A NEW INVARIANT FOR FINITE DIMENSIONAL LEIBNIZ/LIE ALGEBRAS

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Abstract. For an n-dimensional Leibniz/Lie algebra $\mathfrak{h}$ over a field k we introduce a new invariant $\mathcal{A}(\mathfrak{h})$, called the *universal algebra* of $\mathfrak{h}$, as a quotient of the polynomial algebra $k[X_{ij} \,|\, i, j = 1, \cdots, n]$ through an ideal generated by n^3 polynomials. Furthermore, $\mathcal{A}(\mathfrak{h})$ admits a unique bialgebra structure which makes it an initial object among all bialgebras coacting on $\mathfrak{h}$ through a Leibniz/Lie algebra homomorphism. The bialgebra $\mathcal{A}(\mathfrak{h})$ is the key object in approaching three classical and open problems in Lie algebra theory. First, we prove that the automorphisms group $\mathrm{Aut}_{Lbz}(\mathfrak{h})$ of $\mathfrak{h}$ is isomorphic to the group $U\big(G(\mathcal{A}(\mathfrak{h})^{o})\big)$ of all invertible group-like elements of the finite dual $\mathcal{A}(\mathfrak{h})^{o}$. Secondly, for an abelian group G, we show that there exists a bijection between the set of all G-gradings on $\mathfrak{h}$ and the set of all bialgebra homomorphisms $\mathcal{A}(\mathfrak{h}) \to k[G]$. Finally, for a finite group G, we prove that the set of all actions as automorphisms of G on $\mathfrak{h}$ is parameterized by the set of all bialgebra homomorphisms $\mathcal{A}(\mathfrak{h}) \to k[G]^*$. $\mathcal{A}(\mathfrak{h})$ is also used to prove that there exists a universal commutative Hopf algebra associated to any finite dimensional Leibniz algebra $\mathfrak{h}$.

Introduction

Let A be a unital associative algebra over a field k. M.E. Sweedler's result [23, Theorem 7.0.4] which states that the functor $\mathrm{Hom}(-, A) : \mathrm{CoAlg}_k \to \mathrm{Alg}_k^{\mathrm{op}}$ from the category of coalgebras over k to the opposite of the category of k-algebras has a right adjoint denoted by $\mathrm{M}(-, A)$ proved itself remarkable through its applications. Furthermore, $\mathrm{M}(A, A)$ turns out to be a bialgebra and the final object in the category of all bialgebras that act on A through a module algebra structure. The dual version was considered by Tambara [20] and in a special (graded) case by Manin [17]. To be more precise, [20, Theorem 1.1] proves that if A is a finite dimensional algebra, then the tensor functor $A \otimes - : \mathrm{Alg}_k \to \mathrm{Alg}_k$ has a left adjoint denoted by $a(A, -)$. In the same spirit, $a(A, A)$ is proved to be a bialgebra as well and the initial object in the category of all bialgebras that coact on A through a comodule algebra structure. Both objects are very important: as explain in [17], the Hopf envelope of $a(A, A)$ plays the role of a symmetry group in non-commutative geometry. For further details we refer to [1, 2]. A more general construction, which contains all the above as special cases, was recently considered in [3] in the context of Ω-algebras.

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The starting point of this paper was an attempt to prove the counterpart of Tambara's result at the level of Leibniz algebras. Introduced by Bloh [7] and rediscovered by Loday [14], Leibniz algebras are non-commutative generalizations of Lie algebras. This new concept generated a lot of interest mainly due to its interaction with (co)homology theory, vertex operator algebras, the Godbillon-Vey invariants for foliations or differential geometry. Another important concept for our approach is that of a *current Lie* algebra. Being first introduced in physics [12], current Lie algebras, are Lie algebras of the form $\mathfrak{g} \otimes A$, where $\mathfrak{g}$ is a Lie algebra, A is a commutative algebra and the bracket is given by $[x \otimes a,\, y \otimes b] := [x,\, y] \otimes ab$, for all $x,\, y \in \mathfrak{g}$ and $a,\, b \in A$. They are interesting objects that arise in various branches of mathematics and physics such as the theory of affine Kac-Moody algebras or the structure of modular semisimple Lie algebras (see [25, 26]). Current Leibniz algebras are immediate generalizations: i.e. they are Leibniz algebras of the form $\mathfrak{h} \otimes A$ whose bracket is defined as in the case of Lie algebras, where this time $\mathfrak{h}$ is a Leibniz algebra and A a commutative algebra. By fixing a Leibniz algebra $\mathfrak{h}$, we obtain a functor $\mathfrak{h} \otimes - : {\rm ComAlg}_k \to {\rm Lbz}_k$ from the category of commutative algebras to the category of Leibniz algebras called the current Leibniz algebra functor. Theorem 2.1 proves that the functor $\mathfrak{h} \otimes - : {\rm ComAlg}_k \to {\rm Lbz}_k$ has a left adjoint, denoted by $\mathcal{A}(\mathfrak{h},\, -)$, if and only if $\mathfrak{h}$ is finite dimensional. For an n-dimensional Leibniz algebra $\mathfrak{h}$ and an arbitrary Leibniz algebra $\mathfrak{g}$ with $|I| = {\rm dim}_k(\mathfrak{g})$, $\mathcal{A}(\mathfrak{h},\, \mathfrak{g})$ is a quotient of the usual polynomial algebra $k[X_{si}\, | s = 1, \cdots, n,\, i \in I]$. The commutative algebra $\mathcal{A}(\mathfrak{h},\, \mathfrak{g})$ provides an important tool for studying Leibniz/Lie algebras as it captures all essential information on the two Leibniz/Lie algebras. Note for instance that the characters of this algebra parameterize the set of all Leibniz algebra homomorphisms between $\mathfrak{g}$ and $\mathfrak{h}$ (Corollary 2.3). Theorem 2.1 has obviously a Lie algebra counterpart. In this case, if $\mathfrak{g}$ is a Lie algebra and m a positive integer then the characters of the commutative algebra $\mathcal{A}(\mathfrak{gl}(m, k),\, \mathfrak{g})$ parameterize the space of all m-dimensional representations of $\mathfrak{g}$ (Corollary 2.4). The commutative algebra $\mathcal{A}(\mathfrak{h}) := \mathcal{A}(\mathfrak{h},\, \mathfrak{h})$ is called the *universal algebra* of $\mathfrak{h}$: it is a quotient of the polynomial algebra $M(n) := k[X_{ij}\, |\, i, j = 1, \cdots, n]$ through an ideal generated by n^3 polynomials called the *universal polynomials* of $\mathfrak{h}$. Proposition 2.10 proves that $\mathcal{A}(\mathfrak{h})$ has a canonical bialgebra structure such that the projection $\pi : M(n) \to \mathcal{A}(\mathfrak{h})$ is a bialgebra map. The first main application of the universal (bi)algebra $\mathcal{A}(\mathfrak{h})$ of $\mathfrak{h}$ is given in Theorem 2.14 which provides an explicit description of a group isomorphism between the group of automorphisms of $\mathfrak{h}$ and the group of all invertible group-like elements of the finite dual $\mathcal{A}(\mathfrak{h})^{\rm o}$:

$$ {\rm Aut}_{\rm Lbz}(\mathfrak{h}) \cong U\big(G\big(\mathcal{A}(\mathfrak{h})^{\rm o}\big)\big). $$

We mention that achieving a complete description of the automorphisms group ${\rm Aut}_{\rm Lie}(\mathfrak{h})$ of a given Lie algebra $\mathfrak{h}$ is a classical [8, 13] and notoriously difficult problem intimately related to the structure of Lie algebras (for more details see the recent papers [4, 5, 11] and their references). The unit of the adjunction depicted in Theorem 2.1, denoted by $\eta_{\mathfrak{h}} : \mathfrak{h} \to \mathfrak{h} \otimes \mathcal{A}(\mathfrak{h})$, endows $\mathfrak{h}$ with a right $\mathcal{A}(\mathfrak{h})$-comodule structure and the pair $(\mathcal{A}(\mathfrak{h}),\, \eta_{\mathfrak{h}})$ is the intial object in the category of all commutative bialgebras that coact on the Leibniz algebra $\mathfrak{h}$ (Theorem 2.11). This result allows for two important consequences: Corollary 2.12 proves that for an abelian group G there exists an explicitly described bijection between the set of all G-gradings on $\mathfrak{h}$ and the set of all bialgebra homomorphisms $\mathcal{A}(\mathfrak{h}) \to k[G]$. Secondly, if G is a finite group, Corollary 2.13 shows that there exists a bijection between the set of all actions as automorphisms of G on $\mathfrak{h}$ (i.e. morphisms of groups $G \to \mathrm{Aut}_{\mathrm{Lbz}}(\mathfrak{h})$) and the set of all bialgebra homomorphisms $\mathcal{A}(\mathfrak{h}) \to k[G]^*$. Related to the last two results we mention that there exists a vast literature concerning the classification of all G-gradings on a given Lie algebra (see [6, 10, 18] and their references). On the other hand, the study of actions as automorphisms of a group G on a Lie algebra $\mathfrak{h}$ goes back to Hilbert's invariant theory whose foundation was set at the level of Lie algebras in the classical papers [8, 9, 24]; for further details see [4] and the references therein. Using once again Theorem 2.11 and the existence of a free commutative Hopf algebra on any commutative bialgebra [19, Theorem 65, (2)], we prove in Theorem 2.16 that there exists a universal coacting Hopf algebra on any finite dimensional Leibniz algebra. We point out that, to the best of our knowledge, this is the only universal Hopf algebra associated to a Leibniz algebra appearing in the literature.

1. Preliminaries

All vector spaces, (bi)linear maps, Leibniz, Lie or associative algebras, bialgebras and so on are over an arbitrary field k and $\otimes = \otimes_k$. A Leibniz algebra is a vector space $\mathfrak{h}$, together with a bilinear map $[-, -] : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{h}$ satisfying the Leibniz identity for any $x, y, z \in \mathfrak{h}$:

$$[x, [y, z]] = [[x, y], z] - [[x, z], y] \tag{1}$$

Any Lie algebra is a Leibniz algebra, and a Leibniz algebra $\mathfrak{h}$ satisfying $[x, x] = 0$, for all $x \in \mathfrak{h}$ is a Lie algebra. We shall denote by $\mathrm{Aut}_{\mathrm{Lbz}}(\mathfrak{h})$ (resp. $\mathrm{Aut}_{\mathrm{Lie}}(\mathfrak{h})$) the automorphisms group of a Leibniz (resp. Lie) algebra $\mathfrak{h}$. Any vector space V is a Leibniz algebra with trivial bracket $[x, y] := 0$, for all $x, y \in V$ – such a Leibniz algebra is called *abelian* and will be denoted by V_0. For two subspaces A and B of a Leibniz algebra $\mathfrak{h}$ we denote by $[A, B]$ the vector space generated by all brackets $[a, b]$, for any $a \in A$ and $b \in B$. In particular, $\mathfrak{h}' := [\mathfrak{h}, \mathfrak{h}]$ is called the derived subalgebra of $\mathfrak{h}$.

We shall denote by Lbz_k, Lie_k and ComAlg_k the categories of Leibniz, Lie and respectively commutative associative algebras. Furthermore, the category of commutative bialgebras (resp. Hopf algebras) is denoted by $\mathrm{ComBiAlg}_k$ (resp. $\mathrm{ComHopf}_k$). If $\mathfrak{h}$ is a Leibniz algebra and A a commutative algebra then $\mathfrak{h} \otimes A$ is a Leibniz algebra with bracket defined for any $x, y \in \mathfrak{h}$ and $a, b \in A$ by:

$$[x \otimes a, y \otimes b] := [x, y] \otimes ab \tag{2}$$

called the *current Leibniz* algebra. Indeed, as A is a commutative and associative algebra, we have:

$$\begin{aligned}
&[\,[x \otimes a, y \otimes b]\,, z \otimes c] - [\,[x \otimes a, z \otimes c]\,, y \otimes b] \\
&= [\,[x, y]\,, z] \otimes abc - [\,[x, z]\,, y] \otimes acb \\
&= \big([\,[x, y]\,, z] - [\,[x, z]\,, y]\,\big) \otimes abc \\
&= [x, [y, z]\,] \otimes abc = [x \otimes a, [y \otimes b, z \otimes c]\,]
\end{aligned}$$

for all $x, y, z \in \mathfrak{h}$ and $a, b, c \in A$, i.e. the Leibniz identity (1) holds for $\mathfrak{h} \otimes A$. For a fixed Leibniz algebra $\mathfrak{h}$, assigning $A \mapsto \mathfrak{h} \otimes A$ defines a functor $\mathfrak{h} \otimes - : \mathrm{ComAlg}_k \to \mathrm{Lbz}_k$

from the category of commutative k-algebras to the category of Leibniz algebras called the current Leibniz algebra functor. If $f : A \to B$ is an algebra map then $\mathrm{Id}_{\mathfrak{h}} \otimes f : \mathfrak{h} \otimes A \to \mathfrak{h} \otimes B$ is a morphism of Leibniz algebras.

For basic concepts on category theory we refer the reader to [16] and for unexplained notions pertaining to Hopf algebras to [21, 23].

2. Universal constructions and applications

Our first resut is the Leibniz algebra counterpart of [20, Theorem 1.1].

Theorem 2.1. *Let $\mathfrak{h}$ be a Leibniz algebra. Then the current Leibniz algebra functor $\mathfrak{h} \otimes - : \mathrm{ComAlg}_k \to \mathrm{Lbz}_k$ has a left adjoint if and only if $\mathfrak{h}$ is finite dimensional. Moreover, if $\mathfrak{h} \neq 0$ the functor $\mathfrak{h} \otimes -$ does not admit a right adjoint.*

Proof. Assume first that $\mathfrak{h}$ is a finite dimensional Leibniz algebra and $\dim_k(\mathfrak{h}) = n$. Fix $\{e_1, \cdots, e_n\}$ a basis in $\mathfrak{h}$ and let $\{\tau_{i,j}^s \,|\, i, j, s = 1, \cdots, n\}$ be the structure constants of $\mathfrak{h}$, i.e. for any i, $j = 1, \cdots, n$ we have:

$$[e_i,\, e_j]_{\mathfrak{h}} = \sum_{s=1}^{n} \tau_{i,j}^s \, e_s. \tag{3}$$

In what follows we shall explicitly construct a left adjoint of the current Leibniz algebra functor $\mathfrak{h} \otimes -$, denoted by $\mathcal{A}(\mathfrak{h}, -) : \mathrm{Lbz}_k \to \mathrm{ComAlg}_k$. Let $\mathfrak{g}$ be a Leibniz algebra and let $\{f_i \,|\, i \in I\}$ be a basis of $\mathfrak{g}$. For any i, $j \in I$, let $B_{i,j} \subseteq I$ be a finite subset of I such that for any i, $j \in I$ we have:

$$[f_i,\, f_j]_{\mathfrak{g}} = \sum_{u \in B_{i,j}} \beta_{i,j}^u \, f_u. \tag{4}$$

Let $k[X_{si} \,|\, s = 1, \cdots, n,\, i \in I]$ be the usual polynomial algebra and define

$$\mathcal{A}(\mathfrak{h},\, \mathfrak{g}) := k[X_{si} \,|\, s = 1, \cdots, n,\, i \in I]/J \tag{5}$$

where J is the ideal generated by all polynomials of the form

$$P_{(a,i,j)}^{(\mathfrak{h},\, \mathfrak{g})} := \sum_{u \in B_{i,j}} \beta_{i,j}^u \, X_{au} - \sum_{s,t=1}^{n} \tau_{s,t}^a \, X_{si} X_{tj}, \quad \text{for all } a = 1, \cdots, n \text{ and } i,\, j \in I. \tag{6}$$

We denote by $x_{si} := \widehat{X_{si}}$ the class of X_{si} in the algebra $\mathcal{A}(\mathfrak{h},\, \mathfrak{g})$; thus the following relations hold in the commutative algebra $\mathcal{A}(\mathfrak{h},\, \mathfrak{g})$:

$$\sum_{u \in B_{i,j}} \beta_{i,j}^u \, x_{au} = \sum_{s,t=1}^{n} \tau_{s,t}^a \, x_{si} x_{tj}, \quad \text{for all } a = 1, \cdots, n, \text{ and } i,\, j \in I. \tag{7}$$

Now we consider the map:

$$\eta_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{h} \otimes \mathcal{A}(\mathfrak{h},\, \mathfrak{g}), \quad \eta_{\mathfrak{g}}(f_i) := \sum_{s=1}^{n} e_s \otimes x_{si}, \quad \text{for all } i \in I. \tag{8}$$

We shall prove first that $\eta_{\mathfrak{g}}$ is a Leibniz algebra homomorphism. Indeed, for any $i,\, j \in I$ we have:

$$\begin{aligned}
&[\eta_{\mathfrak{g}}(f_i),\, \eta_{\mathfrak{g}}(f_j)]_{\mathfrak{h}\otimes\mathcal{A}(\mathfrak{h},\,\mathfrak{g})} = \Big[\sum_{s=1}^{n} e_s \otimes x_{si},\, \sum_{t=1}^{n} e_t \otimes x_{tj}\Big]_{\mathfrak{h}\otimes\mathcal{A}(\mathfrak{h},\,\mathfrak{g})} = \sum_{s,t=1}^{n} \, [e_s,\, e_t]_{\mathfrak{h}} \otimes x_{si}x_{tj}\\
&= \sum_{a=1}^{n} e_a \otimes \underline{\big(\sum_{s,\,t=1}^{n} \tau_{s,t}^{a}\, x_{si}x_{tj}\big)} \overset{(7)}{=} \sum_{a=1}^{n} e_a \otimes \big(\sum_{u\in B_{i,j}} \beta_{i,j}^{u}\, x_{au}\big) = \sum_{u\in B_{i,j}} \beta_{i,j}^{u}\, \eta_{\mathfrak{g}}(f_u)\\
&= \eta_{\mathfrak{g}}([f_i,\, f_j]_{\mathfrak{g}})
\end{aligned}$$

Now we prove that for any Leibniz algebra $\mathfrak{g}$ and any commutative algebra A the map defined below is bijective:

$$\gamma_{\mathfrak{g},\,A}\ : \mathrm{Hom}_{\mathrm{Alg}_k}\,(\mathcal{A}(\mathfrak{h},\,\mathfrak{g}),\, A) \to \mathrm{Hom}_{\mathrm{Lbz}_k}\,(\mathfrak{g},\, \mathfrak{h}\otimes A), \quad \gamma_{\mathfrak{g},\,A}(\theta) := \big(\mathrm{Id}_{\mathfrak{h}} \otimes \theta\big)\circ\eta_{\mathfrak{g}} \tag{9}$$

To this end, let $f : \mathfrak{g} \to \mathfrak{h}\otimes A$ be a Leibniz algebra homomorphism. We have to prove that there exists a unique algebra homomorphism $\theta : \mathcal{A}(\mathfrak{h},\,\mathfrak{g}) \to A$ such that the following diagram is commutative:

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "\eta_{\mathfrak{g}}"] \arrow[dr, "f"'] & \mathfrak{h}\otimes\mathcal{A}(\mathfrak{h},\,\mathfrak{g}) \arrow[d, "\mathrm{Id}_{\mathfrak{h}}\otimes\theta"] \\
& \mathfrak{h}\otimes A
\end{tikzcd}
\qquad i.e.\ \ f = \big(\mathrm{Id}_{\mathfrak{h}} \otimes \theta\big)\circ \eta_{\mathfrak{g}}. \tag{10}$$

Let $\{d_{si}\,|\, s = 1, \cdots, n, i \in I\}$ be a family of elements of A such that for any $i \in I$ we have:

$$f(f_i) = \sum_{s=1}^{n} e_s \otimes d_{si} \tag{11}$$

A straightforward computation shows that for all $i,\, j \in I$ we have:

$$f\big([f_i,\, f_j]_{\mathfrak{g}}\big) = \sum_{a=1}^{n} e_a \otimes \big(\sum_{u\in B_{ij}} \beta_{i,j}^{u}\, d_{au}\big) \text{ and } [f(f_i),\, f(f_j)]_{\mathfrak{h}\otimes A} = \sum_{a=1}^{n} e_a \otimes \big(\sum_{s,t=1}^{n} \tau_{s,t}^{a}\, d_{si}d_{tj}\big)$$

Since $f : \mathfrak{g} \to \mathfrak{h}\otimes A$ is a Leibniz algebra homomorphism, it follows that the family of elements $\{d_{si}\,|\, s = 1, \cdots, n, i \in I\}$ need to fullfil the following relations in A:

$$\sum_{u\in B_{ij}} \beta_{i,j}^{u}\, d_{au} = \sum_{s,t=1}^{n} \tau_{s,t}^{a}\, d_{si}d_{tj}, \quad \text{for all } i,\, j \in I \text{ and } a = 1, \cdots, n. \tag{12}$$

The universal property of the polynomial algebra yields a unique algebra homomorphism $v : k[X_{si}\,|s = 1, \cdots, n,\, i \in I] \to A$ such that $v(X_{si}) = d_{si}$, for all $s = 1, \cdots, n$ and $i \in I$. It can be easily seen that $\mathrm{Ker}(v) \supseteq J$, where J is the ideal generated by all polynomials listed in (6). Indeed, for any $i,\, j \in I$ and $a = 1, \cdots, n$ we have

$$v\big(P_{(a,i,j)}^{(\mathfrak{h},\,\mathfrak{g})}\big) = v\big(\sum_{u\in B_{i,j}} \beta_{i,j}^{u}\, X_{au} - \sum_{s,t=1}^{n} \tau_{s,t}^{a}\, X_{si}X_{tj}\big) = \sum_{u\in B_{i,j}} \beta_{i,j}^{u}\, d_{au} - \sum_{s,t=1}^{n} \tau_{s,t}^{a}\, d_{si}d_{tj} \overset{(12)}{=} 0.$$

Thus, there exists a unique algebra homomorphism $\theta : \mathcal{A}(\mathfrak{h},\, \mathfrak{g}) \to A$ such that $\theta(x_{si}) = d_{si}$, for all $s = 1, \cdots, n$ and $i \in I$. Furthermore, for any $i \in I$ we have:

$$\big(\mathrm{Id}_{\mathfrak{h}} \otimes \theta\big) \circ \eta_{\mathfrak{g}}(f_i) = \big(\mathrm{Id}_{\mathfrak{h}} \otimes \theta\big)\big(\sum_{s=1}^{n} e_s \otimes x_{si}\big) = \sum_{s=1}^{n} e_s \otimes d_{si} \overset{(11)}{=} f(f_i).$$

Therefore, we have $\big(\mathrm{Id}_{\mathfrak{h}} \otimes \theta\big) \circ \eta_{\mathfrak{g}} = f$ as desired. Next we show that θ is the unique morphism with this property. Let $\tilde{\theta} : \mathcal{A}(\mathfrak{h},\, \mathfrak{g}) \to A$ be another algebra homomorphism such that $\big(\mathrm{Id}_{\mathfrak{h}} \otimes \tilde{\theta}\big) \circ \eta_{\mathfrak{g}}(f_i) = f(f_i)$, for all $i \in I$. Then, $\sum_{s=1}^{n} e_s \otimes \tilde{\theta}(x_{si}) = \sum_{s=1}^{n} e_s \otimes d_{si}$, and hence $\tilde{\theta}(x_{si}) = d_{si} = \theta(x_{si})$, for all $s = 1, \cdots, n$ and $i \in I$. Since $\{x_{si} \,|\, s = 1, \cdots, n, i \in I\}$ is a system of generators for the algebra $\mathcal{A}(\mathfrak{h},\, \mathfrak{g})$ we obtain $\tilde{\theta} = \theta$. All in all, we have proved that the map $\gamma_{\mathfrak{g},\, A}$ given by (9) is bijective.

Next we show that assigning to each Leibniz algebra $\mathfrak{g}$ the commutative algebra $\mathcal{A}(\mathfrak{h},\, \mathfrak{g})$ defines a functor $\mathcal{A}(\mathfrak{h},\, -) : \mathrm{Lbz}_k \to \mathrm{ComAlg}_k$. First, let $\alpha : \mathfrak{g}_1 \to \mathfrak{g}_2$ be a Leibniz algebra homomorphism. Using the bijectivity of the map defined by (9) for the Leibniz algebra homomorphism $f := \eta_{\mathfrak{g}_2} \circ \alpha$, yields a unique algebra homomorphism $\theta : \mathcal{A}(\mathfrak{h},\, \mathfrak{g}_1) \to \mathcal{A}(\mathfrak{h},\, \mathfrak{g}_2)$ such that the following diagram is commutative:

$$\begin{tikzcd} \mathfrak{g}_1 \arrow[r, "\eta_{\mathfrak{g}_1}"] \arrow[d, "\alpha"'] & \mathfrak{h} \otimes \mathcal{A}(\mathfrak{h},\, \mathfrak{g}_1) \arrow[d, "\mathrm{Id}_{\mathfrak{h}} \otimes \theta"] \\ \mathfrak{g}_2 \arrow[r, "\eta_{\mathfrak{g}_2}"] & \mathfrak{h} \otimes \mathcal{A}(\mathfrak{h},\, \mathfrak{g}_2) \end{tikzcd} \qquad i.e. \ \big(\mathrm{Id}_{\mathfrak{h}} \otimes \theta\big) \circ \eta_{\mathfrak{g}_1} = \eta_{\mathfrak{g}_2} \circ \alpha \tag{13}$$

We denote this unique morphism θ by $\mathcal{A}(\mathfrak{h},\, \alpha)$ and the functor $\mathcal{A}(\mathfrak{h},\, -)$ is now fully defined. Furthermore, the commutativity of the diagram (13) shows the naturality of $\gamma_{\mathfrak{g},\, A}$ in $\mathfrak{g}$. It can now be easily checked that $\mathcal{A}(\mathfrak{h},\, -)$ is indeed a functor and that $\gamma_{\mathfrak{g},\, A}$ is also natural in A. To conclude, the functor $\mathcal{A}(\mathfrak{h},\, -)$ is a left adjoint of the current Leibniz algebra functor $\mathfrak{h} \otimes -$.

Conversely, assume that the functor $\mathfrak{h} \otimes - : \mathrm{ComAlg}_k \to \mathrm{Lbz}_k$ has a left adjoint. In particular, $\mathfrak{h} \otimes -$ preserves arbitrary products. Now recall that in both categories ComAlg_k and Lbz_k products are constructed as simply the products of the underlying vector spaces. Imposing the condition that $\mathfrak{h} \otimes -$ preserves the product of a countable number of copies of the base field k will easily lead to the finite dimensionality of $\mathfrak{h}$.

Assume, now that the functor $\mathfrak{h} \otimes - : \mathrm{ComAlg}_k \to \mathrm{Lbz}_k$ has a right adjoint. This implies that $\mathfrak{h} \otimes -$ preserves coproducts. Now, since in the category ComAlg_k of commutative algebras the coproduct of two commutative algebras is given by their tensor product, it follows that for any commutative algebras A and B there exists an isomorphism of Leibniz algebras $\mathfrak{h} \otimes (A \otimes B) \cong (\mathfrak{h} \otimes A) \sqcup (\mathfrak{h} \otimes B)$, where we denote by $\sqcup$ the coproduct of two current Leibniz algebras. In particular, for $A = B := k$, we obtain that $\mathfrak{h} \sqcup \mathfrak{h} \cong \mathfrak{h}$, that is $\mathfrak{h} = 0$. The proof is now complete. □

Remark 2.2. Theorem 2.1 remains valid in the special case of Lie algebras: if $\mathfrak{h}$ is a finite dimensional Lie algebra, the current Lie algebra functor $\mathfrak{h} \otimes - : \mathrm{ComAlg}_k \to \mathrm{Lie}_k$ has a left adjoint $\mathcal{A}(\mathfrak{h},\, -)$ which is constructed as in the proof of Theorem 2.1. We point out, however, that the polynomials defined in (6) take a rather simplified form. The skew symmetry fulfilled by the bracket of a Lie algebra imposes the following restrictions on the structure constants:

$$\tau_{i,i}^{s}=0 \text{ and } \tau_{i,j}^{s}=-\tau_{j,i}^{s} \text{ for all } i,j,s=1,\cdots,n.$$

The commutative algebra $\mathcal{A}(\mathfrak{h},\,\mathfrak{g})$ constructed in the proof of Theorem 2.1 provides an important tool for studying Lie/Leibniz algebras as it captures most of the essential information on the two Lie/Leibniz algebras. Indeed, note for instance that the characters of this algebra (i.e. the algebra homomorphisms $\mathcal{A}(\mathfrak{h},\,\mathfrak{g})\to k$) parameterize the set of all Leibniz algebra homomorphisms between the two algebras. This follows as an easy consequence of the bijection described in (9) by taking $A:=k$:

Corollary 2.3. *Let $\mathfrak{g}$ and $\mathfrak{h}$ be two Leibniz algebras such that $\mathfrak{h}$ is finite dimensional. Then the following map is bijective:*

$$\gamma\,:\mathrm{Hom}_{\mathrm{Alg_k}}\left(\mathcal{A}(\mathfrak{h},\,\mathfrak{g}),\,k\right)\to \mathrm{Hom}_{\mathrm{Lbz_k}}\left(\mathfrak{g},\,\mathfrak{h}\right),\quad \gamma(\theta):=\big(\mathrm{Id}_{\mathfrak{h}}\otimes\theta\big)\circ\eta_{\mathfrak{g}}. \tag{14}$$

In particular, by applying Corollary 2.3 for $\mathfrak{h}:=\mathfrak{gl}(m,k)$ and an arbitrary Lie algebra $\mathfrak{g}$ we obtain:

Corollary 2.4. *Let $\mathfrak{g}$ be a Lie algebra and m a positive integer. Then there exists a bijective correspondence between the space of all m-dimensional representations of $\mathfrak{g}$ and the space of all algebra homomorphisms $\mathcal{A}(\mathfrak{gl}(m,k),\,\mathfrak{g})\to k$.*

Examples 2.5. 1. If $\mathfrak{h}$ and $\mathfrak{g}$ are abelian Leibniz algebras then $\mathcal{A}(\mathfrak{h},\,\mathfrak{g})\cong k[X_{si}\,|s=1,\cdots,n,\,i\in I]$, where $n=\dim_k(\mathfrak{h})$ and $|I|=\dim_k(\mathfrak{g})$.

2. Let $\mathfrak{h}$ be an n-dimensional Leibniz algebra with structure constants $\{\tau_{i,j}^{s}\,|\,i,j,s=1,\cdots,n\}$. Then $\mathcal{A}(\mathfrak{h},\,k)\cong k[X_1,\cdots,X_n]/J$, where J is the ideal generated by the polynomials $\sum_{s,t=1}^{n}\tau_{s,t}^{a}X_sX_t$, for all $a=1,\cdots,n$.

3. Let $\mathfrak{g}$ be a Leibniz algebra. Then $\mathcal{A}(k,\,\mathfrak{g})\cong S(\mathfrak{g}/\mathfrak{g}')$, the symmetric algebra of $\mathfrak{g}/\mathfrak{g}'$, where $\mathfrak{g}'$ is the derived subalgebra of $\mathfrak{g}$. In particular, if $\mathfrak{g}$ is perfect (that is $\mathfrak{g}'=\mathfrak{g}$), then $\mathcal{A}(k,\,\mathfrak{g})\cong k$.

Indeed, the functor $\mathcal{A}(k,\,-)$ is a left adjoint for the tensor functor $k\otimes-:\mathrm{ComAlg}_k\to\mathrm{Lbz}_k$; since the tensor product is also taken over k this functor is isomorphic to the functor $(-)_0:\mathrm{ComAlg}_k\to\mathrm{Lbz}_k$, which sends any commutative algebra A to the abelian Leibniz algebra $A_0:=A$. We shall prove that the functor $\mathfrak{g}\mapsto S(\mathfrak{g}/\mathfrak{g}')$ is a left adjoint of $(-)_0$. The uniqueness of adjoint functors [16] will then lead to the desired algebra isomorphism $\mathcal{A}(k,\,\mathfrak{g})\cong S(\mathfrak{g}/\mathfrak{g}')$.

Let $\mathfrak{g}$ be a Leibniz algebra and define $\overline{\eta_{\mathfrak{g}}}:\mathfrak{g}\to S(\mathfrak{g}/\mathfrak{g}')$ as the composition $\overline{\eta_{\mathfrak{g}}}:=i\circ\pi$, where $\pi:\mathfrak{g}\to\mathfrak{g}/\mathfrak{g}'$ is the usual projection and $i:\mathfrak{g}/\mathfrak{g}'\hookrightarrow S(\mathfrak{g}/\mathfrak{g}')$ is the canonical inclusion of the vector space $\mathfrak{g}/\mathfrak{g}'$ in its symmetric algebra. We shall prove now that the following map is bijective for any commutative algebra A and any Leibniz algebra $\mathfrak{g}$:

$$\overline{\gamma_{\mathfrak{g},\,A}}\,:\mathrm{Hom}_{\mathrm{Alg_k}}\left(S(\mathfrak{g}/\mathfrak{g}'),\,A\right)\to\mathrm{Hom}_{\mathrm{Lbz_k}}\left(\mathfrak{g},\,A_0\right),\quad \overline{\gamma_{\mathfrak{g},\,A}}\,(\theta):=\theta\circ\overline{\eta_{\mathfrak{g}}} \tag{15}$$

This shows that the functor $\mathfrak{g}\mapsto S(\mathfrak{g}/\mathfrak{g}')$ is a left adjoint of $(-)_0$. Indeed, let $f:\mathfrak{g}\to A_0$ be a Leibniz algebra homomorphism, i.e. f is a k-linear map such that $f([x,\,y])=0$, for all $x,\,y\in\mathfrak{g}$. That is Ker(f) contains $\mathfrak{g}'$, the derived algebra of $\mathfrak{g}$. Thus, there exists a unique k-linear map $\overline{f} : \mathfrak{g}/\mathfrak{g}' \to A$ such that $\overline{f} \circ \pi = f$. Now, using the universal property of the symmetric algebra we obtain that there exists a unique algebra homomorphism $\theta : S(\mathfrak{g}/\mathfrak{g}') \to A$ such that $\theta \circ i = \overline{f}$, and hence $\overline{\gamma_{\mathfrak{g},\, A}}\,(\theta) = f$. Therefore, the map $\overline{\gamma_{\mathfrak{g},\, A}}$ is bijective and the proof is now finished.

Definition 2.6. Let $\mathfrak{g}$ and $\mathfrak{h}$ be Leibniz algebras with $\mathfrak{h}$ finite dimensional. Then the commutative algebra $\mathcal{A}(\mathfrak{h},\, \mathfrak{g})$ is called the *universal algebra* of $\mathfrak{h}$ and $\mathfrak{g}$. When $\mathfrak{h} = \mathfrak{g}$ we denote the universal algebra of $\mathfrak{h}$ simply by $\mathcal{A}(\mathfrak{h})$.

If $\{\tau_{i,j}^s \,|\, i, j, s = 1, \cdots, n\}$ are the structure constants of $\mathfrak{h}$, where n is the dimension of $\mathfrak{h}$, then the polynomials defined for any $a,\, i,\, j = 1, \cdots, n$ by:

$$P_{(a,i,j)}^{(\mathfrak{h})} := \sum_{u=1}^{n} \tau_{i,j}^{u}\, X_{au} - \sum_{s,t=1}^{n} \tau_{s,t}^{a}\, X_{si} X_{tj} \,\in k[X_{ij} \,|\, i, j = 1, \cdots, n] \tag{16}$$

are called the *universal polynomials* of $\mathfrak{h}$. It follows from the proof of Theorem 2.1 that $\mathcal{A}(\mathfrak{h}) = k[X_{ij} \,|\, i, j = 1, \cdots, n]/J$, where J is the ideal generated by the universal polynomials $P_{(a,i,j)}^{(\mathfrak{h})}$, for all $a,\, i,\, j = 1, \cdots, n$. Moreover, if $\{e_1, \cdots, e_n\}$ is a basis in $\mathfrak{h}$ then the canonical map

$$\eta_{\mathfrak{h}} : \mathfrak{h} \to \mathfrak{h} \otimes \mathcal{A}(\mathfrak{h}), \quad \eta_{\mathfrak{h}}(e_i) := \sum_{s=1}^{n} e_s \otimes x_{si} \tag{17}$$

for all $i = 1, \cdots, n$ is a Leibniz algebra homomorphism. The commutative algebra $\mathcal{A}(\mathfrak{h})$ and the family of polynomials $P_{(a,i,j)}^{(\mathfrak{h})}$ are purely algebraic objects that capture the entire information of the Leibniz algebra $\mathfrak{h}$. Moreover, the universal algebra $\mathcal{A}(\mathfrak{h})$ satisfies the following universal property:

Corollary 2.7. *Let $\mathfrak{h}$ be a finite dimensional Leibniz algebra. Then for any commutative algebra A and any Leibniz algebra homomorphism $f : \mathfrak{h} \to \mathfrak{h} \otimes A$, there exists a unique algebra homomorphism $\theta : \mathcal{A}(\mathfrak{h}) \to A$ such that $f = (\mathrm{Id}_{\mathfrak{h}} \otimes \theta) \circ \eta_{\mathfrak{h}}$, i.e. the following diagram is commutative:*

$$\begin{tikzcd} \mathfrak{h} \arrow[r, "\eta_{\mathfrak{h}}"] \arrow[dr, "f"'] & \mathfrak{h} \otimes \mathcal{A}(\mathfrak{h}) \arrow[d, "\mathrm{Id}_{\mathfrak{h}} \otimes \theta"] \\ & \mathfrak{h} \otimes A \end{tikzcd} \tag{18}$$

Proof. Follows straightforward from the bijection given in (9) for $\mathfrak{g} := \mathfrak{h}$. □

Remark 2.8. If $\mathfrak{h}$ is a Lie algebra of dimension n, then the structure constants are subject to the following relations $\tau_{i,i}^s = 0$ and $\tau_{i,j}^s = -\tau_{j,i}^s$, for all $i,\, j,\, s = 1, \cdots, n$. Consequently, we can easily see that the universal polynomials of $\mathfrak{h}$ fulfill the following conditions:

$$P_{(a,i,i)}^{(\mathfrak{h})} = 0 \qquad \text{and} \qquad P_{(a,i,j)}^{(\mathfrak{h})} = -P_{(a,j,i)}^{(\mathfrak{h})} \tag{19}$$

for all $a,\, i,\, j = 1, \cdots n$, $i \neq j$. Thus, in the case of Lie algebras the universal algebra $\mathcal{A}(\mathfrak{h})$ takes a simplified form. We provide further examples in the sequel.

Examples 2.9. 1. Let $\mathfrak{h} := \mathrm{aff}(2,k)$ be the affine 2-dimensional Lie algebra with basis $\{e_1, e_2\}$ and bracket given by $[e_1,\, e_2] = e_1$. Then, we have:

$$\begin{aligned}\mathcal{A}(\mathrm{aff}(2,k)) &\cong k[X_{11}, X_{12}, X_{21}, X_{22}]/(X_{21},\, X_{11} - X_{12}X_{22} + X_{12}X_{21})\\ &\cong k[X, Y, Z]/(X - YZ)\end{aligned}$$

Indeed, the non-zero structure constants of $\mathfrak{h}$ are $\tau^1_{1,2} = 1 = -\tau^1_{2,1}$. Using (19) from the previous remark the only non-zero universal polynomials of the Lie algebra $\mathrm{aff}(2,k)$ are $P_{(1,1,2)} = X_{11} - X_{12}X_{22} + X_{12}X_{21}$, $P_{(2,1,2)} = X_{21}$, $-P_{(1,1,2)}$ and $-P_{(2,1,2)}$. The conclusion now follows.

2. Let $\mathfrak{h} := \mathfrak{sl}(2,k)$ be the Lie algebra with basis $\{e_1, e_2, e_3\}$ and bracket $[e_1,\, e_2] = e_3$, $[e_3,\, e_2] = -2e_2$, $[e_3,\, e_1] = 2e_1$. A routinely computation proves that $\mathcal{A}(\mathfrak{sl}(2,k)) \cong k[X_{ij}\,|\, i,j = 1,2,3]/J$, where J is the ideal generated by the following nine universal polynomials of $\mathfrak{sl}(2,k)$:

$$\begin{aligned}&X_{13} - 2X_{12}X_{31} + 2X_{11}X_{32},\;\; 2X_{11} - 2X_{11}X_{33} + 2X_{13}X_{31},\;\; 2X_{12} - 2X_{13}X_{32} + 2X_{12}X_{33}\\ &X_{23} - 2X_{21}X_{32} + 2X_{22}X_{31},\;\; 2X_{21} - 2X_{23}X_{31} + 2X_{21}X_{33},\;\; 2X_{22} - 2X_{22}X_{33} + 2X_{23}X_{32}\\ &X_{33} - X_{11}X_{22} + X_{12}X_{21},\;\; 2X_{31} - X_{21}X_{13} + X_{11}X_{23},\;\; 2X_{32} - X_{12}X_{23} + X_{13}X_{22}.\end{aligned}$$

We recall that the polynomial algebra $M(n) = k[X_{ij}\,|\, i,j = 1, \cdots, n]$ is a bialgebra with comultiplication and counit given by $\Delta(X_{ij}) = \sum_{s=1}^{n} X_{is} \otimes X_{sj}$ and $\varepsilon(X_{ij}) = \delta_{i,j}$, for any $i,\, j = 1, \cdots, n$. We will prove now that the universal algebra $\mathcal{A}(\mathfrak{h})$ is also a bialgebra.

Proposition 2.10. *Let $\mathfrak{h}$ be a Leibniz algebra of dimension n. Then there exists a unique bialgebra structure on $\mathcal{A}(\mathfrak{h})$ such that the Leibniz algebra homomorphism $\eta_{\mathfrak{h}} : \mathfrak{h} \to \mathfrak{h} \otimes \mathcal{A}(\mathfrak{h})$ becomes a right $\mathcal{A}(\mathfrak{h})$-comodule structure on $\mathfrak{h}$. More precisely, the comultiplication and the counit on $\mathcal{A}(\mathfrak{h})$ are given for any $i,\, j = 1, \cdots, n$ by*

$$\Delta(x_{ij}) = \sum_{s=1}^{n} x_{is} \otimes x_{sj} \quad \text{and} \quad \varepsilon(x_{ij}) = \delta_{i,j} \tag{20}$$

Furthermore, the usual projection $\pi\colon M(n) \to \mathcal{A}(\mathfrak{h})$ becomes a bialgebra homomorphism.

Proof. Consider the Leibniz algebra homomorphism $f : \mathfrak{h} \to \mathfrak{h} \otimes \mathcal{A}(\mathfrak{h}) \otimes \mathcal{A}(\mathfrak{h})$ defined by $f := (\eta_{\mathfrak{h}} \otimes \mathrm{Id}_{\mathcal{A}(\mathfrak{h})}) \circ \eta_{\mathfrak{h}}$. It follows from Corollary 2.7 that there exists a unique algebra homomorphism $\Delta : \mathcal{A}(\mathfrak{h}) \to \mathcal{A}(\mathfrak{h}) \otimes \mathcal{A}(\mathfrak{h})$ such that $(\mathrm{Id}_{\mathfrak{h}} \otimes \Delta) \circ \eta_{\mathfrak{h}} = f$; that is, the following diagram is commutative:

$$\begin{tikzcd} \mathfrak{h} \arrow[r, "\eta_{\mathfrak{h}}"] \arrow[d, "\eta_{\mathfrak{h}}"'] & \mathfrak{h} \otimes \mathcal{A}(\mathfrak{h}) \arrow[d, "\mathrm{Id}_{\mathfrak{h}} \otimes \Delta"] \\ \mathfrak{h} \otimes \mathcal{A}(\mathfrak{h}) \arrow[r, "\eta_{\mathfrak{h}} \otimes \mathrm{Id}_{\mathcal{A}(\mathfrak{h})}"'] & \mathfrak{h} \otimes \mathcal{A}(\mathfrak{h}) \otimes \mathcal{A}(\mathfrak{h}) \end{tikzcd} \tag{21}$$

Now, if we evaluate the diagram (21) at each e_i, for $i = 1, \cdots, n$ we obtain, taking into account (17), the following:

$$\begin{aligned}\sum_{t=1}^{n} e_t \otimes \Delta(x_{ti}) &= (\eta_{\mathfrak{h}} \otimes \mathrm{Id})(\sum_{s=1}^{n} e_s \otimes x_{si}) = \sum_{s=1}^{n}(\sum_{t=1}^{n} e_t \otimes x_{ts}) \otimes x_{si} \\ &= \sum_{t=1}^{n} e_t \otimes (\sum_{s=1}^{n} x_{ts} \otimes x_{si})\end{aligned}$$

and hence $\Delta(x_{ti}) = \sum_{s=1}^{n} x_{ts} \otimes x_{si}$, for all t, $i = 1, \cdots, n$. Obviously, Δ given by this formula on generators is coassociative. In a similar fashion, applying once again Corollary 2.7, we obtain that there exists a unique algebra homomorphism $\varepsilon : \mathcal{A}(\mathfrak{h}) \to k$ such that the following diagram is commutative:

$$\begin{tikzcd} \mathfrak{h} \arrow[r, "\eta_{\mathfrak{h}}"] \arrow[dr, "\mathrm{can}"'] & \mathfrak{h} \otimes \mathcal{A}(\mathfrak{h}) \arrow[d, "\mathrm{Id}_{\mathfrak{h}} \otimes \varepsilon"] \\ & \mathfrak{h} \otimes k \end{tikzcd} \tag{22}$$

where $\mathrm{can} : \mathfrak{h} \to \mathfrak{h} \otimes k$ is the canonical isomorphism, $\mathrm{can}(x) = x \otimes 1$, for all $x \in \mathfrak{h}$. If we evaluate this diagram at each e_t, for $t = 1, \cdots, n$, we obtain $\varepsilon(x_{ij}) = \delta_{i,j}$, for all i, $j = 1, \cdots, n$. It can be easily checked that ε is a counit for Δ, thus $\mathcal{A}(\mathfrak{h})$ is a bialgebra. Furthermore, the commutativity of the above two diagrams imply that the canonical map $\eta_{\mathfrak{h}} : \mathfrak{h} \to \mathfrak{h} \otimes \mathcal{A}(\mathfrak{h})$ defines a right $\mathcal{A}(\mathfrak{h})$-comodule structure on $\mathfrak{h}$. $\square$

We call the pair $(\mathcal{A}(\mathfrak{h}), \eta_{\mathfrak{h}})$, with the coalgebra structure defined in Proposition 2.10, the *universal coacting bialgebra of the Leibniz algebra* $\mathfrak{h}$. It fullfils the following universal property which extends Corollary 2.7:

Theorem 2.11. *Let $\mathfrak{h}$ be a Leibniz algebra of dimension n. Then, for any commutative bialgebra B and any Leibniz algebra homomorphism $f\colon \mathfrak{h} \to \mathfrak{h} \otimes B$ which makes $\mathfrak{h}$ into a right B-comodule there exists a unique bialgebra homomorphism $\theta\colon \mathcal{A}(\mathfrak{h}) \to B$ such that the following diagram is commutative:*

$$\begin{tikzcd} \mathfrak{h} \arrow[r, "\eta_{\mathfrak{h}}"] \arrow[dr, "f"'] & \mathfrak{h} \otimes \mathcal{A}(\mathfrak{h}) \arrow[d, "\mathrm{Id}_{\mathfrak{h}} \otimes \theta"] \\ & \mathfrak{h} \otimes B \end{tikzcd} \tag{23}$$

Proof. As $\mathcal{A}(\mathfrak{h})$ is the universal algebra of $\mathfrak{h}$, there exists a unique algebra homomorphism $\theta\colon \mathcal{A}(\mathfrak{h}) \to B$ such that diagram (23) commutes. The proof will be finished once we show that θ is a coalgebra homomorphism as well. This follows by using again the universal property of $\mathcal{A}(\mathfrak{h})$. Indeed, we obtain a unique algebra homomorphism $\psi\colon \mathcal{A}(\mathfrak{h}) \to B \otimes B$

such that the following diagram is commutative:

$$\begin{tikzcd} \mathfrak{h} \arrow[r, "\eta_{\mathfrak{h}}"] \arrow[dr, "\left(\mathrm{Id}_{\mathfrak{h}}\otimes \Delta_B\circ\theta\right)\circ\eta_{\mathfrak{h}}"'] & \mathfrak{h}\otimes\mathcal{A}(\mathfrak{h}) \arrow[d, "\mathrm{Id}_{\mathfrak{h}}\otimes\psi"] \\ & \mathfrak{h}\otimes B\otimes B \end{tikzcd} \tag{24}$$

The proof will be finished once we show that $(\theta\otimes\theta)\circ\Delta$ makes diagram (24) commutative. Indeed, as $f\colon \mathfrak{h}\to\mathfrak{h}\otimes B$ is a right B-comodule structure, we have:

$$\begin{aligned}
\big(\mathrm{Id}_{\mathfrak{h}}\otimes\ (\theta\otimes\theta)\circ\Delta\big)\circ\eta_{\mathfrak{h}} &= \big(\mathrm{Id}_{\mathfrak{h}}\otimes\theta\otimes\theta\big)\circ\underline{\big(\mathrm{Id}_{\mathfrak{h}}\otimes\Delta\big)\circ\eta_{\mathfrak{h}}}\\
&\overset{(21)}{=} \big(\mathrm{Id}_{\mathfrak{h}}\otimes\theta\otimes\theta\big)\circ(\eta_{\mathfrak{h}}\otimes\mathrm{Id}_{\mathcal{A}(\mathfrak{h})})\circ\eta_{\mathfrak{h}}\\
&= \big(\underline{(\mathrm{Id}_{\mathfrak{h}}\otimes\theta)\circ\eta_{\mathfrak{h}}}\ \otimes\theta\big)\circ\eta_{\mathfrak{h}}\\
&\overset{(23)}{=} \big(f\otimes\theta\big)\circ\eta_{\mathfrak{h}}\\
&= (f\otimes\mathrm{Id}_B)\circ\underline{(\mathrm{Id}_{\mathfrak{h}}\otimes\theta)\circ\ \eta_{\mathfrak{h}}}\\
&\overset{(23)}{=} \underline{(f\otimes\mathrm{Id}_B)\circ f}\\
&= (\mathrm{Id}_{\mathfrak{h}}\otimes\Delta_B)\circ\underline{f}\\
&\overset{(23)}{=} (\mathrm{Id}_{\mathfrak{h}}\otimes\Delta_B)\circ(\mathrm{Id}_{\mathfrak{h}}\otimes\theta)\circ\eta_{\mathfrak{h}}\\
&= (\mathrm{Id}_{\mathfrak{h}}\otimes\Delta_B\circ\theta)\circ\eta_{\mathfrak{h}}
\end{aligned}$$

as desired. Similarly, one can show that $\varepsilon_B\circ\theta=\varepsilon$ and the proof is now finished. □

Next we discuss three consequences of our previous results which highlights the power of the universal coacting bialgebra of a Leibniz algebra. The first application of $\mathcal{A}(\mathfrak{h})$ is related to the classification of all G-gradings on a given Leibniz/Lie algebras. Let G be an abelian group and $\mathfrak{h}$ a Leibniz algebra. We recall that a *G-grading* on $\mathfrak{h}$ is a vector space decomposition $\mathfrak{h}=\oplus_{\sigma\in G}\mathfrak{h}_\sigma$ such that $[\mathfrak{h}_\sigma,\,\mathfrak{h}_\tau]\subseteq\mathfrak{h}_{\sigma\tau}$ for all $\sigma,\,\tau\in G$. For more detail on the problem of classifying G-gradings on Lie algebras see [10] and the references therein. In what follows $k[G]$ denotes the usual group algebra of a group G.

Corollary 2.12. *Let G be an abelian group and $\mathfrak{h}$ a finite dimensional Leibniz algebra. Then there exists a bijection between the set of all G-gradings on $\mathfrak{h}$ and the set of all bialgebra homomorphisms $\mathcal{A}(\mathfrak{h})\to k[G]$.*

The bijection is given such that the G-grading on $\mathfrak{h}$ associated to a bialgebra map θ : $\mathcal{A}(\mathfrak{h})\to k[G]$ is given for any $\sigma\in G$ by:

$$\mathfrak{h}_\sigma := \{x\in\mathfrak{h}\,|\,\big(\mathrm{Id}_{\mathfrak{h}}\otimes\theta\big)\circ\eta_{\mathfrak{h}}(x)=x\otimes\sigma\} \tag{25}$$

Proof. By applying Theorem 2.11 for the commutative bialgebra $B:=k[G]$ yields a bijection between the set of all bialgebra homomorphisms $\mathcal{A}(\mathfrak{h})\to k[G]$ and the set of all Leibniz algebra homomorphisms $f\colon\mathfrak{h}\to\mathfrak{h}\otimes k[G]$ which makes $\mathfrak{h}$ into a right

$k[G]$-comodule. The proof is finished if we show that the latter set is in bijective correspondence with the set of all G-gradings on $\mathfrak{h}$.

Indeed, it is a well known fact in Hopf algebra theory [21, Excercise 3.2.21] that there exists a bijection between the set of all right $k[G]$-comodule structures $f : \mathfrak{h} \to \mathfrak{h} \otimes k[G]$ on the vector space $\mathfrak{h}$ and the set of all vector space decompositions $\mathfrak{h} = \oplus_{\sigma\in G} \mathfrak{h}_\sigma$. The bijection is given such that $x_\sigma \in \mathfrak{h}_\sigma$ if and only if $f(x_\sigma) = x_\sigma \otimes \sigma$, for all $\sigma \in G$. The only thing left to prove is that under this bijection a right coaction $f : \mathfrak{h} \to \mathfrak{h} \otimes k[G]$ is a Leibniz algebra homomorphism if and only if $[\mathfrak{h}_\sigma,\, \mathfrak{h}_\tau] \subseteq \mathfrak{h}_{\sigma\tau}$, for all $\sigma,\, \tau \in G$. To prove this, let $\sigma,\, \tau \in G$ and $x_\sigma \in \mathfrak{h}_\sigma$, $x_\tau \in \mathfrak{h}_\tau$. Then, $[f(x_\sigma),\, f(x_\tau)] = [x_\sigma \otimes \sigma,\, x_\tau \otimes \tau] = [x_\sigma,\, x_\tau] \otimes \sigma\tau$. Thus, we obtain that $f([x_\sigma,\, x_\tau]) = [f(x_\sigma),\, f(x_\tau)]$ if and only if $[x_\sigma,\, x_\tau] \in \mathfrak{h}_{\sigma\tau}$. Hence, $f : \mathfrak{h} \to \mathfrak{h} \otimes k[G]$ is a Leibniz algebra homomorphism if and only if $[\mathfrak{h}_\sigma,\, \mathfrak{h}_\tau] \subseteq \mathfrak{h}_{\sigma\tau}$, for all $\sigma,\, \tau \in G$ and the proof is now finished. □

Let G be a group and $\mathfrak{h}$ a Leibniz algebra. We recall that an *action as automorphisms of G on $\mathfrak{h}$* is a morphism of groups $\varphi : G \to \mathrm{Aut}_{\mathrm{Lbz}}(\mathfrak{h})$.

Corollary 2.13. *Let G be a finite group and $\mathfrak{h}$ a finite dimensional Leibniz algebra with basis $\{e_1, \cdots, e_n\}$. Then there exists a bijection between the set of all actions as automorphisms of G on $\mathfrak{h}$ and the set of all bialgebra homomorphisms $\mathcal{A}(\mathfrak{h}) \to k[G]^*$.*

The bijection is given such that the group homomorphism $\varphi_\theta : G \to \mathrm{Aut}_{\mathrm{Lbz}}(\mathfrak{h})$ associated to a bialgebra homomorphism $\theta : \mathcal{A}(\mathfrak{h}) \to k[G]^*$ is given by:

$$\varphi_\theta(g)(e_i) = \sum_{s=1}^{n} < \theta(x_{si}),\, g > \, e_s \tag{26}$$

for all $g \in G$ and $i = 1, \cdots, n$.

Proof. By applying Theorem 2.11 for the commutative bialgebra $B := k[G]^*$ gives a bijection between the set of all bialgebra homomorphisms $\mathcal{A}(\mathfrak{h}) \to k[G]^*$ and the set of all Leibniz algebra homomorphisms $f\colon \mathfrak{h} \to \mathfrak{h} \otimes k[G]^*$ which make $\mathfrak{h}$ into a right $k[G]^*$-comodule. The proof is finished if we show that the latter set is in bijective correspondence with the set of all group homomorphisms $G \to \mathrm{Aut}_{\mathrm{Lbz}}(\mathfrak{h})$. This follows by a standard argument in Hopf algebra theory, similar to the one used in [22, Lemma 1]. We indicate very briefly how the argument goes, leaving the details to the reader. Indeed, the category of right $k[G]^*$-comodules is isomorphic to the category of left $k[G]$-modules. The left action $\bullet : k[G] \otimes \mathfrak{h} \to \mathfrak{h}$ of the group algebra $k[G]$ on $\mathfrak{h}$ associated to a right coaction $f\colon \mathfrak{h} \to \mathfrak{h} \otimes k[G]^*$ is given by $g \bullet x := < x_{<1>},\, g > \, x_{<0>}$, where we used the $\sum$-notation for comodules, $f(x) = x_{<0>} \otimes x_{<1>} \in \mathfrak{h} \otimes k[G]^*$ (summation understood). We associate to the action $\bullet$ the map $\varphi_\bullet : G \to \mathrm{Aut}_k(\mathfrak{h})$, $\varphi_\bullet(g)(x) := g \bullet x$, for all $g \in G$ and $x \in \mathfrak{h}$. Now, it can be easily checked that $f\colon \mathfrak{h} \to \mathfrak{h} \otimes k[G]^*$ being a Leibniz algebra homomorphism is equivalent to $\varphi_\bullet(g)$ being an automorphism of the Leibniz algebra $\mathfrak{h}$, for all $g \in G$ and the proof is finished. □

Recall that for any bialgebra H the set of group-like elements, denoted by $G(H) := \{g \in H \,|\, \Delta(g) = g \otimes g, \text{ and } \varepsilon(g) = 1\}$, is a monoid with respect to the multiplication of H.

We denote by $H^{\rm o}$, the finite dual bialgebra of H, i.e.:

$$H^{\rm o} := \{f \in H^* \,|\, f(I) = 0, \text{ for some ideal } I \lhd H \text{ with } \dim_k(H/I) < \infty\}$$

It is well known (see for instance [21, pag. 62]) that $G(H^{\rm o}) = {\rm Hom}_{\rm Alg_k}(H,\, k)$, the set of all algebra homomorphisms $H \to k$. Now, we shall give the third application of the universal algebra of a Leibniz algebra.

Theorem 2.14. *Let $\mathfrak{h}$ be a finite dimensional Leibniz algebra with basis $\{e_1, \cdots, e_n\}$ and let $U\big(G\big(\mathcal{A}(\mathfrak{h})^{\rm o}\big)\big)$ be the group of all invertible group-like elements of the finite dual $\mathcal{A}(\mathfrak{h})^{\rm o}$. Then the map defined for any $\theta \in U\big(G\big(\mathcal{A}(\mathfrak{h})^{\rm o}\big)\big)$ and $i = 1, \cdots, n$ by:*

$$\overline{\gamma} : U\big(G\big(\mathcal{A}(\mathfrak{h})^{\rm o}\big)\big) \to {\rm Aut}_{\rm Lbz}(\mathfrak{h}), \qquad \overline{\gamma}(\theta)(e_i) := \sum_{s=1}^{n} \theta(x_{si})\, e_s \tag{27}$$

is an isomorphism of groups.

Proof. By applying Corollary 2.3 for $\mathfrak{g} := \mathfrak{h}$ it follows that the map

$$\gamma : {\rm Hom}_{\rm Alg_k}(\mathcal{A}(\mathfrak{h}),\, k) \to {\rm End}_{\rm Lbz}(\mathfrak{h}), \quad \gamma(\theta) = \big({\rm Id}_{\mathfrak{h}} \otimes \theta\big) \circ \eta_{\mathfrak{g}}$$

is bijective. Based on formula (17), it can be easily seen that γ takes the form given by (27). As we mentioned above we have ${\rm Hom}_{\rm Alg_k}(\mathcal{A}(\mathfrak{h}), k) = G\big(\mathcal{A}(\mathfrak{h})^{\rm o}\big)$. Therefore, since $\overline{\gamma}$ is the restriction of γ to the invertible elements of the two monoids, the proof will be finished once we show that γ is an isomorphism of monoids. We mention that the monoid structure on ${\rm End}_{\rm Lbz}(\mathfrak{h})$ is given by the usual composition of endomorphisms of the Leibniz algebra $\mathfrak{h}$, while $G\big(\mathcal{A}(\mathfrak{h})^{\rm o}\big)$ is a monoid with respect to the convolution product, that is:

$$(\theta_1 \star \theta_2)(x_{sj}) = \sum_{t=1}^{n} \theta_1(x_{st})\theta_2(x_{tj}) \tag{28}$$

for all $\theta_1,\ \theta_2 \in G\big(\mathcal{A}(\mathfrak{h})^{\rm o}\big)$ and $j,\ s = 1, \cdots, n$. Now, for any $\theta_1,\ \theta_2 \in G\big(\mathcal{A}(\mathfrak{h})^{\rm o}\big)$ and $j = 1, \cdots, n$ we have:

$$\begin{aligned}
\big(\gamma(\theta_1) \circ \gamma(\theta_2)\big)(e_j) &= \gamma(\theta_1)\big(\sum_{t=1}^{n} \theta_2(x_{tj}) e_t\big) = \sum_{s,t=1}^{n} \theta_1(x_{st})\theta_2(x_{tj})\, e_s \\
&= \sum_{s=1}^{n} \big(\sum_{t=1}^{n} \theta_1(x_{st})\theta_2(x_{tj})\big)\, e_s = \sum_{s=1}^{n} (\theta_1 \star \theta_2)(x_{sj})\, e_s = \gamma(\theta_1 \star \theta_2)(e_j)
\end{aligned}$$

thus, $\gamma(\theta_1 \star \theta_2) = \gamma(\theta_1) \circ \gamma(\theta_2)$, and therefore γ respects the multiplication. We are left to show that γ also preserves the unit. Note that the unit 1 of the monoid $G\big(\mathcal{A}(\mathfrak{h})^{\rm o}\big)$ is the counit $\varepsilon_{\mathcal{A}(\mathfrak{h})}$ of the bialgebra $\mathcal{A}(\mathfrak{h})$ and we obtain:

$$\gamma(1)(e_i) = \gamma(\varepsilon_{\mathcal{A}(\mathfrak{h})})(e_i) = \sum_{s=1}^{n} \varepsilon_{\mathcal{A}(\mathfrak{h})}(x_{si})\, e_s = \sum_{s=1}^{n} \delta_{si}\, e_s = e_i = {\rm Id}_{\mathfrak{h}}(e_i)$$

Thus we have proved that γ is an isomorphism of monoids and the proof is finished. $\square$

In what follows we construct for any finite dimensional Leibniz algebra $\mathfrak{h}$ a universal commutative Hopf algebra $\mathcal{H}(\mathfrak{h})$ together with a Leibniz algebra homomorphism $\lambda_{\mathfrak{h}}\colon \mathfrak{h} \to \mathfrak{h} \otimes \mathcal{H}(\mathfrak{h})$ which makes $\mathfrak{h}$ into a right $\mathcal{H}(\mathfrak{h})$-comodule. This is achieved by using the free commutative Hopf algebra generated by a commutative bialgebra introduced in [19, Chapter IV]. Recall that assigning to a commutative bialgebra the free commutative Hopf algebra defines a functor $L\colon \mathrm{ComBiAlg}_k \to \mathrm{ComHopf}_k$ which is a left adjoint to the forgetful functor $\mathrm{ComHopf}_k \to \mathrm{ComBiAlg}_k$ ([19, Theorem 65, (2)]). Throughout, we denote by $\mu\colon 1_{\mathrm{ComBiAlg}_k} \to UL$ the unit of the adjunction $L \dashv U$.

Definition 2.15. Let $\mathfrak{h}$ be a finite dimensional Leibniz algebra. The pair $\big(\mathcal{H}(\mathfrak{h}) := L(\mathcal{A}(\mathfrak{h})),\ \lambda_{\mathfrak{h}} := (\mathrm{Id}_{\mathfrak{h}} \otimes \mu_{\mathcal{A}(\mathfrak{h})}) \circ \eta_{\mathfrak{h}}\big)$ is called the *universal coacting Hopf algebra of* $\mathfrak{h}$.

The pair $\big(\mathcal{H}(\mathfrak{h}),\ \lambda_{\mathfrak{h}}\big)$ fulfills the following universal property which shows that it is the initial object in the category of all commutative Hopf algebras that coact on $\mathfrak{h}$.

Theorem 2.16. *Let $\mathfrak{h}$ be a finite dimensional Leibniz algebra. Then, for any commutative Hopf algebra H and any Leibniz algebra homomorphism $f\colon \mathfrak{h} \to \mathfrak{h} \otimes H$ which makes $\mathfrak{h}$ into a right H-comodule there exists a unique Hopf algebra homomorphism $g\colon \mathcal{H}(\mathfrak{h}) \to H$ for which the following diagram is commutative:*

$$\begin{tikzcd} \mathfrak{h} \arrow[r, "\lambda_{\mathfrak{h}}"] \arrow[dr, "f"'] & \mathfrak{h} \otimes \mathcal{H}(\mathfrak{h}) \arrow[d, "\mathrm{Id}_{\mathfrak{h}} \otimes g"] \\ & \mathfrak{h} \otimes H \end{tikzcd} \tag{29}$$

Proof. Let H be a commutative Hopf algebra together with a Leibniz algebra homomorphism $f\colon \mathfrak{h} \to \mathfrak{h} \otimes H$ which makes $\mathfrak{h}$ into right a H-comodule. Using Theorem 2.11 we obtain a unique bialgebra homomorphism $\theta : \mathcal{A}(\mathfrak{h}) \to H$ which makes the following diagram commutative:

$$\begin{tikzcd} \mathfrak{h} \arrow[r, "\eta_{\mathfrak{h}}"] \arrow[dr, "f"'] & \mathfrak{h} \otimes \mathcal{A}(\mathfrak{h}) \arrow[d, "\mathrm{Id}_{\mathfrak{h}} \otimes \theta"] \\ & \mathfrak{h} \otimes H \end{tikzcd} \qquad i.e.\ \ (\mathrm{Id}_{\mathfrak{h}} \otimes \theta) \circ \eta_{\mathfrak{h}} = f. \tag{30}$$

Now the adjunction $L \dashv U$ yields a unique Hopf algebra homomorphism $g\colon L(\mathcal{A}(\mathfrak{h})) \to H$ such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{A}(\mathfrak{h}) \arrow[r, "\mu_{\mathcal{A}(\mathfrak{h})}"] \arrow[dr, "\theta"'] & L(\mathcal{A}(\mathfrak{h})) \arrow[d, "g"] \\ & H \end{tikzcd} \qquad \text{i.e. } g \circ \mu_{\mathcal{A}(\mathfrak{h})} = \theta. \tag{31}$$

We are now ready to show that $g\colon \mathcal{H}(\mathfrak{h}) = L(\mathcal{A}(\mathfrak{h})) \to H$ is the unique Hopf algebra homomorphism which makes diagram (29) commutative. Indeed, putting all the above together yields:

$$\begin{aligned}
(\mathrm{Id}_{\mathfrak{h}} \otimes g) \circ (\mathrm{Id}_{\mathfrak{h}} \otimes \mu_{\mathcal{A}(\mathfrak{h})}) \circ \eta_{\mathfrak{h}} &= (\mathrm{Id}_{\mathfrak{h}} \otimes \underline{g \circ \mu_{\mathcal{A}(\mathfrak{h})}}) \circ \eta_{\mathfrak{h}} \\
&\overset{(31)}{=} \underline{(\mathrm{Id}_{\mathfrak{h}} \otimes \theta) \circ \eta_{\mathfrak{h}}} \\
&\overset{(30)}{=} f.
\end{aligned}$$

Since g is obviously the unique Hopf algebra homomorphism which makes the above diagram commutative, the proof is finished. □

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