Neville Hankins who provided
ct. Ms. Jean Phillips provided

e faithfully supported both this
ughter, Charlotte, and my son,
y the necessary moments and
piration to soldier on. A father
to my lovely wife Minh, a rock
as lit my soul since the day we
of your love in my life. When
eople, language fails to convey

Jonathan E. Martin
Madison, Wisconsin
January 10, 2006

1

Introduction and Review of Mathematical Tools

Objectives

The Earth's atmosphere is majestic in its beauty, awesome in its power, and complex in its behavior. From the smallest drops of dew or the tiniest snowflakes to the enormous circulation systems known as mid-latitude cyclones, all atmospheric phenomena are governed by physical laws. These laws can be written in the language of mathematics and, indeed, must be explored in that vernacular in order to develop a penetrating understanding of the behavior of the atmosphere. However, it is equally vital that a physical understanding accompany the mathematical formalism in this comprehensive development of insight. In principle, if one had a complete understanding of the behavior of seven basic variables describing the current state of the atmosphere (these will be called **basic state variables** in this book), namely u, v, and w (the components of the 3-D wind), T (the temperature), P (the pressure), ϕ (the geopotential), and q (the humidity), then one could describe the future state of the atmosphere by considering the equations that govern the evolution of each variable. It is not, however, immediately apparent what form these equations might take. In this book we will develop those equations in order to develop an understanding of the basic dynamics that govern the behavior of the atmosphere at middle latitudes on Earth.

In this chapter we lay the foundation for that development by reviewing a number of basic conceptual and mathematical tools that will prove invaluable in this task. We begin by assessing the troubling but useful notion that the air surrounding us can be considered a continuous fluid. We then proceed to a review of useful mathematical tools including vector calculus, the Taylor series expansion of a function, centered difference approximations, and the relationship between the Lagrangian and Eulerian derivatives. We then examine the notion of estimating using scale analysis and conclude the chapter by considering the basic kinematics of fluid flows.

Mid-Latitude Atmospheric Dynamics Jonathan E. Martin

1.1 Fluids and the Nature of Fluid Dynamics

Our experience with the natural world makes clear that physical objects manifest themselves in a variety of forms. Most of these physical objects (and every one of them with which we will concern ourselves in this book) have **mass**. The mass of an object can be thought of as a measure of its substance. The Earth's atmosphere is one such object. It certainly has mass[1] but differs from, say, a rock in that it is not solid. In fact, the Earth's atmosphere is an example of a general category of substances known as fluids. A fluid can be colloquially defined as any substance that takes the shape of its container. Aside from the air around us, another fluid with which we are all familiar is water. A given mass of liquid water clearly adopts the shape of any container into which it is poured. The given mass of liquid water just mentioned, like the air around us, is actually composed of discrete molecules. In our subsequent discussions of the behavior of the atmospheric fluid, however, we need not concern ourselves with the details of the molecular structure of the air. We can instead treat the atmosphere as a continuous fluid entity, or **continuum**. Though the assumption of a continuous fluid seems to fly in the face of what we recognize as the underlying, discrete molecular reality, it is nonetheless an insightful concept. For instance, it is much more tenable to consider the flow of air we refer to as the wind to be a manifestation of the motion of such a continuous fluid. Any 'point' or 'parcel' to which we refer will be properly considered as a very small volume element that contains large numbers of molecules. The various basic state variables mentioned above will be assumed to have unique values at each such 'point' in the continuum and we will confidently assume that the variables and their derivatives are continuous functions of physical space and time. This means, of course, that the fundamental physical laws governing the motions of the atmospheric fluid can be expressed in terms of a set of partial differential equations in which the basic state variables are the dependent variables and space and time are the independent variables. In order to construct these equations, we will rely on some mathematical tools that you may have seen before. The following section will offer a review of a number of the more important ones.

1.2 Review of Useful Mathematical Tools

We have already considered, in a conceptual sense only, the rather unique nature of fluids. A variety of mathematical tools must be brought to bear in order to construct rigorous descriptions of the behavior of these fascinating fluids. In the following section we will review a number of these tools in some detail. The reader familiar with any of these topics may skip the treatments offered here and run no risk of confusion later. We will begin our review by considering elements of vector analysis.

[1] The Earth's atmosphere has a mass of 5.265×10^{18} kg!

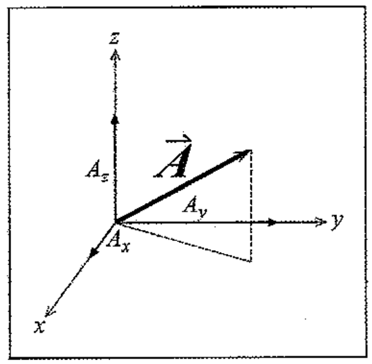

Figure 1.1 The 3-D representation of a vector, $\vec{A}$. The components of $\vec{A}$ are shown along the coordinate axes

1.2.1 Elements of vector calculus

Many physical quantities with which we are concerned in our experience of the universe are described entirely in terms of *magnitude*. Examples of these types of quantities, known as **scalars**, are area, volume, money, and snowfall total. There are other physical quantities such as velocity, the force of gravity, and slopes to topography which are characterized by both magnitude and direction. Such quantities are known as **vectors** and, as you might guess, any description of the fluid atmosphere necessarily contains reference to both scalars and vectors. Thus, it is important that we familiarize ourselves with the mathematical descriptions of these quantities, a formalism known as vector analysis.[2]

Employing a Cartesian coordinate system in which the three directions (x, y, and z) are mutually orthogonal (i.e. perpendicular to one another), an arbitrary vector, $\vec{A}$, has components in the x, y, and z directions labeled A_x, A_y, and A_z, respectively. These components themselves are scalars since they describe the magnitude of vectors whose directions are given by the coordinate axes (as shown in Figure 1.1). If we denote the direction vectors in the x, y, and z directions as $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$, respectively (where the ^ symbol indicates the fact that they are vectors with magnitude 1 in the respective directions – so-called **unit vectors**), then

$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k} \tag{1.1a}$$

is the component form of the vector, $\vec{A}$. In a similar manner, the component form of an arbitrary vector $\vec{B}$ is given by

$$\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}. \tag{1.1b}$$

[2] Vector analysis is generally considered to have been invented by the Irish mathematician Sir William Rowan Hamilton in 1843. Despite its enormous value in the physical sciences, vector analysis was met with skepticism in the nineteenth century. In fact, Lord Kelvin wrote, in the 1890s, that vectors were 'an unmixed evil to those who have touched them in any way . . vectors . . have never been of the slightest use to any creature'. Remember, no matter how great a thinker one may be, one cannot always be right!

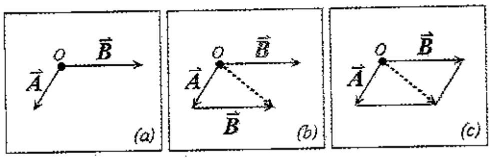

Figure 1.2 (a) Vectors $\vec{A}$ and $\vec{B}$ acting upon a point O. (b) Illustration of the tail-to-head method for adding vectors $\vec{A}$ and $\vec{B}$. (c) Illustration of the parallelogram method for adding vectors $\vec{A}$ and $\vec{B}$

The vectors $\vec{A}$ and $\vec{B}$ are equal if $A_x = B_x$, $A_y = B_y$, and $A_z = B_z$. Furthermore, the magnitude of a vector $\vec{A}$ is given by

$$\left|\vec{A}\right| = \left(A_x^2 + A_y^2 + A_z^2\right)^{1/2} \tag{1.2}$$

which is simply the 3-D Pythagorean theorem and can be visually verified with the aid of Figure 1.1.

Vectors can be added to and subtracted from one another both by graphical methods as well as by components. Graphical addition is illustrated with the aid of Figure 1.2. Imagine that the force vectors $\vec{A}$ and $\vec{B}$ are acting at point O as shown in Figure 1.2(a). The total force acting at O is equal to the sum of $\vec{A}$ and $\vec{B}$. Graphical construction of the vector sum $\vec{A} + \vec{B}$ can be accomplished either by using the tail-to-head method or the parallelogram method. The tail-to-head method involves drawing $\vec{B}$ at the head of $\vec{A}$ and then connecting the tail of $\vec{A}$ to the head of the redrawn $\vec{B}$ (Figure 1.2b). Alternatively, upon constructing a parallelogram with sides $\vec{A}$ and $\vec{B}$, the diagonal of the parallelogram between $\vec{A}$ and $\vec{B}$ represents the vector sum, $\vec{A} + \vec{B}$ (Figure 1.2c).

If we know the component forms of both $\vec{A}$ and $\vec{B}$, then their sum is given by

$$\vec{A} + \vec{B} = (A_x + B_x)\hat{\imath} + (A_y + B_y)\hat{\jmath} + (A_z + B_z)\hat{k}. \tag{1.3a}$$

Thus, the sum of $\vec{A}$ and $\vec{B}$ is found by simply adding like components together. It is clear from considering the component form of vector addition that addition of vectors is commutative ($\vec{A} + \vec{B} = \vec{B} + \vec{A}$) and associative ($(\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C})$).

Subtraction is simply the opposite of addition so $\vec{B}$ can be subtracted from $\vec{A}$ by simply adding $-\vec{B}$ to $\vec{A}$. Graphical subtraction of $\vec{B}$ from $\vec{A}$ is illustrated in Figure 1.3. Notice that $\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$ results in a vector directed from the head of $\vec{B}$ to the head of $\vec{A}$ (the lighter dashed arrow in Figure 1.3). Component subtraction involves

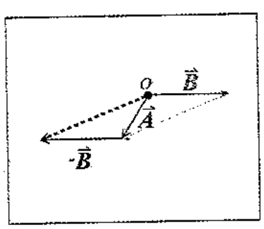

Figure 1.3 Graphical subtraction of vector $\vec{B}$ from vector $\vec{A}$

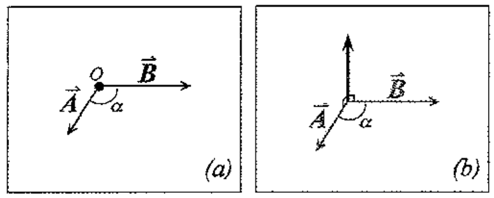

Figure 1.4 (a) Vectors $\vec{A}$ and $\vec{B}$ with an angle α between them. (b) Illustration of the relationship between vectors $\vec{A}$ and $\vec{B}$ (gray arrows) and their cross-product, $\vec{A} \times \vec{B}$ (bold arrow). Note that $\vec{A} \times \vec{B}$ is perpendicular to both $\vec{A}$ and $\vec{B}$

subtracting like components and is given by

$$\vec{A} - \vec{B} = (A_x - B_x)\hat{\imath} + (A_y - B_y)\hat{\jmath} + (A_z - B_z)\hat{k}. \tag{1.3b}$$

Vector quantities may also be multiplied in a variety of ways. The simplest vector multiplication involves the product of a vector, $\vec{A}$, and a scalar, F. The resulting expression for $F\vec{A}$ is given by

$$F\vec{A} = FA_x\hat{\imath} + FA_y\hat{\jmath} + FA_z\hat{k}, \tag{1.4}$$

a vector with direction identical to the original vector, $\vec{A}$, but with a magnitude F times larger than the original magnitude.

It is also possible to multiply two *vectors* together. In fact, there are two different vector multiplication operations. One such method renders a scalar as the product of the vector multiplication and is thus known as the **scalar** (or **dot**) product. The dot product of the vectors $\vec{A}$ and $\vec{B}$ shown in Figure 1.4(a) is given by

$$\vec{A} \cdot \vec{B} = |A|\,|B| \cos\alpha \tag{1.5}$$

where α is the angle between $\vec{A}$ and $\vec{B}$. Clearly this product is a scalar. Using this formula, we can determine a less mystical form of the dot product of $\vec{A}$ and $\vec{B}$. Given that $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$ and $\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}$, the dot product is given by

$$\vec{A} \cdot \vec{B} = (A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}) \cdot (B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}) \tag{1.6}$$

which expands to the following nine terms:

$$\begin{aligned}\vec{A} \cdot \vec{B} = {} & A_xB_x(\hat{\imath} \cdot \hat{\imath}) + A_xB_y(\hat{\imath} \cdot \hat{\jmath}) + A_xB_z(\hat{\imath} \cdot \hat{k}) \\ & + A_yB_x(\hat{\jmath} \cdot \hat{\imath}) + A_yB_y(\hat{\jmath} \cdot \hat{\jmath}) + A_yB_z(\hat{\jmath} \cdot \hat{k}) \\ & + A_zB_x(\hat{k} \cdot \hat{\imath}) + A_zB_y(\hat{k} \cdot \hat{\jmath}) + A_zB_z(\hat{k} \cdot \hat{k}).\end{aligned}$$

Now, according to (1.5), $\hat{\imath} \cdot \hat{\imath} = \hat{\jmath} \cdot \hat{\jmath} = \hat{k} \cdot \hat{k} = 1$ since the angle between like unit vectors is 0°. However, the dot products of all other combinations of the unit vectors are zero since the unit vectors are mutually orthogonal. Thus, only three terms survive out of the nine-term expansion of $\vec{A} \cdot \vec{B}$ to yield

$$\vec{A} \cdot \vec{B} = A_xB_x + A_yB_y + A_zB_z. \tag{1.7}$$

Given this result, it is easy to show that the dot product is commutative ($\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$) and distributive ($\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$).

Two vectors can also be multiplied together to produce another vector. This vector multiplication operation is known as the **vector** (or **cross-**)product and is signified

$$\vec{A} \times \vec{B}.$$

The magnitude of the resultant vector is given by

$$|A|\,|B| \sin \alpha \tag{1.8}$$

where α is the angle between the vectors. Note that since the resultant of the cross-product is a vector, there is also a direction to be discerned. The resultant vector is in a plane that is perpendicular to the plane that contains $\vec{A}$ and $\vec{B}$ (Figure 1.4b). The direction in that plane can be determined by using the **right hand rule**. Upon curling the fingers of one's right hand in the direction from $\vec{A}$ to $\vec{B}$, the thumb points in the direction of the resultant vector, $\vec{A} \times \vec{B}$, as shown in Figure 1.4(b). Because the resultant direction depends upon the order of multiplication, the cross-product has different properties than the dot product. It is not commutative ($\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$; instead $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$) and it is not associative ($\vec{A} \times (\vec{B} \times \vec{C}) \neq (\vec{A} \times \vec{B}) \times \vec{C}$) but it is distributive ($\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$).

Given the vectors $\vec{A}$ and $\vec{B}$ in their component forms, the cross-product can be calculated by first setting up a 3×3 determinant using the unit vectors as the first row, the components of $\vec{A}$ as the second row, and the components of $\vec{B}$ as the third row:

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}. \tag{1.9a}$$

Evaluating this determinant involves evaluating three 2×2 determinants, each one corresponding to a unit vector $\hat{i}$, $\hat{j}$, *or* $\hat{k}$. For the $\hat{i}$ component of the resultant vector, only the components of $\vec{A}$ and $\vec{B}$ in the $\hat{j}$ and $\hat{k}$ columns are considered. Multiplying the components along the diagonal (upper left to lower right) first, and then subtracting from that result the product of the terms along the anti-diagonal (lower left to upper right) yields the $\hat{i}$ component of the vector $\vec{A} \times \vec{B}$, which equals $(A_y B_z - A_z B_y)\hat{i}$. The same operation done for the $\hat{k}$ component yields $(A_x B_y - A_y B_x)\hat{k}$. For the $\hat{j}$ component, the first and third columns are used to form the 2×2 determinant and since the columns are non-consecutive, the result must be multiplied by -1 to yield $-(A_x B_z - A_z B_x)\hat{j}$. Adding these three components together yields

$$\vec{A} \times \vec{B} = (A_y B_z - A_z B_y)\hat{i} + (A_z B_x - A_x B_z)\hat{j} + (A_x B_y - A_y B_x)\hat{k}. \tag{1.9b}$$

Vectors, just like scalar functions, can be differentiated as long as the rules of vector addition and multiplication are obeyed. One simple example is Newton's second law (which we will see again soon) that states that an object's momentum will not change unless a force is applied to the object. In mathematical terms,

$$\vec{F} = \frac{d}{dt}(m\vec{V}) \tag{1.10}$$

where m is the object's mass and $\vec{V}$ is its velocity. Using the chain rule of differentiation on the right hand side of (1.10) renders

$$\vec{F} = m\frac{d\vec{V}}{dt} + \vec{V}\frac{dm}{dt} \text{ or } \vec{F} = m\vec{A} + \vec{V}\frac{dm}{dt} \tag{1.11}$$

where $\vec{A}$ is the object's acceleration. Exploitation of the second term of this expansion is what made Einstein famous!

Let us consider a more general example. Consider a velocity vector defined as $\vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$. In such a case, the acceleration will be given by

$$\frac{d\vec{V}}{dt} = \frac{du}{dt}\hat{i} + u\frac{d\hat{i}}{dt} + \frac{dv}{dt}\hat{j} + v\frac{d\hat{j}}{dt} + \frac{dw}{dt}\hat{k} + w\frac{d\hat{k}}{dt}. \tag{1.12}$$

The terms involving derivatives of the unit vectors may seem like mathematical baggage but they will be extremely important in our subsequent studies. Physically, such terms will be non-zero only when the coordinate axes used to reference motion are not fixed in space. Our reference frame on a rotating Earth is clearly not fixed and so we will eventually have to make some accommodation for the acceleration of our rotating reference frame. Thus, all six terms in the above expansion will be relevant in our examination of the mid-latitude atmosphere.

The last stop on the review of vector calculus is perhaps the most important one and will examine a tool that is extremely useful in fluid dynamics. We will often need to describe both the magnitude and direction of the derivative of a scalar field. In order to do so we employ a mathematical operator known as the **del operator**, defined as

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}. \tag{1.13}$$

If we apply this partial differential del operator to a scalar function or field, the result is a vector that is known as the **gradient** of that scalar. Consider the 2-D plan view of an isolated hill in an otherwise flat landscape. If the elevation at each point in the landscape is represented on a 2-D projection, a set of elevation contours results as shown in Figure 1.5. Such contours are lines of equal height above sea level, Z. Given such information, we can determine the gradient of elevation, ∇Z, as

$$\nabla Z = \frac{\partial Z}{\partial x}\hat{i} + \frac{\partial Z}{\partial y}\hat{j}.$$

Note that the gradient vector, ∇Z, points up the hill *from low values of elevation to high values*. At the top of the hill, the derivatives of Z in both the x and y

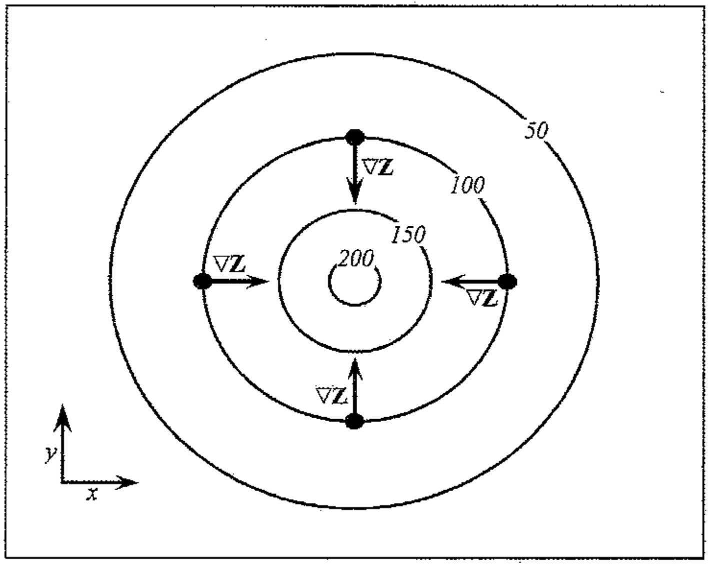

Figure 1.5 The 2-D plan view of an isolated hill in a flat landscape. Solid lines are contours of elevation (Z) at 50m intervals. Note that the gradient of Z points from low to high values of the scalar Z

directions are zero so there is no gradient vector there. Thus the gradient, ∇Z, not only measures magnitude of the elevation difference but assigns that magnitude a direction as well. Any scalar quantity, Φ, is transformed into a vector quantity, $\nabla\Phi$, by the del operator. In subsequent chapters in this book we will concern ourselves with the gradients of a number of scalar variables, among them temperature and pressure.

The del operator may also be applied to vector quantities. The dot product of ∇ with the vector $\vec{A}$ is written as

$$\nabla \cdot \vec{A} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})$$
$$\nabla \cdot \vec{A} = \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) \tag{1.14}$$

which is a scalar quantity known as the **divergence of $\vec{A}$**. Positive divergence physically describes the tendency for a vector field to be directed away from a point whereas negative divergence (also known as **convergence**) describes the tendency for a vector field to be directed toward a point. Regions of convergence and divergence in the atmospheric fluid are extremely important in determining its behavior.

The cross-product of ∇ with the vector $\vec{A}$ is given by

$$\nabla \times \vec{A} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \times (A_x\hat{i} + A_y\hat{j} + A_z\hat{k}). \tag{1.15a}$$

The resulting vector can be calculated using the determinant form we have seen previously,

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \tag{1.15b}$$

where the second row of the 3×3 determinant is filled by the components of ∇ and the third row is filled by the components of $\vec{A}$. This vector is known as the **curl of** $\vec{A}$. The curl of the velocity vector, $\vec{V}$, will be used to define a quantity called **vorticity** which is a measure of the rotation of a fluid.

Quite often in a study of the dynamics of the atmosphere, we will encounter second-order partial differential equations. Some of these equations will contain a mathematical operator (which will operate on scalar quantities) known as the **Laplacian** operator. The Laplacian is the **divergence of the gradient** and so takes the form

$$\textit{Laplacian} = \nabla \cdot (\nabla F) = \nabla^2 F = \left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2} \right). \tag{1.16}$$

It is also possible to combine the vector $\vec{A}$ with the del operator to form a new operator that takes the form

$$\vec{A} \cdot \nabla = A_x \frac{\partial}{\partial x} + A_y \frac{\partial}{\partial y} + A_z \frac{\partial}{\partial z}$$

and is known as the scalar invariant operator. This operator, which can be used with both vector and scalar quantities, is important because it is used to describe a process known as **advection**, a ubiquitous topic in the study of fluids.

1.2.2 The Taylor series expansion

It is sometimes convenient to estimate the value of a continuous function, $f(x)$, about the point $x = 0$ with a power series of the form

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n. \tag{1.17}$$

The fact that this can actually be done might appear to be an assumption so we must identify conditions for which this assumption is true. These conditions are that (1) the polynomial expression (1.17) passes through the point $(0, f(0))$ and (2) its first n derivatives match the first n derivatives of $f(x)$ at $x = 0$. Implicit in this second condition is the fact that $f(x)$ is differentiable at $x = 0$. In order for these conditions to be met, the coefficients $a_0, a_1, \ldots, a_n$ must be chosen properly. Substituting $x = 0$ into (1.17) we find that $f(0) = a_0$. Taking the first derivative of

(1.17) with respect to x and substituting $x = 0$ into the resulting expression we get $f'(0) = a_1$. Taking the second derivative of (1.17) with respect to x and substituting $x = 0$ into the result leaves $f''(0) = 2a_2$, or $f''(0)/2 = a_2$. If we continue to take higher order derivatives of (1.17) and evaluate each of them at $x = 0$ we find that, in order that the n derivatives of (1.17) match the n derivatives of $f(x)$, the coefficients, a_n, of the polynomial expression (1.17) must take the general form

$$a_n = \frac{f^n(0)}{n!}.$$

Thus, the value of the function $f(x)$ at $x = 0$ can be expressed as

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^n(0)}{n!}x^n. \tag{1.18}$$

Now, if we want to determine the value of $f(x)$ near the point $x = x_0$, the above expression can be generalized into what is known as the Taylor series expansion of $f(x)$ about $x = x_0$, given by

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^n(x_0)}{n!}(x - x_0)^n. \tag{1.19}$$

Since the dependent variables that describe the behavior of the atmosphere are all continuous variables, use of the Taylor series to approximate the values of those variables will prove to be a nifty little trick that we will exploit in our subsequent analyses. Most often we consider Taylor series expansions in which the quantity $(x - x_0)$ is very small in order that all terms of order 2 and higher in (1.19), the so-called **higher order terms**, can be effectively neglected. In such cases, we will approximate the given functions as

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0).$$

1.2.3 Centered difference approximations to derivatives

Though the atmosphere is a continuous fluid and its observed state at any time *could theoretically* be represented by a continuous function, the reality is that actual observations of the atmosphere are only available at discrete points in space and time. Given that much of the subsequent development in this book will arise from consideration of the spatial and temporal variation of observable quantities, we must consider a method of approximating derivative quantities from discrete data. One such method is known as **centered differencing**[3] and it follows directly from the prior discussion of the Taylor series expansion.

[3] Centered differencing is a subset of a broader category of such approximations known as **finite differencing**.

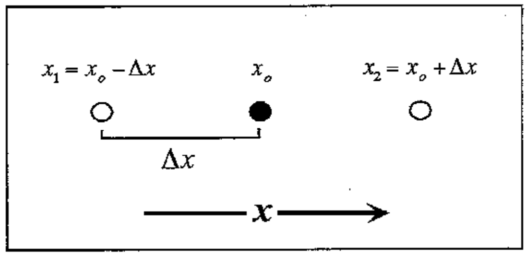

Figure 1.6 Points x_1 and x_2 defined with respect to a central point x_0

Consider the two points x_1 and x_2 in the near vicinity of a central point, x_0, as illustrated in Figure 1.6. We can apply (1.19) at both points to yield

$$f(x_1) = f(x_0 - \Delta x) = f(x_0) + f'(x_0)(-\Delta x) + \frac{f''(x_0)}{2!}(-\Delta x)^2 + \cdots + \frac{f^n(x_0)}{n!}(-\Delta x)^n \tag{1.20a}$$

and

$$f(x_2) = f(x_0 + \Delta x) = f(x_0) + f'(x_0)(\Delta x) + \frac{f''(x_0)}{2!}(\Delta x)^2 + \cdots + \frac{f^n(x_0)}{n!}(\Delta x)^n. \tag{1.20b}$$

Subtracting (1.20a) from (1.20b) produces

$$f(x_0 + \Delta x) - f(x_0 - \Delta x) = 2f'(x_0)(\Delta x) + 2f'''(x_0)\frac{(\Delta x)^3}{6} + \cdots. \tag{1.21}$$

Isolating the expression for $f'(x_0)$ on one side then leaves

$$f'(x_0) = \frac{f(x_0 + \Delta x) - f(x_0 - \Delta x)}{2\Delta x} - f'''(x_0)\frac{(\Delta x)^2}{6} - \cdots$$

which, upon neglecting terms of second order and higher in Δx, can be approximated as

$$f'(x_0) \approx \frac{f(x_0 + \Delta x) - f(x_0 - \Delta x)}{2\Delta x}. \tag{1.22}$$

The foregoing expression represents the centered difference approximation to $f'(x)$ at x_0 accurate to second order (i.e. the neglected terms are at least quadratic in Δx).

Adding (1.20a) to (1.20b) gives a similarly approximated expression for the second derivative as

$$f''(x_0) \approx \frac{f(x_0 + \Delta x) - 2f(x_0) + f(x_0 - \Delta x)}{\Delta x^2}. \tag{1.23}$$

Such expressions will prove quite useful in evaluating a number of relationships we will encounter later.

1.2.4 Temporal changes of a continuous variable

The fluid atmosphere is an ever evolving medium and so the fundamental variables discussed in Section 1.1 are ceaselessly subject to temporal changes. But what does it really mean to say 'The temperature has changed in the last hour'? In the broadest sense this statement could have two meanings. It could mean that the temperature of an individual air parcel, moving past the thermometer on my back porch, is changing as it migrates through space. In this case, we would be considering the change in temperature experienced *while moving with a parcel of air.* However, the statement could also mean that the temperature of the air parcels currently in contact with my thermometer is lower than that of air parcels that used to reside there but have since been replaced by the importation of these colder ones. In this case we would be considering the changes in temperature as measured *at a fixed geographic point.* These two notions of temporal change are clearly not the same, but one might wonder if and how they are physically and mathematically related. We will consider a not so uncommon example to illustrate this relationship.

Imagine a winter day in Madison, Wisconsin characterized by biting northwesterly winds which are importing cold arctic air southward out of central Canada. From the fixed geographical point of my back porch, the temperature (or potential temperature) drops with the passage of time. If, however, I could ride along with the flow of the air, I would likely find that the temperature does not change over the passage of time. In other words, a parcel with $T = 270°\mathrm{K}$ passing my porch at 8 a.m. still has $T = 270°\mathrm{K}$ at 2 p.m. even though it has traveled nearly to Chicago, Illinois by that time. Therefore, *the steady drop in temperature I observe at my porch is a result of the continuous importation of colder air parcels from Canada.* Phenomenologically, therefore, we can write an expression for this relationship we've developed:

$$\begin{array}{c} \textit{Change with Time} \\ \textit{Following an Air} \\ \textit{Parcel} \end{array} = \begin{array}{c} \textit{Change with Time} \\ \textit{at a Fixed} \\ \textit{Location} \end{array} - \begin{array}{c} \textit{Rate of Importation} \\ \textit{of Temperature by} \\ \textit{Movement of Air.} \end{array} \qquad (1.24)$$

This relationship can be made mathematically rigorous. Doing so will assist us later in the development of the equations of motion that govern the mid-latitude atmosphere. The change following the air parcel is called the **Lagrangian** rate of change while the change at a fixed point is called the **Eulerian** rate of change. We can quantify the relationship between these two different views of temporal change by considering an arbitrary scalar (or vector) quantity that we will call Q. If Q is a function of space and time, then

$$Q = Q(x, y, z, t)$$

and, from the differential calculus, the *total differential of* Q is

$$dQ = \left(\frac{\partial Q}{\partial x}\right)_{y,z,t} dx + \left(\frac{\partial Q}{\partial y}\right)_{x,z,t} dy + \left(\frac{\partial Q}{\partial z}\right)_{x,y,t} dz + \left(\frac{\partial Q}{\partial t}\right)_{x,y,z} dt \tag{1.25}$$

where the subscripts refer to the independent variables that are held constant whilst taking the indicated partial derivatives. Upon dividing both sides of (1.25) by dt, the total differential of t which represents a time increment, the resulting expression is

$$\frac{dQ}{dt} = \left(\frac{\partial Q}{\partial t}\right)\frac{dt}{dt} + \left(\frac{\partial Q}{\partial x}\right)\frac{dx}{dt} + \left(\frac{\partial Q}{\partial y}\right)\frac{dy}{dt} + \left(\frac{\partial Q}{\partial z}\right)\frac{dz}{dt} \tag{1.26}$$

where the subscripts on the partial derivatives have been dropped for convenience. The rates of change of x, y, or z with respect to time are simply the component velocities in the x, y, or z directions. We will refer to these velocities as u, v, and w and define them as $u = dx/dt$, $v = dy/dt$, and $w = dz/dt$, respectively. Substituting these expressions into (1.26) yields

$$\frac{dQ}{dt} = \left(\frac{\partial Q}{\partial t}\right) + u\left(\frac{\partial Q}{\partial x}\right) + v\left(\frac{\partial Q}{\partial y}\right) + w\left(\frac{\partial Q}{\partial z}\right) \tag{1.27}$$

which can be rewritten in vector notation as

$$\frac{dQ}{dt} = \left(\frac{\partial Q}{\partial t}\right) + \vec{V}\cdot\nabla Q \tag{1.28}$$

where $\vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$ is the 3-D vector wind. The three terms in (1.27) involving the component winds and derivatives of Q physically represent the horizontal and vertical transport of Q by the flow. Thus, we see that dQ/dt corresponds to the Lagrangian rate of change noted in (1.24). The Eulerian rate of change is represented by $\partial Q/\partial t$. The rate of importation by the flow (recall it was subtracted from the Eulerian change on the RHS of (1.24)) is represented by $-\vec{V}\cdot\nabla Q$ (*minus* the dot product of the velocity vector and the gradient of Q). In subsequent discussions in this book, $-\vec{V}\cdot\nabla Q$ will be referred to as **advection of** Q. Next we show that the mathematical expression $-\vec{V}\cdot\nabla Q$ actually describes the rate of importation of Q by the flow.

Consider the isotherms (lines of constant temperature) and wind vector shown in Figure 1.7. The gradient of temperature (∇T) is a vector that always points from lowest temperatures to highest temperatures as indicated. The wind vector, clearly drawn in Figure 1.7 so as to transport warmer air toward point A, is directed opposite to ∇T. Recall that the dot product is given by $\vec{V}\cdot\nabla T = |\vec{V}||\nabla T|\cos\alpha$ where α is the angle between the vectors $\vec{V}$ and ∇T. Given that the angle between $\vec{V}$ and ∇T is 180° in Figure 1.7, the dot product $\vec{V}\cdot\nabla T$ returns a negative value. Therefore, the sign of $\vec{V}\cdot\nabla T$ does not accurately reflect the reality of the physical situation depicted in Figure 1.7 – that is, that importation of *warmer air* is occurring at point A.

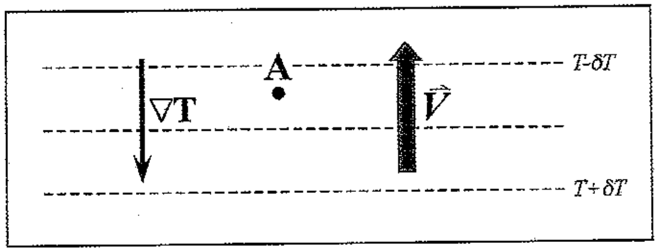

Figure 1.7 Isotherms (dashed lines) and wind vector $\vec{V}$ (filled arrow) surrounding point A. The thin black arrow is the horizontal temperature gradient vector

Thus, we define temperature advection, a measure of the rate (and sign) of importation of temperature to point A, as $-\vec{V} \cdot \nabla T$. The physical situation depicted in Figure 1.7, therefore, is said to be characterized by positive temperature (or warm air) advection.

To round out this discussion, we now return to the example that motivated the mathematical development: measuring the temperature change on my back porch. Rearranging (1.28) and substituting T (temperature) for Q we get

$$\left(\frac{\partial T}{\partial t}\right) = \frac{dT}{dt} - \vec{V} \cdot \nabla T$$

which shows that the Eulerian (fixed location) change is equal to the sum of the Lagrangian (parcel following) change and advection. In the prior example we imagined a temperature drop at my back porch. We also surmised that the temperature of individual air parcels did not undergo any change as the day wore on. Thus, the advective change at the porch must be negative – there must be negative temperature advection, or cold air advection (i.e. $-\vec{V} \cdot \nabla T < 0$), occurring in Madison on this day. Clearly, the situation of northwesterly winds importing cold air southward out of Canada fits the bill.

1.3 Estimating with Scale Analysis

In many fluid dynamical problems, it is convenient and insightful to estimate which physical terms are likely to contribute most to a particular process under study. For instance, in assessing the threat to coastal property in Hawaii in the face of a major tsunami, it is not likely that the ambient wind speed will figure into the problem in any significant way. In the development of the equations of motion in subsequent chapters, a variety of physical processes will be confronted, each of which has some bearing on the behavior of the fluid atmosphere. At many junctures, however, we will attempt to simplify those equations by estimating the magnitude of the mathematical terms that comprise them. A formal process known as scale analysis is employed in such an exercise. Here we illustrate, with a very simple example, the power of scale analysis as an analytical tool.

Imagine you are charged with filling an Olympic-sized swimming pool with water. Your boss wants to know how long it will take to get the job done and asks you for an estimate of the completion time. In order to make a reasonable approximation, you need to know a number of physical characteristics of the problem. You certainly need to know the volume of the pool and the flow rate you can expect from the hose you will use to fill the pool. You might want to know if there are cracks in the pool walls through which seepage might occur. Though it is surely physically relevant, you probably guess that you needn't concern yourself with the evaporation rate of water from the surface of the filling pool.

All four of the above-mentioned physical characteristics can be measured with varying degrees of accuracy. The volume is likely to be a fairly accurate measurement as is the flow rate from the hose. Seepage rate and evaporation rates, however, are likely to be quite difficult to measure accurately. Imagine we do, in fact, make some measurements of each of these characteristics, assigning an estimated (but characteristic) rate to each of the last three. The flow rate is found to be approximately $100\ \mathrm{m^3\,h^{-1}}$, the evaporation rate $0.001\ \mathrm{m^3\,h^{-1}}$, the seepage rate $0.000\,01\ \mathrm{m^3\,h^{-1}}$. It is clear upon comparison of the three that the flow rate is the most important process (it is five to seven orders of magnitude larger than the others). Therefore, we could say that, subject to some small amount of error, the time needed to fill the pool is equal to

$$t_{fill} \approx \frac{\mathit{Volume\ of\ the\ Pool}}{\mathit{Flow\ Rate}}.$$

We will achieve a similar simplification of the equations of motion by similarly estimating the scale of various terms that appear in those equations.

1.4 Basic Kinematics of Fluids

As can be readily discerned from inspection of any satellite animation of clouds or water vapor, the wind field varies in the x and y directions. Therefore, there are x and y derivatives of the horizontal wind components, u and v. In fact, there are only four such derivatives: $\partial u/\partial x$ and $\partial u/\partial y$ along with $\partial v/\partial x$ and $\partial v/\partial y$. Let us consider all possible sums of these four derivatives with the stipulation that each sum must include a derivative of u with respect to one direction and a derivative of v with respect to the other. Under this condition there are only four independent, linear combinations of x and y derivatives of the horizontal wind, namely $\partial u/\partial x \pm \partial v/\partial y$ and $\partial v/\partial x \pm \partial u/\partial y$. We will now consider what these derivative combinations describe about the fluid flow and we will do it by considering Taylor series expansions of the functions $u(x,y)$ and $v(x,y)$. Since u and v are continuous functions of x and y

space, the expansion of each about some arbitrary point in space (say $(x,y) = (0, 0)$) becomes

$$u(x, y) = u_0 + \left(\frac{\partial u}{\partial x}\right)_0 x + \left(\frac{\partial u}{\partial y}\right)_0 y + \left(\frac{\partial^2 u}{\partial x^2}\right)_0 \frac{x^2}{2} + \left(\frac{\partial^2 u}{\partial y^2}\right)_0 \frac{y^2}{2} + \textit{Higher Order Terms} \qquad (1.29a)$$

$$v(x, y) = v_0 + \left(\frac{\partial v}{\partial x}\right)_0 x + \left(\frac{\partial v}{\partial y}\right)_0 y + \left(\frac{\partial^2 v}{\partial x^2}\right)_0 \frac{x^2}{2} + \left(\frac{\partial^2 v}{\partial y^2}\right)_0 \frac{y^2}{2} + \textit{Higher Order Terms.} \qquad (1.29b)$$

If we neglect the terms of order 2 and greater (the so-called higher order terms), which is eminently defensible because they are generally very small, we have

$$u - u_0 = \left(\frac{\partial u}{\partial x}\right)_0 x + \left(\frac{\partial u}{\partial y}\right)_0 y \qquad (1.30a)$$

$$v - v_0 = \left(\frac{\partial v}{\partial x}\right)_0 x + \left(\frac{\partial v}{\partial y}\right)_0 y \qquad (1.30b)$$

where we have written $u(x, y)$ and $v(x, y)$ more conveniently as u and v, respectively.

Returning to our four independent linear combinations of x and y derivatives of the wind field, we next assign names to each combination. We will let $\partial u/\partial x + \partial v/\partial y = D$ where D is the **divergence**. We will let $\partial u/\partial x - \partial v/\partial y = F_1$ where F_1 is the **stretching deformation**. We will let $\partial v/\partial x + \partial u/\partial y = F_2$ where F_2 is the **shearing deformation**. Finally, we will let $\partial v/\partial x - \partial u/\partial y = \zeta$ where ζ is the **vorticity**. Given these definitions, we can rewrite (1.30a) and (1.30b) in terms of these quantities as

$$u - u_0 = \frac{1}{2}(D + F_1)x - \frac{1}{2}(\zeta - F_2)y = \frac{1}{2}(Dx + F_1 x - \zeta y + F_2 y) \qquad (1.31a)$$

$$v - v_0 = \frac{1}{2}(\zeta + F_2)x + \frac{1}{2}(D - F_1)y = \frac{1}{2}(\zeta x + F_2 x + Dy - F_1 y). \qquad (1.31b)$$

By assuming that u_0 and v_0 (the u and v velocities at our arbitrary origin point) are both zero we can quite readily use the expressions (1.31a) and (1.31b) to investigate what each of the four derivative fields looks like physically. We will consider each quantity in isolation even though, in nature, they all can occur simultaneously in a given observed flow.

Figure 1.

1.4.1 Pure vorticity

In order to examine what a f
and (1.31b) and set D, F_1,
(1.31a) and (1.31b) become
plot u and v for the case of p
that a field of pure positive
counterclockwise flow about

1.4.2 Pure divergence

An example of pure positiv
equal to zero while $D = 1$.
$v = \frac{1}{2}y$, respectively. Figure
all directions away from the
origin. Such a picture is con
Notice that if we had assum
toward the origin – consiste
fact, we will refer to negative
studies.

1.4.3 Pure stretching def

Pure stretching deformatio
$F_1 = 1$. In this case, (1.31a)

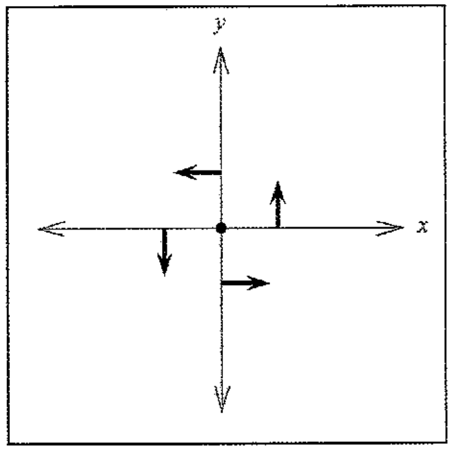

Figure 1.8 A field of pure, positive vorticity ($\zeta = 1$)

1.4.1 Pure vorticity

In order to examine what a flow with pure positive vorticity looks like, we use (1.31a) and (1.31b) and set D, F_1, and F_2 equal to zero while letting $\zeta = 1$. In such a case, (1.31a) and (1.31b) become $u = -\frac{1}{2}y$ and $v = \frac{1}{2}x$. Employing a Cartesian grid we plot u and v for the case of pure vorticity at a number of points in Figure 1.8. We find that a field of pure positive vorticity (recall that we set $\zeta = 1$) describes a *circular, counterclockwise* flow about the origin.

1.4.2 Pure divergence

An example of pure positive divergence occurs in a flow when ζ, F_1, and F_2 are all equal to zero while $D = 1$. In such a case, (1.31a) and (1.31b) become $u = \frac{1}{2}x$ and $v = \frac{1}{2}y$, respectively. Figure 1.9 illustrates the resulting flow field: a fluid moving *in all directions away from the origin*, at speeds proportional to the distance from the origin. Such a picture is consistent with the colloquial sense of the word 'divergence'. Notice that if we had assumed a value of $D = -1$ instead, we would get fluid moving toward the origin – consistent with the colloquial sense of the word 'convergence'. In fact, we will refer to negative divergence as convergence quite often in our subsequent studies.

1.4.3 Pure stretching deformation

Pure stretching deformation is obtained by setting D, ζ, and F_2 equal to zero while $F_1 =1$. In this case, (1.31a) and (1.31b) become $u = \frac{1}{2}x$ and $v = -\frac{1}{2}y$, respectively.

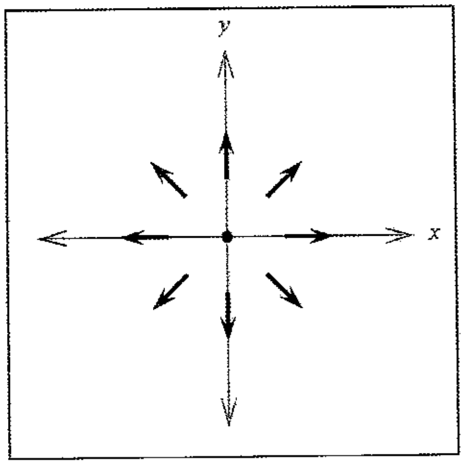

Figure 1.9 A field of pure, positive divergence ($D = 1$)

Figure 1.10 demonstrates that the resulting flow field is stretched along the x-axis and compressed along the y-axis. In fact, these two axes have special names: the flow is stretched along the **axis of dilatation** while it is compressed along the **axis of contraction.** It is important to distinguish between deformation and convergence as they are commonly confused. If we consider the area of a fluid element bounded by curve C embedded within a field of pure convergence (Figure 1.11a) we see immediately that the area will become progressively smaller under the influence of the convergent flow. If the same fluid parcel were placed in a field of pure stretching deformation, however, the shape of the originally square fluid element would be

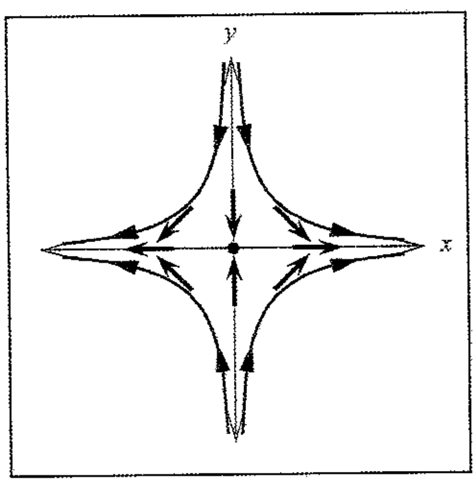

Figure 1.10 A field of pure, positive stretching deformation ($F_1 = 1$). The dark solid lines are streamlines of the deformation field. The x-axis serves as the axis of dilatation and the y-axis is the axis of contraction

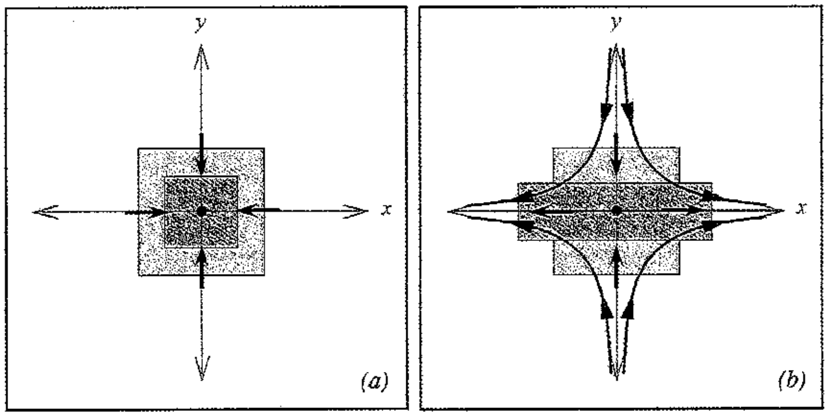

Figure 1.11 (a) A fluid element in a field of pure convergence. The lighter square represents the initially square element. Note that the area of the fluid element is decreased in a field of convergence. (b) A fluid element in a field of pure stretching deformation. The original square is deformed into a rectangle whose area is the same as that of the square

deformed into a rectangle while preserving its area (Figure 1.11b). The proof that the area is unchanged is left to the reader as an exercise. This essential physical distinction between convergence and deformation (specifically, confluence) is made manifest to the driver of an automobile using an entrance ramp to a highway. The flow of traffic is **confluent** (i.e. resembles the flow in the vicinity of the x-axis in Figure 1.10) between the entrance ramp and the highway but it is certainly not convergent. If it were, the number of accidents would be staggering!

1.4.4 Pure shearing deformation

Pure shearing deformation is obtained by letting $F_2 = 1$ while setting D, ζ, and F_1 equal to zero. By doing so, (1.31a) and (1.31b) become $u = \frac{1}{2}y$ and $v = \frac{1}{2}x$. The resulting flow field (Figure 1.12) looks like the stretching deformation rotated counterclockwise by 45°. So, how do we tell the difference between the stretching and shearing deformations and is the difference even important physically? It turns out that most often we are concerned with the **total deformation** without regard to the separate expressions for F_1 and F_2. The total deformation is given by

$$F = \left(F_1^2 + F_2^2\right)^{\frac{1}{2}} \tag{1.32}$$

where F represents the resultant magnitude of what appears to be a deformation vector with components, F_1 and F_2. It is clear that with a rotation of 45° of the coordinate axes, we can transform $F_1 = 1$ into $F_1' = 0$ and $F_2 = 0$ into $F_2' = 1$.

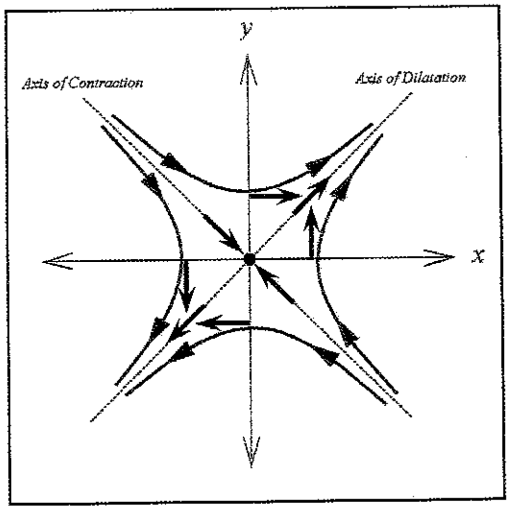

Figure 1.12 A field of pure, positive shearing deformation ($F_2 = 1$). The dark solid lines are streamlines of the deformation field. The axes of dilatation and contraction are indicated by the dashed lines

Thus, deformation is **rotationally variant.** In fact, if one rotates the coordinate axes by the angle

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{F_2}{F_1}\right) \tag{1.33}$$

then the resultant deformation has its axis of dilatation at an angle θ counterclockwise from the original x-axis. It is clear that any rotation of the x- and y-axes will have no effect whatever on the vorticity or divergence. As a result, these two properties of the flow are known as **rotationally invariant** or **Galilean invariant.** This characteristic vests the vorticity and divergence with considerable power in explaining the behavior of fluids, as we will see.

1.5 Mensuration

Before we embark upon our investigation of the forces that govern the behavior of the fluid atmosphere, we must explicitly lay out the units with which we will measure the quantities of interest. Throughout the remainder of the text we will employ the **Système Internationale** (SI) units shown in Table 1.1.

Table 1.1 Standard SI units

Property	Name	Symbol
Length	Meter	m
Mass	Kilogram	kg
Time	Second	s
Temperature	Kelvin	K

Table 1.2 Important SI derived units

Property	Name	Symbol
Frequency	Hertz	Hz (s^{-1})
Force	Newton	N (kg m s^{-2})
Pressure	Pascal	Pa (N m^{-2})
Energy	Joule	J (N m)
Power	Watt	W (J s^{-1})

Additionally, a number of derived quantities will be referenced throughout our study and they are shown in Table 1.2.

Despite the fact that we will refer to temperature in °C (or occasionally in °F when using an older diagram to illustrate a point), it is important to remember to use SI units in all calculations you may have to make.

Selected References

A complete reference list is provided in the Bibliography at the end of the book.

Spiegel, M. R., *Vector Analysis and an Introduction to Tensor Analysis*, is an outstanding, concise text on vector calculus with nearly 500 solved problems.

Thomas and Finney, *Calculus and Analytic Geometry*, provides additional detail on the Taylor series and fundamental calculus.

Hess, *Introduction to Theoretical Meteorology*, discusses the basic kinematics of fluids.

Saucier, *Principles of Meteorological Analysis*, is another fine reference on kinematics.

Problems

1.1. Let $\vec{A} = \nabla\phi = 8x\hat{i} + 3y^2\hat{j}$. If you know that $\phi(1,1) = 8$ and $\phi(0,1) = 4$, derive a functional expression for $\phi(x, y)$.

1.2. Prove the vector identities in (a) – (c) letting $\vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$ and $\nabla = \dfrac{\partial}{\partial x}\hat{i} + \dfrac{\partial}{\partial y}\hat{j} + \dfrac{\partial}{\partial z}\hat{k}$:

(a) $\nabla \cdot (\nabla \times \vec{V}) = 0$
(b) $(\vec{V} \cdot \nabla)\vec{V} = (1/2)\nabla(\vec{V} \cdot \vec{V}) - \vec{V} \times (\nabla \times \vec{V})$
(c) $\nabla \cdot (f\vec{V}) = f(\nabla \cdot \vec{V}) + \vec{V} \cdot \nabla f$
(d) Prove that $\hat{k} \times (\hat{k} \times \vec{A}) = -\vec{A}$ where $\vec{A} = A_1\hat{i} + A_2\hat{j}$.
(e) Use the 'right hand rule' to verify (d) graphically.

1.3. The symbol $\vec{A}_{\vec{B}}$ stands for the projection of vector $\vec{A}$ onto vector $\vec{B}$. In other words, $\vec{A}_{\vec{B}}$ represents the component of $\vec{A}$ that is parallel to $\vec{B}$. Derive an expression for $\vec{A}_{\vec{B}}$ in terms of the vectors $\vec{A}$ and $\vec{B}$.

1.4. Show that a field of pure deformation (i.e. the combination of both components, F_1 and F_2) has no divergence and no vorticity.

1.5. Consider Figure 1.1A which shows isotherms (dashed lines) in fields of pure vorticity, pure convergence (negative divergence), and deformation. The vector ∇T has both magnitude and direction.

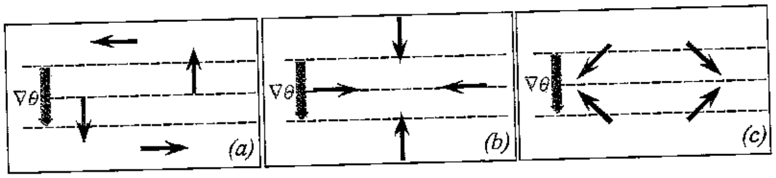

Figure 1.1A

(a) Do you think the vorticity can change both the direction and magnitude of ∇T? Does the orientation of the isotherms affect the answer to the first question? Explain.

(b) Do you think the convergence can change both the direction and magnitude of ∇T? Does the orientation of the isotherms affect the answer to the first question? Explain.

(c) Do you think the deformation can change both the direction and magnitude of ∇T? Does the orientation of the isotherms affect the answer to the first question? Explain.

1.6. Consider a fluid element with area, $A = \delta x\, \delta y$.

(a) Derive an expression for the time rate of change of this area, dA/dt. (Hint:

$$\frac{d}{dt}(\delta F) = \delta\left(\frac{dF}{dt}\right)$$

where F is any variable.)

(b) What kinematic field is represented by

$$\frac{1}{A}\frac{dA}{dt}?$$

Defend your choice.

(c) Describe (with a word) the type of flow that will result in a decrease in A. Defend your choice with a diagram and accompanying explanation.

1.7. Find the angle between the surfaces $2x^2 - y^2 + z^2 = 9$ and $3z = x^2 - 4y^2 + 5$ at the point $(2, 1, -2)$.

1.8. If $\nabla\phi = 2xyz^2\hat{i} + x^2z^2\hat{j} + 2x^2yz\hat{k}$, find $\phi(x, y, z)$ if $\phi(1, -2, 2) = 4$.

1.9. Prove that $\nabla^2(\alpha\beta) = \alpha\nabla^2\beta + 2\nabla\alpha \cdot \nabla\beta + \beta\nabla^2\alpha$ where αgnd β are scalar functions.

1.10. An automobile equipped with a thermometer is heading southward at 100 km h^{-1}, bound for a location 300 km away. During transit, the temperature drops to −5°C at the origin. If the temperature at departure was measured to be 0°C and the temperature tendency measured along the journey is +5°C h^{-1}, what temperature should the travelers expect at their destination?

1.11. Demonstrate that $\vec{A} \cdot (\vec{B}$

1.12. A car is driving straight
pressure decreases towa
at the service station if
50 Pa/3 h?

1.13. Imagine a stably stratifi
What must be the relatio
Give a physical explanati

Solutions

1.1. $\phi(x, y) = 4x^2 + y^3 + 3$

1.7. $\alpha = 46.06°$

1.8. $\phi(x, y, z) = x^2yz^2 + 12$

1.10. The temperature at the d

1.12. The pressure falls at a rat

1.13. $w = \frac{-\vec{V}\cdot\nabla T}{(\partial T/\partial z)}$

1.11. Demonstrate that $\vec{A} \cdot (\vec{B} \times \vec{C}) = -\vec{B} \cdot (\vec{A} \times \vec{C})$.

1.12. A car is driving straight southward, past a service station, at 100 km h^{-1}. The surface pressure decreases toward the southeast at 1 Pa km^{-1}. What is the pressure tendency at the service station if the pressure measured by the car is decreasing at a rate of 50 Pa/3 h?

1.13. Imagine a stably stratified, steady-state flow in which temperature (T) is conserved. What must be the relationship between horizontal advection of T and vertical motion? Give a physical explanation of this relationship.

Solutions

1.1. $\phi(x, y) = 4x^2 + y^3 + 3$

1.7. $\alpha = 46.06°$

1.8. $\phi(x, y, z) = x^2 yz^2 + 12$

1.10. The temperature at the destination will be 15°C.

1.12. The pressure falls at a rate of 87.38 Pa h^{-1}.

1.13. $w = \frac{-\vec{V} \cdot \nabla T}{(\partial T / \partial z)}$