UNC | GILLINGS SCHOOL OF GLOBAL PUBLIC HEALTH

# Doubly robust estimation under a randomly censored covariate

Brian Richardson

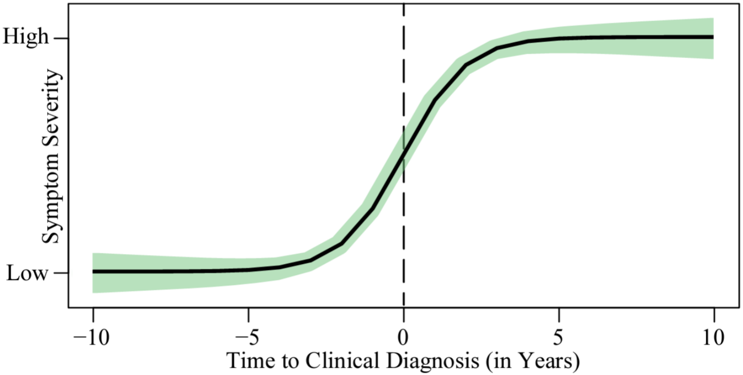
# Acknowledgements

Seong-Ho Lee, PhD

Tanya Garcia, PhD

Yanyuan Ma, PhD

# Huntington's Disease and Censored Covariates




Lotspeich et. al. (2024)

# Censored Covariates: a Simple Example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  $\mathrm{E}(Y) = \beta_0 + \beta_1 X$

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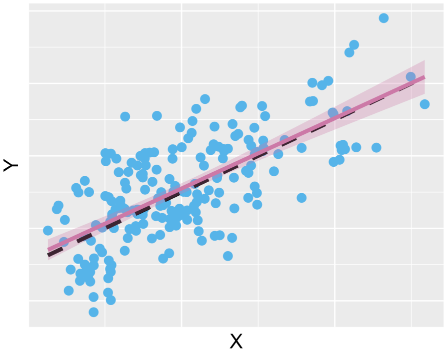


- Regression model:
  $\mathrm{E}(Y) = \beta_0 + \beta_1 X$
- Estimate $\boldsymbol{\beta} = (\beta_0, \beta_1)^T$ by solving:

$$\sum_{i=1}^{n} \underbrace{(Y_i - \beta_0 - \beta_1 X_i)(1, X_i)^T}_{\text{score function: } \mathbf{S}^F_{\boldsymbol{\beta}}} = \mathbf{0}$$

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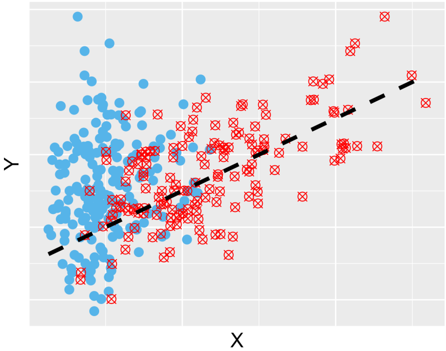


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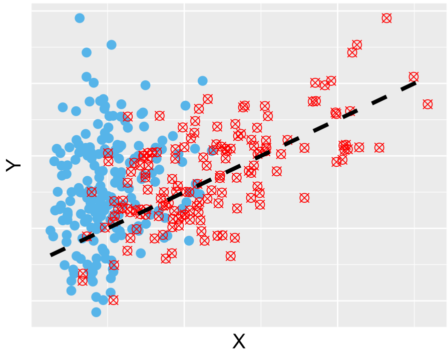


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- $\Delta = I(X \leq C)$

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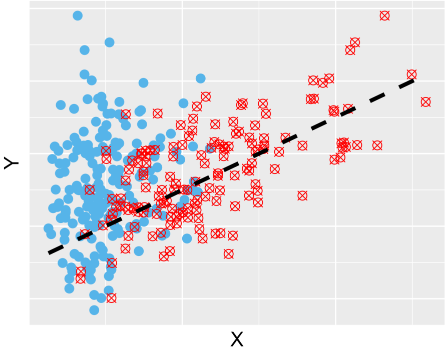
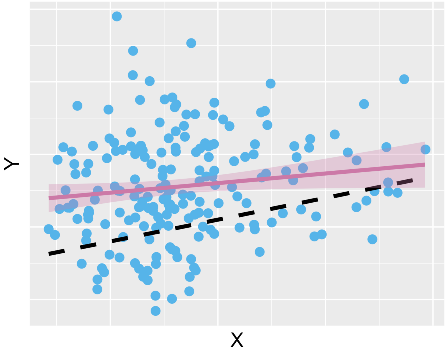



Problem: $X$ is censored by a **random censoring time** $C$

- $W = \min(X, C)$
- $\Delta = I(X \leq C)$
- assume: $C \perp\!\!\!\perp (X, Y)$

# A Naive Approach

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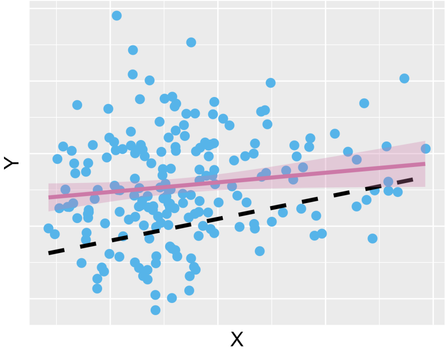


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- Estimate $\boldsymbol{\beta}$ by solving:

$$\sum_{i=1}^{n} \mathbf{S}_{\boldsymbol{\beta}}^{F}(Y_i, \boxed{W_i}, \boldsymbol{\beta}) = \mathbf{0}$$

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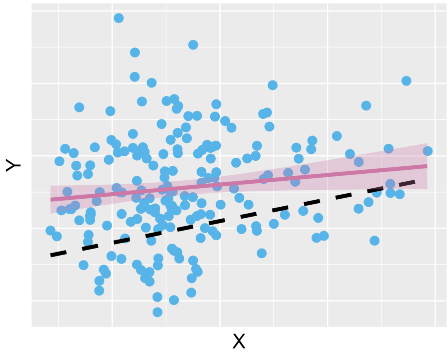


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- Estimate $\boldsymbol{\beta}$ by solving:

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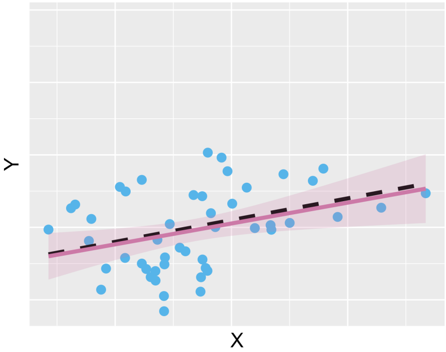
✗ Inconsistent estimator of $\boldsymbol{\beta}$

# Complete Case Analysis




Only use observations where $X$ is *uncensored*

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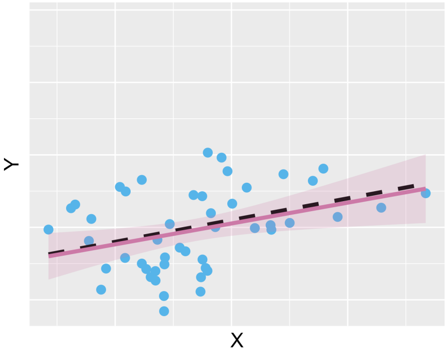


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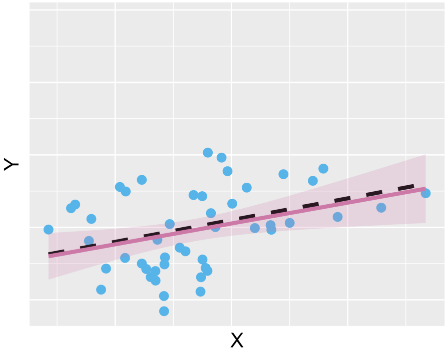


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✓ Consistent

✗ Inefficient

# Maximum Likelihood Estimation (MLE)

$$f_{Y,W,\Delta}(y, w, \delta, \boldsymbol{\beta}, \boxed{\boldsymbol{\alpha}}) \propto \underbrace{\{f_{Y|X}(y, w, \boldsymbol{\beta})\}^{\delta}}_{\text{uncensored}} \underbrace{\left\{\int_{w}^{\infty} f_{Y|X}(y, x, \boldsymbol{\beta}) \boxed{f_X(x, \boldsymbol{\alpha})} dx\right\}^{1-\delta}}_{\text{censored}}$$

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# Maximum Likelihood Estimation (MLE)

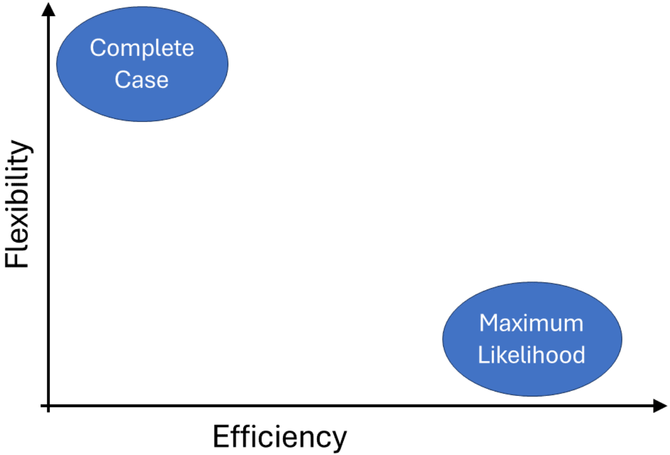
**Pros**

- ✓ consistent estimator of $\boldsymbol{\beta}$
- ✓ fully efficient

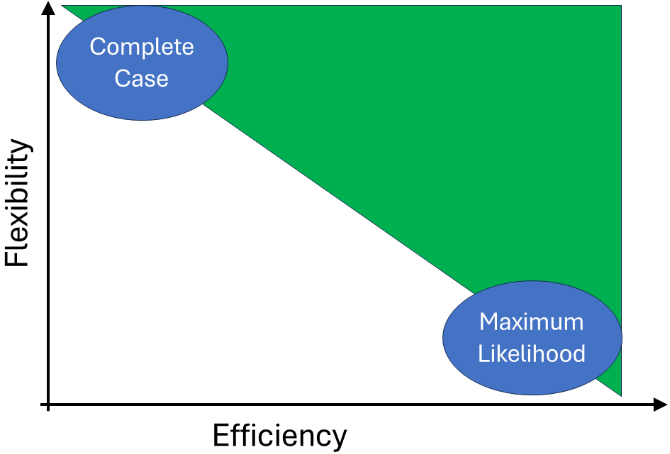
**Cons**

- ✗ requires model for **nuisance parameter** $f_X$
- ✗ inconsistent estimator when model for $f_X$ is incorrect

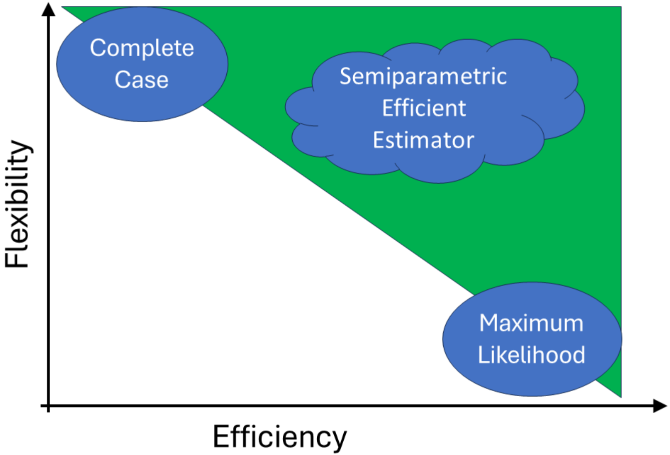
# Existing Methods



# Existing Opportunity



# A New Approach



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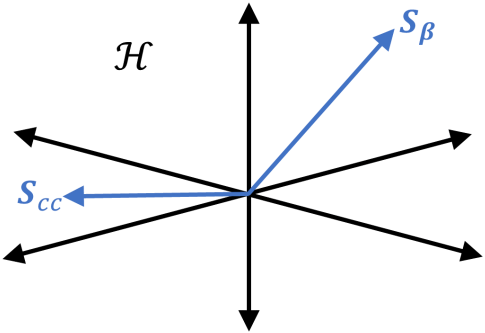
- **semiparametric**: infinite dimensional nuisance parameters $f_X, f_C$
- Geometric approach from Tsiatis (2006)

# The Semiparametric Recipe

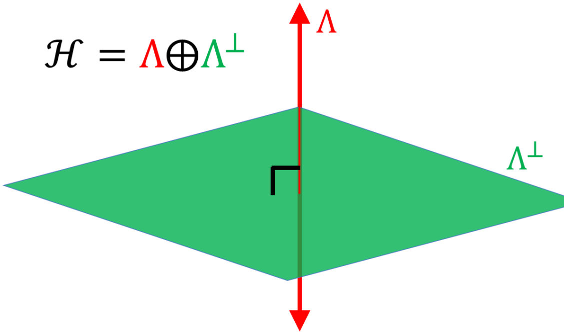
- **Hilbert space** of estimating functions
- **covariance inner product** $\langle \boldsymbol{h}, \boldsymbol{g} \rangle = \mathrm{E}(\boldsymbol{h}^T \boldsymbol{g})$



# The Semiparametric Recipe

- construct $\Lambda$ using **nuisance scores**
- orthogonal complement $\Lambda^{\perp}$

$$\mathcal{H} = \Lambda \oplus \Lambda^{\perp}$$



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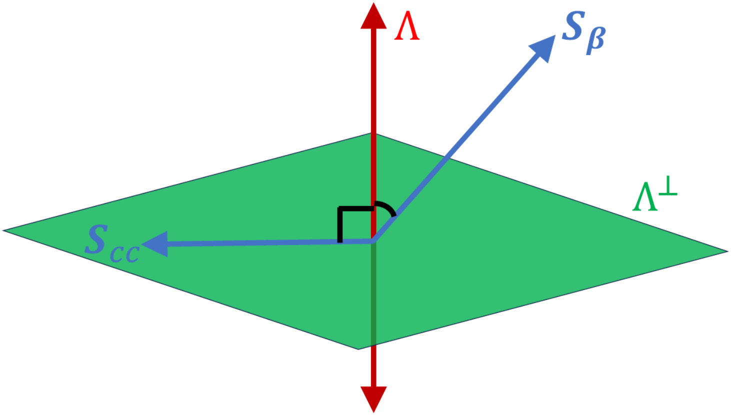

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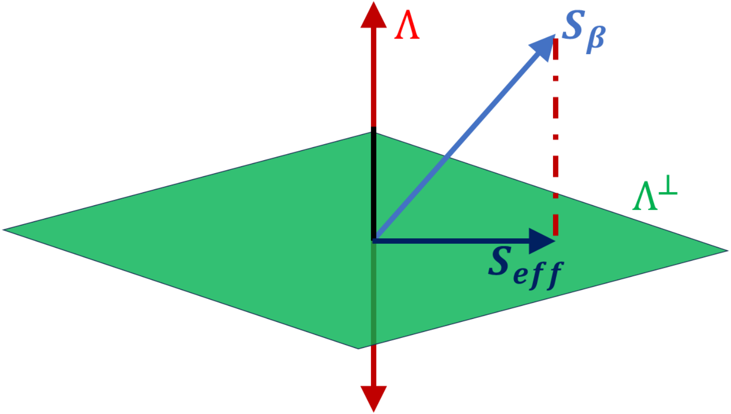

# Properties of the Proposed Estimator

The **semiparametric efficient estimator** $\widehat{\boldsymbol{\beta}}_{\text{eff}}$ is the solution to

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- ✓ **Doubly Robust**: $\widehat{\boldsymbol{\beta}}_{\text{eff}}$ is consistent if at least one of $f_X$, $f_C$ is correctly specified
- ✓ **Locally Efficiency**: If $f_X$, $f_C$ are *both* correctly specified, then $\widehat{\boldsymbol{\beta}}_{\text{eff}}$ achieves the **semiparametric efficiency bound**

# Simulation Setup

- $Y|X \sim N(\beta_0 + \beta_1 X, \sigma^2)$

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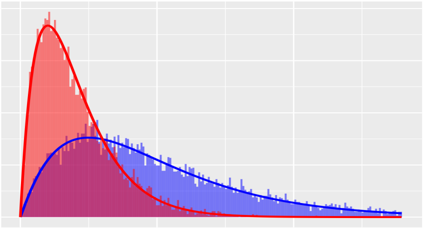
- $X$, $C \sim$ gamma distributions
- $X$, $C$ possibly **misspecified** as exponential

$X$, $C$ correct



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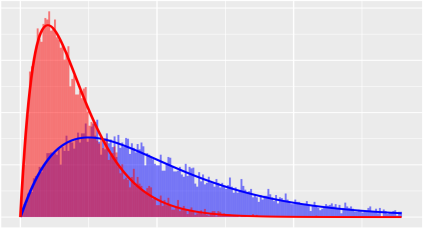

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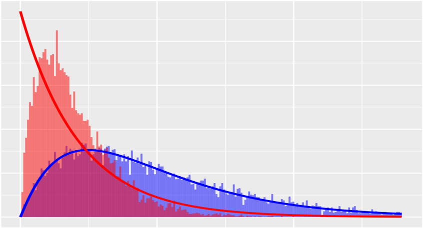

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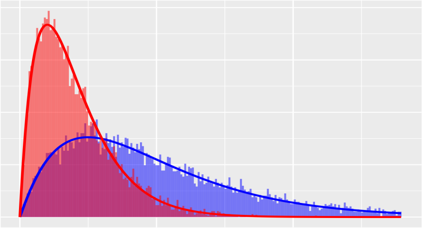

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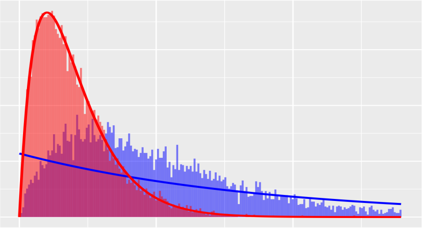

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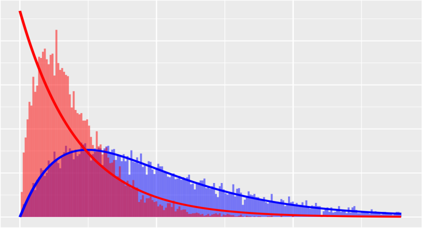

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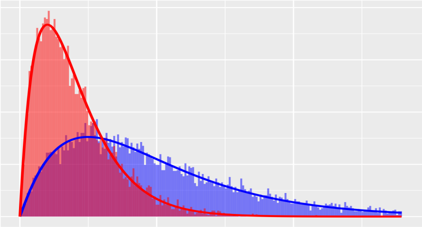

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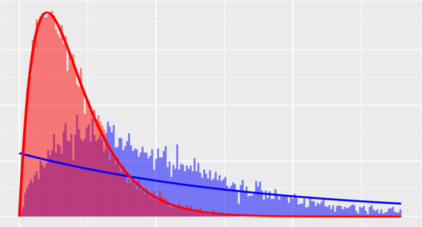

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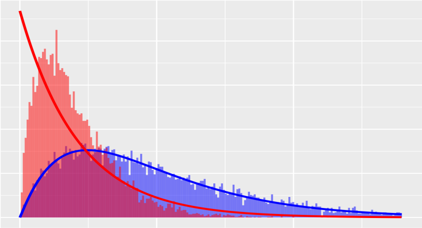

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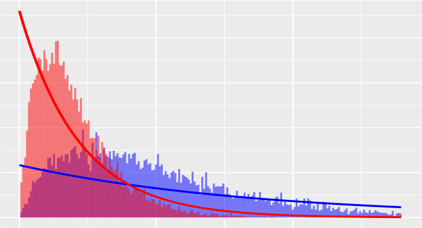
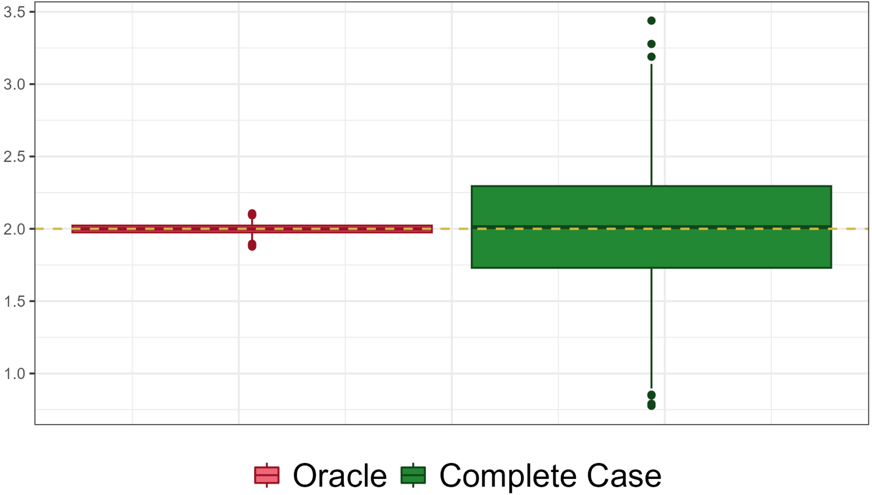


# Simulation Results



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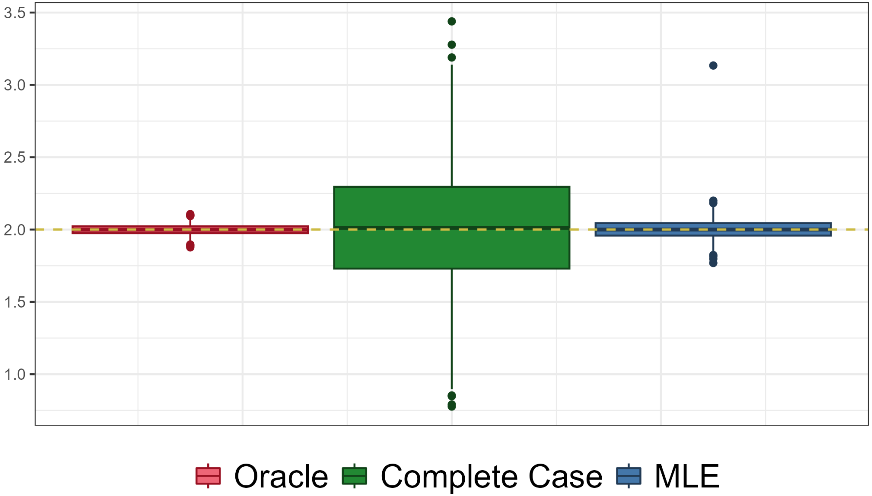

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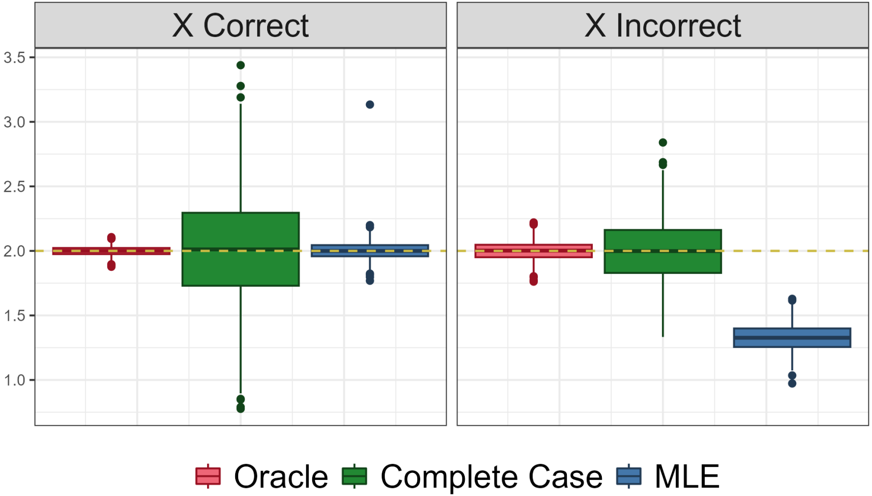

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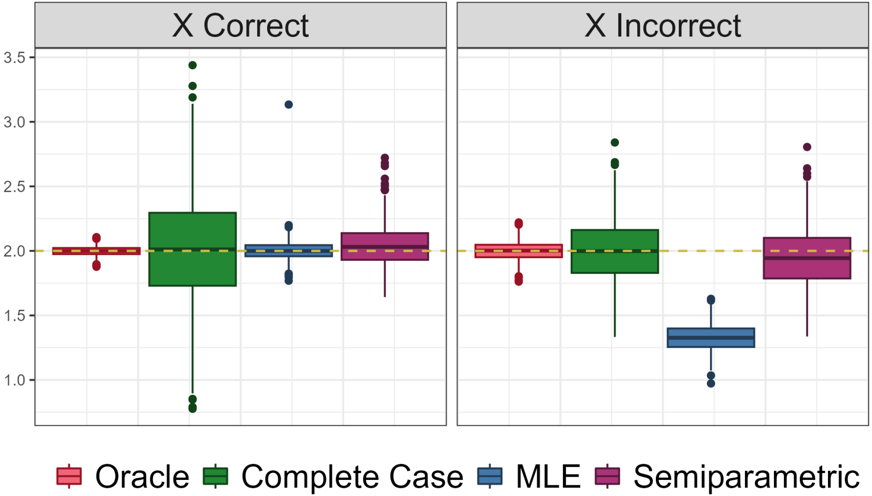

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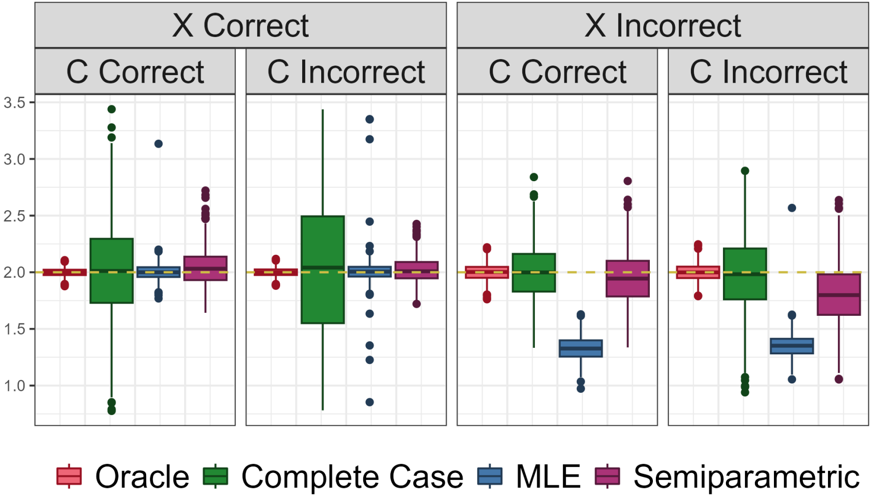

# Generalizations

The methods presented here extend to:

- Nonlinear $\mathrm{E}(Y|X) = m(X, \boldsymbol{\beta})$

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- Additional uncensored covariates $\mathbf{Z}$
  - $\mathrm{E}(Y|X, \mathbf{Z}) = m(X, \mathbf{Z}, \boldsymbol{\beta})$
  - Nuisance distributions become $f_{X|\mathbf{Z}}, f_{C|\mathbf{Z}}, f_{\mathbf{Z}}$

# SPARCC: Semiparametric Censored Covariate Estimation

R package available at https://github.com/brian-d-richardson/sparcc

# Thank you! Any questions?

Brian Richardson
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