CS-2102
Discrete Math
Problem Set 6, Due Friday March 24, 2017 by 5 pm.

Remember that you can work one other person. Be sure to give adequate explanation for your solutions, lack of clarity and completeness may result in points being deducted. Answers given with no work or justification may have points taken off.

(4 pts) 1. Prove that 3, 5, and 7 are the only prime numbers of the form p, $p+2$, and $p+4$.

(10 pts, 4 pts each for parts a and c, 2 points for part b) 2. Prove the following statements.

(a) Let p be a prime number and a and b be integers. If $p|ab$, then $p|a$ or $p|b$. Hint 1: Think about ways we have discussed to handle proofs of conditional statements when the conclusion is an "or" statement. Hint 2: Think about how the Fundamental Theorem of Arithmetic might help here.

(b) Let p be a prime number and n an integer. If $p|n^2$, then $p|n$.

(c) Let p be a prime number, then $\sqrt{p}$ is irrational.

(4 pts) 3. Prove or Disprove that $p_1p_2 \ldots p_n + 1$ is prime for every positive integer n, $p_1, p_2, \ldots, p_n$ are the n smallest prime numbers. For example when $n = 3$ then $p_1p_2p_3 + 1 = 2 \cdot 3 \cdot 5 + 1 = 31$ which is prime.

(4 pts, 1 each) 4. Find the integer a such that

(a) $a \equiv 43 \ mod \ 23$ and $-22 \leq a \leq 0$.

(b) $a \equiv 24 \ mod \ 31$ and $-15 \leq a \leq 15$.

(c) $a \equiv -15 \ mod \ 27$ and $-26 \leq a \leq 0$.

(d) $a \equiv -11 \ mod \ 21$ and $90 \leq a \leq 110$.

(4 pts, 2 pts each) 5. Find counterexamples to each of these statements about congruences.

(a) If $ac \equiv bc \ mod \ m$, where a, b, c, and m are integers with $m \geq 2$, then $a \equiv b \ mod \ m$.

(b) If $a \equiv b \ mod \ m$ and $c \equiv d \ mod \ m$, where $a, b, c, d, m \in \mathbb{Z}$ with c and d positive and $m \geq 2$, then $a^c \equiv b^d \ mod \ m$.

(10 pts, 5 pts each) 6. Prove the following statements

(a) If $2^p - 1$ is a prime number, then p is a prime number. Hint use the identity

$$2^{ab} - 1 = (2^a - 1)\left(2^{a(b-1)} + 2^{a(b-2)} + 2^{a(b-3)} + \cdots + 2^a + 1\right)$$

(b) Let a and n be integers with $n > 1$ and $a > 1$. If $a^n - 1$ is a prime number, then $a = 2$ and n is a prime number.

(4 pts, 1 pt each) 7. Find all positive integers m for which the following statements are ture.

(a) $13 \equiv 5 \bmod m$

(b) $10 \equiv 1 \bmod m$

(c) $-7 \equiv 6 \bmod m$

(d) $100 \equiv -5 \bmod m$