Measurement spaces

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Abstract

The question of what should be meant by a measurement is tackled from a mathematical perspective whose physical interpretation is that a measurement is a process via which a finite amount of classical information is generated. This motivates a mathematical definition of space of measurements that consists of a topological stably Gelfand quantale whose open sets represent measurable physical properties. It also accounts for the distinction between quantum and classical measurements, and for the emergence of "classical observers." The latter have a relation to groupoid C*-algebras, and link naturally to Schwinger's notion of selective measurement.

Keywords: Measurement problem, selective measurements, classical observers, locally compact locales, stably Gelfand quantales, groupoid C*-algebras

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1 Introduction

The notion of measurement in quantum mechanics is deeply problematic because of the apparent need to understand it in terms of concepts like "system" and "apparatus" which ultimately are ill-defined and therefore provide a shaky conceptual foundation for one of the pillars of modern physics. This explains many of the efforts devoted to the development of realist variants that either try to explain or do away with the "collapse of the wave function," such as Bohmian mechanics [5], objective collapse theories [3, 18, 40], the many-worlds interpretation [11, 16, 60], or decoherence [23, 61–63]. The following passage from Bell [4] is representative of this:

> "The first charge against 'measurement', in the fundamental axioms of quantum mechanics, is that it anchors there the shifty split of the world into 'system' and 'apparatus'. A second charge is that the word comes loaded with meaning from everyday life, meaning which is entirely inappropriate in the quantum context. (...) In other contexts, physicists have been able to take words from everyday language and use them as technical terms with no great harm done. Take for example, the 'strangeness', 'charm', and 'beauty' of elementary particle physics. No one is taken in by this 'baby talk', as Bruno Touschek called it. Would that it were so with 'measurement'. But in fact the word has had such a damaging effect on the discussion, that I think it should now be banned altogether in quantum mechanics."

Rather than banning the word, in this paper I propose to tackle the question of what should be meant by a measurement from a mathematical perspective, namely by providing a model whose physical interpretation can be loosely described as follows: *by a measurement is meant any finite physical process in the course of which a finite amount of classical information is produced,* where, intuitively, by *finite classical information* is meant that which can be recorded by writing down a finite list of 0s and 1s on a notebook. The model presented in this paper stems from a mathematical rendition of this idea in terms of its structural and logical properties rather than the usual quantitative notions of information theory. The main point of the paper is that the model as a whole can then be regarded as a mathematical *definition* of measurement, one that should have conceptual, if not practical, significance. In particular, no split is assumed a priori between system, apparatus, and the enviroment. Instead, such "macroscopic concepts" emerge from the mathematical structure, thus hinting at how to develop a realist view of physics where measurements are taken to be fundamental.

Concretely, here the concept of a measurement is conveyed by a definition of *space* of measurements whose algebraic structure is based on a multiplication of measurements that represents *composition in time*, such as measuring the spin of a silver atom along z using a Stern–Gerlach apparatus and then measuring its spin along x. In addition, such a space is equipped with a sober topology whose open sets represent *physical properties*, each of which corresponds to a finite amount of classical information. Perhaps not surprisingly, there is a connection to ideas from theoretical computer science where open sets play the role of *finitely observable properties* (of computers running programs) [57]. Additional conditions are imposed via the specialization order of the topology, which is required to possess joins (suprema) that represent *logical disjunctions.* Another part of the algebraic structure is an involution for the multiplication. This can be used in order to define a notion of *reversibility*, and it makes the measurement space a stably Gelfand quantale [46–48]. Such quantales have many *local symmetries* that are encoded by naturally associated étale groupoids [48], so any measurement space that satisfies all these properties is said to be *symmetric.*

An example of a symmetric measurement space is Mulvey's quantale $\operatorname{Max} A$ of closed linear subspaces of a C*-algebra A [25, 35–37], equipped with the lower Vietoris topology [38, 58], and another is the quantale of open sets $\mathcal{O}(G)$ of any locally compact groupoid G [45], equipped with the Scott topology [19, Ch. II]. Whereas the former is regarded as a space of quantum measurements, the latter is an example of a space of classical measurements, in part for reasons which resemble those that justify the appearance of continuous lattices in classical computing — see [56]. There is an interplay between the two types of measurement space: we "quantize" $\mathcal{O}(G)$ by constructing a groupoid C*-algebra of G (at least if G is étale), and, in the opposite direction, we find copies of quantales of the form $\mathcal{O}(G)$ inside $\operatorname{Max} A$, which provide a basis for a notion of *classical observer.* Note that in symmetric measurement spaces there is no associated external notion of time or space. There is only the primitive notion of time which is conveyed by the multiplication, along with the spaces (in fact locales) carried by classical observers, which can be identified with actual topological spaces in the case of $\operatorname{Max} A$.

A related aspect is that, due to the aforementioned abundance of étale groupoids, and in accordance with insights of Ciaglia, Ibort and Marmo [7, 8], the notion of *selective measurement* of Schwinger [54] is naturally encoded in symmetric measurement spaces. This is a relevant fact in the context of this paper because selective measurements provide grounds for a reformulation of quantum mechanics. Ciaglia et al refer to this as the "Schwinger picture" and address its dynamical and statistical aspects in [9, 10].

Structure of the paper. After section 2, which recalls some notions of pointfree topology and related aspects of general topology and logic that are needed in this paper but are not part of the usual set of staples of mathematical physics, section 3 introduces spaces of measurements in a stepwise manner, the main focus being on conveying the underlying intuitive motivations as clearly as possible. Section 4 is more technical and introduces the main classes of examples, namely related to C*-algebras and groupoids; it addresses the difference between quantum and classical measurements, and also a way in which they are related via groupoid C*-algebras. Following this, section 5 studies the notion of classical observer in symmetric measurement spaces. Finally, section 6 explains the relations to Schwinger's selective measurements, and section 7 concludes with a short discussion.

2 Locales and sober spaces

This section recalls basic definitions and facts about locales, including locally compact locales and powerlocales, and related topological notions. The latter include sober spaces, continuous lattices, the Scott topology, powerspaces and the lower Vietoris topology, and relations between locales and propositional logic, in particular intuitionistic and geometric logic.

2.1 Crash course on pointfree topology

Locales. By a *locale* [22] is meant a *complete lattice* L (also called a *sup-lattice* [24]) that satisfies the following distributivity law for all $a \in L$ and $S \subset L$:

$$a \wedge \bigvee S = \bigvee_{s \in S} a \wedge s \ .$$

Locales are often referred to as *pointfree spaces* because the prototypical example of a locale is the topology $\Omega(X)$ of a topological space X, with the binary meets (infima) and arbitrary joins (suprema) given for all $U, V \in \Omega(X)$ and $S \subset \Omega(X)$ by intersections and unions, respectively:

$$U \wedge V = U \cap V \quad \text{and} \quad \bigvee S = \bigcup S \ .$$

Note that any infinite subset $S \subset \Omega(X)$ has a meet $\bigwedge S = \operatorname{int}\bigl(\bigcap S\bigr)$, too, but such infinitary meets do not in general distribute over joins. Accordingly, if we regard a locale L as an algebraic structure in the sense of universal algebra, its algebraic operations are the joins of arbitrary arity (including the constant $0 = \bigvee \emptyset$, which is the least element), the binary meets and the 0-ary meet $1 = \bigwedge \emptyset$ — which is the greatest element. Of course, by iterating binary meets we obtain all the other meets of finite arity.

Homomorphisms and maps. In accordance with the choice of algebraic operations just discussed, by a *homomorphism* of locales $h : M \to L$ is meant a mapping that preserves arbitrary joins (a *homomorphism of sup-lattices*) and finitary meets:

$$h(1) = 1 \ , \quad h(a \wedge b) = h(a) \wedge h(b) \ , \quad h\big(\bigvee S\big) = \bigvee h(S) \ .$$

Again the prototypical example comes from topology: if $f : X \to Y$ is a continuous map of topological spaces the inverse image mapping $f^{-1} : \Omega(Y) \to \Omega(X)$ is a homomorphism of locales.

This motivates the definition of the *category of locales* Loc, whose objects are the locales and whose morphisms $f : L \to M$, which are called *maps of locales*, or *continuous maps (of locales)*, are defined to be homomorphisms f^* in the opposite direction:

$$f^* : M \to L \ .$$

Even though the map f "is" the homomorphism f^*, the latter is usually referred to as the *inverse image homomorphism* of the map f, for obvious reasons.

This presentation is close to that of [22], which mostly I follow in this paper, but note that f^* has a right adjoint $f_* : L \to M$ because f^* preserves arbitrary joins, and it is possible to define maps of locales concretely to be such right adjoints. The latter approach is followed in [41].

Spectra of locales. The assignments $X \mapsto \Omega(X)$ and $f \mapsto f^{-1}$ described above define a functor Ω from the category of topological spaces Top to Loc. In order to obtain a functor in the opposite direction let us first note that the topology of any topological space with a single point is isomorphic to the locale $\mathbf{\Omega} = \{0, 1\}$ with $0 < 1$. So a *point* of a locale L is defined to be a map of locales $p : \mathbf{\Omega} \to L$ or, equivalently, a homomorphism $p^* : L \to \mathbf{\Omega}$. The set of points of L is denoted by $\Sigma(L)$, and for each $a \in L$ we have a subset

$$U_a = \{p \in \Sigma(L) \mid p^*(a) = 1\} \ .$$

The collection $(U_a)_{a \in L}$ satisfies

$$U_1 = \Sigma(L) \ , \quad U_{a \wedge b} = U_a \cap U_b \ , \quad U_{\bigvee S} = \bigcup_{a \in S} U_a \ ,$$

so it is a topology on $\Sigma(L)$. The resulting topological space is called the *spectrum* of L, again denoted by $\Sigma(L)$.

If $f : L \to M$ is a map of locales there is a continuous map

$$\Sigma(f) : \Sigma(L) \to \Sigma(M)$$

which to each point $p \in \Sigma(L)$ assigns the point $\Sigma(f)(p)$ whose inverse image is $p^* \circ f^*$:

$$\begin{tikzcd} L \arrow[dr, "p^*"'] & & M \arrow[ll, "f^*"'] \arrow[dl, "p^* \circ f^*"] \\ & \mathbf{\Omega} & \end{tikzcd}$$

This defines a functor $\Sigma : \mathsf{Loc} \to \mathsf{Top}$, which is right adjoint to $\Omega : \mathsf{Top} \to \mathsf{Loc}$.

An alternative way in which to describe points of a locale L is to consider the elements $\hat{p} \in L$ which are of the form $\hat{p} = \bigvee\{a \in L \mid p^*(a) = 0\}$ for some point p. These can be precisely characterized as being the *prime elements* of L; that is, those elements $\hat{p} \in L$ that satisfy the following two conditions:

$$\hat{p} \neq 1 \;, \quad a \wedge b \leq \hat{p} \Longrightarrow (a \leq \hat{p} \text{ or } b \leq \hat{p}) \text{ for all } a, b \in L \;.$$

Conversely, from a prime element $\hat{p} \in L$ we obtain the point $p \in \Sigma(L)$ defined for all $a \in L$ by

$$p^*(a) = 0 \iff a \leq \hat{p} \;.$$

This establishes a bijection between $\Sigma(L)$ and the set $\mathrm{Prime}(L)$ of prime elements of L. If $f : L \to M$ is a map of locales the right adjoint $f_* : L \to M$ restricts to a map $\mathrm{Prime}(L) \to \mathrm{Prime}(M)$ that "coincides" with $\Sigma(f)$ in the sense that for any point $p \in \Sigma(L)$ we have

$$f_*(\hat{p}) = \widehat{\Sigma(f)(p)} \;.$$

Spatial locales and sober spaces. The locales L such that $L \cong \Omega(X)$ for some space X are called *spatial*, the topological spaces X such that $X \cong \Sigma(L)$ for some locale L are called *sober*, and the adjunction between Ω and Σ restricts to an equivalence of categories between the full subcategories of Top and Loc whose objects are, respectively, the sober spaces and the spatial locales.

Moreover, given a locale L the assignment $a \mapsto U_a$ defines a surjective homomorphism $L \to \Omega(\Sigma(L))$, and L is spatial if and only if this homomorphism is also injective — so we have $L \cong \Omega(\Sigma(L))$. And, given a topological space X, there is a continuous map $X \to \Sigma(\Omega(X))$ which to each $x \in X$ assigns the point $p_x \in \Sigma(\Omega(X))$ defined by

$$p_x^*(U) = 1 \iff x \in U \;.$$

The space X is sober if and only if this is a homeomorphism $X \xrightarrow{\cong} \Sigma(\Omega(X))$.

Alternatively, noticing that the prime elements of $\Omega(X)$ are the complements of the *irreducible closed sets* (the nonempty closed sets C such that $D \cup E = C$ for any two other closed sets D and E implies either $D = C$ or $E = C$), we may identify the points of $\Omega(X)$ with the irreducible closed sets, in which case the point p_x corresponds to the closure $\overline{\{x\}}$. Hence, a space X is T_0 if and only if $\overline{\{x\}} = \overline{\{y\}}$ implies $x = y$ for all $x, y \in X$, and X is sober if and only if it is T_0 and every irreducible closed set is the closure of a singleton. In particular, every Hausdorff space is sober.

Lower powerlocales and powerspaces. Let X be a topological space, denote by $\mathsf{C}(X)$ the set of all the closed sets of X, and for each $U \in \Omega(X)$ define

$$\lozenge U = \{C \in \mathsf{C}(X) \mid C \cap U \neq \emptyset\} \;.$$

The topology generated on $\mathsf{C}(X)$ by the sets $\lozenge U$ is called the *lower Vietoris topology*, so $\Omega(\mathsf{C}(X))$ consists of arbitrary unions of finitary intersections

$$\lozenge U_1 \cap \ldots \cap \lozenge U_k \;.$$

In this paper I shall always write $\mathsf{C}(X)$ for the space of closed sets of X with the lower Vietoris topology. Note that the mapping $\Omega(X) \to \Omega(\mathsf{C}(X))$ which is defined by $U \mapsto \Diamond U$ preserves arbitrary unions:

$$\Diamond(\bigcup_i U_i) = \bigcup_i \Diamond U_i \ .$$

This suggests an analogous construction for locales: given a locale L, the *lower powerlocale* $\mathsf{P_L}(L)$ is presented by generators and relations by using L as the set of generators and by taking the defining relations to be such that the injection of generators $\iota_L : L \to \mathsf{P_L}(L)$ preserves arbitrary joins. In this paper we shall not need the details of this construction; it will suffice to keep in mind its universal property, namely that for any other locale M and any sup-lattice homomorphism $f : L \to M$ there is a unique homomorphism of locales $f^\sharp : \mathsf{P_L}(L) \to M$ that makes the following diagram commute:

$$\begin{tikzcd} L \arrow[r, "\iota_L"] \arrow[dr, "f"'] & \mathsf{P_L}(L) \arrow[d, "f^\sharp"] \\ & M \end{tikzcd}$$

Hence, in particular, the points of the locale $\mathsf{P_L}(L)$ can be identified with the sup-lattice homomorphisms $\phi : L \to \mathbf{\Omega}$ or, equivalently, with the joins of their kernels

$$\bigvee \ker \phi = \bigvee \{a \in L \mid \phi(a) = 0\} \ ,$$

which range over all the elements of L. But in the pointwise order we have $\phi \le \psi$ for such mappings if and only if $\bigvee \ker \phi \ge \bigvee \ker \psi$, so the set of points of $\mathsf{P_L}(L)$ can be identified with L itself, but with the specialization order of the spectrum being the reverse of the order of L.

Hence, if $L = \Omega(X)$ for some topological space X, the spectrum of $\mathsf{P_L}(L)$ can be identified with $\mathsf{C}(X)$ with the specialization order being the inclusion order of closed sets, and it is easy to see that under this identification the spectrum topology is the lower Vietoris topology:

$$\Sigma(\mathsf{P_L}(\Omega(X))) \cong \mathsf{C}(X) \ .$$

This implies that the space $\mathsf{C}(X)$ is sober.

Continuous lattices and locally compact locales. In this paper the expression *locally compact*, for a topological space X, is always used in the strong sense of having a basis of compact neighborhoods; that is, for all open sets $U \subset X$ and all $x \in U$ there is an open set V and a compact set K such that $x \in V \subset K \subset U$. Equivalently, every open set U can be obtained as the following union:

$$U = \bigcup \{V \in \Omega(X) \mid \text{there is } K \text{ compact such that } V \subset K \subset U\} \tag{2.1}$$

This property can be given a precise description in terms of the locale $\Omega(X)$, at least for sober spaces. This will now be recalled along with related notions that will play a role later on. For more details see [19, 22, 41].

Let L be a sup-lattice. For any elements $a, b \in L$ we write $b \ll a$, and say that b *is way below* a (or "b is relatively compact under a"), if for every set $S \subset L$ such that $a \leq \bigvee S$ there is a finite subset $F \subset S$ such that $b \leq \bigvee F$; equivalently, for every directed set $D \subset L$ such that $a \leq \bigvee D$ there is $d \in D$ such that $b \leq d$. (A set $D \subset L$ is *directed* if it is nonempty and for all $x, y \in D$ there is $z \in D$ such that $x \leq z$ and $y \leq z$.)

Example 2.1. Let X be a topological space, and let V and U be open sets. If there exists a compact set K such that $V \subset K \subset U$ then, clearly, $V \ll U$. Conversely, if X is locally compact then $V \ll U$ if and only if $V \subset K \subset U$ for some compact set K. In order to see this, for each $x \in U$ let V_x be an open set and K_x a compact set such that $x \in V_x \subset K_x \subset U$. Then $U = \bigcup_{x \in X} V_x$. Hence, if $V \ll U$ there is a finite subset $F \subset X$ such that $V \subset \bigcup_{x \in F} V_x$, and the compact set $K = \bigcup_{x \in F} K_x$ satisfies $V \subset K \subset U$.

Given $a \in L$, the set $\downarrow\!\!\downarrow(a)$ consists of all $b \in L$ such that $b \ll a$. Then L is said to be a *continuous lattice* if for all $a \in L$ we have, imitating (2.1):

$$a = \bigvee \downarrow\!\!\downarrow(a) \; .$$

It can be shown that if L is both continuous and a distributive lattice then it is a spatial locale, isomorphic to the topology of a locally compact space. Moreover, if X is a sober space then it is locally compact if and only if $\Omega(X)$ is a continuous lattice. For these reasons, continuous locales are referred to as *locally compact locales*.

Example 2.2. Let A be a commutative C*-algebra. The locale of closed ideals $I(A)$ is locally compact because it is isomorphic to the topology of the spectrum of A, which is a locally compact Hausdorff space. More generally, even if A is not commutative, $I(A)$ is a locally compact locale [19, Prop. I-1.21.2].

An element a such that $a \ll a$ is *compact*, and a sup-lattice L is an *algebraic lattice* if every element of L is a join of compact elements. Every algebraic lattice is continuous.

Example 2.3. The compact elements of $\Omega(X)$ are precisely the compact open sets of X, so the topology of any totally disconnected compact Hausdorff space is an algebraic lattice. Other examples of algebraic lattices are obtained from algebraic structures (hence the terminology). For instance, the sup-lattice of all the subgroups of a group is algebraic and its compact elements are the finitely generated subgroups. Similarly, the lattice of ideals of a ring is algebraic. But the lattice of closed ideals of a topological ring is in general not algebraic (cf. Example 2.2).

2.2 Specialization order and the Scott topology

Sober spaces and dcpos. The *specialization order* of a topological space X is the preorder defined by $x \sqsubseteq y$ if $x \in \overline{\{y\}}$. It is a partial order if and only if X is T_0. If X is sober the specialization order is a *directed complete partial order*, or *dcpo*, by which is meant a poset for which every directed set $D \subset X$ has a join $\bigvee D$ in X.

The open sets of the *Scott topology* on a dcpo X are the upwards closed sets which are inaccessible by joins of directed sets:

- $U = \uparrow U = \{x \in X \mid u \leq x \text{ for some } u \in U\}$, and
- if $D \subset X$ is directed and $\bigvee D \in U$ then $D \cap U \neq \emptyset$.

Note that the second condition means precisely that any directed set $D \subset X$, regarded as a net, has limit $\bigvee D$.

The following properties are well known:

Proposition 2.4. *1. The specialization order of a dcpo X equipped with the Scott topology is the order of X itself.*

2. A map between dcpos $f : X \to Y$ is continuous with respect to the Scott topologies of X and Y (i.e., it is Scott-continuous*) if and only if it preserves joins of directed sets; that is, for all directed $D \subset X$ we have*

$$f(\bigvee D) = \bigvee f(D) \, .$$

3. Let X, Y and Z be dcpos. Then $X \times Y$ is a dcpo, and the product topology on $X \times Y$ coincides with the Scott topology. Moreover, a map $f : X \times Y \to Z$ is continuous for this topology if and only if it is continuous in each variable separately.

4. The topology of any sober space is contained in the Scott topology of its specialization order.

5. A continuous map between sober spaces is necessarily Scott-continuous.

Not every subspace of a sober space needs to be sober, but in some situations it is. The following will be relevant in this paper:

Proposition 2.5. *Let X be a sober space whose specialization order is a sup-lattice. Any subspace $Y \subset X$ which is closed under the formation of arbitrary joins is necessarily sober.*

Proof. Let $Y \subset X$ be closed under joins, and let $C \subset Y$ be an irreducible closed set in the relative topology of Y. Then the closure $\overline{C}$ in X is an irreducible closed set of X. In order to prove the latter assertion let us suppose that $\overline{C}$ is not irreducible, and let C_1 and C_2 be closed sets of X such that $C_1 \cup C_2 = \overline{C}$ and $C_1 \neq \overline{C}$ and $C_2 \neq \overline{C}$. Then $(C_1 \cup C_2) \cap Y = \overline{C} \cap Y = C$, so $(C_1 \cap Y) \cup (C_2 \cap Y) = C$. Since C is irreducible in Y we must have $C_1 \cap Y = C$ or $C_2 \cap Y = C$. Then $\overline{C} \subset C_1$ or $\overline{C} \subset C_2$, a contradiction, so $\overline{C}$ is irreducible and therefore $\overline{C} = \overline{\{x\}}$ for some $x \in X$. Hence, $\overline{C}$ is closed under joins because $\overline{\{x\}}$ is the principal ideal $\downarrow(x) = \{y \in X \mid y \leq x\}$, and thus C is closed under joins because it is the intersection $\overline{C} \cap Y$ of two join-closed sets. Let $y = \bigvee C$. Then $C = \downarrow(y) = \overline{\{y\}}$, where $y \in Y$, so we conclude that every irreducible closed set of Y is the closure of a singleton. In addition, Y is T_0 because it is a subspace of a T_0-space, so Y is sober. $\square$

Scott topology on locally compact locales. It is often the case that the theory of continuous lattices is generalized to more general posets, in particular to dcpos, but this will not be needed in this paper. One important property is that any continuous lattice with the Scott topology is necessarily a sober space. In that case it is well known that a basis for the topology consists of all the sets

$$\mathrm{int}(\uparrow(a)) = \twoheaduparrow(a) = \{b \mid a \ll b\} .$$

Alternatively, for a locally compact locale a basis can be given in terms of the compact sets of its spectrum:

Proposition 2.6. *Let X be a topological space. For all compact sets $K \subset X$ the set*

$$N_K := \{U \in \Omega(X) \mid K \subset U\}$$

is open in the Scott topology of $\Omega(X)$. Moreover, if X is locally compact the sets N_K form a basis of the Scott topology.

Proof. N_K is upwards closed by definition, so we need only prove that it is inaccessible by directed joins. Let (U_i) be a directed family in $\Omega(X)$ such that $\bigcup_i U_i \in N_K$. Then, since K is compact and it is covered by (U_i), there is i such that $K \subset U_i$, so $U_i \in N_K$, showing that N_K is Scott open. Now assume that X is locally compact, and let $\mathcal{U}$ be a non-empty Scott open set of $\Omega(X)$. Let $U \in \mathcal{U}$, and let (U_i) be the family of opens for which there exist compact sets K_i such that $U_i \subset K_i \subset U$. Then $U = \bigcup_i U_i$ and, since (U_i) is directed, for some i we have $U_i \in \mathcal{U}$, thus proving that $U \in N_{K_i} \subset \mathcal{U}$. $\square$

2.3 Propositional geometric logic

Locales and propositional logic. Locales can also be regarded as propositional logics in the sense of algebraic logic. For instance, any locale L is an algebraic model of *propositional intuitionistic logic* whereby the interpretation of the logical connectives for conjunction ($\sqcap$), disjunction ($\sqcup$), and implication ($\to$) is given as follows for all $a, b \in L$:

$$a \sqcap b = a \wedge b , \quad a \sqcup b = a \vee b , \quad a \to b = \bigvee\{x \mid x \wedge a \leq b\} .$$

[Equivalently, the implication is defined by the family of equivalences $x \wedge a \leq b \iff x \leq a \to b$, which means that the operator $a \to (-)$ is right adjoint to the meet operator $(-) \wedge a$.]

For the purposes of this paper it is useful to consider instead *propositional geometric logic*, whose roots are in topos theory (see, e.g., [32]) and whose logical connectives are the disjunctions $\bigsqcup_i a_i$ of arbitrary arity and the finitary conjunctions **true** and $a_1 \sqcap \cdots \sqcap a_n$ ($n \geq 1$), interpreted respectively as arbitrary joins and finitary meets:

$$\mathbf{true} = 1 , \quad a_1 \sqcap \cdots \sqcap a_n = a_1 \wedge \cdots \wedge a_n , \quad \bigsqcup_i a_i = \bigvee_i a_i .$$

Then the points of a locale L can be regarded as *logical valuations*: for each $p \in \Sigma(L)$ the inverse image homomorphism $p^* : L \to \mathbf{\Omega}$ is an assignment of the truth value 0 (false) or 1 (true) to each "proposition" $a \in L$ in a way that respects the logical connectives **true**, $\sqcap$, and $\bigsqcup$. This does not imply that the implication connective is preserved by p^*, so the points of L are not the same as logical valuations in the sense of intuitionistic logic.

Observational logic. Propositional geometric logic has a suggestive interpretation in terms of *finitely observable properties* [57], due to which we may also think of propositional geometric logic as being an *observational logic*. This idea originated in computer science and consists of regarding each element a of a locale L as a property of a computational system that can be verified in finite time using finite resources. If $a_1, \ldots, a_n$ are finitely observable properties then so is their conjunction $a_1 \sqcap \cdots \sqcap a_n$, but an infinitary conjunction is in general not finitely observable. This introduces an asymmetry with respect to disjunctions, since in order to verify a disjunction $\sqcup_i a_i$ all it takes is to verify an arbitrary disjunct a_i. Therefore if all the disjuncts are finitely observable then so is the disjunction. Note that this does not require any kind of procedure for choosing a disjunct: the verification of a disjunct, whichever it may be, *is* a verification of the disjunction.

In addition to those seen in section 2.1, there is yet another alternative but equivalent definition of the notion of point of a locale: by a *completely prime filter* F of a locale L is meant a subset $F \subset L$ such that $1 \in F$, $a \wedge b \in F$ for all $a, b \in F$, and $S \cap F \neq \emptyset$ whenever $\bigvee S \in F$. It turns out that a mapping $h : L \to \mathbf{\Omega}$ is a homomorphism if and only if the pre-image $h^{-1}(\{1\})$ is a completely prime filter of L. In the parlance of observational logic, F is a completely prime filter if and only if it consists of all the elements of L which are true under a given logical valuation. So a completely prime filter F can be thought of as a (usually infinitary) logical conjunction of finitely observable properties, namely those properties which are assigned the value 1 by the inverse image homomorphism of the point p that corresponds to F. Of course, such conjunctions cannot in general be represented by meets in L.

If X is a topological space, the set N_x of open neighborhoods of a point $x \in X$ is the completely prime filter of $\Omega(X)$ that corresponds to the point $p_x : \Omega(X) \to \mathbf{\Omega}$. Hence, if X is sober, each point $x \in X$ can be thought of as being a (not finitely observable) property which is obtained by taking the logical conjunction of all the finitely observable properties that are represented by open neighborhoods of x. More generally, if $K \subset X$ is compact, the Scott open N_K (cf. Proposition 2.6) is a similar "conjunction" of the finitely observable properties that are represented by open neighborhoods of K.

3 Measurements

This section is more descriptive than technical. It provides a step by step introduction to the main ideas and definitions that surround measurement spaces, along with related basic notions of quantale theory leading up to involutive quantales and stably Gelfand quantales. The whole section is illustrated by simple examples, some of which are instances of the more general ones that will be seen in later sections.

3.1 Schrödinger's electron

For illustration purposes let us call *Schrödinger's electron* to an experiment in the style of Schrödinger's cat, but in which the cat has been replaced (much more ethically) by an electron that travels through a Stern–Gerlach apparatus. The whole apparatus, including the target that records the deflection of the electron, is inside a closed box and we cannot see what happens inside. However, we know the magnetic field inside is oriented so as to measure spin along z, and we know how long it will take for an electron to be fired and reach the target, so we have a way of knowing in finite time that the experiment has already been performed.

Let us write $\boldsymbol{z}$ in order to denote the measuring process just described. Let us also write $\boldsymbol{z}^{\downarrow}$ and $\boldsymbol{z}^{\uparrow}$, respectively, for the two other measuring processes that can be performed using the same equipment in an open box, so that we can see the deflection of the electron: $\boldsymbol{z}^{\downarrow}$ means the particle went down, and $\boldsymbol{z}^{\uparrow}$ means that it went up.

Similarly, we shall write $\boldsymbol{x}$ for a similar closed box experiment that measures spin along x. Note that this must be different from $\boldsymbol{z}$ because it conveys different information. So we cannot represent measurements like $\boldsymbol{z}^{\downarrow}$ and $\boldsymbol{z}^{\uparrow}$ by rays in the Hilbert space $\mathbb{C}^2$, for in that case we would have $\boldsymbol{z} = \boldsymbol{x} = \mathbb{C}^2$. We shall also write $\boldsymbol{x}^{\downarrow}$ and $\boldsymbol{x}^{\uparrow}$ for the two possible measurements that observe a deflection when measuring spin along x.

This setup will be referred to repeatedly throughout this paper in order to illustrate various concepts.

3.2 Measurements and physical properties

By a *measurement*, or *observation*, will informally be meant any process via which new classical information is recorded in a physical device, such as when 0s and 1s are written on a sheet of paper, or a dial of an electronic instrument indicates a new value, or there is a change in brain synapses, etc. Apart from this intuition, no specific features are ascribed to measurements except for those that will emerge from the mathematical definitions to be seen below.

Any finite amount of classical information which is thus associated to a measurement will be referred to as a *physical property*. I shall model this in a way which is close to the ideas about finitely observable properties described in section 2.3, namely taking measurements to be the points of a sober topological space M whose open sets represent the physical properties. The relation between measurements and properties is expressed simply by saying that if $m \in M$ is a measurement and U is a physical property, then $m \in U$ means that U is *compatible* with m. This does not mean that U is necessarily recorded if m is performed, but only that it *can* be recorded, as the following examples illustrate.

Example 3.1. Let $I = [0, 2] \subset \mathbb{R}$ with the usual topology. This space is meant to represent a two meter long ruler whose markings range from 0 to 2 meters, such that each $x \in I$ is a position measurement (made with an infinitely sharp pointer). This is a Hausdorff space, hence sober. In practice no measurement can be infinitely precise, and the best we can do is, when placing the pointer on the marking that corresponds to $x \in I$, to record an open set U that contains x. In other words, the open neighborhoods of x are the physical properties that are compatible with x. The more precise a measuring device is, the smaller the open sets it will be able to register, but no canonical open set exists for x, and certainly no least open set. So having $x \in U$ should be read as "U can be the physical property recorded by a measuring device that performs the measurement x."

Example 3.2. Consider Schrödinger's electron, as described in section 3.1, and write $Z^{\uparrow}$ and $Z^{\downarrow}$ for the properties that correspond to the information recorded on the target: $Z^{\uparrow}$ means that the electron went up, and $Z^{\downarrow}$ means that it went down. Performing $\boldsymbol{z}$ (i.e., keeping the box closed) yields a superposition of both deflections, so $\boldsymbol{z}$ is compatible with both $Z^{\uparrow}$ and $Z^{\downarrow}$ — and opening the box may lead to either of the properties being recorded. However, $\boldsymbol{z}^{\uparrow}$ is not compatible with $Z^{\downarrow}$, and $\boldsymbol{z}^{\downarrow}$ is not compatible with $Z^{\uparrow}$. So we have

$$\boldsymbol{z} \in Z^{\uparrow} \cap Z^{\downarrow}\,, \quad \boldsymbol{z}^{\uparrow} \in Z^{\uparrow}\,, \quad \boldsymbol{z}^{\downarrow} \in Z^{\downarrow}\,, \quad \boldsymbol{z}^{\uparrow} \notin Z^{\downarrow}\,, \quad \boldsymbol{z}^{\downarrow} \notin Z^{\uparrow}\,.$$

Contrary to the previous example, in this one the space of measurements is not Hausdorff, and there are least open neighborhoods of measurements. So, for instance, a good measuring device will record $Z^{\uparrow} \cap Z^{\downarrow}$ when $\boldsymbol{z}$ is performed, whereas a sloppy device may record, say, $Z^{\uparrow}$, thus registering information that is unnecessarily uncertain because it only tells us that either $\boldsymbol{z}$ or $\boldsymbol{z}^{\uparrow}$ were performed.

With this understanding in mind, a logical *conjunction* of finitely many physical properties $U_1, \ldots, U_n \in \Omega(M)$ $(n \geq 2)$ can in principle be physically measured by repeating the same measurement n times, each time recording one of the properties U_i. This conjunction is represented by the intersection $U_1 \cap \cdots \cap U_n$, whereas a logical *disjunction* of an arbitrary family of physical properties (U_i) is represented by the union $\bigcup_i U_i$. The largest open set is the *trivial property*, which conveys no information at all, and the least open set $\emptyset$ is the *impossible property*, which can never be measured.

The requirement that M be sober means that each measurement m can be identified with the set of all the physical properties that are compatible with m. This means that no distinctions between measurements are made beyond those that can be made on the basis of the physical properties that are compatible with them, so M is a T_0 space, and also that a measurement exists for any logically consistent set of such properties (a completely prime filter of $\Omega(M)$). The latter is a principle of consistency: if from the logical structure of properties it is derived (albeit possibly transfinitely) that a certain measuring process exists, then such a process should really exist.

3.3 Composition of measurements

Although often physics is concerned with measurements that destroy the systems being measured, so that all the information obtained about a type of system can only result from repetitions of the same measurement on systems of that type, it is also the case that nondestructive measurements exist and that additional measurements can be composed with the results of a previous one, such as when doing a spin measurement along x after having performed another along z on the same particle.

Let M be a space of measurements as in the previous section. When a measurement m can be followed in this manner by a measurement n I shall represent the composed measurement by mn, and call it the *product* of m and n. In order for the product of measurements to be a total operation (rather than a partial composition as in a category) I shall also postulate a special measurement $0 \in M$, called the *impossible measurement*, such that writing $mn = 0$ indicates that performing m followed by n is meaningless. Naturally the *absorption laws* $m0 = 0m = 0$ must hold, and no physical properties except the trivial property can contain 0 (since by definition 0 can never be performed), so the only open set that contains 0 is M. I shall also make no distinction between $(mm')m''$ and $m(m'm'')$, thus making the space of measurements a *semigroup with zero*. In addition, it is natural to require the multiplication to be continuous with respect to the topology of physical properties, which leads us to the following general definition:

Definition 3.3. By a *measurement space* M will be meant a sober topological space equipped with a continuous associative binary operation, called the *product*, *multiplication* or *composition*, plus an element 0, called *zero* or *impossible measurement*, which satisfies the absorption laws $m0 = 0m = 0$ for all $m \in M$ and is such that M is the only open set that contains 0. The product of two measurements m and n is denoted by mn.

Example 3.4. Any sober space X equipped with a continuous associative multiplication can be made a measurement space $M = X \amalg \{0\}$ with the (clearly sober) topology

$$\Omega(M) = \Omega(X) \cup \{M\} .$$

The absorption law uniquely extends the multiplication of X to the whole of M, and the extended multiplication is trivially continuous because the only open neighborhood of 0 is M. For instance, any C*-algebra A can be extended in this way to a measurement space (the new zero will not coincide with the zero of A), although as we shall see later there is a better construction of measurement spaces from C*-algebras for the purpose of describing quantum measurements. In particular, the Hausdorff topology of a C*-algebra would not cater for the description of spin measurements as done in Example 3.2.

The following two examples provide two ways of defining a measurement space from any sober space.

Example 3.5. We can define a continuous associative multiplication on any sober space X by defining $mn = n$ (i.e., the multiplication $X \times X \to X$ is just the second projection). In terms of measurements this can be read as saying that we can always compose two measurements m and n in the trivial sense that the first measurement is forgotten and only the second one is taken into account. This, plus adding a zero as in the previous example, allows us to turn any sober space into a measurement space.

Example 3.6. Another way in which to define a continuous associative multiplication from a sober space X is to generate it freely by concatenation; that is, consider the set

$$X^+ = \coprod_{k \in \mathbb{N}} X^k ,$$

and define a product on X^+ by defining the composition of $(m_1, \ldots m_k) \in X^k$ and $(n_1, \ldots, n_l) \in X^l$ to be

$$(m_1, \ldots, m_k, n_1, \ldots, n_l) \in X^{k+l} \ ,$$

which corresponds to the idea of composing measurements just by performing one after another. The topology of each component X^k is the product topology, and the topology $\Omega(X^+)$ is the coproduct topology, which makes each component X^k a clopen set. Then the universal property of the coproduct yields a continuous multiplication on X^+:

$$X^+ \times X^+ \xrightarrow{\cong} \coprod_{k,l \in \mathbb{N}} X^k \times X^l \xrightarrow{\cong} \coprod_{k,l \in \mathbb{N}} X^{k+l} \longrightarrow X^+$$

Adding the impossible measurement to X^+ as in Example 3.4 gives us a measurement space.

3.4 Order and disjunctions

Let M be a measurement space. We have $m \leq n$ in the specialization order of M if and only if every neighborhood of m is also a neighborhood of n. Hence, 0 is the least element in the specialization order, which means that the join $\bigvee \emptyset$ exists. In addition, M is a dcpo because it is sober, and the following properties hold:

Proposition 3.7. *Let M be a measurement space. The multiplication of M satisfies the following distributivity properties for all directed sets S and for $S = \emptyset$:*

$$m(\bigvee S) = \bigvee_{n \in S} mn \ , \quad (\bigvee S)n = \bigvee_{m \in S} mn \ .$$

Proof. The continuity of the multiplication map $M \times M \to M$ implies continuity in each variable, which in turn implies Scott continuity in each variable, thus giving us the required distributivity conditions for directed sets S. For $S = \emptyset$ we have $\bigvee S = 0$, so the distributivity conditions coincide with the absorption laws $m0 = 0m = 0$. □

In particular, letting $S = \{a, b\}$ with $a \leq b$ in the above proposition, we conclude that the multiplication is monotone; that is, $a \leq b$ implies that $ma \leq mb$ and $am \leq bm$ for all $m \in M$.

Example 3.8. The two meter ruler described in Example 3.1 can be turned into a measurement space by defining a multiplication as in Example 3.5 or Example 3.6, and adding a zero as in Example 3.4. However, since the specialization order of any Hausdorff space is discrete, the measurement order either way is just a discrete order with zero attached below, so no directed sets besides singletons and sets of the form $\{0, m\}$ exist.

The immediate interpretation of the specialization order of a measurement space is that $m \leq n$ holds if and only if n is compatible with more physical properties than m. In computer science this would usually be taken to mean that n is a state of a computation which is more "complete" than m because it "has" more properties. However, for measurements the situation is somewhat different.

Example 3.9. Spin measurements along z as in Example 3.2 yield $\boldsymbol{z}^{\downarrow} \leq \boldsymbol{z}$ and $\boldsymbol{z}^{\uparrow} \leq \boldsymbol{z}$, so a fragment of the specialization order of any measurement space that results from adding a multiplication and a zero in the ways seen so far can be represented by the following diagram:

$$
\begin{tikzcd}
 & \boldsymbol{z} \arrow[dl, dash] \arrow[dr, dash] & \\
\boldsymbol{z}^{\downarrow} \arrow[dr, dash] & & \boldsymbol{z}^{\uparrow} \arrow[dl, dash] \\
 & 0 &
\end{tikzcd}
$$

Here we have $\boldsymbol{z} = \boldsymbol{z}^{\downarrow} \vee \boldsymbol{z}^{\uparrow}$. However, contrary to the situation in computer science, in an obvious sense $\boldsymbol{z}$ is less determined than either $\boldsymbol{z}^{\downarrow}$ or $\boldsymbol{z}^{\uparrow}$. The sharpest property which can be recorded when $\boldsymbol{z}$ is measured is $Z^{\uparrow} \cap Z^{\downarrow}$ (cf. Example 3.2), and the measurement $\boldsymbol{z}$ cannot be regarded in the classical sense of lack of knowledge; that is, it is not that either $\boldsymbol{z}^{\downarrow}$ or $\boldsymbol{z}^{\uparrow}$ actually occurred and we do not know which, but rather than $\boldsymbol{z}$ must be regarded as a kind of quantum superposition of $\boldsymbol{z}^{\downarrow}$ and $\boldsymbol{z}^{\uparrow}$, such that neither alternative can be said to hold until a measurement is made that "forces" one of the options. In addition, as mentioned in section 3.1, the Schrödinger electron experiment along the z axis cannot be identified with the analogous experiment along the x axis. Similarly, $\boldsymbol{x}$ is a superposition of two measurements $\boldsymbol{x}^{\downarrow}$ and $\boldsymbol{x}^{\uparrow}$.

This example motivates the interpretation of binary joins (of incomparable measurements) in the specialization order of a measurement space M: if m and n are incomparable then $m \vee n$ is regarded as a *disjunction*. This can also be regarded as a "superposition" of the disjuncts, at least in those cases where the execution of the disjunction measurement is not necessarily accompanied by "decoherence" into any of the disjuncts (as in the Schrödinger electron example). Hence, this can be much like a quantum superposition except that no quantum amplitudes are involved, so all we can ask is whether measurements do or do not belong to a superposition; it is only a *logical superposition*. However, this does not imply that every binary join in the specialization order of an arbitrary measurement space should necessarily be regarded in this way. Disjunctions in the classical sense of lack of knowledge can also be described by joins, as usual in algebraic logic, and the distinction between the two types can be conveyed by lattice-theoretical properties of the specialization order. This will be addressed in section 4.3.

3.5 Complete measurement spaces

The specialization order of a measurement space M is not necessarily a complete lattice, i.e., not every subset $S \subset M$ needs to have a join. In particular, there may be pairs of measurements m and m' such that $m \vee m'$ does not exist. Section 3.4 has put forward an interpretation of $m \vee m'$, if m and m' are incomparable, as being a disjunction of the alternatives m and m'. If the join $m \vee m'$ exists and n is another measurement, it is natural to assume that the composition $(m \vee m')n$ (read "m or m' followed by n") should coincide with $mn \vee m'n$ (read "m followed by n or m' followed by n). Similarly, we should have $n(m \vee m') = nm \vee nm'$ (cf. [1, 44]). As we shall see, all the examples in this paper satisfy these distributivity laws, so we are led to the following definition:

Definition 3.10. By a *complete measurement space* will be meant a measurement space M for which every pair of measurements $m, n \in M$ has a join $m \vee n \in M$, and for which the following distributivity conditions hold for all measurements $m, m', n, n' \in M$:

$$m(n \vee n') = mn \vee mn' \ , \quad (m \vee m')n = mn \vee m'n \ ,$$

By a *morphism* of complete measurement spaces will be meant a continuous map of complete measurement spaces $f : M \to N$ such that

1. $f(mn) = f(m)f(n)$ for all $m, n \in M$;
2. $f(m \vee n) = f(m) \vee f(n)$ for all $m, n \in M$;
3. $f(0_M) = 0_N$.

The resulting category will be referred to as the *category of complete measurement spaces*, denoted by CMSp.

Example 3.11. Let $M = \mathrm{Sub}\, M_n(\mathbb{C})$ be the lattice of linear subspaces of the algebra of $n \times n$ complex valued matrices. This has an associative multiplication that distributes over binary joins and is given by

$$VW = \langle V \cdot W \rangle \ ,$$

where $V \cdot W$ is the pointwise product of complex linear subspaces of $M_n(\mathbb{C})$ and $\langle - \rangle$ means linear span (cf. section 3.6). The joins of the inclusion order are defined by

$$\bigvee_i W_i = \left\langle \bigcup_i W_i \right\rangle \ ,$$

and it is easy to see that the product distributes over them in each variable:

$$V \left\langle \bigcup_i W_i \right\rangle = \left\langle \bigcup_i V W_i \right\rangle \ , \quad \left\langle \bigcup_i V_i \right\rangle W = \left\langle \bigcup_i V_i W \right\rangle \ .$$

The lattice M is algebraic and thus it is a sober space under the Scott topology. Moreover, since the multiplication preserves joins in each variable it is also Scott continuous in each variable, and thus it is continuous as a map $M \times M \to M$. Hence, M is a complete measurement space whose zero is the zero dimensional subspace $\{0\}$.

Example 3.12. Making $n = 2$ in the previous example we obtain a description of the Schrödinger electron experiment of section 3.1. I shall exemplify using only the x and z directions. This uses Pauli spin matrices and projections in an unsurprising way: $\boldsymbol{x}$ and $\boldsymbol{z}$ are unital abelian subalgebras generated by the corresponding spin observables, and $\boldsymbol{z}^{\downarrow}$, $\boldsymbol{z}^{\uparrow}$, $\boldsymbol{x}^{\downarrow}$, and $\boldsymbol{x}^{\uparrow}$ are spanned by the matrices that project onto the respective eigenspaces:

$$
\begin{aligned}
\boldsymbol{z}^{\uparrow} &= \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\rangle \qquad \boldsymbol{z}^{\downarrow} = \left\langle \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle \\
\boldsymbol{z} = D_2(\mathbb{C}) &= \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle \\
\boldsymbol{x}^{\uparrow} &= \left\langle \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\rangle \qquad \boldsymbol{x}^{\downarrow} = \left\langle \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \right\rangle \\
\boldsymbol{x} &= \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle
\end{aligned}
$$

The measurement order is this:

$$
\begin{tikzcd}[column sep=small]
 & \boldsymbol{x} \arrow[dl, dash] \arrow[dr, dash] & & & & \boldsymbol{z} \arrow[dl, dash] \arrow[dr, dash] & \\
\boldsymbol{x}^{\downarrow} \arrow[drrr, dash] & & \boldsymbol{x}^{\uparrow} \arrow[dr, dash] & & \boldsymbol{z}^{\downarrow} \arrow[dl, dash] & & \boldsymbol{z}^{\uparrow} \arrow[dlll, dash] \\
 & & & 0 & & &
\end{tikzcd}
$$

Physical properties such as $Z^{\downarrow}$ and $Z^{\uparrow}$ correspond to Scott open sets that are principal filters of the inclusion order:

$$
\begin{aligned}
Z^{\downarrow} &= \uparrow(\boldsymbol{z}^{\downarrow}) = \{V \in \operatorname{Sub} M_2(\mathbb{C}) \mid z^{\downarrow} \subset V\} \\
Z^{\uparrow} &= \uparrow(\boldsymbol{z}^{\uparrow}) = \{V \in \operatorname{Sub} M_2(\mathbb{C}) \mid z^{\uparrow} \subset V\} .
\end{aligned}
$$

These are the smallest open sets that contain $\boldsymbol{z}^{\downarrow}$ and $\boldsymbol{z}^{\uparrow}$, respectively.

Smallest open sets exist in the above example because we are using the Scott topology, which in particular does not take into account the topology of $M_2(\mathbb{C})$, so measurements along z are discretely separated from measurements along any other direction, no matter how small its angle to the z axis. This means that the Scott topology is not the most appropriate topology for describing the Schrödinger electron experiment, and I have used it here for the purpose of presentation only. This will be discussed again in section 4.1.

3.6 Quantales and completions

Let M be a measurement space. The reason for the terminology "complete" is that, since M is a dcpo with a least element, if $m \vee n$ exists for every pair of elements $m, n \in M$ then $\bigvee S$ exists for every subset $S \subset M$, thus making the specialization order a complete lattice. In particular, then there is a greatest element $1 \in M$ (equivalently, $\{1\}$ is dense in M).

In addition to this, the distributivity conditions of Definition 3.10 together with those of Proposition 3.7 imply that distributivity holds for arbitrary joins, so we conclude that a measurement space is complete if and only if its specialization order is a quantale:

Definition 3.13. By a *quantale* Q [34] is meant a sup-lattice equipped with a semigroup structure whose multiplication distributes over joins in each variable separately; that is, for all $a \in Q$ and $S \subset Q$,

$$a\bigl(\bigvee S\bigr) = \bigvee_{b\in S} ab \;, \quad \bigl(\bigvee S\bigr)b = \bigvee_{a\in S} ab \;.$$

If the multiplication has a unit (i.e., the semigroup is a monoid) the quantale is *unital*. The unit is usually denoted by e, or e_Q. A quantale is *commutative*, or *abelian*, if the multiplication is commutative, and *idempotent* if the multiplication is idempotent.

Example 3.14. Let S be a semigroup. The powerset $\wp(S)$ is a quantale under the inclusion order and pointwise multiplication: for all $X, Y \in \wp(S)$ we have

$$XY = \{st \mid (s,t) \in X \times Y\} \;.$$

If S is a monoid with unit 1 then $\wp(S)$ is a unital quantale with multiplicative unit $e = \{1\}$.

Example 3.15. $\operatorname{Sub} M_n(\mathbb{C})$ is a unital quantale whose unit is $\langle I\rangle$ (cf. Example 3.11).

By similar reasoning as above, one concludes that the morphisms of complete measurement spaces preserve arbitrary joins, hence being homomorphisms of quantales:

Definition 3.16. A *homomorphism* of quantales $h : Q \to R$ is a mapping that preserves joins and the multiplication:

$$h\bigl(\bigvee S\bigr) = \bigvee h(S) \;, \quad h(ab) = h(a)h(b) \;.$$

If h is surjective we say that R is a *quotient* of Q. A homomorphism of unital quantales h is *unital* if it also preserves the unit: $h(e) = e$.

Example 3.17. Every quantale Q is a quotient of a powerset quantale $\wp(S)$ for some semigroup S, where we may take S to be Q itself and define a surjective homomorphism $h : \wp(Q) \to Q$ by $h(X) = \bigvee X$ for all $X \in \wp(Q)$.

Any locale is an example of a commutative and idempotent unital quantale whose multiplication is given by $ab = a \wedge b$ and whose unit is $e = 1$. Partly for this reason, some quantales are often regarded as "noncommutative point-free spaces." And a homomorphism of locales is the same as a homomorphism of unital quantales between locales. A useful proposition to keep in mind is the following:

Proposition 3.18 ([24])**.** *A unital quantale is a locale if and only if it is idempotent and $e = 1$.*

Given a quantale Q, a *nucleus* on Q is a closure operator $j : Q \to Q$ (i.e., j is monotone, and $a \leq j(a)$ and $j(j(a)) = j(a)$ for all $a \in Q$) such that for all $a, b \in Q$ we have

$$j(a)j(b) \leq j(ab) \;.$$

Then the set of fixed points of j,

$$Q_j = \{a \in Q \mid j(a) = a\} = j(Q) \ ,$$

is itself a quantale with multiplication $(a, b) \mapsto a \,\&\, b$ given by

$$a \,\&\, b = j(ab) \ ,$$

and the surjective map $j : Q \to Q_j$ is a homomorphism of quantales. Any quotient of quantales can be represented by a nucleus in this way, for if $h : Q \to R$ is a surjective homomorphism we have $R \cong Q_j$ for the nucleus $j = h_* \circ h$ where h_* is the right adjoint of h. If L is a locale, a nucleus j on L satisfies $j(a \wedge b) = j(a) \wedge j(b)$ for all $a, b \in L$, and L_j is a locale.

Example 3.19. Let A be a $\mathbb{C}$-algebra. The powerset $Q = \wp(A)$ is a quantale under the pointwise multiplication obtained from the multiplication of A as in Example 3.14. The operation $j : Q \to Q$ that to each set $X \in Q$ assigns the linear span $j(X) = \langle X \rangle$ is a nucleus on Q, and Q_j is the quantale $\operatorname{Sub} A$ of linear subspaces of A — with multiplication and joins defined as in Example 3.15.

For these and other basic properties and examples of quantales see [52]. Examples based on C*-algebras and locally compact groupoids will be described in section 4.

An example of application of nuclei arises in the construction of a completion of a measurement space, which I now describe. Recall that the points of a sober space can be identified with its irreducible closed sets. Hence, we can formally add the joins that are missing in a measurement space M by considering a new measurement space whose points are all the nonempty closed sets of M, for these form a sup-lattice whose least element is $\{0\}$. This sup-lattice is denoted by $\widehat{M}$ and it is a quantale (the *completion* of M):

Proposition 3.20. *Let M be a measurement space. Then $\widehat{M}$ is a quantale whose multiplication is defined for all nonempty closed sets $C, D \subset M$ by*

$$CD = \overline{C \cdot D} \ ,$$

where $\cdot$ denotes the pointwise product. Moreover, this quantale structure is that of a complete measurement space if we topologize $\widehat{M}$ as a subspace of $\mathsf{C}(M)$ with the lower Vietoris topology.

Proof. Let Q be the powerset quantale $\wp(M)$. The operator $j : Q \to Q$ defined by $j(X) = \overline{X} \cup \{0\}$ is a nucleus on Q. The multiplication defined on $\widehat{M}$ means precisely that $\widehat{M} = Q_j$. Let us see that this quantale multiplication is continuous in the lower Vietoris topology. Let $C, D \in \widehat{M}$, and let $\mathcal{U} \in \Omega(\mathsf{C}(X))$ be such that $CD \in \mathcal{U}$. I shall show that there exist open sets $\mathcal{V}, \mathcal{W} \in \Omega(\mathsf{C}(X))$ such that $C \in \mathcal{V}$, $D \in \mathcal{W}$, and $\mathcal{V}\mathcal{W} \subset \mathcal{U}$. It suffices to do this for $\mathcal{U} = \Diamond U$ with $U \in \Omega(X)$ (note that since $U \neq \emptyset$ we have $\Diamond U \subset \widehat{M}$). Then the condition $CD \in \Diamond U$ translates to $(C \cdot D) \cap U \neq \emptyset$. Let then $m \in C$ and $n \in D$ be such that $mn \in U$. Due to the continuity of the multiplication of M there are $V, W \in \Omega(M)$ such that $m \in V$, $n \in W$, and $VW \subset U$. Then making $\mathcal{V} = \Diamond V$ and $\mathcal{W} = \Diamond W$ we obtain $C \in \mathcal{V}$ and $D \in \mathcal{W}$. Let $C' \in \mathcal{V}$ and $D' \in \mathcal{W}$, and let $m' \in C' \cap V$ and $n' \in D' \cap W$. Then $m'n' \in C'D' \cap U$, so $C'D' \in \Diamond U$. This shows that $\mathcal{V}\mathcal{W} \subset \Diamond U$, so the multiplication of $\widehat{M}$ is continuous. Finally, in order to see that $\widehat{M}$ is sober, notice that, since M has a least element 0, the singleton $\{0\}$ is an atom of the lattice $\mathsf{C}(M)$, so we can define a nucleus $k : \Omega(\mathsf{C}(M)) \to \Omega(\mathsf{C}(M))$ by setting $k(\emptyset) = k(\{0\})$ and $k(C) = C$ for all $C \neq \emptyset$. Then the points of the locale $L = \Omega(\mathsf{C}(M))_k$ can be identified with the points $p \in \Sigma(\Omega(\mathsf{C}(M)))$ such that $0 = p^*(\emptyset) = p^*(\{0\})$. It follows that $\widehat{M}$ is homeomorphic to $\Sigma(L)$, so $\widehat{M}$ is sober and thus it is a complete measurement space. $\square$

3.7 Reversible measurements and involutive quantales

In the following definition we can regard m^* as a kind of time reversal of m:

Definition 3.21. Let M be a complete measurement space. By an *involution* on M is meant a continuous map $(-)^* : M \to M$ such that for all $m, n \in M$ we have

$$\begin{aligned} m^{**} &= m \\ (mn)^* &= n^* m^* \\ 0^* &= 0 \\ (m \vee n)^* &= m^* \vee n^* . \end{aligned}$$

Note that an involution is necessarily a homeomorphism, and also that it preserves joins of arbitrary subsets, so any complete measurement space with an involution is an involutive quantale:

Definition 3.22. By an *involutive quantale* is meant a quantale Q equipped with a mapping $(-)^* : Q \to Q$ (the *involution*) that satisfies the following conditions for all $a, b, a_i \in Q$:

$$\begin{aligned} a^{**} &= a \\ (ab)^* &= b^* a^* \\ \Big(\bigvee_i a_i\Big)^* &= \bigvee_i a_i^* . \end{aligned}$$

A *homomorphism* of involutive quantales $h : X \to Q$ is a homomorphism of quantales which preserves the involution: for all $a \in X$

$$h(a^*) = h(a)^* .$$

This is also called a $*$*-homomorphism*. By analogy with the terminology for locales, a *map* of involutive quantales $f : Q \to X$ is defined to be a $*$-homomorphism $f^* : X \to Q$, called the *inverse image homomorphism* of f. The right adjoint of f^* is denoted by f_*. A map f if *unital* if f^* is a unital homomorphism, and a *surjection* if f^* is injective.

All the examples of measurement spaces for spin measurements that we have seen thus far are of the form $\mathrm{Sub}\, M_n(\mathbb{C})$ and thus carry involutions obtained pointwise from the matrix adjoints:

$$V^* = \{A^* \mid A \in V\} .$$

This is a particular example of the measurement spaces that are obtained from C*-algebras, as will be seen below.

For any measurement m such that $mm^*m = m$ we also have $m^*mm^* = m^*$ and thus m^* is an inverse of m in the sense of semigroup theory [31]. Such a measurement m can be regarded as being *reversible* in the sense that it can be "undone" by performing m^*.

Another form of reversibility is that of a measurement m such that $mm^*m \leq m$, which essentially means that mm^*m can be regarded as a special way of performing m; in other words, m is less determined than mm^*m in the sense discussed in section 3.4. So this is weaker than true reversibility because in fact m^* is not able to undo m perfectly, as if some trace of the reversal were kept while performing mm^*m. The following specific class of measurement spaces rules out such weak reversals:

Definition 3.23. By a *symmetric measurement space* will be meant a complete measurement space M equipped with an involution such that for all $m \in M$ we have

$$mm^*m \leq m \implies mm^*m = m \; . \tag{3.1}$$

By a *morphism* of symmetric measurement spaces $f : M \to N$ is meant a morphism of complete measurement spaces that also preserves the involution: for all $m \in M$ we have

$$f(m^*) = f(m)^* \; .$$

The ensuing *category of symmetric measurement spaces* is denoted by MSp^*.

The name "symmetric" is motivated by the fact that such a measurement space has many symmetries which are carried by associated pseudogroups and groupoids, as will be seen in section 6. The underlying involutive quantale of a symmetric measurement space is called a *stably Gelfand quantale* [46], and any morphism of symmetric measurement spaces is a $*$-homomorphism.

4 Quantum versus classical

Let us now look at the main types of measurements spaces with which this paper is concerned. While quantum measurements are related to quantales of C*-algebras, the examples of classical measurement spaces are quantales of locally compact groupoids. The interplay between the two types is described in terms of C*-algebras of étale groupoids. Several results are recalled, and new ones are proved.

4.1 Locally convex algebras

Scott versus lower Vietoris. In Example 3.11 we have seen a complete measurement space, $\operatorname{Sub} M_n(\mathbb{C})$, that carries the Scott topology. This topology was used mostly for presentation purposes, and in fact the same construction would carry through for any $\mathbb{C}$-algebra A — the sup-lattice $\operatorname{Sub} A$ is algebraic, so with the Scott topology it is a complete measurement space (cf. Example 3.19). However, the Scott topology forgets important geometric information that is present in the topology of $M_n(\mathbb{C})$, as the following example shows.

Example 4.1. In Example 3.12 the Scott topology enabled the existence of open sets $Z^{\downarrow}$ and $Z^{\uparrow}$ that are principal filters and thus correspond to hypothetical least physical properties that can be recorded when either spin down or up is measured. However, the existence of such open sets shows that the Scott topology is not a physically realistic logic of physical properties because in terms of the directions along which spin can be measured it is akin to regarding $\mathsf{SO}(3)$ as a discrete group, as if we were able to choose the direction of the magnetic field with infinite precision. A way to remedy this is to notice that every linear subspace of $M_2(\mathbb{C})$ is closed in the norm topology, and that we may topologize $\operatorname{Sub} M_2(\mathbb{C})$ as a subspace of $\mathsf{C}(M_2(\mathbb{C}))$ with the lower Vietoris topology. In this way information is retained about the topology of $M_2(\mathbb{C})$, and hence of $\mathsf{SU}(2)$.

Following this example, if A is a topological $\mathbb{C}$-algebra of arbitrary dimension, instead of $\operatorname{Sub} A$ we shall consider the sup-lattice $\operatorname{Max} A$ of *closed* linear subspaces of A, similarly to what is done for C*-algebras in [36]. If A is locally convex and we regard $\operatorname{Max} A$ as a subspace of $\mathsf{C}(A)$ with the lower Vietoris topology, then $\operatorname{Max} A$ is a sober space [51]. I shall explore this fact in order to obtain complete measurement spaces from locally convex algebras.

Topological vector spaces. Before continuing with algebras and quantum systems let us look at mathematical properties of the $\operatorname{Max} A$ construction when A is just a topological vector space. I shall always assume that $\operatorname{Max} A$ carries the lower Vietoris topology, and for each open set $U \subset A$ the following notation for the corresponding subbasic open set of $\operatorname{Max} A$ will be used:

$$\widetilde{U} = \Diamond U \cap \operatorname{Max} A \ .$$

Let us begin with functorial properties. For each pair of topological vector spaces A and B and each continuous linear map $f : A \to B$ let

$$\operatorname{Max} f : \operatorname{Max} A \to \operatorname{Max} B$$

be the mapping that assigns $\overline{f(V)} \in \operatorname{Max} B$ to each $V \in \operatorname{Max} A$. Then, denoting by $\mathsf{TopVect}$ the category of topological vector spaces with continuous linear maps as morphisms, we obtain:

Proposition 4.2. Max *is a functor from* $\mathsf{TopVect}$ *to* Top.

Proof. Let $f : A \to B$ be a continuous linear map of topological vector spaces, and let us show that $\operatorname{Max} f$ is continuous. Let $P \in \operatorname{Max} A$ and let $\mathcal{U} \in \Omega(\operatorname{Max} B)$ be such that

$$\operatorname{Max} f(P) \in \mathcal{U} \ .$$

Let us prove that there is $\mathcal{V} \in \Omega(\operatorname{Max} A)$ such that $P \in \mathcal{V}$ and $\operatorname{Max} f(\mathcal{V}) \subset \mathcal{U}$. It suffices to consider $\mathcal{U} = \widetilde{U}$ for some $U \in \Omega(B)$. Then we have $f(P) \cap U \neq \emptyset$, so there exists $a \in P$ such that $f(a) \in U$. Due to the continuity of f there is $V \in \Omega(A)$ such that $a \in V$ and $f(V) \subset U$. Note that $a \in P \cap V$, so $P \in \widetilde{V}$ and we may take $\mathcal{V}$ to be $\widetilde{V}$. It remains to be seen that $\operatorname{Max} f(\mathcal{V}) \subset \mathcal{U}$. So let $P' \in \mathcal{V}$, and choose $a' \in P' \cap V$. Then $f(a') \in U$, and thus we obtain $\operatorname{Max} f(P') \in \mathcal{U}$.

Now let $g : B \to C$ be another continuous linear map of topological vector spaces. Then for each $P \in \operatorname{Max} A$ we have

$$\operatorname{Max}(g \circ f)(P) = \overline{g(f(P))} = \overline{g(\overline{f(P)})} = \operatorname{Max} g(\operatorname{Max} f(P)) \ ,$$

and, evidently, $\operatorname{Max} \mathrm{id}_A = \mathrm{id}_{\operatorname{Max} A}$, so Max is a functor. $\square$

Now let us see that the formation of joins is topologically well behaved:

Proposition 4.3. *Let A be a topological vector space. For any set J, give $(\operatorname{Max} A)^J$ the product topology. Then the join map $(\operatorname{Max} A)^J \to \operatorname{Max} A$ is continuous.*

Proof. Let $(V_\alpha) \in (\operatorname{Max} A)^J$ and let $U \in \Omega(A)$ be such that $\overline{\sum V_\alpha} \in \widetilde{U}$. Then $\sum V_\alpha \cap U \neq \emptyset$, which means there are $\alpha_1, \ldots, \alpha_n \in J$ and $(v_1, \ldots, v_n) \in \prod_{1=1}^n V_{\alpha_i}$ such that $\sum v_i \in U$. By the continuity of the sum $A^n \to A$, for each i there is a neighborhood U_i of v_i such that $\sum U_i \subset U$. Consider the open set

$$W = \left\{ (V'_\alpha) \in (\operatorname{Max} A)^J \mid V'_{\alpha_i} \in \widetilde{U}_i \text{ for } i = 1, \ldots, n \right\} \ .$$

Then $(V_\alpha) \in W$. Given $(V'_\alpha) \in W$, for each i let $v'_i \in V'_{\alpha_i} \cap U_i$. It follows that $\sum v'_i \in \sum U_i \subset U$, whence $\sum v'_i \in \sum V'_\alpha \cap U$ and therefore $\overline{\sum V'_\alpha} \in \widetilde{U}$. $\square$

Based on this we obtain the following extension property of continuous maps $A \to \operatorname{Max} A$:

Proposition 4.4. *Let A be a topological vector space, and $\alpha : \operatorname{Max} A \to \operatorname{Max} A$ a join preserving homomorphism. The following conditions are equivalent:*

1. *α is continuous;*
2. *the composition $\alpha \circ \mathrm{span} : A \to \operatorname{Max} A$ is continuous.*

Proof. The implication (1)⇒(2) is immediate because $\mathrm{span} : A \to \operatorname{Max} A$ is continuous [50]. In order to prove the converse assume that (2) holds, and let $J \in \operatorname{Max} A$. We shall prove that α is continuous at J. Equip both A^J and $(\operatorname{Max} A)^J$ with the product topology. By the continuity of span and the continuity of joins in $\operatorname{Max} A$ (cf. 4.3), the following mapping j is continuous:

$$\begin{tikzcd}
A^J \arrow[r, "\prod_{a\in J} \alpha\circ\mathrm{span}"] \arrow[rr, bend left, "j = \bigvee \circ (\prod \alpha\circ\mathrm{span})"] & (\operatorname{Max} A)^J \arrow[r, "\bigvee"] & \operatorname{Max} A
\end{tikzcd}$$

For each family $\phi = (b_a)_{a\in J}$ in A^J we have, since α preserves joins,

$$j(\phi) = \bigvee_{a\in J} \alpha(\mathrm{span}(b_a)) = \alpha\Big(\bigvee_{a\in J} \mathrm{span}(b_a)\Big) \ .$$

In particular, if τ is the "tautological family" $(a)_{a\in J}$ we have $j(\tau) = \alpha(J)$. In order to prove that α is continuous at J let U be an open set of A such that

$\alpha(J) \in \widetilde{U}$. This means that $j(\tau) \in \widetilde{U}$ and thus there is a basic open set $\prod_{a\in J} U_a$ of A^J, containing τ, such that

$$j\Bigl(\prod_{a\in J} U_a\Bigr) \subset \widetilde{U}\ . \tag{4.1}$$

For each $a \in J$ we have $a \in U_a$ and thus $J \in \widetilde{U_a}$. Since only finitely many U_a's are proper subsets of A, it follows that:

- the intersection $\mathcal{U} = \bigcap_{a\in J} \widetilde{U_a}$ is an open set of $\operatorname{Max} A$;
- and $J \in \mathcal{U}$.

Hence, in order to conclude that α is continuous at J we need only show that $\alpha(\mathcal{U}) \subset \widetilde{U}$. If $P \in \mathcal{U}$, for each $a \in J$ there is $b_a \in U_a \cap P$, and thus

$$j((b_a)_{a\in J}) = \alpha\Bigl(\bigvee_{a\in J} \operatorname{span}(b_a)\Bigr) \subset \alpha(P)\ .$$

Since by (4.1) we must have $j((b_a)_{a\in J}) \in \widetilde{U}$, the conclusion $\alpha(P) \in \widetilde{U}$ follows. $\square$

Topological algebras. The following facts are well known for C*-algebras [36] but hold equally for arbitrary topological algebras:

Proposition 4.5. *Let A be a topological $\mathbb{C}$-algebra. The sup-lattice $\operatorname{Max} A$ of all the closed linear subspaces of A is a quantale whose multiplication is defined for all $P, Q \in \operatorname{Max} A$ by*

$$PQ = \overline{\langle P \cdot Q\rangle}\ .$$

If B is another topological $\mathbb{C}$-algebra and $f : A \to B$ is a continuous homomorphism the mapping $\operatorname{Max} f : \operatorname{Max} A \to \operatorname{Max} B$ defined by

$$\operatorname{Max} f(P) = \overline{f(P)}$$

is a homomorphism of quantales.

Proof. This follows from the fact that the topological closure operator $V \mapsto \overline{V}$ is a nucleus on the quantale $\operatorname{Sub} A$ (cf. Example 3.19). $\square$

From here on $\operatorname{Max} A$ will always be understood to carry the lower Vietoris topology.

Proposition 4.6. *Let A be a topological $\mathbb{C}$-algebra.*

1. *The quantale multiplication of $\operatorname{Max} A$ is continuous.*
2. *If A is locally convex then $\operatorname{Max} A$ is a complete measurement space.*

Proof. The proof of the continuity of the multiplication is similar to that of Proposition 3.20, except that now it does not suffice to work only with subbasic open sets of $\operatorname{Max} A$. Let $P, Q \in \operatorname{Max} A$, and suppose $\mathcal{U} \in \Omega(\operatorname{Max} A)$ is such that $PQ \in \mathcal{U}$. I shall show that there exist open sets $\mathcal{V}, \mathcal{W} \in \Omega(\operatorname{Max} A)$ such that $P \in \mathcal{V}$, $Q \in \mathcal{W}$, and $\mathcal{V}\mathcal{W} \subset \mathcal{U}$. It suffices to do this for $\mathcal{U} = \widetilde{U}$ with $U \in \Omega(A)$, in which case the condition $PQ \in \widetilde{U}$ translates to $\langle P \cdot Q \rangle \cap U \neq \emptyset$. Let then $a_1, \ldots, a_k \in P$ and $b_1, \ldots, b_k \in Q$ ($k \geq 1$) be such that

$$a_1 b_1 + \cdots + a_k b_k \in U \; .$$

Due to the continuity of the multiplication of A there are

$$V_1, \ldots, V_k, W_1, \ldots, W_k \in \Omega(A)$$

such that $V_1 W_1 + \cdots + V_k W_k \subset U$ and $a_i \in V_i$ and $b_i \in W_i$ for all $i = 1, \ldots, k$. Then making $\mathcal{V} = \widetilde{V_1} \cap \ldots \cap \widetilde{V_k}$ and $\mathcal{W} = \widetilde{W_1} \cap \ldots \cap \widetilde{W_k}$ we obtain $P \in \mathcal{V}$ and $Q \in \mathcal{W}$. Now let $P' \in \mathcal{V}$ and $Q' \in \mathcal{W}$, and choose elements $a'_i \in P' \cap V_i$ and $b'_i \in Q' \cap W_i$ for each $i = 1, \ldots, k$. Then

$$a'_1 b'_1 + \cdots + a'_k b'_k \in P'Q' \cap U \; ,$$

so $P'Q' \in \widetilde{U}$. This shows that $\mathcal{V}\mathcal{W} \subset \widetilde{U}$, so the multiplication of $\operatorname{Max} A$ is continuous. To conclude, if A is locally convex then $\operatorname{Max} A$ is sober [51], so $\operatorname{Max} A$ is a complete measurement space. □

Let us denote the category of locally convex $\mathbb{C}$-algebras (with continuous linear maps as morphisms) by LCAlg.

Corollary 4.7. Max *is a functor from* LCAlg *to* CMSp*.*

Proof. This follows from propositions 4.2, 4.5, and 4.6. □

4.2 Quantum measurements and C*-algebras

Algebraic quantum theory is based on describing quantum systems by means of C*-algebras that contain their observables, and thus it is especially relevant to study measurement spaces associated to C*-algebras. Of course, this has implicitly been done in all the spin examples so far, in terms of their algebra $M_2(\mathbb{C})$.

Let A be a C*-algebra. The quantale $\operatorname{Max} A$ has an involution given for each $V \in \operatorname{Max} A$ by

$$V^* = \{a^* \mid a \in V\} \; ,$$

and it is stably Gelfand [47]. Since the involution of A is continuous, so is the involution of $\operatorname{Max} A$ (due to Proposition 4.2), and thus $\operatorname{Max} A$ is a symmetric measurement space. Moreover, for each $*$-homomorphism $\varphi : A \to B$ of C*-algebras the mapping $\operatorname{Max} \varphi$ is a morphism of symmetric measurement spaces. Denoting by $\mathsf{C^*\text{-}Alg}$ the category of C*-algebras and $*$-homomorphisms we have:

Proposition 4.8. $\operatorname{Max} : \mathsf{C^*\text{-}Alg} \to \mathsf{MSp}^*$ *is a functor.*

Remarkably, at least for unital C*-algebras, this functor is a complete invariant:

Proposition 4.9 ([25])**.** *Let A and B be unital C*-algebras. There is a unital $*$-isomorphism $\varphi : A \to B$ if and only if there exists an isomorphism of unital involutive quantales $\alpha : \operatorname{Max} A \to \operatorname{Max} B$.*

But Max is badly behaved with respect to homomorphisms of unital involutive quantales because not all isomorphisms $\alpha : \operatorname{Max} A \to \operatorname{Max} B$ are of the form $\alpha = \operatorname{Max} \varphi$ for some $*$-isomorphism $\varphi : A \to B$. In particular, in general

$$\operatorname{Aut}(A) \not\cong \operatorname{Aut}(\operatorname{Max} A) ,$$

and a simple example of this is obtained by taking $A = \mathbb{C}^2$ because the involutive quantale $\operatorname{Max} A$ has the following automorphism α which is not of the form $\operatorname{Max} \varphi$ [25]:

$$\begin{aligned} \alpha\big((z,w)\mathbb{C}\big) &= (w,z)\mathbb{C} \quad \text{if } z \neq 0 \text{ and } w \neq 0 \\ \alpha\big((z,0)\mathbb{C}\big) &= (z,0)\mathbb{C} \\ \alpha\big((0,w)\mathbb{C}\big) &= (0,w)\mathbb{C} . \end{aligned} \tag{4.2}$$

However, the lower Vietoris topology rules out this pathology, thus showing that symmetric measurement spaces provide a better complete invariant of unital C*-algebras than discrete involutive quantales do:

Proposition 4.10. *The automorphism $\alpha \in \operatorname{Aut}(\operatorname{Max} \mathbb{C}^2)$ of (4.2) is not continuous with respect to the lower Vietoris topology of $\operatorname{Max} \mathbb{C}^2$.*

Proof. Let us assume that $\operatorname{Max} \alpha$ is continuous and derive a contradiction. We have $\operatorname{Max} \alpha\big((1, \frac{1}{n})\mathbb{C}\big) = (\frac{1}{n}, 1)\mathbb{C}$ for all $n \in \mathbb{N}_{>0}$, and therefore the sequence $(\frac{1}{n}, 1)\mathbb{C}$ in $\operatorname{Max} \mathbb{C}^2$ must converge to $\operatorname{Max} \alpha\big((1,0)\mathbb{C}\big) = (1,0)\mathbb{C}$. Let $D(0)$ and $D(1)$ be the open disks in $\mathbb{C}$ with radius $1/2$ centered in 0 and 1, respectively, and let U be the open set $D(1) \times D(0) \subset \mathbb{C}^2$. Then $(1,0)\mathbb{C} \in \widetilde{U}$ because $(1,0) \in U \cap (1,0)\mathbb{C}$, whereas for each $n \in \mathbb{N}_{>0}$ the condition $(\frac{1}{n}, 1)\mathbb{C} \in \widetilde{U}$ means that there is $\lambda_n \in \mathbb{C}$ such that $\lambda_n(\frac{1}{n}, 1) \in U$, so we have both

$$\left| \frac{\lambda_n}{n} - 1 \right| < \frac{1}{2} \quad \text{and} \quad |\lambda_n| < \frac{1}{2} .$$

This is impossible, so $\operatorname{Max} \alpha$ is not continuous. ☐

These observations suggest that symmetric measurement spaces are able to carry much of the information that C*-algebras do, although understanding the extent to which this is true depends on a better understanding of the functor Max, namely concerning the question of which C*-algebras A yield isomorphisms $\operatorname{Aut}(A) \cong \operatorname{Aut}(\operatorname{Max} A)$, and also regarding the characterization of stably Gelfand quantales that are of the form $\operatorname{Max} A$ — for instance, they are necessarily atomic lattices. This is a difficult problem and it will not be addressed in this paper.

I conclude this section with the following simple but useful proposition, which does not follow from Proposition 4.9 and applies also to non-unital C*-algebras:

Proposition 4.11. *Let A be a C*-algebra. Then A is commutative if and only if $\operatorname{Max} A$ is commutative.*

Proof. It is trivial that if A is commutative then so is $\operatorname{Max} A$. For the converse assume that A is not commutative. Then there is an irreducible representation $\pi : A \to B(H)$ on a Hilbert space with dimension greater than 1. Due to the transitivity theorem there is a copy of $M_2(\mathbb{C})$ in the image $\pi(A)$, so we can find $a, b \in A$ and $x \in H$ such that $\pi(ab)(x) = 0$ and $\pi(ba)(x) \neq 0$. Then the induced principal left $\operatorname{Max} A$-module $P(H)$ (cf. [25]) satisfies $\langle a \rangle \langle b \rangle \cdot \langle x \rangle = \{0\}$ and $\langle b \rangle \langle a \rangle \cdot \langle x \rangle \neq \{0\}$, so $\langle a \rangle \langle b \rangle \neq \langle b \rangle \langle a \rangle$, showing that $\operatorname{Max} A$ is not commutative. ☐

4.3 Classical measurements and groupoids

In the spin examples seen so far a binary join like $\boldsymbol{z}^{\downarrow} \vee \boldsymbol{z}^{\uparrow}$ conveys the idea of quantum superposition in a logical sense, as has been discussed in section 3.4. This enables one to describe an experiment such as the "Schrödinger electron" of section 3.1, in which the join corresponds to an actual measurement in which coherence is maintained. However, in the absence of coherence we are left with a mixture, in which case the probabilities can be given a classical interpretation in terms of lack of knowledge. Hence, in such a situation we are led to regarding each join $m \vee n$ as a *classical logical disjunction* of two possibilities m and n, meaning that either m or n may have been performed but there is lack of information about which. Then, if in addition we interpret $\wedge$ as *logical conjunction*, the conclusion is that the measurement order ought to be a distributive lattice, as usual with the lattice-theoretic description of classical logic (i.e., propositional intuitionistic or geometric logic, as recalled in section 2.3).

However, there is more that can be said about the classical interpretation of joins as disjunctions, namely in the case of infinite disjunctions, as I explain now. Let M be a symmetric measurement space, and $n \in M$ a measurement. First, note that any directed set $D \subset M$ such that $n = \bigvee D \notin D$ can be regarded as a nontrivial approximation to n from below — not only in an order-theoretic sense but also in a topological one because the net $(d)_{d \in D}$ converges to n. The stages of the approximation are the elements of D, and each stage has fewer compatible physical properties than n. For instance, this could mean that the measurement n involves a certain amount of classical computation that lists all the compatible physical properties of n, where the successive stages of the computation are the elements of an increasing sequence (m_i) such that $\bigvee_i m_i = n$. Now suppose there is another measurement m such that $m \ll n$ (cf. section 2.1). Then, necessarily, for some i we must have $m \leq m_i$; that is, everything there is to be known about m will be known after finitely many steps of any such computation. More generally, for any directed set D such that $\bigvee D = n$ we must have $m \leq d$ for some stage $d \in D$. Hence, metaphorically, m is an essential component of n in the sense that approximating n from below forces one to "use" m. Such an interpretation of the way below relation is similar to that which is found in the study of programming languages (see, e.g., [56, Ch. 6]).

Now for a measurement n to be considered of "classical type" it should in principle be possible to determine, even if via an infinite amount of computation, all the classical information which is potentially associated with n. In terms of the interpretation of the way below relation described above, this means that it should be possible to determine, in a sense which is essentially unambiguous, what the essential components of n are, and it should also be possible to approximate n using only the essential components. One way to achieve this is to impose the following condition for all n:

$$n = \bigvee \{m \mid m \ll n\} \ .$$

In other words, M should be a continuous lattice, therefore being a locally compact locale because it is distributive.

Then one possible topology for M is the Scott topology:

Proposition 4.12. *Any stably Gelfand quantale whose order is a continuous lattice is a symmetric measurement space if we equip it with the Scott topology.*

Proof. The quantale multiplication and the involution are Scott continuous because they are join preserving in each variable, and sobriety follows from the lattice continuity. $\square$

The Scott topology on M is a natural choice because when dealing with classical measurements the objections raised at the end of section 3.5 and in section 4.1 should not apply, since we have no changes of basis to account for — cf. Example 4.1. I remark that this choice also gives us, due to Proposition 2.6, a representation of basic physical properties in terms of compact sets of the locale spectrum of M.

The above discussion suggests the following general definition:

Definition 4.13. By a *classical measurement space* is meant a symmetric measurement space M such that the following conditions hold:

1. The measurement order makes M a locally compact locale;
2. The topology of M is the Scott topology.

All the examples of classical measurement spaces in this paper will arise from groupoids. Let us recall basic definitions about groupoids, mostly to fix terminology and notation:

Definition 4.14. A *topological groupoid* G, or simply a *groupoid* since no other kind will be addressed here, consists of topological spaces G_0 (the space of *objects*, or *units*) and G_1 (the space of *arrows*), together with continuous structure maps

$$
\begin{tikzcd}
G \quad = \quad G_2 \arrow[r, "m"] & G_1 \arrow[loop above, "i"] \arrow[r, shift left=2, "r"] \arrow[r, shift right=2, "d"'] & G_0 \arrow[l, "u" description]
\end{tikzcd}
$$

that satisfy the axioms listed below, where: G_2 is the pullback of the *domain map* d and the *range map* r,

$$G_2 = \{(g,h) \in G_1 \times G_1 \mid d(g) = r(h)\} ;$$

the map m is the *multiplication* and i is the *inversion*; the map u is required to be a section of both d and r, so often G_0 is identified with a subspace of G_1 (with u being the inclusion map). With this convention, and writing gh instead of $m(g,h)$, as well as g^{-1} instead of $i(g)$, the axioms satisfied by the structure maps are:

- $d(x) = x$ and $r(x) = x$ for each $x \in G_0$;
- $g(hk) = (gh)k$ for all $(g,h) \in G_2$ and $(h,k) \in G_2$;
- $gd(g) = g$ and $r(g)g = g$ for all $g \in G_1$;
- $d(g) = g^{-1}g$ and $r(g) = gg^{-1}$ for all $g \in G_1$.

The groupoid G is said to be *open* if the map d is open, which is equivalent to requiring that r is open and also equivalent to requiring that m is open. Then by a *locally compact groupoid* will be meant an open groupoid G such that G_1 (and hence also G_0, since it is a quotient of G_1 by an open map) is a locally compact space. Finally, a locally compact groupoid is *second-countable* if G_1 (and hence also G_0) is second-countable.

Example 4.15. Any locally compact groupoid in the sense of [39] is a second-countable locally compact groupoid in the sense of Definition 4.14, and so is any Lie groupoid.

Proposition 4.16. *Let G be a locally compact groupoid. The topology $\Omega(G_1)$ is a classical measurement space, which will be denoted by $\mathcal{O}(G)$.*

Proof. Since the multiplication map of an open groupoid is open, the topology $\Omega(G_1)$ is a quantale under pointwise multiplication (see [45]). It is also involutive under pointwise involution, and it is stably Gelfand because it satisfies

$$U \subset UU^*U$$

for all $U \in \Omega(G_1)$. Hence, since G is locally compact, $\Omega(G_1)$ is a locally compact locale and thus it defines a classical measurement space. $\square$

Example 4.17. Consider an abstract description of a classical system in terms of a locally compact space X (the state space) equipped with a left action $(\alpha, x) \mapsto \alpha \cdot x$ by a locally compact group $\mathit{\Gamma}$ that describes symmetries and the time evolution. The *action groupoid* associated to the action is the locally compact groupoid $G = \mathit{\Gamma} \ltimes X$ defined by

$$\begin{aligned}
G_1 &= \mathit{\Gamma} \times X \quad \text{with the product topology} ,\\
G_0 &= X ,\\
u(x) &= (1, x) ,\\
d(\alpha, x) &= x ,\\
r(\alpha, x) &= \alpha \cdot x ,\\
(\alpha, \beta \cdot x)(\beta, x) &= (\alpha\beta, x) ,\\
(\alpha, x)^{-1} &= (\alpha^{-1}, \alpha \cdot x) .
\end{aligned}$$

The classical measurement space $\mathcal{O}(G)$ describes the measurements that can be performed on the system by composing open sets of X with the transformations induced by the group action.

4.4 Local measurements and commuting observables

In the context of classical physics it is possible to restrict to measurements that are *passive* in the sense that they do not perturb the system being measured. These can be described by the following type of classical measurement space:

Definition 4.18. By a *local measurement space* will be meant a classical measurement space M such that for all $m, n \in M$ we have $mn = m \wedge n$ and $m^* = m$. Equivalently, M is a locally compact locale with trivial involution, equipped with the Scott topology.

Local measurement spaces are very special because the composition of measurements is both commutative and idempotent, thus conveying that the order in which measurements are performed is irrelevant, and that performing the same measurement more than once adds no new information. Moreover, in such a measurement space we can proceed by successive approximations, for if $m \leq n$ we have $nm = m$, which can be interpreted as saying that performing n can be improved by subsequently performing the "sharper" measurement m. Similarly, $mn = m$ implies that after m has been performed no new information is obtained by performing the less precise measurement n. In particular, the least precise measurement 1 is such that $m1 = 1m = m$ for all $m \in M$, so it adds nothing to any other measurement. More generally, for arbitrary m and n we have $mn \leq m$ and $mn \leq n$, meaning that performing both m and n provides sharper information than either m or n alone.

Anyone who runs a collection of experiments entirely based on measurements taken from a local measurement space may naturally be led to regarding measurements as propositions of a propositional logic. Also, even though no underlying state space has been assumed to exist, we can pretend that each measurement is an open set of states, i.e., of locale points. Since a point either belongs to a measurement or it does not, there are no state changes associated to measurements. This conveys the idea of a static universe of states.

A similar situation should occur with a quantum system if we restrict to measurements of a given collection of commuting observables. For instance, anyone who measured the spin of an electron always along z would be excused for wrongly concluding that there are only two spin states, since the measurement space is restricted to the following locale:

$$\begin{tikzcd}[column sep=small]
 & \boldsymbol{z} \arrow[dl, dash] \arrow[dr, dash] & \\
\boldsymbol{z}^{\downarrow} \arrow[dr, dash] & & \boldsymbol{z}^{\uparrow} \arrow[dl, dash] \\
 & 0 &
\end{tikzcd}$$

Hence, a commutative C*-algebra should have an associated local measurement space. Since any such algebra is of the form $C_0(X)$ for a locally compact Hausdorff space X, it is natural to take the associated local measurement space to be $\Omega(X)$, which is a locally compact locale, with the Scott topology. Alternatively, we may take the symmetric measurement space $\operatorname{Max} C_0(X)$, with the lower Vietoris topology as usual, and consider the locale of closed ideals $I(C_0(X))$, which is isomorphic to $\Omega(X)$, regarded as a subspace of $\operatorname{Max} C_0(X)$. The two constructions are equivalent because Proposition 4.21, which will be proved in section 4.5, shows that the spaces $\Omega(X)$ and $I(C_0(X))$ are homeomorphic.

4.5 Étale groupoids and quantizations

By a "quantization problem" will be meant the question of whether a given classical measurement space M can be suitably embedded into one of the form $\operatorname{Max} A$ for a C*-algebra A. Here I shall examine this issue without considering any physical interpretations, mostly in order to establish general terminology and facts that will be used later on. Let us begin with a working definition:

Definition 4.19. Let M be a classical measurement space. A *quantization* of M is a triple (A, s, r) consisting of a C*-algebra A together with a morphism of symmetric measurement spaces $s : M \to \operatorname{Max} A$ and a continuous function $r : \operatorname{Max} A \to M$ such that $r(s(m)) = m$ for all $m \in M$.

I shall focus on measurement spaces $M = \mathcal{O}(G)$ of locally compact groupoids that are also Hausdorff and second-countable.[1] There are well known constructions of C*-algebras from such groupoids provided a Haar system of measures is defined on G, such as the universal C*-algebra $C^*(G)$ or the reduced C*-algebra $C^*_r(G)$, both of which are completions of the convolution algebra $C_c(G)$ of continuous compactly supported functions $f : G_1 \to \mathbb{C}$. More generally, from a Fell bundle $\pi : E \to G$ (see [29]) one obtains C*-algebras as completions of the convolution algebra of continuous compactly supported sections $C_c(G, E)$. In this paper we shall not need such generality, since the algebras of functions are the ones that carry a physical interpretation in terms of probability amplitudes, as will be recalled in section 6.1. In addition I shall restrict to étale groupoids, for which a physical interpretation will be provided in section 6.

Recall that a topological groupoid G is called *étale* if its domain map is a local homeomorphism. This is equivalent to saying that G is open and G_0 is an open subspace of G_1 [45]. Hence, G is étale if and only if $\mathcal{O}(G)$ is a unital quantale whose multiplicative unit is G_0. For example, the action groupoid $\Gamma \ltimes X$ of Example 4.17 is étale if and only if Γ is a discrete group. The *open bisections* of G are the open sets $U \subset G_1$ such that $U^*U \subset G_0$ and $UU^* \subset G_0$, and the set of all the open bisections is denoted by $\mathcal{I}(G)$. If G is étale $\mathcal{I}(G)$ is a basis for the topology of G_1.

Let G be a second-countable locally compact Hausdorff étale groupoid (for instance an étale Lie groupoid). I shall write $\operatorname{supp} f$ and $\operatorname{supp}^\circ f$, respectively, for the *support* and the *open support* of a continuous function $f : G_1 \to \mathbb{C}$:

$$\begin{aligned} \operatorname{supp}^\circ f &= \{x \in G_1 \mid f(x) \neq 0\} , \\ \operatorname{supp} f &= \overline{\operatorname{supp}^\circ f} . \end{aligned}$$

The *convolution algebra* of G,

$$C_c(G) = \{f : G \to \mathbb{C} \mid f \text{ is continuous and } \operatorname{supp} f \text{ is compact}\} ,$$

has multiplication and involution defined for all $x \in G_1$ by

$$f * g(x) = \sum_{x=yz} f(y)g(z) , \quad f^*(x) = \overline{f(x^{-1})} . \tag{4.3}$$

(Note that the above sum has only finitely many nonzero summands.) I shall also use the following notation for all open sets $U \subset G_1$:

$$C_c(U) := \{f \in C_c(G) \mid \operatorname{supp} f \subset U\} .$$

By [47], the *reduced norm* on $C_c(G)$ can be defined to be the unique C*-norm $\|\cdot\|$ such that the following three conditions hold:

[1] To some extent the restriction to Hausdorff groupoids can be dropped, but this would considerably obscure the presentation.

1. $\|f\|_\infty \le \|f\|$ for all $f \in C_c(G)$;
2. $\|f\|_\infty = \|f\|$ if $f \in C_c(U)$ for some $U \in \mathcal{I}(G)$;
3. The completion A of $C_c(G)$ under this norm continuously extends the inclusion $C_c(G) \to C_0(G)$ to an injective function $A \to C_0(G)$.

As usual, I shall refer to the completion A as the *reduced C^*-algebra* of G, and denote it by $C_r^*(G)$. Due to condition 3 above, I shall always regard $C_r^*(G)$ concretely as an algebra of functions on G. Note that if G is compact then $C_r^*(G)$ coincides with $C(G)$ because

$$C_c(G) \subset C_r^*(G) \subset C_0(G) \subset C(G) = C_c(G) \ .$$

Proposition 4.20. *Let G be a second-countable locally compact Hausdorff étale groupoid. The mapping $\overline{C_c(-)} : \mathcal{O}(G) \to \operatorname{Max} C_r^*(G)$ that sends each $U \in \mathcal{O}(G)$ to the closure $\overline{C_c(U)}$ in $C_r^*(G)$ is an injective morphism of symmetric measurement spaces.*

Proof. $\overline{C_c(-)}$ is an injective homomorphism of involutive quantales due to the results of [47]. And, since the topology of $\operatorname{Max} C_r^*(G)$ is contained in the Scott topology, it is continuous because it preserves joins. □

Since $\overline{C_c(G)}$ is injective, the set $\mathcal{O}$ in the following proposition is itself a locale, and it is isomorphic to $\mathcal{O}(G)$:

Proposition 4.21. *Let G be a second-countable locally compact Hausdorff étale groupoid, and define the following subset of $\operatorname{Max} C_r^*(G)$:*

$$\mathcal{O} := \{\overline{C_c(U)} \mid U \in \mathcal{O}(G)\} \ .$$

The lower Vietoris topology on $\mathcal{O}$ coincides with the Scott topology.

Proof. Let $O \in \mathcal{O}(G)$ and let $\mathcal{U} \subset \mathcal{O}$ be a Scott open set containing $\overline{C_c(O)}$. We shall find an open set $U \subset C_r^*(G)$ such that $\overline{C_c(O)} \in \widetilde{U} \cap \mathcal{O} \subset \mathcal{U}$, thus proving that the lower Vietoris topology is equal to the Scott topology on $\mathcal{O}$.

If $O = \emptyset$ then $\overline{C_c(O)} = \{0\}$ and $\mathcal{U} = \mathcal{O}$ because $\mathcal{U}$ is upwards closed, and we can make $U = C_r^*(G)$. So let $O \neq \emptyset$. Then, since G is locally compact and Hausdorff, O is the union of all the open sets V such that $\overline{V} \subset O$ and $\overline{V}$ is compact. The collection of all these open sets V is a directed set, and thus for some such V we have $\overline{C_c(V)} \in \mathcal{U}$. If $V = \emptyset$ we conclude $\mathcal{U} = \mathcal{O}$, so again we can make $U = C_r^*(G)$. Otherwise, by similar reasoning as above, there is an open set W such that $\overline{W} \subset V$ and $\overline{C_c(W)} \in \mathcal{U}$. If $W = \emptyset$ again we can make $U = C_r^*(G)$, so let us assume $W \neq \emptyset$.

Applying Urysohn's lemma we obtain a continuous function on $\overline{V}$ with values in $[0,1]$ such that $\phi(w) = \{1\}$ for all $w \in \overline{W}$ and $\phi(v) = \{0\}$ for all $v \in \partial V$, and we may consider ϕ to be a continuous function on G_1 by making it the null function outside V (this works if $\partial V = \emptyset$, too, in which case $V = O = G_1$ and G_1 is compact). Then ϕ is compactly supported (because $\operatorname{supp}\phi \subset \overline{V}$) and $\operatorname{supp}\phi \subset O$, so $\phi \in \overline{C_c(O)}$. Let U be the open ball

$$B_{1/2}(\phi) = \{f \mid \|f - \phi\| < 1/2\} \ .$$

Then $\overline{C_c(O)} \in \widetilde{U}$ because $\phi \in \overline{C_c(O)} \cap U$. Let us prove that $\widetilde{U} \cap \mathcal{O} \subset \mathcal{U}$. Let $P \in \widetilde{U} \cap \mathcal{O}$. Then $P = \overline{C_c(O')}$ for some $O' \in \mathcal{O}(G)$, and $P \cap U \neq \emptyset$. Let $f \in P \cap U$. Then

$$\|f - \phi\|_\infty \leq \|f - \phi\| < 1/2 ,$$

and thus $W \subset \overline{\operatorname{supp} f} \subset O'$. Hence, $\overline{C_c(W)} \subset \overline{C_c(O')} = P$, and it follows that $P \in \mathcal{U}$ because $\overline{C_c(W)} \in \mathcal{U}$ and $\mathcal{U}$ is upwards closed. $\square$

Now in order to obtain the envisaged quantization we need to find a retraction. The following fact will be needed:

Proposition 4.22. *Let G be a second-countable locally compact Hausdorff étale groupoid. The mapping* $\operatorname{supp}^\circ : C_r^*(G) \to \mathcal{O}(G)$ *is continuous.*

Proof. Let $f \in C_r^*(G)$, and let $\mathcal{U} \subset \mathcal{O}(G)$ be a Scott open set such that

$$\operatorname{supp}^\circ(f) \in \mathcal{U} .$$

From the hypothesis that G is locally compact Hausdorff it follows that $\operatorname{supp}^\circ(f)$ is a directed union of the open sets V such that $\overline{V}$ is compact and $\overline{V} \subset \operatorname{supp}^\circ(f)$, so for some of these open sets we must have $V \in \mathcal{U}$. Let us fix one such V and define

$$\varepsilon = \min\{|f(x)| \mid x \in \overline{V}\} .$$

Then $\varepsilon > 0$, and in order to prove that $\operatorname{supp}^\circ$ is continuous it suffices to show that for the open ball $B_\varepsilon(f) \subset C_r^*(G)$ we have

$$\operatorname{supp}^\circ\bigl(B_\varepsilon(f)\bigr) \subset \mathcal{U} .$$

In order to see this let $g \in B_\varepsilon(f)$ in $C_r^*(G)$. Then

$$\|f - g\|_\infty \leq \|f - g\| < \varepsilon ,$$

so for all $x \in V$ we have $g(x) \neq 0$ because

$$\varepsilon - |g(x)| \leq |f(x)| - |g(x)| \leq |f(x) - g(x)| < \varepsilon .$$

So $V \subset \operatorname{supp}^\circ(g)$, and thus $\operatorname{supp}^\circ(g) \in \mathcal{U}$ because $\mathcal{U}$ is upwards closed. $\square$

Remark 4.23. Note that the involutive quantale $\mathcal{O}(G)$ can be regarded as a convolution "algebra," too, because its elements (open sets of G_1) correspond bijectively to continuous maps $G_1 \to \$$, where $\$$ is the Sierpinsky space, and the pointwise multiplication of open sets is, via the bijection, given by a convolution formula:

$$\phi * \psi(x) = \bigvee_{x=yz} \phi(y) \wedge \psi(z) .$$

Moreover, the whole of $\mathcal{O}(G)$ is obtained by taking joins of compactly supported ϕ's because $\Omega(G_1)$ is a continuous lattice, so $C_r^*(G)$ itself can be regarded as a "quantization" of $\mathcal{O}(G)$ that arises from replacing $\$$ by $\mathbb{C}$. The previous proposition shows that this analogy is consistent with the topologies carried by $C_r^*(G)$ and $\mathcal{O}(G)$.

Let us use the following notation from now on, where $V \in C_r^*(G)$ for a groupoid G:

$$\operatorname{supp}^\circ(V) := \bigcup_{f\in V} \operatorname{supp}^\circ(f)\ . \tag{4.4}$$

The resulting map $\operatorname{supp}^\circ : \operatorname{Max} C_r^*(G) \to \mathcal{O}(G)$ has the following properties:

Proposition 4.24. *Let G be a second-countable locally compact Hausdorff étale groupoid. For all $V, W \in \operatorname{Max} C_r^*(G)$ and all families (V_i) in $\operatorname{Max} C_r^*(G)$ we have*

1. $\operatorname{supp}^\circ\bigl(\bigvee_i V_i\bigr) = \bigcup_i \operatorname{supp}^\circ(V_i)$,
2. $\operatorname{supp}^\circ(VW) \subset \operatorname{supp}^\circ(V)\operatorname{supp}^\circ(W)$,
3. $\operatorname{supp}^\circ(V^*) = (\operatorname{supp}^\circ V)^*$.

Proof. Condition (1) has been proved in [47, Lemma 4.1] in the more general setting of Fell bundles on non-Hausdorff groupoids (cf. [47, Eq. (4.3) and Lemma 4.6]). For future reference I briefly recall the argument here, for Hausdorff groupoids. The mapping $\gamma : \mathcal{O}(G) \to \operatorname{Max} C_r^*(G)$ which is defined for each $U \in \mathcal{O}(G)$ by

$$\gamma(U) = \{f \in C_r^*(G) \mid \operatorname{supp}^\circ(f) \subset U\}$$

satisfies the following equivalence for all $V \in \operatorname{Max} C_r^*(G)$ and $U \in \mathcal{O}(G)$:

$$\operatorname{supp}^\circ(V) \subset U \iff V \subset \gamma(U)\ .$$

This means that $\operatorname{supp}^\circ$ is left adjoint to γ, so $\operatorname{supp}^\circ$ preserves joins as stated.

In order to prove (2) let $f, g \in C_r^*(G)$ and $x \in \operatorname{supp}^\circ(f * g)$. From the convolution formula in (4.3) it follows that for some $(y, z) \in G_2$ such that $x = yz$ we must have both $f(y) \neq 0$ and $g(z) \neq 0$, since otherwise we would obtain $f * g(x) = 0$. So $x \in \operatorname{supp}^\circ(f)\operatorname{supp}^\circ(g)$, showing that $\operatorname{supp}^\circ(f * g) \subset \operatorname{supp}^\circ(f)\operatorname{supp}^\circ(g)$. Then, using (1), we prove (2):

$$\begin{aligned}
\operatorname{supp}^\circ(VW) &= \operatorname{supp}^\circ\Bigl(\bigvee_{f\in V}\langle f\rangle \bigvee_{g\in W}\langle g\rangle\Bigr)\\
&= \operatorname{supp}^\circ\Bigl(\bigvee_{f\in V,\ g\in W}\langle f\rangle\langle g\rangle\Bigr)\\
&= \bigcup_{f\in V,\ g\in W}\operatorname{supp}^\circ(\langle f\rangle\langle g\rangle)\\
&= \bigcup_{f\in V,\ g\in W}\operatorname{supp}^\circ(f * g)\\
&\subset \bigcup_{f\in V,\ g\in W}\operatorname{supp}^\circ(f)\operatorname{supp}^\circ(g)\\
&= \bigcup_{f\in V}\operatorname{supp}^\circ(f)\bigcup_{g\in W}\operatorname{supp}^\circ(g)\\
&= \operatorname{supp}^\circ(V)\operatorname{supp}^\circ(W)\ .
\end{aligned}$$

Finally, observe that if $f \in C_r^*(G)$ we have $f^*(x) = 0$ if and only if $\overline{f(x^{-1})} = 0$, so $\operatorname{supp}^\circ(f^*) = \operatorname{supp}^\circ(f)^*$. From here (3) easily follows, again using (1). ☐

Finally, using the above results, we obtain a quantization of a classical measurement space:

Proposition 4.25. *Let G be a second-countable locally compact Hausdorff étale groupoid. Then*

$$\left(C_r^*(G), \overline{C_c(-)}, \mathrm{supp}^\circ\right)$$

is a quantization of $\mathcal{O}(G)$.

Proof. In Proposition 4.20 it has been shown that

$$\overline{C_c(-)} : \mathcal{O}(G) \to \mathrm{Max}\, C_r^*(G)$$

is a morphism of symmetric measurement spaces, so we need only prove that the following two conditions hold:

1. $\mathrm{supp}^\circ(\overline{C_c(U)}) = U$ for all $U \in \mathcal{O}(G)$;
2. $\mathrm{supp}^\circ : \mathrm{Max}\, C_r^*(G) \to \mathcal{O}(G)$ is continuous.

Let us begin with (1). Let $U \in \mathcal{O}(G)$. If $f \in \overline{C_c(U)}$ we must have a sequence (f_i) in $C_c(U)$ such that $\lim_i f_i = f$. Hence, if $x \notin U$ we have $f(x) = \lim_i f_i(x) = 0$. This shows that $\mathrm{supp}^\circ f \subset U$. On the other hand, if $x \in U$ there is $f \in C_c(G)$ such that $f(x) \neq 0$ and $\mathrm{supp}\, f \subset U$ (because G is locally compact Hausdorff), and thus $\mathrm{supp}^\circ(\overline{C_c(U)}) = U$.

Now let us prove (2). For the sake of clarity, in this part of the proof let us write Supp° for the extended open support map $\mathrm{Max}\, C_r^*(G) \to \mathcal{O}(G)$ that was defined in (4.4). Let

$$\alpha = \overline{C_c(-)} \circ \mathrm{Supp}^\circ : \mathrm{Max}\, C_r^*(G) \to \mathrm{Max}\, C_r^*(G) .$$

This is a sup-lattice homomorphism due to Proposition 4.24, and the restriction $\alpha \circ \mathrm{span}$ coincides with the map

$$\overline{C_c(-)} \circ \mathrm{supp}^\circ : C_r^*(G) \to \mathrm{Max}\, C_r^*(G) ,$$

which is continuous due to Proposition 4.22. Therefore, by Proposition 4.4, α is continuous. Let us denote by $\mathcal{O}$ the image of α, which coincides with the image of $\overline{C_c(-)}$. Due to Proposition 4.21 the topology of $\mathcal{O}$ as a subspace of $\mathrm{Max}\, C_r^*(G)$ coincides with the Scott topology, and thus

$$\overline{C_c(-)} : \mathcal{O}(G) \to \mathcal{O}$$

is a homeomorphism. Hence, Supp° is continuous because α is. $\square$

4.6 Quantization maps

Let (A, s, r) be a quantization of a classical measurement space M. There is a map of involutive quantales

$$p : \mathrm{Max}\, A \to M$$

(recall Definition 3.22) which is defined by $p^* = s$, and in this section I shall study general properties of such maps in the case of quantizations

$$\left(C_r^*(G), \overline{C_c(G)}, \mathrm{supp}^\circ\right)$$

as in Proposition 4.25, in particular focusing on the relation between p and supp°. From here on I adopt the following terminology:

Definition 4.26. Let G be a second-countable locally compact Hausdorff étale groupoid. The *quantization map* of G is the map of involutive quantales

$$p : \operatorname{Max} C_r^*(G) \to \mathcal{O}(G)$$

defined by the condition

$$p^* = \overline{C_c(-)} \ .$$

Remark 4.27. Recall the mapping $\gamma : \mathcal{O}(G) \to C_r^*(G)$ of the beginning of the proof of Proposition 4.24:

$$\gamma(U) = \{f \in C_r^*(G) \mid \operatorname{supp}^\circ(f) \subset U\} \ .$$

This may not necessarily coincide with p^* (cf. section 4.7 below), but at least we have $p^* \leq \gamma$.

We need a few simple results about reduced C*-algebras:

Proposition 4.28. *Let G be a second-countable locally compact Hausdorff étale groupoid, let $f, g \in C_r^*(G)$, and let $U \in \mathcal{I}(G)$.*

1. *If $g \in \overline{C_c(U)}$ then for all $x \in G_1$*

$$fg(x) = \begin{cases} 0 & \text{if } d(x) \notin d(U) \ , \\ f(xz^{-1})g(z) & \text{if } z \in U \text{ and } d(z) = d(x) \ . \end{cases}$$

2. *If $f \in \overline{C_c(U)}$ then for all $x \in G_1$*

$$fg(x) = \begin{cases} 0 & \text{if } r(x) \notin r(U) \ , \\ f(y)g(y^{-1}x) & \text{if } y \in U \text{ and } r(y) = r(x) \ . \end{cases}$$

(Note that y and z above are unique.)

Proof. The first statement is a particular case of [47, Lemma 3.7], and the other can be proved in a similar way. ☐

Now we obtain the envisaged relations between quantization maps and $\operatorname{supp}^\circ$:

Proposition 4.29. *Let G be a second-countable locally compact Hausdorff étale groupoid with quantization map p. Then for all $U \in \mathcal{O}(G)$ and $V \in \operatorname{Max} C_r^*(G)$ we have:*

1. $\operatorname{supp}^\circ(p^*(U)V) = U \operatorname{supp}^\circ(V)$;
2. $\operatorname{supp}^\circ(Vp^*(U)) = \operatorname{supp}^\circ(V)U$.

Proof. (1) It suffices to prove the equation for one dimensional subspaces $V = \langle f \rangle$ because

$$\operatorname{supp}^\circ(p^*(U)V) = \operatorname{supp}^\circ\big(p^*(U) \bigvee_{f \in V} \langle f \rangle\big) = \bigcup_{f \in V} \operatorname{supp}^\circ(p^*(U)\langle f \rangle)$$

and

$$U \operatorname{supp}^\circ(V) = U \operatorname{supp}^\circ\big(\bigvee_{f\in V} \langle f\rangle\big) = \bigcup_{f\in V} U \operatorname{supp}^\circ(\langle f\rangle)\ .$$

It also suffices to prove it for $U \in \mathcal{I}(G)$ because every open set of G_1 is a union of open bisections (U_i), and thus if the equation holds for each U_i it also holds for U:

$$\begin{aligned}\operatorname{supp}^\circ(p^*(U)V) &= \operatorname{supp}^\circ\big(p^*\big(\bigcup_i U_i\big)V\big) = \bigcup_i \operatorname{supp}^\circ(p^*(U_i)V)\\ &= \bigcup_i U_i \operatorname{supp}^\circ V = U \operatorname{supp}^\circ V\ .\end{aligned}$$

So for all $U \in \mathcal{I}(G)$ and $f \in C^*_r(G)$ let us prove

$$\operatorname{supp}^\circ(p^*(U)f) = U \operatorname{supp}^\circ(f)\ . \tag{4.5}$$

Let $x \in U \operatorname{supp}^\circ(f)$. Then $x = yz$ for the unique element

$$y \in r^{-1}(\{x\}) \cap U\ ,$$

and $f(z) \neq 0$. Letting $g \in C_c(U)$ such that $g(y) \neq 0$ we conclude, using Proposition 4.28(2),

$$x \in \operatorname{supp}^\circ(gf) \subset \operatorname{supp}^\circ(p^*(U)f)\ ,$$

and thus

$$U \operatorname{supp}^\circ(f) \subset \operatorname{supp}^\circ(p^*(U)f)\ .$$

For the converse inclusion let $x \in \operatorname{supp}^\circ(p^*(U)f)$. Since $p^*(U) = \bigvee_{g\in C_c(U)}\langle g\rangle$, we obtain

$$\operatorname{supp}^\circ(p^*(U)f) = \bigcup_{g\in C_c(U)} \operatorname{supp}^\circ(gf)\ ,$$

so for some $g \in C_c(U)$ we have $x \in \operatorname{supp}^\circ(gf)$. Again from Proposition 4.28(2) we conclude that $x = yz$ with $y \in \operatorname{supp}^\circ(g)$ and $z \in \operatorname{supp}^\circ(f)$, so

$$x \in \operatorname{supp}^\circ(g) \operatorname{supp}^\circ(f) \subset U \operatorname{supp}^\circ(f)\ .$$

This proves (4.5). The proof of (2) is similar, using Proposition 4.28(1). ◻

Remark 4.30. In the above proposition $\operatorname{Max} C^*_r(G)$ has an $\mathcal{O}(G)$-$\mathcal{O}(G)$-bimodule structure given by "change of ring" along p^*: if $U \in \mathcal{O}(G)$ and $V \in \operatorname{Max} C^*_r(G)$ the left and right actions of U on V are defined by

$$U \cdot V := p^*(U)V \quad \text{and} \quad V \cdot U := Vp^*(U)\ .$$

Hence (since $\operatorname{supp}^\circ$ preserves joins), the proposition is equivalent to stating that $\operatorname{supp}^\circ$ is a homomorphism of $\mathcal{O}(G)$-$\mathcal{O}(G)$-bimodules:

$$\operatorname{supp}^\circ(U \cdot V) = U \operatorname{supp}^\circ(V) \quad \text{and} \quad \operatorname{supp}^\circ(V \cdot U) = \operatorname{supp}^\circ(V)U\ .$$

Note also that these bimodule actions are continuous with respect to the topologies of $\operatorname{Max} C^*_r(G)$ and $\mathcal{O}(G)$, so $\operatorname{supp}^\circ$ is a continuous homomorphism of topological bimodules.

4.7 Localizable groupoids

Let G be a second-countable locally compact Hausdorff étale groupoid. By definition, every $f \in C_r^*(G)$ is a limit $\lim_i f_i$ of a sequence (f_i) in $C_c(G)$, but it is not clear whether it is always possible to find such a sequence if we impose

$$\operatorname{supp} f_i \subset \operatorname{supp}^\circ f$$

for all i. If this is possible we say that f is a *local limit* (because it is approached by continuous compactly supported functions *locally* in $\operatorname{supp}^\circ f$). This motivates the following definition:

Definition 4.31. By a *localizable groupoid* will be meant a second-countable locally compact Hausdorff étale groupoid G such that for all $f \in C_r^*(G)$ we have

$$f \in \overline{C_c(\operatorname{supp}^\circ f)} .$$

Remark 4.32. The terminology "localizable" is taken from [47], where it was applied to C*-completions of convolution algebras of Fell bundles. From the results there it follows that any (not a priori Hausdorff) localizable groupoid G such that G_0 is Hausdorff must have G_1 Hausdorff. This provides another justification for restricting to Hausdorff groupoids in this paper.

The characterization of the class of localizable groupoids is an open problem. The following are sufficient but not necessary conditions for localizability:

Proposition 4.33 ([47])**.** *Let G be a second-countable locally compact Hausdorff étale groupoid.*

1. *If $G_0 = G_1$ (i.e., G is just a space) then G is localizable.*
2. *If G is compact then G is localizable.*

The significance of localizability comes from the following:

Proposition 4.34. *Let G be a second-countable locally compact Hausdorff étale groupoid with quantization map p. The following conditions are equivalent:*

1. *G is localizable.*
2. *For all $V \in \operatorname{Max} C_r^*(G)$ we have $V \subset p^*(\operatorname{supp}^\circ V)$.*
3. *$\operatorname{supp}^\circ : \operatorname{Max} C_r^*(G) \to \mathcal{O}(G)$ is left adjoint to p^*.*
4. *$p^* = \gamma$ (cf. Remark 4.27).*

Proof. (1) $\Longrightarrow$ (2): If G is localizable then (2) holds because for all $f \in V$

$$f \in p^*(\operatorname{supp}^\circ f) \subset p^*(\operatorname{supp}^\circ V) .$$

(2) $\Longrightarrow$ (1): The localizability condition $f \in p^*(\operatorname{supp}^\circ f)$ results from taking $V = \langle f \rangle$ in (2).

(2) $\Longleftrightarrow$ (3): $\operatorname{supp}^\circ$ and p^* are monotone and the retraction condition $\operatorname{supp}^\circ \circ p^* = \mathrm{id}$ provides the counit of the adjunction, so we have an adjunction if and only if (2) holds, for this provides the unit.

(3) $\Longleftrightarrow$ (4): this is immediate because γ is the right adjoint of $\operatorname{supp}^\circ$ (cf. proof of Proposition 4.24). ∎

Remark 4.35. Localizability means that p^* is right adjoint to supp°, so it implies that p^* preserves arbitrary meets:

$$p^*(\bigwedge_i U_i) = \bigcap_i p^*(U_i) \quad \text{for all families } (U_i) \text{ in } \mathcal{O}(G)\ . \tag{4.6}$$

It is not known whether the mere existence of a left adjoint of p^* implies localizability, but it is easy to conclude, since $p^* \le \gamma$ and supp° is left adjoint to γ, that if p^* has a left adjoint $p_!$ then $\mathrm{supp}^\circ \le p_!$.

Assume that G is localizable with quantization map p, and let $V, W \in C_r^*(G)$. In [47] it is proved that if $\mathrm{supp}^\circ V$ and $\mathrm{supp}^\circ W$ are open bisections then for all $U \in \mathcal{O}(G)$

$$\mathrm{supp}^\circ(V\, p^*(U)\, W) = \mathrm{supp}^\circ(V)\ U\, \mathrm{supp}^\circ(W)\ . \tag{4.7}$$

Such a quantization map p is called a *quantic bundle.* This is a strong condition that together with $p^*(G_0)$ being an abelian algebra implies, in the more general setting of C*-algebras of Fell bundles, that the bundles have rank 1. Additionally forcing the bundles to be trivial (so that the reduced C*-algebra can be considered an algebra of complex valued functions rather than sections) requires additional conditions that I will not address here (cf. Remark 6.15).

Finally, recall that a groupoid G is *principal* if for any two units $x, y \in G_0$ there is at most one arrow $z \in G_1$ such that $d(z) = x$ and $r(z) = y$ (G is an equivalence relation). If G is also localizable this is closely related to a stronger condition on the quantization map, which is called a *stable quantic bundle* if (4.7) holds for all $V, W \in \mathrm{Max}\, C_r^*(G)$ without any restrictions:

Proposition 4.36 ([47]). *Let G be a localizable groupoid. If G is principal with discrete orbits then its quantization map is a stable quantic bundle. Conversely, if the quantization map is a stable quantic bundle then G is principal.*

5 Observers

This part of the paper is devoted to the question of how classical observers may arise naturally within symmetric measurements spaces, in particular those of the form $\mathrm{Max}\, A$. The aim is to convey the idea that observers can be regarded as being emergent "entities," hence providing us with a realist model of observations in which measurements are fundamental but observers are not. An additional notion in quantale theory is recalled, namely inverse quantal frames, which are the quantales associated to étale groupoids.

5.1 A short digression

Let us begin with a short, and by no means exhaustive, digression about the problematic role of observers in quantum mechanics. Reliance on "macroscopic observers" is ingrained, albeit never in precise terms, in the Copenhagen interpretation (see [30]), which was the de facto interpretation of quantum mechanics during the first half of the twentieth century, but the question of what should be meant by an observer has always been a daunting one. This has even led von Neumann to propose that the measuring process requires the subjective perception of an observer [59]. Certainly due to the absence of a physical understanding of "subjective perception," or of "observer," von Neumann's proposal has in general been greeted with underwhelming enthusiasm. An amusing and eloquent expression of misgivings concerning "measurers" was voiced by Bell [4]:

> "It would seem that the theory is exclusively concerned about 'results of measurement', and has nothing to say about anything else. What exactly qualifies some physical systems to play the role of 'measurer'? Was the wavefunction of the world waiting to jump for thousands of millions of years until a single-celled living creature appeared? Or did it have to wait a little longer, for some better qualified system... with a PhD?"

Despite this, the subjective expectations of observers play a fundamental role in QBism [6, 17], which provides an account of quantum theory based on Bayesian probabilities that in effect can be regarded as an indirect *definition* of what a subjective observer is.

At the opposite end lie the realist variants of quantum mechanics, which are often portrayed as a means to rid quantum theory of the role played by observers [20] — despite the fact that it is possible to introduce observers, more or less explicitly, in realist formulations of quantum theory such as relational quantum mechanics [53] or the topos models of [12–15, 21]. It should be stressed that realist models are often more than just "interpretations," for they carry subtle modifications which should be experimentally testable, at least in principle. See for instance [2].

A different line of research, which takes measurements as basic — albeit in a traditional sense in which "systems" exist — and does not entail modifications of quantum mechanics, stems from Schwinger's notion of selective measurement [54]. Ciaglia et al [7,8] show that Schwinger's measurements can be regarded as forming finite discrete groupoids, which in turn can actually be thought of as corresponding to observers. This will be taken up again in section 6.

5.2 The spin 1/2 example

Let M be a symmetric measurement space, and let $\mathcal{O}$ be an involutive subquantale of M. This is a set of measurements which is closed under composition, disjunctions and reversals, so it is a symmetric measurement space in its own right because, by Proposition 2.5, the subspace topology is necessarily sober.

By restricting to the measurements in $\mathcal{O}$, one is adopting a restricted point of view on what is being observed. This justifies that subquantales such as $\mathcal{O}$ can be regarded as an approximation to the idea of "observer."

As an example, consider the algebra $M_2(\mathbb{C})$ of spin 1/2 measurements. The lattice

$$\begin{tikzcd} & \boldsymbol{z} \arrow[dl, dash] \arrow[dr, dash] & \\ \boldsymbol{z}^{\downarrow} \arrow[dr, dash] & & \boldsymbol{z}^{\uparrow} \arrow[dl, dash] \\ & 0 & \end{tikzcd}$$

is a locale embedded in $M = \operatorname{Max} M_2(\mathbb{C})$, with multiplication equal to $\wedge = \cap$ and trivial involution $V^* = V$, so it is an involutive subquantale $\mathcal{O} \subset M$. Restricting to the measurements in $\mathcal{O}$ corresponds to making measurements of spin only along z, thereby obtaining limited information about what is being observed. In fact, blindly insisting on performing only these measurements might lead to the conclusion that a classical bit is being observed — cf. section 4.4.

However, should the observer know that the system being observed is in fact a qubit, the need to make sense of arbitrary measurements in M in terms of $\mathcal{O}$ arises. In order to see this, suppose $\boldsymbol{x}^{\uparrow}$ has been performed. In terms of $\mathcal{O}$, this means that the spin along z "is" either $\boldsymbol{z}^{\downarrow}$ or $\boldsymbol{z}^{\uparrow}$; that is, $\boldsymbol{x}^{\uparrow}$ translates to the disjunction $\boldsymbol{z} = \boldsymbol{z}^{\downarrow} \vee \boldsymbol{z}^{\uparrow}$, and so does $\boldsymbol{x}^{\downarrow}$, so we obtain the following join preserving map r_z, which translates measurements along x to measurements along z:

$$\begin{tikzcd} & \boldsymbol{x} \arrow[dl, dash] \arrow[dr, dash] \arrow[rrr, mapsto, dotted, "r_z"] & & & \boldsymbol{z} \arrow[dl, dash] \arrow[dr, dash] & \\ \boldsymbol{x}^{\downarrow} \arrow[dr, dash] \arrow[rrrru, mapsto, dotted, "r_z"] & & \boldsymbol{x}^{\uparrow} \arrow[dl, dash] \arrow[rru, mapsto, dotted, "r_z"] & \boldsymbol{z}^{\downarrow} \arrow[dr, dash] & & \boldsymbol{z}^{\uparrow} \arrow[dl, dash] \\ & 0 \arrow[rrr, mapsto, dotted, "r_z"] & & & 0 & \end{tikzcd}$$

Similar translations should exist for all other $V \in M$, and it makes sense to require the translation function to be continuous with respect to the topology of M. Since any $V \in \mathcal{O}$ should of course translate to itself, we should have $r_z(V) = V$ for all $V \in \mathcal{O}$, thus making $\mathcal{O}$ a topological retract of M.

Furthermore, as regards the involutive quantale structure of M, I argue that the preservation of binary joins should be extended to the whole of M, and that also the involution should be preserved. But multiplications clearly should not: for instance, we have $\boldsymbol{x}^{\downarrow}\boldsymbol{x}^{\uparrow} = 0$ but

$$r_z(\boldsymbol{x}^{\downarrow})r_z(\boldsymbol{x}^{\uparrow}) = \boldsymbol{z}\boldsymbol{z} = \boldsymbol{z}\ ,$$

and thus $r_z(\boldsymbol{x}^{\downarrow}\boldsymbol{x}^{\uparrow}) \subsetneq r_z(\boldsymbol{x}^{\downarrow})r_z(\boldsymbol{x}^{\uparrow})$.

A map r_z with these properties can be obtained by considering the linear surjection

$$\Theta_z : M_2(\mathbb{C}) \to \boldsymbol{z}$$

which is defined by restricting matrices to the main diagonal:

$$\Theta_z(A) = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix} .$$

Then r_z is the surjective continuous join preserving mapping

$$r_z : M \to \mathcal{O}$$

which is defined by setting, for all $V \in M$,

$$r_z(V) = \Theta_z(V)\boldsymbol{z} .$$

Despite not preserving multiplications in general, r_z does so in the restricted situation where at least one of the factors lies in $\mathcal{O}$, as will be seen below.

5.3 Main definitions and examples

The previous example suggests the following definition:

Definition 5.1. Let M be a symmetric measurement space. By an *observer* of M will be meant a pair $(\mathcal{O}, r)$ consisting of an involutive subquantale $\mathcal{O}$ of M together with a topological retraction r of M onto $\mathcal{O}$ such that the following conditions are satisfied for all $m, m \in M$:

1. $r(m \vee n) = r(m) \vee r(n)$,
2. $r(m^*) = r(m)^*$,
3. $r(mn) = mr(n)$ if $m \in \mathcal{O}$,
4. $r(mn) = r(m)n$ if $n \in \mathcal{O}$.

If $\mathcal{O}$ is a classical measurement space then $(\mathcal{O}, r)$ is called a *classical observer*, and if $\mathcal{O}$ is a local measurement space then $(\mathcal{O}, r)$ is called a *local observer*. Moreover, an observer $(\mathcal{O}, r)$ is:

- *full* if $1_{\mathcal{O}} = 1_M$ and $r(mn) \leq r(m)r(n)$ for all $m, n \in M$;
- *faithful* if for all $m \in M$ the condition $r(m^*m) = 0$ implies $m = 0$.

Proposition 5.2. *Let M be a symmetric measurement space, and let $(\mathcal{O}, r)$ be an observer. Then r is a retraction of topological $\mathcal{O}$-$\mathcal{O}$-bimodules.*

Proof. The map r preserves binary joins by definition, it preserves 0 because $0 \in \mathcal{O}$ and r is a retraction, and it preserves directed joins because it is continuous. Hence, it preserves arbitrary joins. The $\mathcal{O}$-module actions are given by multiplication in the obvious way, so they are continuous and make both $\mathcal{O}$ and M topological $\mathcal{O}$-$\mathcal{O}$-bimodules. Moreover, r is a homomorphism of bimodules because by definition of observer it preserves the left and the right actions. ☐

Example 5.3. Let G be a second-countable locally compact Hausdorff étale groupoid with quantization map p, and let

$$\mathcal{O} \subset \operatorname{Max} C_r^*(G) \quad \text{and} \quad r : \operatorname{Max} C_r^*(G) \to \mathcal{O}$$

be defined by

$$\mathcal{O} = p^*(\mathcal{O}(G)) \quad \text{and} \quad r = p^* \circ \operatorname{supp}^\circ .$$

From the results of section 4 it immediately follows that $(\mathcal{O}, r)$ is a full classical observer. Moreover, it is a local observer if and only if $G_1 = G_0$, in which case $\mathcal{O} = I(C_0(G_0))$ and $r(V) = VC_0(G_0)$ for each $V \in \operatorname{Max} C_0(G_0)$.

Proposition 5.4. *Let A be a C^*-algebra and B a sub-C^*-algebra, not necessarily abelian. Let also*

$$\Theta : A \to B$$

be a conditional expectation of A onto B. Then the pair $(I(B), r)$ is a local observer of $\operatorname{Max} A$, where $r : \operatorname{Max} A \to I(B)$ is defined for all $V \in \operatorname{Max} A$ by

$$r(V) = B \operatorname{Max} \Theta(V) B \ .$$

Moreover, this observer is faithful if and only if Θ is a faithful conditional expectation.

Proof. The continuity of r is a consequence of the continuity of Θ (for this implies that $\operatorname{Max} \Theta$ is continuous) and of the continuity of the multiplication by B in $\operatorname{Max} A$. Then r is a retraction because

$$r(J) = B\Theta(J)B = BJB = J$$

for all closed two-sided ideals $J \in I(B)$. Moreover, the closed two-sided ideals are self-adjoint, and it is easy to see that for all V in $\operatorname{Max} A$ we have $r(V^*) = r(V)$, so r is an involution preserving sup-lattice homomorphism. Moreover, $\operatorname{Max} \Theta$ is a homomorphism of $\operatorname{Max} B$-$\operatorname{Max} B$-bimodules because Θ is a homomorphism of B-B-modules, and thus by restricting the actions it is a homomorphism of $I(B)$-$I(B)$-bimodules. And the mapping $q : \operatorname{Max} B \to I(B)$ which is given by $V \mapsto BVB$ is a homomorphism of $I(B)$-$I(B)$-bimodules because $I(B)$ is a comutative quantale: for all $J \in I(B)$ and $V \in \operatorname{Max} B$ we obtain

$$\begin{aligned} q(VJ) &= BVJB = BVBJ = q(V)J \\ q(JV) &= BJVB = JBVB = Jq(V) \ . \end{aligned}$$

Therefore $r = q \circ \operatorname{Max} \Theta$ is a homomorphism of bimodules.

The multiplication of $I(B)$ coincides with $\wedge = \cap$, and the involution on $I(B)$ is trivial, so we have a local observer because $I(B)$ is a locally compact locale.

Finally, if Θ is faithful and for some $V \in \operatorname{Max} A$ we have $r(V^*V) = \{0\}$ then $B \operatorname{Max} \Theta(V^*V)B = \{0\}$ implies $\operatorname{Max} \Theta(V^*V) = \{0\}$ because B has an approximate unit, and thus for all $a \in V$ we have $\Theta(a^*a) = 0$, which implies $a = 0$. Hence, $V = 0$. Conversely, if the observer is faithful and $\Theta(a^*a) = 0$ then $r(\langle a\rangle^*\langle a\rangle) = \{0\}$, and thus $\langle a\rangle = \{0\}$, so Θ is faithful. ☐

The spin 1/2 example of the previous section is a consequence of the previous proposition and, more generally, so is the following:

Example 5.5. Let G be a second-countable locally compact Hausdorff étale groupoid with quantization map p, and let $\Theta : C_r^*(G) \to p^*(G_0)$ be the restriction map defined by $\Theta(f) = f|_{G_0}$ for all $f \in C_r^*(G)$. This is a faithful conditional expectation from $C_r^*(G)$ onto the abelian subalgebra $p^*(G_0) \cong C_0(G_0)$, so a faithful local observer $(\mathcal{O}_0, r_0)$ is defined by

$$\begin{aligned} \mathcal{O}_0 &= p^*(\Omega(G_0)) \ , \\ r_0(V) &= \operatorname{Max} \Theta(V) p^*(G_0) \quad \text{for all } V \in \operatorname{Max} C_r^*(G) \ . \end{aligned}$$

5.4 Adjoint embeddings and localizable observers

The fact that the retraction map r of an observer $(\mathcal{O}, r)$ preserves arbitrary joins means that it has a right adjoint, so we define:

Definition 5.6. Let M be a symmetric measurement space, and let $(\mathcal{O}, r)$ be an observer. The *adjoint embedding* of the observer is the right adjoint of r:

$$r_* : \mathcal{O} \to M \ .$$

Proposition 5.7. *Let M be a symmetric measurement space, and let $(\mathcal{O}, r)$ be an observer. The adjoint embedding $r_* : \mathcal{O} \to M$ satisfies the following properties.*

1. *r_* preserves arbitrary meets (in particular $r_*(1_{\mathcal{O}}) = 1_M$).*
2. *The image $r_*(\mathcal{O})$ is a locale isomorphic to $\mathcal{O}$.*
3. *$a \leq r_*(a)$ for all $a \in \mathcal{O}$.*
4. *$r_*(a^*) = r_*(a)^*$ for all $a \in \mathcal{O}$.*
5. *$r_*(a)r_*(b) \leq r_*(ab)$ for all $a, b \in \mathcal{O}$ if the observer is full.*
6. *The following conditions are equivalent:*
   (a) *$r_*(a) = a$ for all $a \in \mathcal{O}$,*
   (b) *$r_*(\mathcal{O}) = \mathcal{O}$,*
   (c) *$m \leq r(m)$ for all $m \in M$,*
   (d) *r is a closure operator on M,*
   (e) *$\mathcal{O}$ is closed under arbitrary meets in M and for all $m \in M$ we have*

$$r(m) = \bigwedge\{a \in \mathcal{O} \mid m \leq a\} \ .$$

Proof. (1) is equivalent to r_* being a right adjoint map.

(2): r_* is injective because its left adjoint is surjective, so the image of r_* is a locale.

(3) is equivalent to the condition $r(a) \leq a$, which holds because r is a retraction.

(4) holds because the left adjoint r preserves the involution, and (5) holds because, for a full observer, r satisfies the law $r(mn) \leq r(m)r(n)$.

(6a)⇒(6b): Obvious.

(6b)⇒(6c): Assume that $r_*(\mathcal{O}) = \mathcal{O}$, and let $m \in M$. Then $r_*(r(m)) \in \mathcal{O}$, so $r(r_*(r(m))) = r_*(r(m))$. But the adjunction between r and r_* implies that $r \circ r_* \circ r = r$, and thus $r_*(r(m)) = r(m)$. Hence, since the inequality $r_* \circ r \geq \mathrm{id}$ is the unit of the adjunction, we obtain $m \leq r(m)$.

(6c)⇒(6a): Assume that $m \leq r(m)$ for all $m \in M$, and let $a \in \mathcal{O}$. Then, making $m = r_*(a)$, we obtain $r_*(a) \leq r(r_*(a)) \leq a$ due to the inequality $r \circ r_* \leq \mathrm{id}$, which is the counit of the adjunction. Hence, from (3) we obtain $r_*(a) = a$.

(6c) $\iff$ (6d): r is idempotent and monotone, so it is a closure operator on M if and only if $m \leq r(m)$ for all $m \in M$.

(6d) $\iff$ (6e): This is a basic property of closure operators because $\mathcal{O}$ is the set of fixed points of r.

□

Definition 5.8. Let M be a symmetric measurement space. An observer $(\mathcal{O}, r)$ will be called *localizable* if the equivalent conditions of Proposition 5.7(6) hold.

Obviously, the name "localizable" is taken from the analogous terminology for groupoids:

Example 5.9. Let G be a second-countable locally compact Hausdorff étale groupoid with quantization map $p : \operatorname{Max} C_r^*(G) \to \mathcal{O}(G)$, and let $(\mathcal{O}, r)$ be the full classical observer obtained by taking $\mathcal{O} = p^*(\mathcal{O}(G))$ and $r = p^* \circ \operatorname{supp}^\circ$, as in Example 5.3. The map γ of Remark 4.27 is the right adjoint of $\operatorname{supp}^\circ$ and thus $\gamma \circ p_*$ is the adjoint embedding of the observer. Moreover, the observer is localizable if and only if G is a localizable groupoid.

5.5 Inverse quantal frames

If Q is a unital involutive quantale with unit e then by a *partial unit* of Q is meant an element $s \in Q$ such that

$$s^* s \leq e \text{ and } s s^* \leq e \ ,$$

and the set of all the partial units of Q is denoted by $\mathcal{I}(Q)$ [45]. For instance, if $Q = \mathcal{O}(G)$ for an étale groupoid G we have $e = G_0$ and the partial units of Q are the open bisections of G; that is, $\mathcal{I}(Q) = \mathcal{I}(G)$.

Definition 5.10. [45, 48] By an *inverse quantal frame* is meant a unital stably Gelfand quantale which is also a locale (with $\wedge$ different from multiplication in general), and which is covered by its partial units:

$$\bigvee \mathcal{I}(\mathcal{O}) = 1 \ .$$

The *base locale* of an inverse quantal frame Q is the principal ideal

$$Q_0 := \downarrow(e)$$

(this is a locale with multiplication coinciding with $\wedge$); and Q is said to be *spatial* if Q_0 is spatial, and *locally compact* if Q_0 is locally compact.

From the results of [45] it follows that the inverse quantal frames are precisely the quantales associated to *localic* étale groupoids (which we shall not need in this paper), whereas the spatial inverse quantal frames are the quantales of the form $\mathcal{O}(G)$ for a topological étale groupoid G, and the locally compact inverse quantal frames are the quantales $\mathcal{O}(G)$ for a locally compact groupoid G. Hence, a classical measurement space is of the form $\mathcal{O}(G)$ for a locally compact étale groupoid G if and only if it is a unital quantale covered by its partial units.

Definition 5.11. Let M be a symmetric measurement space. A classical observer $(\mathcal{O}, r)$ will be called *étale* if $\mathcal{O}$ is an inverse quantal frame. If M itself is a classical measurement space and an inverse quantal frame then it is said to be an *étale measurement space.*

Proposition 5.12. *Let M be a symmetric measurement space, and $(\mathcal{O}, r)$ an étale observer. Then $(\mathcal{O}_0, r_0)$ is a local observer of M, where for all $m \in M$ we define*

$$r_0(m) = r(m) \wedge e \ .$$

Proof. Let G be a sober locally compact étale groupoid such that $\mathcal{O} \cong \mathcal{O}(G)$ (this determines G up to isomorphism). Then $\mathcal{O}_0$ is a locally compact locale itself, hence a local measurement space. The mapping $(-) \wedge e : \mathcal{O} \to \mathcal{O}_0$ preserves joins and thus it is Scott continuous. Hence, r_0, which is the composition of this map with r, is continuous, and obviously a retraction. Now let $m \in \mathcal{O}$ and $n \in \mathcal{O}_0$. The properties of inverse quantal frames [45] imply that

$$mn \wedge e = (m \wedge e)n \quad \text{and} \quad nm \wedge e = n(m \wedge e) \ ,$$

so the mapping $(-)\wedge$ is a homomorphism of $\mathcal{O}_0$-$\mathcal{O}_0$-bimodules. Therefore r_0 is a homomorphism of $\mathcal{O}_0$-$\mathcal{O}_0$-bimodules because r is a homomorphism of $\mathcal{O}$-$\mathcal{O}$-bimodules. □

Example 5.13. Let G be a second-countable locally compact Hausdorff étale groupoid, let $M = \operatorname{Max} C_r^*(G)$, and let $(\mathcal{O}, r)$ be the observer of Example 5.3. The local observer $(\mathcal{O}_0, r_0)$ of the previous proposition coincides with that of Example 5.5.

5.6 Comparing classical observers

Let us see a way of relating different classical observers which depends on nothing but the fact that the associated retractions are sup-lattice homomorphisms.

Let M be a complete measurement space, and let (L_p, r_p) and (L_q, r_q) be classical observers. If we restrict r_p to L_q we obtain a join preserving mapping

$$L_q \to L_p \ .$$

This is equivalent to a homomorphism of locales

$$\mathsf{P_L}(L_q) \to L_p \ ,$$

which in turn defines a map of locales

$$L_p \to \mathsf{P_L}(L_q) \ .$$

So applying the spectrum functor we obtain a continuous map that sends each point of L_p to a closed set of points of L_q:

$$\boldsymbol{c} : \Sigma(L_p) \to \mathsf{C}(\Sigma(L_q)) \ . \tag{5.1}$$

This continuous map can be regarded as a topological version of change of basis of a vector space: here each "basis vector" $x \in \Sigma(L_p)$ is a "combination" of the "basis vectors" contained in $\boldsymbol{c}(x)$.

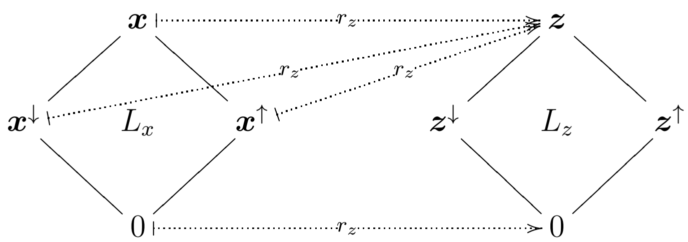
Example 5.14. Let us compute a map $\boldsymbol{c}$ for the local observers of the spin example that we have been using. Letting $\Theta_z : M_2(\mathbb{C}) \to \boldsymbol{z}$ be the map that restricts matrices to the main diagonal,

$$\Theta_z(A) = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix} ,$$

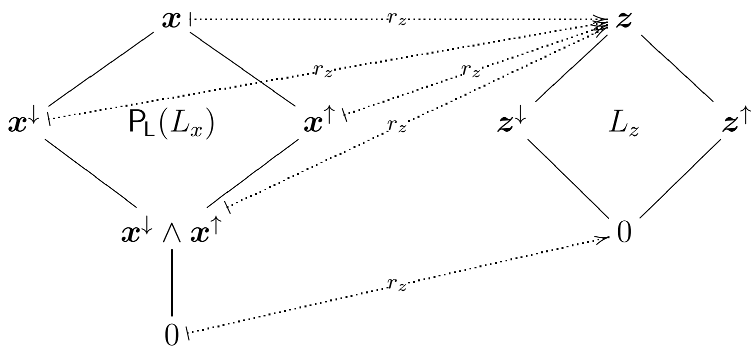
we define $r_z(V) = \Theta_z(V)\boldsymbol{z}$ for all $V \in \operatorname{Max} M_2(\mathbb{C})$. Hence, since

$$\Theta_z(\boldsymbol{x}^{\downarrow}) = \Theta_z(\boldsymbol{x}^{\uparrow}) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a \in \mathbb{C} \right\} ,$$

it follows that $r_z(\boldsymbol{x}^{\downarrow}) = r_z(\boldsymbol{x}^{\uparrow}) = \boldsymbol{z}$, so we obtain the following sup-lattice homomorphism between locales L_x and L_z (cf. section 5.2):



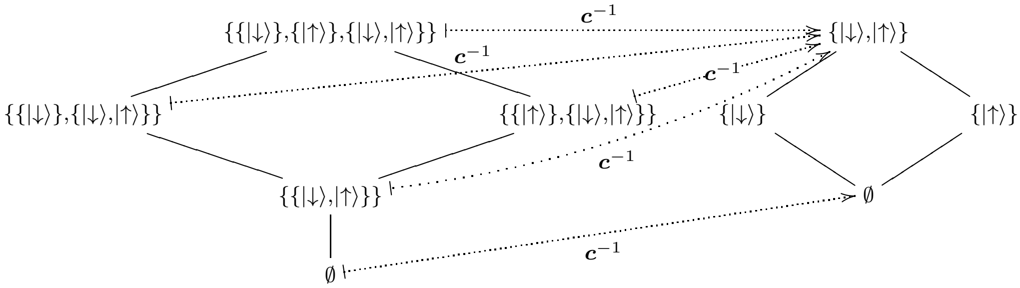
This yields a homomorphism of locales with domain $\mathsf{P_L}(L_x)$:



In order to compute the map $\boldsymbol{c}$ first look at L_x and L_y as being isomorphic to the topology of a discrete two point space $X = \{|\downarrow\rangle, |\uparrow\rangle\}$ and then use the homeomorphism

$$\Sigma(\mathsf{P_L}(\Omega(X))) \cong \mathsf{C}(X) .$$

Then r_z corresponds to $\boldsymbol{c}^{-1}$ in the following diagram:



Hence, the map $\boldsymbol{c}$ is defined by

$$\boldsymbol{c}(|\downarrow\rangle) = \boldsymbol{c}(|\uparrow\rangle) = \{|\downarrow\rangle, |\uparrow\rangle\} .$$

Regarding $\Sigma(L_x)$ and $\Sigma(L_z)$ as bases of eigenvectors of the spin observables $S_x = \frac{\hbar}{2}\sigma_x$ and $S_z = \frac{\hbar}{2}\sigma_z$, respectively, $\boldsymbol{c}$ tells us that each eigenvector of S_z is a superposition of all the eigenvectors of S_x, but without specifying probability amplitudes: a state either does or does not belong to a superposition.

Therefore it is easy to conclude that exactly the same map $\boldsymbol{c}$ would have been obtained if instead of the x direction we had chosen a small angle $\theta \neq 0$ with respect to the z axis. For instance, doing this in the xz plane the corresponding spin observable is

$$S_\theta = \frac{\hbar}{2}(\sigma_x, \sigma_y, \sigma_z) \cdot (\sin\theta, 0, \cos\theta) = \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix},$$

and the measurements in the θ direction give us the locale

$$L_\theta \quad = \quad \begin{tikzcd}[column sep=small] & \boldsymbol{\theta} \arrow[dl, dash] \arrow[dr, dash] & \\ \boldsymbol{\theta}^{\downarrow} \arrow[dr, dash] & & \boldsymbol{\theta}^{\uparrow} \arrow[dl, dash] \\ & 0 & \end{tikzcd}$$

such that $\boldsymbol{\theta} = \langle I, S_\theta\rangle$ and the atoms are spanned by the matrices that project onto the eigenspaces of S_θ:

$$\begin{aligned}
\boldsymbol{\theta} &= \left\{\begin{pmatrix} a + b\cos\theta & b\sin\theta \\ b\sin\theta & a - b\cos\theta \end{pmatrix} \mid a, b \in \mathbb{C}\right\} \\
\boldsymbol{\theta}^{\downarrow} &= \mathbb{C}\begin{pmatrix} \sin^2\frac{\theta}{2} & -\frac{1}{2}\sin\theta \\ -\frac{1}{2}\sin\theta & \cos^2\frac{\theta}{2} \end{pmatrix} \\
\boldsymbol{\theta}^{\uparrow} &= \mathbb{C}\begin{pmatrix} \cos^2\frac{\theta}{2} & \frac{1}{2}\sin\theta \\ \frac{1}{2}\sin\theta & \sin^2\frac{\theta}{2} \end{pmatrix}.
\end{aligned}$$

Hence, for small angles $\theta \neq 0$ the restriction of r_z to L_θ is analogous to that of L_x, so $\boldsymbol{c}$ is the same. This shows that there is a discontinuity as θ approaches 0, for if $\theta = 0$ the restriction of r_z is the identity on L_z, and thus $\boldsymbol{c}(|{\downarrow}\rangle) = \{|{\downarrow}\rangle\}$ and $\boldsymbol{c}(|{\uparrow}\rangle) = \{|{\uparrow}\rangle\}$.

In order to remove this discontinuity we would need to consider the complex measures on $\Sigma(L_x)$ and $\Sigma(L_\theta)$ that contain the probability amplitudes for each eigenstate, whereas $\boldsymbol{c}$ gives us only the supports of the measures.

5.7 An example with spin 1

Let us consider the experiment of section 3.1, but now measuring the spin of a silver atom. The measurement done in the z direction with a closed box will be denoted by $\mathbf{Ag}_z$, whereas the three basic measurements in the same direction, but with the open box, will be denoted by $\mathbf{Ag}_z^-$, $\mathbf{Ag}_z^0$, and $\mathbf{Ag}_z^+$, in correspondence with the three possible recordings on the target.

These measurements can be represented in $\operatorname{Max} M_3(\mathbb{C})$ by taking $\mathbf{Ag}_z$ to be the maximal abelian subalgebra of diagonal matrices in $M_3(\mathbb{C})$, which is generated by the spin observable S_z and the identity matrix I, and the three other measurements are one dimensional subspaces spanned by projection matrices:

$$
\begin{aligned}
\mathbf{Ag}_z &= D_3(\mathbb{C}) \\
\mathbf{Ag}_z^- &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbb{C} \\
\mathbf{Ag}_z^0 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mathbb{C} \\
\mathbf{Ag}_z^+ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mathbb{C} \,.
\end{aligned}
$$

Moreover they are contained in a locale L_z, namely the following Boolean sublattice of $\operatorname{Max} M_3(\mathbb{C})$:

$$
\begin{tikzcd}
 & \mathbf{Ag}_z \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\mathbf{Ag}_z^- \vee \mathbf{Ag}_z^0 \arrow[d, dash] \arrow[dr, dash] & \mathbf{Ag}_z^- \vee \mathbf{Ag}_z^+ \arrow[dl, dash] \arrow[dr, dash] & \mathbf{Ag}_z^0 \vee \mathbf{Ag}_z^+ \arrow[dl, dash] \arrow[d, dash] \\
\mathbf{Ag}_z^- \arrow[dr, dash] & \mathbf{Ag}_z^0 \arrow[d, dash] & \mathbf{Ag}_z^+ \arrow[dl, dash] \\
 & 0 &
\end{tikzcd}
$$

The locale L_z contains the three two dimensional measurements that arise as binary joins of the one dimensional ones. Such superpositions do not arise as actual measurements that can be performed using our toy experimental setup, so we may simply regard them as a formal consequence of having completed the "true" measurement space. Alternatively, a clever setup that allows us to conceal the outcomes of only two beams at a time can be devised, for instance by composing the apparatus with an "inverted" Stern–Gerlach device that recombines only two of the beams. In this way, for instance, the binary join $\mathbf{Ag}_z^- \vee \mathbf{Ag}_z^0$ corresponds to a measurement where no hit is recorded on the target that intersects the spin $+1$ beam.

An observer (L_z, r_z) can be obtained, similarly to the spin-$\frac{1}{2}$ case, by defining r_z from the conditional expectation $\Theta_z : M_3(\mathbb{C}) \to \mathbf{Ag}_z$ that restricts matrices to their main diagonal:

$$
\Theta_z(A) = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix} .
$$

Now let us compare this observer to the analogous one for spin measurements in the x direction. Again we have a locale $L_x \subset \mathrm{Max}\, M_3(\mathbb{C})$,

$$\begin{tikzcd}[column sep=small]
 & \mathbf{Ag}_x \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\mathbf{Ag}_x^- \vee \mathbf{Ag}_x^0 \arrow[d, dash] \arrow[dr, dash] & \mathbf{Ag}_x^- \vee \mathbf{Ag}_x^+ \arrow[dl, dash] \arrow[dr, dash] & \mathbf{Ag}_x^0 \vee \mathbf{Ag}_x^+ \arrow[dl, dash] \arrow[d, dash] \\
\mathbf{Ag}_x^- \arrow[dr, dash] & \mathbf{Ag}_x^0 \arrow[d, dash] & \mathbf{Ag}_x^+ \arrow[dl, dash] \\
 & 0 &
\end{tikzcd}$$

where $\mathbf{Ag}_x$ is generated as a subalgebra of $M_3(\mathbb{C})$ by the identity matrix and the observable

$$S_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} .$$

And the three measurements of dimension one are spanned by the matrices that project onto the spans of the x-spinors

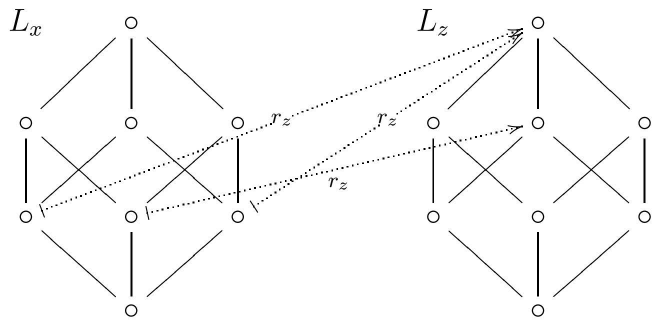
$$\begin{aligned}
|-\rangle_x &= \frac{1}{2} \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} \\
|0\rangle_x &= \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \\
|+\rangle_x &= \frac{1}{2} \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} ,
\end{aligned}$$

so we have

$$\begin{aligned}
\mathbf{Ag}_x^- &= |-\rangle_x \langle -|_x \mathbb{C} = \begin{pmatrix} 1 & -\sqrt{2} & 1 \\ -\sqrt{2} & 2 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix} \mathbb{C} \\
\mathbf{Ag}_x^0 &= |0\rangle_x \langle 0|_x \mathbb{C} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} \mathbb{C} \\
\mathbf{Ag}_x^+ &= |+\rangle_x \langle +|_x \mathbb{C} = \begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 2 & \sqrt{2} \\ 1 & \sqrt{2} & 1 \end{pmatrix} \mathbb{C} .
\end{aligned}$$

The restriction of r_z to L_x yields a mapping which, contrasting with the spin-$\frac{1}{2}$ examples, does not send all the atoms to the top element, for we have

$r_z(\mathbf{Ag}_x^0) = \mathbf{Ag}_z^- \vee \mathbf{Ag}_z^+$:



Hence, identifying both locales with the topology of the discrete space

$$X = \{|-\rangle, |0\rangle, |+\rangle\} ,$$

the continuous map $\boldsymbol{c} : X \to \mathsf{C}(X)$ such that $\boldsymbol{c}^{-1} \cong r_z$ is given by

$$\begin{aligned} \boldsymbol{c}(|-\rangle) &= X \\ \boldsymbol{c}(|0\rangle) &= \{|-\rangle, |+\rangle\} \\ \boldsymbol{c}(|+\rangle) &= X . \end{aligned}$$

Comparing this with the coordinates of the x-spinors, again we see that $\boldsymbol{c}$ gives us the supports of the appropriate complex measures on X.

6 Selective measurements

Now let us look at Schwinger's selective measurements, borrowing from the insights of Ciaglia et al. This will lead to pseudogroups as being an appropriate locus for selective measurements, and to a natural way in which such measurements are associated to projections of symmetric measurement spaces. These pseudogroups are related to the classical observers seen in section 5, and here a preliminary study of a notion of equivalence of observers is obtained in terms of Morita equivalence of both pseudogroups and inverse quantal frames. This includes a few simple new results about Morita equivalence of pseudogroups in symmetric measurement spaces.

6.1 Schwinger's picture

Schwinger [54] proposed a formulation of quantum mechanics in terms of a notion of "selective measurement," based on an idealized notion of physical system for which each physical quantity assumes only a finite set of distinct values. If A is such a physical quantity, $M(a)$ is the measurement that selects those systems whose value of that quantity is a and rejects all others. He also considers a more general type of measurement $M(a', a)$ which disturbs the systems being measured. It selects the systems whose value of A is a, after which those systems emerge in a new state for which the value is a'. So $M(a)$ is identified with $M(a, a)$, and a natural notion of composition of such measurements is given by a multiplication operation such that

$$\begin{aligned} M(a'',a')M(a',a) &= M(a'',a) \\ \text{and} \quad M(a''',a'')M(a',a) &= 0 \quad \text{if} \quad a'' \neq a' \ , \end{aligned}$$

where 0 is the measurement that rejects all systems. (The latter plays a similar role to that of the impossible measurement of measurement spaces.)

Schwinger also defines sums of measurements, for which he gives the following intuitive description:

> "We define the addition of such symbols to signify less specific selective measurements that produce a subensemble associated with any of the values in the summation, none of these being distinguished by the measurement."

The expression "less specific" brings to mind the interpretation of the specialization order in measurement spaces, suggesting that a sum of two selective measurements $M(a',a)$ and $M(a''',a'')$ might be represented by a join

$$M(a',a) \vee M(a''',a'') \ .$$

However, in Schwinger's formulation the sum is that of a $\mathbb{C}$-algebra, and it satisfies the axiom

$$\sum_{a\in X_A} M(a) = 1 \ ,$$

where 1 is said to be the measurement that accepts all systems and X_A is the set of possible values of the physical quantity A.

As pointed out by Ciaglia et al [8], the symbols $M(a',a)$ can be identified with arrows of a discrete groupoid, namely the pair groupoid $G_A = \mathrm{Pair}(X_A)$,

$$X_A \times X_A \times X_A \cong (G_A)_2 \xrightarrow{m\cong\pi_{1,3}} (G_A)_1 = \overset{(a',a)\overset{i}{\mapsto}(a,a')}{\overset{\curvearrowright}{X_A \times X_A}} \underset{r=\pi_1}{\overset{d=\pi_2}{\rightrightarrows}} \!\!\xleftarrow{u=\Delta}\! X_A = (G_A)_0 \ \ ,$$

so Schwinger's collection of measurements can be taken to be the groupoid algebra $\mathfrak{A} = \mathbb{C}G_A \cong M_{|X_A|}(\mathbb{C})$, which has unit 1.

Remark 6.1. Joins of measurements can still be defined in the measurement space $\mathrm{Max}\,\mathfrak{A}$, but note that $\bigvee_{a\in X_A}\langle M(a)\rangle$ is the commutative algebra of diagonal matrices rather than $\langle 1\rangle$. Note also that we may identify the groupoid algebra with the convolution algebra $C_c(G_A) = C(G_A)$, and that G_A has a quantization map

$$p : \mathrm{Max}\,\mathfrak{A} \to \mathcal{O}(G_A)$$

in the sense of Definition 4.26. The image $\mathcal{O} = p^*\big(\mathcal{O}(G_A)\big)$ together with the map

$$p^* \circ \mathrm{supp} : \mathrm{Max}\,\mathfrak{A} \to \mathcal{O}$$

is a localizable classical observer of $\mathrm{Max}\,\mathfrak{A}$ (cf. Example 5.3) to which $\langle 1\rangle$ does not belong, and we have $\bigvee_{a\in X_A}\langle M(a)\rangle = p^*\big((G_A)_0\big)$. We shall return to such observers in section 6.4.

If B is another physical quantity there is another pair groupoid G_B, and a corresponding algebra $\mathfrak{B} = \mathbb{C}G_B$. If both A and B correspond to complete families of observables we should have $\mathfrak{A} \cong \mathfrak{B}$. So G_B can also be embedded into $\mathfrak{A}$, and we can regard G_A and G_B as two different "bases" of $\mathfrak{A}$.

Finally, it is also possible to have measurements $M(b, a)$, where a is a value of the physical quantity A and b is a value of B. An example of such a situation can be a measurement of spin that accepts particles with positive spin along x and changes their state to, say, positive spin along z. Ciaglia et al [8] call such measurements Stern–Gerlach measurements. They further argue that these can be used in order to define a 2-groupoid whose 1-cells are the selective measurements. This structure sums up the kynematical aspects of Schwinger's measurements.

Quantum amplitudes can be defined to be complex valued functions on G_A. These give us the convolution algebra $C(G_A)$, which in this case is isomorphic to $\mathbb{C}G_A$ because G_A is a finite groupoid but in fact should be regarded as being dual to $\mathbb{C}G_A$, with the duality expressed by the pairing function

$$\langle -, - \rangle : C(G_A) \times \mathbb{C}G_A \to \mathbb{C}$$

which is defined by

$$\left\langle f, \sum_i \lambda_i M(a'_i, a_i) \right\rangle = \sum_i \lambda_i f\big(M(a'_i, a_i)\big) \ .$$

Based on the algebra of amplitudes Ciaglia et al [9] handle the dynamical aspects of Schwinger's formulation, and they address the statistical interpretation in [10].

6.2 Pseudogroups

Symmetric pseudogroups. Let A be a physical quantity with possibly infinitely many values that lie in a topological space X. One of the general ideas in this paper is that it is open sets, rather than points, that correspond to observable entities. Hence, a natural generalization of the selective measurements of the form $M(a)$ is to consider instead measurements $M(U)$, where U is an open set of X. Such a measurement is meant to select only those systems that lie in U. Similarly, the selective measurements $M(a, a')$, which correspond to elements of $X \times X$, are not necessarily "physical" and should be replaced by open sets of $X \times X$. However, a question is which topology should be considered on $X \times X$. One possibility is the product topology. This gives us the pair groupoid $\mathrm{Pair}(X)$, which is not étale unless X is a discrete space. So, if we take the product topology, the open sets of X (and X itself) cannot in general be regarded as open sets of the pair groupoid, which means that selective measurements such as the above $M(U)$ do not exist. Instead, any open set V of the pair groupoid that contains U must contain nonidentity arrows, which can be interpreted as saying that it is impossible to perform a "pure selection" of the systems whose value of A lies in U without actually disturbing the systems.

Hence, if we want to allow for the inclusion of nonperturbing selective measurements in our description, we need X to be an open set of the pair groupoid, which means we need to consider a topology that makes the pair groupoid étale. In order to do this, given two open sets $U, V \subset X$ and a homeomorphism

$f : V \to U$ define $M(f)$ to be the selective measurement that selects systems in V and carries them onto U according to the rule specified by f — so $M(f)$ is the graph of f:

$$M(f) := \{(x, y) \in X \times X \mid y \in V \text{ and } x = f(y)\} .$$

These subsets form a basis for a topology on $X \times X$. The groupoid G thus obtained is étale, similarly to the pair groupoid we have $G_0 \cong X$, and the open bisections of G are precisely the sets $M(f)$. Let us fix some terminology and notation:

Definition 6.2. Let X be a topological space. The *étale pair groupoid of X* is the étale groupoid $\mathrm{Pair}^{\circ}(X)$ whose space of objects is X and whose set of arrows is $X \times X$, whose structure maps are the same as those of the usual pair groupoid, but whose topology (on $X \times X$) is generated by the (graphs) of partial homeomorphisms between open subsets of X.

Note that if X is a locally compact space then $\mathrm{Pair}^{\circ}(X)$ is a locally compact groupoid, since $X \times X$ is covered by open sets which are homeomorphic to open sets of X. Similarly, if X is Hausdorff then $\mathrm{Pair}^{\circ}(X)$ is locally Hausdorff, and if X is a manifold (i.e., locally Euclidian and Hausdorff) then so is $\mathrm{Pair}^{\circ}(X)$.

The above construction allows us to shift our attention from the groupoid itself to the measurements $M(f)$, which form a semigroup whose multiplication is defined by

$$M(f)M(g) = \{(x, y) \in X \times X \mid y \in \mathrm{dom}(g),\ g(y) \in \mathrm{dom}(f),\ x = f(g(y))\} .$$

This is an inverse semigroup $\mathcal{I}(X)$, called the *symmetric pseudogroup* on X. Its elements are partially ordered by restriction on their domains, the inverses are given by $M(f)^{-1} = M(f^{-1})$, and the idempotents, which are the identity maps on open subsets of X, form a locale $E(\mathcal{I}(X)) \cong \Omega(X)$.

Abstract pseudogroups. Let X be a topological space. A *pseudogroup of transformations* on X is a subsemigroup $S \subset \mathcal{I}(X)$ that contains all the idempotents of $\mathcal{I}(X)$ and is closed under inverses, and besides is complete in the sense that if $M(f) = \bigvee_i M(f_i)$ in $\mathcal{I}(X)$ and all the $f_i's$ are in S then so is $M(f)$ (this is a sheaf condition). More generally, a pseudogroup of transformations is an instance of the notion of (abstract) *pseudogroup*, by which is meant a complete and infinitely distributive inverse semigroup. A general exposition of inverse semigroups can be found in [31], but in the context of the present paper a simple way to define pseudogroups is to take advantage of inverse quantal frames:

Proposition 6.3 ([45]). *Every pseudogroup is isomorphic as a semigroup, and also order-isomorphic, to the semigroup $\mathcal{I}(Q)$ of partial units of an inverse quantal frame Q. Moreover, Q is unique up to isomorphism.*

The set of idempotents $E(S)$ of a pseudogroup S is necessarily a locale, for if $S = \mathcal{I}(Q)$ for an inverse quantal frame Q then $E(S) = Q_0$. Accordingly, I shall say that S is *spatial* (respectively *locally compact*) if $E(S)$ is a spatial (respectively locally compact) locale.

Given a pseudogroup S, two elements $s, t \in S$ are said to be *compatible* if $st^* \in E(S)$ and $s^*t \in E(S)$; and a *compatible set* is a subset $C \subset S$ for which any two elements $s, t \in C$ are compatible. If $S = \mathcal{I}(Q)$ then the compatible sets are precisely those $C \subset S$ such that $\bigvee C \in S$. Hence, for an arbitrary pseudogroup the compatible sets are precisely the subsets that have joins.

One particular construction of an inverse quantal frame from a pseudogroup S is the quantale $\mathcal{L}^\vee(S)$ of *compatible ideals*, which are the downsets $J \subset S$ that are closed under the formation of joins of compatible sets $C \subset J$. In particular, the principal ideals are compatible. Often the actual construction is not relevant, but it is useful to recall its universal property: for all involutive quantales Q and semigroup homomorphisms $f : S \to Q$ that preserve the joins of compatible subsets of S there is a unique homomorphism of unital involutive quantales

$$f^\sharp : \mathcal{L}^\vee(S) \to Q$$

such that for all $s \in S$ we have $f^\sharp(\downarrow(s)) = f(s)$:

$$\begin{tikzcd}
S \arrow[r, "\downarrow(-)"] \arrow[dr, "f"'] & \mathcal{L}^\vee(S) \arrow[d, "f^\sharp"] \\
& Q
\end{tikzcd}$$

Moreover, the category of inverse quantal frames is equivalent to that of pseudogroups, with the functor $\mathcal{L}^\vee$ being "inverse" to $\mathcal{I}$, and the equivalence restricts to the respective full subcategories of spatial and locally compact objects. For these and other details about pseudogroups and inverse quantal frames see [45] (where pseudogroups were called abstract complete pseudogroups).

From the above facts it follows that any sober étale groupoid G is determined up to isomorphism by its spatial pseudogroup of open bisections $\mathcal{I}(G)$. A concrete construction of the étale groupoid G associated to a spatial pseudogroup S consists of taking arrows to be germs of elements of S at points of the locale $E(S)$. This is analogous to the construction of the space of stalks of a sheaf on a topological space, and the resulting groupoid is denoted by $\mathrm{Germs}(S)$. For general inverse semigroups this is well known [27, 39], and specific details for pseudogroups can be found in [33].

6.3 Measurement space approach

Selective measurements. If one considers a particular symmetric measurement space M to be the space of all the physically meaningful measurements, where are the groupoids of selective measurements to be found? As seen in the previous subsection, a possibility is that these groupoids can be étale pair groupoids $G = \mathrm{Pair}^\circ(X)$, but the actual points and arrows of the groupoids cannot in general be "observed." Instead, I have argued that the measurements can be taken to be the open bisections, which yield symmetric pseudogroups $\mathcal{I}(X)$. Note that for each $f \in \mathcal{I}(X)$ the following conditions hold (the last two are actually equalities):

1. $f^*f \leq \mathrm{id}_X$,

2. $ff^* \leq \mathrm{id}_X$,

3. $f\mathrm{id}_X \leq f$,

4. $\mathrm{id}_X f \leq f$.

To motivate the construction I shall use, let us consider first $M = \operatorname{Max} A$ for a C*-algebra A. Now the space X is replaced by an abelian sub-C*-algebra $B \subset A$ of commuting observables (think of this as $p^*(G_0)$ where p is the quantization map of G), and the "symmetric pseudogroup on B" is defined by mimicking the four conditions above; that is, it is the set $\mathcal{I}(B)$ of those $V \in \operatorname{Max} A$ such that

1. $V^*V \subset B$,

2. $VV^* \subset B$,

3. $VB \subset V$,

4. $BV \subset V$.

More generally, if M is a symmetric measurement space and $b \in Q$ is a projection (i.e., $b = bb = b^*$), we define $\mathcal{I}(b)$ to consist of those measurements $m \in M$ such that

1. $m^*m \leq b$,

2. $mm^* \leq b$,

3. $mb \leq m$,

4. $bm \leq m$.

It is easy to see that $\mathcal{I}(b)$ is closed under multiplication and involution in M. Furthermore, with the order inherited from M it is actually a pseudogroup [48].

Example 6.4. Let G be a second-countable locally compact Hausdorff étale groupoid with quantization map p. Then $p^*(G_0)$ is an abelian sub-C*-algebra of $C_r^*(G)$, and the inverse image homomorphism of p restricts to an injective homomorphism of pseudogroups

$$p^* : \mathcal{I}(G) \to \mathcal{I}(p^*(G_0)) \ .$$

For some groupoids this is an isomorphism, so in those cases the construction of pseudogroups from projections just described enables one to recover a groupoid from its groupoid C*-algebra. This is the case at least when G is localizable and effective [47], where being effective means that the interior of the isotropy bundle of G coincides with G_0 or, equivalently, that the pseudogroup action of $\mathcal{I}(G)$ on G_0 is an injective homomorphism of pseudogroups $\mathcal{I}(G) \to \mathcal{I}(G_0)$. Under the conditions that we are imposing on G, effectiveness is also equivalent to being topologically principal (the set of points of G_0 with trivial isotropy is dense in G_0) [43]. In particular, principal groupoids (equivalence relations) are effective.

Example 6.5. Let $A = M_2(\mathbb{C})$ be the algebra of a qubit. This is the C*-algebra of the finite principal groupoid

$$G = \mathrm{Pair}(\{|0\rangle, |1\rangle\}) ,$$

which is localizable because it is compact, so the quantization map p defines an isomorphism of pseudogroups $p^* : \mathcal{I}(G) \to \mathcal{I}(B)$ for the subalgebra of diagonal matrices $B = D_2(\mathbb{C})$. Concretely, $\mathcal{I}(G)$ contains the partial bijections on $\{|0\rangle, |1\rangle\}$. Each of these can be represented by a partial permutation matrix, which is the support of the corresponding element of $\mathcal{I}(B)$. For instance, the permutation matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

corresponds to the subspace

$$\left\{ \begin{pmatrix} 0 & z_1 \\ z_2 & 0 \end{pmatrix} \mid z_1, z_2 \in \mathbb{C} \right\} .$$

This will be further expanded below in Example 6.7.

Note that the projections in $\operatorname{Max} A$ are the sub-C*-algebras of A, so this general pseudogroup construction applies to nonabelian subalgebras, too.

Stern–Gerlach measurements. Measurements that yield transitions between different groupoids can be described using a similar construction. The following definition gives us the "transitions from b to c":

Definition 6.6. Let M be a symmetric measurement space, and let $b, c \in M$ be projections. The set $\mathcal{I}(c, b)$ consists of those measurements $m \in M$ such that

1. $m^*m \leq b$,
2. $mm^* \leq c$,
3. $mb \leq m$,
4. $cm \leq m$.

Example 6.7. Let $A = M_2(\mathbb{C})$ be the algebra of spin 1/2 measurements, and let $B_z = D_2(\mathbb{C})$. For measurements along z we have the following selective measurements in $\mathcal{I}(B_z)$:

$$\begin{aligned}
M(z^{\uparrow}) &= \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\rangle \\
M(z^{\downarrow}) &= \left\langle \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle \\
M(z^{\downarrow}, z^{\uparrow}) &= \left\langle \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\rangle \\
M(z^{\uparrow}, z^{\downarrow}) &= \left\langle \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\rangle
\end{aligned}$$

These correspond to groupoid arrows, so they are selective measurements in the original sense of Schwinger. Of course, the ability to find actual groupoid arrows in the measurement space Max $M_2(\mathbb{C})$ is merely a feature of this particular example, whose groupoid is finite. In addition to these, in the pseudogroup $\mathcal{I}(B_z)$ we also have

$$\begin{aligned} M(\emptyset) &= \left\langle \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\rangle \\ M(\mathrm{id}_z) &= \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle = M(z^\uparrow) \vee M(z^\downarrow) \\ M(\chi_z) &= \left\langle \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\rangle = M(z^\downarrow, z^\uparrow) \vee M(z^\uparrow, z^\downarrow) \end{aligned}$$

where the last one corresponds to the bijection that swaps the spin values. Note that the idempotents can also be regarded as "state preparations" or "measurement results," as here where $M(z^\downarrow)$ is the preparation and $M(z^\uparrow)$ is the result:

$$\underbrace{\left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\rangle}_{M(z^\uparrow)} \underbrace{\left\langle \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\rangle}_{M(z^\uparrow, z^\downarrow)} \underbrace{\left\langle \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle}_{M(z^\downarrow)} = \underbrace{\left\langle \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\rangle}_{\in \mathcal{I}(B_z)}$$

Now let B_x be the abelian subalgebra of $M_2(\mathbb{C})$ which is generated by the Pauli matrix σ_x:

$$B_x = \left\langle \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \right\rangle .$$

An example that involves a transition from spin along x to spin along z which can be obtained using a Stern–Gerlach apparatus is the following, where now the state preparation is $M(x^\uparrow) \in E(\mathcal{I}(B_x))$:

$$\underbrace{\left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\rangle}_{M(z^\uparrow)} \underbrace{\left\langle \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right\rangle}_{M(z^\uparrow, x^\uparrow)} \underbrace{\left\langle \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\rangle}_{M(x^\uparrow)} = \underbrace{\left\langle \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right\rangle}_{\in \mathcal{I}(B_z, B_x)}$$

Whereas in [8] the structure of a 2-groupoid is proposed for linking the various groupoids of selective measurements, here the "glue" that connects the various pseudogroups $\mathcal{I}(b)$ is provided by a symmetric measurement space M. Its algebraic properties make $\mathcal{I}(c, b)$ an $\mathcal{I}(c)$-$\mathcal{I}(b)$-bimodule in a sense which is useful for defining Morita equivalence of pseudogroups, as I explain now.

First, note that both a left $\mathcal{I}(c)$-action and a right $\mathcal{I}(b)$-action are given by multiplication in M, and $\mathcal{I}(c, b)$ has two inner products

$$\langle -, - \rangle : \mathcal{I}(c, b) \times \mathcal{I}(c, b) \to \mathcal{I}(c) \quad \text{and} \quad [-, -] : \mathcal{I}(c, b) \times \mathcal{I}(c, b) \to \mathcal{I}(b)$$

which for all $m, n \in \mathcal{I}(c, b)$ are defined by

$$\langle m, n \rangle = mn^* \quad \text{and} \quad [m, n] = m^* n .$$

Proposition 6.8. *Let M be a symmetric measurement space, and let $b, c \in M$ be projections. Then for all $m, n, k \in \mathcal{I}(c, b)$, and all $s \in \mathcal{I}(b)$ and $t \in \mathcal{I}(c)$ we have:*

1. $\langle tm, n\rangle = t\langle m, n\rangle$
2. $\langle n, m\rangle = \langle m, n\rangle^*$
3. $\langle m, m\rangle m = m$
4. $[m, ns] = [m, n]s$
5. $[n, m] = [m, n]^*$
6. $m[m, m] = m$
7. $\langle m, k\rangle n = m[k, n]$

Proof. Let $m \in \mathcal{I}(c, b)$. Then $mm^*m \le mb \le m$, so $mm^*m = m$ because M is stably Gelfand. This proves (3) and (6), and all the other properties are immediate. □

Next let us say that any two elements $m, n \in \mathcal{I}(c, b)$ are *compatible* if

$$\langle m, n\rangle \le c \quad \text{and} \quad [m, n] \le b \ ,$$

and let us say that a subset $S \subset \mathcal{I}(c, b)$ is *compatible* if for all $m, n \in S$ the elements m and n are compatible.

Proposition 6.9. *Let M be a symmetric measurement space, and let $b, c \in M$ be projections. Then for all compatible sets $S \subset \mathcal{I}(c, b)$ the join $\bigvee S$ is in $\mathcal{I}(c, b)$. Moreover, the following distributivity properties hold for all $s \in \mathcal{I}(b)$, $t \in \mathcal{I}(c)$, and $m, n \in \mathcal{I}(c, b)$, and for all compatible families (s_i), (t_i), and (m_i), respectively in $\mathcal{I}(b)$, $\mathcal{I}(c)$, and $\mathcal{I}(c, b)$:*

1. $t\left(\bigvee_{i\in I} m_i\right) = \bigvee_{i\in I} tm_i$;
2. $\left(\bigvee_{i\in I} t_i\right) m = \bigvee_{i\in I} t_i m$;
3. $\left(\bigvee_{i\in I} m_i\right) s = \bigvee_{i\in I} m_i s$;
4. $m\left(\bigvee_{i\in I} s_i\right) = \bigvee_{i\in I} ms_i$;
5. $\left\langle\bigvee_{i\in I} m_i, n\right\rangle = \bigvee_{i\in I}\langle m_i, n\rangle$;
6. $\left[n, \bigvee_{i\in I} m_i\right] = \bigvee_{i\in I}[n, m_i]$.

In particular, $\langle 0, y\rangle = 0$ and $[y, 0] = 0$.

Proof. Let $S \subset \mathcal{I}(c, b)$ be compatible, and let $m = \bigvee S$. Then

$$m^*m = \bigvee_{x,y\in S} x^*y \le b \quad \text{and} \quad mm^* = \bigvee_{x,y\in S} xy^* \le c \ .$$

We also have

$$mb = \bigvee_{x\in S} xb \le \bigvee_{x\in S} x = m$$

and, similarly, $cm \le m$, so $m \in \mathcal{I}(c, b)$. The distributivity properties are immediate consequences of the quantale distributivity properties of M. □

This makes $\mathcal{I}(c,b)$ a complete bimodule in the sense of [26], and a Morita equivalence bimodule if the following two additional conditions hold:

Proposition 6.10. *Let M be a symmetric measurement space, and let $b, c \in M$ be projections such that*

$$\bigvee_{m\in\mathcal{I}(c,b)} \langle m,m\rangle = c \quad \textit{and} \quad \bigvee_{m\in\mathcal{I}(c,b)} [m,m] = b\ . \tag{6.1}$$

Then $\mathcal{I}(b)$ and $\mathcal{I}(c)$ are Morita equivalent pseudogroups.

Proof. Due to the results of [26], the two conditions in (6.1) plus those of Proposition 6.8 and Proposition 6.9 make $\mathcal{I}(c,b)$ a Morita equivalence bimodule between the pseudogroups $\mathcal{I}(b)$ and $\mathcal{I}(c)$. $\square$

In general, given arbitrary projections b and c the bimodule $\mathcal{I}(c,b)$ does not make $\mathcal{I}(b)$ and $\mathcal{I}(c)$ Morita equivalent, but it still defines a Morita equivalence of subpseudogroups, as I explain now. Let S be a pseudogroup, and let $f \in E(S)$ be an idempotent. A new pseudogroup $S|_f$ is obtained whose locale of idempotents is $\downarrow(f)$ and whose other elements are those $s \in S$ such that $ss^* \le f$ and $s^*s \le f$. Let us call this an *open subpseudogroup* of S because in terms of the germ groupoid $G = \mathrm{Germs}(S)$ the idempotent f corresponds to an open subspace $U \subset G_0$, and $\mathrm{Germs}(S|_f)$ can be identified with the full subgroupoid of G whose objects lie in U. Then it is not hard to prove that any bimodule of the form $\mathcal{I}(c,b)$ establishes a Morita equivalence of open subpseudogroups of $\mathcal{I}(b)$ and $\mathcal{I}(c)$:

Proposition 6.11. *Let M be a symmetric measurement space, and let $b, c \in M$ be projections. Let*

$$b' = \bigvee_{m\in\mathcal{I}(c,b)} [m,m] \quad \textit{and} \quad c' = \bigvee_{m\in\mathcal{I}(c,b)} \langle m,m\rangle\ .$$

Then the bimodule $\mathcal{I}(c',b')$ coincides with $\mathcal{I}(c,b)$ and it makes $\mathcal{I}(b')$ and $\mathcal{I}(c')$ Morita equivalent.

Proof. Due to Proposition 6.10 it suffices to show that $\mathcal{I}(c,b) = \mathcal{I}(c',b')$. Let $m \in \mathcal{I}(c',b')$. Then

$$mm^* = \langle m,m\rangle \le c' \le c \quad \text{and} \quad m^*m = [m,m] \le b' \le b\ .$$

In addition we have

$$mb = m[m,m]b \le mb'b = mb' \le m$$

and, similarly,

$$cm = c\langle m,m\rangle m \le cc'm = c'm \le m\ ,$$

so $m \in \mathcal{I}(c,b)$. Now let $m \in \mathcal{I}(c,b)$. Then $mb' \le mb \le m$ and $c'm \le m$, and by hypothesis we have $m^*m \le b'$ and $mm^* \le c'$, so $m \in \mathcal{I}(c',b')$. This shows that $\mathcal{I}(c,b) = \mathcal{I}(c',b')$. $\square$

6.4 Observers revisited

Selective observers. As we have seen, each projection b of a symmetric measurement space has an associated pseudogroup $\mathcal{I}(b)$. For example, the commutative algebra of diagonal matrices $D_2(\mathbb{C})$ is a projection of $\operatorname{Max} M_2(\mathbb{C})$, and $\mathcal{I}(D_2(\mathbb{C}))$ is isomorphic to the symmetric pseudogroup on a set with 2 elements (cf. Example 6.5). This contains all the selective measurements of spin relative to the "z basis." Hence, both $D_2(\mathbb{C})$ and its pseudogroup can be regarded as an "observer," so let us link this idea to the notion of classical observer of section 5. In order to do this, take any symmetric measurement space M equipped with an étale observer $(\mathcal{O}, r)$. There are two pseudogroups associated to the multiplicative unit e of $\mathcal{O}$, namely the pseudogroup of partial units $\mathcal{I}(\mathcal{O})$ and the pseudogroup $\mathcal{I}(e)$. Clearly, $\mathcal{I}(\mathcal{O}) \subset \mathcal{I}(e)$, but the converse need not hold. If it does, the observer contains all the "selective measurements" relative to the projection e, leading us to the following working definition:

Definition 6.12. Let M be a symmetric measurement space. By a *selective observer* of M is meant an étale observer $(\mathcal{O}, r)$ such that $\mathcal{I}(\mathcal{O}) = \mathcal{I}(e_{\mathcal{O}})$.

Given a projection b of a symmetric measurement space M, the join-completion of $\mathcal{I}(b)$ is a unital involutive subquantale $\mathcal{O}(b)$ of M whose unit is $e = b$. Obviously, we have $\mathcal{I}(b) = \mathcal{I}(\mathcal{O}(b))$, and thus:

Proposition 6.13. *Let M be a symmetric measurement space. An étale observer $(\mathcal{O}, r)$ is selective if and only if $\mathcal{O} = \mathcal{O}(e_{\mathcal{O}})$.*

A general question is whether a given projection b coincides with the unit projection $e_{\mathcal{O}}$ of a selective observer $(\mathcal{O}, r)$. Some necessary conditions are:

1. $E(\mathcal{I}(b))$ is a locally compact locale;
2. $\mathcal{O} = \mathcal{O}(b)$;
3. $\mathcal{O}$ is a locale.

Note that (1) is always true if $M = \operatorname{Max} A$ for a C*-algebra A. And if (2)–(3) hold then (1) implies that $\mathcal{O}$ is locally compact (with $E(\mathcal{I}(b)) = \mathcal{O}_0$), so it is a classical measurement space. From the results of [48] it also follows that (3) holds if and only if the unique homomorphism of involutive quantales $\varphi : \mathcal{L}^\vee(\mathcal{I}(b)) \to M$ that extends the embedding $\mathcal{I}(b) \to M$ is injective (cf. section 6.2); in other words, the map $p : M \to \mathcal{L}^\vee(\mathcal{I}(b))$ defined by $p^* = \varphi$ is a surjection.

Let us also single out the following property:

4. $\mathcal{O}$ is closed under arbitrary meets in M.

Note that this implies $1_{\mathcal{O}} = 1_M$. If (4) holds, a sup-lattice homomorphism

$$r : M \to \mathcal{O}$$

is canonically obtained as a closure operator:

$$r(m) = \bigwedge \{n \in \mathcal{O} \mid m \leq n\} .$$

Hence, if (1)–(4) hold then all that is needed in order to have a classical observer is to ensure that r is continuous and $\mathcal{O}$-$\mathcal{O}$-equivariant. However, (4) means that the observer is localizable, and for this reason it is not clear whether imposing (4) is a desirable property of selective observers in general, since there is no known classification of localizable groupoids (cf. Example 5.9).

Example 6.14. Let G be a second-countable locally compact Hausdorff étale groupoid with quantization map p, and let $(\mathcal{O}, r)$ be the classical observer of Example 5.3:

$$\mathcal{O} = p^*(\mathcal{O}(G)) \quad \text{and} \quad r = p^* \circ \mathrm{supp}^\circ \ .$$

Example 6.4 shows that if G is also localizable and effective then $(\mathcal{O}, r)$ is a selective observer. Moreover, it satisfies the above condition (4).

Remark 6.15. An interesting problem is that of finding a "groupoid-free" characterization of the subalgebras $p^*(G_0)$ of the above example, but I shall not attempt this here. Note that these are Cartan subalgebras in the sense of Renault [43], but Cartan subalgebras are more general because they are obtained from twisted and not necessarily localizable groupoids. See also [28]. A related question is that of which Cartan subalgebras yield selective observers.

Stern–Gerlach measurements and Morita theory. Stern–Gerlach measurements (cf. section 6.3) yield Morita equivalences of pseudogroups, and therefore also Morita equivalences of their inverse quantal frames (equivalently, of their localic étale groupoids). Let us examine this in relation to selective observers. This is relevant because Morita equivalence of $\mathcal{O}_1$ and $\mathcal{O}_2$ should at least be a part of what it means for two selective observers $(\mathcal{O}_1, r_1)$ and $(\mathcal{O}_2, r_2)$ to be considered equivalent, such as when considering observers induced by complementary pairs of observables like z-spin and x-spin. I shall address this preliminarily, mostly as a direct application of stably Gelfand quantales, without taking into account the role of the retraction maps. A few facts and definitions need to be recalled from [42, 46].

Definition 6.16. [46] Let Q be an inverse quantal frame. A *complete Hilbert Q-module* is a unital left Q-module X equipped with a mapping

$$\langle -, - \rangle : X \times X \to Q$$

which is *left linear* and *Hermitian*, i.e., such that for all $x, y, x_i \in X$ and $a \in Q$ we have

1. $\langle ax, y \rangle = a\langle x, y \rangle$,
2. $\left\langle \bigvee_i x_i, y \right\rangle = \bigvee_i \langle x_i, y \rangle$,
3. $\langle y, x \rangle = \langle x, y \rangle^*$

(i.e., X is a *pre-Hilbert module*), and for which there is a set $\Gamma \subset X$ (of *sections*) such that for all $x \in X$ we have

$$\bigvee_{s \in \Gamma} \langle x, s \rangle s = x \ .$$

The mapping $\langle -, - \rangle$ is called the *inner product*, and Γ is called the *Hilbert basis*.

If G is the étale groupoid of Q, the complete Hilbert Q-modules correspond precisely to the equivariant G-sheaves, so we also call them *Q-sheaves*. It is a property of any Q-sheaf X that it is necessarily a locale [46, 49].

From here on, given two projections b and c of a symmetric measurement space M I shall write $\mathcal{O}(c,b)$ for the join-completion of $\mathcal{I}(c,b)$.

Proposition 6.17. *Let M be a symmetric measurement space, and let c be a projection for which $\mathcal{O}(c)$ is a locale. Let b be a projection satisfying*

$$\bigvee_{m\in\mathcal{I}(c,b)} m^*m = b\ .$$

Then $\mathcal{O}(c,b)$ is an $\mathcal{O}(c)$-sheaf, with inner product given by

$$\langle m,n\rangle = mn^*$$

for all $m,n\in\mathcal{O}(c,b)$, and $\mathcal{I}(c,b)$ is a Hilbert basis. In particular, it follows that $\mathcal{O}(c,b)$ is a locale. Moreover, if $E(\mathcal{I}(c))$ is locally compact so is $\mathcal{O}(c,b)$.

Proof. It is straightforward to verify that the mapping $\langle -,-\rangle$ is left linear and Hermitian, and for all $m\in\mathcal{O}(b,c)$ the Hilbert basis property follows from the condition on the projection b:

$$\bigvee_{n\in\mathcal{I}(c,b)} \langle m,n\rangle n = \bigvee_{n\in\mathcal{I}(c,b)} mn^*n = m\bigvee_{n\in\mathcal{I}(c,b)} n^*n = mb = m\ .$$

So $\mathcal{O}(c,b)$ is an $\mathcal{O}(c)$-sheaf, and thus also a locale. In particular, this makes $\mathcal{O}(c,b)$ a sheaf over the locale $\mathcal{O}(c)_0 = E(\mathcal{I}(c,b))$, so if the latter is locally compact so is $\mathcal{O}(c,b)$. ◻

Now we can address Morita equivalence.

Definition 6.18. [42] Let Q and R be inverse quantal frames. By a *Q-R-bisheaf* will be meant a Q-R-bimodule X that satisfies the following two conditions:

Inner products: X is equipped with two inner products

$$\langle -,-\rangle : X\times X\to Q \qquad \text{and} \qquad [-,-]: X\times X\to R$$

(with $\langle -,-\rangle$ being left Q-linear and $[-,-]$ being right R-linear);

Bisections: There is $\varGamma\subset X$ such that for all $x\in X$

$$\bigvee_{s\in\varGamma}\langle x,s\rangle s = \bigvee_{s\in\varGamma} s[s,x] = x\ .$$

Moreover, a bisheaf X is *biprincipal* if it satisfies both

$$\bigvee_{s\in\varGamma_X}\langle s,s\rangle = e_Q \qquad \text{and} \qquad \bigvee_{s\in\varGamma_X}[s,s] = e_R$$

and for all $s,t,u\in\varGamma_X$ we have

$$\langle s,t\rangle u = s[t,u]\ .$$

Q and R are said to be *Morita equivalent* if there exists a biprincipal Q-R-bisheaf.

If G and H are sober étale groupoids, the biprincipal $\mathcal{O}(G)$-$\mathcal{O}(H)$-bisheaves correspond to the biprincipal bibundles between G and H. Hence, G and H are Morita equivalent if and only if $\mathcal{O}(G)$ and $\mathcal{O}(H)$ are [42]. Similarly, two pseudogroups S and T are Morita equivalent if and only if $\mathcal{L}^{\vee}(S)$ and $\mathcal{L}^{\vee}(T)$ are [26].

The following proposition is an immediate consequence of the above one, and it links Morita equivalence to selective observers and Stern–Gerlach measurements:

Proposition 6.19. *Let b and c be two projections of a symmetric measurement space M. Assume that the following two conditions hold,*

$$\bigvee_{m\in\mathcal{I}(c,b)} m^*m = b \quad \textit{and} \quad \bigvee_{m\in\mathcal{I}(c,b)} mm^* = c$$

(i.e., $\mathcal{I}(c,b)$ is a Morita equivalence bimodule between $\mathcal{I}(c)$ and $\mathcal{I}(b)$), and that $\mathcal{O}(b)$ and $\mathcal{O}(c)$ are locales (and thus inverse quantal frames). Then $\mathcal{O}(c,b)$ is a biprincipal $\mathcal{O}(c)$-$\mathcal{O}(b)$-bisheaf.

Proof. $\mathcal{O}(c,b)$ is a bisheaf due to Proposition 6.17 (the proof for showing that $[-,-]$ makes $\mathcal{O}(c,b)$ a right $\mathcal{O}(b)$-sheaf is analogous to that of $\langle -,-\rangle$), and all the rest is immediate. □

7 Conclusion and discussion

This paper has tackled the question of what is meant by a measurement "from scratch" by defining a model whose physical interpretation is that a measurement is a process via which a finite amount of classical information is recorded. I have tried to convey this idea mathematically in terms of a definition of space of measurements that consists of a sober topological quantale whose open sets represent measurable physical properties, in a way that partly resembles ideas from computer science where open sets represent semidecidable properties of computers running programs [55]. This also accounts for the distinction between quantum and classical measurements, and for the emergence of "classical observers." The latter relate to the mathematical interplay between groupoids and C*-algebras, and link naturally to Schwinger's notion of selective measurement.

Such an approach is an attempt to solidify a concept via a process of familiarization to mathematical structure and properties that are rich enough that something meaningful is conveyed, but also, on the other hand, are sufficiently abstract that we do not end up bogged down in irrelevant details. Ultimately the philosophical purpose behind this is to assess how plausible it is to regard measurements as being fundamental rather than just relative to arbitrary distinctions between system, apparatus, etc. Moreover, given the proximity between measurements and information, this model also brings with it the idea that information itself can be taken to be fundamental, hence yielding an instance of Wheeler's "it from bit" that sidesteps Bell's questions in [4]: "*Whose* information? Information about *what*?"

Concretely, an example of a symmetric measurement space is the involutive quantale $\mathrm{Max}\, A$ of a C*-algebra A, where quantum measurements can be naturally encoded. In this example the topology of measurable physical properties is the lower Vietoris topology, which makes $\operatorname{Max} A$ a sober space as required [51]. Another example is the (again stably Gelfand) quantale $\mathcal{O}(G)$ of a locally compact groupoid G, equipped with the Scott topology. Local compactness makes $\mathcal{O}(G)$ a continuous lattice and underpins the distinction between the logical disjunctions that are of classical type and those of quantum type that are found in $\operatorname{Max} A$. The results of [47] enable one to relate the spaces $\operatorname{Max} A$ to those of the form $\mathcal{O}(G)$ in such a way that the latter can be regarded as emergent classical observers of $\operatorname{Max} A$.

The results of this paper provide an initial step that concerns only a kinematical description of measurements. Additional stages need to follow in order to assess the extent to which measurement spaces can accommodate a useful "picture" of quantum mechanics. At a first glance this should be possible, for due to the properties of stably Gelfand quantales any finite discrete groupoid G can be reconstructed up to isomorphism from a projection of the measurement space $\operatorname{Max} \mathbb{C}G$, and thus so can the algebra of "amplitudes" $C(G)$ in which the dynamics resides. However, a classification of the symmetric measurement spaces that are of the form $\operatorname{Max} A$ for an arbitrary C*-algebra A is currently unavailable (cf. section 4.2), and thus so are, at least for now, purely measurement space based axiomatic approaches to quantum mechanics. Related to this, and of mathematical interest in its own right, is the observation of section 4.2 that the functorial properties of the complete invariant Max of unital C*-algebras seem to improve once the quantales $\operatorname{Max} A$ are regarded as measurement spaces instead of discrete algebraic objects, and the consequences of this ought to be explored. In addition, other mathematical questions and open problems are left unanswered in this paper and deserve further attention. Among these, the classification of localizable groupoids is an open problem (inherited from [47]), and there is room for improvements as regards understanding the relations between projections of symmetric measurement spaces M and selective observers, both for general M and for $M = \operatorname{Max} A$.

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