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BOOK REVIEWS

The Novikov conjecture, geometry and algebra

(OWS - Oberwolfach Seminars 33)

By Matthias Kreck and Wolfgang Lück: 266 pp., €38.00 (CHF64.00),
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The history of the topology of manifolds in the 1950s and 1960s is conveniently encapsulated by the work awarded for the relevant Fields Medals: cobordism (Thom, 1958), exotic differentiable structures on spheres (Milnor, 1962), the generalized Poincaré conjecture and h-cobordism theorem (Smale, 1966), K-theory (Atiyah, 1966) and the topological invariance of the rational Pontrjagin classes (Novikov, 1970). For me, the signature theorem (Hirzebruch) deserves a kind of fantasy Fields Medal (1954, say, or 1958), but in any case has to be mentioned in this encapsulation. The fantasy list could be continued, but this is not the place!

Novikov moved on from working in the topology of manifolds in 1970, while retaining a healthily judgemental interest in the standards of the subject. Novikov [**3**] bequeathed to the subject his enormously influential eponymous conjecture on the homotopy invariance of characteristic numbers of a manifold derived from the fundamental group, predicting the wide variety of methods needed to attack it: geometry, algebra and analysis. The original conjecture may not have appeared to be of great geometric interest, but along with the closely related Borel, Farrell–Jones and Baum–Connes conjectures it is now seen to hold the key to the relation between the local and global properties of manifolds. These conjectures have been a central preoccupation of high-dimensional manifold topologists since 1970. There is an extensive literature of several hundred papers, proving the conjectures for many groups of geometric interest, using a wide variety of methods. There is also one dissenting view (see [**2**]) expressing doubts as to the validity of the conjectures for all groups. There is surely a Fields Medal waiting for the first proof of/counterexample to the Novikov conjecture; I myself am agnostic as to which way it will go.

The book under review was written in connection with a week-long meeting held at the Oberwolfach Mathematics Institute in January 2004 at which the authors introduced the Novikov conjecture to an audience of graduate students and postdoctoral researchers. By all accounts, it was a lively meeting, and the book in question is a fair reflection of this. It may be regarded as a sequel to the Proceedings of the 1993 Oberwolfach conference on the Novikov conjecture [**1**], which recorded the state of play at the time.

It is by no means easy to actually motivate and state the Novikov conjecture for the non-specialist reader. On the geometric side the Novikov conjecture shades into the Borel rigidity conjecture, which is much easier to explain, and is in any case of an earlier (albeit unpublished) vintage. Two-dimensional manifolds (that is, surfaces) are homotopy equivalent if and only if they are homeomorphic. In dimensions at least 3 a homotopy equivalence of manifolds need not be homotopic to a homeomorphism, and hence there is an essential difference between the algebraic and geometric topology of manifolds. Although there are low-dimensional implications and examples, by and large the geometric applications of the conjectures are to manifolds of dimensions at least 5, where surgery theory holds sway.

A manifold is *aspherical* if it is the classifying space of its fundamental group, or equivalently if the higher homotopy groups are trivial. Aspherical manifolds are homotopy equivalent if and only if their fundamental groups are isomorphic. In 1953 Mostow proved that compact solvmanifolds with isomorphic fundamental groups are homeomorphic (subsequently generalized to the 'strong Mostow rigidity'), which led Borel to ask in a 1953 letter to Serre whether any two aspherical manifolds with isomorphic fundamental groups are homeomorphic. (Thanks to Nicolas Monod for this information). The Borel conjecture, never actually made by Borel as such, is that the answer is always yes.

The signature of the intersection form of a manifold is a homotopy invariant, since a homotopy equivalence induces an isomorphism of the intersection forms. The 1953 Hirzebruch theorem equated the signature of a differentiable manifold with a certain combination of the Pontrjagin classes of the tangent bundles. In 1956 Thom extended the signature theorem to piecewise linear manifolds, defining the Pontrjagin classes using the signatures of submanifolds. The Pontrjagin classes are not homotopy invariant in general, since a homotopy equivalence need not preserve the signatures of submanifolds. Novikov's proof of the topological invariance of the rational Pontrjagin classes for compact manifolds had two surprise ingredients: non-compact manifolds such as the Euclidean space $\mathbf{R}^n$, and non-trivial groups such as the free abelian group $\mathbf{Z}^n$, the fundamental group $\pi_1(T^n)$ of the n-torus $T^n = S^1 \times S^1 \times \ldots \times S^1$, with universal cover $\mathbf{R}^n$, along with a cunning trick using T^n to approximate homeomorphisms of differentiable manifolds by differentiable maps that preserve signatures of submanifolds. The original Novikov conjecture predicted that certain combinations of the Pontrjagin classes and the cohomology classes of the fundamental group (the *higher signatures*) are homotopy invariant.

The exciting 1960s developments in high-dimensional manifolds culminated in the 'Browder–Novikov–Sullivan–Wall' surgery theory. Given a homotopy equivalence f of manifolds of dimension at least 5, there is a primary obstruction in the topological K-theory of bundles and a secondary obstruction in the Wall algebraic L-theory of $\mathbf{Z}[\pi]$-valued quadratic forms, for any group π, such that f is homotopic to a homeomorphism if and only if both obstructions vanish. Surgery theory was originally developed in the differentiable and piecewise linear categories, and extended to topological manifolds in 1969 by Kirby and Siebenmann, again using T^n.

The chapters in the book are individually authored by Kreck and Lück, and naturally reflect their somewhat different mathematical styles. Kreck starts off with a motivating problem concerning the classification of particular non-simply connected manifolds of dimensions at most 6, then develops the standard surgery theoretic approach to the Novikov and Borel conjectures in a direct way, culminating in a fairly concrete geometric proof of the Novikov conjecture for finitely generated free abelian groups. By contrast, Lück uses more sophisticated algebraic topology, concentrating on a development of equivariant spectra which unites the Novikov, Borel, Farrell–Jones and Baum–Connes conjectures in a single assembly framework.

As stated in the book, 'The key ingredient for all approaches to the Novikov conjecture is the construction of a so-called *assembly map*'. The name was coined by Quinn (ca. 1970) and refers to a procedure by which a locally defined input is glued together to obtain a globally defined output. The Poincaré duality theorem is a classic example: the locally Euclidean topology of a manifold assembles to an isomorphism between homology and cohomology. The assembly map from a generalized homology theory to L-theory glues together the stalks of a sheaf over a space X of $\mathbf{Z}$-valued quadratic forms to obtain a $\mathbf{Z}[\pi_1(X)]$-valued quadratic form. (This is not in fact the assembly construction used in the book, but there are many ways of constructing assembly maps relevant to the Novikov conjecture, all of which can be derived from this one). Surgery theory reduces the Novikov conjecture to the rational injectivity of the L-theory assembly map. The Borel conjecture is equivalent to the L-theory assembly map being an isomorphism for aspherical manifolds. The fundamental groups of aspherical manifolds are torsion-free, and the assembly map is definitely not an isomorphism for groups with torsion.

The Farrell–Jones conjecture is that a certain refined assembly map is an isomorphism for all groups.

Overall, the book is an excellent introduction to the Novikov and allied conjectures, with many low-dimensional examples and a multitude of exercises. There is a useful table of classes of groups for which the conjectures have been verified, and an extensive bibliography, both now happily out of date. However, it must not be forgotten that this is only an introduction to a field which has very diverse methods. Many techniques such as codimension 1 splitting, controlled topology (a potent mixture of algebra and topology again involving T^n), differential geometry and operator algebras which are used to prove the conjectures in various special cases are mentioned briefly or not at all. The book is perhaps to be regarded as a springboard rather than the water itself!

References

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Asymptotic analysis of random walks: Heavy-tailed distributions

(Encyclopedia of Mathematics and Its Applications 118)

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A random walk is the sequence of partial sums $S_n := \sum_{k=0}^{n} \xi_k$ of a sequence of independent and identically distributed random variables (ξ_n). The prototype is the cumulative gain or loss in a sequence of bets on the toss of a fair coin. This is discrete in both time and space; the corresponding continuous model (in time and space) is Brownian motion. The limit distributions here are normal (Gaussian). There is a wealth of theory for such cases, an exposition of which is promised for a forthcoming companion volume. The theme here is