CS-2102
Discrete Math
Problem Set 4, Due Friday February 24, 2016 by 5 pm.

Remember that you can work one other person. Be sure to give adequate explanation for your solutions, lack of clarity and completeness may result in points being deducted. <u>Answers given with no work or justification may have points taken off.</u>

(5 pts) 1. Use rules of inference to show that if $\forall x\big(P(x) \vee Q(x)\big)$ and $\forall x\Big(\big(\neg P(x) \wedge Q(x)\big) \rightarrow R(x)\Big)$ are true, then $\forall x\big(\neg R(x) \rightarrow P(x)\big)$ is also true, where the domain is the same for each statement. Hint: you don't have to use Resolution for this problem but it helps!

<u>Definition:</u> The floor of x denoted by $\lfloor x \rfloor$, is defined as follows:

$$\lfloor x \rfloor = \text{the unique integer such that } x - 1 < \lfloor x \rfloor \leq x.$$

That is $\lfloor x \rfloor = n \Leftrightarrow x - 1 < n \leq x$.

Definition: The ceiling of x denoted by $\lceil x \rceil$, is defined as follows:

$$\lceil x \rceil = \text{the unique integer } n \text{ such that } x \leq \lceil x \rceil < x + 1.$$

That is $\lceil x \rceil = n \Leftrightarrow x \leq n < x + 1$.

(12 pts, 6 each) 2. Prove the following facts about the floor of odd and even integers.

(a) Prove that for all odd integers, n, $\left\lfloor \frac{n}{2} \right\rfloor = \frac{n-1}{2}$.

(b) Prove that for all even integers, n, $\left\lfloor \frac{n}{2} \right\rfloor = \frac{n}{2}$.

<u>Definition:</u> The set of rational numbers is defined to be $\mathbb{Q} = \left\{ \frac{p}{q} \middle| p \in \mathbb{Z} \text{ and } q \in \mathbb{Z} \text{ with } q \neq 0 \right\}$

(12 pts, 4 each) 3. Prove the following statements.

(a) If r and s are any two rational numbers, then $\frac{r+s}{2}$ is rational.

(b) For all real numbers a and b, if $a < b$ then $a < \frac{a+b}{2} < b$.

(c) Given any two rational numbers r and s with $r < s$, there is another rational number between r and s. (Hint: use the two previous statements.)

(4 pts) 4. Prove that there are infinitely many solutions in positive integers *x, y,* and *z* to the equation $x^2+y^2=z^2$, ie there are infinitely many Pythagorean triples! (Good thing for Fermat that this proof doesn't extend to larger exponents! See Fermat's last theorem). [Hint let $x=m^2-n^2$ and $y=2mn$]

Definition: An integer, p, is prime if and only if $p>1$ and the only integer factors of p are 1 and p.

(6 pts, 2 each) 5. Prove or disprove the following:

(a) There exists an integer *n* such that $-3n^2-14n-8$ is a prime number.
(b) If *m* and *n* are positive integers and *mn* is a perfect square then *m* and *n* are perfect squares.
(c) The difference of the squares of any two consecutive integers is odd. Hint think about what it means for two integers to be consecutive.

(15 pts, 3 each) 6. Prove or disprove the following:

(a) If m and n are perfect squares, then $m+n+2\sqrt{mn}$ is also a perfect square.
(b) If p is a prime number, then 2^p-1 is also a prime number.
(c) Any product of four consecutive integers is one less than a perfect square.
(d) For all nonnegative real numbers a and b, $\sqrt{ab}=\sqrt{a}\cdot\sqrt{b}$. Not that if x is a nonnegative real number, then there is a unique nonnegative real number y, denoted $\sqrt{x}$, such that $y^2=x$.
(e) For all nonegative real numbers a and b, $\sqrt{a+b}=\sqrt{a}+\sqrt{b}$.

(10 pts, 1 for part b and 3 for each of the rest) 7. When expressions of the form $(x-r)(x-s)$ are multiplied out, a quadratic polynomial is obtained. For instance, $(x-2)\big(x-(-7)\big)=(x-2)(x+7)=x^2+5x-14$.

a. What can be said about the coefficients of the polynomial obtained by multiplying out $(x-r)(x-s)$ when both r and s are odd integers? When both r and s are even integers? When one is odd and the other even?
b. It follows from part (a) that $x^2-1253x+255$ cannot be written as a product of two polynomials with integer coefficients. Explain why this is so.
c. Observe that
$$(x-r)(x-s)(x-t)=x^3-(r+s+t)x^2+(rs+rt+st)x-rst$$
Derive a result for cubic polynomials similar to the result in part (a).
d. Can $x^3+7x^2-8x-27$ be written as a product of three polynomials with integer coefficients? Explain.

Definition: We say that integer a divides integer b if and only if there exists an integer k such that $b = ak$. We symbolize this by $a|b \Leftrightarrow \exists k \in \mathbb{Z}, b = ak$, where $a|b$ is short hand for a divides b.

(12 pts, 2 pts each) 8. Determine if the following statements are true or false. Prove the statement directly from the definition(s) if true, and give a counterexample if it is false.

(a) For all integers a, b, and c, if a divides b then a divides bc.
(b) 4 divides the product of any two even integers. (Note the definition of even is: an integer n is even if and only if $\exists k \in \mathbb{Z}$ such that $n = 2k$.)
(c) A sufficient condition for an integer to be divisible by 8 is that it be divisible by 16.
(d) For all integers a, b, and c, if $a|b$ and $a|c$ then $a|(2b - 3c)$.
(e) For all integers a, b, and c, if $ab|c$ then $a|c$ and $b|c$.
(f) For all integers a, b, and c, if $a|(b + c)$ then $a|b$ or a|c.

(6 pts, 3 pts each) 9. Prove the following statements:

a. Suppose x is a real number and $x \neq 0$. Prove that if $\frac{\sqrt[3]{x}+5}{x^2+6} = \frac{1}{x}$ then $x \neq 8$.

b. Suppose x and y are real numbers, and $3x + 2y \leq 5$. Prove that if $x > 1$ then $y < 1$.