APM496 Problem Set 2

Brian Lee 1002750855

1 Problem 1 (Eigenvectors and Eigenvalues)

(a) We define A as

$$A = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 4 \end{pmatrix}$$

This matrix is indeed symmetric.

(b) The eigenvalues as calculated using Python are:

$$\lambda_1 = 5.2143,\ \lambda_2 = 2.4608,\ \lambda_3 = 1.3249$$

The corresponding eigenvalues (i.e. v_i is the eigenvalue corresponding to λ_i) are

$$v_1 = \begin{pmatrix} -0.5207 \\ 0.7392 \\ 0.4271 \end{pmatrix},\ v_2 = \begin{pmatrix} 0.3971 \\ -0.2332 \\ 0.8877 \end{pmatrix},\ v_3 = \begin{pmatrix} -0.7558 \\ -0.6318 \\ 0.1721 \end{pmatrix}$$

(c) From our answer to part (b), we see that all of the eigenvalues of A are positive, $(\lambda_1, \lambda_2, \lambda_3 > 0)$, hence A is positive definite.

2 Problem 2 (Cholesky Decomposition)

(a) Computing L gives us

$$L = \begin{pmatrix} 1.7321 & 0 & 0 \\ -0.5774 & 1.2910 & 0 \\ 0.5774 & -0.5164 & 1.8439 \end{pmatrix}$$

(b) Sampling to get u, we get

$$\mathrm{Cov}(u) = \begin{pmatrix} 0.9981 & 0.0032 & -0.0017 \\ 0.0032 & 1.0002 & -0.0022 \\ -0.0017 & -0.0022 & 1.0002 \end{pmatrix}$$

We see that $\mathrm{Cov}(u)$ is approximately the identity as the diagonals entries are approximately 1 and every off-diagonal entries are approximately 0.

(c) We get

$$\mathrm{Cov}(v) = \begin{pmatrix} 2.9941 & -0.9909 & 0.9897 \\ -0.9909 & 1.9950 & -0.9996 \\ 0.9897 & -0.9996 & 3.9990 \end{pmatrix}$$

which we see is approximately A.

3 Problem 3 (Singular Value Decomposition)

(a) For $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$, we have

$$v^T v = \sum_{i=1}^{n} v_i^2$$

and

$$vv^T = \begin{pmatrix} v_1^2 & v_1v_2 & v_1v_3 & \ldots & v_1v_n \\ v_2v_1 & v_2^2 & v_2v_3 & \ldots & v_2v_n \\ v_3v_1 & v_3v_2 & v_3^2 & \ldots & v_3v_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ v_nv_1 & v_nv_2 & v_nv_3 & \ldots & v_n^2 \end{pmatrix}$$

(b) The rank of vv^T is 1 since every column of vv^T is a scalar multiple of the vector v, i.e. the i-th column of vv^T is v_iv, thus the matrix has rank 1 (in brief, matrix has image $\text{span}\{v\}$).

(c) The rank of $u_iv_i^T$ is 1 since if $u_i = (a_1, \ldots, a_n)$ and $v_i = (b_1 \ldots, b_n)$ then

$$u_i^T v_i = \begin{pmatrix} a_1b_1 & a_1b_2 & \ldots & a_1b_n \\ a_2b_1 & a_2b_2 & \ldots & a_2b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_nb_1 & a_nb_2 & \ldots & a_nb_n \end{pmatrix} = \left(a_1 \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}, a_2 \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}, \ldots, a_n \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \right)$$

So we see that every column of the matrix is a scalar multiple of the vector v_i, hence has one-dimensional image (rank 1). In other words, the image of $u_i^T v_i$ is $\text{span}\{v_i\}$ which is one-dimensional, thus its rank is 1.

(d) For any m by n matrix A, and for any $x \in \mathbb{R}^n$, we get

$$x^T(A^TA)x = (x^TA^T)Ax = (Ax)^TAx = \|Ax\|^2 \geq 0$$

Hence A^TA is positive semi-definite.

(e) Suppose A is an m by n matrix. Recall that the singular values of A correspond to eigenvalues of A^TA. Thus if the number of non-zero singular values is equal to r (counted with multiplicity), this corresponds to r eigenvalues of A^TA (counted with multiplicity), which means that A^TA has rank r.

Recall the rank nullity theorem, which in our case (since our domain is $\mathbb{R}^n$), is of the form

$$n = \text{rank}(A) + \dim(\ker(A))$$

The above also applies for A^TA, which means that the null space of A^TA has dimension $n - r$ by the rank-nullity theorem. Since A has the same null space as A^TA, we get $\dim(\ker A) = n - r$, so by rank-nullity again we get $\text{rank}(A) = n - (n - r) = r$ as desired.