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EC414 HW8

$\det(A-\lambda I)=0$

1 (a)

$$\det\left(\begin{bmatrix}0&2\\2&-3\end{bmatrix}-\begin{bmatrix}\lambda&0\\0&\lambda\end{bmatrix}\right)=0 \longrightarrow \det\begin{bmatrix}-\lambda&2\\2&-3-\lambda\end{bmatrix}=-\lambda(-3-\lambda)-4=0$$

$$\lambda^2+3\lambda-4=0 \rightarrow (\lambda-1)(\lambda+4)=0$$

so $\boxed{\lambda_1=1,\ \lambda_2=-4}$

$$A\vec{v}_1=\lambda_1\vec{v}_1 \rightarrow A\vec{v}_1-\lambda_1\vec{v}_1=0 \rightarrow (A-\lambda_1 I)\vec{v}_1=0$$

$$\left(\begin{bmatrix}0&2\\2&-3\end{bmatrix}-\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)\begin{bmatrix}V_{11}\\V_{12}\end{bmatrix}=0 \longrightarrow \begin{bmatrix}-1&2\\2&-4\end{bmatrix}\begin{bmatrix}V_{11}\\V_{12}\end{bmatrix}=0 \rightarrow -V_{11}+2V_{12}=0$$

so $V_{11}=2V_{12}$ $\quad \boxed{\vec{V}_1=\begin{bmatrix}2\\1\end{bmatrix} \text{ normalized } \frac{\vec{v}_1}{\|\vec{v}_1\|}=\begin{bmatrix}2/\sqrt{5}\\1/\sqrt{5}\end{bmatrix}}$

$$(A-\lambda_2 I)\vec{V}_2=0 \longrightarrow \left(\begin{bmatrix}0&2\\2&-3\end{bmatrix}-\begin{bmatrix}-4&0\\0&-4\end{bmatrix}\right)\begin{bmatrix}V_{21}\\V_{22}\end{bmatrix}=0 \longrightarrow \begin{bmatrix}4&2\\2&1\end{bmatrix}\begin{bmatrix}V_{21}\\V_{22}\end{bmatrix}=0$$

$4V_{21}+2V_{22}=0$ so $4V_{21}=-2V_{22} \longrightarrow 2V_{21}=-V_{22}$

$$\boxed{\vec{V}_2=\begin{bmatrix}-1\\2\end{bmatrix} \text{ normalized } \frac{\vec{V}_2}{\|\vec{v}_2\|}=\begin{bmatrix}-1/\sqrt{5}\\-2/\sqrt{5}\end{bmatrix}}$$

(b)

$$\det\left(\begin{bmatrix}-6&3\\4&5\end{bmatrix}-\begin{bmatrix}\lambda&0\\0&\lambda\end{bmatrix}\right)=0 \longrightarrow \det\begin{bmatrix}-6-\lambda&3\\4&5-\lambda\end{bmatrix}=(-6-\lambda)(5-\lambda)-12=0$$

$$-30+\lambda+\lambda^2-12=\lambda^2+\lambda-42=0$$

$$(\lambda+7)(\lambda-6) \qquad \boxed{\lambda_1=6,\ \lambda_2=-7}$$

$\lambda_1=6$

$$\left(\begin{bmatrix}-6&3\\4&5\end{bmatrix}-\begin{bmatrix}6&0\\0&6\end{bmatrix}\right)\begin{bmatrix}V_{11}\\V_{12}\end{bmatrix}=0 \longrightarrow \begin{bmatrix}-12&3\\4&-1\end{bmatrix}\begin{bmatrix}V_{11}\\V_{12}\end{bmatrix}=0 \longrightarrow -12V_{11}+3V_{12}=0$$

so $V_{12}=4V_{11} \rightarrow$ $\boxed{\vec{V}_1=\begin{bmatrix}1\\4\end{bmatrix} \text{ normalized } \vec{v}_1=\begin{bmatrix}1/\sqrt{17}\\4/\sqrt{17}\end{bmatrix}}$

$$\left(\begin{bmatrix} -6 & 3 \\ 4 & 5 \end{bmatrix} - \begin{bmatrix} -7 & 0 \\ 0 & -7 \end{bmatrix}\right)\begin{bmatrix} V_{21} \\ V_{22} \end{bmatrix} = 0 \longrightarrow \begin{bmatrix} 1 & 3 \\ 4 & 12 \end{bmatrix}\begin{bmatrix} V_{21} \\ V_{22} \end{bmatrix} = 0 \qquad V_{21} + 3V_{22} = 0$$

so $V_{21} = -3V_{22}$ so $\boxed{\vec{V}_2 = \begin{bmatrix} -3 \\ 1 \end{bmatrix} \text{ normalized } \vec{V}_2 = \begin{bmatrix} -3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix}}$

$\boxed{2}$ ⓐ

$$\vec{m} = \frac{1}{4}\begin{bmatrix} -1 + 0 + -2 + 3 \\ -1 + -2 + 2 + -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{2} \end{bmatrix}$$

$$\tilde{X}_1 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} -1 \\ -\frac{1}{2} \end{bmatrix} \qquad \tilde{X}_3 = \begin{bmatrix} -2 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 \\ -1/2 \end{bmatrix} = \begin{bmatrix} -2 \\ \frac{5}{2} \end{bmatrix}$$

$$\tilde{X}_2 = \begin{bmatrix} 0 \\ -2 \end{bmatrix} - \begin{bmatrix} 0 \\ -1/2 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{3}{2} \end{bmatrix} \qquad \tilde{X}_4 = \begin{bmatrix} 3 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 3 \\ -\frac{1}{2} \end{bmatrix}$$

so $\boxed{\tilde{X} = \begin{bmatrix} -1 & -\frac{1}{2} \\ 0 & -\frac{3}{2} \\ -2 & \frac{5}{2} \\ 3 & -\frac{1}{2} \end{bmatrix}}$

ⓑ $\Sigma = \frac{1}{4}\tilde{X}^T\tilde{X}$

$$= \frac{1}{4}\begin{bmatrix} -1 & 0 & -2 & 3 \\ -\frac{1}{2} & -\frac{3}{2} & \frac{5}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} -1 & -\frac{1}{2} \\ 0 & -\frac{3}{2} \\ -2 & \frac{5}{2} \\ 3 & -\frac{1}{2} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1+0+4+9 & \frac{1}{2}+0-5-\frac{3}{2} \\ \frac{1}{2}+0-5-\frac{3}{2} & \frac{1}{4}+\frac{9}{4}+\frac{25}{4}+\frac{1}{4} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 14 & -6 \\ -6 & 9 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} \frac{14}{4} & \frac{-6}{4} \\ \frac{-6}{4} & \frac{9}{4} \end{bmatrix} \longrightarrow \boxed{\Sigma = \begin{bmatrix} \frac{7}{2} & \frac{-3}{2} \\ \frac{-3}{2} & \frac{9}{4} \end{bmatrix}}$$

ⓒ $$\det(\Sigma - \lambda I) = 0 \longrightarrow \det\left(\begin{bmatrix} \frac{7}{2} & -\frac{3}{2} \\ -\frac{3}{2} & \frac{9}{4} \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0 \longrightarrow \det\begin{bmatrix} \frac{7}{2}-\lambda & -\frac{3}{2} \\ -\frac{3}{2} & \frac{9}{4}-\lambda \end{bmatrix}$$

$$\frac{63}{8} - \frac{23}{4}\lambda + \lambda^2 - \frac{9}{4} = 0$$

$$\frac{63}{8} - \frac{46}{8}\lambda + \lambda^2 - \frac{18}{8} = 0 \longrightarrow \lambda^2 - \frac{46}{8}\lambda + \frac{45}{8} = 0 = \left(\lambda - \frac{9}{2}\right)\left(\lambda - \frac{5}{4}\right)$$

so $\boxed{\lambda_1 = \frac{9}{2}, \ \lambda_2 = \frac{5}{4}}$

$$\left(\begin{bmatrix} \frac{7}{2} & -\frac{3}{2} \\ -\frac{3}{2} & \frac{9}{4} \end{bmatrix} - \begin{bmatrix} \frac{9}{2} & 0 \\ 0 & \frac{9}{2} \end{bmatrix}\right)\begin{bmatrix} V_{11} \\ V_{12} \end{bmatrix} = 0 \longrightarrow \begin{bmatrix} -1 & -\frac{3}{2} \\ -\frac{3}{2} & -\frac{9}{4} \end{bmatrix}\begin{bmatrix} V_{11} \\ V_{12} \end{bmatrix} = 0 \quad \text{so} \quad -V_{11} - \frac{3}{2}V_{12} = 0$$

$$V_{11} = -\frac{3}{2}V_{12}$$

$$\text{so} \quad \boxed{\vec{V}_1 = \begin{bmatrix} -3 \\ 2 \end{bmatrix} \longrightarrow \frac{\vec{V}_1}{\|\vec{V}_1\|} = \begin{bmatrix} -3/\sqrt{13} \\ 2/\sqrt{13} \end{bmatrix}}$$

$$\left(\begin{bmatrix} \frac{7}{2} & -\frac{3}{2} \\ -\frac{3}{2} & \frac{9}{4} \end{bmatrix} - \begin{bmatrix} \frac{5}{4} & 0 \\ 0 & \frac{5}{4} \end{bmatrix}\right)\begin{bmatrix} V_{21} \\ V_{22} \end{bmatrix} = 0 \longrightarrow \begin{bmatrix} \frac{9}{4} & -\frac{3}{2} \\ -\frac{3}{2} & 1 \end{bmatrix}\begin{bmatrix} V_{21} \\ V_{22} \end{bmatrix} = 0 \quad \text{so} \quad -\frac{3}{2}V_{21} + V_{22} = 0$$

$$\frac{3}{2}V_{21} = V_{22}$$

$$\text{so} \quad \boxed{\vec{V}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \longrightarrow \frac{\vec{V}_2}{\|\vec{V}_2\|} = \begin{bmatrix} 2/\sqrt{13} \\ 3/\sqrt{13} \end{bmatrix}}$$

(d) largest eigenvalue : $\lambda_1 = \frac{9}{2} \longrightarrow \vec{v}_1 = \begin{bmatrix} -3/\sqrt{13} \\ 2/\sqrt{13} \end{bmatrix}$

$$\text{so} \quad \boxed{C = \begin{bmatrix} -3/\sqrt{13} \\ 2/\sqrt{13} \end{bmatrix}}$$

(e) $Y = \tilde{X}C = \begin{bmatrix} -1 & -\frac{1}{2} \\ 0 & -\frac{3}{2} \\ -2 & \frac{5}{2} \\ -3 & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} -3 \\ 2 \end{bmatrix}\frac{1}{\sqrt{13}} = \begin{bmatrix} 2 \\ -3 \\ 11 \\ -10 \end{bmatrix}\frac{1}{\sqrt{13}}$ so $\boxed{Y = \begin{bmatrix} 2/\sqrt{13} \\ -3/\sqrt{13} \\ 11/\sqrt{13} \\ -10/\sqrt{13} \end{bmatrix}}$