Lecture Notes in Category Theory [Draft]

8th November 2005

These notes are prepared by Mario Cáccamo from a graduate course on Category Theory given by Glynn Winskel in the fall of 1999 at BRICS, Aarhus. Heavy use has been made of latexed notes and exercises by Thomas Mailund and Mikkel Nygaard Hansen, under the editorial hand of Mario Cáccamo and Glynn Winskel. The course was derived from and inspired by Martin Hyland's Cambridge part III Mathematics course given in 1995, when notes were taken by Cocky Hillhorst.

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1 Categories, Functors and Natural Transformations

1.1 Categories

Definition 1.1 (Category) A *category* $\mathcal{C}$ consists of

- a collection of *objects* $\mathbf{ob}(\mathcal{C})$,
- for each pair of objects X, Y a collection $\mathcal{C}(X, Y)$ of *arrows/maps/morphisms*,
- for each triple of objects X, Y, Z a *composition* mapping $\circ_{XYZ} : \mathcal{C}(Y, Z) \times \mathcal{C}(X, Y) \to \mathcal{C}(X, Z)$ and
- for each object X an *identity* id_X in $\mathcal{C}(X, X)$;

satisfying the axioms

$$h \circ_{XZW} (g \circ_{XYZ} f) = (h \circ_{YZW} g) \circ_{XYW} f \qquad f \circ_{XXY} \mathsf{id}_X = f = \mathsf{id}_Y \circ_{XYY} f$$

for any f in $\mathcal{C}(X, Y)$, g in $\mathcal{C}(Y, Z)$ and h in $\mathcal{C}(Z, W)$ where X, Y, Z, W are in $\mathbf{ob}(\mathcal{C})$.

Notation 1.2 Given a category $\mathcal{C}$, we write $X \in \mathcal{C}$ to mean that X is an object in $\mathbf{ob}(\mathcal{C})$. In general we use capital letters from the begining or end of the alphabet for objects: $A, B, \ldots, X, Y, Z$. Given an arrow f in $\mathcal{C}(X, Y)$ we say that X is the *domain* of f and Y is the *codomain* of f. This is emphasised in the diagrammatic notation for categories where f in $\mathcal{C}(X, Y)$ is pictured as $X \xrightarrow{f} Y$; in the same spirit we sometimes write $f : X \to Y$. In general we use small letters for arrows: $f, g, h, \ldots, u, v, \ldots$. For f in $\mathcal{C}(X, Y)$ and g in $\mathcal{C}(Y, Z)$ we write $g \circ f$ for the composition omitting the subscripts; in some cases when the composition is clear we just write $g\,f$ instead of $g \circ f$:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "g\,f"'] & Y \arrow[r, "g"] & Z \end{tikzcd} .$$

Since composition is associative we sometimes omit parenthesis, for example we write $h \circ g \circ f$ to mean either $h \circ (g \circ f)$ or $(h \circ g) \circ f$. We use $\mathbf{arr}(\mathcal{C})$ for $\biguplus_{X,Y \in \mathcal{C}} \mathcal{C}(X, Y)$, the collection of all arrows in $\mathcal{C}$.

Definition 1.3 (Locally Small Category) A category $\mathcal{C}$ is *locally small* iff for each pair $X, Y \in \mathcal{C}$ the collection $\mathcal{C}(X, Y)$ is a set.

The category **Set** of (small) sets as objects and total functions between sets as arrows is a locally small category where composition is defined as composition of functions and identities correspond to the the identity functions.

Definition 1.4 (Small Category) A locally small category is *small* iff the collection of objects is a set.

Simple examples of small categories are **0** and **1**, the empty category and the singleton category respectively. Another example of small category is given by any partial order where the objects are the elements and the arrows are given by pairs of the partial-order relation; the identities correspond to reflexivity and composition to transitivity.

Exercise 1.5 Show that any set can be interpreted as a small category. In general categories of this kind are called *discrete* categories, but not all discrete categories are small.

Notation 1.6 Henceforth, we use $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ . . . to denote locally small categories and $\mathbb{I}$, $\mathbb{J}$, $\mathbb{C}$ . . . to small categories. Given a locally small category $\mathcal{C}$ we call the collection $\mathcal{C}(X, Y)$ the *hom-set* for $X, Y \in \mathcal{C}$.

Definition 1.7 (Isomorphism) An *isomorphism* (or just *iso*) is a pair of arrows

$$\begin{tikzcd} X \arrow[r, shift left, "f"] & Y \arrow[l, shift left, "g"] \end{tikzcd}$$

such that $g \circ f = \mathsf{id}_X$ and $f \circ g = \mathsf{id}_Y$.

Proposition 1.8 Given one arrow of an iso pair, the other arrow is uniquely determined.

Proof: Assume (f, g) and (f, g') are iso pairs, where $f : X \to Y$, then $(g' \circ f) \circ g = \mathsf{id}_X \circ g = g$ and $g' \circ (f \circ g) = g' \circ \mathsf{id}_Y = g'$. From associativity of the composition $(g' \circ f) \circ g = g' \circ (f \circ g)$ and then $g = g'$.

□

Notation 1.9 This proposition allows us to denote an iso pair by just one of the arrows. If $f : X \to Y$ is an arrow of an iso pair, we say that f is an *isomorphism* (or f is an *iso*), and we write $X \overset{f}{\cong} Y$ or $f : X \cong Y$.

Definition 1.10 (Mono) The arrow $f : X \to Y$ in the category $\mathcal{C}$ is *mono* (or *monic*) iff for any pair of arrows g, h

$$\begin{tikzcd} Z \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & X \arrow[r, "f"] & Y \end{tikzcd}$$

we have that $f \circ g = f \circ h$ implies $g = h$.

Exercise 1.11 Show that the composition of monos gives a mono. Show that if the composition $g \circ f$ is mono then f is mono. What are the monos in **Set**?

Definition 1.12 (Epi) The arrow $f : X \to Y$ in the category $\mathcal{C}$ is *epi* iff for any pair of arrows g, h Z and for all $g, h \in \mathcal{C}(Y, Z)$

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & Z \end{tikzcd}$$

we have that $g \circ f = h \circ f$ implies $g = h$.

Exercise 1.13 Show that the composition of epis gives an epi. Show that if the composition $g \circ f$ is epi then g is epi. What are the epis in **Set**?

1.2 Functors

Definition 1.14 (Functor) Let $\mathcal{A}$ and $\mathcal{B}$ be categories. A *functor* $F : \mathcal{A} \to \mathcal{B}$ consists of

- a mapping $F_0 : \mathbf{ob}(\mathcal{A}) \to \mathbf{ob}(\mathcal{B})$ and
- for each pair $X, Y \in \mathcal{A}$ a mapping $F_{XY} : \mathcal{A}(X, Y) \to \mathcal{B}\big(F_0(X), F_0(Y)\big)$;

satisfying the axioms

$$F_{XZ}(g \circ f) = F_{YZ}(g) \circ F_{XY}(f) \qquad F_{XX}(\mathsf{id}_X) = \mathsf{id}_{F_0(X)}$$

for any $f \in \mathcal{C}(X, Y)$ and $g \in \mathcal{C}(Y, Z)$.

Notation 1.15 For a functor $F : \mathcal{A} \to \mathcal{B}$ we usually write just F for F_0 and F_{XY}. Sometimes we write $F\,X$ instead of $F(X)$ for $X \in \mathcal{A}$ and similarly with arrows in $\mathcal{A}$. In general we use capital letters from the middle of the alphabet for functors, *e.g.*, F, G, H, K.

Definition 1.16 (Full Functor) A functor $F : \mathcal{A} \to \mathcal{B}$ is *full* iff for each pair $X, Y \in \mathcal{A}$ the mapping $F : \mathcal{A}(X, Y) \to \mathcal{B}(F\,X, F\,Y)$ is surjective.

Definition 1.17 (Faithful Functor) A functor $F : \mathcal{A} \to \mathcal{B}$ is *faithful* iff for each pair $X, Y \in \mathcal{A}$ the mapping $F : \mathcal{A}(X, Y) \to \mathcal{B}(F\,X, F\,Y)$ is injective.

For a functor F which is both full and faithful (*fully faithful*) the mappings $F : \mathcal{A}(X, Y) \to \mathcal{B}(F\,X, F\,Y)$ are all bijections.

Proposition 1.18 Functors preserve iso: if (f, g) is an iso pair in the category $\mathcal{A}$, and $F : \mathcal{A} \to \mathcal{B}$ is a functor, then $(F\,f, F\,g)$ is an iso pair in $\mathcal{B}$.

Proof: Let $\begin{tikzcd} X \arrow[r, shift left, "f"] & Y \arrow[l, shift left, "g"] \end{tikzcd}$ be iso in $\mathcal{A}$. We are to show that $\begin{tikzcd} F\,X \arrow[r, shift left, "F\,f"] & F\,Y \arrow[l, shift left, "F\,g"] \end{tikzcd}$ is iso in $\mathcal{B}$. This follows from

$$\begin{aligned} (F\,f) \circ (F\,g) &= F(f \circ g) \\ &= F\,\mathsf{id}_Y && \text{as } (f, g) \text{ is an iso pair,} \\ &= \mathsf{id}_{F\,Y}\ . \end{aligned}$$

Analogously, we can conclude $(F\,g) \circ (F\,f) = \mathsf{id}_{F\,X}$.

□

Proposition 1.19 If $F : \mathcal{A} \to \mathcal{B}$ is a full and faithful functor, then F reflects isomorphisms: if $F\,X \cong F\,Y$ then $X \cong Y$.

Proof: Assume $F\,X \cong F\,Y$. Then there exists a pair of arrows f, g

$$\begin{tikzcd} F\,X \arrow[r, shift left, "f"] & F\,Y \arrow[l, shift left, "g"] \end{tikzcd}$$

in $\mathcal{B}$ such that $f \circ g = \mathsf{id}_{F\,Y}$ and $g \circ f = \mathsf{id}_{F\,X}$. By fullness there exists $f' : X \to Y$ and $g' : Y \to X$ in $\mathcal{A}$ such that $F\,f' = f$ and $F\,g' = g$. We show that f' and g' are mutual inverses and so an iso pair:

$$\begin{aligned} F(f' \circ g') &= F\,f' \circ F\,g' \\ &= f \circ g \\ &= \mathsf{id}_{F\,Y} \\ &= F(\mathsf{id}_Y). \end{aligned}$$

By faithfulness we have $f' \circ g' = \mathsf{id}_Y$. Similarly, we can conclude $g' \circ f' = \mathsf{id}_X$.

□

Definition 1.20 (Composition of Functors) Given the functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ their composition $G \circ F : \mathcal{A} \to \mathcal{C}$ is defined as

- $(G \circ F)_0 = G_0 \circ F_0$ and
- $(G \circ F)_{XY} = G_{F_0(X) F_0(Y)} \circ F_{XY}$ for $X, Y \in \mathcal{A}$.

Exercise 1.21 Check that this definition gives rise to a functor.

It is easy to see that the composition of functors is associative. Moreover, for any category $\mathcal{A}$ we can define the *identity functor* $\mathsf{id}_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$ to be the functor which is the identity on both object and arrow parts. These observations allow us to define categories of categories where arrows are functors. In particular, we define the category **Cat** of small categories and **CAT** of locally small categories.

From the observation that functors are arrows in a category of categories we can specialised the general definitions of iso, epi and mono to functors. For example, if a functor F is defined as both a bijection on objects and arrows we can define a functor G to be the inverse of F, and thus F is an isomorphism.

Exercise 1.22 Why is a full and faithful functor not necessarily an isomorphism?

Definition 1.23 (Subcategory) A *subcategory* $\mathcal{A}$ of a category $\mathcal{B}$, written as $\mathcal{A} \subseteq \mathcal{B}$, is a collection of objects and arrows in $\mathcal{B}$ such that:

- for each arrow f in $\mathcal{A}$, we have that the domain and codomain of f are in $\mathcal{A}$,
- for each object A in $\mathcal{A}$, we have the arrow id_A in $\mathcal{A}$, and

- for each pair of composable arrows $f : A \to B$ and $g : B \to C$ in $\mathcal{A}$, we have the composition $g \circ f : A \to C$ in $\mathcal{A}$.

Clearly, the definition above ensures that a subcategory is itself category. The mapping which sends objects and arrows in $\mathcal{A}$ to themselves in $\mathcal{B}$ defines the *inclusion* functor. This functor is faithful and if it is full we say that $\mathcal{A}$ is a *full subcategory* of $\mathcal{B}$.

1.3 Natural Transformations

Definition 1.24 (Natural Transformation) Given the functors $F, G : \mathcal{A} \to \mathcal{B}$, a *natural transformation* $\alpha : F \Rightarrow G$ consists of a collection of arrows $\langle \alpha_C : F\,C \to G\,C \rangle_{C \in \mathcal{A}}$ in $\mathcal{B}$, such that for any arrow $h : X \to Y$ in $\mathbf{arr}(\mathcal{A})$ the diagram

$$\begin{tikzcd}
F\,X \arrow[r, "\alpha_X"] \arrow[d, "F\,h"] & G\,X \arrow[d, "G\,h"] \\
F\,Y \arrow[r, "\alpha_Y"'] & G\,Y
\end{tikzcd}$$

commutes, *i.e.*, $\alpha_Y \circ F\,h = G\,h \circ \alpha_X$. The diagram above is the *naturality square* associated with h.

Notation 1.25 A natural transformation $\alpha : F \Rightarrow G$ for functors $F, G : \mathcal{A} \to \mathcal{B}$ is a family of arrows in $\mathcal{B}$ indexed by objects in $\mathcal{A}$. Each arrow of such a family is a *component* of the natural transformation, and we use subscripts to denote them, *e.g.*, $\alpha_C : F\,C \to G\,C$. We use angle brackets to denote the family of arrows as in the definition $\langle \alpha_C : F\,C \to G\,C \rangle_{C \in \mathcal{A}}$. Naturality squares are often depicted with the arrow which they are associated with on one side, recall that the arrow and the naturality square may be from different categories.

Definition 1.26 (Composition of Natural Transformations) The composition of natural transformations is defined componentwise: given two natural transformations $\alpha : F \Rightarrow G$ and $\beta : G \Rightarrow H$

$$\begin{tikzcd}
\mathcal{A} \arrow[r, bend left=60, "F"] \arrow[r, "G"] \arrow[r, bend right=60, "H"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\beta"] & \mathcal{B}
\end{tikzcd}$$

their composition $\beta \circ \alpha$ at component $A \in \mathcal{A}$ is $(\beta \circ \alpha)_A = \beta_A \circ \alpha_A$.

Exercise 1.27 Check that the componentwise composition of two natural transformations is a natural transformation.

This composition is often called *vertical* composition to distinguish it from the *horizontal* composition of natural transformations – see exercise 1.29 below. Given the functors $F : \mathcal{A} \to \mathcal{B}$ and $K : \mathcal{C} \to \mathcal{D}$ and a natural transformation $\alpha : G \Rightarrow H$

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "F"] & \mathcal{B} \arrow[r, bend left, "G", ""{name=U, below}] \arrow[r, bend right, "H"', ""{name=D}] & \mathcal{C} \arrow[r, "K"] & \mathcal{D}
\arrow[Rightarrow, from=U, to=D, "\alpha"]
\end{tikzcd}$$

we define the composition $\alpha F : G \circ F \Rightarrow H \circ F$ componentwise as $(\alpha F)_A = \alpha_{F(A)}$ for $A \in \mathcal{A}$. Similarly the composition $K \alpha : K \circ G \Rightarrow K \circ H$ at component $B \in \mathcal{B}$ is defined by $(K \alpha)_B = K(\alpha_B)$.

Exercise 1.28 Check that the composition of natural transformations with functors defined above are natural transformations.

Exercise 1.29 Given the natural transformations α and β

$$\begin{tikzcd}
\mathcal{A} \arrow[r, bend left, "F", ""{name=U1, below}] \arrow[r, bend right, "G"', ""{name=D1}] & \mathcal{B} \arrow[r, bend left, "H", ""{name=U2, below}] \arrow[r, bend right, "K"', ""{name=D2}] & \mathcal{C} ,
\arrow[Rightarrow, from=U1, to=D1, "\alpha"]
\arrow[Rightarrow, from=U2, to=D2, "\beta"]
\end{tikzcd}$$

we can define the *horizontal* composition $\beta \star \alpha : H \circ F \Rightarrow K \circ G$ – how? **Hint:** The composition of natural transformations with functors is a degenerate case of the composition $\star$ where the functor stands for the identity natural transformation, *i.e.*, a natural transformation which has identities at each component.

1.4 Functor Categories

Natural transformations between functors can be (vertically) composed and it is easy to check that this composition is associative. Moreover, given a functor $F : \mathcal{A} \to \mathcal{B}$ we can define the identity $\mathsf{id}_F : F \Rightarrow F$ to be the natural transformation with identity at all components: $(\mathsf{id}_F)_C = \mathsf{id}_{F(C)}$. That actually defines a category of functors and natural transformations.

Definition 1.30 (Functor Category) Given the categories $\mathcal{A}$ and $\mathcal{B}$, the *functor category* $[\mathcal{A}, \mathcal{B}]$ is the category where the objects are the functors from $\mathcal{A}$ to $\mathcal{B}$ and the arrows are the natural transformations between them.

As natural transformations are arrows in a functor category we can specialised the notions of iso, mono and epi to them. For example, a natural transformation $\alpha : F \Rightarrow G$ is an iso, called *natural isomorphism*, if there is a natural transformation $\beta : G \Rightarrow F$ such that $\beta \circ \alpha = \mathsf{id}_F$ and $\alpha \circ \beta = \mathsf{id}_G$.

Proposition 1.31 A natural transformation is an isomorphism of functors iff it is an iso at each component.

Exercise 1.32 Prove the proposition above.

Notation 1.33 The category of functors with domain $\mathcal{A}$ and codomain $\mathcal{B}$ is noted by $[\mathcal{A}, \mathcal{B}]$; some authors use $\mathcal{B}^{\mathcal{A}}$. In general we use Greek letters to denote natural transformations. We depict natural transformations as double arrows only when we want to stress we are dealing with natural transformations.

Exercise 1.34 The category $[\mathcal{A}, \mathcal{B}]$ need not be locally small even though $\mathcal{A}$ and $\mathcal{B}$ are locally small. However, when $\mathcal{A}$ is small $[\mathcal{A}, \mathcal{B}]$ is locally small – why?

Exercise 1.35 Let $\mathcal{C}$ be a category, a functor $F : \mathcal{A} \to \mathcal{B}$ induces a functor $F^* : [\mathcal{B}, \mathcal{C}] \to [\mathcal{A}, \mathcal{C}]$ where the action over objects is defined by precomposition with F. What is the action on arrows of F^*?

2 Constructions on Categories

2.1 Opposite Category and Contravariance

Definition 2.1 (Opposite Category) Let $\mathcal{C}$ be a category, we define the opposite category $\mathcal{C}^{\mathsf{op}}$ as:

- $\mathbf{ob}(\mathcal{C}^{\mathsf{op}}) = \mathbf{ob}(\mathcal{C})$, and
- $\mathcal{C}^{\mathsf{op}}(C, D) = \mathcal{C}(D, C)$ for each pair $C, D \in \mathcal{C}^{\mathsf{op}}$.

In words, $\mathcal{C}^{\mathsf{op}}$ is the category with the same collection of objects as $\mathcal{C}$ where the arrows are reversed. Clearly, the identities in $\mathcal{C}^{\mathsf{op}}$ correspond to the identities in $\mathcal{C}$. Some authors prefer to write f^{op} for the arrow f in the opposite category, however, in general there is no reason to make such a distinction. The composition of arrows in $\mathcal{C}^{\mathsf{op}}$ can be expressed in terms of the arrows in $\mathcal{C}$ by reading the composition in the reverse order

$$\begin{tikzcd} \mathcal{C}^{\mathsf{op}} : X \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & Y \arrow[r, "g"] & Z & \mathcal{C} : X & Y \arrow[l, "f"'] & Z \arrow[l, "g"'] \arrow[ll, bend left, "f \circ g"] . \end{tikzcd}$$

A *contravariant functor* $F : \mathcal{C} \to \mathcal{D}$ is essentially a functor which reverses the arrows

$$\begin{tikzcd} A \arrow[d, "f"] & F(A) \\ B \arrow[ur, phantom, "f \longmapsto F(f)"] & F(B) . \arrow[u] \end{tikzcd}$$

In contrast, the functors as defined in section 1.2 are called *covariant.* It is more convenient to identify a contravariant functor $F : \mathcal{C} \to \mathcal{D}$ with a covariant functor $F : \mathcal{C}^{\mathsf{op}} \to \mathcal{D}$.

functions.

Exercise 2.2 Show the opposite construction gives rise to a functor $(-)^{\mathsf{op}} : \mathbf{CAT} \to \mathbf{CAT}$ acting on objects and arrows

$$\begin{tikzcd}
\mathcal{C} \arrow[d, "F"] & \mathcal{C}^{\mathsf{op}} \arrow[d, "F^{\mathsf{op}}"] \\
\mathcal{D} & \mathcal{D}^{\mathsf{op}} .
\end{tikzcd}$$

2.2 Product of Categories

Definition 2.3 (Product of Categories) Given the categories $\mathcal{C}$ and $\mathcal{D}$, we define the product category $\mathcal{C} \times \mathcal{D}$ as:

- $\mathbf{ob}(\mathcal{C} \times \mathcal{D}) = \mathbf{ob}(\mathcal{C}) \times \mathbf{ob}(\mathcal{D})$,
- $\mathcal{C} \times \mathcal{D}\big((C, D), (C', D')\big) = \mathcal{C}(C, C') \times \mathcal{D}(D, D')$ for each $\big((C, D), (C', D')\big) \in \mathcal{C} \times \mathcal{D}$, and
- composition and identities are defined componentwise.

A functor whose domain is a product of categories is called a *bifunctor.*.

In general a mapping M on objetcs and arrows in $\mathcal{C} \times \mathcal{D}$ to objects and arrows in $\mathcal{E}$ determines mappings $M(C, -)$ from $\mathcal{D}$ to $\mathcal{E}$ for each $C \in \mathcal{C}$ defined as follows

$$\begin{tikzcd}
D \arrow[d, "g"] & M(C, D) \arrow[d, "M(C,g)=M(\mathsf{id}_C,g)"] \\
D' & M(C, D') .
\end{tikzcd}$$

Similarly M defines mappings $M(-, D)$ for every $D \in \mathcal{D}$. Notice that M is not necessarily a bifunctor, and from the following proposition we cannot in general conclude that M is a bifunctor even when $M(C, -)$ and $M(-, D)$ are.

Proposition 2.4 Suppose F is a mapping on objects and arrows in $\mathcal{C} \times \mathcal{D}$ such that

- $F(C, D) \in \mathcal{E}$ for any $C \in \mathcal{C}$ and $D \in \mathcal{D}$, and
- $F(f, g) \in \mathcal{E}\big(F(C, D), F(C', D')\big)$ if $f \in \mathcal{C}(C, C')$ and $g \in \mathcal{D}(D, D')$.

F is a bifunctor from $\mathcal{C} \times \mathcal{D}$ to $\mathcal{E}$ iff

1. $F(C, -)$ is a functor from $\mathcal{D}$ to $\mathcal{E}$ for all $C \in \mathcal{C}$,
2. $F(-, D)$ is a functor from $\mathcal{C}$ to $\mathcal{E}$ for all $D \in \mathcal{D}$, and

3. $F(f, g) = F(C', g) \circ F(f, D) = F(f, D') \circ F(C, g)$, *i.e.*, the diagram

$$\begin{tikzcd}
F(C,D) \arrow[r, "F(f,D)"] \arrow[d, "F(C,g)"'] \arrow[dr, "F(f,g)"] & F(C',D) \arrow[d, "F(C',g)"] \\
F(C,D') \arrow[r, "F(f,D')"'] & F(C',D')
\end{tikzcd}$$

commmutes for all $f : C \to C'$ in $\mathcal{C}$ and $g : D \to D'$ in $\mathcal{D}$.

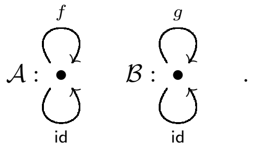
Proof:
"Only-if": if F is a functor then $F(C, -)$ and $F(-, D)$ are functors, and F must satisfy 3 since $(f, g) = (f, \mathsf{id}_{D'}) \circ (\mathsf{id}_C, g) = (\mathsf{id}_{C'}, g) \circ (f, \mathsf{id}_D)$.
"If": we need to prove F satisfies the functor laws. It is easy to show $F(\mathsf{id}_{(C,D)}) = \mathsf{id}_{F(C,D)}$ by using the fact that $F(C, -)$ is a functor (or equivalently that $F(-, D)$ is a functor). It remains to prove F preserves composition; assume $C \xrightarrow{f} C' \xrightarrow{f'} C''$ and $D \xrightarrow{g} D' \xrightarrow{g'} D''$, then

$$\begin{aligned}
F\big((f', g') \circ (f, g)\big) &= F(f' \circ f, g' \circ g) && \text{by comp. in } \mathcal{C} \times \mathcal{D}, \\
&= F(C'', g' \circ g) \circ F(f' \circ f, D) && \text{by 3,} \\
&= \Big(F(C'', g') \circ F(C'', g)\Big) \circ \Big(F(f', D) \circ F(f, D)\Big) && \text{by 1 and 2,} \\
&= F(C'', g') \circ \Big(F(C'', g) \circ F(f', D)\Big) \circ F(f, D) && \text{by associativity,} \\
&= F(C'', g') \circ \Big(F(f', D') \circ F(C', g)\Big) \circ F(f, D) && \text{by 3,} \\
&= F(f', g') \circ F(f, g) && \text{by associativity and twice 3.}
\end{aligned}$$

□

The next example considers a mapping satisfying conditions 1 and 2 but not 3.

Example 2.5 Let $\mathcal{A}$ and $\mathcal{B}$ be one-object categories



Let F be a mapping from $\mathcal{A} \times \mathcal{A}$ to $\mathcal{B}$ which sends $(\bullet, \bullet)$ to the only object in $\mathcal{B}$ and acts on arrows as follows

$$\begin{aligned}
F(f,f) &= g \\
F(\mathsf{id},f) &= \mathsf{id} = F(\bullet,f) \\
F(\mathsf{id},\mathsf{id}) &= \mathsf{id} = F(\bullet,\mathsf{id}) = F(\mathsf{id},\bullet) \\
F(f,\mathsf{id}) &= \mathsf{id} = F(f,\bullet).
\end{aligned}$$

One can check $F(\bullet,-)$ and $F(-,\bullet)$ are functors from $\mathcal{A}$ to $\mathcal{B}$, but F does not satisfy condition 3 of Proposition 2.4

$$F(f,f) = g$$
$$F(\mathsf{id},f) \circ F(f,\mathsf{id}) = \mathsf{id} \circ \mathsf{id} = \mathsf{id}\ .$$

Hence,

$$F\big((\mathsf{id},f) \circ (f,\mathsf{id})\big) = F(f,f) = g \neq F(\mathsf{id},f) \circ F(f,\mathsf{id})\ .$$

Therefore, F is not a functor.

2.3 Natural Transformations between Bifunctors

Given the bifunctors $F,G : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$, consider the natural transformation α

$$\begin{tikzcd}
\mathcal{C} \times \mathcal{D} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "G"'{name=D}] & \mathcal{E} \arrow[Rightarrow, from=U, to=D, "\alpha"]
\end{tikzcd}\ .$$

From the definition of natural transformation α is a family of arrows in $\mathcal{E}$ indexed by objects in $\mathcal{C} \times \mathcal{D}$ such that given an arrow (f,g) in $\mathcal{C} \times \mathcal{D}$ the square

$$\begin{tikzcd}
(C,D) \arrow[d, "{(f,g)}"'] & F(C,D) \arrow[r, "\alpha_{C,D}"] \arrow[d, "{F(f,g)}"'] & G(C,D) \arrow[d, "{G(f,g)}"] \\
(C',D') & F(C',D') \arrow[r, "\alpha_{C',D'}"'] & G(C',D')
\end{tikzcd}$$

commutes. This definition induces two natural transformations by fixing one of the arguments:

- $\alpha_{C,-} : F(C,-) \Rightarrow G(C,-)$ where $(\alpha_{C,-})_D = \alpha_{C,D}$, and
- $\alpha_{-,D} : F(-,D) \Rightarrow G(-,D)$ where $(\alpha_{-,D})_C = \alpha_{C,D}$.

Proposition 2.6 Given the bifunctors $F,G : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$, a family of maps $\langle \alpha_{C,D} : F(C,D) \to G(C,D) \rangle_{(C,D) \in \mathcal{C} \times \mathcal{D}}$ in $\mathcal{E}$ is a natural transformation iff

1. $\alpha_{C,-} = \langle \alpha_{C,D} \rangle_{D \in \mathcal{D}}$ is a natural transformation from $F(C,-)$ to $G(C,-)$ for any $C \in \mathcal{C}$, and

2. $\alpha_{-,D} = \langle \alpha_{C,D} \rangle_{C \in \mathcal{C}}$ is a natural transformation from $F(-, D)$ to $G(-, D)$ for any $D \in \mathcal{D}$.

Proof:
"Only-if": left as exercise.
"If": It follows from exhibing the arrow

$$(C, D) \xrightarrow{(f,g)} (C', D')$$

as the composition

$$(C, D) \xrightarrow{(f,\mathsf{id}_D)} (C', D) \xrightarrow{(\mathsf{id}_{C'},g)} (C', D') \ .$$

Consider the associated squares:

$$\begin{tikzcd}
(C, D) \arrow[d, "(f,\mathsf{id}_D)"] & F(C, D) \arrow[r, "(\alpha_{-,D})_C=\alpha_{C,D}"] \arrow[d, "F(f,\mathsf{id}_D)=F(f,D)"] & G(C, D) \arrow[d, "G(f,\mathsf{id}_D)=G(f,D)"] \\
(C', D) \arrow[d, "(\mathsf{id}_{C'},g)"] & F(C', D) \arrow[r, "(\alpha_{-,D})_{C'}=\alpha_{C',D}=(\alpha_{C',-})_D"] \arrow[d, "F(\mathsf{id}_{C'},g)=F(C',g)"] & G(C', D) \arrow[d, "G(\mathsf{id}_{C'},g)=G(C',g)"] \\
(C', D') & F(C', D') \arrow[r, "(\alpha_{C',-})_{D'}=\alpha_{C',D'}"] & G(C', D')
\end{tikzcd}$$

From 2 we have the commutativity of the upper square and from 1 of the lower square. Hence the outermost square commutes

$$\begin{aligned}
\alpha_{C',D'} \circ F(f, g) &= (\alpha_{C',-})_{D'} \circ F\big((\mathsf{id}_{C'}, g) \circ (f, \mathsf{id}_D)\big) \\
&= (\alpha_{C',-})_D \circ F(\mathsf{id}_{C'}, g) \circ F(f, \mathsf{id}_D) && \text{as } F \text{ is functor,} \\
&= G(\mathsf{id}_{C'}, g) \circ G(f, \mathsf{id}_D) \circ (\alpha_{-,D})_C && \text{by commutativity of the squares,} \\
&= G\big((\mathsf{id}_{C'}, g) \circ (f, \mathsf{id}_D)\big) \circ (\alpha_{-,D})_C && \text{as } G \text{ is functor,} \\
&= G(f, g) \circ \alpha_{C,D} \ .
\end{aligned}$$

Therefore we can conclude α is a natural transformation from F to G.

□

Thus, naturality may be checked in each argument separately when dealing with bifunctors.

Exercise 2.7 Prove the *"Only-if"* part in the proof of Proposition 2.6.

Remark 2.8 (Functoriality and Naturality) Suppose $E(X)$ is an expression, with an argument X, such that given an object $A \in \mathcal{A}$, the expression $E(A)$ stands for an object in the category $\mathcal{C}$; and given an arrow $f \in \mathbf{arr}(\mathcal{A})$, the expression $E(f)$ stands for an arrow in $\mathcal{C}$. We say that $E(X)$ is *functorial* in X, as X ranges over $\mathcal{A}$, if this assignment yields a functor from $\mathcal{A}$ to $\mathcal{C}$. For example, for a category $\mathcal{C}$, the hom-set expression $\mathcal{C}(X, Y)$ is functorial in X ranging over $\mathcal{C}^{\mathsf{op}}$ and Y over $\mathcal{C}$.

Traditionally, arguments in expressions are denoted by place-holders like for example $\mathcal{C}(-, +)$. This notation might lead to some confusion specially in distinguishing the scope where each argument is defined. To save this potential source of confution we sometimes use λ-notation to indicate arguments in expressions. Thus, assuming $E(X)$ is functorial in X of type $\mathcal{A}$ then the expression $\lambda X.\, E(X)$ denotes a functor from $\mathcal{A}$ to $\mathcal{C}$.

We say that the expressions $E_1(X)$ and $E_2(X)$ both functorial in X are *naturally* isomorphic in X or equivantly that there is an isomorphism *natural* in X, iff there is a natural isomorphism between $\lambda X.\, E_1(X)$ and $\lambda X.\, E_2(X)$.

Exercise 2.9 Prove that a bifunctor $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ defines a unique *curried* functor $\lambda X.\, \lambda Y.\, F(X, Y) : \mathcal{A} \to [\mathcal{B}, \mathcal{C}]$. Similarly, show how a natural transformation $\alpha : F \Rightarrow G$ for functors $F, G : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ uniquely defines a natural transformation from $\lambda X.\, \lambda Y.\, F(X, Y)$ to $\lambda X.\, \lambda Y.\, G(X, Y)$.

2.4 Examples

Evaluation Functor: Define the *evaluation* functor $\mathit{eval} : [\mathcal{C}, \mathcal{D}] \times \mathcal{C} \to \mathcal{D}$ for categories $\mathcal{C}$ and $\mathcal{D}$ as follows

$$\begin{tikzcd}
(F, C) \arrow[d, "{(\alpha, g)}"] \arrow[r, mapsto] & F(C) \arrow[d, "F'(g) \circ \alpha_C = \alpha_{C'} \circ F(g)"] \\
(F', C') \arrow[r, mapsto] & F'(C')
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r, mapsto] & {}
\end{tikzcd}$$

where $\alpha : F \Rightarrow F'$ and $g : C \to C'$. The arrow g induces the natural square which justifies the action of *eval* on the arrow (α, g)

$$\begin{tikzcd}
C \arrow[d, "g"] & FC \arrow[r, "\alpha_C"] \arrow[d, "Fg"] & F'C \arrow[d, "F'g"] \\
C' & FC' \arrow[r, "\alpha_{C'}"] & F'C'.
\end{tikzcd}$$

We use Proposition 2.6 to show *eval* is a bifunctor:

1. $\mathit{eval}(F, -) : \mathcal{C} \to \mathcal{D}$ which sends $g : C \to C'$ to $F\,g : F\,C \to F\,C'$ is a functor since F is a functor;

2. $\mathit{eval}(-, C) : [\mathcal{C}, \mathcal{D}] \to \mathcal{D}$ which sends $\alpha : F \Rightarrow F'$ to $\alpha_C : F\,C \to F'\,C$ is a functor since the identity natural transformation has identities in each component and the composition of natural transformations is defined componentwise; and

3. finally we want

   $$eval(\alpha, g) = eval(F', g) \circ eval(\alpha, C) = eval(\alpha, C') \circ eval(F, g)$$

   from the definition of *eval* this is

   $$eval(\alpha, g) = F'\, g \circ \alpha_C = \alpha_{C'} \circ F\, g$$

   and this follows from the naturality square above.

Hom-functor: Recall that the hom-set $\mathcal{C}(C, D)$ is defined to be the set of arrows with domain C and codomain D in the locally small category $\mathcal{C}$. Here we show how hom-sets extend to bifunctors.

The *hom-functor* for a category $\mathcal{C}$ is a functor $\mathcal{C}(-,-) : \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathbf{Set}$ which sends a pair of objects (C, D) to the hom-set $\mathcal{C}(C, D)$ and acts on arrows by sending $(f, g) : (C, D) \to (C', D')$ to

$$\mathcal{C}(C, D) \xrightarrow{\mathcal{C}(f,g) = g \circ - \circ f} \mathcal{C}(C', D')$$

where $\mathcal{C}(f, g)$ is a function mapping any arrow $C \xrightarrow{h} D$ to the arrow $C' \xrightarrow{f} C \xrightarrow{h} D \xrightarrow{g} D'$. Informally, to obtain an arrow from C' to D' out of an arrow from C to D, we need both a way to go from D to D' and a way to go from C' to C.

Exercise 2.10 Check that the definition of the hom-functor above is actually a bifunctor.

From Proposition 2.4 we have a functor $\mathcal{C}(-, D) : \mathcal{C}^{\mathsf{op}} \to \mathbf{Set}$ which acts on objects and arrows

$$\begin{tikzcd}
C \arrow[d, "f"] & \mathcal{C}(C, D) \\
C' & \mathcal{C}(C', D) \arrow[u, "\mathcal{C}(-,D) = - \circ f"'] .
\end{tikzcd}$$

Notice that the arrow f drawn above is in the category $\mathcal{C}$ not $\mathcal{C}^{\mathsf{op}}$; that explains why the direction of the arrow is reversed after applying $\mathcal{C}(-, D)$. In the same manner we can define a functor $\mathcal{C}(C, -) : \mathcal{C} \to \mathbf{Set}$.

Yoneda Functor: Given the category of functors and natural transformations $[\mathcal{C}^{\mathsf{op}}, \mathbf{Set}]$, we define the *Yoneda* functor $\mathcal{Y} : \mathcal{C} \to [\mathcal{C}^{\mathsf{op}}, \mathbf{Set}]$ which takes an object $C \in \mathcal{C}$ to the functor $\mathcal{C}(-, C)$ and acts on arrows as follows

$$\begin{tikzcd}
C \arrow[d, "g"] & \mathcal{Y}\, C = \mathcal{C}(-, C) \arrow[d, "\mathcal{Y}\, g = \langle g \circ - \rangle_{E \in \mathcal{C}}"] \\
D & \mathcal{Y}\, D = \mathcal{C}(-, D).
\end{tikzcd}$$

Exercise 2.11 Check that $\mathcal{Y}$ defines a functor from $\mathcal{C}$ to $[\mathcal{C}^{\mathsf{op}}, \mathbf{Set}]$.

Dually, we define the contravariant version of the Yoneda functor $\mathcal{Y}^o : \mathcal{C}^{\mathsf{op}} \to [\mathcal{C}, \mathbf{Set}]$ which acts on objects and arrows

$$\begin{tikzcd}
C \arrow[d, "g"] & \mathcal{Y}^o\, C = \mathcal{C}(C, -) \\
D & \mathcal{Y}^o\, D = \mathcal{C}(D, -)\ . \arrow[u, "\mathcal{Y}^o\, g = \langle - \circ g \rangle_{E \in \mathcal{C}}"']
\end{tikzcd}$$

Constant Functor: Given $Y \in \mathcal{D}$, we write $\Delta\, Y : \mathcal{C} \to \mathcal{D}$ for the constantly Y functor which acts over objects and arrows as follows:

$$\begin{tikzcd}
C \arrow[d, "f"] & \Delta\, Y(C) = Y \arrow[d, "\Delta\, Y(f) = \mathsf{id}_Y"] \\
C' & \Delta\, Y(C') = Y
\end{tikzcd}$$

This may be extended to the functor $\Delta - : \mathcal{D} \to [\mathcal{C}, \mathcal{D}]$ which sends $g : D \to D'$ to $\Delta g : \Delta D \Rightarrow \Delta D'$ where Δg is just the arrow g at all components.

Exercise 2.12 Check that $\Delta-$ is a functor from $\mathcal{D}$ to $[\mathcal{C}, \mathcal{D}]$.

3 Yoneda Lemma and Universal Properties

3.1 Yoneda Lemma

Proposition 3.1 Let $F : \mathcal{C}^{\mathsf{op}} \to \mathbf{Set}$ be a functor, we have that each element $x \in F(C)$ for some $C \in \mathcal{C}$ induces a natural transformation $\widetilde{x} : \mathcal{Y}\, C = \mathcal{C}(-, C) \Rightarrow F$ whose component at an arbitrary $D \in \mathcal{C}$ is

$$\widetilde{x}_D = F(-)(x)\ . \tag{1}$$

Proof: We need to prove that for an arbitrary arrow $g : E \to D$ in $\mathcal{C}$ the diagram

$$\begin{tikzcd}
\mathcal{C}(D, C) \arrow[r, "\widetilde{x}_D"] \arrow[d, "- \circ g"] & F(D) \arrow[d, "F(g)"] \\
\mathcal{C}(E, C) \arrow[r, "\widetilde{x}_E"] & F(E)
\end{tikzcd}$$

commutes. For any $f \in \mathcal{C}(D, C)$ we have

$$F(f \circ g)(x) = F(g)(F(f)(x))$$

since F is a functor. This gives the commutativity required. Recall F is a contravariant functor and so reverses the order of composition in $\mathcal{C}$.

□

Proposition 3.2 Let $F : \mathcal{C}^{\mathsf{op}} \to \mathbf{Set}$ be a functor; given a natural transformation $\alpha : \mathcal{C}(-, C) \Rightarrow F$ we have $\alpha = \widetilde{\alpha_C(\mathsf{id}_C)}$.

Proof: Given an arbitrary $f \in \mathcal{C}(D, C)$, we need to check that $\alpha_D(f) = F(f)(\alpha_C(\mathsf{id}_C))$. This equality is obtained by chasing the identity id_C around the naturality square determined by f.

□

Theorem 3.3 (Yoneda Lemma) The mapping $\theta_{C,F}$ which takes α in $[\mathcal{C}^{\mathsf{op}}, \mathbf{Set}](\mathcal{Y}\,C, F)$ to $\alpha_C(\mathsf{id}_C) \in F(C)$ is an isomorphism

$$[\mathcal{C}^{\mathsf{op}}, \mathbf{Set}](\mathcal{Y}\,C, F) \overset{\theta_{C,F}}{\cong} F(C)$$

natural in $C \in \mathcal{C}$ and $F \in [\mathcal{C}^{\mathsf{op}}, \mathbf{Set}]$.

Remark 3.4 Notice that the naturality condition in the statement of the Yoneda lemma uses expressions which are functorial in their arguments. The right hand side corresponds to the application of the evaluation functor *eval* : $\mathcal{C}^{\mathsf{op}} \times [\mathcal{C}^{\mathsf{op}}, \mathbf{Set}] \to \mathbf{Set}$. The right hand side is a more complex expression that involves composition of functors:

$$\mathcal{C}^{\mathsf{op}} \times [\mathcal{C}^{\mathsf{op}}, \mathbf{Set}] \xrightarrow{\mathcal{Y}^{\mathsf{op}} \times \mathsf{id}} [\mathcal{C}^{\mathsf{op}}, \mathbf{Set}]^{\mathsf{op}} \times [\mathcal{C}^{\mathsf{op}}, \mathbf{Set}] \xrightarrow{[\mathcal{C}^{\mathsf{op}}, \mathbf{Set}](-,+)} \mathbf{Set}.$$

Proof: We show that $\widetilde{(\ )}$ defined by (1) is the inverse of $\theta_{C,F}$. From Proposition 3.2 we already have that $\widetilde{\theta_{C,F}(\alpha)} = \widetilde{\alpha_C(\mathsf{id}_C)} = \alpha$. Let x be an element in $F(C)$

$$\begin{aligned} \theta_{C,F}(\widetilde{x}) &= \widetilde{x}_C(\mathsf{id}_C) && \text{by definition of } \theta_{C,F}, \\ &= F(\mathsf{id}_C)(x) && \text{by definition of } \widetilde{(\ )}, \\ &= \mathsf{id}_{F(C)}(x) && \text{as } F \text{ is a functor,} \\ &= x\ . \end{aligned}$$

Thus we complete the prove that $\theta_{C,F}$ is an isomorphism of sets.

It remains to prove naturality in $C \in \mathcal{C}$ and $F \in [\mathcal{C}^{\mathsf{op}}, \mathbf{Set}]$. By Proposition 2.6, this is equivalent to proving naturality in C and F separately.

Naturality in C: Given an arrow $f : C \to D$ in $\mathcal{C}$, we need to prove

$$\begin{tikzcd} {[\mathcal{C}^{\mathsf{op}}, \mathbf{Set}](\mathcal{Y}\,C, F)} \arrow[r, "\theta_{C,F}"] & F(C) \\ {[\mathcal{C}^{\mathsf{op}}, \mathbf{Set}](\mathcal{Y}\,D, F)} \arrow[r, "\theta_{D,F}"] \arrow[u, "-\circ(\mathcal{Y}\,f)"'] & F(D) \arrow[u, "F(f)"'] \end{tikzcd}$$

commutes. Given $\beta \in [\mathcal{C}^{\mathsf{op}}, \mathbf{Set}](\mathcal{Y}\,D, F)$, we have

$$\begin{aligned}
\theta_{C,F}(\beta \circ (\mathcal{Y}\,f)) &= (\beta \circ (\mathcal{Y}f))_C(\mathsf{id}_C) && \text{by definition of } \theta_{C,F}, \\
&= (\beta_C \circ (\mathcal{Y}f)_C)(\mathsf{id}_C) && \text{as composition is componentwise,} \\
&= (\beta_C \circ (f \circ -))(\mathsf{id}_C) && \text{by definition of } \mathcal{Y}, \\
&= \beta_C(f) \\
&= F(f)(\beta_D(\mathsf{id}_D))
\end{aligned}$$

which gives the commutativity of the diagram above.

Naturality in F: Given $\alpha \in [\mathcal{C}^{\mathsf{op}}, \mathbf{Set}](\mathcal{Y}\,C, F)$ and $\nu : F \Rightarrow G$. Since the composition of natural transformations is defined componentwise we have

$$(\nu \circ \alpha)_C(\mathsf{id}_C) = \nu_C(\alpha_C(\mathsf{id}_C)) \ ,$$

which proves the commutativity of the diagram

$$\begin{tikzcd}
F \arrow[d, Rightarrow, "\nu"] & {[\mathcal{C}^{\mathsf{op}}, \mathbf{Set}](\mathcal{Y}\,C, F)} \arrow[r, "\theta_{C,F}"] \arrow[d, "\nu\circ-"] & F(C) \arrow[d, "\nu_C"] \\
G & {[\mathcal{C}^{\mathsf{op}}, \mathbf{Set}](\mathcal{Y}\,C, G)} \arrow[r, "\theta_{C,G}"] & G(C)\ .
\end{tikzcd}$$

□

The Yoneda lemma is stated for an arbitrary set-valued functor F. A special case arises when F is itself a hom-functor, say $\mathcal{C}(-, D)$. In this setting the Yoneda lemma tells us that there is an isomorphism between the morphisms in $\mathcal{C}(C, D)$ and the natural transformations in $[\mathcal{C}^{\mathsf{op}}, \mathbf{Set}](\mathcal{Y}C, \mathcal{Y}D)$. As one might expect this isomorphism coincides with the Yoneda functor on arrows.

Corollary 3.5 The Yoneda functor $\mathcal{Y} : \mathcal{C} \to [\mathcal{C}^{\mathsf{op}}, \mathbf{Set}]$ is full and faithful, i.e., given $C, D \in \mathcal{C}$ the Yoneda functor defines a bijection between $(\mathcal{Y}\,D)\,C = \mathcal{C}(C, D)$ and $[\mathcal{C}^{\mathsf{op}}, \mathbf{Set}](\mathcal{Y}C, \mathcal{Y}D)$.

Proof: Given $f \in \mathcal{C}(C, D)$ we show $\widetilde{f} = \mathcal{Y}\,f$. Given an object E in $\mathcal{C}$, from the definition of $\widetilde{(\ )}$ in (1) we have

$$\widetilde{f}_E = \big(\mathcal{C}(-, D)\big)(f) = f \circ - = (\mathcal{Y}f)_E \ .$$

□

Proposition 3.6 Given $C, D \in \mathcal{C}$, we have that $C \cong D$ iff $\mathcal{Y}\,C \cong \mathcal{Y}\,D$ in $[\mathcal{C}^{\mathsf{op}}, \mathbf{Set}]$.

Proof: As $\mathcal{Y}$ is full and faithful—see Proposition 1.19.

□

Exercise 3.7 State the Yoneda Lemma for the Yoneda contravariant functor case.

3.2 Representability

Definition 3.8 (Representation) A *representation* for a functor $F : \mathcal{C}^{\mathsf{op}} \to \mathbf{Set}$ consists of an object $C \in \mathcal{C}$ and a natural isomorphism

$$\mathcal{Y}\, C = \mathcal{C}(-, C) \overset{\alpha}{\cong} F \ .$$

We then say that F is *representable*.

Proposition 3.9 Representations are unique up to isomorphisms: if (A, α) and (B, β) are both representations of a functor $F : \mathcal{C}^{\mathsf{op}} \to \mathbf{Set}$ then $A \cong B$ in $\mathcal{C}$.

Proof: A direct consequence of Proposition 3.6.

□

Remark 3.10 We have a dual definition; a functor $G : \mathcal{C} \to \mathbf{Set}$ is representable iff there exist an object $D \in \mathcal{C}$ and a natural isomorphism $\mathcal{Y}^o\, D = \mathcal{C}(D, -) \overset{\beta}{\cong} G$

Definition 3.11 (Universal Element) Given a functor $F : \mathcal{C}^{\mathsf{op}} \to \mathbf{Set}$, a pair (C, e) where $C \in \mathcal{C}$ and $e \in F(C)$ is a *universal element* of F iff for any $C' \in \mathcal{C}$ and $e' \in F(C')$ there exists a **unique** $f : C' \to C$ in $\mathcal{C}$ such that $e' = F(f)(e)$.

Theorem 3.12 There exists a universal element for a functor $F : \mathcal{C}^{\mathsf{op}} \to \mathbf{Set}$ iff F is representable.

Proof:
"Only-if": Let (C, e) be a universal element for F. From Proposition 3.1 we have that e induces a natural transformation $\widetilde{e} : \mathcal{Y}\, C \Rightarrow F$ whose component at $C' \in \mathcal{C}$ is $\widetilde{e}_{C'} = F(-)(e)$. As e is a universal element we have that $\widetilde{e}_{C'}$ is a bijection. By Proposition 1.31 we conclude that $\widetilde{e}$ is a natural isomorphism.

"if": Suppose $\mathcal{Y}\, C \overset{\alpha}{\cong} F$, then from for any $e' \in F(C')$ for some $C' \in \mathcal{C}$ there exists a unique arrow $\underset{\frown}{f \in} \mathcal{C}(C', C)$ such that $\alpha_{C'}(f) = e'$. From Proposition 3.2 we have that $e' = \alpha_C(\mathsf{id}_C)_{C'}(f) = F(f)(\alpha_C(\mathsf{id}_C))$ and then $(\alpha_C(\mathsf{id}_C), C)$ is a universal element.

□

Theorem 3.13 (Parametrised Representability) Let $F : \mathcal{A} \times \mathcal{B}^{\mathsf{op}} \to \mathbf{Set}$ be a bifunctor such that for every $A \in \mathcal{A}$ there exists a representation $(\mathcal{B}(A), \theta^A)$ for the functor $F(A, -) : \mathcal{B}^{\mathsf{op}} \to \mathbf{Set}$. Then there is a **unique** extension of the mapping $A \mapsto \mathcal{B}(A)$ to a functor $\mathcal{B}(-) : \mathcal{A} \to \mathcal{B}$ such that

$$\mathcal{B}(-, \mathcal{B}(A)) \overset{\theta^A}{\cong} F(A, -)$$

is natural in $A \in \mathcal{A}$.

Proof: By *currying* F we have the functor $\lambda X.\, \lambda Y.\, F(X,Y) : \mathcal{A} \to [\mathcal{B}^{\mathsf{op}}, \mathbf{Set}]$ – see Exercise 2.9. Then, given $f : A \to A'$ we can construct the commuting square

$$\begin{tikzcd}
\mathcal{Y}(\mathcal{B}(A)) \arrow[r, "\cong\theta^{A}"] \arrow[d, "(\theta^{A'})^{-1}\circ F(f,-)\circ\theta^{A}"'] & F(A,-) \arrow[d, "F(f,-)=\langle F(f,B)\rangle_{B\in\mathcal{B}}"] \\
\mathcal{Y}(\mathcal{B}(A')) \arrow[r, "\cong\theta^{A'}"] & F(A',-)\ .
\end{tikzcd}$$

Since $\mathcal{Y}$ is full and faithful, there must exist a **unique** arrow, say $\mathcal{B}(f)$, from $\mathcal{B}(A)$ to $\mathcal{B}(A')$ in $\mathcal{B}$ such that $\mathcal{Y}(\mathcal{B}(f)) = (\theta^{A'})^{-1} \circ F(f,-) \circ \theta^{A}$. We need to check that this extension to $\mathcal{B}(-)$ indeed satisfies the definition of functor:

- Given the arrows $A \xrightarrow{f} A' \xrightarrow{f'} A''$ in $\mathcal{A}$, we have

  $$\begin{aligned}
  \mathcal{Y}\big(\mathcal{B}(f' \circ f)\big) &= (\theta^{A''})^{-1} \circ F(f' \circ f, -) \circ \theta^{A} \\
  &= (\theta^{A''})^{-1} \circ F(f', -) \circ F(f,-) \circ \theta^{A} && \text{as } \lambda X.\, \lambda Y.\, F(X,Y) \text{ is a functor,} \\
  &= (\theta^{A''})^{-1} \circ F(f', -) \circ \theta^{A'} \circ (\theta^{A'})^{-1} \circ F(f,-) \circ \theta^{A} \\
  &= \mathcal{Y}(\mathcal{B}(f')) \circ \mathcal{Y}(B(f)) \\
  &= \mathcal{Y}\big(\mathcal{B}(f') \circ \mathcal{B}(f)\big) && \text{as } \mathcal{Y} \text{ is a functor;}
  \end{aligned}$$

  therefore $\mathcal{Y}(\mathcal{B}(f' \circ f)) = \mathcal{Y}(\mathcal{B}(f) \circ \mathcal{B}(f'))$, and as $\mathcal{Y}$ is faithful we have $\mathcal{B}(f' \circ f) = \mathcal{B}(f') \circ \mathcal{B}(f)$ as required.

- Given the identity id_A in for $A \in \mathcal{A}$ we have

  $$\begin{aligned}
  \mathcal{Y}(\mathcal{B}(\mathsf{id}_A)) &= (\theta^{A})^{-1} \circ F(\mathsf{id}_A, -) \circ \theta^{A} \\
  &= (\theta^{A})^{-1} \circ \mathsf{id}_{F(A,-)} \circ \theta^{A} && \text{as } \lambda X.\, \lambda Y.\, F(X,Y) \text{ is a functor,} \\
  &= \mathsf{id}_{\mathcal{Y}(\mathcal{B}(A))} \\
  &= \mathcal{Y}(\mathsf{id}_{\mathcal{B}(A)}) && \text{as } \mathcal{Y} \text{ is a functor;}
  \end{aligned}$$

  therefore $\mathcal{Y}(\mathcal{B}(\mathsf{id}_A)) = \mathcal{Y}(\mathsf{id}_{\mathcal{B}(A)})$, and as $\mathcal{Y}$ is faithful we have $\mathcal{B}(\mathsf{id}_A) = \mathsf{id}_{\mathcal{B}(A)}$ as required.

$\square$

The composition of the functor $\mathcal{B}(-)$ and $\mathcal{Y}$ gives rise to a functor which is isomorphic to the *curried* version of F where the natural isomorphism is given by $\theta = \langle \theta^{A} \rangle_{A \in \mathcal{A}}$

$$\begin{tikzcd}
\mathcal{A} \arrow[rr, "\lambda X.\,\lambda Y.\,F(X,Y)"] \arrow[rr, bend right=30, "\mathcal{Y}\circ\mathcal{B}(-)"' near end] \arrow[rr, phantom, bend right=15, "\cong\theta\Uparrow"] \arrow[dr, "\mathcal{B}(-)"'] & & {[\mathcal{B}^{\mathsf{op}}, \mathbf{Set}]} \\
& \mathcal{B}^{\mathsf{op}}\ . \arrow[ur, "\mathcal{Y}"'] &
\end{tikzcd}$$

Exercise 3.14 State the dual version of parametrised representability theorem, *i.e.*, where representables are of the form $\mathcal{Y}^o(\mathcal{B}(A))$.

Exercise 3.15 Let $n \in \omega$ and let $\mathit{Prod} : \mathbf{Set} \to \mathbf{Set}$ be the functor which sends the set X to X^n and acts on arrows accordingly, i.e., it sends $f : X \to Y$ to $f^n : X^n \to Y^n$. In this setting, $f^n(x_1, \ldots, x_n) = (fx_1, \ldots, fx_n)$. Suppose $\theta : \mathit{Prod} \Rightarrow \mathit{Prod}$ is a natural isomorphism. Show that there is a permutation π on the sequence $1, \ldots, n$ such that

$$\theta_X(x_1, \ldots, x_n) = (x_{\pi(1)}, \ldots, x_{\pi(n)}) \ .$$

Hint: Use the contravariant Yoneda embedding $\mathcal{Y}^o : \mathbf{Set}^{\mathsf{op}} \to [\mathbf{Set}, \mathbf{Set}]$ noting that $X^n \cong \mathbf{Set}(\vec{n}, X)$, where $\vec{n} = \{1, \ldots, n\}$.

4 Limits and Colimits

4.1 Definition of Limit

We take limits over functors whose domains are small categories ($\mathbb{I}$, $\mathbb{J}$, $\mathbb{K}$ ...) and codomains are locally small categories ($\mathcal{C}$, $\mathcal{D}$ ...). In this setting these functors are sometimes called (small) *diagrams*, *e.g.*, for $F : \mathbb{I} \to \mathcal{C}$ we say F is an $\mathbb{I}$-indexed diagram in $\mathcal{C}$.

Definition 4.1 (Limit Abstractly) A *limit* for a diagram $F : \mathbb{I} \to \mathcal{C}$ is a representation for the functor $[\mathbb{I}, \mathcal{C}](\Delta -, F) : \mathcal{C}^{\mathsf{op}} \to \mathbf{Set}$:

$$\mathcal{C}(-, L) \overset{\theta}{\cong} [\mathbb{I}, \mathcal{C}](\Delta -, F) \ .$$

Notice that the functor $[\mathbb{I}, \mathcal{C}](\Delta -, F) : \mathcal{C}^{\mathsf{op}} \to \mathbf{Set}$ in the definition above is obtained as the composition

$$\mathcal{C}^{\mathsf{op}} \xrightarrow{\Delta -} [\mathbb{I}, \mathcal{C}]^{\mathsf{op}} \xrightarrow{[\mathbb{I}, \mathcal{C}](-, F)} \mathbf{Set}.$$

Take $X \in \mathcal{C}$ in the definition above, then we have $\mathcal{C}(X, \varprojlim_{\mathbb{I}} F) \overset{\theta_X}{\cong} [\mathbb{I}, \mathcal{C}](\Delta X, F)$, *i.e.*, an arrow $f : X \to \varprojlim_{\mathbb{I}} F$ is mapped to a natural transformation $\theta_X(f) = \gamma : \Delta X \Rightarrow F$. Given an arrow $u : I \to J$ in $\mathbb{I}$ we have the associated naturality square in $\mathcal{C}$

$$\begin{tikzcd}
I \arrow[d, "u"] & \Delta X(I) = X \arrow[r, "\gamma_I"] \arrow[d, "\Delta X(u) = \mathsf{id}_X"] & F(I) \arrow[d, "F(u)"] \\
J & \Delta X(J) = X \arrow[r, "\gamma_J"] & F(J)\ .
\end{tikzcd} \tag{2}$$

This diagram can be drawn as

$$\begin{tikzcd} I \arrow[dd, "u"] & & & F(I) \arrow[dd, "F(u)"] \\ & & X \arrow[ru, "\gamma_I"] \arrow[rd, "\gamma_J"] & \\ J & & & F(J); \end{tikzcd}$$

from the commutativity of (2) we have that this triangle commutes, *i.e.*, $\gamma_J = F(u) \circ \gamma_I$. A natural transformation of this kind is called a *cone*.

Definition 4.2 (Cone) Let $F : \mathbb{I} \to \mathcal{C}$ be a diagram. A *cone* α from $X \in \mathcal{C}$ to F (or a *cone* for the $\mathbb{I}$-indexed diagram F in $\mathcal{C}$ with vertex X) is a collection of arrows $\langle \alpha_I : X \to F(I) \rangle_{I \in \mathbb{I}}$ in $\mathcal{C}$ such that for any arrow $u : I \to J$ in $\mathbb{I}$ the diagram

$$\begin{tikzcd} & F(I) \arrow[dd, "F(u)"] \\ X \arrow[ru, "\gamma_I"] \arrow[rd, "\gamma_J"] & \\ & F(J) \end{tikzcd}$$

commutes.

Therefore, the functor $[\mathbb{I}, \mathcal{C}](\Delta -, F)$ takes an object $X \in \mathcal{C}^{\mathsf{op}}$[1] to the set of cones from X to F. Given an arrow $f : X' \to X$ in $\mathcal{C}$ and a cone $\gamma = \langle \gamma_I : X \to F(I) \rangle_{I \in \mathbb{I}}$, we can extend γ to a cone $\gamma' = \langle f \circ \gamma_I : X' \to F(I) \rangle_{I \in \mathbb{I}} = \gamma \circ \Delta f$. This explains how $[\mathbb{I}, \mathcal{C}](\Delta -, F)$ acts on arrows.

From the definition above a limit for a diagram $F : \mathbb{I} \to \mathcal{C}$ is given by a representation $\mathcal{C}(-, L) \overset{\theta}{\cong} [\mathbb{I}, \mathcal{C}](\Delta -, F)$. From the proof of Theorem 3.12, we have that $(\theta_L(\mathsf{id}_L), L)$ is a universal element of $[\mathbb{I}, \mathcal{C}](\Delta -, F)$. Then given a cone $\gamma \in [\mathbb{I}, \mathcal{C}](\Delta X, F)$ there exists a **unique** $f : X \to L$ such that

$$\begin{aligned} \gamma &= \big([\mathbb{I}, \mathcal{C}](\Delta f, F)\big)\big(\theta_L(\mathsf{id}_L)\big) \\ &= \big(\theta_L(\mathsf{id}_L)\big) \circ \Delta f \\ &= \langle (\theta_L(\mathsf{id}_L))_I \circ f \rangle_{I \in \mathbb{I}} \end{aligned}$$

Then each component $\gamma_I : X \to F(I)$ is the composition of arrows

$$X \xrightarrow{\quad f \quad} L \xrightarrow{(\theta_L(\mathsf{id}_L))_I} F(I) \ .$$

This straightforwardly gives a more concrete definition of limit in terms of cones where $\theta_L(\mathsf{id}_L)$ is a limiting cone.

Proposition 4.3 (Limit Concretely) Given a diagram $F : \mathbb{I} \to \mathcal{C}$, a *limit* for F determines and is determined by a cone $\kappa = \langle \kappa_I : L \to F(I) \rangle_{I \in \mathbb{I}}$ from some

---

[1] Recall $\mathbf{ob}(\mathcal{C}^{\mathsf{op}}) = \mathbf{ob}(\mathcal{C})$.

$L \in \mathcal{C}$ to F such that for any other cone $\gamma = \langle \gamma_I : X \to F(I) \rangle_{I \in \mathbb{I}}$ there exists a **unique** arrow $f : X \to L$ such that for every $I \in \mathbb{I}$ the diagram

$$\begin{tikzcd} X \arrow[r, "\gamma_I"] \arrow[d, dashed, "f"'] & F(I) \\ L \arrow[ur, "\kappa_I"] & \end{tikzcd}$$

commutes. The cone κ is called a *limiting cone.*

Proof: Given a representation $\mathcal{C}(-,L) \overset{\theta}{\cong} [\mathbb{I},\mathcal{C}](\Delta-,F)$ from Theorem 3.12 we have that $(\theta_L(\mathsf{id}_L), L)$ is a universal element. To prove the converse we just need to show (κ, L) is a universal element of $[\mathbb{I},\mathcal{C}](\Delta-,F)$, and then by Theorem 3.12 we have the representation $\mathcal{C}(-,L) \overset{\widetilde{\kappa}}{\cong} [\mathbb{I},\mathcal{C}](\Delta-,F)$.

□

Like all universal elements, limiting cones are only determined to within isomorphism:

Proposition 4.4 Let $D : \mathbb{I} \to \mathcal{C}$ be a diagram. Suppose $\varepsilon : \Delta c \Rightarrow D$ and $\varepsilon' : \Delta c' \Rightarrow D$ are both limiting cones. Then $c \cong c'$. Conversely, given a limiting cone $\varepsilon : \Delta c \Rightarrow D$ and an isomorphism $c' \overset{f}{\cong} c$, then $\varepsilon \circ \Delta f : \Delta c' \Rightarrow D$ is a limiting cone.

Definition 4.5 (Complete Category) A category $\mathcal{C}$ is *complete* iff the limits for all diagrams exist in $\mathcal{C}$.

Proposition 4.6 Two results on limiting cones:

(i) Suppose κ is a limiting cone from X to F, and $Y \overset{f}{\cong} X$. Then $\kappa \circ \Delta f$ is a limiting cone from Y to F.

(ii) Suppose κ^X is a limiting cone from X to F and that κ^Y is a limiting cone from Y to F. Then there is an isomorphism $Y \overset{f}{\cong} X$ such that $\kappa^Y = \kappa^X \circ \Delta f$.

Proof: Left as an exercise.

□

Exercise 4.7 Prove the proposition above (use the concrete understanding of limiting cone).

4.2 Examples of Limits

Definition 4.8 (Terminal Object) An object $\mathbf{1}_{\mathcal{C}}$ in a category $\mathcal{C}$ is called a *terminal object* (*final, one*) iff there is exactly one arrow from every $A \in \mathcal{C}$ to $\mathbf{1}_{\mathcal{C}}$.

A limit for a diagram $F : \mathbb{I} \to \mathcal{C}$ where $\mathbb{I}$ is the empty category is a terminal object in $\mathcal{C}$. Notice that this limit might not exist. Any $A \in \mathcal{C}$ is the vertex for an empty cone to F, then there must be a **unique** arrow from A to $\varprojlim_{\mathbb{I}} F = \mathbf{1}_{\mathcal{C}}$.

Exercise 4.9 What is the terminal object in **Set**?

Definition 4.10 (Product) A *product* of two objects A and B in the category $\mathcal{C}$ is an object $A \times B$ in $\mathcal{C}$ together with two *projections* $\pi_1 : A \times B \to A$ and $\pi_2 : A \times B \to B$, such that for any pair of arrows $f : C \to A$ and $g : C \to B$ there is a **unique** (*mediating*) arrow $\langle f, g\rangle : C \to A \times B$ making the diagram

$$\begin{tikzcd}
 & C \arrow[dl, bend right, "f"'] \arrow[d, dashed, "\langle f,g\rangle"'] \arrow[dr, bend left, "g"] & \\
A & A \times B \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & B
\end{tikzcd}$$

commute, *i.e.*, $f = \pi_1 \circ \langle f, g\rangle$ and $g = \pi_2 \circ \langle f, g\rangle$.

Let $\mathbb{I} = \{1, 2\}$ be the discrete category with two objects. A limit for a diagram $F : \mathbb{I} \to \mathcal{C}$ is the product $F(1) \times F(2)$ in $\mathcal{C}$ where the limiting cone gives the projections π_1 and π_2.

One can extend the definition of binary product to general products:

Definition 4.11 Assume $\mathbb{I}$ is a discrete category. Given a diagram $F : \mathbb{I} \to \mathcal{C}$ a *product of* F consists of an object $\prod_{I\in\mathbb{I}} F(I)$ in $\mathcal{C}$, and a collection of *projections* $\pi_I : \prod_{I\in\mathbb{I}} F(I) \to F(I)$ for each $I \in \mathbb{I}$, such that given any $X \in \mathcal{C}$ together with a collection of arrows $\langle f_I : X \to F(I)\rangle_{I\in\mathbb{I}}$ there is a **unique** arrow $\langle f_I\rangle_{I\in\mathbb{I}} : X \to \prod_{I\in\mathbb{I}} F(I)$ such that for each $I \in \mathbb{I}$ the diagram

$$\begin{tikzcd}
X \arrow[r, "f_I"] \arrow[d, dashed, "\langle f_I\rangle_{I\in\mathbb{I}}"'] & F(I) \\
\prod_{I\in\mathbb{I}} F(I) \arrow[ur, "\pi_I"'] &
\end{tikzcd}$$

commutes, *i.e.*, $f_I = \pi_I \circ \langle f_I\rangle_{I\in\mathbb{I}}$.

Definition 4.12 (Pullback) A *pullback* of a pair of arrows $f : A \to C$ and $g : B \to C$ in a category $\mathcal{C}$ is an object $P \in \mathcal{C}$, sometimes written $A \times_C B$, together with a pair of arrows $g' : P \to A$ and $f' : P \to B$ such that

1. $f \circ g' = g \circ f'$, *i.e.*, the diagram

$$\begin{tikzcd}
P \arrow[r, "f'"] \arrow[d, "g'"] & B \arrow[d, "g"'] \\
A \arrow[r, "f"] & C
\end{tikzcd}$$

   commutes, and

2. for any $X \in \mathcal{C}$ together with arrows $h : X \to A$ and $i : X \to B$ for which $f \circ h = g \circ i$, *i.e.*, the diagram

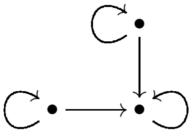
$$\begin{tikzcd}
X \arrow[rrd, bend left, "i"] \arrow[rdd, bend right, "h"'] & & \\
& & B \arrow[d, "g"'] \\
& A \arrow[r, "f"] & C
\end{tikzcd}$$

   commutes, then there exists a **unique** arrow $k : X \to P$ such that $h = g' \circ k$ and $i = f' \circ k$, *i.e.*, the two triangles in

$$\begin{tikzcd}
X \arrow[rrd, bend left, "i"] \arrow[rdd, bend right, "h"'] \arrow[rd, dashed, "k"'] & & \\
& P \arrow[r, "f'"'] \arrow[d, "g'"] & B \arrow[d, "g"'] \\
& A \arrow[r, "f"] & C
\end{tikzcd}$$

   commute.

Exercise 4.13 Let $\mathbb{I}$ be the category represented by the following graph



.

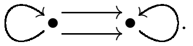
Show that a limit for a diagram $F : \mathbb{I} \to \mathcal{C}$ is a pullback in $\mathcal{C}$.

Definition 4.14 (Equaliser) An *equaliser* of a pair of parallel arrows $A \overset{f}{\underset{g}{\rightrightarrows}} B$ in a category $\mathcal{C}$ is an arrow $m : E \to A$ such that $f \circ m = g \circ m$ and with the universal property that for any arrow $h : X \to A$ such that $f \circ h = g \circ h$ there is a **unique** arrow $k : X \to E$ making $m \circ k = h$, *i.e.*, the triangle in the diagram

$$\begin{tikzcd}
E \arrow[r, "m"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\
X \arrow[u, dashed, "k"] \arrow[ur, "h"'] & &
\end{tikzcd}$$

commutes.

Exercise 4.15 Let $\mathbb{I}$ be the category represented by the following graph



Show that a limit for a diagram $F : \mathbb{I} \to \mathcal{C}$ is an equaliser in $\mathcal{C}$. **Hint:** The equaliser corresponds to one of the arrows in the limiting cone.

Exercise 4.16 Show an equaliser is mono.

4.3 Limits in Set

Consider a diagram $F : \mathbb{I} \to \mathbf{Set}$; from the definition of limit for any $u : I \to J$ the induced triangle

$$\begin{tikzcd}
 & F(I) \arrow[dd, "F(u)"] \\
\varprojlim_{\mathbb{I}} F \arrow[ur, "\kappa_I"] \arrow[dr, "\kappa_J"'] & \\
 & F(J)
\end{tikzcd}$$

commutes, where $\kappa = \langle \kappa_I \rangle_{I \in \mathbb{I}}$ is a limiting cone. Then for any $x \in \varprojlim_{\mathbb{I}} F$ we have that $\big(F(u) \circ \kappa_I\big)(x) = \kappa_J(x)$.

We define our choice of limits in **Set** to be

$$\varprojlim_{\mathbb{I}} F = \big\{ \langle x_I \rangle_{I \in \mathbb{I}} \mid x_I \in F(I) \ \& \ \forall\ u : I \to J\ x_J = (F(u))\, x_I \big\}$$

and the limiting cone $\kappa = \langle \kappa_I \rangle_{I \in \mathbb{I}}$ by letting κ_I be the projection on the I'th component. Clearly, any other cone $\gamma = \langle \gamma_I : X \to F(I) \rangle_{I \in \mathbb{I}}$ factors uniquely through our choice of limit in **Set** by the mapping $z \in X \mapsto \langle \gamma_I(z) \rangle_{I \in \mathbb{I}}$.

Products in Set: Consider the diagram F whose domain is a discrete category with two objects $\{1, 2\}$ and codomain is **Set**. As we have seen before the limit of F is the product $F(1) \times F(2)$ in **Set**. Using the definition of limits in **Set** above we obtain

$$\varprojlim_{\{1,2\}} F = \{ \langle x_1, x_2 \rangle \mid x_1 \in F(1) \wedge x_2 \in F(2) \}$$

which is the usual definition of product of sets.

Pullbacks in Set: Let us assume F is a diagram in **Set** whose limit gives a pullback for the arrows

$$\begin{tikzcd}
& A \arrow[d, "f"] \\
B \arrow[r, "g"] & C
\end{tikzcd}$$

(the identities arrows are not drawn). From the definition of limits in **Set**, the pullback P is

$$\begin{aligned} P &= \{\langle a, c, b\rangle \mid a \in A \wedge b \in B \wedge c \in C \wedge f(a) = c \wedge g(b) = c\} \\ &= \{\langle a, b\rangle \mid a \in A \wedge b \in B \wedge f(a) = g(b)\}. \end{aligned}$$

Exercise 4.17 What are equalisers in **Set**?

Exercise 4.18 The set $[\mathbb{I}, \mathcal{C}](\Delta\, X, F)$ is itself (isomorphic to) a limit, of which diagram?

4.4 Limits as Products and Equalisers

Assume a category $\mathcal{C}$ with all products. Given a small diagram $F : \mathbb{I} \to \mathcal{C}$, then a cone γ uniquely defines a mediating arrow $\langle \gamma_I \rangle_{I\in\mathbb{I}} : X \to \prod_{\mathbb{I}} F(I)$ such that for every $I \in \mathbb{I}$ the diagram

$$\begin{tikzcd}
& F(I) \\
X \arrow[ur, "\gamma_I"] \arrow[r, "\gamma=\langle\gamma_I\rangle_{I\in\mathbb{I}}"'] & \prod_{I\in\mathbb{I}} F(I) \arrow[u, "\pi_I"]
\end{tikzcd}$$

commutes. This is just a consequence of the definition of product.

We can also define in $\mathcal{C}$ the product $\prod_{u:I\to J\in\mathbf{arr}(\mathbb{I})} F(J)$: the product of $F(J)$ indexed by arrows $u : I \to J \in \mathbf{arr}(\mathbb{I})$. The projections of this product are indexed by the arrows in $\mathbb{I}$.

We have two mediating arrows from $\prod_{I\in\mathbb{I}} F(I)$ to $\prod_{u:I\to J\in\mathbf{arr}(\mathbb{I})} F(J)$:

1. $\langle \pi_J \rangle_{u:I\to J\in\mathbf{arr}(\mathbb{I})}$ such that for an arbitrary $u : I \to J$ the diagram

$$\begin{tikzcd}
& F(I) \arrow[r, "F(u)"] & F(J) \\
\prod_{I\in\mathbb{I}} F(I) \arrow[urr, "\pi_J"] \arrow[rr, "\langle\pi_J\rangle_{u:I\to J\in\mathbf{arr}(\mathbb{I})}"] & & \prod_{u:I\to J\in\mathbf{arr}(\mathbb{I})} F(J) \arrow[u, "\pi_u"']
\end{tikzcd}$$

commutes; and

2. $\langle F(u) \circ \pi_I \rangle_{u:I \to J \in \mathbf{arr}(\mathbb{I})}$ such that for an arbitrary $u : I \to J$ the diagram

$$\begin{tikzcd}
 & F(I) \arrow[r, "F(u)"] & F(J) \\
\prod_{I \in \mathbb{I}} F(I) \arrow[ur, "\pi_I"] \arrow[rr, "\langle F(u) \circ \pi_I \rangle_{u:I \to J \in \mathbf{arr}(\mathbb{I})}"] & & \prod_{u:I \to J \in \mathbf{arr}(\mathbb{I})} F(J) \arrow[u, "\pi_u"']
\end{tikzcd}$$

commutes.

Theorem 4.19 A category $\mathcal{C}$ has all small limits iff $\mathcal{C}$ has all products and equalisers.

Proof:
"Only-if": Obvious since products and equalisers are limits.
"If": Given a diagram $F : \mathbb{I} \to \mathcal{C}$, a cone γ from X to F equalises the mediating arrows $\langle \pi_J \rangle_{u:I \to J \in \mathbf{arr}(\mathbb{I})}$ and $\langle F(u) \circ \pi_I \rangle_{u:I \to J \in \mathbf{arr}(\mathbb{I})}$. On the other hand, any arrow equalising these arrows determines a cone on F. So a limit of F is an equaliser for $\langle \pi_J \rangle_{u:I \to J \in \mathbf{arr}(\mathbb{I})}$ and $\langle F(u) \circ \pi_I \rangle_{u:I \to J \in \mathbf{arr}(\mathbb{I})}$.

□

4.5 Definition of Colimit

Definition 4.20 (Colimit Abstractly) A *colimit* for a diagram $F : \mathbb{I} \to \mathcal{D}$ is a representation for $[\mathbb{I}, \mathcal{D}](F, \Delta -) : \mathcal{D} \to \mathbf{Set}$:

$$\mathcal{D}(C, -) \overset{\theta}{\cong} [\mathbb{I}, \mathcal{D}](F, \Delta -) \ .$$

The object C is often written as $\varinjlim_{\mathbb{I}} F$ or $\mathrm{colim}_{\mathbb{I}} F$.

The component at X of the natural isomorphism θ in the definition above takes arrows in $\mathcal{D}(C, X)$ to cones from F to X (or a *cocone* for the $\mathbb{I}$-indexed diagram F in $\mathcal{D}$ with vertex X). A cone β from F to X is a collection of arrows $\langle \beta_I : F(I) \to X \rangle_{I \in \mathbb{I}}$ such that for any $u : I \to J$ in $\mathbb{I}$ the diagram

$$\begin{tikzcd}
F(I) \arrow[d, "F(u)"'] \arrow[dr, "\beta_I"] & \\
F(J) \arrow[r, "\beta_J"] & X
\end{tikzcd}$$

commutes.

Proposition 4.21 (Colimit Concretely) Given a diagram $F : \mathbb{I} \to \mathcal{D}$, a *colimit* for F determines and is determined by a cone $\kappa = \langle \kappa_I : F(I) \to C \rangle_{I \in \mathbb{I}}$

from F to some $C \in \mathcal{D}$ such that given any other cone $\beta = \langle \beta_I : F(I) \to C \rangle_{I \in \mathbb{I}}$ there exists a **unique** arrow $g : C \to X$ such that for every $I \in \mathbb{I}$ the diagram

$$\begin{tikzcd}
F(I) \arrow[r, "\beta_I"] \arrow[d, "\kappa_I"] & X \\
C \arrow[ru, dashed, "g"'] &
\end{tikzcd}$$

commutes.

Proof: Left as exercise.

□

Exercise 4.22 Prove the proposition above by using the dual of Theorem 3.12.

All the remarks we made for limits are applied to colimits in the dual version.

Definition 4.23 (Cocomplete Category) A category $\mathcal{D}$ is cocomplete iff the colimits for all diagrams exist in $\mathcal{D}$.

4.6 Examples of Colimits

Definition 4.24 (Initial Object) An object $\mathbf{0}_{\mathcal{D}}$ in a category $\mathcal{D}$ is called an *initial object* (*zero*) iff there is exactly one arrow from $\mathbf{0}_{\mathcal{D}}$ to every $A \in \mathcal{D}$.

A colimit for a diagram $F : \mathbb{I} \to \mathcal{D}$ where $\mathbb{I}$ is the empty category is an initial object in $\mathcal{D}$. Notice that this colimit might not exist. Any $A \in \mathcal{D}$ is the vertex for an empty cone from F, then there must be a **unique** arrow from $\varinjlim_{\mathbb{I}} F = \mathbf{0}_{\mathcal{D}}$ to A.

Definition 4.25 (Coproduct) A *coproduct* of two objects A and B in the category $\mathcal{D}$ is an object $A+B$ in $\mathcal{D}$ together with two *injections* $\iota_1 : A \to A+B$ and $\iota_2 : B \to A+B$, such that for any pair of arrows $f : A \to C$ and $g : B \to C$ there is exactly one arrow $[f,g] : A+B \to C$ such that the diagram

$$\begin{tikzcd}
A \arrow[r, "\iota_1"] \arrow[rd, bend right, "f"'] & A+B \arrow[d, dashed, "{[f,g]}"'] & B \arrow[l, "\iota_2"'] \arrow[ld, bend left, "g"] \\
& C &
\end{tikzcd}$$

commutes, *i.e.*, $f = [f,g] \circ \iota_1$ and $g = [f,g] \circ \iota_2$.

Let $\mathbb{I} = \{1,2\}$ be the discrete category with two objects. A colimit for a diagram $F : \mathbb{I} \to \mathcal{D}$ is a coproduct $F(1) + F(2)$ in $\mathcal{D}$ where the limiting cone defines the injections ι_1 and ι_2.

We can extend this to general coproducts:

Definition 4.26 Assume $\mathbb{I}$ is a discrete category. Given a diagram $F : \mathbb{I} \to \mathcal{D}$ a colimit for F is called a coproduct and consists of an object $\sum_{I\in\mathbb{I}} F(I)$ and a cone $\langle \iota_I : F(I) \to \sum_{I\in\mathbb{I}} F(I)\rangle_{I\in\mathbb{I}}$ such that for any other cone $\langle f_I : F(I) \to X\rangle_{I\in\mathbb{I}}$ there exists a **unique** arrow $[f_I]_{I\in\mathbb{I}}$ such that for each $I \in \mathbb{I}$ the diagram

$$\begin{tikzcd}
F(I) \arrow[r, "\iota_I"] \arrow[dr, "f_I"'] & \sum_{I\in\mathbb{I}} F(I) \arrow[d, dashed, "{[f_I]_{I\in\mathbb{I}}}"] \\
 & X
\end{tikzcd}$$

commutes.

Definition 4.27 (Pushout) A *pushout* of a pair of arrows $f : C \to A$ and $g : C \to B$ is an object P together with a pair of arrows $f' : B \to P$ and $g' : A \to P$ such that

1. $f' \circ g = g' \circ f$, *i.e.*, the diagram

$$\begin{tikzcd}
P & B \arrow[l, "f'"] \\
A \arrow[u, "g'"'] & C \arrow[l, "f"'] \arrow[u, "g"']
\end{tikzcd}$$

   commutes, and

2. for any object X together with arrows $h : A \to X$ and $i : B \to X$ for which $h \circ f = i \circ g$, *i.e.*, the diagram

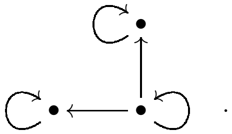
$$\begin{tikzcd}
X & & \\
 & & B \arrow[ull, bend right, "i"'] \\
 & A \arrow[uu, bend left, "h"] & C \arrow[l, "f"'] \arrow[u, "g"']
\end{tikzcd}$$

   commutes, then there exists a **unique** arrow $k : P \to X$ such that $h = k \circ g'$ and $i = k \circ f'$, *i.e.*, the two triangles in

$$\begin{tikzcd}
X & & \\
 & P \arrow[ul, dashed, "k"'] & B \arrow[l, "f'"] \arrow[ull, bend right, "i"'] \\
 & A \arrow[u, "g'"'] \arrow[uul, bend left, "h"] & C \arrow[l, "f"'] \arrow[u, "g"']
\end{tikzcd}$$

   commute.

Exercise 4.28 Let $\mathbb{I}$ be a category defined by the graph



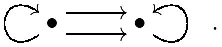
Show that colimit for a diagram $F : \mathbb{I} \to \mathcal{D}$ is a pushout in $\mathcal{D}$.

Definition 4.29 (Coequaliser) A *coequaliser* of a pair of parallel arrows $A \overset{f}{\underset{g}{\rightrightarrows}} B$ in a category $\mathcal{D}$ is an arrow $m : B \to E$ such that $m \circ f = m \circ g$ with the universal property that for any arrow $h : B \to X$ such that $h \circ f = h \circ g$ there is a **unique** arrow $k : E \to X$ making $k \circ m = h$, *i.e.*, the triangle in the diagram

$$\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "m"] \arrow[dr, "h"'] & E \arrow[d, dashed, "k"] \\
 & & X \ .
\end{tikzcd}$$

commutes.

Exercise 4.30 Let $\mathbb{I}$ be the category defined by the graph



Show that a colimit for a diagram $F : \mathbb{I} \to \mathcal{D}$ is a coequaliser in $\mathcal{D}$.

4.7 Colimits in Set

Consider a diagram $F : \mathbb{I} \to \mathbf{Set}$, from the definition of colimits we have that for any $u : I \to J$ the triangle

$$\begin{tikzcd}
F(I) \arrow[dr, "\kappa_I"] \arrow[dd, "F(u)"] & \\
 & \varinjlim_{\mathbb{I}} F \\
F(J) \arrow[ur, "\kappa_J"'] &
\end{tikzcd}$$

commutes, where $\langle \kappa_I \rangle_{I \in \mathbb{I}}$ is a colimiting cone. Then for any $x \in F(I)$ we have $\kappa_I(x) = (\kappa_j \circ F(u))(x)$.

We define our choice of colimits in **Set** to be $\varinjlim_{\mathbb{I}} F = \biguplus_{I \in \mathbb{I}} F(I)/ \sim$, where the relation $\sim$ represents the constraints imposed by the arrows in $\mathbb{I}$. We define the relation $\sim$ as the least equivalence relation on $\biguplus_{I \in \mathbb{I}} F(I) \times \biguplus_{I \in \mathbb{I}} F(I)$ such that

$$(I, x) \sim (J, y) \Leftrightarrow \exists\ u : I \to J \in \mathbf{arr}(\mathbb{I})\ (F(u))x = y\ . \qquad (3)$$

For $(I, x) \in \biguplus_{I \in \mathbb{I}} F(I)$ we let $[(I, x)]_\sim$ denote the equivalence class of $\sim$ containing (I, x). The colimiting cone is given by the collection of injection arrows $\kappa_I : x \mapsto [(I, x)]_\sim$.

Exercise 4.31 Show this choice indeed gives a colimit in **Set**.

Exercise 4.32 What are coproducts, pushouts and coequalisers in **Set**?

4.8 Limits with Parameters

A special case of parametrised representability arises when considering limits of diagrams as representations:

Proposition 4.33 (Parametrised Limits) Let $F : \mathbb{I} \times \mathcal{A} \to \mathcal{C}$ be a functor such that for every $A \in \mathcal{A}$ the functor (small diagram in $\mathcal{C}$) $F(-, A) : \mathbb{I} \to \mathcal{C}$ has as limit the representation $\mathcal{C}(-, \varprojlim_{\mathbb{I}} F(-, A) \overset{\theta^A}{\cong} [\mathbb{I}, \mathcal{C}](\Delta -, F(-, A))$. Then the mapping $A \mapsto \varprojlim_{\mathbb{I}} F(-, A)$ extends uniquely to a functor $\lambda W.\, \varprojlim_{\mathbb{I}} F(-, W) : \mathcal{A} \to \mathcal{C}$ such that

$$\mathcal{C}\big(X, \varprojlim_{\mathbb{I}} F(-, A)\big) \overset{(\theta^A)_X}{\cong} [\mathbb{I}, \mathcal{C}]\big(\Delta X, F(-, A)\big) \tag{4}$$

natural in $A \in \mathcal{A}$ and $X \in \mathcal{C}$.

Proof: This proposition is just a corollary of Theorem **??** (Parametrised Representability) applied to the functor $\lambda A, X.\, [\mathbb{I}, \mathcal{C}](\Delta X, F(-, A)) : \mathcal{A} \times \mathcal{C}^{\mathsf{op}} \to \mathbf{Set}$.

$\square$

Given the functor $F : \mathbb{I} \times \mathcal{A} \to \mathcal{C}$ in the proposition, an arrow $g : A \to B$ in $\mathcal{A}$ has associated the naturality square

$$\begin{tikzcd}
A \arrow[d, "g"] & \mathcal{C}\big(X, \varprojlim_{\mathbb{I}} F(-, A)\big) \arrow[r, "\cong\theta^A_X"] \arrow[d, "\varprojlim_{\mathbb{I}} F(-,g)\circ -"] & {[\mathbb{I}, \mathcal{C}]}\big(\Delta\, X, F(-, A)\big) \arrow[d, "F(-,g)\circ -"] \\
B & \mathcal{C}\big(X, \varprojlim_{\mathbb{I}} F(-, B)\big) \arrow[r, "\cong\theta^B_X"] & {[\mathbb{I}, \mathcal{C}]}\big(\Delta\, X, F(-, B)\big)
\end{tikzcd}$$

for an arbitrary $X \in \mathcal{C}$. Taking $X = \varprojlim_{\mathbb{I}} F(-, A)$ and chasing the identity $\mathsf{id}_{\varprojlim_{\mathbb{I}} F(-,A)}$ around the diagram we have

$$\theta^B_{\varprojlim_{\mathbb{I}} F(-,A)}\big(\varprojlim_{\mathbb{I}} F(-, g)\big) = F(-, g) \circ \theta^A_{\varprojlim_{\mathbb{I}} F(-,A)}(\mathsf{id}_{\varprojlim_{\mathbb{I}} F(-,A)}),$$

i.e., $\varprojlim_{\mathbb{I}} F(-, g)$ is the mediating arrow associated with the cone $F(-, g) \circ \theta^A_{\varprojlim_{\mathbb{I}} F(-,A)}(\mathsf{id}_{\varprojlim_{\mathbb{I}} F(-,A)})$.

Exercise 4.34 Verify that $F(-, g) \circ \theta^A_{\varprojlim_{\mathbb{I}} F(-,A)}(\mathsf{id}_{\varprojlim_{\mathbb{I}} F(-,A)})$ is a cone from $\varprojlim_{\mathbb{I}} F(-, A)$ to $F(-, B)$.

An important special case of Proposition 4.33 arises by taking $\mathcal{A} = [\mathbb{I}, \mathcal{C}]$ and F to be the evaluation functor $\mathit{eval} : \mathbb{I} \times [\mathbb{I}, \mathcal{C}] \to \mathcal{C}$ where $\mathcal{C}$ has all limits for $\mathbb{I}$-indexed diagrams. Hence, from the definition of the evaluation functor, for every $G \in [\mathbb{I}, \mathcal{C}]$ there is a representation

$$\mathcal{C}\big(-, \varprojlim_{\mathbb{I}} \mathit{eval}(-, G)\big) \cong [\mathbb{I}, \mathcal{C}]\big(\Delta -, \mathit{eval}(-, G)\big).$$

Notice that $\mathit{eval}(-, G) = G$. Then from parametrised limits we have a functor $\lambda W. \varprojlim_{\mathbb{I}} \mathit{eval}(-, W) = \varprojlim_{\mathbb{I}} - : [\mathbb{I}, \mathcal{C}] \to \mathcal{C}$, the *limit functor*, such that

$$\mathcal{C}(X, \varprojlim_{\mathbb{I}} F) \cong [\mathbb{I}, \mathcal{C}](\Delta\, X, F)$$

natural in $X \in \mathcal{C}$ and $F \in [\mathbb{I}, \mathcal{C}]$. From the observation above, given a natural transformation $\alpha : F \Rightarrow G$ the arrow $\varprojlim_{\mathbb{I}} \alpha$ is just the mediating arrow associated with the cone resulting from the composition of the limiting cone for $\varprojlim_{\mathbb{I}} F$ with α.

Remark 4.35 The definition of the limit functor is given w.r.t. a particular choice of representation for each diagram $F \in [\mathbb{I}, \mathcal{C}]$; or equivalently w.r.t. a particular choice of limiting cone for each diagram. The limit functors asscociated with different choices of representation are, however, all isomorphic.

Remark 4.36 In general there are categories where limits for some diagrams are missing. We can still define the limit functor, however, by restricting to the subcategory of diagrams for which all limits exists. Given a category $\mathcal{A}$ for which not necessarily all limits of type $\mathbb{I}$ exist, we define the full subcategory $\mathit{Diag}[\mathbb{I}, \mathcal{A}] \subseteq [\mathbb{I}, \mathcal{A}]$ of diagrams with limits in $\mathcal{A}$ and restrict the limit functor to range over such diagrams. Notice that if $\mathcal{A}$ is not the empty category then $\mathit{Diag}[\mathbb{I}, \mathcal{A}]$ contains at least the constant diagrams since their limits always exist in $\mathcal{A}$.

Exercise 4.37 State the dual of the Proposition 4.33 for colimits and conclude that if $\mathcal{C}$ has all limits for $\mathbb{I}$-indexed diagrams there is a colimit functor $\varinjlim_{\mathbb{I}} - : [\mathbb{I}, \mathcal{C}] \to \mathcal{C}$.

Exercise 4.38 Check that the definition of $\varprojlim_{\mathbb{I}} -$ satisfies the axioms in the definition of functor. In principle that is not necessary since parametrised representability ensures that the extension of the mapping $F \mapsto \varprojlim_{\mathbb{I}} F$ is a functor!

5 Ends

5.1 Dinaturality

We have considered bifunctors and natural transformations between them in § 2.2. We now study special kinds of bifunctors of mixed-variance, e.g. with domain $\mathcal{C}^{\mathsf{op}} \times \mathcal{C}$, and *dinatural transformations* between them. A main motivation for dinatural transformations is to present *ends* and dually *coends* as representations for suitable functors.

Definition 5.1 (Dinatural Transformation) Let $F, G : \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathcal{D}$ be functors. A *dinatural transformation* $\alpha : F \ddot{\to} G$ consists of a family $\langle \alpha_U : F(U,U) \to G(U,U) \rangle_{U \in \mathcal{C}}$ of arrows in $\mathcal{D}$ such that for every arrow $f : V \to U$ in $\mathcal{C}$ the diagram

$$\begin{tikzcd}
& F(U,U) \arrow[r, "\alpha_U"] & G(U,U) \arrow[dr, "G(f{,}\mathsf{id}_U)"] & \\
F(U,V) \arrow[ur, "F(\mathsf{id}_U{,}f)"] \arrow[dr, "F(f{,}\mathsf{id}_V)"'] & & & G(V,U) \\
& F(V,V) \arrow[r, "\alpha_V"'] & G(V,V) \arrow[ur, "G(\mathsf{id}_V{,}f)"'] &
\end{tikzcd}$$

commutes.

Recall that $F(\mathsf{id}_U, f) = F(U, f)$ where $F(U, -) : \mathcal{C} \to \mathbf{Set}$ is the functor obtained by fixing the object U in the first argument of F (see § 2.2). Henceforth we use the more compact $F(U, f)$ instead of $F(\mathsf{id}_U, f)$. Similarly, we use $F(f, V)$ instead of $F(f, \mathsf{id}_V)$.

Every ordinary natural transformation gives rise to a dinatural transformation: Let η

$$\begin{tikzcd}
\mathcal{C}^{\mathsf{op}} \times \mathcal{C} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "G"'{name=D}] \arrow[Rightarrow, from=U, to=D, "\eta"] & \mathcal{E}
\end{tikzcd}$$

be a natural transformation then the family of arrows $\langle \eta_{U,U} : F(U,U) \to G(U,U) \rangle_{U \in \mathcal{C}}$ is a dinatural transformation. Given the arrow $f : V \to U$ we construct the diagram

$$\begin{tikzcd}
& F(U,U) \arrow[rr, "\eta_{U,U}"] \arrow[dr, phantom, "(1)"] & & G(U,U) \arrow[dr, "G(f{,}U)"] & \\
F(U,V) \arrow[ur, "F(U{,}f)"] \arrow[rr, "\eta_{U,V}"] \arrow[dr, "F(f{,}V)"'] & & G(U,V) \arrow[ur, "G(U{,}f)"] \arrow[dr, "G(f{,}V)"'] \arrow[rr, phantom, "(3)"] & & G(V,U) \\
& F(V,V) \arrow[rr, "\eta_{V,V}"'] \arrow[ur, phantom, "(2)"] & & G(V,V) \arrow[ur, "G(V{,}f)"'] &
\end{tikzcd}$$

where

(1) is the naturality square associated with $(\mathsf{id}_U, f) : (U, V) \to (U, U)$,

(2) is the naturality square associated with $(f, \mathsf{id}_V) : (U, V) \to (V, V)$, and

(3) commutes since G is a bifunctor and then $G(f, f) = G(V, f) \circ G(f, V) = G(f, U) \circ G(U, f)$ (see § 2.2).

Therefore,

$$\begin{aligned} G(f,U) \circ \eta_{U,U} \circ F(U,f) &= G(f,U) \circ G(U,f) \circ \eta_{U,V} && \text{(1) commutes,} \\ &= G(V,f) \circ G(f,V) \circ \eta_{U,V} && \text{(3) commutes,} \\ &= G(V,f) \circ \eta_{V,V} \circ F(f,V) && \text{(2) commutes.} \end{aligned}$$

Thus $\langle \eta_{U,U} \rangle_{U \in \mathcal{C}}$ is a dinatural transformation.

In general dinatural transformations do not compose. But dinatural transformations obtained from natural transformations compose with arbitrary dinatural transformations.

Exercise 5.2 Given $\alpha : F' \Rightarrow F$, $\beta : F \mathrel{\ddot{\rightarrow}} G$ and $\gamma : G \Rightarrow G'$ (assume functors of the right type), show that

1. $\beta \circ \langle \alpha_{U,U} \rangle_U$;
2. $\langle \gamma_{U,U} \rangle_U \circ \beta$; and
3. $\langle \gamma_{U,U} \rangle_U \circ \beta \circ \langle \alpha_{U,U} \rangle_U$

are dinatural transformations, where the composition is defined componentwise.
Hint: For 1 consider the diagram

$$\begin{tikzcd}
& F'(U,U) \arrow[rr, "\alpha_{U,U}"] & & F(U,U) \arrow[rr, "\beta_U"] \arrow[dr, "F(f,U)"] & & G(U,U) \arrow[dr, "G(f,U)"] & \\
F'(U,V) \arrow[ur, "F'(U,f)"] \arrow[rr, "\alpha_{U,V}"] \arrow[dr, "F'(f,V)"'] & & F(U,V) \arrow[ur, "F(U,f)"] \arrow[dr, "F(f,V)"'] & & F(V,U) & & G(V,U) \\
& F'(V,V) \arrow[rr, "\alpha_{V,V}"'] & & F(V,V) \arrow[ur, "F(V,f)"'] \arrow[rr, "\beta_V"'] & & G(V,V) \arrow[ur, "G(V,f)"'] &
\end{tikzcd}$$

The composition of dinatural transformations with natural transformations is functorial, i.e., we have a functor

$$\mathbf{Dinat}(-,+) : [\mathcal{C}^{\mathsf{op}} \times \mathcal{C}, \mathcal{D}]^{\mathsf{op}} \times [\mathcal{C}^{\mathsf{op}} \times \mathcal{C}, \mathcal{D}] \to \mathbf{Set}$$

where $-$ and $+$ are place-holders used to stress the mixed-variance of the functor. $\mathbf{Dinat}(-,+)$ acts on functors and natural transformations as follows:

$$\begin{tikzcd}
F & G \arrow[d, Rightarrow, "\varphi"] & \mathbf{Dinat}(F,G) \arrow[d, "\mathbf{Dinat}(\eta,\varphi)=\varphi\circ-\circ\eta"] \\
H \arrow[u, Rightarrow, "\eta"] & K & \mathbf{Dinat}(H,K)\ .
\end{tikzcd}$$

5.2 Special Cases of Dinatural Transformations

In the next examples we consider functors $F, G : \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathcal{D}$ and dinatural transformations from F to G.

One-side Constant: Assume $F = \Delta\, X$ is a constant functor for some $X \in \mathcal{D}$ and consider a dinatural transformation $\alpha : \Delta\, X \ddot{\to} G$. Then given $f : V \to U$ in $\mathcal{C}$ the diagram

$$\begin{tikzcd}
& X \arrow[r, "\alpha_U"] & G(U,U) \arrow[dr, "G(f,U)"] & \\
X \arrow[ur, "\mathsf{id}_X"] \arrow[dr, "\mathsf{id}_X"'] & & & G(V,U) \\
& X \arrow[r, "\alpha_V"'] & G(V,V). \arrow[ur, "G(V,f)"'] &
\end{tikzcd}$$

commutes. This diagram can be drawn as

$$\begin{tikzcd}
& G(U,U) \arrow[dr, "G(f,U)"] & \\
X \arrow[ur, "\alpha_U"] \arrow[dr, "\alpha_V"'] & & G(V,U) \\
& G(V,V). \arrow[ur, "G(V,f)"'] &
\end{tikzcd}$$

Such an α is called an *extra* natural transformation, a *super* natural transformation, or a *wedge* from X to G.

Likewise, if G is a constant functor we can consider the dinatural transformation $\beta : F \ddot{\to} \Delta\, X$ for some $X \in \mathcal{D}$. Then we have for any $f : V \to U$ that the diagram

$$\begin{tikzcd}
& F(U,U) \arrow[dr, "\beta_U"] & \\
F(U,V) \arrow[ur, "F(U,f)"] \arrow[dr, "F(f,V)"'] & & X \\
& F(V,V) \arrow[ur, "\beta_V"'] &
\end{tikzcd}$$

commutes. This is a wedge from F to X.

As an example of the first case consider the set-valued functors: $\Delta\,\{\bullet\}$ and $\mathcal{C}(-,+)$. The collection α whose component at U is the function $\bullet \mapsto \mathsf{id}_U$ is a dinatural transformation from $\Delta\{\bullet\}$ to $\mathcal{C}(-,+)$. For any $f : V \to U$ in $\mathcal{C}$ we construct the diagram

$$\begin{tikzcd}
& \mathcal{C}(U,U) \arrow[dr, "-\circ f"] & \\
\{\bullet\} \arrow[ur, "\alpha_U"] \arrow[dr, "\alpha_V"'] & & \mathcal{C}(V,U) \\
& \mathcal{C}(V,V)\ . \arrow[ur, "f\circ -"'] &
\end{tikzcd}$$

By chasing $\bullet$ in the diagram it is easy to check that the diagram commutes.

An example of the second case is given by the family of arrows $\langle ev^B : \mathbf{Set}(B,A) \times B \to A\rangle_{B\in\mathbf{Set}}$ for $A \in \mathbf{Set}$ where the component ev^B is defined by $(g,b) \mapsto g(b)$. Given an arrow $f : D \to C$ in $\mathbf{Set}$ we construct the diagram

$$\begin{tikzcd}
& \mathbf{Set}(C,A)\times C \arrow[dr, "ev^C"] & \\
\mathbf{Set}(C,A)\times D \arrow[ur, "\mathbf{Set}(C,A)\times f"] \arrow[dr, "-\circ f\times D"'] & & A \\
& \mathbf{Set}(D,A)\times D\ . \arrow[ur, "ev^D"'] &
\end{tikzcd}$$

By chasing $(k,x) \in \mathbf{Set}(C,A) \times D$ in the diagram above

$$(k,x) \xmapsto{ev^C \circ \mathbf{Set}(C,A)\times f} k(f(x)), \text{ and}$$

$$(k,x) \xmapsto{ev^D \circ (-\circ f\times D)} (k\circ f)(x) = k(f(x)).$$

Therefore $\langle ev^B\rangle_{B\in\mathbf{Set}}$ is a dinatural transformation from $\big(\mathbf{Set}(-,A)\times +\big)$ to $\Delta\, A$.

"Dummy" in One Argument: A functor $F : \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathcal{D}$ is *dummy* in its first argument if there exists a functor $F_0 : \mathcal{C} \to \mathcal{D}$ such that F is the composition

$$\mathcal{C}^{\mathsf{op}} \times \mathcal{C} \xrightarrow{\pi_2} \mathcal{C} \xrightarrow{F_0} \mathcal{D}$$

where $\pi_2 : \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathcal{C}$ is the projection to the second argument. Analogously, a functor $G : \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathcal{D}$ is dummy in its second argument if there exists a functor $G_0 : \mathcal{C}^{\mathsf{op}} \to \mathcal{D}$ such that G is the composition

$$\mathcal{C}^{\mathsf{op}} \times \mathcal{C} \xrightarrow{\pi_1} \mathcal{C}^{\mathsf{op}} \xrightarrow{G_0} \mathcal{D}$$

where $\pi_1 : \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathcal{C}^{\mathsf{op}}$ is the projection to the first argument.

Suppose $F, G : \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathcal{D}$ are both dummy in the first argument and $\alpha : F \mathrel{\ddot{\to}} G$ is a dinatural transformation. Then for any $f : V \to U$ in $\mathcal{C}$ the diagram

$$\begin{tikzcd}
& F(U,U) = F_0(U) \arrow[r, "\alpha_U"] & G(U,U) = G_0(U) \arrow[dr, "G(f,U)=G_0(U)"] & \\
F(U,V) = F_0(V) \arrow[ur, "F(U,f)=F_0(f)"] \arrow[dr, "F(f,V)=F_0(V)"'] & & & G(V,U) = G_0(U) \\
& F(V,V) = F_0(V) \arrow[r, "\alpha_V"'] & G(V,V) = G_0(V) \arrow[ur, "G(V,f)=G_0(f)"'] &
\end{tikzcd}$$

commutes. Hence the diagram

$$\begin{tikzcd}
F_0(U) \arrow[r, "\alpha_U"] & G_0(U) \\
F_0(V) \arrow[u, "F_0(f)"] \arrow[r, "\alpha_V"] & G_0(V) \arrow[u, "G_0(f)"']
\end{tikzcd}$$

commutes and α defines a natural transformation from F_0 to G_0.

If F is dummy in its second argument and G is dummy in its first argument, a dinatural transformation $\alpha : F \mathrel{\ddot{\to}} G$ is a family $\langle \alpha_U : F_0(U) \to G_0(U) \rangle_{U \in \mathcal{C}}$ of arrows in $\mathcal{D}$ where $F_0 : \mathcal{C}^{\mathsf{op}} \to \mathcal{D}$ and $G_0 : \mathcal{C} \to \mathcal{D}$ are functors such that given $f : V \to U$ in $\mathcal{C}$ the diagram

$$\begin{tikzcd}
F_0(U) \arrow[r, "\alpha_U"] & G_0(U) \arrow[d, "G_0(f)"] \\
F_0(V) \arrow[u, "F_0(f)"] \arrow[r, "\alpha_V"'] & G_0(V)
\end{tikzcd}$$

commutes. The case where F is dummy in its first argument and G is dummy in its second argument is similar.

5.3 Definition of End

We take ends over functors whose domains are products of the form $\mathbb{I}^{\mathsf{op}} \times \mathbb{I}$, where $\mathbb{I}$ is a small category.

Definition 5.3 (End Abstractly) An *end* for a functor $F : \mathbb{I}^{\mathsf{op}} \times \mathbb{I} \to \mathcal{D}$ is a representation for $\mathbf{Dinat}\big(\Delta -, F\big) : \mathcal{D}^{\mathsf{op}} \to \mathbf{Set}$

$$\mathcal{D}(-, end\, F) \overset{\theta}{\cong} \mathbf{Dinat}\big(\Delta -, F\big).$$

The set $\mathbf{Dinat}\big(\Delta X, F\big)$ for $X \in \mathcal{D}$ is the set of wedges from X to F. We can recover a more concrete characterisation of ends as we have done with limits. Given a wedge $\gamma \in \mathbf{Dinat}\big(\Delta X, F\big)$ by Theorem 3.12 there exists a **unique** arrow $h : X \to \mathit{end}\, F$ such that

$$\begin{aligned}\gamma &= \mathbf{Dinat}\big(\Delta h, F\big)\big(\theta_{\mathit{end}\,F}(\mathsf{id}_{\mathit{end}\,F})\big)\\ &= \big(\theta_{\mathit{end}\,F}(\mathsf{id}_{\mathit{end}\,F})\big) \circ \Delta h\\ &= \langle \big(\theta_{\mathit{end}\,F}(\mathsf{id}_{\mathit{end}\,F})\big)_I \circ h \rangle_{I \in \mathbb{I}}\ .\end{aligned}$$

In other words, the *universal* or *ending* wedge $\theta_{\mathit{end}\,F}(\mathsf{id}_{\mathit{end}\,F})$ is uniquely extended by the arrow h resulting in the wedge γ. Given $f : V \to U$ the situation is illustrated in the following commutative diagram where $\kappa = \theta_{\mathit{end}\,F}(\mathsf{id}_{\mathit{end}\,F})$

$$\begin{tikzcd}
 & & F(U,U) \arrow[dr, "F(f,U)"] & \\
D \arrow[r, dashed, "h"] \arrow[urr, bend left, "\gamma_U"] \arrow[drr, bend right, "\gamma_V"'] & \mathit{end}\, F \arrow[ur, "\kappa_U"'] \arrow[dr, "\kappa_V"] & & F(V,U) \\
 & & F(V,V). \arrow[ur, "F(V,f)"'] &
\end{tikzcd}$$

Proposition 5.4 (End Concretely) An *end* for a functor determines and is determined by a universal wedge.

Proof: By using Theorem 3.12 following the same reasoning done with limits.

□

Notation 5.5 In general we write $\int_I F(I,I)$ for the object $\mathit{end}\, F \in \mathcal{D}$, where $F : \mathbb{I}^{\mathsf{op}} \times \mathbb{I} \to \mathcal{D}$; the properties of ends shall justify this notation.

Limits are Ends: Let $F_0 : \mathbb{I} \to \mathcal{D}$ be a functor with limit the representation $\mathcal{D}(-, \varprojlim_{\mathbb{I}} F_0) \overset{\theta}{\cong} [\mathbb{I}, \mathcal{D}](\Delta_{\mathbb{I}}-, F_0)$. The functor F_0 induces a functor $F : \mathbb{I}^{\mathsf{op}} \times \mathbb{I} \to \mathcal{D}$ dummy in its first argument. The natural transformations from $\Delta_{\mathbb{I}}\, X$ to F_0 are the dinatural transformations from $\Delta_{\mathbb{I}^{\mathsf{op}} \times \mathbb{I}}\, X$ to F – see § 5.2. Thus we have

$$\mathcal{D}(X, \varprojlim_{\mathbb{I}} F_0) \overset{\theta_X}{\cong} [\mathbb{I}, \mathcal{D}](\Delta_{\mathbb{I}}\, X, F_0) = \mathbf{Dinat}\big(\Delta_{\mathbb{I}^{\mathsf{op}} \times \mathbb{I}}\, X, F\big)$$

natural in X, *i.e.*, $(\varprojlim_{\mathbb{I}} F_0, \theta)$ is a representation for $\mathbf{Dinat}\big(\Delta_{\mathbb{I}^{\mathsf{op}} \times \mathbb{I}}-, F\big)$ and consequently $\varprojlim_{\mathbb{I}} F_0 = \int_I F(I,I)$.

Notation 5.6 This justifies writing $\int_I F_0(I)$ for $\varprojlim_{\mathbb{I}} F_0$.

Ends are Limits: Let $F : \mathbb{I}^{\mathsf{op}} \times \mathbb{I} \to \mathcal{D}$ be a functor with end the representation $\mathcal{D}(-, \int_I F(I,I)) \overset{\theta}{\cong} \mathbf{Dinat}\big(\Delta -, F\big)$. We construct a category $\mathbb{I}^\S$ and a functor $d^\S : \mathbb{I}^\S \to \mathbb{I}^{\mathsf{op}} \times \mathbb{I}$ such that $\int_I F(I,I) = \varprojlim_{\mathbb{I}^\S}(F \circ d^\S)$. The category $\mathbb{I}^\S$ is built from the objects and arrows in the category $\mathbb{I}$ as follows:

- $\mathbf{ob}(\mathbb{I}^\S) = \mathbf{ob}(\mathbb{I}) \cup \mathbf{arr}(\mathbb{I})$; and
- $\mathbf{arr}(\mathbb{I}^\S)$ is the collection of identities and for every $f : V \to U$ in $\mathbb{I}$ the arrows

$$\begin{tikzcd}
U \arrow[dr] & \\
& f \\
V. \arrow[ur] &
\end{tikzcd}$$

The only possible composition in $\mathbb{I}^\S$ is with identities.

Now, we define the functor $d^\S : \mathbb{I}^\S \to \mathbb{I}^{\mathsf{op}} \times \mathbb{I}$ acting on objects and arrows

$$\begin{tikzcd}
U \arrow[dr] & & & & (U,U) \arrow[dr, "{(f,U)}"] & \\
& f & {} \arrow[r, mapsto, "d^\S"] & {} & & (V,U) \\
V \arrow[ur] & & & & (V,V)\ . \arrow[ur, "{(V,f)}"'] &
\end{tikzcd}$$

Observe (left as an exercise) that the cones in $[\mathbb{I}^\S, \mathcal{D}](\Delta_{\mathbb{I}^\S} X, F \circ d^\S)$ are exactly the wedges in $\mathbf{Dinat}\big(\Delta_{\mathbb{I}^{\mathsf{op}} \times \mathbb{I}} X, F\big)$:

$$\mathcal{D}(-, \textstyle\int_I F(I,I)) \overset{\theta}{\cong} \mathbf{Dinat}\big(\Delta_{\mathbb{I}^{\mathsf{op}} \times \mathbb{I}} -, F\big) = [\mathbb{I}^\S, \mathcal{D}](\Delta_{\mathbb{I}^\S} -, F \circ d^\S)\ .$$

Therefore $\int_I F(I,I)$ is a limit for $F \circ d^\S$.

Exercise 5.7 Show that $[\mathbb{I}^\S, \mathcal{D}](\Delta_{\mathbb{I}^\S} -, F \circ d^\S) = \mathbf{Dinat}\big(\Delta_{\mathbb{I}^{\mathsf{op}} \times \mathbb{I}} -, F\big)$.

As a consequence of these two results we can deduce that if $\mathcal{D}$ has all limits then $\mathcal{D}$ has all ends; we also have a choice for ends in **Set** given by the choice for limits and a parametricity result.

5.4 Ends in Set

Let $F : \mathbb{I}^{\mathsf{op}} \times \mathbb{I} \to \mathbf{Set}$ be a functor. A choice for the end of F is given by

$$\begin{aligned}
&\textstyle\int_I F(I,I) = \varprojlim_{\mathbb{I}^\S}(F \circ d^\S) \\
&= \big\{ \langle x_A \rangle_{A \in \mathbb{I}^\S} \mid x_A \in (F \circ d^\S)(A) \ \&\ \forall f : V \to U \in \mathbf{arr}(\mathbb{I}) \big(F(f,U)(x_U) = x_f \ \&\ F(V,f)(x_V) = x_f \big) \big\} \\
&= \big\{ \langle x_U \rangle_{U \in \mathbb{I}} \mid x_U \in F(U,U) \ \&\ \forall f : V \to U \in \mathbf{arr}(\mathbb{I}) \ :\ F(f,U)(x_U) = F(V,f)(x_V) \big\}
\end{aligned} \tag{5}$$

where the projections form the ending wedge.

5.5 Ends with Parameters

Proposition 5.8 (Parametrise Ends) Let $F : \mathbb{I}^{\mathsf{op}} \times \mathbb{I} \times \mathcal{B} \to \mathcal{D}$ be a functor such that for every $B \in \mathcal{B}$ the end $\int_I F(I, I, B)$ of the functor $F(-, +, B) : \mathbb{I}^{\mathsf{op}} \times \mathbb{I} \to \mathcal{D}$ exists in $\mathcal{D}$. Then the function $B \mapsto \int_I F(I, I, B)$ extends uniquely to a functor from $\mathcal{B}$ to $\mathcal{D}$ such that

$$\mathcal{D}\big(D, \textstyle\int_I F(I, I, B)\big) \cong \mathbf{Dinat}\big(\Delta\, D, F(-, +, B)\big)$$

natural in $D \in \mathcal{D}$ and $B \in \mathcal{B}$.

Proof: The proof follows from parametrised representability (Theorem **??**) applied to the functor $\lambda B, D.\, \mathbf{Dinat}\big(\Delta D, F(-, +, B)\big)$.

$\square$

Exercise 5.9 Define the "end functor" *end* $-$: $[\mathbb{I}^{\mathsf{op}} \times \mathbb{I}, \mathcal{C}] \to \mathcal{C}$ where $\mathcal{C}$ has all ends of type $\mathbb{I}$.

6 Limits and Ends

6.1 Some Useful Isomorphisms

Curry and Uncurry: Letting $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$, the isomorphism *curry* : $[\mathcal{A} \times \mathcal{B}, \mathcal{C}] \cong [\mathcal{A}, [\mathcal{B}, \mathcal{C}]]$ with inverse *uncurry* (see exercise 2.9) gives an isomorphism

$$[\mathcal{A} \times \mathcal{B}, \mathcal{C}](F, G) \overset{\textbf{curry}}{\cong} [\mathcal{A}, [\mathcal{B}, \mathcal{C}]](\lambda A.\, \lambda B.\, F(A, B), \lambda A.\, \lambda B.\, G(A, B)) \tag{6}$$

which sends a natural transformation $\alpha : F \Rightarrow G$ to $\langle\langle \alpha_{A,B} \rangle_{B \in \mathcal{B}}\rangle_{A \in \mathcal{A}}$. Its inverse, *uncurry*, acts by sending β to $\langle (\beta_A)_B \rangle_{A,B \in \mathcal{A} \times \mathcal{B}}$. The isomorphism is natural in F and G.

Exercise 6.1 Swap?

Limits and Ends as Representations: Assume $\mathcal{D}$ has all limits, as a consequence of parametrised representability for limits there is an isomorphism

$$\mathcal{D}(X, \textstyle\int_{\mathbb{I}} F(I)) \overset{\theta_{X,F}}{\cong} [\mathbb{I}, \mathcal{D}](\Delta\, X, F)$$

natural in $X \in \mathcal{D}$ and $F \in [\mathbb{I}, \mathcal{D}]$ – see § 4.8. Moreover, from Theorem 3.12 the isomorphism is defined by

$$f \in \mathcal{D}\big(X, \textstyle\int_I F(I)\big) \xmapsto{\theta_{X,F}} \kappa \circ \Delta f \in [\mathbb{I}, \mathcal{D}](\Delta\, X, F) \tag{7}$$

where $\kappa = \theta_{\int_I F(I), F}(\mathsf{id}_{\int_I F(I)})$ is the limiting cone of F.

The case for ends goes similarly. As $\mathcal{D}$ has all limits then it has all ends, then from parametrised representability for ends there is an isomorphism

$$\mathcal{D}(X, \textstyle\int_I G(I,I)) \overset{\sigma_{X,G}}{\cong} \mathbf{Dinat}\big(\Delta\, X, G\big)$$

natural in $X \in \mathcal{D}$ and $G \in [\mathbb{I}^{\mathsf{op}} \times \mathbb{I}, \mathcal{D}]$, where $\sigma_{X,G}$ is defined by

$$f \in \mathcal{D}\big(X, \textstyle\int_I G(I,I)\big) \xmapsto{\sigma_{X,G}} \gamma \circ \Delta f \in \mathbf{Dinat}\big(\Delta X, G\big) \tag{8}$$

where $\gamma = \sigma_{\int_I G(I,I), G}(\mathsf{id}_{\int_I G(I,I)})$ is the universal wedge of G.

Dinaturality Formula: Letting $F, G : \mathbb{I}^{\mathsf{op}} \times \mathbb{I} \to \mathcal{D}$, then

$$\mathbf{Dinat}\big(F, G\big) = \textstyle\int_I \mathcal{D}\big(F(I,I), G(I,I)\big) \tag{9}$$

provided we make the specific choice for ends of set-valued functors as given by (5). The right hand side of this formula is the end taken over the functor $\lambda I^-, I^+.\, \mathcal{D}\big(F(I^+, I^-), G(I^-, I^+)\big)$. The two sides of (9) are the same (left as an exercise) functorially in F and G (by virtue of the identity isomorphism).

Exercise 6.2 Prove the dinaturality formula by using our explicit choice for ends in **Set**.

Naturality Formula: We can specialise the dinaturality formula to an end characterisation of natural transformations. Letting $F, G : \mathbb{I} \to \mathcal{D}$, then

$$\mathbf{Nat}\big(F, G\big) = [\mathbb{I}, \mathcal{D}](F, G) = \textstyle\int_I \mathcal{D}\big(F(I), G(I)\big) \tag{10}$$

where the end of the right hand side is taken over the functor $\lambda I^-, I^+.\, F(I^-, I^+)$. Again with respect to the specific choice for ends in **Set** the two sides are the same functorially in F and G.

6.2 Limits in Functor Categories

Let $F : \mathbb{I} \times \mathbb{J} \to \mathcal{D}$ be a functor; if for each $J \in \mathbb{J}$ the functor $F(-, J) : \mathbb{I} \to \mathcal{D}$ has a limit in $\mathcal{D}$ then by parametrised limits (Theorem 4.33) there is a functor $\lambda J.\, \int_I F(I, J) : \mathbb{J} \to \mathcal{D}$. This limit functor is in fact isomorphic to the limit $\int_I \lambda J.\, F(I, J)$. In other words, limits in functor categories are obtained pointwise provided the pointwise limits exist.

Proposition 6.3 Assume a functor $F : \mathbb{I} \times \mathbb{J} \to \mathcal{D}$ is such that for each object $J \in \mathbb{J}$ there is a limiting cone κ^J for $F(-, J)$. Then $\big\langle \langle \kappa_I^J \rangle_{J \in \mathbb{J}} \big\rangle_{I \in \mathbb{I}}$ is a limiting cone for $\lambda I.\, \lambda J.\, F(I, J)$ with vertex $\lambda J.\, \int_I F(I, J)$. Assuming every functor F in $[\mathbb{I} \times \mathbb{J}, \mathcal{D}\ ]$ has a limit $\int_I F(I, J)$ for all object $J \in \mathbb{J}$, the isomorphism

$$\textstyle\int_I \lambda J.\, F(I, J) \cong \lambda J.\, \int_I F(I, J)\ ,$$

determined by universality, is natural in F.

Proof: We show $[\mathbb{J},\mathcal{D}]\big(Y, \lambda J.\ \int_I F(I,J)\big) \cong [\mathbb{I},[\mathbb{J},\mathcal{D}]]\big(\Delta_{\mathbb{I}} Y, \lambda I.\,\lambda J.\,F(I,J)\big)$ natural in $Y \in [\mathbb{I},\mathcal{D}]$. This natural isomorphism is obtained as the composition of the following natural isomorphisms:

$$
\begin{aligned}
[\mathbb{J},\mathcal{D}]\big(Y, \lambda J.\ \textstyle\int_I F(I,J)\big) &= \textstyle\int_J \mathcal{D}\big(Y(J), \int_I F(I,J)\big) && \text{by naturality formula (10),}\\
&\cong \textstyle\int_J [\mathbb{I},\mathcal{D}]\big(\lambda I.\,Y(J), \lambda I.\,F(I,J)\big) && \text{by representation of limits,}\\
&= [\mathbb{J},[\mathbb{I},\mathcal{D}]]\big(\lambda J.\,\lambda I.\,\Delta(Y(J)), \lambda J.\,\lambda I.\,F(I,J)\big) && \text{by naturality formula (10),}\\
&\cong [\mathbb{I}\times\mathbb{J},\mathcal{D}]\big(\lambda I,J.\,Y(J), \lambda I,J.\,F(I,J)\big) && \text{by uncurrying and swapping,}\\
&\cong [\mathbb{I},[\mathbb{J},\mathcal{D}]]\big(\lambda I.\,\lambda J.\,(Y\ J), \lambda I.\,\lambda J.\,F(I,J)\big) && \text{by currying.}
\end{aligned}
$$

As each isomorphism is natural in $Y \in [\mathbb{J},\mathcal{D}]$ we obtain the required representation. A natural transformation $\varphi \in [\mathbb{J},\mathcal{D}]\big(Y, \lambda J.\ \int_I F(I,J)\big)$ is sent to $\big\langle\langle \kappa_I^J \circ \varphi_J\rangle_{J\in\mathbb{J}}\big\rangle_{I\in\mathbb{I}}$ as shown below:

$$
\varphi = \langle\varphi_J\rangle_{J\in\mathbb{J}} \longmapsto \langle\kappa^J \circ \Delta\varphi_J\rangle_{J\in\mathbb{J}} \longmapsto \langle\kappa_I^J \circ \varphi_J\rangle_{I,J\in\mathbb{I}\times\mathbb{J}} \longmapsto \big\langle\langle\kappa_I^J \circ \varphi_J\rangle_{J\in\mathbb{J}}\big\rangle_{I\in\mathbb{I}}\ .
$$

In particular the identity $\mathsf{id}_{\lambda J.\ \int_I F(I,J)}$ is sent to $\big\langle\langle\kappa_I^J\rangle_{J\in\mathbb{J}}\big\rangle_{I\in\mathbb{I}}$ which by Theorem 3.12 is a limiting cone for $\lambda I.\,\lambda J.\,F(I,J)$. As each isomorphism is natural in F as well, by parametrised representability, we have

$$
\textstyle\int_I \lambda J.\,F(I,J) \cong \lambda J.\ \int_I F(I,J)
$$

natural in F.

□

6.3 Fubini Theorem

Proposition 6.4 Let $F : \mathbb{I}\times\mathbb{J} \to \mathcal{D}$ be a functor and assume $\int_I F(I,J)$ exists for any object J in $\mathbb{J}$, then

$$
\textstyle\int_J \int_I F(I,J) \cong \int_{(I,J)} F(I,J)
$$

meaning if one limit exists so does the other. If $\mathcal{D}$ is complete the isomorphism is natural in F.

Proof: By parametrised representability and since $\int_I F(I,J)$ exists for any $J \in \mathbb{J}$ we have the functor $\lambda J.\ \int_I F(I,J)$. By representation of limits

$$
\begin{aligned}
\mathcal{D}\big(-, \textstyle\int_J \int_I F(I,J)\big) &\cong [\mathbb{J},\mathcal{D}]\big(\Delta-, \lambda J.\ \textstyle\int_I F(I,J)\big) \text{ and}\\
\mathcal{D}\big(-, \textstyle\int_{(I,J)} F(I,J)\big) &\cong [\mathbb{I}\times\mathbb{J},\mathcal{D}]\big(\Delta-, \lambda I,J.\,F(I,J)\big)\ .
\end{aligned}
$$

We prove that the right hand sides are isomorphic:

$$
\begin{aligned}
[\mathbb{J},\mathcal{D}]\big(\Delta V, \lambda J.\ \textstyle\int_I F(I,J)\big) &\cong \textstyle\int_J \mathcal{D}\big(V, \int_I F(I,J)\big) && \text{by naturality formula (10),}\\
&\cong \textstyle\int_J [\mathbb{I},\mathcal{D}]\big(\lambda I.\,V, \lambda I.\,F(I,J)\big) && \text{by representation for limits,}\\
&\cong [\mathbb{J},[\mathbb{I},\mathcal{D}]]\big(\lambda J.\,\lambda I.\,V, \lambda J.\,\lambda I.\,F(I,J)\big) && \text{by naturality formula (10),}\\
&\cong [\mathbb{I}\times\mathbb{J},\mathcal{D}]\big(\lambda I,J.\,V, \lambda I,J.\,F(I,J)\big) && \text{by uncurring and swapping.}
\end{aligned}
$$

Notice that each isomorphism is natural in $V \in \mathcal{D}$ and $F \in [\mathbb{I} \times \mathbb{J}, \mathcal{D}]$ (provided we assume $\mathcal{D}$ is complete).

□

Corollary 6.5 (Fubini for Limits) Let $F : \mathbb{I} \times \mathbb{J} \to \mathcal{D}$ be a functor such that

- $\int_I F(I,J)$ exists for all object J in $\mathbb{J}$; and
- $\int_J F(I,J)$ exists for all object I in $\mathbb{I}$,

then

$$\int_J \int_I F(I,J) \cong \int_I \int_J F(I,J) \ ,$$

meaning if one side exists so does the other. Assuming $\mathcal{D}$ is complete the isomorphism is natural in F.

Proof: Apply the Proposition 6.4 twice.

□

As ends are limits we also have a Fubini theorem for ends:

Corollary 6.6 (Fubini for Ends) Given $F : \mathbb{I}^{\mathsf{op}} \times \mathbb{I} \times \mathbb{J}^{\mathsf{op}} \times \mathbb{J} \to \mathcal{D}$ such that

- $\int_I F(I, I, J^-, J^+)$ exists for all J^-,$J^+ \in \mathbb{J}$; and
- $\int_J F(I^-, I^+, J, J)$ exists for all I^-,$I^+ \in \mathbb{I}$,

then

$$\int_I \int_J F(I,I,J,J) \cong \int_J \int_I F(I,I,J,J) \ ,$$

meaning if one side exists so does the other. Moreover this isomorphism is natural in F if $\mathcal{D}$ is complete.

7 Coends

7.1 Definition of Coend

Definition 7.1 (Coend Abstractly) A *coend* for $F : \mathbb{I}^{\mathsf{op}} \times \mathbb{I} \to \mathcal{D}$ is a representation for $\mathbf{Dinat}\big(F, \Delta -\big) : \mathcal{D} \to \mathbf{Set}$

$$\mathcal{D}\big(\int^I F(I,I), -\big) \overset{\theta}{\cong} \mathbf{Dinat}\big(F, \Delta -\big) \ .$$

An element of the set $\mathbf{Dinat}\big(F, \Delta X\big)$ is a wedge from F to X. By using the dual form of Theorem 3.12 we can recover a concrete definition for coends in terms of couniversal wedges. Thus given a wedge $\alpha \in \mathbf{Dinat}\big(F, \Delta X\big)$ there exists a **unique** arrow $h : \int^I F(I,I) \to X$ such that $\alpha = \Delta\, h \circ \theta_{\int^I F(I,I)}\Big(\mathsf{id}_{\int^I F(I,I)}\Big)$

where $\theta_{\int^I F(I,I)}\Big(\mathsf{id}_{\int^I F(I,I)}\Big)$ is the couniversal wedge. Therefore, given an arrow $f : V \to U$ in $\mathbb{I}$ we have that in the diagram

$$\begin{tikzcd}
& F(U,U) \arrow[rdd, "\kappa_U"] \arrow[rrdd, bend left, "\alpha_U"] & & \\
& & (1) & \\
F(U,V) \arrow[uur, "{F(U,f)}"] \arrow[ddr, "{F(f,V)}"'] & & \int^I F(I,I) \arrow[r, dashed, "h"] & X \\
& & (2) & \\
& F(V,V) \arrow[ruu, "\kappa_V"'] \arrow[rruu, bend right, "\alpha_V"'] & &
\end{tikzcd}$$

(1) and (2) commutes, where $\kappa = \theta_{\int^I F(I,I)}\Big(\mathsf{id}_{\int^I F(I,I)}\Big)$.

By means of the dinaturality 9 formula we can define coends in terms of ends, given a functor $F : \mathbb{I}^{\mathsf{op}} \times \mathbb{I} \to \mathcal{D}$

$$\mathcal{D}\big(\textstyle\int^I F(I,I), V\big) \cong \mathbf{Dinat}\big(F, \Delta V\big) = \textstyle\int_I \mathcal{D}(F(I,I), V) \tag{11}$$

natural in V (and also natural in F if $\mathcal{D}$ is a complete category).

The relation between colimits and coends allows to extend our choice for colimits of set-valued functors to coends in **Set**. Thus, given a functor $F : \mathbb{I}^{\mathsf{op}} \times \mathbb{I} \to \mathbf{Set}$

$$\textstyle\int^I F(I,I) = \biguplus_{I\in\mathbb{I}} F(I,I)/\sim \tag{12}$$

where the relation $\sim$ is the least equivalence relation on $\biguplus_{I\in\mathbb{I}} F(I,I) \times \biguplus_{I\in\mathbb{I}} F(I,I)$ such that

$$(U,x) \sim (V,y) \text{ if } \exists f : V \to U \in \mathbf{arr}(\mathbb{I})\, \exists z \in F(U,V)\,.\,(x = F(U,f)z)\,\&\,(y = F(f,V)z)\,.$$

The universal wedge is given by the collection of injection arrows $x \mapsto [(U,x)]_\sim$.

The results for ends have a dual version for coends: parametricity coends, Fubini Theorem and pointwise coends in functor categories.

7.2 Density Formula

Given an object X in the functor category $[\mathbb{C}^{\mathsf{op}}, \mathbf{Set}]$, by Yoneda Lemma (Theorem 3.3) and naturality formula 10

$$X(U) \cong [\mathbb{C}^{\mathsf{op}}, \mathbf{Set}]\big(\mathcal{Y}U, X\big) = \textstyle\int_V \mathbf{Set}\big((\mathcal{Y}U)V, X(V)\big). \tag{13}$$

Notation 7.2 Henceforth we shall denote hom-sets in **Set** with square brackets, for instance $[A,B]$ instead of $\mathbf{Set}(A,B)$. Rewriting (13) to $X(U) \cong \int_V [\mathbb{C}(V,U), X(V)]$.

We now present a coend formula for X. The *density formula* expresses $X(U)$ in terms of the coend

$$X(U) \cong \textstyle\int^W \mathbb{C}(U,W) \times X(W)$$

natural in $U \in$ and $X \in [\mathbb{C}^{\mathsf{op}}, \mathbf{Set}]$. Notice that the right hand side is a coend for the functor $\lambda W^-, W^+.\, \mathbb{C}(U, W^+) \times X(W^-) : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbf{Set}$ where $U \in \mathbb{C}$.

Notation 7.3 In a category $\mathcal{C}$ with all small coproducts, for $S \in \mathbf{Set}$ and $A \in \mathcal{C}$, the *copower* $S \cdot A$ is the coproduct $\sum_{s \in S} A$.

Let us rewrite the density formula using copowers:

$$\begin{aligned}
X &\cong \lambda U.\, \int^{W} X(W) \times \mathbb{C}(U,W) \\
&\cong \int^{W} \lambda U.\, X(W) \times \mathbb{C}(U,W) && \text{as coends are computed pointwise,} \\
&\cong \int^{W} X(W) \cdot (\lambda U.\, \mathbb{C}(U,W)) && (14) \\
&= \int^{W} X(W) \cdot \mathcal{Y} W
\end{aligned}$$

where (14) is justified by

$$\sum_{s \in X(W)} (\lambda U.\, \mathbb{C}(U,W)) \cong \lambda U. \sum_{s \in X(W)} \mathbb{C}(U,V) = \lambda U.\, X(W) \times \mathbb{C}(U,W)\ .$$

Proof of the density formula: Let $Y \in [\mathbb{C}^{\mathsf{op}}, \mathbf{Set}]$

$$\begin{aligned}
[\mathbb{C}^{\mathsf{op}}, \mathbf{Set}]\big(\int^{W} X(W) \cdot \mathcal{Y} W, Y\big) &= \int_{U} [(\int^{W} X(W) \cdot \mathcal{Y} W) U, Y(U)] && \text{by naturality formula (10),} \\
&\cong \int_{U} [\int^{W} X(W) \times \mathbb{C}(U,W), Y(U)] && \text{as coends are computed pointwise,} \\
&\cong \int_{U} \int_{W} [X(W) \times \mathbb{C}(U,W), Y(U)] && \text{by (11),} \\
&\cong \int_{U} \int_{W} [X(W), [\mathbb{C}(U,W), Y(U)]] && \text{by currying,} \\
&\cong \int_{W} \int_{U} [X(W), [\mathbb{C}(U,W), Y(U)]] && \text{by Fubini,} \\
&\cong \int_{W} [X(W), \int_{U} [\mathbb{C}(U,W), Y(U)]] && \text{by dinaturality formula,} \\
&\cong \int_{W} [X(W), Y(W)] && \text{by Yoneda Lemma,} \\
&= [\mathbb{C}^{\mathsf{op}}, \mathbf{Set}](X, Y) && \text{by dinaturality formula,}
\end{aligned}$$

natural in Y. By Proposition 3.6 we have $X \cong \int^{W} X(W) \cdot \mathcal{Y} W$.

7.3 Recasting the Density Formula

The density formula can be rewritten in two ways. Both make use of a construction called the *category of elements* of a set-valued functor.

Definition 7.4 Let $G : \mathcal{A} \to \mathbf{Set}$ be a functor. Define its *category of elements* $\mathbf{els}(G)$ (sometimes $\int G$ though this clashes with our use for ends and coends) to be the category consisting of

- $\mathbf{ob}(\mathbf{els}(G)) = \biguplus_{A \in \mathcal{A}} G(A)$; and
- arrows $h : (A, z) \to (A', z')$ where $h : A \to A'$ is an arrow in $\mathcal{A}$ such that $G\,h(z) = z'$.

Often we will take the category of elements defined by a contravariant functor $G : \mathbb{C}^{\mathsf{op}} \to \mathbf{Set}$, in this case $h : (C, z) \to (C', z')$ is an arrow of $\mathbf{els}(G)$ if $h : C' \to C$ is an arrow in $\mathcal{C}$ and $(G\,h)(z) = z'$.

Note that a product of sets $X(W) \times \mathbb{C}(U, W)$ can be written as a coproduct in two ways

$$\sum_{x \in X(W)} \mathbb{C}(U, W) = \int^{x \in X(W)} \mathbb{C}(U, W) \tag{15}$$

or

$$\sum_{f \in \mathcal{C}(U,W)} X(W) = \int^{f \in \mathbb{C}(U,W)} X(W) \tag{16}$$

where we treat the sets $X(W)$ and $\mathbb{C}(W, U)$ as discrete categories (i.e. all maps are identities) when using the "integral" notation (here representing colimits).

From the density formula, we have

$$X(U) \cong \int^{W} X(W) \times \mathbb{C}(U, W) = \int^{W} \int^{x \in X(W)} \mathbb{C}(U, W)\ .$$

By using $\mathbf{els}(X)$ we can replace the "double integral" by a "single integral"

$$\int^{W} \int^{x \in X(W)} \mathbb{C}(U, W) \cong \int^{(W,x) \in \mathbf{els}(X)} \mathbb{C}(U, W)\ . \tag{17}$$

Notice that while the left hand side is a coend for the functor $\lambda W^-, W^+.\ \int^{x \in X(W^-)} \mathbb{C}(U, W^+)$ the right hand side is a colimit for the functor $(W, x) \mapsto \mathbb{C}(U, W)$ from $\mathbf{els}(X)$ to $\mathbf{Set}$.

Exercise 7.5 Considering our particular choice for coends and colimits in $\mathbf{Set}$ prove that the isomorphism (17) holds and conclude that it is natural in U.

Hence, $X(U) \cong \int^{(W,x) \in \mathbf{els}(X)} \mathbb{C}(U, W)$, so

$$X \cong \lambda U. \int^{(W,x) \in \mathbf{els}(X)} \mathbb{C}(U, W) \cong \int^{(W,x) \in \mathbf{els}(X)} \lambda U.\, \mathbb{C}(U, W)\ , \tag{18}$$

exhibiting X as a colimit of representables. Similarly, by considering (16)

$$X(U) \cong \int^{W} \int^{f \in \mathbb{C}(U,W)} X(W) \cong \int^{(W,f) \in \mathbf{els}(\mathbb{C}(U,-))} X(W).$$

Exercise 7.6 Show this isomorphism by considering the concrete construction of coends and colimits in $\mathbf{Set}$.

8 Limit Preservation from Naturality

8.1 Preservation of Limits

Definition 8.1 (Preservation of Limits) Let $G : \mathcal{C} \to \mathcal{D}$ be a functor. The functor G preserves limits for a diagram $D : \mathbb{I} \to \mathcal{C}$ if whenever $\kappa : \Delta c \Rightarrow D$ is a limit then the cone $G\kappa : \Delta G(c) \Rightarrow G \circ D$, got by composition with G, is also a limit.

Clearly, if composition with G sends one limit for a diagram $D : \mathbb{I} \to \mathcal{C}$ to a limit for $G \circ D$ then it sends any other limit for D to a limit for $G \circ D$. Most often we talk of the functor G preserving $\mathbb{I}$-limits, or being $\mathbb{I}$-continuous; this means that G preserves limits of all diagrams in $[\mathbb{I}, \mathcal{C}]$. Sometimes we are interested in a subcategory of diagrams $\mathcal{K} \subseteq [\mathbb{I}, \mathcal{C}]$ for which limits exist in $\mathcal{C}$. Just as above we can define a functor $\varprojlim_{\mathbb{I}} : \mathcal{K} \to \mathcal{C}$. And, as later, talk about a functor preserving $\mathcal{K}$-limits or being $\mathcal{K}$-continuous.

Proposition 8.2 The hom-functor $\mathcal{C}(C, -) : \mathcal{C} \to \mathbf{Set}$ preserves all limits.

Proof: Suppose $F : \mathbb{I} \to \mathcal{C}$ is a diagram with limiting cone $\kappa = \langle \kappa_I \rangle_{I \in \mathbb{I}}$, as a consequence of our choice of limits in **Set** (a special case of naturality formula (10)):

$$\mathcal{C}(C, \textstyle\int_I F(I)) \cong [\mathbb{I}, \mathcal{C}](\Delta C, F) = \textstyle\int_I \mathcal{C}(C, F(I)).$$

That does not ensure that $\mathcal{C}(C, -)\,\kappa$ is limiting. Let us examine the diagram for κ_I

$$\begin{tikzcd}
\mathcal{C}(C, \int_I F(I)) \arrow[r, "\mathcal{C}(C,\kappa_I)=\kappa_I\circ -"] \arrow[d, "\cong\theta"] & \mathcal{C}(C, F(I)) \\
\int_I \mathcal{C}(C, F(I)) \arrow[ur, "\pi_I"'] &
\end{tikzcd}$$

given $f \in \mathcal{C}(C, \int_I F(I))$, we have from the representation for limits (7)

$$f \xmapsto{\pi_I \circ \theta} \kappa_I \circ f \ .$$

Thus $\pi_I \circ \theta = \kappa_I \circ - = \mathcal{C}(C, \kappa_I)$, and by Proposition 4.6*(i)* we conclude $\langle \mathcal{C}(C, \kappa_I) \rangle_{I \in \mathbb{I}}$ is a limiting cone in **Set**.

□

Clearly, if F and G are isomorphic functors then F preserves $\mathbb{I}$-indexed limits iff G does. Hence from Proposition 8.2, we conclude that representable functors preserve limits.

Proposition 8.3 Given a diagram $F \in [\mathbb{I}, \mathcal{C}]$ and a functor $G : \mathcal{C} \to \mathcal{D}$ such that the limits of F and $G \circ F$ exists, then G preserves the limit of F iff the mediating arrow defined by universality of limit is an isomorphism.

Proof: Left as exercise.

□

Exercise 8.4 Prove the result above by using Proposition 4.6

Let $G : \mathcal{C} \to \mathcal{D}$ be a functor. Given a cone $\gamma : \Delta \varprojlim_{\mathbb{I}} D \Rightarrow D$ for a diagram $D : \mathbb{I} \to \mathcal{C}$ the natural transformation $G\gamma : \Delta G(\varprojlim_{\mathbb{I}} D) \Rightarrow G \circ D$ obtained by composition is a cone as well. Thus, given a limiting cone $\varepsilon : \Delta \varprojlim_{\mathbb{I}} G{\circ}D \Rightarrow G{\circ}D$ there is a unique mediating morphism $m : G(\varprojlim_{\mathbb{I}} D) \to d$ such that the diagram

$$\begin{tikzcd}
\Delta G(\varprojlim_{\mathbb{I}} D) \arrow[r, "\Delta m"] \arrow[dr, "G\gamma"'] & \Delta \varprojlim_{\mathbb{I}} G \circ D \arrow[d, "\varepsilon"] \\
 & G \circ D
\end{tikzcd}$$

commutes. Requiring that G preserves limits of D is equivalent to insisting that the mediating arrow defined by $G\gamma$ is an isomorphism; in fact some authors use this as the definition of preservation of limits.

In order to prove that a functor $G : \mathcal{C} \to \mathcal{D}$ preserves limits of a diagram $D : \mathbb{I} \to \mathcal{C}$ it is not enough to exhibit an isomorphism

$$G(\varprojlim_{\mathbb{I}} D) \cong \varprojlim_{\mathbb{I}} G \circ D.$$

Indeed, the action of G over arrows may result in a family that is not universal. As an example consider the category *count* of countably infinite sets and functions. Clearly the objects of *count* are all isomorphic. There is a functor $\_ + 1 : \mathit{count} \to \mathit{count}$ that acts over sets by adding a new element: given $X \in \mathit{count}$ then $X + 1 = X \cup \{X\}$ and given a function $f : X \to Y$, the function $f + 1$ sends $a \in X$ to $f(a)$ and $\{X\}$ to $\{Y\}$. There is an isomorphism

$$(X \times Y) + 1 \cong (X + 1) \times (Y + 1),$$

but this functor does not preserve products; the arrow $\pi_X + 1 : (X \times Y) + 1 \to X + 1$ is not a projection.

If the categories $\mathcal{C}$ and $\mathcal{D}$ have enough limits the expressions

$$G(\varprojlim_{\mathbb{I}} D) \quad \text{and} \quad \varprojlim_{\mathbb{I}}(G \circ D)$$

are both functorial in D. For every D there is a mediating arrow m_D defined by $G\varepsilon$ where ε is the universal cone associated to $\varprojlim_{\mathbb{I}} D$. The family $\langle m_D \rangle_D$ is natural, this follows directly from the universality of the mediating arrows. Thus if G is $\mathcal{K}$-continuous there is a canonical isomorphism

$$G(\varprojlim_{\mathbb{I}} D) \cong \varprojlim_{\mathbb{I}}(G \circ D)$$

natural in $D \in \mathcal{K}$.

An isomorphism $G(\varprojlim_{\mathbb{I}} D) \cong \varprojlim_{\mathbb{I}}(G \circ D)$ natural in D is not always unique. Consider, for instance, the category $\mathbf{1}$ with one object, say $\star$, and the identity arrow. The functor category $[\mathbf{1},\mathbf{1}\ ]$ has only one object: the “constant” functor

$\Delta\star$. The limit for this functor is the object $\star$ itself where the limiting cone is the identity. We can extend $\mathbf{1}$ with an extra morphism

$$\begin{tikzcd} \star \arrow[loop left, "\mathsf{id}"] & {} \arrow[r, hook, "\iota"] & {} & \star \arrow[loop left, "\mathsf{id}"] \arrow[loop right, "f"] \end{tikzcd}$$

where f is also an isomorphism, *i.e.*$f \circ f = \mathsf{id}$. The inclusion functor ι clearly preserves the limit of the diagram $\Delta\star$. The mediating arrow is given by the identity on $\star$ which is an isomorphism and trivially natural. The arrow f, however, gives another isomorphism

$$\iota(\varprojlim_{\mathbf{1}} \Delta\star) \cong \varprojlim_{\mathbf{1}} \iota \circ \big(\Delta \star\big)$$

—naturality here is trivial as well.

Often checking the isomorphism between the limiting objects follows from a fairly direct calculation, while proving that a functor preserves a limiting cone can involve a fair amount of bookkeeping. We wish to determine under which conditions having an isomorphism

$$G(\varprojlim_{\mathbb{I}} D) \cong \varprojlim_{\mathbb{I}}(G \circ D)$$

is enough to ensure that G preserves limits of D.

We shall first investigate two extreme cases, one when diagrams are connected and the other when they are discrete. We later combine the results for general limits.

8.2 Connected Diagrams

We first consider a special case: preservation of connected limits, *i.e.*limits of connected diagrams. A category $\mathcal{E}$ is *connected* if it is nonempty and for any pair of objects $a, b \in \mathcal{E}$ there is a chain of arrows

$$a \to e_1 \leftarrow e_2 \to \ldots \to e_n \leftarrow b.$$

Observe that if $\mathbb{I}$ is connected then $\mathcal{C}$ has limits for all $\mathbb{I}$-indexed constant diagrams.

Proposition 8.5 Let $\mathbb{J}$ be a connected small category and $D : \mathbb{J} \to \mathcal{C}$ a diagram. For every limiting cone $\gamma : \Delta c \Rightarrow D$ the arrow $\varprojlim_{\mathbb{J}} \gamma$ is an isomorphism.

Proof: By definition $\varprojlim_{\mathbb{J}} \gamma$ is the unique arrow making the diagram

$$\begin{tikzcd} \Delta \varprojlim_{\mathbb{J}} \Delta c \arrow[r, "\Delta \varprojlim_{\mathbb{J}} \gamma"] \arrow[d, "\beta"] & \Delta \varprojlim_{\mathbb{J}} D \arrow[d, "\kappa"] \\ \Delta c \arrow[r, "\gamma"'] & D \end{tikzcd}$$

commute, where β and κ are the chosen limiting cones associated to $\varprojlim_{\mathbb{J}} \Delta c$ and $\varprojlim_{\mathbb{J}} D$ respectively. As $\mathbb{J}$ is connected there is an isomorphism $\varprojlim_{\mathbb{I}} \Delta c \overset{f}{\cong} c$ such that $\beta = \Delta f$. As γ and κ are limiting cones by Proposition 4.4 there is an isomorphism $g : c \cong \varprojlim_{\mathbb{J}} D$ such that $\gamma = \kappa \circ \Delta g$. Hence,

$$\kappa \circ \Delta(g \circ f) = \kappa \circ \Delta g \circ \Delta f = \gamma \circ \Delta f = \gamma \circ \beta = \kappa \circ \Delta \varprojlim_{\mathbb{J}} \gamma.$$

Since κ is limiting,

$$\varprojlim_{\mathbb{J}} \gamma = g \circ f.$$

Thus as f and g are isomorphisms it follows that $\varprojlim_{\mathbb{J}} \gamma$ is an isomorphism as well.

$\square$

The following main theorem of this section, establishes that a natural isomorphism is enough to ensure preservation of limits of *connected* diagrams.

Theorem 8.6 Let $\mathbb{J}$ be a small connected category. Let $G : \mathcal{C} \to \mathcal{D}$ be a functor with all $\mathbb{J}$-limits. The functor G preserves $\mathbb{J}$-limits if and only if there is an isomorphism

$$G(\varprojlim_{\mathbb{J}} D) \cong \varprojlim_{\mathbb{J}} (G \circ D)$$

natural in $D \in [\mathbb{J}, \mathcal{C}]$.

Proof: The *"only-if"* part from the general fact that limit preservation implies that the mediating arrows are isomorphisms. To show the *"if"* part, let $\mathbb{J}$ be a connected small category and assume there is an isomorphism

$$G(\varprojlim_{\mathbb{J}} D) \overset{\theta_D}{\cong} \varprojlim_{\mathbb{J}} (G \circ D)$$

natural in D. Given a diagram $D : \mathbb{J} \to \mathcal{C}$ and a limiting cone $\gamma : \Delta c \Rightarrow D$ there is a unique morphism $m : G(c) \to \varprojlim_{\mathbb{J}} G \circ D$ such that the diagram

$$\begin{tikzcd}
\Delta G(c) \arrow[r, "\Delta m"] \arrow[dr, "G\gamma"'] & \Delta \varprojlim_{\mathbb{J}} (G \circ D) \arrow[d, "\varepsilon"] \\
 & G \circ D
\end{tikzcd}$$

commutes, where ε is the chosen limiting cone for $G \circ D$. We verify that m is an isomorphism as required.

The limiting cone $\gamma : \Delta c \Rightarrow D$ induces the naturality square

$$\begin{tikzcd}
G\big(\varprojlim_{\mathbb{J}} \Delta c\big) \arrow[r, "\theta_{\Delta c}", "\cong"'] \arrow[d, "G(\varprojlim_{\mathbb{J}} \gamma)"'] & \varprojlim_{\mathbb{J}}(\Delta G(c)) \arrow[d, "\varprojlim_{\mathbb{J}} G\gamma"] \\
G(\varprojlim_{\mathbb{J}} D) \arrow[r, "\theta_D", "\cong"'] & \varprojlim_{\mathbb{J}}(G \circ D).
\end{tikzcd}$$

Since $\mathbb{J}$ is connected by Lemma 8.5 the arrow $\varprojlim_{\mathbb{J}} \gamma$ is an isomorphism and so $G(\varprojlim_{\mathbb{J}} \gamma)$ is an isomorphism as well. From the naturality square above we can conclude $\varprojlim_{\mathbb{J}} G\gamma$ is an isomorphism.

By definition $\varprojlim_{\mathbb{J}} G\gamma$ is the unique mediating arrow making the diagram

$$\begin{tikzcd}
\Delta \varprojlim_{\mathbb{J}}(\Delta G(c)) \arrow[r, "\Delta \varprojlim_{\mathbb{J}} G\gamma"] \arrow[d, "\Delta h"'] & \Delta \varprojlim_{\mathbb{J}}(G \circ D) \arrow[d, "\varepsilon"] \\
\Delta G(c) \arrow[r, "G\gamma"'] & G \circ D
\end{tikzcd}$$

commute where Δh is the chosen limiting cone associated with $\varprojlim_{\mathbb{J}}(G \circ \Delta c)$. Since $\mathbb{J}$ is connected h is an isomorphism. Hence,

$$G\gamma = \varepsilon \circ (\Delta \varprojlim_{\mathbb{J}} G\gamma) \circ (\Delta h)^{-1} = \varepsilon \circ \Delta((\varprojlim_{\mathbb{J}} G\gamma) \circ h^{-1}) \ .$$

By uniqueness of the mediating arrow,

$$m = (\varprojlim_{\mathbb{J}} G\gamma) \circ h^{-1}$$

and m is an isomorphism. Thus G preserves the limits of D.

□

It is stressed that the statement of Theorem 8.6 above refers to any natural isomorphism and not necessarily to the canonical natural transformation defined from the limit. The theorem establishes that if there exists such a natural isomorphism then the canonical natural transformation is indeed a natural isomorphism as well.

We can relax the conditions of this theorem to consider the case where not all $\mathbb{J}$-limits exist. Take instead a full subcategory $\mathcal{K} \subseteq [\mathbb{J}, \mathcal{C}]$ of diagrams whose limits exist in $\mathcal{C}$ and such that $\mathcal{K}$ includes all constant diagrams.

Corollary 8.7 Let $\mathbb{J}$ be a connected small category and $\mathcal{K}$ be a full subcategory of $[\mathbb{J},\mathcal{C}\ ]$ including all constant diagrams and such that $\mathcal{C}$ is $\mathcal{K}$-complete. Then G is $\mathcal{K}$-continuous if and only if

1. for every $D \in \mathcal{K}$, $G \circ D$ has a limit in $\mathcal{D}$; and
2. there exists an isomorphism $\varprojlim_{\mathbb{J}}(G \circ D) \cong G(\varprojlim_{\mathbb{J}} D)$ natural in $D \in \mathcal{K}$.

Proof: We use the proof of Theorem 8.6 within the subcategory $\mathcal{K}$. Notice that the expression $\varprojlim_{\mathbb{J}} D$ is functorial in D but the domain is $\mathcal{K}$ instead of $[\mathbb{J},\mathcal{C}\ ]$. As the indexing category $\mathbb{J}$ is connected the limits for constants diagrams exist in $\mathcal{C}$.

□

Connectivity is a significant constraint on diagrams. There are, however, many applications where connected limits (and colimits) are central and then the result above (and its dual) can be useful.

Theorem 8.6 does not necessarily hold when the indexing category is not connected. For example, consider the functor category $[\mathbf{2},\mathbf{1}\ ]$ were $\mathbf{2}$ is the two-objects discrete category. This functor category has a unique object: the constant diagram $\Delta\star$. Now consider the functor $G : \mathbf{1} \to \mathbf{Set}$ selecting a countable infinite set, say the natural numbers $\mathbf{N}$. As $\star \times \star = \star$ in $\mathbf{1}$, where the projections are given by the identity, we have

$$G(\star \times \star) \cong G(\star) \times G(\star)$$

natural in $\star$ since $\mathbf{N} \cong \mathbf{N} \times \mathbf{N}$. The pair $(\mathsf{id}_{\mathbf{N}}, \mathsf{id}_{\mathbf{N}})$, however, is not a product.

8.3 Products

Clearly, Theorem 8.6 cannot be applied to products; then the index category is discrete, an extreme example of lack of connectivity.

Given a discrete category $\mathbb{K}$ a diagram $D : \mathbb{K} \to \mathcal{C}$ can be regarded as a tuple of objects $\langle x_k \rangle_{k\in\mathbb{K}}$ in $\mathcal{C}$ where $x_k = D(k)$. A cone for this functor is any family of arrows (called projections) $\langle f_k : x \to x_k \rangle_{k\in\mathbb{K}}$ for some object x. Notice that as the index category is discrete there is no commutativity to check and naturality comes for free. We say that a family $\langle f_k : x \to x_k \rangle_{k\in\mathbb{K}}$ is a $\mathbb{K}$-product (or often just product) when it is a limiting cone. Because with a discrete index category naturality is automatic, we can strengthen Proposition 4.4; we retain a product when objects in the diagram vary to within isomorphism.

Proposition 8.8 Let $f_k : x \to x_k$ and $f'_k : x' \to x'_k$, for $k \in \mathbb{K}$, where $\mathbb{K}$ is a discrete category. Suppose there are isomorphisms $s : x' \cong x$, and $s_k : x'_k \cong x_k$ indexed by $k \in \mathbb{K}$, such that

$$\begin{tikzcd}
x' \arrow[r, "\overset{s}{\cong}"] \arrow[d, "f'_k"'] & x \arrow[d, "f_k"] \\
x'_k \arrow[r, "\cong", "s_k"'] & x_k
\end{tikzcd}$$

commutes for all $k \in \mathbb{K}$. Then $\langle f_k : x \to x_k \rangle_{k\in\mathbb{K}}$ is a product in $\mathcal{C}$ iff $\langle f'_k : x' \to x'_k \rangle_{k\in\mathbb{K}}$ is a product in $\mathcal{C}$.

We now study the conditions for a functor to preserve products. In the next section we see how these same conditions are enough to ensure preservation of limits in the general case. In a category with terminal object $\top$ we use $! : c \to \top$ to denote the unique arrow from c to $\top$. We will use the following simple fact about products with a terminal object.

Proposition 8.9 Let $\mathcal{C}$ be a category with a terminal object $\top$. A pair

$$\langle f : x' \to x, ! : x' \to \top \rangle$$

is a product iff f is an isomorphism.

As special limits, $\mathbb{K}$-products extend to functors once a choice of $\mathbb{K}$-product $\langle \pi_k^D : \prod_{k \in \mathbb{K}} x_k \to x_k \rangle_{k \in \mathbb{K}}$ is made for each diagram $D = \langle x_k \rangle_{k \in \mathbb{K}}$. As is traditional we have written the chosen limit object for a diagram $\langle x_k \rangle_{k \in \mathbb{K}}$ as $\prod_{k \in \mathbb{K}} x_k$, and write $x \times y$ for a diagram $\langle x, y \rangle$.

Theorem 8.10 Let $\mathcal{C}$, $\mathcal{D}$ be categories with finite products. The functor $G : \mathcal{C} \to \mathcal{D}$ preserves binary products if

1. G preserves terminal objects and
2. there is an isomorphism

   $$G(x \times y) \cong G(x) \times G(y)$$

   natural in $x, y \in \mathcal{C}$.

Proof: Assume that G preserves terminal objects and that the isomorphism

$$G(x \times y) \overset{s_{x,y}}{\cong} G(x) \times G(y)$$

is natural in x, y. Let $\top$ be a terminal object of $\mathcal{C}$. There is a morphism $! : y \to \top$ in $\mathcal{C}$. This morphism determines the commuting naturality square in the diagram

$$\begin{tikzcd}
& G(x \times y) \arrow[r, "\overset{s_{x,y}}{\cong}"] \arrow[dl, "G(\pi_1^{x,y})"'] \arrow[d, "G(\mathsf{id}_x \times !)"] & G(x) \times G(y) \arrow[d, "G(\mathsf{id}_x) \times G(!)"'] \arrow[dr, "\pi_1^{Gx,Gy}"] & \\
G(x) & G(x \times \top) \arrow[l, "G(\pi_1^{x,\top})"] \arrow[r, "\overset{s_{x,\top}}{\cong}"] & G(x) \times G(\top) \arrow[r, "\pi_1^{Gx,G\top}"'] & G(x) \ .
\end{tikzcd}$$

The left triangle commutes since it is obtained by applying G to the commuting triangle

$$\begin{tikzcd}
& x \times y \arrow[dl, "\pi_1^{x,y}"'] \arrow[d, "\mathsf{id}_x \times !"] \\
x & x \times \top \ . \arrow[l, "\pi_1^{x,\top}"]
\end{tikzcd}$$

By Proposition 8.9 the morphism $\pi_1^{x,\top}$ is an isomorphism and so $G(\pi_1^{x,\top})$ is an isomorphism as well. The right triangle commutes as products are special limit functors.

By assumption $G(\top)$ is a terminal object and so from Proposition 8.9 the arrow $\pi_1^{Gx,G\top}$ is an isomorphism. Thus the composition

$$s_1 = \pi_1^{Gx,G\top} \circ s_{x,\top} \circ G(\pi_1^{x,\top})^{-1}$$

forms an isomorphism such that

$$\begin{tikzcd}
G(x \times y) \arrow[r, "s_{x,y}", "\cong"'] \arrow[d, "G(\pi_1^{x,y})"'] & G(x) \times G(y) \arrow[d, "\pi_1^{Gx,Gy}"] \\
G(x) \arrow[r, "s_1", "\cong"'] & G(x)
\end{tikzcd}$$

commutes.

We can follow the same argument with y instead of x. Then by Proposition 8.8 the pair

$$\langle G(\pi_1^{x,y}), G(\pi_2^{x,y}) \rangle$$

is a product. (Notice that the mediating arrow defined by $\langle G(\pi_1^{x,y}), G(\pi_2^{x,y}) \rangle$ is an isomorphism but does not necessarily coincide with $s_{x,y}$.)

□

We generalise the last theorem to $\mathbb{K}$-products where the naturality of the isomorphism is required within a subcategory $\mathcal{K} \subseteq [\mathbb{K}, \mathcal{C}]$ of product diagrams.

Theorem 8.11 Let $\mathbb{K}$ be a discrete category and $\mathcal{C}$, $\mathcal{D}$ be categories with terminal object. Let $\mathcal{K} \subseteq [\mathbb{K}, \mathcal{C}]$ be the full subcategory of diagrams for which products exist in $\mathcal{C}$. The functor $G : \mathcal{C} \to \mathcal{D}$ preserves $\mathbb{K}$-products of tuples in $\mathcal{K}$ if

1. whenever $\langle x_k \rangle_k \in \mathcal{K}$ then a product of $\langle G(x_k) \rangle_k \in \mathcal{K}$ exists in $\mathcal{D}$,
2. G preserves terminal objects, and
3. there is an isomorphism

   $$G(\textstyle\prod_{k \in \mathbb{K}} x_k) \xrightarrow[\cong]{s_{\langle x_k \rangle_k}} \textstyle\prod_{k \in \mathbb{K}} G(x_k)$$

   natural in $\langle x_k \rangle_k \in \mathcal{K}$.

Proof: This generalises the proof of Theorem 8.10 above to $\mathbb{K}$-products of tuples within $\mathcal{K}$. It follows by fixing one component at a time and mapping all other components to the terminal object $\top$.

□

8.4 General Limits

A small category $\mathbb{I}$ can be decomposed into its connected components. We write $\mathbb{I} = \sum_{k \in \mathbb{K}} \mathbb{I}_k$ for this decomposition where $\mathbb{I}_k$'s are the connected components of $\mathbb{I}$—this assumes that $\mathbb{K}$ is a discrete category. A connected component $\mathbb{I}_k$ is a full subcategory of $\mathbb{I}$ and there is an inclusion functor $\iota_k : \mathbb{I}_k \to \mathbb{I}$. This functor defines by pre-composition the *"restriction"* functor

$$\_ \circ \iota_k : [\mathbb{I}, \mathcal{C}] \to [\mathbb{I}_k, \mathcal{C}].$$

If the category $\mathbb{I}$ is connected then we have

$$[\mathbb{I}, \mathcal{C}](\Delta c, \Delta d) \cong \mathcal{C}(c, d)$$

and the diagonal functor is full and faithful.

Proposition 8.12 Let $\mathbb{I} = \sum_{k \in \mathbb{K}} \mathbb{I}_k$ be a small category with $\mathbb{I}_k$, where $k \in \mathbb{K}$, its connected components. There is an isomorphism

$$[\mathbb{I}, \mathcal{C}]\big(H, F\big) \cong \prod_{k \in \mathbb{K}} [\mathbb{I}_k, \mathcal{C}]\big(H \circ \iota_k, F \circ \iota_k\big)$$

natural in $H, F \in [\mathbb{I}, \mathcal{C}]$ where $\iota_k : \mathbb{I}_k \to \mathbb{I}$ is the inclusion functor.

Proof: The isomorphism takes a natural transformation $\alpha : H \Rightarrow F$ and splits it in the natural transformations $\alpha\, \iota_k : H \circ \iota_k \Rightarrow F \circ \iota_k$. Conversely a collection of natural transformations $\langle \beta_k : H \circ \iota_k \Rightarrow F \circ \iota_k \rangle_{k \in \mathbb{K}}$ gives a natural transformation $\beta : H \Rightarrow F$. This construction is clearly a bijection and it is preserved through pre- and post-composition and thus is natural in both variables.

□

A limit can be decomposed into a product of connected limits provided these exist:

Proposition 8.13 Let $\mathbb{I} = \sum_{k \in \mathbb{K}} \mathbb{I}_k$ be a small category with $\mathbb{I}_k$, where $k \in \mathbb{K}$, its connected components. Let $D : \mathbb{I} \to \mathcal{C}$ be a functor. Assume a limiting cone γ_k with limit object $\varprojlim_{\mathbb{I}_k} (D \circ \iota_k)$ for each $k \in \mathbb{K}$ and a product $\prod_{k \in \mathbb{K}} \varprojlim_{\mathbb{I}_k} (D \circ \iota_k)$ with projections π_k. Then, for $k \in \mathbb{K}$ and $i \in \mathbb{I}_k$, the arrows

$$\textstyle\prod_{k \in \mathbb{K}} \varprojlim_{\mathbb{I}_k} (D \circ \iota_k) \xrightarrow{\pi_k} \varprojlim_{\mathbb{I}_k} (D \circ \iota_k) \xrightarrow{\gamma_i^k} D(i)$$

form the components of a limiting cone for D.

Proof: From Proposition 8.12 there is an isomorphism

$$[\mathbb{I}, \mathcal{C}]\big(\Delta c, D\big) \cong \prod_{k \in \mathbb{K}} [\mathbb{I}_k, \mathcal{C}]\big(\Delta c, D \circ \iota_k\big)$$

natural in c. Hence,

$$\begin{aligned}
[\mathbb{I},\mathcal{C}]\big(\Delta c, D\big) &\cong \prod_{k\in\mathbb{K}} [\mathbb{I}_k,\mathcal{C}]\big(\Delta c, D\circ\iota_k\big) \\
&\cong \prod_{k\in\mathbb{K}} \mathcal{C}\big(c, \varprojlim_{\mathbb{I}_k}(D\circ\iota_k)\big) && \text{the limit as a representation,} \\
&\cong \mathcal{C}\big(c, \prod_{k\in\mathbb{K}} \varprojlim_{\mathbb{I}_k}(D\circ\iota_k)\big) && \text{since hom-functor preserves limits,}
\end{aligned}$$

all isomorphisms being natural in c. This provides the limit of D as a representation. The limiting cone is obtained as the associated univeral element:

$$\mathsf{id} \longmapsto \langle \pi_k\rangle_{k\in\mathbb{K}} \longmapsto \langle \gamma^k \circ \Delta\pi_k\rangle_{k\in\mathbb{K}} \ .$$

□

The task now is to combine the results on products (Theorem 8.11) and on connected diagrams (Theorem 8.6) to treat preservation of more general limits. In order to do so we use two embeddings of functor categories. Assuming $\mathcal{C}$ has terminal object $\top$, the first embedding is the right adjoint of $\_\circ\iota_k$:

$$\begin{tikzcd}
{[\mathbb{I},\mathcal{C}]} \arrow[r, shift left, "\_\circ\iota_k"] \arrow[r, phantom, "\perp"] & {[\mathbb{I}_k,\mathcal{C}]} \arrow[l, shift left, "\_^{+}"]
\end{tikzcd} \ . \tag{19}$$

Given $H : \mathbb{I}_k \to \mathcal{C}$, the functor $H^+ : \mathbb{I} \to \mathcal{C}$ is such that it acts as H over the component $\mathbb{I}_k$ and as the constant functor $\Delta\top$ otherwise. The unit of the adjunction above is defined for $D \in [\mathbb{I},\mathcal{C}]$ as

$$(\eta_D)_i = \begin{cases} \mathsf{id}_{D(i)} & \text{if } i\in\mathbb{I}_k \\ ! & \text{otherwise (the unique arrow from } D(i) \text{ to } \top,) \end{cases}$$

which is clearly universal.

Proposition 8.14 Let $\mathbb{I}_k$ be a connected component of $\mathbb{I}$. Assume categories $\mathcal{C}$ and $\mathcal{D}$ with terminal objects and a functor $G : \mathcal{C} \to \mathcal{D}$ that preserves terminal objects.

1. there is an isomorphism $\varprojlim_{\mathbb{I}} H^+ \cong \varprojlim_{\mathbb{I}_k} H$, and
2. there is an isomorphism $\varprojlim_{\mathbb{I}} G\circ H^+ \cong \varprojlim_{\mathbb{I}_k} G\circ H$.

In both cases we mean that if one side of the isomorphism exists then so does the other. The isomorphisms are natural in $H \in \mathcal{K}$ for a subcategory $\mathcal{K} \subseteq [\mathbb{I}_k,\mathcal{C}]$ such that $\mathcal{C}$ is $\mathcal{K}$-complete.

Proof: For 1 consider the chain of isomorphisms

$$\begin{aligned}
\mathcal{C}(c, \varprojlim_{\mathbb{I}} H^+) &\cong [\mathbb{I}, \mathcal{C}]\big(\Delta c, H^+\big) && \text{the limit as a representation,} \\
&\cong [\mathbb{I}_k, \mathcal{C}]\big((\Delta c) \circ \iota_k, H\big) && \text{by the adjunction (19),} \\
&= [\mathbb{I}_k, \mathcal{C}]\big(\Delta c, H\big) && \\
&\cong \mathcal{C}(c, \varprojlim_{\mathbb{I}_k} H) && \text{the limit as a representation,}
\end{aligned}$$

all natural in c and $H \in \mathcal{K}$. As the Yoneda embedding is full and faithful it follows that there is an isomorphism

$$\varprojlim_{\mathbb{I}} H^+ \cong \varprojlim_{\mathbb{I}_k} H$$

natural in $H \in \mathcal{K}$.

For 2 observe that since G preserves the terminal objects it is possible to define an adjunction as (19) with $\mathcal{D}$ as codomain where $G \circ H^+ \cong (G \circ H)^+$. Thus we have

$$\begin{aligned}
\mathcal{D}(d, \varprojlim_{\mathbb{I}} G \circ H^+) &\cong [\mathbb{I}, \mathcal{D}]\big(\Delta d, G \circ H^+\big) && \text{the limit as a representation,} \\
&\cong [\mathbb{I}, \mathcal{D}]\big(\Delta d, (G \circ H)^+\big) && \\
&\cong [\mathbb{I}_k, \mathcal{D}]\big((\Delta d) \circ \iota_k, G \circ H\big) && \text{by the adjunction (19),} \\
&= [\mathbb{I}_k, \mathcal{D}]\big(\Delta d, G \circ H\big) && \\
&\cong \mathcal{D}(d, \varprojlim_{\mathbb{I}_k} G \circ H) && \text{the limit as a representation,}
\end{aligned}$$

all natural in d and H. It follows that there is an isomorphism

$$\varprojlim_{\mathbb{I}} G \circ H^+ \cong \varprojlim_{\mathbb{I}_k} G \circ H$$

natural in H.

□

There is a less obvious embedding $\mathbb{\Delta} : [\mathbb{K}, \mathcal{C}] \to [\mathbb{I}, \mathcal{C}]$ where $\mathbb{K}$ is the discrete category whose objects are identified with the connected components of $\mathbb{I}$. Given a tuple $\langle x_k \rangle_{k \in \mathbb{K}}$, the functor $\mathbb{\Delta}\langle x_k \rangle_{k \in \mathbb{K}} : \mathbb{I} \to \mathcal{C}$ acts as the constant Δx_k over the objects and arrows in $\mathbb{I}_k$.

Proposition 8.15 Let $G : \mathcal{C} \to \mathcal{D}$ be a functor and $\mathbb{I} = \sum_{k \in \mathbb{K}} \mathbb{I}_k$ be a small category with $\mathbb{I}_k$, where $k \in \mathbb{K}$, its connected components:

1. there is an isomorphism $\varprojlim_{\mathbb{I}} \mathbb{\Delta}\langle x_k \rangle_{k \in \mathbb{K}} \cong \prod_{k \in \mathbb{K}} x_k$, and
2. there is an isomorphism $\varprojlim_{\mathbb{I}} G \circ \mathbb{\Delta}\langle x_k \rangle_{k \in \mathbb{K}} \cong \prod_{k \in \mathbb{K}} G(x_k)$.

In both cases it is meant that if one side of the isomorphism exists then so does the other. The isomorphisms are natural in $\langle x_k \rangle_{k \in \mathbb{K}} \in \mathcal{K}$ for a subcategory of $\mathbb{K}$-tuples $\mathcal{K} \subseteq [\mathbb{K}, \mathcal{C}]$ such that $\mathcal{C}$ is $\mathcal{K}$-complete.

Proof: For 1,

$$
\begin{aligned}
\mathcal{C}(c, \varprojlim_{\mathbb{I}} \mathbb{\Delta}\langle x_k \rangle_{k \in \mathbb{K}}) &\cong [\mathbb{I}, \mathcal{C}]\big(\Delta c, \mathbb{\Delta}\langle x_k \rangle_{k \in \mathbb{K}}\big) && \text{the limit as a representation,} \\
&\cong \prod_{k \in \mathbb{K}} [\mathbb{I}_k, \mathcal{C}]\big(\Delta c \circ \iota_k, \mathbb{\Delta}\langle x_k \rangle_{k \in \mathbb{K}} \circ \iota_k\big) && \text{by Proposition 8.12,} \\
&= \prod_{k \in \mathbb{K}} [\mathbb{I}_k, \mathcal{C}]\big(\Delta c, \Delta_{\mathbb{I}_k} x_k\big) && \text{by definition of } \mathbb{\Delta}, \\
&= [\mathbb{K}, \mathcal{C}]\big(\Delta c, \langle x_k \rangle_{k \in \mathbb{K}}\big) \\
&\cong \mathcal{C}(c, \varprojlim_{\mathbb{K}} \langle x_k \rangle_{k \in \mathbb{K}}) && \text{the limit as a representation,}
\end{aligned}
$$

all natural in c and $\langle x_k \rangle_{k \in \mathbb{K}}$. Thus $\varprojlim_{\mathbb{I}} \mathbb{\Delta}\langle x_k \rangle_{k \in \mathbb{K}}$ is isomorphic to $\prod_{k \in \mathbb{K}} x_k = \varprojlim_{\mathbb{K}} \langle x_k \rangle_{k \in \mathbb{K}}$ with naturality following from Yoneda. In a similar way, using the identity

$$G \circ \Delta x = \Delta G(x) \ ,$$

we can prove 2.

□

Now we can reduce the preservation of general limits to naturality.

Theorem 8.16 Let $\mathcal{C}$, $\mathcal{D}$ be complete categories. A functor $G : \mathcal{C} \to \mathcal{D}$ is continuous if and only if for any small category $\mathbb{I}$ there is an isomorphism

$$G(\varprojlim_{\mathbb{I}} D) \cong \varprojlim_{\mathbb{I}} (G \circ D) \tag{20}$$

natural in $D \in [\mathbb{I}, \mathcal{C}]$.

Proof: The *"only-if"* part follows as usual. For the *"if"* part first observe that G trivially preserves terminal objects: take $\mathbb{I}$ to be the empty category. Let $\mathbb{I} = \sum_{k \in \mathbb{K}} \mathbb{I}_k$ be a non-empty small category with $\mathbb{I}_k$, where $k \in \mathbb{K}$, its connected components. By Proposition 8.13, any diagram $D : \mathbb{I} \to \mathcal{C}$ has a limiting cone with components

$$\textstyle\prod_{k \in \mathbb{K}} \varprojlim_{\mathbb{I}_k} (D \circ \iota_k) \xrightarrow{\pi_k} \varprojlim_{\mathbb{I}_k} (D \circ \iota_k) \xrightarrow{\gamma_i^k} D(i)$$

with projections π_k and where γ^k is the limiting cone associated with $\varprojlim_{\mathbb{I}_k} (D \circ \iota_k)$. So in order to prove that $G\varepsilon$ is limiting, it is enough to verify that the cone with components

$$G(\gamma_i^k) \circ G(\pi_k)$$

is limiting. For this it suffices to show that $\langle G(\pi_k)\rangle_{k\in\mathbb{K}}$ is a product and that for every k the cone $G\gamma^k$ is limiting.

However,

$$\begin{aligned}
G(\prod_{k\in\mathbb{K}} x_k) &\cong G(\varprojlim_{\mathbb{I}} \Delta\langle x_k\rangle_k) && \text{by Proposition 8.15,}\\
&\cong \varprojlim_{\mathbb{I}}(G\circ \Delta\langle x_k\rangle_k) && \text{by assumption (20),}\\
&\cong \prod_{k\in\mathbb{K}} G(x_k) && \text{by Proposition 8.15,}
\end{aligned}$$

all natural in $\langle x_k\rangle_k \in [\mathbb{K},\mathcal{C}]$. So G preserves $\mathbb{K}$-products by Theorem 8.11 and $\langle G(\pi_k)\rangle_{k\in\mathbb{K}}$ above is a product.

Similarly,

$$\begin{aligned}
G(\varprojlim_{\mathbb{I}_k} H) &\cong G(\varprojlim_{\mathbb{I}} H^+) && \text{by Proposition 8.14,}\\
&\cong \varprojlim_{\mathbb{I}}(G\circ H^+) && \text{by assumption (20),}\\
&\cong \varprojlim_{\mathbb{I}}(G\circ H^+) && \text{by Proposition 8.14,}
\end{aligned}$$

all natural in $H\in[\mathbb{I}_k,\mathcal{C}]$. Thus G preserves $\mathbb{I}_k$-limits by Theorem 8.6 and $G(\gamma^k)$ above is a limit for $G(D\circ\iota_k)$.

□

The proof of the theorem above can be carried out under more liberal assumptions, to cover the preservation of $\mathbb{I}$-limits, for a particular small category $\mathbb{I}$.

Theorem 8.17 Let $\mathbb{I}$ be a small category. Suppose categories $\mathcal{C}$ and $\mathcal{D}$ are categories with terminal objects and all $\mathbb{I}$-limits. A functor $G:\mathcal{C}\to\mathcal{D}$ preserves $\mathbb{I}$-limits if

1. G preserves terminal objects, and
2. there is an isomorphism $G(\varprojlim_{\mathbb{I}} D)\cong \varprojlim_{\mathbb{I}}(G\circ D)$, natural in $D\in[\mathbb{I},\mathcal{C}]$.

Proof: Let $\mathbb{I}=\sum_{k\in\mathbb{K}}\mathbb{I}_k$ with $\mathbb{I}_k$, for $k\in\mathbb{K}$, being its connected components. Having $\mathbb{I}$-limits implies having $\mathbb{I}_k$-limits for $k\in\mathbb{K}$ (Proposition 8.14), and $\mathbb{K}$-products (Proposition 8.15). Whereupon the proof can be conducted as for Theorem 8.16.

□

8.4.1 Examples

Proposition 8.18 The Yoneda embedding $\mathcal{Y} : \mathcal{C} \to [\mathcal{C}^{\mathsf{op}}, \mathbf{Set}]$ preserves limits.

Proof: As a consequence of Propositions 8.2 and 6.3 there is an isomorphism

$$(\mathcal{Y} \textstyle\int_I F(I))(C) \cong (\int_I \mathcal{Y}\, F(I))(C)$$

natural in $C \in \mathcal{C}$ and $F \in \mathit{Diag}[\mathbb{I}, \mathcal{C}]$. As $[\mathcal{C}^{\mathsf{op}}, \mathbf{Set}]$ is complete the functor $\mathcal{Y} \circ F$ has limit for all F, then from Corollary 8.17 we can conclude that $\mathcal{Y}$ preserves limits.

$\square$

Proposition 8.19 The bifunctor $\mathit{eval} : [\mathbb{J}, \mathcal{D}] \times \mathbb{J} \to \mathcal{D}$ induces a functor $\mathit{eval}(-, J) : [\mathbb{J}, \mathcal{D}] \to \mathcal{D}$ for every $J \in \mathbb{J}$. Suppose $F : \mathbb{I} \to [\mathbb{J}, \mathcal{D}]$ is a functor such that $\lambda I.\, (F\, I)J$ has a limit for all $J \in \mathbb{J}$; then $\mathit{eval}(-, J)$ preserves the limit of F.

Proof: Again we use Proposition 6.3 to conclude that there is an isomorphism

$$\big(\textstyle\int_I F(I)\big) J \cong \int_I \big(F(I)J\big)$$

natural in F and J. By definition $\mathit{eval}(\int_I F(I), J) = (\int_I F(I))(J)$ and $\int_I \mathit{eval}(F(I), J) = \int_I \big(F(I)J\big)$. Thus, from Corollary 8.17 we have $\mathit{eval}(-, J)$ preserves the limit of F.

$\square$

8.5 Preservation of colimits

Of course, we have dual results concerning the preservation of colimits. The main theorems are:

Theorem 8.20 Suppose the category $\mathbb{I}$ is small and connected. Suppose categories $\mathcal{C}, \mathcal{D}$ have initial objects and all $\mathbb{I}$-colimits.

A functor $G : \mathcal{C} \to \mathcal{D}$ preserves $\mathbb{I}$-colimits iff there is an isomorphism

$$G(\varinjlim_{\mathbb{I}} D) \cong \varinjlim_{\mathbb{I}}(G \circ D)\ ,$$

natural in $D \in [\mathbb{I}, \mathcal{C}]$.

Theorem 8.21 Suppose the category $\mathbb{I}$ is small. Suppose categories $\mathcal{C}, \mathcal{D}$ have all $\mathbb{I}$-colimits. Suppose that G sends initial objects to initial objects.

A functor $G : \mathcal{C} \to \mathcal{D}$ preserves $\mathbb{I}$-colimits iff there is an isomorphism

$$G(\varinjlim_{\mathbb{I}} D) \cong \varinjlim_{\mathbb{I}}(G \circ D)\ ,$$

natural in $D \in [\mathbb{I}, \mathcal{C}]$.

Theorem 8.22 Suppose categories $\mathcal{C}, \mathcal{D}$ are cocomplete.

A functor $G : \mathcal{C} \to \mathcal{D}$ preserves all colimits iff for all small categories $\mathbb{I}$ there is an isomorphism

$$G(\varinjlim_{\mathbb{I}} D) \cong \varinjlim_{\mathbb{I}}(G \circ D) \ ,$$

natural in $D \in [\mathbb{I}, \mathcal{C}]$.

9 Adjunctions

9.1 Definition of Adjunctions

Definition 9.1 (Adjunction) Let $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ be functors where $\mathcal{C}$ and $\mathcal{D}$ are categories. An *adjunction* in which F is the *left adjoint* and G is the *right adjoint* consists of an isomorphism

$$\mathcal{D}(F(X), Y) \overset{\theta_{X,Y}}{\cong} \mathcal{C}(X, G(Y))$$

natural in $X \in \mathcal{C}$ and $Y \in \mathcal{D}$. When such a natural isomorphism exists we write $F \dashv G$.

By (the dual form of) parametrised representability (Theorem **??**), it is sufficient to specify an adjunction via a representation for the functor $\mathcal{C}(X, G(-))$

$$\mathcal{D}(F[X], -) \overset{\theta^X}{\cong} \mathcal{C}(X, G(-))$$

where $F[X] \in \mathcal{D}$ for each $X \in \mathcal{C}$. Then, the mapping which sends X to $F[X]$ extends uniquely to a functor $F : \mathcal{C} \to \mathcal{D}$ such that

$$\mathcal{D}(F(X), Y) \overset{\theta^X_Y}{\cong} \mathcal{C}(X, G(Y))$$

natural in $X \in \mathcal{C}$ and $Y \in \mathcal{D}$, *i.e.*, so F is the left adjoint of G.

From (the dual form of) Theorem 3.12, the representation for $\mathcal{C}(X, G(-))$ defines a universal element $\eta_X = \theta^X_{F[X]}(\mathsf{id}_{F[X]})$, *i.e.*, for any $f \in \mathcal{C}(X, G(Y))$ where $Y \in \mathcal{D}$ there exists a **unique** $\overline{f} \in \mathcal{D}(F[X], Y)$ such that

$$\begin{aligned} f &= \big(\mathcal{C}(X, G(-))(\overline{f})\big)(\eta_X) \\ &= \big(G(\overline{f}) \circ -\big)(\eta_X) \\ &= G(\overline{f}) \circ \eta_X . \end{aligned}$$

Then the diagram

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[dr, "f"] & G(F[X]) \arrow[d, dashed, "G(\overline{f})"] & F[X] \arrow[d, dashed, "\overline{f}"] \\
& G(Y) & Y
\end{tikzcd}$$

commutes. The collection of universal elements $\langle \eta_X \rangle_{X \in \mathcal{C}}$ is called the *unit* of the adjunction. Thus we obtain the *freeness* condition sufficient to establish an adjunction.

Definition 9.2 (Freeness Condition) Let $G : \mathcal{D} \to \mathcal{C}$ be a functor. Say a pair $(F[X], \eta_X)$, where $F[X]$ is an object of $\mathcal{D}$ and $\eta_X : X \to G(F[X])$ is a map in $\mathcal{C}$, is *free* over X with respect to G iff for any map $f : X \to G(Y)$ in $\mathcal{C}$ with $Y \in \mathcal{D}$, there is a **unique** map $\overline{f} : F[X] \to Y$ in $\mathcal{D}$ such that $f = G(\overline{f}) \circ \eta_x$.

Suppose $G : \mathcal{D} \to \mathcal{C}$ is a functor, a necessary and sufficient condition for there to be a functor $F : \mathcal{C} \to \mathcal{D}$ such that $F \dashv G$ is that for each object $X \in \mathcal{C}$ there is an object $F[X] \in \mathcal{D}$ and an arrow $\eta_X : X \to G(F[X])$ in $\mathcal{C}$ such that $(F[X], \eta_X)$ is free over X. The family $\langle \eta_X \rangle_{X \in \mathcal{C}}$ is called the *unit* of the adjunction.

Exercise 9.3 Show that the action of the induced functor F over an arrow $f : C \to C'$ in $\mathcal{C}$ is the unique arrow determined by the freeness condition of η_C for the arrow $\eta_{C'} \circ f$. Then conclude that the collection $\langle \eta_X \rangle_{X \in \mathcal{C}}$ is a natural transformation from $\mathsf{id}_{\mathcal{C}}$ to $G \circ F$.

Dually, suppose $F : \mathcal{C} \to \mathcal{D}$ is a functor, a necessary and sufficient condition for there to be a functor $G : \mathcal{D} \to \mathcal{C}$ such that $F \dashv G$ is that for each object $Y \in \mathcal{D}$ there is an object $G[Y] \in \mathcal{C}$ and an arrow $\varepsilon_Y : F(G[Y]) \to Y$ in $\mathcal{D}$ such that $(G[Y], \varepsilon_Y)$ is *co-free* over Y with respect to F, i.e. the diagram

$$\begin{tikzcd}
G[Y] & Y & F(G[Y]) \arrow[l, "\varepsilon_Y"'] \\
X \arrow[u, dashed, "\overline{g}"'] & & F(X) \arrow[ul, "g"] \arrow[u, dashed, "F(\overline{g})"']
\end{tikzcd}$$

commutes. The family of arrows $\langle \varepsilon_Y \rangle_{Y \in \mathcal{D}}$ is called the *co-unit* of the adjunction.

9.2 The Naturality Laws for Adjunctions

Given an adjunction $F \dashv G$ for functors $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ we denote the natural isomorphism

$$\mathcal{D}(F(X), Y) \cong \mathcal{C}(X, G(Y))$$

by $\overline{(-)}$ in both directions. Traditionally this correspondence is represented by

$$\frac{F(X) \xrightarrow{g} Y}{X \xrightarrow{\overline{g}} G(Y)} \quad \text{and} \quad \frac{X \xrightarrow{f} G(Y)}{F(X) \xrightarrow{\overline{f}} Y} \; .$$

Given that both the isomorphism and its inverse are written as $\overline{(-)}$ we must have $\overline{\overline{f}} = f$ and $\overline{\overline{g}} = g$. Given an arrow h we called the arrow $\overline{h}$ the *transpose* of h. Using this notation, we investigate the consequences of naturality.

Naturality in X: For $h : X' \to X$ in $\mathcal{C}$ the square

$$\begin{tikzcd}
X & \mathcal{D}(F(X),Y) \arrow[d, "-\circ F\,h"] & \cong & \mathcal{C}(X,G(Y)) \arrow[d, "-\circ h"] \\
X' \arrow[u, "h"] & \mathcal{D}(F(X'),Y) & \cong & \mathcal{D}(X',G(Y))
\end{tikzcd}$$

commutes, *i.e.*, given $f \in \mathcal{D}(F(X),Y)$

$$\frac{F(X')\xrightarrow{F\,h}F(X)\xrightarrow{f}Y}{X'\xrightarrow{h}X\xrightarrow{\overline{f}}G(Y)}\ .$$

Reading from top to bottom ($\downarrow$) $\overline{f \circ F\,h} = \overline{f} \circ h$, and reading from bottom to top ($\uparrow$) $\overline{g \circ h} = \overline{g} \circ F\,h$.

Naturality in Y: For $k : Y \to Y'$ in $\mathcal{D}$ the square

$$\begin{tikzcd}
Y \arrow[d, "k"] & \mathcal{D}(F(X),Y) \arrow[d, "k\circ-"] & \cong & \mathcal{C}(X,G(Y)) \arrow[d, "G\,k\circ-"] \\
Y' & \mathcal{D}(F(X),Y') & \cong & \mathcal{D}(X,G(Y'))
\end{tikzcd}$$

commutes, *i.e.*, given $f \in \mathcal{D}(F(X),Y)$

$$\frac{F(X)\xrightarrow{f}Y\xrightarrow{k}Y'}{X\xrightarrow{\overline{f}}G(Y)\xrightarrow{G\,k}G(Y')}\ .$$

Reading from top to bottom ($\downarrow$) $\overline{k \circ f} = G\,k \circ \overline{f}$, and reading from bottom to top ($\uparrow$) $\overline{G\,k \circ g} = k \circ \overline{g}$.

9.3 The Triangle Identities

Assume the functors $\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "F"] & \mathcal{D} \arrow[l, shift left, "G"] \end{tikzcd}$.

Proposition 9.4 Given an adjunction $F \dashv G$ with unit η and counit ε the diagrams of functors and natural transformations

$$\begin{tikzcd}
G \arrow[r, "\eta\,G"] \arrow[dr, "\mathsf{id}_G"'] & G\,F\,G \arrow[d, "G\,\varepsilon"] & F \arrow[r, "F\,\eta"] \arrow[dr, "\mathsf{id}_F"'] & F\,G\,F \arrow[d, "\varepsilon\,F"] \\
 & G & & F
\end{tikzcd}$$

commute. They are called the *triangle identities.*

Proof: Recall from exercise 9.3 (and its dual) that the unit and counit of an adjunction are natural transformations. We prove only the left hand identity above, the other follows analogously. Given $Y \in \mathcal{D}$, we have

$$\begin{array}{c} G(Y) \xrightarrow{\eta_{G(Y)}} G(F(G(Y))) \xrightarrow{G(\varepsilon_Y)} G(Y) \\ \hline F(G(Y)) \xrightarrow{\overline{\eta_{G(Y)}} = \mathsf{id}_{G(Y)}} F(G(Y)) \xrightarrow{\varepsilon_Y} Y, \end{array}$$

and by definition $\overline{\varepsilon_Y} = \mathsf{id}_{G(Y)}$. Then $\overline{\overline{G(\varepsilon_Y) \circ \eta_{G(Y)}}} = \mathsf{id}_{G(Y)}$, *i.e.*, $G(\varepsilon_Y) \circ \eta_{G(Y)} = \mathsf{id}_{G(Y)}$.

□

Theorem 9.5 (Adjunction) A pair of natural transformations $(\eta : \mathsf{id}_{\mathcal{C}} \Rightarrow G \circ F, \varepsilon : F \circ G \Rightarrow \mathsf{id}_{\mathcal{D}})$ satisfying the triangular identities define an adjunction as follows:

- Given an arrow $f : X \to G(Y)$ in $\mathcal{C}$ the transpose $\overline{f}$ is $\varepsilon_Y \circ F(f)$, and
- given an arrow $g : F(X) \to Y$ in $\mathcal{D}$ the transpose $\overline{g}$ is $\eta_X \circ G(f)$.

Proof: We need to check these functions give a natural bijection. Naturality is inherited from naturality of η and ε, *e.g.*, given an arrow $h : X' \to X$ in $\mathcal{C}$ and $g \in \mathcal{D}(F(X), Y)$

$$\begin{aligned} \overline{g} \circ h &= G(g) \circ \eta_X \circ h && \text{by def. of } (\overline{-}), \\ &= G(g) \circ G(F(h)) \circ \eta_{X'} && \text{by naturality of } \eta, \\ &= G(g \circ F(h)) \circ \eta_{X'} \\ &= \overline{g \circ F(h)}. \end{aligned}$$

Now, we check the functions are mutually inverse. Given $f : X \to G(Y)$ in $\mathcal{C}$

$$\begin{aligned} \overline{\overline{f}} &= G(\varepsilon_Y \circ F(f)) \circ \eta_X && \text{by def. of } (\overline{-}), \\ &= G(\varepsilon_Y) \circ G(F(f)) \circ \eta_X \\ &= G(\varepsilon_Y) \circ \eta_{G(Y)} \circ f && \text{by naturality of } \eta, \\ &= f && \text{by triangle identity.} \end{aligned}$$

Similarly given an arrow $g : F(X) \to Y$ we can prove $\overline{\overline{g}} = g$.

□

Corollary 9.6 An adjunction $F \dashv G$ is completely determined by its unit and counit.

Proof: Since the unit and counit are natural transformations satisfying the triangle identities and the definition of the transpose in the theorem above corresponds to the freeness conditions.

□

9.4 Limits and Adjunctions

Theorem 9.7 Suppose the functors $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ form an adjunction $F \dashv G$. Then G preserves all limits which exist in $\mathcal{D}$, and F preserves all colimits which exist in $\mathcal{C}$.

Proof: Assume $D : \mathbb{I} \to \mathcal{D}$ is a diagram whose limit exists in $\mathcal{D}$. Composing isomorphisms

$$\begin{aligned}
\mathcal{C}\big(U, G(\textstyle\int_I D(I))\big) &\cong \mathcal{D}\big(F(U), \textstyle\int_I D(I)\big) && \text{by the adjunction,}\\
&\cong \textstyle\int_I \mathcal{D}\big(F(U), D(I)\big) && \text{by Proposition 8.2,}\\
&\cong \textstyle\int_I \mathcal{C}\big(U, G(D(I))\big) && \text{by the adjunction,}\\
&\cong \mathcal{C}\big(U, \textstyle\int_I G(D(I))\big) && \text{by representation of limits,}
\end{aligned}$$

by Proposition 3.6 we have

$$G(\textstyle\int_I D(I)) \cong \textstyle\int_I G(D(I))$$

natural in $U \in \mathcal{C}$ and $D \in \mathit{Diag}[\mathbb{I}, \mathcal{D}]$ (see Remark 4.36). It follows that the limit for $G \circ D$ exists for any $D \in \mathit{Diag}[\mathbb{I}, \mathcal{D}]$ and thus from Theorem 8.17 we can conclude that G preserves limits. Dually F preserves colimits.

□

Exercise 9.8 Complete the proof by showing that left adjoints preserve colimits.

Exercise 9.9 Show $m : A \to B$ is monic in a category $\mathcal{D}$ iff

$$\begin{tikzcd}
A \arrow[r, "\mathsf{id}_A"] \arrow[d, "\mathsf{id}_A"'] & A \arrow[d, "m"] \\
A \arrow[r, "m"'] & B
\end{tikzcd}$$

is a pullback. Deduce right adjoints preserve monics. Why do left adjoints preserve epis?

9.5 Representation Functors

Assume a functor $F : \mathbb{C} \to \mathcal{D}$ where $\mathcal{D}$ has all small colimits (i.e. $\mathcal{D}$ is cocomplete). We can obtain an adjunction

$$\begin{tikzcd}
{[\mathbb{C}^{\mathsf{op}}, \mathbf{Set}]} \arrow[r, bend right, "F_!"'] \arrow[r, phantom, "\top"] & \mathcal{D} \arrow[l, bend right, "F^*"']
\end{tikzcd}$$

The functor $F_!$ is got by extending F along the Yoneda embedding $\mathcal{Y}$ (an example of a left Kan extension):

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "\mathcal{Y}"] \arrow[dr, "F"'] & {[\mathbb{C}^{\mathsf{op}}, \mathbf{Set}]} \arrow[d, dashed, "F_!"] \\
 & \mathcal{D}.
\end{tikzcd}$$

Define $F_! = \lambda X.\, \int^C X(C) \cdot F(C)$ and $F^* = \lambda Y.\, \lambda C.\, \mathcal{D}(F(C), Y)$. To show $F_! \dashv F^*$:

$$\begin{aligned}
\mathcal{D}(F_!(X), Y) &= \mathcal{D}\big(\textstyle\int^C X(C) \cdot F(C), Y\big) && \text{by definition,} \\
&= \textstyle\int_C \mathcal{D}(X(C) \cdot F(C), Y) && \text{by representation for coends,} \\
&= \textstyle\int_C \mathcal{D}(\sum_{x \in X(C)} F(C), Y) && \text{by definition of copower,} \\
&\cong \textstyle\int_C \prod_{x \in X(C)} \mathcal{D}(F(C), Y) && \qquad\qquad\qquad\qquad (21) \\
&\cong \textstyle\int_C [X(C), \mathcal{D}(F(C), Y)] \\
&\cong \textstyle\int_C [X(C), (F^*(Y))(C)] && \text{by definition of } F^*, \\
&\cong [\mathbb{C}^{\mathsf{op}}, \mathbf{Set}](X, F^*(Y)) && \text{by naturality formula,}
\end{aligned}$$

where all the isomorphisms are natural in $X \in [\mathbb{C}^{\mathsf{op}}, \mathbf{Set}]$ and $Y \in \mathcal{D}$.

Note that treating the set $X(C)$ as a discrete category we have

$$\sum_{x \in X(C)} F(C) = \int^{x \in X(C)} F(C) \quad \text{and} \quad \textstyle\prod_{x \in X(C)} F(C) = \int_{x \in X(C)} F(C),$$

so step (21) is a special case of the representation for colimits.

In particular, presheaf categories $\widehat{\mathbb{D}} = [\mathbb{D}^{\mathsf{op}}, \mathbf{Set}]$ have all colimits (they are obtained pointwise from colimits in $\mathbf{Set}$). So given $F : \mathbb{C} \to \widehat{\mathbb{D}}$ we obtain

$$\begin{tikzcd}
\widehat{\mathbb{C}} \arrow[r, bend right, "F_!"'] \arrow[r, phantom, "\top"] & \widehat{\mathbb{D}}. \arrow[l, bend right, "F^*"']
\end{tikzcd}$$

Note that F might arise from a functor $F_0 : \mathbb{C} \to \mathbb{D}$ by composition with Yoneda:

$$F : \mathbb{C} \xrightarrow{F_0} \mathbb{D} \xrightarrow{\mathcal{Y}} \widehat{\mathbb{D}}.$$

9.6 Cartesian-closed Categories

Definition 9.10 (Cartesian Closed Categories) A *cartesian-closed category* (ccc) is a category $\mathcal{C}$ with

- all finite products specifically given (so there is a terminal object $\mathbf{1}_{\mathcal{C}}$ and binary products $A \times B$ for all $A, B \in \mathcal{C}$); and

- a representation for the functor $\mathcal{C}(- \times B, C)$:

$$\mathcal{C}(-, [B, C]) \overset{\theta}{\cong} \mathcal{C}(- \times B, C)$$

  for all $B, C \in \mathcal{C}$.

Notation 9.11 The object $[B, C] \in \mathcal{C}$ is called the *exponential* of B and C and sometimes is written as C^B.

By parameterised representability (Theorem **??**) for a ccc $\mathcal{C}$ there is a unique way to extend the mapping $(B, C) \mapsto [B, C]$ to an *exponentiation* functor $[-, +]$ from $\mathcal{C}^{\mathsf{op}} \times \mathcal{C}$ to $\mathcal{C}$ such that

$$\mathcal{C}(A, [B, C]) \cong \mathcal{C}(A \times B, C)$$

natural in A, B, C. Notice that from this isomorphism we have $- \times B \dashv [B, -]$. In fact, provided $\mathcal{C}$ has specified finite products an adjunction with $- \times B \dashv [B, -]$ for all $B \in \mathcal{C}$, conversely determines a cartesian closed structure on $\mathcal{C}$.

Exercise 9.12 What are the unit and counit of $- \times B \dashv [B, -]$?

9.7 Presheaf Categories

Proposition 9.13 Presheaf categories are cartesian-closed.

Proof: The category $\widehat{\mathbb{C}} = [\mathbb{C}^{\mathsf{op}}, \mathbf{Set}]$ has products, computed pointwise, like all limits:

$$(X \times Y)(U) \cong X(U) \times Y(U)$$

natural in $U \in \mathcal{C}$. We can make an educated guess at the form of the exponentiation $[Y, Z]$, for $Y, Z \in \widehat{\mathbb{C}}$. It should satisfy:

$$\begin{aligned} [Y, Z](U) &\cong \widehat{\mathbb{C}}(\mathcal{Y}U, [Y, Z]) && \text{by the Yoneda Lemma (Theorem 3.3)} \\ &\cong \widehat{\mathbb{C}}(\mathcal{Y}U \times Y, Z) && \text{by the adjunction property.} \end{aligned}$$

So we define $[Y, Z] = \lambda U.\, \widehat{\mathbb{C}}(\mathcal{Y}U \times Y, Z)$ and check that $\widehat{\mathbb{C}}(X, [Y, Z]) \cong \widehat{\mathbb{C}}(X \times Y, Z)$ is natural in X:

$$
\begin{aligned}
\widehat{\mathbb{C}}(X, [Y, Z]) &\cong \textstyle\int_U [X(U), [Y, Z](U)] && \text{by naturality formula,}\\
&\cong \textstyle\int_U [X(U), \widehat{\mathbb{C}}(\mathcal{Y}\,U \times Y, Z)] && \text{by definition of } [Y, Z],\\
&\cong \textstyle\int_U \big[X(U), \int_V [\mathbb{C}(V, U) \times Y(V), Z(V)]\big] && \text{by naturality formula,}\\
&\cong \textstyle\int_U \int_V \big[X(U), [\mathbb{C}(V, U) \times Y(V), Z(V)]\big] && \text{by representation for ends,}\\
&\cong \textstyle\int_U \int_V \big[X(U) \times \mathbb{C}(V, U), [Y(V), Z(V)]\big] && \text{by currying and uncurrying,}\\
&\cong \textstyle\int_V \int_U \big[X(U) \times \mathbb{C}(V, U), [Y(V), Z(V)]\big] && \text{by Fubini (Corollary 6.6),}\\
&\cong \textstyle\int_V \big[\int^U X(U) \times \mathbb{C}(V, U), [Y(V), Z(V)]\big] && \text{by representation of coends,}\\
&\cong \textstyle\int_V \big[X(V), [Y(V), Z(V)]\big] && \text{by the density formula,}\\
&\cong \textstyle\int_V [X(V) \times Y(V), Z(V)] && \text{by uncurrying,}\\
&\cong \textstyle\int_V [(X \times Y)(V), Z(V)] && \text{product is got pointwise (Proposition 6.3,}\\
&\cong \widehat{\mathbb{C}}(X \times Y, Z) && \text{by naturality formula,}
\end{aligned}
$$

all natural in $X \in \widehat{\mathcal{C}}$.

$\square$

Theorem 9.14 (Presheaf categories are free colimit completions) The category of presheaves $\widehat{\mathbb{C}} = [\mathbb{C}^{\mathsf{op}}, \mathbf{Set}]$ with the Yoneda embbeding $\mathcal{Y} : \mathbb{C} \to \widehat{\mathbb{C}}$ is the free colimit completion of $\mathbb{C}$, i.e. for any functor $F : \mathbb{C} \to \mathcal{D}$, where $\mathcal{D}$ is a category with all colimits there is a **unique** colimit-preserving functor $F_! : \widehat{\mathbb{C}} \to \mathcal{D}$ such that the diagram

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "\mathcal{Y}"] \arrow[dr, "F"'] & \widehat{\mathbb{C}} \arrow[d, dashed, "F_!"] \\
& \mathcal{D}
\end{tikzcd}$$

commutes, *i.e.*, $F_! \circ \mathcal{Y} \cong F$.

Proof: Let us define $F_! = \lambda X.\, \int^C X(C) \cdot F(C)$. As shown in § 9.5 there is an adjunction $F_! \dashv \lambda D.\, \mathcal{D}(F(-), D)$. By Theorem 9.7 as a left adjoint $F_!$ preserves colimits.

The adjunction $F_! \dashv \lambda D.\, \mathcal{D}(F(-), D)$ is given by an isomorphism

$$\mathcal{D}(F_!(X), D) \cong \widehat{\mathbb{C}}(X, \mathcal{D}(F(-), D))$$

natural in $X \in \widehat{\mathbb{C}}$ and $D \in \mathcal{D}$. Setting X to be a representable $\mathcal{Y}\,V$:

$$
\begin{aligned}
\mathcal{D}(F_!(\mathcal{Y}\,V), D) &\cong \widehat{\mathbb{C}}(\mathcal{Y}\,V, \mathcal{D}(F(-), D))\\
&\cong \mathcal{D}(F(V), D) && \text{by the Yoneda Lemma (Theorem 3.3)}
\end{aligned}
$$

natural in D and V. Then $F_!(\mathcal{Y}\,V) \cong F(V)$, natural in $V \in \mathbb{C}$, *i.e.*, $F_! \circ \mathcal{Y} \cong F$.

It remains to prove the uniqueness of $F_!$. Suppose that $H : \widehat{\mathbb{C}} \to \mathcal{D}$ is a colimit-preserving functor such that $H \circ \mathcal{Y} \cong F$. Then,

$$\begin{aligned} H(X) &\cong H\left(\int^C X(C) \cdot \mathcal{Y}\, C\right) && \text{by the density formula,} \\ &\cong \int^C X(C) \cdot H(\mathcal{Y}\, C) && \text{as H preserves colimits,} \\ &\cong \int^C X(C) \cdot F(C) && \text{since } H \circ \mathcal{Y} \cong F, \\ &= F_!(X) && \text{by definition of } F_!, \end{aligned} \tag{22}$$

natural in $X \in \widehat{\mathbb{C}}$. Hence $H \cong F_!$, as required to complete the proof. The following expands step 23

$$\begin{aligned} H\left(\int^C X(C) \cdot \mathcal{Y}\, C\right) &\cong \int^C H(X(C) \cdot \mathcal{Y}\, C) && \text{as coends are colimits,} \\ &\cong \int^C X(C) \cdot H(\mathcal{Y}\, C) && \text{as copowers are coproducts,} \end{aligned}$$

and H preserves colimits.

□

10 Monads and their Algebras

10.1 Definition of Monad

Definition 10.1 A *monad* on a category $\mathcal{C}$ is a functor $T : \mathcal{C} \to \mathcal{C}$ equipped with two natural transformations $\eta : \mathsf{id}_{\mathcal{C}} \Rightarrow T$ and $\mu : T^2 \Rightarrow T$ such that the diagrams of functors and natural transformations

$$\begin{tikzcd} T^3 \arrow[r, "T\,\mu"] \arrow[d, "\mu\, T"'] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"] & T \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, "T\,\eta"] \arrow[dr, "\mathsf{id}_T"'] & T^2 \arrow[d, "\mu"] & T \arrow[l, "\eta\, T"'] \arrow[dl, "\mathsf{id}_T"] \\ & T & \end{tikzcd} \tag{23}$$

commute. Sometimes we write a monad as a tripe (T, η, μ).

Example 10.2 (Action of a Monoid?) If $[M, *, e]$ is a monoid in **Set** then $M \times - :$ **Set** $\to$ **Set** is a monad where the function $x \mapsto (e, x)$ for $x \in X$ defines η_X, and the function $(m_1, (m_2, x)) \mapsto (m_1 * m_2, x)$ for $m_1, m_2 \in M$ and $x \in X$ defines μ_X.

Example 10.3 (Monads for Posets) closure operators

Theorem 10.4 Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors. An adjunction $F \dashv G$ with unit η and counit ε defines a monad $(G\,F, \eta, G\,\varepsilon\,F)$ on $\mathcal{C}$.

Proof: By naturality of ε for any $B \in \mathcal{D}$ the square

$$\begin{tikzcd}
F\,G\,F\,G\,B \arrow[r, "\varepsilon_{F\,G\,B}"] \arrow[d, "F\,G\,\varepsilon_B"] & F\,G\,B \arrow[d, "\varepsilon_B"] \\
F\,G\,B \arrow[r, "\varepsilon_B"] & B
\end{tikzcd}$$

commutes. By pre-composing with F and post-composing with G we obtain the required square defined in **??**. The triangles in **??** follow similarly from the triangular identities.

□

10.2 Algebra for a Monad

Definition 10.5 (T-algebra) Suppose (T, η, μ) is a monad on $\mathcal{C}$. An algebra for T or T-*algebra* is an arrow $\theta : T\,A \to A$ in $\mathcal{C}$ such that the diagrams

$$\begin{tikzcd}
T^2\,A \arrow[r, "\mu_A"] \arrow[d, "T\,\theta"'] & T\,A \arrow[d, "\theta"] \\
T\,A \arrow[r, "\theta"] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "\eta_A"] \arrow[dr, "\mathsf{id}_A"'] & T\,A \arrow[d, "\theta"] \\
 & A
\end{tikzcd}
\tag{24}$$

commute.

Definition 10.6 (Map of T-algebras) Let $\theta : TA \to A$ and $\psi : TB \to B$ be T-algebras for a monad T on $\mathcal{C}$. A *map* from θ to ψ is an arrow $f : A \to B$ in $\mathcal{C}$ such that the diagram

$$\begin{tikzcd}
T\,A \arrow[r, "T\,f"] \arrow[d, "\theta"'] & T\,B \arrow[d, "\psi"] \\
A \arrow[r, "f"] & B
\end{tikzcd}
\tag{25}$$

commutes.

Proposition 10.7 (Eilenberg-Moore Category) Given a monad (T, η, μ) on $\mathcal{C}$, there is a category $\mathcal{C}^T$ where the objects are T-algebras and arrows are maps of T-algebras

Proof: Left as an exercise.

□

Exercise 10.8 Verify the definition above gives a category.

There is a "forgetful" functor $G^T : \mathcal{C}^T \to \mathcal{C}$ acting on objects and arrows

$$\begin{tikzcd}
\theta : T\,A \to A \arrow[d, "f"] & & A \arrow[d, "f"] \\
\psi : T\,B \to B & \longmapsto & B.
\end{tikzcd}$$

The functor G^T has a left adjoint $F^T : \mathcal{C} \to \mathcal{C}^T$ defined by

- $F^T(A) = \mu_A : T^2\,A \to T\,A$ for $A \in \mathcal{C}$; and
- $F^T(f) = T(f)$ for $f \in \mathbf{arr}(\mathcal{C})$.

Exercise 10.9 Check that $\mu_A : T^2\,A \to T\,A$ is a T-algebra and that given an arrow $f : A \to B$ in $\mathcal{C}$ the arrow $T(f)$ is a map of T-algebras.

Proposition 10.10 There is an adjunction $F^T \dashv G^T$.

Proof: Since $G^T \circ F^T = T$ we can define the unit of the adjunction to be the natural transformation η defined by the monad. The square in the definition of T-algebra 25 establishes tha any T-algebra $\theta : T\,A \to A$ is itself a map from $F^T \circ G^T(A) = \mu_A : T^2\,A \to T\,A$ to A. Moreover, this transformation is gives rise to a natural transformation $\varepsilon : F^T \circ G^T \Rightarrow \mathsf{id}_{\mathcal{C}^T}$ – left as exercise. Now we check that η and ε satisfy the triangular identities:

- the identity $\theta \circ \eta_A$ in the definition of T-algebra gives $G^T\,\varepsilon \circ \eta\,G^T = \mathsf{id}_{G^T}$; and
- the identity $\mu_A \circ T\,\eta_A$ in the definition of monad gives $\varepsilon\,F^T \circ F^T\,\nu = \mathsf{id}_{F^T}$.

$\square$

Exercise 10.11 Prove that the collection of arrows $\langle \varepsilon_{\theta : T\,A \to A} = \theta \rangle_{\theta : T\,A \to A \in \mathcal{C}^T}$ is a natural transformation from $F^T \circ G^T$ to $\mathsf{id}_{\mathcal{C}^T}$.

Theorem 10.12 Every monad is defined by its T-algebras: the monad induced by the adjunction $F^T \dashv G^T$ is T istself.

Proof: From proposition 11.4 the adjunction $F^T \dashv G^T$ with unit η and counit ε defines a monad $(G^T\,F^T, \eta, G^T\,\varepsilon\,F^T)$. Clearly, $G^T\,F^T = T$ and η is the unit of T. Given $A \in \mathcal{C}$ the component $(G^T\,\varepsilon\,F^T)_A = \mu_A$.

$\square$

Definition 10.13 A T-algebra θ isomorphic to $F^T(A) = \mu_A : T^2\,A \to T\,A$ for some $A \in \mathcal{C}$ is called a *free T-algebra*.

Proposition 10.14 (Kleisli Category) Given a monad (T, η, μ) on $\mathcal{C}$ the following definition gives a category:

- $\mathbf{ob}(\mathcal{C}_T) = \mathbf{ob}(\mathcal{C})$,
- $\mathcal{C}_T(A, B) = \mathcal{C}(A, T\,B)$,
- for $f \in \mathcal{C}_T(A, B)$ and $g \in \mathcal{C}_T(B, C)$ the composition $g \circ f \in \mathcal{C}_T(A, C)$ is defined to be $\mu_C \circ T\,g \circ f \in \mathcal{C}(A, T\,C)$, and
- the identity $\mathsf{id}_A \in \mathcal{C}_T(A, A)$ is $\eta_A \in \mathcal{C}(A, T\,A)$.

This is the *Kleisli category* defined by T and is written as $\mathcal{C}_T$.

Proof: That this definition gives indeed a category follows from the indentities in the definition of monad and naturality; the details are left as exercise.

□

Exercise 10.15 Prove the proposition above.

Proposition 10.16 The full subcategory induced by the free subalgebras is equivalent to $\mathcal{C}_T$.

Proof: Let $\mathit{Free}^T \subseteq \mathcal{C}^T$ be the full subctegory where the objects are the free T-algebras. Define the functor $H : \mathcal{C}_T \to \mathit{Free}^T$ as follows:

- $H(C) = \mu_C : T^2\,C \to T\,C$ for $C \in \mathcal{C}_T$, and
- $H(f) = \mu_D \circ T(f)$ for $f \in \mathcal{C}_T(C, D)$.

That $H(f)$ is a T-algebras follows from naturality of μ and the identity $\mu \circ \mu\,T = \mu \circ T\mu$ in the definition of monad. Similarly we can show that H preserves identities and composition defining indeed a functor.

By definition of Free^T the functor H is esentially surjective . Moreover H is full and faithful:

> H is full: given map of T-algebras $g : H(C) \to H(D)$ we have the arrow $g \circ \eta_C : C \to T\,D$ in $\mathcal{C}$ sucht that
>
> $$\begin{aligned} H(g \circ \eta_C) &= \mu_D \circ T\,g \circ T\,\eta_C \\ &= g \circ \mu_C \circ T\,\eta_C && g \text{ is a } T\text{-algebra}, \\ &= g \circ \mathsf{id}_{T(C)} && \text{by triangle in } \mathbf{??}, \\ &= g. \end{aligned}$$
>
> H is faithful: given $f, g : C \to T\,D$ such that $H(f) = H(g)$, then
>
> $$\begin{aligned} f &= \mu_D \circ \eta_{T\,D} \circ f && \text{by triangle in } (\mathbf{??}), \\ &= \mu_D \circ T\,f \circ \eta_C && \text{by naturality of } \eta, \\ &= \mu_D \circ T\,g \circ \eta_C && \text{by assumption } H(f) = H(g), \\ &= \mu_D \circ \eta_{T\,D} \circ g && \text{by naturality of } \eta, \\ &= g && \text{by triangle in } (\mathbf{??}). \end{aligned}$$

Therefore, H is an equivalence.

$\square$

We have the diagram of categories and functors:

$$\begin{tikzcd}
\mathcal{C}_T \arrow[r, "H"] & \mathcal{C}^T \arrow[dl, shift left, "G^T"] \\
\mathcal{C}. \arrow[ur, shift left, "F^T"] \arrow[ur, phantom, "\dashv"] &
\end{tikzcd}$$

The equivalence $H : \mathcal{C}_T \to \mathit{Free}^T \subseteq \mathcal{C}^T$ induces a functor $S : \mathit{Free}^T \to \mathcal{C}_T$ which is an equivalence as well. Moreover by Proposition **??** there is an adjunction $S \dashv H$.

Corollary 10.17 The functor $G^T \circ H : \mathcal{C}_T \to \mathcal{C}$ where H is the equivalence from $\mathcal{C}_T$ to the full subcategory of free algebras has a left adjoint. Furthermore the monad defined by this adjunction is T.

Proof: By composing the adjunction $S \dashv H$ with the adjunction obtained by restricting $F^T \dashv G^T$ to the full subcategory of free algebras we obtaine an adjunction $G_T = F_T = S \circ F^T \dashv G^T \circ H = G_T$ – see **??**. By using the axioms in the definition of Monad it easy to check that $G_T \circ F_T = T$, since the unit and counit of $S \dashv H$ are natural isomorphisms we conclude that the monad given by $F_T \dashv G_T$ is indeed T.

$\square$