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Problem Set 5

1) $Q = \{x \mid \exists a b \in \mathbb{Z}$ with $b \neq 0$ where $x = \frac{a}{b}$

i) x is rational- Let x represent a rational number by the given definition, therefore, $x = \frac{a}{b}$, where $b \neq 0$, thus i) is true.

ii) $\frac{x}{2}$ is rational. Again let x represent a Q, where $x = \frac{a}{b}$ and $b \neq 0$, thus:

$x = \frac{\frac{a}{b}}{2}$ or $x = \frac{a}{2b}$ and since we know $a, b \in \mathbb{Z}$

we know an integer times an integer will result in an integer and since we know $b \neq 0$, we know $2b$ will result in some integer and because we proved that there still exists an $a, b \in \mathbb{Z}$, we know x is still rational. So ii) is true.

iii) $3x-1$ is rational. Let x represent a rational number by the given definition, where $x = \frac{a}{b}$, and $a, b \in \mathbb{Z}$ and $b \neq 0$.

$3(\frac{a}{b}) - 1$

$\frac{3a}{3b} - 1$ since we know a and $b \in \mathbb{Z}$, we know an int times an int results in an int. We also know that $\frac{3a}{3b}$ is rational and that 1 is rational. And the difference of 2 rational numbers is rational, therefore all 3 statements about the real number x are equivalent.

2) proof by contradiction:
we can say now that none of the $\mathbb{R}$ numbers $a_1, a_2 \ldots a_n$ is greater than or equal to the average of these numbers.
let the average be denoted as A where:

$$A = \frac{a_1 + a_2 \ldots + a_n}{n}$$

we say that $a_1 < A$, $a_2 < A \ldots a_n < A$

or

$$A \cdot n > a_1 + a_2 \ldots + a_n$$

from here we take A and replace it with its equivalence $\frac{a_1 + a_2 \ldots + a_n}{n}$ where we have:

$$\left(\frac{a_1 + a_2 \ldots a_n}{n}\right) \cdot n > a_1 + a_2 \ldots + a_n$$

Distribute:

★ $a_1 + a_2 \ldots a_n > a_1 + a_2 \ldots + a_n$
here we have a contradiction which means $\neg P \rightarrow F$, which proves that the initial given hypothesis, must be true.

3) Find a counter example to the statement:
Every Positive integer can be written as the sum of the squares of 3 integers.

Initial perfect squares are: 0, 1, 4, 9, 16

1 = 1 + 0 + 0
2 = 1 + 1 + 0
4 = 4 + 0 + 0
5 = 4 + 1 + 0
6 = 4 + 1 + 1
→ 7 = 4 + x + x

At the number 7, we have our first error as the numbers 0, 1, 4 are the only perfect perfect squares able to go into 7, however no possible combination of the 3 add up to 7.

4) a) Harmonic Mean = $\frac{2xy}{x+y}$ Arithmetic Mean: $\frac{x+y}{2}$
let Harmonic mean be denoted HM and Arithmitic mean, AM

HM	points (x, y)	AM
1	1, 1	1
4/3	1, 2	9/6
6/4	1, 3	8/4
8/5	1, 4	25/10

based on the above, we found that the AM $\geq$ HM, thus we can make our conjecture:

$$\frac{x+y}{2} \geq \frac{2xy}{x+y} \quad \text{then} \quad x+y \geq \frac{4xy}{x+y}$$

$$x^2 + 2xy + y^2 \geq 4xy \rightarrow \underline{x^2 + y^2 \geq 2xy} \text{ or } \frac{(x^2+y^2)}{2xy} \geq 1$$

we prove our conjecture by plugging in random variables for (x, y). Test 1: (6, 3) → 45 $\geq$ 36
Test 2: (1, 12) → 145 $\geq$ 24
Test 3: (3, 2) → 13 $\geq$ 12

Which proves our conjecture is true. QED

4) b. Quadratic mean $= \sqrt{\frac{x^2+y^2}{2}}$ Arithmetic $= \frac{x+y}{2}$

let QM denote Quadratic mean, and AM = Arithmitic mean.

QM	points	AM
1	1, 1	1
$\sqrt{5/2}$	1, 2	3/2
$\sqrt{5}$	1, 3	2
$\sqrt{17/2}$	1, 4	5/2
2	2, 2	2

from this we know that QM > AM

$$\frac{x+y}{2} \leq \sqrt{\frac{x^2+y^2}{2}}$$ so...

$$\frac{(x+y)^2}{(2)^2} \leq \frac{x^2+y^2}{2} \rightarrow \left(\frac{x^2+2xy+y^2}{4} \leq \frac{x^2+y^2}{2}\right)\cdot 2$$

then

$$x^2+2xy+y^2 \leq 2x^2+2y^2$$

$$2xy \leq x^2+y^2$$

$$0 \leq x^2-2xy+y^2 \quad \text{or} \quad (x-y)^2 \geq 0$$

we test our conjecture: $(x-y)^2 \geq 0$

test 1: $(1,2) \rightarrow 1 \geq 0$

test 2: $(3,4) \rightarrow 1 \geq 0$

test 3: $(2,7) \rightarrow 25 \geq 0$

thus we can say that our conjecture holds true.

5) prove: if x and y are $\mathbb{R}$, then $|x|-|y| \leq |x-y|$

Proof by Cases:

Case 1: let x and y be positive

let a represent the $\mathbb{R}^+$, x

let b represent the $\mathbb{R}^+$, y

thus $|a|-|b| = a-b$

thus $|a-b| = a-b$

since $|a|-|b|=|a-b|$, we can say $|x|-|y| \leq |x-y|$ is true.

Case 2: let x and y be negative

let a represent the $\mathbb{R}^-$, x

let b represent the $\mathbb{R}^-$, y

thus $|a|-|b| = |(-a)|-|(-b)| \Longrightarrow a-b$

thus $|a-b| = |(-a)-(-b)| \Longrightarrow |-a+b|$

since $a-b < |-a+b|$, we can say $|x|-|y| \leq |x-y|$ is true.

Case 3: let x be positive and y be negative

let a represent a $\mathbb{R}^+$, x

let b represent a $\mathbb{R}^-$, y

thus

$|a|-|b| = |a|-|-b| \Longrightarrow a-b$

$|a-b| = |a-(-b)| \Longrightarrow |a+b|$

since $a-b < |a+b|$, we can say $|x|-|y| \leq |x-y|$ is true.

Since these 3 cases encompass all Real numbers, we can say: if x and y are $\mathbb{R}$, then $|x|-|y| \leq |x-y|$ is true.

PROVE

6.) $\forall x \in \mathbb{Z}$ that $x^2 \bmod 3 = 0$ or $x^2 \bmod 3 = 1$

all x values are integer:

QR theorem $\rightarrow a = bq + r \quad 0 \leq r \leq b \quad a = x,\ b = 3$

$x \bmod 3 \rightarrow x = 3q + r \quad r = 0, 1, \text{or } 2$

Proof by cases:

Case 1: $r = 0$

$x = 3q + 0$

$x^2 = 9q^2$

$9q^2 \bmod 3 = 0$

Case 2: $r = 1$

$x = 3q + 1$

$x^2 = 9q^2 + 6q + 1$

$9q^2 + 6q + 1 \bmod 3 = 1$

Case 3: $r = 2$

$x = 3q + 2$

$x^2 = 9q^2 + 12q + 4 = 9q^2 + 12q + 3 + 1$

$9q^2 + 12q + 4 \bmod 3 = 1$

These three cases demonstrate all possibilities and proves the theorem since in each case $x^2 \bmod 3$ equals 0 or 1

7.) a. For all integers a, b, and c, if $a|b$ and $a|c$ then $a|b+c$ and $a|b-c$

$b = ak$ $c = am$ and $b+c = at$ and $b-c = ap$

Case1 $ak + am = at$ Case2 $ak - am = ap$

a can be divided from both sides $a(k+m) = at$ $a(k-m) = ap$

a clearly divides $b+c$ and $b-c$ when $a|b$ and $a|c$ is true

b. When a nonnegative integer n ends in the digit 0 or 5, there will be some nonnegative integer p so that $n = 10p + d$ where d is the last digit

n can be rewritten as $n = 5 \cdot 2 \cdot p + d$ to show that $10x$ is always divisible by 5. There are two proof cases:

$n = 5 \cdot 2x + 0$ or $n = 5 \cdot 2x + 5$

$5|n$ because $5|n$ because

$5 \cdot (2x) + 0 = 5k$ where k is an integer is true $5 \cdot (2x) + 5 = 5k$ where k is an integer is true

5 divides into the n values in the cases demonstrated which encompass the statement given and therefore proves the statement

C. If the decimal representation ends in $d_1 d_0$ and $4 | d_1 \cdot 10 + d_0$, the $4|n$. n is $d_k d_{k-1} d_{k-2} \ldots d_0$

$d_k \to d_2$ can be represented as $d_k 10^k + d_{k-1} 10^{k-1} \ldots + d_2 10^2$

this sum of d_k to d_2 will always be divisible by 4 since 4 factors into factors of 100 easily. Since 4 will clearly divide into $d_k \to d_2$ digits, then if $4 | d_1 \cdot 10 + d_0$ is true, $4|n$ will certainly be true since n is just the sum of these two digit representations.

7.) d. Prove that for any nonnegative integer n, if the sum of the digits of n is divisible by 3, then n is divisible by 3

n has digits $d_k d_{k-1} \ldots d_0$

n can be written as $n = d_k 10^k + d_{k-1} 10^{k-1} \ldots + d_1 10^1 + d_0 10^0$

$\rightarrow n = d_k(10^k - 1) + d_{k-1}(10^{k-1} - 1) \ldots + d_0(10^0 - 1) + d_k + d_{k-1} \ldots + d_0$

$10^k - 1$ where k is an integer will always result in a number with every digit as a 9 or it will be 0

Since every version of $10^k - 1$ (like $10^{k-1} - 1$) will be a number with digits of 9, they will all certainly be divisible by 3. Therefore, the only remaining factor is whether $d_k + d_{k-1} + \ldots d_0$ is divisible by 3. If that is true, $3 | n$ because $n = d_k(10^k - 1) + d_{k-1}(10^{k-1} - 1) + \ldots d_1 10^1 + d_k + \ldots + d_0$ will certainly be divisible by 3 since the condition of the sum of digits being divisible by 3 is true and $d_k(10^k - 1) + d_{k-1}(10^{k-1} - 1) + \ldots d_0(10^0 - 1)$ is always divisible by 3.

Therefore, the statement is true.

8. a. For any integer n, n^2-2 is not divisible by 4
This means that $4 \nmid n^2-2$ or $n^2-2 \neq 4a$ where a is an integer. So $n^2 \neq 4a+2 \rightarrow \sqrt{n^2} \neq \sqrt{4a+2}$
$n \neq \sqrt{4a+2}$ because 4 times an integer a plus 2 will always result in an integer whose square root does not result in a integer n.

a	1	2	3	4	5	6	7	8	9	10
$\sqrt{4a+2}$	$\sqrt{6}$	$\sqrt{10}$	$\sqrt{14}$	$\sqrt{18}$	$\sqrt{22}$	$\sqrt{26}$	$\sqrt{30}$	$\sqrt{34}$	$\sqrt{38}$	$\sqrt{42}$

b. For all prime numbers a, b, and c, $a^2+b^2 \neq c^2$
A prime number can only be divided by 1 and itself and must be a whole number greater than 1.
Proof by contradiction $a^2+b^2=c^2$, or $a^2=c^2-b^2$
$a^2=(c-b)(c+b)$ ∴ a^2 must be greater than 0 and positive because it is the square of a prime. $(c+b)$ is the sum of two primes and must be positive, To make $a^2=(c-b)(c+b)$ true, $(c-b)$ must be greater or equal to 1
$(c-b) \geq 1$ Proof by cases
Case 1: $(c-b)=1$, c and b are prime so the only time this is true is when $c=3$, $b=2$. Therefore, a would $=\sqrt{5}$. fails because $\sqrt{5}$ is irrational
Case 2: $(c-b) > 1$, for $a^2=(c-b)(c+b)$ to be true, $(c+b)$ and $(c-b) > 1$. for a^2, a must $=(c-b)=(c+b)$, but this fails as $-b \neq b$. So it is proven for prime numbers that $a^2+b^2 \neq c^2$

c. a,b,c are integers and $a^2+b^2=c^2$, then at least one of a or b is even
Proof by contradiction:
Suppose a and b are both odd so $a=2k+1$ and $b=2c+1$ where k and c are integers, this representation for a and b will always be odd. $c^2=a^2+b^2 \rightarrow c^2=(2k+1)^2+(2c+1)^2$
$$c^2=4k^2+4k+1+4c^2+4c+1$$
$$c^2=4(k^2+k+c^2+c)+2$$
let $k^2+k+c^2+c=p$ an integer because sums of integers and squares of integers is always an integer, so
$$c^2=4p+2 \rightarrow c^2-2=4p \text{ so } 4 \mid c^2-2$$
Since c^2-2 is divisible by 4, c^2 cannot be divisible by 4 since two less than a divisible number by 4 will not divide by an integer

$\forall$ real numbers x, if x is irrational, then -x is irrational.

9) (a) The negative of any irrational number is irrational.

1) Contraposition: if the negative of a number x is rational, then x is rational

$\neg p \rightarrow F$

2) Contradiction: The negative of any rational number is irrational.

proof by contraposition:

let there be an integer that is rational. Then use the definition of a rational. $n \in Q$ if $\exists p, q \in \mathbb{Z}$ where $n = \frac{p}{q}$ and $q \neq 0$

so let there exist n integer where $n = {}^{-p}/q$ or $(-p)/q$ and since we know $-p$ and q are integers where $q \neq 0$, we know $-x$ is rational, which proves the original statement true.

proof by contradiction:

we assume $\neg p$, so we assume x is rational then prove to see if its negation is irrational

so let $x \in Q$ if $\exists p, q \in \mathbb{Z}$, where $x = \frac{p}{q}$ and $q \neq 0$

start by negating x. $x = p/q$

then $-1 \cdot x = \frac{-1}{1} \cdot \left(\frac{p}{q}\right)$ thus we have

$-x = {}^{(-1)(p)}/q$ or $-x = {}^{(-p)}/q$ and

since we know $-p$ and q are integers with $q \neq 0$ we can say $\neg p \rightarrow$ False. which proves are original statement true.

9) (b) : for all integers a, b, and c, if $a \mid b$ and $a \nmid c$, then $a \nmid (b+c)$

1) Contraposition: if $a \mid (b+c)$, then $a \nmid b$ or $a \mid c$

2) Contradiction: if $a \nmid b$ or $a \mid c$, then $a \nmid (b+c) \to F$

proof by contraposition:
assume, $\neg q$, so assume $\frac{b+c}{a}$, so $\exists k \in \mathbb{Z}$ st.
$b+c = ka$ then prove $\neg q$
$b = ka - c$
so $a \nmid b \Rightarrow \frac{ka-c}{a}$ then

$\frac{ka}{a} - \frac{c}{a} \Rightarrow k - \frac{c}{a}$ which proves $b \nmid a$ true

and since it is an or-statement we only need to prove the one condition.

proof by contradiction:
assume $\neg p$, so $a \nmid b$ or $a \mid c$, then $a \nmid (b+c)$

starting with $\neg p$ we start with $a \mid c$, so assume $\frac{c}{a}$, so $\exists k \in \mathbb{Z}$ st $ka = c$ then prove q

$a \nmid (b+c) \Rightarrow \frac{b}{a} + \frac{ka}{a} \Rightarrow \frac{b}{a} + k$ which shows $a \nmid (b+c)$ is false because $a \nmid b$ then $\exists k$