APM496 Problem Set 2

Brian Lee 1002750855

1 Problem 1 (Eigenvectors and Eigenvalues)

(a) We define A as

$$A = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 4 \end{pmatrix}$$

This matrix is indeed symmetric.

(b) The eigenvalues as calculated using Python are:

$$\lambda_1 = 5.2143,\ \lambda_2 = 2.4608,\ \lambda_3 = 1.3249$$

The corresponding eigenvalues (i.e. v_i is the eigenvalue corresponding to λ_i) are

$$v_1 = \begin{pmatrix} -0.5207 \\ 0.7392 \\ 0.4271 \end{pmatrix},\ v_2 = \begin{pmatrix} 0.3971 \\ -0.2332 \\ 0.8877 \end{pmatrix},\ v_3 = \begin{pmatrix} -0.7558 \\ -0.6318 \\ 0.1721 \end{pmatrix}$$

(c) From our answer to part (b), we see that all of the eigenvalues of A are positive, $(\lambda_1, \lambda_2, \lambda_3 > 0)$, hence A is positive definite.

2 Problem 2 (Cholesky Decomposition)

(a) Computing L gives us

$$L = \begin{pmatrix} 1.7321 & 0 & 0 \\ -0.5774 & 1.2910 & 0 \\ 0.5774 & -0.5164 & 1.8439 \end{pmatrix}$$

(b) Sampling to get u, we get

$$\mathrm{Cov}(u) = \begin{pmatrix} 0.9981 & 0.0032 & -0.0017 \\ 0.0032 & 1.0002 & -0.0022 \\ -0.0017 & -0.0022 & 1.0002 \end{pmatrix}$$

We see that $\mathrm{Cov}(u)$ is approximately the identity as the diagonals entries are approximately 1 and every off-diagonal entries are approximately 0.

(c) We get

$$\mathrm{Cov}(v) = \begin{pmatrix} 2.9941 & -0.9909 & 0.9897 \\ -0.9909 & 1.9950 & -0.9996 \\ 0.9897 & -0.9996 & 3.9990 \end{pmatrix}$$

which we see is approximately A.

3 Problem 3 (Singular Value Decomposition)

(a) For $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$, we have

$$v^T v = \sum_{i=1}^{n} v_i^2$$

and

$$vv^T = \begin{pmatrix} v_1^2 & v_1 v_2 & v_1 v_3 & \ldots & v_1 v_n \\ v_2 v_1 & v_2^2 & v_2 v_3 & \ldots & v_2 v_n \\ v_3 v_1 & v_3 v_2 & v_3^2 & \ldots & v_3 v_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ v_n v_1 & v_n v_2 & v_n v_3 & \ldots & v_n^2 \end{pmatrix}$$

(b) Assuming $v \neq 0$, the rank of vv^T is 1 since every column of vv^T is a scalar multiple of the vector v, i.e. the i-th column of vv^T is $v_i v$, thus the matrix has rank 1 (in brief, matrix has image $\text{span}\{v\}$).

(c) Since $u_i = \sigma_i^{-1} A v_i$,

$$u_i v_i^T = \sigma_i^{-1} A (v_i v_i^T)$$

Now for any matrices C, D we have $\text{rank}(DC) \leq \min\{\text{rank}(D), \text{rank}(C)\}$, hence

$$\text{rank}(u_i v_i^T) \leq \min\{\text{rank}(A)\, \text{rank}(v_i v_i^T)\} \leq 1$$

and since $u_i v_i^T \neq 0$ for $1 \leq i \leq r$, we get that $\text{rank}(u_i v_i^T) \geq 1$, which implies

$$\text{rank}(u_i v_i^T) = 1$$

(d) For any m by n matrix A, and for any $x \in \mathbb{R}^n$, we get

$$x^T (A^T A) x = (x^T A^T) A x = (Ax)^T A x = \|Ax\|^2 \geq 0$$

Hence $A^T A$ is positive semi-definite.

(e) Suppose A is an m by n matrix. Recall that the singular values of A correspond to eigenvalues of $A^T A$. Thus if the number of non-zero singular values is equal to r (counted with multiplicity), this corresponds to r eigenvalues of $A^T A$ (counted with multiplicity), which means that $A^T A$ has rank r.

Recall the rank nullity theorem, which in our case (since our domain is $\mathbb{R}^n$), is of the form

$$n = \text{rank}(A) + \dim(\ker(A))$$

The above also applies for $A^T A$, which means that the null space of $A^T A$ has dimension $n - r$ by the rank-nullity theorem. Since A has the same null space as $A^T A$, we get $\dim(\ker A) = n - r$, so by rank-nullity again we get $\text{rank}(A) = n - (n - r) = r$ as desired. Thus the number of non-zero singular values of A is equal to the rank of A.