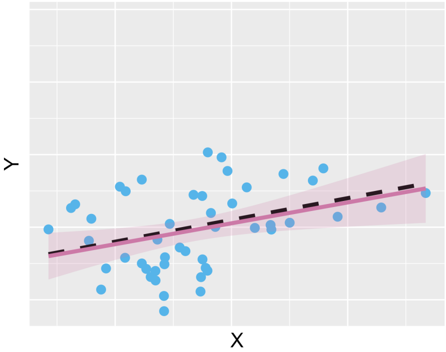

Only use observations where X is *uncensored*

✓ Consistent

✗ Inefficient

Maximum Likelihood Estimation (MLE)

$$f_{Y,W,\Delta}(y, w, \delta, \boldsymbol{\beta}, \boxed{\boldsymbol{\alpha}}) \propto \underbrace{\{f_{Y|X}(y, w, \boldsymbol{\beta})\}^{\delta}}_{\text{uncensored}} \underbrace{\left\{\int_{w}^{\infty} f_{Y|X}(y, x, \boldsymbol{\beta}) \boxed{f_X(x, \boldsymbol{\alpha})} dx\right\}^{1-\delta}}_{\text{censored}}$$

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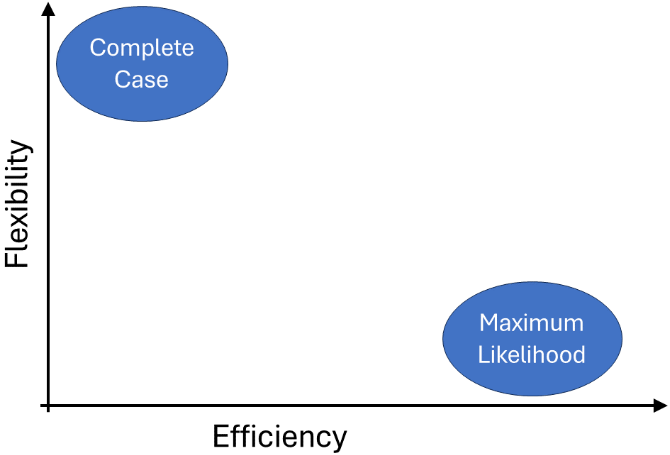
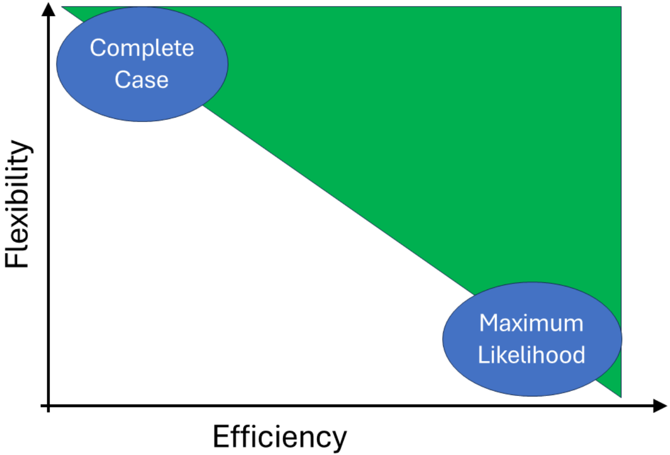
✗ inconsistent estimator when model for **nuisance parameter** f_X is incorrect

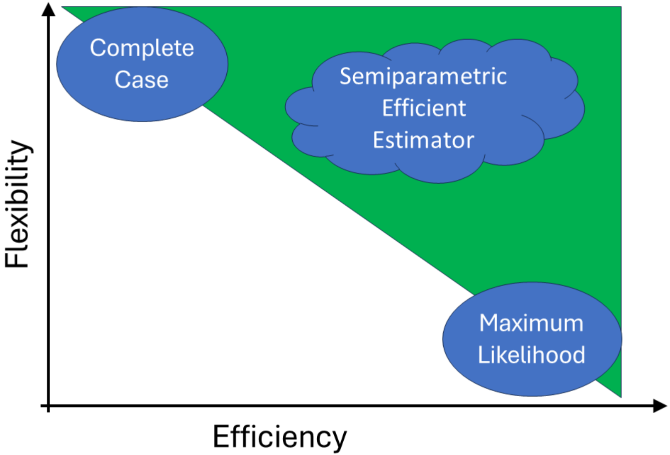
Existing Methods



Existing Opportunity



A New Approach



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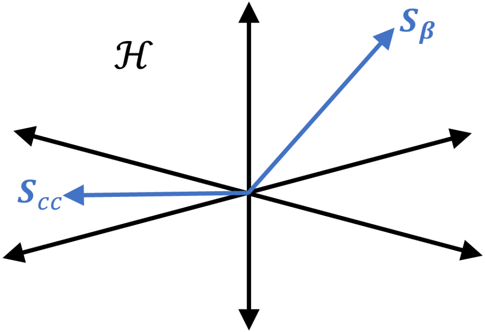
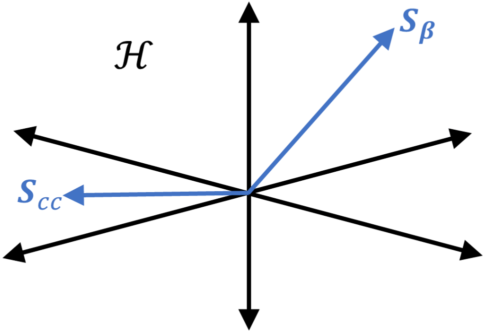
- Geometric approach from Tsiatis, *Semiparametric theory and missing data*

The Semiparametric Recipe



- **Hilbert space** of estimating functions

The Semiparametric Recipe



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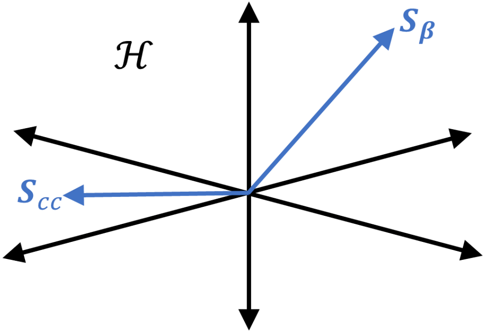
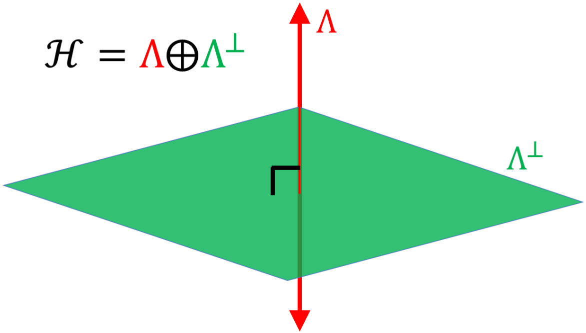


- **Hilbert space** of estimating functions
- **covariance inner product**
  $\langle \boldsymbol{h}, \boldsymbol{g} \rangle \equiv \mathrm{E}(\boldsymbol{h}^T \boldsymbol{g})$
- **orthogonal** ⇔ **uncorrelated**

$$\boldsymbol{h} \perp \boldsymbol{g} \iff \langle \boldsymbol{h}, \boldsymbol{g} \rangle = 0$$

The Semiparametric Recipe

$$\mathcal{H} = \Lambda \oplus \Lambda^{\perp}$$



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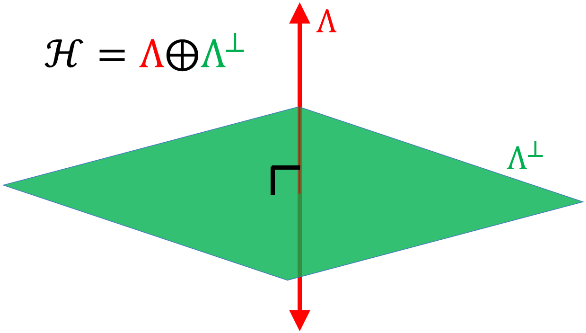

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$$\partial \log f_{Y,W,\Delta}(y, w, \delta, \boldsymbol{\beta}, \boldsymbol{\eta}) / \partial \boldsymbol{\eta}$$

- orthogonal complement $\Lambda^{\perp}$

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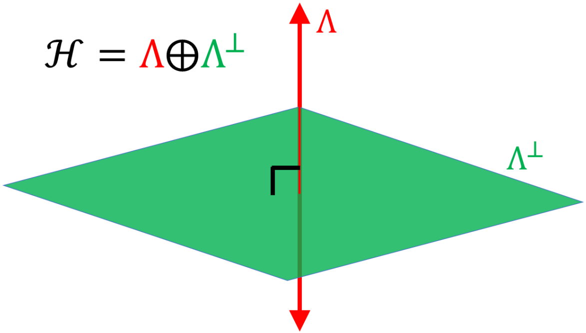

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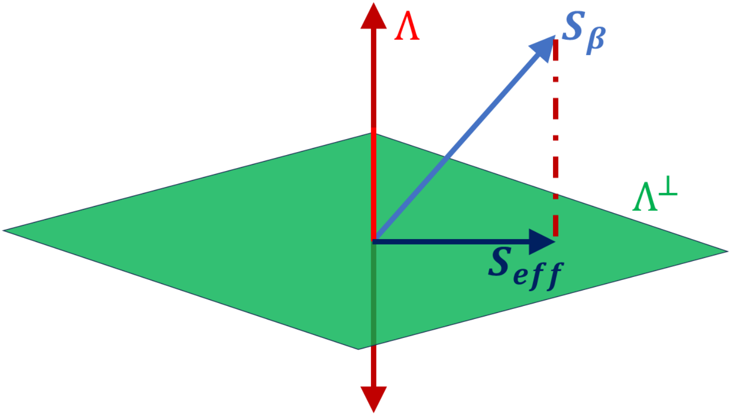

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The **semiparametric efficient estimator** $\widehat{\boldsymbol{\beta}}_{\text{eff}}$ is the solution to

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✓ **Doubly Robust**: $\widehat{\boldsymbol{\beta}}_{\text{eff}}$ is consistent if at least one of f_X, f_C is correctly specified

✓ **Locally Efficiency**: If f_X, f_C are *both* correctly specified, then $\widehat{\boldsymbol{\beta}}_{\text{eff}}$ achieves the **semiparametric efficiency bound**

Simulation Setup

- $Y|X \sim N(\beta_0 + \beta_1 X, \sigma^2)$

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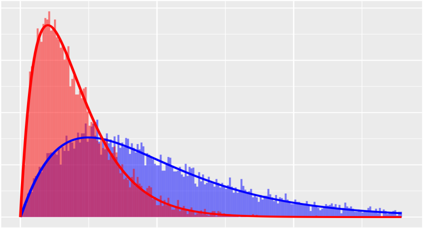
- X, $C \sim$ gamma distributions
- X, C possibly **misspecified** as exponential

X, C correct



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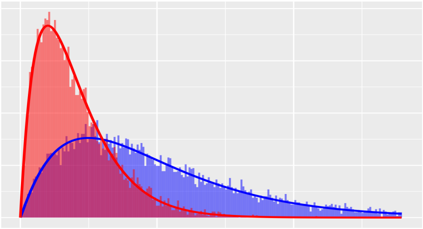
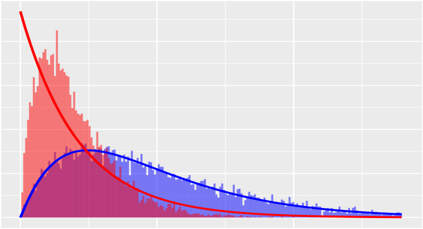


X correct, C incorrect



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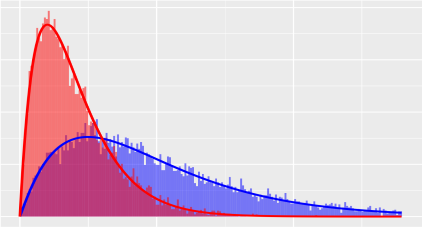

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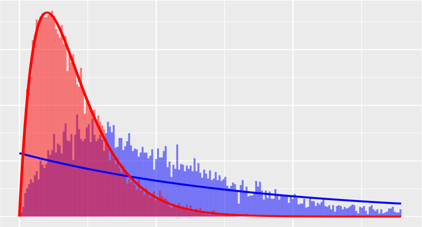

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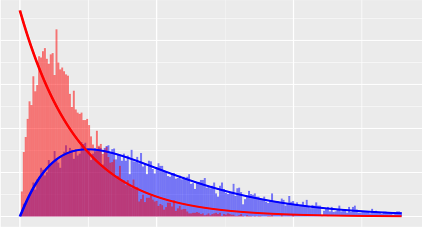

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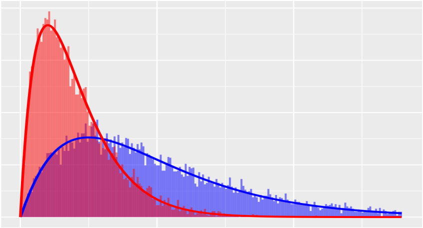

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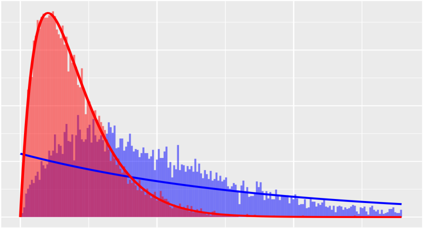

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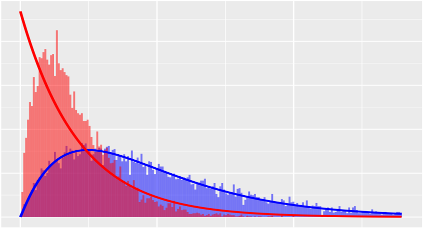

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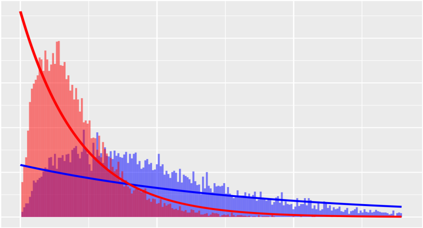
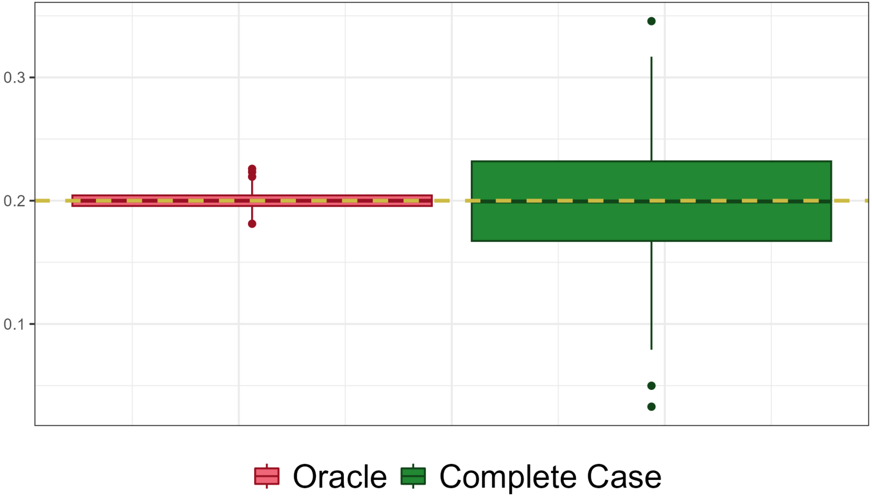


Simulation Results



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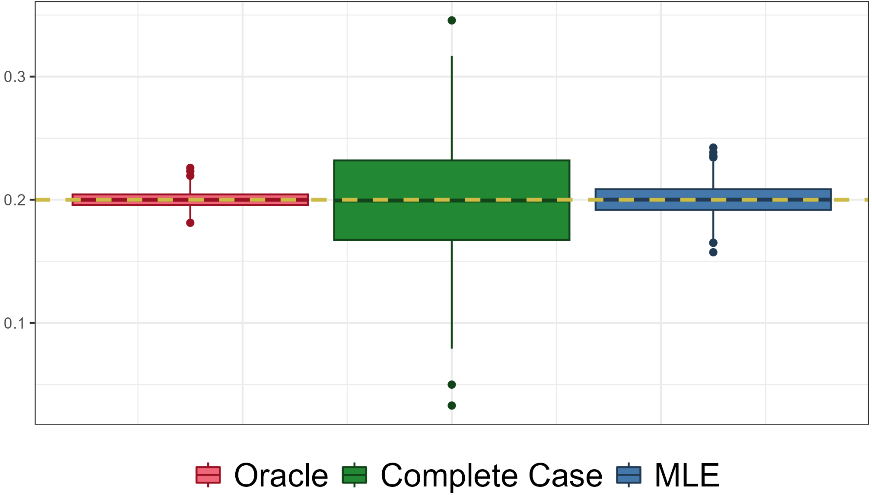

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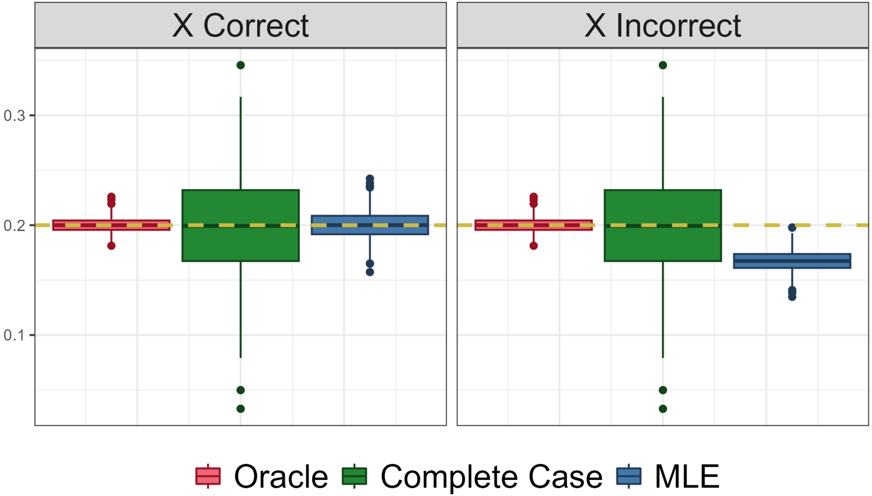

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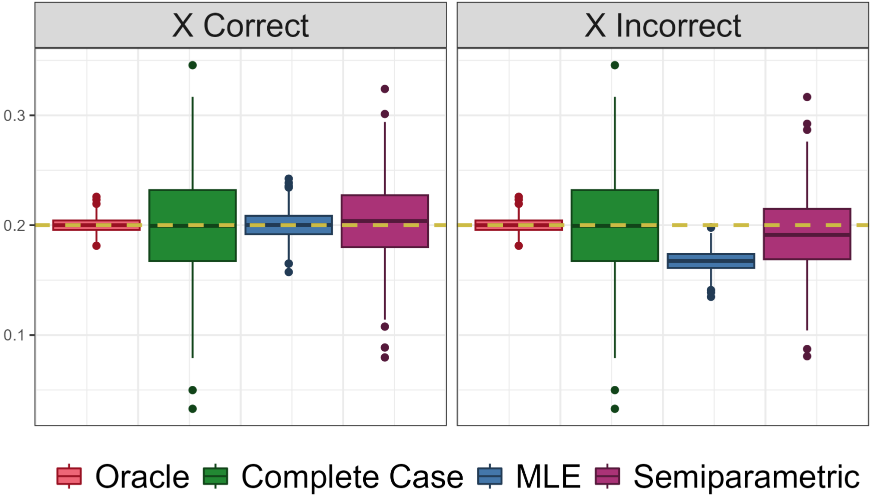

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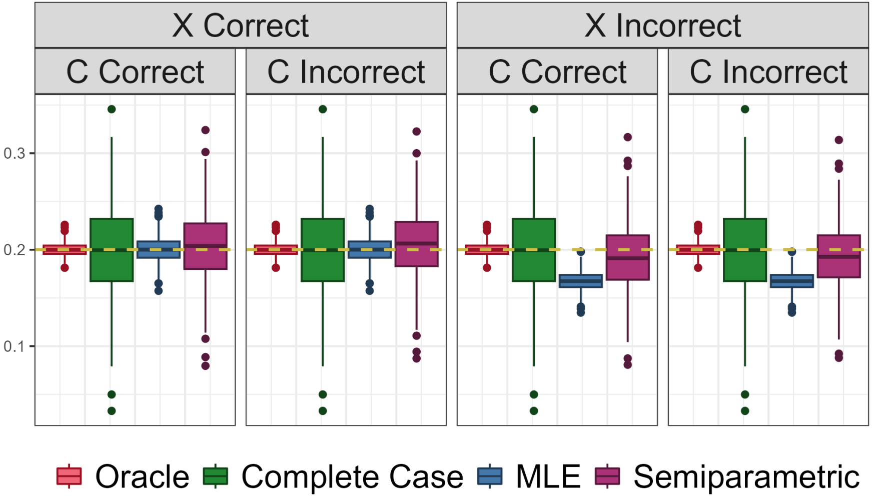

Generalizations

The methods presented here extend to:

- Nonlinear $\mathrm{E}(Y|X) = m(X, \boldsymbol{\beta})$

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- Nonlinear $\mathrm{E}(Y|X) = m(X, \boldsymbol{\beta})$
- Additional uncensored covariates $\mathbf{Z}$
  - $\mathrm{E}(Y|X, \mathbf{Z}) = m(X, \mathbf{Z}, \boldsymbol{\beta})$
  - Nuisance distributions become $f_{X|\mathbf{Z}}, f_{C|\mathbf{Z}}, f_{\mathbf{Z}}$

SPARCC: Semiparametric Censored Covariate Estimation

R package available at https://github.com/brian-d-richardson/sparcc

Appendix I: MLE Score Function

$$\mathbf{S}_{\boldsymbol{\beta}}(y, w, \delta, \mathbf{z}, \boldsymbol{\beta}) = \underbrace{\delta \mathbf{S}^{\mathrm{F}}_{\boldsymbol{\beta}}(y, w, \mathbf{z}, \boldsymbol{\beta})}_{\text{uncensored}} + \underbrace{(1-\delta)\frac{\mathrm{E}\{\mathrm{I}(X > w)\mathbf{S}^{\mathrm{F}}_{\boldsymbol{\beta}}(y, X, \mathbf{z}, \boldsymbol{\beta}) \mid y, \mathbf{z}\}}{\mathrm{E}\{\mathrm{I}(X > w) \mid y, \mathbf{z}\}}}_{\text{censored}}$$

Appendix II: Efficient Score Function

$$
\begin{aligned}
\mathbf{S}_{\text{eff}}(y, w, \delta, \mathbf{z}, \boldsymbol{\beta}) \equiv\ & \delta\{\mathbf{S}^{\mathrm{F}}_{\boldsymbol{\beta}}(y, w, \mathbf{z}, \boldsymbol{\beta}) - \boxed{\mathbf{a}(w, z, \boldsymbol{\beta})}\} \\
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\end{aligned}
$$

where $\mathbf{a}(x, \mathbf{z}, \boldsymbol{\beta})$ satisfies

$$
\begin{aligned}
&\mathrm{E}\{\mathrm{I}(x \le C) \mid \mathbf{z}\}\mathbf{a}(x, \mathbf{z}, \boldsymbol{\beta}) + \mathrm{E}\left[\mathrm{I}(x > C)\frac{\mathrm{E}\{\mathrm{I}(X > C)\mathbf{a}(X, \mathbf{z}, \boldsymbol{\beta}) \mid Y, C, \mathbf{z}\}}{\mathrm{E}\{\mathrm{I}(X > C) \mid Y, C, \mathbf{z}\}} \mid x, \mathbf{z}\right] \\
&= \mathrm{E}\left[\mathrm{I}(x > C)\frac{\mathrm{E}\{\mathrm{I}(X > C)\mathbf{S}^{\mathrm{F}}_{\boldsymbol{\beta}}(Y, X, \mathbf{z}, \boldsymbol{\beta}) \mid Y, C, \mathbf{z}\}}{\mathrm{E}\{\mathrm{I}(X > C) \mid Y, C, \mathbf{z}\}} \mid x, \mathbf{z}\right]
\end{aligned}
$$

Thank you! Any questions?

Brian Richardson
✉: brichson@ad.unc.edu