Foundations of the WKB Approximation for Models of Cochlear Mechanics in 1- and 2-D: Extensive Computations

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1 Introduction

In the main text of the tutorial, results regarding WKB approximate solutions were presented along with outlines of derivations. Most notably, two instructive derivations for the higher-order 2-D WKB approximate solutions were outlined in the appendices.

The purpose of this document is to provide the extensive computations involved in these derivations, along with certain other details that were deemed too intricate and lengthy to appear in the main text. While these details may appear superfluous to some readers, they are important to present (a) for completeness in creating a self-contained exposition, and (b) for potential modification should a modeler desire to introduce additional effects to a model and still arrive at a WKB solution/WKB projections. These derivations are presented in the order of their appearance in the main text.

2 The Series Solution Method

In the main text's Appendix A, I outlined a method by which a power series approximation can be used to arrive at a higher-order solution for the pressure BVP. Here, I provide the extremely detailed computations involved in the derivation.

2.1 Deriving the BVP in A

In Appendix A of the main text, we consider a BVP for the auxiliary pressure variable Q:

$$\frac{\partial^2 Q}{\partial x^2} + K^2 \frac{\partial^2 Q}{\partial \zeta^2} = 0, \tag{1}$$

$$\left.\frac{\partial Q}{\partial \zeta}\right|_{\zeta=H} = 0, \tag{2}$$

$$\left.\frac{\partial Q}{\partial \zeta}\right|_{\zeta=0} + Hf^2(x)Q(x,0) = 0. \tag{3}$$

We also assume Q has the form

$$Q(x,\zeta) = A(x,\zeta;K)e^{jKg(x)} \cosh\left[\kappa(x)(H-\zeta)\right]. \tag{4}$$

The derivatives of Q in terms of A are

$$\frac{\partial Q}{\partial \zeta} = \frac{\partial A}{\partial \zeta} e^{jKg} \cosh a - \kappa A e^{jKg} \sinh a, \tag{5}$$

$$\frac{\partial^2 Q}{\partial \zeta^2} = \frac{\partial^2 A}{\partial \zeta^2} e^{jKg} \cosh a - \kappa \frac{\partial A}{\partial \zeta} e^{jKg} \sinh a - \kappa \frac{\partial A}{\partial \zeta} e^{jKg} \sinh a + \kappa^2 A e^{jKg} \cosh a, \tag{6}$$

$$\frac{\partial Q}{\partial x} = \frac{\partial A \cosh a}{\partial x} e^{jKg} + jKg' A e^{jKg} \cosh a, \tag{7}$$

$$\frac{\partial^2 Q}{\partial x^2} = \frac{\partial^2 A \cosh a}{\partial x^2} e^{jKg} + 2jKg' e^{jKg} \frac{\partial A \cosh a}{\partial x} - K^2 g'^2 A e^{jKg} \cosh a + jKg'' A e^{jKg} \cosh a. \tag{8}$$

The boundary conditions for A at $\zeta = H$ (where $a = 0$) are

$$\left.\frac{\partial Q}{\partial \zeta}\right|_{\zeta=H} = \left.\frac{\partial A}{\partial \zeta}\right|_{\zeta=H} e^{jKg} = 0, \tag{9}$$

$$\left.\frac{\partial A}{\partial \zeta}\right|_{\zeta=H} = 0. \tag{10}$$

The boundary condition at $\zeta = 0$ is

$$\begin{aligned}\left.\frac{\partial Q}{\partial \zeta}\right|_{\zeta=0} + Hf^2(x)Q(x,0) = \left.\frac{\partial A}{\partial \zeta}\right|_{\zeta=0} e^{jKg} \cosh a(x,0) - \kappa A e^{jKg} \sinh a(x,0) +& \\ + Hf^2 A(x,0) e^{jKg} \cosh a(x,0) = 0,&\end{aligned} \tag{11}$$

which simplifies to

$$\left.\frac{\partial A}{\partial \zeta}\right|_{\zeta=0} - \kappa A(x,0)\tanh a(x,0) + Hf^2 A(x,0) = 0. \tag{12}$$

Finally,the PDE in A is

$$\begin{aligned}\frac{\partial^2 Q}{\partial x^2} + K^2\frac{\partial^2 Q}{\partial \zeta^2} = \frac{\partial^2 A\cosh a}{\partial x^2}e^{jKg} + 2jKg'e^{jKg}\frac{\partial A\cosh a}{\partial x} - K^2 g'^2 A e^{jKg}\cosh a + \\ + jKg'' A e^{jKg}\cosh a + K^2\frac{\partial^2 A}{\partial \zeta^2}e^{jKg}\cosh a - K^2\kappa\frac{\partial A}{\partial \zeta}e^{jKg}\sinh a + \\ + K^2\kappa^2 A e^{jKg}\cosh a - K^2\kappa\frac{\partial A}{\partial \zeta}e^{jKg}\sinh a = 0.\end{aligned} \tag{13}$$

After some simplification, this is

$$\begin{aligned}K^2\left[(\kappa^2 - g'^2)A\cosh a + \frac{\partial^2 A}{\partial \zeta^2}\cosh a - 2\kappa\frac{\partial A}{\partial \zeta}\sinh a\right] + \\ + jK\left[g'' A\cosh a + 2g'\frac{\partial A\cosh a}{\partial x}\right] + \frac{\partial^2 A\cosh a}{\partial x^2} = 0.\end{aligned} \tag{14}$$

2.2 Creating the system of PDEs in A

In Appendix A of the main text, with the BVP in A determined, we consider a series ansatz

$$A(x,\zeta;K) = A_0(x) + \sum_{n=1}^{\infty}\frac{1}{(jK)^n}A_n(x,\zeta) \tag{15}$$

and use it to find an approximate solution for the BVP.

Plugging the series ansatz in to the PDE in A (Eqn 13) gives

$$\begin{aligned}\Bigg[(\kappa^2 - g'^2)\cosh a\left(K^2 A_0 - jKA_1 + \sum_{n=2}^{\infty}\frac{1}{(jK)^{n-2}}A_n\right) + \\ + \cosh a\left(-jK\frac{\partial^2 A_1}{\partial \zeta^2} + \sum_{n=2}^{\infty}\frac{1}{(jK)^{n-2}}\frac{\partial^2 A_n}{\partial \zeta^2}\right) + \\ -2\kappa\sinh a\left(-jK\frac{\partial A_1}{\partial \zeta} + \sum_{n=2}^{\infty}\frac{1}{(jK)^{n-2}}\frac{\partial A_n}{\partial \zeta}\right)\Bigg] + \\ + \Bigg[g''\cosh a\left(jKA_0 + \sum_{n=1}^{\infty}\frac{1}{(jK)^{n-1}}A_n\right) + 2g'\sinh a\left(jKA_0 + \sum_{n=1}^{\infty}\frac{1}{(jK)^{n-1}}A_n\right) + \\ +2g'\cosh a\left(jK\frac{dA_0}{dx} + \sum_{n=1}^{\infty}\frac{1}{(jK)^{n-1}}\frac{\partial A_n}{\partial x}\right)\Bigg] + \Bigg[\cosh a\left(A_0 + \sum_{n=1}^{\infty}\frac{1}{(jK)^n}A_n\right) + \\ +2\sinh a\left(\frac{dA_0}{dx} + \sum_{n=1}^{\infty}\frac{1}{(jK)^n}\frac{\partial A_n}{\partial x}\right) + \cosh a\left(\frac{d^2 A_0}{dx^2} + \sum_{n=1}^{\infty}\frac{1}{(jK)^n}\frac{\partial^2 A_n}{\partial x^2}\right)\Bigg] = 0.\end{aligned} \tag{16}$$

We have to separate this leviathan into its components in powers of jK. The single term in K^2 gives

$$(\kappa^2 - g'^2)\cosh a A_0 = 0, \tag{17}$$

or

$$g'^2(x) = \kappa^2(x). \tag{18}$$

Next, consider the jK terms:

$$-(\kappa^2 - g'^2)A_0 - \cosh a \frac{\partial^2 A_1}{\partial \zeta^2} + 2\kappa \sinh a \frac{\partial A_1}{\partial \zeta} + g'' A_0 \cosh a + 2g' \sinh a A_0 + 2g' \cosh a \frac{dA_0}{dx} = 0. \tag{19}$$

With the aid of Eqn 18, we can see the first of these summands is 0. This simplifies the expression to

$$\begin{aligned} \cosh a \frac{\partial^2 A_1}{\partial \zeta^2} - 2\kappa \sinh a \frac{\partial A_1}{\partial \zeta} &= g'' A_0 \cosh a + 2g' \sinh a A_0 + 2g' \cosh a \frac{dA_0}{dx} \\ &= g'' A_0 \cosh a + 2g' \frac{\partial A_0 \cosh a}{\partial x}. \end{aligned} \tag{20}$$

As for the rest of the terms, we have a system of infinitely many equations. Again, by Eqn 18,the terms leading with $\kappa^2 - g'^2$ are 0. The relation is

$$\begin{aligned} \cosh a \frac{\partial^2 A_n}{\partial \zeta^2} - 2\kappa \sinh a \frac{\partial A_n}{\partial \zeta} &= (g'' \cosh a + 2g' \sinh a) A_{n-1} + 2g' \cosh a \frac{\partial A_{n-1}}{\partial x} + \\ &\quad + \cosh a \frac{\partial A_{n-2}}{\partial x} + 2 \sinh a \frac{\partial A_{n-2}}{\partial x} + \cosh a \frac{\partial^2 A_{n-2}}{\partial x^2} \\ &= g'' \cosh a + 2g' \frac{\partial A_{n-1} \cosh a}{\partial x} + \frac{\partial^2 A_{n-2} \cosh a}{\partial x^2}, \qquad n \geq 2. \end{aligned} \tag{21}$$

In theory, this will allow us to solve for any order of approximation desired so long as we can solve for A_0. In practice, we will only solve for A_0 to make a first approximation.

Now consider boundary conditions. At $\zeta = H$, all terms must have a zero derivative (trivial for A_0 which has no ζ dependence). That is,

$$\left. \frac{\partial A_n}{\partial \zeta} \right|_{\zeta = H} = 0, \quad n \geq 1. \tag{22}$$

As for $\zeta = 0$, plugging in the series ansatz to the boundary condition gives

$$(Hf^2 - \kappa \tanh a) A_0 + \sum_{n=1}^{\infty} \frac{1}{(jK)^n} \left(\frac{\partial A_n}{\partial \zeta} + (Hf^2 - \kappa \tanh a) A_n \right) = 0 \tag{23}$$

at $\zeta = 0$. Recalling that $a(x, 0) = \kappa H$, the A_0 term gives

$$\kappa \tanh \kappa H = Hf^2. \tag{24}$$

As for the other terms, the above relationship simplifies the summands to

$$\left. \frac{\partial A_n}{\partial \zeta} \right|_{\zeta = 0} = 0, \quad n \geq 1. \tag{25}$$

2.3 Solving the PDE for A_0

We now look to solve the BVP up to order 0. Recalling that $a(x,\zeta) = \kappa(x)(H-\zeta)$, we can note a relation that helps to simplify the left-hand side of the equation:

$$\begin{aligned}
\operatorname{sech} a \frac{\partial}{\partial \zeta}\left[\cosh^2 a \frac{\partial A_1}{\partial \zeta}\right] &= \operatorname{sech} a \cosh^2 a \frac{\partial^2 A_1}{\partial \zeta^2} + \operatorname{sech} a \frac{\partial \cosh^2 a}{\partial \zeta}\frac{\partial A_1}{\partial \zeta} \\
&= \cosh a \frac{\partial^2 A_1}{\partial \zeta^2} + \operatorname{sech} a (2\cosh a \sinh a)\frac{\partial a}{\partial \zeta}\frac{\partial A_1}{\partial \zeta} \\
&= \cosh a \frac{\partial^2 A_1}{\partial \zeta^2} - 2\kappa \sinh a \frac{\partial A_1}{\partial \zeta}.
\end{aligned} \tag{26}$$

This is precisely the left-hand side of Eqn **??**. That is,

$$\operatorname{sech} a \frac{\partial}{\partial \zeta}\left[\cosh^2 a \frac{\partial A_1}{\partial \zeta}\right] = g'' A_0 \cosh a + 2g' \frac{\partial A_0 \cosh a}{\partial x}. \tag{27}$$

Multiplying by $\cosh a$ on both sides of the equation and integrating from 0 to ζ,

$$\begin{aligned}
&\int_0^\zeta \frac{\partial}{\partial \xi}\left[\cosh^2 a \frac{\partial A_1}{\partial \xi}\right] d\xi = \cosh^2 a \frac{\partial A_1}{\partial \zeta} \\
&\quad = \int_0^\zeta g'' A_0 \cosh^2 a + 2g' \cosh a \frac{\partial A_0 \cosh a}{\partial x}\; d\xi \\
&\quad = g'' A_0 \int_0^\zeta \cosh^2 a \; d\xi + 2g' \frac{dA_0}{dx}\int_0^\zeta \cosh^2 a \; d\xi + g' A_0 \int_0^\zeta 2\cosh a \frac{\partial}{\partial x}\cosh a \; d\xi \\
&\quad = \left(g'' A_0 + 2g' \frac{dA_0}{dx} + g' A_0 \frac{\partial}{\partial x}\right)\int_0^\zeta \cosh^2 a \; d\xi,
\end{aligned} \tag{28}$$

where in the last step I have applied the product rule of derivatives in reverse.

The only integral to compute now is that of the squared hyperbolic cosine. Using the double-argument formula for cosh,

$$\begin{aligned}
\int_0^\zeta \cosh^2 a \; d\xi &= \frac{1}{2}\int_0^\zeta \cosh\left[2\kappa(x)(H-\xi)\right] + 1 \; d\xi \\
&= \left[\frac{-1}{4\kappa}\sinh\left[2\kappa(x)(H-\xi)\right] + \frac{\xi}{2}\right|_{\xi=0}^{\zeta} \\
&= \frac{\zeta}{2} - \frac{1}{2\kappa}\left[\sinh\left[\kappa(x)(H-\xi)\right]\cosh\left[\kappa(x)(H-\xi)\right]\right|_{\xi=0}^{\zeta} \\
&= \frac{\zeta}{2} - \frac{1}{2\kappa}\left[\sinh a \cosh a - \sinh \kappa H \cosh \kappa H\right].
\end{aligned} \tag{29}$$

Plugging this into Eqn 28 yields a simplified relationship between A_0 and A_1 with a useful property – plugging in $\zeta = H$ facilitates solution for only terms involving $A_0(x)$. Because A_0 is ζ-independent, this gives full knowledge of A_0.

The boundary condition in Eqn **??** gives that the left-hand side of Eqn 28 is 0. Plugging H into Eqn 28 (using the integral identity in Eqn 29),

$$\left(g''A_0 + 2g'\frac{dA_0}{dx} + g'A_0\frac{d}{dx}\right)\left(\frac{H}{2} + \frac{1}{2\kappa}\sinh\kappa H\cosh\kappa H\right) = 0. \tag{30}$$

This is a separable first order *ordinary* differential equation for A_0 in x. To solve it, begin by defining

$$S(x) = \frac{1}{2\kappa}\sinh\kappa H\cosh\kappa H. \tag{31}$$

Eqn **??** gives $g' = \pm\kappa$, so $g'' = \pm\kappa''$. The $\pm$ ambiguity can be seen to be inconsequential by plugging either signed solution into Eqn 30. Using these identities and the above-defined S, Eqn 30 becomes

$$\left(g''A_0 + 2g'\frac{dA_0}{dx} + g'A_0\frac{d}{dx}\right)\left(\frac{H}{2} + S\right) = 2\kappa\frac{dA_0}{dx}\left(\frac{H}{2} + S\right) + A_0\left(\kappa'\left(\frac{H}{2} + S\right) + \kappa S'\right) = 0. \tag{32}$$

Separation of variables gives

$$\frac{dA_0}{dx}A_0^{-1} = -\frac{\kappa'}{2\kappa} - \frac{1}{2}\frac{S'}{S + \frac{H}{2}}. \tag{33}$$

Integration of both sides gives

$$\ln A_0 = C + \frac{-1}{2}\ln\kappa + \frac{-1}{2}\ln\left(S + \frac{H}{2}\right) = C + \ln\left(\left(\kappa S + \frac{\kappa H}{2}\right)^{-1/2}\right), \tag{34}$$

where C is an arbitrary constant.

Exponentiating both sides and substituting back in for S, we get

$$A_0 = C\left(\frac{\kappa H}{2} + \frac{1}{2}\sinh\kappa H\cosh\kappa H\right)^{-1/2} = C(\kappa H + \sinh\kappa H\cosh\kappa H)^{-1/2}, \tag{35}$$

where C is an arbitrary constant that absorbs the $\sqrt{2}$ term.

2.4 Finding the Constants for Pressure

In the main text we solve for pressure in terms of arbitrary constants and must find their values using boundary conditions. Computing first the integral of pressure at $x = 0$ across z and writing $\kappa(0) = \kappa_0$,

$$\begin{aligned}\frac{1}{h}\int_0^h P(0,z)\ dz &= (\kappa_0 h + \sinh\kappa_0 h\cosh\kappa_0 h)^{-1/2}[C_+ + C_-]\int_0^h \cosh\left[\kappa_0(h-z)\right]\ dz \\ &= (\kappa h + \sinh\kappa h\cosh\kappa h)^{-1/2}[C_+ + C_-]\frac{1}{\kappa_0}\sinh\kappa_0 h = P_{OW}.\end{aligned} \tag{36}$$

This gives the relationship

$$C_+ + C_- = P_{OW}\frac{\kappa_0 h(\kappa_0 h + \sinh\kappa_0 h\cosh\kappa_0 h)^{1/2}}{\sinh\kappa_0 h}. \tag{37}$$

As for the second condition, we need the derivative of P in x. This is no easy task, unless we make the WKB assumption that $|\kappa'| \ll |\kappa|$. In taking the derivative, the three-term product rule will give factors of k' for all terms other than the derived exponential. Thus, this is the only term that matters, giving:

$$\begin{aligned}
\frac{\partial P}{\partial x} &= \frac{\partial}{\partial x}(\kappa h + \sinh \kappa h \cosh \kappa h)^{-1/2} \cosh\left[\kappa(h-z)\right]\left[C_+ e^{j\int_0^x \kappa(\xi)\, d\xi} + C_- e^{-j\int_0^x \kappa(\xi)\, d\xi}\right] \\
&= (\kappa h + \sinh \kappa h \cosh \kappa h)^{-1/2} \cosh\left[\kappa(h-z)\right]\left[j\kappa C_+ e^{j\int_0^x \kappa(\xi)\, d\xi} - j\kappa C_- e^{-j\int_0^x \kappa(\xi)\, d\xi}\right] + \\
&\quad + \frac{-1}{2}\frac{\kappa' h + \kappa' h(\sinh^2 \kappa h + \cosh^2 \kappa h)}{(\kappa h + \sinh \kappa h \cosh \kappa h)^{3/2}}\left[C_+ e^{j\int_0^x \kappa(\xi)\, d\xi} + C_- e^{-j\int_0^x \kappa(\xi)\, d\xi}\right] + \\
&\quad + (\kappa h + \sinh \kappa h \cosh \kappa h)^{-1/2} \kappa'(h-z) \sin h[\kappa(h-z)]\left[C_+ e^{j\int_0^x \kappa(\xi)\, d\xi} + C_- e^{-j\int_0^x \kappa(\xi)\, d\xi}\right] \\
&\approx (\kappa h + \sinh \kappa h \cosh \kappa h)^{-1/2} \cosh\left[\kappa(h-z)\right]\left[j\kappa C_+ e^{j\int_0^x \kappa(\xi)\, d\xi} - j\kappa C_- e^{-j\int_0^x \kappa(\xi)\, d\xi}\right].
\end{aligned} \tag{38}$$

This quantity is 0 at $x = L$, i.e.

$$C_+ = C_- e^{-2j\int_0^L \kappa(\xi)\, d\xi}. \tag{39}$$

Combining this with Eqn 37,

$$C_+ + C_- = C_-\left(1 + e^{-2j\int_0^L \kappa(\xi)\, d\xi}\right) = P_{OW} \frac{\kappa_0 h(\kappa_0 h + \sinh \kappa_0 h \cosh \kappa_0 h)^{1/2}}{\sinh \kappa_0 h}, \tag{40}$$

$$C_- = P_{OW} \frac{\kappa_0 h(\kappa_0 h + \sinh \kappa_0 h \cosh \kappa_0 h)^{1/2}}{\sinh \kappa_0 h} \frac{1}{1 + e^{-2j\int_0^L \kappa(\xi)\, d\xi}}, \tag{41}$$

$$C_+ = C_- e^{-2j\int_0^L \kappa(\xi)\, d\xi} = P_{OW} \frac{\kappa_0 h(\kappa_0 h + \sinh \kappa_0 h \cosh \kappa_0 h)^{1/2}}{\sinh \kappa_0 h} \frac{e^{-2j\int_0^L \kappa(\xi)\, d\xi}}{1 + e^{-2j\int_0^L \kappa(\xi)\, d\xi}}. \tag{42}$$

Finally, these are the constants in the pressure equation. This gives the model equation free of arbitrary parameters:

$$\begin{aligned}
P(x,z) = \frac{P_{OW} \kappa_0 h \cosh\left[\kappa(x)(h-z)\right]}{\sinh k_0 h} &\sqrt{\frac{\kappa_0 h + \sinh \kappa_0 h \cosh \kappa_0 h}{\kappa(x) h + \sinh \kappa(x) h \cosh \kappa(x) h}} \times \\
&\times \frac{e^{-j\int_0^x \kappa(\xi)\, d\xi} + e^{\int_0^x \kappa(\xi)\, d\xi - 2j\int_0^L \kappa(\xi)\, d\xi}}{1 + e^{-2j\int_0^L \kappa(\xi)\, d\xi}}
\end{aligned} \tag{43}$$

3 The Variational Method

In the main text's Appendix B, I outlined a method by which the Euler-Lagrange equations can be used to arrive at a higher-order solution for the pressure BVP. Here, I provide the extremely detailed computations involved in the derivation.

3.1 Computing the Kinetic Energy in the Fluid

To compute the Lagrangian, we must compute the kinetic energy density in the fluid. To compute the fluid kinetic energy at position x, the two-dimensional fluid velocity must be considered over the whole cross-section (z from 0 to h). There is kinetic energy in both chambers of fluid, so the result for energy in a single chamber must be multiplied by 2. This is

$$
\begin{aligned}
T_f &= = 2\frac{1}{2\pi}\int_0^{2\pi}\int_0^h \frac{1}{2}\rho(\mathcal{R}[\dot{u}]^2 + \mathcal{R}[\dot{w}]^2)\ dzd\omega t \\
&= \frac{1}{2}\int_0^h \rho\left(\frac{\omega^2 W^2}{k^2\sinh^2 kh}k^2\cosh^2[k(z-h)] + \frac{\omega^2 W^2}{k^2\sinh^2 kh}k^2\sinh^2[k(z-h)]\right)\ dz \\
&= \frac{\rho\omega^2 W^2}{2\sinh^2 kh}\int_0^h (\cosh^2[k(z-h)] + \sinh^2[k(z-h)])\ dz \\
&= \frac{\rho\omega^2 W^2}{2\sinh^2 kh}\int_0^h \cosh[2k(z-h)]\ dz \\
&= \frac{\rho\omega^2 W^2}{2\sinh^2 kh}\left[\frac{1}{2k}\sinh[2k(z-h)]\right|_0^h \\
&= \frac{\rho\omega^2 W^2}{4k\sinh^2 kh}(0 - \sinh(-2kh)) \\
&= \frac{\rho\omega^2 W^2}{4k\sinh^2 kh}2\sinh kh\cosh kh,
\end{aligned}
\tag{44}
$$

Using the definition of equivalent height, this simplifies to

$$
T_f = \frac{1}{2}h_e\rho\omega^2 W^2. \tag{45}
$$

3.2 Eliminating the Constant in the Displacement Expression

To find an expression for pressure free of arbitrary constants, we must eliminate the unknown C in the initially derived equation. I denote the average displacement at the stapes as δ_{st}. This is related to the x-direction motion by averaging u over the cross-section at $x = 0$. The x-direction motion is

$$
\begin{aligned}
u &= \int \dot{u}\ dt \\
&= \frac{1}{j\omega}\frac{\partial\phi}{\partial x} \\
&= \frac{1}{j\omega}\frac{\partial}{\partial x}\left[-\frac{j\omega W}{k\sinh kh}\cosh\left[k(z-h)\right]e^{-jkx}\right] \\
&= \frac{1}{j\omega}\left[-\frac{(j\omega W)(-jk)}{k\sinh kh}\cosh\left[k(z-h)\right]e^{-jkx}\right] \\
&= \frac{jW}{\sinh kh}\cosh\left[k(z-h)\right]e^{-jkx}.
\end{aligned}
\tag{46}
$$

where I used the formula for ϕ in Eqn **??** (with constant k). Note that the above expression relies on W, which is still parameterized by an unknown constant C.

Then δ_{st} is the average u at $x = 0$:

$$
\begin{aligned}
\delta_{st} &= \frac{1}{h}\int_0^h u(0,z)\ dz \\
&= \int_0^h \frac{jW_0}{h\sinh k_0 h}\cosh\left[k_0(z-h)\right]\ dz \\
&= \frac{jW_0}{h\sinh k_0 h}\int_0^h \cosh\left[k_0(z-h)\right]\ dz \\
&= \frac{jW_0}{h\sinh k_0 h}\left[\frac{1}{k_0}\sinh\left[k_0(z-h)\right]\right|_{z=0}^{h} \\
&= \frac{jW_0}{hk_0},
\end{aligned}
\tag{47}
$$

with the 0 subscript indicating evaluation at $x = 0$.

The way to get rid of the unknown constant C is to take the quotient between ϕ, which we are solving for, and the known stapes displacement. Using the formula for W in Eqn **??** and the formula for ϕ, we can write:

$$\begin{aligned}
\frac{\phi}{\delta_{st}} &= -\frac{j\omega \cosh\left[k(z-h)\right] W e^{-jkx}}{k \sinh kh} \frac{hk_0}{jW_0} \\
&= -\frac{\omega h k_0 \cosh\left[k(z-h)\right] e^{-jkx}}{k \sinh kh} \frac{W}{W_0} \\
&= -\frac{\omega h k_0 \cosh\left[k(z-h)\right] e^{-jkx}}{k \sinh kh} \sqrt{\frac{f_{k,0}}{f_k}} \\
&= -\frac{\omega h k_0}{k \sinh kh} \sqrt{\frac{\tanh k_0 h + k_0 h \operatorname{sech}^2 k_0 h}{k_0^2 \tanh^2 k_0 h}} \sqrt{\frac{k^2 \tanh^2 kh}{\tanh kh + kh \operatorname{sech}^2 kh}} \cosh\left[k(z-h)\right] e^{-jkx} \\
&= -\frac{\omega h k_0}{k \sinh kh} \frac{k \tanh kh}{k_0 \tanh k_0 h} \sqrt{\frac{\tanh k_0 h + k_0 h \operatorname{sech}^2 k_0 h}{\tanh kh + kh \operatorname{sech}^2 kh}} \cosh\left[k(z-h)\right] e^{-jkx} \\
&= -\frac{\omega h}{\cosh kh \tanh k_0 h} \sqrt{\frac{\tanh k_0 h + k_0 h \operatorname{sech}^2 k_0 h}{\tanh kh + kh \operatorname{sech}^2 kh}} \cosh\left[k(z-h)\right] e^{-jkx},
\end{aligned} \tag{48}$$

where $f_{k,0}$ is the k-derivative of f at $x = 0$.

3.3 Model Equations in Terms of Average Pressure

In the main text, we write most equations in terms of pressure. We wish to rewrite the found velocity potential equation in terms of pressure. The average pressure at the oval window, P_{OW}, can be found by averaging Eqn **??** over z at $x = 0$. Noting that at $x = 0$ the expression inside the square root becomes 1, we have

$$\begin{aligned}
P_{OW} &= \delta_{st} \frac{2\rho\omega^2 h}{\cosh k_0 h \tanh k_0 h} \frac{1}{h} \int_0^h \cosh\left[k_0(z-h)\right]\, dz \\
&= \delta_{st} \frac{2\rho\omega^2}{\cosh k_0 h \tanh k_0 h} \left[\frac{1}{k_0} \sinh\left[k_0(z-h)\right] \right]\Bigg|_{z=0}^{h} \\
&= \delta_{st} \frac{2\rho\omega^2 \sinh k_0 h}{k_0 \cosh k_0 h \tanh k_0 h} \\
&= \delta_{st} \frac{2\rho\omega^2}{k_0}.
\end{aligned} \tag{49}$$

Plugging in to Eqn **??**,

$$P(x,z,t) = P_{OW} \frac{k_0 h}{\cosh kh \tanh k_0 h} \sqrt{\frac{\tanh k_0 h + k_0 h \operatorname{sech}^2 k_0 h}{\tanh kh + kh \operatorname{sech}^2 kh}} \cosh\left[k(z-h)\right] e^{-jkx}. \tag{50}$$

4 The WKB Subspace

In the main text's Sec 6, I outlined a method by which pressure waveforms can be projected onto WKB basis waves. In doing so, I presented two incomplete computations. These are laid out in detail below.

4.1 Computing the Wronskian

Recall that the WKB basis functions are given by

$$W_+ = \sqrt{\frac{1}{k}} e^{-j \int_0^x k(\xi)\, d\xi}. \tag{51}$$

$$W_- = \sqrt{\frac{1}{k}} e^{j \int_0^x k(\xi)\, d\xi}, \quad x \in I. \tag{52}$$

The derivatives of these functions are

$$W_+' = -jk\sqrt{\frac{1}{k}} e^{-j \int_0^x k(\xi)\, d\xi} + \frac{-k^{-3/2}}{2} e^{-j \int_0^x k(\xi)\, d\xi}. \tag{53}$$

$$W_-' = jk\sqrt{\frac{1}{k}} e^{j \int_0^x k(\xi)\, d\xi} + \frac{-k^{-3/2}}{2} e^{j \int_0^x k(\xi)\, d\xi} \tag{54}$$

The Wronskian is defined as

$$\mathcal{D} = \mathbf{det} \begin{pmatrix} W_+ & W_- \\ W_+' & W_-' \end{pmatrix} = W_+ W_-' - W_- W_+'. \tag{55}$$

Using the equations above, I have

$$\begin{aligned}
\mathcal{D} = &\left(\sqrt{\frac{1}{k}} e^{-j \int_0^x k(\xi)\, d\xi}\right)\left(jk\sqrt{\frac{1}{k}} e^{j \int_0^x k(\xi)\, d\xi} + \frac{-k'k^{-3/2}}{2} e^{j \int_0^x k(\xi)\, d\xi}\right) \\
&- \left(\sqrt{\frac{1}{k}} e^{j \int_0^x k(\xi)\, d\xi}\right)\left(-jk\sqrt{\frac{1}{k}} e^{-j \int_0^x k(\xi)\, d\xi} + \frac{-k'k^{-3/2}}{2} e^{-j \int_0^x k(\xi)\, d\xi}\right) \\
&= j + \frac{k'}{2k^2} + j - \frac{k'}{2k^2} \\
&= 2j.
\end{aligned}$$

4.2 Computing the Projection

To compute the projection operator for the WKB basis functions, note that the system of equations in $\psi_\pm$ is

$$\begin{pmatrix} W_+ & W_- \\ W_+' & W_-' \end{pmatrix} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = \begin{pmatrix} p \\ \partial p/\partial x \end{pmatrix}. \tag{56}$$

This is solved by inverting the 2×2 matrix, involving the determinant which is, in this case, the Wronskian $\mathcal{D} = 2j$ (derived above). The solution is

$$\begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = \frac{1}{2j} \begin{pmatrix} W_-' & -W_- \\ -W_+' & W_+ \end{pmatrix} \begin{pmatrix} p \\ \partial p/\partial x \end{pmatrix}. \tag{57}$$

I can write the two equations governing this solution separately as

$$\psi_+ = \frac{1}{2j}\left(W_-' - W_- \frac{\partial}{\partial x}\right)p, \tag{58}$$

$$\psi_- = \frac{-1}{2j}\left(W_+' - W_+ \frac{\partial}{\partial x}\right)p. \tag{59}$$

Compactly, this is

$$\psi_\pm = \frac{\pm 1}{2j}\left(W_\mp' - W_\mp \frac{\partial}{\partial x}\right)p \tag{60}$$

The components of pressure are simply these values multiplied by the corresponding basis functions, or

$$\mathcal{P}_\pm[p] = p_\pm = \psi_\pm W_\pm = \frac{\pm W_\pm}{2j}\left(W_\mp' - W_\mp \frac{\partial}{\partial x}\right)p \tag{61}$$

I will handle this term by term. The terms $W_\pm W_\mp'$ appear in the computation of the Wronskian above, and are

$$W_\pm W_\mp' = \pm j + \frac{k'}{2k^2}. \tag{62}$$

The products of the basis functions are easily computed as the exponentials simply cancel and we have

$$W_\pm W_\mp = \frac{1}{k}. \tag{63}$$

Together, I get

$$\begin{aligned} p_\pm &= \frac{\pm W_\pm}{2j}\left(W_\mp' - W_\mp \frac{\partial}{\partial x}\right)p \\ &= \frac{\pm 1}{2j}\left(\pm j + \frac{k'}{2k^2} - \frac{1}{k}\frac{\partial}{\partial x}\right)p \\ &= \frac{1}{2}\left(1 \pm \frac{k'}{2jk^2} \pm \frac{j}{k}\frac{\partial}{\partial x}\right)p. \end{aligned} \tag{64}$$

5 Finding the Wavenumber Near the Point of Discontinuity

In this section I provide a derivation for the chosen wavenumber near the point at which the WKB assumption breaks down (Eqn 77 in Sec 7.3 of the main text). We know k lives in the fourth quadrant, so we write $z = kh = a - j\beta$ where $a, \beta > 0$. I begin by finding the real and imaginary parts of $\tanh(a - j\beta)$.

Beginning with the formula for hyperbolic tangent in terms of the trigonometric tangent,

$$\begin{aligned} \tanh(a - j\beta) &= j \tan j(a - j\beta) \\ &= j \frac{\sin(\beta + ja)}{\cos(\beta + ja)} \\ &= \frac{j \sin\beta \cos ja + j \cos\beta \sin ja}{\cos\beta \cos ja - \sin\beta \sin ja}. \end{aligned} \tag{65}$$

Similarly, $\cos jz = \cosh z$ and $\sin jz = j \sinh z$, so

$$\begin{aligned} \tanh(a - j\beta) &= \frac{j \sin\beta \cosh a - \cos\beta \sinh a}{\cos\beta \cosh a - j \sin\beta \sinh a} \\ &= \frac{\cos\beta \cosh a + j \sin\beta \sinh a}{\cos\beta \cosh a + j \sin\beta \sinh a} \frac{j \sin\beta \cosh a - \cos\beta \sinh a}{\cos\beta \cosh a - j \sin\beta \sinh a} \\ &= \frac{(\cos\beta \cosh a + j \sin\beta \sinh a)(j \sin\beta \cosh a - \cos\beta \sinh a)}{\cos^2\beta \cosh^2 a + \sin^2\beta \sinh^2 a} \end{aligned} \tag{66}$$

I will handle the numerator, N, and the denominator, D, separately. For the numerator,

$$\begin{aligned} N &= -(\cos^2\beta + \sin^2\beta) \cosh a \sinh a + j(\cosh^2 a - \sinh^2 a) \cos\beta \sin\beta \\ &= -\cosh a \sinh a + j \cos\beta \sin\beta \\ &= \frac{-\sinh 2a + j \sin 2\beta}{2}. \end{aligned} \tag{67}$$

On the other hand, the denominator is

$$\begin{aligned} D &= \cos^2\beta \cosh^2 a + \sin^2\beta \sinh^2 a \\ &= \frac{1 + \cos 2\beta}{2} \frac{1 + \cosh 2a}{2} + \frac{1 - \cos 2\beta}{2} \frac{\cosh 2a - 1}{2} \\ &= \frac{1 + \cos 2\beta + \cosh 2a + \cos 2\beta \cosh 2a - 1 + \cos 2\beta + \cosh 2a - \cos 2\beta \cosh 2a}{4} \\ &= \frac{\cos 2\beta + \cosh 2a}{2}. \end{aligned} \tag{68}$$

The quotient N/D split into its real and imaginary parts is thereby

$$\tanh(a - j\beta) = \frac{-\sinh 2a}{\cos 2\beta + \cosh 2a} + j \frac{\sin 2\beta}{\cos 2\beta + \cosh 2a}. \tag{69}$$

The object of interest is $z \tanh z$. The real and imaginary parts of this quantity are

$$\begin{aligned} (a - j\beta) \tanh(a - j\beta) &= (a - j\beta) \left(\frac{-\sinh 2a}{\cos 2\beta + \cosh 2a} + j \frac{\sin 2\beta}{\cos 2\beta + \cosh 2a} \right) \\ &= \frac{\beta \sin 2\beta - a \sinh 2a}{\cos 2\beta + \cosh 2a} + j \frac{a \sin 2\beta + \beta \sinh 2\alpha}{\cos 2\beta + \cosh 2a} \end{aligned} \tag{70}$$

The goal is to solve for a and β. Consider the following formula from the main text:

$$z \tanh z = \frac{-j}{\gamma}. \tag{71}$$

Equating the real and imaginary parts gives two equations:

$$a \sinh 2a - \beta \sin 2\beta = 0, \tag{72}$$

$$\frac{a \sin 2\beta + \beta \sinh 2\beta}{\cosh 2a + \cos 2\beta} = \gamma^{-1}. \tag{73}$$

We can solve these for a and β under a few assumptions. Numerical studies of the root-finding problem have shown that as x increases, the roots with the smallest magnitude negative imaginary parts tend towards $-\pi j/2$. This motivates the assumption that $0 < a \ll 1$ and $\beta = \pi/2 - \epsilon$ with $0 < \epsilon \ll 1$.

Now we solve for a and β. The following relations and the Maclaurin approximations up to second order are of use:

$$\sin(\pi - x) = \sin x, \quad \cos(\pi - x) = -\cos x \tag{74}$$

$$\sin x \approx x, \quad \cosh x \approx 1 + \frac{x^2}{2}. \tag{75}$$

The imaginary part formula (Eqn 73) gives a first approximation for a:

$$\begin{aligned} \frac{1}{\gamma} &= \frac{a \sin 2\beta + \beta \sinh 2a}{\cosh 2a + \cos 2\beta} \\ &\approx \frac{a \sin(\pi - 2\epsilon) + 2a\beta}{1 + 2a^2 + \cos(\pi - 2\epsilon)} \\ &\approx \frac{2a\epsilon + a\pi - 2a\epsilon}{1 + 2a^2 - 1} \\ &= \frac{a\pi}{2a^2} = \frac{\pi}{2a}. \end{aligned} \tag{76}$$

The final approximation is thereby

$$a = \frac{\pi}{2}\gamma. \tag{77}$$

We can plug this in to the real part equation (Eqn 72) to find a first approximation of β:

$$\begin{aligned} 0 &= a \sinh 2a - \beta \sin 2\beta \\ &\approx 2a^2 - \left(\frac{\pi}{2} - \epsilon\right) \sin \pi - 2\epsilon \\ &\approx 2a^2 - 2\epsilon\left(\frac{\pi}{2} - \epsilon\right) \\ &\approx 2a^2 - \epsilon\pi, \end{aligned} \tag{78}$$

where I have used the approximation that $\epsilon^2 \approx 0$. Plugging in the formula for a (Eqn 77),

$$\begin{aligned} \epsilon &= 2\frac{\pi^2\gamma^2}{4}\frac{1}{\pi} \\ &= \frac{\pi}{2}\gamma^2. \end{aligned} \tag{79}$$

Knowing that $\beta = \pi/2 - \epsilon$,

$$\beta = \frac{\pi}{2}(1 - \gamma^2). \tag{80}$$

Finally recalling that $z = kh = a - j\beta$, k_d at the point of discontinuity is

$$k_d \approx \frac{\pi}{2h}\gamma - j\frac{\pi}{2h}(1 - \gamma^2), \quad \gamma = \frac{R}{2\rho h\omega_d}. \tag{81}$$