Lemma (Lemma 2.1.2)**.** For every type A and every $x, y, z : A$ there is a function

$$(x = y) \longrightarrow (y = z) \longrightarrow (x = z)$$

Proof 2: Induction on p. Define

$$D : \prod_{(x,y:A)} (x = y) \to U$$
$$D(x, y, p) :\equiv \prod_{(z:A)} \prod_{(q:y=z)} (x = z)$$

Then

$$D(x, x, \mathtt{refl}_x) \equiv \prod_{(z:A)} \prod_{(q:x=z)} (x = z)$$

Thus, we have

$$d :\equiv \lambda x.\lambda z.\lambda q.q : \prod_{(x:A)} D(x, x, \mathtt{refl}_x)$$

So by induction, we have a function

$$f : \prod_{(x,y:A)} \prod_{(p:x=y)} D(x, y, p) \equiv \prod_{(x,y:A)} \prod_{(p:x=y)} \prod_{(z:A)} \prod_{(q:y=z)} (x = z)$$

such that

$$f(x, x, \mathtt{refl}_x) \equiv \lambda z.\lambda q.q$$

In summary, we make the following judgement:

$$\mathtt{ind}_{=_A}\left(\prod_{(z:A)} \prod_{(q:y=z)} (x = z), \lambda x.\lambda z.\lambda q.q\right)$$
$$: \prod_{(x,y:A)} \prod_{(p:x=y)} \prod_{(z:A)} \prod_{(q:y=z)} (x = z)$$

□

Proof 3: Induction on q. Define

$$D : \prod_{(y,z:A)} (y = z) \to U$$
$$D(y, z, p) :\equiv \prod_{(x:A)} \prod_{(q:x=y)} (x = z)$$

Then
$$D(y, y, \mathtt{refl}_y) \equiv \prod_{(x:A)} \prod_{(q:x=y)} (x = y)$$
Thus, we have
$$d :\equiv \lambda y.\lambda x.\lambda q.q : \prod_{(y:A)} D(y, y, \mathtt{refl}_y)$$
So by induction, we have a function
$$f : \prod_{(y,z:A)} \prod_{(p:y=z)} D(y, z, p) \equiv \prod_{(y,z:A)} \prod_{(p:y=z)} \prod_{(x:A)} \prod_{(q:x=y)} (x = z)$$
such that
$$f(y, y, \mathtt{refl}_y) \equiv \lambda x.\lambda q.q$$
In summary, we make the following judgement:
$$\mathtt{ind}_{=_A}\left(\prod_{(x:A)} \prod_{(q:x=y)} (x = y), y.\lambda x.\lambda q.q\right) : \prod_{(y,z:A)} \prod_{(p:y=z)} \prod_{(x:A)} \prod_{(q:x=y)} (x = z)$$
□

The first proof, namely the one involving induction on both p and q appears in the text.