Categorical and Algebraic Aspects of the Intuitionistic Modal Logic IEL$^-$ and its predicate extensions

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Abstract

The modal intuitionistic epistemic logic **IEL**$^-$ was proposed by S. Artemov and T. Protopopescu as the intuitionistic version of belief logic [3]. We construct the modal lambda calculus which is Curry-Howard isomorphic to **IEL**$^-$ as the type-theoretical representation of applicative computation widely known in functional programming. We also provide a categorical interpretation of this modal lambda calculus considering coalgebras associated with a monoidal functor on a cartesian closed category. Finally, we study Heyting algebras and locales with corresponding operators. Such operators are used in point-free topology as well. We study compelete semantics à la Kripke-Joyal for predicate extensions of **IEL**$^-$ and **IEL** using Dedekind-MacNeille completions and cover systems introduced by Goldblatt [31]. The paper extends the conference paper published in the LFCS'20 volume [59].

Keywords— Intuitionistic modal logic, Modal type theory, Functional programming, Locales, Prenucleus, Cover systems

1 Introduction

1.1 Intutionistic modal logic and Heyting algebras with operators

Intuitionistic modal logic study extensions of intuitionistic logic with modal operators. One may consider such extensions in two perspectives. The first perspective is a consideration of intuitionistic modal logic as the branch of modal logic. Here, intuitionistic modalities might be interpreted as a constructive necessity, provability in Heyting arithmetics, intuitionistic knowledge, etc. The second perspective is the modal type theory that provides us with a more computational interpretation of intuitionistic modalities. In type theory, each value in an arbitrary computation is annotated with the relevant data type and modalised type might be one of them.

The first perspective arises to Prior who introduced the system called **MIPC** [57] to investigate modal counterparts of intuitionistic predicate monadic logic. The relation between intuitionistic modalities and quantifiers was further developed by Bull [15] and by Ono [53]. Monadic Heyting algebras were studied comprehensively by Bezhanishvili as well, see, for instance, [8].

Fischer-Servi was the first who was seeking to provide an intuitionistic analogue of the minimal normal modal logic consisting both $\Box$ and $\Diamond$ as mutually inexpressible connectives [60].

The question of intuitionistic epistemic modalities was initially by Williamson [67] considering the problem of an intuitionist knowledge in means of the capability of verification. This direction was further developed by Artemov and Protopopescu, see [3] and [58].

We also emphasise briefly the direction related to Heyting algebras with operators. Heyting algebras with Fischer-Servi modal operators have a nice topological duality piggybacked on Esakia representation

[22]. This duality that provides the characterisation of general descriptive frames for extensions of intuitionistic modal logic containing the Fischer-Servi system, see the paper by Palmigiano [54]. The class of Heyting algebras with operators is Heyting algebras with nucleus that were discovered by Macnab [49]. We discuss nuclei closely in Section 5, here we merely claim that logic of Heyting algebras with nucleus and their predicate extensions were investigated by Bezhanishvili and Ghilardi [9]; Goldblatt [28] [31]; Fairtlough, Mendler, and Walton [24] [26].

We refer the reader to this paper by Wolter and Zakharyaschev [69], the paper by Božić and Došen [14], and the monograph by Simpson [62], where the underlying results of the model-theoretic results related to intuitionistic modal logic are descibed.

1.2 Modalities from a computational perspective

The second perspective we emphasised is related to intuitionistic modalities in means of Curry-Howard correspondence. Curry-Howard correspondence provides bridges between intuitionistic proofs and programs understood in a type-theoretical sense [52] [63].

Modal lambda calculi that corresponds to certain intuitionistic modal logics were studied by Artemov [2]; Bierman and de Paiva [13]; Davies and Pfenning [18]; Fairtlough and Mendler[25], etc.

The categorical semantics of modal type theory was studies by de Paive and Ritter [19]. Modal operators are also studied within the context of Homotopy Type Theory, see the recent paper by Rijke, Shulman, and Spitters [64].

One may find a proof of concept for modal types in functional programming. Let us observe a sort of computation called monadic. A monad is a concept in functional programming that was initially implemented in the functional language called Haskell. Computational monads were examined type-theoretically by Moggi [51]. Very informally, a monad is a method of structuring a computation as a linearly connected chain of actions within such types as the list or the input/output (`IO`). Such sequences are often called a *pipeline* in which one passes a value from an external world and yield a result after the series of actions. Computational monads might be considered logically within intuitionistic modal logic.

Functional programming languages such as Haskell, Idris or Purescript have specific type classes [1] for computation within an environment. By *computational context* (or, *environment*), we mean some, roughly speaking, type-level map f, where f is a “function” from $*$ to $*$: such a type-level map takes a simple type which has kind $*$ and yields another simple type of kind $*$. For a more detailed description of the type system with kinds implemented in Haskell see [63].

Here, the underlying type class is `Functor` which has the following formal definition:

```
class Functor f where
  fmap :: (a -> b) -> f a -> f b
```

`Functor` provides a generalisation of higher-order functions as `map`. `map` merely yields an image of a list by a given function. Let us take a look at its implementation:

```
map :: (a -> b) -> [a] -> [b]
map f []     = []
map f (x:xs) = f x : (map f xs)
```

The first line claims that `map` is a two-argument function. The arguments of `map` are a unary function of type $a \to b$ and a list of elements of a. The result of the `map` function is a list of b. This line of the piece of code is the so-called type-signature. Type-signature describes the behaviour of the function in terms of types of input and output.

The next two lines describe the recursive implementation of `map`. At first, we tell that an image of the empty list is empty. This part is the termination condition of a recursion. After that, we consider the case with a non-empty list. A non-empty list is a list obtained by adding an element to the top of the list. Suppose one has a list xs and x is an element of type a. In the case of non-empty list $x : xs$, one needs to call `map` recursively on the tail xs. We also apply a given function f to the head x. Finally, we add $f\,x$ to the top of the list `map` $f\ xs$ which is an image of the tail xs.

[1] In Haskell, type class is a general interface for some special group of data types.

The list data type is one of the instances of a functor. Generally, `Functor` provides a uniform method to carry unary functions through parametrised types such as list one. In other words, the notion of a functor in functional programming is a counterpart of the category-theoretic functor.

One may extend a functor to the so-called monad which is a functional programming counterpart of Kleisli triples. In Haskell-like languages, one also has the type class called `Monad`, a type class of an abstract data type of action in some computational environment. Here we define the `Monad` type class as follows:

```
class Functor m => Monad m where
  return :: a -> m a
  (>>=)  :: m a -> (a -> m b) -> m b
```

`Monad` is a type class that extends `Functor` with two methods called `return` and `(>>=)` (a monadic bind).

Monads present a uniform technique for miscellaneous computations such as computation with a mutable state, many-valued computation, side effect input-output computation, etc. All those computations are arranged in the same fashion as pipelines. Historically, monads were implemented in Haskell to process side-effects that arise in input/output world. The advantage of a monad is an ability to isolate side-effects within a monad remaining the relevant code purely functional. That is, one has a tool to describe a sequence of actions, where the result of each step depends on the previous ones somehow. In other words, one has so-called monadic binding by which such a sequence of actions with dependencies performs.

Monadic metalanguage proposed by Moggi [51] is the modal lambda calculus that describes a computation within an abstract monad. From a proof-theoretical point of view, this modal extension of the simply-typed lambda calculus is Curry-Howard isomorphic to lax logic. The typing rules for modalities of this metalanguage correspond to the `return` and the monadic bind methods. From a logical point of view, this extension is Curry-Howard isomorphic to lax logic, the logic of Heyting algebras with a nucleus operator which we discussed earlier.

Let us take a look at the example of a monad. There is a parametrised data type `Maybe` in Haskell. The main application of `Maybe` is making a partial function total:

```
data Maybe a = Nothing | Just a
```

The data type consists of two constructors. Suppose we deal with some computation that might terminate with some failure. `Nothing` is a flag that claims this failure arose. The second constructor `Just` stores some value of type a, a successful result of a considered computation.

For example, one needs to extract the first element of a list. There might be an error if a given array is empty. This problem could be solved with the `Maybe` data type:

```
safeHead :: [a] -> Maybe a
safeHead []     = Nothing
safeHead (x:xs) = Just x
```

The `Maybe` instance of `Monad` is the following one:

```
instance Monad Maybe where
  return = Just
  (Just x) >>= f = f x
  Nothing  >>= f = Nothing
```

Here, the `return` method merely embeds any value of type a into the type `Maybe` a via the `Just` constructor. The implementation of a monadic bind for `Maybe` is also quite simple. Suppose one has a function f of type $a \to$ `Maybe` b and some value x of type `Maybe` a. Here we match on x. If x is `Nothing`, then the monadic bind yields `Nothing`. Otherwise, we extract the value of type a and apply a given function to the extracted value.

The monad interface for `Maybe` allows one to perform sequences of actions, where some values might be undefined. If all values are well defined on each step, then the result of an execution is a term of the form `Just n`. Otherwise, if something went wrong and we have no required value somewhere, then the computation halts with `Nothing`. The other examples of `Monad` instances have more or less the same explanation since the monadic interface was proposed for a side-effect processing.

Let us discuss why `Applicative` class was introduced since this class is comparatively recent. This class was proposed by Paterson and McBride to describe effectful programming in an applicative style [50]. One may consider the `Applicative` type class as the intermediate one between `Functor` and `Monad`. See this paper to have a more clear understanding of the connection between applicative functors ans monads [46].

Here is the precise definition of the `Applicative` class:

```
class Functor f => Applicative f where
  pure :: a -> f a
  (<*>) :: f (a -> b) -> f a -> f b
```

The main aim of an applicative functor is a generalisation the action of a functor for functions of arbitrary arity, for instance:

```
liftA2
  :: Applicative f
  => (a -> b -> c)
  -> f a -> f b -> f c
liftA2 f x y = ((pure f) <*> x) <*> y
```

`liftA2` is a version of `fmap` for arbitrary two-argument function. It is clear that one may implement `liftA3`, `liftA4`, and `liftAn` for each $n < \omega$. In the case of lists, `liftA2` passes two-argument function, two lists, and yields the list obtained by applying to every possible pair the first element of which is an element of the first list and the second element belongs to the second list.

In this paper, we consider applicative computation type-theoretically, which is weaker than the monadic one.

As we will see further, the modal axioms of $\mathbf{IEL}^{-}$ and types of the `Applicative` methods in Haskell-like languages are syntactically similar. We investigate the relationship between intuitionistic epistemic logic $\mathbf{IEL}^{-}$ and applicative computation constructing the type system which is Curry-Howard isomorphic to $\mathbf{IEL}^{-}$.

This type system consists of the rules for simply-typed lambda-calculus extended via the special modal rules. We assume that the proposed type system axiomatises applicative computation. We provide a proof-theoretical view of this kind of computation in functional programming and prove such metatheoretical properties as strong normalisation and confluence. The initial idea to consider applicative functors type-theoretically belongs to Krishnaswami [41] and we are going to develop his ideas considering the $\mathbf{IEL}^{-}$ from a computational perspective. Litak et. al. [47] also made an observation that the logic $\mathbf{IEL}^{-}$ might be treated as a logic of an applicative functor as well [2]. In further sections, we study semantical questions of the $\mathbf{IEL}^{-}$ and its related systems: categorical semantics for the provided modal lambda calculus and cover semantics for quantified versions of intuitionistic modal logic with $\mathbf{IEL}^{-}$-like modalities.

2 The intutionistic modal logic IEL^{-}

The intuitionistic modal logic $\mathbf{IEL}^{-}$ was proposed by S. Artemov and T. Protopopescu [3]. According to the authors, $\mathbf{IEL}^{-}$ represents beliefs agreed with BHK-semantics of intuitionistic logic. $\mathbf{IEL}^{-}$ is a weaker version of the system **IEL** that represents knowledge as provably consistent intuitionistic belief.

The logic $\mathbf{IEL}^{-}$ is defined by following axioms and derivation rules:

Definition 1. *Intuitionistic epistemic logic* $\mathbf{IEL}^{-}$*:*

[2]John Connor (the City College of New York) also connected the intuitionistic epistemic logic $\mathbf{IEL}^{-}$ with propositional truncation in Homotopy Type Theory. Those results were presented at the category theory seminar, the CUNY Graduate Centre. At the moment, there is only a video of this talk on YouTube.

1. $(\varphi \to (\psi \to \theta)) \to ((\varphi \to \psi) \to (\varphi \to \theta))$

2. $\varphi \to (\psi \to \varphi)$

3. $\varphi \to (\psi \to (\varphi \wedge \psi))$

4. $\varphi_1 \wedge \varphi_2 \to \varphi_i$, $i = 1, 2$

5. $(\varphi \to \theta) \to ((\psi \to \theta) \to (\varphi \vee \psi \to \theta))$

6. $\varphi_i \to \varphi_1 \vee \varphi_2$, $i = 1, 2$

7. $\bot \to \varphi$

8. $\bigcirc(\varphi \to \psi) \to (\bigcirc\varphi \to \bigcirc\psi)$

9. $\varphi \to \bigcirc\psi$

10. *The Modus Ponens rule: from $\varphi \to \psi$ and φ infer ψ*

The last modal axiom is also called *co-reflection*. One may consider this axiom as the principle which connects intuitionistic truth and intuitionistic knowledge. From a Kripkean point of view, the logic $\mathbf{IEL}^-$ is the logic of all frames $\langle W, \leqslant, E\rangle$, where $\langle W, R\rangle$ is a partial order and E is a binary "knowledge" relation, a subrelation of $\leqslant$. The relation E should satisfy the following conditions:

1. $E(w) \subseteq\uparrow w$ for each $w \in W$.
2. $E(u) \subseteq E(w)$, if wRu

A model for $\mathbf{IEL}^-$ is a quadruple $\mathcal{M} = \langle W, \leqslant, E, \vartheta\rangle$, an extended intuitionistic Kripke model with the additional forcing relation for modal formulas defined via the relation E. Here, the $\bigcirc$ connective has the "necessity" semantics:

$$\mathcal{M}, x \Vdash \bigcirc\varphi \Leftrightarrow \forall y \in E(x)\ \ \mathcal{M}, y \Vdash \varphi.$$

The logic $\mathbf{IEL}$, the full epistemic intuitionistic logic, extends $\mathbf{IEL}^-$ as $\mathbf{IEL} = \mathbf{IEL}^- \oplus \bigcirc\varphi \to \neg\neg\varphi$. This additional axiom is often called the *intuitionistic relfection principle*. An $\mathbf{IEL}$-frame is an $\mathbf{IEL}^-$ frame with the condition $E(u) \neq \varnothing$ for each $u \in W$. One has the following theorem proved by Artemov and Protopopescu [3] by the standard Henkin construction with the canonical model on prime theories:

Theorem 1.
Let $\mathcal{L} \in \{\mathbf{IEL}^-, \mathbf{IEL}\}$, then $\mathrm{Log}(\mathrm{Frames}(\mathcal{L})) = \mathcal{L}$

V. Krupski and A. Yatmanov investigated proof-theoretical and algorithmic aspects of the stronger logic $\mathbf{IEL}$. In this paper [42], they provided the sequent calculus for $\mathbf{IEL}$ and proved that the derivability problem of this calculus is PSPACE-complete. $\mathbf{IEL}^-$ is also decidable, since this logic has FMP that was shown by Wolter and Zakharyaschev [68].

For further pusposes, we define the natural deduction calculus for $\mathbf{IEL}^-$ that we call $\mathbf{NIEL}^-$. For simplicity, we restrict our language to $\to$, $\wedge$, and $\bigcirc$.

Definition 2. *The natural deduction calculus* $\mathbf{NIEL}^-$ *for* $\mathbf{IEL}^-$ *is an extension of the intuitionistic natural deduction calculus with the additional inference rules for modality:*

$$\frac{}{\Gamma, \varphi \vdash \varphi}\ \mathbf{ax}$$

$$\frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \to \psi} \to_I \qquad \frac{\Gamma \vdash \varphi \to \psi \qquad \Gamma \vdash \varphi}{\Gamma \vdash \psi} \to_E$$

$$\frac{\Gamma \vdash \varphi \qquad \Gamma \vdash \psi}{\Gamma \vdash \varphi \wedge \psi} \wedge_I \qquad \frac{\Gamma \vdash \varphi_1 \wedge \varphi_2}{\Gamma \vdash \varphi_i} \wedge_E, i = 1, 2$$

$$\frac{\Gamma \vdash \varphi}{\Gamma \vdash \bigcirc\varphi} \bigcirc_{I_1} \qquad \frac{\Gamma \vdash \bigcirc\vec{\varphi} \qquad \vec{\varphi} \vdash \psi}{\Gamma \vdash \bigcirc\psi} \bigcirc_{I_2}$$

The first modal rule allows one to derive co-reflection and its consequences. The second modal rule is a counterpart of $\bigcirc_I$ rule in natural deduction calculus for constructive **K** (see [40]). We will denote $\Gamma \vdash \bigcirc\varphi_1, \dots, \Gamma \vdash \bigcirc\varphi_n$ and $\varphi_1, \dots, \varphi_n \vdash \psi$ as $\Gamma \vdash \bigcirc\overrightarrow{\varphi}$ and $\overrightarrow{\varphi} \vdash \psi$ respectively for brevity.

It is straightforward to check that the second modal rule is equivalent to the **K** $\bigcirc$-rule:

$$\frac{\Gamma \vdash \varphi}{\bigcirc\Gamma \vdash \bigcirc\varphi}$$

Let us show that one may translate $\mathbf{NIEL}^{-}$ into $\mathbf{IEL}^{-}$ as follows:

Lemma 1. $\Gamma \vdash_{\mathbf{NIEL}^{-}} \varphi \Rightarrow \mathbf{IEL}^{-}_{\to,\wedge,\bigcirc} \vdash \bigwedge\Gamma \to \varphi$.

Proof. Induction on the derivation. Let us consider the modal cases.

1. If $\Gamma \vdash_{\mathbf{NIEL}^{-}} \varphi$, then $\mathbf{IEL}^{-}_{\to,\wedge,\bigcirc} \vdash \bigwedge\Gamma \to \bigcirc\varphi$.

(1)	$\bigwedge\Gamma \to \varphi$	assumption
(2)	$\varphi \to \bigcirc\varphi$	co-reflection
(3)	$(\bigwedge\Gamma \to \varphi) \to ((\varphi \to \bigcirc\varphi) \to (\bigwedge\Gamma \to \bigcirc\varphi))$	IPC theorem
(4)	$(\varphi \to \bigcirc\varphi) \to (\bigwedge\Gamma \to \bigcirc\varphi)$	from (1), (3) and MP
(5)	$\bigwedge\Gamma \to \bigcirc\varphi$	from (2), (4) and MP

2. If $\Gamma \vdash_{\mathbf{NIEL}^{-}} \bigcirc\overrightarrow{\varphi}$ and $\overrightarrow{A} \vdash \psi$, then $\mathbf{IEL}^{-}_{\to,\wedge,\bigcirc} \vdash \bigwedge\Gamma \to \bigcirc\psi$.

(1)	$\bigwedge\Gamma \to \bigcirc\varphi_1, \dots, \bigwedge\Gamma \to \bigcirc\varphi_n$	assumption
(2)	$\bigwedge\Gamma \to \bigwedge\limits_{i=1}^{n} \bigcirc\varphi_i$	$\mathbf{IEL}^{-}$ theorem
(3)	$\bigwedge\limits_{i=1}^{n} \bigcirc\varphi_i \to \bigcirc \bigwedge\limits_{i=1}^{n} \varphi_i$	$\mathbf{IEL}^{-}$ theorem
(4)	$\bigwedge\Gamma \to \bigcirc \bigwedge\limits_{i=1}^{n} \varphi_i$	from (2), (3) and transitivity
(5)	$\bigwedge\limits_{i=1}^{n} \varphi_i \to \psi$	assumption
(6)	$(\bigwedge\limits_{i=1}^{n} \varphi_i \to \psi) \to \bigcirc(\bigwedge\limits_{i=1}^{n} \varphi_i \to \psi)$	co-reflection
(7)	$\bigcirc(\bigwedge\limits_{i=1}^{n} \varphi_i \to \psi)$	from (5), (6) and MP
(8)	$\bigcirc \bigwedge\limits_{i=1}^{n} \varphi_i \to \bigcirc\psi$	from (7) and normality
(9)	$\bigwedge\Gamma \to \bigcirc\psi$	from (4), (8) and transitivity

□

Lemma 2. *If* $\mathbf{IEL}^{-}_{\to,\wedge,\bigcirc} \vdash A$*, then* $\mathbf{NIEL}^{-} \vdash A$.

Proof. By straightforward derivation of modal axioms in $\mathbf{NIEL}^{-}$. We will consider those derivations via terms below. □

It is clear that one may enrich the observed natural deduction calculus with the well-known inference rules for disjunction and bottom and prove the same lemmas as above.

We build further the typed lambda-calculus based on the $\mathbf{NIEL}^{-}$ by proof-assignment in the inference rules.

3 Modal Lambda Calculus based on the IEL^- logic

Let us define terms and types for the desired modal lambda calculus.

Definition 3. *The set of terms:*
Let $\mathbb{V} = \{x, y, z, \dots\}$ be the set of variables. The set $\Lambda_\bigcirc$ of terms is generated by the following grammar:

$$\Lambda_\bigcirc ::= \mathbb{V} \mid (\lambda \mathbb{V}.\Lambda_\bigcirc) \mid (\Lambda_\bigcirc \Lambda_\bigcirc) \mid (\langle \Lambda_\bigcirc, \Lambda_\bigcirc \rangle) \mid (\pi_1 \Lambda_\bigcirc) \mid (\pi_2 \Lambda_\bigcirc) \mid (\mathbf{pure}\ \Lambda_\bigcirc) \mid (\mathbf{let}\ \bigcirc\ \mathbb{V}^* = \Lambda^*_\bigcirc\ \mathbf{in}\ \Lambda_\bigcirc)$$

where $\mathbb{V}^*$ and $\Lambda^*_\bigcirc$ denote the set of finite sequences of variables $\cup_{i<\omega} \mathbb{V}^i$ and the set of finite sequences of terms $\cup_{i<\omega} \Lambda^i_\bigcirc$. In the term $(\mathbf{let}\ \bigcirc\ \overrightarrow{x} = \overrightarrow{M}\ \mathbf{in}\ N)$, the sequence of variables $\overrightarrow{x}$ and the sequence of terms $\overrightarrow{M}$ should have the same length. Otherwise, the term is not well-formed.

As we discuss below, the terms of the form $\mathbf{let}\ \bigcirc\ \overrightarrow{x} = \overrightarrow{M}\ \mathbf{in}\ N$ correspond to the special local binding.

Definition 4. *The set of types:*
Let $\mathbb{T} = \{p_0, p_1, \dots\}$ be the set of atomic types. The set $\mathbb{T}_\bigcirc$ of types is generated by the grammar:

$$\mathbb{T}_\bigcirc ::= \mathbb{T} \mid (\mathbb{T}_\bigcirc \to \mathbb{T}_\bigcirc) \mid (\mathbb{T}_\bigcirc \times \mathbb{T}_\bigcirc) \mid (\bigcirc \mathbb{T}_\bigcirc)$$

A context is defined standardly [52][63] as a sequence of type declarations $\Gamma = \{x_0 : \varphi_1, \dots, x_n : \varphi_{n-1}\}$, where x_i is a variable and φ_i is a type for each $i < n < \omega$.

Definition 5. *The modal lambda calculus $\lambda_{\mathbf{IEL}^-}$:*

$$\frac{}{\Gamma, x : \varphi \vdash x : \varphi}\ ax$$

$$\frac{\Gamma, x : \varphi \vdash M : \psi}{\Gamma \vdash \lambda x.M : \varphi \to \psi}\ \to_i \qquad \frac{\Gamma \vdash M : \varphi \to \psi \qquad \Gamma \vdash N : \varphi}{\Gamma \vdash MN : \psi}\ \to_e$$

$$\frac{\Gamma \vdash M : \varphi \qquad \Gamma \vdash N : \psi}{\Gamma \vdash \langle M, N \rangle : \varphi \times \psi}\ \times_i \qquad \frac{\Gamma \vdash M : \varphi_1 \times \varphi_2}{\Gamma \vdash \pi_i M : \varphi_i}\ \times_e,\ i = 1, 2$$

$$\frac{\Gamma \vdash M : \varphi}{\Gamma \vdash \mathbf{pure}\ M : \bigcirc\varphi}\ \bigcirc_I \qquad \frac{\Gamma \vdash \overrightarrow{M} : \bigcirc\overrightarrow{\varphi} \qquad \overrightarrow{x} : \overrightarrow{A} \vdash N : \psi}{\Gamma \vdash \mathbf{let}\ \bigcirc\ \overrightarrow{x} = \overrightarrow{M}\ \mathbf{in}\ N : \bigcirc\psi}\ let_\bigcirc$$

$\Gamma \vdash \overrightarrow{M} : \bigcirc\overrightarrow{\varphi}$ is a short form for the sequence $\Gamma \vdash M_1 : \bigcirc\varphi_1, \dots, \Gamma \vdash M_n : \bigcirc\varphi_n$ and $\overrightarrow{x} : \overrightarrow{\varphi} \vdash N : \psi$ is a short form for $x_1 : \varphi_1, \dots, x_n : \varphi_n \vdash N : B$. We use this short form instead of $\mathbf{let}\ \bigcirc\ x_1, \dots, x_n = M_1, \dots, M_n\ \mathbf{in}\ N$. The $\bigcirc_I$-typing rule is the same as $\bigcirc$-introduction in monadic metalanguage [55]. $\bigcirc_I$ injects an object of type A into $\bigcirc$. According to this rule, it is clear that the type constructor **pure** reflects the method `pure` in the `Applicative` class.

The rule $\mathrm{let}_\bigcirc$ is similar to the $\bigcirc$-rule in typed lambda calculus for intuitionistic normal modal logic **IK**, which is introduced in [38]. Informally, one may read $\mathbf{let}\ \bigcirc\ \overrightarrow{x} = \overrightarrow{M}\ \mathbf{in}\ N$ as a simultaneous local binding in N, where each free variable of a term N should be binded with term of modalised type from $\overrightarrow{M}$. In other words, we modalise all free variables of term N and ‘substitute‘ them to terms from the sequence $\overrightarrow{M}$.

As a matter of fact, our calculus extends the typed lambda calculus for **IK** with $\bigcirc_I$-rule with the co-reflection rule which allows one to modalise any type in an arbitrary context.

Here are some examples:

$$\dfrac{\dfrac{x : \varphi \vdash x : \varphi}{x : \varphi \vdash \mathbf{pure}\, x : \bigcirc\varphi}\ \bigcirc_I}{\vdash (\lambda x.\mathbf{pure}\, x) : \varphi \to \bigcirc\varphi}\ \to_I$$

$$\dfrac{f : \bigcirc(\varphi \to \psi) \vdash f : \bigcirc(\varphi \to \psi) \qquad x : \bigcirc\varphi \vdash x : \bigcirc\varphi \qquad \dfrac{g : \varphi \to \psi \vdash g : \varphi \to \psi \qquad y : \varphi \vdash \varphi : \psi}{g : \varphi \to \psi, y : \varphi \vdash gy : \psi} \to_e}{\dfrac{\dfrac{f : \bigcirc(\varphi \to \psi), x : \bigcirc\varphi \vdash \mathbf{let} \bigcirc g, y = f, x \,\mathbf{in}\, gy : \bigcirc\psi}{f : \bigcirc(\varphi \to \psi) \vdash \lambda x.\mathbf{let} \bigcirc g, y = f, x \,\mathbf{in}\, gy : \bigcirc\varphi \to \bigcirc\psi} \to_I}{\vdash \lambda f.\lambda x.\mathbf{let} \bigcirc g, y = f, x \,\mathbf{in}\, gy : \bigcirc(\varphi \to \psi) \to \bigcirc\varphi \to \bigcirc\psi} \to_I} \mathrm{let}_{\bigcirc}$$

Here we provided the derivations for modal axioms of $\mathbf{IEL}^-$. In fact, we proved Lemma 2 using proof-assignment via terms.

Now we define free variables and substitutions:

Definition 6. *The set $FV(M)$ of free variables for a term M:*

1. $FV(x) = \{x\}$;
2. $FV(\lambda x.M) = FV(M) \backslash \{x\}$;
3. $FV(MN) = FV(M) \cup FV(N)$;
4. $FV(\langle M, N \rangle) = FV(M) \cup FV(N)$;
5. $FV(\pi_i M) = FV(M)$, $i = 1, 2$;
6. $FV(\mathbf{pure}\ M) = FV(M)$;
7. $FV(\mathbf{let} \bigcirc \overrightarrow{x} = \overrightarrow{M} \,\mathbf{in}\, N) = \cup_{i=1}^{n} FV(M)$, *where* $n = |\overrightarrow{M}|$.

Definition 7. *Substitution:*

1. $x[x := N] = N$, $x[y := N] = x$;
2. $(MN)[x := N] = M[x := N]N[x := N]$;
3. $(\lambda x.M)[y := N] = \lambda x.M[y := N]$, $y \in FV(M)$;
4. $(M, N)[x := P] = (M[x := P], N[x := P])$;
5. $(\pi_i M)[x := P] = \pi_i(M[x := P])$, $i = 1, 2$;
6. $(\mathbf{pure}\, M)[x := P] = \mathbf{pure}\, (M[x := P])$;
7. $(\mathbf{let} \bigcirc \overrightarrow{x} = \overrightarrow{M} \,\mathbf{in}\, N)[y := P] = \mathbf{let} \bigcirc \overrightarrow{x} = (\overrightarrow{M}[y := P]) \,\mathbf{in}\, N$.

Substitutions and free variables for terms of the kind $\mathbf{let} \bigcirc \overrightarrow{x} = \overrightarrow{M} \,\mathbf{in}\, N$ are defined similarly to [38]. That is, we do not take into account free variables of N because those variables occur in the list $\overrightarrow{x}$ and are eliminated by the assignment $\overrightarrow{x} = \overrightarrow{M}$.

The reduction rules are the following ones:

Definition 8. *β-reduction rules for $\lambda_{\mathbf{IEL}^-}$.*

1. $(\lambda x.M)N \to_\beta M[x := N]$
2. $\pi_1\langle M, N \rangle \to_\beta M$
3. $\pi_2\langle M, N \rangle \to_\beta N$
4. $\mathbf{let} \bigcirc \overrightarrow{x}, y, \overrightarrow{z} = \overrightarrow{M}, \mathbf{let} \bigcirc \overrightarrow{w} = \overrightarrow{N} \,\mathbf{in}\, Q, \overrightarrow{P}$ *in* $R \to_\beta$
   $\mathbf{let} \bigcirc \overrightarrow{x}, \overrightarrow{w}, \overrightarrow{z} = \overrightarrow{M}, \overrightarrow{N}, \overrightarrow{P} \,\mathbf{in}\, R[y := Q]$
5. $\mathbf{let} \bigcirc \overrightarrow{x} = \mathbf{pure}\, \overrightarrow{M} \,\mathbf{in}\, N \to_\beta \mathbf{pure}\, N[\overrightarrow{x} := \overrightarrow{M}]$
6. $\mathbf{let} \bigcirc \_\_ = \_\_ \,\mathbf{in}\, M \to_\beta \mathbf{pure}\, M$, *where* $\_\_$ *is an empty sequence of terms*

If M reduces to N by one of these rules, then we will write $M \to_r N$. A multistep reduction $\twoheadrightarrow_r$ is a reflexive transitive closure of $\to_r$. $=_r$ is a symmetric closure of $\twoheadrightarrow_r$.

Now we formulate the standard lemmas for contexts.

Proposition 1. *The generation lemma for $\bigcirc_I$.*
Let $\Gamma \vdash \textbf{pure } M : \bigcirc\varphi$, then $\Gamma \vdash M : \varphi$.

Proof. Straightforwardly. □

Lemma 3. *Basic lemmas.*

1. *If $\Gamma \vdash M : \varphi$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash M : \varphi$*
2. *If $\Gamma \vdash M : \varphi$, then $\Delta \vdash M : \varphi$, where $\Delta = \{x : \psi \mid (x : \psi) \in \Gamma \ \& \ x \in FV(M)\}$*
3. *If $\Gamma, x : \varphi \vdash M : \phi$ and $\Gamma \vdash N : \varphi$, then $\Gamma \vdash M[x := N] : \psi$*

Proof.
The items 1-2 are proved by induction on the derivation of $\Gamma \vdash M : \varphi$. The item 3 is proved by induction on the derivation of $\Gamma \vdash N : \psi$.
□

Theorem 2. *Subject reduction*
If $\Gamma \vdash M : \varphi$ and $M \twoheadrightarrow_r N$, then $\Gamma \vdash N : \varphi$.

Proof. Induction on the derivation $\Gamma \vdash M : \varphi$ and on the generation of $\rightarrow_\beta$. The general statement follows from transitivity of $\twoheadrightarrow_\beta$ and Proposition 1 and Lemma 3. □

Theorem 3.
$\twoheadrightarrow_\beta$ is strongly normalising

Proof. Follows from Theorem 5 below, so far as reduction in monadic metalanguage is strongly normalising [7] and $\lambda_{\mathbf{IEL}^-}$ is sound with respect to this system. □

Theorem 4.
$\twoheadrightarrow_r$ is confluent.

Proof.
By Newman's lemma [63], if a given relation is strongly normalising and locally confluent, then this relation is confluent. It is sufficient to show that a multistep reduction is locally confluent.

Lemma 4. *Local confluence*
If $M \rightarrow_r N$ and $M \rightarrow_r Q$, then there exists some term P, such that $N \twoheadrightarrow_r P$ and $Q \twoheadrightarrow_r P$.

Proof. Let us consider the following critical pairs and show that they are joinable:

1.

$$\begin{tikzcd}
\textbf{let } \bigcirc x = (\textbf{let } \bigcirc \overrightarrow{y} = \textbf{pures } \overrightarrow{N} \textbf{ in } P) \textbf{ in } M \arrow[d, "\beta"'] \arrow[dr, "\beta"] & \\
\textbf{let } \bigcirc \overrightarrow{y} = \textbf{pure } \overrightarrow{N} \textbf{ in } M[x := P] & \textbf{let } \bigcirc x = \textbf{pure } P[\overrightarrow{y} := \overrightarrow{N}] \textbf{ in } M
\end{tikzcd}$$

$$\begin{array}{ll}
\textbf{let } \bigcirc \overrightarrow{y} = \textbf{pure } \overrightarrow{N} \textbf{ in } M[x := P] \rightarrow_\beta & \\
\qquad\qquad \textbf{pure } M[x := P][\overrightarrow{y} := \overrightarrow{N}] & \\
\textbf{let } \bigcirc x = \textbf{pure } P[\overrightarrow{y} := \overrightarrow{N}] \textbf{ in } M \rightarrow_\beta & \\
\qquad\qquad \textbf{pure } M[x := P[\overrightarrow{y} := \overrightarrow{N}]] \equiv & \\
\text{Since } x \notin \overrightarrow{y} & \\
\qquad\qquad \textbf{pure } M[x := P][\overrightarrow{y} := \overrightarrow{N}] &
\end{array}$$

2.

$$\begin{tikzcd}[column sep=small]
\textbf{let } \bigcirc x = (\textbf{let } \bigcirc \underline{\quad} = \underline{\quad} \textbf{ in } N) \textbf{ in } M \arrow[d, "\beta"] \arrow[dr, "\beta"] & \\
\textbf{let } \bigcirc \underline{\quad} = \underline{\quad} \textbf{ in } M[x := N] & \textbf{let } \bigcirc x = \textbf{pure } N \textbf{ in } M
\end{tikzcd}$$

$$\textbf{let } \bigcirc \underline{\quad} = \underline{\quad} \textbf{ in } M[x := N] \to_\beta \textbf{let } \bigcirc (M[x := N])$$
$$\textbf{let } \bigcirc x = \textbf{pure } N \textbf{ in } M \to_\beta \textbf{pure } (M[x := N])$$

□

One may consider four critical pairs analysed in the confluence proof for lambda-calculus based on the intuitionistic normal modal logic **IK** [38]. Those pairs are joinable in the observed calculus as well.

□

3.1 Relation with the monadic metalanguage

The monadic metalanguage is the modal lambda-calculus based on the categorical semantics of computation suggested by Moggi [51]. As we mentioned above, the monadic metalanguage might be considered as the type-theoretical representation of computation with an abstract data type of action. In fact, the monadic metalanguage is a type-theoretical formulation for monadic computation implemented in Haskell. Here we show that $\lambda_{\mathbf{IEL}^-}$ is sound with respect to the monadic metalanguage.

Definition 9. *The monadic metalanguage*

The monadic metalanguage extends the simply-typed lambda calculus with the additional typing rules:

$$\dfrac{\Gamma \vdash M : \varphi}{\Gamma \vdash \textbf{val}\, M : \nabla\varphi}\ \nabla_I \qquad\qquad \dfrac{\Gamma \vdash M : \nabla\varphi \qquad \Gamma, x : \varphi \vdash N : \nabla\psi}{\Gamma \vdash \textbf{let val}\, x = M \textbf{ in } N : \nabla\psi}\ let_\nabla$$

The reduction rules are the following ones (in addition to the standard rule for abstraction and application):

1. $\textbf{let val}\, x = \textbf{val}\, M \textbf{ in } N \to_\beta N[x := M]$;
2. $\textbf{let val}\, x = (\textbf{let val}\, y = N \textbf{ in } P) \textbf{ in } M \to_\beta \textbf{let val}\, y = N \textbf{ in } (\textbf{let val}\, x = P \textbf{ in } M)$;
3. $\textbf{let val}\, x = M \textbf{ in } x \to_\eta M$.

Let us define the translation $\ulcorner . \urcorner$ from $\lambda_{\mathbf{IEL}^-}$ to the monadic metalanguage:

1. $\ulcorner p_i \urcorner = p_i$, where p_i is atomic
2. $\ulcorner \varphi \to \psi \urcorner = \ulcorner \varphi \urcorner \to \ulcorner \psi \urcorner$
3. $\ulcorner \bigcirc\varphi \urcorner = \nabla \ulcorner \varphi \urcorner$

1. $\ulcorner x \urcorner = x$, x is a variable
2. $\ulcorner \lambda x.M \urcorner = \lambda x.\ulcorner M \urcorner$
3. $\ulcorner M\ N \urcorner = \ulcorner M \urcorner \ulcorner N \urcorner$
4. $\ulcorner \textbf{pure}\, M \urcorner = \textbf{val}\, \ulcorner M \urcorner$
5. $\ulcorner \textbf{let } \bigcirc \overrightarrow{x} = \overrightarrow{M} \textbf{ in } N \urcorner = \textbf{let val}\, \overrightarrow{x} = \ulcorner \overrightarrow{M} \urcorner \textbf{ in } \ulcorner N \urcorner$

where $\textbf{let val}\, \overrightarrow{x} = \ulcorner \overrightarrow{M} \urcorner \textbf{ in } N$ denotes $\textbf{let val}\, x_1 = \ulcorner M_1 \urcorner \textbf{ in } (\ldots \textbf{ in } (\textbf{let val}\, x_n = \ulcorner M_n \urcorner \textbf{ in } N) \ldots)$

It is clear that, if $\Gamma = \{x_1 : \varphi_1, \ldots, x_n : \varphi_n\}$ is a context, then $\ulcorner \Gamma \urcorner = \{x_1 : \ulcorner \varphi_1 \urcorner, \ldots, x_n : \ulcorner \varphi_n \urcorner\}$. Let us denote $\vdash_{\lambda_{\mathbf{IEL}^-}}$ as the derivability relation in $\lambda_{\mathbf{IEL}^-}$ in order to distinguish from the derivability in the monadic metalanguage.

Lemma 5.
If $\Gamma \vdash_{\lambda_{\mathbf{IEL}^-}} M : A$, *then* $\ulcorner\Gamma\urcorner \vdash \ulcorner M \urcorner : \ulcorner A \urcorner$ *in the monadic metalanguage.*

Proof. By induction on $\Gamma \vdash_{\lambda_{\mathbf{IEL}^-}} M : A$. One may prove the cases of $\Box_I$ and $\mathrm{let}_{\Box}$ as follows:

$$\frac{\ulcorner\Gamma\urcorner \vdash \ulcorner M\urcorner : \ulcorner A\urcorner}{\ulcorner\Gamma\urcorner \vdash \mathbf{val}\,\ulcorner M\urcorner : \nabla\ulcorner A\urcorner}$$

$$\frac{\ulcorner\Gamma\urcorner \vdash \ulcorner\overrightarrow{M}\urcorner : \nabla\ulcorner\overrightarrow{A}\urcorner \qquad \dfrac{\overrightarrow{x} : \ulcorner\overrightarrow{A}\urcorner \vdash \ulcorner N\urcorner : \ulcorner B\urcorner}{\overrightarrow{x} : \ulcorner\overrightarrow{A}\urcorner \vdash \mathbf{val}\,\ulcorner N\urcorner : \nabla\ulcorner B\urcorner}}{\ulcorner\Gamma\urcorner \vdash \mathbf{let\ val}\,\overrightarrow{x} = \ulcorner\overrightarrow{M}\urcorner\ \mathbf{in\ val}\,\ulcorner N\urcorner : \nabla\ulcorner B\urcorner}$$

□

Now one may formulate the following lemma:

Lemma 6.

1. $\ulcorner M[x := N]\urcorner = \ulcorner M\urcorner[x := \ulcorner N\urcorner]$;
2. $M \twoheadrightarrow_r N \Rightarrow \ulcorner M\urcorner \twoheadrightarrow_\beta \ulcorner N\urcorner$;

Proof.

1. Induction on the structure of M.
2. By the induction on $\rightarrow_r$:

   (a) For simplicity, we will consider the case with only one variable in **let** ◯ local binding, that can be easily extended to an arbitrary number of variables in local binding:
   $\ulcorner\mathbf{let}\ \bigcirc\, x = (\mathbf{let}\ \bigcirc\, \overrightarrow{y} = \overrightarrow{N}\ \mathbf{in}\ P)\ \mathbf{in}\ M\urcorner =$
   $\mathbf{let\ val}\, x = (\mathbf{let\ val}\,\overrightarrow{y} = \ulcorner\overrightarrow{N}\urcorner\ \mathbf{in\ val}\,\ulcorner P\urcorner)\ \mathbf{in\ val}\,\ulcorner M\urcorner \rightarrow_\beta$
   $\mathbf{let\ val}\,\overrightarrow{y} = \ulcorner\overrightarrow{N}\urcorner\ \mathbf{in}\ (\mathbf{let\ val}\, x = \ulcorner P\urcorner\ \mathbf{in\ val}\,\ulcorner M\urcorner) \rightarrow_\beta$
   $\mathbf{let\ val}\,\overrightarrow{y} = \ulcorner\overrightarrow{N}\urcorner\ \mathbf{in\ val}\,\ulcorner M\urcorner[x := \ulcorner P\urcorner] = \ulcorner\mathbf{let}\ \bigcirc\,\overrightarrow{y} = \overrightarrow{N}\ \mathbf{in}\ M[x := P]\urcorner$

   (b)
   $\ulcorner\mathbf{let}\ \bigcirc\,\overrightarrow{x} = \mathbf{pure}\,\overrightarrow{N}\ \mathbf{in}\ M\urcorner = \mathbf{let\ val}\,\overrightarrow{x} = \mathbf{val}\,\ulcorner\overrightarrow{N}\urcorner\ \mathbf{in\ val}\ulcorner M\urcorner \rightarrow_\beta$
   $\mathbf{val}\,\ulcorner M\urcorner[\overrightarrow{x} := \ulcorner\overrightarrow{N}\urcorner] = \ulcorner\mathbf{pure}\, M[\overrightarrow{x} := \overrightarrow{N}]\urcorner$

   (c) $\ulcorner\mathbf{let}\ \bigcirc\, x = M\ \mathbf{in}\ x\urcorner = \mathbf{let\ val}\, x = \ulcorner M\urcorner\ \mathbf{in\ val}\, x \rightarrow_\eta \ulcorner M\urcorner$

□

Theorem 5.
$\mathbf{IEL}^-$ *is sound with respect to the monadic metalanguage.*

Proof. Follows from the lemmas above. □

3.2 Categorical semantics

In this subsection, we provide a categorical semantics for the modal lambda calculus proposed above considering the co-reflection principle coalgebraically. Here we need a bit of category theory to investigate the categorical interpretation of this calculus. We recall the required definitions first. For the abstract definitions of category, functor, natural transformation see the book by Goldblatt [32] or the book by MacLane and Moerdijk [48].

We piggyback the construction used in the proof of the completeness for simply-typed lambda-calculus, see [1] and [44] to have comprehensive details.

Definition 10. *A category $\mathcal{C}$ is called cartesian closed if this category has products $A \times B$, exponentials B^A and a terminal object $\mathbb{1}$ such that the universal product and exponentiation properties hold.*

Following to Bellin et.al. [6] and Kakutani [38] [39], we interpret a modal operator as a monoidal endofunctor on a cartesian closed category. Monoidal endofunctors are introduced as morphisms of those categories that respect monoidal structure. Here we refer to the work by Eilenberg and Kelly for precise details [21]. Here we define a monoidal endofunctor on a cartesian closed category as the underlying notion.

Definition 11. *Let $\mathcal{C}$ be a cartesian closed category and $\mathbf{F} : \mathcal{C} \to \mathcal{C}$ an endofunctor. $\mathbf{F}$ is called monoidal if there exists a natural transformation m consists of components $m_{A,B} : \mathbf{F}A \times \mathbf{F}B \to \mathbf{F}(A \times B)$ and a natural transformation $u : \mathbb{1} \to \mathbf{F}\mathbb{1}$ such that the well-known diagrams commute (MacLane pentagon and triangle identity).*

Coalgebraic techniques are widely used in logic and computer science, see [12] [43] [66]. The abstract definition of a coalgebra is the following one:

Definition 12. *Let $\mathcal{C}$ be a category and $\mathbf{F} : \mathcal{C} \to \mathcal{C}$ an endofunctor. If $A \in \mathrm{Ob}(\mathcal{C})$, then an $\mathbf{F}$-coalgebra is a pair $\langle A, \alpha \rangle$, where $\alpha \in \mathrm{Hom}_{\mathcal{C}}(A, \mathbf{F}A)$. An $\mathbf{F}$-coalgebra homomorpism from $\langle A, \alpha \rangle$ to $\langle A, \beta \rangle$ is a map $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ such that the following square commutes:*

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d, "f"'] & \mathbf{F}A \arrow[d, "\mathbf{F}f"] \\
B \arrow[r, "\beta"'] & \mathbf{F}B
\end{tikzcd}$$

Given a natural transformation $\alpha : \mathrm{Id}_{\mathcal{C}} \to \mathbf{F}$, one may associate an $\mathbf{F}$-coalgebra $\langle A, \alpha_A \rangle$ for each $A \in \mathrm{Ob}(\mathcal{C})$. Homomorphisms of such coalgebras are defined by naturality.

Definition 13. *Let $\mathcal{C}$ be a cartesian closed category, $\mathbf{F} : \mathcal{C} \to \mathcal{C}$ a monoidal functor on $\mathcal{C}$, and $\alpha : \mathrm{Id}_{\mathcal{C}} \to \mathbf{F}$ a natural transformation. An $\mathbf{IEL}^-$-category is a pair $\langle \mathcal{C}, \mathbf{F}, \alpha \rangle$ such that the following coherence conditions hold:*

1. *$u = \alpha_{\mathbb{1}}$, where $\alpha_{\mathbb{1}}$*
2. *$m_{A,B} \circ (\alpha_A \times \alpha_B) = \alpha_{A \times B}$, i.e. the following diagram commutes:*

$$\begin{tikzcd}
A \times B \arrow[r, "\alpha_A \times \alpha_B"] \arrow[dr, "\alpha_{A \times B}"'] & \mathbf{F}A \times \mathbf{F}B \arrow[d, "m_{A,B}"] \\
& \mathbf{F}(A \times B)
\end{tikzcd}$$

The following construction is more or less describes the standard construction of semantical characterisation of typed lambda-calculus [1] [44]. First of all, let us define semantic brackets $[\![.]\!]$, a semantic translation from λ_{IEL^-} to the $\mathbf{IEL}^-$-category $\langle \mathcal{C}, \mathbf{F}, \alpha \rangle$. Suppose one has an assignment $\hat{\ }$ that maps every primitive type to some object of $\mathcal{C}$. Such semantic brackets $[\![.]\!]$ have the following inductive definition:

1. $[\![p_i]\!] := \hat{p_i}$
2. $[\![\varphi \to \psi]\!] := [\![\varphi]\!]^{[\![\psi]\!]}$
3. $[\![\varphi \times \psi]\!] := [\![\varphi]\!] \times [\![\psi]\!]$
4. $[\![\bigcirc\varphi]\!] = \mathbf{F}[\![\varphi]\!]$

We extend this interpretaion for contexts by induction too:

1. $[\![\quad]\!] = \mathbb{1}$, where $\mathbb{1}$ is a terminal object of a given CCC
2. $[\![\Gamma, x : \varphi]\!] = [\![\Gamma]\!] \times [\![\varphi]\!]$

Here is the interpretation of typing rules to define an interpretation for typing assignments $\Gamma \vdash M : A$ in means of arrows like $[\![\Gamma \vdash M : \varphi]\!] := [\![M]\!] : [\![\Gamma]\!] \to [\![\varphi]\!]$.

$$\dfrac{}{\pi_2 : [\![\Gamma]\!] \times [\![\varphi]\!] \to [\![\varphi]\!]} \qquad \dfrac{[\![M]\!] : [\![\Gamma]\!] \to [\![\varphi]\!]}{[\![M]\!] \circ \eta_{[\![\varphi]\!]} : [\![\Gamma]\!] \to \mathbf{F}[\![\varphi]\!]}$$

$$\dfrac{[\![M]\!] : [\![\Gamma]\!] \times [\![\varphi]\!] \to [\![\psi]\!]}{\Lambda([\![M]\!]) : [\![\Gamma]\!] \to [\![\psi]\!]^{[\![\varphi]\!]}} \qquad \dfrac{[\![M]\!] : [\![\Gamma]\!] \to [\![\psi]\!]^{[\![\varphi]\!]} \qquad [\![N]\!] : [\![\Gamma]\!] \to [\![\varphi]\!]}{\langle [\![M]\!], [\![N]\!] \rangle \circ \epsilon_{[\![\varphi]\!],[\![B]\!]} : [\![\Gamma]\!] \to [\![\psi]\!]}$$

$$\dfrac{[\![M]\!] : [\![\Gamma]\!] \to [\![\varphi]\!] \qquad [\![N]\!] : [\![\Gamma]\!] \to [\![\psi]\!]}{\langle [\![M]\!], [\![N]\!] \rangle : [\![\Gamma]\!] \to [\![\varphi]\!] \times [\![\psi]\!]} \qquad \dfrac{[\![M]\!] : [\![\Gamma]\!] \to [\![\varphi_1]\!] \times [\![\varphi_2]\!]}{[\![M]\!] \circ \pi_i : [\![\Gamma]\!] \to [\![\varphi_i]\!]}\ i = 1, 2$$

$$\dfrac{\langle [\![M_1]\!], \dots, [\![M_n]\!] \rangle : [\![\Gamma]\!] \to \prod\limits_{i=1}^{n} \mathbf{F}[\![\varphi_i]\!] \qquad [\![N]\!] : \prod\limits_{i=1}^{n} [\![\varphi_i]\!] \to [\![\psi]\!]}{\mathbf{F}([\![N]\!]) \circ m_{[\![\varphi_1]\!],\dots,[\![\varphi_n]\!]} \circ \langle [\![M_1]\!], \dots, [\![M_n]\!] \rangle : [\![\Gamma]\!] \to \mathbf{F}[\![\psi]\!]}$$

An interpretation for **let**$\bigcirc$-rule is similar to interpretation for $\Box$-rule in term calculus for intutionistic **K** [6]. The semantic brackets respect substitution and reduction:

Lemma 7.

1. *$[\![M[x_1 := M_1, \dots, x_n := M_n]]\!] = [\![M]\!] \circ \langle [\![M_1]\!], \dots, [\![M_n]\!] \rangle$*
2. *Let $\Gamma \vdash M : A$ and $M \to_r N$, then $[\![\Gamma \vdash M : A]\!] = [\![\Gamma \vdash N : A]\!]$*

Proof.

1. By simple induction on M. Let us check only the modal cases.

$[\![\Gamma \vdash (\mathbf{pure}\, M)[\vec{x} := \vec{M}] : \bigcirc\varphi]\!] = [\![\Gamma \vdash \mathbf{pure}\,(M[\vec{x} := \vec{M}]) : \bigcirc\varphi]\!] =$
$\eta_{[\![A]\!]} \circ [\![(M[\vec{x} := \vec{M}])]\!] = \alpha_{[\![A]\!]} \circ ([\![M]\!] \circ \langle [\![M_1]\!], \dots, [\![M_n]\!] \rangle) =$
$(\alpha_{[\![A]\!]} \circ [\![M]\!]) \circ \langle [\![M_1]\!], \dots, [\![M_n]\!] \rangle = [\![\Gamma \vdash \mathbf{pure}\, M : \bigcirc\varphi]\!] \circ \langle [\![M_1]\!], \dots, [\![M_n]\!] \rangle$

$[\![\Gamma \vdash (\mathbf{let} \bigcirc\ \vec{x} = \vec{M}\ \mathbf{in}\ N)[\vec{y} := \vec{P}] : \bigcirc\psi]\!] = [\![\Gamma \vdash \mathbf{let} \bigcirc \vec{x} = (\vec{M}[\vec{y} := \vec{P}])\ \mathbf{in}\ N : \bigcirc\psi]\!] =$
$\mathbf{F}([\![N]\!]) \circ m_{[\![\varphi_1]\!],\dots,[\![\varphi_n]\!]} \circ [\![\Gamma \vdash (\vec{M}[\vec{y} := \vec{P}]) : \bigcirc\vec{\varphi}]\!] =$
$\mathbf{F}([\![N]\!]) \circ m_{[\![\varphi_1]\!],\dots,[\![\varphi_n]\!]} \circ [\![\vec{M}]\!] \circ \langle [\![P_1]\!], \dots, [\![P_n]\!] \rangle =$
$[\![\Gamma \vdash \mathbf{let} \bigcirc \vec{x} = \vec{M}\ \mathbf{in}\ N : \bigcirc\varphi]\!] \circ \langle [\![P_1]\!], \dots, [\![P_n]\!] \rangle$

2. The cases with β-reductions for $let_{\bigcirc}$ are shown in [38]. Those cases are similar to ours. Let us consider the cases with the **pure** terms. Those cases immediately follows from the coherence conditions of an $\mathbf{IEL}^{-}$-category and the previous item of this lemma.

   (a) $[\![\Gamma \vdash \mathbf{let} \bigcirc \vec{x} = \mathbf{pure}\, \vec{M}\ \mathbf{in}\ N : \bigcirc\psi]\!] = [\![\Gamma \vdash \mathbf{pure}\, N[\vec{x} := \vec{M}] : \bigcirc\psi]\!]$

   $[\![\Gamma \vdash \mathbf{let} \bigcirc \vec{x} = \mathbf{pure}\, \vec{M}\ \mathbf{in}\ N : \Box B]\!] =$
   $\mathbf{F}([\![N]\!]) \circ m_{[\![\varphi_1]\!],\dots,[\![\varphi_n]\!]} \circ \langle \alpha_{[\![\varphi_1]\!]} \circ [\![M_1]\!], \dots, \alpha_{[\![\varphi_n]\!]} \circ [\![M_n]\!] \rangle =$
   $\mathbf{F}([\![N]\!]) \circ m_{[\![\varphi_1]\!],\dots,[\![\varphi_n]\!]} \circ (\alpha_{[\![\varphi_1]\!]} \times \dots \times \alpha_{[\![\varphi_n]\!]}) \circ \langle [\![M_1]\!], \dots, [\![M_n]\!] \rangle =$
   $\mathbf{F}([\![N]\!]) \circ \alpha_{[\![\varphi_1]\!] \times \dots \times [\![\varphi_n]\!]} \circ \langle [\![M_1]\!], \dots, [\![M_n]\!] \rangle = \alpha_{[\![\psi]\!]} \circ [\![N]\!] \circ \langle [\![M_1]\!], \dots, [\![M_n]\!] \rangle =$
   $\alpha_{[\![\psi]\!]} \circ [\![\Gamma \vdash N[\vec{x} := \vec{M}] : \psi]\!] = [\![\Gamma \vdash \mathbf{pure}\,(N[\vec{x} := \vec{M}]) : \bigcirc\psi]\!]$

   (b) $[\![\vdash \mathbf{let} \bigcirc \_\_ = \_\_\ \mathbf{in}\ M : \bigcirc\varphi]\!] = [\![\vdash \mathbf{pure}\, M : \bigcirc\varphi]\!]$

   $[\![\vdash\vdash \mathbf{let} \bigcirc = \_\_\ \mathbf{in}\ M : \bigcirc\varphi]\!] =$
   $\mathbf{F}([\![M]\!]) \circ u = \mathbf{F}([\![M]\!]) \circ \alpha_{\mathbb{1}} = \alpha_{[\![A]\!]} \circ [\![M]\!] = [\![\vdash \mathbf{pure}\, M : \bigcirc\varphi]\!]$

$\Box$

The following soundness theorem follows from the lemma above and the whole construction:

Theorem 6. *Soundness*
Let $\Gamma \vdash M : \varphi$ and $M =_r N$, then $[\![\Gamma \vdash M : \varphi]\!] = [\![\Gamma \vdash N : \varphi]\!]$

The completeness theorem is proved via the syntactic model. We will consider term model for the simply-typed lambda-calculus with $\times$ and $\to$ standardly described in [1] [44].

Let us define a binary relation on lambda-terms$\sim_{\varphi,\psi} \subseteq (\mathbb{V} \times \Lambda_{\Box})^2$ as :

$$(x, M) \sim_{\varphi,\psi} (y, N) \Leftrightarrow x : \varphi \vdash M : \psi \ \& \ y : \varphi \vdash N : \psi \ \& \ M =_r N[y := x];$$

We will denote equivalence class as $[x, M]_{\varphi,\psi} = \{(y, N) \mid (x, M) \sim_{\varphi,\psi} (y, N)\}$ (we will drop indices below). Let us recall the definition of the category $\mathcal{C}(\lambda)$, a model structure for the simply-typed lambda calculus.

A category $\mathcal{C}(\lambda)$ has the class of objects defined as $\mathrm{Ob}_{\mathcal{C}(\lambda)} = \{\hat{\varphi} \mid \varphi \in \mathbb{T}\} \cup \{\mathbb{1}\}$. For $\hat{\varphi}, \hat{\psi} \in \mathrm{Ob}_{\mathcal{C}(\lambda)}$, the set of morphism has the form $\mathrm{Hom}_{\mathcal{C}(\lambda)}(\hat{\varphi}, \hat{\psi}) = \{[x, M] \mid x : \varphi \vdash M : \psi\}$. Let $[x, M] \in \mathrm{Hom}_{\mathcal{C}(\lambda)}(\hat{\varphi}, \hat{\psi})$ and $[y, N] \in \mathrm{Hom}_{\mathcal{C}(\lambda)}(\hat{\psi}, \hat{\theta})$, then $[y, M] \circ [x, M] = [x, N[y := M]]$. Identity morphisms are $id_{\hat{\varphi}} = [x, x]$.

The category $\mathcal{C}(\lambda)$ is cartesian closed since $\mathbb{1}$ is a terminal object such that $\mathrm{Hom}_{\mathcal{C}(\lambda)}(\mathbb{1}, \hat{\varphi}) = \{[\blacksquare, M] \mid \ \vdash M : \varphi \text{is provable}\}$; $\widehat{\varphi \times \psi} = \hat{\varphi} \times \hat{\psi}$; and $\widehat{\varphi \to \psi} = \hat{\psi}^{\hat{\varphi}}$. Canonical projections are defined as $[x, \pi_i x] \in \mathrm{Hom}_{\mathcal{C}(\lambda)}(\hat{\varphi}_1 \times \hat{\varphi}_2, \hat{\varphi}_i)$ for $i = 1, 2$. The evaluation arrow is a morphism $ev_{\hat{\varphi},\hat{\psi}} = [x, (\pi_1 x)(\pi_2 x)] \in \mathrm{Hom}_{\mathcal{C}(\lambda)}(\hat{\psi}^{\hat{\varphi}} \times \hat{\varphi}, \hat{\psi})$.

Let us define a map $\mathbf{F} : \mathcal{C}(\lambda) \to \mathcal{C}(\lambda)$, such that forall $[x, M] \in \mathrm{Hom}_{\mathcal{C}(\lambda)}(\hat{\varphi}, \hat{\psi}), \boxdot([x, M]) = [y, \mathbf{let} \bigcirc x = y \ \mathbf{in} \ M] \in \mathrm{Hom}_{\mathcal{C}(\lambda)}(\boxdot\hat{\varphi}, \boxdot\hat{\psi})$. The following functoriality condition might be easily checked with the reduction rules:

1. $\boxdot(g \circ f) = \boxdot g \circ \boxdot f$;
2. $\boxdot(id_{\hat{A}}) = id_{\boxdot\hat{A}}$.

We define the following maps. $\eta : Id_{\mathcal{C}(\lambda)} \to \mathbf{F}$ such that for each $\hat{\varphi} \in \mathrm{Ob}_{\mathcal{C}(\lambda)}$ one has $\eta_{\hat{\varphi}} = [x, \mathbf{pure} \ x] \in \mathrm{Hom}_{\mathcal{C}(\lambda)}(\hat{A}, \boxdot\hat{A})$. We express a monoidal transformation as $m_{\hat{\varphi},\hat{\psi}} : \mathbf{F}\hat{\varphi} \times \mathbf{F}\hat{\psi} \to \mathbf{F}(\hat{\varphi} \times \hat{\psi})$ such that one has $m_{\hat{\varphi},\hat{\psi}} = [p, \mathbf{let} \bigcirc x, y = \pi_1 p, \pi_2 p \ \mathbf{in} \ \langle x, y\rangle] \in \mathrm{Hom}_{\mathcal{C}(\lambda)}(\boxdot\hat{\varphi} \times \boxdot\hat{\psi}, \boxdot(\hat{\varphi} \times \hat{\psi}))$. Also we express $u_{\mathbb{1}}$ as $[\blacksquare, \mathbf{let} \bigcirc \_\_ = \_\_ \ \mathbf{in} \ \blacksquare]$.

$\mathbf{F}$ is a monoidal endofunctor, see, e.g. [39]. Let us check the required coherence conditions:

Lemma 8.

1. $\mathbf{F}(f) \circ \alpha_{\varphi} = \alpha_{\beta} \circ f$
2. $(m_{\hat{\varphi},\hat{\psi}}) \circ (\alpha_{\varphi} \times \alpha_{\beta}) = \alpha_{\varphi \times \alpha_{\beta}}$
3. $u_{\mathbb{1}} = \eta_{\mathbb{1}}$

Proof.

1. $\eta_{\hat{\psi}} \circ f = [y, \mathbf{pure} \ y] \circ [x, M] = [x, \mathbf{pure} \ y[y := M]] = [x, \mathbf{pure} \ M]$
   From the other hand, one has:
   $\boxdot f \circ \eta_{\hat{A}} =$
   $[z, \mathbf{let} \bigcirc x = z \ \mathbf{in} \ M] \circ [x, \mathbf{pure} \ x] = [x, \mathbf{let} \bigcirc x = z \ \mathbf{in} \ M[z := \mathbf{pure} \ x]] =$
   $[x, \mathbf{let} \bigcirc x = \mathbf{pure} \ x \ \mathbf{in} \ M] = [x, \mathbf{pure} \ M[x := x]] = [x, \mathbf{pure} \ M]$
2. $m_{\hat{A},\hat{B}} \circ (\eta_{\hat{A}} \times \eta_{\hat{B}}) = \eta_{\hat{A} \times \hat{B}}$

   $m_{\hat{A},\hat{B}} \circ (\eta_{\hat{A}} \times \eta_{\hat{B}}) =$
   $[q, \mathbf{let} \bigcirc x, y = \pi_1 q, \pi_2 q \ \mathbf{in} \ \langle x, y\rangle] \circ [p, \langle \mathbf{pure} \ (\pi_1 p), \mathbf{pure} \ (\pi_2 p)\rangle] =$
   $[p, \mathbf{let} \bigcirc x, y = \pi_1 q, \pi_2 q \ \mathbf{in} \ \langle x, y\rangle[q := \langle \mathbf{pure} \ (\pi_1 p), \mathbf{pure} \ (\pi_2 p)\rangle]] =$
   $[p, \mathbf{let} \bigcirc x, y = \pi_1(\langle \mathbf{pure} \ (\pi_1 p), \mathbf{pure} \ (\pi_2 p)\rangle), \pi_2(\langle \mathbf{pure} \ (\pi_1 p), \mathbf{pure} \ (\pi_2 p)\rangle) \ \mathbf{in} \ \langle x, y\rangle] =$
   $[p, \mathbf{let} \bigcirc x, y = \mathbf{pure} \ (\pi_1 p), \mathbf{pure} \ (\pi_2 p) \ \mathbf{in} \ \langle x, y\rangle] =$
   $[p, \mathbf{pure} \ (\langle x, y\rangle[x := \pi_1 p, y := \pi_2 p])] = [p, \mathbf{pure} \ \langle \pi_1 p, \pi_2 p\rangle] = [p, \mathbf{pure} \ p] = \eta_{\hat{A} \times \hat{B}}$
3. Immediately.

□

We summaries the results of the previous constructions and lemmas that standardly implies the categorical completeness.

Lemma 9. $\langle \mathcal{C}(\lambda), \boxdot, \eta\rangle$ *is an* $\mathbf{IEL}^-$*-category*

4 Prenuclear algebras and their representation

4.1 The background on locales, nuclei, and localic cover systems

A Heyting algebra is a bounded distributive lattice $\mathcal{H} = \langle H, \wedge, \vee, \bot, \top \rangle$ with the operation $\Rightarrow$ such that the following quasi-identities hold:

$$a \wedge b \leqslant c \text{ iff } a \leqslant b \Rightarrow c$$

Recall that a *locale* is a complete lattice $\mathcal{L} = \langle L, \wedge, \bigvee \rangle$ with the infinite distributive law:

$$a \wedge \bigvee B = \bigvee \{a \wedge b \mid b \in B\} \text{ for each } B \subseteq L$$

The notion of a locale coincides with the notion of a complete Heyting algebra since an implication might uniquely defined for each $a, b \in L$ as

$$a \Rightarrow b = \bigvee \{c \in \mathcal{L} \mid a \wedge c \leqslant b\}$$

Here we note that the categories of complete Heyting algebras and locales are not the same since their classes of morphisms are different. We don't take into consideration these categories, so here we assume that locale and complete Heyting algebra are synonymical terms [3].

A locale is a central object in such discipline as point-free topology [4], where a locale is a lattice-theoretic counterpart of a topological space. The aim of this discipline is to study point-set topology concerning topological spaces only with the structure of their topologies as lattices without mentioning points. For the further discussion see [36] [37] [48] [56]. In usual point-set topology one often interested in subspaces. In point-free topology, subspaces are characterised via operators on a locale called *nuclei*. A nucleus on a Heyting algebra is a multiplicative closure operator or a completion operator according to the Dragalin's terminology [20]. More precisely:

Definition 14. *A nucleus on a Heyting algebra $\mathcal{H}$ is a monotone map $j : \mathcal{L} \to \mathcal{L}$ such that*

1. $a \leqslant ja$
2. $ja = jja$
3. $j(a \wedge b) = ja \wedge jb$

One may consider a nucleus operator as a lattice-theoretic analogue of a Lawvere-Tierney topology that generalises the notion of a Grothendieck topology on a presheaf topos. In its turn, Lawvere-Tierney topology provides a modal operator often called a geometric modality [45]. Here, one may read $j\varphi$ as "it is locally the case that φ". The logic of Heyting algebras with a nucleus operator was studied by Goldblatt from a Kripkean and topos-theoretic perspectives, see [28] and [32] as well.

It is also well-known that the set of fixpoints of a nucleus on a Heyting algebra is a Heyting subalgebra. From a perspective of point-free topology, nuclei characterise sublocales [56]. Moreover, those operators play a huge role in a locale representation. In this monograph [20], Dragalin showed that any complete Heyting algebra is isomorphic to the locale of fixpoints of a nucleus operator on the locale of up-sets. Moreover, any spatial locale (the lattice of open sets of a topological space) is isomorphic to the complete Heyting algebra of fixpoints of a nucleus operator generated by a suitable Dragalin frame. We recall that a Dragalin frame is a structure that generalises both Kripke and Beth semantics of intuitionistic logic. Bezhanishvili and Holliday strengthened this result for arbitrary complete Heyting algebras, see [10] and [11] as well. The alternative way of a locale representation was studied by Goldblatt [31] [33] with cover systems. Perhaps, Dragalin frames and Goldblatt cover systems may be connected to each other somehow, but it seems that the relationship between them is not investigated yet [5].

[3] There is a third synonym for locales and complete Heyting algebras called *frame*, but this term is slightly overused within our context.

[4] Such a topology if often called *pointless*, but we find the point-free topology term appropriate

[5] This note is based on the recent conversation between Prof. Valentin Shehtman and the author

We examine Goldblatt's framework closely. First of all, let us recall some helpful notions. Let $\langle P, \leqslant\rangle$ be a poset. A subset $A \subseteq P$ is called *upwardly closed*, if $x \in A$ and $x \leqslant y$ implies $y \in A$. For $A \subseteq P$, $\uparrow A = \{x \in P \mid \exists y \in A\, y \leqslant x\}$. If $x \in P$, then the *cone at* x is an up-set $\uparrow x = \uparrow \{x\}$. A subset $Y \subseteq P$ refines a subset $X \subseteq P$ if $Y \subseteq \uparrow X$. By $\mathrm{Up}(P, \leqslant)$ we will mean the poset (in fact, the locale) of all upwardly closed subsets of a partial order $\langle P, \leqslant\rangle$. It is also clear that the set of all upwardly closed sets forms a locale.

Here we consider triples $\mathcal{S} = \langle P, \leqslant, \rhd\rangle$, where $\langle P, \leqslant\rangle$ is a poset and $\rhd$ is a binary relation between P and $\mathcal{P}(P)$. Given $x \in P$ and $C \subseteq P$, then we say that x is *covered* by C (C is an x-cover), if $x \rhd C$ ($C \lhd x$). Cover systems were presented to study local truth that comes from a topological and topos-theoretic intuition. A statement is locally true concerning some object as topological space or an open subset if this object has an open cover in each member of which the statement is true. For instance, such a statement might be a local equality of continuous maps, see [28] and also [29]. An abstract cover system has the following definition:

Definition 15. *A triple $\mathcal{S} = \langle P, \leqslant, \rhd\rangle$ as above is called cover system, if the following axioms hold for $x \in P$:*

1. *(Existence) There exists an x-cover C such that $C \subseteq \uparrow x$*
2. *(Transitivity) Let $x \rhd C$ and for each $y \in C$ $y \rhd C_y$, then $x \rhd \bigcup_{y \in C} C_y$*
3. *(Refinement) If $x \leqslant y$, then any x-cover might be refined to a y-cover. That is, $C \rhd x$ implies that there exists an y-cover C' such that $C' \subseteq C$*

Let $\mathcal{S}$ be a cover system, let us define an operator $j : \mathcal{P}(P) \to \mathcal{P}(P)$ as

$$jX = \{x \in P \mid \exists C\, x \rhd C \subseteq X\}$$

If $x \in jX$ is called a local member of X. A subset $X \subseteq P$ is called localised if $jX \subseteq X$. A localised up-set is called a *proposition*. $\mathrm{Prop}(\mathcal{S})$ is the set of all propositions of a cover system. Goldblatt showed that such an operator is a closure operator on a locale of all up-sets [31] that follows from the axioms of a cover system. According to this observation, a subset X is a proposition iff $X = \uparrow X = jX$.

Definition 16. *A cover system is called localic, if the following axiom hold*

Every x-cover can be refined to an x-cover that is included in $\uparrow x$.

That is, $x \rhd C$ implies that there exists $x \rhd C'$ such that $C' \subseteq \uparrow C$ and $C' \subseteq \uparrow x$

This localic axiom makes an j-operator a nucleus. That is, if $\mathcal{S} = \langle P, \leqslant, \rhd\rangle$ is a localic cover system, then $\mathrm{Prop}(\mathcal{S})$ is a sublocale of $\mathrm{Up}(P, \leqslant)$ since the set of fixpoints of nucleus is a sublocale of $\mathrm{Up}(P, \leqslant)$. Here we strengthen the fourth axiom of a localic cover system as:

Every x-cover is included in $\uparrow x$

Such a local cover system is called a *strictly localic cover system.* The stronger fourth axiom is built in such generalisation of open-covers systems as Grothendieck topology and cover schemes [5] [48].

The representation theorem for an abritrary locale is the following one [31]:

Theorem 7. *Let $\mathcal{L}$ be a locale, then there exists a strictly localic cover system $\mathcal{S}$ such that* $\mathcal{L} \cong \mathrm{Prop}(\mathcal{S})$

Proof. We provide a proof sketch in order to remain the paper self-contained.

Given a locale $\mathcal{L} = \langle L, \bigvee, \wedge\rangle$. Let us define $\mathcal{S}_{\mathcal{L}} = \langle L, \sqsubseteq, \rhd\rangle$ such that $x \sqsubseteq y$ iff $y \leqslant x$ and $x \rhd C$ iff $x = \bigvee C$ in $\mathcal{L}$. Then $\mathcal{S}_{\mathcal{L}}$ is a localic cover system. The strictness follows from the fact that if $x \rhd C$, that is, $x = \bigvee C$, then $C \subseteq (x] = \{y \mid y \sqsubseteq x\}$. Every cone $(x] = \uparrow x$ is localised, thus, $(x]$ is a proposition. It is not to so difficult to see that an arbitrary proposition of $\mathcal{S}_{\mathcal{L}}$ has the form of downset of $\sqsubseteq$.

An isomorphism itself is established with the map $x \mapsto (x]$. $\square$

As a consequence, one has a uniform embedding for arbitrary Heyting algebras as follows:

Theorem 8. *Every Heyting algebra is isomorphic to a subalgebra of propositions of a suitable strictly localic cover system.*

Proof. Every Heyting algebra has a Dedekind-Macneille completion $\mathcal{H} \hookrightarrow \mathcal{H}'$, where $\mathcal{H}'$ is a locale, see [31]. But $\mathcal{H}'$ is isomorphic to the locale of propositions of a strictly localic cover system $\mathcal{S}_{\mathcal{H}'}$. $\square$

Strictly localic cover systems provides an alternative model structures for intuitionistic predicate logic. Let $\mathcal{S} = \langle P, \leqslant, \rhd \rangle$ be a strictly localic cover system and D be a non-empty, a domain of individuals. Let V be a valuation function that maps each k-ary predicate letter P to $V(P) : D^k \to \text{Prop}(\mathcal{S})$. To interpret variable, we use D-assignments that have the form of infinite sequences $\sigma = \langle \sigma_0, \sigma_1, \dots, \sigma_n, \dots \rangle$, where $\sigma_i \in D$ for each $i < \omega$. A D-assignment maps each variable x_i to the corresponding σ_i. Given an assignment σ and $d \in D$, then $\sigma(d/n)$ is a D-assignment obtained from σ replacing σ_n to d

By IPL-model we will mean a structure $\mathfrak{M} = \langle \mathcal{S}, D, V \rangle$, where $\mathcal{S}$ is a strictly localic cover system, D is a domain of individuals, and V is a D-valuation. Given a D-assignment and $x \in \mathcal{S}$, let us define the truth relation $\mathfrak{S}, x, \sigma \models \varphi$ inductively:

1. $\mathfrak{M}, x, \sigma \Vdash P(x_{n_1}, \dots, x_{n_k})$ iff $x \in V(P)(\sigma_{n_1}, \dots, \sigma_{n_k})$
2. $\mathfrak{M}, x, \sigma \Vdash \varphi \wedge \psi$ iff $\mathfrak{M}, x, \sigma \Vdash \varphi$ and $\mathfrak{M}, x, \sigma \Vdash \psi$
3. $\mathfrak{M}, x, \sigma \Vdash \varphi \wedge \psi$ iff there exists an x-cover C such that for each $y \in C$ $\mathfrak{M}, x, \sigma \Vdash \varphi$ or $\mathfrak{M}, x, \sigma \Vdash \psi$
4. $\mathfrak{M}, x, \sigma \Vdash \varphi \to \psi$ iff for all $y \in \uparrow x$, if $\mathfrak{M}, y, \sigma \Vdash \varphi$ implies $\mathfrak{M}, y, \Vdash \psi$
5. $\mathfrak{M}, x, \sigma \Vdash \forall x_n \varphi$ iff for each $d \in D$, $\mathfrak{M}, x, \sigma(d/n) \Vdash \varphi$
6. $\mathfrak{M}, x, \sigma \Vdash \exists x_n \varphi$ iff there exist an x-cover C and $d \in D$ such that for each $y \in C$ one has $\mathfrak{M}, y, \sigma(d/n) \Vdash \varphi$

With each formula φ one may associate a truth set $||\varphi||^{\mathfrak{M}}_{\sigma}$ defined in means of locale operations on $\text{Prop}(\mathcal{S})$:

1. $||P(x_{n_1}, \dots, x_{n_k})||^{\mathfrak{M}}_{\sigma} = V(P)(\sigma_{n_1}, \dots, \sigma_{n_k})$
2. $||\varphi \wedge \psi||^{\mathfrak{M}}_{\sigma} = ||\varphi||^{\mathfrak{M}}_{\sigma} \cap ||\psi||^{\mathfrak{M}}_{\sigma}$
3. $||\varphi \vee \psi||^{\mathfrak{M}}_{\sigma} = j(||\varphi||^{\mathfrak{M}}_{\sigma} \cup ||\psi||^{\mathfrak{M}}_{\sigma})$
4. $||\varphi \to \psi||^{\mathfrak{M}}_{\sigma} = ||\varphi||^{\mathfrak{M}}_{\sigma} \Rightarrow ||\psi||^{\mathfrak{M}}_{\sigma}$
5. $||\forall x_n \varphi||^{\mathfrak{M}}_{\sigma} = \bigwedge\limits_{d \in D} ||\varphi||^{\mathfrak{M}}_{\sigma(d/n)}$
6. $||\exists x_n \varphi||^{\mathfrak{M}}_{\sigma} = j(\bigvee\limits_{d \in D} ||\varphi||^{\mathfrak{M}}_{\sigma(d/n)})$

where j is an associated nucleus on the locale of $\mathcal{S}$-propositions.

Thus, one has the completeness theorem:

Theorem 9. *Intuitionistic first-order logic is sound and complete with respect to IPL-models.*

Here we note that this constuction admits generalisations and provide complete semantics for predicate substructural logics, see, e. g., [30].

Now we define modal cover systems. Suppose one has a localic cover system $\mathcal{S} = \langle S, \leqslant, \rhd \rangle$. We seek to extend $\mathcal{S}$ with a binary relation R on S that yields an operator on $\mathcal{P}(S)$:

$$\langle R \rangle A = \{x \in S \mid \exists y \in A \, xRy\} = R^{-1}(A)$$

Definition 17. *A quadruple $\mathcal{M} = \langle S, \leqslant, \rhd, R \rangle$ is called a modal cover system, if a triple $\langle S, \leqslant, \rhd \rangle$ is a strictly localic cover system and the following conditions hold:*

1. *(Confluence) If $x \leqslant y$ and xRz, then there exists w such that yRw and $z \leqslant w$*
2. *(Modal localisation) If there exists C such that $x \rhd C \subseteq \langle R \rangle A$, then there exists $y \in R(x)$ with a y-cover included in X*

The first condition is a general requirement for intuitionistic modal logic that allows $\langle R \rangle A$ to be up-set whenever A is an up-set. The modal localisation claims that $\mathrm{Prop}(\mathcal{M})$ is closed under $\langle R \rangle$.

There is a representation theorem for locale with monotone operators, see [31] to have a proof in detail:

Theorem 10. *Let $\mathcal{L}$ be a locale and $m : \mathcal{L} \to \mathcal{L}$ a monotone map on $\mathcal{L}$, then the algebra $\langle \mathcal{L}, m \rangle$ is isomorphic to the algebra $\langle \mathrm{Prop}(\mathcal{S}_{\mathcal{L}}), \langle R_m \rangle) \rangle$*

Proof. As we already know by Theorem 7, $\mathcal{L} = \langle L, \bigvee, \wedge \rangle$ is isomorphic to the locale $\mathrm{Prop}(\mathcal{S}_{\mathcal{L}})$. $\mathcal{S}_{\mathcal{L}} = \langle L, \sqsubseteq, \rhd \rangle$ is a strictly localic cover system, where $x \sqsubseteq y$ iff $y \leqslant x$ and $x \rhd C$ iff $x = \bigvee C$ in $\mathcal{L}$. We recall that this isomorphism was established with map $x \mapsto (x] = \{y \in L | y \sqsubseteq x\}$. Let us put $x R_m y$ iff $x \leqslant my$. The relation is well-defined and the confluence and modal localisation conditions holds. Here the key observation $(ma] = \langle R_m \rangle (a]$ gives the desired isomorphism. □

Goldblatt introduced modal cover systems to provide semantics for quantified lax logic and intuitionistic counterparts of the modal predicate logics **K** and **S**4 [31]. In the next subsection, we introduce the similar cover systems to provide complete semantics for intuitionistic predicate modal logics with $\mathbf{IEL}^{-}$-like modalities.

4.2 Prenuclei operators

In this subsection, we discuss prenuclei operators, overview their use cases and provide representation for Heyting algebra with equipped prenucleus operators via modal localic cover systems.

A weaker version of nuclei operators is quite helpful in point-free topology. Let us discuss a prenucleus operator.

Definition 18.

Let $\mathcal{H}$ be a Heyting algebra, a prenucleus on $\mathcal{H}$ is an operator monotone $j : \mathcal{H} \to \mathcal{H}$ such that for each $a, b \in \mathcal{H}$:

1. $a \leqslant ja$
2. $ja \wedge b \leqslant j(a \wedge b)$.

A prenucleus is called multiplitcative, if it distributives over finite infima.

By *prenuclear algebra*, we will mean a pair $\langle \mathcal{H}, j \rangle$, where j is a prenucleus on $\mathcal{H}$. A prenuclear algebra is localic, when its Heyting reduct is a locale. A prenuclear algebra is *multiplitcative*, if its prenucleus is multiplicative. Simmons calls multiplicative prenuclei merely as prenuclei [61], but this term is more spread for operators as defined above, see, e.g. [56]. We introduce the term "multiplicative prenucleus" in order to distinguish all those operators from each other since we are going to consider both of them.

Prenuclei operators are applied in point-free topology in factorising locales considering sublocales as quotients. See the paper by Banaschewski [4] and the monograph by Picado and Pultr [56] for the discussion in detail. Here we note that one may generate nucleus from the generated by a sequence prenuclei parametrised over ordinals.

Multiplicative prenuclei are used in the study of lattices of nuclei. Given a locale, its nuclei form a complete Heyting algebra with the pointwise ordering. In this locale, meets are defined pointwise, but joins are slightly awkward to have an explicit definition. The reason is pointwise joins are not idempotent generally. Multiplicative prenuclei were introduced to have a suitable description of nuclei join in such a locale. Multiplicative prenuclei form a complete locale as well. Such operators are closed under composition and pointwise directed joins. Thus, one may define joins of nuclei in means of so-called nuclear reflection, an approximation of nucleus via prenuclei. Here we refer the reader to this paper [23], where this aspect is explained more comprehensively.

The other aspect of multiplicative prenuclei were studied by Haykazyan and Simmons [35] [61]. They consider the special multiplicative prenucleus. On a bounded distributive lattice $\mathcal{L}$, one may introduce a preorder $\preceq$ defined as follows for each $a, b \in \mathcal{L}$:

$$a \preceq b \Leftrightarrow \forall c \in \mathcal{L} \ a \vee c = \top \Rightarrow b \vee c = \top$$

If this preorder on a locale is agreed with the parent order, then this complete Heyting algebra is called subfit. This preorder also has an associated map $\xi : a \mapsto \bigvee\{b \in \mathcal{L} \mid b \preceq a\}$ as it is observed by Coquand [16]. ξ is a prenucleus on an arbitrary locale as it is shown by Simmons [61], where he studies certain properties of nuclei on the locale of ideals of a complete Heyting algebra. Moreover, one may associate a certain nucleus with the prenucleus ξ to measure the subfitness of a locale.

Let us define prenuclear cover systems to have a suitable representation for prenuclear algebras.

Definition 19. *Let $\mathcal{S} = \langle S, \preceq, \rhd, R\rangle$ be a modal cover system, then $\mathcal{S}$ is called prenuclear, if the following two conditions hold:*

1. *R is reflexive*
2. *Let $x, y \in S$ such that xRy, then there exists $z \in \uparrow y$ such that $x \leqslant z$ and $x \in R(z)$*

One may visualise the second condition with the following diagram:

$$\begin{tikzcd}
x \arrow[r, dashed, "R", "\leqslant"'] \arrow[d, "R"'] & \exists z \\
y \arrow[ur, dashed, "\leqslant"'] &
\end{tikzcd}$$

This lemma claims that a prenuclear cover system is well-defined as follows, the similar statement was proved by Goldblatt for nuclear cover systems [31]:

Lemma 10. *Let $\mathcal{S} = \langle P, \leqslant, \rhd, R\rangle$ be a prenuclear cover system, then $\langle R\rangle$ is a prenucleus on* $\operatorname{Prop}(\mathcal{S})$*, that is for each $A, B \in \operatorname{Prop}(\mathcal{S})$:*

1. *$A \subseteq \langle R\rangle A$*
2. *$A \cap \langle R\rangle B \subseteq \langle R\rangle(A \cap B)$*

Proof. The condition $A \subseteq \langle R\rangle A$ holds according to the standard modal logic argument.

Let us check the second condition. Let $A \cap \langle R\rangle B$, then $x \in A$ and $x \in R(y)$ for some $y \in B$. xRy implies there exists $z \in \uparrow y$ such that xRz and $x \leqslant z$. A is an up-set, then $z \in A$, so $z \in A \cap B$, but xRz, thus, $x \in \langle R\rangle(A \cap B)$. ☐

The lemma above allows one to extend the representation of arbitrary modal cover system described in the proof of Theorem 10 to prenuclear ones:

Theorem 11. *Every localic prenuclear algebra is isomorphic to the algebra of propositions of some modal prenuclear localic cover system.*

Proof. Let $\mathcal{L} = \langle L, \bigvee, \wedge\rangle$ be a locale and $\mathfrak{L} = \langle \mathcal{L}, \iota\rangle$ a localic prenuclear algebra. Then $\mathcal{S}_{\mathfrak{L}} = \langle L, \sqsubseteq, \rhd, R_\iota\rangle$ is a modal cover system, where $xR_\iota y$ iff $x \leqslant \iota y$. Let us ensure that this cover system is prenuclear one. The relation is clearly reflexive, $xR_\iota x$ follows the inflationary condition. The second prenuclear cover system axiom also holds. $xR_\iota y$, then $x \leqslant \iota y$. Let us put $z = x \wedge y$, then $xR_\iota z$ since $x \leqslant x \wedge \iota y \leqslant \iota(x \wedge y)$. $y \sqsubseteq z$ holds obviously. ☐

To embed an arbitrary prenuclear algebra, Heyting reduct of which is non-necessarily complete Heyting algebra, one need to preserve prenucleus under Dedekind-MacNeille completion. First of all, we recall what completion is. Given a bounded lattice $\mathcal{L}$, a *completion* of $\mathcal{L}$ is a complete lattice $\overline{\mathcal{L}}$ that contains $\mathcal{L}$ as a sublattice. A completion $\overline{\mathcal{L}}$ is called *Dedekind-McNeille* if every element of $\overline{\mathcal{L}}$ is both a join and meet of elements of $\mathcal{L}$, that is for each $a \in \overline{\mathcal{L}}$ (see [17] to read more about lattice completions):

$$a = \bigvee\{b \in \mathcal{L} \mid a \leqslant b\} = \bigwedge\{b \in \mathcal{L} \mid b \leqslant a\}$$

The class of all Heyting algebras is closed under Dedekind-MacNeille completions: if $\mathcal{H}$ is a Heyting algebra, then $\overline{\mathcal{H}}$ is a locale. An implication in an arbitrary Heyting algebra $\mathcal{H}$ is extended as follows, where $a, b \in \overline{\mathcal{H}}$:

$$a \Rightarrow b = \bigwedge\{c \to d \mid a \geqslant c \in \mathcal{H} \,\&\, d \leqslant b \in \mathcal{H}\}$$

Completions of Heyting algebras are interesting topic itself, we refer the reader to these papers [27] [34] for further discussion.

Given a lattice $\mathcal{L}$ and $f : \mathcal{L} \to \mathcal{L}$ a monotone function on this lattice, let us define maps $f^{\circ}, f^{\bullet} : \overline{\mathcal{L}} \to \overline{\mathcal{L}}$ as follows for $a \in \overline{\mathcal{L}}$:

$$f^{\circ}(a) = \bigvee\{f(x) \mid a \geqslant x \in \mathcal{L}\}$$
$$f^{\bullet}(a) = \bigwedge\{f(x) \mid a \leqslant x \in \mathcal{L}\}$$

f° and $f^{\bullet}$ both extend f and $f^{\circ} \leqslant f^{\bullet}$ in means of the pointwise order. Generally, neither f° nor $f^{\bullet}$ multiplicative, if f is. One the other hand, if f is a multiplicative function on a Heyting algebra, so f° is, see [65].

Lemma 11. *Let ι be a prenucleus on a Heyting algebra $\mathcal{H}$, then $\iota^{\bullet}$ is a prenucleus on $\overline{\mathcal{H}}$*

Proof. The proof is similar for the similar statement for nuclei [31]. ι is inflationary, so $\iota^{\bullet}$ is, it is readily checked. Let us check that $a \wedge \iota^{\bullet} b \leqslant \iota^{\bullet}(a \wedge b)$ for each $a, b \in \overline{\mathcal{H}}$. □

One may prove the following representation theorem for Heyting algebra with prenuclei operators combining Theorem 11 and Lemma 11:

Theorem 12. *Every prenuclear algerba is isomorphic to the subalgerba to the algebra of propositions of some modal prenuclear localic cover system*

Let us consider the multiplicative case. We cannot easily apply Dedekind-MacNeille completion as in Lemma 11 for multiplicative prenuclear algebras, since lower extensions preserve multiplicativy and upper extensions preserve inflationarity. To simplify the issue, we reformulate multiplicative prenuclear algebras as follows:

Proposition 2. *Let $\mathcal{H}$ be a Heyting algebra and j a function that preserves finite infima, then for each $a, b \in \mathcal{H}$ one has $a \leqslant ja$ iff $a \wedge jb \leqslant j(a \wedge b)$*

Proof. Both implications are quite simple. One has $a = a \wedge \top = a \wedge j\top \leqslant j(a \wedge \top) = ja$. On the other hand, $a \wedge jb \leqslant ja \wedge jb = j(a \wedge b)$ □

Lemma 12. *Let $\mathcal{H}$ be a Heyting algebra and ι a multiplicative prenucleus on $\mathcal{H}$, then ι° is a multiplicative nucleus on $\overline{\mathcal{H}}$*

Proof. According to Proposition 2, one may equivalently replace the inflationary condition to $j\top = \top$ and $a \wedge \iota b \leqslant \iota(a \wedge b)$. In fact, one needs to check that the inequation $x \wedge \iota^{\circ} y \leqslant \iota^{\circ}(x \wedge y)$ holds for each $x, y \in \overline{\mathcal{H}}$. One has:

$$x \wedge \iota^{\circ} y = \bigvee\{a \in \mathcal{H} \mid a \leqslant x\} \wedge \bigvee\{\iota b \in \mathcal{H} \mid b \in \mathcal{H}, b \leqslant y\} =$$
$$\bigvee\{a \wedge \iota b \mid a \leqslant x, b \leqslant y, a, b \in \mathcal{H}\} \leqslant \bigvee\{\iota(a \wedge b) \mid a \leqslant x, b \leqslant y, a, b \in \mathcal{H}\} \leqslant$$
$$\bigvee\{\iota c \mid c \in \mathcal{H}, c \leqslant x \wedge y\} = \iota^{\circ}(x \wedge y)$$

ι° is multiplicative since ι is multiplicative. Thus, ι° is a multiplicative prenucleus on $\mathcal{H}$. □

Let us define a suitable cover system.

Definition 20. *Let $\mathcal{M} = \langle S, \leqslant, \rhd, R\rangle$ be a modal cover system, then $\mathcal{M}$ is called multiplicative prenuclear if the following conditions hold:*

1. *R is serial, that is, for each $x \in S$ there exists $y \in S$ such that xRy*
2. *if xRy and xRz then there exists $w \in \uparrow x \cap \uparrow y$ such that xRw*
3. *Let $x, y \in S$ such that xRy, then there exists $z \in \uparrow y$ such that $x \leqslant z$ and $x \in R(z)$*

One may consider a multiplicative prenuclear frame as an $R_{\Box}$-reduct of a CK-modal cover system [31] with the additional principle that corresponds to the second postulate of a prenuclear cover system. Strictly speaking, such a cover system desribes the logic modal axioms of which are $\bigcirc\top$, $\bigcirc p \wedge \bigcirc q \Rightarrow \bigcirc(p \wedge q)$, and $p \wedge \bigcirc q \Rightarrow \bigcirc(p \wedge q)$ plus the $\bigcirc$-monotonicity rule. It is not so hard to see that this logic is deductively equivalent to $\mathbf{IEL}^{-}$ over intuitionistic logic.

Lemma 13. *Let $\mathcal{M} = \langle S, \leqslant, \rhd, R\rangle$ be a multiplicative prenuclear cover system, then $\langle R\rangle$ is a multiplicative prenucleus on* $\mathrm{Prop}(\mathcal{S})$

Proof. $\langle R\rangle$ is clearly serial. The multiplicativity follows from the second postulate of a multiplicative prenuclear cover system. The third equation is proved similarly to Lemma 10. □

Theorem 13.

1. *Every localic multiplicative prenuclear algebra is representable as a modal locale of the propositions of a suitable modal cover system.*
2. *Every multiplicative prenuclear algerba is isomorphic to the subalgerba to the algebra of propositions of some multiplicative prenuclear localic cover system*

Proof.

1. The proof is similar to Theorem 11 concerning Lemma 13.
2. Follows from the previous item and Lemma 12.

□

Finally, we consider **IEL**-cover systems and corresponding multiplicative prenuclear algebras, where the equation $j\bot = \bot$ is satisfied. We call such multiplicative prenuclear algebras *dense*. In particular, $j\bot = \bot$ implies $j^{\circ}\bot = \bot$ [65]. Thus, if an operator on a Heyting algebra is a dense multiplicative prenucleus, so its lower Dedekind-MacNeille completion is.

An **IEL**-cover system is a multiplicative prenuclear system $\mathcal{S} = \langle P, \leqslant, \rhd, R\rangle$ such that for each $x, y \in S$ if xRy and $y \rhd \varnothing$ implies $x \rhd \varnothing$. This condition yields $\langle R\rangle\varnothing = \varnothing$.

Thus, one may immediately extend Theorem 13:

Theorem 14.

1. *Every localic dense multiplicative prenuclear algebra is isomorphic to the algerba of propositions of some* **IEL***-cover system*
2. *Every dense multiplicative prenuclear algebra is isomorphic to the subalgebra of propositions of some* **IEL***-cover system*

4.3 Completeness theorems

In this subsection, by $\mathbf{IEL}^{-}_{-}$ we mean the set of formulas defined as

$$\mathbf{IPC} \oplus \varphi \to \bigcirc\varphi \oplus \varphi \wedge \bigcirc\psi \to \bigcirc(\varphi \wedge \psi)$$

Let us define first the intuionistic modal predicate logic $\mathbf{QIEL}^{-}_{-}$ as an extension of intiuitionistic predicate logic with modal axioms that correspond to the conditions of a prenucleus operators. Here we deal with a signature consisting of predicate symbols of an arbitrary arity without function symbols and individual constants.

1. $\mathbf{IEL}^{-}_{-}$-axioms
2. $\forall x\varphi \to \varphi(t/x)$
3. $\varphi(t/x) \to \exists x\varphi$
4. Inference rules are Modus Ponens, Bernays rules, and $\bigcirc$-monotonicity

Then $\mathbf{QIEL}^- = \mathbf{QIEL}^-_- \oplus \bigcirc(\varphi \to \psi) \to (\bigcirc\varphi \to \bigcirc\psi)$ and $\mathbf{QIEL} = \mathbf{QIEL}^- \oplus \neg \bigcirc \bot$, where $\neg\varphi = \varphi \to \bot$.

In this section we show that the logics above are complete with respect to their suitable cover systems. Let $\mathcal{L}$ be a logic above $\mathbf{QIEL}^-$, let us define their models. Let $\mathcal{C}$ be a prenuclear cover system $\mathcal{M} = \langle S, \leqslant, \rhd, R\rangle$, V a valuation function, and D a set of individuals, then an $\mathcal{L}$-cover model is a triple $\mathfrak{M} = \langle \mathcal{M}, V, D\rangle$. Given a D-assignment and $x \in S$, a modal operator has the following semantics:

$$\mathfrak{M}, x, \sigma \models \bigcirc\varphi \text{ iff there exists } y \in R(x) \text{ such that } \mathcal{M}, y, \sigma \models \varphi$$

Indeed, one may reformulate the truth condition above in means of an $\langle R\rangle$-operator on the locale of propositions, then we have:

$$|| \bigcirc \varphi||^{\mathfrak{M}}_{\sigma} = \langle R\rangle||\varphi||^{\mathfrak{M}}_{\sigma}$$

The completeness theorem converts the Lindenbaum-Tarksi algebra to a suitable locale with a certain operator via Dedekind-MacNeille completion. After that, we represent this algebra as an algebra of propositions of a localic system by the representation theorem we proved. More precisely, one has:

Theorem 15. *Let $\mathcal{L} \in \{\mathbf{IEL}^-_-, \mathbf{IEL}^-, \mathbf{IEL}\}$, then $\mathbf{Q}\mathcal{L}$ is sound and complete with repsect to their cover systems*

Proof. Let us consider the $\mathbf{IEL}^-_-$-case, the rest two cases are shown in the same fashion via relevant representations and Dedekind-MacNeille completions. Let Fm be the set of all formulas and $\mathcal{V}$ the set of all variables, then one has an equivalence relation $\varphi \sim \psi$ $\mathbf{IEL}^-_- \vdash \varphi \to \psi$ and $\mathbf{IEL}^-_- \vdash \psi \to \varphi$. Then, one has an ordering on $\mathrm{Fm}/\sim$ defined as $|\varphi| \leqslant |\psi|$ iff $\vdash_{\mathbf{IEL}^-_-} \varphi \to \psi$. The operations on $\mathrm{Fm}/\sim$ are defined as:

$$\begin{aligned} |\varphi \wedge \psi| &= |\varphi| \wedge |\psi| \\ |\varphi \vee \psi| &= |\varphi| \vee |\psi| \\ |\varphi \to \psi| &= |\varphi| \Rightarrow |\psi| \\ |\forall x\varphi| &= \bigwedge_{x \in V} |\varphi| \\ |\exists x\varphi| &= \bigvee_{x \in V} |\varphi| \\ | \bigcirc \varphi| &= \bigcirc|\varphi| \\ \top &= |\varphi|, \text{ where } \mathbf{IEL}^-_- \vdash \varphi \end{aligned}$$

This algebra is clearly prenuclear one, but its Heyting reduct is not necessarily a complete one. By Lemma 12, one may embed the Lindenbaum-Tarksi algebra $\mathcal{L}_{\mathbf{IEL}^-_-}$ to the prenucleus $\bigcirc^{\bullet}$ on $\overline{\mathcal{F}/\sim}$. A localic prenuclear algebra $\langle\overline{\mathcal{F}/\sim}, \bigcirc^{\bullet}\rangle$ is isomorphic to some prenuclear cover system. Thus, by Theorem 11, one has an isomorphism $f : \langle\overline{\mathcal{F}/\sim}, \bigcirc^{\bullet}\rangle \cong \langle \mathrm{Prop}(\mathcal{S}_{\mathbf{IEL}^-_-}), \langle R_{\bigcirc}\rangle, \rangle$, where $\mathcal{S}_{\mathbf{IEL}^-_-}$ is an obtained prenuclear cover system. Let define an $\mathbf{IEL}^-_-$ cover model $\mathfrak{M} = \langle \mathcal{S}_{\mathbf{IEL}^-_-}, D, V\rangle$ putting $D = \mathrm{V}$ and a valuation V is defined as $V(P)(x_{n_1}, \dots, x_{n_k}) = f(|P(x_{n_1}, \dots, x_{n_k})|)$. Here, a D-assignment σ is merely an identity function. Here, the key observations are let φ be a formula, then $||\varphi||^{\mathfrak{M}}_{\sigma} = f|\varphi|$ that might be shown by easy induction on φ. Then, if φ is true in every $\mathbf{IEL}^-_-$-model, then in $\mathfrak{M}$ defined as above $||\varphi||^{\mathfrak{M}}_{\sigma} = \top = S$, thus, $f|\varphi| = \top$. Hence, the value of φ in the Lindenbaum-Tarski equals to $\top$, hence, $\mathbf{IEL}^-_- \vdash \varphi$. Thus, $\mathbf{IEL}^-_-$ is sound and complete with respect models on prenuclear cover systems.

The $\mathbf{IEL}^-$ ($\mathbf{IEL}$) case follows from the same construction using Theorem 13 (Theorem 14).

□

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