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On The Algebraic L-theory of Δ-sets

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Abstract: The algebraic L-groups $L_*(\mathbb{A}, X)$ are defined for an additive category $\mathbb{A}$ with chain duality and a Δ-set X, and identified with the generalized homology groups $H_*(X; \mathbb{L}_\bullet(\mathbb{A}))$ of X with coefficients in the algebraic L-spectrum $\mathbb{L}_\bullet(\mathbb{A})$. Previously such groups had only been defined for simplicial complexes X.
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Introduction

A 'Δ-set' X in the sense of Rourke and Sanderson [9] is a simplicial set without degeneracies. A simplicial complex is a Δ-set; conversely, the second barycentric (aka derived) subdivision of a Δ-set is a simplicial complex, and the homotopy theory of Δ-sets is the same as the homotopy theory of simplicial complexes. However, Δ-sets are sometimes more convenient than simplicial complexes: they are generally smaller, and the quotient of a Δ-set by a group action is again a Δ-set. In this paper we extend the algebraic L-theory of simplicial complexes of Ranicki [6] to Δ-sets.

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In the original formulation of Wall [10] the surgery obstruction theory of high-dimensional manifolds involved the algebraic L-groups $L_*(R)$ of a ring with involution R, which are the Witt groups of quadratic forms over R and their automorphisms. The subsequent development of the theory in [6] viewed $L_*(R)$ as the cobordism groups of R-module chain complexes with quadratic Poincaré duality, constructed a spectrum $\mathbb{L}_\bullet(R)$ with homotopy groups $L_*(R)$, and also introduced the algebraic L-groups $L_*(R, X)$ of a simplicial complex X. An element of $L_n(R, X)$ is a cobordism class of directed systems over X of R-module chain complexes with an n-dimensional quadratic Verdier-type duality. The groups $L_*(R, X)$ were identified with the generalized homology groups $H_*(X; \mathbb{L}_\bullet(R))$, and the algebraic L-theory assembly map $A : L_*(R, X) \to L_*(R[\pi_1(X)])$ was defined and extended to the algebraic surgery exact sequence

$$\cdots \longrightarrow L_n(R, X) \xrightarrow{A} L_n(R[\pi_1(X)]) \longrightarrow \mathcal{S}_n(R, X) \longrightarrow L_{n-1}(R, X) \longrightarrow \cdots$$

with $\mathcal{S}_n(R, X)$ the cobordism groups of the $R[\pi_1(X)]$-contractible directed systems. In particular, the 1-connective version gave an algebraic interpretation of the exact sequence of the topological version of the Browder-Novikov-Sullivan-Wall surgery theory: if the polyhedron $\|X\|$ of a finite simplicial complex X has the homotopy type of a closed n-dimensional topological manifold then $\mathcal{S}_{n+1}(\mathbb{Z}, X)$ is the structure set of closed n-dimensional topological manifolds M with a homotopy equivalence $M \simeq \|X\|$.

The Verdier-type duality of [6] used the dual cells in the barycentric subdivision of a simplicial complex X to define the dual of a directed system over X of R-modules to be a directed system over X of R-module chain complexes. The Δ-set analogues of dual cells introduced by us in Ranicki and Weiss [8] are used here to define a Verdier-type duality for directed systems of R-modules over a Δ-set X, which is used to define the generalized homology groups $L_*(R, X) = H_*(X; \mathbb{L}_\bullet(R))$ and an algebraic surgery exact sequence as in the simplicial complex case.

The algebraic L-theory of Δ-sets is used in Macko and Weiss [5], and its multiplicative properties are investigated in Laures and McClure [3].

1. Functor categories

In this section, X denotes a category with the following property. For every object x, the set of morphisms to x (with unspecified source) is finite; moreover, given morphisms $f : y \to x$ and $g : z \to x$ in X, there exists at most one morphism $h : y \to z$ such that $gh = f$.

Let $\mathbb{A}$ be an additive category with zero object $0 \in \mathrm{Ob}(\mathbb{A})$.

Definition 1.1. (i) A function

$$M \ : \ \mathrm{Ob}(X) \to \mathrm{Ob}(\mathbb{A}) \ ; \ x \mapsto M(x)$$

is *finite* if $M(x) = 0$ for all but a finite number of objects x in $\mathbb{A}$.
The direct sum $\sum\limits_{x\in \mathrm{Ob}(X)} M(x)$ will be written as $\sum\limits_{x\in X} M(x)$.
(ii) A functor $F : X \to \mathbb{A}$ is *finite* if the function $F : \mathrm{Ob}(X) \to \mathrm{Ob}(\mathbb{A})$ is finite. $\square$

Definition 1.2. (i) The *contravariant functor category* $\mathbb{A}_*[X]$ is the additive category of finite contravariant functors $F : X \to \mathbb{A}$. The morphisms in $\mathbb{A}_*[X]$ are the natural transformations.
(ii) The *covariant functor category* $\mathbb{A}^*[X]$ is the additive category of covariant functors $F : X \to \mathbb{A}$. The morphisms in $\mathbb{A}^*[X]$ are the natural transformations. We write $\mathbb{A}^*_f[X]$ for the full subcategory whose objects are the finite functors in $\mathbb{A}^*[X]$. $\square$

Remark 1.3. We use the terminology $\mathbb{A}^*[X]$ for the *covariant* functor category because it behaves contravariantly in the variable X. Indeed a functor $g : X \to Y$ induces a functor $\mathbb{A}^*[Y] \to \mathbb{A}^*[X]$ by composition with g. Our reasons for using the terminology $\mathbb{A}_*[X]$ for the *contravariant* functor category are similar, but more complicated. Below we introduce a variation denoted $\mathbb{A}_*(X)$ which behaves covariantly in X. $\square$

For the remainder of this section we shall only consider the contravariant functor category $\mathbb{A}_*[X]$, but every result also has a version for the covariant functor category $\mathbb{A}^*[X]$ (or $\mathbb{A}^*_f[X]$ in some cases).

Definition 1.4. (i) A chain complex in an additive category $\mathbb{A}$

$$C \ : \ \ldots \longrightarrow C_{n+1} \xrightarrow{d} C_n \xrightarrow{d} C_{n-1} \longrightarrow \ldots \ (d^2 \ = \ 0)$$

is *finite* if $C_n = 0$ for all but a finite number of $n \in \mathbb{Z}$.
(ii) Let $\mathbb{B}(\mathbb{A})$ be the additive category of finite chain complexes in $\mathbb{A}$ and chain maps. □

A finite chain complex C in $\mathbb{A}_*[X]$ is just an object in $\mathbb{B}(\mathbb{A})_*[X]$, and likewise for chain maps, so that

$$\mathbb{B}(\mathbb{A}_*[X]) \;=\; \mathbb{B}(\mathbb{A})_*[X] \;.$$

Definition 1.5. A chain map $f : C \to D$ of chain complexes in $\mathbb{A}_*[X]$ is a *weak equivalence* if each

$$f[x] \;:\; C[x] \;\to\; D[x] \;(x \in X)$$

is a chain equivalence in $\mathbb{A}$. □

A morphism $f : C \to D$ in $\mathbb{B}(\mathbb{A}_*[X])$ which is a chain equivalence is also a weak equivalence, but in general a weak equivalence need not be a chain equivalence – see 1.11 for a more detailed discussion.

Definition 1.6. Let x be an object in X.
(i) The *under category* x/X is the category with objects the morphisms $f : x \to y$ in X, and morphisms $g : f \to f'$ the morphisms $g : y \to y'$ in X such that $gf = f'$

$$\begin{tikzcd} & x \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ y \arrow[rr, "g"] & & y' \end{tikzcd}$$

The *open star* of x is the set of objects in x/X

$$\mathrm{st}(x) \;=\; \mathrm{Ob}(x/X) \;=\; \{x \to y\} \;.$$

(ii) The *over category* X/x is the category with morphisms $f : y \to x$ in X as its objects, and so that morphisms $g : f \to f'$ are the morphisms $g : y \to y'$ in X such that $f = f'g$

$$\begin{tikzcd} y \arrow[rr, "g"] \arrow[dr, "f"'] & & y' \arrow[dl, "f'"] \\ & x & \end{tikzcd}$$

The *closure* of x is the set of objects in X/x

$$\mathrm{cl}(x) \;=\; \mathrm{Ob}(X/x) \;=\; \{y \to x\} \;.$$

Because of our standing assumptions on X, the over category X/x is isomorphic to a finite poset. $\square$

In the applications of the contravariant functor category $\mathbb{A}_*[X]$ to topology we shall be particularly concerned with the subcategory of functors satisfying the following property.

Definition 1.7. A contravariant functor

$$F \ : \ X \ \to \ \mathbb{A} \ ; \ x \mapsto F[x]$$

in $\mathbb{A}_*[X]$ is *induced* if there exists a finite function $x \mapsto F(x) \in \mathrm{Ob}(\mathbb{A})$ and a natural isomorphism

$$F[x] \cong \bigoplus_{x \to y} F(y) \ .$$

The sum ranges over $\mathrm{st}(x)$, and since the function $x \mapsto F(x)$ is finite, $F[x]$ is only a sum of a finite number of non-zero objects in $\mathbb{A}$.
Similarly a covariant functor

$$F \ : \ X \ \to \ \mathbb{A} \ ; \ x \mapsto F[x]$$

in $\mathbb{A}^*[X]$ is *induced* if there exists a function $x \mapsto F(x) \in \mathrm{Ob}(\mathbb{A})$ and a natural isomorphism

$$F[x] \cong \bigoplus_{y \to x} F(y) \ .$$

The full subcategories of the functor categories $\mathbb{A}_*[X]$, respectively $\mathbb{A}^*[X]$, with objects the induced functors $F : X \to \mathbb{A}$ are equivalent, as we shall prove below, to the following categories.

Definition 1.8. Let $\mathbb{A}_*(X)$ be the additive category whose objects are functions $x \mapsto F(x)$ such that $F(x) = 0$ for all but a finite number of objects x. A morphism $f : E \to F$ in $\mathbb{A}_*(X)$ is a collection of morphisms $f(\phi) : E(x) \to F(y)$ in $\mathbb{A}$, one for each morphism $\phi : x \to y$ in X. The composite of the morphisms

$$f \ = \ \{f(\phi)\} \ : \ M \to N \ , \ g \ = \ \{g(\theta)\} \ : \ N \to P$$

is the morphism

$$gf \ = \ \{gf(\psi)\} \ : \ M \to P$$

with

$$gf(\psi : x \to z) \;=\; \sum_{\phi:x\to y,\theta:y\to z,\theta\phi=\psi} g(\theta)f(\phi) \;:\; M(x) \to P(z)\ .$$

We can view an object F of $\mathbb{A}_*(X)$ as an object in $\mathbb{A}_*[X]$ by writing

$$F[x] = \bigoplus_{x\to y} F(y).$$

A morphism $\theta : w \to x$ in X induces a morphism $F[x] \to F[w]$ in $\mathbb{A}$ which maps the summand $F(y)$ corresponding to some $\phi : x \to y$ identically to the summand $F(y)$ corresponding to the composition $\phi\theta : w \to y$.

Let $\mathbb{A}^*(X)$ be the additive category whose objects are functions $x \mapsto F(x)$. A morphism $f : E \to F$ in $\mathbb{A}_*(X)$ is a collection of morphisms $f(\phi) : E(y) \to F(x)$ in $\mathbb{A}$, one for each morphism $\phi : x \to y$ in X. Again we can view an object F of $\mathbb{A}^*(X)$ as an object in $\mathbb{A}^*[X]$ by writing

$$F[x] = \bigoplus_{y\to x} F(y).$$

Proposition 1.9. (i) *For any object M in $\mathbb{A}_*(X)$ and any object N in $\mathbb{A}_*[X]$*

$$\mathrm{Hom}_{\mathbb{A}_*[X]}(M, N) \;=\; \sum_{x\in X} \mathrm{Hom}_{\mathbb{A}}(M(x), N[x])\ .$$

(ii) *For any objects L, M in $\mathbb{A}_*(X)$*

$$\mathrm{Hom}_{\mathbb{A}_*[X]}(L, M) \;=\; \sum_{x\to y} \mathrm{Hom}_{\mathbb{A}}(L(x), M(y))\ .$$

(iii) *The additive category $\mathbb{A}_*(X)$ is equivalent to the full subcategory of the contravariant functor category $\mathbb{A}_*[X]$ with objects the induced functors.*

Proof. (i) A morphism $f : M \to N$ in $\mathbb{A}_*[X]$ is determined by the composite morphisms in $\mathbb{A}$

$$M(x) \xrightarrow{\text{inclusion}} M[x] \xrightarrow{\;f[x]\;} N[x]\ (x \in X)$$

(ii) By (i), a morphism $f : L \to M$ in $\mathbb{A}_*[X]$ is determined by the composite morphisms in $\mathbb{A}$

$$L(x) \xrightarrow{\text{inclusion}} L[x] \xrightarrow{\;f[x]\;} M[x] \;=\; \sum_{x\to y} M(y)\ (x \in X)\ .$$

(iii) Every object M in $\mathbb{A}_*(X)$ determines an induced contravariant functor

$$X \to \mathbb{A} \; ; \; x \mapsto M[x] \; = \; \sum_{x \to y} M(y) \; ,$$

i.e. an object in $\mathbb{A}_*[X]$, and every induced functor is naturally equivalent to one of this type. □

Proposition 1.10. *The following conditions on a chain map $f : C \to D$ in $\mathbb{A}_*(X)$ are equivalent:*

(a) *f is a chain equivalence,*
(b) *each of the component chain maps in $\mathbb{A}$*

$$f(1_x) \; : \; C(x) \to D(x) \; (x \in X)$$

is a chain equivalence,
(c) *$f : C \to D$ is a weak equivalence in $\mathbb{A}_*[X]$, that is, $C[x] \to D[x]$ is a chain equivalence for all x.*

Proof. The proof given in Proposition 2.7 of Ranicki and Weiss [8] in the case when $\mathbb{A}$ is the additive category of R-modules (for some ring R) works for an arbitrary additive category. □

Remark 1.11. Every chain equivalence of chain complexes in $\mathbb{A}_*[X]$ is a weak equivalence. By 1.10 every weak equivalence of degreewise induced finite chain complexes in $\mathbb{A}_*[X]$ is a chain equivalence. See Ranicki and Weiss [8, 1.13] for an explicit example of a weak equivalence of finite chain complexes in $\mathbb{A}_*[X]$ which is not a chain equivalence. It is proved in [8, 2.9] that every finite chain complex C in $\mathbb{A}_*[X]$ is weakly equivalent to one in $\mathbb{A}_*(X)$. □

2. Δ-sets

Let Δ be the category with objects the sets

$$[n] \; = \; \{0, 1, \ldots, n\} \; (n \geqslant 0)$$

and morphisms $[m] \to [n]$ order-preserving injections. Every such morphism has a unique factorization as the composite of the order-preserving injections

$$\partial_i \ : \ [k-1] \to [k] \ ; \ j \mapsto \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geqslant i \ . \end{cases}$$

Definition 2.1. (Rourke and Sanderson [9]) A Δ-*set* is a contravariant functor

$$X \ : \ \Delta \to \{\text{sets and functions}\} \ ; \ [n] \mapsto X^{(n)} \ .$$

□

Equivalently, a Δ-set X can be regarded as a sequence $X^{(n)}$ $(n \geqslant 0)$ of sets, together with face maps

$$\partial_i \ : \ X^{(n)} \ \to \ X^{(n-1)} \ (0 \leqslant i \leqslant n)$$

such that

$$\partial_i \partial_j \ = \ \partial_{j-1} \partial_i \text{ for } i < j \ .$$

The elements $x \in X^{(n)}$ are the n-*simplices* of X.

Definition 2.2. (Rourke and Sanderson [9])
(i) The *realization* of a Δ-set X is the *CW* complex

$$\|X\| \ = \ \coprod_{n=0}^{\infty} (X^{(n)} \times \Delta^n)/\sim$$

with

$$\begin{aligned} &\Delta^n \ = \ \{(s_0, s_1, \dots, s_n) \in \mathbb{R}^n \,|\, 0 \leqslant s_i \leqslant 1, \ \sum_{i=0}^{n} s_i = 1\} \ , \\ &\partial_i \ : \ \Delta^{n-1} \hookrightarrow \Delta^n \ ; \ (s_0, s_1, \dots, s_{n-1}) \mapsto (s_0, s_1, \dots, s_{i-1}, 0, s_{i+1}, \dots, s_n) \ , \\ &(x, \partial_i s) \sim (\partial_i x, s) \ (x \in X^{(n)}, s \in |\Delta^{n-1}|) \ . \end{aligned}$$

(ii) There is one n-cell $x(\Delta^n) \subseteq \|X\|$ for each n-simplex $x \in X$, with *characteristic map*

$$x \ : \ \Delta^n \to \|X\| \ ; \ (s_0, s_1, \dots, s_n) \mapsto (x, (s_0, s_1, \dots, s_n)) \ .$$

The *boundary* $x(\partial\Delta^n) \subseteq \|X\|$ is the image of

$$\begin{aligned} \partial\Delta^n = &\ \bigcup_{i=0}^{n} \partial_i |\Delta^{n-1}| \\ = &\ \{(s_0, s_1, \dots, s_n) \in \mathbb{R}^n \,|\, 0 \leqslant s_i \leqslant 1, \ \sum_{i=0}^{n} s_i = 1, \ s_i = 0 \text{ for some } i\} \end{aligned}$$

and the *interior* $x(\overset{\circ}{\Delta}{}^n) \subseteq \|X\|$ is the image of

$$\begin{aligned}\overset{\circ}{\Delta}{}^n = & \ \Delta^n \backslash \partial\Delta^n \\ = & \ \{(s_0, s_1, \ldots, s_n) \in \mathbb{R}^n \,|\, 0 < s_i \leqslant 1,\ \sum_{i=0}^{n} s_i = 1\} \subseteq \Delta^n \ .\end{aligned}$$

The characteristic map $x : \Delta^n \to \|X\|$ is injective on $\overset{\circ}{\Delta}{}^n \subseteq \Delta^n$. □

Example 2.3. Let Δ^n be the Δ-set with

$$(\Delta^n)^{(m)} \ = \ \{\text{morphisms } [m] \to [n] \text{ in } \Delta\} \ (0 \leqslant m \leqslant n) \ .$$

The realization $\|\Delta^n\|$ is the geometric n-simplex Δ^n (as in the above definition). It should be clear from the context whether Δ^n refers to the Δ-set or the geometric realization. □

We regard a Δ-set X as a category, whose objects are the simplices, writing the dimension of an object $x \in X$ as $|x|$, i.e. $|x| = m$ for $x \in X^{(m)}$. A morphism $f : x \to y$ from an m-simplex x to an n-simplex y is a morphism $f : [m] \to [n]$ in Δ such that

$$f^*(y) \ = \ x \in X^{(m)} \ .$$

In particular, for any $x \in X^{(m)}$ with $m \geqslant 1$ there are defined $m+1$ distinct morphisms in X

$$\partial_i \ : \ \partial_i x \to x \ (0 \leqslant i \leqslant m) \ .$$

Example 2.4. (i) Let X be a Δ-set. An object M of $\mathbb{A}_*(X)$ is just an object M of $\mathbb{A}$ with a direct sum decomposition $M = \bigoplus_{x \in X} M(x)$. A morphism $f : M \to N$ in $\mathbb{A}_*(X)$ is a collection of morphisms $f_{xy,\lambda} : M(x) \to N(y)$, one such for every pair of simplices x, y and face operator λ such that $\lambda^* y = x$.
We like to think of a morphism $f : M \to N$ in $\mathbb{A}_*(X)$ as a morphism in $\mathbb{A}$ with additional structure. Source and target of that morphism in $\mathbb{A}$ are $M(X) = \bigoplus_x M(x)$ and $N(X) = \bigoplus_x N(x)$, respectively. For simplices x and y, the xy-component of the morphism $M(X) \to N(X)$ determined by f is

$$\sum_\lambda f_{xy,\lambda}$$

where the sum runs over all λ such that $\lambda^* y = x$.
(ii) If X is a simplicial complex then a morphism in $\mathbb{A}_*(X)$ is just a morphism

$f : M \to N$ in $\mathbb{A}$ between objects with finite direct sum decompositions

$$M = \sum_{x \in X} M(x) \ , \ N = \sum_{y \in X} N(y)$$

such that the components $f(x,y) : M(x) \to N(y)$ are 0 unless $x \leqslant y$.
(iii) The description of $\mathbb{A}_*(X)$ in (ii) also applies in the case of a Δ-set X where, for any two simplices x and y, there is at most one morphism from x to y. In particular it applies when $X = Y'$ is the barycentric subdivision of another Δ-set Y, to be defined in the next section. □

Definition 2.5. Let X be a Δ-set, and let R be a ring.
(i) The *R-coefficient simplicial chain complex of* X is the free (left) R-module chain complex $\Delta(X;R)$ with

$$d = \sum_{i=0}^{n} (-)^i \partial_i \ : \ \Delta(X;R)_n = R[X^{(n)}] \to \Delta(X;R)_{n-1} = R[X^{(n-1)}] \ .$$

The *R-coefficient homology* of X is the homology of $\Delta(X;R)$

$$H_*(X;R) = H_*(\Delta(X;R)) = H_*(\|X\|;R) \ ,$$

noting that $\Delta(X;R)$ is the R-coefficient cellular chain complex of $\|X\|$.
(ii) Suppose that R is equipped with an involution

$$R \to R \ ; \ r \mapsto \overline{r}$$

(e.g. the identity for a commutative ring), allowing the definition of the *dual* of an R-module M to be the R-module

$$M^* = \mathrm{Mod}_R(M,R) \ , \ R \times M^* \to M^* \ ; \ (r,f) \mapsto (x \mapsto f(x)\overline{r}) \ .$$

The *R-coefficient simplicial cochain complex of X*

$$\Delta(X;R)^* = \mathrm{Hom}_R(\Delta(X;R),R)$$

is the R-module cochain complex with

$$d^* = \sum_{i=0}^{n+1} (-)^i \partial_i^* \ : \ \Delta(X;R)^n = R[X^{(n)}]^* \to \Delta(X;R)^{n+1} = R[X^{(n+1)}]^* \ ,$$

The *R-coefficient cohomology* of X is the cohomology of $\Delta(X;R)^*$

$$H^*(X;R) = H^*(\Delta(X;R)^*) = H^*(\|X\|;R) \ ,$$

noting that $\Delta(X;R)^*$ is the R-coefficient cellular cochain complex of $\|X\|$. □

A simplicial complex X is *ordered* if the vertices in any simplex are ordered, with faces having compatible orderings. From now on, in dealing with simplicial complexes we shall always assume an ordering.

Example 2.6. A simplicial complex X can be regarded as a Δ-set, with $X^{(n)}$ the set of n-simplices and

$$\partial_i \ : \ X^{(n)} \to X^{(n-1)} \ ; \ (v_0 v_1 \dots v_n) \mapsto (v_0 v_1 \dots v_{i-1} v_{i+1} \dots v_n) \ .$$

There is one morphism $x \to y$ in X for each face inclusion $x \leqslant y$. The realization $\|X\|$ of X regarded as a Δ-set is the polyhedron of the simplicial complex X, with the characteristic maps $x : \Delta^{|x|} \to \|X\|$ $(x \in X)$ injections. The simplicial chain complex $\Delta(X;R)$ is just the usual R-coefficient simplicial chain complex of X, and $\Delta(X;R)^*$ is the R-coefficient simplicial cochain complex of X. $\square$

Example 2.7. Let X be a Δ-set, and let $x \in X$ be a simplex.
(i) In general, the canonical map

$$\mathrm{Ob}(x/X) \ = \ \mathrm{st}(x) \to \mathrm{Ob}(X) \ ; \ (x \to y) \mapsto y$$

is not injective. The simplices $y \in \mathrm{Ob}(X) \backslash \mathrm{im}(\mathrm{st}(x))$ are the objects of a sub-Δ-set $X \backslash \mathrm{im}(\mathrm{st}(x)) \subset X$. If X is a simplicial complex then $\mathrm{st}(x) \to \mathrm{Ob}(X)$ is injective, and $X \backslash \mathrm{st}(x) \subset X$ is the subcomplex with simplices $y \in X$ such that $x \nleqslant y$.
(ii) The over category $X/x = \{y \to x\}$ (1.6) is a Δ-set with

$$(X/x)^{(n)} \ = \ \{y \to x \,|\, y \in X^{(n)}\} \ (n \geqslant 0) \ .$$

It is isomorphic as a Δ-set to $\Delta^{|x|}$. The forgetful functor

$$X/x \to X \ ; \ (y \to x) \mapsto y$$

is a Δ-map, inducing the characteristic map $\Delta^{|x|} \to \|X\|$. If X is a simplicial complex then $X/x \to X$ is injective, and so is the induced characteristic map. $\square$

Example 2.8. (i) If a group G acts on a Δ-set X the quotient X/G is again a Δ-set, with realization $\|X/G\| = \|X\|/G$. However, if X is a simplicial complex and G acts on X, then X/G is not in general a simplicial complex. See (ii) for an example.
(ii) Suppose $X = \mathbb{R}$, the Δ-set with

$$X^{(0)} \ = \ X^{(1)} \ = \ \mathbb{Z} \ , \ \partial_0(n) = n \ , \ \partial_1(n) = n+1 \ ,$$

and let the infinite cyclic group $G = \mathbb{Z} = \{t\}$ act on X by $tn = n+1$. The quotient Δ-set $S^1 = \mathbb{R}/\mathbb{Z}$ is the circle, with one 0-simplex x_0 and one 1-simplex x_1

$$(S^1)^{(0)} = \{x_0\} \ , \ (S^1)^{(1)} = \{x_1\} \ , \ \partial_0(x_1) = \partial_1(x_1) = x_0 \ .$$

□

Example 2.9. For any space M use the standard n-simplices Δ^n and face inclusions $\partial_i : \Delta^{n-1} \hookrightarrow \Delta^n$ to define the *singular* Δ*-set* $X = M^\Delta$ by

$$X^{(n)} = M^{\Delta^n} \ , \ \partial_i \ : \ X^{(n)} \to X^{(n-1)} \ ; \ x \mapsto x \circ \partial_i \ .$$

We shall say that a singular simplex $x : \Delta^n \to X$ is a face of a singular simplex $y : \Delta^m \to X$ if $x = y \circ \partial_{i_1} \circ \dots \circ \partial_{i_{m-n}}$ for a given face inclusion

$$\partial_{i_1} \circ \dots \circ \partial_{i_{m-n}} \ : \ \Delta^n \hookrightarrow \Delta^m \ ,$$

writing $x \leqslant y$ (and $x < y$ if $x \neq y$). The simplicial chain complex $\Delta(X;R) = S(M;R)$ is just the usual R-coefficient singular chain complex of M, so that

$$H_*(\|X\|;R) = H_*(X;R) = H_*(M;R) \ .$$

Also $\Delta(X;R)^* = S(M;R)^*$ is the R-coefficient singular cochain complex of M, and

$$H^*(\|X\|;R) = H^*(X;R) = H^*(M;R) \ .$$

□

3. The barycentric subdivision

The Δ-set analogue of the barycentric subdivision X' of a simplicial complex X and the dual cells $D(x,X) \subset X'$ $(x \in X)$ makes use of the following standard categorical construction.

Definition 3.1. (i) The *nerve* of a category $\mathcal{C}$ is the simplicial set with one n-simplex for each string $x_0 \to x_1 \to \dots \to x_n$ of morphisms in $\mathcal{C}$, with

$$\partial_i(x_0 \to x_1 \to \dots \to x_n) = (x_0 \to x_1 \to \dots \to x_{i-1} \to x_{i+1} \to \dots \to x_n) \ .$$

(ii) An n-simplex $x_0 \to x_1 \to \dots \to x_n$ in the nerve is *non-degenerate* if none of the morphisms $x_i \to x_{i+1}$ is the identity. □

If the category $\mathcal{C}$ has the property that the composite of non-identity morphisms is a non-identity, then the non-degenerate simplices in the nerve define a Δ-set, which we shall also call the nerve and denote by $\mathcal{C}$.

Definition 3.2. (Rourke and Sanderson [9, §4], Ranicki and Weiss [8, 1.6, 1.7]) Let X be a Δ-set.
(i) The *barycentric subdivision* of X is the Δ-set X' defined by the nerve of the category X.
(ii) The *dual* $x^\perp$ of a simplex $x \in X$ is the nerve of the under category x/X (1.6). An n-simplex in the Δ-set $x^\perp$ is thus a sequence of morphisms in X

$$x \longrightarrow x_0 \longrightarrow x_1 \longrightarrow \cdots \longrightarrow x_n$$

such that $x_0 \to x_1 \to \cdots \to x_n$ is non-degenerate. In particular

$$(x^\perp)^{(0)} \;=\; \{x \to x_0\} \;=\; \mathrm{st}(x)\ .$$

(iii) The *boundary of the dual* $\partial x^\perp$ is the sub-Δ-set of $x^\perp$ consisting of the n-simplices $x \to x_0 \to x_1 \to \cdots \to x_n$ such that $x \to x_0$ is not the identity. □

The under category x/X has an initial object, so that the nerve $x^\perp$ is contractible. The rule $x \to x^\perp$ is contravariant, i.e. every morphism $x \to y$ induces a Δ-map $y^\perp \to x^\perp$.

Lemma 3.3. *The realizations $\|X\|$, $\|X'\|$ of a Δ-set X and its barycentric subdivision X' are homeomorphic, via a homeomorphism $\|X'\| \to \|X\|$ sending the vertex $x \in X = (X')^{(0)}$ to the barycentre*

$$\widehat{x} \;=\; x(\frac{1}{n+1},\frac{1}{n+1},\dots,\frac{1}{n+1}) \in x(\overset{\circ}{\Delta}{}^n) \subseteq \|X\|\ .$$

Proof. It suffices to consider the special case $X = \Delta^n$, so that X and X' are simplicial complexes, and to define a homeomorphism $\|X'\| \to \|X\|$ by $x \mapsto \widehat{x}$ and extending linearly. □

Definition 3.4. Let X be a Δ-set, and let $x \in X$ be a simplex.
(i) The *open star space*

$$\|\mathrm{st}(x)\| \;=\; \bigcup_{y \in x^\perp \backslash \partial x^\perp} \overset{\circ}{\Delta}{}^{|y|} \subseteq \|X'\| \;=\; \|X\|$$

is the subspace of the realization $\|X'\|$ of the barycentric subdivision X' defined by the union of the interiors of the simplices $y \in x^\perp \backslash \partial x^\perp$, i.e.

$$y = (x \to x_0 \to \cdots \to x_n) \in X'$$

with $x \to x_0 = x$ the identity.
(ii) The *homology* of the open star is

$$H_*(\mathrm{st}(x)) = H_*(\Delta(\mathrm{st}(x)))$$

with $\Delta(\mathrm{st}(x))$ the chain complex defined by

$$\Delta(\mathrm{st}(x)) = \Delta(x^\perp, \partial x^\perp)_{*-|x|} .$$

$\square$

Lemma 3.5. *For any simplex $x \in X$ of a Δ-set X the characteristic Δ-map*

$$i : x^\perp \to X' ; (x \to x_0 \to \cdots \to x_n) \mapsto (x_0 \to \cdots \to x_n)$$

is injective on $x^\perp \backslash \partial x^\perp$. The images $i(\partial x^\perp), i(x^\perp) \subseteq X'$ are sub-Δ-sets such that

$$\|i(x^\perp)\| \backslash \|i(\partial x^\perp)\| = \|\mathrm{st}(x)\| \subseteq \|X'\|$$

and there are homology isomorphisms

$$\begin{aligned} H_*(\mathrm{st}(x)) = \ & H_{*-|x|}(x^\perp, \partial x^\perp) \\ \cong \ & H_{*-|x|}(i(x^\perp), i(\partial x^\perp)) \\ \cong \ & H_*(\|X\|, \|X\| \backslash \|\mathrm{st}(x)\|) \\ \cong \ & H_*(\|X\|, \|X\| \backslash \{\widehat{x}\}) . \end{aligned}$$

Proof. The inclusion $(\|X\|, \|X\| \backslash \|\mathrm{st}(x)\|) \hookrightarrow (\|X\|, \|X\| \backslash \{\widehat{x}\})$ is a deformation retraction, and the open star subspace $\|\mathrm{st}(x)\| \subset \|X\|$ has an open regular neighbourhood

$$\|\mathrm{st}(x)\| \times \overset{\circ}{\Delta}{}^{|x|} \subset \|X\|$$

with one-point compactification

$$(\|\mathrm{st}(x)\| \times \overset{\circ}{\Delta}{}^{|x|})^\infty = \|i(x^\perp)\| / \|i(\partial x^\perp)\| \wedge \Delta^{|x|} / \partial \Delta^{|x|} ,$$

so that

$$\begin{aligned} H_*(\|X\|,\|X\|\backslash\{\widehat{x}\}) &\cong H_*(\|X\|,\|X\|\backslash\|\mathrm{st}(x)\|) \\ &\cong \widetilde{H}_*(\|i(x^\perp)\|/\|i(\partial x^\perp)\| \wedge \Delta^{|x|}/\partial\Delta^{|x|}) \\ &\cong H_{*-|x|}(i(x^\perp), i(\partial x^\perp)) \ . \end{aligned}$$

□

Example 3.6. Let X be a simplicial complex. The barycentric subdivision of X is the ordered simplicial complex X' with one n-simplex for each sequence of proper face inclusions $x_0 < x_1 < \cdots < x_n$. By definition, the *dual cell* of a simplex $x \in X$ is the subcomplex $D(x,X) \subseteq X'$ consisting of all the simplices $x_0 < x_1 < \cdots < x_n$ with $x \leqslant x_0$. The *boundary* of the dual cell is the subcomplex $\partial D(x,X) \subseteq D(x,X)$ consisting of all the simplices $x_0 < x_1 < \cdots < x_n$ with $x < x_0$. The Δ-sets associated to $X, X', D(x,X), \partial D(x,X)$ are just the Δ-sets $X, X', x^\perp, \partial x^\perp$ of 3.2, with the characteristic map $i : x^\perp = D(x,X) \to X'$ injective. Moreover, $X\backslash\mathrm{st}(x) \subset X$ is a subcomplex such that

$$\|X\backslash\mathrm{st}(x)\| \ = \ \|X\|\backslash\|\mathrm{st}(x)\|$$

and

$$\Delta(\mathrm{st}(x)) \ = \ \Delta(D(x,X),\partial D(x,X))_{*-|x|} \ \simeq \ \Delta(X, X\backslash\mathrm{st}(x)) \ .$$

□

Example 3.7. Let X be the Δ-set (2.8) with one 0-simplex x_0 and one 1-simplex x_1, with non-identity morphisms

$$x_0 \rightrightarrows x_1$$

and realization $\|X\| = S^1$. The barycentric subdivision X' is the Δ-set with 2 0-simplices and 2 1-simplices:

$$X'^{(0)} \ = \ \{x_0, x_1\} \ , \ X'^{(1)} \ = \ \{ \ x_0 \rightrightarrows x_1 \ \} \ .$$

The duals and their boundaries are given by

$$\begin{aligned} x_0^\perp \ &= \ \{ \ x_0 \longrightarrow x_0 \ , \ \ x_0 \rightrightarrows x_1 \ \} \cup \{ \ x_0 \longrightarrow x_0 \rightrightarrows x_1 \ \} \ , \\ \partial x_0^\perp \ &= \ \{ \ x_0 \rightrightarrows x_1 \ \} \ = \ \{0,1\} \ , \\ x_1^\perp \ &= \ \{ \ x_1 \longrightarrow x_1 \ \} \ , \ \partial x_1^\perp \ = \ \emptyset \ . \end{aligned}$$

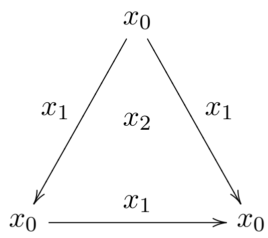
The characteristic map $i : x_0^{\perp} \to X'$ is surjective but not injective, and

$$H_n(x_0^{\perp}, \partial x_0^{\perp}) \;=\; H_n(i(x_0^{\perp}), i(\partial x_0^{\perp})) \;=\; \begin{cases} \mathbb{Z} & \text{if } n = 1 \\ 0 & \text{if } n \neq 1 \; . \end{cases}$$

$\square$

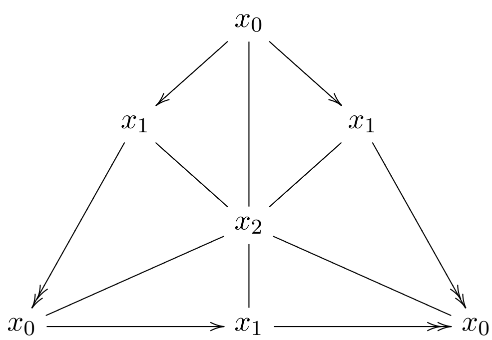
Example 3.8. Let X be the contractible Δ-set with one 0-simplex x_0, one 1-simplex x_1 and one 2-simplex x_2, with non-identity morphisms

$$x_0 \rightrightarrows x_1 \quad , \quad x_1 \;\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}\; x_2 \quad , \quad x_0 \;\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}\; x_2$$



The realization $\|X\|$ is the dunce hat (Zeeman [14]). The barycentric subdivision X' is the Δ-set with three 0-simplices, eight 1-simplices and six 2-simplices:

$$\begin{aligned} X'^{(0)} &= \{x_0, x_1, x_2\} \; , \\ X'^{(1)} &= \{\; x_0 \rightrightarrows x_1 \;\} \cup \{\; x_1 \;\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}\; x_2 \;\} \cup \{\; x_0 \;\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}\; x_2 \;\} \\ X'^{(2)} &= \{\; x_0 \rightrightarrows x_1 \;\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}\; x_2 \;\} \end{aligned}$$



The duals and their boundaries are given by

$$\begin{aligned}
x_0^{\perp} \;&=\; \{\, x_0 \longrightarrow x_0 \,,\;\; x_0 \rightrightarrows x_1 \;\;,\; x_0 \mathrel{\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}} x_2 \,\} \\
&\qquad \cup\{\, x_0 \longrightarrow x_0 \rightrightarrows x_1 \,,\;\; x_0 \longrightarrow x_0 \mathrel{\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}} x_2 \,,\;\; x_0 \rightrightarrows x_1 \mathrel{\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}} x_2 \,\} \\
&\qquad\qquad\qquad \cup\{\, x_0 \longrightarrow x_0 \rightrightarrows x_1 \mathrel{\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}} x_2 \,\}\;, \\
\partial x_0^{\perp} \;&=\; \{\, x_0 \rightrightarrows x_1 \;\;,\; x_0 \mathrel{\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}} x_2 \,\} \cup \{\, x_0 \rightrightarrows x_1 \mathrel{\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}} x_2 \,\}\;,\; \|\partial x_0^{\perp}\| \;\simeq\; S^1 \vee S^1\;, \\
x_1^{\perp} \;&=\; \{\, x_1 \longrightarrow x_1 \;\;,\;\; x_1 \mathrel{\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}} x_2 \,\} \cup \{\, x_1 \longrightarrow x_1 \mathrel{\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}} x_2 \,\}\;, \\
\partial x_1^{\perp} \;&=\; \{\, x_1 \mathrel{\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}} x_2 \,\}\;,\; \|\partial x_1^{\perp}\| \;\simeq\; \{0,1,2\}\;, \\
x_2^{\perp} \;&=\; \{\, x_2 \longrightarrow x_2 \,\}\;,\; \partial x_2^{\perp} \;=\; \emptyset\;.
\end{aligned}$$

The characteristic map $i : x_0^{\perp} \to X'$ is surjective but not injective, with

$$\|i(x_0^{\perp})\| \;\simeq\; \{*\}\;,\; \|i(\partial x_0^{\perp})\| \;\simeq\; S^1 \vee S^1$$

and

$$H_n(x_0^{\perp}, \partial x_0^{\perp}) \;=\; H_n(i(x_0^{\perp}), i(\partial x_0^{\perp})) \;=\; \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{if } n = 2 \\ 0 & \text{if } n \neq 2\;. \end{cases}$$

The characteristic map $i : x_1^{\perp} \to X'$ is neither surjective nor injective, with

$$\|i(x_1^{\perp})\| \;\simeq\; S^1 \vee S^1\;,\; \|i(\partial x_1^{\perp})\| \;\simeq\; \{*\}$$

and

$$H_n(x_1^{\perp}, \partial x_1^{\perp}) \;=\; H_n(i(x_1^{\perp}), i(\partial x_1^{\perp})) \;=\; \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{if } n = 1 \\ 0 & \text{if } n \neq 1\;. \end{cases}$$

□

Definition 3.9. Given a ring R let $\mathrm{Mod}(R)$ be the additive category of left R-modules. For $R = \mathbb{Z}$ write $\mathrm{Mod}(\mathbb{Z}) = \mathrm{Ab}$, as usual. □

Definition 3.10. (Ranicki and Weiss [8, 1.9] for simplicial complexes)
(i) The R-coefficient simplicial chain complex $\Delta(X'; R)$ of the barycentric subdivision X' of a finite Δ-set X is the chain complex in $\mathrm{Mod}(R)_*(X)$ with

$$\Delta(X'; R)(x) \;=\; \Delta(x^{\perp}, \partial x^{\perp}; R)\;,\; \Delta(X'; R)[x] \;=\; \Delta(x^{\perp}; R).$$

Compare example 2.4 case (iii).
(ii) Let $f : Y \to X'$ be a Δ-map from a finite Δ-set Y to the barycentric subdivision X' of a Δ-set X. The R-coefficient simplicial chain complex $\Delta(Y;R)$ is the chain complex in $\mathrm{Mod}(R)_*(X)$ with

$$\Delta(Y;R)(x) \;=\; \Delta(x/f,\partial(x/f);R)\ ,\ \Delta(Y;R)[x] \;=\; \Delta(x/f;R)\ (x\in X)$$

with x/f, $\partial(x/f)$ the Δ-sets defined to fit into strict pullback squares of Δ-sets

$$\begin{tikzcd}
\partial(x/f) \arrow[r] \arrow[d] & x/f \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\
\partial x^{\perp} \arrow[r] & x^{\perp} \arrow[r, "i"] & X'
\end{tikzcd}$$

□

4. The total complex

For a finite chain complex C in $\mathbb{A}_*[X]$, there is defined a chain complex in $\mathbb{A}^*(X)$, called the *total complex* of C.

Definition 4.1. The *total complex* Tot_*C of a finite chain complex C in $\mathbb{A}_*[X]$ is the finite chain complex in $\mathbb{A}^*(X)$ given by

$$(\mathrm{Tot}_*C)(x)_n = C[x]_{n-|x|}$$

with differential $d = d_{C[x]} + \sum_{i=0}^{|x|}(-)^{i+|x|}C(\partial_i x \to x)$. The construction is natural, defining a covariant functor

$$\mathbb{B}(\mathbb{A})_*[X] \to \mathbb{B}(\mathbb{A})^*(X)\ ;\ C \mapsto \mathrm{Tot}_*C\ .$$

□

Remark 4.2. There is a forgetful functor $\mathbb{B}(\mathbb{A})^*_f(X) \to \mathbb{B}(\mathbb{A})$ taking C in $\mathbb{B}(\mathbb{A})^*(X)$ to

$$C(X) = \bigoplus_{x\in X} C(x)\ .$$

Compare example 2.4. The chain complex $(\mathrm{Tot}_*C)(X)$ in $\mathbb{A}$ is the 'realization'

$$\Big(\sum_{x\in X}\Delta(\Delta^{|x|})\otimes_{\mathbb{Z}} C[x]\Big)/\sim$$

with $\sim$ the equivalence relation generated by $a \otimes \lambda^* b \sim \lambda_* a \otimes b$ for a morphism $\lambda : y \to z$ in X, with $a \in \Delta(\Delta^{|y|})$, $b \in C[z]$. $\square$

Example 4.3. The simplicial chain complex $\Delta(X)$ of a finite Δ-set X is $(\mathrm{Tot}_* C)(X)$ for the chain complex C in $\mathrm{Ab}_*[X]$ defined by $C[x] = \mathbb{Z}$ for all x (a constant functor). $\square$

Remark 4.4. There are evident forgetful functors

$$\begin{aligned} \mathbb{B}(\mathbb{A})_*(X) \to \mathbb{B}(\mathbb{A}) \ ; \ C \mapsto C(X) \ , \\ \mathbb{B}(\mathbb{A})^*_f(X) \to \mathbb{B}(\mathbb{A}) \ ; \ C \mapsto C(X) \ . \end{aligned}$$

The diagram

$$\begin{tikzcd} \mathbb{B}(\mathbb{A})_*(X) \arrow[r] \arrow[dr] & \mathbb{B}(\mathbb{A})_*[X] \arrow[r, "\mathrm{Tot}_*"] & \mathbb{B}(\mathbb{A})^*_f(X) \arrow[dl] \\ & \mathbb{B}(\mathbb{A}) & \end{tikzcd}$$

commutes up to natural chain homotopy equivalence: for any finite chain complex C in $\mathbb{A}_*(X)$

$$(\mathrm{Tot}_* C)(X)_n \ = \sum_{x \in X} \sum_{x \to y} C(y)_{n-|x|} \ = \sum_{y \in X} (\Delta(X/y) \otimes_{\mathbb{Z}} C(y))_n$$

with X/y the Δ-set defined in 2.7, which is contractible. $\square$

Proposition 4.5. (i) *For any objects M, N in $\mathbb{A}_*(X)$ the abelian group $\mathrm{Hom}_{\mathbb{A}_*(X)}(M,N)$ is naturally an object in $\mathrm{Ab}^*_f(X)$, with*

$$\begin{aligned} \mathrm{Hom}_{\mathbb{A}_*(X)}(M,N)(x) &= \mathrm{Hom}_{\mathbb{A}}(M(x), [N][x]) \\ &= \sum_{x \to y} \mathrm{Hom}_{\mathbb{A}}(M(x), N(y)) \ (x \in X) \ . \end{aligned}$$

If $f : M' \to M$, $g : N \to N'$ are morphisms in $\mathbb{A}_*(X)$ there is induced a morphism in $\mathrm{Ab}^*(X)$

$$\mathrm{Hom}_{\mathbb{A}_*(X)}(M,N) \to \mathrm{Hom}_{\mathbb{A}_*(X)}(M',N') \ ; \ h \mapsto ghf \ .$$

(ii) *For any objects M, N in $\mathbb{A}^*_f(X)$ the abelian group $\mathrm{Hom}_{\mathbb{A}^*(X)}(M,N)$ is naturally an object in $\mathrm{Ab}_*(X)$, with*

$$\begin{aligned} \mathrm{Hom}_{\mathbb{A}^*(X)}(M,N)(x) &= \mathrm{Hom}_{\mathbb{A}}(M(x), [N][x]) \\ &= \sum_{y \to x} \mathrm{Hom}_{\mathbb{A}}(M(x), N(y)) \ (x \in X) \ . \end{aligned}$$

Naturality as in (i).

Proof. Immediate from 1.9. □

Example 4.6. (i) For a chain complex C in $\mathrm{Ab}_*(X)$ the total complex in $\mathrm{Ab}^*(X)$ of the corresponding chain complex $[C]$ in $\mathrm{Ab}_*[X]$ is given by

$$[C]_*[X] \ = \ \mathrm{Hom}_{\mathrm{Ab}_*(X)}(\Delta(X)^{-*}, C) \ .$$

(ii) For a chain complex D in $\mathrm{Ab}^*(X)$ the total complex in $\mathrm{Ab}_*(X)$ of the corresponding chain complex $[D]$ in $\mathrm{Ab}^*[X]$ is given by

$$[D]^*[X] \ = \ \mathrm{Hom}_{\mathrm{Ab}^*(X)}(\Delta(X), D) \ .$$

□

5. Chain duality in L-theory

In general, it is not possible to extend an involution $T : \mathbb{A} \to \mathbb{A}$ on an additive category $\mathbb{A}$ to the functor category $\mathbb{A}_*(X)$ for an arbitrary category X. An object in $\mathbb{A}_*(X)$ is an induced contravariant functor $F : X \to \mathbb{A}$ and the composite of the contravariant functors

$$X \xrightarrow{F} \mathbb{A} \xrightarrow{T} \mathbb{A}$$

is a covariant functor, not a contravariant functor, let alone an induced contravariant functor. A 'chain duality' on $\mathbb{A}$ is essentially an involution on the derived category of finite chain complexes and chain homotopy classes of chain maps; an involution on $\mathbb{A}$ is an example of a chain duality. Given a chain duality on $\mathbb{A}$ we shall now define a chain duality on the induced functor category $\mathbb{A}_*(X)$, for any Δ-set X, essentially in the same way as was carried out for a simplicial complex X in [6].

Definition 5.1. (Ranicki [6, 1.1]) A *chain duality* (T, e) on an additive category $\mathbb{A}$ is a contravariant additive functor

$$T \ : \ \mathbb{A} \ \to \ \mathbb{B}(\mathbb{A})$$

together with a natural transformation

$$e \ : \ T^2 \to 1 \ : \ \mathbb{A} \to \mathbb{B}(\mathbb{A})$$

such that for each object M in $\mathbb{A}$

(i) $e(T(M)) \circ T(e(M)) = 1 : T(M) \to T^3(M) \to T(M)$,
(ii) $e(M) : T^2(M) \to M$ is a chain equivalence.

□

A chain duality (T, e) on $\mathbb{A}$ extends to a contravariant functor on the bounded chain complex category

$$T \ : \ \mathbb{B}(\mathbb{A}) \ \to \ \mathbb{B}(\mathbb{A}) \ ; \ C \mapsto T(C) \ ,$$

using the double complex construction with

$$T(C)_n \ = \ \sum_{p+q=n} T(C_{-p})_q \ , \ d_{T(C)} \ = \ d_{T(C_{-p})} + (-)^q T(d : C_{-p+1} \to C_{-p}) \ ,$$

and $e(C) : T^2(C) \to C$ a chain equivalence. For any objects M, N in an additive category $\mathbb{A}$ there is defined a $\mathbb{Z}$-module $\mathrm{Hom}_{\mathbb{A}}(M, N)$. Thus for any chain complexes C, D in $\mathbb{A}$ there is defined a $\mathbb{Z}$-module chain complex $\mathrm{Hom}_{\mathbb{A}}(C, D)$, with

$$\mathrm{Hom}_{\mathbb{A}}(C, D)_n \ = \ \sum_{q-p=n} \mathrm{Hom}_{\mathbb{A}}(C_p, D_q) \ , \ d_{\mathrm{Hom}_{\mathbb{A}}(C,D)}(f) \ = \ d_D f + (-)^q f d_C \ .$$

If (T, e) is a chain duality on $\mathbb{A}$ there is defined a $\mathbb{Z}$-module chain map

$$\mathrm{Hom}_{\mathbb{A}}(TC, D) \to \mathrm{Hom}_{\mathbb{A}}(TD, C) \ ; \ f \mapsto e(C)T(f)$$

which is a chain equivalence for finite C.

Example 5.2. An involution (T, e) on $\mathbb{A}$ is a contravariant functor $T : \mathbb{A} \to \mathbb{A}$ with a natural equivalence $e : T^2 \to 1$ such that for each object M in $\mathbb{A}$

$$e(T(M)) \ = \ T(e(M)^{-1}) \ : \ T^3(M) \to T(M) \ .$$

This is essentially the same as a chain duality (T, e) such that $T(M)$ is a 0-dimensional chain complex for each object M in $\mathbb{A}$. □

Definition 5.3. A *chain product* $(\otimes_{\mathbb{A}}, b)$ on an additive category $\mathbb{A}$ is a natural pairing

$$\otimes_{\mathbb{A}} \ : \ \mathrm{Ob}(\mathbb{A}) \times \mathrm{Ob}(\mathbb{A}) \to \{\mathbb{Z}\text{-module chain complexes}\} \ ; \ (M, N) \mapsto M \otimes_{\mathbb{A}} N$$

together with a natural chain equivalence

$$b(M,N)\ :\ M\otimes_{\mathbb{A}}N\to N\otimes_{\mathbb{A}}M$$

such that up to natural isomorphism

$$\begin{aligned}(M\oplus M')\otimes_{\mathbb{A}}N\ &=\ (M\otimes_{\mathbb{A}}N)\oplus(M'\otimes_{\mathbb{A}}N)\ ,\\ M\otimes_{\mathbb{A}}(N\oplus N')\ &=\ (M\otimes_{\mathbb{A}}N)\oplus(M\otimes_{\mathbb{A}}N')\end{aligned}$$

and

$$b(N,M)\circ b(M,N)\ \simeq\ 1\ :\ M\otimes_{\mathbb{A}}N\to M\otimes_{\mathbb{A}}N\ .$$

□

Remark 5.4. The notion of chain product is a linear version of an 'SW-product' in the sense of Weiss and Williams [12], where SW = Spanier-Whitehead.

□

Given an additive category $\mathbb{A}$ with a chain product $(\otimes_{\mathbb{A}},b)$ and chain complexes C,D in $\mathbb{A}$ let $C\otimes_{\mathbb{A}}D$ be the $\mathbb{Z}$-module chain complex defined by

$$\begin{aligned}(C\otimes_{\mathbb{A}}D)_n\ &=\ \sum_{p+q+r=n}(C_p\otimes_{\mathbb{A}}D_q)_r\ ,\\ d_{C\otimes_{\mathbb{A}}D}\ &=\ d_{C_p\otimes_{\mathbb{A}}C_q}+(-)^r(1\otimes_{\mathbb{A}}d_D+(-)^q d_C\otimes_{\mathbb{A}}1)\ .\end{aligned}$$

By the naturality of b there is defined a natural chain equivalence

$$b(C,D)\ :\ C\otimes_{\mathbb{A}}D\to D\otimes_{\mathbb{A}}C\ .$$

Proposition 5.5. *Let $\mathbb{A}$ be an additive category.*
(i) *A chain duality (T,e) on $\mathbb{A}$ determines a chain product $(\otimes_{\mathbb{A}},b)$ on $\mathbb{A}$ by*

$$\begin{aligned}&M\otimes_{\mathbb{A}}N\ =\ \mathrm{Hom}_{\mathbb{A}}(TM,N)\ ,\\ &b(M,N)\ :\ M\otimes_{\mathbb{A}}N\to N\otimes_{\mathbb{A}}M\ ;\\ &\qquad (f:TM\to N)\mapsto(e(M)\circ T(f):TN\to T^2M\to M)\ .\end{aligned}$$

(ii) *If $(\otimes_{\mathbb{A}},b)$ is a chain product on $\mathbb{A}$ such that*

$$M\otimes_{\mathbb{A}}N\ =\ \mathrm{Hom}_{\mathbb{A}}(TM,N)\ ,\ b(M,N)(f)\ =\ e(M)\circ T(f)$$

for some contravariant additive functor $T:\mathbb{A}\to\mathbb{B}(\mathbb{A})$ and natural transformation $e:T^2\to 1:\mathbb{A}\to\mathbb{B}(\mathbb{A})$, then (T,e) is a chain duality on $\mathbb{A}$.

Proof. Immediate from the definitions. □

Example 5.6. Let R be a ring with an involution $R \to R; r \mapsto \overline{r}$. Regard a (left) R-module M as a right R-module by

$$M \times R \to M \ ; \ (x, r) \mapsto \overline{r}x \ .$$

Thus for any R-modules M, N there is defined a $\mathbb{Z}$-module

$$M \otimes_R N \ = \ (M \otimes_{\mathbb{Z}} N)/\{\overline{r}x \otimes y - x \otimes ry \,|\, x \in M, y \in N, r \in R\}$$

with a natural isomorphism

$$b(M,N) \ : \ M \otimes_R N \to N \otimes_R M \ ; \ x \otimes y \mapsto y \otimes x$$

defining a (0-dimensional) chain product $(\otimes_R, b)$ on the R-module category $\mathrm{Mod}(R)$. As in 2.5 use the involution on R to define the contravariant duality functor

$$T \ : \ \mathrm{Mod}(R) \to \mathrm{Mod}(R) \ ; \ M \mapsto M^* = \mathrm{Hom}_R(M, R)$$

with

$$R \times M^* \to M^* \ ; \ (r, f) \mapsto (x \mapsto f(x)\overline{r}) \ .$$

The natural $\mathbb{Z}$-module morphism defined for any R-modules M, N by

$$M \otimes_R N \to \mathrm{Hom}_R(M^*, N) \ ; \ x \otimes y \mapsto (f \mapsto f(x)y)$$

is an isomorphism for f.g. projective M. The R-module morphism defined for any R-module M by

$$e'(M) \ : \ M \to M^{**} \ ; \ x \mapsto (f \mapsto f(x))$$

is an isomorphism for f.g. projective M. Let $\mathrm{Proj}(R) \subset \mathrm{Mod}(R)$ be the full subcategory of f.g. projective R-modules. The natural isomorphisms

$$e(M) \ = \ e'(M)^{-1} \ : \ M^{**} \to M$$

define an involution (T, e) on $\mathrm{Proj}(R)$, corresponding to the restriction to $\mathrm{Proj}(R)$ of the chain product $(\otimes_R, b)$ on $\mathrm{Mod}(R)$. □

Proposition 5.7. (Ranicki [6, 5.1,5.9,7], Weiss [11, 1.5])
A chain duality $(T_{\mathbb{A}}, e_{\mathbb{A}})$ on an additive category $\mathbb{A}$ extends to a chain duality $(T_{\mathbb{A}_*(X)}, e_{\mathbb{A}_*(X)})$ on $\mathbb{A}_*(X)$, for any Δ-set X

$$T_{\mathbb{A}_*(X)} \ : \ \mathbb{A}_*(X) \longrightarrow \mathbb{A}_*[X] \xrightarrow{\mathrm{Tot}_*} \mathbb{B}(\mathbb{A})^*_f(X) \xrightarrow{T_{\mathbb{A}}} \mathbb{B}(\mathbb{A})_*(X)$$

where $T_{\mathbb{A}} : \mathbb{B}(\mathbb{A})^*_f(X) \to \mathbb{B}(\mathbb{A})_*(X)$ *is the extension of the contravariant functor*

$$T_{\mathbb{A}} \ :\ \mathbb{A}^*_f(X) \to \mathbb{B}(\mathbb{A})_*(X)\ ;\ M\ =\ \sum_{x\in X} M(x) \mapsto T_{\mathbb{A}}(M)\ =\ \sum_{x\in X} T_{\mathbb{A}}(M(x))\ .$$

More explicitly, the chain dual of a finite chain complex C in $\mathbb{A}_*(X)$ is given by

$$T_{\mathbb{A}_*(X)}(C)\ =\ T_{\mathbb{A}}(\mathrm{Tot}_*C)\ ,$$

so that

$$\begin{aligned} T_{\mathbb{A}_*(X)}(C)(x)\ &=\ T_{\mathbb{A}}(C[x]_{*-|x|}) \\ &=\ \sum_{x\to y} T_{\mathbb{A}}(C(y)_{*-|x|})\ (x\in X)\ . \end{aligned}$$

Example 5.8. Let $\mathbb{A} = \mathbb{A}(\mathbb{Z})$, the additive category of f.g. free abelian groups.
(i) For any finite chain complex C in $\mathbb{A}_*(X)$, which we also view as a (degreewise) induced chain complex C in $\mathbb{A}_*(X)$, the total complex $\mathrm{Tot}_*(C)$ is given by 4.6 to be

$$\mathrm{Hom}_{\mathbb{A}_*(X)}(\Delta(X)^{-*}, C)\ ,$$

so that the chain dual of C is given by

$$T_{\mathbb{A}_*(X)}(C)\ =\ \mathrm{Hom}_{\mathbb{A}}(\mathrm{Hom}_{\mathbb{A}_*(X)}(\Delta(X)^{-*}, C), \mathbb{Z})\ .$$

(ii) As in 3.10 regard the simplicial chain complex $\Delta(X')$ of the barycentric subdivision X' of a finite Δ-set X as a chain complex in $\mathbb{A}_*(X)$ or in $\mathbb{A}_*[X]$ with

$$\Delta(X')(x)\ =\ \Delta(x^\perp, \partial x^\perp)\ ,\ \Delta(X')[x]\ =\ \Delta(x^\perp)$$

for $x \in X$. The chain dual $T(\Delta(X'))$ is the chain complex in $\mathbb{A}_*(X)$ with

$$T(\Delta(X'))(x)\ =\ \Delta(x^\perp)^{|x|-*}\ (x\in X)\ .$$

□

Remark 5.9. See Fimmel [2] and Woolf [13] for Verdier duality for local coefficient systems on simplicial sets and simplicial complexes. In particular, [13] relates the chain duality of [6, Chapter 5] defined on $\mathrm{Proj}(R)_*(X)$ for a simplicial complex X to the Verdier duality for sheaves of R-module chain complexes over the polyhedron $\|X\|$. □

For any additive category with chain duality $\mathbb{A}$ let $\mathbb{L}_\bullet(\mathbb{A})$ be the quadratic L-theory Ω-spectrum defined in Ranicki [6], with homotopy groups

$$\pi_n(\mathbb{L}_\bullet(\mathbb{A})) \;=\; L_n(\mathbb{A}) \;.$$

It was shown in [6, Chapter 13] that the covariant functor

$$\{\text{simplicial complexes}\} \to \{\Omega\text{−spectra}\} \;;\; X \mapsto \mathbb{L}_\bullet(\mathbb{B}(\mathbb{A}_*(X)))$$

is an unreduced homology theory, i.e. a covariant functor which is homotopy invariant, excisive and sends arbitrary disjoint unions to wedges. More generally :

Proposition 5.10. ([6, 13.7] for simplicial complexes)
(i) *If $\mathbb{A}$ is an additive category with chain duality and X is a Δ-set then $\mathbb{A}_*(X)$ is an additive category with chain duality.*
(ii) *The functor*

$$\{\Delta\text{−sets}\} \to \{\Omega\text{−spectra}\} \;;\; X \mapsto L_*(\mathbb{A},X) \;=\; \mathbb{L}_\bullet(\mathbb{B}(\mathbb{A}_*(X)))$$

is an unreduced homology theory, that is $L_*(\mathbb{A},X) = H_*(X;\mathbb{L}_\bullet(A)$.
(iii) *Let R be a ring with involution, so that $\mathbb{A} = \mathrm{Proj}(R)$ is an additive category of f.g. projective R-modules with the duality involution. If X is a Δ-set and $p : \widetilde{X} \to X$ is a regular cover with group of covering translations π (e.g. the universal cover with $\pi = \pi_1(X)$) the assembly functor*

$$A \;:\; \mathbb{B}(R)_*(X) \to \mathbb{B}(R[\pi]) \;;\; C \mapsto C(\widetilde{X})$$
$$(C(\widetilde{X}) \;=\; \sum_{x\in\widetilde{X}} C(p(x)))$$

is a functor of additive categories with chain duality. The assembly maps A induced in the L-groups fit into an exact sequence

$$\cdots \longrightarrow H_n(X;\mathbb{L}_\bullet(R)) \xrightarrow{A} L_n(R[\pi_1(X)]) \longrightarrow \mathcal{S}_n(R,X) \longrightarrow H_{n-1}(X;\mathbb{L}_\bullet(R)) \longrightarrow \cdots$$

with $\mathcal{S}_n(R,X)$ the cobordism group of the $R[\pi_1(X)]$-contractible $(n-1)$-dimensional quadratic Poincaré complexes in $\mathbb{A}_*(X)$.

Proof. Exactly as for the simplicial complex case, but using the Δ-set duals instead of the dual cells! $\square$

Example 5.11. Let $X = S^1$ be the Δ-set of the circle (2.8, 3.7) with one 0-simplex and one 1-simplex. Given a ring with involution R let the Laurent polynomial extension ring $R[z, z^{-1}]$ have the involution $\overline{z} = z^{-1}$. An n-dimensional quadratic Poincaré complex in $\mathrm{Proj}(R)_*(S^1)$ is an n-dimensional fundamental quadratic Poincaré cobordism over R, with assembly the union n-dimensional quadratic Poincaré complex over $R[z, z^{-1}]$, and the assembly maps

$$A \ : \ H_n(S^1; \mathbb{L}_\bullet(R)) \ = \ L_n(R) \oplus L_{n-1}(R) \to L_n(R[z, z^{-1}])$$

are isomorphisms modulo the usual K-theoretic decorations (Ranicki [7, Chapter 24]. □

Remark 5.12. Proposition 5.10 has an evident analogue for the symmetric L-groups L^*. □

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