CS-2102
Discrete Math
Problem Set 5, Due Friday March 17, 2017 by 5 pm.

Remember that you can work one other person. Be sure to give adequate explanation for your solutions, lack of clarity and completeness may result in points being deducted. <u>Answers given with no work or justification may have points taken off.</u>

(9 pts) 1. Show that these statements about the real number *x* are equivalent: i) x is rational, ii) *x*/2 is rational, iii) 3*x*-1 is rational. The definition of a rational number is:

$$\mathbb{Q} = \left\{ x \middle| \exists a, b \in \mathbb{Z}\ with\ b \neq 0\ where\ x = {}^{a}\!/_{b} \right\}$$

(6 pts) 2. Prove that at least one of the real numbers $a_1, a_2, a_3, \ldots, a_n$ is greater than or equal to the average of these numbers. What kind of proof did you use?

(2 pts) 3. Find a counterexample to the statement that every positive integer can be written as the sum of the squares of three integers.

(6 pts, 3 pts each) 4.

(a) The harmonic mean of two real numbers x and y equals $\frac{2xy}{x+y}$. By computing the arithmetic and harmonic means of different pairs of positive real number, formulate a conjecture about their relative sizes and prove your conjecture.

(b) The quadratic mean of two real number x and y equals $\sqrt{\frac{x^2+y^2}{2}}$. By computing the arithmetic and quadratic means of different pairs of positive real number, formulate a conjecture about their relative sizes and prove your conjecture.

(4 pts) 5. Prove a variation of the triangle inequality, which states that if x and y are real numbers, then $|x| - |y| \leq |x - y|$ (where $|x|$ represents the absolute value of *x*, which equals *x* if $x \geq 0$ and equals –*x* if $x < 0$).

(3 pts) 6. Prove that $\forall x \in \mathbb{Z}$ we have that $x^2 mod\ 3 = 0$ or $x^2 mod\ 3 = 1$.

Definition: Given any nonnegative integer n, the decimal representation of n is an expression of the form: $d_k d_{k-1} \ldots d_2 d_1 d_0$, where k is a nonnegative integer; $d_0, d_1, d_2, \ldots, d_k$ (called the decimal digits of n) are integers from 0 to 9 inclusive; $d_k \neq 0$ unless $n = 0$ and $k = 0$; and
$n = d_k \cdot 10^k + d_{k-1} \cdot 10^{k-1} + \cdots + d_2 \cdot 10^2 + d_1 \cdot 10 + d_0$.
For example $2{,}503 = 2 \cdot 10^3 + 5 \cdot 10^2 + 0 \cdot 10 + 3$.

(18 pts, 3pts for a and 5 pts each for the rest) 7. Prove the following statements:

(a) For all integers a, b, and c, if $a|b$ and $a|c$ then $a|(b + c)$ and $a|(b - c)$.
(b) Prove that if n is any nonnegative integer whose decimal representation ends in 0 or 5, then $5|n$.
(c) Prove that if the decimal representation of a nonnegative integer n ends in $d_1 d_0$ and if $4|d_1 \cdot 10 + d_0$, then $4|n$.
(d) Prove that for any nonnegative integer n, if the sum of the digits of n is divisible by 3, then n is divisible by 3.

(12 pts, 4 pts each) 8. Prove the following statements.

(a) For any integer n, $n^2 - 2$ is not divisible by 4.
(b) For all prime numbers a, b, and c, $a^2 + b^2 \neq c^2$
(c) If a, b, and c are integers and $a^2 + b^2 = c^2$, then at least one of a and b is even.

(8 pts, 4 each) 9. Prove the following statements in two ways: (1) by contraposition and (2) by contradiction. The definition of irrational is any real number that is not rational. (Hint: Write these as if then statements and then write your assumptions for contraposition and contradiction.)

(a) The negative of any irrational number is irrational.
(b) For all integers a, b, and c, if $a|b$ and $a \nmid c$, then $a \nmid (b + c)$.