ON A NOTION OF OPLAX 3-FUNCTOR

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ABSTRACT. We introduce a notion of normalised oplax 3-functor suitable for the elementary homotopy theory of strict 3-categories, following the combinatorics of orientals. We show that any such morphism induces a morphism of simplicial sets between the Street nerves and we characterise those morphisms of simplicial sets coming from normalised oplax 3-functors. This allows us to prove that normalised oplax 3-functors compose. Finally we construct a strictification for normalised oplax 3-functors whose source is a 1-category without split-monos or split-epis.

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Introduction

The homotopy theory of small categories was born with the introduction by Grothendieck of the nerve functor

$$N \colon \mathcal{C}at \to \widehat{\Delta}$$

in [13], where Cat is the category of small categories and $\widehat{\Delta}$ is the category of simplicial sets, allowing us to define a class of weak equivalences in Cat: a functor is a weak equivalence precisely when its nerve is a simplicial weak homotopy equivalence. We call these functors *Thomason equivalences* of Cat. The nerve functor preserves

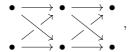
by definition the weak equivalences, *i.e.*, maps Thomason equivalences to simplicial weak equivalences, and therefore there is an induced functor

$$\bar{N} : \operatorname{Ho}(\mathcal{C}at) \longrightarrow \operatorname{Ho}(\widehat{\Delta})$$

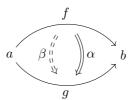
at the level of the homotopy categories.

The first striking result of this theory appears in Illusie's thesis [15] (who credits it to Quillen) and states that this induced functor $\bar{N}\colon \operatorname{Ho}(\mathcal{C}at) \longrightarrow \operatorname{Ho}(\widehat{\Delta})$ is an equivalence of categories. The homotopy inverse of the nerve functor $N\colon \mathcal{C}at \to \widehat{\Delta}$ is not induced by its left adjoint $c\colon \widehat{\Delta} \to \mathcal{C}at$, i.e., the categorical realisation functor, which behaves poorly homotopically, but instead by the category of elements functor $i_\Delta\colon \widehat{\Delta} \to \mathcal{C}at$, mapping a simplicial set X to its category of elements $i_\Delta(X) = \Delta/X$. A careful study of the subtle homotopy theory of small categories by Thomason [27] led him to show another important result: the existence of a model category structure on $\mathcal{C}at$ which is Quillen equivalent to the Kan–Quillen model categories and simplicial sets are not only equivalent as homotopy categories, but actually as weak $(\infty, 1)$ -categories, i.e., as homotopy theories.

One drawback of working with small categories as preferred model for homotopy types is that there are no simple geometric models of simplicial complexes. For instance, the homotopy type of the two-dimensional sphere S^2 is often modelled by the poset



see for instance [4]. This is mainly due to the intrinsic 1-dimensional shape of categories. On the other hand, the homotopy type of S^2 can be easily modelled in a geometric fashion by a small 2-category, namely



This suggests that strict higher categories may provide a more convenient framework for setting up a categorical model for homotopy types and a source of motivation for generalising the homotopy theory of Cat to strict higher categories. In fact, Ara and Maltsiniotis construct a functor \mathcal{O} assigning to any ordered simplicial complex an ∞ -category, see [2, §9].

In a seminal article [24], Street introduced a nerve functor

$$N_{\infty} \colon \infty \text{-} \mathcal{C}at \to \widehat{\Delta}$$

for strict ∞ -categories, allowing one to define and study the homotopy theory of ∞ -Cat, the category of small strict ∞ -categories, as well as of n-Cat, the category of strict n-categories, for every positive integer n. The class of weak equivalences of n-Cat pulled back via the Street nerve shall still be called $Thomason\ equivalences$. This functor is homotopically meaningful, since for instance it sends the above 2-categorical model of S^2 to a simplicial set with the homotopy type of S^2 .

The particular case of small 2-categories was studied by Bullejos and Cegarra [6], Cegarra [8], Chiche [9] and del Hoyo [12]. Their approach stresses on the importance played by (normalised) oplax 2-functors. In fact, it was already noticed that oplax 2-functors are geometrically meaningful, see for instance [25, Section 10]. This is

consequence of the fact that the Street nerve for 2-categories $N_2\colon 2\text{-}\mathcal{C}at\to \widehat{\Delta}$ is faithful but not full; the set of morphisms of simplicial sets between the nerves $N_2(A)$ and $N_2(B)$ of two small 2-categories A and B is in fact in bijection with the set of normalised oplax 2-functors from A to B. Ara and Maltsiniotis [1] provide an abstract framework in which to transfer the Kan–Quillen model category structure on simplicial sets to strict n-categories and showed that this is the case for small 2-categories. Their strategy makes use of normalised oplax 2-functors in order to define some maps needed for a homotopy cobase change property.

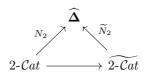
It is therefore natural to study a notion of normalised oplax n-functor which could be used to generalise the results listed above, thus establishing a satisfactory homotopy theory of n-categories. By this we mean showing that n-Cat can be equipped with a Quillen model category structure which is Quillen equivalent to the Kan-Quillen model category structure on simplicial sets. Providing a sensitive definition of such a normalised oplax n-functor for the case n = 3, which is the first one not well understood, is the aim of the present paper.

A normalised oplax n-functor $F\colon A\to B$, with $n\geqslant 1$, should roughly be a morphism of n-graphs which respects the identities on the nose, but that respects compositions of arrows only up to oriented coherences. For example, given a composition $a\xrightarrow{f}b\xrightarrow{g}c$ of two 1-arrows of A, the normalised oplax n-functor F should provide:

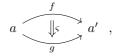
- a 1-arrow $F(a) \xrightarrow{F(gf)} F(c)$ of B,
- two composable 1-arrows $F(a) \xrightarrow{F(f)} F(b)$ and $F(b) \xrightarrow{F(g)} F(c)$ of B,
- a 2-arrow F(g, f): $F(gf) \Rightarrow F(g)F(f)$, which represents the coherence for the composition of f and g.

We observed that a central tool for the elementary homotopy theory of 2-categories is the notion of normalised oplax 2-functor. Moreover, this turns out to be a crucial ingredient in establishing the model category structure \grave{a} la Thomason on 2- $\mathcal{C}at$, too

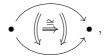
Normalised oplax 2-functors can be composed and hence form a category 2-Cat. There is a canonical nerve functor $\widetilde{N}_2 \colon 2-Cat \to \widehat{\Delta}$ extending the Street nerve for 2-categories, that is, there is a commutative triangle



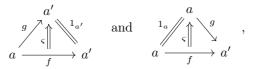
of functors, where the functor $2\text{-}Cat \to 2\text{-}Cat$ is simply the embedding given by the fact that any 2-functor is in particular a normalised oplax 2-functor. The Street nerve N_n is a faithful functor but not full for n>1. In the 2-categorical case, this deficiency is solved by normalised oplax 2-functors: the nerve $N_2: 2\text{-}Cat \to \widehat{\Delta}$ is fully faithful. Following this idea, Street proposes in [26] to define a normalised oplax 3-functor from A to B as a simplicial morphism from $N_3(A)$ to $N_3(B)$. A careful investigation of such a morphism shows that this might not be an optimal definition since in general simplicial morphisms between Street nerves of 3-categories fail to preserve the underlying 3-graph. Indeed, we analyse the case where A is the "categorical 2-disk"



i.e., the 2-category with two parallel 1-cells and a single 2-cell between them, and B is the "invertible categorical 3-disk"



i.e., the 3-category with two parallel 1-cells, two parallel 2-cells between them and a single invertible 3-cell between these 2-cells, and we show that there are more simplicial morphisms than expected between the respective Street nerves. On the one hand, the 2-category A has no compositions and so the normalised oplax 3-functors from A to B should coincide with the strict 3-functors. On the other hand, there are simplicial morphisms from $N_3(A)$ to $N_3(B)$ which do not come from the nerve of any strict 3-functors. This is a consequence of the fact that, for instance, there are two ways to capture the 2-cell ς of A with a 2-simplex of $N_3(A)$, namely



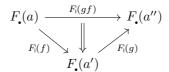
and these two different ways are related by 3-simplices, for instance



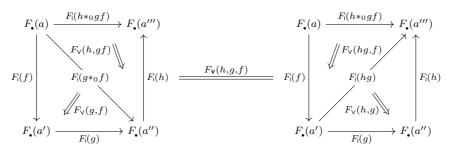
which are sent by any simplicial morphism to 3-simplices of $N_3(B)$ for which the principal 3-cell is invertible, but non necessarily trivial. Said otherwise, the different ways of encoding cells, or simple compositions of cells, with simplices are linked together by higher simplices having the property that the cell of greatest dimension is invertible; these higher simplices act as invertible constraints for morphisms between Street nerves of 3-categories and it is therefore natural to imagine that a normalised oplax 3-functor would correspond to a simplicial morphism for which all these higher simplices acting as constraints have trivial greatest cell, instead of only invertible. In order to determine a substantial set of these constraints, we analyse the simplicial morphism canonically associated to our notion of oplax normalised 3-functor. This provides a simplicial notion of oplax 3-functor preserving the underlying 3-graph, that we call simplicial oplax 3-morphisms. We show that they compose and thus form a category whose objects are small 3-categories.

The standard definition of a normalised oplax 2-functor $F: A \to B$ has objects, 1-cells, 2-cells and composition of 1-cells as datum, that is, to any object, 1-arrow or 2-arrow x of A, we associate an object, a 1-arrow or a 2-arrow, respectively, F(x) of B and for any pair $a \xrightarrow{f} b \xrightarrow{g} c$ of composable 1-arrows of A, we associate a 2-arrow F(g,f) of B going from F(gf) to F(g)F(f). These data must satisfy some normalisation conditions, for instance $F(1_a) = 1_{F(a)}$, for any object a of a, and a cocycle condition, which is a coherence for the composition of three 1-arrows of a, and the vertical and horizontal compatibility for 2-arrows as coherences. Another take on this notion is to see the data, the normalisations and the coherences indexed by Joyal's cellular

category Θ_3 , whose objects are trees of height at most 3. Any tree has a dimension, given by the number of its edges, and a normalised oplax 2-functor can be defined as a set of maps index on the trees \bullet , $|\cdot|$ and \vee of dimension at most 2 for the data, which represents precisely objects, 1-arrows, 2-arrows and compositions of two 1-arrows; the same trees are the indices for the normalisation conditions. Finally the four trees \forall , \forall , \vee and \vee of dimension 3, representing the composition of three composable 1-arrows, the vertical composition of two 2-arrows and the two possible whiskerings of a 2-arrow with a 1-arrow, are the indices for the coherences. More precisely, a normalised oplax 2-functor $F: A \to B$ will consist of a map F_{\bullet} from the objects of A to the objects of B, a map F_{\parallel} from the 1-arrows of A to the 1-arrows of B respecting source and target, i.e., for $f: a \to b$ in A we get $F_{\parallel}(f): F_{\bullet}(a) \to F_{\bullet}(b)$ in B, a map F_{\parallel} from the 2-arrows of A to the 2-arrows of B similarly respecting source and target and a map F_{\vee} that for any pair of composable 1-arrows $a \to b \to c$ of A associates a 2-cell $F_{\vee}(g, f)$



as explained above. The coherence $F_{\mathbb{V}}(h, g, f)$ for a triple of composable 1-arrows of A, say $a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} d$, can be represented by the following diagram



The tree \(\frac{\frac{1}{2}}{2} \) representing the vertical composition of 2-cells plays a key role, which in this low dimensional case is hidden but becomes much clearer in the 3-dimensional case.

We take this latter definition of normalised oplax 2-functor, and the "cellular" point of view behind it, as the starting point for a generalisation of this notion for the case of 3-categories. Indeed, a normalised oplax 3-functor shall consist of a family of maps, the data, indexed by the trees of dimension at most 3 except for \forall , subject to normalisation conditions indexed by these same trees as well as to a set of coherences indexed by the trees of dimension 4 joint with the tree \forall .

induces a canonical simplicial oplax 3-morphism boils down to show that the coherence for the tree \searrow , representing the composition of four 1-arrows, is satisfied by the "obvious" representative for the composition of normalised oplax 3-functors; this proof is highly non-trivial and involves a long and careful study of pastings of all the coherences of the two composed normalised oplax 3-functors. This is just one of the 14 coherences that one would need to check for the composition of normalised oplax 3-functors to be well-defined.

As remarked above, a careful examination of $N_l(F): N_3(A) \to N_3(B)$, the simplicial morphism associated to a normalised oplax 3-functor $F: A \to B$, reveals that certain non-trivial 3-simplices of $N_3(A)$ with trivial principal 3-cell are sent via $N_l(F)$ to 3-simplices of $N_3(B)$ where the principal cell is also trivial. Such simplices were called constraints before and if these constraints are taken as a property, they allow us to give a simplicial definition of a normalised oplax morphism between nerves of 3-categories which preserves their underlying 3-graph. We called simplicial oplax 3-morphisms such morphisms of simplicial sets. Hence, we come equipped with two notions of normalised oplax morphisms for 3-categories: one that is cellular in spirit and the other that is simplicial. The latter has the advantage that it is easy to see that it composes and forms a category; while the advantage of the former is that it allows us to reason on cells or simple composition of cells to define complicated morphisms, instead of describing it for objects of every dimension as for a morphism of simplicial sets. We show how to associate a normalised oplax 3-functor $c_l(F)$ to any simplicial oplax 3-morphism F. In order to do this, we have to check that the "obvious" data that we can associate to a simplicial oplax 3-morphism satisfies the normalisation conditions and the coherences of a normalised oplax 3-functor. The normalisations are simply encoded in the degeneracies, while the coherences are non-trivially encoded by appropriate 4simplices. As one might probably expect, the coherence for the tree representing the horizontal composition of 2-cells is the hardest to prove. We then show that this assignment going from simplicial to cellular is inverse to the map giving a simplicial morphism to any normalised oplax 3-functor. Contrarily to what happens for the 2-categorical case, this is also a non-trivial task. For $F: N_3(A) \to N_3(B)$ a simplicial oplax 3-morphism, we are led to provide, for any 3-simplex x of $N_3(A)$, an explicit description of the principal 3-cell of the 3-simplex $N_l c_l(F)(x)$ in terms of the normalised oplax 3-functor $c_l(F)$. This gives a bijection between the normalised oplax 3-functors from A to B and the simplicial oplax 3-morphisms from $N_3(A)$ to $N_3(B)$. We can then deduce that normalised oplax 3-functors compose and form a category and that this category is isomorphic to that of simplicial oplax 3-morphisms.

The notion of oplax 3-morphism already appears in the literature. Indeed, Gordon–Power–Street in [22] and Gurski in [14] provide similar, although slightly different, general definitions for trimorphisms between tricategories, with oplax variants. However, if we specialise to strict 3-categories we see that these notions are different from our. In fact, the main difference lies in the oriented coherence, by which we mean the datum, associated to the horizontal composition of 2-cells, that in our case expresses instead a relation between the two pieces of data associated to the two possible whiskerings of a 2-cell with a 1-cell; the data of these two whiskering are symmetric, as it is imposed by the algebra of the orientals, which is incompatible with the choice of a prescribed lax/oplax orientation for the horizontal composition of 2-cells, as it is imposed in the definition of trimorphism.

An important and motivating example of normalised oplax 3-functor is given by the "sup" morphism sup: $\Delta/N_3(A) \to A$, where A is a 3-category and $\Delta/N_3(A)$ is the category of elements of its Street nerve. In the 1-categorical case, this morphism

is actually a 1-functor mapping an object (Δ^n, x) of $\Delta/N(A)$, where $x \colon \Delta^n \to A$ is a sequence of n composable arrows of A, to the object x(n) of A and a morphism $f \colon (\Delta^n, y) \to (\Delta^p, y)$ of $\Delta/N(A)$, that we can depict by the triangle

$$\begin{array}{ccc}
\Delta^n & \xrightarrow{f} & \Delta^p \\
& & \swarrow_y & ,
\end{array}$$

to the arrow $y_{\{f(n),p\}}\colon y(f(n))\to y(p)$. This arrow can be seen as a functor

$$\Delta^1 \xrightarrow{\{f(n),p\}} \Delta^p \xrightarrow{y} A$$
.

The sup 1-functor is always a Thomason equivalence and it plays an important role in the elementary homotopy theory of 1-categories. Del Hoyo [12] and Chiche [9] generalised and studied this sup morphism for the case of 2-categories. For instance, for a pair of composable 1-arrows $(\Delta^m, x) \xrightarrow{f} (\Delta^n, y) \xrightarrow{g} (\Delta^p, z)$ of $\Delta/N_2(A)$, we assign the 2-cell of A given by the principal 2-arrow of the 2-functor

$$c_2N_2(\Delta^2) \xrightarrow{\{gf(m),g(n),p\}} c_2N_2(\Delta^p) \xrightarrow{z} A$$
 .

This normalised oplax 2-functor proved to be crucial for the elementary homotopy theory of 2-categories and we provide a 3-dimensional definition with our notion of normalised oplax 3-functor.

We also study the "strictification" of such a morphism. By this we mean the following general procedure: given a 1-category A and a 3-category B, there exists a 3-category \hat{A} and a normalised oplax 3-functor $\eta_A \colon A \to \hat{A}$ such that the set of strict 3-functors from \hat{A} to \hat{B} is in bijection with the set of normalised oplax 3-functors from A to B and moreover this bijection is obtained by pre-composing by η_A . By the correspondence described above between normalised oplax 3-functors and simplicial oplax 3-morphisms, the 3-category \hat{A} is given by $c_3N_3(A)$ and the morphism η_A is just the unit $\eta_A: N_3(A) \to N_3c_3N_3(A)$. A nice description for the 2-categorical case has been given by del Hoyo [12], so in particular we already know how to describe the 1-cells of A. We tackle this problem more generally and we provide a complete description of the ∞ -category $c_{\infty}N_{\infty}(A)$, for any 1-category A without split-monos or split-epis; all posets and the subdivision $c\mathrm{Sd}N(C)$ of any 1-category C have such a property. The objects of $c_{\infty}N(A)$ are the same as those of A and, as we observed above, we already know the 1-cells, which are given by non-degenerate simplices of N(A). We define an ∞ -category $c_{\infty}N(A)(f,g)$, for any pair of parallel 1-arrows f and q of $c_{\infty}N(A)$, as well as "vertical compositions"

$$c_{\infty}N(A)(g,h)\times c_{\infty}N(A)(f,g)\to c_{\infty}N(A)(f,h)$$

of 2-cells and "horizontal compositions"

$$c_{\infty}N(A)(y,z) \times c_{\infty}N(A)(x,y) \to c_{\infty}N(A)(x,z)$$
,

where x, y and z are objects of $c_{\infty}N(A)$, and we check that they satisfy all the axioms of ∞ -category. Finally, we prove that our definition of $c_{\infty}N(A)$ indeed satisfies the expected universal property.

In an appendix, we recall some technicalities that we need in order to develop this last section about strictifications. In fact, we shall need some precise properties about the internal Hom- ∞ -categories of the orientals. We begin by recalling few elements on Steiner's theory of augmented directed complexes. We then give a brief glance at Joyal's Θ category and finally we introduce the orientals together with some results about them that we need in section 5.

Notations and Terminology. The category of small strict n-category will be denoted by n-Cat, for ay $1 \le n \le \infty$. For a definition of n-Cat, see Appendix A. The simplex category shall be denoted by Δ and $\widehat{\Delta}$ shall denote the category of simplicial sets. The functor $\widehat{\Delta} \to Cat$ that to any simplicial set X associate its category of elements Δ/X shall be denoted by i_{Δ} .

All the ∞ -categories will be strict, with the composition operations denoted by $*_i$ and the identity of a cell x denoted by 1_x . That is, if A is an ∞ -category, $0 \le i < j$ are integers and x and y are j-cells of A such that the i-target $t_i(x)$ of x is equal to the i-source $s_i(y)$ of y, then there exists a unique j-cell $y *_i x$ of A which is the i-composition of x and y; similarly, there is a (j+1)-cell 1_x which is the identity of x. We shall often call trivial the identity cells of A. We shall say that two i-cells x and y of A are parallel is they have same source and target; if this is the case, we shall denote by $\underline{\mathsf{Hom}}_A(x,y)$ the ∞ -category whose j-cells, for $j \ge 0$, are the (i+j+1)-cells of A having x as i-source and y as i-target.

The Street nerve from n-categories to simplicial sets shall always be denoted by $N_{\infty} \colon n\text{-}\mathcal{C}at \to \widehat{\Delta}$, for any $1 \leqslant n \leqslant \infty$. This is justified by the fact that we can embed n-categories in ∞ -categories. The left adjoint to the Street nerve shall be denoted by $c_n \colon \widehat{\Delta} \to n\text{-}\mathcal{C}at$.

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1. Normalised oplax 3-functors

- 1.1. **Recall: normalised oplax 2-functors.** We begin this chapter by recalling the classical notion of *normalised oplax 2-functor*.
- **1.1.1.** Let A and B be two 2-categories. A normalised oplax 2-functor $F: A \to B$ is given by:
 - a map $\mathsf{Ob}(A) \to \mathsf{Ob}(B)$ that to any object x of A associates an object F(a) of B;
 - a map $\mathsf{Cell}_1(A) \to \mathsf{Cell}_1(B)$ that to any 1-cell $f \colon x \to y$ of A associates a 1-cell $F(f) \colon F(x) \to F(y)$ of B;
 - a map $\mathsf{Cell}_2(A) \to \mathsf{Cell}_2(B)$ that to any 2-cell $\alpha \colon f \to g$ of A associates a 2-cell $F(\alpha) \colon F(f) \to F(g)$ of B;
 - a map that to any composable 1-cells $x \xrightarrow{f} y \xrightarrow{g} z$ of A associates a 2-cell

$$F(g, f) \colon F(g *_0 f) \to F(g) *_0 F(f)$$

of B.

These data are subject to the following coherences:

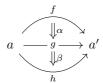
normalisation: for any object x of A (resp. any 1-cell f of A) we have $F(1_x) = 1_{F(x)}$ (resp. $F(1_f) = 1_{F(f)}$); moreover for any 1-cell $f: x \to y$ of A we have

$$F(1_y, f) = 1_{F(f)} = F(f, 1_x);$$

cocycle: for any triple $x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} t$ of composable 1-cells of A we have

$$(F(h) *_0 F(g, f)) *_1 F(h, g *_0 f) = (F(h, g) *_1 F(f)) *_1 F(h *_0 g, f);$$

vertical compatibility: for any pair



of 1-composable 2-cells α and β of A, we have $F(\beta *_1 \alpha) = F(\beta) *_1 F(\alpha)$; horizontal compatibility: for any pair

$$\bullet \xrightarrow{f'} \bullet \xrightarrow{g} \bullet$$

of 0-composable 2-cells α and β of A, we have

$$F(g', f') *_1 F(\beta *_0 \alpha) = (F(\beta) *_0 F(\alpha)) *_1 F(g, f).$$

1.1.2. The coherence given by the compatibility with respect to "horizontal composition" can be equivalently decomposed in two coherences, which correspond to the two possible "whiskerings" of a 2-cell with a 1-cell:

 \vee : for any diagram

$$a \xrightarrow{f} a' \xrightarrow{g} a''$$

of A, we have

$$F(g) *_{0} F(\alpha) *_{1} F(g, f) = F(g, f') *_{1} F(g *_{0} \alpha);$$

 \vee : for any diagram

$$a \xrightarrow{f} a' \xrightarrow{g} a''$$

of A, we have

$$F(g', f) *_1 F(\beta *_0 f) = F(\beta) *_0 F(f) *_1 F(g, f).$$

These two coherences are a particular case of the horizontal coherence of the previous paragraph and reciprocally one checks immediately these two coherences joint with the vertical coherence imply the horizontal coherence.

The advantage of this latter reformulation is that now the coherence datum of a normalised oplax 2-functor is indexed over

- the trees of dimension ≤ 2 for the normalisation;
- the trees of dimension 3 for the other coherences.

This will be the starting point in our generalisation towards a notion of normalised oplax 3-functor. In order to clarify what we mean, we repropose the definition of normalised oplax 2-functor with data and coherences indexed by trees.

- **1.1.3.** A normalised oplax 2-functor $F: A \to B$ consists of the following data:
- •: a map F_{\bullet} that to each object a of A assigns an object $F_{\bullet}(a)$ of B.

: a map F_{l} that to each 1-cell $f: a \to a'$ of A assigns a 1-cell $F_{l}(f): F_{\bullet}(a) \to F_{\bullet}(a')$ of B.

 \bigvee : a map F_{\lor} that to to each pair of 0-composable 1-cells

$$a \xrightarrow{f} a' \xrightarrow{g} a''$$

of A assigns a 2-cell $F_{\vee}(g, f)$

$$F_{\bullet}(a) \xrightarrow{F_{\mathsf{i}}(g *_{0}f)} F_{\bullet}(a'')$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad$$

of B, that is

$$F_{\vee}(g,f)\colon F_{\mathbb{I}}(g*_{0}f)\to F_{\mathbb{I}}(g)*_{0}F_{\mathbb{I}}(f)$$

: a map F_{\dagger} that to each 2-cell $\alpha \colon f \to g$ of A associates a 2-cell $F_{\dagger}(\alpha) \colon F_{\dagger}(f) \to F_{\dagger}(g)$ of B.

These data are subject to the following conditions of normalisation:

.: for any object a of A, we have

$$F_{\mathsf{I}}(1_a) = 1_{F_{\bullet}(a)};$$

 \Box : for any 1-cell f of A we have

$$F(1_f) = 1_{F(f)}$$
;

 \bigvee : for any 1-cell $f: a \to a'$ of A, we have

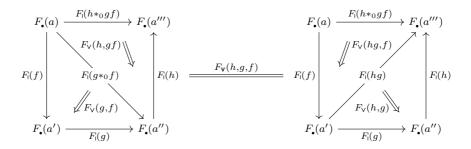
$$F_{\vee}(1_{a'}, f) = F_{\vee}(f, 1_a) = 1_{F_{\mathsf{i}}(f)}$$
.

Finally, we impose the following coherences:

: for any triple of 0-composable 1 cells

$$a \xrightarrow{f} a' \xrightarrow{g} a'' \xrightarrow{h} a'''$$

of A we have



that is we impose the equality

$$F_{l}(h) *_{0} F_{\vee}(g, f) *_{1} F_{\vee}(h, g *_{0} f) = F_{\vee}(h, g) *_{0} F_{l}(f) *_{1} F_{\vee}(h *_{0} g, f);$$

: for any pair

$$a \xrightarrow{f} a'$$

$$\downarrow a \qquad \downarrow b$$

of 1-composable 2-cells α and β of A, we impose

$$F(\beta *_1 \alpha) = F(\beta) *_1 F(\alpha);$$

√: for any whiskering

$$a \xrightarrow{f'} a' \xrightarrow{g} a''$$

of A we impose

$$F_{\mathsf{I}}(g) *_{0} F_{\mathsf{I}}(\alpha) *_{1} F_{\mathsf{V}}(g, f) = F_{\mathsf{V}}(g, f') *_{1} F_{\mathsf{I}}(g *_{0} \alpha);$$

∀: for any whiskering

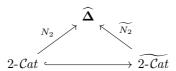
$$a \xrightarrow{f} a' \xrightarrow{g} a''$$

of A we impose

$$F_{\vee}(g', f) *_{1} F_{!}(\beta *_{0} f) = F_{!}(\beta) *_{0} F_{!}(f) *_{1} F_{\vee}(g, f).$$

1.1.4. Given two normalised oplax 2-functors $F \colon A \to B$ and $G \colon B \to C$, there is an obvious candidate for the composition $GF \colon A \to C$ and one checks that this is still a normalised oplax 2-functor; furthermore, the identity functor on a category is clearly an identity element for normalised oplax 2-functor too. Hence, there is a category $2 \cdot Cat$ with small 2-categories as objects and normalised oplax 2-functors as morphisms.

The cosimplicial object $\Delta \to 2\text{-}\mathcal{C}at \to 2\text{-}\mathcal{C}at$ of $2\text{-}\mathcal{C}at$ induces a nerve functor $\widetilde{N_2} \colon \widehat{2\text{-}\mathcal{C}at} \to \widehat{\Delta}$. For any $n \geqslant 0$, the normalised oplax 2-functors $\Delta^n \to A$ correspond precisely to 2-functors $\mathcal{O}_n^{\leqslant 2} \to A$ (see, for instance, [18, Tag 00BE]). Hence, we get a triangle diagram of functors



which is commutative (up to a canonical isomorphism). Moreover, it is a standard fact that the functor \widetilde{N}_2 is fully faithful (see, for instance [7], [17] or [18, Tag 00AU]).

1.2. **Definition of normalised oplax 3-functor.** Let A and B be two 3-categories. We now give the definition of normalised oplax 3-functor $F: A \to B$, which amounts to giving the structure of a family of maps indexed by the objects of Θ of dimension at most 3 subject to a family of relations indexed by the objects of Θ of dimension 4 as well as normalisation for identities of every dimension. A quick description of Θ is provided in Appendix C. The choice of the orientation for the structural maps is strongly guided by the algebra of the orientals and presents therefore a symmetry (or better, a duality) for symmetric trees of Θ , see for instance the structural cells for the trees \vee and \vee denoting the two possible whiskerings of a 2-cell with a 1-cell.

The arboreal rule for indexing structure, normalisation and coherence that we have stated right above has an exception. In fact, we are forced to list the tree Y denoting the vertical composition of two 2-cells among the coherences. This imposes a local strictness on the lax 3-functor, meaning that as a result a normalised 3-functor will induce a strict 2-functor on the hom-2-categories, and it is necessary in order to provide a reasonable coherence for the tree V representing the horizontal

composition of two 2-cells; we shall say more about this in a remark right after the definition. Another explanation for this choice, more simplicial in spirit, will be offered in section 2 (see in particular Remark 2.2.12).

1.2.1 (Data). A normalised oplax 3-functor F from A to B consists of:

•: a map F_{\bullet} that to each object a of A assigns an object $F_{\bullet}(a)$ of B.

l: a map F_{l} that to each 1-cell $f: a \to a'$ of A assigns a 1-cell $F_{l}(f): F_{\bullet}(a) \to F_{\bullet}(a')$ of B;

 \bigvee : a map F_{\lor} that to to each pair of 0-composable 1-cells

$$a \xrightarrow{f} a' \xrightarrow{g} a''$$

of A assigns a 2-cell $F_{\vee}(g, f)$

$$F_{\bullet}(a) \xrightarrow{F_{\mathsf{I}}(g *_{0}f)} F_{\bullet}(a'')$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad$$

of B, that is

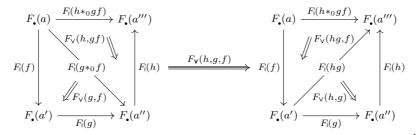
$$F_{\vee}(g, f) : F_{|}(g *_{0} f) \to F_{|}(g) *_{0} F_{|}(f) ;$$

: a map F_{\dagger} that to each 2-cell $\alpha \colon f \to g$ of A associates a 2-cell $F_{\dagger}(\alpha) \colon F_{\dagger}(f) \to F_{\dagger}(g)$ of B;

 \bigvee : a map F_{V} , that to each triple of 0-composable 1 cells

$$a \xrightarrow{f} a' \xrightarrow{g} a'' \xrightarrow{h} a'''$$

of A associates a 3-cell $F_{\mathbb{V}}(h, g, f)$



that is the 3-cell $F_{V}(h, g, f)$ has

$$F_{\rm I}(h) *_{0} F_{\rm V}(g,f) *_{1} F_{\rm V}(h,g *_{0} f)$$

as source and

$$F_{\vee}(h,q) *_{0} F_{|}(f) *_{1} F_{\vee}(h *_{0} q, f)$$

as target;

 \bigvee : a map F_{\bigvee} that to any whiskering

$$a \xrightarrow{f} a' \xrightarrow{g} a''$$

of A associates a 3-cell

$$F_{\vee}(q,\alpha): F_{||}(q) *_{0} F_{||}(\alpha) *_{1} F_{||}(q,f) \to F_{\vee}(q,f') *_{1} F_{||}(q *_{0} \alpha)$$

of B;

 \bigvee : a map F_{\bigvee} , that to each whiskering

$$a \xrightarrow{f} a' \xrightarrow{g} a''$$

of A associates a 3-cell

$$F_{\vee}(\beta,f)\colon F_{\vee}(g',f)*_1F_{!}(\beta*_0f)\to F_{!}(\beta)*_0F_{!}(f)*_1F_{\vee}(g,f)$$
 of $B;$

: a map F_{\dagger} , that to any 3-cell $\gamma: \alpha \to \alpha'$ of A, *i.e.*, any tree of A_{\dagger} , associates a 3-cell

$$F_{\mathbf{i}}(\gamma) \colon F_{\mathbf{i}}(\alpha) \to F_{\mathbf{i}}(\alpha')$$

of B.

1.2.2 (Normalisation). The normalisation requires the following constraints:

.: for any object a of A, we have

$$F_{\mathsf{I}}(1_a) = 1_{F_{\bullet}(a)};$$

| : for any 1-cell f of A we have

$$F_{\mathbf{i}}(1_f) = 1_{F_{\mathbf{i}}(f)};$$

 \bigvee : for any 1-cell $f: a \to a'$ of A, we have

$$F_{\vee}(1_{a'}, f) = F_{\vee}(f, 1_a) = 1_{F_{\downarrow}(f)};$$

: for any 2-cell α of A we have

$$F(1_{\alpha}) = 1_{F(\alpha)}$$
.

 \bigvee : for any pair $a \xrightarrow{f} a' \xrightarrow{g} a''$ of composable 1-cell of A, we have

$$F_{V}(g, f, 1_{a}) = F_{V}(g, 1_{a'}, f) = F_{V}(1_{a''}, g, f) = 1_{F_{V}(g, f)};$$

 \bigvee : for any pair $a \xrightarrow{f} a' \xrightarrow{g} a''$ of composable 1-cells of A, we have

$$F_{\vee}(g, 1_f) = 1_{F_{\vee}(g, f)}$$

and for any 2-cell $\alpha \colon f \to f'$ of A, we have

$$F_{\vee}(1_{a'},\alpha)=1_{F_{\mathbf{i}}(\alpha)};$$

 \bigvee : for any pair $a \xrightarrow{f} a' \xrightarrow{g} a''$ of composable 1-cells of A, we have

$$F_{\backslash I}(1_q, f) = 1_{F_{\backslash I}(q, f)}$$

and for any 2-cell $\beta: g \to g'$ of A, we have

$$F_{\lor}(\beta, 1_{a'}) = 1_{F_{\bullet}(\beta)};$$

1.2.3 (Coherences). These maps are subject to the following relations:

: for any pair of 1-composable 2-cells

$$a \xrightarrow{g \longrightarrow a'} a'$$

of A, *i.e.*, to any tree of A_{Y} , we have an equality

$$F_{\downarrow}(\beta) *_{1} F_{\downarrow}(\alpha) = F_{\downarrow}(\beta *_{1} \alpha)$$

in B. We shall sometimes write $F_{\vee}(\beta, \alpha)$ for the identity 3-cell of this 2-cell above.



: For any quadruple

$$\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet \xrightarrow{h} \bullet \xrightarrow{i} \bullet$$

of 0-composable 1-cells of A we impose that the 3-cells

$$\begin{split} F_{\vee}(i,h) *_0 F_{\mathsf{i}}(g) *_0 F_{\mathsf{i}}(f) *_1 F_{\vee}(ih,g,h) \\ *_2 \\ F_{\mathsf{i}}(i) *_0 F_{\mathsf{i}}(h) *_0 F_{\vee}(g,f) *_1 F_{\vee}(i,h,gf) \end{split}$$

and

$$F_{\forall}(i,h,g) *_{0} F_{|}(f) *_{1} F_{\forall}(i *_{0} h *_{0} g, f) \\ *_{2}$$

$$F_{|}(i) *_{0} F_{\forall}(h,g) *_{0} F_{|}(f) *_{1} F_{\forall}(i,h *_{0} g, f) \\ *_{2}$$

$$F_{|}(i) *_{0} F_{\forall}(h,g,f) *_{1} F_{\forall}(i,h *_{0} g *_{0} f)$$

of B are equal.

: For any triple

$$\bullet \longrightarrow f \longrightarrow \bullet \longrightarrow g \longrightarrow \bullet \xrightarrow{h} h'$$

of 0-composable cells f, g and α of A we impose the 3-cells

$$\begin{split} F_{\mathrm{I}}(\alpha) *_{0} F_{\mathrm{I}}(g) *_{0} F_{\mathrm{I}}(f) *_{1} F_{\mathrm{V}}(h,g,f) \\ *_{2} \\ F_{\mathrm{I}}(h') *_{0} F_{\mathrm{V}}(g,f) *_{1} F_{\mathrm{V}}(\alpha,g *_{0} f) \end{split}$$

and

$$F_{\vee}(\alpha, g) *_{0} F_{\mathsf{i}}(f) *_{1} F_{\vee}(h *_{0} g, f) \\ *_{2} \\ F_{\vee}(h', g) *_{0} F_{\mathsf{i}}(f) *_{1} F_{\vee}(\alpha *_{0} g, f) \\ *_{2} \\ F_{\vee}(h', g, f) *_{1} F_{\mathsf{i}}(\alpha *_{0} g *_{0} f)$$

of B to be equal.

: For any triple

$$\bullet \xrightarrow{f} \bullet \overbrace{\Downarrow}_{\alpha'} \bullet \xrightarrow{h} \bullet$$

of 0-composable cells f, α and h of A, we impose that the 3-cells

$$F_{\vee}(h, g') *_{0} F_{|}(f) *_{1} F_{\vee}(h *_{0} \alpha, f)$$

$$*_{2}$$

$$F_{\vee}(h, g', f) *_{1} F_{|}(h *_{0} \alpha *_{0} f)$$

$$*_{2}$$

$$F_{|}(h) *_{0} F_{\vee}(g', f) *_{1} F_{\vee}(h, \alpha *_{0} f)$$

and

$$F_{\vee}(h,\alpha) *_{0} F_{|}(f) *_{1} F_{\vee}(h *_{0} g, f) \\ *_{2} \\ F_{|}(h) *_{0} F_{|}(\alpha) *_{0} F_{|}(f) *_{1} F_{\vee}(h,g,f) \\ *_{2} \\ F_{|}(h) *_{0} F_{\vee}(\alpha,f) *_{1} F_{\vee}(h,g *_{0} f)$$

of B to be equal.

: For any triple

$$\bullet \underbrace{ \int_{f'}^{g} \alpha }_{f'} \bullet \underbrace{ g }_{g} \longrightarrow \bullet \underbrace{ h }_{h} \longrightarrow \bullet$$

of 0-composable cells α , g and h of A we impose the 3-cells

$$\begin{split} F_{\vee}(h,g) *_{0} F_{\mathsf{I}}(f') *_{1} F_{\vee}(h *_{0} g, \alpha) \\ *_{2} \\ F_{\mathsf{I}}(h) *_{0} F_{\mathsf{I}}(g) *_{0} F_{\mathsf{I}}(\alpha) *_{1} F_{\vee}(h,g,f) \end{split}$$

and

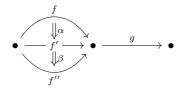
$$F_{\mathbb{V}}(h,g,f') *_{1} F_{\mathbb{I}}(h *_{0} g *_{0} \alpha) \\ *_{2}$$

$$F_{\mathbb{I}}(h) *_{0} F_{\mathbb{V}}(g,f') *_{1} F_{\mathbb{V}}(h,g *_{0} \alpha) \\ *_{2}$$

$$F_{\mathbb{I}}(h) *_{0} F_{\mathbb{V}}(g,\alpha) *_{1} F_{\mathbb{V}}(h,g *_{0} f)$$

of B to be equal.

 \bigvee : For any triple



of cells $\alpha, \, \beta$ and g of A we impose the equality of the 3-cells

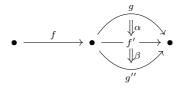
$$F_{\vee}(g, f'') *_1 F_{\vee}(g, \beta *_1 \alpha)$$

and

$$F_{\vee}(g,\beta) *_1 F_{\mathsf{i}}(g *_0 \alpha) \ *_2 \ F_{\mathsf{i}}(g) *_0 F_{\mathsf{i}}(\beta) *_1 F_{\vee}(g,\alpha)$$

of B.

: For any triple



of cells α , β and g of A we impose that the 3-cells

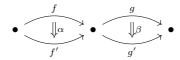
$$F_{\vee}(\beta *_1 \alpha, f)$$

and

$$F_{\downarrow}(\beta) *_{0} F_{\downarrow}(f) *_{1} F_{\downarrow}(\alpha, f) *_{2} F_{\downarrow}(\beta, f) *_{1} F_{\downarrow}(\alpha *_{0} f)$$

of B to be equal.

: Notice first that for any pair



of 0-composable 2-cells α and β of A, we have an equality of 2-cells

$$F_{!}(g' *_{0} \alpha *_{1} \beta *_{0} f) = F_{!}(\beta *_{0} f' *_{1} g *_{0} \alpha)$$

$$Y$$

$$F_{!}(g' *_{0} \alpha) *_{1} F_{!}(\beta *_{0} f)$$

$$F_{!}(\beta *_{0} f') *_{1} F_{!}(g *_{0} \alpha)$$

of B, where the equality in the higher row is just the exchange law. We shall denote by $F_{\rm ex}(\beta,\alpha)$ the identity 3-cell going from $F_{\rm i}(g'*_0\alpha)*_1F_{\rm i}(\beta*_0f)$ to $F_{\rm i}(\beta*_0f')*_1F_{\rm i}(g*_0\alpha)$.

For any pair of 2-cells α and β of A as above, we impose the 3-cells

$$F_{\!\scriptscriptstyle \parallel}(\beta) *_0 F_{\!\scriptscriptstyle \parallel}(f') *_1 F_{\!\scriptscriptstyle \swarrow}(g,\alpha) \ *_2 \ F_{\!\scriptscriptstyle \parallel}(g') *_0 F_{\!\scriptscriptstyle \parallel}(\alpha) *_1 F_{\!\scriptscriptstyle \searrow}(\beta,f)$$

and

$$F_{\vee}(\beta,f') *_{1} F_{!}(g *_{0} \alpha) \ *_{2} \ F_{\vee}(g',f') *_{1} F_{\mathrm{ex}}(\beta,\alpha) \ *_{2} \ F_{\vee}(g',\alpha) *_{1} F_{!}(\beta *_{0} f)$$

of B to be equal. Since $F_{\rm ex}(\beta,\alpha)$ is a trivial 3-cell, this coherence is actually imposing the equality between the 3-cells

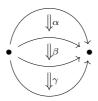
$$F_{\downarrow}(\beta) *_{0} F_{\mid}(f') *_{1} F_{\setminus \prime}(g, \alpha) *_{2} F_{\mid}(g') *_{0} F_{\downarrow}(\alpha) *_{1} F_{\setminus \prime}(\beta, f)$$

and

$$F_{\searrow}(\beta, f') *_1 F_{\downarrow}(g *_0 \alpha) *_2 F_{\swarrow}(g', \alpha) *_1 F_{\downarrow}(\beta *_0 f)$$

of B.

: For any triple



of 1-composable 2-cells $\alpha,\,\beta$ and γ of A we have the equalities between the identity 3-cell

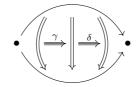
$$F_{\bigvee}(\gamma, \beta *_1 \alpha) *_2 F_{\downarrow}(\gamma) *_1 F_{\bigvee}(\beta, \alpha)$$

and the identity 3-cell

$$F_{\bigvee}(\gamma *_1 \beta, \alpha) *_2 F_{\bigvee}(\gamma, \beta) *_1 F_{\downarrow}(\alpha)$$

of B.

For any pair

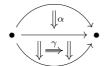


of 2-composable 3-cells γ and δ of A we impose the equality

$$F_{\mathbf{i}}(\delta *_{2} \gamma) = F_{\mathbf{i}}(\delta) *_{2} F_{\mathbf{i}}(\gamma)$$

between these two 3-cells of B.

: For any pair



of 1-composable cells α and γ of A, we impose that the 3-cells

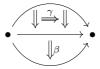
$$F(\gamma) *_1 F(\alpha)$$

and

$$F(\gamma *_1 \alpha)$$

of B are equal.

: For any pair



of 1-composable cells γ and β of A, we impose that the 3-cells

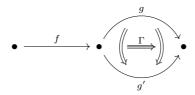
$$F_{\downarrow}(\beta) *_{1} F_{\downarrow}(\gamma)$$

and

$$F_{\mathfrak{t}}(\beta *_1 \gamma)$$

of B are equal.

: For any pair

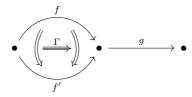


of 0-composable cells f and Γ of A, we impose the equality

$$F_{\vee}(g', f) *_{1} F_{!}(\Gamma *_{0} f) = F_{!}(\Gamma) *_{0} F_{|}(f) *_{1} F_{\vee}(g, f)$$

between these two 3-cells of B.

√ : For any pair



of 0-composable cells Γ and g of A, we impose the equality

$$F_{\vee}(g, f') *_{1} F_{!}(g *_{0} \Gamma) = F_{!}(g) *_{0} F_{!}(\Gamma) *_{1} F_{\vee}(g, f)$$

between these two 3-cells of B.

Remark 1.2.4. Let us clarify a bit better why we need the data for the tree \forall representing the vertical composition of two 2-cells to be trivial. In fact, one would expect that for any pair of 1-composable 2-cells x and y of A, a (normalised) oplax 3-functor would associate a 3-cell

$$F_{\downarrow}(x,y) \colon F_{\downarrow}(x *_1 y) \to F_{\downarrow}(x) *_1 F_{\downarrow}(y).$$

At the same time, for any pair

$$\bullet \xrightarrow{f'} \bullet \xrightarrow{g} \bullet$$

of 0-composable 2-cells α and β of A, the coherence for the tree \bigvee should express a relationship among $F_{\bigvee}(g *_0 \alpha)$, $F_{\bigvee}(\beta *_0 f)$, $F_{\bigvee}(g' *_0 \alpha)$ and $F_{\bigvee}(\beta *_0 f')$. On the one hand, we can compose

$$(F_{\downarrow}(\beta) *_{0} F_{\downarrow}(f') *_{1} F_{\downarrow}(g,\alpha)) *_{2} (F_{\downarrow}(g') *_{0} F_{\downarrow}(\alpha) *_{1} F_{\downarrow}(\beta,f)),$$

getting a 3-cell from

$$F(g') *_0 F(\alpha) *_1 F(g', f) *_1 F(\beta *_0 f)$$

to

$$F_{!}(\beta) *_{0} F_{!}(f') *_{1} F_{\lor}(g, f') *_{1} F_{!}(g *_{0} \alpha).$$

On the other hand, we have the 3-cell

(A)
$$F_{\vee}(g',\alpha) *_1 F_{\downarrow}(\beta *_0 f),$$

with source

$$F_{i}(g') *_{0} F_{i}()\alpha) *_{1} F_{\vee}(g', f) *_{1} F_{i}(\beta *_{0} f)$$

and target

$$F_{\vee}(g', f') *_1 F_{||}(g' *_0 \alpha) *_1 F_{||}(\beta *_0 f),$$

as well as the 3-cell

(B)
$$F_{\backslash \prime}(\beta, f') *_1 F_{!}(g *_0 \alpha),$$

with source

$$F_{\vee}(g', f') *_{1} F_{|}(\beta *_{0} f') *_{1} F_{|}(g *_{0} \alpha)$$

and target

$$F_{!}(\beta) *_{0} F_{!}(f') *_{1} F_{\vee}(g, f') *_{1} F_{!}(g *_{0} \alpha).$$

The 3-cells (A) and (B) are *not* composable, since the target of the first one and the source of the second one are respectively the bottom left and the bottom right 2-cells of the following diagram

of B, where the equality in the higher row is just the exchange law. Unless the data for the tree \forall is trivial, it is impossible to provide a relationship among the 3-cells listed above and involving the whiskerings.

Remark 1.2.5. Gurski defines in [14] a notion of lax trimorphism between tricategories, see Definition 4.11 of loc. cit.; one can easily adapt and dualise suitably the definition and get a notion of normalised oplax trimorphism between tricategories. Consider two strict 3-categories A and B, a normalised oplax 3-functor $F \colon A \to B$ and a normalised oplax trimorphism $G \colon A \to B$. There are two main differences between Gurski's notion of normalised oplax trimorphism and the notion of normalised oplax 3-functor that we presented above.

- The first difference concerns the tree $\stackrel{\vee}{}$. Gurski's notion requires that, for any pair (β,α) of 1-composable 2-cells of A, there is a 3-cell of B going from $G(\beta*_1\alpha)$ to $G(\beta)*_1G(\alpha)$ which is not invertible in general. It is essential in our definition of normaised oplax 3-functor that the tree $\stackrel{\vee}{}$ appears in the coherences. This is slightly unnatural even from our arboreal point of view and it may be reasonable to actually impose this condition on Gurski's normalised oplax trimorphisms, in light of a comparison with normalised oplax 3-functors. Gurski calls this condition local strictness. Notice that it implies in particular that F as well as G induce strict 2-functors on the hom-2-categories.
- The second difference is deeper and somehow irreconcilable. For any pair (β, α) of 0-composable 2-cells of A, the normalised oplax trimorphism provides a 3-cell

$$G_{\backslash \prime}(\beta,\alpha) \colon G(t_1(\beta),t_1(\alpha)) *_1 G(\beta *_0 \alpha) \to (G(\beta) *_0 G(\alpha)) *_1 G(s_1(\beta),s_1(\alpha)).$$

This is incompatible with the algebra of the orientals, as we shall better explain in the following section. In fact, a normalised oplax 3-functor has the tree \vee as a coherence and instead the trees \vee and \vee as part of the data. But these pieces of data are symmetric (or better dual), hence they cannot fit as a particular case of a single lax or oplax datum for \vee .

Example 1.2.6. Let C be a 3-category. We now define a normalised oplax 3-functor $\sup: i_{\Delta}(N_3(C)) \to C$ (cf. Notations and Terminology).

- : The map sup is defined by mapping an object (a, x), where $x : \mathcal{O}_m \to C$, to $x(\langle m \rangle)$.
- : The map \sup_{α} assigns to any morphism $f:(a,x)\to(b,y)$ of $i_{\Delta}(N_3(C))$, where $x:\mathcal{O}_m\to C$ and $y:\mathcal{O}_n\to C$, the 1-cell $x'(\langle f(m),n\rangle)$ of C.
- \vee : The map \sup_{\vee} assigns to any pair of composable morphisms

$$(a,x) \xrightarrow{f} (b,y) \xrightarrow{g} (c,z)$$

of $i_{\Delta}(N_3(C))$, with $x \colon \mathcal{O}_m \to C$, $y \colon \mathcal{O}_n \to C$ and $z \colon \mathcal{O}_p \to C$, the 2-cell $\sup_{\mathcal{V}}(g,f)$ of C given by

$$z(\langle gf(m), g(n), p \rangle)$$

 \forall : The map \sup_{\forall} , assigns to any triple of composable morphisms

$$(a,x) \xrightarrow{f} (b,y) \xrightarrow{g} (c,z) \xrightarrow{h} (d,t)$$

of $i_{\Delta}(N_3(C))$, with $x \colon \mathcal{O}_m \to C$, $y \colon \mathcal{O}_n \to C$, $z \colon \mathcal{O}_p \to C$ and $t \colon \mathcal{O}_q \to C$, the 3-cell $\sup_{\Psi}(h, g, f)$ of C given by

$$t(\langle hgf(m), hg(n), h(p), q \rangle)$$

Notice that by definition we have that $1_{\sup_{\bullet}(a,x)}$ is precisely $\sup_{\bullet}(1_{(a,x)})$ and that the other conditions of normalisation are equally trivial by definition. We now check the coherence for the tree \searrow .

Consider four composable morphisms of $i_{\Delta}(N_3(C))$

$$(a,x) \xrightarrow{f} (b,y) \xrightarrow{g} (c,z) \xrightarrow{h} (d,t) \xrightarrow{i} (e,w)$$
,

with $x: \mathcal{O}_m \to C, y: \mathcal{O}_n \to C, z: \mathcal{O}_p \to C, t: \mathcal{O}_q \to C$ and $w: \mathcal{O}_r \to C$. We have to show that the 3-cells

$$\begin{split} \sup_{\forall (i,h,g) *_0 \sup_{|(f) *_1 \sup_{\forall (i*_0 h *_0 g,f)} *_2} \\ \sup_{|(i) *_0 \sup_{\forall (h,g) *_0 \sup_{|(f) *_1 \sup_{\forall (i,h*_0 g,f)} *_2} \\ \sup_{|(i) *_0 \sup_{\forall (h,g,f) *_1 \sup_{\forall (i,h*_0 g *_0 f)} *_2} \end{split}$$

and

$$\begin{split} \sup_{\bigvee}(i,h) *_0 \sup_{\mathbf{i}}(g) *_0 \sup_{\mathbf{i}}(f) *_1 \sup_{\bigvee}(ih,g,f) \\ *_2 \\ \sup_{\mathbf{i}}(i) *_0 \sup_{\mathbf{i}}(h) *_0 \sup_{\bigvee}(g,f) *_1 \sup_{\bigvee}(i,h,gf) \,. \end{split}$$

But these two 3-cells are precisely the target and the source of the main 4-cell of \mathcal{O}_4 via the ∞ -functor $\mathcal{O}_4 \xrightarrow{\varphi} \mathcal{O}_r \xrightarrow{w} C$, where

$$\varphi = \mathcal{O}_{\{ihgf(m),ihg(n),ih(p),i(q),r\}}$$
.

1.3. From cellular to simplicial. In this subsection we shall show that to any normalised oplax 3-functor $F \colon B \to C$ there is a canonically associated morphism $N_l(F) \colon N_\infty(B) \to N_\infty(C)$ of simplicial sets. Throughout this section we shall sometimes write oplax 3-functor for normalised oplax 3-functor, since no confusion is possible.

1.3.1. Let A be a 1-category and B and C be two 3-categories. Fix two oplax 3-functors $F: A \to B$ and $G: B \to C$. We now define a candidate GF for the composition of F and G and we dedicate the rest of the subsection to prove the coherences.

Since A is a 1-category, the amount of data that we have to provide in order to define an oplax 3-functor, *i.e.*, the trees of dimension less than 3, is limited to the trees \bullet , I , \vee and \vee . The 3-functor GF is defined as follows:

.: The map GF_{\bullet} assigns to any object a of A the object $G_{\bullet}(F_{\bullet}(a))$ of C.

: The map GF_{l} assigns to any 1-cell $f: a \to a'$ of A, i.e., any tree of A_{l} , the 1-cell

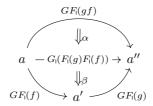
$$G_{\mathsf{I}}(F_{\mathsf{I}}(f)) \colon GF_{\bullet}(a) \to GF_{\bullet}(a')$$

of C.

 \vee : The map GF_{\vee} assigns to any pair of 0-composable 1-cells

$$a \xrightarrow{f} a' \xrightarrow{g} a''$$

of A the 2-cell $GF_{\vee}(q, f)$



of
$$C$$
, where $\alpha = G_{\dagger}(F_{\lor}(g, f))$ and $\beta = G_{\lor}(F_{\dagger}(g), F_{\dagger}(f))$, that is $G_{\lor}(F_{\dagger}(g), F_{\dagger}(f)) *_{1} G_{\dagger}(F_{\lor}(g, f))$.

 \bigvee : The map GF_{\forall} , assigns to any triple of 0-composable 1 cells

$$a \xrightarrow{f} a' \xrightarrow{g} a'' \xrightarrow{h} a'''$$

of A the 3-cell $GF_{\mathbb{V}}(h,g,f)$ defined as

$$G_{\vee}(\mathit{F_l}h,\mathit{F_l}g) *_0 G\mathit{F_l}(f) *_1 G_{\vee}(\mathit{F_l}(h,g),\mathit{F_l}f) *_1 G_{\nmid}(\mathit{F_l}(hg,f))$$

$$G_{\Psi}(F_{\mathsf{I}}h,F_{\mathsf{I}}g,F_{\mathsf{I}}f) *_{1} G_{\mathsf{f}}(F_{\Psi}(h,g,f))$$

$$GF_{\mathsf{I}}(h) *_0 G_{\vee}(F_{\mathsf{I}}g,F_{\mathsf{I}}f) *_1 G_{\vee}(F_{\mathsf{I}}h,F_{\vee}(g,f)) *_1 G_{\mathsf{I}}(F_{\vee}(h,gf)) \,.$$

Notice that for the 3-cell $G_{\dagger}(F_{\forall}(h,g,f))$ in the second line we are implicitly using the coherence G_{\forall} , as we use the equality

$$G_{!}(F_{!}h *_{0} F_{\vee}(g,f)) *_{1} G_{!}(F_{\vee}(h,gf)) = G_{!}(F_{!}h *_{0} F_{\vee}(g,f) *_{1} F_{\vee}(h,gf))$$

as well as the equality

$$G_{\downarrow}(F_{\lor}(h,g) *_{0} F_{\downarrow}f) *_{1} G_{\downarrow}(F_{\lor}(hg,f)) = G_{\downarrow}(F_{\lor}(h,g) *_{0} F_{\downarrow}f *_{1} F_{\lor}(hg,f))$$

of 2-cells of A, which are respectively the source and target of the 3-cell $G_{\mathfrak{t}}(F_{\mathbb{V}}(h,g,f))$.

- **1.3.2.** The conditions of normalisation for the composite GF are tedious but straightforward. We give here a few examples and we leave the other similar verifications to the reader.
- \Box : For any 0-cell a of A we have

$$GF_{\rm I}(1_a) = s_1^0(GF_{\rm I}(a)) = 1_{GF_{\rm I}(a)}$$
 .

 \bigvee : For any 1-cell $f: a \to a'$ of A, we have

$$G_{\vee}(F_{l}(f), F_{l}(1_{a})) = G_{\vee}(F_{l}(f), 1_{F(a)}) = 1_{G(F_{l}(f))} = 1_{G(F_{l}(f))}$$

and also

$$G_{!}(F_{\lor}(f, 1_a)) = G_{!}(1_{F_{!}(f)}) = 1_{G_{!}(F_{!}(f))} = 1_{GF_{!}(f)},$$

so that

$$GF_{\vee}(f, 1_a) = 1_{GF(f)} *_1 1_{GF(f)} = 1_{GF(f)}$$
.

 \checkmark : For any pair $a \xrightarrow{f} a' \xrightarrow{g} a''$ of 1-cells of A, we have

$$G_{\vee}(F_{\vee}(g,f),F_{|}(1_a)) = G_{\vee}(F_{\vee}(g,f),1_{F_{\bullet}(a)}) = 1_{G_{\bullet}(F_{\vee}(g,f))}$$

and

$$G_{\mathbb{V}}\big(\mathit{F}_{\mathsf{I}}(g),\mathit{F}_{\mathsf{I}}(f),\mathit{F}_{\mathsf{I}}(1_{a})\big) = G_{\mathbb{V}}\big(\mathit{F}_{\mathsf{I}}(g),\mathit{F}_{\mathsf{I}}(f),1_{\mathit{F}_{\bullet}(a)}\big) = 1_{G_{\mathsf{I}}(\mathit{F}_{\mathsf{I}}(g),\mathit{F}_{\mathsf{I}}(f))}$$

and

$$G_{\sharp}\big(F_{\mathbb{V}}(g,f,1_a)\big) = G_{\sharp}(1_{F_{\mathbb{V}}(g,f)}) = 1_{G_{\sharp}(F_{\mathbb{V}}(g,f))}$$

and

$$G_{\bigvee}\big(F_{\mathsf{I}}(g),F_{\bigvee}(f,1_a)\big)=G_{\bigvee}\big(F_{\mathsf{I}}(g),1_{F(f)}\big)=1_{F_{\bigvee}(g,f)}\,.$$

Hence, we get that

$$GF_{\mathbb{V}}(g,f,1_a) = 1_{G_{\mathbb{I}}(F_{\mathbb{I}}(g),F_{\mathbb{I}}(f))} *_{1} 1_{G_{\mathbb{I}}(F_{\mathbb{V}}(g,f))} = 1_{GF_{\mathbb{V}}(g,f)} \, .$$

1.3.3. Since A is a 1-category, the only coherence we have to prove is the coherence associated to the tree \checkmark . Consider four composable 1-cells of A

$$\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet \xrightarrow{h} \bullet \xrightarrow{i} \bullet .$$

We have to show that the 3-cells

$$\begin{split} GF_{\mathbb{V}}(i,h,g) *_{0} GF_{\mathbb{I}}(f) *_{1} GF_{\mathbb{V}}(i *_{0} h *_{0} g, f) \\ *_{2} \\ GF_{\mathbb{I}}(i) *_{0} GF_{\mathbb{V}}(h,g) *_{0} GF_{\mathbb{I}}(f) *_{1} GF_{\mathbb{V}}(i,h *_{0} g, f) \\ *_{2} \\ GF_{\mathbb{I}}(i) *_{0} GF_{\mathbb{V}}(h,g,f) *_{1} GF_{\mathbb{V}}(i,h *_{0} g *_{0} f) \end{split}$$

and

$$\begin{split} GF_{\vee}(i,h) *_{0} GF_{\mathsf{I}}(g) *_{0} GF_{\mathsf{I}}(f) *_{1} GF_{\vee}(ih,g,f) \\ *_{2} \\ GF_{\mathsf{I}}(i) *_{0} GF_{\mathsf{I}}(h) *_{0} GF_{\vee}(g,f) *_{1} GF_{\vee}(i,h,gf) \end{split}$$

of C are equal. The five 3-cells involved in this compositions are:

(a) the 3-cell $GF_{\mathbb{V}}(h, g, f)$ of C, which is defined as

$$\begin{split} G_{\vee}(F_{|}h,F_{|}g) *_{0} GF_{|}(f) *_{1} G_{\vee}(F_{\vee}(h,g),F_{|}f) *_{1} G_{|}(F_{\vee}(hg,f)) \\ *_{2} \\ G_{\vee}(F_{|}h,F_{|}g,F_{|}f) *_{1} G_{|}(F_{\vee}(h,g,f)) \\ *_{2} \end{split}$$

$$GF_{\rm I}(h) *_{0} G_{\rm V}(F_{\rm I}g,F_{\rm I}f) *_{1} G_{\rm V}(F_{\rm I}h,F_{\rm V}(g,f)) *_{1} G_{\rm I}(F_{\rm V}(h,gf))\,,$$

which is a suitably whiskered 1-composition of the 3-cells

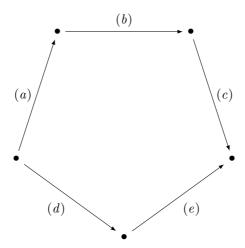
$$(a.1)$$
 $G_{\vee}(F_{\mathsf{I}}h, F_{\vee}(g,f)),$

(a.2)
$$G_{\forall}(F_{l}h, F_{l}g, F_{l}f) *_{1} G_{!}(F_{\forall}(h, g, f)),$$

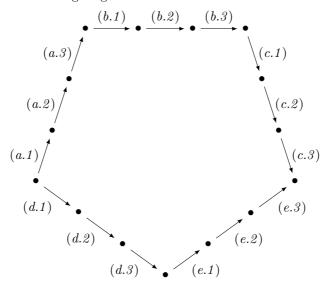
(a.3)
$$G_{\vee}(F_{\vee}(h,g),F_{|}f)$$
, of C ;

```
(b) the 3-cell GF_{\vee}(i, h *_0 g, f) of C, which is defined as
            G_{\vee}(\mathit{Fii},\mathit{Fihg}) *_{0} G\mathit{Fi}(f) *_{1} G_{\vee}(\mathit{F}_{\vee}(i,hg),\mathit{Fi}f) *_{1} G_{\dagger}(\mathit{F}_{\vee}(ihg,f))
                                   G_{\mathbb{V}}(\mathit{Fii},\mathit{Fihg},\mathit{Fif}) *_{1} G_{\sharp}(\mathit{F}_{\mathbb{V}}(i,hg,f))
           GF_{\mathsf{I}}(i) *_{0} G_{\vee}(F_{\mathsf{I}}hg,F_{\mathsf{I}}f) *_{1} G_{\vee}(F_{\mathsf{I}}i,F_{\vee}(hg,f)) *_{1} G_{\mathsf{I}}(F_{\vee}(i,hgf))\,,
       which is a suitably whiskered 1-composition of the 3-cells
     (b.1) G_{\vee}(F_{l}i, F_{\vee}(hg, f)),
     (b.2) G_{V}(F_{i}i, F_{i}hg, F_{i}f) *_{1} G_{!}(F_{V}(i, hg, f)),
     (b.3) G_{\vee}(F_{\vee}(i,hg),F_{|}f),
       of C;
(c) the 3-cell GF_{\mathbb{V}}(i,h,g) of C, which is defined as
                G_{\vee}(F_{i}i, F_{i}h) *_{0} GF_{i}(g) *_{1} G_{\vee}(F_{\vee}(i, h), F_{i}g) *_{1} G_{i}(F_{\vee}(ih, g))
                                      G_{\mathbb{V}}(F_{|}i,F_{|}h,F_{|}g)*_{1}G_{\sharp}(F_{\mathbb{V}}(i,h,g))
               GF_{!}(i) *_{0} G_{\lor}(F_{!}h, F_{!}g) *_{1} G_{\lor}(F_{!}i, F_{\lor}(h, g)) *_{1} G_{!}(F_{\lor}(i, hg)),
       which is a suitably whiskered 1-composition of the 3-cells
     (c.1) G_{\lor}(F_{!}i, F_{\lor}(h, g)),
     (c.2) G_{V}(F_{l}i, F_{l}h, F_{l}g) *_{1} G_{t}(F_{V}(i, h, g)),
     (c.3) G_{\vee}(F_{\vee}(i,h),F_{\mid}g),
       of C;
(d) the 3-cell GF_{\mathbb{V}}(i,h,g) of C, which is defined as
            G_{\vee}(F_{|}i,F_{|}h) *_{0} GF_{|}(gf) *_{1} G_{\vee}(F_{\vee}(i,h),F_{|}gf) *_{1} G_{|}(F_{\vee}(ih,gf))
                                   G_{\mathbb{V}}(F_{\mathbb{I}}i,F_{\mathbb{I}}h,F_{\mathbb{I}}gf)*_{1}G_{\sharp}(F_{\mathbb{V}}(i,h,gf))
           GF_{!}(i) *_{0} G_{\lor}(F_{!}h, F_{!}gf) *_{1} G_{\lor}(F_{!}i, F_{\lor}(h, gf)) *_{1} G_{!}(F_{\lor}(i, hgf)),
       which is a suitably whiskered 1-composition of the 3-cells
     (d.1) G_{\vee}(F_i i, F_{\vee}(h, gf)),
     (d.2) G_{\vee}(F_{i}i, F_{i}h, F_{i}gf) *_{1} G_{!}(F_{\vee}(i, h, gf)),
     (d.3) G_{\vee}(F_{\vee}(i,h),F_{\parallel}gf),
       of C:
(e) the 3-cell GF_{\mathbb{V}}(ih, g, f) of C, which is defined as
            G_{\vee}(\mathit{F_i}{ih},\mathit{F_i}{g}) *_0 G\mathit{F_i}(f) *_1 G_{\vee}(\mathit{F_{\vee}}(ih,g),\mathit{F_i}{f}) *_1 G_{\mathsf{i}}(\mathit{F_{\vee}}(ihg,f))
                                   G_{\mathbb{V}}(\mathit{Fi}h,\mathit{F_{\mathsf{I}}}g,\mathit{F_{\mathsf{I}}}f) *_{1} G_{\sharp}(\mathit{F_{\mathbb{V}}}i(h,g,f))
           GF_{|}(ih) *_{0} G_{\vee}(F_{|}g, F_{|}f) *_{1} G_{\vee}(F_{|}ih, F_{\vee}(g, f)) *_{1} G_{|}(F_{\vee}(ih, gf)),
       which is a suitably whiskered 1-composition of the 3-cells
     (e.1) G_{\vee}(F_{l}ih, F_{\vee}(g, f)),
     (e.2) G_{\forall}(F_{i}ih, F_{i}g, F_{i}f) *_{1} G_{\dagger}(F_{\forall}(ih, g, f)),
     (e.3) G_{\lor}(F_{\lor}(ih,g),F_{\i}f),
       of C.
```

In summary, we have to show that the pentagon



of 2-compositions of 3-cells of C is commutative. Using the decomposition of each of these 3-cells as suitably whiskered 1-composition of other 3-cells of C, we have to show that the following diagram



of 2-compositions of 3-cells of C commutes; notice that in the latter diagram the referenced 3-cells of C are not 2-composable: we are making the abuse of denoting each arrow of the diagram with the reference to a particular 3-cell, without the suitable whiskerings making all these 3-cells 2-composable. These whiskerings are written explicitly above.

1.3.4. In order to show that the diagram of the previous paragraph is commutative, we shall decompose it in several smaller diagrams and we shall show that each of them is commutative. This decomposition is displayed in figure 1. There is a duality involving the diagram numbered with (n) and with (n') and the one commutes if and only if the other one does. We shall illustrate this phenomenon with the diagrams (1) and (1') and (2) and (2'), but then we will limit ourself to prove the commutativity of the diagrams of the type (n), leaving the diagrams of type (n') to the reader.

1.3.5 (1). Consider the diagram (1), where again we abuse of notation by forgetting the suitable whiskerings making these 3-cells of C actually 2-composable:

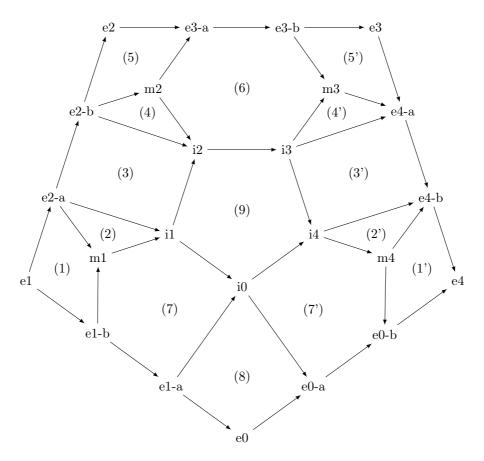
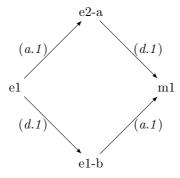


Figure 1. The diagram for the coherence $\vee\!\!\!\vee$.



To be precise, the 3-cell from e1 to e2-a is

$$\begin{split} GF_{\rm l}(i) *_0 GF_{\rm l}(h) *_0 G_{\rm V}(F_{\rm l}g,F_{\rm l}f) *_1 G_{\rm V}(F_{\rm l}h,F_{\rm V}(g,f)) \\ *_1 \\ GF_{\rm l}(i) *_0 G_{\rm l}(F_{\rm V}(h,gf))) *_1 GF_{\rm V}(i,h*_0 g*_0 f) \end{split}$$

and the 3-cell from e1-b to m1 is

$$\begin{split} GF_{\rm l}(i) *_0 GF_{\rm l}(h) *_0 G_{\rm V}(F_{\rm l}g,F_{\rm l}f) *_1 G_{\rm V}(F_{\rm l}h,F_{\rm V}(g,f)) \\ *_1 \\ G_{\rm V}(F_{\rm l}i,F_{\rm l}(h) *_0 F_{\rm l}(gf)) *_1 G_{\rm l}(F_{\rm l}i *_0 F_{\rm V}(h,gf)) *_1 G_{\rm l}(F_{\rm l}(i,hgf)) \,, \end{split}$$

while the 3-cell from e1 to e1-b is

$$\begin{split} GF_{\mathbf{i}}(i) *_{0} \left(GF_{\mathbf{i}}(h) *_{0} GF_{\mathbf{v}}(g,f) *_{1} G_{\mathbf{v}}(F_{\mathbf{i}}h,F_{\mathbf{i}}gf) \right) \\ *_{1} \\ G_{\mathbf{v}}(F_{\mathbf{i}},F_{\mathbf{v}}(h,gf)) *_{1} G_{\mathbf{i}}(F_{\mathbf{v}}(i,hgf)) \end{split}$$

and the 3-cell from e2-a to m1 is

$$\begin{split} GF_{\rm l}(i) *_0 \left(GF_{\rm l}(h) *_0 G_{\rm V}(F_{\rm l}g,F_{\rm l}f) *_1 G_{\rm V}(F_{\rm l}h,F_{\rm l}g *_0 F_{\rm l}f) *_1 G_{\rm l}(F_{\rm l}h *_0 F_{\rm V}(g,f)) \right) \\ *_1 \\ G_{\rm V}(F_{\rm l}i,F_{\rm V}(h,gf)) *_1 G_{\rm l}(F_{\rm V}(i,hgf)) \,. \end{split}$$

The commutativity of the diagram is simply an instance of the interchange law. Indeed, the source and the target of the 3-cell of ${\cal C}$

$$GF_1(h) *_0 G_{\vee}(F_1g, F_1f) *_1 G_{\vee}(F_1h, F_{\vee}(g, f))$$

are the 2-cells of C

$$GF_{\downarrow}(h) *_{0} GF_{\lor}(g, f) *_{1} G_{\lor}(F_{\downarrow}h, F_{\downarrow}gf)$$

and

$$GF_{\rm I}(h) *_0 G_{\rm V}(F_{\rm I}g,F_{\rm I}f) *_1 G_{\rm V}(F_{\rm I}h,F_{\rm I}g*_0 F_{\rm I}f) *_1 G_{\rm I}(F_{\rm I}h*_0 F_{\rm V}(g,f))$$

respectively; while the source and target of the 3-cell

$$G_{\vee}(F_{|}i,F_{\vee}(h,gf))*_{1}G_{|}(F_{\vee}(i,hgf))$$

of C are the 2-cells

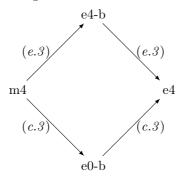
$$GF(i) *_0 G_{\dagger}(F_{\lor}(h, gf))) *_1 GF_{\lor}(i, h *_0 g *_0 f)$$

and

$$G_{\vee}(F_!i,F_!(h)*_0F_!(gf))*_1G_!(F_!i*_0F_{\vee}(h,gf))*_1G_!(F_!(i,hgf))$$

respectively.

1.3.6 (1'). Consider the diagram (1'), where we abuse of notation by forgetting the suitable whiskerings making these 3-cells of C actually 2-composable:



To be precise, the 3-cell from e4-b to e4 is

$$G_{\vee}(\mathit{Fii},\mathit{Fih}) *_{0} G\mathit{Fi}(g) *_{0} G\mathit{Fi}(f) *_{1} G_{\vee}(\mathit{F}_{\vee}(i,h),\mathit{Fig}) \\ *_{1}$$

$$G_{i}(F_{\vee}(ih,g))) *_{0} GF_{i}(f) *_{1} GF_{\vee}(i *_{0} h *_{0} g, f)$$

and the 3-cell from m4 to e0-b is

$$G_{\vee}(F_{i}i, F_{i}h) *_{0} GF_{i}(g) *_{0} GF_{i}(f) *_{1} G_{\vee}(F_{\vee}(i, h), F_{i}g)$$

$$*_{1}$$

$$G_{\vee}(F_{i}i *_{0} F_{i}(hg), F_{i}(f)) *_{1} G_{i}(F_{\vee}(ih, g) *_{0} F_{i}f) *_{1} G_{i}(F_{i}(ihg, f)),$$

while the 3-cell from e0-b to e4 is

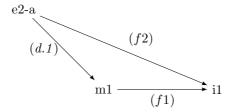
$$\begin{split} \left(GF_{\vee}(i,h) *_0 GF_{\mathsf{I}}(g) *_1 G_{\vee}(F_{\mathsf{I}}ih,F_{\mathsf{I}}g)\right) *_0 GF_{\mathsf{I}}(i) \\ *_1 \\ G_{\vee}(F_{\vee}(ih,g),F_{\mathsf{I}}f) *_1 G_{\mathsf{I}}(F_{\vee}(ihg,f)) \end{split}$$

and the 3-cell from m4 to e4-b is

$$\begin{split} \left(G_{\vee}(F_{!}i,F_{!}h) *_{0} GF_{!}(g) *_{1} G_{\vee}(F_{!}i *_{0} F_{!}h,F_{!}g) *_{1} G_{!}(F_{\vee}(i,h) *_{0} F_{!}g) \right) *_{0} GF_{!}(f) \\ *_{1} \\ G_{\vee}(F_{\vee}(ih,g),F_{!}f) *_{1} G_{!}(F_{\vee}(ihg,f)) \, . \end{split}$$

Analogously to the previous case, the commutativity of the diagram is simply an instance of the interchange law.

1.3.7 (2). Consider the diagram (2), where we always adopt the same notational abuse:



where

$$(f1) \hspace{3.1em} G_{\vee}(\mathit{Fi},\mathit{Fi}h *_{0} F_{\vee}(g,f))$$

is the principal 3-cell of (f1) and

(f2)
$$G_{\vee}(F_i, F_i h *_0 F_{\vee}(g, f) *_1 F_{\vee}(h, gf))$$

is the principal 3-cell of (f2). More precisely, the 3-cell of C from e2-a to m1 is

$$GF_{\mathsf{i}}(i) *_{0} (GF_{\mathsf{i}}(h) *_{0} G_{\vee}(F_{\mathsf{i}}g, F_{\mathsf{i}}f) *_{1} G_{\vee}(F_{\mathsf{i}}h, F_{\mathsf{i}}g *_{0} F_{\mathsf{i}}f)) *_{1} G_{\vee}(F_{\mathsf{i}}i, F_{\vee}(h, gf)) *_{1} G_{\vee}(F_{\mathsf{i}}i, F_{\vee}(h, gf)) *_{1} G_{\mathsf{i}}(F_{\vee}(i, hqf)),$$

the 3-cell from m1 to i1 is

$$GF_{!}(i) *_{0} (GF_{!}(h) *_{0} G_{\vee}(F_{!}g, F_{!}f) *_{1} G_{\vee}(F_{!}h, F_{!}g *_{0} F_{!}f)) \\ *_{1}$$

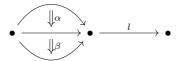
$$G_{\vee}(F_{!}i, F_{!}h *_{0} F_{\vee}(g, f)) *_{1} G_{!}(F_{!}i *_{0} F_{\vee}(h, gf)) \\ *_{1}$$

$$G_{!}(F_{\vee}(i, hgf))$$

and the 3-cell from e2-a to i1 is

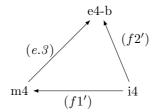
$$\begin{split} GF_{\mathbf{i}}(i) *_{0} & \left(GF_{\mathbf{i}}(h) *_{0} G_{\vee}(F_{\mathbf{i}}g, F_{\mathbf{i}}f) *_{1} G_{\vee}(F_{\mathbf{i}}h, F_{\mathbf{i}}g *_{0} F_{\mathbf{i}}f) \right) \\ *_{1} \\ & G_{\vee}\left(F_{\mathbf{i}}i, F_{\mathbf{i}}h *_{0} F_{\vee}(g, f) *_{1} F_{\vee}(h, gf) \right) \\ *_{1} \\ & G_{\mathbf{i}}(F_{\vee}(i, hgf)) \, . \end{split}$$

The 3-cells appearing in the middle lines of these 1-compositions are precisely the 3-cells of the coherence for G for the tree \bigvee for the pasting diagram



where $\alpha = F_{\lor}(h,gf), \ \beta = F_{|}(h) *_{0} F_{\lor}(g,f)$ and $l = F_{|}(i)$. Hence the diagram commutes.

1.3.8 (2'). Consider the diagram (2'), where we always adopt the same notational abuse:



where the 3-cell of C from m4 to e4-b is

$$\begin{split} \left(G_{\vee}(F_{!}i,F_{!}h) *_{0} GF_{!}(g) *_{1} G_{\vee}(F_{!}i *_{0} F_{!}h,F_{!}g)\right) *_{0} GF_{!}(f) \\ *_{1} \\ G_{!}(F_{\vee}(i,h) *_{0} F_{!}g) *_{0} GF_{!}(f) *_{1} G_{\vee}(F_{\vee}(ih,g),F_{!}f) \\ *_{1} \\ G_{!}(F_{\vee}(ihg,f)) \,, \end{split}$$

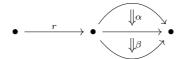
the 3-cell from i4 to m4 is

$$\begin{array}{c} \left(G_{\vee}(F_{!}i,F_{!}h) *_{0} GF_{!}(g) *_{1} G_{\vee}(F_{!}i *_{0} F_{!}h,F_{!}g)\right) *_{0} GF_{!}(f) \\ *_{1} \\ G_{\vee}(F_{\vee}(i,h) *_{0} F_{!}g,F_{!}f) *_{1} G_{!}(F_{\vee}(ih,g) *_{0} F_{!}f) \\ *_{1} \\ G_{!}(F_{\vee}(ihg,f)) \end{array}$$

and the 3-cell from i4 to e4-b is

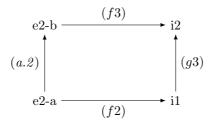
$$\begin{split} \left(G_{\vee}(F_{!}i,F_{!}h) *_{0} GF_{!}(g) *_{1} G_{\vee}(F_{!}i *_{0} F_{!}h,F_{!}g) \right) *_{0} GF_{!}(f) \\ *_{1} \\ G_{\vee}\left(F_{\vee}(ih,g) *_{1} F_{\vee}(i,h) *_{0} F_{!}g,F_{!}f \right) \\ *_{1} \\ G_{!}(F_{\vee}(ihg,f)) \, . \end{split}$$

The 3-cells appearing in the middle lines of these 1-compositions are precisely the 3-cells of the coherence for G for the tree \bigvee for the pasting diagram



where $\alpha = F_{\vee}(ih, g)$, $\beta = F_{\vee}(i, h) *_{0} F_{l}(g)$ and $r = F_{l}(f)$. Hence the diagram commutes.

1.3.9 (3). Consider the diagram (3):



where

$$(f3) \qquad \qquad G_{\vee}(\mathit{Fii}, \mathit{F}_{\vee}(h,g) *_{0} \mathit{F}_{|f} *_{1} \mathit{F}_{\vee}(hg,f))$$

is the principal 3-cell of (f3) and

$$(g3) GF_{\mathsf{I}}(i) *_{0} G_{\mathsf{V}}(F_{\mathsf{I}}h, F_{\mathsf{I}}g, F_{\mathsf{I}}f) *_{1} G_{\mathsf{V}}(F_{\mathsf{I}}i, F_{\mathsf{I}}h F_{\mathsf{I}}g F_{\mathsf{I}}f) *_{1} G_{\mathsf{I}}(F_{\mathsf{I}}(i) *_{0} F_{\mathsf{V}}(h, g, f))$$

is the principal 3-cell of (g3). More precisely, the 3-cell from e2-a to e2-b is

$$G_{\mathbb{V}}(F_{\mathbb{I}}h, F_{\mathbb{I}}g, F_{\mathbb{I}}f) *_{1} G_{\mathfrak{f}}(F_{\mathbb{V}}(h, g, f))$$
 $*_{1}$
 $GF_{\mathbb{V}}(i, hgf);$

the 3-cell from e2-a to i1 is

$$\begin{split} GF_{!}(i) *_{0} & \left(GF_{!}(h) *_{0} G_{\vee}(F_{!}g, F_{!}f) *_{1} G_{\vee}(F_{!}h, F_{!}g *_{0} F_{!}f) \right) \\ *_{1} \\ & G_{\vee}(F_{!}i, F_{\vee}(h, gf) *_{1} F_{!}h *_{0} F_{\vee}(g, f)) \\ *_{1} \\ & G_{!}(F_{\vee}(i, hgf)) \, ; \end{split}$$

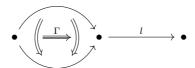
the 3-cell from e2-b to i2 is

$$\begin{split} GF_{\rm I}(i) *_0 & \left(G_{\rm V}(F_{\rm I}hF_{\rm I}g) *_0 GF_{\rm I}(f) *_1 G_{\rm V}(F_{\rm I}hF_{\rm I}g,F_{\rm I}f) \right) \\ *_1 \\ & G_{\rm V} \Big(F_{\rm I}(i), F_{\rm V}(hg,f) *_1 F_{\rm V}(h,g) *_0 F_{\rm I}(f) \Big) \\ *_1 \\ & G_{\rm I}(F_{\rm V}(i,ghf)) \, ; \end{split}$$

the 3-cell from i1 to i2 is

$$\begin{split} GF_{\mathbf{i}}(i) *_{0} G_{\mathbf{V}} & \big(F_{\mathbf{i}}(h), F_{\mathbf{i}}(g), F_{\mathbf{i}}(f) \big) \\ *_{1} \\ G_{\mathbf{V}} & \big(F_{\mathbf{i}}(i), F_{\mathbf{i}}(h) *_{0} F_{\mathbf{i}}(g) *_{0} F_{\mathbf{i}}(f) \big) \\ *_{1} \\ G_{\mathbf{i}} & \big(F_{\mathbf{i}}(i) *_{0} F_{\mathbf{V}}(h, g, f) \big) \\ *_{1} \\ G_{\mathbf{i}} & \big(F_{\mathbf{V}}(i, hgf) \big) \, . \end{split}$$

The coherence for the tree \bigvee applied to the pasting diagram



where $\Gamma = F_{\mathbb{V}}(h, g, f)$ and $l = F_{\mathbb{V}}(i)$ gives the equality

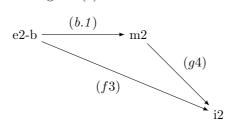
$$\begin{split} G_{\bigvee} \big(F_{\!\!\!1}(i), F_{\!\!\!\!\vee}(h,g) *_{0} F_{\!\!\!\!1}(f) *_{1} F_{\!\!\!\vee}(hg,f) \big) \\ *_{2} \\ GF_{\!\!\!\!1}(i) *_{0} G_{\!\!\!\!\uparrow} \big(F_{\!\!\!\!\vee}(h,g,f) \big) \\ = \\ G_{\!\!\!\uparrow} \big(F_{\!\!\!\uparrow}(i) *_{0} F_{\!\!\!\!\vee}(h,g,f) \big) \\ *_{2} \\ G_{\bigvee} \big(F_{\!\!\!\downarrow}(i), F_{\!\!\!\downarrow}(h) *_{0} F_{\!\!\!\vee}(g,f) *_{1} F_{\!\!\!\vee}(h,gf) \big) \end{split}$$

of 3-cells of C; 1-precomposing, *i.e.*, whiskering by $*_1$, this equality on both sides by the 2-cell $G_{\dagger}(F_{\lor}(i,hgf))$ and 1-post-composing, *i.e.*, whiskering by $*_1$ again, both sides by the 3-cell

$$GF_{\mathsf{I}}(i) *_{0} G_{\mathsf{V}}(F_{\mathsf{I}}(h), F_{\mathsf{I}}(g), F_{\mathsf{I}}(f))$$
,

the two sides of the equality are precisely the two paths of diagram (3), which is therefore commutative.

1.3.10 (4). Consider the diagram (4)



where the principal 3-cell of (g4) is

$$(g4) G_{\vee}(F_{\mathsf{I}}(i), F_{\vee}(h, g) *_{0} F_{\mathsf{I}}(f)).$$

More precisely, we have already seen that the 3-cell from e2-b to i2 is set to be

$$\begin{split} GF_{\mathbf{i}}(i) *_{0} & \left(G_{\vee}(F_{\mathbf{i}}(h)F_{\mathbf{i}}(g)) *_{0} GF_{\mathbf{i}}(f) *_{1} G_{\vee}(F_{\mathbf{i}}(h)F_{\mathbf{i}}(g), F_{\mathbf{i}}(f)) \right) \\ *_{1} \\ & G_{\vee}\left(F_{\mathbf{i}}(i), F_{\vee}(hg,f) *_{1} F_{\vee}(h,g) *_{0} F_{\mathbf{i}}(f) \right) \\ *_{1} \\ & G_{\mathbf{i}}(F_{\vee}(i,ghf)) \, ; \end{split}$$

the 3-cell from m2 to i2 is

$$GF_{\mathsf{I}}(i) *_{0} (G_{\mathsf{V}}(F_{\mathsf{I}}(h), F_{\mathsf{I}}(g)) *_{0} GF_{\mathsf{I}}(f) *_{1} G_{\mathsf{V}}(F_{\mathsf{I}}(h) F_{\mathsf{I}}(g), F_{\mathsf{I}}(f)))$$

$$*_{1}$$

$$G_{\mathsf{V}}(F_{\mathsf{I}}(i), F_{\mathsf{V}}(h, g) *_{0} F_{\mathsf{I}}(f))$$

$$*_{1}$$

$$G_{\mathsf{I}}(F_{\mathsf{I}}(i) *_{0} F_{\mathsf{V}}(hg, f))$$

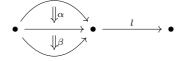
$$*_{1}$$

$$G_{\mathsf{I}}(F_{\mathsf{V}}(i, hgf));$$

the 3-cell of C from e2-b to m2 is

$$\begin{split} GF_{\mathbf{i}}(i) *_{0} &\left(G_{\vee}(F_{\mathbf{i}}(h), F_{\mathbf{i}}(g)) *_{0} GF_{\mathbf{i}}(f) *_{1} G_{\vee}(F_{\mathbf{i}}(h) F_{\mathbf{i}}(g), F_{\mathbf{i}}(f))\right) \\ *_{1} \\ GF_{\mathbf{i}}(i) *_{0} G_{\mathbf{i}}(F_{\vee}(h, g) *_{0} F_{\mathbf{i}}(f)) \\ *_{1} \\ G_{\vee}(F_{\mathbf{i}}(i), F_{\vee}(hg, f)) \\ *_{1} \\ G_{\mathbf{i}}(F_{\vee}(i, hgf)) \,. \end{split}$$

The coherence for the tree \bigvee applied to the pasting diagram



of C, where $\alpha = F_{\lor}(hg,f), \ \beta = F_{\lor}(h,g) *_0 F_{\mid}(f)$ and $l = F_{\mid}(i)$ gives the equality

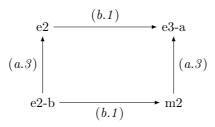
$$\begin{split} G_{\vee}\big(F_{\mathsf{I}}(i),F_{\vee}(h,g) *_{0} F_{\mathsf{I}}(f) *_{1} F_{\vee}(hg,f)\big) &= \\ G_{\vee}\big(F_{\mathsf{I}}(i),F_{\vee}(h,g) *_{0} F_{\mathsf{I}}(f)\big) *_{1} G_{\dagger}\big(F_{\mathsf{I}}(i) *_{0} F_{\vee}(hg,f)\big) \\ *_{2} \\ GF_{\mathsf{I}}(i) *_{0} G_{\dagger}\big(F_{\vee}(hg,f)\big) *_{1} G_{\vee}\big(F_{\mathsf{I}}(i),F_{\vee}(h,g) *_{0} F_{\mathsf{I}}(f)\big) \end{split}$$

of 3-cells of C; 1-precomposing, *i.e.*, whiskering by $*_1$, both members of the equality by the 2-cell $G_{\dagger}(F_{\lor}(i,hgf))$ and 1-post-composing, *i.e.*, whiskering by $*_1$, by the 2-cell

$$GF_{\mathsf{I}}(i) *_{0} (G_{\vee}(F_{\mathsf{I}}(h), F_{\mathsf{I}}(g)) *_{0} GF_{\mathsf{I}}(f) *_{1} G_{\vee}(F_{\mathsf{I}}(h) F_{\mathsf{I}}(g), F_{\mathsf{I}}(f)))$$

we get precisely the 3-cells of diagram (4), which therefore commutes.

1.3.11 (5). Consider the diagram (5)



More precisely, we already know that the 3-cell from e2 to e3-a is

$$GF_{\mathbf{i}}(i) *_{0} (GF_{\mathbf{v}}(h,g) *_{0} GF_{\mathbf{i}}(f))$$
 $*_{1}$
 $GF_{\mathbf{i}}(i) *_{0} G_{\mathbf{v}}(F_{\mathbf{i}}(hg), F_{\mathbf{i}}(f))$
 $*_{1}$
 $G_{\mathbf{v}}(F_{\mathbf{i}}(i), F_{\mathbf{v}}(hg, f))$
 $*_{1}$
 $G_{\mathbf{i}}(F_{\mathbf{v}}(i, hgf)),$

the 3-cell from e2-b to e2 is

$$GF_{\mathbf{i}}(i) *_{0} G_{\vee}(F_{\mathbf{i}}(h), F_{\mathbf{i}}(g)) *_{0} GF_{\mathbf{i}}(f) \\ *_{1} \\ GF_{\mathbf{i}}(i) *_{0} G_{\vee}(F_{\vee}(h, g), F_{\mathbf{i}}f) \\ *_{1} \\ G_{\mathbf{i}}(F_{\vee}(h, g) *_{0} F_{\mathbf{i}}(f)) \\ *_{1} \\ GF_{\vee}(i, hqf)$$

and the 3-cell from e2-b to m2 is

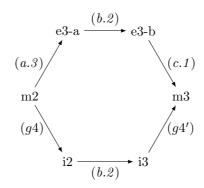
$$GF_{\mathsf{I}}(i) *_{0} \left(G_{\vee}(F_{\mathsf{I}}(h), F_{\mathsf{I}}(g)) *_{0} GF_{\mathsf{I}}(f)\right) \\ *_{1} \\ GF_{\mathsf{I}}(i) *_{0} \left(G_{\vee}(F_{\mathsf{I}}(h) F_{\mathsf{I}}(g), F_{\mathsf{I}}(f)) *_{1} G_{\mathsf{I}}(F_{\vee}(h, g) *_{0} F_{\mathsf{I}}(f)\right) \\ *_{1} \\ G_{\vee}(F_{\mathsf{I}}(i), F_{\vee}(hg, f)) \\ *_{1} \\ G_{\mathsf{I}}(F_{\vee}(i, hgf)) :$$

the 3-cell from m2 to e3-a is

$$\begin{split} GF_{\rm l}(i) *_0 G_{\lor} \big(F_{\rm l}(h), F_{\rm l}(g) \big) *_0 GF_{\rm l}(f) \\ *_1 \\ GF_{\rm l}(i) *_0 G_{\lor} (F_{\lor}(h,g), F_{\rm l}f) \\ *_1 \\ G_{\lor} \big(F_{\rm l}(i), F_{\rm l}(hg) *_0 F_{\rm l}(f) \big) \\ *_1 \\ G_{\dagger} \big(F_{\rm l}(i) *_0 F_{\lor}(hg,f) \big) *_1 G_{\dagger} \big(F_{\lor}(i,hgf) \big) \,. \end{split}$$

It is clear from this explicit description of the 3-cells involved that diagram (5) is commutative by virtue of the interchange law.

1.3.12 (6). Consider the diagram (6)



where the principal 3-cell of (g4') is

$$(g4') G_{\backslash \backslash}(F_{\mathsf{I}}(i) *_{0} F_{\backslash \backslash}(h,g), F_{\mathsf{I}}(f)).$$

More precisely, the 3-cell from i3 to m3 is

$$\begin{split} GF_{\mathbf{i}}(i) *_{0} G_{\vee}\big(F_{\mathbf{i}}(h), F_{\mathbf{i}}(g)\big) *_{0} GF_{\mathbf{i}}(f) \\ *_{1} \\ G_{\vee}\big(F_{\mathbf{i}}(i), F_{\mathbf{i}}(h) *_{0} F_{\mathbf{i}}(g)\big) *_{0} GF_{\mathbf{i}}(f) *_{1} G_{\vee}\big(F_{\mathbf{i}}(i) *_{0} F_{\vee}(h, g), F_{\mathbf{i}}(f)\big) \\ *_{1} \\ G_{\mathbf{i}}\big(F_{\vee}(i, hg) *_{0} F_{\mathbf{i}}(f)\big) *_{1} G_{\mathbf{i}}\big(F_{\vee}(ihg, f)\big) \,; \end{split}$$

the 3-cell from i2 to i3 is

$$GF_{!}(i) *_{0} G_{\vee}(F_{!}(h), F_{!}(g)) *_{0} GF_{!}(f)$$

$$*_{1}$$

$$G_{\vee}(F_{!}(i), F_{!}(h) *_{0} F_{!}(g), F_{!}(f)) *_{1} G_{!}(F_{!}(h) *_{0} F_{!}(h, g) *_{0} F_{!}(f))$$

$$*_{1}$$

$$G_{!}(F_{\vee}(i, hg, f));$$

we already know that the 3-cell from m2 to i2 is

$$GF_{!}(i) *_{0} G_{\vee}(F_{!}(h), F_{!}(g)) *_{0} GF_{!}(f)$$

$$*_{1}$$

$$GF_{!}(i) *_{0} G_{\vee}(F_{!}(h) *_{0} F_{!}(g), F_{!}(f)) *_{1} G_{\vee}(F_{!}(i), F_{\vee}(h, g) *_{0} F_{!}(f))$$

$$*_{1}$$

$$G_{!}(F_{!}(i) *_{0} F_{\vee}(hq, f)) *_{1} G_{!}(F_{\vee}(i, hqf));$$

the 3-cell from e3-b to m3 is

$$\begin{split} GF_{\mathbf{i}}(i) *_{0} G_{\vee} \big(F_{\mathbf{i}}(h), F_{\mathbf{i}}(g) \big) *_{0} GF_{\mathbf{i}}(f) \\ *_{1} \\ G_{\vee} \big(F_{\mathbf{i}}(i), F_{\vee}(h,g) \big) *_{0} GF_{\mathbf{i}}(f) *_{1} G_{\vee} \big(F_{\mathbf{i}}(i) *_{0} F_{\mathbf{i}}(h,g), F_{\mathbf{i}}(f) \big) \\ *_{1} \\ G_{\mathbf{i}} \big(F_{\vee}(i,hg) *_{0} F_{\mathbf{i}}(f) \big) *_{1} G_{\mathbf{i}} \big(F_{\vee}(ihg,f) \big) \, ; \end{split}$$

we already know that the 3-cell from e3-a to e3-b is

$$\begin{split} GF_{\mathbf{i}}(i) *_{0} G_{\vee}\big(F_{\mathbf{i}}(h), F_{\mathbf{i}}(g)\big) *_{0} GF_{\mathbf{i}}(f) \\ *_{1} \\ GF_{\mathbf{i}}(i) *_{0} G_{\mathbf{i}}\big(F_{\mathbf{i}}(h,g)\big) *_{0} F_{\mathbf{i}}(f) *_{1} G_{\vee}\big(F_{\mathbf{i}}(i), F_{\mathbf{i}}(hg), F_{\mathbf{i}}(f)\big) \\ *_{1} \\ G_{\mathbf{i}}\big(F_{\vee}(i,hg,f)\big) \,; \end{split}$$

we also know that the 3-cell from m2 to e3-a is

$$GF_{|}(i) *_{0} G_{\vee}(F_{|}(h), F_{|}(g)) *_{0} GF_{|}(f)$$

$$*_{1}$$

$$GF_{|}(i) *_{0} G_{\vee}(F_{\vee}(h, g), F_{|}(f)) *_{1} G_{\vee}(F_{|}(i), F_{|}(hg) *_{0} F_{|}(f))$$

$$*_{1}$$

$$G_{|}(F_{|}(i) *_{0} F_{\vee}(hg, f)) *_{1} G_{|}(F_{\vee}(i, hgf)).$$

Notice that there is a complete duality between the 3-cell from m2 to e3-a and the 3-cell from e3-b to m3 and also between the 3-cell from m2 to i2 and the 3-cell from

i3 to m3. The coherence for the tree \checkmark applied to the pasting diagram

$$\bullet \xrightarrow{k} \bullet \overbrace{\qquad \qquad } \bullet \xrightarrow{l} \bullet$$

of C with $k = F_l(f)$, $\alpha = F_{\vee}(h, g)$ and $l = F_l(i)$ gives the equality between the 3-cell

$$G_{\vee}\big(F_{\mathsf{I}}(i),F_{\mathsf{I}}(h)*_{0}F_{\mathsf{I}}(g)\big)*_{0}GF_{\mathsf{I}}(f)*_{1}G_{\vee}\big(F_{\mathsf{I}}(i)*_{0}F_{\vee}(h,g),F_{\mathsf{I}}(f)\big)\\ *_{1}\\G_{\vee}\big(F_{\mathsf{I}}(i),F_{\mathsf{I}}(h)*_{0}F_{\mathsf{I}}(g),F_{\mathsf{I}}(f)\big)*_{1}G_{\mathsf{I}}\big(F_{\mathsf{I}}(h)*_{0}F_{\mathsf{I}}(h,g)*_{0}F_{\mathsf{I}}(f)\big)\\ *_{1}\\GF_{\mathsf{I}}(i)*_{0}G_{\vee}\big(F_{\mathsf{I}}(h)*_{0}F_{\mathsf{I}}(g),F_{\mathsf{I}}(f)\big)*_{1}G_{\vee}\big(F_{\mathsf{I}}(i),F_{\vee}(h,g)*_{0}F_{\mathsf{I}}(f)\big)$$

of C and the 3-cell

$$G_{\vee}(F_{|}(i), F_{\vee}(h, g)) *_{0} GF_{|}(f) *_{1} G_{\vee}(F_{|}(i) *_{0} F_{|}(h, g), F_{|}(f))$$

$$*_{1}$$

$$GF_{|}(i) *_{0} G_{|}(F_{|}(h, g)) *_{0} F_{|}(f) *_{1} G_{\vee}(F_{|}(i), F_{|}(hg), F_{|}(f))$$

$$*_{1}$$

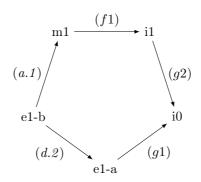
$$GF_{|}(i) *_{0} G_{\vee}(F_{\vee}(h, g), F_{|}(f)) *_{1} G_{\vee}(F_{|}(i), F_{|}(hg) *_{0} F_{|}(f))$$

of C. We get the diagram (6) by 1-precomposing, *i.e.*, whiskering by $*_1$, both terms of this equality with the 2-cell

$$GF_{l}(i) *_{0} G_{\vee}(F_{l}(h), F_{l}(g)) *_{0} GF_{l}(f)$$

of C and by 1-post-composing with the 3-cell $G_{\dagger}(F_{\forall}(i,hg,f))$, i.e., a "vertical composition" of 3-cells. Hence, the diagram is commutative.

1.3.13 (7). Consider the diagram (7)



where

$$(g1) G_{\vee}(F_{\mathsf{I}}(i) *_{0} F_{\mathsf{I}}(h), F_{\vee}(g, f))$$

is the principal 3-cell of (g1) and

(g2)
$$G_{V}(F_{I}(i), F_{I}(h), F_{I}(g) *_{0} F_{I}(f))$$

is the principal 3-cell of (g2). More precisely, the 3-cell of C from e1-a to i0 is

$$GF_{\mathsf{I}}(i) *_{0} GF_{\mathsf{I}}(h) *_{0} G_{\vee} \big(F_{\mathsf{I}}(g), F_{\mathsf{I}}(f) \big) \\ *_{1} \\ G_{\vee} \big(F_{\mathsf{I}}(i), F_{\mathsf{I}}(h) \big) *_{0} G_{\mathsf{I}} \big(F_{\mathsf{I}}(g) *_{0} F_{\mathsf{I}}(f) \big) *_{1} G_{\vee} \big(F_{\mathsf{I}}(i) *_{0} F_{\mathsf{I}}(h), F_{\vee}(g, f) \big) \\ *_{1} \\ G_{\mathsf{I}} \big(F_{\vee}(i, h) *_{0} F_{\mathsf{I}}(gf) \big) *_{1} G_{\mathsf{I}} \big(F_{\vee}(ih, gf) \big) ;$$

the 3-cell from e1-b to e1-a, as we know, is

$$\begin{split} GF_{\mathbf{i}}(i) *_{0} GF_{\mathbf{i}}(h) *_{0} G_{\vee} \big(F_{\mathbf{i}}(g), F_{\mathbf{i}}(f) \big) \\ *_{1} \\ GF_{\mathbf{i}}(i) *_{0} GF_{\mathbf{i}}(h) *_{0} G_{\mathbf{j}} \big(F_{\vee}(g,f) \big) *_{1} G_{\vee} \big(F_{\mathbf{i}}(i), F_{\mathbf{i}}(h), F_{\mathbf{i}}(gf) \big) \\ *_{1} \\ G_{\mathbf{j}} \big(F_{\vee}(i,h,gf) \big) \, ; \end{split}$$

the 3-cell from i1 to i0 is

$$\begin{split} GF_{\mathbf{i}}(i) *_{0} GF_{\mathbf{i}}(h) *_{0} G_{\vee}\big(F_{\mathbf{i}}(g), F_{\mathbf{i}}(f)\big) \\ *_{1} \\ G_{\vee}\big(F_{\mathbf{i}}(i), F_{\mathbf{i}}(h), F_{\mathbf{i}}(g) *_{0} F_{\mathbf{i}}(f)\big) *_{1} G_{\mathbf{i}}\big(F_{\mathbf{i}}(i) *_{0} F_{\mathbf{i}}(h) *_{0} F_{\vee}(g, f)\big) \\ *_{1} \\ G_{\mathbf{i}}\big(F_{\vee}(i, h, gf)\big) \,; \end{split}$$

the 3-cell from m1 to i1, as we know, is

$$\begin{split} GF_{\mathbf{i}}(i) *_{0} GF_{\mathbf{i}}(h) *_{0} G_{\vee} \big(F_{\mathbf{i}}(g), F_{\mathbf{i}}(f) \big) \\ *_{1} \\ GF_{\mathbf{i}}(i) *_{0} G_{\vee} \big(F_{\mathbf{i}}(h), F_{\mathbf{i}}(g) *_{0} F_{\mathbf{i}}(f) \big) *_{1} G_{\vee} \big(F_{\mathbf{i}}(i), F_{\mathbf{i}}(h) *_{0} F_{\vee}(g, f) \big) \\ *_{1} \\ G_{\mathbf{i}} \big(F_{\vee}(i, h) *_{0} F_{\mathbf{i}}(gf) \big) *_{1} G_{\mathbf{i}} \big(F_{\vee}(ih, gf) \big) \, ; \end{split}$$

finally we know that the 3-cell from e1-b to m1 is

$$GF_{!}(i) *_{0} GF_{!}(h) *_{0} G_{\vee}(F_{!}(g), F_{!}(f))$$

$$*_{1}$$

$$GF_{!}(i) *_{0} G_{\vee}(F_{!}(h), F_{\vee}(g, f)) *_{1} G_{\vee}(F_{!}(i), F_{!}(h) *_{0} F_{!}(gf))$$

$$*_{1}$$

$$G_{!}(F_{\vee}(i, h) *_{0} F_{!}(gf)) *_{1} G_{!}(F_{\vee}(ih, gf)).$$

The coherence for the tree \bigvee applied to the pasting diagram

of C, with $\alpha = F_{\vee}(g, f)$, $k = F_{|}(h)$ and $l = F_{|}(i)$ give the equality

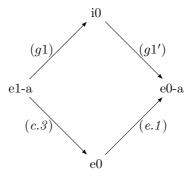
$$G_{\vee}\big(F_{\mathsf{I}}(i),F_{\mathsf{I}}(h)\big) *_{0} G_{\mathsf{I}}\big(F_{\mathsf{I}}(g) *_{0} F_{\mathsf{I}}(f)\big) *_{1} G_{\vee}\big(F_{\mathsf{I}}(i) *_{0} F_{\mathsf{I}}(h),F_{\vee}(g,f)\big) \\ *_{1} \\ GF_{\mathsf{I}}(i) *_{0} GF_{\mathsf{I}}(h) *_{0} G_{\mathsf{I}}\big(F_{\vee}(g,f)\big) *_{1} G_{\vee}\big(F_{\mathsf{I}}(i),F_{\mathsf{I}}(h),F_{\mathsf{I}}(gf)\big) \\ = \\ G_{\vee}\big(F_{\mathsf{I}}(i),F_{\mathsf{I}}(h),F_{\mathsf{I}}(g) *_{0} F_{\mathsf{I}}(f)\big) *_{1} G_{\mathsf{I}}\big(F_{\mathsf{I}}(i) *_{0} F_{\mathsf{I}}(h) *_{0} F_{\vee}(g,f)\big) \\ *_{1} \\ GF_{\mathsf{I}}(i) *_{0} G_{\vee}\big(F_{\mathsf{I}}(h),F_{\mathsf{I}}(g) *_{0} F_{\mathsf{I}}(f)\big) *_{1} G_{\vee}\big(F_{\mathsf{I}}(i),F_{\mathsf{I}}(h) *_{0} F_{\vee}(g,f)\big) \\ *_{1} \\ GF_{\mathsf{I}}(i) *_{0} G_{\vee}\big(F_{\mathsf{I}}(h),F_{\vee}(g,f)\big) *_{1} G_{\vee}\big(F_{\mathsf{I}}(i),F_{\mathsf{I}}(h) *_{0} F_{\mathsf{I}}(gf)\big)$$

of 3-cells of C. If we 1-post-compose, *i.e.*, we whisker with $*_1$, with the 2-cell

$$GF_{l}(i) *_{0} GF_{l}(h) *_{0} G_{\vee}(F_{l}(g), F_{l}(f))$$

of C and we 1-precompose with the 3-cell $G_{\dagger}(F_{\forall}(i,h,gf))$, *i.e.*, we perform a "vertical composition" of 3-cells, both members of the equality above, then we get precisely diagram (7), which therefore commutes.

1.3.14 (8). Consider the diagram (8)



where the principal 3-cell of (g1') is

$$(1.3.14.1) G_{\backslash I}(F_{\vee}(i,h),F_{|}(g)*_{0}F_{|}(f)).$$

More precisely, the 3-cell of C from e0 to e0-a, as we know, is

$$\begin{split} GF_{\mathbf{i}}(i) *_{0} GF_{\mathbf{i}}(h) *_{0} G_{\vee}\big(F_{\mathbf{i}}(g), F_{\mathbf{i}}(f)\big) \\ *_{1} \\ G_{\mathbf{i}}\big(F_{\vee}(i,h)\big) *_{0} G_{\mathbf{i}}\big(F_{\mathbf{i}}(g) *_{0} F_{\mathbf{i}}(f)\big) *_{1} G_{\vee}\big(F_{\vee}(i,h), F_{\mathbf{i}}(g) *_{0} F_{\mathbf{i}}(f)\big) \\ *_{1} \\ G_{\mathbf{i}}\big(F_{\vee}(ih,gf)\big) \,; \end{split}$$

the 3-cell from e1-a to e0 is, as we know,

$$GF_{!}(i) *_{0} GF_{!}(h) *_{0} G_{\vee}(F_{!}(g), F_{!}(f)) \\ *_{1}$$

$$G_{!}(F_{!}(i) *_{0} F_{!}(h)) *_{0} G_{!}(F_{\vee}(g, f)) *_{1} G_{\vee}(F_{\vee}(i, h) *_{0} F_{!}(gf)) \\ *_{1}$$

$$G_{!}(F_{\vee}(ih, gf));$$

the 3-cell from e0 to e0-a is

$$GF_{!}(i) *_{0} GF_{!}(h) *_{0} G_{\vee}(F_{!}(g), F_{!}(f))$$

$$*_{1}$$

$$G_{\vee}(F_{\vee}(i, h), F_{!}(g) *_{0} F_{!}(f)) *_{1} G_{!}(F_{!}(ih) *_{0} F_{\vee}(g, f))$$

$$*_{1}$$

$$G_{!}(F_{\vee}(ih, gf));$$

finally the 3-cell from e1-a to e0, as we know, is

$$\begin{split} GF_{\rm l}(i) *_0 GF_{\rm l}(h) *_0 G_{\vee} \big(F_{\rm l}(g), F_{\rm l}(f)\big) \\ *_1 \\ G_{\vee} \big(F_{\rm l}(i) *_0 F_{\rm l}(h), F_{\vee}(g,f)\big) *_1 G_{\rm l} \big(F_{\vee}(i,h) *_0 F_{\rm l}(gf)\big) \\ *_1 \\ G_{\rm l} \big(F_{\vee}(ih,gf)\big) \, . \end{split}$$

The coherence for the tree \bigvee applied to the pasting diagram

$$\bullet \qquad \downarrow \alpha \qquad \bullet \qquad \downarrow \beta \qquad \bullet$$

of C, where $\alpha = F_{\vee}(g, f)$ and $\beta = F_{\vee}(h, i)$, gives us the equality

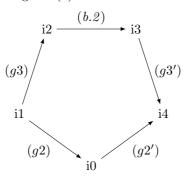
$$G_{\dagger}(F_{\lor}(i,h)) *_{0} G_{\dagger}(F_{\exists}(g) *_{0} F_{\exists}(f)) *_{1} G_{\lor}(F_{\lor}(i,h), F_{\exists}(g) *_{0} F_{\exists}(f)) *_{1} G_{\lor}(F_{\exists}(i) *_{0} F_{\exists}(h)) *_{0} G_{\dagger}(F_{\lor}(g,f)) *_{1} G_{\lor}(F_{\lor}(i,h) *_{0} F_{\exists}(gf)) = G_{\lor}(F_{\lor}(i,h), F_{\exists}(g) *_{0} F_{\exists}(f)) *_{1} G_{\dagger}(F_{\exists}(ih) *_{0} F_{\lor}(g,f)) *_{1} G_{\lor}(F_{\exists}(i) *_{0} F_{\exists}(h), F_{\lor}(g,f)) *_{1} G_{\dagger}(F_{\lor}(i,h) *_{0} F_{\exists}(gf))$$

of 3-cells of C. By 1-post-composing with the 2-cell

$$GF_{l}(i) *_{0} GF_{l}(h) *_{0} G_{\vee}(F_{l}(g), F_{l}(f))$$

of C and 1-precomposing with the 2-cell $G_{\dagger}(F_{\lor}(ih, gf))$ we get the commutativity of the diagram (8).

1.3.15 (9). Consider the diagram (9)



where

$$G_{\mathbb{V}}\big(F_{\mathbb{I}}(i) *_{0} F_{\mathbb{I}}(h), F_{\mathbb{I}}(g), F_{\mathbb{I}}(f)\big) \\ *_{1} \\ G_{\mathbb{I}}\big(F_{\mathbb{V}}(ih, g, f)\big)$$

is the principal 3-cell of (g2') and

$$G_{\mathbb{V}}\big(F_{\mathbb{I}}(i), F_{\mathbb{I}}(h) *_{0} F_{\mathbb{I}}(g), F_{\mathbb{I}}(f)\big) \\ *_{1} \\ G_{\mathbb{I}}\big(F_{\mathbb{V}}(i, hg, f)\big)$$

is the principal 3-cell of (g3'). More precisely, the 3-cell of C from i0 to i4 is

$$G_{\vee}(F_{|}(i), F_{|}(h)) *_{0} GF_{|}(g) *_{0} GF_{|}(f) *_{1} G_{\vee}(F_{|}(i) *_{0} F_{|}(h), F_{|}(g), F_{|}(f))$$

$$*_{1}$$

$$G_{i}(F_{\vee}(i, h) *_{0} F_{|}(g) *_{0} F_{|}(f)) *_{1} G_{i}(F_{\vee}(ih, g, f));$$

the 3-cell from i1 to i0, as we know, is

$$GF_{\rm l}(i) *_0 GF_{\rm l}(h) *_0 G_{\rm V} \left(F_{\rm l}(g), F_{\rm l}(f)\right) *_1 G_{\rm V} \left(F_{\rm l}(i), F_{\rm l}(h), F_{\rm l}(g) *_0 F_{\rm l}(f)\right) \\ *_1$$

$$G_{\dagger}(F_{!}(i) *_{0} F_{!}(h) *_{0} F_{\lor}(g, f)) *_{1} G_{\dagger}(F_{\lor}(i, h, gf));$$

the 3-cell from i3 to i4 is

$$G_{\mathbb{V}}(F_{\mathbb{I}}(i), F_{\mathbb{I}}(h), F_{\mathbb{I}}(g)) *_{0} GF_{\mathbb{I}}(f) *_{1} G_{\mathbb{V}}(F_{\mathbb{I}}(i) *_{0} F_{\mathbb{I}}(H) *_{0} F_{\mathbb{I}}(g), F_{\mathbb{I}}(f)) *_{1} G_{\mathbb{I}}(F_{\mathbb{V}}(ihg, f));$$

the 3-cell from i2 to i3, as we know, is

$$GF_{\mathsf{I}}(i) *_{0} G_{\mathsf{V}}(F_{\mathsf{I}}(h), F_{\mathsf{I}}(g)) *_{0} GF_{\mathsf{I}}(f) *_{1} G_{\mathsf{V}}(F_{\mathsf{I}}(i), F_{\mathsf{I}}(h) *_{0} F_{\mathsf{I}}(g), F_{\mathsf{I}}(f))$$
 $*_{1}$

$$G_{\dagger}(F_{\dagger}(h) *_{0} F_{\dagger}(h,g) *_{0} F_{\dagger}(f)) *_{1} G_{\dagger}(F_{\lor}(i,hg,f));$$

finally we also already know that the 3-cell from i1 to i2 is

$$\begin{split} GF_{|}(i) *_{0} G_{\vee} & \left(F_{|}(h), F_{|}(g), F_{|}(f) \right) *_{1} G_{\vee} \left(F_{|}(i), F_{|}(h) *_{0} F_{|}(g) *_{0} F_{|}(f) \right) \\ & *_{1} \\ G_{!} & \left(F_{|}(i) *_{0} F_{\vee}(h, g, f) \right) *_{1} G_{!} \left(F_{\vee}(i, hgf) \right). \end{split}$$

Notice that diagram (9) is actually formed by "vertical compositions" of these 3-cells; that is, the first lines of these 3-cells are 2-composable and the same holds for

the 3-cells of the second line. Observe that in order to 2-compose the 3-cell from i1 to i0 with the 3-cell from i0 to i4 we use the relations

$$\begin{split} G_{\dagger} \big(F_{\!\!\!|}(i) *_0 F_{\!\!\!|}(h) *_0 F_{\!\!\!|}(g,f) \big) *_1 G_{\dagger} \big(F_{\!\!\!|}(i,h) *_0 F_{\!\!\!|}(g *_0 f) \big) \\ &= \\ G_{\dagger} \big(F_{\!\!\!|}(i) *_0 F_{\!\!\!|}(h) *_0 F_{\!\!\!|}(g,f) *_1 F_{\!\!\!|}(i,h) *_0 F_{\!\!\!|}(g *_0 f) \big) \\ &= \\ G_{\dagger} \big(F_{\!\!\!|}(i,h) *_0 F_{\!\!\!|}(g) *_0 F_{\!\!\!|}(f) *_1 F_{\!\!\!|}(i *_0 h) *_0 F_{\!\!\!|}(g,f) \big) \\ &= \\ G_{\dagger} \big(F_{\!\!\!|}(i,h) *_0 F_{\!\!\!|}(g) *_0 F_{\!\!\!|}(f) \big) *_1 G_{\dagger} \big(F_{\!\!\!|}(i *_0 h) *_0 F_{\!\!\!|}(g,f) \big) \,, \end{split}$$

where the first and the last equality are instances of the coherence for the tree \bigvee , *i.e.*, the 0-composition of 2-cells, and the equality in the middle is simply given by the interchange law. Now, the coherence for the tree \bigvee applied to the pasting diagram

$$\bullet \stackrel{f}{\longrightarrow} \bullet \stackrel{g}{\longrightarrow} \bullet \stackrel{h}{\longrightarrow} \bullet \stackrel{i}{\longrightarrow} \bullet$$

of 0-composable 1-cells of A gives us the equality

$$F_{\vee}(i,h) *_{0} F_{|}(g) *_{0} F_{|}(f) *_{1} F_{\vee}(ih,g,h)$$

$$*_{2}$$

$$F_{|}(i) *_{0} F_{|}(h) *_{0} F_{\vee}(g,f) *_{1} F_{\vee}(i,h,gf)$$

$$=$$

$$F_{\vee}(i,h,g) *_{0} F_{|}(f) *_{1} F_{\vee}(i *_{0} h *_{0} g,f)$$

$$*_{2}$$

$$F_{|}(i) *_{0} F_{\vee}(h,g) *_{0} F_{|}(f) *_{1} F_{\vee}(i,h *_{0} g,f)$$

$$*_{2}$$

$$F_{|}(i) *_{0} F_{\vee}(h,g,f) *_{1} F_{\vee}(i,h *_{0} g *_{0} f)$$

of 3-cells of B. Applying G_t to both terms of this equality and using the coherences

for the trees
$$\bigvee$$
, \bigvee and \bigvee we get the following equality $G_{!}(F_{\lor}(i,h) *_{0} F_{!}(g) *_{0} F_{!}(f)) *_{1} G_{!}(F_{\lor}(ih,g,h))$
 $*_{2}$
 $G_{!}(F_{!}(i) *_{0} F_{!}(h) *_{0} F_{\lor}(g,f)) *_{1} G_{!}(F_{\lor}(i,h,gf))$
 $=$
 $G_{!}(F_{\lor}(i,h,g) *_{0} F_{!}(f)) *_{1} G_{!}(F_{\lor}(i *_{0} h *_{0} g,f))$
 $*_{2}$
 $G_{!}(F_{!}(i) *_{0} F_{\lor}(h,g) *_{0} F_{!}(f)) *_{1} G_{!}(F_{\lor}(i,h *_{0} g,f))$
 $*_{2}$
 $G_{!}(F_{!}(i) *_{0} F_{\lor}(h,g,f)) *_{1} G_{!}(F_{\lor}(i,h *_{0} g *_{0} f));$

applying instead the coherence for the tree $\,$ to the pasting scheme given by the four 0-composable 1-cells of B

$$\bullet \xrightarrow{F_{\mathbf{i}}(f)} \bullet \xrightarrow{F_{\mathbf{i}}(g)} \bullet \xrightarrow{F_{\mathbf{i}}(h)} \bullet \xrightarrow{F_{\mathbf{i}}(i)} \bullet$$

we get the equality

$$\begin{split} G_{\vee}\big(F_{\mathsf{I}}(i),F_{\mathsf{I}}(h)\big) *_{0} GF_{\mathsf{I}}(g) *_{0} GF_{\mathsf{I}}(f) *_{1} G_{\vee}\big(F_{\mathsf{I}}(i) *_{0} F_{\mathsf{I}}(h),F_{\mathsf{I}}(g),F_{\mathsf{I}}(f)\big) \\ *_{1} \\ GF_{\mathsf{I}}(i) *_{0} GF_{\mathsf{I}}(h) *_{0} G_{\vee}\big(F_{\mathsf{I}}(g),F_{\mathsf{I}}(f)\big) *_{1} G_{\vee}\big(F_{\mathsf{I}}(i),F_{\mathsf{I}}(h),F_{\mathsf{I}}(g) *_{0} F_{\mathsf{I}}(f)\big) \\ = \\ G_{\vee}\big(F_{\mathsf{I}}(i),F_{\mathsf{I}}(h),F_{\mathsf{I}}(g)\big) *_{0} GF_{\mathsf{I}}(f) *_{1} G_{\vee}\big(F_{\mathsf{I}}(i) *_{0} F_{\mathsf{I}}(H) *_{0} F_{\mathsf{I}}(g),F_{\mathsf{I}}(f)\big) \\ *_{2} \\ GF_{\mathsf{I}}(i) *_{0} G_{\vee}\big(F_{\mathsf{I}}(h),F_{\mathsf{I}}(g)\big) *_{0} GF_{\mathsf{I}}(f) *_{1} G_{\vee}\big(F_{\mathsf{I}}(i),F_{\mathsf{I}}(h) *_{0} F_{\mathsf{I}}(g),F_{\mathsf{I}}(f)\big) \\ *_{1} \\ GF_{\mathsf{I}}(i) *_{0} G_{\vee}\big(F_{\mathsf{I}}(h),F_{\mathsf{I}}(g),F_{\mathsf{I}}(f)\big) *_{1} G_{\vee}\big(F_{\mathsf{I}}(i),F_{\mathsf{I}}(h) *_{0} F_{\mathsf{I}}(g) *_{0} F_{\mathsf{I}}(f)\big) \,. \end{split}$$

Notice that the 1-composition line by line of the 3-cells of these two equalities give precisely the 3-cells defining diagram (9), which by the interchange law is therefore commutative.

The previous paragraph ends the proof of the coherence presented in paragraph 1.3.3, hence achieving the following proposition.

Proposition 1.3.16. Let $F: A \to B$ and $G: B \to C$ be two normalised oplax 3-functors. If A is a 1-category, then the data of GF defined in paragraph 1.3.1 define a normalised oplax 3-functor.

Remark 1.3.17. It is suggestive, in light of the coherences described above, to decorate the polygons of figure (1) with Stasheff trees. We give such a representation in figure (2) (the unlabelled polygons being commutative by exchange law), but we do not pursue this informal approach any further.

Proposition 1.3.18. Let $u: A \to A'$, $F: A \to B$ and $G: B \to C$ be normalised oplax 3-functors, where A and a' are 1-categories. Then G(Fu) = (GF)u.

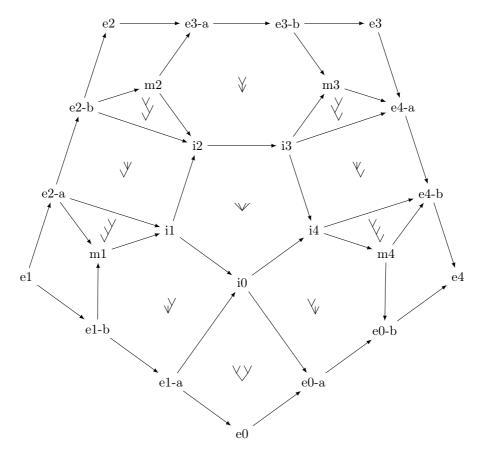
Proof. Notice that a normalised oplax 3-functor $u: A' \to A$ between 1-categories is simply a 1-functor and one immediately checks that the equality G(Fu) = (GF)u of normalised oplax 3-functor is verified.

Lemma 1.3.19. Let B be a 3-category. The n-simplices of $N_{\infty}(B)$ are in bijection with the oplax normalised 3-functors $\Delta^n \to B$.

Proof. For any 3-category B the simplicial set $N_{\infty}(B)$ is 4-coskeletal (see [24]) and so we have to check that the set of normalised oplax 3-functors $x \colon \Delta^n \to B$ are in bijection with the set of morphisms of simplicial sets $x \colon \operatorname{Sk}_4(\Delta^n) \to N_{\infty}(B)$; it is enough to define the latter ones on the i-simplices of Δ^n , with $i = 0, 1, \ldots, 4$. But such a definition corresponds precisely to the data \bullet , I , \vee and \forall joint with the coherence \swarrow of a normalised oplax 3-functor from the 1-category Δ^i to the 3-category B.

Theorem 1.3.20. Let $G: B \to C$ be a normalised oplax 3-functor. Then there is a morphism of simplicial sets $N_l(G): N_{\infty}(B) \to N_{\infty}(C)$, where, for any $n \ge 0$, an n-simplex of $N_{\infty}(B)$ corresponding to a normalised oplax 3-functor $x: \Delta^n \to B$ is sent to the n-simplex of C corresponding to the normalised oplax 3-functor $Gx: \Delta^n \to C$.

Proof. To any normalised oplax 3-functor $G \colon B \to C$, we have seen along this section that we can define a composition $Gx \colon \Delta^n \to C$ which is still a normalised oplax 3-functor and so it canonically corresponds to an n-simplex $N_lG_n(x)$ of $N_\infty(C)$.



The functoriality of this correspondence, given by the preceding proposition, implies the naturality of the functions N_lG_n , which assemble to a morphism of simplicial sets $N_l(G) \colon N_{\infty}(B) \to N_{\infty}(C)$.

Example 1.3.21. Let C be a small 3-category and consider the normalised oplax 3-functor $i_{\Delta}(N_3(C)) \to C$ defined in Example 1.2.6. One checks that the associated morphism of simplicial sets $N_l(\sup) : Ni_{\Delta}(N_3(C)) \to N_3(C)$ coincides with the morphism of simplicial sets called $\tau_{N_3(C)}$ in paragraph 7.3.14 of [10]. Hence, Proposition 7.3.15 of loc. cit. implies that $N_l(\sup)$ is a simplicial weak equivalence.

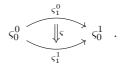
2. The simplicial definition

It is expected that a good notion of normalised oplax 3-functor would satisfy the following property: for any 3-categories A and B, the set of normalised oplax 3-functors from A to B is in bijection with the set of simplicial morphisms from $N_3(A)$ to $N_3(B)$. Nevertheless, a careful investigation of this latter notion shows that they might not be optimal as they fail to preserve the underlying 3-graph. Indeed, we will analyse the case where A is the "2-disk", i.e., the 2-category with two parallel 1-cells and a single 2-cell between them, and B is the "invertible 3-disk", i.e., the 3-category with two parallel 1-cells, two parallel 2-cells between them and a single invertible 3-cell between these 2-cells, and we show that there are more simplicial morphisms than expected between the respective Street nerves. On the one hand, the 2-category A has no compositions and so the normalised oplax 3-functors from A to B should coincide with the strict 3-functors. On the

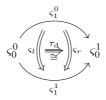
other hand, there are simplicial morphisms from $N_3(A)$ to $N_3(B)$ which do not come from the nerve of strict 3-functors. This is a consequence of the fact that, for instance, there are two ways to capture the 2-cell of A with a 2-simplex of $N_3(A)$ and these two different ways are related by 3-simplices which are sent by any simplicial morphism $N_3(A) \to N_3(B)$ to 3-simplices of $N_3(B)$ for which the main 3-cell is invertible. Said otherwise, the different ways to encode cells, or simple compositions of cells, with simplices are linked together by higher simplices with the property of having the cell of greatest dimension invertible; these higher simplices act as invertible constraints for morphisms between Street nerves of 3-categories and it is therefore natural to imagine that a normalised oplax 3-functor would correspond to a simplicial morphism for which all these higher simplices acting as constraints have trivial greatest cell, instead of only invertible. In order to identify such constraints, we shall examine in further detail the nerve $N_l(F)$ of any normalised oplax 3-functor F.

2.1. Case study: D_2 .

2.1.1. Consider the 2-category D_2



We know from paragraphs 2.2.2 that the simplicial set $N_{\infty}(D_2)$ has at least two non-degenerate 2-simplices ς_l and ς_r , at least two non-degenerate 3-simplices that we shall call τ_u and τ_d and also at least two non-degenerate 4-simplices that we shall name x_{τ} and y_{τ} . In fact, Ozornova and Rovelli have shown in [21] that these are the only non-degenerate *i*-simplices, for i=2,3,4. Therefore we get an explicit description of the 3-category $\tau_{\leqslant 3}^i c_{\infty} N_{\infty}(D_2)$ given by



where the inverse of the 3-cell τ_d is given by the 3-cell $\tau_u : \varsigma_r \to \varsigma_l$. We shall call D_3^{\sharp} this 3-category. This is motivated by the fact that the 2-skeleton of this 3-category is equal to that of D_3 , but the top dimensional cell is invertible.

2.1.2. There are no compositions of cells in the 3-category D_2 and therefore a good notion of oplax 3-functor F with source D_2 and target a 3-category B should coincide with a strict 3-functor, since there is no composition to "laxify" in D_2 . This is not the case if we set the oplax 3-functors from A to B, where A and B are two small 3-categories, to be the set

$$\operatorname{Hom}_{\widehat{\Delta}}(N_{\infty}(A),N_{\infty}(B)) \cong \operatorname{Hom}_{3\text{-}\mathcal{C}at}(c_{3}N_{\infty}(A),B)\,.$$

Indeed, let $A = D_2$ and $B = D_3^{\sharp}$ and let us restrict our attention to the 3-functors mapping $\varsigma_1^{\varepsilon}$ to $\varsigma_1^{\varepsilon}$, for $\varepsilon = 0, 1$, *i.e.*, mapping the top cell ς of D_2 to a non-trivial 2-cell of D_3^{\sharp} . We have precisely two such 3-functors: one sends ς to ς_l and the other to ς_r . Nevertheless, is we consider the 3-functors in

$$\operatorname{Hom}_{3\text{-}\mathcal{C}at}(c_3N_\infty(A),B)\cong\operatorname{Hom}_{3\text{-}\mathcal{C}at}(\operatorname{D_3}^\sharp,\operatorname{D_3}^\sharp)$$

mapping $\varsigma_1^{\varepsilon}$ to $\varsigma_1^{\varepsilon}$, then we count four of them and they are determined by their behaviour with respect to the 3-cell τ_d : there are two of them sending τ_d to the

identity of ς_l and to the identity of ς_r respectively, which are the (mates of the) nerve of the 3-functors from D₂ to D₃^{\sharp} we considered above; furthermore, there are two 3-functors, corresponding to the automorphisms of the 3-category D₃^{\sharp}, mapping τ_d to itself and to τ_u respectively.

- 2.2. The nerve of a normalised oplax 3-functor. Let A and B be two small 3-categories and consider a morphism $F \colon N_{\infty}(A) \to N_{\infty}(B)$ of simplicial sets. In this subsection we shall study some of the constraints to which the morphism F is subject. As explained above, here by *constraint* we mean an invertible and not necessarily trivial cell of B, normally a 3-cell, which is the principal cell of a 3-simplex F(x), where on the other hand the 3-cell of A defined by x is a trivial cell of A. The term constraints is due to the fact that, when trying to extract a cellular form of oplax 3-functor from such a morphism, these particular 3-simplices act as additional data which do not respect the underlying 3-graphs or as invertible coherences.
- **2.2.1.** Any object a of A, that is 0-simplex of $N_{\infty}(A)$, is mapped to an object F(a) of B and any 1-cell $f: a \to a'$ of A, that is 1-simplex of $N_{\infty}(A)$, is mapped to a 1-cell $F(f): F(a) \to F(a')$ of B.

Encoding the behaviour of higher cells in a morphism between the nerve of two 3-categories requires choices and leads to a web of coherences which increasingly become hard to control. The prototypical example of such a phenomenon is given by the way a simplicial morphism between nerves of 3-categories encodes a 2-cell. There are two different way of encoding a 2-cell α of A as a 2-simplex. The main 2-cells of B of the images under F of these two 2-simplices are possibly two different 2-cells; this can be read as the fact that simplicial morphisms do not respect the underlying 3-graph in general. Nonetheless, these two different 2-cells of B can be proven to be linked one another by an invertible 3-cell of B. This is described in detail in the next paragraph.

2.2.2. Consider a 2-cell



of A. The simplicial set $N_{\infty}(A)$ encodes the cell α in two different 2-simplices, namely

$$lpha_l := egin{array}{ccccc} a & & f & a' \\ & & \downarrow lpha & g & \end{array} \quad ext{and} \quad lpha_r := egin{array}{cccc} a & & f & a' \\ & & \downarrow lpha & a' & \end{array} \quad .$$

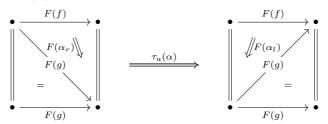
These two 2-simplices of $N_{\infty}(A)$ are linked together by the following two non-degenerate 3-simplices



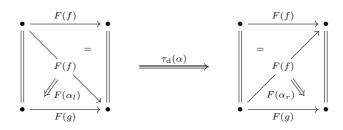
and



The images under F of these two 3-simplices of $N_{\infty}(A)$ give the following two 3-simplices of $N_{\infty}(B)$:



and



Remark 2.2.3. If B is a 2-category, then the 2-cells $F(\alpha_l)$ and $F(\alpha_r)$ coincide.

2.2.4. The 3-cells $\tau_{\rm u}(\alpha)$ and $\tau_{\rm d}(\alpha)$ of B described in the preceding paragraph turn out to be connected by two non-degenerate 4-simplices of $N_{\infty}(B)$, that we shall call x_{τ} and y_{τ} . The first one, displayed in diagram (3a), witnesses the relation

$$\tau_{\rm d}(\alpha) *_2 \tau_{\rm u}(\alpha) = 1_{F(\alpha_r)}$$
.

The second one, displayed in diagram (3b), witnesses instead the relation

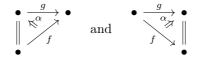
$$\tau_{\mathbf{u}}(\alpha) *_{2} \tau_{\mathbf{d}}(\alpha) = 1_{F(\alpha_{l})},$$

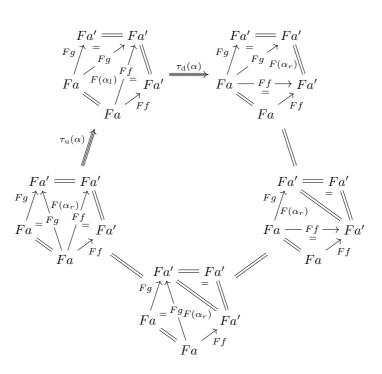
so that in fact $\tau_{\rm u}(\alpha)$ and $\tau_{\rm d}(\alpha)$ are two invertible 3-cells of B.

2.2.5. Let $F: A \to B$ be a normalised oplax 3-functor. Consider a 2-cell $\alpha: f \to g$ of A and the two normalised oplax 3-functors $L: \Delta^2 \to A$ and $R: \Delta^2 \to A$ defined by mapping

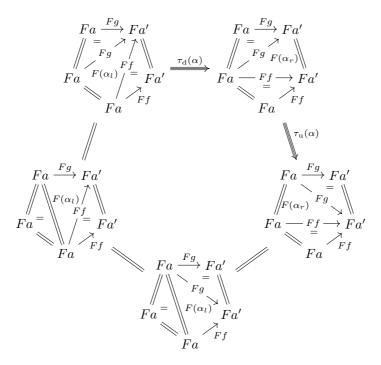
$$\begin{split} \langle 01 \rangle \mapsto \mathbf{1}_{s_0(\alpha)} &= a, & \langle 01 \rangle \mapsto g, \\ \langle 12 \rangle \mapsto g, & \langle 12 \rangle \mapsto \mathbf{1}_{t_0(\alpha)} &= a', \\ \langle 02 \rangle \mapsto f, & \langle 02 \rangle \mapsto f, \\ \langle 12 \rangle *_0 \langle 01 \rangle \mapsto g, & \langle 12 \rangle *_0 \langle 01 \rangle \mapsto g, \\ L_{\vee}(\langle 12 \rangle, \langle 01 \rangle) &= \alpha, & R_{\vee}(\langle 12 \rangle, \langle 01 \rangle) &= \alpha, \end{split}$$

respectively, that we can depict as





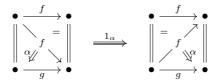
(a) The 4-simplex x_{τ} .



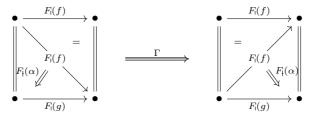
(b) The 4-simplex y_{τ} .

Figure 3. The 4-simplices governing $\tau_{\rm u}$ and $\tau_{\rm d}$.

Now, we can view L and R as the 3-face and 0-face respectively of the normalised oplax 3-functor $T \colon \Delta^3 \to A$ that we can represent as



The conditions of normalisations impose that the image under F of such a diagram of A, *i.e.*, the image of the normalised oplax 3-functor $FT: \Delta^3 \to B$, must be



where

$$\Gamma = FT_{\Psi}(\langle 23 \rangle, \langle 12 \rangle, \langle 01 \rangle)$$
.

Now, the four 3-cells of B appearing in the definition of Γ are (see 1.3.1)

$$F_{\bigvee}(\alpha, 1_a) = 1_{F_{\downarrow}(\alpha)}$$

$$F_{V}(1_{a'}, g, 1_a) = 1_{F_{I}(g)},$$

and

$$F(1_{\alpha}) = 1_{F(\alpha)}$$

and

$$F_{\vee}(1_{a'},\alpha)=1_{F_{\downarrow}(\alpha)}$$
.

Hence, for any 2-cell α of A, we have that the 3-cells $\tau_{\rm d}(\alpha)$ and $\tau_{\rm u}(\alpha)$ of B associated to the morphism of simplicial sets $N_l(F)$: $N_{\infty}(A) \to N_{\infty}(B)$ are both trivial.

2.2.6. Consider two composable 1-cells

$$a \xrightarrow{f} a' \xrightarrow{g} a''$$

of A. The simplicial set $N_{\infty}(A)$ encodes the composition of f and g with the 2-simplex

$$a \xrightarrow{g*_1 f} a''$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \qquad \qquad \downarrow \qquad \qquad$$

The morphism F maps this 2-simplex to a 2-simplex

where we call $F_{g,f}$ the 2-cell of B filling the triangle, i.e., having $F(g *_0 f)$ as source and $Fg *_0 Ff$ as target; we shall often write Fgf for the 1-cell $F(g *_0 f)$ of B.

If the morphism of simplicial sets F is the nerve of a normalised oplax 3-funtor $G \colon A \to B$, then by definition $F_{g,f} = G_{\vee}(g,f)$ (see 1.3.1).

2.2.7. Consider a 2-cell

$$a \xrightarrow{\int_{\alpha} a''} a''$$

of A. The simplicial set $N_{\infty}(A)$ can encode the cell α with the 2-simplices α_l and α_r described above in paragraph 2.2.2, but also with the 2-simplex

$$\bar{\alpha} := \begin{array}{c} a \xrightarrow{f} a'' \\ \downarrow a \nearrow_h \\ a' \end{array}$$

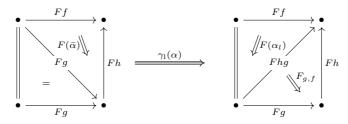
of $N_{\infty}(A)$. These three 2-simplices of $N_{\infty}(A)$ are tied together by the two 3-simplices described in paragraph 2.2.2, but also by the following two 3-simplices



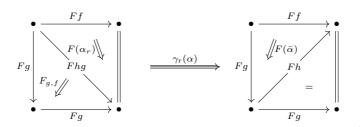
and



of $N_{\infty}(A)$, whose image under F gives the following two 3-simplices of $N_{\infty}(B)$:



and



Remark 2.2.8. If B is a 2-category, then the 2-cells $F_{g,f} *_1 F(\alpha_l)$, $F_{g,f} *_1 F(\alpha_r)$ and $F(\bar{\alpha})$ coincide.

2.2.9. The 3-cells $\gamma_{\rm l}(\alpha)$ and $\gamma_{\rm r}(\alpha)$ of B described in the preceding paragraph turn out to be connected by two non-degenerate 4-simplices of $N_{\infty}(B)$, that we shall call x_{γ} and y_{γ} . The first one, displayed in diagram 4a, witnesses the relation

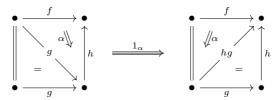
$$\gamma_{l}(\alpha) *_{2} \gamma_{r}(\alpha) = F_{q,f} *_{1} \tau_{u}(\alpha)$$
.

The second one, displayed in diagram 4b, witnesses instead the relation

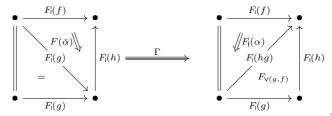
$$\gamma_{\mathbf{r}}(\alpha) *_{2} (F_{g,f} *_{1} \tau_{\mathbf{d}}(\alpha)) *_{2} \gamma_{\mathbf{l}}(\alpha) = 1_{F\bar{\alpha}}.$$

We already know by paragraph 2.2.4 that $\tau_{\rm u}(\alpha)$ and $\tau_{\rm d}(\alpha)$ are two invertible 3-cells of B, inverses of each others. Hence we obtain that $\gamma_{\rm l}(\alpha)$ and $\gamma_{\rm r}(\alpha)$ are invertible 3-cells of B, with $F_{g,f} *_1 \tau_{\rm d}(\alpha) *_2 \gamma_{\rm l}(\alpha)$ as inverse of $\gamma_{\rm r}(\alpha)$.

2.2.10. Let $F: A \to B$ be a normalised oplax 3-functor. Consider the normalised oplax 3-functor $T: \Delta^3 \to A$ given by



The conditions of normalisations impose that the image under F of T is



Moreover, the four main 3-cells of Γ are by definition (see 1.3.1):

$$\begin{split} F_{\bigvee}(1_{hg},1_a) &= 1_{F_{\!\!\!\mid}(hg)}\,, \\ F_{\!\!\!\mid}(h,g,1_a) &= 1_{F_{\!\!\mid}(g,f)}\,, \\ F_{\!\!\mid}(1_\alpha) &= 1_{F_{\!\!\mid}(\alpha)} \end{split}$$

and

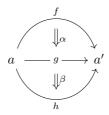
$$F_{\vee}(h, 1_g) = 1_{F_{\vee(g,f)}}$$
.

Hence the 3-cell Γ is trivial. This is equivalent to saying that for any diagram

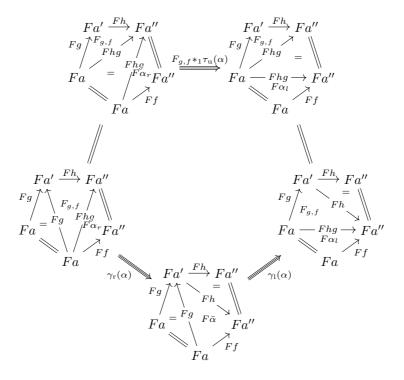
$$a \underbrace{\downarrow^{\alpha}}_{g \rightarrow a'} a''$$

of A, the 3-cells $\gamma_1(\alpha)$ and $\gamma_r(\alpha)$ associated to the morphism of simplicial set $N_l(F): N_{\infty}(A) \to N_{\infty}(B)$ are trivial.

2.2.11. Consider two 1-composable 2-cells



of A. We have a 3-simplex $\sigma_{\alpha,\beta}$



(a) The 4-simplex x_{γ} .

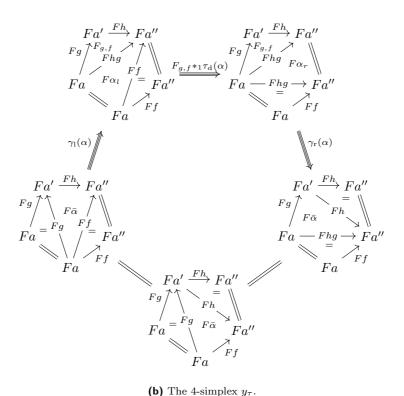


Figure 4. The 4-simplices governing γ_1 and γ_r .

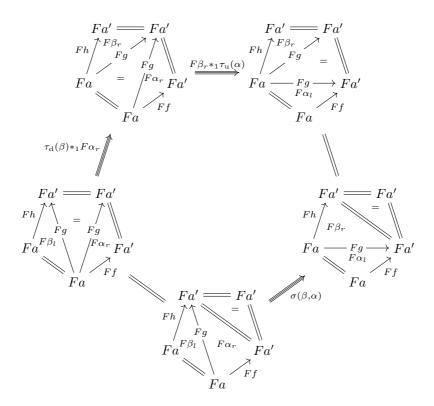
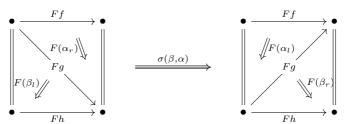


Figure 5. The 4-simplex x_{σ}



of $N_{\infty}(A)$ whose image under F is given by the following 3-simplex $F(\sigma_{\alpha,\beta})$ of $N_{\infty}(B)$

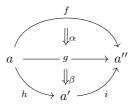


There is a close relationship between the 3-cell $\sigma(\beta, \alpha)$ and the 3-cells $\tau_{\rm u}(\alpha)$ and $\tau_{\rm d}(\beta)$ of B, as displayed by the 4-simplex x_{σ} in figure 5. In particular, $\sigma(\beta, \alpha)$ is an invertible 3-cell of B. Being more precise, we have

$$\sigma(\beta, \alpha) = \tau_{\rm d}(\beta) *_{1} \tau_{\rm u}(\alpha).$$

Remark 2.2.12. If $F: A \to B$ be a normalised oplax 3-functor, then it follows by paragraph 2.2.5 and by the relation of the previous paragraph that given any pair (β, α) of 1-composable 2-cells of A, the 3-cell $\sigma(\beta, \alpha)$ of B associated to the morphism of simplicial sets $N_l(F): N_{\infty}(A) \to N_{\infty}(B)$ is trivial. This can be taken as a further justification for the choice of listing the datum associated to the tree \forall , representing the vertical composition of 2-cells, as a coherence and not as a structural cell in the definition of normalised oplax 3-functor. Indeed, any preferred direction (lax/oplax) would be incompatible with the combinatorics dictated by the simplicial sets; more precisely, it would be irreconcilable with the combinatorics of the orientals and thus with the data encoded by morphisms of simplicial sets between Street nerve of 3-categories.

2.2.13. The images of vertical compositions of cells encode a great deal of information into the form of coherences, *i.e.*, invertible cells of B. An example of critical importance for the following sections is presented in this paragraph. Consider two 1-composable 2-cells



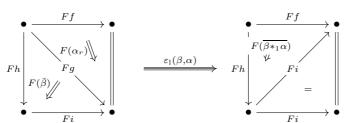
of A, to which we can associate the following two 3-simplices



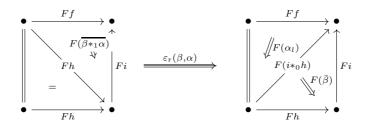
and



of $N_{\infty}(A)$. These are mapped under F to the following two 3-simplices of $N_{\infty}(B)$:



and



2.2.14. Under the assumptions of the preceding paragraph, we can construct the two 4-simplices x_{ε} and y_{ε} of $N_{\infty}(B)$ displayed in figure 6.

We have the following equalities of 3-cells of B:

$$\begin{split} & \gamma_{\mathrm{r}}(\beta) *_{1} F \alpha_{l} *_{2} F_{i,h} *_{1} \sigma(\beta,\alpha) *_{2} \gamma_{\mathrm{l}}(\beta) *_{1} F \alpha_{r} \\ & = \gamma_{\mathrm{r}}(\beta) *_{1} F \alpha_{l} *_{2} (F_{i,h} *_{1} \tau_{\mathrm{d}}(\beta) *_{1} \tau_{\mathrm{u}}(\alpha)) *_{2} \gamma_{\mathrm{l}}(\beta) *_{1} F \alpha_{r} \\ & = (\gamma_{\mathrm{r}}(\beta) *_{1} F_{i,h} *_{2} \tau_{\mathrm{d}}(\beta) *_{2} \gamma_{\mathrm{l}}(\beta)) *_{1} \tau_{\mathrm{u}}(\alpha) \\ & = 1_{F\beta_{r}*_{1}F\alpha_{l}} *_{1} \tau_{\mathrm{u}}(\alpha) = F\bar{\beta} *_{1} \tau_{\mathrm{u}}(\alpha) \,, \end{split}$$

where the first equality follows by paragraph 2.2.11, the second one by the exchange law and the third one by paragraph 2.2.9. Hence the 4-simplex x_{ε} depicted in figure 6a witnesses the relation

$$\varepsilon_{\mathbf{r}}(\beta, \alpha) *_{2} \varepsilon_{1}(\beta, \alpha) = F \bar{\beta} *_{1} \tau_{\mathbf{u}}(\alpha),$$

that by paragraph 2.2.4 is equivalent to saying that the 3-cell

$$F\bar{\beta} *_{1} \tau_{\mathrm{d}}(\alpha) *_{2} \varepsilon_{\mathrm{r}}(\beta, \alpha) *_{2} \varepsilon_{\mathrm{l}}(\beta, \alpha)$$

is precisely the identity of the 2-cell

and

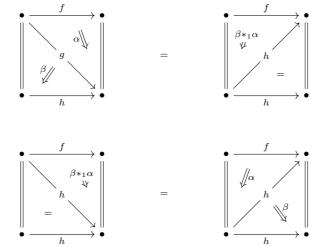
$$F_{i,h} *_1 F \bar{\beta} *_1 F \alpha_l$$
.

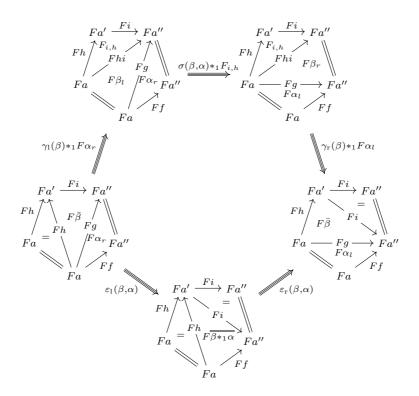
Moreover, the 4-simplex y_{ε} depicted in figure 6b gives us that the 3-cell

$$\varepsilon_{l}(\beta, \alpha) *_{2} F\beta_{r} *_{1} \tau_{d}(\alpha) *_{2} \varepsilon_{r}(\beta, \alpha)$$

of B is an identity cell, too. Therefore both the 3-cells $\varepsilon_l(\beta, \alpha)$ and $\varepsilon_r(\beta, \alpha)$ are invertible.

2.2.15. Consider two 1-composable 2-cells α and β of A as in paragraph 2.2.11 and the two 3-simplices





(a) The 4-simplex x_{ε} .

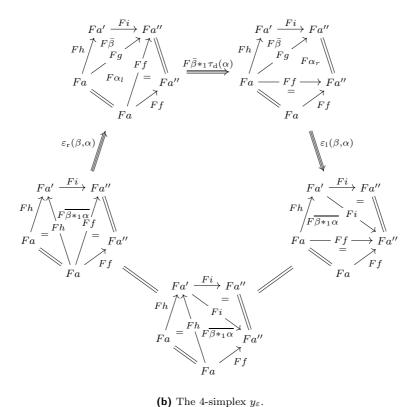
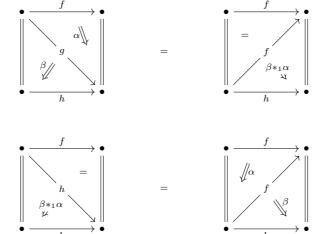


Figure 6. The 4-simplices governing $\varepsilon_1(\beta, \alpha)$ and $\varepsilon_1(\beta, \alpha)$.

and

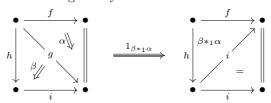
of $N_{\infty}(A)$. It follows immediately from the previous paragraph that the main 3-cells of the images by F of the two 3-simplices above is trivial.

There are two others 3-simplices of $N_{\infty}(A)$ that is natural to consider, that is

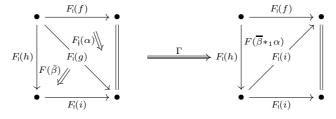


Let us denote by $\omega_{\rm r}(\beta,\alpha)$ and $\omega_{\rm l}(\beta,\alpha)$ respectively the main 3-cells of B of the images under F of the 3-simplices of $N_{\infty}(A)$ above. Both these 3-cells of B are invertible. The proof for $\omega_{\rm r}(\beta,\alpha)$ is given by the 4-simplex x_{ω} of $N_{\infty}(B)$ depicted in figure 7 and the proof for $\omega_{\rm l}(\beta,\alpha)$ is completely similar and we leave it to the reader. One can actually check that the four 3-cells $\varepsilon_{\rm l}(\beta,\alpha)$, $\varepsilon_{\rm r}(\beta,\alpha)$, $\omega_{\rm l}(\beta,\alpha)$ and $\omega_{\rm r}(\beta,\alpha)$ are all tied together by a 5-simplex of B, that we will not need and so we are not going to describe.

2.2.16. Let $F: A \to B$ be a normalised oplax 3-functor and consider the normalised oplax 3-functor $T: \Delta^3 \to A$ given by



Following the definition given in paragraph 1.3.1, the conditions of normalisations impose that the image under F of T is



where paragraph 2.2.10 and Remark 2.2.12 give

$$F(\bar{\alpha}) = F_{\vee(q,f)} *_1 F_{\downarrow}(\alpha)$$

as well as

$$F(\overline{\beta} *_1 \alpha) = F_{\vee (g,f)} *_1 F_{!}(\beta) *_1 F_{!}(\alpha).$$

Moreover, the four main 3-cells of Γ are by definition

$$F_{\backslash \prime}(1_i,h) = 1_{F_{\vee}(i,h)},$$

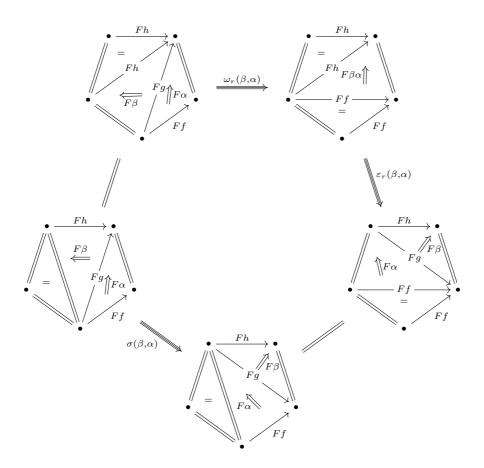


Figure 7. The 4-simplex x_{ω} , showing that $\omega_{\rm r}(\beta, \alpha)$ is invertible.

$$F_{V}(1_{a''}, i, h) = 1_{F_{V}(i,h)},$$

$$F_{\mathbf{i}}(1_{\beta*_1\alpha}) = 1_{F_{\mathbf{i}}(\beta)*_1F_{\mathbf{i}}(\alpha)}$$

and

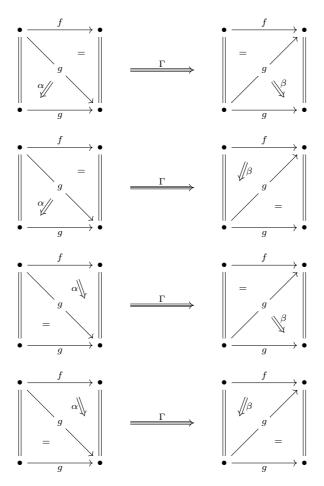
$$F_{\bigvee}(1_{a^{\prime\prime}},\beta)=1_{F_{\mathbf{i}(\beta)}}.$$

Hence the 3-cell Γ is trivial, which implies that the 3-cells $\varepsilon_l(\beta, \alpha)$, $\varepsilon_r(\beta, \alpha)$, $\omega_l(\beta, \alpha)$ and $\omega_r(\beta, \alpha)$ associated to the morphism of simplicial sets $N_l(F)$: $N_{\infty}(A) \to N_{\infty}(B)$ are all trivial.

2.2.17. The last piece of information we want to analyse in this section is the behaviour of the morphism $F \colon N_{\infty}(A) \to N_{\infty}(B)$ with respect to the 3-cells of A. Let



be a 3-cell of A. We have several ways of encoding Γ as a 3-simplex of $N_{\infty}(A)$. In particular, we have the following 3-simplices:



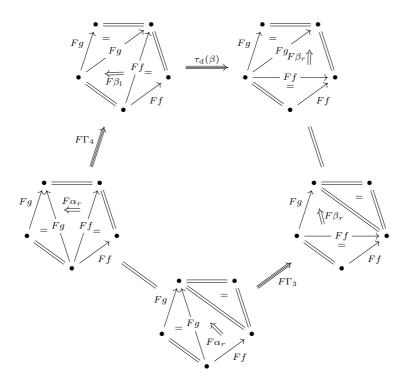
We claim that all these 3-simplices of $N_{\infty}(A)$ are sent under F to 3-simplices of $N_{\infty}(B)$ such that their principal 3-cell is invertible, principal cells that we shall call $F\Gamma_i$, i=1,2,3,4, respectively. The 4-simplex z_{Γ} of $N_{\infty}(B)$ depicted in (8a) shows this claim for the last two 3-simplices, while the 4-simplex y_{Γ} depicted in figure (8b) proves the claim for the middle two. We leave to the reader the easy assignment of describing a 4-simplex of $N_{\infty}(B)$ showing the claim for the first two 3-simplices above.

Remark 2.2.18. Notice that the 3-cells $F\Gamma_i$, i=1,2,3,4, of the preceding paragraph are linked together by whiskering with invertible 2-cells studied in this subsection, such as $\tau_{\rm d}$ and $\tau_{\rm u}$. If F is actually the image of a normalised oplax 3-functor G, then we have shown that these 2-cells are trivial and therefore in this case the four 3-cells $N_l(G)\Gamma_i$ are in fact all equal to $G_i(\Gamma)$.

3. Simplicial oplax 3-morphisms

The previous subsection shows that if we consider a normalised oplax 3-functor $F: A \to B$, then its nerve $N_l(F)$ has the property that some particular (non-degenerate) 3-simplices of $N_{\infty}(A)$ with trivial principal 3-cell are sent to 3-simplices of $N_{\infty}(3)$ where the principal 3-cell is also trivial.

In this section we consider the class of morphisms of simplicial sets between Street nerve of 3-categories having precisely this property and we show that they form a subcategory of simplicial sets. In fact, we will prove that they are canonically equivalent to normalised oplax 3-functors. As a first main step towards this



(a) The 4-simplex z_{Γ} .

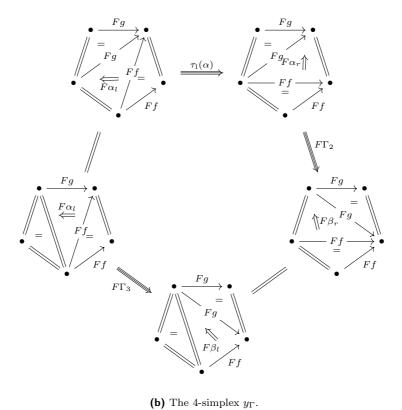


Figure 8. The 4-simplices governing the images of Γ .

correspondence, in the following subsection we shall show how to associate a normalised oplax 3-functor to a simplicial morphism between nerves of 3-categories satisfying these trivialising properties.

Throughout this subsection, we shall make heavy use of the notations introduced in the preceding subsection.

Definition 3.0.1. Let A and B be two small 3-categories. We say that a morphism $F \colon N_{\infty}(A) \to N_{\infty}(B)$ is a *simplicial oplax* 3-morphism if the following conditions are satisfied:

- (a) for any 2-cell α of A, the 3-cell $\tau_{\rm d}(\alpha)$ of B is trivial;
- (b) for any 2-cell α : $f \to h *_0 g$ of A, the 3-cell $\gamma_1(\alpha)$ of B is trivial;
- (c) for any pair of 1-composable 2-cells α and β of A as in 2.2.13, the 3-cell $\varepsilon_1(\beta,\alpha)$ of B is trivial.

Remark 3.0.2. It is clear that the definition above can be framed within stratified simplicial sets. However, the author sees little or no advantage in pursuing this point of view, since the 3-simplices involved are very particular and no lifting property is present.

Remark 3.0.3. The relations described in paragraphs 2.2.4 and 2.2.11 tell us that under condition (a) above also the 3-cells $\tau_{\rm u}(\gamma)$ and $\sigma(\beta,\alpha)$ of B are also trivial, for any choice of 2-cells α , β and γ of A, such that the first two are 1-composable.

Assuming conditions (a) and (b) and using what we just observed in the relations of paragraph 2.2.9 we get immediately that the 3-cell $\gamma_{\rm r}(\alpha)$ is trivial.

If $F: N_{\infty}(A) \to N_{\infty}(B)$ is a simplicial oplax 3-morphism, putting together all we have said right above and the relations analysed in paragraphs 2.2.14 and 2.2.15 gives us that the 3-cells $\varepsilon_{\rm r}(\beta, \alpha)$, $\omega_{\rm r}(\beta, \alpha)$ and $\omega_{\rm l}(\beta, \alpha)$ are trivial, for any appropriate choice of 3-cells α and β of A.

In short, all the invertible 3-cells of B we described in the previous subsection are actually trivial whenever F is a simplicial oplax 3-morphism.

3.0.4. Let $F: N_{\infty}(A) \to N_{\infty}(B)$ be a morphism of simplicial sets satisfying condition (a). It follows from the previous remark that for any 2-cell α of A, the 2-cells $F(\alpha_l)$ and $F(\alpha_r)$ of B coincide. Whenever this happens we shall then simply write $F(\alpha)$, or more often just $F\alpha$ for this 2-cell of B. Furthermore, the relations observed in paragraph 2.2.17 give us that for any 3-cell $\Gamma: \alpha \to \beta$ of A, the four ways we described in that paragraph to encode the image of Γ via F are all the same 3-cell of B, that we shall then call $F(\Gamma)$ or simply $F\Gamma$.

Example 3.0.5. For any normalised oplax 3-functor $F: A \to B$, the previous subsection shows that its nerve $N_l(F)$ is a simplicial oplax 3-morphism.

We now check that simplicial oplax 3-morphisms are closed under composition. Let $F \colon A \to B$ and $G \colon B \to C$ be two simplicial oplax 3-morphisms.

3.0.6. Let $\alpha \colon f \to g$ be a 2-cell of A. By assumption, we have the 3-simplex

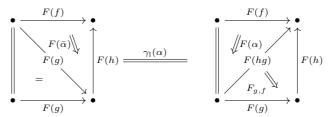


of $N_{\infty}(B)$. Setting $\beta = F(\alpha_l) = F(\alpha_r)$ and applying G to the two 3-simplices above, we get the following 3-simplex of $N_{\infty}(C)$:

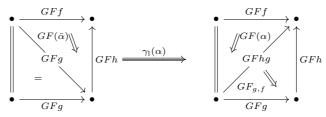


Since $\beta_l = F(\alpha_l)$ and $\beta_r = F(\alpha_r)$ by definition, we have that $G(\beta_l) = GF(\alpha_l)$ and $G(\beta_r) = GF(\alpha_r)$. Thus the morphism $GF \colon N_{\infty}(A) \to N_{\infty}(C)$ of simplicial sets satisfies condition (a) of the definition of simplicial oplax 3-morphisms.

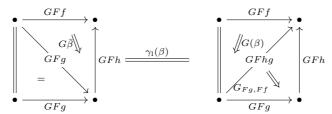
3.0.7. Let $\alpha: f \to h *_0 g$ be a 2-cell of A. By assumption, we have a 3-simplex



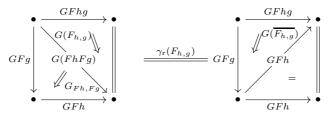
of $N_{\infty}(B)$, where the main 3-cell is trivial as the morphism F verifies condition (b). We have to show that the 3-cell of the 3-simplex



of $N_{\infty}(C)$ is trivial. The 2-cell $\beta = F_{g,f} *_1 F\alpha$ of B has Ff as source and $Fh *_0 Fg$ as target. So applying the morphism $G: N_{\infty}(B) \to N_{\infty}(C)$ we get a 3-simplex



of $N_{\infty}(C)$ where the main 3-cell is trivial by condition (b). Notice that by definition $G\bar{\beta}=GF\bar{\alpha}$ and $G\beta=G(F_{h,g}*_1F\alpha)$; this latter 2-cell of C is equal to $G(F_{h,g})*_1GF\alpha$ by condition (c). Moreover, condition (b) also entails that the following 3-simplex



of $N_{\infty}(C)$ is trivial, as we have observed in Remark 3.0.3. The 4-simplex of $N_{\infty}(C)$ displayed in figure 9 allows to conclude.

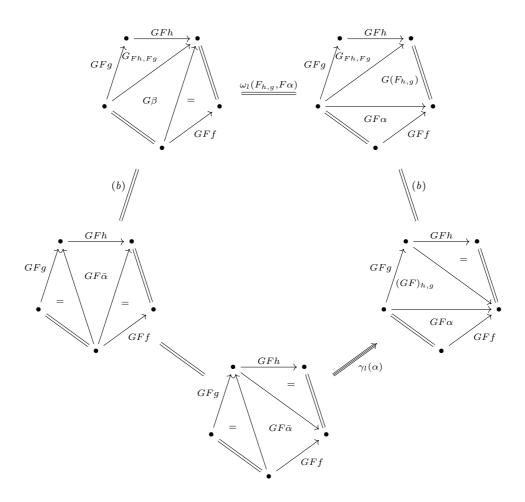
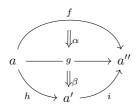
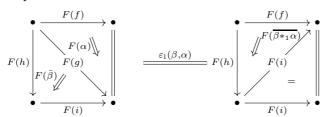


Figure 9. The 3-cell $\gamma_1(\alpha)$ is trivial.

3.0.8. Consider two 1-composable 2-cells



of A. The 3-simplex



of $N_{\infty}(B)$ has trivial main 3-cell $\varepsilon_1(\beta,\alpha)$ by condition (c). Thus $F\overline{\beta}*_1\alpha$ is equal to $F\overline{\beta}*_1F\alpha$ (which by the preceding paragraph we also know to be equal to the

3-cell $F_{h,i} *_1 F \beta *_1 F \alpha$). Now, applying the morphism G with condition (c) at hand we get the 3-simplex



with trivial main 3-cell $\varepsilon_l(F\beta, F\alpha)$. Noticing that the 2-simplex $GF\bar{\beta}$ of $N_{\infty}(C)$ is precisely $GF\bar{\beta}$ and that the 2-simplex $GF\bar{\beta}*_1\alpha$ of $N_{\infty}(C)$ is equal to $GF\bar{\beta}*_1\alpha$, we conclude that the morphism $GF: N_{\infty}(A) \to N_{\infty}(B)$ satisfies condition (c).

Theorem 3.0.9. The class of simplicial oplax 3-morphisms is stable under composition.

Proof. This follows immediately from paragraphs 3.0.6, 3.0.7 and 3.0.8.

Definition 3.0.10. We shall denote by $3\text{-}Cat_{\Delta}$ the subcategory of $\widehat{\Delta}$ whose objects are the nerves of the small 3-categories and whose morphisms are simplicial oplax 3-functors. This shall be called the category of *small 3-categories and simplicial oplax 3-morphisms*.

Remark 3.0.11. The nerve of any 3-functor $u: A \to B$ is clearly a simplicial oplax 3-morphism, since the Street nerve of a 3-functor sends 3-simplices of A with trivial principal 3-cell to 3-simplices of B with trivial principal 3-cell; hence, the nerve induces a faithful functor $3-Cat \hookrightarrow 3-Cat_{\Delta}$.

- 3.1. Simplicial to cellular. Let $F: A \to B$ be a simplicial oplax 3-morphism.
- **3.1.1** (Data). We now associate to F the data of a normalised oplax 3-functor.
- .: The map F_{\bullet} that to each object a of A, *i.e.*, any 0-simplex of $N_{\infty}(A)_0$, assigns an object $F_{\bullet}(a)$ of B, *i.e.*, a 0-simplex of $N_{\infty}(B)$, is simply defined to be F_0 .
- : The map F_{\mid} that to each 1-cell $f: a \to a'$ of A, i.e., any 1-simplex of $N_{\infty}(A)_1$, assigns a 1-cell $F(f): F_{\bullet}(a) \to F_{\bullet}(a')$, i.e., a 1-simplex of $N_{\infty}(B)$, is simply defined to be $F_1: N_{\infty}(A)_1 \to N_{\infty}(B)_1$.
- \vee : The map F_{\vee} that to each pair of 0-composable 1-cells

$$a \xrightarrow{f} a' \xrightarrow{g} a''$$

of A assigns a 2-cell $F_{\vee}(g, f)$

$$F_{\bullet}(a) \xrightarrow{F_{l}(g*_{0}f)} F_{\bullet}(a'')$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad$$

of B, that is

$$F_{\vee}(g, f) : F_{|}(g *_{0} f) \to F_{|}(g) *_{0} F_{|}(f)$$

is defined as $F_{\vee}(g,f) := F_{g,f}$ (see paragraph 2.2.6).

: The map F_1 that to each 2-cell $\alpha \colon f \to g$ of A associates a 2-cell

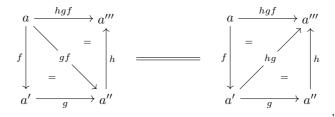
$$F_{\mathsf{I}}(\alpha) \colon F_{\mathsf{I}}(f) \to F_{\mathsf{I}}(g)$$

of B is defined to be $F_i(\alpha) = F(\alpha)$, with the notation of paragraph 3.0.4.

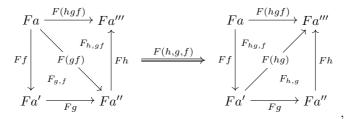
 \bigvee : We define the map F_{V} by mapping any triple of 0-composable 1 cells

$$a \xrightarrow{f} a' \xrightarrow{g} a'' \xrightarrow{h} a'''$$

of A, i.e., any 3-simplex of $N_{\infty}(A)$ of the form



to the main 3-cell $F_{\forall}(h, g, f)$ of the 3-simplex of B image of the above 3-simplex of A by the morphism F:



where $F_{h,gf} = F_{\vee}(h,g *_0 f)$ and $F_{hg,f} = F_{\vee}(h *_0 g,f)$, so that the 3-cell $F_{\vee}(h,g,f)$ has

$$F_{l}(h) *_{0} F_{\vee}(g, f) *_{1} F_{\vee}(h, g *_{0} f)$$

as source and

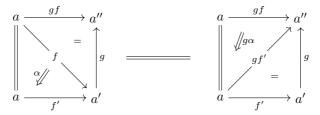
$$F_{\vee}(h,g) *_{0} F_{\mathsf{I}}(f) *_{1} F_{\vee}(h *_{0} g, f)$$

as target.

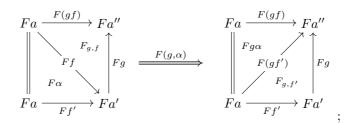
: Consider a whiskering

$$a \xrightarrow{f} a' \xrightarrow{g} a''$$

of A and the following associated 3-simplex



of $N_{\infty}(A)$, where we wrote $g\alpha$ for $g*_0\alpha$. We define $F_{\vee}(g,\alpha)$ to be the main 3-cell of the image under F of the above 3-simplex:



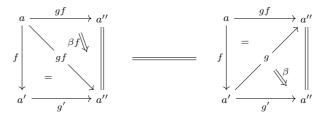
so that

$$F_{\vee}(\alpha,g)\colon F_{!}(g)\ast_{0}F_{!}(\alpha)\ast_{1}F_{\vee}(g,f)\to F_{\vee}(g,f')\ast_{1}F_{!}(g\ast_{0}\alpha)$$
 as a 3-cell of B .

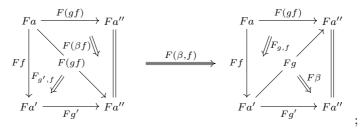
: Consider a whiskering

$$a \xrightarrow{f} a' \xrightarrow{g} a''$$

of A and the following associated 3-simplex



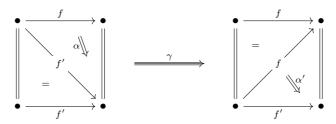
of SN(A), where we wrote βf for $\beta *_0 f$. We define $F_{\vee}(\beta, f)$ as the main 3-cell of the image under F of the above 3-simplex:



so that

$$F_{\vee}(\beta,f)\colon F_{\vee}(g',f)*_1F_{\dagger}(\beta*_0f)\to F_{\dagger}(\beta)*_0F_{\lnot}(f)*_1F_{\vee}(g,f)$$
 as a 3-cell of B .

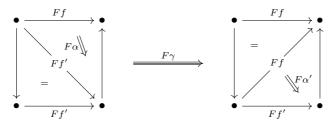
: Consider a 3-cell $\gamma: \alpha \to \alpha'$ of A, and the following associated 3-simplex



of $N_{\infty}(A)$. The 3-cell

$$F_{\mathbf{i}}(\gamma) \colon F_{\mathbf{i}}(\alpha) \to F_{\mathbf{i}}(\alpha')$$

of B is defined to be the main 3-cell of the image under F of the above 3-simplex:



These data satisfy the normalisation conditions.

- **3.1.2** (Normalisation). In this paragraph we shall commit the abuse of denoting the main cell of a simplex of $N_{\infty}(B)$ by the simplex itself. The normalisation is an immediate consequence of the degeneracies of F.
- .: for any object a of A, we have

$$F_1(1_a) = s_1^0(F_{\bullet}a) = F_{\sigma_{\bullet}^0}(a) = 1_{F_{\bullet}(a)};$$

| : for any 1-cell f of A, we have

$$F_{!}(1_f) = s_2^0(F_{!}f) = F_{\sigma_2^0}(f) = 1_{F_{!}(f)};$$

 \bigvee : for any 2-cell $f: a \to a'$ of A, we have

: for any 2-cell α of A we have

$$F_{\mathbf{i}}(1_{\alpha}) = s_3^0(\alpha_r) = F_{\sigma_3^0}(\alpha_r) = 1_{F_{\mathbf{i}}(\alpha)};$$

 \bigvee : for any pair $a \xrightarrow{f} a' \xrightarrow{g} a''$ of 1-cells of A, we have

$$F_{V}(g, f, 1_{a}) = s_{3}^{0}(F_{V}(g, f)) = 1_{F_{V}(g, f)},$$

$$F_{V}(g, 1_{a'}, f) = s_{3}^{1}(F_{V}(g, f)) = 1_{F_{V}(g, f)},$$

$$F_{V}(1_{a''}, g, f) = s_{3}^{2}(F_{V}(g, f)) = 1_{F_{V}(g, f)};$$

$$\bigvee$$
: for any pair $a \xrightarrow{f} a' \xrightarrow{g} a''$ of 1-cells of A, we have

$$F_{\vee}(g, 1_f) = s_3^0(F_{\vee}(g, f)) = 1_{F_{\vee}(g, f)},$$

and for any 2-cell $\alpha \colon f \to f'$ of A, we have

$$F_{\checkmark}(1'_a,\alpha) = s_3^2(F_{\dagger}(\alpha)) = 1_{F_{\dagger}(\alpha)};$$

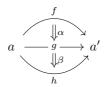
: this normalisation is dual to the previous one;

Checking the coherences is much strenuous. The next subsection is devoted to establish that they hold, thus proving the following result.

Theorem 3.1.3. Let $F: A \to B$ be a simplicial oplax 3-morphism. With the data defined right above, F is a normalised oplax 3-functor from A to B.

3.2. Coherences. Let $F:A\to B$ be a simplicial oplax 3-morphism. In this subsection we are going to check that the data of F_{\bullet} , $F_{||}$, and $F_{||}$ defined above satisfy the set of coherences for a normalised oplax 3-functor defined in paragraph 1.2.3. We shall do so by showing that every coherence can be encoded in a particular 4-simplex of $N_{\infty}(B)$, image of a 4-simplex of $N_{\infty}(A)$. We shall only draw the former, the latter being clear. Moreover, we shall omit the various whiskerings when denoting the 3-cells of these 4-simplices, which are nevertheless clear from the picture.

: For any pair of 1-composable 2-cells



of A, all the images by F of the related 3-simplices have a trivial main 3-cell of B; indeed, this is the case for the 3-cells $\varepsilon_{\rm l}(\beta,\alpha)$, $\varepsilon_{\rm r}(\beta,\alpha)$, $\omega_{\rm l}(\beta,\alpha)$ and $\omega_{\rm l}(\beta,\alpha)$ of B. Thus we have an *identity* 3-cell

$$F_{\downarrow}(\beta) *_{1} F_{\downarrow}(\alpha) \to F_{\downarrow}(\beta *_{1} \alpha)$$

of B, establishing the coherence for \forall .

: Consider a quadruple

$$\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet \xrightarrow{h} \bullet \xrightarrow{i} \bullet$$

of 0-composable 1-cells of A. The 4-simplex of $N_{\infty}(B)$ in figure 10, where $F_{i,hqf} = F_{\vee}(i,ghf)$ by definition, shows that the 3-cells

$$F_{\vee}(i,h) *_{0} F_{|}(g) *_{0} F_{|}(f) *_{1} F_{\vee}(ih,g,h)$$

$$*_{2}$$

$$F_{|}(i) *_{0} F_{|}(h) *_{0} F_{\vee}(g,f) *_{1} F_{\vee}(i,h,gf)$$

and

$$\begin{split} F_{\mathbb{V}}(i,h,g) *_{0} F_{\mathbb{I}}(f) *_{1} F_{\mathbb{V}}(i *_{0} h *_{0} g, f) \\ *_{2} \\ F_{\mathbb{I}}(i) *_{0} F_{\mathbb{V}}(h,g) *_{0} F_{\mathbb{I}}(f) *_{1} F_{\mathbb{V}}(i,h *_{0} g, f) \\ *_{2} \\ F_{\mathbb{I}}(i) *_{0} F_{\mathbb{V}}(h,g,f) *_{1} F_{\mathbb{V}}(i,h *_{0} g *_{0} f) \end{split}$$

of B are equal. This establishes the coherence

: Consider a triple

$$\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet \underbrace{\downarrow \alpha}_{h'} \bullet$$

of 0-composable cells f, g and α of A. The 4-simplex of $N_{\infty}(B)$ depicted in figure 11 ensures that the 3-cells

$$\begin{split} F_{\mathrm{I}}(\alpha) *_{0} F_{\mathrm{I}}(g) *_{0} F_{\mathrm{I}}(f) *_{1} F_{\mathrm{V}}(h,g,f) \\ *_{2} \\ F_{\mathrm{I}}(h') *_{0} F_{\mathrm{V}}(g,f) *_{1} F_{\mathrm{V}}(\alpha,g *_{0} f) \end{split}$$



Figure 10. Establishing the coherence ihgf.

and

$$\begin{split} F_{\bigvee}(\alpha,g) *_0 F_{\mathsf{I}}(f) *_1 F_{\bigvee}(h *_0 g, f) \\ *_2 \\ F_{\bigvee}(h',g) *_0 F_{\mathsf{I}}(f) *_1 F_{\bigvee}(\alpha *_0 g, f) \\ *_2 \\ F_{\bigvee}(h',g,f) *_1 F_{\mathsf{I}}(\alpha *_0 g *_0 f) \end{split}$$

of B are equal. This establishes the coherence \bigvee .

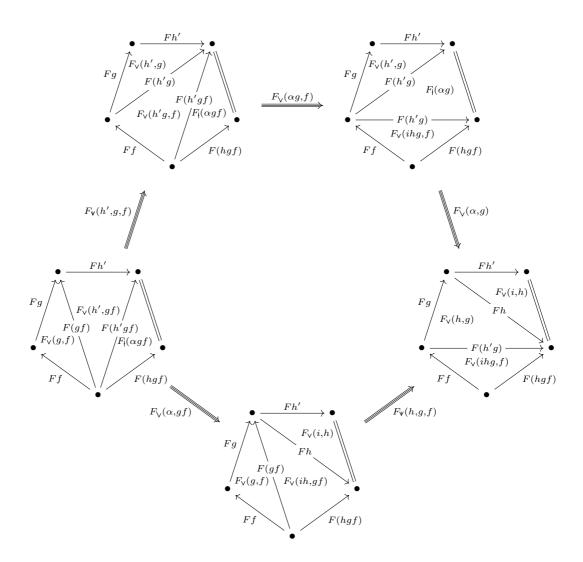
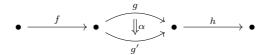


Figure 11. Establishing the coherence αgf .

: Consider a triple



of 0-composable cells f, α and h of A. The proof of the coherence for this cellular pasting diagram is quite involved and relies on the construction and analysis of four 4-simplices of $N_{\infty}(B)$. The main such 4-simplex is depicted in figure 12, where

$$\beta = F_{\vee}(g',f) *_1 F_{\dagger}(\alpha *_0 f) \quad \text{and} \quad \gamma = F_{\vee}(h *_0 g,f) *_1 F_{\dagger}(h *_0 \alpha *_0 f) \,,$$

and shows that the 3-cell

$$F_{\vee}(h,\alpha) *_{0} F_{|}(f) *_{1} F_{\vee}(h *_{0} g, f)$$

$$*_{2}$$

$$F_{|}(h) *_{0} F_{|}(\alpha) *_{0} F_{|}(f) *_{1} F_{\vee}(h, g, f)$$

$$*_{2}$$

$$F_{|}(h) *_{0} \Omega *_{1} F_{\vee}(h, g *_{0} f)$$

of B is equal the following 3-cell

$$F_{\vee}(h,g') *_{0} F_{|}(f) *_{1} \Phi *_{2} \Psi$$

of B. Now, the 4-simplices depicted in figures 13, 14 and 15 entail the equalities

$$\Omega = F_{\vee}(\alpha, f) ,$$

$$\Phi = F_{\vee}(h *_{0} \alpha, f)$$

and

$$\Psi = \mathit{F}_{V}(h, g', f) *_{1} \mathit{F}_{!}(h *_{0} \alpha *_{0} f) *_{2} \mathit{F}_{!}(h) *_{0} \mathit{F}_{V}(g', f) *_{1} \mathit{F}_{V}(h, \alpha *_{0} f) \, .$$

We can then conclude that the 3-cell (3.2) of B is equal to

$$F_{\vee}(h, g') *_{0} F_{|}(f) *_{1} F_{\vee}(h *_{0} \alpha, f)$$

$$*_{2}$$

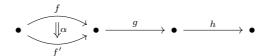
$$F_{\vee}(h, g', f) *_{1} F_{|}(h *_{0} \alpha *_{0} f)$$

$$*_{2}$$

$$F_{|}(h) *_{0} F_{\vee}(g', f) *_{1} F_{\vee}(h, \alpha *_{0} f),$$

thereby establishing the coherence of \checkmark .

: Consider a triple



of 0-composable cells α , g and h of A. The 4-simplex of $N_{\infty}(B)$ depicted in figure 16, totally symmetric to 11, shows that the 3-cells

$$\begin{split} F_{\mathbb{V}}(h,g,f') *_{1} F_{\mathbb{I}}(h *_{0} g *_{0} \alpha) \\ *_{2} \\ F_{\mathbb{I}}(h) *_{0} F_{\mathbb{V}}(g,f') *_{1} F_{\mathbb{V}}(h,g *_{0} \alpha) \\ *_{2} \\ F_{\mathbb{I}}(h) *_{0} F_{\mathbb{V}}(g,\alpha) *_{1} F_{\mathbb{V}}(h,g *_{0} f) \end{split}$$

and

$$\begin{split} F_{\vee}(h,g) *_0 F_{|}(f') *_1 F_{\vee}(h *_0 g, \alpha) \\ *_2 \\ F_{|}(h) *_0 F_{|}(g) *_0 F_{|}(\alpha) *_1 F_{\vee}(h,g *_0 f) \end{split}$$

of B are equal, therefore establishing the coherence \checkmark

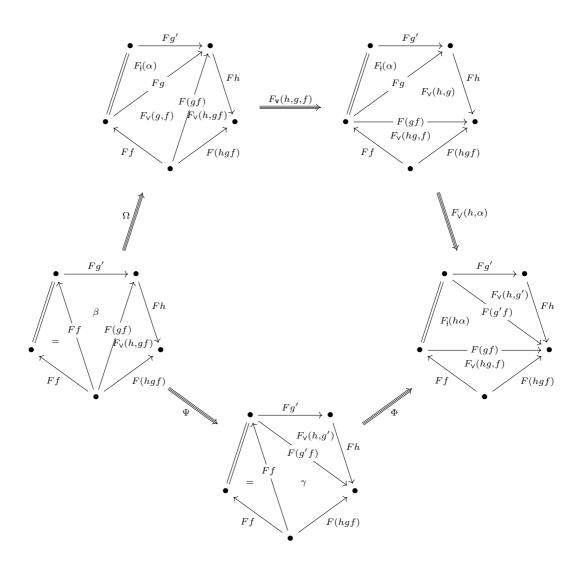
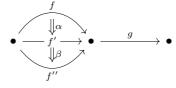


Figure 12. Establishing the coherence $h\alpha f$.

 \bigvee : Consider a triple



of cells α , β and g of A as in the drawing. The 4-simplex of $N_{\infty}(B)$ depicted in figure 17 shows that the 3-cells

$$\begin{split} F_{\bigvee}(g,\beta) *_{1} F_{|}(g *_{0} \alpha) *_{2} \\ F_{|}(g) *_{0} F_{|}(\beta) *_{1} F_{\bigvee}(g,\alpha) *_{2} \\ F_{|}(g) *_{0} F_{\bigvee}(\beta,\alpha) *_{1} F_{\bigvee}(g,f) \end{split}$$

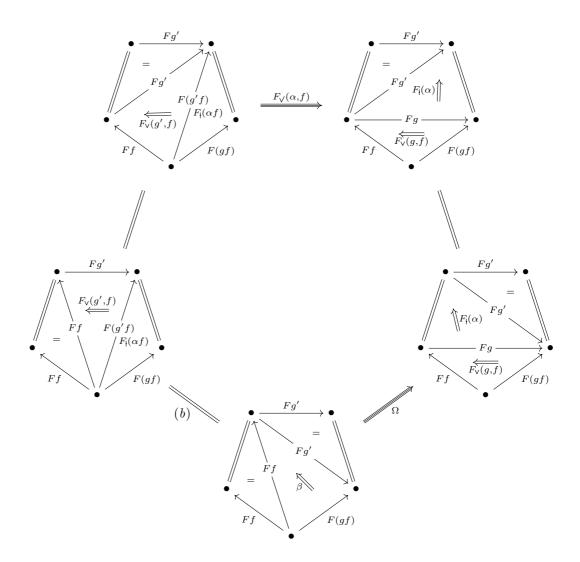


Figure 13. The 3-cell Ω of B.

and

$$F_{\vee}(g,f^{\prime\prime})*_{1}F_{\vee}(g*_{0}\beta,g*_{0}\alpha) *_{2}F_{\vee}(g,\beta*_{1}\alpha)$$

of B are equal. Since $F_{\gamma}(\beta,\alpha)$ and $F_{\gamma}(g*_0\beta,g*_0\alpha)$ are trivial by condition (c), the 4-simplex actually exhibits the equality of the 3-cells

$$F_{\checkmark}(g,\beta) *_1 F_{\downarrow}(g *_0 \alpha) *_2 F_{\downarrow}(g) *_0 F_{\downarrow}(\beta) *_1 F_{\checkmark}(g,\alpha)$$

and

$$F_{\lor}(g,f'')*_1F_{\lor}(\beta,\alpha)$$

of B.

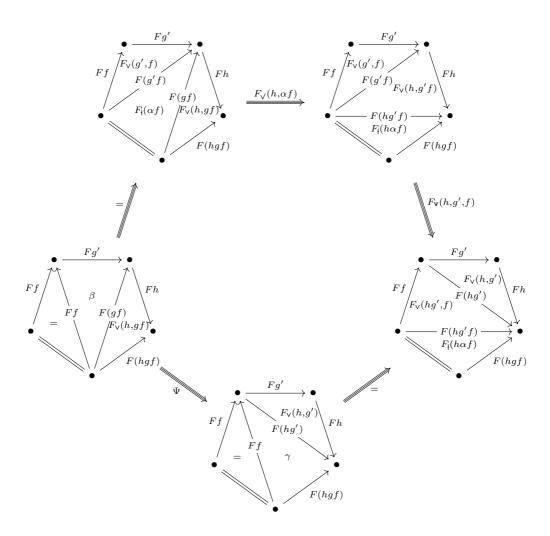
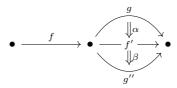


Figure 14. The 3-cell Ψ of B.

: Consider a triple



of cells α , β and g of A as in the drawing. The 4-simplex of $N_{\infty}(B)$ displayed in figure 18, completely dual to the 4-simplex 17, shows that the 3-cells

$$\begin{split} & \left(F_{\uparrow}(\beta,\alpha) *_0 F_{\mid}(f) *_1 F_{\lor}(g,f)\right) *_2 \\ & \left(F_{\mid}(\beta) *_0 F_{\mid}(f) *_1 F_{\lor}(\alpha,f)\right) *_2 \\ & \left(F_{\lor}(\beta,f) *_1 F_{\mid}(\alpha *_0 f)\right) \end{split}$$

and

$$F_{\vee}(\beta *_1 \alpha, f) *_2 F_{\vee}(g'', f) *_1 F_{\vee}(\beta *_0 f, \alpha *_0 f)$$

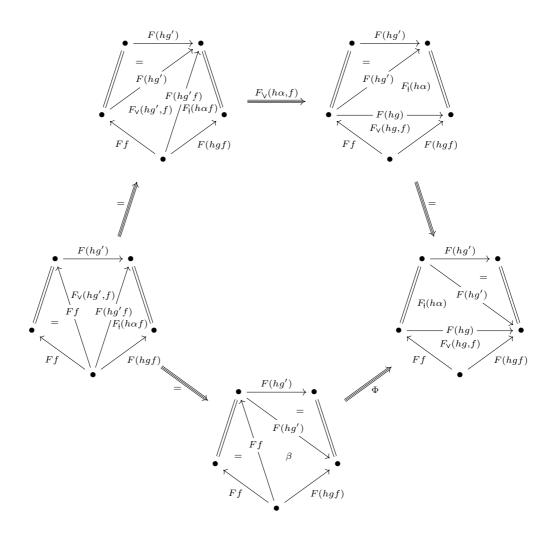


Figure 15. The 3-cell Φ of B.

of B are equal. Since the 3-cells $F_{\gamma}(\beta, \alpha)$ and $F_{\gamma}(\beta *_0 f, \alpha *_0 f)$ are trivial by condition (c), the 4-simplex is actually imposing the equality of the 3-cells

$$F_{\mathsf{I}}(\beta) *_{0} F_{\mathsf{I}}(f) *_{1} F_{\bigvee}(\alpha, f) \ *_{2} \ F_{\bigvee}(\beta, f) *_{1} F_{\mathsf{I}}(\alpha *_{0} f)$$

and

$$F_{\vee}(\beta *_1 \alpha, f)$$

of B.

✓ : Consider a pair

of 0-composable 2-cells α and β of A. The 4-simplex of $N_{\infty}(B)$ depicted in figure 19 shows that the 3-cells

$$F_{\vee}(\beta,f') *_{1} F_{\dagger}(g *_{0} \alpha) *_{2} F_{\vee}(g',f') *_{1} F_{\mathrm{ex}}(\beta,\alpha) *_{2} F_{\vee}(g',\alpha) *_{1} F_{\dagger}(\beta *_{0} f)$$

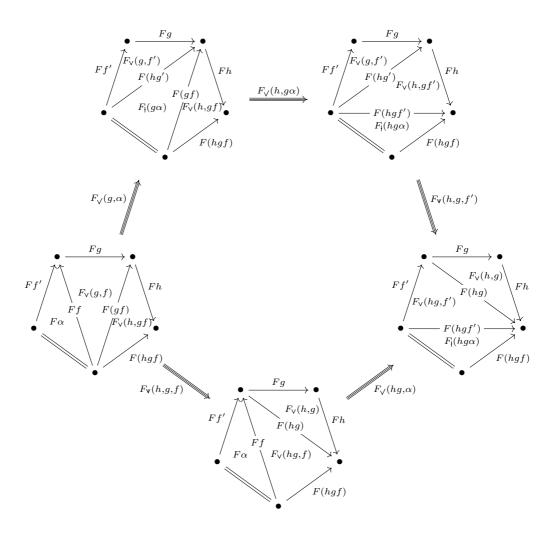


Figure 16. Establishing the coherence $hg\alpha$.

and

$$F_{\downarrow}(\beta) *_{0} F_{\downarrow}(f') *_{1} F_{\downarrow\prime}(g, \alpha) *_{2} F_{\downarrow}(g') *_{0} F_{\downarrow}(\alpha) *_{1} F_{\downarrow\prime}(\beta, f)$$

of B are equal. Here we denote by $F_{\rm ex}(\beta,\alpha)$ the identity 3-cell going from $F_{\dagger}(g'*_0\alpha)*_1F_{\dagger}(\beta*_0f)$ to $F_{\dagger}(\beta*_0f')*_1F_{\dagger}(g*_0\alpha)$ that we get from the composition of the following pair of trivial 3-cells

$$F_{\dagger}(g'*_0\alpha *_1\beta *_0f) = F_{\dagger}(\beta *_0f'*_1g *_0\alpha)$$

$$F_{\Upsilon}(g'*_0\alpha ,\beta *_0f)$$

$$F_{\dagger}(g'*_0\alpha) *_1F_{\dagger}(\beta *_0f)$$

$$F_{\dagger}(\beta *_0f') *_1F_{\Upsilon}(g *_0\alpha)$$

of B, where the equality in the upper row is just the exchange law.

Since $F_{\rm ex}(\beta,\alpha)$ is a trivial 3-cell, the 4-simplex is actually imposing the equality between the 3-cells

$$F_{\backslash \prime}(\beta,f') *_1 F_{\downarrow}(g*_0\alpha) *_2 F_{\backslash \prime}(g',\alpha) *_1 F_{\downarrow}(\beta*_0f)$$

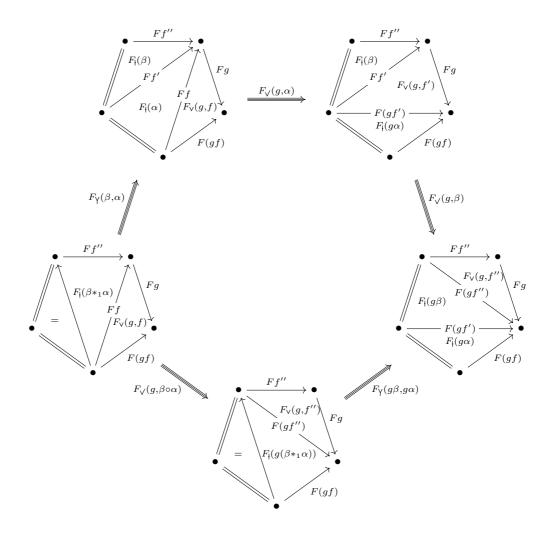


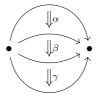
Figure 17. Establishing the coherence $g\beta\alpha$.

and

$$F_{\mathsf{I}}(\beta) *_{0} F_{\mathsf{I}}(f') *_{1} F_{\bigvee}(g,\alpha) *_{2} F_{\mathsf{I}}(g') *_{0} F_{\mathsf{I}}(\alpha) *_{1} F_{\bigvee}(\beta,f)$$

of B, thereby establishing the coherence \bigvee .

: Consider a triple



of 1-composable 2-cells α , β and γ of A. The simplicial oplax 3-morphism F trivially satisfies the coherence associated to this tree, which is the trivial

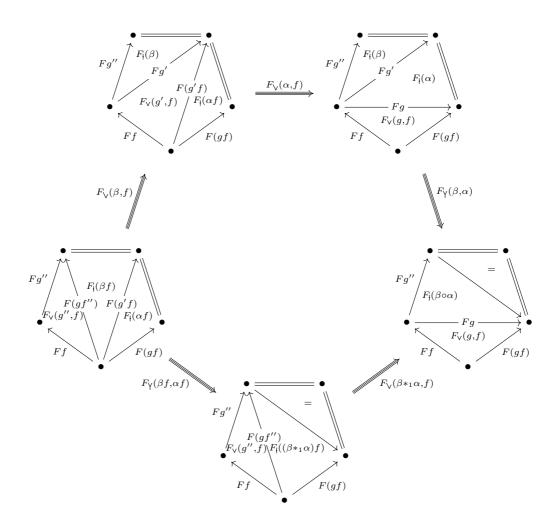


Figure 18. Establishing the coherence $\beta \alpha f$.

equality between the following identity 3-cell

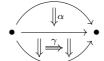
$$F_{Y}(\gamma *_{1} \beta, \alpha) *_{2} F_{Y}(\gamma, \beta) *_{1} F_{\dagger}(\alpha)$$

and

$$F_{\bigvee}(\gamma,\beta*_1\alpha)*_2F_{\downarrow}(\gamma)*_1F_{\bigvee}(\beta,\alpha)$$

of B. This coherence is encoded in the 4-simplex of $N_{\infty}(B)$ depicted in figure 20.

: Consider a pair



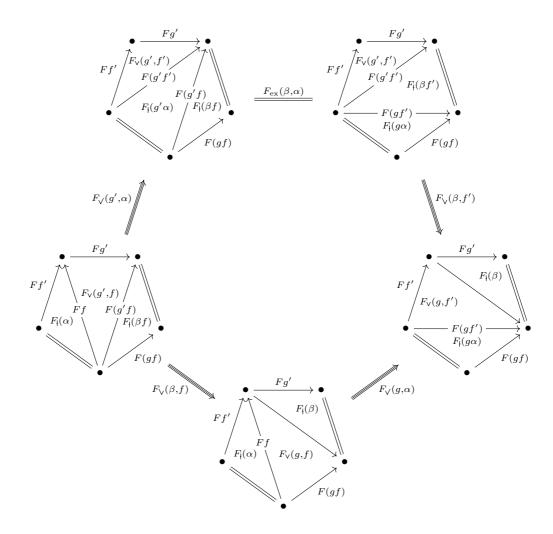


Figure 19. Establishing the coherence $\beta\alpha$.

of 1-composable cells α and γ of A. The 4-simplex of $N_{\infty}(B)$ depicted in figure 21 shows that the 3-cells

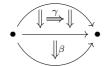
$$F(\gamma) *_1 F(\alpha)$$

and

$$F(\gamma *_1 \alpha)$$

of ${\cal B}$ are equal, which establishes the coherence for this tree.

: For any pair



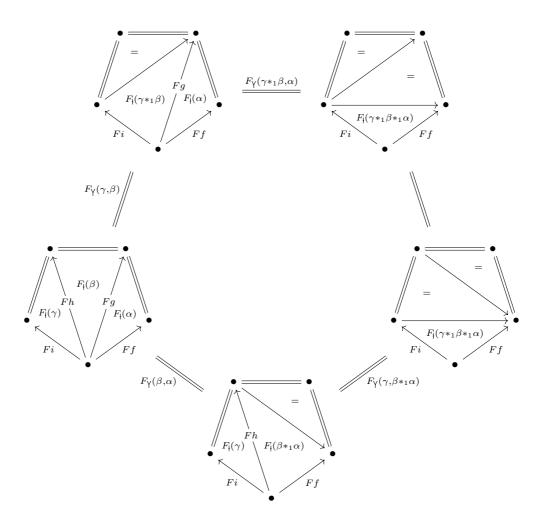


Figure 20. Representing the trivial coherence $\gamma \beta \alpha$

of 1-composable cells γ and β of A, there is a 4-simplex of $N_{\infty}(B)$ dual to the one depicted in 21 showing the equality between the 3-cells

$$F(\beta) *_1 F(\gamma)$$

and

$$F_{\sharp}(\beta *_1 \gamma)$$

of B and thus establishing the coherence for the tree \checkmark .

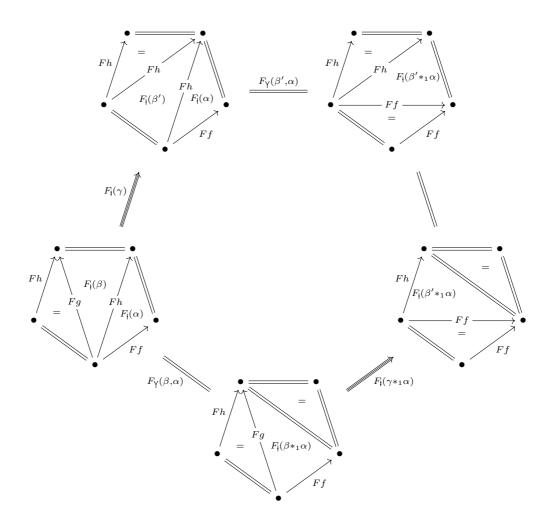
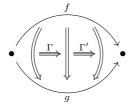


Figure 21. Establishing the coherence $\gamma \alpha$.

: Consider a pair



of 2-composable 3-cells $\Gamma\colon \alpha\to\beta$ and $\Gamma'\colon\beta\to\delta$ of A. The 4-simplex of $N_\infty(B)$ displayed in figure 22 shows that we have the equality

$$F_{\sharp}(\delta *_2 \gamma) = F_{\sharp}(\delta) *_2 F_{\sharp}(\gamma)$$

between these two 3-cells of B.

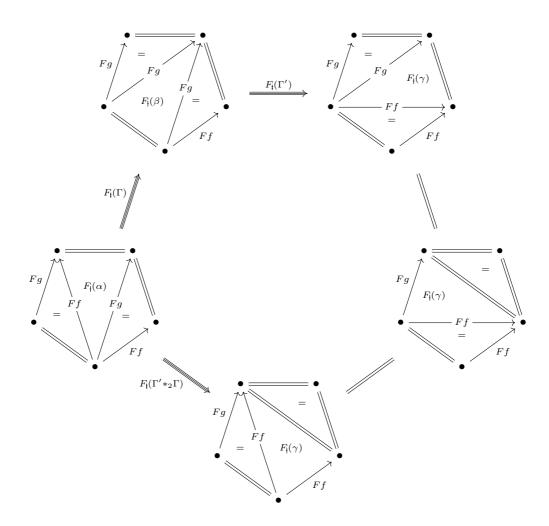
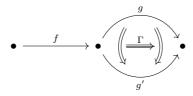


Figure 22. Establishing the coherence $\Gamma'\Gamma$.

∨ : Consider a pair



of 0-composable cells f and $\Gamma: \alpha \to \beta$ of A. The 4-simplex of $N_{\infty}(B)$ depicted in figure 23 shows that the equality

$$F_{\bigvee}(\beta,f) *_1 F_{\sharp}(\Gamma *_0 f) = \Delta *_0 F_{!}(f) *_1 F_{\bigvee}(\alpha,f) \,.$$

The 3-cell Δ is in fact equal to $F_{!}(\Gamma)$, as the 4-simplex of $N_{\infty}(B)$ depicted in figure 24 shows. Hence, the coherence for this tree is verified.

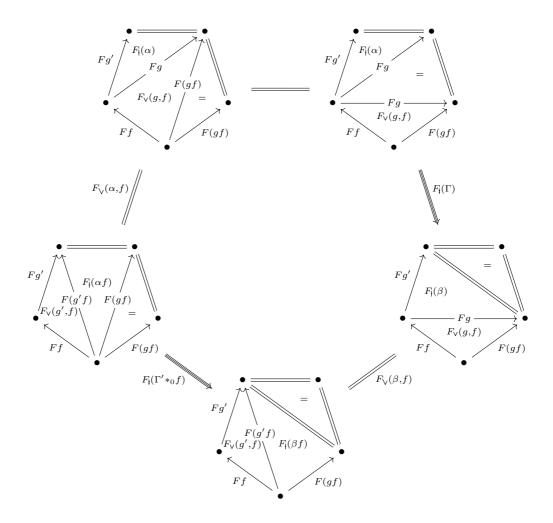
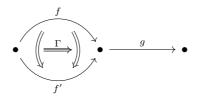


Figure 23. Establishing the coherence Γf .

: Consider a pair



of 0-composable cells Γ : $\alpha \to \beta$ and g of A. There is a 4-simplex of $N_{\infty}(B)$ dual to the one depicted in figure 23 showing that the following equality

$$F_{\bigvee}(g,\beta) *_1 F_{\sharp}(g *_0 \Gamma) = F_{\mathrm{I}}(g) *_0 F_{\sharp}(\Gamma) *_1 F_{\bigvee}(g,\alpha)$$

of 3-cells of B holds and thus establishing the coherence \bigvee .

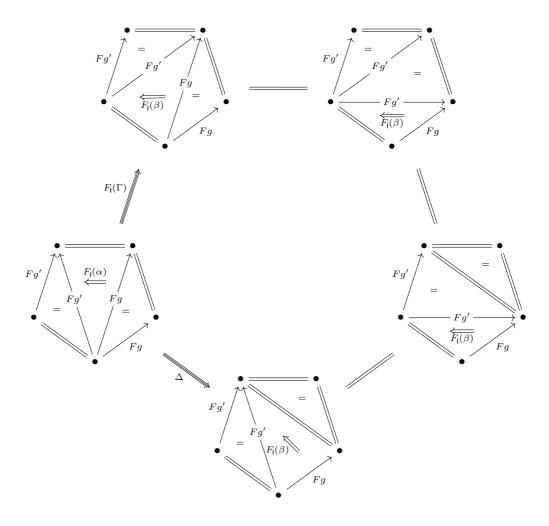


Figure 24. the 3-cell Δ of B.

4. Correspondence

The results of the preceding sections give us an application that associates to a simplicial oplax 3-morphism F a normalised oplax 3-functor, that we shall denote by $c_l(F)$ as well as an application in the opposite direction assigning to each normalised oplax 3-functor G a simplicial oplax 3-morphism $N_l(G)$. These two applications are actually inverses of one another, so that we have a precise bijective correspondence between oplax 3-functors and simplicial oplax 3-morphisms. The aim of this section is to check this statement.

4.0.1. It results immediately from the definitions and from conditions (a), (b) and (c) that given any normalised oplax 3-functor $F: A \to B$, we have an equality $F = c_l N_l(F)$. For instance, for the tree \bigvee we have that to any triple (h, g, f) of composable 1-cells of A, the simplicial oplax 3-morphism $N_l(F): N_\infty(A) \to N_\infty(B)$ associates the 3-simplex

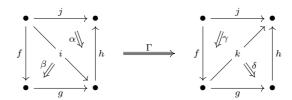
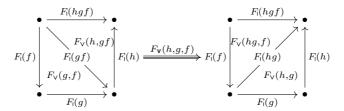


Figure 25. The 3-simplex x of $N_{\infty}(A)$.



of $N_{\infty}(A)$; by definition, we set $c_l N_l(F)_{\forall}(h, g, f)$ to be the main 3-cell of this 3-simplex, i.e., $F_{\forall}(h, g, f)$.

4.0.2. Let $F: N_{\infty}(A) \to N_{\infty}(B)$ be a simplicial oplax 3-morphism. We have seen in subsection 3.1 that there is a canonically associated normalised oplax 3-functor $c_l(F): A \to B$. Moreover, in subsection 1.3 we have shown that given any n-simplex of $N_{\infty}(A)$ in the form of a normalised oplax 3-functor $x: \Delta^n \to A$, for $n \ge 0$, we get an n-simplex $N_l c_l F(x)$ of $N_{\infty}(B)$ given by the composition $F \circ x$. We want to check that $F(x) = N_l c_l F(x)$. Since $N_{\infty}(B)$ is 4-coskeletal (see [24, Theorem 5.2]), it is enough to check that $F(x) = N_l c_l F(x)$ for all n-simplices x of $N_{\infty}(A)$ with $0 \le n \le 4$.

The result is trivially verified for 0-simplices and 1-simplices. Consider a 2-simplex \boldsymbol{x}

$$\begin{array}{c}
\bullet \\
f \\
\uparrow \alpha \\
\downarrow g \\
gf
\end{array}$$

of $N_{\infty}(A)$. The 1-skeletons of F(x) and $N_l c_l F(x)$ coincide and the 2-cell of B filling the 2-simplex $N_l c_l F(x)$ is defined by $c_l F_{\vee}(g,f) *_1 c_l F_{||}(\alpha)$. Condition (b) states precisely that this is the 2-cell of B filling the 2-simplex F(x).

Consider a 3-simplex x of $N_{\infty}(A)$ as depicted in figure 25. By the definition given in paragraph 1.3.1, the main 3-cell of the 3-simplex $N_l c_l F(x)$ of $N_{\infty}(B)$ is defined as

$$c_{l}F_{\vee}(h,g) *_{0} c_{l}F(f) *_{1} c_{l}F_{\vee}(\delta,f) *_{1} c_{l}F_{\dagger}(\gamma)$$

$$*_{2}$$

$$c_{l}F_{\vee}(h,g,f) *_{1} c_{l}F_{\dagger}(\Gamma)$$

$$*_{2}$$

$$c_{l}F(h) *_{0}F_{\vee}(g,f) *_{1} c_{l}F_{\vee}(h,\beta) *_{1} c_{l}F_{\dagger}(\alpha).$$

We already know that the 2-skeleton of F(x) and $N_l c_l F(x)$ coincide, so we are left with showing that the main 3-cell Ψ of the 3-simplex F(x) of $N_{\infty}(B)$ corresponds to the above 3-cell. This demands a careful analysis of some 4-simplices encoding valuable information for an explicit description of F(x), which will be carried over in the next paragraph. We end this paragraph by remarking that once the equivalence of translation of 3-simplices is verified, then the correspondence for 4-simplices

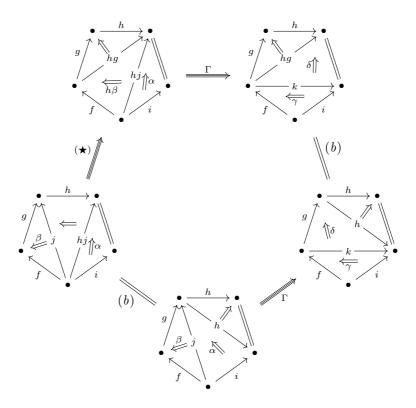
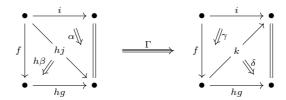


Figure 26. The 4-simplex y of $N_{\infty}(A)$

follows easily, since it is completely characterised by the 2-composition of 3-cells in B.

4.0.3. Consider a 3-simplex x of $N_{\infty}(A)$ as depicted in figure 25 and call Ψ the main 3-cell of the 3-simplex F(x) of $N_{\infty}(B)$. The 4-simplex of $N_{\infty}(A)$ depicted in figure 26 shows that Ψ is given by the 2-composition of the image of the 3-cell denoted by (\bigstar) , whiskered with $c_lF(\alpha)$, followed by the image of the main 3-cell of the following 3-simplex

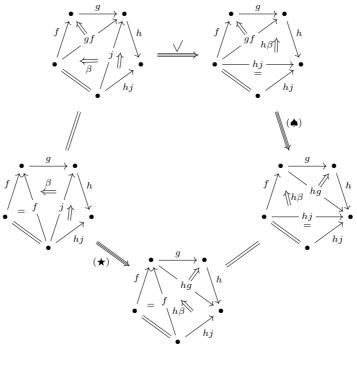


of $N_{\infty}(A)$, which is a 3-cell of $N_{\infty}(B)$ that we shall call Φ , whiskered by $c_l F_{\vee}(h,g)$. The 4-simplices depicted in figure 27a show that the image under F of the 3-cell (\bigstar) is given by

$$c_l F_{\downarrow}(\spadesuit) *_2 c_l F_{\lor}(g, f) *_1 c_l F_{\lor/}(h, \beta)$$

and in turn the 4-simplex of $N_{\infty}(A)$ depicted in figure 27b entails that

$$c_l F_{\downarrow}(\spadesuit) = c_l F_{\lor}(h, g, f) *_1 c_l F_{\downarrow}(h\beta)$$
.



(a) The 3-arrow (★).

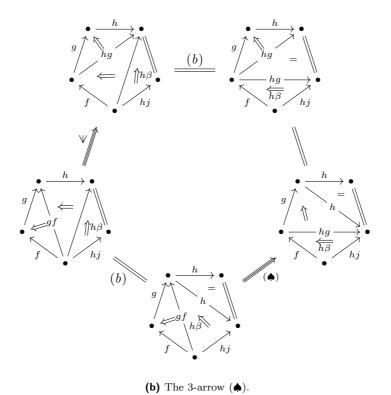


Figure 27. The 3-cells (\bigstar) and (\spadesuit) .

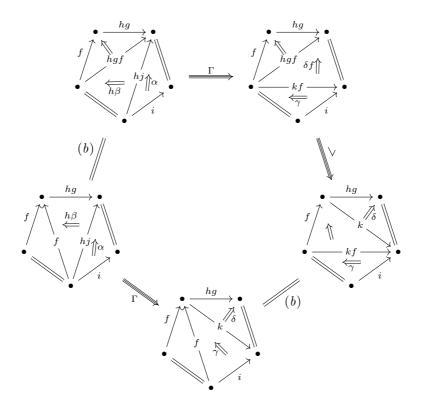


Figure 28. The 4-simplex z of $N_{\infty}(A)$.

Therefore we get that Ψ is given by the composition

$$c_{l}F_{\vee}(h,g) *_{0} c_{l}F_{\mathsf{i}}(f) *_{1} \Phi$$

$$*_{2}$$

$$c_{l}F_{\vee}(h,g,f) *_{1} c_{l}F_{\mathsf{i}}(h *_{0} \beta) *_{1} c_{l}F_{\mathsf{i}}(\alpha)$$

$$*_{2}$$

$$c_{l}F(h) *_{0} F_{\vee}(g,f) *_{1} c_{l}F_{\vee}(h,\beta) *_{1} c_{l}F_{\mathsf{i}}(\alpha).$$

Finally, the 4-simplex z of $N_{\infty}(A)$ depicted in figure 28, joint with the 4-simplex z' of $N_{\infty}(A)$ depicted in figure 29, where we have denoted by $\beta + \alpha$ and $\delta + \gamma$ the evident whiskered compositions, and another one totally similar but dual show that the 3-cell Φ of B is in fact

$$\begin{split} c_{l}F_{\vee}(h,g) *_{0} c_{l}F_{|}(f) *_{1} c_{l}F_{\vee}(\delta,f) *_{1} c_{l}F_{|}(\gamma) \\ *_{2} \\ c_{l}F_{\vee}(h,g) *_{0} c_{l}F_{|}(f) *_{1} c_{l}F_{\vee}(h *_{0} g,f) *_{1} c_{l}F_{|}(\Gamma) \,. \end{split}$$

Using the interchange law we immediately deduce from the above that the 3-cell Ψ of B is precisely the 3-cell detailed in (4.0.2).

4.0.4. Given two oplax 3-functors $F: A \to B$ and $G: B \to C$, we now check that the "obvious" candidate for the composite oplax 3-functor $G \circ F$ corresponds via the bijections established above to the composite of the simplicial oplax 3-morphisms associated to F and G. That is, we shall show that $G \circ F = c_l(N_l G \circ N_l F)$. Hence, we will deduce that oplax 3-functors admit a composition operation and that the

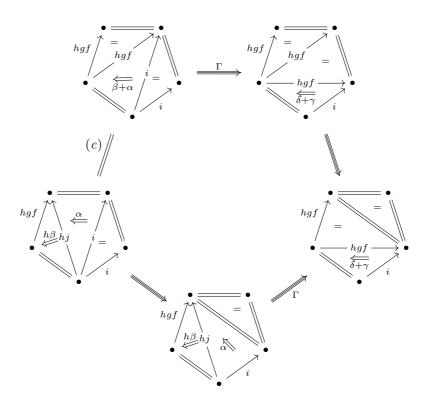


Figure 29. The 4-simplex z' of $N_{\infty}(A)$.

category of 3-categories and oplax 3-functors and 3-categories and simplicial oplax 3-morphisms are isomorphic.

We already know from section 1.3 that:

.: for any object a of A, we have $GF_{\bullet}(a) = G_{\bullet}(F_{\bullet}(a))$;

: for any 1-cell $f: a \to a'$ of A, we have $GF_i(f) = G_i(F_i(f))$;

 \vee : for any pair of 0-composable 1-cells

$$a \xrightarrow{f} a' \xrightarrow{g} a''$$
 of A , we have $GF_{\vee}(g,f) = G_{\vee}\big(F_{\!\!\!\mid}(g),F_{\!\!\!\mid}(f)\big) *_1 G_{\!\!\mid}\big(F_{\!\!\!\mid}(g,f)\big);$
: for any 2-cell $\alpha\colon f\to g$ of A , we have $GF_{\!\!\mid}(\alpha)=G_{\!\!\mid}\big(F_{\!\!\mid}(\alpha)\big);$
 \checkmark : for any triple of 0-composable 1-cells

of
$$A$$
, we have that $GF_{\forall}(h,g,f)$ is the 3-cell
$$G_{\vee}(F_{\mid}h,F_{\mid}g)*_{0}GF_{\mid}(F_{\mid}f)*_{1}G_{\vee}(F_{\vee}(h,g),F_{\mid}f)$$

$$*_{2}$$

$$G_{\forall}(F_{\mid}h,F_{\mid}g,F_{\mid}f)*_{1}G_{\dagger}(F_{\forall}(h,g,f))$$

$$*_{2}$$

 $a \xrightarrow{f} a' \xrightarrow{g} a'' \xrightarrow{h} a'''$

$$GF_{\mathsf{I}}(h) *_{0} G_{\vee}(F_{\mathsf{I}}g,F_{\mathsf{I}}f) *_{1} G_{\vee}(F_{\mathsf{I}}h,F_{\vee}(g,f)) *_{1} G_{\mathsf{I}}(F_{\vee}(h,gf)) \,.$$

As for the remaining trees, we have:

: for any whiskering

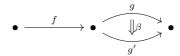
$$\bullet \xrightarrow{f'} \bullet \xrightarrow{g} \bullet$$

of A, we define $GF_{\checkmark}(g,\alpha)$ to be the 3-cell

$$\begin{split} G_{\vee}\big(F_{\mathsf{I}}(g),F_{\mathsf{I}}(f')\big) *_{1} G_{\dagger}\big(F_{\vee}(g,\alpha)\big) \\ *_{2} \\ G_{\vee}\big(F_{\mathsf{I}}(g),F_{\dagger}(\alpha)\big) *_{1} G_{\dagger}\big(F_{\vee}(g,f)\big) \end{split}$$

of C.

: for any whiskering



of A, the 3-cell $GF_{\backslash \prime}(\beta, f)$ is defined to be

$$\begin{split} G_{\bigvee} & \left(F_{\mathbf{i}}(\beta), F_{\mathbf{i}}(f) \right) *_{1} G_{\mathbf{i}} \left(F_{\bigvee}(g, f) \right) \\ & *_{2} \\ G_{\bigvee} & \left(F_{\mathbf{i}}(g'), F_{\mathbf{i}}(f) \right) *_{1} G_{\mathbf{i}} \left(F_{\bigvee}(\beta, f) \right) . \end{split}$$

: for any 3-cell $\gamma: \alpha \to \alpha'$ of A, we set

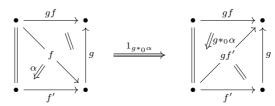
$$GF_{\mathfrak{t}}(\gamma) = G_{\mathfrak{t}}(F_{\mathfrak{t}}(\gamma))$$
.

4.0.5. We have to show that the definition we have given for the composition for the trees \vee , \vee and \dagger agree with the data encoded by the composition of the associated simplicial morphisms.

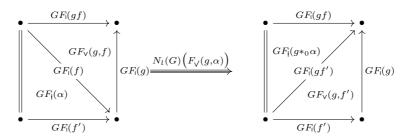
v: given a whiskering

$$\bullet \xrightarrow{f'} \bullet \xrightarrow{g} \bullet$$

of A, consider the 3-simplex x



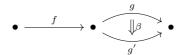
of $N_{\infty}(A)$. This is sent by $N_l(G)N_l(F)$ to the 3-simplex



of $N_{\infty}(C)$, where we denoted by $N_l(G)(F_{\vee}(g,\alpha))$ the image under $N_l(G)$ of the main 3-cell $F_{\vee}(g,\alpha)$ of the 3-simplex $N_l(F)(x)$. By paragraph 4.0.3, we find that the 3-cell $N_{\infty}(G)(F_{\vee}(g,\alpha))$ of C is precisely

$$\begin{split} G_{\vee}\big(\mathit{F}_{\mathsf{I}}(g),\mathit{F}_{\mathsf{I}}(f')\big) *_{1} G_{\dagger}\big(\mathit{F}_{\vee}(g,\alpha)\big) \\ *_{2} \\ G_{\vee}\big(\mathit{F}_{\mathsf{I}}(g),\mathit{F}_{\mathsf{I}}(\alpha)\big) *_{1} G_{\dagger}\big(\mathit{F}_{\vee}(g,f)\big) \,; \end{split}$$

: given a whiskering



of A, an argument dual with respect to the previous point gives us that this is indeed $N_{\infty}(G)(F_{\vee}(\beta, f))$.

: they trivially agree by definition.

Summing up the results of this chapter we get the following theorem.

Theorem 4.0.6. The class of small 3-categories and normalised oplax 3-functors are organised in a category $\widetilde{3\text{-}Cat}$, which is isomorphic via the functor N_l to the category $3\text{-}Cat_{\Delta}$ of 3-categories and simplicial oplax 3-morphisms.

5. STRICTIFICATION

In this section we are going to explicitly describe the ∞ -category $c_{\infty}(A)$, where A is a 1-category without split-monos and split-epis.

5.0.1. We say that a 1-category A is *split-free* if it does not have any split-monos or split-epis.

Example 5.0.2. Any poset is a split-free category. Moreover, for any category A, the category $c\operatorname{Sd}N(A)$ is split-free.

We fix a side-free category A.

5.0.3. We now define a reflexive ∞ -graph \tilde{A} associated to the 1-category A. The objects of \tilde{A} are precisely the objects of A. For any pair of objects (a,a') of A, we then define a reflexive ∞ -graph $\tilde{A}(a,a')$ whose objects, *i.e.*, the 1-cells of \tilde{A} having a as source and a' as target, are given by the set of non-degenerate simplices $x \colon \Delta^n \to A$ of $N_1(A)$ such that $x_0 = a$ and $x_n = a'$, for $n \ge 0$; that is to say, the objects of $\tilde{A}(a,a')$ are the tuples (f_1,\ldots,f_n) of composable non-trivial arrows of A such that $s(f_1) = a$ and $t(f_n) = a'$, with $n \ge 0$. The 0-tuple, where necessarily a = a', corresponds to the non-degenerate simplex $\Delta^0 \to A$ pointing at a, and it is by definition the identity 1-cell of the object a of \tilde{A} .

Consider two objects x and y of $\tilde{A}(a,a')$, *i.e.*, two tuples $x=(f_1,\ldots,f_m)$ and $y=(g_1,\ldots,g_n)$ as described above. We define $\tilde{A}(x,y)$ as follows:

m = 0: if $x: \Delta^0 \to A$ is a 0-simplex of A, we set $\tilde{A}(x,y)$ to be the final ∞ -category \mathcal{O}_0 ;

m=1: we define

$$\tilde{A}((f),(g_1,\ldots,g_p)),$$

where $s(f) = s(g_1)$ and $t(f) = t(g_p)$, to be the ∞ -category

$$\tilde{A}((f),(g_1,\ldots,g_p)) = \mathcal{O}_{\omega}(\langle 0,p\rangle,\langle 0,1\rangle + \langle 1,2\rangle + \cdots + \langle p-1,p\rangle).$$

m>1: otherwise, we set $\tilde{A}(x,y)$ to be the ∞ -category

$$\prod \tilde{A}((f_1),(g_1,\ldots,g_{\varphi(1)})) \times \cdots \times \tilde{A}((f_m),(g_{\varphi(m-1)+1},\ldots,g_n)),$$

where the sum runs over all the arrows $\varphi \colon \Delta^m \to \Delta^n$ of Δ which are:

- (a) strictly increasing;
- (b) we have $\varphi(0) = 0$ and $\varphi(m) = n$;
- (c) such that, for all $1 < i \le m$, we have

$$g_{\varphi(i)} *_0 \dots *_0 g_{\varphi(i-1)+1} = f_i$$

in the category A.

These conditions ensure that $s(f_i) = s(g_{\varphi(i-1)+1})$ and $t(f_i) = t(g_{\varphi(i)})$, for all $1 \leq i \leq m$, so that in particular we have $x_i = y_{\varphi(i)}$ for every $0 \leq i \leq m$; notice that the condition imposing that φ is an active morphism, i.e., $\varphi(0) = 0$ and $\varphi(m) = n$ is actually implied by the others. We shall sometimes write the above sum as

$$\coprod_{\varphi} \tilde{A}_{\varphi}(x,y) .$$

Using the canonical isomorphism of ∞ -categories described in Proposition A.4 of [2], we shall often identify the ∞ -category $\tilde{A}_{\varphi}(x,y)$ with

$$\mathcal{O}_{\omega}(\langle 0, \varphi(1) \rangle + \cdots + \langle \varphi(m-1), \varphi(m) \rangle, \langle 0, 1 \rangle + \cdots + \langle n-1, n \rangle)$$
.

Note that if the index of the sum above is empty, that is there is no arrow $\varphi \colon \Delta^m \to \Delta^n$ satisfying conditions (a) and (c), then $\tilde{A}(x,y)$ is set to be the empty ∞ -category. This happens in particular every time m > n. Observe also that condition (c) entails that if there is a cell between (f_1, \ldots, f_m) and (g_1, \ldots, g_n) , then necessarily

$$f_m *_0 \ldots *_0 f_1 = g_n *_0 \ldots *_0 g_1.$$

For any 1-cell $x = (f_1, \ldots, f_n)$ of \tilde{A} , the identity of x is given by the only trivial 2-cell of the ∞ -category

$$\tilde{A}(f_1, f_1) \times \cdots \times \tilde{A}(f_n, f_n)$$
.

Indeed, observe that for any arrow f of A, the only morphism satisfying (a) and (c) is $phi = 1_{\Delta^n}$, so that $\tilde{A}(f, f)$ is isomorphic to $\mathcal{O}_{\omega}(\langle 0, 1 \rangle, \langle 0, 1 \rangle)$, which is the terminal ∞ -category.

Remark 5.0.4. Without the hypothesis on the category A, that is in the general situation in which we have split-monos and split-epis, the definition of the hom- ∞ -category $\tilde{A}(f), (g_1, \ldots, g_n)$ is more complicated. This is due to the fact that, although the simplex (g_1, \ldots, g_n) is non-degenerate, there could be two consecutive arrows, say g_i and g_{i+1} which compose to the identity. When introducing the operations on the ∞ -graph \tilde{A} , this becomes a serious issue.

5.0.5. In this paragraph we want to endow the reflexive ∞ -graph $\tilde{A}(a,a')$ with the structure of an ∞ -category, for any pair (a, a') of objects of A. In order to do so, for any $x = (f_1, \ldots, f_\ell)$, $y = (g_1, \ldots, g_m)$ and $z = (h_1, \ldots, h_n)$ of $\tilde{A}(a, a')$, we want to define an ∞ -functor

$$\tilde{A}(y,z) \times \tilde{A}(x,y) \to \tilde{A}(x,z)$$
.

Without any loss of generality, we can suppose $\ell \leq m \leq n$ (see the preceding paragraph) and consider the case $\ell > 0$, since the other cases are trivial. Let us fix two morphisms $\varphi \colon \Delta^{\ell} \to \Delta^m$ and $\psi \colon \Delta^m \to \Delta^n$ satisfying conditions (a) and (c) of the previous paragraph. We set

$$\Phi(i) = \varphi(i) - \varphi(i-1)$$
 and $\Psi(j) = \psi(j) - \psi(j-1)$

for any $1 \le i \le \ell$ and $1 \le j \le m$. We have to give an ∞ -functor which has

$$\tilde{A}((g_1),(h_1,\ldots,h_{\psi(1)})\times\cdots\times\tilde{A}((g_m),(h_{\psi(m-1)+1},\ldots,h_n))$$

$$(5.0.5.1)$$
 ×

$$\tilde{A}((f_1),(g_1,\ldots,g_{\varphi(1)})\times\cdots\times\tilde{A}((f_\ell),(g_{\varphi(\ell-1)+1},\ldots,g_m))$$

as source, which by definition is the ∞ -category

$$\prod_{i=1}^{m} \mathcal{O}_{\omega} (\langle 0, \Psi(i) \rangle, \langle 0, 1 \rangle + \dots + \langle \Psi(i) - 1, \Psi(i) \rangle) \times$$

$$\prod_{i=1}^{\ell} \mathcal{O}_{\omega} (\langle 0, \Phi(i) \rangle, \langle 0, 1 \rangle + \dots + \langle \Phi(i) - 1, \Phi(i) \rangle).$$

Notice that for every $1 \leqslant p \leqslant \ell$ we have

$$f_p = g_{\varphi(p)} *_0 g_{\varphi(p)-1} *_0 \dots *_0 g_{\varphi(p-1)+1},$$

and for every $1 \leq q \leq m$ we have

$$g_q = h_{\psi(q)} *_0 h_{\psi(q)-1} *_0 \dots *_0 h_{\psi(q-1)+1}$$

so that in fact

$$f_p = h_{\psi(\varphi(p))} *_0 h_{\psi(\varphi(p))-1} *_0 \dots *_0 h_{\psi(\varphi(p)-1)+1}$$

$$*_0 h_{\psi(\varphi(p)-1)} *_0 h_{\psi(\varphi(p)-1)-1} *_0 \dots *_0 h_{\psi(\varphi(p)-2)+1}$$

$$*_0 \dots$$

$$*_0 h_{\psi(\varphi(p-1)+1)} *_0 h_{\psi(\varphi(p-1)+1)-1} *_0 \dots *_0 h_{\psi(\varphi(p-1))+1},$$

for every $1 \leq p \leq \ell$. Now, for every $1 \leq i \leq m$, we have that the ∞ -category

$$\mathcal{O}_{\omega}(\langle 0, \Psi(i) \rangle, \langle 0, 1 \rangle + \cdots + \langle \Psi(i) - 1, \Psi(i) \rangle)$$

is canonically isomorphic by Corollary D.0.10 to the ∞ -category

$$\mathcal{O}_{\omega}(\langle \psi(i-1), \psi(i) \rangle, \langle \psi(i-1), \psi(i-1) + 1 \rangle + \cdots + \langle \psi(i) - 1, \psi(i) \rangle)$$
.

In order to simplify the notations, let us set

$$b_i = \langle \psi(i-1), \psi(i) \rangle$$
 , $c_i = \sum_{k=0}^{\Psi(i)-1} \langle \psi(i-1) + k, \psi(i-1) + k + 1 \rangle$,

for $1 \leq i \leq m$, and also

$$b = b_1 + b_2 + \dots + b_m$$
 and $c = c_1 + c_2 + \dots + c_m$

There is a canonical ∞ -functor

$$\prod_{i=1}^{m} \mathcal{O}_{\omega}(b_i, c_i) \to \mathcal{O}_{\omega}(b, c)$$

given by "horizontal composition" $*_0$, *i.e.*, mapping a tuple (x_1, \ldots, x_m) of p-cells to the p-cell $x_1 *_0 x_2 *_0 \ldots *_0 x_m$ of $\mathcal{O}_{\omega}(b, c)$. Proposition D.0.7 actually shows that this ∞ -functor is an isomorphism of ∞ -categories. The same argument entails that the ∞ -category

$$\prod_{i=1}^{\ell} \mathcal{O}_{\omega} (\langle 0, \varphi(i) \rangle, \langle 0, 1 \rangle + \dots + \langle \varphi(i) - 1, \varphi(i) \rangle)$$

is canonically isomorphic to $\mathcal{O}_{\omega}(a',b')$ via the "horizontal composition" $*_0$, where we have set

$$a' = \sum_{i=1}^{\ell} \langle \varphi(i-1), \varphi(i) \rangle$$
 and $b' = \sum_{i=1}^{m} \langle i-1, i \rangle$.

Applying the increasing morphism ψ and setting

$$a = \sum_{i=1}^{\ell} \langle \psi \varphi(i-1), \psi \varphi(i) \rangle$$

we get, again by Corollary D.0.10, a canonical isomorphism of ∞ -categories $\mathcal{O}_{\omega}(a',b') \cong \mathcal{O}_{\omega}(a,b)$. The ∞ -category in (5.0.5) is thus canonically isomorphic to the ∞ -category

$$\mathcal{O}_{\omega}(b,c) \times \mathcal{O}_{\omega}(a,b)$$
.

On the other hand, the target ∞ -category of the ∞ -functor we are set to construct is

$$\tilde{A}(x,z) = \prod_{i=1}^{\ell} \tilde{A}((f_i), (h_{\psi\varphi(i-1)+1}, \dots, h_{\psi\varphi(i)}),$$

which is by definition

$$\prod_{i=1}^{\ell} \mathcal{O}_{\omega} (\langle 0, \psi \varphi(i) \rangle, \langle 0, 1 \rangle + \dots + \langle \psi \varphi(i) - 1, \psi \varphi(i) \rangle).$$

The same argument used above gives us that this ∞ -category is canonically isomorphic to the ∞ -category

$$\mathcal{O}_{\omega}(a,c)$$
.

We then define the ∞ -functor

$$\mathcal{O}_{\omega}(b,c) \times \mathcal{O}_{\omega}(a,b) \to \mathcal{O}_{\omega}(a,c)$$

to be the "vertical composition" $*_1$, *i.e.*, a pair of *p*-cells (x,y) of the source is mapped to the *p*-cell $x *_1 y$ of $\mathcal{O}_{\omega}(a,c)$.

Alternatively, for any $1 \leq i \leq \ell$ we can consider the ∞ -category

$$\tilde{A}((f_i),(g_{\varphi(i-1)+1},\ldots,g_{\varphi(i)-1},g_{\varphi(i)})),$$

which is defined as

$$\mathcal{O}_{\omega}(\langle 0, \Phi(i) \rangle, \langle 0, 1 \rangle + \cdots + \langle \Phi(i) - 1, \Phi(i) \rangle)$$
.

The latter is canonical isomorphic by Corollary D.0.10 to

$$\mathcal{O}_{\omega}(\langle \varphi(i-1), \varphi(i) \rangle, \langle \varphi(i-1), \varphi(i-1) + 1 \rangle + \cdots + \langle \varphi(i) - 1, \varphi(i) \rangle),$$

which in turn is isomorphic to the ∞ -category

$$\mathcal{O}_{\omega}(\langle \psi\varphi(i-1), \psi\varphi(i) \rangle, \langle \psi(\varphi(i-1)), \psi(\varphi(i-1)+1) \rangle + \cdots + \langle \psi(\varphi(i)-1), \psi(\varphi(i)) \rangle)$$
.

If we set $a_i = \langle \psi \varphi(i-1), \psi \varphi(i) \rangle$ for every $1 \leq i \leq \ell$, then we can write the above ∞ -category as

$$\mathcal{O}_{\omega}(a_i,b_{\omega(i-1)+1}+b_{\omega(i-1)+2}\cdots+b_{\omega(i)})$$
.

Similarly, for any $1 \leq i \leq m$ we have seen above that the ∞ -category

$$\tilde{A}((g_i),(h_{\psi(i-1)},\ldots,h_{\psi(i)}))$$

is canonically isomorphic to the ∞ -category

$$\mathcal{O}_{\omega}(\langle \psi(i-1), \psi(i) \rangle, \langle \psi(i-1), \psi(i-1) + 1 \rangle + \dots + \langle \psi(i) - 1, \psi(i) \rangle),$$

which we can denote by $\mathcal{O}_{\omega}(b_i, c_i)$. Therefore, for a fixed $1 \leq p \leq \ell$ we have that the ∞ -category

$$\prod_{k=\varphi(p-1)+1}^{\varphi(p)} \tilde{A}((g_k), (h_{\psi(k-1)+1}, \dots, h_{\psi(k)}))$$

$$\tilde{A}((f_p),(g_{\varphi(p-1)+1},\ldots,g_{\varphi(p)}))$$

is canonically isomorphic to the ∞ -category

$$\prod_{k=\varphi(p-1)+1}^{\varphi(p)} \mathcal{O}_{\omega}(b_k, c_k) \times \mathcal{O}_{\omega}(a_p, b_{\varphi(p-1)+1} + \dots + b_{\varphi(p)}).$$

Using Proposition D.0.7, we obtain

$$\prod_{k=\varphi(p-1)+1}^{\varphi(p)} \mathcal{O}_{\omega}(b_k, c_k) \cong \mathcal{O}_{\omega}(b_{\varphi(p-1)+1} + \dots + b_{\varphi(p)}, c_{\varphi(p-1)+1} + \dots + c_{\varphi(p)})$$

and hence the former ∞-category is canonically isomorphic to

$$\mathcal{O}_{\omega}(b_{\varphi(p-1)+1} + \dots + b_{\varphi(p)}, c_{\varphi(p-1)+1} + \dots + c_{\varphi(p)}) \times \\ \mathcal{O}_{\omega}(a_p, b_{\varphi(p-1)+1} + \dots + b_{\varphi(p)}).$$

We set

$$B_p = b_{\varphi(p-1)+1} + \dots + b_{\varphi(p)}$$
 and $C_p = c_{\varphi(p-1)+1} + \dots + c_{\varphi(p)}$

Applying the "vertical composition" $*_1$ to this product of ∞ -categories we get an ∞ -functor

$$\mathcal{O}_{\omega}(B_p, C_p) \times \mathcal{O}_{\omega}(a_p, B_p) \to \mathcal{O}_{\omega}(a_p, C_p)$$
.

Finally, the "horizontal composition" $*_0$ provides us with an ∞ -functor

$$\prod_{p=1}^{\ell} \mathcal{O}_{\omega}(a_p, C_p) \to \mathcal{O}_{\omega}(a, c) \,,$$

since $a = a_1 + \cdots + a_\ell$ and $c = C_1 + \cdots + C_\ell$.

These two approaches are equivalent by virtue of the exchange law between $*_0$ and $*_1$.

This endows the reflexive ∞ -graph $\tilde{A}(a,a')$ with the structure of an ∞ -category.

5.0.6. In this paragraph we put an ∞ -category structure on the reflexive ∞ -graph \tilde{A} . In order to do this, we shall define, for any objects a, a' and a'' of \tilde{A} , an ∞ -functor

$$\tilde{A}(a', a'') \times \tilde{A}(a, a') \to \tilde{A}(a, a'')$$
.

As ∞ -categories are categories enriched in ∞ -categories, an ∞ -functor F between two ∞ -categories C and D can be given by a map $F_0: C_0 \to D_0$ on objects and a family of ∞ -functors $C(c,c') \to D(Fc,Fc')$, indexed by the pairs of objects (c,c') of C, satisfying the axioms described in paragraph A.0.1.

In light of the above, we have to provide a map

$$\tilde{A}(a',a'')_0 \times \tilde{A}(a,a')_0 \to \tilde{A}(a,a'')_0$$

that we define by sending a pair (y, x) with $x: \Delta^m \to A$ and $y: \Delta^n \to A$ to the concatenation simplex

$$y \cdot x \colon \varDelta^{m+n} \to A \quad , \quad \langle i, i+1 \rangle \mapsto \begin{cases} x_{\{i, i+1\}} \,, & \text{if } i < m, \\ y_{\{i-m, i+1-m\}} \,, & \text{if } i \geqslant m. \end{cases}$$

Furthermore, for any choice of objects (y,x) and (t,z) of $\tilde{A}(a',a'')_0 \times \tilde{A}(a,a')_0$, we have to provide an ∞ -functor

$$\tilde{A}(y,t) \times \tilde{A}(x,z) \to \tilde{A}(y \cdot x, t \cdot z)$$
.

Notice that if either $\tilde{A}(x,z)$ or $\tilde{A}(y,t)$ are empty, then the same holds for $\tilde{A}(y\cdot x,t\cdot z)$. If x (resp. y) is a 0-simplex, then so is z (resp. t) and the ∞ -functor above is simply the identity on $\tilde{A}(y,t)$ (the identity on $\tilde{A}(x,z)$). We can therefore suppose that $\tilde{A}(x,z)$ and $\tilde{A}(y,t)$ are non-empty and that x and y are not trivial.

Following the reasoning of the previous paragraph, we know that there are integers

$$0 = i_0 < i_1 < \dots < i_m$$
 and $0 = j_0 < j_1 < \dots < j_n$

such that, if we set

$$a = \sum_{k=0}^{m-1} \langle i_k, i_{k+1} \rangle, \qquad c = \sum_{p=0}^{i_m-1} \langle p, p+1 \rangle,$$
$$b = \sum_{k=0}^{n-1} \langle j_k, j_{k+1} \rangle, \qquad d = \sum_{p=0}^{j_n-1} \langle p, p+1 \rangle,$$

then we have canonical isomorphisms

$$\tilde{A}(x,z) \cong \mathcal{O}_{\omega}(a,c)$$
 and $\tilde{A}(y,t) \cong \mathcal{O}_{\omega}(b,d)$

of ∞ -categories. Moreover, setting

$$b' = \sum_{k=0}^{n-1} \langle i_m + j_k, i_m + j_{k+1} \rangle$$
 and $d' = \sum_{p=0}^{j_n-1} \langle i_m + p, i_m + p + 1 \rangle$,

we have by Corollary D.0.10 a canonical isomorphism

$$\mathcal{O}_{\omega}(b,d) \cong \mathcal{O}_{\omega}(b',d')$$

and by the same argument we can build a further canonical isomorphism

$$\tilde{A}(y \cdot x, t \cdot z) \cong \mathcal{O}_{\omega}(a + b', c + d')$$

of ∞ -categories. We are thus left to provide an ∞ -functor

$$\mathcal{O}_{\omega}(b',d') \times \mathcal{O}_{\omega}(a,c) \to \mathcal{O}_{\omega}(a+b',c+d')$$
,

which we set to be the "horizontal composition" by $*_0$. Notice that by Proposition D.0.7, this ∞ -functor is in fact an isomorphism.

The identity axioms are trivial from the definition and the associativity follows immediately from the associativity of the "horizontal composition" $*_0$ as an operation of the ∞ -category \mathcal{O}_{ω} .

Lemma 5.0.7. Let a and a' be two objects of A and consider two elements

$$x = (f_1, \dots, f_m)$$
 and $y = (g_1, \dots, g_n)$

of $\tilde{A}(a,a')$. Then there is a zig-zag of 2-cells linking x to y if and only if

$$f_m *_0 \dots *_0 f_1 = g_n *_0 \dots *_0 g_1$$

in A.

Proof. This is trivially true if x, and then also y, is a trivial cell of A. So let us suppose m > 0 and n > 0.

On the one hand, condition (c) immediately implies that two 1-cells x and y as above are connected by a zig-zag of 2-cells only if

$$f_m *_0 \dots *_0 f_1 = g_n *_0 \dots *_0 g_1$$
.

On the other hand, let

$$h = f_m *_0 \dots *_0 f_1 = g_n *_0 \dots *_0 g_1$$

and consider the 1-cell z=(h) of \tilde{A} . It results immediately from the structure of the oriental \mathcal{O}_{ω} that the ∞ -categories

$$\tilde{A}((h),x) = \mathcal{O}_{\omega}(\langle 0,m\rangle,\langle 0,1\rangle + \dots + \langle m-1,m\rangle)$$

and

$$\tilde{A}((h), y) = \mathcal{O}_{\omega}(\langle 0, n \rangle, \langle 0, 1 \rangle + \dots + \langle n - 1, n \rangle)$$

are non-empty; hence x and y are connected by a zig-zag of length two.

Corollary 5.0.8. We have $\tau_{\leq 1}^{i}(\tilde{A}) \cong A$.

Proof. We have a canonical ∞ -functor $\varepsilon_A \colon \tilde{A} \to A$ which is the identity on objects and that maps a 1-cell $x = (f_1, \dots, f_n)$ to $f_n *_0 \dots *_0 f_1$ if n > 0 and a 0-simplex $a \colon \Delta^0 \to A$ to the identity of a in A. The identity is clearly preserved, the functoriality follows by the definition of 0-composition of 1-cells of \tilde{A} by concatenation and moreover the assignment is well-defined by the previous lemma. We are left with showing that for any pair of objects (a, a') of A, the map

$$\tau^{\mathbf{i}}_{\leq 0}(A)(a,a') \to A(a,a')$$

is a bijection. It is clearly surjective, since for any morphism $f: a \to a'$ of A we have $\varepsilon_A(f) = f$ (and similarly if f is an identity cell of A). It results from the previous lemma that this map is also injective, hence completing the proof of the corollary.

5.0.9. We now turn to constructing a normalised oplax 3-functor

$$\eta_A \colon A \to \tau^{\mathrm{i}}_{\leq 3}(\tilde{A})$$
.

- : The map $(\eta_A)_{\bullet}$ is defined to be the identity map on objects.
- : The map $(\eta_A)_{\mid}$ assigns to any non-trivial morphism $f: a \to a'$ of A the 1-cell $(f): \Delta^1 \to A$ of \tilde{A} and to any identity 1_a of A the trivial 1-cell $a: \Delta^0 \to A$ of \tilde{A}
- \vee : The map $(\eta_A)_{\vee}$ assigns to any pair of composable morphisms

$$a \xrightarrow{f} a' \xrightarrow{g} a''$$

of A the unique 2-cell $(\eta_A)_{\vee}(g, f)$ of \tilde{A} with source $(g*_0 f)$ and target (f, g), i.e., the unique element (0, 1, 2) of the set

$$\tilde{A}((g *_0 f), (g, f)) = \mathcal{O}_{\omega}(\langle 0, 2 \rangle, \langle 0, 1 \rangle + \langle 1, 2 \rangle).$$

 \forall : The map $(\eta_A)_{\forall}$, assigns to any triple of composable morphisms

$$a \xrightarrow{f} a' \xrightarrow{g} a'' \xrightarrow{h} a'''$$

of A the unique 3-cell $(\eta_A)_{\forall}(h,g,f)$ of \tilde{A} with 1-source $(h*_0g*_0f)$ and 1-target (h,g,f), i.e., the unique arrow (0,1,2,3) of the 1-category

$$\tilde{A}((h *_0 g *_0 f), (f, g, h))\mathcal{O}_{\omega}(\langle 0, 3 \rangle, \langle 0, 1 \rangle + \langle 1, 2 \rangle + \langle 2, 3 \rangle).$$

Notice that by definition we have that $1_{(\eta_A),(a)}$, that is the 1-cell $a: \Delta^0 \to A$, is precisely $(\eta_A)(1_a)$; the other conditions of normalisation are trivial. We are left with checking the coherence for the tree \checkmark .

Consider four composable morphisms of A

$$\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet \xrightarrow{h} \bullet \xrightarrow{i} \bullet .$$

We have to show that the 3-cells

$$\begin{split} (\eta_A)_{\forall}(i,h,g) *_0 &(\eta_A)_{\mathsf{I}}(f) *_1 (\eta_A)_{\vee}(i *_0 h *_0 g, f) \\ *_2 \\ (\eta_A)_{\mathsf{I}}(i) *_0 (\eta_A)_{\vee}(h,g) *_0 (\eta_A)_{\mathsf{I}}(f) *_1 (\eta_A)_{\forall}(i,h *_0 g, f) \\ *_2 \\ (\eta_A)_{\mathsf{I}}(i) *_0 (\eta_A)_{\forall}(h,g,f) *_1 (\eta_A)_{\vee}(i,h *_0 g *_0 f) \end{split}$$

and

$$(\eta_{A})_{\vee}(i,h) *_{0} (\eta_{A})_{\mathsf{I}}(g) *_{0} (\eta_{A})_{\mathsf{I}}(f) *_{1} (\eta_{A})_{\forall}(ih,g,f)$$

$$*_{2}$$

$$(\eta_{A})_{\mathsf{I}}(i) *_{0} (\eta_{A})_{\mathsf{I}}(h) *_{0} (\eta_{A})_{\vee}(g,f) *_{1} (\eta_{A})_{\forall}(i,h,gf)$$

of $\tau_{\leq 3}^{i}(A)$ are equal, which is equivalent to exhibiting a zig-zag of 4-cells connecting them. In fact, they are precisely the target and the source of the unique 2-cell of the 2-category

$$\tilde{A}((i *_0 h *_0 g *_0 f), (f, g, h, i)),$$

i.e., the cell $\langle 0, 1, 2, 3, 4 \rangle$ of the 2-category

$$\mathcal{O}_{\omega}(\langle 0, 4 \rangle, \langle 0, 1 \rangle + \langle 1, 2 \rangle + \langle 2, 3 \rangle + \langle 3, 4 \rangle).$$

- **5.0.10.** The construction of the preceding paragraph is in fact the cellular version of the truncation of a simplicial morphism $N_{\infty}(A) \to N_{\infty}(\tilde{A})$, which we shall still denote by η_A . We shall dedicate the rest of the chapter to define such a map and moreover show that it is the unit map of the adjoint pair (c_{∞}, N_{∞}) applied to the simplicial set $N_{\infty}(A)$, so that in particular $\tilde{A} \cong c_{\infty}N_{\infty}(A)$.
- **5.0.11.** For any object $\langle i \rangle$ of \mathcal{O}_m , with $0 \leq i \leq m$, we set $\tilde{y}(\langle i \rangle) = y(i)$. For any $0 < i \leq m$, we denote by f_i the arrow $y_{\{i-1,i\}}$ of A and consider the 1-cell

$$a = \langle i_0, i_1 \rangle + \langle i_1, i_2 \rangle + \cdots + \langle i_{k-1}, i_k \rangle$$

of \mathcal{O}_m , with $0 \leq i_0 < i_1 < \cdots < i_k \leq m$, that we can see as a strictly increasing morphism $a: \Delta^k \to \Delta^m$. To this 1-cell, it is canonically associated the k-simplex $z: \Delta^k \to A$ of $N_{\infty}(A)$ defined by

$$z_{\{p,p+1\}} = f_{i_{p+1}} *_0 \dots *_0 f_{i_p+1},$$

that is to say z = ya. This is a non-degenerate k-simplex of A and thus defines a 1-cell of \tilde{A} . Hence, we set $\tilde{y}(a) = z = ya$.

5.0.12. Consider two 1-cells a and b of \mathcal{O}_m that we can write as two non-degenerate, that is strictly increasing, simplices

$$a \colon \Delta^p \to \Delta^m$$
 and $b \colon \Delta^q \to \Delta^m$,

and suppose they are such that a(0) = b(0) and a(p) = b(q). More explicitly, the 1-cells a and b of \mathcal{O}_m correspond respectively to the 1-cells

$$a = \langle a_0, a_1 \rangle + \dots + \langle a_{p-1}, a_p \rangle$$

and

$$b = \langle b_0, b_1 \rangle + \dots + \langle b_{a-1}, b_a \rangle \rangle$$

such that $a_0 = b_0$ and $a_p = b_q$, where we have set $a_i = a(i)$, for $0 \le i \le p$ and $b_j = b(j)$ for $0 \le j \le q$. It results from Lemma 10.4 of [2] that there is a 2-cell from a to b if and only if there exists a strictly increasing morphism $\varphi \colon \Delta^p \to \Delta^q$ of Δ such that $a = b\varphi$. Notice that if such a morphism φ exists, than it is unique, as b is a monomorphism. We suppose that this is the case and we define an ∞ -functor

$$\tilde{x}_{a,b} \colon \mathcal{O}_m(a,b) \longrightarrow \tilde{A}(\tilde{y}(a), \tilde{y}(b))$$
.

The source of this ∞ -functor is the ∞ -category

$$\mathcal{O}_m(a,b) = \mathcal{O}_m(\langle a_0, a_1 \rangle + \dots + \langle a_{p-1}, a_p \rangle, \langle b_0, b_1 \rangle + \dots + \langle b_{q-1}, b_q \rangle)),$$

that by virtue of Proposition D.0.7 is canonically isomorphic to

$$\prod_{i=1}^{p} \mathcal{O}_{m}(\langle a_{i-1}, a_{i} \rangle, \langle b_{\varphi(i-1)+1}, b_{\varphi(i-1)+2} \rangle + \dots + \langle b_{\varphi(i)-1}, b_{\varphi(i)} \rangle),$$

while the target ∞ -category $\tilde{A}\big(\tilde{y}(a),\tilde{y}(b)\big) = \tilde{A}(ya,yb)$ is a sum of ∞ -categories $\tilde{A}_{\psi}(ya,yb)$ indexed on strictly increasing morphisms $\psi\colon \Delta^p\to\Delta^q$ of Δ such that $(ya)_{\{i-1,i\}}=(yb)_{\{\varphi(i-1),\dots,\varphi(i)\}}$, that is to say verifying $ya=yb\psi$. Any strictly increasing morphism $\varphi\colon \Delta^p\to\Delta^q$ such that $a=b\varphi$ trivially verifies $ya=yb\varphi$ and we observed above that there is at most one such morphism. Therefore, if such a morphism φ exists, than it appears as index in the sum of ∞ -categories defining $\tilde{A}(ya,yb)$ and we have

$$\tilde{A}_{\varphi}(ya, yb) \cong \mathcal{O}_{\omega}(\langle 0, \varphi(1) \rangle + \dots + \langle \varphi(p-1), \varphi(p) \rangle, \langle 0, 1 \rangle + \dots + \langle q-1, q \rangle).$$

Now, this ∞ -category is equal to the ∞ -category

$$\mathcal{O}_q(\langle 0, \varphi(1) \rangle, + \dots + \langle \varphi(p-1), \varphi(p) \rangle, \langle 0, 1 \rangle + \dots + \langle q-1, q \rangle)$$

and the injective morphism $b: \Delta^q \to \Delta^m$ induces by Corollary D.0.10 a canonical isomorphism between the latter ∞ -category and $\mathcal{O}_m(a,b)$. We set the ∞ -functor

$$\tilde{y}_{a,b} \colon \mathcal{O}_m(a,b) \to \tilde{A}(ya,yb)$$

to be the composition $\mathcal{O}_m(a,b) \to \tilde{A}_{\varphi}(ya,yb)$ of the isomorphisms we have just described followed by the embedding $\tilde{A}_{\varphi}(ya,yb) \to \tilde{A}(ya,yb)$.

We have to check that for any triple (a, b, c) of composable 1-cells of \mathcal{O}_m we have a commutative diagram

(5.0.12.1)
$$\mathcal{O}_{m}(b,c) \times \mathcal{O}_{m}(a,b) \xrightarrow{*_{1}} \mathcal{O}_{m}(a,c)$$

$$\tilde{y}_{b,c} \times \tilde{y}_{a,b} \downarrow \qquad \qquad \downarrow \tilde{y}_{a,c}$$

$$\tilde{A}(\tilde{y}(b), \tilde{y}(c)) \times \tilde{A}(\tilde{y}(a), \tilde{y}(b)) \xrightarrow{*_{1}} \tilde{A}(\tilde{y}(a), \tilde{y}(c))$$

of ∞ -categories. Suppose that we have $a \colon \Delta^p \to \Delta^m$, $b \colon \Delta^q \to \Delta^m$ and $c \colon \Delta^r \to \Delta^m$, with $1 \leqslant p \leqslant q \leqslant r \leqslant m$, and that $\varphi \colon \Delta^p \to \Delta^q$ and $\psi \colon \Delta^q \to \Delta^r$ are the unique morphisms of Δ such that $a = \varphi b$ and $b = \psi c$. Thus we get

$$\tilde{A}_{\varphi}(\tilde{y}(a), \tilde{y}(b)) = \mathcal{O}_{\omega}(\langle 0, \varphi(1) \rangle + \dots + \langle \varphi(p-1), \varphi(p) \rangle, \langle 0, 1 \rangle + \dots + \langle q-1, q \rangle)$$
 and

$$\tilde{A}_{\psi}(\tilde{y}(b), \tilde{y}(c)) = \mathcal{O}_{\omega}(\langle 0, \psi(1) \rangle + \dots + \langle \psi(q-1), \psi(q) \rangle, \langle 0, 1 \rangle + \dots + \langle r-1, r \rangle)$$

$$\tilde{A}_{\psi\varphi}\big(\tilde{y}(a),\tilde{y}(c)\big) = \mathcal{O}_{\omega}\big(\langle 0,\psi\varphi(1)\rangle + \dots + \langle \psi\varphi(p-1),\psi\varphi(p)\rangle,\langle 0,1\rangle + \dots + \langle r-1,r\rangle\big).$$
 We set

$$a' = \sum_{i=1}^{p} \langle \psi \varphi(i-1), \psi \varphi(i) \rangle$$
 and $b' = \sum_{i=1}^{q} \langle \psi(i-1), \psi(i) \rangle$.

Remember that the ∞ -functor $tA_{\psi}(\tilde{y}(b), \tilde{y}(c)) \times \tilde{A}_{\varphi}(\tilde{x}(a), \tilde{x}(b)) \to \tilde{A}_{\psi\varphi}(\tilde{y}(a), \tilde{y}(c))$ is defined by making use of the canonical isomorphism between $\tilde{A}_{\varphi}(\tilde{y}(a), \tilde{y}(b))$ and the ∞ -category $\mathcal{O}_{\omega}(a', b')$. We thus have canonical isomorphisms

$$\tilde{A}_{\varphi}(\tilde{y}(a), \tilde{y}(b)) \cong \mathcal{O}_{r}(a', b')$$
, $\tilde{A}_{\psi}(\tilde{y}(b), \tilde{y}(c)) \cong \mathcal{O}_{r}(b', \langle 0, 1 \rangle + \dots + \langle r - 1, r \rangle)$

and

$$\tilde{A}_{\psi\varphi}(\tilde{y}(a), \tilde{y}(c)) \cong \mathcal{O}_r(a', \langle 0, 1 \rangle + \dots + \langle r - 1, r \rangle).$$

Moreover, the morphism $c \colon \Delta^r \to \Delta^m$ induces by Corollary D.0.10 canonical isomorphisms

$$\mathcal{O}_r(a',b') \cong \mathcal{O}_m(a,b)$$
 , $\mathcal{O}_r(b',\langle 0,1\rangle + \dots + \langle r-1,r\rangle) \cong \mathcal{O}_m(b,c)$

and

$$\mathcal{O}_r(a', \langle 0, 1 \rangle + \dots + \langle r - 1, r \rangle) \cong \mathcal{O}_m(a, c)$$
.

Under this isomorphisms, we claim that the square

$$\mathcal{O}_{r}(b',c') \times \mathcal{O}_{r}(a',b') \xrightarrow{*_{1}} \mathcal{O}_{r}(a',c')$$

$$\downarrow \qquad \qquad \downarrow$$

$$\mathcal{O}_{m}(b,c) \times \mathcal{O}_{m}(a,b) \xrightarrow{*_{1}} \mathcal{O}_{m}(a,c)$$

of ∞ -categories is commutative, where we have set $c' = \langle 0, 1 \rangle + \cdots + \langle r - 1, r \rangle$. Indeed, consider two k-cells α in $\mathcal{O}_r(a',b')$ and β in $\mathcal{O}_r(b',c')$ and suppose that they are 1-composable, k > 0. We can express α and β as homogeneous elements of $\mathsf{c}(\Delta^r)_{k+1}$ and thus as sums of atoms, say

$$\alpha = \sum_{i=0}^{s} \alpha_i$$
 and $\beta = \sum_{i=0}^{t} \beta_i$.

The operation $*_1$ at this level is simply the sum $\alpha + \beta$ and the morphism $\mathsf{c}(\Delta^r)_{k+1} \to \mathsf{c}\Delta^m_{k+1}$ induced by $c \colon \Delta^r \to \Delta^m$ sends an atom $\langle j_0, \ldots, j_{k+1} \rangle$ of the source to the atom $\langle c(j_0), \ldots, c(j_{k+1}) \rangle$ of the target. The morphism $\mathsf{c}(c) \colon \mathsf{c}(\Delta^r) \to \mathsf{c}\Delta^m$ respects sums, since it is a morphism of augmented directed complexes and therefore the square above commutes. Observe that (up to canonical isomorphisms of the factors in the line below) picking the inverses to the vertical isomorphisms of ∞ -categories of the commutative square above gives the following commutative square

$$\mathcal{O}_{m}(b,c) \times \mathcal{O}_{m}(a,b) \xrightarrow{*_{1}} \mathcal{O}_{m}(a,c)$$

$$\tilde{y}_{b,c} \times \tilde{y}_{a,b} \downarrow \qquad \qquad \downarrow \tilde{y}_{a,c}$$

$$\tilde{A}_{\psi}(\tilde{y}(b), \tilde{y}(c)) \times \tilde{A}_{\varphi}(\tilde{y}(a), \tilde{y}(b)) \xrightarrow{*_{1}} \tilde{A}_{\psi\varphi}(\tilde{y}(a), \tilde{y}(c))$$

of ∞ -categories. Since the square of embeddings

$$\begin{split} \tilde{A}_{\psi} \left(\tilde{y}(b), \tilde{y}(c) \right) \times \tilde{A}_{\varphi} \left(\tilde{y}(a), \tilde{y}(b) \right) & \stackrel{*_{1}}{\longrightarrow} \tilde{A}_{\psi \varphi} \left(\tilde{y}(a), \tilde{y}(c) \right) \\ \downarrow & \downarrow \\ \tilde{A}_{\psi} \left(\tilde{y}(b), \tilde{y}(c) \right) \times \tilde{A}_{\varphi} \left(\tilde{y}(a), \tilde{y}(b) \right) & \stackrel{*_{1}}{\longrightarrow} \tilde{A}_{\psi \varphi} \left(\tilde{y}(a), \tilde{y}(c) \right) \end{split}$$

is obviously commutative, we obtain the commutativity of the square depicted in (5.0.12.1). Hence, we have checked that the assignment

$$\tilde{y} \colon \mathcal{O}_m(\langle i \rangle, \langle j \rangle) \to \tilde{A}(y(i), y(j))$$

defines an ∞ -functor for any object $\langle i \rangle$ and $\langle j \rangle$ of \mathcal{O}_m .

In order to conclude that $\tilde{y} \colon \mathcal{O}_m \to \tilde{A}$ is an ∞ -functor, it remains to show that for any $0 \leqslant i < j < k \leqslant m$ the square

$$\mathcal{O}_{m}(\langle j \rangle, \langle k \rangle) \times \mathcal{O}_{m}(\langle i \rangle, \langle j \rangle) \xrightarrow{*_{0}} \mathcal{O}_{m}(\langle i \rangle, \langle k \rangle)$$

$$\downarrow^{\tilde{y}_{j,k} \times \tilde{y}_{i,j}} \qquad \qquad \downarrow^{\tilde{y}_{i,k}}$$

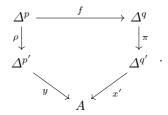
$$\tilde{A}(y(j), y(k)) \times \tilde{A}(y(i), y(j)) \xrightarrow{*_{0}} \tilde{A}(y(i), y(k))$$

of ∞ -categories is commutative. The proof uses the same strategy adopted in paragraph 5.0.6. On the objects, that is for any choice of composable 1-cells a and b of \mathcal{O}_m , with s(a)=i, t(a)=s(b)=j and t(b)=k, the commutativity of the above diagram is equivalent to the equality $yb \cdot ya = y(b \cdot a)$, which is clearly verified. Moreover, for any (a,b) and (c,d) in $\mathcal{O}_m(\langle i \rangle, \langle j \rangle) \times \mathcal{O}_m(\langle j \rangle, \langle k \rangle)$ such that the image of a is contained in the image of c and the image of c is contained in the image of c (the other cases being trivial), one easily checks the commutativity of the square

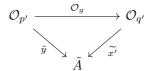
$$\mathcal{O}_{m}(b,d) \times \mathcal{O}_{m}(a,c) \xrightarrow{*_{0}} \mathcal{O}_{m}(b \cdot a, d \cdot c)
\downarrow \tilde{y}_{b \cdot a, d \cdot c}
\tilde{A}(yb, yd) \times \tilde{A}(ya, yc) \xrightarrow{*_{0}} \tilde{A}(y(b \cdot a), y(d \cdot c))$$

by reducing to the atoms, as we did for the square (5.0.12.1).

5.0.13. In this paragraph we show that the assignment sending a functor $x \colon \Delta^n \to A$ to the ∞ -functor $\tilde{x} \colon \mathcal{O}_n \to \tilde{A}$ defines a morphism of simplicial sets $N_{\infty}(A) \to N_{\infty}(\tilde{A})$. Let $f \colon \Delta^p \to \Delta^q$ be a morphism in Δ and $x \colon \Delta^q \to A$ a functor. We have to show that the equality $\tilde{x}\mathcal{O}_f = xf$ holds true. Consider the Eilenberg–Zilber decompositions (π, x') of x and (ρ, y) of xf, where $x' \colon \Delta^{q'} \to A$ and $y \colon \Delta^{p'} \to A$. We can depict the situation with the following diagram



The morphism πf of Δ admits a decomposition of a degeneracy $e \colon \Delta^p \to \Delta^\ell$ followed by a face $g \colon \Delta^\ell \to \Delta^{q'}$. Now, the composition $x'g \colon \Delta^\ell \to A$ is a non-degenerate element of $N_\infty(A)$ and so we must have $\Delta^\ell = \Delta^{p'}$, $e = \rho$ and x'g = y, by the uniqueness of the Eilenberg–Zilber decomposition. We have to show that $\widetilde{y} = \widetilde{x'}\mathcal{O}_g$, so that the following triangle



of ∞ -functors is commutative. It trivially commutes at the level of objects. For any injective map $a \colon \Delta^n \to \Delta^{p'}$, we clearly have ya = x'ga, so that by definition $\tilde{y}(a) = \tilde{x'}\mathcal{O}_g(a)$ and therefore the triangle is commutative on 1-cells. Let a and b be two parallel 1-cells of $\mathcal{O}_{p'}$, say $a \colon \Delta^m \to \Delta^{p'}$ and $b \colon \Delta^n \to \Delta^{p'}$. Observe that $\mathcal{O}_{p'}(a,b)$ is empty if and only if $\mathcal{O}_{q'}(ma,mb)$ is empty if and only if $\tilde{A}(\tilde{y}(a),\tilde{y}(b))$ is so. Otherwise, there exist a unique monomorphism $\varphi \colon \Delta^m \to \Delta^n$ of Δ , an integer

 $r \geqslant 0$, an unique injective morphism $h: \Delta^r \to \Delta^{p'}$ of Δ and 1-cells a' and b' of \mathcal{O}_r such that $b\varphi = a$, ha' = a, hb' = b and

$$b' = \langle 0, 1 \rangle + \langle 1, 2 \rangle + \dots + \langle r - 1, r \rangle.$$

By definition, $\tilde{A}(\tilde{y}(a), \tilde{y}(b)) = \mathcal{O}_r(a', b')$ and we have a commutative square of isomorphism

$$\mathcal{O}_{p'}(a,b) \xrightarrow{(\mathcal{O}_g)_{a,b}} \mathcal{O}_{q'}(ma,mb)$$

$$(\mathcal{O}_h)_{a',b'}$$

$$\mathcal{O}_r(a',b')$$

of ∞ -categories by Corollary D.0.10 and this immediately implies that the triangle

$$\mathcal{O}_{p'}(a,b) \xrightarrow{\mathcal{O}_g} \mathcal{O}_{q'}(ma,mb)$$

$$\tilde{A}_{\varphi}(\tilde{y}(a),\tilde{y}(b))$$

of ∞ -categories commutes, as $\tilde{y}_{a,b}$ is defined as the inverse of $(\mathcal{O}_h)_{a',b'}$ and $\widetilde{x'}_{ma,mb}$ as the inverse of $(\mathcal{O}_{hg})_{a',b'}$. This concludes the proof, showing that the assignment $\eta_A \colon N_{\infty}(A) \to N_{\infty}(\tilde{A})$ is indeed a morphism of simplicial sets.

5.0.14. We now want to show that the morphism $\eta_A \colon N_{\infty}(A) \to N_{\infty}(\tilde{A})$ is the counit of the adjoint pair (c_{∞}, N_{∞}) for the object $N_{\infty}(A)$. This is equivalent to say that the precomposition by η_A induces a bijection

$$\operatorname{Hom}_{\infty\text{-}\operatorname{Cat}}(\tilde{A},B) \cong \operatorname{Hom}_{\widehat{\mathbf{A}}}(N_{\infty}(A),N_{\infty}(B))$$

of sets for any ∞ -category B. In turn, this bijection means that for any morphism of simplicial sets $F: N_{\infty}(A) \to N_{\infty}(B)$ there exists a unique ∞ -functor $\bar{F}: \tilde{A} \to B$ such that the triangle

$$(5.0.14.1) N_{\infty}(A) \xrightarrow{F} N_{\infty}(B)$$

$$\downarrow^{\eta_A} \qquad N_{\infty}(\tilde{F})$$

$$N_{\infty}(\tilde{A})$$

is commutative. This would show in particular that $\tilde{A} \cong c_{\infty}N_{\infty}(A)$. We shall first prove the uniqueness and then the existence of such an ∞ -functor \bar{F} .

5.0.15 (Uniqueness). Suppose a functor $G: \tilde{A} \to B$ such that $F = N_{\infty}(G)\eta_A$ exists. Object-wise, the functor G must coincide with F_0 . A 1-cell of \tilde{A} is a tuple $a = (f_1, \ldots, f_n)$ of non-trivial composable arrows of A. Let f be the composite $f_n *_0 \ldots *_0 f_1$ of the arrows which are components of a; we can view f as a non-degenerate 1-simplex $f: \Delta^1 \to N_{\infty}(A)$. Then G(a) must be equal to F(f). For observe that $N_{\infty}(\varepsilon_A)\eta_A$ is the identity on $N_{\infty}(A)$ and $\varepsilon_A(a) = f$.

Let a and b be two 1-cells of \tilde{A} , say $a: \Delta^m \to A$ and $b: \Delta^n \to A$. We can suppose that there is an injective morphism $\varphi: \Delta^m \to \Delta^n$ of Δ such that $b\varphi = a$, otherwise $\tilde{A}(a,b)$ is empty; we fix such a morphism φ . By definition,

$$\tilde{A}_{\varphi}(a,b) = \mathcal{O}_n(\langle \varphi(0), \varphi(1) \rangle + \dots + \langle \varphi(m-1), \varphi(m) \rangle, \langle 0, 1 \rangle + \dots + \langle n-1, n \rangle)$$
 and we have an ∞ -functor

$$F(b)\colon \mathcal{O}_n\to B$$
.

Hence, the ∞ -functor $\tilde{A}_{\varphi}(a,b) \to B(Ga,Gb)$ is the composition of the following ∞ -functors

$$\tilde{A}_{\varphi}(a,b) \stackrel{\cong}{\longrightarrow} \mathcal{O}_n(a',b') \stackrel{G(b)}{\longrightarrow} B(Ga,Gb)$$
,

where we have set

$$a' = \langle \varphi(0), \varphi(1) \rangle + \dots + \langle \varphi(m-1), \varphi(m) \rangle$$
 and $b' = \langle 0, 1 \rangle + \dots + \langle n-1, n \rangle$.

Varying φ this gives a unique ∞ -functor $\tilde{A}(a,b) \to B(Ga,Gb)$, thus proving the uniqueness of G.

5.0.16 (Existence). The previous paragraph already shows how the functor $\bar{F}: \tilde{A} \to B$ must be define, if it exists. It remains to check that this assignment is indeed an ∞ -functor.

Let x and y be objects of \tilde{A} and $a: \Delta^{\ell} \to A$, $b: \Delta^m \to A$ and $c: \Delta^n \to A$ be 1-cells of $\tilde{A}(x,y)$. Without loss of generality, we can suppose that there are injective morphisms $\varphi: \Delta^{\ell} \to \Delta^m$ and $\psi: \Delta^m \to \Delta^n$ such that $c\psi = b$ and $b\varphi = a$. We set

$$a' = \langle \psi \varphi(0), \psi \varphi(1) \rangle + \dots + \langle \psi \varphi(\ell - 1), \psi \varphi(\ell) \rangle,$$

$$b' = \langle \psi(0), \psi(1) \rangle + \dots + \langle \psi(m - 1), \psi(m) \rangle,$$

$$c' = \langle 0, 1 \rangle + \dots + \langle n - 1, n \rangle.$$

We have a diagram

$$\tilde{A}_{\psi}(b,c) \times \tilde{A}_{\varphi}(a,b) \xrightarrow{*_{1}} \tilde{A}_{\psi\varphi}(a,c)$$

$$\cong \downarrow \qquad \qquad \downarrow \cong$$

$$\mathcal{O}_{n}(b',c') \times \mathcal{O}_{n}(a',b') \xrightarrow{*_{1}} \mathcal{O}_{n}(a',c') ,$$

$$\downarrow^{F(c)} \qquad \qquad \downarrow^{F(c)}$$

$$B(\bar{F}b,\bar{F}c) \times B(\bar{F}a,\bar{F}b) \xrightarrow{*_{1}} B(\bar{F}a,\bar{F}c)$$

where the upper square commutes by definition and the lower square commutes by the ∞ -functoriality of $F(c) \colon \mathcal{O}_n \to B$. Making the morphisms φ and ψ varying among the index defining the sum $\tilde{A}(a,b)$ and $\tilde{A}(a,b)$, we get a commutative square

$$\tilde{A}(b,c) \times \tilde{A}(a,b) \xrightarrow{*_{1}} \tilde{A}_{\psi\varphi}(a,c)
\bar{F}_{b,c} \times \bar{F}_{a,b} \downarrow \qquad \qquad \downarrow \bar{F}_{a,c}
B(\bar{F}b,\bar{F}c) \times B(\bar{F}a,\bar{F}b) \xrightarrow{*_{1}} B(\bar{F}a,\bar{F}c)$$

of ∞ -categories.

Let x, y and z be three objects of \tilde{A} . We have to show that the square of ∞ -categories

$$\begin{split} \tilde{A}(y,z) \times \tilde{A}(x,y) & \xrightarrow{\quad *_0 \quad} \tilde{A}_{\psi\varphi}(x,z) \\ \bar{F}_{y,z} \times \bar{F}_{x,y} \Big\downarrow & & \downarrow \bar{F}_{x,z} \\ B(\bar{F}y,\bar{F}z) \times B(\bar{F}x,\bar{F}y) & \xrightarrow{\quad *_0 \quad} B(\bar{F}x,\bar{F}z) \end{split}$$

is commutative. As for the objects, that is the 1-cells of \tilde{A} and B, it is clear: indeed, for any $a\colon x\to y$ and $b\colon y\to z$ of \tilde{A} , we have on the one hand that $b\ast_0 a$ is just the concatenation of the simplices a and b of A, while on the other hand $\bar{F}(c)$ applied to a 1-cell $c=(f_1,\ldots,f_n)$ of \tilde{A} gives image under F of the composition $F(f_n\ast_0\ldots\ast_0 f_1)$ of its components. Therefore we have to check that, for any choice (b,a) and (d,c) of elements of $\tilde{A}(y,z)\times \tilde{A}(x,y)$, the square

$$\begin{split} \tilde{A}(b,d) \times \tilde{A}(a,c) & \xrightarrow{\quad *_0 \quad} \tilde{A}_{\psi\varphi}(b \cdot a, d \cdot c) \\ \bar{F}_{b,d} \times \bar{F}_{a,c} \downarrow & \downarrow \bar{F}_{b \cdot a, d \cdot c} \\ B(\bar{F}b, \bar{F}d) \times B(\bar{F}a, \bar{F}c) & \xrightarrow{\quad *_0 \quad} B(\bar{F}(b \cdot a), \bar{F}(d \cdot c)) \end{split}$$

of ∞ -categories is commutative. This is completely analogous to the case of the "vertical composition" $*_1$ that we showed above: we reduce to components $\tilde{A}_{\psi}(b,d)$ and $\tilde{A}_{\varphi}(a,c)$, for which there is a diagram

$$\begin{split} \tilde{A}_{\psi}(b,d) \times \tilde{A}_{\varphi}(a,c) & \xrightarrow{*_{0}} \tilde{A}_{\psi \cdot \varphi}(a,c) \\ \cong \downarrow & \downarrow \cong \\ \mathcal{O}_{n}(b',d') \times \mathcal{O}_{n}(a',c') & \xrightarrow{*_{0}} \mathcal{O}_{n}(b' \cdot a',d' \cdot c') \\ F(d \cdot c) \downarrow & \downarrow F(d \cdot c) \\ B(\bar{F}b,\bar{F}d) \times B(\bar{F}a,\bar{F}c) & \xrightarrow{*_{0}} B(\bar{F}(b \cdot a),\bar{F}(d \cdot c)) \end{split}$$

of ∞ -categories in which the upper square commutes by definition and the lower square by ∞ -functoriality of $F(d \cdot c) : \mathcal{O}_n \to B$, and finally we conclude by varying among all the morphisms φ and ψ indexing the coproducts $\tilde{A}(a,c)$ and $\tilde{A}(b,d)$. This achieves the proof of the existence of the ∞ -functor $\bar{F} : \tilde{A} \to B$ and so this establishes the lifting problem depicted in (5.0.14.1). Equivalently, the precomposition by $\eta_A : N_\infty(A) \to N_\infty(\tilde{A})$ gives a bijection

$$\operatorname{Hom}_{\infty\text{-}\operatorname{Cat}}(\tilde{A},B) \cong \operatorname{Hom}_{\widehat{\Delta}}(N_{\infty}(A),N_{\infty}(B))$$
,

from which we deduce the isomorphism $\tilde{A} = c_{\infty} N_{\infty}(A)$.

Theorem 5.0.17. Let A be a split-free category. Then the ∞ -category \tilde{A} defined in paragraph 5.0.6 is isomorphic to the ∞ -category $c_{\infty}N_{\infty}(A)$.

Example 5.0.18. Let C be a 3-category and consider the normalised oplax 3-functor $\sup: i_{\Delta}(N_3(C)) \to C$ defined in example 1.2.6. For any 1-category A, the category $c\mathrm{Sd}N(A)$ is split-free and moreover it is shown in Theorem 32 of [11] that the canonical morphism $c\mathrm{Sd}N(A) \to A$ is a Thomason equivalence. Hence, we get a diagram

$$c\operatorname{Sd}N(i_{\Delta}(N_3(C))) \to i_{\Delta}(N_3(C)) \to C$$

whose composition if still a normalised oplax 3-functor by Theorem 4.0.6. Now, the category $C' = c \operatorname{Sd}N(i_{\Delta}(N_3(C)))$ is split-free and therefore we get a span

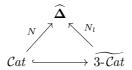
$$C' \leftarrow \tau^{\mathrm{i}}_{\leqslant 3} \widetilde{C'} \to C$$

of 3-functors. We conjecture that both the 3-functors above are Thomason equivalences. Since we observed in Example 1.3.21 that the morphism $N_l(\sup)$ of simplicial sets is a simplicial weak equivalence and we observed above that the functor $C' \to i_{\Delta}(N_3(C))$ is a Thomason equivalence, by a 2-out-of-3 argument one of these 3-functor is a Thomason equivalence if and only if the other is so.

This is a partial generalisation to 3-categories of the approach used by Chiche in [9] to show that the minimal fundamental localiser of $2\text{-}\mathcal{C}at$ is given by the class of Thomason equivalences, thus showing that 2-categories intrinsically model homotopy types. In order to generalise this result to higher category, one would need to prove that both the 3-functors of the above span are aspherical, *i.e.*, they satisfy the 3-categorical generalisation of Quillen's Theorem A. The author does not even know if this is true for 2-categories. In fact, Chiche avoids this problem introducing a notion of asphericity for oplax 2-functors, that seems out of reach for higher dimension. We can nonetheless say something interesting about the homotopy theory of normalised 3-functors, as pointed out in the following remark.

Remark 5.0.19. The nerve functor $N_l: 3-Cat \to \widehat{\Delta}$ allows us to define a class of weak equivalences on 3-Cat, that we call Thomason equivalences. More precisely, a

normalised oplax 3-functor is a Thomason equivalence if and only if its image via N_l is a weak homotopy equivalence. Since the triangle



commutes, a classical result of Illusie-Quillen tells us that the composite functor

$$\widehat{\Delta} \xrightarrow{i_{\Delta}} \widetilde{3\text{-}Cat} \xrightarrow{N_l} \widehat{\Delta}$$

is weakly homotopy equivalent to the identity on simplicial sets. Moreover, Example 1.3.21 gives us that a normalised oplax 3-functor $u: A \to B$ is a Thomason equivalence if and only if the functor $i_{\Delta}(N_l(u)): i_{\Delta}(N_l(A)) \to i_{\Delta}(N_l(B))$ is so. Hence, the composite functor

$$\widetilde{3\text{-}\mathcal{C}at} \xrightarrow{N_l} \widehat{\Delta} \xrightarrow{i_{\Delta}} \widetilde{3\text{-}\mathcal{C}at}$$

is homotopic to the identity functor on 3- \widetilde{Cat} . We conclude that the nerve functor $N_l \colon \widetilde{3\text{-}Cat} \to \widehat{\Delta}$ induces an equivalence at the level of the underlying homotopy categories.

APPENDIX A. STRICT HIGHER CATEGORIES

A.0.1. Let \mathcal{V} be a category. A \mathcal{V} -graph X is the data of a set X_0 of objects and, for any x, y in X_0 , an object X(x, y) of \mathcal{V} . A morphism of \mathcal{V} -graphs $f \colon X \to Y$ is given by a function $f_0 \colon X_0 \to Y_0$ between the objects as well as morphisms

$$f_{x,y} \colon X(x,y) \to Y(fx,fy)$$

of \mathcal{V} for any x, y in X_0 . We denote by \mathcal{V} - $\widehat{\mathbb{G}}_1$ the category of \mathcal{V} -graphs.

Let I be an object of \mathcal{V} . A reflexive (\mathcal{V}, I) -graph X, or simply reflexive \mathcal{V} -graph if the object I is clear from the context, is a \mathcal{V} -graph endowed with a morphism $k_x \colon I \to \mathsf{Hom}_{\mathcal{V}}(x,x)$ for any element x in X_0 . A morphism of reflexive \mathcal{V} -graphs $f \colon X \to Y$ is a morphism of \mathcal{V} -graphs such that $f_{x,x} k_x = k_{fx}$ for any x in X_0 . We denote by $\mathcal{V} \cdot \widehat{\mathbb{O}}_1$ the category of reflexive \mathcal{V} -graphs.

Let $(\mathcal{V}, \otimes, I)$ be a monoidal category. A $(\mathcal{V}, \otimes, I)$ -category A, or simply \mathcal{V} -category if the monoidal structure is clear, is a reflexive (\mathcal{V}, I) -graph endowed with morphisms

$$\operatorname{\mathsf{Hom}}_A(b,c)\otimes\operatorname{\mathsf{Hom}}_A(a,b)\to\operatorname{\mathsf{Hom}}_A(a,c)$$

of \mathcal{V} , for any objects a, b and c in A_0 , satisfying the associativity axioms

for any a, b, c and d in A_0 , and the identity axioms

and

$$\begin{array}{ccc} \operatorname{Hom}_A(a,b) \otimes I & \xrightarrow{\operatorname{Hom}_A(a,b) \otimes k_a} & \operatorname{Hom}_A(a,b) \otimes \operatorname{Hom}_A(a,a) \\ \cong & & \downarrow \\ \operatorname{Hom}(a,b) & \xrightarrow{} & \operatorname{Hom}(a,b) \end{array}$$

for any a and b in A_0 . A V-category is also widely known as V-enriched category or category enriched in V. A morphism of V-categories $\varphi \colon A \to B$, also called V-enriched functor, is a morphism of the underlying reflexive V-graphs which moreover commutes with compositions morphisms. We denote by V-Cat the category of V-categories. It is easy to see that if the category V has finite products, then the category V-Cat also has finite products.

A.0.2. For n > 1, the category (n + 1)-Cat of (n + 1)-categories can be inductively defined as the category of reflexive n-Cat-graphs. This provides a canonical functor n- $Cat \to (n + 1)$ -Cat which admits a left adjoint τ^n . The category ∞ -Cat can be defined as the limit of the tower

$$\ldots \longrightarrow (n+1)\text{-}\mathcal{C}at \xrightarrow{\tau^n} n\text{-}\mathcal{C}at \xrightarrow{\tau^{n-1}} (n-1)\text{-}\mathcal{C}at \xrightarrow{\tau^{n-2}} \ldots$$

APPENDIX B. STEINER THEORY

In this section we present the theory of augmented directed complexes, introduced by Steiner in [23]. We follow closely the exposition given by Ara and Maltsiniotis in [2] and [3].

- **B.0.1.** Unless explicitly stated, in this section we shall always write "chain complex" to mean "chain complex of abelian groups in non-negative degrees with homological indexing". We remind that a homogeneous element of a chain complex K is an element of a group K_n for some $n \ge 0$. If x is an homogeneous element of K, we shall call the degree of x the unique $n \ge 0$ for which k belongs to K_n and we shall denote it by |x|.
- **B.0.2.** An augmented directed complex is a triple (K, K^*, e) where

$$K = \cdots \xrightarrow{d_{n+1}} K_n \xrightarrow{d_n} K_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} K_1 \xrightarrow{d_1} K_0$$

is a chain complex, $e: K_0 \to \mathbb{Z}$ is an augmentation (so that we have $e d_1 = 0$) and $K^* = (K_i^*)_{i \ge 0}$ is a graded set such that for any $i \ge 0$ the set K_i^* is a submonoid of the abelian group K_i . We shall call *positivity submonoids* of K the submonoids K_i^* , with $i \ge 0$.

We will often denote an augmented directed complex simply by its underlying chain complex, especially if the augmented and positivity structures are clear.

Remark B.0.3. We warn the reader that we do not ask any compatibility of the submonoids of positivity with respect to the differentials.

B.0.4. Let (K, K^*, e) be an augmented directed complex. For any $i \ge 0$, the submonoid K_i^* induces a preorder relation \le on K_i , compatible with the abelian group structure, defined by

$$x \leqslant y \iff y - x \in K_i^*$$
.

In particular, we have

$$K_i^* = \{ x \in K_i : x \geqslant 0 \}.$$

More precisely, for an augmented complex (K, e) the additional structure given by the collection of submonoids K^* is equivalent to endow, for all $i \ge 0$, each abelian group K_i with a preorder relation compatible with the abelian group structure.

- **B.0.5.** A morphism of augmented directed complexes f from (K, K^*, e) to (K', K'^*, e') is a morphism of augmented chain complexes which moreover respects the submonoids of positivity. That is, $f: K \to K'$ is a morphism of chain complexes such that $e' f_0 = e$ and for any $i \ge 0$ the image $f(K_i^*)$ of the submonoid K_i^* under f is contained in $K_i^{\prime*}$. This latter condition can be stated equivalently by saying that the morphism $f_i: K_i \to K'_i$ preserves the preorder \leq on K_i for all $i \geq 0$. We shall denote by C_{ad} the category of augmented directed complexes.
- **B.0.6.** A basis of an augmented directed complex K is a graded set $B = (B_i)_{i>0}$ such that, for any $i \geq 0$, the set B_i is both a basis for the \mathbb{Z} -module K_i and a set of generators for the submonoid K_i^* of K_i . We shall often identify a basis $B = (B_i)_{i \ge 0}$ to the set $\coprod_{i\geqslant 0} B_i$

If an augmented directed complex has a basis, for any $i \geq 0$ the preorder relation of positivity on K_i defined in paragraph B.0.4 is a partial order relation and the elements B_i of the basis are the minimal elements of the poset $(K_i^* \setminus \{0\}, \leq)$; in particular, if a basis of K exists then it is unique. When an augmented directed complex has a basis, we shall say that the complex is with basis.

B.0.7. Let K be an augmented directed complex with basis B. For any i > 0, a ihomogeneous element x can be written uniquely as a linear combination of elements of B_i

$$x = \sum_{b \in B_i} x_b \, b$$

with integral coefficients. The support of x, denoted by supp(x), is the (finite) set of elements of the basis appearing in this linear combination with non-zero coefficient. We can write the i-homogeneous elements uniquely as the difference of two positive *i*-homogeneous elements with disjoint supports $x = x_{+} - x_{-}$, where

$$x_{+} = \sum_{\substack{b \in B_{i} \\ x_{b} > 0}} x_{b} b$$
 and $x_{-} = -\sum_{\substack{b \in B_{i} \\ x_{b} < 0}} x_{b} b$.

B.0.8. Let K be an augmented directed complex with basis $B = (B_i)_{i \ge 0}$. For $i \ge 0$ and x in K_i , we define a matrix

$$\langle x \rangle = \begin{pmatrix} \langle x \rangle_0^0 & \langle x \rangle_1^0 & \dots & \langle x \rangle_{i-1}^0 & \langle x \rangle_i^0 \\ \langle x \rangle_0^1 & \langle x \rangle_1^1 & \dots & \langle x \rangle_{i-1}^1 & \langle x \rangle_i^1 \end{pmatrix},$$

where the elements $\langle x \rangle_k^{\varepsilon}$ are inductively defined by:

- $\langle x \rangle_i^0 = x = \langle x \rangle_i^1$;

• $\langle x \rangle_{k-1}^0 = d(\langle x \rangle_k^0)_-$ and $\langle x \rangle_{k-1}^1 = d(\langle x \rangle_k^1)_+$, for $0 < k \leqslant i$. We say that the basis B of K is unital if, for any $i \geq 0$ and any x in B_i , we have the equality $e(\langle x \rangle_0^0) = 1 = e(\langle x \rangle_0^1)$.

We shall say that an augmented directed complex K is with unital basis is it is with basis and its unique basis is unital.

B.0.9. Let K be an augmented directed complex with basis B. For $i \geq 0$, we denote by \leq_i the smallest preorder relation on $B = \coprod_i B_i$ satisfying

$$x \leq_i y$$
 if $|x| > i, |y| > i$ and supp $(\langle x \rangle_i^1) \cap \text{supp}(\langle y \rangle_i^0) \neq \emptyset$.

We say that the basis B is loop-free if, for any $i \ge 0$, the preorder relation $\le i$ is a partial order relation.

We shall call Steiner complex an augmented directed complex K with unital and loop-free basis B.

B.0.10. Let K be an augmented directed complex with basis $B = \coprod_{i \geqslant 0} B_i$. We shall denote by $\leq_{\mathbb{N}}$ the smallest preorder relation on B satisfying

$$x \leq_{\mathbb{N}} y$$
 if $x \in \text{supp}(d(y)_{-})$ or $y \in \text{supp}(d(x)_{+})$,

where we fixed by convention d(b) = 0 if b belongs to B_0 . We shall say that a basis B is strongly loop-free if the preorder relation $\leq_{\mathbb{N}}$ is actually a partial order

We shall call an augmented directed complex K a strong Steiner complex if it is with basis and its unique basis is unital and strongly loop-free.

Proposition B.0.11 (Steiner). Let K be an augmented directed complex with basis B. If the basis B is strongly loop-free, then it is loop-free.

B.0.12. We define a functor

$$\nu \colon \mathcal{C}_{\mathrm{ad}} \to \infty\text{-}\mathcal{C}at$$

as follows.

Let K be an augmented directed complex. For $i \ge 0$, the i-cells of $\nu(K)$ are the matrices

$$\begin{pmatrix} x_0^0 & \dots & x_{i-1}^0 & x_i^0 \\ x_0^1 & \dots & x_{i-1}^1 & x_i^1 \end{pmatrix}$$

such that

- (a) x_k^{ε} belongs to K_k^* for $\varepsilon = 0, 1$ and $0 \leqslant k \leqslant i$; (b) $d(x_k^{\varepsilon}) = x_{k-1}^1 x_{k-1}^0$ for $\varepsilon = 0, 1$ and $0 < k \leqslant i$; (c) $e(x_0^{\varepsilon}) = 1$ for $\varepsilon = 0, 1$;
- $(d) x_i^0 = x_i^1.$

Let us describe the ∞ -categorical structure. Let

$$x = \begin{pmatrix} x_0^0 & \dots & x_{i-1}^0 & x_i^0 \\ x_0^1 & \dots & x_{i-1}^1 & x_i^1 \end{pmatrix}$$

be an i-cell of $\nu(K)$ for $i \ge 0$. If i > 0 we define the source and the target of x to be respectively

$$s(x) = \begin{pmatrix} x_0^0 & \dots & x_{i-2}^0 & x_{i-1}^0 \\ x_0^1 & \dots & x_{i-2}^1 & x_{i-1}^0 \end{pmatrix} \quad \text{and} \quad t(x) = \begin{pmatrix} x_0^0 & \dots & x_{i-2}^0 & x_{i-1}^1 \\ x_0^1 & \dots & x_{i-2}^1 & x_{i-1}^1 \end{pmatrix}.$$

The identity of x is given by the matrix

$$1_x = \begin{pmatrix} x_0^0 & \dots & x_{i-1}^0 & x_i^0 & 0 \\ x_0^1 & \dots & x_{i-1}^1 & x_i^1 & 0 \end{pmatrix}.$$

Finally if

$$y = \begin{pmatrix} y_0^0 & \dots & y_{i-1}^0 & y_i^0 \\ y_0^1 & \dots & y_{i-1}^1 & y_i^1 \end{pmatrix}$$

is another i-cell which is j-composable with x, with $i > j \ge 0$, then we set

$$x *_{j} y = \begin{pmatrix} y_{0}^{0} & \dots & y_{j}^{0} & x_{j+1}^{0} + y_{j+1}^{0} & \dots & x_{i}^{0} + y_{i}^{0} \\ x_{0}^{1} & \dots & x_{j}^{1} & x_{j+1}^{1} + y_{j+1}^{1} & \dots & x_{i}^{1} + y_{i}^{1} \end{pmatrix}.$$

One checks that this indeed defines an ∞ -category.

If x is an i-cell of $\nu(K)$, $i \ge 0$, then we shall denote by x_k^{ε} the component of the matrix defining x, for $0 \le k \le i$ and $\varepsilon = 0, 1$. We shall simply name by x_i the element $x_i^0 = x_i^1$ and for k > i and $\varepsilon = 0, 1$ we set $x_k^{\varepsilon} = 0$.

Let $f \colon K \to K'$ be a morphism of augmented directed complexes. The collection of functions

$$\begin{pmatrix} x_0^0 & \dots & x_{i-1}^0 & x_i^0 \\ x_0^1 & \dots & x_{i-1}^1 & x_i^1 \end{pmatrix} \mapsto \begin{pmatrix} f(x_0^0) & \dots & f(x_{i-1}^0) & f(x_i^0) \\ f(x_0^1) & \dots & f(x_{i-1}^1) & f(x_i^1) \end{pmatrix}$$

defines an ∞ -functor $\nu(f) \colon \nu(K) \to \nu(K')$

Remark B.0.13. Steiner shows in [23] that the functor ν admits a left adjoint λ , that we will not define since we will not need it. In particular, the composition $N = N_{\infty} \circ \nu \colon \mathcal{C}_{ad} \to \widehat{\Delta}$ defines a nerve functor for augmented directed complexes and this has a left adjoint given by $\mathbf{c} = \lambda \circ c_{\infty} \colon \widehat{\Delta} \to \infty\text{-}\mathcal{C}at$.

B.0.14. Let K be an augmented directed complex with basis B. For any element x of K_i , one easily checks that the matrix

$$\langle x \rangle = \begin{pmatrix} \langle x \rangle_0^0 & \langle x \rangle_1^0 & \dots & \langle x \rangle_{i-1}^0 & \langle x \rangle_i^0 \\ \langle x \rangle_0^1 & \langle x \rangle_1^1 & \dots & \langle x \rangle_{i-1}^1 & \langle x \rangle_i^1 \end{pmatrix},$$

as defined in paragraph B.0.8, is an *i*-cell of $\nu(K)$ if and only if the element x belongs to K_i^* and we have the equalities $e(\langle x \rangle_0^0) = 1 = e(\langle x \rangle_0^1)$. Setting $\langle x \rangle_i^\varepsilon = x_i^\varepsilon$ for $k \leq i$ and $\langle x \rangle_k^\varepsilon = 0$ for all k > i, $\varepsilon = 0, 1$, these notations are compatible with those of paragraph B.0.12 whenever $\langle x \rangle$ is an *i*-cell of $\nu(K)$.

If the basis B of K is unital, then for any element x of the basis the matrix defined by $\langle x \rangle$ is a cell of $\nu(K)$. In this case, we call the cell $\langle x \rangle$ of $\nu(X)$ the atom associated to x.

Theorem B.0.15. The functors

$$\lambda \colon \infty\text{-}\mathcal{C}\mathit{at} \to \mathcal{C}_{\mathrm{ad}} \quad \mathit{and} \quad \nu \colon \mathcal{C}_{\mathrm{ad}} \to \infty\text{-}\mathcal{C}\mathit{at}$$

define a pair of adjoint functors.

Proof. This is Theorem 2.11 of [23].

Theorem B.0.16 (Steiner). For any Steiner complex K, the counit morphism

$$\lambda(\nu(K)) \to K$$

is an isomorphism. In particular, the restriction of the functor $\nu \colon \mathcal{C}_{ad} \to \infty\text{-}\mathcal{C}at$ to the category of Steiner complexes is fully faithful

B.0.17. We shall call $Steiner \infty$ -category (resp. $strong\ Steiner \infty$ -category) an ∞ -category in the essential image of the restriction of the functor $\nu: \mathcal{C}_{ad} \to \infty$ - $\mathcal{C}at$ to the full subcategory of Steiner complexes (resp. strong Steiner complexes). The preceding theorem states that the functor ν induces an equivalence of categories between the category of Steiner complexes and that of Steiner ∞ -categories (resp. between the category of strong Steiner complexes and that of strong Steiner ∞ -categories).

Appendix C. Joyal's Θ category

C.0.1. For any $i \ge 0$, we shall denote by Cell_i the set of i-cells of an ∞ -category and by D_i the ∞ -category corepresenting the functor $\mathsf{Cell}_i \colon \infty\text{-}\mathcal{C}at \to \mathcal{S}et$ mapping an ∞ -category A to the set of its i-cells. In fact, this ∞ -category is an i-category having a single non-trivial i-cell that we shall call its $principal\ cell$. For any $0 \le k \le i$ the i-category D_i has exactly two non-trivial k-cells which are the k-dimensional iterated source and target of its principal cell. This is how the graphs of D_i (without identities) for i=0,1,2 look like:

$$D_0 = \bullet$$
 , $D_1 = \bullet \longrightarrow \bullet$, $D_2 = \bullet \longrightarrow \bullet$, $D_3 = \bullet \longrightarrow \bullet$.

For i > 0, the natural transformations source and target $\mathsf{Cell}_i \to \mathsf{Cell}_{i-1}$ induces ∞ -functors $\sigma^i, \tau^i \colon \mathsf{D}_{i-1} \to \mathsf{D}_i$. Explicitly the ∞ -functor σ_i (resp. τ_i) sends the principal cell of D_{i-1} to the source (resp. the target) of the principal cell of D_i .

For $0 \le j < i$ we shall denote by $\sigma_j^i, \tau_j^i : D_j \to D_i$ the ∞ -functors corepresented by the natural transformations s_i^j and t_i^j respectively, *i.e.*, the ∞ -functors

$$\sigma_j^i = \sigma^i \dots \sigma^{j+2} \sigma^{j+1}$$
 and $\tau_j^i = \tau^i \dots \tau^{j+2} \tau^{j+1}$.

C.0.2. Let $\ell > 0$ and $i_1, \ldots, i_\ell, j_1, \ldots, j_{\ell-1}$ be a collection of positive integers satisfying the inequalities

$$i_k > j_k < i_{k+1}$$
, for $0 < k < \ell$.

We shall often organise these integers in a matrix, called *matrix of dimensions*, of the following form

$$\begin{pmatrix} i_1 & i_2 & \dots & i_{\ell-1} & i_\ell \ j_1 & j_2 & \dots & j_{\ell-1} \end{pmatrix}$$

and associate to it the diagram

in ∞ -Cat. We shall call globular sum the colimit of such a diagram and we shall simply denote it by

$$D_{i_1} \coprod_{D_{j_1}} D_{i_2} \coprod_{D_{j_2}} \cdots \coprod_{D_{j_{\ell-1}}} D_{i_\ell}$$
.

We shall call globular pasting scheme any ∞ -category that we get this way.

C.0.3. Consider a matrix of dimensions

$$egin{pmatrix} i_1 & i_2 & \dots & i_{\ell-1} & i_\ell \ j_1 & j_2 & \dots & j_{\ell-1} \end{pmatrix}$$
 .

The dimension of the globular pasting scheme T

$$D_{i_1} \coprod_{D_{j_1}} D_{i_2} \coprod_{D_{j_2}} \cdots \coprod_{D_{j_{\ell-1}}} D_{i_{\ell}}$$

is given by the number

$$\sum_{1 \leqslant k \leqslant \ell} i_k - \sum_{0 < k < \ell} j_k = i_1 - j_1 + i_2 - j_2 + \dots + i_{\ell-1} - j_{\ell-1} + i_{\ell}.$$

The height of the globular pasting scheme T is defined as the number

$$ht(T) = \max_{1 \le k \le \ell} (i_k).$$

- **C.0.4.** Joyal's Θ category is the full subcategory of ∞ - $\mathcal{C}at$ spanned by globular pasting schemes. We shall denote by Θ_+ the full subcategory of ∞ - $\mathcal{C}at$ obtained by adding the empty ∞ -category to Θ .
- **C.0.5.** The height defined in the previous paragraph defines a canonical grading on the objects of Θ . For any integer $n \geq 0$, we denote by Θ_n the full subcategory of Θ spanned by the objects of height at most n. We observe that Θ_0 is the category whose only object is D_0 and whose only morphism is the identity and that Θ_1 is canonically isomorphic to the category Δ of simplices; we thus get a canonical embedding $\Delta \hookrightarrow \Theta$.

Remark C.0.6. The category Θ was first introduced by Joyal in [16], with a definition more geometric in spirit. Berger [5] and Makkai-Zawadowski [19] later independently showed that the two definitions are actually equivalent. Another equivalent definition is due to Oury [20].

C.0.7. Another convenient and graphical description of the category Θ can be given in terms of planar rooted trees. Let \mathcal{T} be the category of presheaves in finite linearly ordered set on the poset of non-negative integers, i.e., an element X of \mathcal{T} is a sequence of finite linearly ordered sets $(X_n)_{n\geqslant 0}$ equipped with order-preserving maps $X_n \to X_{n-1}$ for all n > 0. A planar rooted tree, or simply tree, is an object T of \mathcal{T} such that T_0 is a singleton and for which T_i is eventually empty for i big enough. The greatest i for which T_i is non-empty will be called the *height* of the

Let us now sketch the correspondence between objects of Θ and trees. Instead of giving a formal framework, we are going to present some examples upon which one can easily build the intuition behind this bijection. For any $i \ge 0$, we associate to the object D_i the linear tree T of height i, that is, for which T_k is a singleton for all $0 \le k \le i$, and we depict it as



So for instance we have

$$\mathrm{D}_0 = ullet \quad , \quad \mathrm{D}_1 = ullet \quad , \quad \mathrm{D}_2 = ullet \quad , \quad \mathrm{D}_3 = ullet$$

The height i of the tree determines the dimension of the principal cell.

The element of Δ corresponds precisely to the trees of height at most 1, for instance

$$\Delta^0 = ullet$$
 , $\Delta^1 = ullet$, $\Delta^2 = ullet$, $\Delta^3 = ullet$

More generally, given an object
$$S$$
 of Θ with matrix of dimensions
$$\begin{pmatrix} i_1 & i_2 & \dots & i_{\ell-1} & i_{\ell} \\ j_1 & j_2 & \dots & j_{\ell-1} \end{pmatrix},$$

the corresponding tree is drawn inductively as follows. The terms i_k and i_{k+1} correspond to the height of two linear trees that are glued together, i.e., share the same root and then fork from height j_k . We proceed from right to left, so that the description agrees with the usual way of writing the composition of cells. As a first example, consider the following tables of dimensions

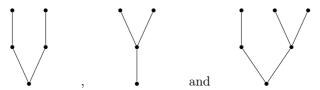
$$\begin{pmatrix} 2 & 2 \\ 0 & \end{pmatrix} , \begin{pmatrix} 2 & 2 \\ 1 & \end{pmatrix} , \begin{pmatrix} 2 & 2 & 2 \\ 0 & 1 \end{pmatrix},$$

and the associated objects of Θ

$$D_2 \coprod_{D_0} D_2 \quad , \quad D_2 \coprod_{D_1} D_2 \quad , \quad D_2 \coprod_{D_0} D_2 \coprod_{D_1} D_2 \, ,$$

that is the globular pasting schemes

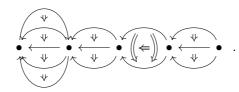
These objects correspond respectively to trees



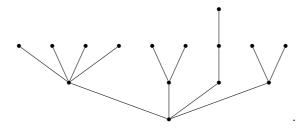
As another more involved example, consider the matrix of dimensions

$$\begin{pmatrix} 2 & 2 & 2 & 2 & 2 & 2 & 2 & 3 & 2 & 2 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

and the corresponding globular pasting scheme



The associated tree is given by



APPENDIX D. ORIENTALS

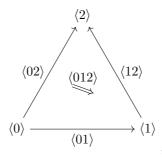
In this short section we present a visual intuition of the first few orientals. Then for every poset E we give a description of the ∞ -category $c_{\infty}N(E)$, that we shall simply denote by \mathcal{O}_E and call the *oriental of* E, and we deduce some of its basic properties, following closely [2, §6].

D.0.1. Fix an integer $n \geq 0$. For any $0 \leq i \leq n$ the *i*-chains of the strong Steiner complex $\mathsf{c}(\Delta^n)$ are the elements of the free abelian groups generated by the elements of the set $(\Delta^n)_i^{\mathrm{nd}}$, the set of non-degenerate *i*-simplices of the representable simplicial set Δ^n ; that is, the generators of $\mathsf{c}(\Delta^n)$ are the *i*-tuples (j_0, j_1, \ldots, j_i) of non-negative integers such that $0 \leq j_\ell < j_{\ell+1} \leq n$ for all $\ell = 0, \ldots, i-1$. For any such element of the basis of $\mathsf{c}(\Delta^n)_i$, we shall write $\langle j_0 j_1 \ldots j_i \rangle$ for the corresponding atom, instead of the more pedantic $\langle (j_0, j_1, \ldots, j_i) \rangle$ (see paragraph B.0.8).

D.0.2. The ∞ -categories \mathcal{O}_0 and D_0 are isomorphic. They are both terminal objects for the category ∞ - $\mathcal{C}at$ of small ∞ -categories and they corepresent the functor mapping any ∞ -category A to the set $\mathsf{Ob}\,A$ of its objects.

The ∞ -categories \mathcal{O}_1 and D_1 are isomorphic, too. They are both generated as ∞ -graphs by $\bullet \longrightarrow \bullet$ and they corepresent the functor mapping any ∞ -category A to the set $\mathsf{Cell}_1(A)$ of its 1-cells.

The ∞ -category \mathcal{O}_2 is a free ∞ -category, generated by the ∞ -graph



so that the 2-cell $\langle 012 \rangle$ has $\langle 02 \rangle$ as source and $\langle 12 \rangle *_0 \langle 01 \rangle$ as target. With the notations as above and as in paragraph B.0.8, we have

$$\langle i \rangle = \begin{pmatrix} (i) \\ (i) \end{pmatrix}$$
, for $i = 0, 1, 2$

for the objects,

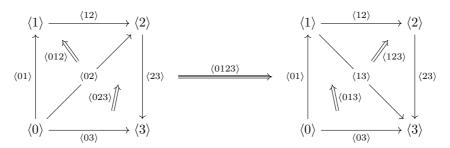
$$\langle ij \rangle = \begin{pmatrix} (i) & (i,j) \\ (j) & (i,j) \end{pmatrix}$$
 , for $i,j=0,1,2$ and $i < j$

for the 1-cells and

$$\langle 012 \rangle = \begin{pmatrix} (0) & (0,2) & (0,1,2) \\ (2) & (0,1) + (1,2) & (0,1,2) \end{pmatrix}$$

for the 2-cell.

The ∞ -category \mathcal{O}_3 is a free ∞ -category generated by the ∞ -graph



so that the 3-cell $\langle 0123 \rangle$ has the 2-cell

$$(\langle 23 \rangle *_0 \langle 012 \rangle) *_1 \langle 023 \rangle$$

as source and the 2-cell

$$(\langle 123 \rangle *_0 \langle 01 \rangle) *_1 \langle 013 \rangle$$

as target. Indeed we have

$$\langle 0123 \rangle = \begin{pmatrix} (0) & (01) + (12) + (23) & (012) + (023) & (0123) \\ (3) & (03) & (123) + (013) & (0123) \end{pmatrix}$$

The ∞ -category \mathcal{O}_4 is freely generated by the diagram displayed in figure 30, where we omitted the brackets $\langle \cdot \rangle$ for reasons of space.

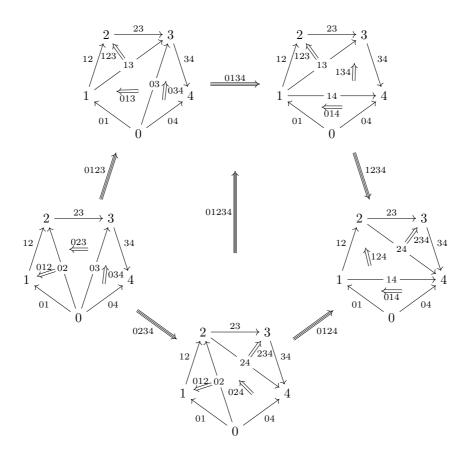


Figure 30. The ∞ -category \mathcal{O}_4 .

D.0.3. Let E be a poset. For any $p \ge 0$ the non-degenerated p-simplices of $N_{\infty}(E)$ are strictly increasing maps $\Delta^p \to E$, *i.e.*, the set $N_{\infty}(E)_p$ of non-degenerate p-simplices of the nerve of E consists of (p+1)-tuples

$$(x_0, x_1, \ldots, x_p)$$

of elements of E such that $x_i < x_{i+1}$ for all i = 0, 1, ..., p-1. The abelian group $(cN_{\infty}(E))_p$ (resp. abelian monoid $(cN_{\infty}(E))_p^*$) is freely generated by the set $N_{\infty}(E)_p^{\text{nd}}$. The differential is defined by

$$d(i_0, i_1, \dots, i_p) = \sum_{k=0}^{p} (-1)^k (i_0, i_1, \dots, \widehat{i_k}, \dots, i_p), \quad p > 0,$$

where $(i_0, \ldots, \widehat{i_k}, \ldots, i_p) = (i_0, \ldots, i_{k-1}, i_{k+1}, \ldots, i_p)$, and the augmentation by $e(i_0) = 1$.

Theorem D.0.4. The augmented directed complex $cN_{\infty}(E)$ is a strong Steiner complex.

Proof. This follows immediately from Theorem 8.6 of [2]. Indeed, using the notation of *loc. cit.*, for any poset E the simplicial set $N_{\infty}(E)$ is canonically isomorphic to $k^*(E, \xi E)$ (cf. paragraph 8.4 of [2]).

D.0.5. The previous theorem shows that the ∞ -category $\nu c N_{\infty}(E)$ is a strong Steiner category and it is more precisely freely generated by the atoms

$$\langle x_0 x_1 \dots x_p \rangle$$
, $p \geqslant 0$ and $x_0 < x_1 < \dots < x_p$.

This ∞ -category associated to the poset E shall be called the *oriental of* E and shall be denoted by \mathcal{O}_E .

Lemma D.0.6. The functor $\mathcal{O} \colon \mathcal{O}rd \to \infty\text{-}\mathcal{C}at$ associating to any poset E its oriental $\infty\text{-}category \mathcal{O}_E$ preserves monomorphisms.

Proof. This is Proposition 9.6 of [2].

We report here an important result of [2] giving a very explicit description of the "horizontal composition" of cells of the orientals.

Proposition D.0.7. Let $n \ge 1$, $m \ge 1$ and i_0, i_1, \dots, i_m be integers such that

$$0 = i_0 < i_1 < \dots < i_{m-1} < i_m = n$$
.

Then, the ∞ -functor

$$\prod_{k=1}^m \operatorname{Hom}_{\mathcal{O}_n}(a_k, b_k) \longrightarrow \operatorname{Hom}_{\mathcal{O}_n}(a, b) ,$$

where

$$a_k = \begin{pmatrix} (i_{k-1}) & (i_{k-1}, i_k) \\ (i_k) & (i_{k-1}, i_k) \end{pmatrix}, \quad b_k = \begin{pmatrix} (i_{k-1}) & \sum\limits_{i_{k-1} < l \leqslant i_k} (l-1, l) \\ (i_k) & \sum\limits_{i_{k-1} < l \leqslant i_k} (l-1, l) \end{pmatrix}, \quad 1 \leqslant k \leqslant m ,$$

$$a = \begin{pmatrix} (0) & \sum_{k=1}^{m} (i_{k-1}, i_k) \\ (n) & \sum_{k=1}^{m} (i_{k-1}, i_k) \end{pmatrix}, \quad b = \begin{pmatrix} (0) & \sum_{l=1}^{n} (l-1, l) \\ (n) & \sum_{l=1}^{n} (l-1, l) \end{pmatrix},$$

defined by the "horizontal composition" $*_0$ of \mathcal{O}_n

$$(x_1, x_2, \ldots, x_m) \longmapsto x_1 *_0 x_2 *_0 \cdots *_0 x_m$$

is an isomorphism of ∞ -categories.

Proof. This is Proposition A.4 of [2].

In section 5 we shall need few more properties of the hom- ∞ -categories of the oriental \mathcal{O}_E of a poset E that are proven in [2].

Proposition D.0.8. Let E be a poset and $s: \Delta^n \to E$ a non-degenerate n-simplex of N(E), with n > 0. Consider the 1-cell S of \mathcal{O}_E defined by

$$S = \sum_{i=0}^{n-1} \langle s_i, s_{i+1} \rangle.$$

Then the ∞ -functor $\mathcal{O}_s \colon \mathcal{O}_n \to \mathcal{O}_E$ induces an isomorphism

$$\underline{\mathsf{Hom}}_{\mathcal{O}_n}(\langle 0, n \rangle, \langle 0, 1 \rangle + \dots + \langle n - 1, n \rangle) \to \underline{\mathsf{Hom}}_{\mathcal{O}_E}(\langle s_0, s_n \rangle, S)$$

of ∞ -categories.

Proof. This is a particular case of Proposition 1.5 of [2], since it is clear that the ∞ -categories \mathcal{O}_n and what they denote by $\mathcal{O}(S)$ are canonically isomorphic. \square

Corollary D.0.9. Let E be a poset and s: $\Delta^n \to E$ a non-degenerate n-simplex of N(E), with n > 0. Consider the 1-cell S of \mathcal{O}_E defined by

$$S = \sum_{i=0}^{n-1} \langle s_i, s_{i+1} \rangle.$$

Then for any 1-cell

$$f = \sum_{i=0}^{m-1} \langle j_i, j_{i+1} \rangle$$

with $i_0 = 0$ and $i_m = n$ we have that the ∞ -functor $\iota_s = \mathcal{O}_s \colon \mathcal{O}_n \to \mathcal{O}_E$ induces an isomorphism

$$\underline{\mathsf{Hom}}_{\mathcal{O}_n}(f,\langle 0,1\rangle+\cdots+\langle n-1,n\rangle)\to \underline{\mathsf{Hom}}_{\mathcal{O}_E}(\iota_s(f),S)$$

of ∞ -categories.

Proof. This is an equivalent formulation of Proposition 1.5 of [2] and follows immediately from the preceding two propositions. \Box

Corollary D.0.10. Let $j: E \hookrightarrow F$ an inclusion of posets. Then for any two parallel 1-cells f and g of E, the ∞ -functor $\iota_j = \mathcal{O}_j: \mathcal{O}_E \to \mathcal{O}_F$ induces an isomorphism

$$\underline{\mathsf{Hom}}_{\mathcal{O}_E}(f,g) \to \underline{\mathsf{Hom}}_{\mathcal{O}_E}(\iota_j(f),\iota_j(g))$$

of ∞ -categories.

Proof. This follows immediately from the previous proposition by considering the simplex $\bar{g}: \Delta^n \to E$ of N(E), where

$$g = \sum_{i=0}^{n-1} \langle \bar{g}_i, \bar{g}_{i+1} \rangle. \qquad \Box$$

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