

# INTERNAL SPLIT OPFIBRATIONS AND COFUNCTORS

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**ABSTRACT.** Split opfibrations are functors equipped with a suitable choice of opcartesian lifts. The purpose of this paper is to characterise internal split opfibrations through separating the structure of a suitable choice of lifts from the property of these lifts being opcartesian. The underlying structure of an internal split opfibration is captured by an internal functor equipped with an internal cofunctor, while the property may be expressed as a pullback condition, akin to the simple condition on an internal functor to be an internal discrete opfibration. Furthermore, this approach provides two additional characterisations of internal split opfibrations, via the décalage construction and strict factorisation systems. For small categories, this theory clarifies several aspects of delta lenses which arise in computer science.

## 1. INTRODUCTION

Split opfibrations are functors equipped with a functorial choice of opcartesian lifts, and they have been widely studied since the seminal work of [Gra66]. In the paper [Str74], split opfibrations were generalised to an arbitrary 2-category  $\mathcal{K}$ , and this definition readily specialises to the case  $\mathcal{K} = \mathbf{Cat}(\mathcal{E})$ , for a fixed category  $\mathcal{E}$  with pullbacks, to yield internal split opfibrations. However, there are other ways to define internal split opfibrations, including in [Joh77] as internal categories in the category of internal diagrams, or in the recent paper [vG18] using lax codescent objects.

This paper aims to develop a new approach to internal split opfibrations, using the theory of *cofunctors* from [Agu97]. The structure of an *internal lens* (Definition 3.1) is introduced as an internal functor  $f: A \rightarrow B$  equipped with an internal cofunctor  $\varphi: B \rightarrowtail A$ , and is equivalent to a commutative diagram of internal functors,

$$\begin{array}{ccc} & \Lambda & \\ \varphi \swarrow & & \searrow \overline{\varphi} \\ A & \xrightarrow{f} & B \end{array} \quad (*)$$

where  $\varphi$  is identity-on-objects and  $\overline{\varphi}$  is an internal discrete opfibration. An internal split opfibration is defined as an internal lens satisfying a certain property (Definition 4.1). The main result (Theorem 4.7) will be to characterise internal split opfibrations via a natural condition on  $(*)$  with respect to the right décalage comonad on  $\mathbf{Cat}(\mathcal{E})$ . An internal lens with a certain strict factorisation system is also shown to characterise internal split opfibrations (Theorem 5.6), providing a link to the explicit axiomatisation in [Joh77].

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The motivation for this paper arises from both theoretical and applied concerns. In part, there was a goal to find a defining property of internal split opfibrations which closely resembles the remarkably simple condition on an internal functor to be an internal discrete opfibration (Definition 2.5). However, split opfibrations are functors with *additional structure*, and it becomes necessary to unravel this definition to discover the appropriate internal structure with the desired property. This key idea is realised by separating an internal split opfibration into three parts: an internal functor, an internal cofunctor (Definition 2.10), and the defining property expressed as a pullback condition.

Internal cofunctors are a kind of morphism between internal categories. They were developed in the thesis [Agu97, Chapter 4], and generalise the earlier concept of comorphisms between vector bundles (and other related structures) introduced in [HM93]. In the ordinary case (Example 2.9), a cofunctor between categories is a functorial *lifting* of morphisms in the opposite direction to the assignment on objects, thus providing an ideal structure to describe the *splitting* of a split opfibration. An important result in [HM93] is that every cofunctor  $\varphi: B \nrightarrow A$  may be represented as a span of internal functors,

$$\begin{array}{ccc} & \Lambda & \\ \overline{\varphi} \swarrow & & \searrow \varphi \\ B & & A \end{array}$$

where  $\overline{\varphi}$  is an internal discrete opfibration and  $\varphi$  is identity-on-objects (Proposition 2.16). This reveals how cofunctors may be naturally understood as morphisms between categories, and clarifies the abstract composition of cofunctors (Remark 2.12) through the concrete composition of spans.

There is also another natural way of understanding cofunctors as morphisms between categories. Recall that for a category  $\mathcal{E}$  with pullbacks, an internal category is a formal monad in the bicategory  $\mathbf{Span}(\mathcal{E})$  of spans. An internal functor corresponds to a lax monad morphism whose 1-cell component is a *right adjoint* in  $\mathbf{Span}(\mathcal{E})$ . Dually, an internal cofunctor corresponds to a lax monad morphism whose 1-cell component is a *left adjoint*. In this sense, cofunctors are dual to functors, and may be unified under the notions of two-dimensional partial map, as in [LS02, Appendix], or Mealy morphism, as in [Par12], however this relationship is not pursued further in this paper.

An internal functor suitably equipped with an internal cofunctor is called an internal lens (Definition 3.1) between internal categories. When  $\mathcal{E} = \mathbf{Set}$ , this morphism between small categories is known as a *delta lens* (Example 3.3) introduced in the paper [DXC11]. The general notion of a lens, whose name derives from *focusing* on a database view, now covers many different structures in the literature (see [JR16] for some examples) since the influential paper [FGM<sup>+</sup>07]. Delta lenses were originally defined in computer science as an algebraic framework for *bidirectional model transformations*, and their relationship to split opfibrations is important for applications such as *least-change view updating* [JRW12]. This paper is motivated by the goal to better understand the precise relationship between delta lenses and split opfibrations, through generalising the result in [JR13] that split opfibrations are delta lenses whose lifts are opcartesian.

Internal split opfibrations are defined as internal lenses with a certain property, expressed as a pullback condition (Definition 4.1). In the recent paper [AU17, Section 6], split opfibrations were instead defined via additional structure on a delta lens, using the syntax of directed containers from [ACU14]. Generalising the approach in [AU17], internal split opfibrations may also be described via additional structure on an internal lens, and this characterisation is shown (Proposition 4.2) to be equivalent to Definition 4.1, while also specialising to the familiar definition in the  $\mathcal{E} = \mathbf{Set}$  case.

Given the category  $\mathbf{Lens}(\mathcal{E})$  of internal categories and internal lenses, there is an identity-on-objects functor  $U: \mathbf{Lens}(\mathcal{E}) \rightarrow \mathbf{Cat}(\mathcal{E})$  which forgets the internal cofunctor structure. The right décalage construction (Definition 4.5) is a comonad on  $\mathbf{Cat}(\mathcal{E})$ , and it is natural to ask if this comonad restricts along  $U$  to an endofunctor on  $\mathbf{Lens}(\mathcal{E})$ . The main result of this paper (Theorem 4.7) characterises internal split opfibrations as exactly those internal lenses for which the right décalage construction yields an internal lens.

A classical result regarding split opfibrations is that every morphism in the domain factorises uniquely into a chosen opcartesian lift followed by a “vertical” morphism in the fibre (Example 5.5), thus yielding a strict factorisation system (Definition 5.1) as in [RW02]. This strict factorisation system may be generalised to internal lenses (Proposition 5.4) in the form of additional structure. Surprisingly, the property of an internal lens having this strict factorisation system provides another characterisation of internal split opfibrations (Theorem 5.6). This result also reveals an equivalence between the definition of internal split opfibration in this paper and the explicit axiomatisation via structure on an internal functor appearing as an exercise in [Joh77, Chapter 2].

**Structure of the paper.** Section 2 recalls the relevant concepts from internal category theory for the convenience of the reader, and does not contain any original results. Section 3 introduces internal lenses, and includes basic properties and examples. Section 4 provides a new definition of internal split opfibrations and their characterisation via décalage. Section 5 explores the connection to strict factorisation systems, and the earlier axiomatisation of Johnstone. Section 6 provides some concluding remarks.

**Further reading.** The recent papers [Str17] and [LR19] provide an excellent overview of internal category theory and fibrations, respectively. Modern textbooks containing background material on both internal categories and fibrations include [Bor94], [BW12], and [Joh02]. An elementary introduction to lenses, from the perspective of category theory, may be found in the thesis [Cla18] by the current author.

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## 2. INTERNAL CATEGORIES, FUNCTORS, AND COFUNCTORS

This section reviews the background material on internal categories and functors, in a fixed category  $\mathcal{E}$  with pullbacks, and establishes the notation for the rest of the paper. This section also adapts the definition of an internal cofunctor from [Agu97] to this setting, and includes an internal version of the result [HM93, Theorem 5.8] which states that every cofunctor may be represented as a span of functors, with left leg a discrete opfibration and right leg an identity-on-objects functor.

**Definition 2.1.** An *internal category*  $A$  is a diagram,

$$\begin{array}{ccccc} & & \longleftarrow d_0 & \longrightarrow & \\ & \longleftarrow d_0 & \longrightarrow & \longleftarrow d_0 & \\ A_0 & \xrightarrow{i_0} & A_1 & \xleftarrow{d_1} & A_2 & \xleftarrow{d_2} & A_3 \\ & \longleftarrow d_1 & \longrightarrow & \longleftarrow d_1 & \\ & & \longrightarrow i_1 & \longrightarrow & \\ & & \longleftarrow d_2 & \longrightarrow & \end{array}$$

where the objects  $A_2$  and  $A_3$  are defined by the pullbacks,

$$\begin{array}{ccc} & A_2 & \\ d_2 \swarrow & \downarrow & \searrow d_0 \\ A_1 & & A_1 \\ & d_0 \searrow & \swarrow d_1 \\ & A_0 & \end{array} \quad \begin{array}{ccc} & A_3 & \\ d_3 \swarrow & \downarrow & \searrow d_0 \\ A_2 & & A_2 \\ & d_0 \searrow & \swarrow d_2 \\ & A_1 & \end{array} \quad (2.1)$$

and where the *identity map*  $i_0: A_0 \rightarrow A_1$  and *composition map*  $d_1: A_2 \rightarrow A_1$  are determined by the commutative diagrams,

$$\begin{array}{ccc} & A_0 & \\ 1 \swarrow & \downarrow i_0 & \searrow 1 \\ A_0 & \xleftarrow{d_1} & A_1 & \xrightarrow{d_0} & A_0 \end{array} \quad \begin{array}{ccccc} A_1 & \xleftarrow{d_2} & A_2 & \xrightarrow{d_0} & A_1 \\ d_1 \downarrow & & \downarrow d_1 & & \downarrow d_0 \\ A_0 & \xleftarrow{d_1} & A_1 & \xrightarrow{d_0} & A_0 \end{array} \quad (2.2)$$

and satisfy the *unitality* and *associativity* axioms given by the commutative diagrams:

$$\begin{array}{ccc} & A_1 & \\ i_1 \swarrow & \downarrow 1 & \searrow i_0 \\ A_2 & & A_2 \\ & d_1 \searrow & \swarrow d_1 \\ & A_1 & \end{array} \quad \begin{array}{ccc} & A_3 & \\ d_2 \swarrow & \downarrow & \searrow d_1 \\ A_2 & & A_2 \\ & d_1 \searrow & \swarrow d_1 \\ & A_0 & \end{array} \quad (2.3)$$

An internal category is often depicted by its underlying directed graph consisting of the *object of objects*  $A_0$ , the *object of morphisms*  $A_1$ , the *domain map*  $d_1: A_1 \rightarrow A_0$ , and the *codomain map*  $d_0: A_1 \rightarrow A_0$ . The morphisms  $i_0, i_1: A_1 \rightarrow A_2$  and  $d_1, d_2: A_3 \rightarrow A_2$  appearing in (2.3) are defined using the universal property of the pullback.

**Definition 2.2.** Let  $A$  and  $B$  be internal categories. An *internal functor*  $f: A \rightarrow B$  is a pair of morphisms,

$$f_0: A_0 \longrightarrow B_0 \quad f_1: A_1 \longrightarrow B_1$$

satisfying the commutative diagrams for a directed graph homomorphism,

$$\begin{array}{ccccc}
 A_0 & \xleftarrow{d_1} & A_1 & \xrightarrow{d_0} & A_0 \\
 f_0 \downarrow & & \downarrow f_1 & & \downarrow f_0 \\
 B_0 & \xleftarrow{d_1} & B_1 & \xrightarrow{d_0} & B_0
 \end{array} \tag{2.4}$$

and which respect the identity and composition maps:

$$\begin{array}{ccc}
 A_0 & \xrightarrow{i_0} & A_1 \\
 f_0 \downarrow & & \downarrow f_1 \\
 B_0 & \xrightarrow{i_0} & B_1
 \end{array}
 \qquad
 \begin{array}{ccc}
 A_2 & \xrightarrow{d_1} & A_1 \\
 f_2 \downarrow & & \downarrow f_1 \\
 B_2 & \xrightarrow{d_1} & B_1
 \end{array} \tag{2.5}$$

The morphism  $f_2: A_2 \rightarrow B_2$  is defined using the universal property of the pullback.

*Remark 2.3.* Let  $\mathbf{Cat}(\mathcal{E})$  denote the category of internal categories and internal functors in a fixed category  $\mathcal{E}$  with pullbacks. It is well-known that  $\mathbf{Cat}(\mathcal{E})$  has pullbacks which are computed component-wise.

There are two classes of internal functors which are of particular interest, and the following lemma will be useful for many of the main results.

**Definition 2.4.** An internal functor  $f: A \rightarrow B$  is called *isomorphism-on-objects* if the morphism  $f_0$  is an isomorphism. In particular, if  $f_0 = 1$  the functor is called *identity-on-objects*.

**Definition 2.5.** An *internal discrete opfibration* is an internal functor  $f: A \rightarrow B$  such that the following commutative diagram, appearing in (2.4), is a pullback:

$$\begin{array}{ccc}
 & A_1 & \\
 d_1 \swarrow & & \searrow f_1 \\
 A_0 & & B_1 \\
 f_0 \searrow & & \swarrow d_1 \\
 & B_0 &
 \end{array} \tag{2.6}$$

Dually, an *internal discrete fibration* is an internal functor  $f: A \rightarrow B$  such that the right-hand square in (2.4) is a pullback.

**Lemma 2.6.** Let  $\mathcal{C}$  denote the class of *isomorphism-on-objects* internal functors (resp. *internal discrete opfibrations*). Then  $\mathcal{C}$  satisfies the following conditions:

- (i)  $\mathcal{C}$  is closed under composition;
- (ii)  $\mathcal{C}$  contains the isomorphisms;
- (iii)  $\mathcal{C}$  is stable under pullback by internal functors;
- (iv) for all composable pairs of internal functors  $f: A \rightarrow B$  and  $g: B \rightarrow C$ , if both  $g$  and  $g \circ f$  are in  $\mathcal{C}$ , then  $f$  is in  $\mathcal{C}$ .

*Remark 2.7.* An internal discrete opfibration is an internal functor with a certain *property*, however this property may also be specified by *unique structure*. Thus an internal discrete opfibration is equivalent to internal functor equipped with a morphism,

$$\varphi_1: A_0 \times_{B_0} B_1 \longrightarrow A_1$$

making the following diagrams commute:

$$\begin{array}{ccc} & A_0 \times_{B_0} B_1 & \\ \pi_0 \swarrow & \downarrow \varphi_1 & \searrow \pi_1 \\ A_0 & \xleftarrow{d_1} A_1 & \xrightarrow{f_1} B_1 \end{array} \qquad \begin{array}{ccc} & A_0 \times_{B_0} B_1 & \\ \langle d_1, f_1 \rangle \nearrow & & \searrow \varphi_1 \\ A_1 & \xrightarrow{1} & A_1 \end{array}$$

This relationship between property and structure will later be important when characterising internal split opfibrations.

**Example 2.8** ( $\mathcal{E} = \mathbf{Set}$ ). A discrete opfibration is a functor  $f: A \rightarrow B$  between small categories such that for each pair  $(a \in A, u: fa \rightarrow b \in B)$  there exists a *unique* morphism  $\varphi(a, u): a \rightarrow a'$  in  $A$  such that  $f\varphi(a, u) = u$ .

$$\begin{array}{ccc} A & & a \xrightarrow{\varphi(a, u)} a' \\ f \downarrow & & \vdots \qquad \qquad \vdots \\ B & & fa \xrightarrow{u} b \end{array}$$

Note that the codomain  $a'$  of the morphism  $\varphi(a, u)$  is also a function of the pair  $(a, u)$ , and this may be made explicit through writing  $a' = p(a, u)$ .

Internal cofunctors will now be introduced as a kind of morphism between internal categories. They generalise both internal discrete opfibrations and isomorphism-on-objects internal functors. Since the definition of an internal cofunctor is more involved than that of an internal functor, it is beneficial to first introduce the definition for small categories.

**Example 2.9** ( $\mathcal{E} = \mathbf{Set}$ ). A *cofunctor*  $\varphi: B \nrightarrow A$  between small categories consists of a function on objects  $a \in A \mapsto \varphi a \in B$  together with a function on pairs,

$$(a \in A, u: \varphi a \rightarrow b \in B) \quad \longmapsto \quad \varphi(a, u): a \rightarrow p(a, u)$$

satisfying the axioms:

- (1)  $\varphi p(a, u) = b$
- (2)  $\varphi(a, 1_{\varphi a}) = 1_a$
- (3)  $\varphi(a, v \circ u) = \varphi(p(a, u), v) \circ \varphi(a, u)$

A cofunctor may be understood as a functional *lifting* arrows from  $B$  to  $A$ :

$$\begin{array}{ccc} A & & a \xrightarrow{\varphi(a, u)} p(a, u) \\ \varphi \uparrow & & \vdots \qquad \qquad \vdots \\ B & & \varphi a \xrightarrow{u} b \end{array}$$

Note that the codomain  $p(a, u)$  of the morphism  $\varphi(a, u)$  is given explicitly as a function of the pair  $(a, u)$ .

**Definition 2.10.** Let  $A$  and  $B$  be internal categories. An *internal cofunctor*  $\varphi: B \rightarrow A$  is a pair of morphisms,

$$\varphi_0: A_0 \longrightarrow B_0 \quad \varphi_1: \Lambda_1 \longrightarrow A_1$$

with the objects  $\Lambda_1$  and  $\Lambda_2$  defined by the pullbacks,

$$\begin{array}{ccc} & \Lambda_1 & \\ d_1 \swarrow & \downarrow & \searrow \bar{\varphi}_1 \\ A_0 & & B_1 \\ \varphi_0 \searrow & & \swarrow d_1 \\ & B_0 & \end{array} \quad \begin{array}{ccc} & \Lambda_2 & \\ d_2 \swarrow & \downarrow & \searrow \bar{\varphi}_2 \\ \Lambda_1 & & B_2 \\ \bar{\varphi}_1 \searrow & & \swarrow d_2 \\ & B_1 & \end{array} \quad (2.7)$$

satisfying commutative diagrams with respect to the domain and codomain maps,

$$\begin{array}{ccccc} & \Lambda_1 & \xrightarrow{\bar{\varphi}_1} & B_1 & \\ d_1 \swarrow & \downarrow \varphi_1 & & \downarrow d_0 & \\ A_0 & \xleftarrow{d_1} A_1 & \xrightarrow{d_0} A_0 & \xrightarrow{\varphi_0} B_0 & \end{array} \quad (2.8)$$

and with respect to the identity and composition maps:

$$\begin{array}{ccc} A_0 & \xrightarrow{i_0} & \Lambda_1 \\ \parallel & & \downarrow \varphi_1 \\ A_0 & \xrightarrow{i_0} & A_1 \end{array} \quad \begin{array}{ccc} \Lambda_2 & \xrightarrow{d_1} & \Lambda_1 \\ \varphi_2 \downarrow & & \downarrow \varphi_1 \\ A_2 & \xrightarrow{d_1} & A_1 \end{array} \quad (2.9)$$

It is useful to define the morphism  $p_0 = d_0\varphi_1: \Lambda_1 \rightarrow A_0$ . The morphisms  $i_0: A_0 \rightarrow \Lambda_1$ ,  $d_1: \Lambda_2 \rightarrow \Lambda_1$ , and  $\varphi_2: \Lambda_2 \rightarrow A_2$  are defined using the universal property of the pullback:

$$\begin{array}{ccccccc} & A_0 & & \Lambda_2 & & \Lambda_2 & & \Lambda_2 \\ & \downarrow \varphi_0 & & \downarrow d_1 & & \downarrow \bar{\varphi}_2 & & \downarrow \bar{\varphi}_2 \\ 1 \swarrow & \Lambda_1 & & \Lambda_1 & & \Lambda_1 & & \Lambda_1 \\ & \downarrow d_1 & & \downarrow d_1 & & \downarrow p_0 & & \downarrow \varphi_2 \\ & \downarrow \varphi_0 & & \downarrow \varphi_0 & & \downarrow \varphi_0 & & \downarrow d_0 \\ & B_0 & & B_0 & & B_0 & & A_0 \end{array}$$

*Notation 2.11.* In contrast to Remark 2.7, the above definition uses the notation  $\Lambda_1$  for the pullback  $A_0 \times_{B_0} B_1$  as it is more compact (likewise for  $\Lambda_2$ ). The above definition also uses the suggestive notation  $d_1: \Lambda_1 \rightarrow A_0$  and  $\bar{\varphi}_1: \Lambda_1 \rightarrow B_1$  for the pullback projections  $\pi_0: A_0 \times_{B_0} B_1 \rightarrow A_0$  and  $\pi_1: A_0 \times_{B_0} B_1 \rightarrow B_1$ , respectively, as this will be particularly useful for the following results.

*Remark 2.12.* Let  $\mathbf{Cof}(\mathcal{E})$  denote the category of internal categories and internal cofunctors in a fixed category  $\mathcal{E}$  with pullbacks. Given internal cofunctors  $\gamma: C \rightrightarrows B$  and  $\varphi: B \rightrightarrows A$ , the composite internal cofunctor  $\varphi \circ \gamma: C \rightrightarrows A$  consists of the pair of morphisms:

$$\gamma_0 \varphi_0: A_0 \longrightarrow C_0 \quad \varphi_1 \langle \pi_0, \gamma_1(\varphi_0 \times 1) \rangle: A_0 \times_{C_0} C_1 \longrightarrow A_1 \quad (2.10)$$

**Example 2.13.** An internal discrete opfibration  $A \rightarrow B$  induces an internal cofunctor  $B \rightrightarrows A$ , while an isomorphism-on-objects internal functor  $A \rightarrow B$  induces an internal cofunctor  $A \rightrightarrows B$ .

Let  $\mathbf{Span}_{\text{iso}}(\mathbf{Cat}(\mathcal{E}))$  denote the *category* whose objects are internal categories and whose morphisms are *isomorphism classes* of spans of internal functors. There exists a faithful, identity-on-objects functor  $\mathbf{Cof}(\mathcal{E}) \rightarrow \mathbf{Span}_{\text{iso}}(\mathbf{Cat}(\mathcal{E}))$  which assigns each internal cofunctor to an isomorphism class of spans, which have left leg an internal discrete opfibration and right leg an isomorphism-on-objects internal functor. In practice, an internal cofunctor will always be identified with a chosen representative of this isomorphism class, whose right leg is an identity-on-object internal functor. This representative span for a cofunctor will now be constructed.

**Lemma 2.14.** *Given an internal cofunctor  $\varphi: B \rightrightarrows A$ , there exists an internal category, denoted  $\Lambda$ , defined by the diagram:*

$$\begin{array}{ccccc} & \longleftarrow p_0 \longrightarrow & & \longleftarrow p_0 \longrightarrow & \\ A_0 & \xrightarrow{i_0} & \Lambda_1 & \xleftarrow{d_1} & \Lambda_2 \\ & \longleftarrow d_1 \longrightarrow & & \longleftarrow d_2 \longrightarrow & \end{array}$$

*Proof.* We prove that axiom (2.2) for an internal category is satisfied, and suppress details required to prove (2.3).

First, to show  $\Lambda_2$  is the pullback of the morphisms  $d_1, p_0: \Lambda_1 \rightarrow A_0$ , consider the following diagrams which are equal by the construction of  $p_0: \Lambda_2 \rightarrow \Lambda_1$  in Definition 2.10:

$$\begin{array}{ccc} \Lambda_2 \xrightarrow{p_0} \Lambda_1 \xrightarrow{\bar{\varphi}_1} B_1 & & \Lambda_2 \xrightarrow{\bar{\varphi}_2} B_2 \xrightarrow{d_0} B_1 \\ d_2 \downarrow \quad d_1 \downarrow \quad \lrcorner \quad \downarrow d_1 & = & d_2 \downarrow \quad \lrcorner \quad d_2 \downarrow \quad \lrcorner \quad \downarrow d_1 \\ \Lambda_1 \xrightarrow{p_0} A_0 \xrightarrow{\varphi_0} B_0 & & \Lambda_1 \xrightarrow{\bar{\varphi}_1} B_1 \xrightarrow{d_0} B_0 \end{array}$$

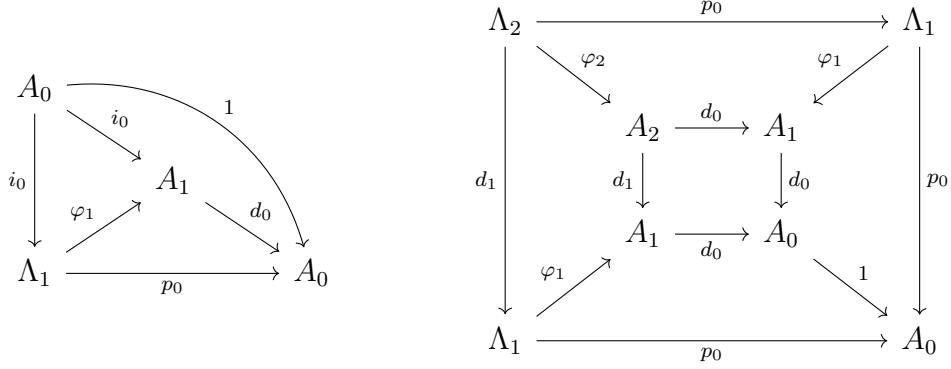
Using the pullback pasting lemma, the remaining square must be a pullback, as required.

To show the identity map  $i_0: A_0 \rightarrow \Lambda_1$  and the composition map  $d_1: \Lambda_2 \rightarrow \Lambda_1$  are well-defined, first notice by their construction in Definition 2.10 that the following diagrams commute:

$$\begin{array}{ccc} & A_0 & \\ \swarrow 1 & \downarrow i_0 & \\ A_0 & \xleftarrow{d_1} & \Lambda_1 \end{array} \quad \begin{array}{ccc} \Lambda_1 & \xleftarrow{d_2} & \Lambda_2 \\ d_1 \downarrow & & \downarrow d_1 \\ A_0 & \xleftarrow{d_1} & \Lambda_1 \end{array}$$



The counterparts to the above diagrams are obtained by pasting as follows:



Thus the axiom (2.2) is satisfied as required.  $\square$

**Example 2.15** ( $\mathcal{E} = \mathbf{Set}$ ). For a cofunctor  $\varphi: B \rightarrowtail A$  between small categories as in Example 2.9, the category  $\Lambda$  has objects  $a \in A$  and morphisms given by pairs depicted:

$$a \xrightarrow{(a, u)} p(a, u)$$

Thus  $\Lambda$  may be understood as the category of *chosen lifts* in  $A$  with respect to  $\varphi$ .

**Proposition 2.16.** *Given an internal cofunctor  $\varphi: B \rightarrowtail A$  there is a span of internal functors,*

$$\begin{array}{ccc} & \Lambda & \\ \bar{\varphi} \swarrow & & \searrow \varphi \\ B & & A \end{array} \quad (2.11)$$

with left leg an internal discrete opfibration and right leg an identity-on-objects internal functor.

*Proof.* The internal discrete opfibration  $\bar{\varphi}: \Lambda \rightarrow B$  is determined by the pair of morphisms  $(\varphi_0, \bar{\varphi}_1)$  which satisfy the commutative diagrams:

$$\begin{array}{ccccc} A_0 & \xleftarrow{d_1} & \Lambda_1 & \xrightarrow{p_0} & A_0 \\ \varphi_0 \downarrow & & \downarrow \bar{\varphi}_1 & & \downarrow \varphi_0 \\ B_0 & \xleftarrow{d_1} & B_1 & \xrightarrow{d_0} & B_0 \end{array} \quad \begin{array}{ccccc} A_0 & \xrightarrow{i_0} & \Lambda_1 & & \\ \varphi_0 \downarrow & & \downarrow \bar{\varphi}_1 & & \\ B_0 & \xrightarrow{i_0} & B_1 & & \end{array} \quad \begin{array}{ccccc} \Lambda_2 & \xrightarrow{d_1} & \Lambda_1 & & \\ \bar{\varphi}_2 \downarrow & & \downarrow \bar{\varphi}_1 & & \\ B_2 & \xrightarrow{d_1} & B_1 & & \end{array}$$

The internal identity-on-objects functor  $\varphi: \Lambda \rightarrow A$  is determined by the pair of morphisms  $(1_{A_0}, \varphi_1)$  which satisfy the commutative diagrams:

$$\begin{array}{ccccc} A_0 & \xleftarrow{d_1} & \Lambda_1 & \xrightarrow{p_0} & A_0 \\ 1 \downarrow & & \downarrow \varphi_1 & & \downarrow 1 \\ A_0 & \xleftarrow{d_1} & A_1 & \xrightarrow{d_0} & A_0 \end{array} \quad \begin{array}{ccccc} A_0 & \xrightarrow{i_0} & \Lambda_1 & & \\ 1 \downarrow & & \downarrow \varphi_1 & & \\ A_0 & \xrightarrow{i_0} & A_1 & & \end{array} \quad \begin{array}{ccccc} \Lambda_2 & \xrightarrow{d_1} & \Lambda_1 & & \\ \varphi_2 \downarrow & & \downarrow \varphi_1 & & \\ A_2 & \xrightarrow{d_1} & A_1 & & \end{array}$$

These commutative diagrams all appear in Definition 2.10, thus the proof is complete.  $\square$

The above proposition shows that every internal cofunctor may be understood as a chosen representative span of an isomorphism class in  $\mathbf{Span}_{\text{iso}}(\mathbf{Cat}(\mathcal{E}))$ . Given an arbitrary span in a suitable isomorphism class, there is a converse which states that there exists a corresponding internal cofunctor.

**Proposition 2.17.** *Given a span of internal functors,*

$$\begin{array}{ccc} & X & \\ h \swarrow & & \searrow g \\ B & & A \end{array}$$

*with left leg a discrete opfibration and right leg an isomorphism-on-objects functor, there is an internal cofunctor  $B \rightharpoonup A$ .*

*Proof.* Since  $g: X \rightarrow A$  is isomorphism-on-objects, define the composite morphism:

$$\begin{array}{ccc} A_0 & \xrightarrow{\varphi_0} & B_0 \\ & \searrow g_0^{-1} \quad \nearrow h_0 & \\ & X_0 & \end{array}$$

This morphism may be used to define the pullback  $\Lambda_1$  as in (2.7). Since  $h: X \rightarrow B$  is an internal discrete opfibration, there exists a morphism  $g_0^{-1} \times 1: \Lambda_1 \rightarrow X_1$  using the universal property of the pullback, which may be used to define the composite morphism:

$$\begin{array}{ccc} \Lambda_1 & \xrightarrow{\varphi_1} & A_1 \\ & \searrow g_0^{-1} \times 1 \quad \nearrow g_1 & \\ & X_1 & \end{array}$$

The corresponding internal cofunctor  $\varphi: B \rightharpoonup A$  is constructed from the pair of morphisms  $(\varphi_0, \varphi_1)$ . We suppress the details required to show that the axioms for a cofunctor are satisfied.  $\square$

Representing internal cofunctors as spans of internal functors also shows how the apparently complicated composition in  $\mathbf{Cof}(\mathcal{E})$  given by (2.10) may be understood simply as span composition:

$$\begin{array}{ccccc} & & \Omega \times_B \Lambda & & \\ & \swarrow & \downarrow \vee & \searrow & \\ & \Omega & & \Lambda & \\ \bar{\gamma} \swarrow & & \gamma \searrow & & \bar{\varphi} \swarrow \quad \searrow \varphi \\ C & & B & & A \end{array} \tag{2.12}$$

This composition is well-defined by Lemma 2.6, and using Proposition 2.17 it may be shown that the construction  $\mathbf{Cof}(\mathcal{E}) \rightarrow \mathbf{Span}_{\text{iso}}(\mathbf{Cat}(\mathcal{E}))$  is functorial.

### 3. INTERNAL LENSES

This section introduces internal lenses between internal categories, in a fixed category  $\mathcal{E}$  with pullbacks, generalising the concept of a delta lens from [DXC11]. Internal lenses

may be represented as commutative diagrams of internal functors, and are closed under composition and stable under pullback.

**Definition 3.1.** Let  $A$  and  $B$  be internal categories. An *internal lens*  $(f, \varphi): A \rightleftharpoons B$  consists of an internal functor  $f: A \rightarrow B$  and an internal cofunctor  $\varphi: B \rightharpoonup A$  satisfying the commutative diagrams:

$$\begin{array}{ccc} & A_0 & \\ \swarrow & & \searrow \varphi_0 \\ A_0 & \xrightarrow{f_0} & B_0 \end{array} \qquad \begin{array}{ccc} & \Lambda_1 & \\ \swarrow \varphi_1 & & \searrow \bar{\varphi}_1 \\ A_1 & \xrightarrow{f_1} & B_1 \end{array} \quad (3.1)$$

*Remark 3.2.* Let  $\mathbf{Lens}(\mathcal{E})$  denote the category of internal categories and internal lenses in a fixed category  $\mathcal{E}$  with pullbacks. Given internal lenses  $(f, \varphi): A \rightleftharpoons B$  and  $(g, \gamma): B \rightleftharpoons C$ , the composite internal lens  $(g \circ f, \varphi \circ \gamma): A \rightleftharpoons C$  is given by the composite internal functor  $g \circ f: A \rightarrow C$  and the composite internal cofunctor  $\varphi \circ \gamma: C \rightharpoonup A$ . One may check that (3.1) holds.

**Example 3.3** ( $\mathcal{E} = \mathbf{Set}$ , see [DXC11]). A *delta lens*  $(f, \varphi): A \rightleftharpoons B$  between small categories consists of a functor  $f: A \rightarrow B$  together with a function,

$$(a \in A, u: fa \rightarrow b \in B) \quad \longmapsto \quad \varphi(a, u): a \rightarrow p(a, u)$$

satisfying the axioms:

- (1)  $f\varphi(a, u) = u$
- (2)  $\varphi(a, 1_{fa}) = 1_a$
- (3)  $\varphi(a, v \circ u) = \varphi(p(a, u), v) \circ \varphi(a, u)$

A delta lens is exactly an internal lens in  $\mathbf{Set}$ , and may be understood as functor equipped with a functorial *lifting* of morphisms from the codomain to the domain. Unlike a split opfibration however, there is no requirement for these lifts to be opcartesian. Based on the diagram for a cofunctor in Example 2.9, the following illustrates the behaviour of a delta lens:

$$\begin{array}{ccc} A & & a \xrightarrow{\varphi(a, u)} p(a, u) \\ \varphi \uparrow \Big\| f & & \vdots \qquad \qquad \vdots \\ B & & fa \xrightarrow{u} b \end{array}$$

**Example 3.4** (See [FGM<sup>+</sup>07, JRW10]). Let  $\mathcal{E}$  have finite limits. A *very well-behaved lens* consists of a pair of morphisms in  $\mathcal{E}$ ,

$$f: A \longrightarrow B \qquad p: A \times B \longrightarrow A$$

satisfying three commutative diagrams:

$$\begin{array}{ccc} & A \times B & \\ p \swarrow & & \searrow \pi_1 \\ A & \xrightarrow{f} & B \end{array} \qquad \begin{array}{ccc} & A \times B & \\ \langle 1_A, f \rangle \swarrow & & \searrow p \\ A & \xrightarrow{1_A} & A \end{array} \qquad \begin{array}{ccc} A \times B \times B & \xrightarrow{p \times 1_B} & A \times B \\ \pi_{0,2} \downarrow & & \downarrow p \\ A \times B & \xrightarrow{p} & A \end{array}$$

A very well-behaved lens is exactly an internal lens between internal codiscrete categories. It consists of the internal functor between the codiscrete categories on  $A$  and  $B$ , and the internal cofunctor where  $\Lambda_1 = A \times B$  and  $\varphi_1 = \langle \pi_0, p \rangle: A \times B \rightarrow A \times A$ . The special case of a delta lens between small codiscrete categories was shown in [JR16]. It was shown in [JRW10] that the morphism  $f: A \rightarrow B$  in a very well-behaved lens must be equivalent to a product projection with  $A \cong C \times B$ .

**Example 3.5.** Let  $\mathcal{E}$  have finite limits, and let  $A$  and  $B$  be internal monoids, considered as internal categories whose object of objects is terminal. An internal lens  $(f, \varphi): A \rightleftharpoons B$  is exactly a monoid homomorphism  $f: A \rightarrow B$  with a chosen right inverse  $\varphi: B \rightarrow A$ . For arbitrary internal categories, any isomorphism-on-objects internal functor with a chosen right inverse forms an internal lens.

**Example 3.6.** An internal lens  $(f, \varphi): A \rightleftharpoons B$  is an internal discrete opfibration if and only if  $\varphi_1: \Lambda_1 \rightarrow A_1$  is an isomorphism (see Remark 2.7). Internal lenses may be understood as being more general than internal discrete opfibrations while being less general than internal cofunctors.

The next two results follow almost immediately from the corresponding results for cofunctors (Proposition 2.16 and Proposition 2.17) and the definition of an internal lens.

**Proposition 3.7.** *Given an internal lens  $(f, \varphi): A \rightleftharpoons B$  there is a commutative diagram of internal functors,*

$$\begin{array}{ccc} & \Lambda & \\ \varphi \swarrow & & \searrow \bar{\varphi} \\ A & \xrightarrow{f} & B \end{array} \quad (3.2)$$

where  $\varphi$  is a faithful, identity-on-objects functor and  $\bar{\varphi}$  is a discrete opfibration.

**Proposition 3.8.** *Given a commutative diagram of internal functors,*

$$\begin{array}{ccc} & X & \\ g \swarrow & & \searrow h \\ A & \xrightarrow{f} & B \end{array}$$

where  $g$  is an isomorphism-on-objects functor and  $h$  is a discrete opfibration, there is an internal lens  $A \rightleftharpoons B$ .

The above propositions allow one to identify an internal lens with a diagram (3.2), and prove results concerning internal lenses using the properties of isomorphisms-on-objects internal functors and internal discrete opfibrations stated in Lemma 2.6. In Theorem 4.7, internal split opfibrations will be characterised in this way.

*Remark 3.9.* Given internal lenses  $(f, \varphi): A \rightleftharpoons B$  and  $(g, \gamma): B \rightleftharpoons C$ , their composite may be computed by constructing pullbacks in  $\mathbf{Cat}(\mathcal{E})$ :

$$\begin{array}{ccccc}
 & & \Lambda \times_B \Omega & & \\
 & \swarrow & \downarrow \vee & \searrow & \\
 & \Lambda & & \Omega & \\
 \varphi \swarrow & & \bar{\varphi} \searrow & \gamma \swarrow & \bar{\gamma} \searrow \\
 A & \xrightarrow{f} & B & \xrightarrow{g} & C
 \end{array}$$

**Proposition 3.10** (Stability under pullback). *Given a pullback square of internal functors,*

$$\begin{array}{ccc}
 A \times_B C & \xrightarrow{\pi_1} & C \\
 \pi_0 \downarrow & \lrcorner & \downarrow g \\
 A & \xrightarrow{f} & B
 \end{array}$$

*if  $(f, \varphi): A \rightleftharpoons B$  is an internal lens, then  $\pi_1$  has the structure of an internal lens.*

*Proof.* Given that  $(f, \varphi)$  is an internal lens, using Proposition 3.7 we may construct the following diagram of internal functors:

$$\begin{array}{ccccc}
 \Lambda \times_B C & \xrightarrow{\varphi \times 1} & A \times_B C & \xrightarrow{\pi_1} & C \\
 \pi_0 \downarrow & \lrcorner & \pi_0 \downarrow & \lrcorner & \downarrow g \\
 \Lambda & \xrightarrow{\varphi} & A & \xrightarrow{f} & B
 \end{array}$$

By Lemma 2.6, both identity-on-objects functors and discrete opfibrations are stable under pullback along internal functors, thus  $\varphi \times 1: \Lambda \times_B C \rightarrow A \times_B C$  is identity-on-objects and  $\pi_1: A \times_B C \rightarrow C$  is a discrete opfibration. By applying Proposition 3.8, the internal functor  $\pi_1: A \times_B C \rightarrow C$  has the structure of an internal lens.  $\square$

#### 4. INTERNAL SPLIT OPFIBRATIONS AND DÉCALAGE

This section defines an internal split opfibration, in a fixed category  $\mathcal{E}$  with pullbacks, as an internal lens with a certain property. This definition is shown to be equivalent to a previous characterisation of split opfibrations in [AU17], and the main theorem will further characterise internal split opfibrations using the right décalage construction.

To prepare for the main definition, consider an internal lens  $(f, \varphi): A \rightleftharpoons B$  and construct the following pullback:

$$\begin{array}{ccc}
 & \Lambda_1 \times_{A_1} A_2 & \\
 \pi_0 \swarrow & \downarrow \vee & \searrow \pi_1 \\
 \Lambda_1 & & A_2 \\
 \varphi_1 \searrow & & \swarrow d_2 \\
 & A_1 &
 \end{array} \tag{4.1}$$

The projection  $\pi_1: \Lambda_1 \times_{A_1} A_2 \rightarrow A_2$  may be pasted with the commutative diagram in (2.5) for the composition map of an internal functor to obtain:

$$\begin{array}{ccc}
 \Lambda_1 \times_{A_1} A_2 & & \\
 \swarrow d_1 \pi_1 & \downarrow \pi_1 & \searrow f_2 \pi_1 \\
 & A_2 & \\
 \swarrow d_1 & & \searrow f_2 \\
 A_1 & & B_2 \\
 \searrow f_1 & & \swarrow d_1 \\
 & B_1 &
 \end{array}
 =
 \begin{array}{ccc}
 \Lambda_1 \times_{A_1} A_2 & & \\
 \swarrow d_1 \pi_1 & & \searrow f_2 \pi_1 \\
 A_1 & & B_2 \\
 \searrow f_1 & & \swarrow d_1 \\
 & B_1 &
 \end{array}
 \quad (4.2)$$

To reiterate, the diagram (4.2) commutes for any internal lens  $(f, \varphi): A \rightleftharpoons B$ .

**Definition 4.1.** An *internal split opfibration* is an internal lens  $(f, \varphi): A \rightleftharpoons B$  such that the following commutative diagram, constructed in (4.2), is a pullback:

$$\begin{array}{ccc}
 \Lambda_1 \times_{A_1} A_2 & & \\
 \swarrow d_1 \pi_1 & & \searrow f_2 \pi_1 \\
 A_1 & & B_2 \\
 \searrow f_1 & & \swarrow d_1 \\
 & B_1 &
 \end{array}$$

The above definition mirrors Definition 2.5, which states that an internal discrete opfibration is an internal functor  $f: A \rightarrow B$  such that the commutative diagram (2.6) is a pullback.

While an internal lens may have the *property* of being an internal split opfibration, this property may also be specified by *unique structure*, as was done in Remark 2.7 for internal discrete opfibrations. This characterisation in terms of unique structure will be described Proposition 4.2 and allows for a straightforward verification that the above definition correctly generalises the  $\mathcal{E} = \mathbf{Set}$  case.

There is also the question of proving that internal categories and internal split opfibrations form a category, denoted  $\mathbf{SOpf}(\mathcal{E})$ , and also that internal split opfibrations are stable under pullback. An answer is found through a natural characterisation of internal split opfibrations via the right décalage comonad in Theorem 4.7, which utilises the equivalent representation of an internal lens given by (3.2). This characterisation also provides motivation for *why* the commuting diagram (4.2) is used to define an internal split opfibration.

Finally, one might protest that the real benefit in defining internal discrete opfibrations using a pullback condition is that object of morphisms of the domain is endowed with the corresponding universal property, which is not apparent for internal split opfibrations as given in Definition 4.1. This is addressed in the following section, where it is shown that an internal split opfibration may also be characterised as an internal lens with a particular strict factorisation system on the domain, which endows the object of morphisms with a suitable universal property.

First, to unpack the definition of an internal split opfibration consider the pullback:

$$\begin{array}{ccc}
 & A_1 \times_{B_1} B_2 & \\
 \pi_0 \swarrow & \downarrow \vee & \searrow \pi_1 \\
 A_1 & & B_2 \\
 & \swarrow f_1 & \nwarrow d_1 \\
 & B_1 &
 \end{array} \tag{4.3}$$

The following result characterises internal split opfibrations in terms of unique structure on an internal lens. This characterisation is essentially the same as given in [AU17], except formulated using internal category theory rather than directed containers.

**Proposition 4.2.** *An internal lens  $(f, \varphi): A \rightrightarrows B$  is an internal split opfibration if and only if there exists a morphism,*

$$\psi: A_1 \times_{B_1} B_2 \longrightarrow A_1$$

*satisfying the following four commutative diagrams:*

◇ *This diagram specifies a “domain condition”:*

$$\begin{array}{ccccc}
 A_1 \times_{B_1} B_2 & \xrightarrow{\psi} & A_1 & & \\
 d_1 \times d_2 \downarrow & & & & \downarrow d_1 \\
 \Lambda_1 & \xrightarrow{\varphi_1} & A_1 & \xrightarrow{d_0} & A_0
 \end{array} \tag{4.4}$$

◇ *This diagram specifies a “uniqueness condition”:*

$$\begin{array}{ccccc}
 \Lambda_1 \times_{A_1} A_2 & \xrightarrow{\pi_1} & A_2 & \xrightarrow{\langle d_1, f_2 \rangle} & A_1 \times_{B_1} B_2 \\
 \pi_1 \downarrow & & & & \downarrow \psi \\
 A_2 & \xrightarrow{\quad d_0 \quad} & A_1 & & 
 \end{array} \tag{4.5}$$

◇ *This diagram specifies a “lifting condition”:*

$$\begin{array}{ccc}
 A_1 \times_{B_1} B_2 & \xrightarrow{\psi} & A_1 \\
 \pi_1 \downarrow & & \downarrow f_1 \\
 B_2 & \xrightarrow{d_0} & B_1
 \end{array} \tag{4.6}$$

◇ *This diagram specifies a “composition condition”:*

$$\begin{array}{ccc}
 & A_1 \times_{B_1} B_2 & \\
 \hat{\psi} \swarrow & & \searrow \pi_0 \\
 A_2 & \xrightarrow{d_1} & A_1
 \end{array} \tag{4.7}$$

The pullback  $\Lambda_1 \times_{A_1} A_2$  is from (4.1), the pullback  $A_1 \times_{B_1} B_2$  is from (4.3), and the morphism  $\widehat{\psi}$  is defined using (4.4) via the universal property of the pullback:

$$\begin{array}{ccccc}
 & & A_1 \times_{B_1} B_2 & & \\
 & d_1 \times d_2 \swarrow & \downarrow \widehat{\psi} & \searrow \psi & \\
 \Lambda_1 & & A_2 & & \\
 \varphi_1 \downarrow & d_2 \swarrow & \downarrow \checkmark & \searrow d_0 & \\
 A_1 & & & & A_1 \\
 & d_0 \swarrow & & \searrow d_1 & \\
 & & A_0 & & 
 \end{array}$$

*Proof.* First notice from Definition 4.1 that an internal lens  $(f, \varphi): A \rightrightarrows B$  is an internal split opfibration if and only if there is an isomorphism  $\Lambda_1 \times_{A_1} A_2 \cong A_1 \times_{B_1} B_1$ . For any internal lens the diagram (4.2) commutes, thus following composite morphism exists by the universal property of the pullback:

$$\Lambda_1 \times_{A_1} A_2 \xrightarrow{\pi_1} A_2 \xrightarrow{\langle d_1, f_2 \rangle} A_1 \times_{B_1} B_2 \quad (4.8)$$

Therefore an internal lens is an internal split opfibration if and only if (4.8) has an inverse. Given a morphism  $\psi: A_1 \times_{B_1} B_2 \rightarrow A_1$  satisfying axiom (4.4) there exists a morphism,

$$\langle d_1 \times d_2, \widehat{\psi} \rangle: A_1 \times_{B_1} B_2 \longrightarrow \Lambda_1 \times_{A_1} A_2$$

which is the required inverse if axioms (4.5), (4.6), and (4.7) are satisfied. Conversely, an inverse to (4.8) exists exactly when there is a morphism  $\psi$  which satisfies the axioms in the statement of Proposition 4.2. The necessary diagram-chasing has been excluded.  $\square$

The following example shows that the above characterisation of internal split opfibrations yields the usual definition for  $\mathcal{E} = \mathbf{Set}$ , and explains the meaning of the names given to the axioms above.

**Example 4.3** ( $\mathcal{E} = \mathbf{Set}$ ). A delta lens  $(f, \varphi): A \rightrightarrows B$  (see Example 3.3) is a split opfibration if for all commutative diagrams of the form,

$$\begin{array}{ccc}
 fa & \xrightarrow{u} & b \\
 & \searrow f w & \swarrow v \\
 & & fa'
 \end{array} \quad \text{in } B$$

for all  $w: a \rightarrow a'$  in  $A$ , there exists a unique morphism  $\psi(w, u, v): p(a, u) \rightarrow a'$  such that  $f\psi(w, u, v) = v$  and the following diagram commutes:

$$\begin{array}{ccc}
 a & \xrightarrow{\varphi(a, u)} & p(a, u) \\
 & \searrow w & \swarrow \psi(w, u, v) \\
 & & a'
 \end{array} \quad \text{in } A$$

The domain of  $\psi(w, u, v)$  is determined by (4.4) and the uniqueness by (4.5). The other two conditions on  $\psi(w, u, v)$  are determined exactly by (4.6) and (4.7), respectively. Thus  $\psi$  provides a global way of stating that the chosen lifts given by  $\varphi$  are *f-opcartesian*.



**Example 4.4.** Every internal discrete opfibration is an internal split opfibration.

There is an intrinsic way of characterising internal split opfibrations with respect to the right décalage construction on  $\mathbf{Cat}(\mathcal{E})$ , using the properties of isomorphism-on-objects internal functors and internal discrete opfibrations from Lemma 2.6.

**Definition 4.5.** The *right décalage* construction is a comonad  $D$  on  $\mathbf{Cat}(\mathcal{E})$  which assigns each internal functor  $f: A \rightarrow B$  to an internal functor  $Df: DA \rightarrow DB$  given by:

$$\begin{array}{ccccc} A_1 & \xleftarrow{d_1} & A_2 & \xrightarrow{d_0} & A_1 \\ f_1 \downarrow & & f_2 \downarrow & & \downarrow f_1 \\ B_1 & \xleftarrow{d_1} & B_2 & \xrightarrow{d_0} & B_1 \end{array} \quad (4.9)$$

The counit  $\varepsilon: D \Rightarrow 1$  for the comonad assigns each internal category  $A$  to an internal discrete fibration  $\varepsilon_A: DA \rightarrow A$  given by:

$$\begin{array}{ccccc} A_1 & \xleftarrow{d_1} & A_2 & \xrightarrow{d_0} & A_1 \\ d_1 \downarrow & & d_2 \downarrow & \lrcorner & \downarrow d_1 \\ A_0 & \xleftarrow{d_1} & A_1 & \xrightarrow{d_0} & A_0 \end{array} \quad (4.10)$$

**Example 4.6** ( $\mathcal{E} = \mathbf{Set}$ ). Given a small category  $A$ , the right décalage  $DA$  has objects given by arrows  $w: a' \rightarrow a$  in  $A$  and morphisms  $u: w \rightsquigarrow v$  given by commutative triangles:

$$\begin{array}{ccc} a' & \xrightarrow{u} & a'' \\ & \searrow w & \swarrow v \\ & a & \end{array}$$

That is, the right décalage  $DA$  is given by the coproduct of the slice categories of  $A$ . The counit  $\varepsilon_A: DA \rightarrow A$  sends a morphism  $u: w \rightsquigarrow v$  in  $DA$ , depicted above, to the morphism  $u: a' \rightarrow a''$  in  $A$ .

Given an internal lens  $(f, \varphi): A \rightleftharpoons B$  with representation in  $\mathbf{Cat}(\mathcal{E})$  given by (3.2), construct the following diagram using the counit of the right décalage comonad:

$$\begin{array}{ccccc} \Lambda \times_A DA & \xrightarrow{\pi_1} & DA & \xrightarrow{Df} & DB \\ \pi_0 \downarrow & \lrcorner & \downarrow \varepsilon_A & & \downarrow \varepsilon_B \\ \Lambda & \xrightarrow{\varphi} & A & \xrightarrow{f} & B \\ & \searrow \varphi & & & \end{array} \quad (4.11)$$

Notice that the projection  $\pi_1$  is an identity-on-objects internal functor, therefore it is natural to ask: for which internal lenses  $(f, \varphi): A \rightleftharpoons B$  is the composite  $Df \circ \pi_1$  an internal discrete opfibration?

**Theorem 4.7.** *An internal lens  $(f, \varphi): A \rightleftharpoons B$  is an internal split opfibration if and only if the internal functor  $Df \circ \pi_1: \Lambda \times_A DA \rightarrow DB$ , defined in (4.11), is an internal discrete opfibration.*

*Proof.* The pullback  $\Lambda \times_A DA$  has object of objects  $A_1$  and object of morphisms  $\Lambda_1 \times_{A_1} A_2$  as defined in (4.1). The projection  $\pi_1: \Lambda \times_A DA \rightarrow DA$  is an identity-on-objects internal functor given by:

$$\begin{array}{ccccc} A_1 & \xleftarrow{d_1 \pi_1} & \Lambda_1 \times_{A_1} A_2 & \xrightarrow{d_0 \pi_1} & A_1 \\ 1 \downarrow & & \downarrow \pi_1 & & \downarrow 1 \\ A_1 & \xleftarrow{d_1} & A_2 & \xrightarrow{d_0} & A_1 \end{array}$$

Post-composing this projection by the internal functor  $Df: DA \rightarrow DB$ , defined in (4.9), yields an internal functor given by:

$$\begin{array}{ccccc} A_1 & \xleftarrow{d_1 \pi_1} & \Lambda_1 \times_{A_1} A_2 & \xrightarrow{d_0 \pi_1} & A_1 \\ f_1 \downarrow & & \downarrow f_2 \pi_1 & & \downarrow f_1 \\ B_1 & \xleftarrow{d_1} & B_2 & \xrightarrow{d_0} & B_1 \end{array}$$

By Definition 2.5, this internal functor is an internal discrete opfibration if and only if the commutative diagram (4.2) is a pullback, which holds if and only if the internal lens  $(f, \varphi): A \rightleftharpoons B$  is an internal split opfibration by Definition 4.1.  $\square$

**Corollary 4.8.** *If  $(f, \varphi): A \rightleftharpoons B$  is an internal split opfibration, then the internal functor  $Df: DA \rightarrow DB$ , defined in (4.9), has the structure of an internal lens whose internal cofunctor component consists of the morphisms:*

$$f_1: A_1 \longrightarrow B_1 \qquad \hat{\psi}: A_1 \times_{B_1} B_2 \longrightarrow A_2$$

*Proof.* If  $(f, \varphi): A \rightleftharpoons B$  is an internal split opfibration, then from Theorem 4.7 there is a commutative diagram of internal functors,

$$\begin{array}{ccc} & \Lambda \times_A DA & \\ \pi_1 \swarrow & & \searrow Df \circ \pi_1 \\ DA & \xrightarrow{Df} & DB \end{array}$$

where  $\pi_1$  is an identity-on-objects internal functor and  $Df \circ \pi_1$  is an internal discrete opfibration. Applying Proposition 3.8 and Proposition 4.2 yields the result.  $\square$

Theorem 4.7 shows that the property an internal lens must satisfy to be an internal split opfibration is derived from the property of an internal functor induced from the right décalage comonad to be an internal discrete opfibration. Furthermore, Corollary 4.8 shows that the unique structure used to characterise internal split opfibrations in Proposition 4.2 is equivalent to giving the internal functor  $Df: DA \rightarrow DB$  a particular internal lens structure. Thus an internal split opfibration may be described by a suitable pair of internal lenses, making it straightforward to prove the well-known results that they are closed under composition and stable under pullback (see Remark 3.9 and Proposition 3.10).

*Remark 4.9.* Let  $\mathbf{SOpf}(\mathcal{E})$  be the category of internal categories and internal split opfibrations, together with the faithful forgetful functor  $\mathbf{SOpf}(\mathcal{E}) \rightarrow \mathbf{Lens}(\mathcal{E})$ . By Corollary 4.8, the right décalage comonad restricts to a functor  $D: \mathbf{SOpf}(\mathcal{E}) \rightarrow \mathbf{Lens}(\mathcal{E})$ .

## 5. INTERNAL SPLIT OPFIBRATIONS AND STRICT FACTORISATION SYSTEMS

This section characterises an internal split opfibration via a strict factorisation system on the domain of an internal lens, corresponding to the known result in **Set** that every morphism in the domain factorises uniquely into a *chosen* opcartesian lift followed by a *vertical* morphism. It is shown that this equivalent formulation of an internal split opfibration agrees with the definition given in [Joh77, Chapter 2, Exercise 6].

**Definition 5.1.** A *strict factorisation system*  $(E, M)$  on an internal category  $A$  consists of a pair of faithful, identity-on-objects internal functors,

$$e: E \longrightarrow A \qquad m: M \longrightarrow A$$

together with the pullback,

$$\begin{array}{ccc} & E_1 \times_{A_0} M_1 & \\ \pi_0 \swarrow & \downarrow \smash{\scriptstyle \vee} & \searrow \pi_1 \\ E_1 & & M_1 \\ d_0 \searrow & & \swarrow d_1 \\ & A_0 & \end{array} \quad (5.1)$$

such that the morphism  $d_1(e_1 \times m_1): E_1 \times_{A_0} M_1 \rightarrow A_1$  is an isomorphism.

**Example 5.2** ( $\mathcal{E} = \mathbf{Set}$ ). A strict factorisation system  $(E, M)$  on a small category  $A$  consists of wide subcategories  $E$  and  $M$  of  $A$  such that every morphism  $w \in A$  has a strictly unique factorisation  $w = m \circ e$  with  $m \in M$  and  $e \in E$ .

Given an internal lens  $(f, \varphi): A \rightrightarrows B$ , the goal is to give a strict factorisation system  $(\Lambda, V)$  on the internal category  $A$ , where the internal functor  $\varphi: \Lambda \rightarrow A$  provides the left class, and the right class  $j: V \rightarrow A$  is defined as follows.

**Definition 5.3.** Given an internal functor  $f: A \rightarrow B$ , there is a faithful, identity-on-objects internal functor  $j: V \rightarrow A$  constructed by the following pullback,

$$\begin{array}{ccc} & V & \xrightarrow{\widehat{f}} \\ j \swarrow & \downarrow \smash{\scriptstyle \vee} & \searrow \\ A & & B_0 \\ f \searrow & & \swarrow i_0 \\ & B & \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} & V_1 & \xrightarrow{\widehat{f}_1} \\ j_1 \swarrow & \downarrow \smash{\scriptstyle \vee} & \searrow \\ A_1 & & B_0 \\ f_1 \searrow & & \swarrow i_0 \\ & B_1 & \end{array} \quad (5.2)$$

where  $i: B_0 \rightarrow B$  is the inclusion of the discrete category of objects, and  $V$  is the *internal category of vertical morphisms* for an internal functor  $f$ . Explicitly, the internal functor  $j: V \rightarrow A$  is given by:

$$\begin{array}{ccccc} A_0 & \xleftarrow{d_1} & V_1 & \xrightarrow{d_0} & A_0 \\ 1 \downarrow & & \downarrow j_1 & & \downarrow 1 \\ A_0 & \xleftarrow{d_1} & A_1 & \xrightarrow{d_0} & A_0 \end{array} \quad (5.3)$$

To define the strict factorisation system  $(\Lambda, V)$  on  $A$  for an internal lens  $(f, \varphi): A \rightleftharpoons B$ , first construct the following pullback:

$$\begin{array}{ccccc}
 & \Lambda_1 \times_{A_0} V_1 & & & \\
 \pi_0 \swarrow & \downarrow \smile & \searrow \pi_1 & & \\
 \Lambda_1 & & & V_1 & \\
 p_0 \searrow & & \swarrow d_1 & & \\
 & A_0 & & & 
 \end{array} \tag{5.4}$$

The following result provides the unique structure such that  $d_1(\varphi_1 \times j_1): \Lambda_1 \times_{A_0} V_1 \rightarrow A_1$  is an isomorphism.

**Proposition 5.4.** *If  $(f, \varphi): A \rightleftharpoons B$  is an internal lens, then  $(\Lambda, V)$  is a strict factorisation system on  $A$  consisting of the internal functors  $\varphi: \Lambda \rightarrow A$  and  $j: V \rightarrow A$  if and only if there exists an endomorphism,*

$$\chi: A_1 \longrightarrow A_1$$

satisfying the following four commutative diagrams:

◇ This diagram specifies a “domain condition”:

$$\begin{array}{ccccc}
 A_1 & \xrightarrow{\chi} & A_1 & & \\
 \langle d_1, f_1 \rangle \downarrow & & & & \downarrow d_1 \\
 \Lambda_1 & \xrightarrow{\varphi_1} & A_1 & \xrightarrow{d_0} & A_0
 \end{array} \tag{5.5}$$

◇ This diagram specifies a “uniqueness condition”:

$$\begin{array}{ccccc}
 \Lambda_1 \times_{A_0} V_1 & \xrightarrow{\varphi_1 \times j_1} & A_2 & \xrightarrow{d_1} & A_1 \\
 \pi_1 \downarrow & & & & \downarrow \chi \\
 V_1 & \xrightarrow{j_1} & & & A_1
 \end{array} \tag{5.6}$$

◇ This diagram specifies a “fibre condition”:

$$\begin{array}{ccccc}
 A_1 & \xrightarrow{\chi} & A_1 & & \\
 f_1 \downarrow & & & & \downarrow f_1 \\
 B_1 & \xrightarrow{d_0} & B_0 & \xrightarrow{i_0} & B_1
 \end{array} \tag{5.7}$$

◇ This diagram specifies a “composition condition”:

$$\begin{array}{ccc}
 & A_1 & \\
 \widehat{\chi} \swarrow & & \searrow 1 \\
 A_2 & \xrightarrow{d_1} & A_1
 \end{array} \tag{5.8}$$

The pullback  $\Lambda_1 \times_{A_0} V_1$  is defined in (5.4), and the morphism  $\hat{\chi}$  is defined using (5.5) via the universal property of the pullback:

$$\begin{array}{ccc}
 & A_1 & \\
 \langle d_1, f_1 \rangle \swarrow & \downarrow \hat{\chi} & \searrow \chi \\
 \Lambda_1 & A_2 & \\
 \varphi_1 \downarrow & \swarrow d_2 & \searrow d_0 \\
 A_1 & \checkmark & A_1 \\
 & \searrow d_0 & \swarrow d_1 \\
 & A_0 &
 \end{array}$$

*Proof.* By Definition 5.1 the pair  $(\Lambda, V)$  is a strict factorisation system on  $A$  if and only if the morphism  $d_1(\varphi_1 \times j_1): \Lambda_1 \times_{A_0} V_1 \rightarrow A_1$  has an inverse.

Given a morphism  $\chi: A_1 \rightarrow A_1$  satisfying axioms (5.5) and (5.7) there is a morphism,

$$\langle \langle d_1, f_1 \rangle, \langle \chi, d_0 f_1 \rangle \rangle: A_1 \longrightarrow \Lambda_1 \times_{A_0} V_1$$

which is the required inverse if axioms (5.6) and (5.8) are satisfied. Conversely, an inverse to  $d_1(\varphi_1 \times j_1): \Lambda_1 \times_{A_0} V_1 \rightarrow A_1$  exists exactly when there is a morphism  $\chi$  which satisfies the axioms in the statement of Proposition 5.4. The necessary diagram-chasing has been excluded.  $\square$

The following example shows that the above characterisation yields the expected strict factorisation system on a small category  $A$  given by a *pre-opcartesian morphism* followed by a vertical morphism, and explains the meaning of the names given to the axioms above.

**Example 5.5** ( $\mathcal{E} = \mathbf{Set}$ ). A delta lens  $(f, \varphi): A \rightleftharpoons B$  (see Example 3.3) has a strict factorisation system  $(\Lambda, V)$  if for all morphisms  $w: a \rightarrow a'$  in  $A$  there exists a unique morphism  $\chi(w): p(a, fw) \rightarrow a'$  such that  $f\chi(w) = 1_{a'}$  and  $\chi(w) \circ \varphi(a, fw) = w$ .

$$\begin{array}{ccc}
 & & p(a, fw) \\
 & \nearrow \varphi(a, fw) & \downarrow \chi(w) \\
 A & a \xrightarrow{w} a' & \\
 \varphi \uparrow \parallel f & \vdots & \vdots \\
 B & fa \xrightarrow{fw} fa' &
 \end{array}$$

The domain of  $\chi(w)$  corresponds to (5.5) and the uniqueness to (5.6). The other two conditions on  $\chi(w)$  correspond exactly to (5.7) and (5.8), respectively. Thus  $\chi$  provides a global way of characterising the chosen lifts  $\varphi(a, fw)$  as *pre-opcartesian morphisms*.

The axioms of a delta lens imply that the chosen pre-opcartesian morphisms compose to give a chosen pre-opcartesian morphism. This means that a delta lens with the strict factorisation system as given in Example 5.5 is a split opfibration. Internally, there is also a strong similarity between Proposition 4.2 and Proposition 5.4, in terms of additional structure with axioms placed on an internal lens, which motivates the following theorem.

**Theorem 5.6.** *An internal lens  $(f, \varphi): A \rightleftharpoons B$  is an internal split opfibration if and only if the pair  $(\Lambda, V)$  is a strict factorisation system on  $A$ .*

*Proof.* Given an internal split opfibration  $(f, \varphi): A \rightleftharpoons B$  equipped with a morphism  $\psi: A_1 \times_{A_0} B_2 \rightarrow A_1$  as in Proposition 4.2, define a morphism  $\chi': A_1 \rightarrow A_1$  as the following composite:

$$A_1 \xrightarrow{\langle 1, i_1 f_1 \rangle} A_1 \times_{B_1} B_2 \xrightarrow{\psi} A_1 \quad (5.9)$$

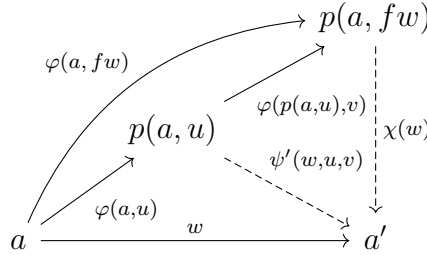
It may be shown that  $\chi'$  satisfies the axioms of Proposition 5.4 and thus yields a strict factorisation system  $(\Lambda, V)$  on  $A$ .

Conversely, given an internal lens  $(f, \varphi): A \rightleftharpoons B$  equipped with an endomorphism  $\chi: A_1 \rightarrow A_1$  as in Proposition 5.4, define a morphism  $\psi': A_1 \times_{B_1} B_2 \rightarrow A_1$  as the following composite:

$$A_1 \times_{B_1} B_2 \xrightarrow{\langle \varphi_1 p_0 \langle d_1 \times d_2, \pi_1 \rangle, \chi \pi_0 \rangle} A_2 \xrightarrow{d_1} A_1 \quad (5.10)$$

It may be shown that  $\psi'$  satisfies the axioms of Proposition 4.2 and thus yields an internal split opfibration.  $\square$

**Example 5.7** ( $\mathcal{E} = \mathbf{Set}$ ). Given a split opfibration as in Example 5.5, the function (5.10) is defined as  $\psi'(w, u, v) = \chi(w) \circ \varphi(p(a, u), v)$  as depicted below:



Theorem 5.6 is important as it characterises internal split opfibrations as internal lenses  $(f, \varphi): A \rightleftharpoons B$  where  $A_1 \cong \Lambda_1 \times_{A_1} V_1$ , giving the object of morphisms the universal property of the pullback (5.4). This mirrors Definition 2.5 which states that internal discrete opfibrations are internal functors  $f: A \rightarrow B$  where  $A_1 \cong \Lambda_1 = A_0 \times_{B_0} B_1$ .

**Corollary 5.8.** *If  $(f, \varphi): A \rightleftharpoons B$  is an internal split opfibration, then the following diagrams commute:*

$$\begin{array}{ccc} \Lambda_1 \xrightarrow{\varphi_1} A_1 & A_2 \xrightarrow{\langle \chi d_1 (\chi \times \varphi \langle d_1, f_1 \rangle), \chi d_0 \rangle} A_2 & \\ p_0 \downarrow & d_1 \downarrow & \\ A_0 \xrightarrow{i_0} A_1 & A_1 \xrightarrow{\chi} A_1 & \end{array} \quad \begin{array}{ccc} & A_1 & \\ \chi \swarrow & & \searrow d_0 \\ A_1 & \xrightarrow{d_0} & A_0 \end{array} \quad \begin{array}{ccc} & A_1 & \\ \chi \swarrow & & \searrow \chi \\ A_1 & \xrightarrow{\chi} & A_1 \end{array}$$

*Proof.* These diagrams may be derived using the axioms in Proposition 5.4.  $\square$

**Remark 5.9.** The definition of an internal split opfibration in [Joh77, Ch. 2, Ex. 6] is equivalent to an internal lens  $(f, \varphi): A \rightleftharpoons B$  equipped with a morphism  $\chi: A_1 \rightarrow A_1$  satisfying the diagrams in Proposition 5.4 and Corollary 5.8. Therefore by Theorem 5.6, the definition of internal split opfibrations provided in Definition 4.1 agrees with the characterisation in [Joh77] while requiring fewer axioms.

## 6. CONCLUDING REMARKS

This paper has introduced several characterisations of internal split opfibrations in terms of both property and structure. The core of this approach has been the treatment of internal split opfibrations as *morphisms* between internal categories, given by an internal functor and an internal cofunctor which form an internal lens. This is unlike many previous approaches, which instead consider split opfibrations as *objects* over a fixed base category. The treatment of internal split opfibrations as morphisms is not only of theoretical interest, but is important for the application of delta lenses in computer science where compositionality plays a central role.

Another significant component of this treatment of internal split opfibrations is that it does not use any of the 2-categorical features of  $\mathbf{Cat}(\mathcal{E})$ . Instead the paper utilises two special classes of morphisms in  $\mathbf{Cat}(\mathcal{E})$  – isomorphism-on-objects internal functors and internal discrete opfibrations – which satisfy the conditions of Lemma 2.6, together with a comonad on  $\mathbf{Cat}(\mathcal{E})$ , given by the right décalage construction. Consequently, the main characterisation of internal split opfibrations in Theorem 4.7 may be further generalised by replacing  $\mathbf{Cat}(\mathcal{E})$  with a category  $\mathcal{C}$  with pullbacks, together with suitable classes of morphisms  $(\mathcal{B}, \mathcal{D})$  and a comonad  $D$ , however further work is needed to identify interesting examples in this general framework.

The theory of internal lenses and internal split opfibrations developed in this paper clarifies and extends many results concerning delta lenses in the computer science literature. For example, Proposition 3.7 provides a simple diagrammatic characterisation of the delta lens axioms in Example 3.3, which further clarifies composition of delta lenses by Remark 3.9. The characterisations of internal split opfibrations presented also provide the necessary and sufficient conditions for a delta lens to be “least-change”, an important property for applications. Future work will apply these results to recent research on symmetric lenses [JR17, JR19] with the goal of finding similar conditions for universality.

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