Question 3

Over all real numbers, find the minimum value of a positive real number, y such that

$y = \text{sqrt}((x+6)^2 + 25) + \text{sqrt}((x-6)^2 + 121)$

Start by differentiating the equation

$d/dx [\sqrt{(x + 6)^2 + 25}] + d/dx [\sqrt{(x - 6)^2 + 121}] =$

$(1/2 ((x + 6)^2 + 25)^{-½} * d/dx [(x + 6)^2 + 25]) + (1/2 ((x - 6)^2 +121)^{-1/2} * d/dx [(x - 6)^2+ 121])$

$= ((d/dx [(x + 6)^2] + d/dx [25])/ 2 \sqrt{(x + 6)^2 + 25}) + ((d/dx [(x - 6)^2] + d/dx [121])/ 2 \sqrt{(x - 6)^2 + 121})$

$= (2 (x + 6)*d/dx [x + 6] + 0)/ (2 \sqrt{(x + 6)^2 + 25}) + (2 (x - 6) *d/dx [x - 6] + 0/ (2 \sqrt{(x - 6)^2 + 121}))$

$= (d/dx [x] + d/dx [6]) (x + 6) /\sqrt{(x + 6)^2 + 25} + (d/dx [x] + d/dx [- 6]) (x - 6) /\sqrt{(x - 6)^2 + 121}$

$= (1 + 0) (x + 6)/ \sqrt{(x + 6)^2 + 25} + (1 + 0) (x - 6) /\sqrt{(x - 6)^2 + 121}$

$= x + 6/ \sqrt{(x + 6)^2 + 25} + x - 6/ \sqrt{(x - 6)^2 + 121}$

Equate to zero

$x + 6/ \sqrt{(x + 6)^2 + 25} + x - 6/ \sqrt{(x - 6)^2 + 121} = 0$

$x + 6/ \sqrt{(x + 6)^2 + 25} = -(x - 6)/ \sqrt{(x - 6)^2 + 121}$

Square both sides

$(x + 6)^2/ (x + 6)^2 + 25 = (x - 6)^2/ (x - 6)^2 + 121$

Cross multiply

$(x-6)^2*[(x+6)^2+25] = (x+6)^2*[(x-6)^2+121]$

Simplify

$(x-6)^2*(x+6)^2 + 25(x-6)^2 = (x+6)^2*(x-6)^2 + 121(x+6)^2$

$25(x-6)^2 = 121(x+6)^2$
$25x^2 - 300x + 900 = 121x^2 + 1452x + 4356$
$25x^2 - 121x^2 - 300x - 1452x + 900 - 4356 = 0$
$-96x^2 - 1752x - 3456 = 0$

Factorize
$-24(4x + 9)(x+16) = 0$
$4x+9 = 0$ or $x+16 = 0$
$x = -9/4$ or $x = -16$

Check which x tends to 0 by replacing x in the derivative: $x + 6/ \sqrt{((x + 6)^2 + 25)} + x - 6/ \sqrt{((x - 6)^2 + 121)}$:

When $x = -16$,
$(-16 + 6/ \sqrt{((-16 + 6)^2 + 25)}) + (-16 - 6/ \sqrt{((-16 - 6)^2 + 121)})$
$= (-10/\sqrt{125}) + (-22/\sqrt{605})$
$= -0.89-0.89$
$= -1.78$

When $x = -9/4(-2.25)$,
$((-2.25 + 6)/ \sqrt{((-2.25 + 6)^2 + 25)}) + ((-2.25 - 6)/ \sqrt{((-2.25 - 6)^2 + 121)})$
$= (3.75/\sqrt{39.06}) + (-8.25/\sqrt{189.06})$
$= 0.6-0.6$
$= 0$

Plugging in $x = -9/4$ (-2.25) into original equation to get positive y value,

$y = \text{sqrt}((x+6)^2 + 25) + \text{sqrt}((x-6)^2 + 121)$
$y = \text{sqrt}((-2.25+6)^2 + 25) + \text{sqrt}((-2.25-6)^2 + 121)$
$y = \text{sqrt}(39.06) + \text{sqrt}(189.06)$
$y = 6.25 + 13.74$
$y = 19.99$