Induction, Coinduction, and Fixed Points in Programming Languages (PL) Type Theory

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Abstract

Recently we presented a concise survey of the formulation of the induction and coinduction principles, and some concepts related to them, in programming languages type theory and four other mathematical disciplines. The presentation in type theory involved the separate formulation of these concepts, first, in the theory of types of functional programming languages and, next, in the theory of types of object-oriented programming languages.

In this article we show that separating these two formulations helps demonstrate some of the fundamental differences between structural subtyping, predominant in functional programming languages, and nominal subtyping, predominant in object-oriented programming languages—including differences concerning type negation and concerning the existence of inductive types, of coinductive types, and of approximations thereof. In the article we also motivate mutual coinduction and mutual coinductive types, and their approximations, and we discuss in brief the potential relevance of these concepts to object-oriented programming (OOP) type theory.

1 Introduction

Applications of the induction and coinduction principles, and of concepts related to them, in scientific fields are plenty, including ones in economics and econometrics [35], in mathematical physics [38], in computer science, and in many other areas of mathematics itself. Their applications in computer science, in particular, include ones in programming language semantics—which we touch upon in this article—as well as in relational database theory (e.g., recursive or iterated joins) and in concurrency theory.¹

Fixed points, induction and coinduction have been formulated and studied in various subfields of mathematics, usually using different vocabulary in each field. The interested reader is invited to check our concise comparative survey in [8].

Of particular interest to programming languages researchers, in this article we give attention to the fundamental conceptual difference between *structural* type theory, where type equality and type inclusion are defined based on type structures but not on type names, and *nominal* type theory, where type equality and type inclusion are defined based on the names of types in addition to their structures. Given their difference in how each particularly defines the *inclusion* relation between types (*i.e.*, the subtyping relation), the conceptual difference between structural typing and nominal typing expresses itself, prominently, when fixed points and related concepts are formulated in each of structural type theory and nominal type theory.

As such, in this article we present the formulation of these concepts in the theory of types of functional programming languages (which is largely structurally-typed) in §2.1 (FP Type Theory), then we follow that by presenting their formulation in the type theory of object-oriented programming languages (which is largely nominally-typed) in §2.2 (OOP Type Theory).

In §7 of [8] we summarized the article by presenting tables that collect the formulations given in the article. Based on the tabular comparison in §7 of [8] and based on the discussions in §2

¹For more details on the use of induction, coinduction, and fixed points in computer science, and for concrete examples of how they are used, the reader is invited to check relevant literature on the topic, e.g., [26, 39, 15, 42, 32].

(and in §3 of [8]), we discuss in §3 (Struct. Type Theory vs. Nom. Type Theory) some of the fundamental differences between structural typing and nominal typing, particularly ones related to type negation and coinductive types, and we also discuss some consequences of these fundamental differences.

2 Programming Languages Theory

Given the 'types as sets' view of types in programming languages, this section builds on the *set*-theoretic presentation in §3 of [8] to present the induction and coinduction principles using the jargon of programming languages type theory. The presentation allows us to demonstrate and discuss the influence structural and nominal typing have on the theory of type systems of functional programming languages (which mostly use structural typing) and object-oriented programming languages (which mostly use nominal typing).

2.1 Inductive and Coinductive Functional Data Types

Formulation

Let \mathbb{D} be the set of *structural* types in functional programming.²

Let \subseteq ('is a subset/subtype of') denote the *structural subtyping/inclusion* relation between structural data types, and let : ('has type/is a member of/has structural property') denote the *structural typing* relation between structural data values and structural data types.

Now, if $F: \mathbb{D} \to \mathbb{D}$ is a polynomial (with powers) datatype constructor³, *i.e.*, if

$$\forall S, T \in \mathbb{D}.S \subseteq T \implies F(S) \subseteq F(T)$$
,

then an inductively-defined type/set μ_F , the smallest F-closed set, exists in \mathbb{D} , and μ_F is also the smallest fixed point of F, and a coinductively-defined type/set ν_F , the largest F-consistent set, exists in \mathbb{D} , and ν_F is also the largest fixed point of F.⁴

Further, for any type $P \in \mathbb{D}$ (where P, as a structural type, expresses a structural property of data values) we have:

• (structural induction, and recursion) $F(P) \subseteq P \implies \mu_F \subseteq P$

$$(i.e., \forall p : F(P).p : P \implies \forall p : \mu_F.p : P),$$

which, in words, means that if the (structural) property P is preserved by F (i.e., if P is F-closed), then all data values of the inductive type μ_F have property P (i.e., $\mu_F \subseteq P$). Furthermore, borrowing terminology from category theory (see §6 of [8]), a recursive function $f: \mu_F \to P$ that maps data values of the inductive type μ_F to data values of type P (i.e., having structural property P) is the unique catamorphism (also called a fold) from μ_F to P (where μ_F is viewed as an initial F-algebra and P as an F-algebra), and

• (structural coinduction, and corecursion) $P \subseteq F(P) \implies P \subseteq \nu_F$

$$(i.e., \forall p : P.p : F(P) \implies \forall p : P.p : \nu_F),$$

which, in words, means that if the (structural) property P is reflected by F (i.e., if P is F-consistent), then all data values that have property P are data values of the coinductive

 $^{^2}$ By construction/definition, the poset of structural types $\mathbb D$ under the inclusion/structural subtyping ordering relation is always a complete lattice. This point is discussed in more detail below.

³That is, F is one of the +, \times , or \rightarrow data type constructors (*i.e.*, the summation/disjoint-union/variant constructor, the product/record/labeled-product constructor, or the continuous-function/exponential/power constructor, respectively) or is a composition of these constructors. By their definitions in domain theory [43, 40, 27, 30, 12, 23, 19], these structural datatype constructors, and their compositions, are monotonic (also called *covariant*) datatype constructors (except for the first type argument of \rightarrow , for which \rightarrow is an *anti-monotonic/contravariant* constructor, but that otherwise "behaves nicely" [34]).

⁴See Table 1 of [8] for the definitions of μ_F and ν_F .

type ν_F (i.e., $P \subseteq \nu_F$).

Furthermore, borrowing terminology from category theory, a corecursive function $f: P \to \nu_F$ that maps data values of type P (i.e., having structural property P) to data values of the coinductive type ν_F is the unique anamorphism from P to ν_F (where P is viewed as an F-coalgebra and ν_F as a final F-coalgebra).

Notes

- To guarantee the existence of μ_F and ν_F in \mathbb{D} for all type constructors F, and hence to guarantee the ability to reason easily—i.e., inductively and coinductively—about functional programs, the domain \mathbb{D} of types in functional programming is *deliberately* constructed to be a complete lattice under the inclusion ordering. This is achieved by limiting the type constructors used in constructing \mathbb{D} and over \mathbb{D} to *structural* type constructors only (i.e., to the constructors +, \times , \to and their compositions, in addition to basic types such as Unit, Bool, Top, Nat and Int).
 - For example, the *inductive* type of lists of integers in functional programming is defined structurally (*i.e.*, using +, \times , and structural induction) as

$$L_i \simeq \mathtt{Unit} + \mathtt{Int} \times L_i,$$

which defines the type L_i as (isomorphic/equivalent to) the summation of type Unit (which provides the value unit as an encoding for the empty list) to the product of type Int with type L_i itself.

— In fact the three basic types Bool, Nat and Int can also be defined structurally. For example, in a functional program we may structurally define type Bool using the definition Bool ~ Unit+Unit (for false and true), structurally define type Nat using the definition Nat ~ Unit+Nat (for 0 and the successor of a natural number), and, out of other equally-valid choices, structurally define type Int using the definition Int ~ Nat+Unit+Nat (for negative integers, zero, and positive integers).

References

See [39, 22, 15, 47].

2.2 Object-Oriented Type Theory

The accurate and precise understanding of the generic subtyping relation in mainstream OOP languages such as Java, C#, C++, Kotlin and Scala, and the proper mathematical modeling of the OO subtyping relation in these languages, is one of our main research interests. Due to the existence of features such as wildcard types, type erasure, and bounded generic classes (where classes⁵ play the role of type constructors), the mathematical modeling of the generic subtyping relation in mainstream OOP languages is a hard problem that, in spite of much effort, seems to still have not been resolved, at least not completely nor satisfactorily, up to the present moment [45, 25, 6, 4, 9, 7].

The majority of mainstream OO programming languages are class-based, and subtyping (<:) is a fundamental relation in OO software development. In industrial-strength OOP, *i.e.*, in statically-typed class-based OO programming languages such as Java, C#, C++, Kotlin and Scala, class names are used as type names, since class names—which objects carry at runtime—are assumed to be associated with behavioral class contracts by developers of OO software. Hence, the decision of equality between types in these languages takes type names in consideration—hence, nominal typing. In agreement with the nominality of typing in these OO languages, the fundamental subtyping

⁵The notion of *class* in this article includes that of an abstract class, of an interface, and of an enum in Java [24]. It also includes similar "type-constructing" constructs in other nominally-typed OO languages, such as traits in Scala [36]. And a *generic class* is a class that takes a type parameter (An example is the generic interface List in Java—that models lists/sequences of items—whose type parameter specifies the type of items in a list).

relation in these languages is also a nominal relation. Accordingly, subtyping decisions in the type systems of these OO languages make use of the inherently-nominal inheritance declarations (i.e., that are explicitly declared between class names) in programs written using these languages.⁶ (For a more detailed overview of nominal typing versus structural typing in OOP see [1].)

Formulation

Let <: ('is a subtype of') denote the *nominal subtyping* relation between nominal data types (*i.e.*, class types), and let : ('has type') denote the *nominal typing* relation between nominal data values (*i.e.*, objects) and nominal data types.

Further, let \mathbb{T} be the set of nominal types in object-oriented programming, ordered by the nominal subtyping relation, and let $F: \mathbb{T} \to \mathbb{T}$ be a type constructor over \mathbb{T} (e.g., a generic class).

A type $P \in \mathbb{T}$ is called an 'F-supertype' if its F-image is a subtype of it, i.e., if

and P is said to be preserved by F. (An F-supertype is sometimes also called an F-closed type, F-lower bounded type, F-large type, inductive type, or algebraic type). The root or top of the subtyping hierarchy, if it exists (in \mathbb{T}), is usually called Object or All , and it is an F-supertype for all generic classes F. In fact the top type, when it exists, is the greatest F-supertype for all F.

A type $P \in \mathbb{T}$ is called an 'F-subtype' if it is a subtype of its F-image, i.e., if

$$P <: F(P)$$
,

and P is said to be reflected by F. (An F-subtype is sometimes also called an F-consistent type, F-(upper) bounded type⁸, F-small type, coinductive type, or coalgebraic type). The bottom of the subtyping hierarchy, if it exists (in \mathbb{T}), is usually called Null or Nothing, and it is an F-subtype for all generic classes F. In fact the bottom type, when it exists, is the least F-subtype for all F.

A type $P \in \mathbb{T}$ is called a fixed point (or 'fixed type') of F if it is equal to its F-image, i.e., if

$$P = F(P)$$
.

As such, a fixed point of F is simultaneously an F-supertype and an F-subtype. (Such fixed types/points are rare in OOP practice).

Now, if F is a covariant generic class (i.e., a types-generator) 9 , i.e., if

$$\forall S, T \in \mathbb{T}.S <: T \implies F(S) <: F(T),$$

and if μ_F , the 'least F-supertype' exists in \mathbb{T} , and μ_F is also the least fixed point of F, and if ν_F , the 'greatest F-subtype', exists in \mathbb{T} , and ν_F is also the greatest fixed point of F, then, for any type $P \in \mathbb{T}$ we have:

$$I_1 \sqsubseteq I_2 \implies F(I_1) <: F(I_2),$$

meaning that if interval type I_1 is a subinterval-of (or contained-in) interval type I_2 then the instantiation of generic class F with I_1 (i.e., the parameterized type $F(I_1)$, usually written as $F\langle I_1\rangle$) is always a subtype-of the instantiation of class F with I_2 (i.e., of the parameterized type $F(I_2)$, usually written as $F\langle I_2\rangle$). As such, a generic class in Java is not exactly an endofunction over types but is rather what may be called an "indirect endofunction," since it generates/constructs types not directly from types but from interval types that are themselves derived from types (see [9, 10]), and the generic class, as a function, is monotonic/covariant with respect to the containment relation over these interval types (i.e., it "generates a subtype when provided with a subinterval").

⁶Type/contract inheritance that we discuss in this article is the same thing as the inheritance of behavioral interfaces (APIs) from superclasses to their subclasses that mainstream OO software developers are familiar with. As is empirically familiar to OO developers, the subtyping relation in class-based OO programming languages is in one-to-one correspondence with API (and, thus, type/contract) inheritance from superclasses to their subclasses [46]. Formally, this correspondence is due to the nominality of the subtyping relation [2].

⁷Unlike poset \mathbb{D} in §2.1 (of structural types under the structural subtyping relation), poset \mathbb{T} (of nominal types under the nominal subtyping relation) is *not* guaranteed to be a complete lattice.

⁸From which comes the name *F*-bounded generics in object-oriented programming.

⁹Generic classes in Java are in fact always monotonic/covariant, not over types ordered by subtyping but over interval types ordered by containment. (See [10].) In particular, for any generic class F in Java we have

¹⁰See Table 2 of [8] for the definitions of μ_F and ν_F in the (rare) case when T happens to be a complete lattice.

• (induction) $F(P) <: P \implies \mu_F <: P$ (i.e., $\forall p : F(P) . p : P \implies \forall p : \mu_F . p : P$),

which, in words, means that if the contract (i.e., behavioral type) P is preserved by F (i.e., P is an F-supertype), then the inductive type μ_F is a subtype of P, and

• (coinduction) $P <: F(P) \implies P <: \nu_F$ (i.e., $\forall p : P.p : F(P) \implies \forall p : P.p : \nu_F$),

which, in words, means that if the contract (i.e., behavioral type) P is reflected by F (i.e., P is an F-subtype), then P is a subtype of the coinductive type ν_F .

Notes

- As discussed earlier and in §2.1, in structural type theory type expressions express only structural properties of data values, *i.e.*, how the data values of the type are structured and constructed. In nominal type theory type names are associated with formal or informal contracts, called *behavioral contracts*, which express behavioral properties of the data values (*e.g.*, objects) in addition to their structural properties.
 - To demonstrate, in a pure structural type system a record type that has, say, one member (e.g., type plane = { fly() }, type bird = { fly() } and type insect = { fly() }) is semantically equivalent to any other type that has the same member (i.e., type plane is equivalent to type bird and to type insect)—in other words, in a pure structural type system these types are 'interchangeable for all purposes'.
 - On the other hand, in a pure nominal type system any types that have the same structure but have different names (e.g., types plane, bird and insect) are considered distinct types that are not semantically equivalent, since their different names (e.g., 'plane' versus 'bird' versus 'insect') imply the possibility, even likelihood, that data values of each type maintain different behavioral contracts, and thus of the likelihood of different use considerations for the types and their data values.^{11,12}
- In industrial-strength OO programming languages (such as Java, C#, C++, Kotlin and Scala) where types are nominal types rather than structural ones and, accordingly, where subtyping is a nominal relation, rarely is poset \mathbb{T} a lattice under the subtyping relation <:, let alone a complete lattice. Further, many type constructors (i.e., generic classes) in these languages are not covariant. As such, μ_F and ν_F rarely exist in $\mathbb{T}^{.13}$ Still, the notion of a pre-fixed point (or of an F-algebra) of a generic class F and the notion of a post-fixed point (or of an F-coalgebra) of F, under the names F-supertype and F-subtype respectively, do have relevance in OO type theory, e.g., when discussing F-bounded generics [7].¹⁴

type money=float
type distance=float

does not help in a purely structural type system, however, since the types float, money, and distance are structurally equivalent. On the other hand, in a purely nominal type system the declarations of types money and distance do have the desired effect, since the non-equivalence of the types is implied by their different names.

¹²Further, when (1) the functional components of data values are (mutually) recursive, which is typical for methods of objects in OOP [5], and when (2) data values (i.e., objects) are autognostic data values (i.e., have a notion of self/this, which is an essential feature of mainstream OOP [20])—which are two features of OOP that necessitate recursive types—then the semantic differences between nominal typing and structural typing become even more prominent, since type names and their associated contracts gain more relevance as expressions of the richer recursive behavior of the more complex data values. (For more details, see [3] and [39, §19.3].)

¹³Annihilating the possibility of reasoning inductively or coinductively about nominal OO types.

 $^{^{11}}$ For another example, a float used for monetary values (e.g., in financial transactions) should normally not be confused with (i.e., equated to) a float used for measuring distances (e.g., in scientific applications). Declaring

 $^{^{14}}$ In fact, owing to our research interests (hinted at in the beginning of §2.2), inquiries in [7] have been a main initial motivation for writing this note/article. In particular, we have noted that if F is a generic class in a Java

- While not immediately obvious (nor widely-known), but, as a set of pairs of types, the subtyping relation in nominally-typed OOP is in fact a coinductive set, i.e., is a coinductively-defined subset of the set of all pairs of class types [39, Ch. 21]. That is because subtyping between two types holds in nominally-typed OOP as long as there is no (finite) reason for it not to hold. The following is in fact how javac, the standard Java compiler, type-checks Java programs: During type checking javac assumes that a subtyping relation between two given types holds unless the type checker can (finitely) prove, using the explicitly specified subtyping declarations in a Java program, that the relation cannot hold.¹⁵
 - In particular, to the best of our knowledge, the Java language specification does not stipulate that the subtyping relation holds between two types only if the relation can be (finitely) proven to hold between the two types (see [24, $\S4.10$]). This missing "disclaimer", which is usually stipulated in the definition of similar relations but that seems to be intentionally missing in the definition of the subtyping relation between class types (a.k.a., reference types) in Java, allows for the subtyping relation in Java to be coinductive.

References

See [18, 17, 16, 14, 22, 31, 45, 7].

3 Structural Type Theory versus Nominal Type Theory

3.1 Existence of Fixed Points

The discussion in §2.1, together with that in §§3 and 5 of [8], demonstrates that FP type theory, with its structural types and structural subtyping rules being motivated by mathematical reasoning about programs (using induction or coinduction), is closer in its flavor to set theory (and first-order logic/predicate calculus), since structural type theory assumes and requires the existence of fixed points μ_F and ν_F in $\mathbb D$ for all type constructors F. (For a discussion of the importance of structural typing in FP see [29, 33] and [39, §19.3].)

On the other hand, the discussion in §2.2, together with that in §§6 and 2 of [8], demonstrates that OOP type theory, with its nominal types and nominal subtyping being motivated by the association of nominal types with behavioral contracts, is closer in its flavor to category theory and order theory, since nominal type theory does *not* assume or require the existence of fixed points μ_F and ν_F in $\mathbb O$ for all type constructors F. (For a discussion of why nominal typing and nominal subtyping matter in OOP see [3] and [39, §19.3].)

As such, we conclude that the theory of data types of functional programming languages is more similar in its views and its flavor to the views and flavor of set theory and first-order logic, while the theory of data types of object-oriented programming languages is more similar in its views and its flavor to those of category theory and order theory. This conclusion adds further supporting

program then a role similar to the role played by the coinductive type ν_F is played by the wildcard type F<?>, since, by the subtyping rules of Java (discussed in [7], and illustrated vividly in earlier publications such as [9, 6]), every F-subtype (i.e., every parameterized type constructed using F—called an instantiation of F—and every subtype thereof) is a subtype of the type F<?>. On the other hand, in Java there is not a non-Null type (not even type F<Null>; see [7]) that plays a role similar to the role played above by the inductive type μ_F (i.e., a type that is a subtype of all F-supertypes, which are all instantiations of F and all supertypes thereof). This means that in Java greatest post-fixed points (i.e., greatest F-subtypes) that are not greatest fixed points do exist, while non-bottom least pre-fixed points (i.e., least F-supertypes) do not exist. Also, since $\mathbb T$ is rarely a complete lattice, greatest fixed points, generally-speaking, do not exist in Java, neither do least fixed points. These same observations apply more-or-less to other nominally-typed OOP languages similar to Java, such as C#, C++, Kotlin and Scala. (See further discussion in Footnote 18 in \S of [8].)

¹⁵The subtyping relation in Java seems to be even a little bit more complex than a coinductive set. Due to the existence of wildcard/interval types, the Java subtyping relation, together with the containment relation between wildcard/interval types [10], seems to be an instance of a mutually coinductive set (i.e., the coinductive counterpart of a mutually inductive set, which we did not get to discuss in this article but may do in the future.)

evidence to our speculation (e.g., in [6, 4]) that category theory is more suited than set theory for the accurate understanding of mainstream object-oriented type systems.

3.2 Type Negation, Coinductive Nominal Types, and Free Types

In §3 of [11] we noted that negation of a mathematical object (e.g., a logical statement, a set, or a type) is useful in defining coinductive objects. We also noted that negation in set theory amounts to set complementation, which is useful in defining coinductive sets in terms of inductive sets. We wondered also about defining negation in other categories, such as the category of structural types (ordered by structural subtyping) in functional programming languages and the category of nominal types (ordered by nominal subtyping) in object-oriented programming languages. We resume this discussion here.

In object-oriented programming for example (see $\S 2.2$), we may try to define negation of a class type as follows. First, let's define an "implication" type (which may also be called an 'exponential type') from type x to type y as

$$(x \Rightarrow y) \doteq \bigvee_{x \land a <: y} a \tag{1}$$

i.e., as the join of all types whose meet with type x is a subtype of type y, ¹⁶ then (as done in functional programming) let's define negation of a type x as

$$\neg x \doteq (x \Rightarrow \bot)$$

which in other words means defining type negation as

$$\neg x \doteq \bigvee_{x \land a = \bot} a ,$$

i.e., defining the negation of a type as the join of all types parallel to the type, ¹⁷ or more precisely as the join ("lub") of those types whose meet with the type is the bottom type.

As such, noting (see §2.2) that \bot actually denotes type Null (sometimes also called Nothing or Void) and \top denotes type Object (or All) then the negation of \top is \bot (since \bot is the *only* type whose meet with \top is equal to \bot), and the negation of \bot is \top (since the meet of \bot and *every* type, including \top , is \bot).

However, due to the general lack of (true) union types in most OOP languages¹⁸, for most types (i.e., ones other than \top and \bot) the negation of a nominal type x will usually be type \top . This makes the negation of class types, if defined as suggested above, not quite interesting or useful. (Also, as observed in §2.2, the subtyping relation in mainstream OOP languages rarely has fixed points of type generators/constructors).

As such, in set theory the complement $\neg x$ of a set x (as a negation of x) can also be defined as

$$\neg x \doteq (x \Rightarrow \phi) \doteq \bigcup_{x \cap a = \phi} a$$

i.e., as the union of all subsets of U that are disjoint with respect to x. This definition, although it does not (explicitly) mention U, is equivalent to the standard definition of set complementation, namely the definition

$$\neg x \doteq U - x \doteq U \backslash x.$$

¹⁶Formula (1) comes from the study of Heyting algebras (which model constructive/intuitionistic logic). The formula is valid for defining complementation in set theory (where join \vee is interpreted as set union \cup , meet \wedge is interpreted as set intersection \cap , ≤ is interpreted as set inclusion ⊆, and ⊥ is interpreted as the empty set ϕ), since the inclusion lattice in set theory is a Boolean algebra and every Boolean algebra is a Heyting algebra.

¹⁷The term parallel here is used in an order-theoretic sense. In an ordered set (P, \leq) two elements $a, b \in P$ are parallel if neither $a \leq b$ holds nor $b \leq a$ holds (i.e., a and b are "independent elements" of P). This is usually denoted by writing $a \parallel b$ [21, 28, 41]. (Although not widely-known, but LATEX has a command \parallel for inputting the symbol \parallel).

¹⁸In fact the lub() function—defined in the Java language specification—does not compute the least upper bound of a set of types (which itself is an approximation of the true union of the set of types) but it 'only approximates a least upper bound' [24, §4.10.4].

An alternative way to defining the negation of a type in OOP in a more genuine object-oriented way (*i.e.*, not via defining implication/exponential types) is to allow OO developers to choose which types they wish to negate, rather than trying to define a negation "automatically" for every type. An OO developer may define a "negation" of a type (in a genuine OO way, if they wished to have such a negative type) by them simply extending the negative type from a supertype of the negated type (or from one of its supertypes, depending on which part of the inherited behavioral contract is being negated).

For example, say we have class Window that is extended by class ColoredWindow. An OO developer can define the negation of ColoredWindow by declaring a class NonColoredWindow that extends class Window. (As is standard in nominal typing, it is up to the developer to ensure that a NonColoredWindow is indeed not colored, and in fact also that a ColoredWindow is indeed one.)

Negative types, according to this genuine nominal/OO way of defining them, are totally under the control of software developers, unlike the case in structural/FP typing. It may be noted also that type negation in a genuine OO way is not exclusive. For example, generally-speaking there is nothing that prevents the OO developer from declaring yet a third subclass of class Window that is not a ColoredWindow yet is also not a NonColoredWindow. (That third class may or may not be useful. The option of declaring it, or not, is available to the developer however. In other words, it is the developer, not the language, who makes the decision as to defining the class or not, based on his or her need for the class.)

The more genuinely OO way of defining negative types presented above, arguably, is better than the FP/structural way (via $T \Rightarrow \bot$), since it gives more control and offers more flexibility to developers. It should be noted, though, that there is a similarity between both ways: they both define type negation by depending (directly, in OOP, and indirectly, in FP) on 'parallel types' (see Footnote 17 on the preceding page).

Having considered negative nominal types, now what about *coinductive nominal types*? Understanding these types (as counterparts of coinductive structural types in structural type theory and of coinductive sets in set theory) was the initial motivation for us considering type negation in the first place (see §3 of [11]). The starting point for considering such (strange-named, but familiar we assert) types will be Equation (1) of [11], which, let's recall, is the equation that defines coinductive sets (in terms of inductive ones and negation) as

$$\nu_F = \neg \mu_{\neg F \neg}.$$

Equation (1) uses three negations, and we have just defined an intuitive notion of type negation for nominal types. Our view of coinductive nominal types will be a combination of both. In particular, we take each negation in Equation (1) separately to define a negative nominal type (that may or may not exist). As such, for a particular generic class F (i.e., F is a nominal type generator) we define its corresponding coinductive nominal type ν_F (if it exists) as the negative of (an approximation of, since nominal types do not have fixed points) the inductive type $\mu_{F^{\delta}}$ of a "dual" generic class F^{δ} (which, also, may or may not exist) that is applied to negations of types (if all these negations exist) that can be passed to the generic class F as type arguments (!!).

Sounds too complex? (Due to the three negations?) Rest assured, in conclusion, and to make a long story short, intuitively-speaking the sought after *coinductive nominal type* ν_F is roughly (*i.e.*, is approximated by) the familiar type $F\langle?\rangle$ —an observation we earlier made (in Footnote 14 on page 5) without any discussion of type negation. In other words, we have

$$\nu_F \approx F \langle ? \rangle$$

(again due to the absence of fixed points of generic classes in nominal type theory, a true coinductive nominal type in fact never exists). The type $F\langle?\rangle$ is also called the 'free type' corresponding to the generic class F.¹⁹

¹⁹The name 'free type' comes from category theory. The concept of a free type corresponding to a generic class is similar (in a precise category theoretic sense) to the *free monoid* corresponding to a set and to the *free category* (a *quiver*) corresponding to a graph [44]. The free type corresponding to a generic class is defined as the parameterized type formed by instantiating the generic class with the wildcard type?. For example, the free type corresponding to

In light of the discussion (in §3 of [11]) of the intuitions behind coinductive sets ("good/constructible/consistent vs. bad/inconstructible/inconsistent", and finitely-constructible vs. infinitely-constructible) and that coinductive sets contain all elements (all data values, in case of coinductive structural data types) that can be constructed using a generator (i.e., ones that do not "break the rules" of the generator/constructor), it is not surprising that, for a generic class F, the free type $F\langle?\rangle$ is the best approximation of the coinductive nominal type ν_F . That is because the free type $F\langle?\rangle$ is intuitively understood (e.g., by OO software developers and OO language designers) as the type that contains all objects that can be instances of (recall Java's instanceof operator) the generic class F (i.e., contains all objects that can be constructed using the generic class F). Further, in light of OOP not having a goal of inductive logical reasoning (i.e., finite structural reasoning) about objects, it is not surprising that the inductive nominal type μ_F is not supported (nor even an approximation for it) in most OOP languages. If defined, such a type or its approximation will contain only objects (in the OO sense, not the mathematical/category theoretic one) that can be finitely constructed using F—which is not of much value (actually, makes little sense) to most OO software developers²⁰.

3.3 Structural Induction, Nominal Coinduction, and The Future

The discussion regarding coinductive nominal types and their approximations, combined with our observation regarding the coinductiveness of the subtyping relation (as a set of pairs) in nominal type systems (see notes of §2.2), make us conclude that fundamental differences exist between structural type theory (in FP) and nominal type theory (in OOP) regarding, first, the existence (in FP) versus nonexistence (in OOP) of fixed points (and pre-/post-fixed points) of type constructors (in FP) and generic classes (in OOP) and, second, regarding the dominance of (singular) induction and inductive definitions (in FP) versus the dominance of (mutual) coinduction and coinductive definitions (in OOP).

In light of the existence of circularity and recursive definitions in OOP at multiple levels, *i.e.*, at the level of values (e.g., via this) and at the level of types (e.g., between class definitions, and noting the circular dependency of parameterized types on interval types and vice versa), and in light of other work also discussing coinductive definitions and the use of coinduction in OOP and in nominal type systems (e.g., see [39, p.312] and [13, 45]), we observe that:

"Induction reigns supreme" in functional programming, while in object-oriented programming "coinduction reigns supreme".

This observation should not be viewed as a mathematical or academic statement but rather as an expression of practically-motivated fundamental features of OOP and FP. The main practical value of functional programming and the main motivation behind its very existence and behind its continued usage is to enable precise mathematical reasoning about software. The main practical value of object-oriented programming and the main motivation behind its existence and behind its continued usage is the accurate and intuitive modeling of typically highly-interconnected real or imaginary parts of our world. We, humans, know well how to reason, mathematically, about objects built up from few basic ones (*i.e.*, inductively-defined objects), yet we also need software to accurately simulate and model parts of our interrelated world—one in which, to overcome circularity and interrelatedness, we are used to holding statements about objects as facts as long as these statements cannot be disproven (*i.e.*, is an inherently coinductive world).

As such, the co-existence of FP (with its mainly inductive type systems) and OOP (with its mainly coinductive type systems), and our continued need for both, is a reflection of our reality,

generic class List is the type List<?>. Free types and (Java) type erasure both form an adjunction—they are a pair of adjoint functors between 'classes & subclassing' and 'types & subtyping'. (In order theory, an adjunction is also called a Galois connection. See [4] for more details on this adjunction.) Based on the discussion in the main text, free types and coinductive nominal types agree, i.e., are the same concept. More precisely, free types are the best approximation of (true) coinductive nominal types. In category-theoretic terms free types are, precisely, instances of final coalgebras. (See Table 2 of [8].)

²⁰That is, to developers and designers of industrial-strength and main stream OO software, who are content as long as a type that contains all objects that can be constructed using F (regardless of whether they can be constructed "finitely or infinitely")—namely the free type F $\langle ? \rangle$ —is defined.

i.e., of what we currently know how to do and of what we currently need to do. It is our opinion that this current state of affairs is not a perpetual one, but is one that particularly invites for the further development of the mathematical methods used to reason about mutually coinductive definitions (and, more generally and more precisely, about mutually-defined post-fixed points) to reach or exceed the same level of mathematical maturity of the mathematical methods and tools used to reason about singular inductive definitions (i.e., about least fixed points). In summary, the current state of affairs invites PL theorists and researchers to make clear and transparent what is currently totally opaque, ²¹ thereby elegantly combining the accurate mathematical-reasoning benefits of functional programming with the accurate world-modeling benefits of object-oriented programming.

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²¹See, for example, the "totally opaque" code on p.58 of [37]—a standard reference on practical functional programming—which nevertheless has a strong *object-oriented* flavor!

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