Modified numerals and maximality

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Abstract

In this paper, we describe and attempt to solve a novel puzzle arising from the interpretation of modified numerals like *less than five* and *between two and five*. The puzzle is this: such modified numerals seem to mean different things depending on whether they combine with distributive or non-distributive predicates. When they combine with distributive predicates, they intuitively impose a kind of upper bound, whereas when they combine non-distributive predicates, they do not (they sometimes even impose a lower bound). We propose and explore in detail four solutions to this puzzle, each involving some notion of *maximality*, but differing in the *type* of maximality involved ('standard' maximality vs. 'informativity-based' maximality) and in the *source* of maximality (lexically encoded in the meaning of the numeral modifier vs. non-lexical). While the full range of data we consider do not conclusively favor one theory over the other three, we do argue that overall the evidence (i) goes *against* the view that modified numerals lexically encode a 'standard' maximality operator, and (ii) suggests the need for a pragmatic blocking mechanism that filters out readings (logical forms) of sentences that are generated by the grammar but intuitively unavailable.

1 Introduction

The goal of the this paper is to solve a puzzle that arises regarding the interpretation of certain types of modified numerals, such as *less than five* and *between two and five*, and, in so doing, to shed light on the role that the notion of maximality plays in the interpretation of plural expressions. The puzzle is the following. As Buccola (2015) and Spector (2014) recently observed, these expressions seem to be interpreted differently depending on whether they combine with distributive or non-distributive predicates. One way to present the puzzle is to point out that while (1a) and (1b) are intuitively equivalent, (2a) and (2b) are not.

(1) a. Less than five people came.
b. It is not the case that more than four people came.

(2) a. Less than five guests drank over twenty bottles of beers between them.
b. It is not the case that more that four guests drank over twenty bottles of beers between them.

In a similar way, while (3a) entails (3b), (4a) does not entail (4b):

(3) a. Between two and five people came.
b. It is not the case that more than five people came.

(4) a. Between two and five guests drank over twenty bottles of beers between them.
b. It is not the case that more than five guests drank over twenty bottles of beers between them.

(2a) and (4a) illustrate the so-called *cumulative* reading of transitive predicates, which is forced here by the phrase *between them*. What (2a) (resp. (4a)) means is that there exists a group of guests of cardinality less than five (resp. of cardinality between two and five) such that this group, in total, drank over twenty beers. This, of course, does not exclude the existence of some larger group drinking, in total, even more beer.

A similar pattern arises when the relevant expressions combine with certain collective predicates, as illustrated in (5):

(5) a. Less than five/between two and four boys lifted a piano together.
b. It is not the case that more more than four boys lifted a piano together.

In this case, (5a) does not entail (5b). For instance, (5a), but not (5b), is true in a situation where a group of four boys lifted a piano together and some other group of ten boys lifted another piano together.[1]

Focusing just on *less than five*, it seems natural, on the basis of (2a) and (5a), to entertain the following lexical entry:

(6) $[\![\text{less than five}]\!] = \lambda P\,.\,\lambda Q\,.$ there is a group x such that x has at most four members and $P(x)$ and $Q(x)$

As we shall see, however, such a proposal has disastrous consequences for cases like (1a) and (3a)—it leads to what has been called *Van Benthem's problem*. For those cases, the following would work:

(7) $[\![\text{less than five}]\!] = \lambda P\,.\,\lambda Q\,.$ there is no group x such that x has more than four members and $P(x)$ and $Q(x)$

This, however, would incorrectly predict that (2a) and (5a) entail (2b) and (5b), respectively. We thus face a somewhat surprising situation, where the very same expression seems to be interpreted differently depending on its linguistic environment, and it is easy to formulate lexical entries that work for one of the two cases, but not one that works for both.

Let us observe furthermore that the entry given in (6) makes *less than five* monotone decreasing with respect to both its restrictor and its nuclear scope. This is not so, however, for the lexical entry given in (7)—(7) treats such modified numerals as existential quantifiers. So, on the basis of (6), but not on (7), *less than five* is predicted to license weak negative polarity items in both its restrictor and its nuclear scope (Ladusaw 1979). The following

[1] One could try to explain this intuition in terms of an analysis where the indefinite takes widest scope (Klaus Abels, p.c.). Under an LF amounting to, 'There is a piano such that it is not the case that more than four boys lifted it', (5a) would not entail (5b). Note, however, that the presence of an indefinite is not crucial to the judgment, though it makes it clearer for pragmatic reasons. A sentence such as *Less than five boys lifted the piano together* is true even if the piano was lifted twice, once by four boys and another time by ten boys.

contrast regarding NPI licensing is therefore in line with truth-conditional judgments:

(8) a. Less than ten soldiers visited any castle.
b. *Less than ten soldiers surrounded any castle.

In (8a), the subject combines with a distributive predicate, and if we use the lexical entry in (7), *any* occurs in a downward-entailing environment (DE environment for short), and is therefore licensed. In (8b), however, the subject combines with a collective predicate, and on the basis of the lexical entry in (6), *any* is not in a DE environment, hence fails to be licensed. Of course, it is again mysterious why we would need two distinct rules of interpretation for modified numerals, one for each case. This contrast in acceptability judgments thus illustrates the very same puzzle as the one we highlighted in terms of truth-conditional intuitions.

In this paper, we explore four different accounts of this puzzle, each one involving some notion of *maximality*. Each account makes different commitments about (i) which notion of maximality is operative, and (ii) how exactly maximality comes into play in the interpretation of modified numerals. We will first discuss two accounts in which the underlying notion of maximality is the 'standard' one, i.e. based on the natural ordering of numbers. On these two accounts, sentences containing the relevant modified numerals are ambiguous between two readings. The puzzle is resolved by resorting to a pragmatic blocking mechanism, which filters out one of the readings in certain situations. These two accounts differ regarding the way the ambiguity is derived, i.e. on the source of maximality: one account takes maximality to be part of the intrinsic, lexical meaning of the relevant numeral modifiers, while the other posits a separate (non-lexical) and optional maximality component. The two accounts make slightly different predictions for some complex cases.

The other two accounts we discuss rely on an underlying notion of maximality based on maximal informativeness, i.e. maximal logical strength. They too differ in the source of maximality (lexical vs. non-lexical), and they too will need to be supplemented with a pragmatic blocking mechanism. Their main strength, however, is that they do a better job of predicting which readings of modified numerals are accessible in which syntactic environments. While this makes them conceptually appealing, they also run into serious empirical problems, which can be solved only by quite *ad hoc* measures.

The paper is organized as follows. Section 2 provides some necessary background and presents the basic puzzle in a more explicit way. Sections 3, 4, 5, 6, and 7 present and discuss two accounts based on the standard notion of maximality. Section 8 discusses alternative accounts based on a notion of maximality in terms of logical strength. Section 9 provides an overall assessment of the four proposals we will have considered. Section 10 concludes.

2 Background

In this section we present a simple theory of numerals and plurals and show how it leads to some well-known problems for expressions like *less than four* and *between three and five*. This discussion forms the backdrop against which the main proposals in this paper are elaborated.

2.1 Technical background and conventions

We follow Link (1983) and subsequent work in assuming that our domain of entities, D_e, includes both ordinary individuals like John (j), Mary (m), this table (t), and that chair (c), as well as pluralities (sums) of individuals, such as John and Mary ($j \sqcup m$), Mary and that chair ($m \sqcup c$), and so on.[2] This domain is closed under sum formation[3] and is partially ordered by the *part of* relation, $\sqsubseteq$, induced by the sum formation operation.[4] An *atom* is an individual with no proper subpart,[5] and the cardinality of a plurality x, notated as $|x|$, is the number of atoms that are part of x.[6] Except where indicated (section 8.3), we take the standard view that there is no null individual, i.e. there is no individual x such that $|x| = 0$. For a more detailed overview, see Champollion and Krifka 2015 and Nouwen 2015.

We assume also a type d for degrees, whose domain we take (for the purposes of this paper) to be the set of natural numbers, $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$.

When deriving the meaning of a sentence, we give the sentence together with the appropriate logical form (LF) and the derived truth conditions. For typographical convenience, the types of functions are written $(\sigma\tau)$ (rather than, say, $\langle \sigma, \tau \rangle$), and we always drop the outermost parentheses. Subscripts on (occurrences of) variables indicate the type of the variable, e.g. x_e has type e. If $f_{\sigma t}$ is a boolean function (for some type σ), and x_σ is in the domain of f, then for convenience we write $f(x)$ for $f(x) = 1$, and $\neg f(x)$ for $f(x) = 0$. We also talk interchangeably about boolean functions and the sets they characterize, so that, for example, $P_{dt} \neq \varnothing$ means $\{n_d : P(n)\} \neq \varnothing$, and $f_{et} \subseteq g_{et}$ means $\{x_e : f(x)\} \subseteq \{x_e : g(x)\}$, and so on. We use the symbol $\subsetneq$ to indicate the *proper subset* relation.

2.2 Bare numerals, plurals, and distributivity

We start with the idea that numerals are interpreted as intersective adjectives, specifically as predicates of pluralities (Verkuyl 1981; Hoeksema 1983; Partee 1987; Landman 2004; Geurts 2006). For example, *three* can be taken to denote a function that maps a plurality x to true just in case the number of atomic parts of x is equal to 3. This view explains straightforwardly the truth conditions of a sentence with a predicative numeral, as in (10a), which comes out true just in case the plurality denoted by *the guests* has exactly three atomic parts.[7]

[2] We use the terms *group*, *plurality*, and *sum* interchangeably, with no theoretical distinction between them.

[3] $\forall x, y[x, y \in D_e \rightarrow x \sqcup y \in D_e]$.

[4] $\forall x, y[x \sqsubseteq y \leftrightarrow x \sqcup y = y]$.

[5] $\forall x[\mathbf{atom}(x) \leftrightarrow \forall y[y \sqsubseteq x \rightarrow y = x]]$. Note: In section 8.3, we will entertain the possibility of adding to the domain of individuals D_e a *null individual*, which will be part of every other individual. Doing so requires that the notion of *atom* be redefined, as we will point out in footnote 61.

[6] $|x|$ is shorthand for $|\{y : y \sqsubseteq x \wedge \mathbf{atom}(y)\}|$.

[7] Admittedly, the predicative use of numerals in English is less pervasive than in other languages. For example, (10a) sounds only marginally natural to us, and (i) even less so.

(i) ??The books on that table are three.
(Cf. There are three books on that table.)

(9) $[\![\text{three}]\!] = \lambda x_e \,.\, |x| = 3$

(10) a. The guests are three.
b. [the guests] [are three]
c. $|\,[\![\text{the guests}]\!]\,| = 3$

In non-predicative positions, numerals intersect (like intersective adjectives) with other predicates, such as *students*.[8] The result is a new, conjunctive predicate.

(11) a. $[\![\text{three}]\!] = \lambda x_e \,.\, |x| = 3$
b. $[\![\text{students}]\!] = \lambda x_e \,.\, \mathbf{students}(x)$
c. $[\![\text{three students}]\!] = \lambda x_e \,.\, |x| = 3 \wedge \mathbf{students}(x)$

If a subject NP and a VP are both predicates of individuals, then they too are intersected, and the resulting predicate is existentially closed. For concreteness, assume that a silent determiner, $\varnothing$, does this (Link 1987).[9]

(12) $[\![\varnothing]\!] = \lambda P_{et} \,.\, \lambda Q_{et} \,.\, \exists x[P(x) \wedge Q(x)]$

Example (13) illustrates.

(13) a. Three students smiled.
b. [$\varnothing$ [three students]] smiled
c. $\exists x[|x| = 3 \wedge \mathbf{students}(x) \wedge \mathbf{smiled}(x)]$

It is important to note that (13c) does not entail any upper bound on the number of students who smiled. Thus, for example, (13a) is predicted to be consistent with, and in fact entailed by, ten students having smiled. The reason is that, if there is a plurality of ten students who smiled, then there is also a plurality of three students who smiled: just pick any three (distinct) students among the ten students who smiled, and those three are all students who smiled.

The more precise reason for such an entailment is that *students* and *smiled* are both distributive predicates.[10] For our purposes, distributivity can be defined as in (14).[11] Thus, *smiled* is distributive, because if, say, Ann, Bill, and Carol smiled ($\mathbf{smiled}(a \sqcup b \sqcup c)$), then it intuitively follows that Ann and Bill smiled ($\mathbf{smiled}(a \sqcup b)$), that Ann and Carol smiled ($\mathbf{smiled}(a \sqcup c)$), that Carol smiled ($\mathbf{smiled}(c)$), and so on. And the same holds for *students*.

(14) **Distributivity**
P is distributive iff $\forall x, y[[P(x) \wedge y \sqsubseteq x] \rightarrow P(y)]$.
(…iff, if P is true of a plurality x, then P is true of every part of x.)

This ensures that the argument in (15) is valid.

[8] See, for example, the rule of *Predicate Modification* in Heim and Kratzer 1998.
[9] We could also assume an optional rule of global existential closure (Heim 1982).
[10] In the terminology of Krifka 1989, they have *divisive reference*.
[11] We say that an (object-language) expression like *smiled* is 'distributive' to mean that its extension is distributive in the sense of (14). The same holds, *mutatis mutandis*, for cumulative expressions below.

(15) $\exists x[|x| = 10 \wedge \textbf{students}(x) \wedge \textbf{smiled}(x)]$ (Ten students smiled.)
$\forall x, y[[\textbf{students}(x) \wedge y \sqsubseteq x] \rightarrow \textbf{students}(y)]$ (*students* is distributive)
$\forall x, y[[\textbf{smiled}(x) \wedge y \sqsubseteq x] \rightarrow \textbf{smiled}(y)]$ (*smiled* is distributive)
$\therefore\ \exists y[|y| = 3 \wedge \textbf{students}(y) \wedge \textbf{smiled}(y)]$ (Three students smiled.)

It is well known, however, that sentences like (13a) are most often interpreted as implying an upper bound, i.e. *three* often seems to mean *exactly three*. One approach consists in viewing this stronger meaning as the result of pragmatic strengthening, i.e. as a kind of scalar implicature that results from the fact that numerals form a scale (e.g. via Gricean reasoning). There are, however, many reasons why such an account cannot be the full story. In particular, the 'strong' (or 'exactly', or 'two-sided') meaning of numerals tends to be easily accessible even in syntactic environments where standard scalar items tend to lose their strengthened meaning (see, e.g., Horn 2006, Geurts 2006, Spector 2013 for a survey, and Kennedy 2015 for a recent proposal). There is also psycholinguistic evidence (Musolino 2004; Papafragou and Musolino 2003; Marty, Chemla, and Spector 2013) suggesting that the strong meaning of numerals is acquired and processed differently from the strengthened meaning of scalar items. The consensus view seems to be that numerals, in some way or another, are ambiguous between the 'at least' (or 'one-sided') reading we are assuming here and a stronger reading that implies an upper bound. At this point, however, we only consider the reading derived in (13), but we return to this issue in section 4.

Now, based on our assumptions, when a numerical phrase like *three students* combines instead with a non-distributive predicate, like *lift the piano* on its collective interpretation, as in (16), then the numeral is predicted to get an *exactly three* interpretation—in a certain sense. What (16) is predicted to mean, on its collective interpretation, is that a group of exactly three students lifted the piano together. It is not expected to be verified by a group of, say, ten students having lifted the piano together, but it is still consistent with such a scenario.[12]

(16) a. Three students lifted the piano.
b. [∅ [three students]] [lifted the piano]
c. $\exists x[|x| = 3 \wedge \textbf{students}(x) \wedge \textbf{lifted}(x)]$

More precisely, *lift the piano*, on its collective interpretation, lacks the distributive property that we saw with *smile*: if, say, Ann, Bill, and Carol (collectively) lifted the piano ($\textbf{lifted}(a \sqcup b \sqcup c)$), it does not necessarily follow that Ann and Bill lifted the piano ($\textbf{lifted}(a \sqcup b)$). Thus, the argument in (17) is *not* valid.

(17) $\exists x[|x| = 10 \wedge \textbf{students}(x) \wedge \textbf{lifted}(x)]$ (Ten students lifted the piano.)
$\forall x, y[[\textbf{students}(x) \wedge y \sqsubseteq x] \rightarrow \textbf{students}(y)]$ (*students* is distributive)
$\not\therefore\ \exists y[|y| = 3 \wedge \textbf{students}(y) \wedge \textbf{lifted}(y)]$ (Three students lifted the piano.)

These predictions indeed accord with our judgments about the meaning of (16).

[12] For simplicity, we analyze transitive predicates like *lifted the piano* and *surrounded the castle* as 1-place rather than 2-place predicates.

2.3 Modified numerals and Van Benthem's problem

A straightforward extension of the intersective theory of numerals to modified numerals might maintain that modified numerals, like bare numerals, are predicates of pluralities, as in the example lexical entries below. On this view, the numeral modifier simply changes the relation between the numeral and the plurality's cardinality ($>$ in the case of *more than*, $<$ in the case of *less than*, etc.), and we get existential closure on top as usual.

(18) $[\![\text{more than three}]\!] = \lambda x_e \,.\, |x| > 3$

(19) $[\![\text{less than three}]\!] = \lambda x_e \,.\, |x| < 3$

(20) $[\![\text{between three and five}]\!] = \lambda x_e \,.\, 3 \leq |x| \leq 5$

This approach works fine for expressions like *more than three*, as (21) illustrates. However, it runs into serious trouble with expressions like *less than three* and *between three and five*. Consider (22a). This sentence is expected to be true just in case a group of less than three students smiled. There are two problems with these truth conditions. First, since we are assuming that there is no null individual (i.e. every plurality has a cardinality of at least 1), (22c) entails that at least one student smiled;[13] however, (22a), unlike (21a), is judged true even if no students smiled.[14] Call this the *existential entailment problem*. Second, (22c) (like in the bare numeral case) does not entail any upper bound. Thus, for example, (22a) is predicted to be consistent with, and in fact entailed by, ten students having smiled, just as in the bare numeral case, as the argument in (23) illustrates. And yet (22a) is clearly judged false if three or more students smiled. This latter problem was first pointed out by Van Benthem (1986) and has since become known as *Van Benthem's problem*. What these two problems together amount to is that (22a) should have a reading along the lines of, 'Some student(s) smiled', which of course intuitively it does not.[15]

(21) a. More than three students smiled.
b. [∅ [[more than three] students]] smiled
c. $\exists x[|x| > 3 \wedge \textbf{students}(x) \wedge \textbf{smiled}(x)]$

(22) a. Less than three students smiled.
b. [∅ [[less than three] students]] smiled
c. $\exists x[|x| < 3 \wedge \textbf{students}(x) \wedge \textbf{smiled}(x)]$

(23)
$\exists x[|x| = 10 \wedge \textbf{students}(x) \wedge \textbf{smiled}(x)]$	(Ten students smiled.)
$\forall x, y[[\textbf{students}(x) \wedge y \sqsubseteq x] \rightarrow \textbf{students}(y)]$	(*students* is distributive)
$\forall x, y[[\textbf{smiled}(x) \wedge y \sqsubseteq x] \rightarrow \textbf{smiled}(y)]$	(*smiled* is distributive)
$\therefore\ \exists y[|y| < 3 \wedge \textbf{students}(y) \wedge \textbf{smiled}(y)]$	(Less than three students smiled.)

[13] In fact, if we assume that the plural noun *students* has only pluralities of cardinality 2 or more in its extension, then this sentence is expected to entail that at least *two* students smiled. Following Hoeksema (1983), Van Eijk (1983), Krifka (1989), Sauerland (2003), Spector (2007), and Zweig (2009), among others, we assume that plural nouns include atomic individuals in their extension.

[14] Sentence (22a) may of course trigger the implicature that some student(s) smiled, but this should not be viewed as part of the literal meaning of the sentence. For example, *Every student read less than three books* is true even if some of the relevant students read no books at all.

[15] Marty, Chemla, and Spector (2015) present experimental evidence suggesting that, though not detectable introspectively, this reading does influence speakers' behavior in a truth-value judgment task.

A similar problem arises with (24a), which, under the LF in (24b), is predicted to have the truth-conditions in (24c), which are equivalent to (24d).

(24) a. Between three and five students smiled.
b. [∅ [[between three and five] students]] smiled
c. $\exists x[3 \leq |x| \leq 5 \wedge \textbf{students}(x) \wedge \textbf{smiled}(x)]$
d. $\exists x[|x| \geq 3 \wedge \textbf{students}(x) \wedge \textbf{smiled}(x)]$

In this case, an existential entailment is predicted, but this is not problematic, since for sure (24a) entails (24d). However, as with (22), the predicted reading for (24a) fails to entail an upper bound, which is again an instance of Van Benthem's problem.

Returning to (22a), its truth conditions are better represented as in (25a), or equivalently, (25b).[16]

(25) a. $\neg\exists x[|x| \geq 3 \wedge \textbf{students}(x) \wedge \textbf{smiled}(x)]$
b. $\max(\lambda n \,.\, \exists x[|x| = n \wedge \textbf{students}(x) \wedge \textbf{smiled}(x)]) < 3$
(We must assume that $\max(\varnothing) = 0$ to allow (22a) to be true in a context where no students smiled, i.e. to avoid an existential entailment problem. See section 3 for further discussion.)

According to these truth conditions, no plurality x of students who smiled has a cardinality of 3 or greater, i.e. contains three or more atoms. This, in and of itself, does not exclude the existence of several distinct two-member pluralities of students who smiled, as long as no greater plurality smiled. However, such a state of affairs is in fact impossible, because *students* and *smile* are not only distributive, they are also *cumulative* in the sense of Krifka 1989: for example, if Ann smiled ($\textbf{smiled}(a)$) and Bill smiled ($\textbf{smiled}(b)$), then it follows that Ann and Bill smiled ($\textbf{smiled}(a \sqcup b)$).[17]

(26) **Cumulativity**
P is cumulative iff $\forall x, y[[P(x) \wedge P(y)] \rightarrow P(x \sqcup y)]$.
(…iff, if P is true of two pluralities x and y, then P is true of the plurality formed by x and y.)

So, if there were two distinct two-member pluralities of students who smiled, say $a \sqcup b$ and $b \sqcup c$, then there would also be a plurality of smiling students with at least three members (in this case, $a \sqcup b \sqcup c$), which would contradict (25). In sum, since *students* and *smiled* are both cumulative, (25) correctly amounts to saying that either zero, one, or two students smiled, and no more than that.

These two properties (distributivity and cumulativity) follow immediately if one assumes that the only way in which a predicate such as *smiled* can be applied to non-atomic individuals is by pluralizing a more primitive version of *smiled* which is defined only for atoms. On this view, we start from a predicate *smiled* which has only atoms in its

[16] In a completely parallel way, the truth conditions we need for (24a) are as follows:

(i) $3 \leq \max(\lambda n \,.\, \exists x[|x| = n \wedge \textbf{students}(x) \wedge \textbf{smiled}(x)]) \leq 5$

[17] Cumulativity so defined is a property of 1-place predicates. In section 5.2, we also talk about the so-called 'cumulative reading' of sentences involving *transitive* (2-place) predicates and plural arguments. There, cumulativity (being now a property of 2-place predicates) has a somewhat different sense.

extension, and a pluralizing operator, usually notated as $*$, turns this predicate into one that is defined for both atomic and non-atomic individuals, by closing its extension under the sum operation (cf. Link 1983 and subsequent work; see also section 8.3):

(27) Let E be a set of individuals (atomic or non-atomic). Then the *closure under sum* of E, written $E^{\sqcup}$, is the smallest set F such that:

a. $E \subseteq F$, and

b. $\forall x \forall y[x \in F \land y \in F \to x \sqcup y \in F]$.

($E^{\sqcup}$ is the set of all sums that can be formed by summing up members of E.)

(28) $[\![*]\!] = \lambda P_{et} \,.\, \lambda x_e \,.\, x \in \{y : P(y)\}^{\sqcup}$

($[\![*\alpha]\!]$ is the set of all sums that can be formed by summing up members of $[\![\alpha]\!]$.)

When *smiled* combines with an expression that denotes a non-atomic individual, it first needs to be pluralized, i.e. is parsed as **smiled*, and (the denotation of) **smiled* is now cumulative in the sense of (26).[18]

Having established that the truth conditions in (25) are what we want for (22a) (plus the assumption that *students* and *smiled* are both cumulative), the problem is that there is no way to arrive at these truth conditions under the assumptions made so far: none of the lexical entries, compositional rules, or predicate properties involve any maximality operator or negation. This example thus illustrates that we need to introduce, in some way or another, a maximality component into the semantics of modified numeral constructions to explain the upper bound facts associated with sentences like (22a) (and its counterpart with *between three and five*), for instance along the lines of (7).[19]

2.4 Non-distributive predicates and Van Benthem's non-problem

When we move to non-distributive predicates, however, Van Benthem's problem and the existential entailment problem both become non-problems. That is, for sentences such as (29a), an LF analogous to (22b), as in (29b), delivers what seem to be the intuitively correct truth conditions.

(29) a. Less than four boys lifted the piano.[20]

b. [$\varnothing$ [[less than three] boys]] [lifted the piano]

c. $\exists x[|x| < 4 \land \mathbf{boys}(x) \land \mathbf{lifted}(x)]$

First, as we mentioned, (29a), on its collective reading, is intuitively consistent with groups of more than three boys having lifted the piano, as predicted by (29c). Second, (29a), on its collective reading, intuitively entails that at least some student(s) lifted the piano. That

[18]For all x, y, if $[\![\text{*smiled}]\!]\,(x)$ and $[\![\text{*smiled}]\!]\,(y)$, then since $[\![\text{*smiled}]\!]$ is by definition (cf. (28)) closed under sum, we have $[\![\text{*smiled}]\!]\,(x \sqcup y)$.

[19]As we will discuss, (7) is not the most standard way of incorporating a maximality component. Rather, modified numerals are most often treated as generalized quantifiers over degrees, i.e. as combining with predicates of degrees (see, e.g., Hackl 2000). This is not relevant at this point, however.

[20]Context: A group of boys want to see how many of them it takes to lift the piano. Specifically, they want to know if less than four of them can do it. Different groups take turns trying to lift the piano. In the end, one group of five, one group of four, and one particularly strong group of three boys manage to lift it. In this context, (29) is both felicitous and true.

is, (29a), on its collective reading (and unlike on its distributive reading), is judged false in a scenario where no boys at all lifted the piano.[21] This entailment is captured by existential statement in (29c), since we are assuming there is no null individual.

Furthermore, as expected on the intersective theory, there is, as far as we can tell, no collective reading of (29a) that has (30) as its truth conditions. (We will return to this reading in section 7.)

(30) $\max(\lambda n\,.\,\exists x[|x| = n \wedge \mathbf{boys}(x) \wedge \mathbf{lifted}(x)]) < 4$

2.5 Summary of the puzzle

Let us refer to truth conditions (readings) of the form

(31) $\exists x[|x| < n \wedge P(x)]$

as non-upper-bounded truth conditions (readings) with an existential entailment: they entail that at least some x has property P (e.g. being students who smiled, or being boys who together lifted the piano), and in general, they are consistent with other groups larger than x having property P.[22]

Let us refer to truth conditions (readings) of the form

(32) $\max(\lambda m\,.\,\exists x[|x| = m \wedge P(x)]) < n$

as upper-bounded truth conditions (readings) with no existential entailment: they entail an upper bound n on the number of individuals with property P (whenever P is cumulative), and in general, they are consistent with no individuals having property P.

The facts just observed suggest that any theory of modified numerals needs to generate both upper-bounded truth conditions with no existential entailment, as well as non-upper-bounded truth conditions with existential entailments. However, it needs to do so in such a way that only upper-bounded truth conditions with no existential entailments are assigned in some cases (distributive), and only non-upper-bounded truth conditions with existential entailments are assigned in other cases (collective). In the rest of this paper, we propose and compare four different accounts for deriving such readings. We focus mainly on *less than n*, in part for ease of exposition, and in part because it (more so than *between m and n*) teases apart the different accounts, except in section 9, where discussion of *between m and n* becomes important in the overall assessment of the accounts we will have proposed.

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[21] Context: A group of boys want to see how many of them it takes to lift the piano. Different groups take turns trying to lift the piano, but in the end, *no* group manages to lift it. In this context, (29), on its collective reading, is judged false.

[22] We say in general because P might, for example, be rigidly true of at most one plurality, in which case, if P is true of a plurality x, then there is no plurality y distinct from x (hence, no plurality y larger than x) which P is true of. An example of such a predicate is *ate that whole pizza*, since only one group can ever eat a particular pizza in its entirety. But in cases like this (*Less than four boys ate that whole pizza together*), the upper-bound inference (that no group more than three boys ate that whole pizza together) clearly comes from a particular property of the predicate, not from the structure of the sentence.

2.6 Semantic housekeeping

Before we move on, we need to take care of some semantic housekeeping. To be able to derive the meanings of expressions like *less than three* compositionally from the meanings of *less than* and *three* (that is, to be able to specify categorematically the meaning of *less than*), we need to alter our lexical entry for *three*. *Three* cannot simply denote a function of type *et* because *less than* needs access to the '3' part of the meaning of *three*. Thus, we let *three* simply denote the number (degree) 3, and similarly for the other numerals. On this basis, it is easy to define some plausible denotations for *less than* (and other numeral modifiers). Some examples are given below.[23]

(33) $[\![\text{three}]\!] = 3$

(34) $[\![\text{less than}]\!] =$

a. $\lambda n_d \,.\, \lambda x_e \,.\, |x| < n$

b. $\lambda n_d \,.\, \lambda P_{dt} \,.\, \exists m[m < n \wedge P(m)]$

c. $\lambda n_d \,.\, \lambda P_{dt} \,.\, \exists m[m < n \wedge \max(P) = m]$

To derive the type *et* predicate that we originally posited as the meaning of *three*, we assume a silent syntactic operator `isCard`, which maps a degree-denoting numeral n to a predicate of pluralities, namely that predicate which characterizes the set of pluralities x whose cardinality is n. The intersective use of *three* is now derived syntactically (at LF) as `isCard` *three*, which we write as $three_{\texttt{isCard}}$ for convenience.[24, 25]

(35) $[\![\texttt{isCard}]\!] = \lambda n_d \,.\, \lambda x_e \,.\, |x| = n$

(36) $[\![n_{\texttt{isCard}}]\!] = [\![\texttt{isCard}\ n]\!] = \lambda x_e \,.\, |x| = [\![n]\!]$

3 Lexical maximality and scope ambiguity

In this section we present what we consider to be a fairly 'standard' account of *less than*, at least compared to the other accounts presented in this paper. We follow Heim (2000) and Hackl (2000) (among many others) in assuming that the lexical semantics of *less than*, given in (37), involves a maximality operator, max. We therefore refer to this account as *L(exical)Max*.

[23]Of course the choice between different possibilities has to be made not only on the basis of the behavior of modified numerals, but as part of a more general theory of comparative constructions. A number of works on comparatives assume that comparative quantifiers have a lexical entry that includes a maximality component, along the lines of (34c) (see, e.g., Heim 2000 and, especially, Hackl 2000 and the references therein). Some of these accounts also assume that *less* should be decomposed into *little* + *-er* and that the maximality component is introduced by *-er* (Rullmann 1995; Heim 2006).

[24]Our `isCard` is essentially a Bresnan-style *many* (Bresnan 1973); our choice of name is simply to mnemonically contrast `isCard` from two other syntactic operators to appear later (`isMax` in section 4 and `isMaxInf` in section 8.5).

[25]Note that the way we have chosen to implement the 'shift' of a numeral from a degree to a predicate of individuals is arbitrary. For example, we could also have posited a semantic typeshifting operation (Partee 1987). For convenience, we will loosely say that n '(type)shifts' to $n_{\texttt{isCard}}$, even though our particular implementation is not, strictly speaking, a typeshifting rule.

(37) $[\![\text{less than}]\!] = \lambda n_d \,.\, \lambda P_{dt} \,.\, \exists m[m < n \wedge \max(P) = m]$

A precise definition of max is provided in (38). Notice that, to avoid any existential entailment problems (explained below), we assume that $\max(\varnothing) = 0$ (see also the discussion in section 8.2).[26]

(38) $$\max(P_{dt}) = \begin{cases} m \,.\, P(n) \wedge \forall m[P(m) \rightarrow m \leq n] & \text{if } \exists n P(n) \\ 0 & \text{otherwise} \end{cases}$$

As a generalized quantifier over degrees, we assume that *less than three* can move (in fact, must move, for type reasons), creating a predicate of degrees in its scope. Example (39) illustrates one possibility.

(39) a. Less than three students smiled.
b. [less than three] [λn [[$\varnothing$ [$n_{\texttt{isCard}}$ students]] smiled]]
c. $\max(\lambda n \,.\, \exists x[|x| = n \wedge \textbf{students}(x) \wedge \textbf{smiled}(x)]) < 3$

In the above derivation, *less than three* moves to take scope over $\varnothing$. More precisely, it raises (QRs) and adjoins to the S node of the sentence. Its trace is then shifted to an intersective numeral (type *et*). The reading thus derived is the (attested) upper-bounded reading. Moreover, it (correctly) makes no existential entailment for the following reason: if no students attended, then the degree predicate

(40) λn [[$\varnothing$ [$n_{\texttt{isCard}}$ students]] attended]

has an empty extension (it does not even contain 0, since we are assuming there is no null individual, hence no x such that $0_{\texttt{isCard}}\ x$ is true), but by the definition of max in (38), the maximum of an empty degree predicate is zero, which is less than five. Thus, (39) is true even if no students smiled.

Less than three could also move but remain below $\varnothing$, i.e. QR into AP (or equivalently, into NP), as in (41).[27] The simplification in (41c) is due to the fact that for every sum x, we

[26]Nothing really hinges on the use of a 'max operator' at all; it simply makes exposition and comparison of accounts easier. We could instead have:

(i) $[\![\text{less than}]\!] = \lambda n_d \,.\, \lambda P_{dt} \,.\, \neg\exists m[m \geq n \wedge P(m)]$

The only difference is in the case where the degree predicate P has no maximum (it extends indefinitely toward infinity): entry (i) makes $[\![\text{less than}]\!]\,(n)(P)$ return false for every n, whereas entry (37), which does not cover the case where P has no unique greatest element, makes it undefined. This difference may turn out to be important for a sentence like (ii), which is intuitively false, rather than a case of presupposition failure. Moreover, we cannot simply stipulate in the definition of max that it (somehow) projects 'false' if P has no unique greatest element, since (iii) is intuitively true, not false. However, we leave these issues aside for the remainder of the paper.

(ii) Less than five prime numbers are odd.

(iii) It is not the case that less than five prime numbers are odd.

[27]We assume here that NPs and APs, like VPs, have an internal subject position that can be abstracted over, as proposed in Heim and Kratzer 1998, ch. 8. See Buccola 2015 for details of this implementation. Henceforth, we show only the case where the degree predicate quantifies into AP; quantifying into NP yields a logically

have $\max(\lambda n\,.\,|x| = n) = |x|$, since every sum has exactly one cardinality—an important equivalence that will come up several more times in this paper.

(41) a. Less than three students smiled.
b. [$\varnothing$ [[λx [[less than three] [λn [x $n_{\texttt{isCard}}$]]]] students]] smiled
c. $\exists x[\max(\lambda n\,.\,|x| = n) < 3 \wedge \textbf{students}(x) \wedge \textbf{smiled}(x)]$
$\equiv \exists x[|x| < 3 \wedge \textbf{students}(x) \wedge \textbf{smiled}(x)]$

We see, then, that when *less than n* scopes below $\varnothing$, we get a non-upper-bounded reading with an existential entailment. And when *less than n* scopes above $\varnothing$, we get an upper-bounded reading with no existential entailment. Since nothing about the predicates constrains where *less than n* lands, LMax incorrectly derives both types of readings in both distributive and collective cases.

Notice, however, what the truth conditions of the non-upper-bounded reading in the distributive case, (41c), actually mean. It follows from the distributivity of *students* and *smiled* that

(42) $\exists x[|x| < 3 \wedge \textbf{students}(x) \wedge \textbf{smiled}(x)]$

is equivalent to

(43) $\exists x[\textbf{students}(x) \wedge \textbf{smiled}(x)]$

The entailment from (42) to (43) follows from simply conjunction elimination. The reverse entailment follows because if a plurality of students x smiled, then by distributivity, every part of x, including the atoms (singleton pluralities, with cardinality $1 < 3$) of x, are students who smiled, too.

From this observation we make the following generalization: The numeral being modified (*three*) does no semantic work: we would derive the exact same truth conditions with any other numeral (greater than 1) in its place, for example if *three* were replaced by *two* or by *four*.

On this basis, we propose that a pragmatic economy constraint blocks such LFs. One way to formulate such a constraint is given in (44).[28]

equivalent LF, thanks to the following truth-conditional equivalence:

(i) $\exists x[\max(\lambda n\,.\,|x| = n \wedge [\![\text{NP}]\!](x)) < 3 \wedge [\![\text{VP}]\!](x)]$ (quantifying into NP)
$\equiv \exists x[\max(\lambda n\,.\,|x| = n) < 3 \wedge [\![\text{NP}]\!](x) \wedge [\![\text{VP}]\!](x)]$ (quantifying into AP)
$\equiv \exists x[|x| < 3 \wedge [\![\text{NP}]\!](x) \wedge [\![\text{VP}]\!](x)]$

[28] The idea to specifically capitalize on the fact that in the infelicitous LFs a numeral is in some sense making no semantic contribution was suggested to Benjamin Spector by Emmanuel Chemla (p.c.). We might have wanted to formulate a more general constraint that would be easier to motivate in terms of Gricean maxims. A natural idea would be to interpret the maxim of *manner* as implying that a given LF is infelicitous if it is truth-conditionally equivalent to some simpler LF. On this view, a sentence of the form *Between n and m NP VP* could not be assigned an LF which would express the proposition 'at least *n* NP VP', because this proposition could be expressed by a simpler LF. While appealing, such a principle does not make clear predictions short of an explicit complexity metric (see Katzir 2007 for an attempt to define such a metric). It might also rule out as infelicitous certain felicitous sentences (for instance, the sentence *I read a book that was very interesting* would be ruled out because it is equivalent to and more complex than *I read a very interesting book*). (cont'd next page)

(44) **Pragmatic economy constraint**
An LF ϕ containing a numeral n is infelicitous if, for some m distinct from n, ϕ is truth-conditionally equivalent to $\phi[n \mapsto m]$ (the result of substituting m for n in ϕ).

Given the argumentation above, this pragmatic constraint filters out the unattested, non-upper-bounded readings with existential entailments in the distributive case.

Moreover, it correctly does *not* filter out those same, attested readings in the collective case. Consider, for example, the truth conditions in (45c) for (45a) on its collective reading. Since *lifted the piano* here is non-distributive, the conjunct $|x| < 4$ actually has semantic import and cannot be dropped. In other words, the numeral *four* in (45b) actually does semantic work: if we replaced *four* with *three*, we would get stronger truth conditions, and if we replaced *four* with *five*, we would get weaker truth conditions. Hence, this reading is not blocked by the constraint in (44).

(45)
a. Less than four students lifted the piano.
b. [$\varnothing$ [[λx [[less than four] [λn [x $n_{\texttt{isCard}}$]]]] students]] [lifted the piano]
c. $\exists x[\textbf{students}(x) \wedge \max(\lambda n\,.\,|x| = n) < 4 \wedge \textbf{lifted}(x)]$
$\equiv \exists x[|x| < 4 \wedge \textbf{students}(x) \wedge \textbf{lifted}(x)]$

More generally, when P is non-distributive, then

(46) $\exists x[|x| < n \wedge P(x)]$

is *not* equivalent to the following, for any $m \neq n$:

(47) $\exists x[|x| < m \wedge P(x)]$

because there is no entailment relation at all between the m and n alternatives when P is non-distributive. And it is also not equivalent to

(48) $\exists x P(x)$

because P may only be true of pluralities with cardinalities greater than n.

To recap, the semantics proposed in this section, i.e. LMax, generates two readings for basic sentences with *less than n*: (i) an upper-bounded reading with no existential entailment, when *less than n* scopes over $\varnothing$, and (ii) a non-upper-bounded reading with an existential entailment, when *less than n* scopes below $\varnothing$. The unattested, existential, non-upper-bounded reading in distributive cases (which was shown to face an existential entailment problem and Van Benthem's problem) is blocked by the pragmatic economy constraint in (44).

Furthermore, we might have considered a weaker version of our condition, where ϕ would be ruled out as infelicitous if, for *every* m distinct from n, ϕ is truth conditionally equivalent to $\phi[n \mapsto m]$. This would arguably be a more direct implementation of the intuition that what is wrong with the relevant LFs is that some numeral is in some sense doing no semantic work. However, with such a weaker constraint, we might fail to exclude the existential reading for (24a) (*Between 3 and 5 students smiled*), which is equivalent to *At least 3 students smiled*, since replacing 5 with 2 in (24a) would make the resulting sentence (i.e. *Between 3 and 2 students smiled*) either contradictory or equivalent to *At least two students smiled*, depending on how exactly *between* is treated.

At this point, however, we do predict an ambiguity for modified numerals in non-distributive contexts. That is, (45a) could also receive the following parse:

(49) a. Less than four students lifted the piano.
b. [less than four] [λn [[$\varnothing$ [$n_{\texttt{isCard}}$ students]] [lifted the piano]]]
c. $\max(\lambda n \,.\, \exists x[|x| = n \wedge \textbf{students}(x) \wedge \textbf{lifted}(x)]) < 4$
$\equiv \neg\exists x[|x| \geq 4 \wedge \textbf{students}(x) \wedge \textbf{lifted}(x)]$

On such a reading, the sentence would be true even if no piano-lifting occurred. It would also be true if several groups of students lifted the piano, so long as none of these groups contained more than three members. As we said, it is not clear to us that such a reading exists, a point we will return to in section 7.

4 Maximality as an optional operation

In this section we sever maximality from the semantics of *less than* and instead posit an optional, autonomous maximization operation. This operation is inspired by the sort of operation that may be responsible for upper-bounded interpretations of bare numerals (cf. the discussion in section 2.2). We therefore start by showing how the operation is applied in the bare numeral case, and then move to modified numerals. Since the operation can be thought of as a (silent) syntactic operator, we refer to this theory as *S(yntactic)Max*.[29]

4.1 Maximality and bare numerals

We build on our 'semantic housekeeping' (section 2.6) by assuming not only the operator `isCard`, repeated in (50), but also an operator `isMax`, given in (51). Whereas `isCard` maps a type d numeral to a type et predicate, `isMax` maps a numeral to a type $(dt)t$ generalized quantifier over degrees. Note that the metalanguage operator 'max' used in (51) is the same as the standard one we defined in (38) in section 3. Thus, `isMax` combines with a numeral n and a degree predicate P and returns true iff n is the maximum of P (or P is empty and n is zero). In the same way that we write $\textit{three}_{\texttt{isCard}}$ for `isCard` *three*, we will write $\textit{three}_{\texttt{isMax}}$ for `isMax` *three*.[30]

(50) $[\![\texttt{isCard}]\!] = \lambda n_d \,.\, \lambda x_e \,.\, |x| = n$

(51) $[\![\texttt{isMax}]\!] = \lambda n_d \,.\, \lambda P_{dt} \,.\, \max(P) = n$

(52) $[\![n_{\texttt{isMax}}]\!] = [\![\texttt{isMax}\ n]\!] = \lambda P_{dt} \,.\, \max(P) = [\![n]\!]$

Consider the example in (53). Two LFs are given. In the first LF, (53b-i), *three* has shifted to $\textit{three}_{\texttt{isCard}}$, the intersective meaning of *three*, and the truth conditions are derived as we already saw in (13) in section 2.

[29] The reader may also think of SMax as short for *SeparateMax*, since the implementation need not be syntactic (cf. fn. 30).

[30] Just like with `isCard`, our implementation of maximization in terms of a silent syntactic expression is arbitrary. We could also, for example, posit a semantic typeshifting operation. Again, for convenience, we loosely say that n '(type)shifts' to $n_{\texttt{isMax}}$, even though our implementation does not involve any actual typeshifting operation.

(53) a. Three students smiled.
b. (i) [$\varnothing$ [three$_{\texttt{isCard}}$ students]] smiled
(ii) $\exists x[|x| = 3 \wedge \textbf{students}(x) \wedge \textbf{smiled}(x)]$
c. (i) three$_{\texttt{isMax}}$ [λn [[$\varnothing$ [$n_{\texttt{isCard}}$ students] smiled]]]
(ii) $\max(\lambda n\,.\,\exists x[|x| = n \wedge \textbf{students}(x) \wedge \textbf{smiled}(x)]) = 3$

In the second LF, (53c-i), *three* is shifted to *three*$_{\texttt{isMax}}$, which (for type reasons) raises above existential closure, creating a degree predicate in its scope, with its trace shifted to an intersective numerical variable. This derives the upper-bounded truth conditions in (53c-ii). This approach to the interpretation of bare numerals is close to a number of proposals (see, e.g., Geurts 2006; Kennedy 2015) in which various typeshifting operations are proposed in order to account for the different attested readings of numerals. It is particularly close to Kennedy 2015, in which *three*$_{\texttt{isMax}}$ is taken to be the basic meaning of numerals. The analysis in (53c) is exactly what Kennedy (2015) proposes for the interpretation of bare numerals.

At this point we should note that it is also possible to derive the non-upper-bounded reading of numerals in distributive contexts even on the basis of *three*$_{\texttt{isMax}}$, by making sure that *three*$_{\texttt{isMax}}$ occurs in the scope of the empty existential quantifier (following the logic of Van Benthem's problem):[31]

(54) a. Three students smiled.
b. [$\varnothing$ [[λx [three$_{\texttt{isMax}}$ [λn [x $n_{\texttt{isCard}}$]]]] students]] smiled
c. $\exists x[\max(\lambda n\,.\,|x| = n) = 3 \wedge \textbf{students}(x) \wedge \textbf{smiled}(x)]$
$\equiv \exists x[|x| = 3 \wedge \textbf{students}(x) \wedge \textbf{smiled}(x)]$

4.2 Maximality and modified numerals

Since we are severing maximality from *less than*, the lexical meaning of the latter, given in (55), looks weaker than what we saw in section 3. There, *less than three*, for example, combined with a predicate of degrees P and returned true iff $\max(P) < 3$, i.e. iff either P is true of some $m < 3$ and moreover m is the maximum of P, or else P is empty. Now, we require only that P be true of some $m < 3$, and not that m also be the maximum of P; the maximality part will come separately, via the operator `isMax` in (51).[32]

(55) $[\![\text{less than}]\!] = \lambda n_d\,.\,\lambda P_{dt}\,.\,\exists m[m < n \wedge P(m)]$

First consider (56). In the given LF, *less than three* scopes above existential closure, creating a degree predicate in its scope, with its trace shifted to an intersective denotation. Due to the distributivity of *smiled*, the derived truth conditions are weak existential truth conditions, which amount to saying that some student(s) smiled. At this point, we invoke

[31] Notice that we again rely on the equivalence between $\max(\lambda n\,.\,|x| = n)$ and $|x|$, which in turn ensures the equivalence between *three*$_{\texttt{isMax}}$ *[λn [x $n_{\texttt{isCard}}$]]* and *x three*$_{\texttt{isCard}}$.

[32] The new meaning for *less than* in (55) is not strictly weaker than the one in (37) in section 3, since the latter, but not the former, returns true when the degree predicate is empty.

the pragmatic constraint in (44) from section 3 to block this LF.[33]

(56) a. Less than three students smiled.
b. [less than three] [λn [[$\varnothing$ [$n_{\texttt{isCard}}$ students]] smiled]]
c. $\exists n[n < 3 \wedge \exists x[|x| = 3 \wedge \textbf{students}(x) \wedge \textbf{smiled}(x)]]$
$\equiv \exists x[|x| < 3 \wedge \textbf{students}(x) \wedge \textbf{smiled}(x)]$

Now consider (57). The given LF involves two movements of *less than three*. It first raises above existential closure, creating a degree predicate in its scope, with its trace ($m_{\texttt{isCard}}$) shifted to an intersective denotation, as is familiar by now. It then raises a second time, creating a second degree predicate, with its second trace shifted to an isMax denotation ($n_{\texttt{isMax}}$). The resulting truth conditions are upper-bounded with no existential entailment, i.e. precisely the truth conditions we want for a sentence like this with the distributive predicate *smiled*.

(57) a. Less than three students smiled.
b. [less than three] [λn [$n_{\texttt{isMax}}$ [λm [[$\varnothing$ [$m_{\texttt{isCard}}$ students]] smiled]]]]
c. $\exists n[n < 3 \wedge \max(\lambda m \,.\, \exists x[|x| = m \wedge \textbf{students}(x) \wedge \textbf{smiled}(x)]) = n]$
$\equiv \max(\lambda m \,.\, \exists x[|x| = m \wedge \textbf{students}(x) \wedge \textbf{smiled}(x)]) < 3$

SMax can thus generate the same set of truth conditions as the previous theory; however, it does so not by scope ambiguity, but rather by optional application of maximization.

Of course, in principle nothing prevents *less than three* from scoping below existential closure in this theory as well. (58b) illustrates this case with no maximization, and (59b) with maximization. It turns out that both LFs derive the exact same, weak, existential truth conditions as were already derived in (56).[34] The reason that the same truth conditions are derived is twofold. First, the lexical meaning of *less than* on this theory is a simple existential statement (about degrees); thus, the existential of *less than* and that of the silent determiner *commute*, meaning that there is no scope dependency between *less than three* and $\varnothing$; thus, (56b) and (58b) are equivalent. Second, when max ($n_{\texttt{isMax}}$) takes lowest scope, it has no semantic effect, due to the by now familiar equivalence between $\max(\lambda n \,.\, |x| = n)$ and $|x|$ (cf. the LF in (54b); see also fn. 31); thus, (58b) and (59b) are equivalent. It follows that (56b), (58b), and (59b) are all equivalent.

(58) a. Less than three students smiled.
b. [$\varnothing$ [[λx [[less than three] [λn [x $n_{\texttt{isCard}}$]]]] students]] smiled
c. $\exists x[\textbf{students}(x) \wedge \textbf{smiled}(x) \wedge \exists n[n < 3 \wedge |x| = n]]$
$\equiv \exists x[|x| < 3 \wedge \textbf{students}(x) \wedge \textbf{smiled}(x)]$

(59) a. Less than three students smiled.
b. [$\varnothing$ [[λx [[less than three] [λn [$n_{\texttt{isMax}}$ [λm [x $m_{\texttt{isCard}}$]]]]]] students]] smiled
c. $\exists x[\textbf{students}(x) \wedge \textbf{smiled}(x) \wedge \exists n[n < 3 \wedge \max(\lambda m \,.\, |x| = m) = n]]$
$\equiv \exists x[\textbf{students}(x) \wedge \textbf{smiled}(x) \wedge \max(\lambda m \,.\, |x| = m) < 3]$

[33] Of course, this sort of LF is exactly right for the analogous sentence with a non-distributive predicate, such as *lifted the piano*, and that LF will not be blocked.

[34] Again, these LFs would be suitable for the analogous sentence with a non-distributive predicate, such as *lifted the piano*, and would not be blocked.

$$\equiv \exists x[|x| < 3 \wedge \mathbf{students}(x) \wedge \mathbf{smiled}(x)]$$

The upshot is that SMax can derive non-upper-bounded, existential truth conditions (which are required for the collective cases) in several ways, and it can derive upper-bounded truth conditions as well (which are required for the distributive cases) with the scope order *less than three* $> n_{\texttt{isMax}} > \varnothing$. And the same pragmatic constraint we posited in section 3 can be invoked to block the unattested LFs in (56b), (58b), and (59b).

5 Further support

Before considering data that potentially tease apart LMax and SMax, we first provide further support that such analyses are on the right track. We discuss in turn (i) a subclass of collective predicates which license 'downward inferences' (inferences from groups to subgroups) and thus pattern like distributive predicates with respect to modified numerals, and (ii) issues that arise with cumulative predicates.

5.1 Collective predicates with downward inferences

To recap so far, we've seen that a sentence like (60), with the distributive predicate *smiled*, has an upper-bounded reading with no existential entailment, which can be paraphrased as, 'The maximum number of students who smiled, if any, is less than three.' It does not have the non-upper-bounded, existential reading, 'A group of less than three students smiled.' By contrast, (61), with the collective predicate *lift the piano*, does have the non-upper-bounded, existential reading, 'A group of less than four students lifted the piano.'

(60) Less than three students smiled.

(61) Less than four students lifted the piano.

Now consider (62), with the collective predicate *gathered*. Intuitively, (62) works just like (60), and not like (61): (62) is true iff the maximum number of students who gathered is less than four. That is, if (a group of) four or more students gathered, then (62) is false.

(62) Less than four students gathered.

We argue that our pragmatic constraint actually predicts this. The argument relies on the fact that collective predicates like *gather*, despite being collective, do in fact allow a kind of weak downward inference. Suppose that Ann, Bill, Carol, and Dan gathered. Then we cannot conclude that, say, Ann gathered, for *Ann gathered* is an unacceptable sentence, and it is unclear what it would mean for a single individual to gather. However, intuitively, we *can* conclude that Ann, Bill, and Carol gathered; that Ann, Bill, and Dan gathered; that Ann and Bill gathered; etc. In other words, although *gather* does not distribute all the way down to the atoms of a plurality, it does seem to distribute down to (at least some of) the non-atomic subparts of a plurality.

What is important, as far as our pragmatic constraint is concerned, is that if a group x gathered, then we can always find some subgroup y of x such that $|y| = 2$ and y gathered.

Let us call this property *weak distributivity*.[35]

(63) **Weak distributivity**
P is weakly distributive iff $\forall x[[P(x) \wedge |x| \geq 2] \rightarrow \exists y[y \sqsubseteq x \wedge |y| = 2 \wedge P(y)]]$.
(... iff, if P is true of a non-atomic plurality x, then P is true of some subpart of x with two atomic parts.)

Now, the inference from *Ten students gathered* to *Two students gathered* is predicted to be valid, as shown in (64). This prediction is in line with our judgments.[36]

(64)
$\exists x[|x| = 10 \wedge \textbf{students}(x) \wedge \textbf{gathered}(x)]$ (Ten students gathered.)
$\forall x, y[[\textbf{students}(x) \wedge y \sqsubseteq x] \rightarrow \textbf{students}(y)]$ (*students* is (strongly) distr.)
$\forall x[[\textbf{gathered}(x) \wedge |x| \geq 2] \rightarrow \exists y[y \sqsubseteq x \wedge |y| = 2 \wedge \textbf{gathered}(y)]]$
(*gathered* is weakly distr.)
$\therefore\ \exists y[|y| = 2 \wedge \textbf{students}(y) \wedge \textbf{gathered}(y)]$ (Two students gathered.)

What we would like now is for our pragmatic economy constraint in (44) to block any LF of (62) that corresponds to the existential, non-upper-bounded reading represented in (65).

(65) $\exists x[|x| < 4 \wedge \textbf{students}(x) \wedge \textbf{gathered}(x)]$

If the weak distributivity of *gathered* is taken into consideration, then (65) is equivalent to (66).

(66) $\exists x[\textbf{students}(x) \wedge \textbf{gathered}(x)]$

The entailment from (65) to (66) follows from conjunction elimination. The reverse entailment follows because if a plurality of students x gathered, then, since *gathered* has only non-atomic sums in its extension, $|x| \geq 2$, and so by weak distributivity of *gathered* (and of *students*, which is also strongly distributive), there is a subpart y of x with cardinality $2 < 4$ who are students who gathered.

What this reasoning illustrates is that the numeral *four* (in an LF of the kind in question) has no semantic import. The same truth conditions would be derived (by similar reasoning) if *four* were replaced, say, by *three* or by *five*. Thus, this reading is ruled out by our pragmatic constraint, leaving only the upper-bounded reading.

Other collective predicates that seem to work like *gather*, i.e. that are weakly distributive in the sense of (63), are *be neighbors*, *hold hands*, *know each other*, and *be similar*. As expected, the sentences in (67) only have upper-bounded readings.

(67) a. Less than four students are neighbors.
b. Less than four students were holding hands.

[35] On this definition, every (strongly) distributive predicate (in the sense of (14)) is weakly distributive, but not vice versa.

[36] More precisely, we find this inference to just as natural as the one from (ia) to (ib).

(i) a. Ten students attended.
b. Two students attended.

c. Less than four students know each other.
d. Less than four students are similar.

5.2 A predicted asymmetry with cumulative transitive predicates

Let us return to our example in (2a), repeated below as (68).

(68) Less than five guests drank over twenty bottles between them.

This is an instance of the so-called *cumulative reading* of transitive predicates, which is forced here by the phrase *between them*. The important point here is that on the cumulative reading of *drank*, there is no downward inference regarding the subject argument. That is, (69a) below does not entail (69b):

(69) a. Peter, Sue, and Mary drank over twenty bottles between them.
b. Peter and Sue drank over twenty bottles between them.

Given the absence of any downward inference, the subject of such a cumulative predicate will behave just as if it were the subject of a collective predicate that does not allow any downward inference. As a result, the existential, non-upper-bounded reading of (68) will (correctly) not be blocked by our pragmatic economy constraint.

Now, let us see what we predict if a modified numeral of the relevant sort (i.e. *less than n* or *between m and n*) occurs in the object position of a cumulative predicate. The predictions we make depend on the specific treatment of cumulative predicates. We will argue that, contrary to what is often assumed, the object of a cumulative predicate licenses downward inferences,[37] so that we expect an upper-bounded reading to be forced for modified numerals in object position.

Let us first consider the following simple sentence:

(70) These twenty boys danced with those ten girls.

On the cumulative reading, this is standardly analyzed as meaning that every one of these twenty boys danced with at least one of those ten girls, and that for every one of those ten girls, at least one of these twenty boys danced with her. This leads to the following kind of entry for $dance_{cumul}$ on the cumulative reading, defined in terms of a primitive lexical entry for *dance* that is only defined for atoms.

(71)
$$\begin{aligned}&[\![\text{dance}_{\text{cumul}}]\!]\\ &= \lambda Y_e\,.\,\lambda X_e\,.\,\forall x[[\mathbf{atom}(x) \wedge x \sqsubseteq X] \rightarrow \exists y[\mathbf{atom}(y) \wedge y \sqsubseteq Y \wedge [\![\text{dance}]\!]\,(y)(x)]]\\ &\qquad \wedge \forall y[[\mathbf{atom}(y) \wedge y \sqsubseteq Y] \rightarrow \exists x[\mathbf{atom}(x) \wedge x \sqsubseteq X \wedge [\![\text{dance}]\!]\,(y)(x)]]\end{aligned}$$

On such a view, there is no downward inference either on the subject side or on the object side, and as a result we would expect no difference for modified numerals depending on whether they occur in subject or in object position.

[37] This observation has also been independently made by Viola Schmitt (Schmitt 2015) and to some extent (in our view anyway) by Seth Cable (class notes on Krifka 1999, available at `http://people.umass.edu/scable/LING720-FA10/Handouts/Krifka-1999.pdf`).

However, there is some evidence that (71) leads to truth conditions that are too strong. This can easily be seen if we use another type of predicate:

(72) These twenty chickens laid those ten eggs between them.

On the basis of a lexical entry such as (71) for *lay*, (72) would be predicted to entail that every one of the twenty chickens laid one of the ten eggs, which entails that some eggs were laid by two different chickens—an impossible state of affairs. That is, (72) would be a contextual contradiction. However, we observe that it is not. Rather, it simply entails that each of the ten eggs was laid by one of the twenty chickens. This suggests a much weaker lexical entry:

(73) $$[\![\text{lay}_{\text{cumul}}]\!] \\ = \lambda Y_e \,.\, \lambda X_e \,.\, \forall y[[\textbf{atom}(y) \wedge y \sqsubseteq Y] \rightarrow \exists x[\textbf{atom}(x) \wedge x \sqsubseteq X \wedge [\![\text{lay}]\!]\,(y)(x)]]$$

Now, on such a view, the cumulative reading of *lay* does actually license a downward inference on the object side. Informally, X *laid* Y now means that every atomic part of Y was laid by an atomic part of X. So suppose that X *laid* Y is true and that Y' is a proper subpart of Y. Since every atomic member of Y' is also an atomic member of Y, every atomic member of Y' must have been laid by an atomic part of X, and therefore X *laid* Y' is true as well.

Given this, it is now predicted that our pragmatic economy constraint will block the existential, non-upper-bounded reading of a modified numeral in object position, i.e. force an upper-bounded reading. This is exactly what we observe: (74a) is interpreted as being equivalent to (74b).

(74) a. These chickens laid less than ten eggs between them.
b. It is not the case that these chickens laid more than nine eggs between them.

We also have an explanation for the following contrast:

(75) a. Less than ten chickens laid more than twenty eggs between them.
$\rightsquigarrow$ Non-upper-bounded reading for the subject: compatible with twenty chickens having laid more than twenty eggs between them.
b. More than ten chickens laid less than twenty eggs between them.
$\rightsquigarrow$ Upper-bounded reading for the object: entails that there is a group of more than ten chickens who did not laid more than 19 eggs between them.[38]

6 Discussion: teasing apart LMax and SMax

So far, our two proposals (LMax and SMax) are on a par. Both generate upper-bounded and non-upper-bounded truth conditions, both rely on a pragmatic blocking mechanism

[38] We have in mind here an LF where *more than ten* scopes above *less than twenty*, and both degree phrases scope out of their respective DPs (hence, take scope over their respective silent determiners). It is unclear to us whether an LF where *less than twenty* scopes above *more than ten* is also available. (Such an LF would seem to violate Kennedy's generalization; see Kennedy 1997 and Heim 2000.)

to rule out one of the two readings in distributive contexts, and both predict an ambiguity in non-distributive contexts (which we will discuss in section 7). At this point, do we have any reason to choose one over the other? Note that both proposals rely on well-motivated mechanisms.

In this section, we will discuss two kinds of cases where the two theories happen to make distinct predictions. We think that the first case provides quite a compelling argument in favor of the SMax view. Regarding the second type of case, we will see that it is very hard to get clear truth-conditional data. Finally, we will also discuss a possible conceptual motivation for the SMax account.

6.1 Upward scalar degree predicates

Consider the following sentence:

(76) Ten eggs are sufficient to make an omelet for all these people.

This seems to express a generic statement to the effect that someone who has more or less any (group of) ten eggs has enough eggs to make an omelet for all these people (see Buccola in prep. for a detailed discussion of the interpretation of bare and modified numerals in generic sentences). Consider now the degree predicate

(77) λn [n eggs are sufficient to make an omelet for all these people][39]

As previously discussed in Beck and Rullmann 1999, this predicate is *upward scalar*, i.e. if it holds of a number n, it holds of $n+1$ as well.[40] For instance, (76) entails (78):

(78) Eleven eggs are sufficient to make an omelet for all these people.

Consider now the following sentence:

(79) Less than ten eggs are sufficient to make an omelet for all these people.

This sentence is not particularly hard to understand, and appears to mean that there is a number n smaller than 10 such that having n eggs is sufficient to make an omelet for all these people.

Now, it seems to us that this reading cannot be generated under the LMax account. The problem for the LMax account is the following. First, if the modified numeral takes widest scope, as in (80), we end up with a sentence which is either false or undefined.

(80) [Less than ten] [λn [n eggs are sufficient to make an omelet for all these people]]

39This LF is not completely explicit because it does not specify whether the numerical variable should be understood as $n_{\texttt{isCard}}$ or $n_{\texttt{isMax}}$, whether $\varnothing$ occurs, etc. This is because we do not want (and do not need) to commit ourselves to a specific analysis of such generic sentences. Buccola (in prep.) suggests that in such cases the numerical variable is under the scope of a silent *generic* determiner. What matters for us is that the relevant degree predicate is upward scalar.

40To round out the terminology, a predicate like *λn [∅ [*$n_{\texttt{isCard}}$ *students] smiled]* is *downward scalar*, since, if it holds of a number n, it holds of $n-1$ as well (down to 1). A predicate like *λn [∅ [*$n_{\texttt{isCard}}$ *soldiers] [surrounded the castle]]* is *non-scalar* since, if it holds of a number n, we can conclude neither that it holds of $n+1$ nor that it holds of $n-1$; that is, it is neither downward nor upward scalar.

On the LMax account, (80) is true if the *maximal* number n which instantiates the property expressed by the degree predicate λn *[n eggs are sufficient sufficient to make an omelet for all these people]* is smaller than 10. But because the degree predicate is upward scalar, it cannot have a maximum, and therefore the proposition expressed by (80) is either false or undefined (depending on how exactly maximality failure is treated, which does not concern us here; cf. fn. 26).[41]

On the SMax account, on the other hand, the very same LF expresses the proposition that for some number n smaller than 10, n eggs are sufficient to make an omelet for all these people, which seems to be exactly what we want.

Now, is there a way to generate the reading we want under the LMax account, by means of an LF in which *less than ten* takes scope below a silent determiner (following the LMax strategy of deriving non-upper-bounded, existential readings by having the modified numeral scope lower than the silent determiner)? Such an LF would be the following:[42]

(81) [$\varnothing$ [[λx [[less than ten] [λn [x $n_{\mathtt{isCard}}$]]]] eggs]] [are sufficient to make an omelet for all these people]

What this would mean is that there exists a specific plurality consisting of less than ten eggs such that this plurality is sufficient to make an omelet for all these people. Such a statement is strictly weaker than the generic statement that a sentence like (78) expresses, namely that there is a number n, smaller than 10, such that more or less *any* group of n eggs is sufficient. This LF therefore does not generate what seems to be the most salient reading of the sentence.[43]

We conclude that the interpretation of (78) provides quite a compelling argument in favor of SMax over LMax, but we will revisit the significance of this conclusion in section 9.1, where we show that SMax seems unable to capture parallel facts associated with *between n and m*.

[41] In section 8, we discuss a theory where maximal informativity is used instead of standard maximality, following the insights of Beck and Rullmann (1999) regarding degree questions that contain upward scalar predicates. This theory will assign correct truth conditions to LFs like (80).

[42] We may also entertain the possibility of an LF where the whole subject is interpreted 'existentially' below the modal *sufficient*, assuming some covert *have* somewhere in the structure. The intended meaning would be something like, 'It is sufficient to be in a situation where one has less than ten eggs to make an omelet for all these people', where 'has less than ten eggs' is in principle ambiguous between an upper-bounded reading and a non-upper-bounded reading. On both readings, though, the resulting LF would entail that *any* number below 10 is sufficient—which is very unlikely to be ever true. So the resulting reading would be different from the one we perceive for (78).

[43] One may argue that the statement that there is a group of n eggs such that this group of eggs is sufficient contextually entails that any group of n eggs is sufficient, hence that the non-generic statement expressed by (81) is contextually equivalent to the perceived reading. However, such an argument would not carry over the following structurally identical case:

(i) Less than ten good arguments are sufficient to win the case.

It is not normally the case that if there are, say, three good arguments such that these arguments suffice (together) to win the case, then *any* three good arguments suffice (maybe some arguments are more likely to convince the judge than others). Yet (i) is easily understood as meaning that for some n smaller than 10 more or less *any* n good arguments are sufficient to win the case.

6.2 Split scope

A key difference between the LMax account and the SMax account is the following. On the LMax account, the 'maximality component' is part of the very meaning of modified numerals, while on the SMax account, it is part of the meaning of the numerical variable bound by the modified numeral phrase. It follows that while the SMax account can generate all the readings that the LMax account can, the reverse is not true. Let us first illustrate this point in an abstract and schematic way. On the SMax account, LFs of the following type are in principle available:

(82) [less than ten] [λn [Op [$n_{\texttt{isMax}}$ [λm [$\ldots$ [$\varnothing$ [$m_{\texttt{isCard}}$ $\ldots$]] $\ldots$]]]]]

The crucial point here is that between *less than ten* and $n_{\texttt{isMax}}$, an operator, *Op*, intervenes. Such an LF is unavailable on the LMax account: if *less than ten* takes widest scope, then so does the maximality component, because this component is part of the meaning of the modified numeral. We can show the relevance of this point by considering now a real sentence:

(83) I have to take less than five pills.

Let us schematize the surface form of this sentence as follows, where $\Box$ is a necessity operator standing for *have to*.

(84) $\Box$ [I take less than five pills]

On the LMax account, *less than ten* has to move to combine with a predicate of degrees, and (84) is ambiguous depending on where exactly the numerical phrase *less than ten* lands. We assume here that the numerical phrase cannot be interpreted lower than the silent indefinite, $\varnothing$, due to our pragmatic blocking mechanism.[44] There are then two possible LFs, depending on whether *less than ten* scopes below or above the necessity modal, as illustrated in (85):

(85) a. Low-scope for *less than ten*:
 (i) $\Box$ [less than ten] [λn [[$\varnothing$ [$n_{\texttt{isCard}}$ pills]] [λx [I take x]]]]
 (ii) 'In every permissible world, the number of pills I take is smaller than five.'
 b. Wide-scope for *less than ten*:
 (i) [less than ten] [λn [$\Box$ [[$\varnothing$ [$n_{\texttt{isCard}}$ pills]] [λx [I take x]]]]]
 (ii) $\max(\lambda n\,.\,\text{I have to take } n \text{ pills or more}) < 10$
 $\rightsquigarrow$ 'The number n such that I have to take n pills and don't have to take more than n pills is smaller than 10.'

On the low-scope reading, (84) simply states that I am forbidden to take more than ten pills. On the wide-scope reading, the sentence states that the minimal required number of pills I have to take is smaller than ten—on this reading the sentence is compatible with a

[44]This point relies on the fact that *take* is distributive on its object argument. In other words, the predicate λx *[I take x]* is distributive: if I take x, then I take every part of x.

situation where I am allowed to take 100 pills.[45]

Now, on the SMax account, both of these readings are generated (depending on whether maximization applies or not). However, a third ('split-scope') reading is predicted to be available, whose LF is given in (86a):

(86) a. [less than ten] [λn [$\Box$ [$n_{\texttt{isMax}}$ [λm [[$\varnothing$ [$m_{\texttt{isCard}}$ pills]] [λx [I take x]]]]]]]

b. 'There is a number n, smaller than 10, such that I have to take *exactly* n pills.'

The resulting reading states that there is a certain number, smaller than 10, such that I have to take exactly that number of pills. Now, does this reading exist? At an intuitive level, we seem to access this reading in the following type of discourse:

(87) I visited the doctor yesterday, and I don't remember everything he told me. He prescribed a certain medication. I have to take less than ten pills, but I don't remember how many exactly.

One can certainly understand (87) as implying that there is a number n smaller than 10 such that I have to take exactly n pills. This, however, is not sufficient to establish that the LF in (86a) is really available. The reason is the following. In normal situations, doctors prescribe a specific number of pills. Suppose that this information is taken for granted by the speaker and addressee of (87). Now, assume that *I have to take less than ten pills* is parsed as in (85a). Then it simply means that in every permissible world the number of pills I take is smaller than ten (the 'low-scope' reading). While this low-scope reading is strictly weaker than the reading corresponding to (86a), together with the contextual knowledge that there is a number n such that I have to take exactly n pills, it turns out to be contextually equivalent to (86a). That is, if, in every permissible world, I take less than ten pills (meaning of (85a)), and if, furthermore, I take the same number of pills in every permissible world (contextual information), then it must be the case that there is a specific number n, smaller than 10, such that in every permissible world I take exactly n pills (meaning of (86a)).

The problem we face here is that the type of reading we are trying to probe, viz. (86a), is not very plausible in general. For instance, if we replace *take less than ten pills* with *solve less than ten problems*, the corresponding reading would imply that there is a specific number n such that I have to solve n problems and am forbidden to solve more, which is a highly unlikely state of affairs. But the kind of contexts that make readings corresponding to (86a) plausible also tend to make this reading contextually equivalent to the low-scope reading (corresponding to (85a)).[46]

[45]While such a reading is not pragmatically plausible in this case, structurally identical sentences clearly license the wide-scope reading. Consider, for instance, (i), inspired by Heim 2000:

(i) Fortunately, we have to write less than 10 pages.
Intended: 'Fortunately, the minimal required length of the paper is less than 10 pages.'

[46]One could argue that the sluice (*I don't remember how many exactly*) independently forces, for syntactic reasons, a wide-scope construal of its antecedent, *less than ten pills*, which would be consistent with Romero 1998 and Johnson 2001. Yet the whole argument would then depend on a specific syntactic analysis of sluicing, which might be open to challenges.

One potential way around this problem is to construct an example where the entailment relations between putative readings are reversed. This we can do by replacing the upward-entailing modal *have to* with the downward-entailing modal *be forbidden to*. And of course we have to create an appropriate context. Here is our attempt.

(88) a. Context: Peter belongs to a weird cult, in which a certain number (maybe 13) is viewed as evil, and it is absolutely forbidden (on pain of death, say) to invite exactly that number of people. Any other number is OK.
b. Peter is forbidden to invite between 10 and 20 people, but I don't remember the exact number.

Now, the intended (split-scope) reading is, 'There is a number n between 10 and 20 such that Peter is forbidden to invite exactly n people.' This reading is true, say, if Peter can invite any number of people distinct from 13 but is forbidden to invite exactly 13 people. Importantly, in such a situation, the low-scope reading is false since Peter can invite 12 people, for instance. As for the potential wide-scope reading ('There is a number n between 10 and 20 such that n is the maximum of the extension of λn *[Peter is forbidden to invite n or more people]*'), it should be either false or a presupposition failure, depending on how max is defined, simply because either the degree predicate is empty, or if it is non-empty, then it has no maximal element (see also fn. 26).

So, to the extent that (88b) could be considered appropriate, it would provide evidence for a parse of the following sort:

(89) a. [between 10 and 20] [λn [forbidden [$n_{\mathtt{isMax}}$ [λm [[$\varnothing$ [$m_{\mathtt{isCard}}$ people]] [λx [Peter invites x]]]]]]]
b. 'There is a number n, between 10 and 20, such that Peter is forbidden to invite *exactly n* people.'

We are dealing here with very delicate judgments for which introspective intuitions might be unreliable, but the informants we consulted seem to be able to access such a reading, which would go in favor of SMax.

6.3 A potential conceptual motivation for SMax

Let us now turn to a potential conceptual motivation for the SMax account. The argument is based on the case of modified numerals of the form *between m and n*. Ideally, we would like the denotation of such a phrase to be fully determined by the meaning of the numerals and that of the preposition *between*. Now, *between* is a locative preposition. In most of its uses, it refers to a spatial relationship that can be approximated by Euclidean geometrical concepts, without any reference to an ordering relation between points in terms of which a notion of maximality can be defined. In general terms, *between x and y* is felicitous as long as the denotations of x and y belong to a domain of entities for which a notion of betweenness is defined, most typically locations.[47] To say that a location z is between Paris

[47] Geometrical betweenness can be axiomatized without any reference to an ordering relation, hence in such a way that no notion of 'maximality' arises—though it's possible to define an ordering relation based on betweenness. See for instance Hilbert's axioms of betweenness in Hilbert 1899.

and Lyon is just to say that z lies on the segment whose endpoints are Paris and Lyon. Saying that John lives between Paris and Lyon amounts to saying that the place where he lives is (approximately) on this segment. There is no clear sense in which a notion of maximality plays any role here. On this basis, a natural characterization of the meaning of *between x and y* is the following:

(90) $[\![\text{between } x \text{ and } y]\!] = \lambda P \,.\, \exists z[z \text{ is between } x \text{ and } y \wedge P(z)]$

Let us see what this delivers in the case of *John lives between Paris and Lyon*:

(91)
a. John lives between Paris and Lyon.
b. [between Paris and Lyon] [λz [John lives at z]]
c. 'There is a place z between Paris and Lyon such that John lives at z.'

Now, if we are dealing with a naturally ordered set of entities, then the ordering in turn defines a natural relation of betweenness. In the case of numbers, for example, betweenness is defined in terms of the ordering of numbers, so that z is between x and y iff $x \leq z \leq y$ or $y \leq z \leq x$. So we have, for example:

(92) $$\begin{aligned}[\![\text{between 5 and 10}]\!] &= \lambda P \,.\, \exists z[z \text{ is between } 5 \text{ and } 10 \wedge P(z)] \\ &= \lambda P \,.\, \exists z[5 \leq z \leq 10 \wedge P(z)]\end{aligned}$$ [48]

But this, of course, is what the SMax theory assumes. To sum up, there is a conceptual motivation for severing the maximality component from the meaning of modified numerals which are built on the basis of locative prepositions. These considerations, however, do not extend to modified numerals like *less than n*.

7 Is the pragmatic blocking mechanism sufficient?

So far, we are able to account for the fact that, in certain environments, modified numerals can only give rise to an upper-bounded reading. The gist of the explanation is that the other reading (namely the existential, non-upper-bounded reading) is ruled out by our pragmatic economy constraint in certain environments. We do, however, predict a genuine ambiguity in cases where the constraint does not rule out the existential, non-upper-bounded reading. Consider in this light the following sentence:

(93) Less than ten soldiers surrounded the castle.

In the introduction, we argued that sentences like (93) can *only* receive the existential, non-upper-bounded reading, and we observed that for many speakers replacing *the castle* with the NPI *any castle* leads to a deviant sentence, which seems to add further support

[48] The question arises whether the ternary relation 'z is between x and y' is 'inclusive', i.e. does it hold when $z = x$ or $z = y$? According to Hilbert's axiomatization of geometrical betweenness, it does not hold. However, it seems to us that *John read between 3 and 10 books* is true if John read exactly 3 or exactly 10 books. Furthermore, even though (91) seems false if John happens to live in Paris, we suspect that this is due to an implicature, rather than to the literal truth conditions of the sentence. Thus, a sentence such as *Every restaurant in this guide is located between Paris and Lyon* does not strike us as false if some of the relevant restaurants are in Paris or in Lyon.

that the upper-bounded reading is unavailable. This, however, is not expected if (93) is ambiguous, as we currently predict, between the existential and the upper-bounded readings. Under the parses given in (94a) and in (94b) the sentence should be able to be interpreted as in (94c).

(94) a. Generation of the upper-bounded reading under the LMax account:
   (i) [less than ten] [λn [[$\varnothing$ [$n_{\texttt{isCard}}$ soldiers]] [surrounded the castle]]]
   (ii) $\max(\lambda n\,.\,\exists x[|x| = n \wedge \textbf{soldiers}(x) \wedge \textbf{surrounded}(x)]) < 10$

b. Generation of the upper-bounded reading under the SMax account:
   (i) [less than ten] [λn [$n_{\texttt{isMax}}$ [λm [[$\varnothing$ [$m_{\texttt{isCard}}$ soldiers]] [surrounded the castle]]]]]
   (ii) $\exists n[n < 10 \wedge \max(\lambda m\,.\,\exists x[|x| = m \wedge \textbf{soldiers}(x) \wedge \textbf{surrounded}(x)]) = n]$
   $\equiv \max(\lambda m\,.\,\exists x[|x| = m \wedge \textbf{soldiers}(x) \wedge \textbf{surrounded}(x)]) < 10$

c. 'No group of more than nine soldiers surrounded the castle.'

Somehow we need to rule out this reading. In section 8, we will develop a theory where a non-existential, upper-bounded interpretation is simply not generated at all when the modified numeral occurs in a position that does not license a downward inference. As we shall see, the theory in question can itself be implemented in various ways.

Before turning to this alternative theory, however, we would like to discuss a case where the ambiguity predicted by the LMax and SMax accounts seem to be detectable for at least some speakers. The relevant examples involve the predicate *form a stable coalition* (thanks to Philippe Schlenker for suggesting this predicate). Consider the following sentences:

(95) a. I will despair of politics only if less than 50 MPs form a stable coalition.
b. Unfortunately, less than 50 MPs formed a stable coalition.
c. When less than 350 MPs form a stable coalition, there cannot be a stable government.

(96) Surprisingly, less than 50 MPs formed a stable coalition and these MPs were able to counterbalance the much larger coalition that 360 other MPs formed at the same time.

The intended readings for (95a), (95b) and (95c) are, respectively, the following:

(97) a. I will despair of politics only if no coalition with more than 49 MPs is formed.
b. Unfortunately, there was no coalition formed by more than 49 MPs.
c. When no coalition with more than 349 MPs is formed, there cannot be a stable government.

In these three cases, the intended reading is one where the modified numeral receives the upper-bounded interpretation. To the extent that these readings are available for the sentences in (95), it appears that a modified numeral that combines with *form a coalition* can give rise to an upper-bounded interpretation, despite the fact that *form a coalition* does not seem to license downward inferences. (If a group of MPs formed a stable coalition, it does not follow that any proper subgroup did.)

In the case of (96), however, an upper-bounded reading would lead to a contextual contradiction, and the only possible reading is the existential, non-upper-bounded one, paraphrased in (98).

(98) Surprisingly, there was a coalition formed which contained less than 50 MPs, and the members of this coalition were able to counterbalance the much larger coalition formed by 360 other MPs.

If both types of readings are available, we have evidence for a genuine ambiguity. We should note, however, that judgments are far from uniform: only some speakers accept the interpretations of the sentences in (95) suggested in (97). Furthermore, even for those speakers who access these readings (such as one of the two authors of this paper), they seem to be possible only with some predicates. This might suggest that the relevant predicates are themselves ambiguous between a meaning that licenses downward inferences (contrary to appearances) and a meaning that does not. Given all these observations, it it thus worth investigating an alternative theory in which the relevant sentences are not predicted to be ambiguous. We now turn to one such theory.

8 Informativity-based maximality

In this section we explore the question of whether a theory of overgeneration together with pragmatic blocking is really the best route, or whether there is a way to directly generate exactly the right reading for each type of sentence. We elaborate on a suggestion by Philippe Schlenker (p.c.), which is reminiscent of works in which it is argued, in relation with various semantic phenomena, that the standard notion of maximality should be replaced by a notion based on maximal informativity, i.e. logical strength (Dayal 1996; Rullmann 1995; Beck and Rullmann 1999; Fox and Hackl 2007; Schlenker 2012; von Fintel, Fox, and Iatridou 2014).

We first present the basic idea. We then discuss some non-trivial complications that arise and propose an explicit implementation that addresses these complications. Finally, we will see that when we move from standard maximality to a notion of maximality defined in terms of logical strength, we still have to decide whether the 'maximality component' is part of the intrinsic meaning of modified numerals or rather is independent of it (possibly by locating it on numerical variables). We will then sketch an account where maximality, now thought of as 'maximal informativity', is severed from modified numerals, and discuss potential motivations for such an account.

8.1 A semantics based on maximal informativity: the basic idea

The proposal we will develop here is based on a fairly simple idea. Whether we get an upper-bounded or a non-upper-bounded reading for modified numerals depends on whether the degree predicate they combine with has a certain logical property, namely the property of licensing downward inferences (*downward scalarity* in the sense of Beck and Rullmann 1999; cf. fn. 40). If we manage to define a semantics for modified numerals that somehow refers to this property, we might be able to solve the puzzle.

The approach we will present requires that we now view degree predicates as representing *properties* of degrees, i.e. functions from worlds to (characteristic functions of) sets of degrees. Consider the predicate of degrees in (99). While, in any given world, its denotation is a type dt function, its *intension* is a type $s(dt)$ function. We are going to devise a semantics for modified numerals where they take as an argument an $s(dt)$ function, i.e. the intension of a degree predicate.

(99) λn [[$\varnothing$ [$n_{\texttt{isCard}}$ students]] smiled]

We start with the following auxiliary notion (we identify degrees with numbers, as we are only concerned in this paper with numerical expressions):

(100) For any property of numbers P (i.e. function of type $s(dt)$), a number n is P-maximal in a world w iff the following two conditions hold:
  a. $P(w)(n) = 1$.
  b. There is no number m such that both $P(w)(m) = 1$ and $[\lambda v\,.\,P(v)(m) = 1] \subsetneq [\lambda v\,.\,P(v)(n) = 1]$.[49]

More informally (allowing ourselves to ignore the *use*/*mention* distinction, i.e. conflating expressions with their semantic values):

(101) For any property of numbers P, a number n is P-maximal in w iff:
  a. $P(n)$ is true in w.
  b. There is no number m such that both $P(m)$ is true in w, and the proposition expressed by $P(m)$ asymmetrically entails the proposition expressed by $P(n)$.

In plain English, for a number n to be P-maximal in w, the proposition $P(n)$ must both be true in w and be such that there is no more informative proposition of the form $P(m)$ that is also true in w. Note that the underlying notion of maximality we use here is quite *weak*. For a number n to be P-maximal in w, it is not necessary that the proposition $P(n)$ be the (unique) most informative true proposition of this form. If for some other number m, $P(m)$ is true and $P(n)$ does not entail $P(m)$, n can still be P-maximal, so long as $P(m)$ does not entail $P(n)$ (i.e. if there is simply no entailment in either direction). For this reason, it is perfectly possible in principle for two numbers m and n to both count as P-maximal in a given world—a fact that will prove important.[50]

Let us now introduce the following notation:

(102) For any property of numbers P, world w, and number n:
  $\text{max}_{\text{inf}}(P)(w)(n) = 1$ iff n is P-maximal in w

We can now provide the following entry for *less than*:[51, 52]

[49]Recall that the symbol $\subsetneq$ denotes the *proper subset* relation.

[50]In this respect, the notion of maximal informativity that we use here is weaker than the one found in some other works that rely on maximal informativity (e.g. Dayal 1996; Rullmann 1995; Beck and Rullmann 1999; Fox and Hackl 2007; Abrusán 2007; Abrusán and Spector 2011; von Fintel, Fox, and Iatridou 2014).

[51]It may be helpful to compare this definition to the one for the LMax account in (37).

[52]We are treating modified numerals here as *intensional operators*, because they need to have access to the *intension* of their argument, given the definition of max_{inf}. We are assuming an intensional system in which

(103) $[\![\text{less than}]\!]^w = \lambda n_d \,.\, \lambda P_{s(dt)} \,.\, \exists m[m < n \wedge \max_{\text{inf}}(P)(w)(n) = 1]$

As a result, we have:

(104) $[\![\text{less than three}]\!]^w = \lambda P_{s(dt)} \,.\, \exists m[m < 3 \wedge \max_{\text{inf}}(P)(w)(n) = 1]$

Note that thanks to the rule of intensional functional application (cf. fn. 52; this rule amounts to treating the argument of an intensional operator as denoting its intension instead of its standard denotation), *less than three* can directly combine with a predicate of type dt, so that we can keep intact the lexical entries and types that we used in the previous sections.

As we will now see, such a lexical entry captures the basic contrast between predicates that license downward inferences (typically distributive predicates) and predicates that do not license downward inferences (typically non-distributive predicates).

Let us first consider what happens for a sentence such as (105).

(105)
a. Less than three students smiled.
b. [less than three] [λm [[$\varnothing$ [$m_{\texttt{isCard}}$ students]] smiled]]

Less than three combines with an expression of type dt, namely:

(106) λm [[$\varnothing$ [$m_{\texttt{isCard}}$ students]] smiled]

In a world w, this expression denotes (the characteristic function of) the set of numbers m such that there are m smiling students in w. Because *less than three* wants an argument of type $s(dt)$, however, intensional functional application (cf. fn. 52) has to be used, and so the argument of $[\![\text{less than three}]\!]^w$ is $\lambda v \,.\, [\![\lambda m \text{ [[}\varnothing\text{ [}m_{\texttt{isCard}}\text{ students]] smiled]}]\!]^v$. The sentence

denotations are relativized to a world parameter (which occurs as a superscript to the right of the denotation brackets). We do not assume any world variable in the syntax. Furthermore, predicates and generalized quantifiers still retain a purely extensional type (et and $(et)t$, respectively), but we now need two rules of functional application: the standard one and the one that Heim and Kratzer (1998, ch. 12) call *Intensional Functional Application*, which is needed to interpret intensional operators (i.e. operators that take the intension of their complement as their argument). That is, our two rules of composition are the following:

(i)
a. **Standard Functional Application**
If the daughters of a node α are β and γ, with γ of type σ and β of type $\sigma\tau$, then:
$[\![\alpha]\!]^w = [\![\beta]\!]^w([\![\gamma]\!]^w)$.
b. **Intensional Functional Application**
If the daughters of a node α are β and γ, with γ of type σ and β of type $(s\sigma)\tau$, then:
$[\![\alpha]\!]^w = [\![\beta]\!]^w(\lambda v_s \,.\, [\![\gamma]\!]^v)$.

With this in place, we can keep the very same lexical entries as before for numerals viewed as cardinality predicates ($n_{\texttt{isCard}}$) and the null indefinite determiner ($\varnothing$). We just need to add the world parameter on the left-hand side of the definitions, and sometimes as an argument on the right-hand side :

(ii)
a. $[\![\varnothing]\!]^w = \lambda P_{et} \,.\, \lambda Q_{et} \,.\, \exists x[P(x) \wedge Q(x)]$
b. $[\![n_{\texttt{isCard}}]\!]^w = [\![\texttt{isCard}\ n]\!]^w = \lambda x_e \,.\, |x| = [\![n]\!]^w$
c. $[\![\text{students}]\!]^w = \lambda x \,.\, \textbf{students}(w)(x) = 1$

Note that even though the object language expression *students* is still treated as an expression of type et, the metalanguage expression **students** now represents a function of type $s(et)$. '$\textbf{students}(w)(x)$' can be read as, 'x are students in w.'

ends up true in a world w just in case there is a number k, smaller than 3, such that the following holds:

(107) $\text{max}_{\text{inf}}(\lambda v \,.\, [\![\lambda m\ [[\varnothing\ [m_{\texttt{isCard}}\ \text{students}]]\ \text{smiled}]]\!]^{v})(w)(k) = 1$

Now, when does this hold? Let $P = \lambda v \,.\, [\![\lambda m\ [[\varnothing\ [m_{\texttt{isCard}}\ \text{students}]]\ \text{smiled}]]\!]^{v}$. Then (107) holds if k is P-maximal in w, i.e. if the two conditions in (108) are met:

(108) a. There are k smiling students in w.
b. There is no k' such that the proposition that there are k' smiling students is both true in w and asymmetrically entails the proposition that there are k smiling students.

Now, because *smile* is both distributive and cumulative in the sense of (14) and (26), the two statements in (109) are equivalent:

(109) a. The proposition that there are k' smiling students asymmetrically entails the proposition that there are k smiling students.
b. $k' > k$.

Thanks to this equivalence, we can restate (108) as follows:

(110) a. There are k smiling students in w.
b. There is no k' such that $k' > k$ and there are k' smiling students in w.

So, for k to be P-maximal, it should simply be the greatest number n such that $P(n)$ is true. The sentence is then predicted to mean that the greatest number n such that there are n smiling students is smaller than 3, which gives us our desired upper bound.[53]

Of course this result will generalize to all predicates which, like *smile*, allow downward inferences with respect to their argument. In all such cases, we get exactly the same result as the one that follows from a lexical entry based on standard maximality, because standard maximality and maximality based on logical strength happen to be equivalent in such cases.

Let us now turn to a typical non-distributive predicate like *surround the castle*, as in the following sentence:

(111) a. Less than ten soldiers surrounded the castle.
b. [less than ten] [λm [[$\varnothing$ [$m_{\texttt{isCard}}$ soldiers]] [surrounded the castle]]]

The compositional steps in the calculation of the meaning of (111) are the same as in the previous case. So the sentence ends up true in a world w just in case there is a number k, smaller than 10, which meets the two following conditions:

(112) a. There are k soldiers surrounding the castle in w.
b. There is no k' such that the proposition that there are k' soldiers surrounding the castle is both true in w and asymmetrically entails the proposition that

[53] Given the particular way we have formulated max_{inf}, there is also an undesired existential entailment here, an issue we return to in section 8.2.

there are k soldiers surrounding the castle.

Now, however, it turns our that the second clause, (112b), is trivially true, simply because, given two distinct numbers m and n, there is no entailment in either direction between the two following propositions:

(113) a. There are m soldiers surrounding the castle.
b. There are n soldiers surrounding the castle.

For instance, there are situations where 15 soldiers form a circle around the castle and it's not the case that 10 soldiers form a circle around the castle; and conversely, there are situations where 10 soldiers form a circle around the castle and it's not the case that 15 soldiers form a circle around the castle.

As a result, we have the following:

(114) (111) is true in w just in case there is a number k, smaller than 10, such that there are k soldiers surrounding the castle.

This is, of course, equivalent to the existential, non-upper-bounded reading that we want to derive for such cases.

At this point, it seems that this approach, which we will dub the *Lexical approach based on Maximal Informativity* (LMaxInf for short), is our best candidate. It predicts no ambiguity, and it predicts the intuitively correct truth-conditions both when the modified numeral combines with a predicate that licenses downward inferences and one that does not.[54] However, we will see that there are several complications. First, the LMaxInf proposal, as we have formulated it so far, encounters problems in cases where the relevant predicates (e.g. *smile* and *surround the castle*) have an empty extension. Second, it will turn out that just as in the LMax approach, scope interactions actually give rise to an ambiguity between the upper-bounded and the non-upper-bounded readings in distributive contexts, so that our pragmatic economy condition is still needed after all. Finally, with the LMaxInf account too, we may consider an alternative proposal where the maximality component (which is now cashed out in terms of logical strength) is severed from the meaning of modified numerals (we will dub this approach the SMaxInf approach). The very same considerations that potentially motivate the SMax account will still be relevant. We end by discussing some cases where an approach based on standard maximality might make better predictions.

8.2 Existential entailment problems: What to do about empty extensions?

Suppose that in fact no student smiled. Then we want a sentence such as (105), repeated below as (115), to count as true (see also fn. 14).

[54] LMaxInf also generates the correct truth conditions for sentences where the modified numeral combines with an upward scalar predicate, such as *Less than ten eggs are sufficient*, as we will show in section 9. We defer discussion of upward scalar cases until then, where they become a deciding factor in an overall assessment of the accounts elaborated in this paper.

(115) a. Less than three students smiled.
b. [less than three] [λm [[$\varnothing$ [$m_{\texttt{isCard}}$ students]] smiled]]

What does the LMaxInf proposal deliver for this case? As we know, relative to a world w the sentence is predicted to mean that there is a number n, smaller than 3, such that n is in the extension of

(116) λm [$\varnothing$ [$m_{\texttt{isCard}}$ students] smiled]

(call it P for short), and n is P-maximal in w. Given our assumptions, however, if there is no smiling student in w, then P is empty in w. This is because of our ontological assumption that there exists no individual x such that the cardinality predicate $0_{\texttt{isCard}}$ holds of x: an individual always has at least one atomic individual as a part. As a result of P being empty, *no* number n is P-maximal in w, not even 0, and so the sentence is predicted to be false. (Cf. the discussion of empty degree predicates in section 3.)

As we did with standard maximality in section 3, we can again stipulate that if P has an empty extension in a world w, then the number 0 is P-maximal in w.[55] We would thus now define our metalanguage expression $\max_{\text{inf}}$ as follows:

(117) For any function P of type $s(dt)$, number n, and world w, n is P-maximal in w ($\max_{\text{inf}}(P)(w)(n) = 1$) iff one of the following two conditions holds:
a. (i) $P(w)(n) = 1$.
(ii) There is no m such that both $P(w)(m) = 1$ and $[\lambda v \,.\, P(v)(m) = 1] \subsetneq [\lambda v \,.\, P(v)(n)]$.
b. (i) There is no number k such that $P(w)(k) = 1$.
(ii) $n = 0$.

Using this definition of $\max_{\text{inf}}$ in the lexical entry for *less than* given in (103), we now correctly predict that (115) is true if no students smiled: in this case, $P(w)(k) = 0$ for all k (so we are in condition (117b)), and so there *is* an $n < 5$ such that $\max_{\text{inf}}(P)(w)(n) = 1$, namely $n = 0$ because $\max_{\text{inf}}(P)(w)(0) = 1$ here.

However, this move now has an undesirable outcome for non-distributive predicates. It is now predicted that (111), repeated below as (118), is true in a world where no group of soldiers surrounded the castle.

(118) a. Less than ten soldiers surrounded the castle
b. [less than ten] [λm [[$\varnothing$ [$m_{\texttt{isCard}}$ soldiers]] [surrounded the castle]]]

That is, (118) is predicted to be true if either no soldiers surrounded the castle or some group of less than ten soldiers did (which is compatible with another, larger group of soldiers surrounding the castle). However, as we argued in section 2.4, (118) licenses an existential inference, i.e. that some soldier(s) surrounded the castle.[56]

[55] In section 3, we subdivided the definition of max into two cases (one case where P is empty, the other where P is non-empty) and stipulated that when P is non-empty, then $\max(P) = 0$.

[56] We actually *can* derive the existential reading, namely by scoping *less than ten* below the silent determiner, as we will see in section 8.4. The point here is that we *also* derive an unattested, non-existential reading, too. Thus, we incorrectly predict an ambiguity. Moreover, a similar point could be made with the following

We are thus faced with a dilemma: either we do not modify our definition of $\max_{\text{inf}}$ along the lines of (117), and then we fail to capture that (115) is true if no students smiled, or we do, and then we fail to capture the existential entailment of a sentence such as (118).[57] We know of no non-*ad-hoc* solution to this problem, but in the next section we offer a solution based on the introduction of a null individual into our ontology and some additional stipulations regarding the denotations of distributive and non-distributive predicates.

8.3 Null individuals?

We may try to solve the problem we have just pointed out by adding to our ontology a *null individual*. We sketch here what we take to be a conceivable approach to the problem, but point out that there are non-trivial holes to be filled in for such an approach to work (beyond the fact that it is also quite a stipulative approach).

First, instead of assuming that the set of pluralities in our ontology is a join-semilattice, we now take it to be a complete lattice, i.e. as having a bottom element, which we call the *null individual*, notated as **0** (see Landman 2004 for another proposal that incorporates the null individual into the ontology). The null individual is a part of every individual,[58] and the sum consisting of the null individual and any individual x is simply x.[59] Crucially, **0** does not have any part distinct from itself,[60] but is also not itself an atomic individual,[61] so that we have $|\mathbf{0}| = 0$.[62]

Second, we assume that predicates such as *smile*, which are distributive, necessarily include the null individual in their extension (i.e. the null individual is in their extension in every world), and that non-distributive predicates never have the null individual in their extension.

Finally, pluralized nouns, but not singular nouns, also necessarily include the null individual in their extension. As we will discuss shortly, it is not obvious to us how to derive this outcome in a way that is not entirely *ad hoc* and thus reduces the conceptual appeal of the LMaxInf account. But let us put this aside for a moment.

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sentence:

(i) a. Less than ten eggs are sufficient to make an omelet for all these people.
  b. [less than ten] [λn [n eggs are sufficient ...]]

Our revised definition of $\max_{\text{inf}}$ would fail to derive the entailment that there is some number of eggs that are sufficient. That is, (i) would be predicted to be true in a world where it is impossible to make omelets with (any number of) eggs. We feel that such a prediction is incorrect. For example, *Less than ten eggs are sufficient to build a house* seems false to us (relative to our actual world), not true.

[57] More precisely, we predict an unattested ambiguity by deriving a non-existential reading (see fn. 56).

[58] $\forall x[\mathbf{0} \sqsubseteq x]$.

[59] $\forall x[\mathbf{0} \sqcup x = x \wedge x \sqcup \mathbf{0} = x]$.

[60] $\forall x[x \sqsubseteq \mathbf{0} \rightarrow x = \mathbf{0}]$.

[61] The notion of *atom* must therefore be redefined so that an atom is a non-null individual whose only proper subpart is the null individual: $\forall x[\mathbf{atom}(x) \leftrightarrow [x \neq \mathbf{0} \wedge \forall y[y \sqsubseteq x \rightarrow [y = x \vee y = \mathbf{0}]]]]$.

[62] The set of pluralities is now isomorphic to the power set of the set of atoms, where the null individual is identified with the empty set, an atomic individual is identified with a singleton set, and the *part of* relation is identified with set-theoretic inclusion. This is because the power set of a set defines a complete lattice, where the ordering (*part of*) relation is set-theoretic inclusion, the *join* of two sets is their union, and the *meet* of two sets is their intersection.

We also need to modify the lexical entries of quantifiers such as *no* and *some* as follows:

(119) a. $[\![\text{some}]\!]^w = \lambda P\,.\,\lambda Q\,.\,P \cap Q \nsubseteq \{\mathbf{0}\}$
b. $[\![\text{no}]\!]^w = \lambda P\,.\,\lambda Q\,.\,P \cap Q \subseteq \{\mathbf{0}\}$

With all this in place, if we revert back to our original definition of $\max_{\text{inf}}$ in (100), keeping exactly the same lexical entries for the silent existential determiner ($\varnothing$), for numerals ($n_{\texttt{isCard}}$), and for modified numerals, we now get what we want. Consider again (115), repeated as (120).

(120) a. Less than three students smiled.
b. [less than three] [λm [[$\varnothing$ [$m_{\texttt{isCard}}$ students]] smiled]]

If nobody smiled in a world w, then the following expression is true in w

(121) [$\varnothing$ [$m_{\texttt{isCard}}$ students]] [smiled]

because **0** is in the denotations of both *students* and *smiled* in w. Therefore, the number 0 has the property denoted by

(122) λm [[$\varnothing$ [$m_{\texttt{isCard}}$ students]] smiled]

Finally, 0 is the only number that has this property, and is thus maximal relative to it in w, which thereby makes the whole sentence true (because $0 < 3$).

Now, on the other hand, things will be different for a case such as (118), repeated below as (123).

(123) a. Less than ten soldiers surrounded the castle.
b. [less than ten] [λm [[$\varnothing$ [$m_{\texttt{isCard}}$ soldiers]] [surrounded the castle]]]

If no group of soldiers surrounded the castle in a world w, then, since **0** is not in the extension of *surrounded the castle* in w, the degree predicate

(124) λm [[$\varnothing$ [$m_{\texttt{isCard}}$ soldiers]] [surrounded the castle]]

has an empty extension in w, and because of this the existential statement introduced by *less than ten* is false.

While this may seem like the beginning of a solution, it is important to see that there is no straightforward way to derive what we want by minimally modifying a standard plural semantics framework. What we need, on the one hand, is an operation which, starting from a 'basic' denotation for a predicate like *smile*, which includes only atomic individuals, returns the distributive predicate *smile*′ whose denotation now includes the null individual and every plurality whose atomic parts are in the extension of *smile*. On the other hand, we need to *prevent* the null individual from entering the denotation of a predicate like *surround the castle*. A straightforward proposal would be to define the *star operator* (cf. section 2.3, specifically (26) and (28)) in such a way that it is responsible for introducing the null individual into a predicate's extension. We would then have:[63]

[63] Recall that if E is a set of individuals (atomic or non-atomic), $E^{\sqcup}$ denotes the closure of E under sum (cf. (27)).

(125) $[\![*]\!]^w = \lambda P_{et} \,.\, \lambda x_e \,.\, x = \mathbf{0} \vee x \in \{y : P(y) = 1\}^{\sqcup}$
($[\![*\alpha]\!]^w$ is the set of all sums, including the null individual, that can be formed by summing up members of $[\![\alpha]\!]^w$.)

A problem with this proposal, if unconstrained, is that nothing would prevent the star operator from applying to a non-distributive predicate such as *surrounded the castle*. In fact, this is even something that we want, for the following reason. If two distinct groups of people surrounded the castle, namely $\{a, b, c, d\}$ and $\{e, f, g, h, i\}$, then it is intuitively correct to say that the plurality consisting of these two groups surrounded the castle, which suggests that *surrounded the castle* is cumulative.[64] That is, if the soldiers from army #1 surrounded the castle and the soldiers from army #2 did as well, then it seems true to say *The soldiers from armies #1 and #2 surrounded the castle*. The denotation we would get when we apply the star operator is as follows:

(126) a. $[\![\text{surround the castle}]\!]^w = \{a \sqcup b \sqcup c \sqcup d, \quad e \sqcup f \sqcup g \sqcup h \sqcup i\}$
b. $[\![*[\text{surround the castle}]]\!]^w = \{\mathbf{0}\} \cup \{a \sqcup b \sqcup c \sqcup d, \quad e \sqcup f \sqcup g \sqcup h \sqcup i\}^{\sqcup}$
$= \{\mathbf{0}, \quad a \sqcup b \sqcup c \sqcup d, \quad e \sqcup f \sqcup g \sqcup h \sqcup i, \quad a \sqcup b \sqcup c \sqcup d \sqcup e \sqcup f \sqcup g \sqcup h \sqcup i\}$

Now, the resulting predicate is cumulative in the sense of (26), but it is not distributive. So we will still predict a non-upper-bounded meaning. However, because the null individual always belongs to the extension of **[surround the castle]*, we lose the existential entailment we want to derive, hence do not solve the problem. Now, maybe *less than ten soldiers* should be allowed to combine with the lexical, non-starred version of *surrounded the castle*, in which case an existential entailment would still be predicted (because if the star operator does not apply, the predicate cannot have the null individual in its extension). This would in fact predict an ambiguity, i.e. the existence of two non-upper-bounded readings, one with an existential entailment and one without. It is far from clear that such an ambiguity is desirable. This line of analysis, therefore, does not really solve the problem. Note also that on such a view, nothing would prevent *less than three students* from combining with the non-starred version of *smiled*. The non-starred version of *smiled* only has atomic individuals in its extension, and as a result is neither distributive nor cumulative, and so *Less than three students smiled* would now come out as equivalent to 'at least one person smiled' (because now the 'maximal informativeness' clause of our entry for *less than* would become trivially satisfied, just as it is with non-distributive predicates in general). We would then need our pragmatic constraint after all to rule out this reading. (There are in fact independent reasons why we still need our pragmatic constraint, as we will discuss in section 8.4 and in fn. 72.)[65]

[64] Or maybe we would rather want to restrict the application of the star operator to lexical predicates. In such a case, we would need to apply the so-called double star operator to the transitive verb *surround*, which can be seen as a variant of a star operator that applies to relations instead of applying to 1-place predicates (see Champollion 2014 for a survey, and the references therein). But this does not change anything regarding our main point. For our proposal to work, we would also need the double star operator to introduce the null individual into the denotation of intuitively 'distributive' 2-place predicates, e.g. *see*, so that *Less than three boys saw the movie* comes out true if no boy saw the movie.

[65] Instead of having the null individual introduced by a star operator, which applies indiscriminately to both *smiled* and *surrounded the castle* (hence, fails to derive an existential entailment in the case of *surrounded the castle*), perhaps the null individual is introduced by a distributive operator Δ, which forces 'atomic

To conclude this section, there does not seem to us to be a straightforward solution to the problem faced by the LMaxInf account, namely that one cannot easily predict at the same time the absence of an existential entailment in distributive contexts and the presence of an existential entailment in non-distributive contexts. Nevertheless, we would like to put this problem aside and see, in section 9, whether we can marshal independent evidence in favor of the LMaxInf account—namely, whether there are facts that are better predicted by the LMaxInf account. Before discussing this, however, we would like to discuss two other issues. First, we will see that Van Benthem's problem does not in fact disappear in the LMaxInf account. Second, we will discuss another variant of the LMaxInf approach, in which maximality is severed from the meaning of modified numerals.

8.4 No escape from Van Benthem's problem: pragmatic blocking is still needed

Consider again the sentence *Less than three students smiled*. So far, when presenting the LMaxInf account, we only considered LFs in which *less than three* takes widest scope. Without further assumptions, however, the LF in (127) is also expected to be possible. In fact, whether we use standard maximality or maximal informativity, in basic cases the situation is the same as in section 3. Namely, depending on the relative scope of the modified numeral and the silent indefinite determiner, we get or do not get the upper-bounded reading. To see why, consider the LF in (127), which is exactly the same as the LF in (41) discussed for the LMax account.[66]

(127) $[\varnothing\ [[\lambda x\ [[\text{less than three}]\ [\lambda n\ [x\ n_{\texttt{isCard}}]]]]\ \text{students}]]\ \text{smiled}$

This LF is true in a world w iff the following holds:

(128) $$\exists x[\exists m[m < 3 \wedge \max_{\text{inf}}(\lambda v\,.\,\lambda n\,.\,|x| = n)(w)(m) = 1] \wedge \textbf{students}(w)(x) \wedge \textbf{smiled}(w)(x)]$$

For any given individual x and any world v, the function $\lambda n\,.\,|x| = n$ corresponds to the singleton set containing the unique number k such that x has exactly k atomic parts. This number, because it is unique, trivially counts as maximal relative to the predicate in v. In other words, $\max_{\text{inf}}(\lambda v\,.\,\lambda n\,.\,|x| = n)(w)(k) = 1$ iff $|x| = k$ (in w).[67] As a result, the LF in (127) is equivalent to:

distributivity':

(i) $\Delta = \lambda P_{et}\,.\,\lambda x_e\,.\,x = \mathbf{0} \vee \forall y[[\textbf{atom}(x) \wedge y \sqsubseteq x] \rightarrow P(y)]$

The point is that with an intrinsically collective predicate P (i.e. a predicate that has no atom in its extension), $\Delta(P)$ will always (in every world) be the singleton set $\{\mathbf{0}\}$ (which we may assume is disallowed for pragmatic reasons), so that Δ will only apply, in practice, to predicates like *smiled* or *lift the piano*, but not predicates like *surround the castle*. In that way, only 'pure' distributive readings would fail to generate an existential inference. We leave aside for further research an examination of this idea and of its interactions with the rest of the of our assumptions.

[66] Recall that we are assuming here that *less than three* quantifies into AP. (Quantifying into NP still yields a logically equivalent LF.)

[67] This is exactly analogous to the fact exploited several times earlier, in section 3 and section 4 (for standard maximality), that $\max(\lambda n\,.\,|x| = n) = k$ iff $|x| = k$.

(129) $\exists x[\exists m[m < 3 \wedge |x| = m] \wedge \textbf{students}(w)(x) \wedge \textbf{smiled}(w)(x)]$
$\equiv \exists x[|x| < 3 \wedge \textbf{students}(w)(x) \wedge \textbf{smiled}(w)(x)]$

which expresses the proposition that there is a plurality of less than three students who smiled, i.e. the non-upper-bounded, existential reading. This LF is, as before, ruled out by our pragmatic economy constraint.[68] We thus see that the unattested, non-upper-bounded reading is derived after all even in the LMaxInf account. This means that our pragmatic economy principle cannot be dispensed with even in this approach.[69] However, a potential benefit of the LMaxInf approach is that it generates no ambiguity when modified numerals combine with non-distributive predicates (or more precisely, predicates that do not license downward inferences).

8.5 Severing maximal informativity from modified numerals: the SMaxInf account

Much like a theory based on standard maximality can assume either that the maximality component is or isn't part of the meaning of modified numerals, the same choice arises for a theory based on maximal informativity. Instead of locating the maximality component in the meaning of modified numerals, there is an alternative where it is part of the meaning of numerical variables. More specifically, we can proceed in a way analogous to what we did in section 4, except that instead of the operator `isMax`, we now rely on an operator `isMaxInf`, defined as follows (we now need to make the operator intensional, in that its second argument has to be the intension of a degree predicate).

(130) $[\![\texttt{isMaxInf}]\!]^w = \lambda n_d \,.\, \lambda P_{s(dt)} \,.\, \text{max}_{\text{inf}}(P)(w)(n) = 1$

As usual, we write $n_{\texttt{isMaxInf}}$ for `isMaxInf` n:

(131) $[\![n_{\texttt{isMaxInf}}]\!]^w = [\![\texttt{isMaxInf}\ n]\!]^w = \lambda P_{s(dt)} \,.\, \text{max}_{\text{inf}}(P)(w)([\![n]\!]^w) = 1$

Numerical variables of the form $n_{\texttt{isMaxInf}}$ are thus treated as intensional operators which can combine with expressions of type dt thanks to the rule of intensional functional application (cf. fn. 52).

As for modified numerals, we now revert back to a basic, existential lexical entry, as we did in (55) in section 4:

[68] Note that under the null individual approach, both the plural noun *students* and the predicate *smiled* would necessarily include $\mathbf{0}$ in their extension. Since $|\mathbf{0}| = 0$, the LF in (129) would be necessarily true, i.e. would now express a tautology. It would still, of course, be ruled out by our pragmatic economy constraint. However, one might wonder, then, whether our pragmatic economy constraint could not be replaced by a general ban against tautologous LFs (or at least certain types of tautologous LFs, cf. Gajewski 2002). This, however, would not take care of cases involving modified numerals of the form *between n and m*, for which the corresponding LF would *not* express a tautology (except perhaps in the special case where *n* is *zero*).

[69] To avoid this result, we could follow Hackl 2000 in proposing that our silent determiner, $\varnothing$, is parameterized for degrees:

(i) $[\![\varnothing]\!]^w = \lambda n_d \,.\, \lambda P_{et} \,.\, \lambda Q_{et} \,.\, \exists x[|x| = n \wedge P(x) \wedge Q(x)]$

This would force *less than three* to scope above $\varnothing$ in order to create a type *d* degree trace that $\varnothing$ can combine with (see Hackl 2000 for details). Nevertheless, we would still need our pragmatic economy constraint to rule out sentences like *?More than ten eggs are sufficient to make an omelet*, as we will argue in section 9.1 (see fn. 72).

(132) $[\![\text{less than}]\!]^w = \lambda n_d \,.\, \lambda P_{dt} \,.\, \exists m[m < n \wedge P(m)]$

With this in place, the following two LFs for *Less than ten people smiled* and *Less than ten soldiers surrounded the castle* generate the right truth conditions (putting aside the special case where the relevant predicates have an empty extension):

(133) a. Less than ten students smiled.
b. [less than ten] [λn [$n_{\texttt{isMaxInf}}$ [λm [[$\varnothing$ [$m_{\texttt{isCard}}$ students]] smiled]]]]

(134) a. Less than ten students surrounded the castle.
b. [less than ten] [λn [$n_{\texttt{isMaxInf}}$ [λm [[$\varnothing$ [$m_{\texttt{isCard}}$ students]] [surrounded the castle]]]]]

These two LFs are identical up to the choice of the predicate, but while the first one delivers an upper-bounded reading, the second does not. In (133), the degree predicate

(135) λn [$n_{\texttt{isMax}}$ [λm [[$\varnothing$ [$m_{\texttt{isCard}}$ students]] smiled]]]

denotes the property that a number n has if n students smiled and the proposition that n students smiled is not entailed by any other true proposition of the same form. The only n that can have such a property is the total number of students who smiled. As before, the fact that *smiled* is distributive is crucial to get this result (because of the role played by entailment). As a result, the LF ends up entailing that the number of students who smiled is smaller than ten. If *smiled* is replaced with a non-distributive predicate such as *surrounded the castle*, the condition that cares about entailment becomes vacuous (because there is simply no entailment relation whatsoever between $\varnothing$ $n_{\texttt{isCard}}$ *soldiers surrounded the castle* and $\varnothing$ $m_{\texttt{isCard}}$ *soldiers surrounded the castle*, if m and n are distinct), and so we get the non-upper-bounded meaning.

Finally, another LF is available, where `isMaxInf` is not used. This LF always gives rise to a non-upper-bounded reading. The pragmatic economy constraint, which we must retain, rules out this reading in distributive contexts.

Now, are there reasons to prefer this approach, which we may dub the SMaxInf approach, to the LMaxInf approach? All the differences in predictions between the LMax and the SMax approaches that we discussed in section 6.2, which pertained to cases where SMax predicts additional scope ambiguities ('split scope'), carry over to the comparison between the LMaxInf and the SMaxInf approaches. The relevant cases give rise to very delicate judgments, but they seem to point in the direction of severing maximality from the meaning of modified numerals (i.e. either SMax or SMaxInf). The argument in favor of SMax over LMax that we gave in section 6.1, which was based on upward scalar predicates, does *not*, however, carry over to SMaxInf, as we will discuss in the next section.

9 Comparison of accounts

In the end, we have discussed four theories, all of which can be characterized in terms of two binary factors. One factor is the notion of maximality that the theory relies on (standard maximality vs. logical maximality). The other factor is whether the maximality component is viewed as part of the intrinsic, lexical meaning of modified numerals, or as separate from them. All four theories need to appeal to a pragmatic blocking mechanism in order to rule out the non-upper-bounded reading when needed.[70] Let us describe the four theories we have developed in terms of these factors, in the order in which we presented them.

(136) a. [+Standard Maximality, +Lexical Maximality] = LMax
b. [+Standard Maximality, −Lexical Maximality] = SMax
c. [−Standard Maximality, +Lexical Maximality] = LMaxInf
d. [−Standard Maximality, −Lexical Maximality] = SMaxInf

Regarding the first factor (type of maximality), one advantage of using logical maximality is that we do not generate upper-bounded readings in collective contexts, which seems to us to be a good outcome (cf. our discussion in section 7). A significant problem, though, is that it is difficult to provide a theory based on logical maximality which correctly takes care of cases where a degree predicate has an empty extension (cf. section 8.2, section 8.3). In the rest of this section, we consider other cases where our theories make different predictions, and try to assess in this light which theory fares best.

9.1 Upward scalar predicates revisited

In section 6.1, we gave what we take to be quite a compelling argument against LMax, based on sentences involving upward scalar predicates. This argument gives us a reason to prefer the choice '−Lexical Maximality', *if* the other choice is '+Standard Maximality'. It is worth pointing out, however, that the argument we gave then has no way to distinguish between LMaxInf and SMaxInf. Consider again the following LF for a sentence such as *Less than ten eggs are sufficient to make an omelet for all these people*:

(137) [less than ten] [λn [n eggs are sufficient ...]]

On the SMax and SMaxInf accounts, this LF does not include the maximality component, and the sentence ends up meaning that for some n smaller than 10, n eggs are sufficient, which is the reading we wish to derive. But on the LMaxInf account, exactly the same reading is predicted. This is so because of the following fact:

(138) Assume that P is an upward scalar degree predicate (i.e. for every n, $P(n)$ entails $P(n+1)$). Then the following two statements are equivalent:
a. $P(n)$ asymmetrically entails $P(m)$.
b. $n < m$.

As a result, if P is upward scalar, a number n is P-maximal in a given world w if it is the *smallest* number of which P holds in w. So the LF in (137) ends up meaning that the smallest number n such that n eggs are sufficient is smaller than ten. This statement is in fact equivalent to simply saying that there is a number n, smaller than 10, such that having

[70] Except for a version of LMaxInf where the silent determiner is parameterized for degrees (cf. fn. 69). However, as we will argue in this section, even that account will eventually need to be supplemented with a pragmatic blocking mechanism (see fn. 72).

n eggs is sufficient (because this of course entails that the smallest such number is at least as small, hence is itself smaller than 10).

But do upward scalar predicates provide us with the means to adjudicate between standard maximality and logical maximality? Well, they do, but the facts seem to go in both directions. In this case, considerations of a non-monotone numerical phrase based on *between* proves crucial. Consider the following sentence:

(139) Between 5 and 10 eggs are sufficient to make an omelet for all these people

Consider first the LMaxInf and SMaxInf accounts. Both predict that (139) can have the reading given in (140a), which is equivalent to (140b).

(140) a. The smallest number n such that n eggs are sufficient is included in the interval $[5, 10]$.
b. To make an omelet, it is necessary to have at least 5 eggs, and it is not necessary to have more than ten.

It seems to us that this reading is indeed accessible. Importantly, however, this reading cannot be captured by the LMax and SMax accounts, because the maximality component in these accounts cannot scope over an upward scalar predicate without generating maximality failure (see section 6.1).[71] More precisely, on LMax approach, the sentence should be either false or undefined; similar for the SMax approach if maximization applies. If, on the SMax approach, maximization does not apply, then the predicted reading is, 'for some number n between 5 and 10, n eggs are sufficient'. But note that this is equivalent to 'ten eggs are sufficient', hence should be ruled out by our pragmatic economy constraint. In sum, the SMax and LMax accounts both predict that (139) should sound deviant (the only readings generated either involve maximality failure or are blocked by the pragmatic economy constraint), and yet it seems to be fine.

Now, it is worth pointing out that the following *does* sound odd, a fact that we would like to explain.

(141) ?More than ten eggs are sufficient to make an omelet.

On the SMaxInf account, the following LF should be available:

(142) [more than ten] [λn [$n_{\texttt{isMaxInf}}$ [λm [m eggs are sufficient ...]]]]

The predicted reading for this LF is given in (143a), which is equivalent to (143b) (assuming a straightforward lexical entry for *more than*, modeled on *less than*, with $<$ replaced by $>$):

(143) a. The smallest number n such that n eggs are sufficient ... is greater than 10.
b. It is necessary to have more than 10 eggs to make an omelet for all these people.

[71] On the LMax account, the maximality component could maybe be interpreted within the scope of *sufficient*, yielding something like, 'There is a number n between 5 and 10 such that it is sufficient to have *exactly* n eggs'; this, however, is not the same as what we have in (140) (in fact, it would be true even if just one egg were sufficient; see also fn. 42).

This, of course, is a perfectly good proposition, and our pragmatic economy principle does not appear to rule out the LF in (142), and so SMaxInf does not account for the oddness of (141). In this case, LMaxInf might fare better. This is because nothing forces the LMaxInf approach to provide parallel lexical entries to *less than* and *more than*. While on the LMaxInf account, the entry for *less than* refers to maximal informativity (cf. (103)), the entry for *more than* does not have to be stated in the same way. That is, the LMaxInf account could entertain the following entry for *more than*:

(144) $[\![\text{more than}]\!]^{w} = \lambda n_d \,.\, \lambda P_{dt} \,.\, \exists m[m > n \wedge P(m)]$

Then (141) would have the parse in (145) below, which means that for some number $n > 10$, n eggs are sufficient.

(145) [more than ten] [λn [n eggs are sufficient ...]]

A little bit of reasoning will show that given (144), the LF in (145) is equivalent to 'for some number n, n eggs are sufficient'. Here is why. Suppose that for some number n, n eggs are sufficient. Then, whether or not n is smaller or greater than 10, for any $m > n$, m eggs are sufficient (because λk *[k eggs are sufficient]* is upward scalar), and so for some $k > 10$, k eggs are sufficient. As a result, replacing *ten* with another numeral (say, *nine* or *eleven*) does not alter the predicted truth conditions, in violation of our pragmatic economy principle, and so this LF is blocked. Thus, LMaxInf has an explanation for the unacceptability of (141).[72]

Likewise, both LMax and SMax predict (141) to be odd. Under LMax, either the LF in (145) suffers from maximality failure or, if the lexical entry in (144) is chosen, the pragmatic economy constraint is violated. Under SMax, we use the lexical entry in (144), and then there is a choice between the LF in (145), which violates our pragmatic economy constraint, and an LF identical to (142), except that $n_{\texttt{isMaxInf}}$ is replaced with $n_{\texttt{isMax}}$. This latter choice results again in maximality failure. So the bottom line is that the oddness of (141) and its perceived interpretation seem to be explainable under the LMax, SMax, and LMaxInf accounts, but are inexplicable under the SMaxInf account.[73]

9.2 Overall assessment

When we compare our four theories, the evidence we gathered seems to us to warrant conflicting conclusions. The availability of the reading for (79) (*Less than ten eggs are sufficient* ...) discussed in section 6.1 rules out LMax as a viable candidate. Furthermore, to the extent that (139) (*Between 5 and 10 eggs are sufficient* ...) sounds fine, it provides evidence against SMax (and, once again, against LMax) and in favor of using logical maximality, i.e. LMaxInf or SMaxInf. To the extent that (141) (*More than 10 eggs are sufficient* ...) sounds odd, it provides evidence against SMaxInf. So on this basis, LMaxInf should be the winner. However, the judgments regarding 'split-scope' data discussed in section

[72]This argument shows that even if we manage to prevent LMaxInf from deriving non-upper-bounded readings in distributive contexts, e.g. by adopting a silent determiner parameterized for degrees (see fn. 69), it is nevertheless still necessary (or at least advantageous) for LMaxInf to appeal to pragmatic blocking to explain the unacceptability of (141).

[73]Importantly, the entry in (144) would also work for acceptable cases like *More than three students smiled* and *More than ten soldiers surrounded the castle* on all three accounts (LMax, SMax, and LMaxInf).

6.2, though delicate, go in favor of a theory where maximality is severed from the meaning of modified numerals (i.e. either SMax or SMaxInf). Note also that Marty, Chemla, and Spector (2015) present experimental evidence suggesting that the non-upper-bounded reading is somehow accessed by naïve speakers even when the relevant modified numerals occur in a distributive context. This would make some sense in a theory where this reading is generated by the grammar but is ruled out by a pragmatic economy constraint (namely, every one of our four proposals except LMaxInf). Finally, while the appeal of a theory based on maximal informativity is that it explains the absence of the upper-bounded reading in collective contexts, we have seen that it is not straightforward to formulate such an account in a way that assigns the correct truth value in cases where the modified numeral combines with a predicate that has an empty denotation.

10 Conclusion

In this paper, we have (i) uncovered a new empirical puzzle pertaining to the interpretation of modified numerals, and (ii) investigated a family of approaches to this puzzle. While each of the four proposals we developed is able to predict the main empirical generalization, namely the contrast between distributive and non-distributive environments regarding the interpretation of modified numerals, they differ from each other both at a conceptual level and in terms of their detailed predictions. Each of them has strengths and weaknesses, and we did not provide definitive evidence in favor of one over all the others. We hope that this paper can serve as a starting point for further investigations.

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