Foundations of the WKB Approximation for Models of Cochlear Mechanics in 1- and 2-D: Extensive Computations

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1 Introduction

In the main text of the tutorial, results regarding WKB approximate solutions were presented along with outlines of derivations. Most notably, an instructive derivation for the higher-order 2-D WKB approximate solutions was outlined in the appendix, and a second derivation of this same formula was referenced.

The purpose of this document is to provide the extensive computations involved in these derivations, along with certain other details that were deemed too intricate and lengthy to appear in the main text. While these details may appear superfluous to some readers, they are important to present (a) for completeness in creating a self-contained exposition, and (b) for potential modification should a modeler desire to introduce additional effects to a model and still arrive at a WKB solution/WKB projections. These derivations are presented in the order of their appearance in the main text.

2 The Series Solution Method

In the main text's Appendix A, I outlined a method by which a power series approximation can be used to arrive at a higher-order solution for the pressure BVP, modified from that of Viergever [1] and inspired by Keller [2]. Here, I provide the extremely detailed computations involved in the derivation.

2.1 Deriving the BVP in A

In Appendix A of the main text, we consider a BVP for the auxiliary pressure variable Q:

$$\frac{\partial^2 Q}{\partial x^2} + K^2 \frac{\partial^2 Q}{\partial \zeta^2} = 0, \tag{1}$$

$$\left.\frac{\partial Q}{\partial \zeta}\right|_{\zeta=H} = 0, \tag{2}$$

$$\left.\frac{\partial Q}{\partial \zeta}\right|_{\zeta=0} + Hf^2(x)Q(x,0) = 0. \tag{3}$$

We also assume Q has the form

$$Q(x,\zeta) = A(x,\zeta;K)e^{jKg(x)} \cosh\left[\kappa(x)(H-\zeta)\right]. \tag{4}$$

The derivatives of Q in terms of A are

$$\frac{\partial Q}{\partial \zeta} = \frac{\partial A}{\partial \zeta} e^{jKg} \cosh a - \kappa A e^{jKg} \sinh a, \tag{5}$$

$$\frac{\partial^2 Q}{\partial \zeta^2} = \frac{\partial^2 A}{\partial \zeta^2} e^{jKg} \cosh a - \kappa \frac{\partial A}{\partial \zeta} e^{jKg} \sinh a - \kappa \frac{\partial A}{\partial \zeta} e^{jKg} \sinh a + \kappa^2 A e^{jKg} \cosh a, \tag{6}$$

$$\frac{\partial Q}{\partial x} = \frac{\partial A \cosh a}{\partial x} e^{jKg} + jKg' A e^{jKg} \cosh a, \tag{7}$$

$$\frac{\partial^2 Q}{\partial x^2} = \frac{\partial^2 A \cosh a}{\partial x^2} e^{jKg} + 2jKg' e^{jKg} \frac{\partial A \cosh a}{\partial x} - K^2 g'^2 A e^{jKg} \cosh a + jKg'' A e^{jKg} \cosh a. \tag{8}$$

The boundary conditions for A at $\zeta = H$ (where $a = 0$) are

$$\left.\frac{\partial Q}{\partial \zeta}\right|_{\zeta=H} = \left.\frac{\partial A}{\partial \zeta}\right|_{\zeta=H} e^{jKg} = 0, \tag{9}$$

$$\left.\frac{\partial A}{\partial \zeta}\right|_{\zeta=H} = 0. \tag{10}$$

The boundary condition at $\zeta = 0$ is

$$\begin{aligned}\left.\frac{\partial Q}{\partial \zeta}\right|_{\zeta=0} + Hf^2(x)Q(x,0) = \left.\frac{\partial A}{\partial \zeta}\right|_{\zeta=0} e^{jKg} \cosh a(x,0) - \kappa A e^{jKg} \sinh a(x,0) + \\ + Hf^2 A(x,0) e^{jKg} \cosh a(x,0) = 0,\end{aligned} \tag{11}$$

which simplifies to

$$\left.\frac{\partial A}{\partial \zeta}\right|_{\zeta=0} - \kappa A(x,0)\tanh a(x,0) + Hf^2 A(x,0) = 0. \tag{12}$$

Finally,the PDE in A is

$$\begin{aligned}\frac{\partial^2 Q}{\partial x^2} + K^2\frac{\partial^2 Q}{\partial \zeta^2} = \frac{\partial^2 A\cosh a}{\partial x^2}e^{jKg} + 2jKg' e^{jKg}\frac{\partial A\cosh a}{\partial x} - K^2 g'^2 A e^{jKg}\cosh a +\\ + jKg'' A e^{jKg}\cosh a + K^2\frac{\partial^2 A}{\partial \zeta^2}e^{jKg}\cosh a - K^2\kappa\frac{\partial A}{\partial \zeta}e^{jKg}\sinh a +\\ + K^2\kappa^2 A e^{jKg}\cosh a - K^2\kappa\frac{\partial A}{\partial \zeta}e^{jKg}\sinh a = 0.\end{aligned} \tag{13}$$

After some simplification, this is

$$\begin{aligned}K^2\left[(\kappa^2 - g'^2)A\cosh a + \frac{\partial^2 A}{\partial \zeta^2}\cosh a - 2\kappa\frac{\partial A}{\partial \zeta}\sinh a\right] +\\ + jK\left[g'' A\cosh a + 2g'\frac{\partial A\cosh a}{\partial x}\right] + \frac{\partial^2 A\cosh a}{\partial x^2} = 0.\end{aligned} \tag{14}$$

2.2 Creating the system of PDEs in A

In Appendix A of the main text, with the BVP in A determined, we consider a series ansatz

$$A(x,\zeta;K) = A_0(x) + \sum_{n=1}^{\infty}\frac{1}{(jK)^n}A_n(x,\zeta) \tag{15}$$

and use it to find an approximate solution for the BVP.

Plugging the series ansatz in to the PDE in A (Eqn 13) gives

$$\begin{aligned}&\left[(\kappa^2 - g'^2)\cosh a\left(K^2 A_0 - jKA_1 - \sum_{n=2}^{\infty}\frac{1}{(jK)^{n-2}}A_n\right) +\right.\\ &+\cosh a\left(-jK\frac{\partial^2 A_1}{\partial \zeta^2} - \sum_{n=2}^{\infty}\frac{1}{(jK)^{n-2}}\frac{\partial^2 A_n}{\partial \zeta^2}\right) +\\ &\left.-2\kappa\sinh a\left(-jK\frac{\partial A_1}{\partial \zeta} - \sum_{n=2}^{\infty}\frac{1}{(jK)^{n-2}}\frac{\partial A_n}{\partial \zeta}\right)\right] +\\ &+\left[g''\cosh a\left(jKA_0 + \sum_{n=1}^{\infty}\frac{1}{(jK)^{n-1}}A_n\right) + 2g'\sinh a\left(jKA_0 + \sum_{n=1}^{\infty}\frac{1}{(jK)^{n-1}}A_n\right) +\right.\\ &\left.+2g'\cosh a\left(jK\frac{dA_0}{dx} + \sum_{n=1}^{\infty}\frac{1}{(jK)^{n-1}}\frac{\partial A_n}{\partial x}\right)\right] + \left[\cosh a\left(A_0 + \sum_{n=1}^{\infty}\frac{1}{(jK)^n}A_n\right) +\right.\\ &\left.+2\sinh a\left(\frac{dA_0}{dx} + \sum_{n=1}^{\infty}\frac{1}{(jK)^n}\frac{\partial A_n}{\partial x}\right) + \cosh a\left(\frac{d^2 A_0}{dx^2} + \sum_{n=1}^{\infty}\frac{1}{(jK)^n}\frac{\partial^2 A_n}{\partial x^2}\right)\right] = 0.\end{aligned} \tag{16}$$

We have to separate this leviathan into its components in powers of jK. The single term in K^2 gives

$$(\kappa^2 - g'^2)\cosh a A_0 = 0, \tag{17}$$

or

$$g'^2(x) = \kappa^2(x). \tag{18}$$

Next, consider the jK terms:

$$-(\kappa^2 - g'^2)A_0 - \cosh a \frac{\partial^2 A_1}{\partial \zeta^2} + 2\kappa \sinh a \frac{\partial A_1}{\partial \zeta} + g'' A_0 \cosh a + 2g' \sinh a A_0 + 2g' \cosh a \frac{dA_0}{dx} = 0. \tag{19}$$

With the aid of Eqn 18, we can see the first of these summands is 0. This simplifies the expression to

$$\begin{aligned} \cosh a \frac{\partial^2 A_1}{\partial \zeta^2} - 2\kappa \sinh a \frac{\partial A_1}{\partial \zeta} &= g'' A_0 \cosh a + 2g' \sinh a A_0 + 2g' \cosh a \frac{dA_0}{dx} \\ &= g'' A_0 \cosh a + 2g' \frac{\partial A_0 \cosh a}{\partial x}. \end{aligned} \tag{20}$$

As for the rest of the terms, we have a system of infinitely many equations. Again, by Eqn 18,the terms leading with $\kappa^2 - g'^2$ are 0. The relation is

$$\begin{aligned} \cosh a \frac{\partial^2 A_n}{\partial \zeta^2} - 2\kappa \sinh a \frac{\partial A_n}{\partial \zeta} &= (g'' \cosh a + 2g' \sinh a)A_{n-1} + 2g' \cosh a \frac{\partial A_{n-1}}{\partial x} + \\ &\quad + \cosh a \frac{\partial A_{n-2}}{\partial x} + 2 \sinh a \frac{\partial A_{n-2}}{\partial x} + \cosh a \frac{\partial^2 A_{n-2}}{\partial x^2} \\ &= g'' \cosh a A_{n-1} + 2g' \frac{\partial A_{n-1} \cosh a}{\partial x} + \frac{\partial^2 A_{n-2} \cosh a}{\partial x^2}, \qquad n \geq 2. \end{aligned} \tag{21}$$

In theory, this will allow us to solve for any order of approximation desired so long as we can solve for A_0. In practice, we will only solve for A_0 to make a first approximation.

Now consider boundary conditions. At $\zeta = H$, all terms must have a zero derivative (trivial for A_0 which has no ζ dependence). That is,

$$\left.\frac{\partial A_n}{\partial \zeta}\right|_{\zeta=H} = 0, \quad n \geq 1. \tag{22}$$

As for $\zeta = 0$, plugging in the series ansatz to the boundary condition gives

$$\big(Hf^2 - \kappa \tanh a(x,0)\big)A_0 + \sum_{n=1}^{\infty} \frac{1}{(jK)^n} \left(\frac{\partial A_n}{\partial \zeta} + (Hf^2 - \kappa \tanh a)A_n \right|_{\zeta=0} = 0 \tag{23}$$

at $\zeta = 0$. Recalling that $a(x,0) = \kappa H$, the A_0 term gives

$$\kappa \tanh \kappa H = Hf^2. \tag{24}$$

As for the other terms, the above relationship simplifies the summands to

$$\left.\frac{\partial A_n}{\partial \zeta}\right|_{\zeta=0} = 0, \quad n \geq 1. \tag{25}$$

2.3 Solving the PDE for A_0

We now look to solve the BVP up to order 0. Recalling that $a(x,\zeta) = \kappa(x)(H-\zeta)$, we can note a relation that helps to simplify the left-hand side of the equation:

$$\begin{aligned}
\operatorname{sech} a \frac{\partial}{\partial \zeta}\left[\cosh^2 a \frac{\partial A_1}{\partial \zeta}\right] &= \operatorname{sech} a \cosh^2 a \frac{\partial^2 A_1}{\partial \zeta^2} + \operatorname{sech} a \frac{\partial \cosh^2 a}{\partial \zeta}\frac{\partial A_1}{\partial \zeta} \\
&= \cosh a \frac{\partial^2 A_1}{\partial \zeta^2} + \operatorname{sech} a (2\cosh a \sinh a)\frac{\partial a}{\partial \zeta}\frac{\partial A_1}{\partial \zeta} \\
&= \cosh a \frac{\partial^2 A_1}{\partial \zeta^2} - 2\kappa \sinh a \frac{\partial A_1}{\partial \zeta}.
\end{aligned} \tag{26}$$

This is precisely the left-hand side of Eqn 20. That is,

$$\operatorname{sech} a \frac{\partial}{\partial \zeta}\left[\cosh^2 a \frac{\partial A_1}{\partial \zeta}\right] = g'' A_0 \cosh a + 2g' \frac{\partial A_0 \cosh a}{\partial x}. \tag{27}$$

Multiplying by $\cosh a$ on both sides of the equation and integrating from 0 to ζ,

$$\begin{aligned}
&\int_0^\zeta \frac{\partial}{\partial \xi}\left[\cosh^2 a \frac{\partial A_1}{\partial \xi}\right] d\xi = \cosh^2 a \frac{\partial A_1}{\partial \zeta} \\
&\quad = \int_0^\zeta g'' A_0 \cosh^2 a + 2g' \cosh a \frac{\partial A_0 \cosh a}{\partial x} \; d\xi \\
&\quad = g'' A_0 \int_0^\zeta \cosh^2 a \; d\xi + 2g' \frac{dA_0}{dx}\int_0^\zeta \cosh^2 a \; d\xi + g' A_0 \int_0^\zeta 2\cosh a \frac{\partial}{\partial x}\cosh a \; d\xi \\
&\quad = \left(g'' A_0 + 2g' \frac{dA_0}{dx} + g' A_0 \frac{\partial}{\partial x}\right)\int_0^\zeta \cosh^2 a \; d\xi,
\end{aligned} \tag{28}$$

where in the last step I have recognized the term in the integral as the derivative of $\cosh^2 a$.

The only integral to compute now is that of the squared hyperbolic cosine. Using the double-argument formula for cosh,

$$\begin{aligned}
\int_0^\zeta \cosh^2 a \; d\xi &= \frac{1}{2}\int_0^\zeta \cosh\left[2\kappa(x)(H-\xi)\right] + 1 \; d\xi \\
&= \left[\frac{-1}{4\kappa}\sinh\left[2\kappa(x)(H-\xi)\right] + \frac{\xi}{2}\right|_{\xi=0}^{\zeta} \\
&= \frac{\zeta}{2} - \frac{1}{2\kappa}\left[\sinh\left[\kappa(x)(H-\xi)\right]\cosh\left[\kappa(x)(H-\xi)\right]\right|_{\xi=0}^{\zeta} \\
&= \frac{\zeta}{2} - \frac{1}{2\kappa}\left[\sinh a \cosh a - \sinh \kappa H \cosh \kappa H\right].
\end{aligned} \tag{29}$$

Plugging this into Eqn 28 yields a simplified relationship between A_0 and A_1 with a useful property – plugging in $\zeta = H$ facilitates solution for only terms involving $A_0(x)$. Because A_0 is ζ-independent, this gives full knowledge of A_0.

The boundary condition in Eqn 22 gives that the left-hand side of Eqn 28 is 0. Plugging H into Eqn 28 (using the integral identity in Eqn 29),

$$\left(g'' A_0 + 2g' \frac{dA_0}{dx} + g' A_0 \frac{d}{dx}\right)\left(\frac{H}{2} + \frac{1}{2\kappa} \sinh \kappa H \cosh \kappa H\right) = 0. \tag{30}$$

This is a separable first order *ordinary* differential equation for A_0 in x. To solve it, begin by defining

$$S(x) = \frac{1}{2\kappa} \sinh \kappa H \cosh \kappa H. \tag{31}$$

Eqn 18 gives $g' = \pm\kappa$, so $g'' = \pm\kappa''$. The $\pm$ ambiguity can be seen to be inconsequential by plugging either signed solution into Eqn 30. Using these identities and the above-defined S, Eqn 30 becomes

$$\left(g'' A_0 + 2g' \frac{dA_0}{dx} + g' A_0 \frac{d}{dx}\right)\left(\frac{H}{2} + S\right) = 2\kappa \frac{dA_0}{dx}\left(\frac{H}{2} + S\right) + A_0\left(\kappa'\left(\frac{H}{2} + S\right) + \kappa S'\right) = 0. \tag{32}$$

Separation of variables gives

$$\frac{dA_0}{dx} A_0^{-1} = -\frac{\kappa'}{2\kappa} - \frac{1}{2} \frac{S'}{S + \frac{H}{2}}. \tag{33}$$

Integration of both sides gives

$$\ln A_0 = C + \frac{-1}{2} \ln \kappa + \frac{-1}{2} \ln\left(S + \frac{H}{2}\right) = C + \ln\left(\left(\kappa S + \frac{\kappa H}{2}\right)^{-1/2}\right), \tag{34}$$

where C is an arbitrary constant.

Exponentiating both sides and substituting back in for S, we get

$$A_0 = C\left(\frac{\kappa H}{2} + \frac{1}{2} \sinh \kappa H \cosh \kappa H\right)^{-1/2} = C(\kappa H + \sinh \kappa H \cosh \kappa H)^{-1/2}, \tag{35}$$

where C is an arbitrary constant that absorbs the $\sqrt{2}$ term.

2.4 Finding the Constants for Pressure

In the main text we solve for pressure in terms of arbitrary constants and must find their values using boundary conditions. Computing first the integral of pressure at $x = 0$ across z and writing $k(0) = k_0$,

$$\begin{aligned} \frac{1}{h} \int_0^h P(0, z)\ dz &= (k_0 h + \sinh k_0 h \cosh k_0 h)^{-1/2} [C_+ + C_-] \int_0^h \cosh\left[k_0(h - z)\right]\ dz \\ &= (kh + \sinh kh \cosh kh)^{-1/2} [C_+ + C_-] \frac{1}{k_0} \sinh k_0 h = P_{OW}. \end{aligned} \tag{36}$$

This gives the relationship

$$C_+ + C_- = P_{OW} \frac{k_0 h (k_0 h + \sinh k_0 h \cosh k_0 h)^{1/2}}{\sinh \kappa_0 h}. \tag{37}$$

As for the second condition, we need the derivative of P in x. This is no easy task, unless we make the WKB assumption that $|k'| \ll |k|^2$. In taking the derivative, the three-term product rule will give factors of

k' for all terms saved the derived exponential, which will have a factor of k. This is the only term that I keep, giving:

$$\begin{aligned}\frac{\partial P}{\partial x} &= \frac{\partial}{\partial x}(kh + \sinh kh \cosh kh)^{-1/2} \cosh\left[k(h-z)\right]\left[C_+ e^{j\int_0^x k(\xi)\,d\xi} + C_- e^{-j\int_0^x k(\xi)\,d\xi}\right] \\ &\approx (kh + \sinh kh \cosh kh)^{-1/2} \cosh\left[k(h-z)\right]\left[jkC_+ e^{j\int_0^x k(\xi)\,d\xi} - jkC_- e^{-j\int_0^x k(\xi)\,d\xi}\right].\end{aligned} \tag{38}$$

This quantity is 0 at $x = L$, i.e.

$$C_+ = C_- e^{-2j\int_0^L k(\xi)\,d\xi}. \tag{39}$$

Combining this with Eqn 37,

$$C_+ + C_- = C_-\left(1 + e^{-2j\int_0^L k(\xi)\,d\xi}\right) = P_{OW}\frac{k_0h(k_0h + \sinh k_0h \cosh k_0h)^{1/2}}{\sinh k_0h}, \tag{40}$$

$$C_- = P_{OW}\frac{k_0h(k_0h + \sinh k_0h \cosh k_0h)^{1/2}}{\sinh k_0h}\frac{1}{1 + e^{-2j\int_0^L k(\xi)\,d\xi}}, \tag{41}$$

$$C_+ = C_- e^{-2j\int_0^L k(\xi)\,d\xi} = P_{OW}\frac{k_0h(k_0h + \sinh k_0h \cosh k_0h)^{1/2}}{\sinh k_0h}\frac{e^{-2j\int_0^L k(\xi)\,d\xi}}{1 + e^{-2j\int_0^L k(\xi)\,d\xi}}. \tag{42}$$

Finally, these are the constants in the pressure equation. This gives the model equation free of arbitrary parameters:

$$\begin{aligned}P(x,z) = \frac{P_{OW}k_0h\cosh\left[k(x)(h-z)\right]}{\sinh k_0h}&\sqrt{\frac{k_0h + \sinh k_0h \cosh k_0h}{k(x)h + \sinh k(x)h \cosh k(x)h}}\times \\ &\times\frac{e^{-j\int_0^x k(\xi)\,d\xi} + e^{j\int_0^x k(\xi)\,d\xi - 2j\int_0^L k(\xi)\,d\xi}}{1 + e^{-2j\int_0^L k(\xi)\,d\xi}}\end{aligned} \tag{43}$$

3 The Variational Method

In the main text I referenced a method by which the Euler-Lagrange equations can be used to arrive at a higher-order solution for the pressure BVP. Here, I provide the extremely detailed computations involved in the derivation modified from that of Steele and Taber [3].

The method is inspired by Whitman's treatment of waves in fluids [4], and consists of the following steps:

1. Solve the BVP in velocity potential under the assumption that all parameters are constant.
2. Find the Lagrangian of this system.
3. Write the Euler-Lagrange equations for the system in its parameters previously assumed to be constant.
4. Solve these ODEs to find a non-constant form of these parameters, yielding a higher-order approximation.

3.1 Solving the Laplace Equation with Constant Parameters

Step (1) follows from the method of separation of variables. This time, however, the BVP will be solved in terms of velocity potential which also satisfies the Laplace equation. We begin by assuming that the displacement at the basilar membrane is a wave *with only one mode*, and travels only in the base-to-apex direction. That is, transverse displacement w at $z = 0$ is

$$w(x, 0) = We^{-jkx}, \quad \dot{w}(x, 0) = j\omega We^{-jkx} \tag{44}$$

where W and k are assumed constant for now. Looking ahead, these will be the variables of our Euler-Lagrange equations in step (3).

A solution for pressure was already shown in Sec V of the main text, and with constant k (and ignoring the basal-traveling wave) we arrive again at:

$$\phi(x, z) = A \cosh\left[k(z - h)\right] e^{-jkx}$$

The value of (constant) A can be found via application of the boundary condition at $z = 0$ using Eqn 44,

$$\left.\frac{\partial \phi}{\partial z}\right|_{z=0} = j\omega We^{-jkx}. \tag{45}$$

Plugging into the formula for velocity potential gives

$$-Ak \sinh kh = j\omega W,$$

giving

$$\phi(x, z) = \frac{-j\omega W}{k \sinh kh} \cosh\left[k(z - h)\right] e^{-jk}. \tag{46}$$

As little has been done differently from the derivation in Sec V of the main text thus far, it is easily shown that the same equations hold for the dispersion relation, effective height and pressure focusing (See Sec VB of the main text). That is, the following equations from the main text still hold in this derivation:

$$k \tanh kh = -2j\omega\rho Y_{OC}. \tag{47}$$

$$h_e(k) = \frac{1}{k \tanh kh}, \quad Z_{OC} = -2j\omega\rho h_e, \tag{48}$$

$$\alpha(x) = \frac{k(x)h(x)}{\tanh\left[k(x)h(x)\right]}. \tag{49}$$

3.2 Computing the Lagrangian with Constant Parameters

Proceeding with step (2), we look to find the Lagrangian of our system. The system's (time-averaged) energy can be separated into three components – the potential energy in the OCC V, the kinetic energy in the OCC T_{OC} and the kinetic energy in the fluid T_f. We write the impedance at the OCC as a standard linear point-impedance, having a mass M, stiffness S and resistance R:

$$Z_{OC} = -\frac{p(x,0)}{\dot{w}(x,0)} = j\omega M + R + \frac{S}{j\omega}.$$

In general, all of these quantities may be x-dependent. Lagrangian mechanics assumes a lossless system [5], and thus the impact of resistance will be ignored and simply "added in" at the end of the derivation. As such, we will define an undamped OCC impedance by $Z_u = j\omega M + \frac{S}{j\omega}$

To compute the Lagrangian, we must compute the kinetic energy density in the fluid. At position x, the two-dimensional fluid velocity must be considered over the whole cross-section (z from 0 to h). There is kinetic energy in both chambers of fluid, so the result for energy in a single chamber must be multiplied by 2. Denoting the real part operator as $\mathcal{R}$, this is

$$\begin{aligned}
T_f &= = 2\frac{1}{2\pi}\int_0^{2\pi}\int_0^h \frac{1}{2}\rho(\mathcal{R}[\dot{u}]^2 + \mathcal{R}[\dot{w}]^2)\ dz d\omega t \\
&= \frac{1}{2}\int_0^h \rho\left(\frac{\omega^2 W^2}{k^2\sinh^2 kh}k^2\cosh^2[k(z-h)] + \frac{\omega^2 W^2}{k^2\sinh^2 kh}k^2\sinh^2[k(z-h)]\right) dz \\
&= \frac{\rho\omega^2 W^2}{2\sinh^2 kh}\int_0^h (\cosh^2\left[k(z-h)\right] + \sinh^2\left[k(z-h)\right])\ dz \\
&= \frac{\rho\omega^2 W^2}{2\sinh^2 kh}\int_0^h \cosh[2k(z-h)]\ dz \\
&= \frac{\rho\omega^2 W^2}{2\sinh^2 kh}\left[\frac{1}{2k}\sinh[2k(z-h)]\right|_0^h \\
&= \frac{\rho\omega^2 W^2}{4k\sinh^2 kh}(0 - \sinh(-2kh)) \\
&= \frac{\rho\omega^2 W^2}{4k\sinh^2 kh}2\sinh kh\cosh kh,
\end{aligned} \tag{50}$$

where I have used

$$\int_0^{2\pi}\cos^2\omega t\ d\omega t = \int_0^{2\pi}\sin^2\omega t\ d\omega t = \pi.$$

Using the definition of equivalent height, this simplifies to

$$T_f = \frac{1}{2}h_e\rho\omega^2 W^2. \tag{51}$$

The potential energy density is given by

$$\begin{aligned} V &= \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2}S\mathcal{R}[w(x,0,t)]^2 d\omega t \\ &= \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2}SW^2\cos^2\omega t d\omega t \\ &= \frac{SW^2}{4}. \end{aligned} \tag{52}$$

The kinetic energy of the OCC is computed as

$$\begin{aligned} T_{OC} &= \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2}M\mathcal{R}[\dot{w}(x,0,t)]^2 d\omega t \\ &= \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2}M\omega^2W^2\sin^2\omega t d\omega t \\ &= \frac{M\omega^2W^2}{4}. \end{aligned} \tag{53}$$

The total kinetic energy density is the sum of the fluid and membrane contributions. This lets us write the Lagrangian:

$$\mathcal{L}(k,W) = T - V = \frac{f(k)W^2}{4}, \tag{54}$$

where

$$f(k) = 2h_e(k)\rho\omega + M\omega^2 - S. \tag{55}$$

The last two summands in the above equation resemble the impedance Z_u, allowing us to write the function f above as

$$f(k) = 2\rho\omega^2 h_e(k) - j\omega Z_u. \tag{56}$$

Guided by the the WKB approximation, we now replace the phase term by defining

$$\theta = \int_0^x k(\xi)d\xi. \tag{57}$$

This phase term *would have* appeared if we had performed a first-order WKB approximation in x (see also [6, 7]). Under this definition, $\theta' = k$ so we can rewrite the Lagrangian as

$$\mathcal{L}(\theta,W) = \frac{f(\theta')W^2}{4}. \tag{58}$$

3.3 The Euler-Lagrange Equations

The Euler-Lagrange equations, derived from Hamilton's principle, are PDEs that relate the Lagrangian to its parameters [5]. For any parameter of the Lagrangian ψ, the corresponding Euler-Lagrange equation is

$$\frac{\partial\mathcal{L}}{\partial\psi} - \frac{d}{dx}\frac{\partial\mathcal{L}}{\partial\psi'} = 0. \tag{59}$$

In the present case, the Lagrangian parameters are the phase (θ) and amplitude (W) of the transverse displacement at the OCC (see Eqn 58). Beginning with W, we can quickly see that the Lagrangian has explicit dependence on W, but no explicit dependence on W'. Thereby, the Euler-Lagrange equation simplifies to

$$\frac{\partial \mathcal{L}}{\partial W} = 0, \tag{60}$$

Solution of this equation simply yields the dispersion relation of Eqn 47, adding no new information to the problem.

As for θ, we see that $\mathcal{L}$ has explicit dependence on θ' but no explicit dependence on θ, so

$$\frac{\partial \mathcal{L}}{\partial \theta} = 0. \tag{61}$$

The Euler-Lagrange equation thereby simplifies to

$$\frac{d}{dx}\frac{\partial \mathcal{L}}{\partial \theta'} = \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial k} = 0, \tag{62}$$

as $\theta' = k$.

Next is the Euler-Lagrange equation in k, Eqn 62, which when combined with Eqn 58 yields:

$$\frac{d}{dx}\frac{\partial \mathcal{L}}{\partial k} = \frac{W^2}{4}\frac{d}{dx}\frac{\partial f}{\partial k} \tag{63}$$

$$= \frac{d}{dx}\frac{W^2}{4} f_k \tag{64}$$

$$= 0. \tag{65}$$

Integrating both sides gives

$$W = C f_k^{-1/2} \tag{66}$$

where C is some arbitrary constant. This equation, which gives the displacement amplitude as a function of x, is called the transport equation. The derivative of f with respect to k, f_k is

$$f_k = -2\rho\omega^2 \frac{\tanh kh + kh\,\mathrm{sech}^2 kh}{k^2 \tanh^2 kh}. \tag{67}$$

3.4 Eliminating the Constant and Solving for Velocity Potential

To find an expression for pressure free of arbitrary constants, we must eliminate the unknown C in Eqn 66. I denote the average displacement at the stapes as δ_{st}. This is related to the x-direction motion by averaging u

over the cross-section at $x = 0$. Plugging W into Eqn 46, the x-direction motion is

$$
\begin{aligned}
u &= \int \dot{u}\ dt \\
&= \frac{1}{j\omega}\frac{\partial \phi}{\partial x} \\
&= \frac{1}{j\omega}\frac{\partial}{\partial x}\left[-\frac{j\omega W}{k\sinh kh}\cosh\left[k(z-h)\right]e^{-jkx}\right] \\
&= \frac{1}{j\omega}\left[-\frac{(j\omega W)(-jk)}{k\sinh kh}\cosh\left[k(z-h)\right]e^{-jkx}\right] \\
&= \frac{jW}{\sinh kh}\cosh\left[k(z-h)\right]e^{-jkx}.
\end{aligned}
\tag{68}
$$

where I used the formula for ϕ in Eqn 46 (with constant k). Note that the above expression relies on W, which is still parameterized by an unknown constant C.

Then δ_{st} is the average u at $x = 0$:

$$
\begin{aligned}
\delta_{st} &= \frac{1}{h}\int_0^h u(0,z)\ dz \\
&= \int_0^h \frac{jW_0}{h\sinh k_0 h}\cosh\left[k_0(z-h)\right]\ dz \\
&= \frac{jW_0}{h\sinh k_0 h}\int_0^h \cosh\left[k_0(z-h)\right]\ dz \\
&= \frac{jW_0}{h\sinh k_0 h}\left[\frac{1}{k_0}\sinh\left[k_0(z-h)\right]\right|_{z=0}^{h} \\
&= \frac{jW_0}{hk_0},
\end{aligned}
\tag{69}
$$

with the 0 subscript indicating evaluation at $x = 0$.

To eliminate the unknown constant C, to take the quotient between ϕ, which we are solving for, and the known stapes displacement. Using the formula for W in Eqn 66 and the formula for ϕ, we can write:

$$
\begin{aligned}
\frac{\phi}{\delta_{st}} &= -\frac{j\omega \cosh\left[k(z-h)\right] W e^{-jkx}}{k \sinh kh} \frac{hk_0}{jW_0} \\
&= -\frac{\omega h k_0 \cosh\left[k(z-h)\right] e^{-jkx}}{k \sinh kh} \frac{W}{W_0} \\
&= -\frac{\omega h k_0 \cosh\left[k(z-h)\right] e^{-jkx}}{k \sinh kh} \sqrt{\frac{f_{k,0}}{f_k}} \\
&= -\frac{\omega h k_0}{k \sinh kh} \sqrt{\frac{\tanh k_0 h + k_0 h \operatorname{sech}^2 k_0 h}{k_0^2 \tanh^2 k_0 h}} \sqrt{\frac{k^2 \tanh^2 kh}{\tanh kh + kh \operatorname{sech}^2 kh}} \cosh\left[k(z-h)\right] e^{-jkx} \\
&= -\frac{\omega h k_0}{k \sinh kh} \frac{k \tanh kh}{k_0 \tanh k_0 h} \sqrt{\frac{\tanh k_0 h + k_0 h \operatorname{sech}^2 k_0 h}{\tanh kh + kh \operatorname{sech}^2 kh}} \cosh\left[k(z-h)\right] e^{-jkx} \\
&= -\frac{\omega h}{\cosh kh \tanh k_0 h} \sqrt{\frac{\tanh k_0 h + k_0 h \operatorname{sech}^2 k_0 h}{\tanh kh + kh \operatorname{sech}^2 kh}} \cosh\left[k(z-h)\right] e^{-jkx},
\end{aligned}
\tag{70}
$$

where $f_{k,0}$ is the k-derivative of f at $x = 0$.

Multiplying by stapes displacement, we have

$$
\phi = -\delta_{st} \frac{\omega h}{\cosh kh \tanh k_0 h} \sqrt{\frac{\tanh k_0 h + k_0 h \operatorname{sech}^2 k_0 h}{\tanh kh + kh \operatorname{sech}^2 kh}} \cosh\left[k(z-h)\right] e^{-j\theta}. \tag{71}
$$

Using the relationship between displacement and pressure, the same equation can be written for pressure in terms of average pressure at the oval window P_{OW} to arrive at

$$
p(x,z) = P_{OW} \frac{k_0 h}{\cosh kh \tanh k_0 h} \sqrt{\frac{\tanh k_0 h + k_0 h \operatorname{sech}^2 k_0 h}{\tanh kh + kh \operatorname{sech}^2 kh}} \cosh\left[k(z-h)\right] e^{-j\int_0^x k(\xi)d\xi}, \tag{72}
$$

precisely the “higher order WKB solution.”

4 The WKB Subspace

In the main text's Sec 6, I outlined a method by which pressure waveforms can be projected onto WKB basis waves modified from Shera and Zweig [8]. In doing so, I presented two incomplete computations. These are laid out in detail below.

4.1 Computing the Wronskian

Recall that the WKB basis functions are given by

$$W_+ = \sqrt{\frac{1}{k}} e^{-j\int_0^x k(\xi)\, d\xi}. \tag{73}$$

$$W_- = \sqrt{\frac{1}{k}} e^{j\int_0^x k(\xi)\, d\xi}, \quad x \in I. \tag{74}$$

The derivatives of these functions are

$$W_+' = -jk\sqrt{\frac{1}{k}} e^{-j\int_0^x k(\xi)\, d\xi} + \frac{-k' k^{-3/2}}{2} e^{-j\int_0^x k(\xi)\, d\xi}. \tag{75}$$

$$W_-' = jk\sqrt{\frac{1}{k}} e^{j\int_0^x k(\xi)\, d\xi} + \frac{-k' k^{-3/2}}{2} e^{j\int_0^x k(\xi)\, d\xi} \tag{76}$$

The Wronskian is defined as

$$\mathcal{D} = \mathbf{det}\begin{pmatrix} W_+ & W_- \\ W_+' & W_-' \end{pmatrix} = W_+ W_-' - W_- W_+'. \tag{77}$$

Using the equations above, I have

$$\begin{aligned}
\mathcal{D} &= \left(\sqrt{\frac{1}{k}} e^{-j\int_0^x k(\xi)\, d\xi}\right)\left(jk\sqrt{\frac{1}{k}} e^{j\int_0^x k(\xi)\, d\xi} + \frac{-k' k^{-3/2}}{2} e^{j\int_0^x k(\xi)\, d\xi}\right) \\
&- \left(\sqrt{\frac{1}{k}} e^{j\int_0^x k(\xi)\, d\xi}\right)\left(-jk\sqrt{\frac{1}{k}} e^{-j\int_0^x k(\xi)\, d\xi} + \frac{-k' k^{-3/2}}{2} e^{-j\int_0^x k(\xi)\, d\xi}\right) \\
&= j - \frac{k'}{2k^2} + j + \frac{k'}{2k^2} \\
&= 2j.
\end{aligned}$$

4.2 Computing the Projection

To compute the projection operator for the WKB basis functions, note that the system of equations in $\psi_\pm$ is

$$\begin{pmatrix} W_+ & W_- \\ W_+' & W_-' \end{pmatrix}\begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = \begin{pmatrix} p \\ \partial p/\partial x \end{pmatrix}. \tag{78}$$

This is solved by inverting the 2×2 matrix, involving the determinant which is, in this case, the Wronskian $\mathcal{D} = 2j$ (derived above). The solution is

$$\begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = \frac{1}{2j}\begin{pmatrix} W_-' & -W_- \\ -W_+' & W_+ \end{pmatrix}\begin{pmatrix} p \\ \partial p/\partial x \end{pmatrix}. \tag{79}$$

I can write the two equations governing this solution separately as

$$\psi_+ = \frac{1}{2j}\left(W'_- - W_- \frac{\partial}{\partial x}\right)p, \tag{80}$$

$$\psi_- = \frac{-1}{2j}\left(W'_+ - W_+ \frac{\partial}{\partial x}\right)p. \tag{81}$$

Compactly, this is

$$\psi_\pm = \frac{\pm 1}{2j}\left(W'_\mp - W_\mp \frac{\partial}{\partial x}\right)p \tag{82}$$

The components of pressure are simply these values multiplied by the corresponding basis functions, or

$$\mathcal{P}_\pm[p] = p_\pm = \psi_\pm W_\pm = \frac{\pm W_\pm}{2j}\left(W'_\mp - W_\mp \frac{\partial}{\partial x}\right)p \tag{83}$$

I will handle this term by term. The terms $W_\pm W'_\mp$ appear in the computation of the Wronskian above, and are

$$W_\pm W'_\mp = \pm j - \frac{k'}{2k^2}. \tag{84}$$

The product of the basis functions are easily computed as the exponentials simply cancel and we have

$$W_\pm W_\mp = \frac{1}{k}. \tag{85}$$

Together, I get

$$\begin{aligned} p_\pm &= \frac{\pm W_\pm}{2j}\left(W'_\mp - W_\mp \frac{\partial}{\partial x}\right)p \\ &= \frac{\pm 1}{2j}\left(\pm j - \frac{k'}{2k^2} - \frac{1}{k}\frac{\partial}{\partial x}\right)p \\ &= \frac{1}{2}\left(1 \pm \frac{jk'}{2k^2} \pm \frac{j}{k}\frac{\partial}{\partial x}\right)p. \end{aligned} \tag{86}$$

5 Finding the Wavenumber Near the Point of Discontinuity

In this section I provide a derivation for the chosen wavenumber near the point at which the WKB assumption breaks down (Eqn 77 in Sec 7.3 of the main text). We know k lives in the fourth quadrant, so we write $z = kh = a - j\beta$ where $a, \beta > 0$. I begin by finding the real and imaginary parts of $\tanh{(a - j\beta)}$.

Beginning with the formula for hyperbolic tangent in terms of the trigonometric tangent,

$$\begin{aligned}\tanh{(a - j\beta)} &= -j \tan j(a - j\beta) \\ &= -j \frac{\sin{(\beta + ja)}}{\cos{(\beta + ja)}} \\ &= -\frac{j \sin\beta \cos ja + j \cos\beta \sin ja}{\cos\beta \cos ja - \sin\beta \sin ja}.\end{aligned} \tag{87}$$

Similarly, $\cos jz = \cosh z$ and $\sin jz = j \sinh z$, so

$$\begin{aligned}\tanh{(a - j\beta)} &= -\frac{j \sin\beta \cosh a - \cos\beta \sinh a}{\cos\beta \cosh a - j \sin\beta \sinh a} \\ &= -\frac{\cos\beta \cosh a + j \sin\beta \sinh a}{\cos\beta \cosh a + j \sin\beta \sinh a} \frac{j \sin\beta \cosh a - \cos\beta \sinh a}{\cos\beta \cosh a - j \sin\beta \sinh a} \\ &= -\frac{(\cos\beta \cosh a + j \sin\beta \sinh a)(j \sin\beta \cosh a - \cos\beta \sinh a)}{\cos^2\beta \cosh^2 a + \sin^2\beta \sinh^2 a}\end{aligned} \tag{88}$$

I will handle the numerator, N, and the denominator, D, separately. For the numerator,

$$\begin{aligned}N &= -(\cos^2\beta + \sin^2\beta) \cosh a \sinh a + j(\cosh^2 a - \sinh^2 a) \cos\beta \sin\beta \\ &= -\cosh a \sinh a + j \cos\beta \sin\beta \\ &= \frac{-\sinh 2a + j \sin 2\beta}{2}.\end{aligned} \tag{89}$$

On the other hand, the denominator is

$$\begin{aligned}D &= \cos^2\beta \cosh^2 a + \sin^2\beta \sinh^2 a \\ &= \frac{1 + \cos 2\beta}{2} \frac{1 + \cosh 2a}{2} + \frac{1 - \cos 2\beta}{2} \frac{\cosh 2a - 1}{2} \\ &= \frac{1 + \cos 2\beta + \cosh 2a + \cos 2\beta \cosh 2a - 1 + \cos 2\beta + \cosh 2a - \cos 2\beta \cosh 2a}{4} \\ &= \frac{\cos 2\beta + \cosh 2a}{2}.\end{aligned} \tag{90}$$

The quotient $-N/D$ split into its real and imaginary parts is thereby

$$\tanh{(a - j\beta)} = \frac{\sinh 2a}{\cos 2\beta + \cosh 2a} - j \frac{\sin 2\beta}{\cos 2\beta + \cosh 2a}. \tag{91}$$

The object of interest is $z \tanh z$. The real and imaginary parts of this quantity are

$$\begin{aligned}(a - j\beta) \tanh{(a - j\beta)} &= (a - j\beta)\left(\frac{\sinh 2a}{\cos 2\beta + \cosh 2a} - j \frac{\sin 2\beta}{\cos 2\beta + \cosh 2a}\right) \\ &= \frac{a \sinh 2a - \beta \sin 2\beta}{\cos 2\beta + \cosh 2a} - j \frac{a \sin 2\beta + \beta \sinh 2a}{\cos 2\beta + \cosh 2a}\end{aligned} \tag{92}$$

The goal is to solve for a and β. Consider the following formula from the main text:

$$z \tanh z = \frac{-j}{\gamma}. \tag{93}$$

Equating the real and imaginary parts gives two equations:

$$a \sinh 2a - \beta \sin 2\beta = 0, \tag{94}$$

$$\frac{a \sin 2\beta + \beta \sinh 2a}{\cosh 2a + \cos 2\beta} = \gamma^{-1}. \tag{95}$$

We can solve these for a and β under a few assumptions. Numerical studies of the root-finding problem have shown that as x increases, the roots with the smallest magnitude negative imaginary parts tend towards $-\pi j/2$. This motivates the assumption that $0 < a \ll 1$ and $\beta = \pi/2 - \epsilon$ with $0 < \epsilon \ll 1$.

Now we solve for a and β. The following relations and the Maclaurin approximations up to second order are of use:

$$\sin(\pi - x) = \sin x, \quad \cos(\pi - x) = -\cos x \tag{96}$$

$$\sin x \approx x, \quad \sinh x \approx x, \quad \cosh x \approx 1 + \frac{x^2}{2}. \tag{97}$$

The imaginary part formula (Eqn 95) gives a first approximation for a:

$$\begin{aligned} \frac{1}{\gamma} &= \frac{a \sin 2\beta + \beta \sinh 2a}{\cosh 2a + \cos 2\beta} \\ &\approx \frac{a \sin(\pi - 2\epsilon) + 2a\beta}{1 + 2a^2 + \cos(\pi - 2\epsilon)} \\ &\approx \frac{2a\epsilon + a\pi - 2a\epsilon}{1 + 2a^2 - 1} \\ &= \frac{a\pi}{2a^2} = \frac{\pi}{2a}. \end{aligned} \tag{98}$$

The final approximation is thereby

$$a = \frac{\pi}{2}\gamma. \tag{99}$$

We can also find a first approximation of β:

$$\begin{aligned} 0 &= a \sinh 2a - \beta \sin 2\beta \\ &\approx 2a^2 - \left(\frac{\pi}{2} - \epsilon\right) \sin(\pi - 2\epsilon) \\ &\approx 2a^2 - 2\epsilon\left(\frac{\pi}{2} - \epsilon\right) \\ &\approx 2a^2 - \epsilon\pi, \end{aligned} \tag{100}$$

where I have used the approximation that $\epsilon^2 \approx 0$. Plugging in the formula for a (Eqn 99),

$$\begin{aligned} \epsilon &= 2\frac{\pi^2\gamma^2}{4}\frac{1}{\pi} \\ &= \frac{\pi}{2}\gamma^2. \end{aligned} \tag{101}$$

Knowing that $\beta = \pi/2 - \epsilon$,

$$\beta = \frac{\pi}{2}(1 - \gamma^2). \tag{102}$$

Finally recalling that $z = kh = a - j\beta$, k_d at the point of discontinuity is

$$k_d \approx \frac{\pi}{2h}\gamma - j\frac{\pi}{2h}(1 - \gamma^2), \quad \gamma = \frac{R}{2\rho h\omega_d}. \tag{103}$$

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