CS-2102
Discrete Math
Problem Set 2, Due Friday February 10, 2016 by 5 pm.

Remember that you can work one other person. Be sure to give adequate explanation for your solutions, lack of clarity and completeness may result in points being deducted. Answers given with no work or justification may have points taken off.

(6 pts, 2 pts each) 1. Rewrite each statement without using quantifiers or variables. Indicate which are true and which are false, and justify your answers as best as you can.

Let the domain of x be the set of integers, and let Odd(x) be "x is odd," Prime(x) be "x is prime," and Square(x) be "x is a perfect square."

For example $\forall x(Square(x) \rightarrow Odd(x))$ would be "If an integer is a perfect square then it is odd". This is of course false as 4 is a perfect square that is not odd.

(a) $\exists x\big(Prime(x) \wedge \neg Odd(x)\big)$

(b) $\forall x\big(Prime(x) \rightarrow \neg Square(x)\big)$

(c) $\exists x\big(Odd(x) \wedge Square(x)\big)$

(4 pts, 1 pts each) 2. For each of the following sentences, write the negation of the sentence, but place the ¬ symbol as far to the right as possible. Then rewrite the negation in English.

For example, for the sentence

$$\forall x \in \mathbb{Z}, x\ is\ odd$$

The negation would be

$$\exists x \in \mathbb{Z}, \neg(x\ is\ odd)$$

Which in English is "There is an integer that is not odd."

(a) $\forall x \in \mathbb{Z}, x < 0.$

(b) $\exists x \in \mathbb{Z}, x = x + 1.$

(c) $\exists x \in \mathbb{N}, x > 10.$

(d) $\forall x \in \mathbb{N}, x > 10.$

(4 pts, 1 pt each) 3. Write a formal negation for each of the following statements (By formal we mean use quantifiers and variables. You may use English words):

(a) $\forall$ fish x, x has gills.

(b) $\forall$ computers c, c has a CPU.

(c) $\exists$ a movie m such that m is over 6 hours long.

(d) $\exists$ a band b such that b has won at least 10 Grammy awards.

(12 pts, 2 each) 4. Let S be the set of students at UVA, let M be the set of movies that have ever been released, and let $V(s, m)$ be "student s has seen movie m." Rewrite each of the following statements without using the symbol ∀, and ∃, or variables.

(a) $\exists s \in S \text{ such that } V(s, Casablanca)$.
(b) $\forall s \in S, V(s, Star\ Wars)$. (This had better be true for everyone in the class!)
(c) $\forall s \in S, \exists m \in M \text{ such that } V(s, m)$.
(d) $\exists m \in M \text{ such that } \forall s \in S, V(s, m)$.
(e) $\exists s \in S, \exists t \in S, and\ \exists m \in M \text{ such that } s \neq t\ and\ V(s, m) \wedge V(t, m)$.
(f) $\exists s, \in \mathrm{S}, \exists t \in \mathrm{S}, \text{and } \forall m \in \mathrm{M} \text{ such that } s \neq t \text{ and } \mathrm{V}(s, m) \rightarrow \mathrm{V}(t, m)$.

(2 pts) 5. The computer scientists Richard Conway and David Gries once wrote:

"The absence of error messages during translation of a computer program is only a necessary and not a sufficient condition for reasonable [program] correctness"

Rewrite this statement without using the words *necessary* or *sufficient*.

(16 pts, 2 each) 6. Indicate which of the following statements are true and which are false. Justify your answer as best you can. Note that $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ and $\mathbb{R}^+$ is the set of all positive real numbers.

(a) $\forall x \in \mathbb{Z}^+, \exists y \in \mathbb{Z}^+ \text{ such that } x = y + 1$.
(b) $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z} \text{ such that } x = y + 1$.
(c) $\exists x \in \mathbb{R} \text{ such that } \forall y \in \mathbb{R}, x = y + 1$.
(d) $\forall x \in \mathbb{R}^+, \exists y \in \mathbb{R}^+ \text{ such that } xy = 1$.
(e) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} \text{ such that } xy = 1$.
(f) $\forall x \in \mathbb{Z}^+ \text{ and } \forall y \in \mathbb{Z}^+, \exists z \in \mathbb{Z}^+ \text{ such that } z = x - y$.
(g) $\forall x \in \mathbb{Z} \text{ and } \forall y \in \mathbb{Z}, \exists z \in \mathbb{Z} \text{ such that } z = x - y$.
(h) $\exists u \in \mathbb{R}^+ \text{ such that } \forall v \in \mathbb{R}^+, uv < v$.

(18 pts, 3 pts each) 7. Determine whether the following are logically equivalent. Justify your answers. Note that A is a preposition that does not depend on the value of x.

(a) $\forall x\big(P(x) \rightarrow Q(x)\big)$ and $\forall x P(x) \rightarrow \forall x Q(x)$
(b) $\exists x\big(P(x) \rightarrow Q(x)\big)$ and $\forall x P(x) \rightarrow \exists x Q(x)$
(c) $\forall x\big(P(x) \leftrightarrow Q(x)\big)$ and $\forall x P(x) \leftrightarrow \forall x Q(x)$
(d) $\forall x\big(A \rightarrow P(x)\big)$ and $A \rightarrow \forall x P(x)$
(e) $\forall x(P(x) \rightarrow A)$ and $\exists x P(x) \rightarrow A$
(f) $\forall x P(x) \vee \forall x Q(x)$ and $\forall\big(P(x) \vee Q(x)\big)$

(10 pts, 2 pts each) 8. Determine the truth value of each of these statements if the domain of each variable consists of all real numbers.

(a) $\forall x \exists y (x^2 = y)$
(b) $\forall x \exists y (x = y^2)$
(c) $\forall x \big(x \neq 0 \rightarrow \exists y (xy = 1)\big)$
(d) $\exists x \forall y (y \neq 0 \rightarrow xy = 1)$
(e) $\forall x \forall y \exists z \left(z = \frac{x+y}{2}\right)$

(10 pts, 2 pts each) 9. Definition for upper bounds for a subset of $\mathbb{R}$: Let A be a subset of the real numbers. A number b is called an upper bound for the set A provided that for each element x in A, $x \leq b$.

(a) Write this definition in symbolic form by completing the following: Let A be a subset of the real numbers. A number b is called an upper bound for the set A provided that ...
(b) Give examples of three different upper bounds for the set $A = \{x \in \mathbb{R} | 1 \leq x \leq 3\}$.
(c) Does the set $B = \{x \in \mathbb{R} | x > 0\}$ have an upper bound? Explain.
(d) Give examples of three different real numbers that are not upper bounds for the set $A = \{x \in \mathbb{R} | 1 \leq x \leq 3\}$.
(e) Complete the following in symbolic form: Let A be a subset of the real numbers. A number b is not an upper bound for the set A provided that ...