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EC414 HW1

1 ⓐ $1-0.16-0.08-0.37-0.06-0.21 = 0.12$

$\boxed{\text{'?' value} = 0.12}$

ⓑ $P(B=1 \mid A=\text{flower}) = \dfrac{P(B=1 \cap A=\text{flower})}{P(A=\text{flower})} = \dfrac{0.16}{0.16+0.37} = 0.301887$

$\boxed{P(B=1 \mid A=\text{flower}) = 0.302}$

ⓒ $P(B=2) = 0.37 + 0.06 + 0.21 \rightarrow \boxed{P(B=2) = 0.64}$

ⓓ $P(A=\text{circle} \mid B=2) = \dfrac{P(A=\text{circle} \cap B=2)}{P(B=2)} = \dfrac{0.21}{0.64} = 0.328125$

$\boxed{P(A=\text{circle} \mid B=2) = 0.328}$

2 ⓐ $E[X] = \mu \quad , E[Y] = \lambda$

$E[A] = 2E[X] + E[Y] \longrightarrow \boxed{E[A] = 2\mu + \lambda}$

$E[B] = E[X] - 2E[Y] \longrightarrow \boxed{E[B] = \mu - 2\lambda}$

ⓑ $E[X^2] = \sigma^2 + \mu^2 \qquad E[Y^2] = \lambda^2 + \lambda$

$Var(A) = E[A^2] - E[A]^2$

$A^2 = 4X^2 + 4XY + Y^2$

$E[A^2] = 4E[X^2] + 4E[X]E[Y] + E[Y^2]$

$= 4(\sigma^2 + \mu^2) + 4(\mu)(\lambda) + (\lambda^2 + \lambda) = 4\sigma^2 + 4\mu^2 + 4\mu\lambda + \lambda^2 + \lambda$

$Var(A) = 4\sigma^2 + 4\mu^2 + 4\mu\lambda + \lambda^2 + \lambda - (2\mu + \lambda)^2$

$(4\mu^2 + 4\mu\lambda + \lambda^2)$

$\boxed{Var(A) = 4\sigma^2 + \lambda}$

$Var(B) = E[B^2] - E[B]^2$

$B^2 = X^2 - 4XY + 4Y^2$

$E[B^2] = E[X^2] - 4E[X]E[Y] + 4E[Y^2]$

$E[B^2] = (\sigma^2+\mu^2) - 4(\mu)(\lambda) + 4(\lambda^2+\lambda) = \sigma^2 + \mu^2 - 4\mu\lambda + 4\lambda^2 + 4\lambda$

$Var(B) = \sigma^2 + \mu^2 - 4\mu\lambda + 4\lambda^2 + 4\lambda - (\mu - 2\lambda)^2$

$(\mu^2 - 4\mu\lambda + 4\lambda^2)$

$\boxed{Var(B) = \sigma^2 + 4\lambda}$

(c) Covariance measures how linearly related 2 random variables are. If Covariance is 0, then the 2 random variables are independent, and if Covariance is 1, then the 2 random variables have a linear relationship. Correlation measures how strongly related 2 random variables are to each other. If 2 random variables are independent, then they are uncorrelated (but the opposite is not always true).

(d) $Cov(A,B) = Cov(2X+Y, X-2Y) = 2Cov(X,X) - 4Cov(X,Y) + Cov(Y,X) - 2Cov(Y,Y)$

$= 2Var(X) - 3Cov(X,Y) - 2Var(Y)$

$Var[X] = \sigma^2$, $Var[Y] = \lambda$, $Cov(X,Y) = E[XY] - E[X]E[Y]$

$= E[X]E[Y] - E[X]E[Y]$

$= \mu\lambda - \mu\lambda = 0$

$\boxed{Cov(A,B) = 2\sigma^2 - 2\lambda}$

(e) 2 Random Variables are independent if observing an event in one Random Variable does not affect the probability of observing the event in the other random variable

A and B are not independent because $Cov(A,B) \neq 0$

3 (a) $\theta_{ML} = \underset{\theta}{\operatorname{argmax}} \prod_{i=1}^{n} \theta e^{-\theta x_i}$

$$= \theta^n e^{-\theta \sum_{i=1}^{n} x_i}$$

$$\ln(P(x|\theta)) = \underset{\theta}{\operatorname{argmax}}\ n\ln(\theta) - \theta \sum_{i=1}^{n} x_i$$

$$\frac{d(\ln(P(x|\theta)))}{d\theta} = \frac{n}{\theta} - \sum_{i=1}^{n} x_i$$

$$\frac{n}{\theta} - \sum_{i=1}^{n} x_i = 0 \rightarrow \boxed{\theta_{MLE} = \frac{n}{\sum_{i=1}^{n} x_i}}$$

(b) $\theta_{ML} = \underset{\theta}{\operatorname{argmax}} \prod_{i=1}^{n} \frac{1}{\theta}$

$$= \frac{1}{\theta^n} = \theta^{-n}$$ ← decreasing function. We want smallest θ such that $\theta > x_i$. So θ has to be the max of $x_1, \dots, x_n$

$$\ln(P(x|\theta)) = \underset{\theta}{\operatorname{argmax}} -n\ln(\theta)$$

$$\frac{d(\ln(P(x|\theta)))}{d\theta} = -\frac{n}{\theta}$$ ← This is <0 for $\theta > 0$

So $\boxed{\theta_{MLE} = \max_{i=1}^{n} \{x_i\} = x_n}$

(c) $\theta_{ML} = \underset{\theta}{\operatorname{argmax}} \prod_{i=1}^{n} \frac{2x_i}{\theta^2} \cdot e^{\frac{-x_i^2}{\theta^2}}$

$$= \underset{\theta}{\operatorname{argmax}} \prod_{i=1}^{n} \frac{2x_i}{\theta^2} e^{-\sum_{i=1}^{n} x_i^2 / \theta^2}$$

$$\ln(P(x|\theta)) = \underset{\theta}{\operatorname{argmax}}\ \underbrace{\ln\left(\prod_{i=1}^{n} \frac{2x_i}{\theta^2}\right)}_{= \ln\left(\frac{2x_1}{\theta^2}\right) + \ln\left(\frac{2x_2}{\theta^2}\right) + \dots + \ln\left(\frac{2x_n}{\theta^2}\right)} - \frac{\sum_{i=1}^{n} x_i^2}{\theta^2}$$

$$= \underset{\theta}{\operatorname{argmax}} \sum_{i=1}^{n} \ln\left(\frac{2x_i}{\theta^2}\right) - \frac{\sum_{i=1}^{n} x_i^2}{\theta^2}$$

$$= \underset{\theta}{\arg\max} \sum_{i=1}^{n} \left(\ln(2x_i) - \ln(\theta^2) \right) - \frac{\sum_{i=1}^{n} x_i^2}{\theta^2}$$

$$= \underset{\theta}{\arg\max} \sum_{i=1}^{n} \ln(2x_i) - n\ln(\theta^2) - \frac{\sum_{i=1}^{n} x_i^2}{\theta^2}$$

$$\frac{d(\ln(P(x|\theta)))}{d\theta} = -\frac{n}{\theta^2} \cdot 2\theta - \left(\frac{-2\sum_{i=1}^{n} x_i^2}{\theta^3} \right)$$

$$= \frac{-2n}{\theta} + \frac{2\sum_{i=1}^{n} x_i^2}{\theta^3}$$

Set to 0 + solve for θ: $\frac{2\sum_{i=1}^{n} x_i^2}{\theta^3} = \frac{2n}{\theta}$

$$\sum_{i=1}^{n} x_i^2 = n\theta^2 \Rightarrow \frac{1}{n}\sum_{i=1}^{n} x_i^2 = \theta^2$$

so $\boxed{\theta_{MLE} = \sqrt{\frac{1}{n}\sum_{i=1}^{n} x_i^2}}$

(d) $P(\theta) = \text{Uniform}(a,b) = \frac{1}{b-a}$ ← not dependent on θ

$P(x|\theta) = \text{Exp}(\theta) = \theta e^{-\theta x}$ $x \geq 0$

$$\theta_{MAP} = \underset{\theta}{\arg\max} \prod_{i=1}^{n} \theta e^{-\theta x} \cdot \left(\frac{1}{b-a}\right)$$

$$= \underset{\theta}{\arg\max} \left[\ln\left(\theta^n e^{-\theta \sum_{i=1}^{n} x_i}\right) + \ln\left(\frac{1}{b-a}\right) \right]$$

$$= \underset{\theta}{\arg\max} \left[\ln\theta^n - \theta\sum_{i=1}^{n} x_i + \ln\left(\frac{1}{b-a}\right) \right]$$

$$\frac{\partial}{\partial\theta}\left(\ln\theta^n - \theta\sum_{i=1}^{n} x_i + \ln\left(\frac{1}{b-a}\right) \right) = \frac{n}{\theta} - \sum_{i=1}^{n} x_i$$

so $\boxed{\hat{\theta}_{MAP} = \frac{n}{\sum_{i=1}^{n} x_i} \quad \text{(same as } \hat{\theta}_{MLE} \text{ from part (a))}}$