Nonlinear equations

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Nonlinear equations

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a nonlinear real-valued function in a single variable x. We are interested in finding the solutions of the equation $f(x) = 0$, i.e., the roots of $f(x)$.

Examples

- Let us look at three functions (polynomials):
  - $f_1(x) = x^4 - 12x^3 + 47x^2 - 60x$
  - $f_2(x) = x^4 - 12x^3 + 47x^2 - 60x + 24$
  - $f_3(x) = x^4 - 12x^3 + 47x^2 - 60x + 24.1$
- Finding the zeros of these polynomials is not an easy task:
  - $f_1(x)$ has real zeros 0, 3, 4, and 5.
  - $f_2(x)$ has real zeros 1, 0.8883.
  - $f_3(x)$ has no real zeros at all.

Nonlinear equations

Consider the nonlinear equation $f(x) = 0$.

- Suppose the solution exists and is unique, then the problem becomes how we can find it.
- We will study iterative methods for finding the solution; namely, generate a sequence $x_0, x_1, x_2, \ldots$ that converges to the solution.
- Iterative methods:
  - bisection method
  - Newton's method
  - secant method

Bisection method (method of interval halving)

- **An observation:** If $f(x)$ is a continuous function on an interval $[a, b]$, and $f(a)$ and $f(b)$ have different signs such that $f(a)f(b) < 0$, then $f(x)$ must have a zero in (a, b). (ensured by the IVT for continuous functions)
- **The basic idea:** Assume that $f(a)f(b) < 0$.
  - Compute $c = (a + b)/2$.
  - If $f(c) = 0$, then c is a zero of $f(x)$.
  - If $f(a)f(c) < 0$, then the zero is in $[a, c]$; otherwise the zero is in $[c, b]$. The length of the new interval is half of the original one.
  - Repeat the process until the interval is very small then any point in the interval can be used as approximations of the zero.

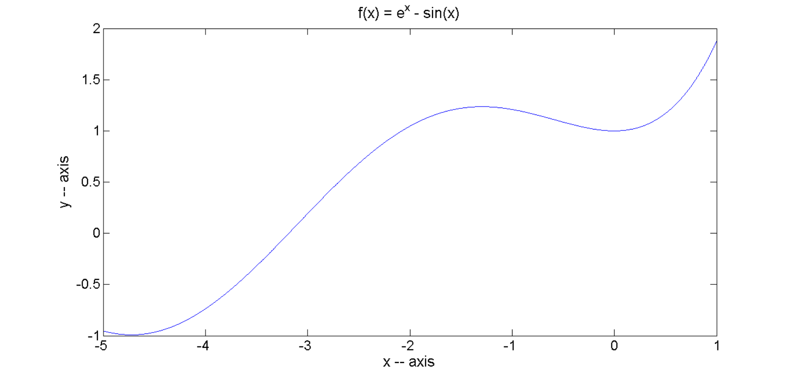
What do we need?

- We need an intial interval $[a, b]$.
- We need some stopping criteria: given $\epsilon > 0, \delta > 0$, and M
  - If $|f(c)| < \epsilon$, we stop.
  - If $|b - a| < \delta$, we stop.
  - If $k > M$, we stop to avoid infinite loop.

An example

Use the bisection method to find the solution of $e^x = \sin(x)$.

Let $f(x) = e^x - \sin(x)$ and its graph is shown below. The initial interval can be chosen as $[-4, -3]$ (what if we don't konw its graph? a larger initial interval should be guessed).



An example $(f(x) := e^x - \sin(x) = 0)$

```
% matlab code for bisection method
a = -4; b = -3; M = 30; delta = 1e-6; epsilon = 1e-6;
fa = exp(a)-sin(a); fb = exp(b)-sin(b);
format long
for k = 1 : M
    c = (a+b)/2; fc = exp(c)-sin(c);
    if b-a < delta  &&  abs(fc) < epsilon
        break
    end
    if sign(fc) ~= sign(fa)
        b = c;
        fb = fc;
    else
        a = c;
        fa = fc;
    end
end
[k, c, fc]
format short
```

Other methods?

Some major problems with the bisection method

- Finding the initial interval is not easy.
- Often slow.
- Doesn't work for system of nonlinear equations.

Newton's method

Suppose $f(x)$ is differentiable on its domain. Consider the following iteration:

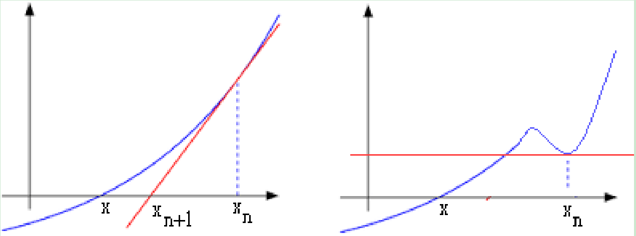
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, n = 0, 1, 2, \ldots$$

Note that we need an initial guess x_0 to start the iteration.

The geometrical interpretation

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, n = 0, 1, 2, \ldots$$

$\Big(y = f(x_n) + f'(x_n)(x - x_n)$ is the tangent line of f at $(x_n, f(x_n))\Big)$



- We see that x_{n+1} is a better approximation of x than x_n.
- What is the geometrical meaning of $f'(x_n) = 0$?

Some stopping criteria

- Using residual information $f(x_k)$:
  - if $|f(x_k)| < \epsilon$ then stop
  - if $|f(x_k)| < \epsilon |f(x_0)|$ then stop
- Using the step size information $|x_{k+1} - x_k|$:
  - if $|x_{k+1} - x_k| < \delta$ then stop
  - if $\dfrac{|x_{k+1} - x_k|}{|x_{k+1}|} < \delta$ then stop
- Maximum number of iterations M.

An example ($f(x) := x^2 - 2 = 0$)

```
% matlab code for Newton's method
x = 1; M = 10; delta = 1e-3; epsilon = 1e-3;
format long
for n = 1 : M
    xold = x;
    %x = x - (x^2-2)/(2*x)
    x = x/2 + 1/x;
    fx = x^2 - 2;
    if abs(xold-x) < delta && abs(fx) < epsilon
        break
    end
end
[n, x, fx]
format short
```

Quiz

Use Newton's method to find the solution of $f(x) = 0$:

1. $f(x) = e^x - \cos(x)$ with $x_0 = -1$.
2. $f(x) = x^3 - 3x - 1$ with $x_0 = 2$.

Use $M = 10; delta = 1e-3; epsilon = 1e-3;$