SOME REMARKS ON MULTICATEGORIES AND ADDITIVE CATEGORIES

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ABSTRACT.

Categories are coreflectively embedded in multicategories via the "discrete cocone" construction, the right adjoint being given by the monoid construction. Furthermore, the adjunction lifts to the "cartesian level": preadditive categories are coreflectively embedded (as theories for many-sorted modules) in cartesian multicategories (general algebraic theories). In particular, one gets a direct link between two ways of considering modules over a rig, namely as additive functors valued in commutative monoids or as models of the theory generated by the rig itself.

1. Introduction

The present work develops around the idea that a (semi)module over a rig \mathcal{C} is a model of the theory whose operations are finite sequences of arrows in \mathcal{C} , giving linear combinations:

$$\langle \lambda_1, \dots, \lambda_n \rangle : \langle a_1, \dots, a_n \rangle \mapsto \lambda_1 a_1 + \dots + \lambda_n a_n \qquad ; \qquad \langle \rangle : \langle \rangle \mapsto 0$$
 (1)

(Recall that a rig, a ring without negatives, is a monoid enriched in commutative monoids.) Indeed, the rig \mathcal{C} generates a one-object cartesian multicategory \mathcal{C}_{\bullet} such that modules over \mathcal{C} are morphisms of cartesian multicategories $\mathcal{C}_{\bullet} \to \mathbf{Set}$, as we are going to illustrate below in this introduction.

The relation between monoidal categories and multicategories is well-known (see for instance [Lambek, 1989], [Hermida, 2000] and [Leinster, 2003]); in particular there is a functor $(-)_{\otimes}: \mathbf{Cat}^{\otimes} \to \mathbf{Mlt}$ from the category of monoidal categories and lax monoidal functors to the category of multicategories and (multi)functors, with

$$\mathcal{M}_{\otimes}(X_1, \dots, X_n; X) \cong \mathcal{M}(X_1 \otimes \dots \otimes X_n; X)$$
 (2)

that restricts to an equivalence $\mathbf{Cat}^{\otimes} \simeq \mathbf{Rep}$, where $\mathbf{Rep} \subset \mathbf{Mlt}$ is the full subcategory of representable multicategories.

Less known is the important fact that also categories (with functors) are embedded in Mlt via the discrete cocone functor $(-)_{\bullet}$: Cat \rightarrow Mlt:

$$\mathcal{C}_{\bullet}(X_1,\dots,X_n;X) \cong \mathcal{C}(X_1;X) \times \dots \times \mathcal{C}(X_n;X)$$

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So *n*-arrows in \mathcal{C}_{\bullet} are sequences $\langle \lambda_1, \dots, \lambda_n \rangle$ of concurrent arrows in \mathcal{C} with the obvious composition, e.g.

$$\langle \lambda, \mu, \nu \rangle (\langle \alpha, \beta \rangle, \langle \gamma, \langle \gamma \rangle) = \langle \lambda \alpha, \lambda \beta, \nu \gamma \rangle$$

and for any functor $F: \mathcal{C} \to \mathcal{D}$

$$F_{\blacktriangleright}\langle\lambda_1,\cdots,\lambda_n\rangle = \langle F\lambda_1,\cdots,F\lambda_n\rangle$$

This functor (whose existence I learned from [Hermida, 2000]; see also [Leinster, 2003]) is pivotal in our work, although it does not seem to have received much attention in the literature. We will denote by $\mathbf{Cat}_{\bullet} \subset \mathbf{Mlt}$ the full subcategory of those multicategories isomorphic to \mathcal{C}_{\bullet} for some category \mathcal{C} , so that there is an equivalence $\mathbf{Cat}_{\bullet} \simeq \mathbf{Cat}$.

Note that $\mathbf{Sum} := \mathbf{Cat}_{\bullet} \cap \mathbf{Rep} \subset \mathbf{Mlt}$ is equivalent to the category of categories with finite coproducts (and all functors); indeed, \mathcal{C}_{\bullet} is representable iff \mathcal{C} has finite coproducts. Note also that multicategories of the form \mathcal{M}_{\otimes} and \mathcal{C}_{\bullet} give two main instances of promonoidal categories [Day et al., 2005].

The notion of cartesian multicategory we refer to is that in [T. Fiore, 2005], which actually does not even mention explicitly multicategories (see however [Gould, 2008]). It is essentially a multicategory \mathcal{M} with an action of mappings (over obj \mathcal{M}) $\mathbf{n} = \{1, \dots, n\} \rightarrow \mathbf{m} = \{1, \dots, m\}$ on its hom-sets which is compatible with composition (see Section 3). We denote by **fpMlt** the category of cartesian multicategories and their morphisms, with the obvious underlying functor to (plain) multicategories $U: \mathbf{fpMlt} \rightarrow \mathbf{Mlt}$. We will sometimes refer to cartesian multicategories and their morphisms as multicategories with an "fp-structure" and "fp-functors" respectively.

So as monoidal categories and categories give two main instances of multicategories, finite product categories and preadditive categories (that is enriched in commutative monoids) give rise, for somewhat opposite reasons that we will illustrate presently, to two main instances of cartesian multicategories. More is true: the full subcategory $\mathbf{fpRep} \subset \mathbf{fpMlt}$ above (with respect to U) \mathbf{Rep} is equivalent to the category of finite product categories and finite product preserving functors while the full subcategory $\mathbf{fpCat}_{\bullet} \subset \mathbf{fpMlt}$ above \mathbf{Cat}_{\bullet} is equivalent to the category of preadditive categories and additive functors (propositions 3.6 and 3.8). Then $\mathbf{fpSum} = \mathbf{fpCat}_{\bullet} \cap \mathbf{fpRep} \subset \mathbf{fpMlt}$ is equivalent to the category of additive categories; see the figure on page 14.

Let us come back to our one-object case. If \mathcal{C} is a rig, the operad \mathcal{C}_{\bullet} has the following fp-structure: bijections $\mathbf{n} \to \mathbf{n}$ act obviously on sequences $\langle \lambda_1, \dots, \lambda_n \rangle$, while the effect of monotone mappings is to sum pairs or sequences in \mathcal{C} and to insert zeros. For instance, if

$$p: \{1, 2, 3\} \to \{1, 2, 3\}$$
 ; $1 \mapsto 1, 2 \mapsto 1, 3 \mapsto 3$ (3)

then

$$p(\alpha, \beta, \gamma) = (\alpha + \beta, 0, \gamma)$$

On the other hand, the finite product category \mathbf{Set} gives rise to the cartesian multicategory \mathbf{Set}_{\times} with

$$\mathbf{Set}_{\times}(X_1, \dots, X_n; X) = \mathbf{Set}(X_1 \times \dots \times X_n; X)$$

and with actions of mappings $\mathbf{n} \to \mathbf{m}$ given by composition with diagonals and projections; for instance for $p: \{1,2,3\} \to \{1,2,3\}$ as in (3) and for $f: X \times X \times X \to Y$ in **Set** we have

$$pf(x,y,z) = f(x,x,z)$$

Now, to give a morphism $\overline{(-)}: \mathcal{C}_{\bullet} \to \mathbf{Set}_{\times}$ in **fpMlt** amounts to

1. giving a set X with a commutative monoid structure; indeed, if we define

$$\mathbf{m}_n = \overline{\langle 1, \dots, 1 \rangle}$$

with n 1's, then $m_2: X \times X \to X$ and $m_0: 1 \to X$ clearly define a monoid (since $\overline{(-)}$ is a morphism of multicategories) which is commutative (since it also preserves the fp-structure relative to bijections);

2. giving (functorially) a monoid endomorphism $\overline{\lambda}: X \to X$ for any $\lambda \in \mathcal{C}$; this follows from the equalities in $\mathcal{C}_{\blacktriangleright}$

$$\langle 1, 1 \rangle (\langle \lambda \rangle, \langle \lambda \rangle) = \langle \lambda \rangle (\langle 1, 1 \rangle)$$
 ; $\langle \lambda \rangle = \langle \lambda \rangle (\langle \lambda \rangle)$

which become in **Set** the monoid morphism conditions

$$m_2(\overline{\lambda}, \overline{\lambda}) = \overline{\lambda} m_2$$
 ; $m_0 = \overline{\lambda} m_0$

Note that from the equality $\langle \alpha, \beta \rangle = \langle 1, 1 \rangle (\langle \alpha \rangle, \langle \beta \rangle)$ in \mathcal{C}_{\bullet} one gets, using the sum notation for the monoid structure on X,

$$\overline{\langle \alpha, \beta \rangle}(x, y) = \overline{\alpha}x + \overline{\beta}y$$
 ; $\overline{\langle \rangle} = 0$

that is the arrows in \mathcal{C}_{\bullet} become indeed "linear combinations" in X;

3. such that the fp-structure is preserved also for the non-bijective mappings; for instance, if p is as in (3)

$$p\overline{\langle \alpha, \beta, \gamma \rangle} = \overline{p\langle \alpha, \beta, \gamma \rangle} = \overline{\langle \alpha + \beta, 0, \gamma \rangle}$$

that is

$$\overline{\alpha}x + \overline{\beta}x + \overline{\gamma}z = \overline{(\alpha + \beta)}x + \overline{0}y + \overline{\gamma}z$$

Summarizing, to give a morphism $\overline{(-)}: \mathcal{C}_{\bullet} \to \mathbf{Set}_{\times}$ in **fpMlt** amounts to give a commutative monoid (X, +, 0) in **Set** which has a module structure on \mathcal{C} :

$$\overline{1}x = x$$
 ; $\overline{\alpha}\beta x = \overline{\alpha}(\overline{\beta}x)$

$$\overline{\alpha}0 = 0$$
 ; $\overline{\alpha}(x+y) = \overline{\alpha}x + \overline{\alpha}y$

$$\overline{0}x = 0$$
 ; $\overline{(\alpha + \beta)}x = \overline{\alpha}x + \overline{\beta}x$

Thus, for a rig \mathcal{C} , we have a correspondence

$$\frac{\text{fpMlt}(\mathcal{C}_{\bullet}, \text{Set})}{\mathcal{C} - \text{Mod}}$$

$$\frac{\text{cMonCat}(\mathcal{C}, \text{cMon}(\text{Set}))}{\text{cMonCat}(\mathcal{C}, \text{cMon}(\text{Set}))}$$
(4)

which is in fact just an instance of an adjunction (see Proposition 3.5):

$$(-)$$
, \dashv cMon $(-)$: fpMlt \rightarrow cMonCat (5)

1.1. Summary. In Section 2 we present the tensored structure of the 2-category Mlt of multicategories, (multi)functors and natural transformations, which implies the "plain version" of the adjunction (5):

$$(-)$$
 \vdash \dashv $Mon(-): Mlt \to Cat$ (6)

In Section 3 we move to the "cartesian level", discussing the relationships between cartesian multicategories, (pre)additive categories and finite product categories.

In Section 4 we briefly consider the results from a fibrational point of view: from (6) we get in particular that (strong) indexed monoidal categories correspond to functors $C_{\bullet} \to \mathbf{Cat}_{\times}$ in \mathbf{Mlt} , and thus (ignoring pseudo-issues for simplicity) also to multicategory fibrations (in the sense of [Hermida, 2004]) over C_{\bullet} :

$$\frac{\operatorname{Cat}(\mathcal{C}, \operatorname{Mon}(\operatorname{Cat}_{\times}))}{\operatorname{Mlt}(\mathcal{C}_{\blacktriangleright}, \operatorname{Cat}_{\times})} \tag{7}$$

$$\frac{\operatorname{mltfb}/\mathcal{C}_{\blacktriangleright}}{}$$

We conclude by presenting some elementary examples of multicategories arising in this way as the domain of a fibration over \mathcal{C}_{\bullet} for an indexed monoidal category $\mathcal{C} \to \mathbf{Mon}(\mathbf{Cat})$.

The aim of the present paper is mainly expository, so that some proofs are omitted or just sketched; for a specific but not central fact (Proposition 3.8) I have to refer to "folklore". A more detailed account will follow.

1.2. Terminology and notations. Following other authors, we omit the prefix "semi" which is sometimes used to refer to commutative monoids (rather than abelian groups) enrichments. Thus, "preadditive" categories and "additive" functors are **cMon**enriched categories and functors, while "additive" categories are those preadditive categories with biproducts (again with additive functors, that necessarily preserve biproducts).

As already seen, we use the following notation which may be a little confusing: \mathbf{fpCat}_{\bullet} is the category of those cartesian multicategories whose underlying multicategory has the form \mathcal{C}_{\bullet} (with all fp-functors). It is equivalent to the category of preadditive categories. On the other hand, \mathbf{fpRep} is the category of representable cartesian multicategories and is

equivalent to \mathbf{Cat}^{\times} , the category of finite product categories and finite product preserving functors. In fact, it is this equivalence that suggests to extend the "fp" prefix (for "finite product") to all the "cartesian level". A similar notation is used in [Gould, 2008].

So the prefix "fp" (and the term "cartesian" itself) refer to multicategories rather than to categories, allowing us to emphasize the correspondence between the cartesian and the plain levels (see the figure on page 14).

2. The plain level

We assume that the reader is familiar with the 2-category **Mlt** of multicategories, functors and natural transformations; [Leinster, 2003] is a good reference.

The discrete cocone functor $(-)_{\bullet}$: Cat \rightarrow Mlt mentioned in the introduction is in fact a full and faithful 2-functor. For fullness, let $F: \mathcal{C}_{\bullet} \rightarrow \mathcal{D}_{\bullet}$ be a functor in Mlt, $(\lambda, \mu) \in \mathcal{C}_{\bullet}(X, Y; Z)$, and $F(\lambda, \mu) = (\lambda', \mu') \in \mathcal{D}_{\bullet}(FX, FY; FZ)$; then

$$\lambda' = F(\lambda, \mu)(\mathrm{id}_{FX}, \langle \rangle) = F(\lambda, \mu)(F\mathrm{id}_X, F(\gamma)) = F(\langle \lambda, \mu \rangle(\mathrm{id}_X, \langle \gamma \rangle)) = F\lambda$$

so that in fact $F = (\underline{F})_{\bullet}$, where $\underline{F} : \mathcal{C} \to \mathcal{D}$ is the "underlying" functor.

It is also easy to see that $(-)_{\bullet}$ preserves products. In particular, 1_{\bullet} is terminal in Mlt. The next proposition says that the 2-category Mlt is tensored and that the tensor of $\mathcal{M} \in \mathbf{Mlt}$ by $\mathcal{C} \in \mathbf{Cat}$ is provided by the cartesian product in Mlt of \mathcal{M} and \mathcal{C}_{\bullet} :

2.1. Proposition. There are natural isomorphisms

$$\frac{\mathbf{Cat}(\mathcal{C}, \mathbf{Mlt}(\mathcal{M}, \mathcal{N}))}{\mathbf{Mlt}(\mathcal{C}_{\bullet} \times \mathcal{M}, \mathcal{N})}$$
(8)

Proof. Straightforward calculations.

Thus, for any $\mathcal{M} \in \mathbf{Mlt}$ we have an adjunction

$$(-) \times \mathcal{M} - Mlt(\mathcal{M}, -) : Mlt \rightarrow Cat$$
 (9)

The most important particular case is obtained when $\mathcal{M} = 1_{\bullet}$; recall that, for $\mathcal{N} \in \mathbf{Mlt}$, the category $\mathbf{Mlt}(1_{\bullet}, \mathcal{N})$ can be identified as the category $\mathbf{Mon}(\mathcal{N})$ of monoids in \mathcal{N} :

2.2. Corollary. The monoid functor is right adjoint to the discrete cocone functor:

$$(-)$$
, $\dashv Mon(-): Mlt \rightarrow Cat$ (10)

- 2.3. Remark. Then we could say that C_{\bullet} is the universal multicategory containing a C-indexed monoid.
- 2.4. REMARK. The counit of the adjunction takes an arrow $\langle \lambda_1, \dots, \lambda_n \rangle : \langle X_1, \dots, X_n \rangle \to X$ in $\mathbf{Mon}(\mathcal{M})_{\bullet}$ to $\mathbf{m}_n^X(\lambda_1, \dots, \lambda_n)$ in \mathcal{M} .
- 2.5. REMARK. Since $(-)_{\blacktriangleright}$ is fully faithful, $\mathbf{Mon}(\mathcal{C}_{\blacktriangleright}) \cong \mathcal{C}$. When \mathcal{C} has finite sums we find again the classical fact that in the monoidal category $(\mathcal{C}, +, 0)$ every object carries a unique monoid structure. The converse also holds; see for instance [Heunen, 2008].
- 2.6. COROLLARY. Cat, is coreflective in Mlt.
- 2.7. Remark. This is related to an old result in [Fox, 1976]: the commutative monoid construction provides a universal way to endowe a symmetric monoidal category with finite sums. Indeed, an intermediate level between cartesian and plain multicategories is that of symmetric multicategories (see [Leinster, 2003], [Lambek, 1989] and [Gould, 2008]): multicategories in **sMlt** are defined as cartesian multicategories except that only bijective mappings are supposed to act on the hom-sets. Any \mathcal{C}_{\bullet} has an ovious s-structure and it is easy to see that there is a symmetric version of (10):

$$(-)$$
, \dashv cMon $(-)$: sMlt \rightarrow Cat (11)

so that Cat, is coreflective in sMlt. Thus, Fox's result can be seen as stating that the latter restricts to representable symmetric multicategories: finite sums categories are coreflective in symmetric monoidal categories.

Another particular case of (9) is obtained when $\mathcal{M} = 1$, the multicategory with just an arrow; it is immediate to observe that $\mathbf{Mlt}(1,\mathcal{N})$ is the "underlying" category $\underline{\mathcal{N}}$ of \mathcal{N} , with $\underline{\mathcal{N}}(X;Y) = \mathcal{N}(X;Y)$, while $\mathcal{C}_{\bullet} \times 1$ is the "linear" multicategory $\mathcal{C}_{!}$, with $\mathcal{C}_{!}(X;Y) = \mathcal{C}(X;Y)$ and $\mathcal{C}_{!}(X_{1},\dots,X_{n};Y) = \emptyset$ for $n \neq 1$:

2.8. Corollary. The underlying functor is right adjoint to the linear functor:

$$(-)_!\dashv\underline{(-)}:Mlt\to Cat$$

Since $(-)_!: \mathbf{Cat} \to \mathbf{Mlt}$ is also clearly a full and faithful 2-functor, \mathbf{Cat} is coreflectively embedded in two ways in \mathbf{Mlt} . Anyway, in the present work the prominent role is played by the discrete cocone embedding.

Note that $(-) \circ (-)_{\blacktriangleright} : \mathbf{Cat} \to \mathbf{Cat}$ is the identity, so that both (-) and $\mathbf{Mon}(-)$ provide left inverses for the discrete cocone functor.

3. The cartesian level

3.1. Cartesian multicategory has appeared in various guises (and names) in the literature; see e.g. [Lambek, 1989], where

they are called "Gentzen multicategories". The idea is that they are to finite product categories what multicategories are to monoidal categories. So they should fulfill at least the condition that representable cartesian multicategories are finite product categories, that is the tensor product is cartesian. An elegant definition is given in [T. Fiore, 2005] (without mentioning explicitly multicategories) and taken over in [Gould, 2008] for operads. We report here the essence of that notion.

Let **N** the full subcategory of **Set** which has objects $\mathbf{0}, \mathbf{1}, \mathbf{2}, \cdots$, with $\mathbf{n} = \{1, 2, \cdots, n\}$ (so that $\mathbf{0} = \emptyset$) and consider, for a multicategory \mathcal{M} , the obvious comma category $\mathbf{N}/\text{obj}\mathcal{M}$.

Now, the domain of an n-arrow in \mathcal{M} is in fact a mapping $\alpha : \mathbf{n} \to \text{obj } \mathcal{M}$, that is an object of $\mathbf{N}/\text{obj }\mathcal{M}$; thus, for any fixed codomain object $X \in \mathcal{M}$ we have a mapping

$$\operatorname{obj}(\mathbf{N}/\operatorname{obj}\mathcal{M}) \to \operatorname{obj}\mathbf{Set}$$
 ; $\alpha \mapsto \mathcal{M}(\alpha, X) = \mathcal{M}(X_1, \dots, X_n, X)$

To give an fp-structure on \mathcal{M} means to extend these mappings to functors

$$(-)_X: \mathbf{N}/\mathrm{obj}\,\mathcal{M} \to \mathbf{Set}$$

in a way compatible with composition.

For instance, let obj $\mathcal{M} = \{A, B, C, D, \cdots\}$ and let $\alpha : \mathbf{3} \to \text{obj } \mathcal{M}$ and $\beta : \mathbf{3} \to \text{obj } \mathcal{M}$ be the objects of $\mathbb{N}/\text{obj } \mathcal{M}$ given by

$$\alpha: \quad 1 \mapsto A, \quad 2 \mapsto B, \quad 3 \mapsto A \quad ; \quad \beta: \quad 1 \mapsto B, \quad 2 \mapsto A, \quad 3 \mapsto C$$

If $p: \alpha \to \beta$ in $\mathbb{N}/\text{obj } \mathcal{M}$ is given by

$$p: 1 \mapsto 2, 2 \mapsto 1, 3 \mapsto 2$$

then an fp-structure on \mathcal{M} gives in particular a mapping

$$p_D: \mathcal{M}(A, B, A; D) \to \mathcal{M}(B, A, C; D)$$

(We will hereafter omit the subscript indicating the codomain object.)

As for the compatibility conditions with respect to composition, there are two of them. The first one is pretty obvious:

$$f(p_1f_1, \dots, p_nf_n) = (p_1 + \dots + p_n)f(f_1, \dots, f_n)$$

that is to compose f with arrows f_i acted upon by p_i is the same as composing f with the f_i and then acting on it with the obvious "sum" of maps in $\mathbb{N}/\text{obj}\mathcal{M}$.

The second compatibility condition concerns the case when it is f that is acted upon by a mapping p:

$$(pf)(f_1, \dots, f_n) = p'(f(f_{p1}, \dots, f_{pn}))$$

where p' is a suitably defined map in $\mathbb{N}/\text{obj}\mathcal{M}$.

Let us illustrate it by referring to the above instance. Let

$$f: \langle A, B, A \rangle \to D$$
 ; $a: \langle A_1, A_2 \rangle \to A$; $b: \langle B_1, B_2 \rangle \to B$; $c: \langle C_1, C_2 \rangle \to C$

be arrows in \mathcal{M} and let p be defined as above. Then

$$(pf)(b,a,c) = p'(f(a,b,a)) : \langle B_1, B_2, A_1, A_2, C_1, C_2 \rangle \to D$$

where $f(a,b,a): \langle A_1,A_2,B_1,B_2,A_1,A_2 \rangle \to D$ and $p': \gamma \to \delta$ (for the obvious γ and δ) is given by the following mapping $\mathbf{6} \to \mathbf{6}$:

$$p':$$
 $1 \mapsto 3$, $2 \mapsto 4$, $3 \mapsto 1$, $4 \mapsto 2$, $5 \mapsto 3$, $6 \mapsto 4$

The key cases are in fact those of the mappings p over $\mathbf{v}_n : \mathbf{n} \to \mathbf{1}$ (contractions or diagonals), over $\mathbf{d}_{1,2} : \mathbf{1} \to \mathbf{2}$ (weakenings or projections) and $\mathbf{s} : \mathbf{2} \to \mathbf{2}$, or more generally bijections $\mathbf{n} \to \mathbf{n}$ (exchange):

$$v_2: \mathcal{M}(X, X; Z) \to \mathcal{M}(X; Z)$$
 ; $v_0: \mathcal{M}(; Z) \to \mathcal{M}(X; Z)$
$$d_1: \mathcal{M}(X; Z) \to \mathcal{M}(X, Y; Z)$$
 ;
$$d_2: \mathcal{M}(Y; Z) \to \mathcal{M}(X, Y; Z)$$

$$s: \mathcal{M}(X, Y; Z) \to \mathcal{M}(Y, X; Z)$$

(Note that $\mathbf{0} \to \text{obj } \mathcal{M}$ is initial in $\mathbf{N}/\text{obj } \mathcal{M}$, while $A: \mathbf{1} \to \text{obj } \mathcal{M}$ receives exactly one map from the constantly A-valued object $\mathbf{n} \to \text{obj } \mathcal{M}$ and none from the other ones.)

A functor $\mathcal{M} \to \mathcal{M}'$ induces an obvious functor $\mathbf{N}/\mathrm{obj}\,\mathcal{M} \to \mathbf{N}/\mathrm{obj}\,\mathcal{M}'$ and maps of cartesian multicategories are of course those functors which commute with the actions of $\mathbf{N}/\mathrm{obj}\,\mathcal{M}$ and $\mathbf{N}/\mathrm{obj}\,\mathcal{M}'$. We so obtain the 2-category **fpMlt** of cartesian multicategories with the obvious forgetful functor $U:\mathbf{fpMlt}\to\mathbf{Mlt}$.

3.2. Remark. Cartesian multicategories can be seen as (many-sorted) algebraic theories with the same expressive power as (many-sorted) Lawvere theories. Indeed, there is an equivalence between cartesian multicategories and Lawvere theories ([T. Fiore, 2005]; see also [Lambek, 1989]). In this perspective, fp-functors $\mathcal{M} \to \mathbf{Set}$ (or more generally $\mathcal{M} \to \mathcal{N}$) are the (\mathcal{N} -valued) "algebras" or "models" of \mathcal{M} . See also [Leinster, 2003] for a discussion of those algebraic theories (namely the "strongly regular" ones, whose equations are "linear") which can be expressed by plain operads.

We have already sketched in the introduction how cartesian multicategories arise both from finite product categories and from preadditive categories. Let us state this fact more formally. We denote by \mathbf{Cat}^{\times} the category of finite product categories and finite product preserving functors, while $\mathbf{cMonCat}$ is the category of categories enriched in $\mathbf{cMon} = \mathbf{cMon(Set)}$, that is preadditive categories and additive functors.

3.3. Proposition. There are fully faithful functors

$$(-)_{\times}: \mathbf{Cat}^{\times} \to \mathbf{fpMlt}$$
 ; $(-)_{\bullet}: \mathbf{cMonCat} \to \mathbf{fpMlt}$

PROOF. Given a finite product category $\mathcal{C} \in \mathbf{Cat}^{\times}$ we get $\mathcal{C}_{\times} \in \mathbf{fpMlt}$ with

$$\mathcal{C}_{\times}(X_1,\dots,X_n;X) \cong \mathcal{C}(X_1 \times \dots \times X_n;X)$$

and with contractions and weakenings given by diagonals and projections:

$$v_{2}: \mathcal{C}(X \times X; Z) \to \mathcal{C}(X; Z) \qquad ; \qquad v_{2}: f \mapsto f \circ \Delta$$

$$v_{0}: \mathcal{C}(1; Z) \to \mathcal{C}(X; Z) \qquad ; \qquad v_{0}: x \mapsto x \circ !_{X}$$

$$d_{1}: \mathcal{C}(X; Z) \to \mathcal{C}(X \times Y; Z) \qquad ; \qquad d_{1}: f \mapsto f \circ \pi_{1}$$

$$s: \mathcal{C}(X \times Y; Z) \to \mathcal{C}(Y \times X; Z) \qquad ; \qquad d_{1}: f \mapsto f \circ \langle \pi_{2}, \pi_{1} \rangle$$

Given a preadditive category $\mathcal{C} \in \mathbf{cMonCat}$ we get $\mathcal{C}_{\bullet} \in \mathbf{fpMlt}$ with

$$\mathcal{C}_{\bullet}(X_1,\dots,X_n;X) \cong \mathcal{C}(X_1;X) \times \dots \times \mathcal{C}(X_n;X)$$

as in Section 2 and with contractions and weakenings given by sums and zero's insertions:

$$v_{2}: \mathcal{C}(X; Z) \times \mathcal{C}(X; Z) \to \mathcal{C}(X; Z) \qquad ; \qquad v_{2}: \langle f, g \rangle \mapsto f + g$$

$$v_{0}: 1 \to \mathcal{C}(X; Z) \qquad ; \qquad v_{0}: * \mapsto 0_{X, Z}$$

$$d_{1}: \mathcal{C}(X; Z) \to \mathcal{C}(X; Z) \times \mathcal{C}(Y; Z) \qquad ; \qquad d_{1}: f \mapsto \langle f, 0 \rangle$$

$$s: \mathcal{C}(X; Z) \times \mathcal{C}(Y; Z) \to \mathcal{C}(Y; Z) \times \mathcal{C}(X; Z) \qquad ; \qquad s: \langle f, g \rangle \mapsto \langle g, f \rangle$$

Both the assignments are easily seen to extend to faithful functors. For the fullness of $(-)_{\times}$ note that if $F: \mathcal{C}_{\times} \to \mathcal{D}_{\times}$ is an fp-functor then the underlying $\underline{F}: \mathcal{C} \to \mathcal{D}$ preserves projections, since

$$\underline{F}\pi_1^{X,Y} = F(\mathbf{d}_1 \mathbf{i} \mathbf{d}_X) = \mathbf{d}_1 \mathbf{i} \mathbf{d}_{FX} = \pi_1^{\underline{F}X,\underline{F}Y}$$

so that $F = (\underline{F})_{\times}$, with $\underline{F} : \mathcal{C} \to \mathcal{D}$ in \mathbf{Cat}^{\times} .

As for the fullness of $(-)_{\blacktriangleright}$, let $F: \mathcal{C}_{\blacktriangleright} \to \mathcal{D}_{\blacktriangleright}$ be an fp-functor with underlying $\underline{F}: \mathcal{C} \to \mathcal{D}$. Recall from Section 2 that $F = (\underline{F})_{\blacktriangleright}$, that is $F\langle f, g \rangle = \langle \underline{F}f, \underline{F}g \rangle$, so that it is enough to show that \underline{F} is additive:

$$\underline{F}(f+g) = F(\mathbf{v}_2\langle f,g\rangle) = \mathbf{v}_2 F\langle f,g\rangle = \mathbf{v}_2 \langle \underline{F}f,\underline{F}g\rangle = \underline{F}f + \underline{F}g$$

In the other direction we have:

3.4. PROPOSITION. cMon(-) gives a functor $fpMlt \rightarrow cMonCat$.

PROOF. If \mathcal{M} is a cartesian multicategory, then $\mathbf{cMon}(\mathcal{M})$ is a preadditive category. This can be seen as in the case $\mathcal{M} = \mathbf{Set}$ but using explicitly the fp-structure of \mathcal{M} . In particular, if $f, g: X \to Y$ are maps in $\mathbf{cMon}(\mathcal{M})$ their sum is given by

$$f + g = v_2 m_2^Y(f, g)$$

Again, the assignment is easily seen to extend to a functor.

3.5. Proposition. There is an adjunction

$$(-)$$
, \dashv cMon $(-)$: fpMlt \rightarrow cMonCat (12)

PROOF. Just follow the trace given in the introduction, with minor modifications to take in account the many-sorted case and the generality of \mathcal{M} (with respect to **Set**).

Now we characterize (up to equivalence) the codomain of the functors of Proposition 3.3. Recall that in the introduction we defined \mathbf{fpCat}_{\bullet} and \mathbf{fpRep} as the full subcategories of \mathbf{fpMlt} over \mathbf{Cat}_{\bullet} and \mathbf{Rep} respectively, so that $(-)_{\bullet}$ and $(-)_{\times}$ are in fact functors

$$(-)_{\bullet}: cMonCat \rightarrow fpCat_{\bullet}$$
; $(-)_{\times}: Cat^{\times} \rightarrow fpRep$

3.6. Proposition. The adjunction (12) restricts to an equivalence

$$fpCat \simeq cMonCat$$

PROOF. If $C_{\bullet} \in \mathbf{Cat}_{\bullet}$ has an fp-structure then contractions v_0 and v_2 give a monoid structure on the hom-sets of C that distributes over compositions because of the compatibility conditions. Thus any $\mathcal{M} \in \mathbf{fpCat}_{\bullet}$ is indeed isomorphic to C_{\bullet} for a preadditive C_{\bullet}

- 3.7. COROLLARY. fpCat, is coreflective in fpMlt.
- 3.8. Proposition. There is an equivalence

$$fpRep \simeq Cat^{\times}$$

More precisely, if a representable multicategory \mathcal{M} has an fp-structure then it is unique and the representing tensor is in fact a product (so that $\mathcal{M} \cong \mathcal{C}_{\times}$ for $\mathcal{C} \in \mathbf{Cat}^{\times}$).

PROOF. As stated at the beginning of this section, the proposition should be true for any adequate definition of cartesian multicategory. I do not have myself a complete proof but the missing step seems to be folklore. First note that, for any $\mathcal{M} \in \mathbf{fpRep}$, the representing tensor product is symmetric (because of the acting bijections, in particular the exchange $s: \mathbf{2} \to \mathbf{2}$). Furthermore, for any multicategory \mathcal{M} the hom-sets functors $\mathcal{M}^n: \underline{\mathcal{M}}^{\mathrm{op}} \times \cdots \times \underline{\mathcal{M}}^{\mathrm{op}} \times \underline{\mathcal{M}} \to \mathbf{Set}$ (where $\underline{\mathcal{M}}$ is the underlying category) induce (via the diagonal $\underline{\mathcal{M}}^{\mathrm{op}} \to \underline{\mathcal{M}}^{\mathrm{op}} \times \cdots \times \underline{\mathcal{M}}^{\mathrm{op}}$) "diagonal" endoprofunctors \mathcal{M}^n_{Λ} on $\underline{\mathcal{M}}$:

$$\mathcal{M}^n_{\Delta}(X,Y)$$
 = $\mathcal{M}(X,\cdots,X;Y)$

Now, because of the compatibility conditions, the contraction and exchange mappings v_n and s are profunctor morphisms:

$$\mathcal{M}^n_{\Delta} \to \mathcal{M}^1_{\Delta} = \underline{\mathcal{M}} \qquad ; \qquad \mathcal{M}^2_{\Delta} \to \mathcal{M}^2_{\Delta}$$

with $v_2s = v_2$ (in fact we have a functor $\mathbf{N} \to \mathbf{Set}^{\underline{\mathcal{M}}^{op} \times \underline{\mathcal{M}}}$); thus one gets (Yoneda) a natural commutative comonoid structure

$$X \to X \otimes X$$
 ; $X \to 1$

on each object $X \in \mathcal{M}$. The missing step is then the dual of the following statement:

Given a symmetric monoidal category (C, \otimes, I) , if the forgetful functor $\mathbf{cMon}(C) \to C$ is a split epi in \mathbf{Cat} then I is initial and \otimes is the coproduct in C.

Among the papers which cite (without proof) this fact there are [Lamarche, 2007], [M. Fiore, 2007] and [Pavlovic, 2008]; an ingenious example by Jeff Egger shows that the commutativity hypothesis is necessary.

3.9. Corollary. The full subcategory

$$fpSum := fpCat \cap fpRep \subset fpMlt$$

of those cartesian multicategories which are over Sum can be identified with both

- 1. preadditive categories with finite sums (and additive functors);
- 2. categories with finite sums and with finite products which are naturally isomorphic (and finite (bi)product preserving functors).

Thus, we find again the result that additive categories (preadditive categories with biproducts) can be characterized both as preadditive categories with sums and as categories with finite sums and products and with natural isomorphisms $X + Y \rightarrow X \times Y$ (see [Lack, 2009]). Furthermore, the additive functors between additive categories coincide with the biproduct preserving functors.

3.10. Remark. The forgetful functor U has a left adjoint (see [Gould, 2008] for the case of operads):

$$F \dashv U : \mathbf{fpMlt} \to \mathbf{Mlt}$$

In particular, if $\mathcal{C}_{\bullet} \in \mathbf{Cat}_{\bullet}$, $F\mathcal{C}_{\bullet}$ gives the usual free preadditive category on $\mathcal{C} \in \mathbf{Cat}$.

4. Multifibrations

From the adjunction (-), \dashv Mon(-): Mlt \rightarrow Cat of Corollary 2.2, we get in particular isomorphisms

$$\frac{\operatorname{Cat}(\mathcal{C}, \operatorname{Mon}(\operatorname{Cat}_{\times}))}{\operatorname{Mlt}(\mathcal{C}_{\bullet}, \operatorname{Cat}_{\times})}$$
(13)

Now, an obvious generalization of the Grothendieck construction shows that there is a correspondence between (covariant) fibrations of multicategories (see [Hermida, 2004]) over \mathcal{M} and (pseudo)functors $\mathcal{M} \to \mathbf{Cat}_{\times}$:

$$\frac{\mathbf{Mlt}(\mathcal{M}, \mathbf{Cat}_{\times})}{\mathbf{mltfb}/\mathcal{M}}$$
 (14)

Thus we have a correspondence

$$\frac{\mathbf{Cat}(\mathcal{C}, \mathbf{Mon}(\mathbf{Cat}_{\times}))}{\mathbf{mltfb}/\mathcal{C}_{\bullet}}$$
 (15)

(more precisely, above the line we should consider pseudofunctors from \mathcal{C} to the category representable multicategories with functors that preserve universal n-arrows) which can be seen directly as follows.

Given $\mathcal{M}: \mathcal{C} \to \mathbf{Mon}(\mathbf{Cat}_{\times})$ and $a_i \in \mathcal{M}X_i$, arrows $\langle a_1, \dots, a_n \rangle \to a$ in the domain $\widehat{\mathcal{M}}$ of the corresponding fibration are sequences of arrows $\lambda_i : X_i \to X$ in \mathcal{C} plus a morphism $\lambda_1 a_1 \otimes \dots \otimes \lambda_n a_n \to a$ in $\mathcal{M}X$. Convesely, suppose we have a fibration over \mathcal{C}_{\bullet} and an object $X : 1 \to \mathcal{C}$; the corresponding $X_{\bullet} : 1_{\bullet} \to \mathcal{C}_{\bullet}$ induces, by pullback, a fibration over 1_{\bullet} , giving a monoidal category $\mathcal{M}X$; then, as in the standard case, one extends this to a (pseudo)functor $\mathcal{M}: \mathcal{C} \to \mathbf{Mon}(\mathbf{Cat}_{\times})$.

4.1. REMARK. If \mathcal{C} has finite sums, \mathcal{C}_{\bullet} is representable (that is fibered over 1_{\bullet}) so that the domain of the fibration corresponding to an indexed monoidal category is also representable; thus we get as a particular case of (7) (the dual of) the correspondence in [Shulman, 2008], for a finite product category \mathcal{C} , between \mathcal{C}^{op} -indexed monoidal categories and monoidal fibrations over \mathcal{C} .

Of course, (15) restricts to discrete (op)fibrations:

$$\frac{\operatorname{Cat}(\mathcal{C}, \operatorname{Mon}(\operatorname{Set}_{\times}))}{\operatorname{mltdf}/\mathcal{C}_{\bullet}}$$
(16)

The idea is that a functor $\mathcal{M}: \mathcal{C} \to \mathbf{Mon}(\mathbf{Set}_{\times})$ provides a way to combine (rather than just to compare, as in the case of functors $\mathcal{C} \to \mathbf{Set}$) objects of various sorts (elements $a_i \in \mathcal{M}(X_i), X_i \in \mathcal{C}$) producing another object $\lambda_1 a_1 + \cdots + \lambda_n a_n$ (where $\lambda_i \in \mathcal{C}$ expresses how a_i is used). The domain $\widehat{\mathcal{M}}$ of the resulting fibration is the multicategory of such combinations and the functor $\widehat{\mathcal{M}} \to \mathcal{C}_{\bullet}$ gives the "abstract shape" of each concrete combination in $\widehat{\mathcal{M}}$.

Let us see some examples.

4.2. LINEAR COMBINATIONS. A (single sorted) instance is given by the case of a module over a rig (say a vector space) discussed in the introduction. In this case, arrows in $\widehat{\mathcal{M}}$ are true linear combinations and $\widehat{\mathcal{M}}(a_1,\dots,a_n;a)$ is not empty iff a is spanned by the a_i . Composition in $\widehat{\mathcal{M}}$ is just the natural idea that a combination of combinations gives a combination. The fact that \mathcal{M} is actually a module (that is \mathcal{C} is a rig and $\mathcal{M}: \mathcal{C} \to \mathbf{cMon}(\mathbf{Set}_{\times})$ is additive) is reflected in the fact that $\widehat{\mathcal{M}}$ has itself an fp-structure and $\widehat{\mathcal{M}} \to \mathcal{C}_{\bullet}$ preserves it.

4.3. COMBINATIONS OF FIGURES. The isometry group \mathcal{C} of the plane X acts on "figures" $A \subseteq X$; it in fact acts on the monoid $(\mathcal{P}X, \cup, \emptyset)$. Thus we obtain the multicategory $\widehat{\mathcal{M}}$ with figures as objects and arrows

$$\langle \lambda_1, \dots, \lambda_n \rangle : \langle A_1, \dots, A_n \rangle \to \lambda_1 A_1 \cup \dots \cup \lambda_n A_n \qquad ; \qquad \langle \rangle : \langle \rangle \to \emptyset$$
 (17)

Note that the underlying discrete opfibration (of categories) $\widehat{\mathcal{M}} \to \mathcal{C}$ corresponds to the action of \mathcal{C} on the set $\mathcal{P}X$ and is the usual symmetry groupoid: $\lambda : A \to \lambda A$.

4.4. TANGRAM GAME. We can modify the above example by defining the monoid $(\mathcal{P}X, \cup', \emptyset)$ where $A \cup' B = A \cup B$ if A and B are disjoint, and $A \cup' B = X$ if they are not. We so obtain the multicategory $\widehat{\mathcal{M}}'$ with figures as objects and multiarrows

$$\langle \lambda_1, \dots, \lambda_n \rangle : \langle A_1, \dots, A_n \rangle \to A \qquad ; \qquad \langle \rangle : \langle \rangle \to \emptyset$$
 (18)

where A is a disjoint union $\lambda_1 A_1 \cup \cdots \cup \lambda_n A_n$ (if A is not X). Passing to the associated multiposet we get $\langle A_1, \cdots, A_n \rangle \vdash A$ iff A can be obtained by combining the A_i as in the Tangram game.

4.5. COVERINGS. Following example 4.3 above, we can exploit the posetal structure of the monoid $(\mathcal{P}X, \supseteq; \cup, \varnothing)$, making it a non-discrete monoidal category. Now $\langle \lambda_1, \dots, \lambda_n \rangle : \langle A_1, \dots, A_n \rangle \to A$ is an arrow in $\widehat{\mathcal{M}}$ iff $A \subseteq \lambda_1 A_1 \cup \dots \cup \lambda_n A_n$.

Passing to the associated multiposet we get $\langle A_1, \dots, A_n \rangle \vdash A$ iff A can be covered by figures congruent with the A_i .

5. Conclusions

We have seen that, within cartesian multicategories, an fp-functor $\mathcal{C}_{\bullet} \to \mathcal{D}_{\bullet}$ is just an additive functor, an fp-functor $\mathcal{C}_{\times} \to \mathcal{D}_{\times}$ is just a finite product preserving functor and an fp-functor $\mathcal{C}_{\bullet} \to \mathbf{Set}_{\times}$ (more generally $\mathcal{C}_{\bullet} \to \mathcal{D}_{\times}$ or $\mathcal{C}_{\bullet} \to \mathcal{M}$) is just a \mathcal{C} -module.

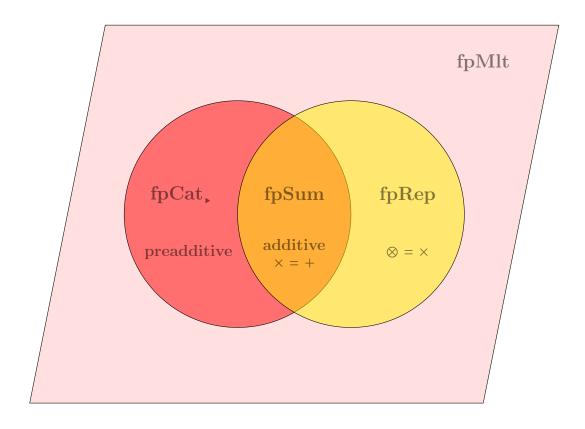
Among algebraic theories (cartesian multicategories), those of the form \mathcal{C}_{\bullet} for a preadditive \mathcal{C} thus occupy a special place. Their models (generalized modules) in any cartesian multicategory \mathcal{M} correspond to additive functors $\mathcal{C} \to \mathbf{cMon}(\mathcal{M})$. The (logically) non-linear axiom

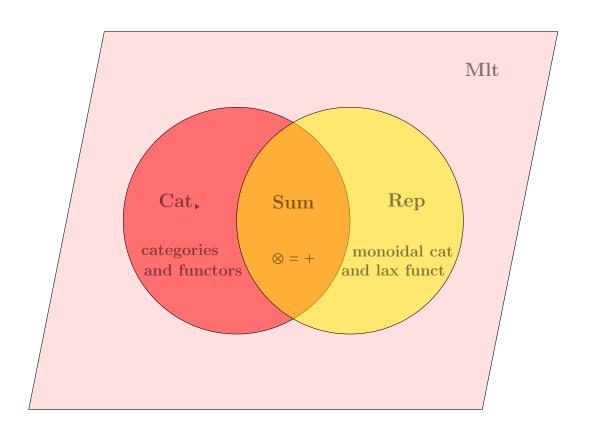
$$(\lambda + \mu)x = \lambda x + \mu x$$

can be seen either as expressing the fact that $\mathcal{C}_{\bullet} \to \mathcal{M}$ preserves the cartesian structure or as expressing the additivity of $\mathcal{C} \to \mathbf{cMon}(\mathcal{M})$.

Similarly, on the plain level, among "linear" or "strongly regular" theories (multicategories) those of the form \mathcal{C}_{\bullet} for a category \mathcal{C} occupy a special place. Their models in any multicategory \mathcal{M} correspond to functors $\mathcal{C} \to \mathbf{Mon}(\mathcal{M})$.

Finally, we have shown how natural instances of (promonoidal) multicategories arise as (the domain of) the fibration corresponding to a (strong) indexed monoidal category.





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