THE ω-CATEGORIFICATION OF ALGEBRAIC THEORIES

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ABSTRACT. Batanin and Leinsters work on globular operads has provided one of many potential definitions of a weak ω-category. Through the language of globular operads they construct a monad whose algebras encode weak ω-categories. The purpose of this work is to show how to construct a similar monad which will allow us to formulate weak ω-categorifications of any equational algebraic theory. We first review the classical theory of operads and PROs. We then present how Leinsters globular operads can be extended to a theory of globular PROs via categorical enrichment over the category of collections. It is then shown how a process called globularization allows us to construct from a classical PRO P a globular PRO whose algebras are those algebras for P which are internal to the category of strict ω-categories and strict ω-functors. Leinsters notion of a contraction structure on a globular operad is then extended to this setting of globular PROs in order to build a monad whose algebras are globular PROs with contraction over the globularization of the classical PRO P. Among these PROs with contraction over P is the globular PRO whose algebras are by construction the fully weakened ω-categorifications of the algebraic theory encoded by P.

1. Introduction

The purpose of this work is to provide a framework and procedure for the categorification of general equational algebraic theories such as monoids, groups, quandles, rings, etc. The notion of a categorification of an algebraic structure was first introduced by Crane[4]. His original formulation consisted of constructing from an algebraic structure, such as a Hopf algebra, a new algebraic structure with analogous operations one categorical dimension higher, such as a Hopf category[5]. Such a construction, from a given model of an algebraic theory, requires choices and is in most nontrivial cases not functorial. But as we will see, when this process is performed at the level of theories, it can be made functorial in a nontrivial and universal way.

There are many various ways one could categorify an abstract theory. One approach is to provide a definition of your algebraic theory T as an obect in the category **Set** equipped with various functions whose composition satisfies certain relations. These relations then endow any such set with the structure of a T-algebra. We can then internalize this construction in the category **Cat**, so that the underlying set of the T-algebra is now replaced with an underlying category. Moreover, the operations in the theory are now replaced with functors. This is precisely what is meant by raising the categorical dimension by one step (in this case from 0 to 1).

2010 Mathematics Subject Classification: 18C10,18C15,18C20,18D05,18D10,18D15,18D20,18D50.
Key words and phrases: Operads, PROs, Higher Categories, Globular Sets, Categorification.

The drawback to this procedure is that such a construction gives us a notion of categorification that is strict, in the sense that all relations satisfied by the composition of functors strictly hold by equality. In many context this is not quite the desired construction. We often want a notion of categorification that satisfies its defining relations weakly, in the sense that the equations imposed on the operations hold up to an equivalence defined by morphisms one categorical dimension higher. But this leaves us with a new problem. Once we construct morphisms one categorical dimension higher (in this case natural transformations between compositions of functors), we must determine what coherence conditions these new morphisms must satisfy when composed in various ways. And even if appropriate coherence conditions can be determined, we are now left with a construction that gives a strict 2-categorification of the theory T. We can similarly make our strict 2-categorification weak by replacing the coherence condition equations with 3-morphisms. But once again we must determine what higher coherence conditions these 3-morphisms must satisfy. This in turn leaves us with a strict 3-categorifiction, leading to an infinite sequence of analogous problems.

In principle we can continue to climb this ladder of categorical dimension hopping forever. Unfortunately, each step becomes conceptually and computationally more difficult than the last. It is hence more desirable to have a limiting construction that categorifies T in each finite categorical dimension all at once. This is precisely the approach taken in higher category theory when attempting to defining notions of weak ω-categories. Batanin[1] and Leinster[11] have provided globular notions of weak ω-categories. After building up a language of globular operads, they then present globular ω-categories as algebras for the initial globular operad constructed via a particular monad.

The process presented in the present work extends Batanin and Leinster's constructions to a formulation of how to build weak ω-categorification of any equational algebraic theory T. This is achieved by extending the theory of globular operads to a theory of globular PROs (a nonsymetric version of MacLane's notion of a PROP[12]). We shall first develop this theory of globular PROs. Once this notion has been made precise, a process for turning the classical PRO P_T for the theory T into a globular PRO $\mathcal{P}_T$ will be presented. This globular PRO $\mathcal{P}_T$ will have the property that its algebras in **Glob**, the category of globular sets, are algebras for P internal to the category of strict ω-categories and strict ω-functors. Leinster's notion of contractibility will then be used to construct an initial globular PRO with contraction over $\mathcal{P}_T$. This initial globular PRO with contraction over $\mathcal{P}_T$ will then be, by construction, the globular PRO whose algebras are weak ω-categorifications of the theory T.

Throughout we shall use the phrase 'classical operads' to mean nonsymmetric operads in **Set**. We will not in this paper consider operads equipped with a symmetric group action. We will often refer to nonsymmetric operads simply as operads, with no other further adjectives attached. The use of 'classical' is intended only to distinguish between the globular and non-globular cases.

2. Basic Definitions

The first goal of this work is to develop a sufficient categorical description of a *globular PRO*. We shall start by describing how classical operads and non-globular PROs can be described in terms of a composition tensor product on graded sets. We will then describe in what follows how such structures can be made globular. The reason we begin with the non-globular setting is because many of the details carry over to the globular constructions. This has the benefit of providing useful intuition for the globular constructions, as well as allowing us to omit certain details when working in the globular setting that are clear in light of their non-globular counterpart.

We begin by recalling several basic definitions and constructions. Further details can found in May's[14] or Leinster's[11] work.

2.1. Definition. *A nonsymmetric operad O consists of a sequence of sets $\{O(n)\}_{n\in\mathbb{N}}$ whose n-th entry is called the set of n-ary operations, an identity operation $\mathbb{1}_O \in O(1)$, and for all $n, k_1, k_2, ..., k_n \in \mathbb{N}$ a composition operation*

$$\circ : O(n) \times \prod_{i=1}^{n} O(k_i) \to O(\sum_{i=1}^{n} k_i)$$

such that

$$\theta_0 \circ (\theta_1 \circ (\theta_{1_1}, ..., \theta_{1_k}), ..., \theta_n \circ (\theta_{n_1}, ..., \theta_{n_l})) = (\theta_0 \circ (\theta_1, ..., \theta_n)) \circ (\theta_{1_1}, ..., \theta_{1_k}, ..., \theta_{n_1}, ..., \theta_{n_l})$$

and

$$\theta_0 \circ (\mathbb{1}_O, \mathbb{1}_O, ..., \mathbb{1}_O) = \theta_0 = \mathbb{1}_O \circ \theta_0$$

for all $\theta_i \in \{O(n)\}_{n\in\mathbb{N}}$ whenever the compositions are well-defined.

2.2. Definition. *A homomorphism of operads $f : O \to P$ is a sequence of maps $\{f_n : O(n) \to P(n)\}_{n\in\mathbb{N}}$ that preserves both the identity operation and composition maps in the obvious sense.*

2.3. Definition. *A graded set is a set X equipped with a function $x : X \to \mathbb{N}$ called the arity map.*

A graded set may also be thought of as a countably indexed family of sets in which for each $n \in \mathbb{N}$ the fiber $X_n := x^{-1}(n)$ is the set of 'n-ary' elements. In the following we will often represent a graded set $x : X \to \mathbb{N}$ by its underlying set X.

2.4. Definition. *Let $x : X \to \mathbb{N}$ and $y : Y \to \mathbb{N}$ be graded sets. A morphism of graded sets between them is a function $f : X \to Y$ which makes the following triangle commute:*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "x"'] & & Y \arrow[dl, "y"] \\ & \mathbb{N} & \end{tikzcd}$$

Although the category of graded sets is the slice category $\mathbf{Set}/\mathbb{N}$, we shall denote it by $\mathbf{GrdSet}$.

2.5. Definition. *A monad $(T : \mathcal{C} \to \mathcal{C}, \mu : T^2 \Rightarrow T, \eta : \mathbb{1} \Rightarrow T)$ on a category $\mathcal{C}$ is a triple consisting of an endofunctor $T : \mathcal{C} \to \mathcal{C}$ together with two natural transformations μ and η which make the diagrams*

$$\begin{tikzcd}
T(T(T(X))) \arrow[r, "T(\mu_X)"] \arrow[d, "\mu_{T(X)}"'] & T(T(X)) \arrow[d, "\mu_X"] \\
T(T(X)) \arrow[r, "\mu_X"] & T(X)
\end{tikzcd}
\qquad
\begin{tikzcd}
T(T(X)) \arrow[dr, "\mu_{T(X)}"'] & T(X) \arrow[l, "T(\eta_X)"'] \arrow[r, "\eta_{T(X)}"] \arrow[d, "\mathbb{1}_{T(X)}"'] & T(T(X)) \arrow[dl, "\mu_{T(X)}"] \\
 & T(X) &
\end{tikzcd}$$

commute for all objects $X \in \mathcal{C}$.

A monad can be briefly defined as a monoid object in a 2-category $\mathcal{C}^{\mathcal{C}}$ of endofunctors from a fixed category $\mathcal{C}$ to itself.

2.6. Definition. *An algebra $(X, h : T(X) \to X)$ for a monad $(T : \mathcal{C} \to \mathcal{C}, \mu : T^2 \Rightarrow T,$ $\eta : \mathbb{1} \Rightarrow T)$ is a pair consisting of an object $X \in \mathcal{C}$ together with a morphism $h : T(X) \to X$ in $\mathcal{C}$ called the structure map which makes the diagrams*

$$\begin{tikzcd}
T(T(X)) \arrow[r, "T(h)"] \arrow[d, "\mu_X"'] & T(X) \arrow[d, "h"] \\
T(X) \arrow[r, "h"] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[dr, "\mathbb{1}_X"'] & T(X) \arrow[d, "h"] \\
 & X
\end{tikzcd}$$

commute.

In what follows we will freely interchange the set $\mathbb{N}$ with $T(\{*\})$ where $(T : \mathbf{Set} \to \mathbf{Set},$ $\mu : T^2 \Rightarrow T, \eta : \mathbb{1}_{\mathbf{Set}} \Rightarrow T)$ is the monoid monad on **Set** which sends a set X to the underlying set $T(X)$ of the free monoid on X. This shift in perspective (of thinking about $\mathbb{N}$ as the free monoid on the one point set $\{*\}$) will help us later to more naturally generalize this construction to the globular setting. Moreover, we will shortly make use of the fact that the monad T is cartesian[10].

2.7. Definition. *A monad $(T : \mathcal{C} \to \mathcal{C}, \mu : T^2 \Rightarrow T, \eta : \mathbb{1}_{\mathcal{C}} \Rightarrow T)$ is a cartesian monad if all naturality squares for μ and η are pullback squares and T preserves all pullbacks.*

The category **GrdSet** has a monoidal category structure different from the natural cartesian and cocartesian structures. We shall here denote this third monoidal product by $\Box$.

2.8. Definition. *Let $x : X \to \mathbb{N}$ and $y : Y \to \mathbb{N}$ be a pair of graded sets. Their composition tensor product $x \Box y : X \Box Y \to \mathbb{N}$ is defined by the diagram*

$$\begin{tikzcd}
X \Box Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & T(Y) \arrow[r, "T(y)"] \arrow[d, "T(!_Y)"] & T^2(\{*\}) \arrow[r, "\mu_{\{*\}}"] & T(\{*\}) \\
X \arrow[r, "x"] & T(\{*\}) & &
\end{tikzcd}$$

where $!_Y : Y \to \{*\}$ is the unique map from Y to the terminal one point set. The underlying graded set $X\square Y$ is the pullback of x and $T(!_Y)$ with the arity function $x\square y$ defined to be the composition along the top row.

This definition makes $X\square Y$ the graded set whose elements are pairs (a,ψ) consisting of an element $a \in X$ and a word ψ of elements from Y with the property that the arity of a agrees with the length of ψ given by $T(!_Y)$. We then think of the elements of $X\square Y$ as composable pairs consisting of a word of elements from Y that can be 'plugged into' a single element from X. Moreover, the arity of each pair is given as the sum of the arities of the entries in ψ (as elements of Y).

2.9. THEOREM. *The product $\square$ together with the graded set $i : \{*\} \hookrightarrow T(\{*\})$ gives* **GrdSet** *the structure of a monoidal category.*

PROOF. The associator for **GrdSet** with respect to $\square$ can be obtained as follows. Consider the graded sets $x : X \to T(\{*\})$, $y : Y \to T(\{*\})$, and $z : Z \to T(\{*\})$. Then construct the graded sets

$$x\square y : X\square Y \to T(\{*\})$$
$$y\square z : Y\square Z \to T(\{*\})$$
$$(x\square y)\square z : (X\square Y)\square Z \to T(\{*\})$$
$$x\square(y\square z) : X\square(Y\square Z) \to T(\{*\})$$

as described above. By definition, this means that the diagrams defining $(x\square y)\square z$ and $x\square y$ can be configured together in the following way:

$$\begin{tikzcd}
(X\square Y)\square Z \arrow[rrr, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & & & T(Z) \arrow[r, "T(z)"] \arrow[d, "T(!_Z)"] & T^2(\{*\}) \arrow[r, "\mu_{\{*\}}"] & T(\{*\}) \\
X\square Y \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] \arrow[dr, phantom, "\lrcorner", very near start] & T(Y) \arrow[r, "T(y)"] \arrow[d, "T(!_Y)"] & T^2(\{*\}) \arrow[r, "\mu_{\{*\}}"] & T(\{*\}) & & \\
X \arrow[r, "x"] & T(\{*\}) & & & &
\end{tikzcd}$$

Note then that the top pullback square can be factored into three iterated pullback squares to obtain the following diagram:

$$
\begin{tikzcd}
(X\square Y)\square Z \arrow[r, "\phi"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & T(Y\square Z) \arrow[r, "T(\pi_2)"] \arrow[d, "T(\pi_1)"] \arrow[dr, phantom, "\lrcorner", very near start] & T^2(Z) \arrow[r, "\mu_Z"] \arrow[d, "T^2(!_Z)"] \arrow[dr, phantom, "\lrcorner", very near start] & T(Z) \arrow[r, "T(z)"] \arrow[d, "T(!_Z)"] & T^2(\{*\}) \arrow[r, "\mu_{\{*\}}"] & T(\{*\}) \\
X\square Y \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] \arrow[dr, phantom, "\lrcorner", very near start] & T(Y) \arrow[r, "T(y)"] \arrow[d, "T(!_Y)"] & T^2(\{*\}) \arrow[r, "\mu_{\{*\}}"] & T(\{*\}) & & \\
X \arrow[r, "x"] & T(\{*\}) & & & &
\end{tikzcd}
$$

Here, since T is a cartesian monad, the top right square must be a pullback because it is a naturality square for μ. The top middle square is a pullback because it is the image of a pullback square under T, which preserves all pullbacks. The top left square is then the pullback square which must exist by the fact that the single pullback we started with can be factored in this way by the right and middle pullbacks just described. Now observe that the top left and bottom pullback squares together must form a pullback. But then the pair of maps

$$x \circ \pi_1 : X\square(Y\square Z) \to T(\{*\})$$

$$T(!_Y) \circ T(\pi_1) \circ \pi_2 : X\square(Y\square Z) \to T(\{*\})$$

give another pullback of the same cospan, inducing a map $\overline{\alpha}_{X,Y,Z} : X\square(Y\square Z) \to (X\square Y)\square Z$ which is the desired associator. It remains to see that this map preserves the arity map for these two graded sets. To see this, consider the following diagram:

$$
\begin{tikzcd}
T(Z) \arrow[r, "T(z)"] & T^2(\{*\}) \arrow[dr, "\mu_{\{*\}}"] & \\
T^2(Z) \arrow[u, "\mu_Z"] \arrow[r, "T^2(z)"] & T^3(\{*\}) \arrow[u, "\mu_{T(\{*\})}"'] \arrow[d, "T(\mu_{\{*\}})"'] & T(\{*\}) \\
T(Y\square Z) \arrow[u, "T(\pi_2)"] \arrow[r, "T(y\square z)"] & T^2(\{*\}) \arrow[ur, "\mu_{\{*\}}"'] &
\end{tikzcd}
$$

The bottom square is T applied to the diagram which defines the arity map $y\square z$. The top square commutes by the naturality of μ. The right square commutes by the associativity condition on μ as the multiplication transformation for T as a monad. The commutativity of the outer edges of this diagram then gives that

$$\mu_{\{*\}} \circ T(y\square z) = \mu_{\{*\}} \circ T(z) \circ \mu_Z \circ T(\pi_2)$$

which together with the fact that

$$\phi \circ \overline{\alpha}_{X,Y,Z} = \pi_2 : X\Box(Y\Box Z) \to T(Y\Box Z)$$

shows that

$$x\Box(y\Box z) = \mu_{\{*\}} \circ T(y\Box z) \circ \pi_2$$
$$= \mu_{\{*\}} \circ T(z) \circ \mu_Z \circ T(\pi_2) \circ \phi \circ \overline{\alpha}_{X,Y,Z} = (x\Box y)\Box z \circ \overline{\alpha}_{X,Y,Z}$$

ensuring that the associator preserves arities, thus giving an isomorphism of graded sets.

The monoidal identity for $\Box$ is the graded set $i : \{*\} \hookrightarrow \mathbb{N}$ where i is simply the inclusion of the generator $*$ into the set $T(\{*\})$. To see that this is the correct monoidal identity for $\Box$, notice that for any graded set X, $X\Box\{*\} = \{(a,n)|a \in X, n \in T(\{*\}), x(a) = n\}$. In other words, $X\Box\{*\}$ consists of pairs, an element from X together with its arity. This means that for each $X \in \mathbf{GrdSet}$ the X component of the right unitor $\rho_X^{\mathbf{GrdSet}} : X\Box\{*\} \to X$ is simply first projection with its inverse given by the graded set inclusion map $\overline{\rho}_X^{\mathbf{GrdSet}} : X \hookrightarrow X\Box\{*\}$ which couples each element in X with its arity. By swapping the two variables we get that $\{*\}\Box X = \{(*,\psi)|\psi \in T(X), T(!_X)(\psi) = *\}$. Hence $\{*\}\Box X$ consists of pairs, the singleton $\{*\}$ and a word of length one from $T(X)$. But words of length one in $T(X)$ are exactly the elements of X. This then implies that the X component of the left unitor $\lambda_X^{\mathbf{GrdSet}} : \{*\}\Box X \to X$ must be second projection with inverse given by the graded set inclusion map $\overline{\lambda}_X^{\mathbf{GrdSet}} : X \hookrightarrow \{*\}\Box X$ which couples each element in X with $*$.

It then remains only to show that the pentagon coherence condition follows. But this is clear from the fact that each component of the associator is given by a universal construction. The triangle identities follow immediately from the fact that each component of the left and right unitors is simply a projection map. ∎

2.10. Theorem. *A nonsymmetric operad is a monoid in* **GrdSet** *with respect to the monoidal product* $\Box$.

Proof. A monoid in **GrdSet** consists of an underlying graded set $x : X \to \mathbb{N}$, thought of as a set of 'operations', together with a composition function $m : X\Box X \to X$ and a unit function $e : \{*\} \to X$ from **GrdSet**, all of which must satisfy the usual associativity and unital conditions expressed by asserting that the diagrams

$$\begin{tikzcd}
(X\Box X)\Box X \arrow[r, "\alpha^{\mathbf{GrdSet}}_{X,X,X}"] \arrow[d, "m\Box\mathbb{1}_X"'] & X\Box(X\Box X) \arrow[r, "\mathbb{1}_X\Box m"] & X\Box X \arrow[d, "m"] \\
X\Box X \arrow[rr, "m"'] & & X
\end{tikzcd}$$

$$\begin{tikzcd}
\{*\}\Box X \arrow[r, "e\Box\mathbb{1}_X"] \arrow[dr, "\lambda^{\mathbf{GrdSet}}_X"'] & X\Box X \arrow[d, "m"] & X\Box\{*\} \arrow[l, "\mathbb{1}_X\Box e"'] \arrow[dl, "\rho^{\mathbf{GrdSet}}_X"] \\
& X &
\end{tikzcd}$$

commute. We denote n-ary operations of the underlying set X by X_n, which is simply the fiber over n along the arity map x. These fibers then form the sequence of sets $\{X_n\}_{n\in\mathbb{N}}$ for a nonsymmetric operad. The function m keeps track of how to compose strings of elements in X with an element $a \in X$ of the appropriate arity. The function e distinguishes an element of X which will behave like an identity operation on the elements of X. The associativity and unit commutative diagrams for a monoid internal to a category then ensure that these graded set maps endow X with the needed associative and unital operadic composition with respect to the $\square$ product. Conversely, given a nonsymmetric operad O, its underlying graded set can be obtained by constructing a map whose fiber over n is exactly the nth term in the sequence $\{O(n)\}_{n\in\mathbb{N}}$. By construction, each well-defined composition in O can be identified with an element of $O\square O$. Hence the composition map $m : O\square O \to O$ is defined by sending each composable pair to their composite in O. As O has a distinguished identity element, the map $e : \{*\} \to O$ sends * to this distinguished element. The associativity and unital conditions required of O then guarantee that the needed commutative diagram conditions are satisfied. ∎

3. The Internal Hom in **GrdSet**

In this section we will recall several topos theoretic constructions in order to better understand the category **GrdSet**. Further details can be found in [13], [6], and [7]. Let $f : A \to B$ be a set map. It is classically known that there is an induced functor $f^* : \mathbf{Set}/B \to \mathbf{Set}/A$ between slice categories called a *change of base* functor. It takes a set map $\chi : X \to B$ and returns the pullback map $f^*(\chi) : X{}_\chi{\times}_f A \to A$ of χ along f. It is furthermore known that for each f the functor f^* has both a left and right adjoint. On objects its left adjoint $\Sigma_f : \mathbf{Set}/A \to \mathbf{Set}/B$ is simply composing an object of $\mathbf{Set}/A$ with f resulting in an object in $\mathbf{Set}/B$. The right adjoint $\Pi_f : \mathbf{Set}/A \to \mathbf{Set}/B$ is however a bit more complicated. Nonetheless, when our base category is **Set**, it has a fairly straight forward description as follows. Let $\psi : Y \to A$ be any morphism in $\mathbf{Set}/A$. The map $\Pi_f(\psi) : \Gamma \to B$ is constructed by specifying the fiber over each point as follows. Take an element $b \in B$ and consider its fiber A_b along the map f. Each element $c \in A_b$ then has a fiber Y_c sitting above it along the map ψ. We can then define the fiber Γ_b along the map $\Pi_f(\psi)$ to be the product $\displaystyle\prod_{c\in A_b} Y_c$. Following this construction for each $b \in B$ gives the complete map from $\Gamma := \displaystyle\coprod_{b\in B}\prod_{c\in A_b} Y_c$ to B.

Now consider the functor $-\square B : \mathbf{GrdSet} \to \mathbf{GrdSet}$ for the graded set $b : B \to \mathbb{N}$. We can write $-\square B$ as the following composition:

$$-\square B = \Sigma_{\mu_{\{*\}}}\Sigma_{T(b)}T(!_B)^*$$

Notice that this functor takes the graded set $a : A \to \mathbb{N}$ to the graded set $a\square b : A\square B \to \mathbb{N}$, where the arity map $a\square b$ is exactly the image of $\Sigma_{\mu_{\{*\}}}\Sigma_{T(b)}T(!_B)^*(a)$. We see in the diagram below that this is exactly the topmost horizontal composition in the diagram used earlier to define the composition tensor product $\square$ in **GrdSet**.

$$\begin{tikzcd} A\square B \arrow[r, "T(!_B)^*"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & T(B) \arrow[r, "T(b)"] \arrow[d, "T(!_B)"] & T(\mathbb{N}) \arrow[r, "\mu_{\{*\}}"] & \mathbb{N} \\ A \arrow[r, "a"] & \mathbb{N} & & \end{tikzcd}$$

Writing the functor $-\square B$ in this way, just as with $-\times B$ above, allows us to immediately compute its right adjoint $[B,-]$: **GrdSet** $\to$ **GrdSet** by taking the right adjoint of each factor in the composition and reversing the order in which they are composed. This then leads to the following formula.

$$[B,-] = \Pi_{T(!_B)}T(b)^*\mu^*_{\{*\}}$$

We shall first consider how the composite $T(b)^*\mu^*_{\{*\}}$ acts on a graded set $a : A \to \mathbb{N}$. Recall that the map $T(b)^*\mu^*_{\{*\}}(a)$ is given as the topmost edge in the following double pullback diagram.

$$\begin{tikzcd} (A_a\times_{\mu^*_{\{*\}}}T(\mathbb{N}))_{\mu^*_{\{*\}}(a)}\times_{T(b)}T(B) \arrow[r, "T(b)^*\mu^*_{\{*\}}(a)"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & T(B) \arrow[d, "T(b)"] \\ A_a\times_{\mu^*_{\{*\}}}T(\mathbb{N}) \arrow[r, "\mu^*_{\{*\}}(a)"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & T(\mathbb{N}) \arrow[d, "\mu_{\{*\}}"] \\ A \arrow[r, "a"] & \mathbb{N} \end{tikzcd}$$

More concretely, for every $\beta \in T(B)$ there is a fiber over it along $T(b)^*\mu^*_{\{*\}}(a)$ living in the set $A_a\times_{\mu^*_{\{*\}}}T(\mathbb{N})$ consisting of pairs (α, t) with $\alpha \in A$ and t a word of natural numbers such that the arity of α is the sum of the arities in each slot of the 'operation' t. Moreover, the letters of t give the arities of the letters in β respectively. We now apply $\Pi_{T(!_B)}$ to $T(b)^*\mu^*_{\{*\}}(a)$ to get the internal hom in **GrdSet**. Recall from above that the map $\Pi_{T(!_B)}T(b)^*\mu^*_{\{*\}}(a) : \Gamma \to \mathbb{N}$ is constructed by specifying the fiber over each point. So take any $n \in \mathbb{N}$ and consider its fiber $T(B)_n$ along the map $T(!_B) : T(B) \to \mathbb{N}$. Each element $\beta \in T(B)_n$ then has a fiber $((A_a\times_{\mu^*_{\{*\}}}T(\mathbb{N}))_{\mu^*_{\{*\}}(a)}\times_{T(b)}T(B))_\beta$ sitting above it along the map $T(b)^*\mu^*_{\{*\}}(a)$. We can then define the fiber Γ_n along the map $\Pi_T(!_B)T(b)^*\mu^*_{\{*\}}(a)$ to be the following product:

$$\prod_{\beta\in T(B)_n}((A_a\times_{\mu^*_{\{*\}}}T(\mathbb{N}))_{\mu^*_{\{*\}}(a)}\times_{T(b)}T(B))_\beta$$

Following this construction for each $n \in \mathbb{N}$ gives the complete map. Thus **GrdSet** is right closed with respect to the composition tensor product.

Via this construction, we can now compute the internal hom $H_{B,A} : [B, A] \to \mathbb{N}$ in **GrdSet** between any two graded sets $a : A \to \mathbb{N}$ and $b : B \to \mathbb{N}$. We can think of the n-ary elements of the underlying set $[B, A]$ as follows. An element $\beta \in T(B)_n$ can be thought of as a choice of n elements $\{\beta_1, \beta_2, ..., \beta_n\}$ from B juxtaposed together such that $\beta = \beta_1\beta_2...\beta_n$. This allows us to think of the fiber

$$[B, A]_n = \prod_{\beta \in T(B)_n} \{((p, w), \beta) | p \in A, w \in T(\mathbb{N})_n, a(p) = \sum_i w_i, T(b)(\beta) = w\}$$

instead as the set

$$[B, A]_n \cong \prod_{\beta \in T(B)_n} \{(p, \beta) | p \in A, a(p) = \sum_i b(\beta_i)\}$$

up to isomorphism. Hence, an element $\gamma \in [B, A]_n$ may be thought of as a choice of elements from A to correspond to each string of n elements from B in such a way as to preserve arities, making γ a map of graded sets. In other words, a 'map' in the internal hom is essentially a thing that takes n elements from the source and picks an element of the target that has arity equal to the sum of the arities of the n elements from the source.

We now have the following theorem.

3.1. Theorem. *The category* **GrdSet** *has a closed monoidal structure with respect to the monoidal product* $\square$*.*

4. The Tautological Classical Operad

Consider a graded set $x : X \to \mathbb{N}$. We shall now construct the tautological operad on X, denoted again by $taut(X)$. First consider the graded set $taut(X) := [X, X]$ obtained using the internal hom in **GrdSet**. The underlying graded set for the tautological operad on X can be thought of as abstractly encoding all the possible operations that take some number of elements from X to a single output from X.

4.1. Theorem. *The graded set* $taut(X)$ *admits the structure of a nonsymmetric operad.*

Proof. We must show that $taut(X)$ is a monoid in **GrdSet** with respect to the tensor product $\square$. The operad identity is given by the map $e : \{*\} \to [X, X]$ which is constructed as the currying of the left unitor $\lambda_X : \{*\}\square X \to X$ for the monoidal structure in **GrdSet**. The composition map $m : [X, X]\square[X, X] \to [X, X]$ is similarly constructed in the following way. Consider the unit $\eta_A : [A, -]\square A \Rightarrow \mathbb{1}_{\mathbf{GrdSet}}$ of the hom-tensor adjunction in **GrdSet** between $-\square A$ and $[A, -]$, which we can think of as *evaluation at* A. It has components $\eta_A^B : [A, B]\square A \to B$ for each graded set A. We then get a map

$$\kappa : [X, X]\square([X, X]\square X) \to [X, X]\square X \to X$$

which is the composite $\kappa := \eta^X_X(\mathbb{1}_X \Box \eta^X_X)$. The operad multiplication for $[X,X]$ is then the currying of the map κ. It then remains only to show that $taut(X) : [X,X] \to \mathbb{N}$ together with $e : \{*\} \to [X,X]$ and $m : [X,X]\Box[X,X] \to [X,X]$ satisfy the commutative diagrams required of a monoid object in **GrdSet**. This can be seen by first currying the maps in the relevant diagrams and checking to see that these new curried diagrams, whose commutativity is equivalent with that of the originals, commute. Here we have dropped all parentheses by MacLane's coherence theorem applied to **GrdSet**. We first consider the diagram

$$\begin{tikzcd}
{[X,X]\Box[X,X]\Box[X,X]\Box X} \arrow[r, "m\Box\mathbb{1}_{[X,X]}\Box\mathbb{1}_X"] \arrow[d, "\mathbb{1}_{[X,X]}\Box m\Box\mathbb{1}_X"'] \arrow[dr, "\mathbb{1}_{[X,X]}\Box\mathbb{1}_{[X,X]}\Box\eta^X_X"] & {[X,X]\Box[X,X]\Box X} \arrow[r, "\mathbb{1}_{[X,X]}\Box\eta^X_X"] & {[X,X]\Box X} \arrow[dd, "\eta^X_X"] \\
 & {[X,X]\Box[X,X]\Box X} \arrow[ur, "m\Box\mathbb{1}_X"] \arrow[d, "\mathbb{1}_{[X,X]}\Box\eta^X_X"] & \\
{[X,X]\Box[X,X]\Box X} \arrow[r, "\mathbb{1}_{[X,X]}\Box\eta^X_X"] & {[X,X]\Box X} \arrow[r, "\eta^X_X"] & X
\end{tikzcd}$$

whose commutativity is equivalent with that of the diagram asserting associativity of our multiplication m. Note that the commutativity of the bottom left and right squares follows from the way we defined m as the currying of two sequential multiplication operations. The remaining top square then commutes by the functoriality of the composition tensor product. We next consider the diagram

$$\begin{tikzcd}
{\{*\}\Box[X,X]\Box X} \arrow[r, "e\Box\mathbb{1}_{[X,X]}\Box\mathbb{1}_X"] \arrow[d, "\mathbb{1}_{\{*\}}\Box\eta^X_X"'] & {[X,X]\Box[X,X]\Box X} \arrow[r, "\mathbb{1}_{[X,X]}\Box\eta^X_X"] & {[X,X]\Box X} \arrow[d, "\eta^X_X"] \\
{\{*\}\Box X} \arrow[urr, "e\Box\mathbb{1}_X"] \arrow[rr, "\lambda_X"'] & & X
\end{tikzcd}$$

which comes from currying the maps from the needed left-sided unit diagram. Here the top left square commutes by the functoriality of the composition tensor product. The bottom right triangle commutes by the definition of e. We finally consider the diagram

$$\begin{tikzcd}
{[X,X]\square\{*\}\square X} \arrow[r, "\mathbb{1}_{[X,X]}\square e\square \mathbb{1}_X"] \arrow[d, "\mathbb{1}_{[X,X]}\square\lambda_X"'] & {[X,X]\square[X,X]\square X} \arrow[r, "\mathbb{1}_{[X,X]}\square\eta^X_X"] \arrow[dl, "\mathbb{1}_{[X,X]}\square\eta^X_X"] & {[X,X]\square X} \arrow[d, "\eta^X_X"] \\
{[X,X]\square X} \arrow[rr, "\eta^X_X"'] & & X
\end{tikzcd}$$

which comes from currying the maps from the needed right-sided unit diagram. The top left triangle of this diagram commutes by the definition of e. The bottom right square commutes trivially. It therefore follows that $(taut(X), m, e)$ is a monoid in **GrdSet**. ■

Note that the operad identity map $e : \{*\} \to [X,X]$ is the canonical morphism that maps the singleton $*$ to the element of [X,X] corresponding to the identity set map on X, while the composition map $m : [X,X]\square[X,X] \to [X,X]$ is the canonical map which takes a pair $(a,\omega) \in [X,X]\square[X,X]$ and composes each of the letters from the word $\omega \in T([X,X])$ with each of the respective inputs for the 'operation' $a \in [X,X]$. Hence we always have a natural way to equip $taut(X)$ with the structure of a nonsymmetric operad in **GrdSet**. This construction allows us to make the following definition.

4.2. Definition. *Let $o : O \to \mathbb{N}$ be an operad. A* graded algebra for O *is a morphism of operads $f : O \to taut(X)$ for some graded set $x : X \to \mathbb{N}$. An* algebra in **Set** for O *is a graded algebra for O such that the graded set X is concentrated in degree 0 (i.e. the arity map factors as $x = [0]\circ !_X$, where $[0] : \{*\} \to T(\{*\})$ is the 'name of zero' map which identifies the empty word in $T(\{*\})$.*

In the situation above, we say that the operad O acts on the graded set X. Note here that the algebras for an operad in this sense would be more general than that of the algebras for a classically defined operad if we did not require that X be concentrated over zero. In order for the operad to act on the set X as an ordinary set, as opposed to a graded set, we must think of ordinary sets as being graded sets concentrated over zero. In this way the n-th component of the tautological operad defined on the graded set $x : X \to \mathbb{N}$ concentrated over $0 \in \mathbb{N}$ corresponds exactly to the classic definition of the tautological operad on the set X in the sense that $[X,X]_n$ would consist of pairings of a length n word from X, whose letters each have arity zero (i.e. n elements from the set X), to an element of X whose arity is the sum of the arities of the n letters which comprised the source word, which is also zero. Hence $[X,X]_n$ can be thought of as associating to each string of n elements from X a single element of X. Thus $[X,X]_n \cong \mathbf{Set}(X^n, X)$ in **Set**.

Before moving on, we immediately get the following theorem by defining algebras this way (via precomposition on the defining operad homomorphism).

4.3. Theorem. *An algebra for an operad O is an algebra for every operad P which maps to O. In particular, an algebra for O is an algebra for every sub-operad of O.*

5. PROs

Having seen that classical operads are simply monoids with respect to a particular monoidal product, it is natural to ask if such a structure exists which also admits coarities for each of our abstract operations. Note that any algebraic structure whose axioms require two instances of the same variable on the same side of an equation cannot be represented by an operad. For example, a group G requires that for all $g \in G$

$$g \cdot g^{-1} = e$$

with e being the identity element. But capturing such a relation using abstract operations requires the use of a diagonal map $\Delta : G \times G \to G$, inversion map $(-)^{-1} : G \to G$, multiplication map $m : G \times G \to G$, identity identification map $e : \{*\} \to G$, and the unique set map $!_G : G \to \{*\}$ to the terminal one point set such that the diagram

$$\begin{tikzcd}
 & G \times G \arrow[rr, "\mathbb{1}_G \times (-)^{-1}"] & & G \times G \arrow[dr, "m"] & \\
G \arrow[ur, "\Delta"] \arrow[rr, "!_G"'] & & \{*\} \arrow[rr, "e"'] & & G
\end{tikzcd}$$

commutes. But the diagonal map is not something which can exists among the operations of an operad. In what follows we will be working primarily with PROs (whose name is short for product category), a generalization of nonsymetric operads which allows for operations like the diagonal. As with operads, we avoid axiomatizing a symmetric group action, preferring PROs to MacLane's notion of PROPs [12] (whose name is short for product and permutation category).

5.1. Definition. *A PRO P is a strict monoidal category whose object set is isomorphic to $\mathbb{N}$ such that the monoidal product $+ : P \times P \to P$ is identified with addition of natural numbers at the level of objects.*

Given a PRO P, we may think of the morphisms in $P(n, m)$ as operations of arity n and coarity m. Until an algebra for a PRO is specified, the objects $\mathbb{N}$ behave as placeholders for the arities and coarities of these operations, so that they may be composed with one another. Once an algebra is specified for P, these 'slots' will be filled with elements from the underlying set of the algebra, justifying the use of the name operations for the morphisms of our PRO.

Before moving on, it should be noted that PROs are also a special type of monoid object. As monoidal categories they are precisely the strict monoids in **Cat**, with respect to the cartesian product in **Cat**, which have the extra property that their monoid of objects is isomorphic to $(\mathbb{N}, +, 0)$. Moreover, the special case of a PRO whose monoidal product $+$ is precisely the cartesian product on morphisms is called a *cartesian PRO*. As we will later see, this is a special case of a more general construction which we will here call an *enriched cartesian PRO*.

6. Algebras

The closed structure on **GrdSet** with respect to $\square$ allows us to use the following alternative definition for algebras.

6.1. Definition. *An algebra A for an operad O is a set A, thought of as a degenerate graded set concentrated in degree 0, together with a graded set homomorphism $\omega : O\square A \to A$ which makes the diagrams*

$$\begin{tikzcd}
(O\square O)\square A \arrow[r, "\alpha^{\mathbf{GrdSet}}_{O,O,A}"] \arrow[d, "\circ^{O}\square \mathbb{1}_A"'] & O\square(O\square A) \arrow[r, "\mathbb{1}_O\square\omega"] & O\square A \arrow[d, "\omega"] \\
O\square A \arrow[rr, "\omega"'] & & A
\end{tikzcd}$$

$$\begin{tikzcd}
& O\square A \arrow[dr, "\omega"] & \\
\{*\}\square A \arrow[ur, "e\square \mathbb{1}_A"] \arrow[rr, "\lambda^{\mathbf{GrdSet}}_A"'] & & A
\end{tikzcd}$$

commute.

The previous two definitions of an algebra for an operad are equivalent. As seen in the explicit description of the internal hom in **GrdSet**, when the set A is perceived as a graded set concentrated in degree 0, we get that $[A, A]_n = \mathbf{Set}(A^n, A)$, where $[A, A]_n$ is the fiber over n of the graded set produced by the internal hom. Moreover, the tautological operad on A is precisely the internal hom $[A, A]$, which consists of the union of all such fibers. The action map $\omega : O\square A \to A$ from the first definition may then be curried, via the adjunction between $-\square A$ and $[A, -]$, to get a map $[\omega] : O \to [A, A]$. The curried versions of the two diagrams from the first definition then impose upon $[\omega]$ the structure of an operad homomorphism.

We now define the analogous constructions for PROs. Note that although this structure can be phrased in terms of $\square$, doing so requires the language of duoidal enriched categories. This approach will be taken later when working in the globular setting. However, for the sake of clarity, we here give the more intuitive notion in terms of the cartesian product of sets.

6.2. Definition. *An algebra for a PRO P in* **Set** *is a set A , together with, for all $n, m \in \mathbb{N}$, a family of functions*

$$\omega_{n,m} : P(n, m) \times A^n \to A^m$$

which make the following diagrams commute

$$\begin{tikzcd}
{[P(m,r)\times P(n,m)]\times A^n} \arrow[r, "\alpha^{\mathbf{Set}}_{P(m,r),P(n,m),A^n}"] \arrow[d, "\circ_{n,m,r}"'] & {P(m,r)\times[P(n,m)\times A^n]} \arrow[r, "\mathbb{1}_{P(m,r)}\times\omega_{n,m}"] & {P(m,r)\times A^m} \arrow[d, "\omega_{m,r}"] \\
{P(n,r)\times A^n} \arrow[rr, "\omega_{n,r}"'] & & A^r
\end{tikzcd}$$

$$\begin{tikzcd}
{[P(n,m)\times P(r,s)]\times[A^n\times A^r]} \arrow[r, "+_{n,m,r,s}\times\mathbb{1}_{n+r}"] \arrow[d, "\boxtimes_{P(n,m),P(r,s),A^n,A^r}"'] & {P(n+r,m+s)\times A^{n+r}} \arrow[d, "\omega_{n+r}"] \\
{[P(n,m)\times A^n]\times[P(r,s)\times A^r]} \arrow[r, "\omega_{n,m}\times\omega_{r,s}"'] & A^m\times A^s\cong A^{m+s}
\end{tikzcd}$$

$$\begin{tikzcd}
& {P(n,n)\times A^n} \arrow[dr, "\omega_{n,n}"] & \\
\{*\}\times A^n \arrow[ur, "j_n\times\mathbb{1}_{A^n}"] \arrow[rr, "\lambda^{\mathbf{Set}}_{A^n}"'] & & A^n
\end{tikzcd}$$

for all $n, m, r, s \in \mathbb{N}$, where the set map $\boxtimes_{X,Y,Z,W} : [X\times Y]\times[Z\times W] \to [X\times Z]\times[Y\times W]$ is the interchange morphism in **Set** *which swaps the second factor with the third and reassociates accordingly, while $j_n : \{*\} \to P(n,n)$ identifies which element of $P(n,n)$ is the identity operation.*

Just as with operads, there is a tautological PRO on an object $X \in \mathcal{C}$. This will allow us to give X the structure of an algebra via a representation homomorphism.

6.3. Definition. *Given a set A, the tautological PRO on A, denoted by $Taut(A)$, is the PRO which has as its set of objects all successive cartesian powers $A^n = \prod_{i=1}^{n} A$ of the underlying set A for all $n \in \mathbb{N}$, which can be naturally identified with $\mathbb{N}$. Under this identification, the hom-sets $Taut(A)(m,n)$ in $Taut(A)$ are the hom-sets* $\mathbf{Set}(A^n, A^m)$*. Composition in the PRO is simply the composition induced from* **Set**. *The monoidal product is induced by the product structure on* **Set**, *which may be identified with addition of natural numbers since $A^n \times A^m \cong A^{n+m}$.*

6.4. Definition. *An algebra for a PRO P is a strict monoidal functor $F : P \to Taut(A)$ for some set A.*

We once again have two equivalent notions of an algebra whose equivalence can be seen via currying the action maps $\omega_{n,m}$ to obtain components of the desired monoidal functor. In either description, we say that the PRO P *acts on the object A*. We once again get another immediate theorem regarding induced algebras for a PRO.

6.5. Theorem. *An algebra for a PRO P is an algebra for every other PRO Q which maps to P. In particular, an algebra for P is an algebra for every sub-PRO of P.*

7. Collections

We begin this section by recalling the notion of a globular set. To do so requires the following category $\mathbb{G}$, known as the *globe category*. The category $\mathbb{G}$ has $\mathbb{N}$ as its set of objects. Its morphisms are generated by $\sigma_n : n \to n+1$ and $\tau_n : n \to n+1$ for all $n \in \mathbb{N}$ subject to the relations $\sigma_{n+1} \circ \sigma_n = \tau_{n+1} \circ \sigma_n$ and $\sigma_{n+1} \circ \tau_n = \tau_{n+1} \circ \tau_n$.

7.1. Definition. *A* globular set *is a contravariant functor $G : \mathbb{G} \to$ **Set**. The category* **Glob** *of globular sets is the category of presheaves on $\mathbb{G}$.*

Another integral piece of structure needed to define globular operads is the free strict ω-category monad $\mathcal{T} : \mathbf{Glob} \to \mathbf{Glob}$. Just like the monad T above, $\mathcal{T}$ is cartesian. This fact, as well as a more detailed explanation of its construction and use, can be found in [11]. Briefly, this monad takes a globular set $\mathcal{X}$ and returns the underlying globular set of the free strict ω-category generated by $\mathcal{X}$. In other words, it takes a globular set $\mathcal{X}$ and constructs the globular set $\mathcal{T}(\mathcal{X})$ consisting of all possible pasting diagrams, or as we will often describe them, 'globular words' built out of the cells of $\mathcal{X}$. The motivation for calling such a pasting diagram a word is that a pasting diagram, all of whose cells are cells in $\mathcal{X}$, can be thought of as a generalization of the notion of a word in some set Y (i.e. a string of concatenated elements from Y). The main difference between the two notions is that a globular word can be built out of concatenation of cells along any of their boundary cells, as opposed to the classical setting in which elements, or letters, can only be composed as horizontal strings. So in this way we can think of words on a set (in either setting) as simply an element, or cell, in the underlying object of the free monoid, or ω-category, on the respective notion of set.

Throughout this work we use the terms 'pasting diagram' and 'pasting scheme' in the sense of Power[15]. However, all cells of the pasting schemes and diagrams are globes. For our purposes we will be specifically interested in the globular set $\mathcal{T}(\mathbf{1})$ generated by the terminal globular set $\mathbf{1}$ which has exactly one cell in each dimension. It is precisely $\mathcal{T}(\mathbf{1})$ which allows us to generalize our notion of the arity of an operation. In the classical case, the arity of an element is the length of the word in $T(\{*\})$ over which the operation sits with respect to the operad's graded set structure. In this more general context, the arity of a cell in a globular set $\mathcal{X}$ is a pasting scheme specified by a globular cell in $\mathcal{T}(\mathbf{1})$.

More precisely, we can equip a globular set $\mathcal{X}$ with a morphism $x : \mathcal{X} \to \mathcal{T}(\mathbf{1})$ which specifies globular arities via cells in $\mathcal{T}(\mathbf{1})$ that are named by pasting schemes. We may in turn think of the pasting schemes which name the cell in $\mathcal{T}(\mathbf{1})$ as the possible arity shapes with which the globular cells in $\mathcal{X}$ may be equipped.

7.2. Definition. *A* collection *is a globular set $\mathcal{X}$ equipped with a globular set homomorphism $x : \mathcal{X} \to \mathcal{T}(\mathbf{1})$ called the* arity map.

By abuse of notation, just as with graded sets, we will often represent a collection by simply writing its underlying globular set $\mathcal{X}$.

7.3. Definition. *Let $x : \mathcal{X} \to \mathcal{T}(\mathbf{1})$ and $y : \mathcal{Y} \to \mathcal{T}(\mathbf{1})$ be a pair of collections. A* collection homomorphism *between them is a globular set map $f : \mathcal{X} \to \mathcal{Y}$ which makes the triangle*

$$\begin{tikzcd}
\mathcal{X} \arrow[rr, "f"] \arrow[dr, "x"'] & & \mathcal{Y} \arrow[dl, "y"] \\
& \mathcal{T}(\mathbf{1}) &
\end{tikzcd}$$

commute.

We shall use **Col** to denote the category of collections. Note that **Col** is simply the slice category **Glob**$/\mathcal{T}(\mathbf{1})$. Furthermore, **Col**, as with **GrdSet**, has a monoidal structure with respect to a composition tensor product $\Box : \mathbf{Col} \times \mathbf{Col} \to \mathbf{Col}$ defined analogously as follows:

7.4. Definition. *Let $x : \mathcal{X} \to \mathcal{T}(\mathbf{1})$ and $y : \mathcal{Y} \to \mathcal{T}(\mathbf{1})$ be a pair of collections. Their composition tensor product $x \Box y : \mathcal{X} \Box \mathcal{Y} \to \mathcal{T}(\mathbf{1})$ is defined by the diagram:*

$$\begin{tikzcd}
\mathcal{X}\Box\mathcal{Y} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{T}(\mathcal{Y}) \arrow[r, "\mathcal{T}(y)"] \arrow[d, "\mathcal{T}(!_{\mathcal{Y}})"] & \mathcal{T}^2(\mathbf{1}) \arrow[r, "\mu_{\mathbf{1}}"] & \mathcal{T}(\mathbf{1}) \\
\mathcal{X} \arrow[r, "x"] & \mathcal{T}(\mathbf{1}) & &
\end{tikzcd}$$

where $!_{\mathcal{Y}} : \mathcal{Y} \to \mathbf{1}$ is the unique map from $\mathcal{Y}$ to the terminal globular set. The underlying globular set $\mathcal{X}\Box\mathcal{Y}$ is the pullback of x and $\mathcal{T}(!_{\mathcal{Y}})$ with the arity globular set map $x\Box y$ defined to be the composition along the top row.

This definition makes $X\Box Y$ the unique collection whose cells are pairs (a, ψ) consisting of a k-cell $a \in X$ and a 'globular word' ψ of k-cells from Y indexed by the arity of a. In $X\Box Y$, the 'globular letters' in the globular word $\psi \in \mathcal{T}(Y)$ may be compatibly 'glued together' via the shape of $x(a) \in \mathcal{T}(\mathbf{1})$ in the sense that each globular letter of ψ is a k-cell whose arity shape under y can replace a particular k-cell in the pasting scheme which names the cell $x(a) \in \mathcal{T}(\mathbf{1})$. We can thus think of the cells of $X\Box Y$ as composable pairs specified by a cell of X and a 'word' of cells from Y, each of whose 'letters' may be plugged into a sub k-cell of the k-dimensional pasting scheme which names the arity cell $x(a)$.

Furthermore, the arity for a composable pair in $X\Box Y$ may be thought of as the 'sum' of the arities of each letter in ψ 'glued together' in the shape of the pasting scheme which names $x(a)$. More precisely, note that the map $\mathcal{T}(y)$ takes a word of cells from Y and returns a word of arity cells (i.e. a cell in $\mathcal{T}^2(\mathbf{1})$ which is named by a pasting diagram of pasting schemes). The component at $\mathbf{1}$ of the unit transformation μ for $\mathcal{T}$ then takes this globular word of arities and returns the cell in $\mathcal{T}(\mathbf{1})$ which is named by the pasting scheme we would get if we strictly pasted together this diagram of schemes. We can think of the cells in $\mathcal{T}^2(\mathbf{1})$ as being named by factorizations of pasting schemes. From this perspective, $\mu_{\mathbf{1}}$ essentially reduces this factorization by specifying the cell in $\mathcal{T}(\mathbf{1})$ which is named by the strict pasting composition specified by the factorization.

The analogous arity construction for graded sets consisted of equipping a composable pair (a, ψ) in a graded set $X\Box Y$ with the arity obtained by sending the word ψ to the word in $T^2(\{*\})$ whose letters were the arities of the letters in ψ. As each of these arities was named by a natural number, the component at $\{*\}$ of the unit transformation μ for the monoid monad T simply concatenated this word of arities into a single arity which is named by the sum of the natural numbers which named each letter. The essential change in the globular setting is that the notion of 'sum' has become the more general notion of strictly pasting a globular pasting scheme. In fact, a graded set may be identified with a collection whose underlying globular set consists only of 1-cells. In which case the classical and globular construction of the $\Box$ product are the same.

8. Special Collections

There are four particular collections that are worth noting. The first is the terminal collection $\mathbb{1} : \mathcal{T}(\mathbf{1}) \to \mathcal{T}(\mathbf{1})$. This collection is the unit for the cartesian product in **Col**. To see why, recall that the cartesian product in a slice category is defined via the pullback of a sliced object along another. Hence the cartesian product of a collection $a : A \to \mathcal{T}(\mathbf{1})$ against $\mathbb{1}$ creates an isomorphic collection whose underlying globular set consists of the elements of A paired together with their arity shape specified by a.

There is also a collection $I : \mathbf{1} \hookrightarrow \mathcal{T}(\mathbf{1})$ whose arity map is simply the inclusion of generators. This collection is important because it is the unit for $\Box$ in **Col**. Note that applying $\Box$ to $a : A \to \mathcal{T}(\mathbf{1})$ with I on the right gives a collection whose underlying globular set consists of pairs whose entries are an element of A together with the globular pasting scheme from $\mathcal{T}(\mathbf{1})$ which represents its arity. Similarly, tensoring with I on the left gives a collection whose underlying globular set consists of pairs whose entries are a generic n-cell from $\mathbf{1}$ together with an n-cell from A. As each of these collections is isomorphic to $a : A \to \mathcal{T}(\mathbf{1})$, the collection $I : \mathbf{1} \hookrightarrow \mathcal{T}(\mathbf{1})$ must be the unit for $\Box$ in **Col**. Moreover, when enriching over **Col** with respect to $\Box$, this collection can be used to distinguish elements in a particular hom-object. For example, if $a : A \to \mathcal{T}(\mathbf{1})$ is a hom-object collection, then a cell x of A may be distinguished by a collection morphism $[x] : \mathbf{1} \to A$.

Note that the previous two collections are units for $\times$ and $\square$ respectively. As previously noted, these units give **Col** two different monoidal structures. Note that the $\square$ unit I is a sub-object of the cartesian unit $\mathbb{1}$, suggesting that these two monoidal structures should have some sort of nice interaction. We shall see shortly that the precise way in which these two monoidal structures interact will play a part in the construction of globular PROs.

The third collection worth noting is the initial collection $\{\} : \emptyset \to \mathcal{T}(\mathbf{1})$ whose arity map is the vacuous mapping from the empty globular set. We may think of this as the empty collection. For any collection $a : A \to \mathcal{T}(\mathbf{1})$, it follows that both the cartesian and $\square$ product (on either side) with $\{\}$ is simply $\{\}$. With respect to these two products, $\{\}$ essentially behaves like multiplication by 0.

One final special collection worth noting is given by the globular set map $[id] : \mathbf{1} \to \mathcal{T}(\mathbf{1})$. Note that among the many cells in $\mathcal{T}(\mathbf{1})$ there are the underlying globular cells of identity morphisms created when $\mathcal{T}$ produces the underlying globular set of the free strict ω-category on $\mathbf{1}$. Among these identities are the following special identities. There is the underlying 1-cell of the identity on the single vertex in $\mathbf{1}$. This identity map then has an identity 2-cell that sits over it. And over this identity 2-cell there is an identity 3-cell that sits above it. Continuing this process, we see that there is an inclusion of the terminal globular set $\mathbf{1}$ into $\mathcal{T}(\mathbf{1})$ whose cells are exactly the iterated identities on the single 0-cell. This sub-object can be thought of as the globular ω-analogue of the additive identity $0 \in \mathbb{N}$ from the graded set case. The map $[id]$ is then the globular set map which sends each of the single n-cells in $\mathbf{1}$ to the corresponding iterated identity cell of dimension n from the construction just described. The map $[id]$ is, in this way, an identification of this 'tower' of iterated identities as the particular identities for each n-dimensional pasting composition.

9. Globular Operads

We can now see that **Col** has the following monoidal structure.

9.1. THEOREM. *The product $\square$ together with the collection $I : \mathbf{1} \hookrightarrow \mathcal{T}(\mathbf{1})$ gives* **GrdSet** *the structure of a monoidal category.*

PROOF. The proof is exactly analogous to that of the graded set case. ■

9.2. DEFINITION. *A* globular operad *is a monoid in* **Col** *with respect to the monoidal product $\square$.*

Seeing why this data should intuitively make sense as an operad is analogous to seeing how monoids in **GrdSet** with respect to the $\square$ tensor product are classical nonsymmetric operads. We may even recover the classical notion of a nonsymmetric operad from this more general construction by realizing that a graded set is essentially a degenerate collection in which the underlying globular set consists only of 1-cells, each of which has as its boundary the same single 0-cell.

10. The Internal Hom in **Col**

Let $\phi : \mathcal{A} \to \mathcal{B}$ be a globular set map. Just as with the construction of classical operads above, we again get an induced change of base functor $\phi^* : \mathbf{Glob}/\mathcal{B} \to \mathbf{Glob}/\mathcal{A}$ between slice categories. It takes a globular set map and returns its pullback along ϕ. Once again the functor ϕ^* has both a left and right adjoint. Its left adjoint $\Sigma_\phi : \mathbf{Glob}/\mathcal{A} \to \mathbf{Glob}/\mathcal{B}$ is also composition with ϕ. Its right adjoint $\Pi_\phi : \mathbf{Glob}/\mathcal{A} \to \mathbf{Glob}/\mathcal{B}$ is again a bit complicated to describe in general. More detail on the general construction of Π_ϕ can again be found in [13]. We nonetheless know that such a functor must exist and can intuitively think of the fibered globular sets in the image of $\Pi_\phi(\xi : \mathcal{X} \to \mathcal{A})$ as the globular set fibered over $\mathcal{B}$ of generalized sections of the globular set map ξ, by analogy to the construction of defining the right adjoint in the graded set case. More precisely, in a category whose objects have elements, such as **Glob**, this right adjoint fortunately has a relatively nice description. Let $\psi : \mathcal{Y} \to \mathcal{A}$ be any morphism in $\mathbf{Glob}/\mathcal{A}$. The map $\Pi_\phi(\psi) : \Gamma \to \mathcal{B}$ is constructed by specifying the fiber over each point. Take an element $b \in \mathcal{B}$ and consider its fiber A_b along the map ϕ. Each element $c \in \mathcal{A}_b$ then has a fiber $\mathcal{Y}_c$ sitting above it along the map ψ. We can then define the fiber Γ_b along the map $\Pi_\phi(\psi)$ to be the product $\prod_{c \in \mathcal{A}_b} Y_c$. Following this construction for each $b \in \mathcal{B}$ gives the complete map from $\Gamma := \coprod_{b \in \mathcal{B}} \prod_{c \in \mathcal{A}_b} \mathcal{Y}_c$ to $\mathcal{B}$.

Now consider the functor $-\Box\mathcal{B} : \mathbf{Col} \to \mathbf{Col}$ for the collection $b : \mathcal{B} \to \mathcal{T}(\mathbf{1})$. We can write $-\Box\mathcal{B}$ as the following composition:

$$-\Box\mathcal{B} = \Sigma_{\mu_\mathbf{1}} \Sigma_{\mathcal{T}(b)} \mathcal{T}(!_\mathcal{B})^*$$

Note that this functor takes the collection $a : \mathcal{A} \to \mathcal{T}(\mathbf{1})$ to the collection $a\Box b : \mathcal{A}\Box\mathcal{B} \to \mathcal{T}(\mathbf{1})$, where the arity map $a\Box b$ is exactly the image of $\Sigma_{\mu_\mathbf{1}} \Sigma_{\mathcal{T}(b)} \mathcal{T}(!_\mathcal{B})^*(a)$. Again as before, this is exactly composition in the augmented pullback diagram used to define the composition tensor product $\Box$ in **Col**. Writing the functor $-\Box\mathcal{B}$ in this way allows us to compute the appropriate right adjoint $[\mathcal{B}, -] : \mathbf{Col} \to \mathbf{Col}$ by taking the right adjoint of each factor in the composition and reversing the order in which they are composed. This then leads to the following formula:

$$[\mathcal{B}, -] = \Pi_{\mathcal{T}(!_\mathcal{B})} \mathcal{T}(b)^* \mu_\mathbf{1}^*$$

We shall again consider first how the composite $\mathcal{T}(b)^* \mu_\mathbf{1}^*$ acts on a collection $a : \mathcal{A} \to \mathcal{T}(\mathbf{1})$. Recall that the map $\mathcal{T}(b)^* \mu_\mathbf{1}^*(a)$ is given as the topmost edge in the appropriate double pullback diagram to get the globular set map

$$\mathcal{T}(b)^* \mu_\mathbf{1}^*(a) : (\mathcal{A}_a \times_{\mu_\mathbf{1}^*} \mathcal{T}^2(\mathbf{1}))_{\mu_\mathbf{1}^*(a)} \times_{\mathcal{T}(b)} \mathcal{T}(\mathcal{B}) \to \mathcal{T}(\mathcal{B})$$

which is simply second projection. We can intuitively think of this map as associating to each cell $\beta \in \mathcal{T}(\mathcal{B})$, which is named by a pasting diagram labeled by cells in $\mathcal{B}$, a pair $(\alpha, t) \in \mathcal{A} \times \mathcal{T}(\mathcal{T}(\mathbf{1}))$ consisting of cell $\alpha \in \mathcal{A}$ and cell of cells t of shape $\sigma \in \mathcal{T}(\mathbf{1})$ (i.e. a globular word of cells in $\mathcal{T}(\mathbf{1})$ indexed by the diagram of shape σ) such that the shape of α is the same as the shape of the cell obtained by gluing together the cells in t per the pasting formula given by σ. Moreover, the unlabeled cells of t each have the same shape as the corresponding cells which make up the labeled diagram β. We then apply $\Pi_{\mathcal{T}(!_{\mathcal{B}})}$ to $\mathcal{T}(b)^*\mu_{\mathbf{1}}^*(a)$ to get the desired internal hom.

Via this construction, we can now compute the internal hom $\mathcal{H}_{\mathcal{B},\mathcal{A}} : [\mathcal{B}, \mathcal{A}] \to \mathcal{T}(\mathbf{1})$ in **Col** between any collections $b : \mathcal{B} \to \mathcal{T}(\mathbf{1})$ and $a : \mathcal{A} \to \mathcal{T}(\mathbf{1})$. Although there are some subtle technical differences from the graded set case which are lurking in the construction encoded by the functor $\Pi_{\mathcal{T}(!_{\mathcal{B}})}$, we can intuitively think of cells in each fiber $[A, B]_\sigma$ of our internal hom in the following way. Recall that the internal hom is constructed as the object of general sections of the globular set map $\mathcal{T}(b)^*\mu_{\mathbf{1}}^*(a)$ defined above. Moreover, a cell $\beta \in \mathcal{T}(\mathcal{B})_\sigma$ can be thought of as a choice of cells $\{\beta_\tau\}_{\tau\in\sigma}$ from $\mathcal{B}$ glued together along halves of their boundaries as prescribed by the pasting formula given by the pasting diagram σ. Or rather, we can think of them as a coloring of the diagram σ by cells in $\mathcal{B}$. This allows us to think of a cell $\gamma \in [\mathcal{B}, \mathcal{A}]_\sigma$ as a choice of a cell $\alpha \in \mathcal{A}$ to correspond to each coloring of the diagram σ by cells of $\mathcal{B}$ so that the shape of α is the same as the shape of the diagram obtained by gluing the cells $\{\beta_\tau\}_{\tau\in\sigma}$ together via the pasting formula given by σ. In other words, a 'map' in the internal hom is roughly a thing that takes a coloring of the diagram $\sigma \in \mathcal{T}(\mathbf{1})$ by cells from the source and picks a cell of the target that has the same arity shape as the cells from the source after all the pasting compositions prescribed by the diagram σ have been performed.

We can thus conclude this section with the following theorem.

10.1. THEOREM. *The category* **Col** *has a closed monoidal structure with respect to the monoidal product* $\Box$.

11. The Tautological Globular Operad

Consider the collection $x : \mathcal{X} \to \mathcal{T}(\mathbf{1})$. We shall now construct the tautological globular operad on $\mathcal{X}$, denoted $Gtaut(\mathcal{X})$, analogously to the construction in the case of graded sets. We again define $Gtaut(\mathcal{X}) := [\mathcal{X}, \mathcal{X}]$ via the internal hom construction in **Col**. The underlying collection for the tautological globular operad on $\mathcal{X}$ can be thought of as abstractly encoding all the possible operations that take a coloring of a pasting diagram of shape $\sigma \in \mathcal{T}(\mathbf{1})$ by globular cells from $\mathcal{X}$ to a single globular cell from $\mathcal{X}$ whose shape is the same as the 'word of cells' after each of the pasting compositions prescribed by σ are evaluated to give a composed cell in $\mathcal{X}$. But since they are constructed using the internal hom, rather than the set valued hom, these 'maps' from $\mathcal{X}$ to $\mathcal{X}$ naturally fiber over $\mathcal{T}(\mathbf{1})$ so that we can again place a canonical operad structure on $Gtaut(\mathcal{X})$. The operad identity is given by the map $\iota : \mathbf{1} \to [\mathcal{X}, \mathcal{X}]$ which maps each single k-cell of $\mathbf{1}$ to the the respective k-cell of $[\mathcal{X}, \mathcal{X}]$ which corresponds to the identity operation on k-cells of $\mathcal{X}$. This map ι can be constructed canonically as the currying of the left unitor $\lambda_{\mathcal{X}} : \mathbf{1}\Box\mathcal{X} \to \mathcal{X}$ for the monoidal structure in **Col**. The composition map $\nu : [\mathcal{X},\mathcal{X}]\Box[\mathcal{X},\mathcal{X}] \to [\mathcal{X},\mathcal{X}]$ is the canonical map which takes a pair $(a, w) \in [\mathcal{X},\mathcal{X}]\Box[\mathcal{X},\mathcal{X}]$ and composes each of the letters from the word $w \in \mathcal{T}([\mathcal{X},\mathcal{X}])$ with each of the respective inputs for the operation $a \in [\mathcal{X},\mathcal{X}]$. It can be canonically constructed as follows. Consider the counit $\epsilon^{\mathcal{A}} : [\mathcal{A}, -]\Box\mathcal{A} \Rightarrow \mathbb{1}_{\mathbf{Col}}$ of the hom-tensor adjunction in **Col** between $-\Box\mathcal{A}$ and $[\mathcal{A}, -]$, which has components $\epsilon^{\mathcal{B}}_{\mathcal{A}} : [\mathcal{A},\mathcal{B}]\Box\mathcal{A} \to \mathcal{B}$ for collection $\mathcal{A}$. We then get a map

$$\mathcal{K} : ([\mathcal{X},\mathcal{X}]\Box[\mathcal{X},\mathcal{X}])\Box\mathcal{X} \to [\mathcal{X},\mathcal{X}]\Box([\mathcal{X},\mathcal{X}]\Box\mathcal{X}) \to [\mathcal{X},\mathcal{X}]\Box\mathcal{X} \to \mathcal{X}$$

which is the composite $\mathcal{K} := \epsilon^{\mathcal{X}}_{\mathcal{X}} \circ (\mathbb{1}_{\mathcal{X}}\Box\epsilon^{\mathcal{X}}_{\mathcal{X}}) \circ \alpha_{[\mathcal{X},\mathcal{X}],[\mathcal{X},\mathcal{X}],\mathcal{X}}$. The operad multiplication for $[\mathcal{X},\mathcal{X}]$ is then the currying of the map $\mathcal{K}$.

11.1. Theorem. *Given a collection $x : \mathcal{X} \to \mathcal{T}(\mathbf{1})$, the collection Gtaut($\mathcal{X}$) : $[\mathcal{X},\mathcal{X}] \to \mathcal{T}(\mathbf{1})$ admits the structure of a globular operad.*

Proof. We need only to show that for $Gtaut(\mathcal{X}) : [\mathcal{X},\mathcal{X}] \to \mathcal{T}(\mathbf{1})$ the collection morphisms $\iota : \mathbf{1} \to [\mathcal{X},\mathcal{X}]$ and $\nu : [\mathcal{X},\mathcal{X}]\Box[\mathcal{X},\mathcal{X}] \to [\mathcal{X},\mathcal{X}]$ satisfy the commutative diagrams required of a monoid object in **Col**. This can be seen, just as in the graded set case, by first currying the maps in the relevant diagrams and checking to see that these new curried diagrams, whose commutativity is equivalent with that of the originals, do in fact commute. The details of which are the same as those from the graded set case above. ■

12. Algebras for a Globular Operad

The notion of an algebra for a globular operad is completely analogous to that of algebras for classical operads. Here graded sets are replaced with collections and our notion of the tautological operad is a bit more complicated. Nonetheless, the general structure is essentially the same.

12.1. Definition. *A collection $x : \mathcal{X} \to \mathcal{T}(\mathbf{1})$ is said to be degenerate if the arity map factors as $x = [id]\circ!_{\mathcal{X}}$, where $[id] : \mathbf{1} \to \mathcal{T}(\mathbf{1})$ is the map which identifies the unique copy of $\mathbf{1}$ in $\mathcal{T}(\mathbf{1})$ consisting exclusively of iterated identities on the single 0-cell, and $!_{\mathcal{X}} : \mathcal{X} \to \mathbf{1}$ is the unique map from $\mathcal{X}$ to the terminal globular set $\mathbf{1}$.*

12.2. Definition. *An algebra $\mathcal{A}$ for a globular operad $(\mathcal{O}, \circ, e)$ is a globular set $\mathcal{A}$, thought of as a degenerate collection, together with a collection homomorphism $\omega : \mathcal{O}\Box\mathcal{A} \to \mathcal{A}$ which makes the diagrams*

$$\begin{tikzcd}
(\mathcal{O}\Box\mathcal{O})\Box\mathcal{A} \arrow[r, "\alpha^{\mathbf{Col}}_{\mathcal{O},\mathcal{O},\mathcal{A}}"] \arrow[d, "\circ^{\mathcal{O}}\Box\mathbb{1}_{\mathcal{A}}"'] & \mathcal{O}\Box(\mathcal{O}\Box\mathcal{A}) \arrow[r, "\mathbb{1}_{\mathcal{O}}\Box\omega"] & \mathcal{O}\Box\mathcal{A} \arrow[d, "\omega"] \\
\mathcal{O}\Box\mathcal{A} \arrow[rr, "\omega"'] & & \mathcal{A}
\end{tikzcd}$$

$$\begin{tikzcd}
 & \mathcal{O}\Box\mathcal{A} \arrow[dr, "\omega"] & \\
\mathbf{1}\Box\mathcal{A} \arrow[ur, "e\Box\mathbb{1}_{\mathcal{A}}"] \arrow[rr, "\lambda^{\mathbf{Col}}_{\mathcal{A}}"'] & & \mathcal{A}
\end{tikzcd}$$

commute.

We can alternatively use $Gtaut(\mathcal{X})$ as defined above to define algebras as a representation of our globular operad.

12.3. Definition. *Let $o : \mathcal{O} \to \mathcal{T}(\mathbf{1})$ be a globular operad. An $\mathcal{O}$-module is a globular operad homomorphism $f : \mathcal{O} \to Gtaut(\mathcal{A})$ for some collection $a : \mathcal{A} \to \mathcal{T}(\mathbf{1})$. An algebra for $\mathcal{O}$ is an $\mathcal{O}$-module such that the collection is degenerate.*

Just as we saw in the non-globular case, these are equivalent definitions of an algebra as seen by currying the map ω via the adjunction between $-\Box\mathcal{A}$ and $[\mathcal{A}, -]$, to get a collection map $[\omega]$ which has the structure of a globular operad homomorphism. We again say in both cases that the operad $\mathcal{O}$ acts on the collection $\mathcal{X}$. Moreover, when an operad acts on a collection $x = [id]\circ !_{\mathcal{X}} : \mathcal{X} \to \mathcal{T}(\mathbf{1})$ we get a collection that is really a globular set in disguise. Because each globular cell in $\mathcal{X}$ sits above one of the iterated identity cells in $\mathcal{T}(\mathbf{1})$ described above, it can be thought of as a collection whose globular cells have 'empty arity'. Because of the way $\Box$ is defined on collections, the arity of a composite cell is the arity shape of the original cell expanded to include the shapes of the cells which were plugged in to the original cell. One can think about pasting composition as if the arity shape of the original cell is a reduction of arity of the composition, pretending that the arity shape of the cells plugged in can be strictly composed as morphisms in a strict ω-category. But when we compose collection cells that sit over one of these special identities, the arity of the composite does not expand in this typical way. It is analogous to the graded set case in which the cells concentrated over zero, when composed as graded set elements, amount to a composite arity of a sum of zeros. Hence, the arity of everything in the graded set remains zero both before and after composition. Similarly, in a collections of the form $x = [id]\circ !_{\mathcal{X}} : \mathcal{X} \to \mathcal{T}(\mathbf{1})$, the arity of any cell in $\mathcal{X}$ is simply the iterative identity of the appropriate dimension, both before and after applying any well-defined pasting operation.

We again get an immediate result regarding induced algebras, this time in the context of globular operads.

12.4. Theorem. *An algebra for a globular operad $\mathcal{O}$ is an algebra for every globular operad $\mathcal{P}$ which maps to $\mathcal{O}$. In particular, an algebra for $\mathcal{O}$ is an algebra for every globular sub-operad.*

13. Cartesian-Duoidal Enriched Categories

What we shall eventually define to be a globular PRO turns out to be a monoidal category enriched over a special type of category. Before defining globular PROs we must first recall the structure of a duoidal category as presented by Batanin and Markl[2] and describe the enrichment structure in general. Let us begin this construction with the following definition.

13.1. Definition. *A duoidal category is a nonuple* $(\mathcal{D}, \otimes, I, \odot, U, \delta, \phi, \theta, \boxplus)$ *consisting of a category* $\mathcal{D}$*, a pair of 2-variable functors* $\otimes : \mathcal{D} \times \mathcal{D} \to \mathcal{D}$ *and* $\odot : \mathcal{D} \times \mathcal{D} \to \mathcal{D}$*, a pair of unit objects* I *and* U*, three morphism* $\delta : I \to I \odot I$*,* $\phi : U \otimes U \to U$*, and* $\theta : I \to U$ *in* $\mathcal{D}$*, and a natural transformation*

$$\boxplus : \otimes(\odot(-,-), \odot(-,-)) \Rightarrow \odot(\otimes(-,-) \otimes (-,-))$$

given by components

$$\boxplus_{A,B,C,D} : [A \odot B] \otimes [C \odot D] \to [A \otimes C] \odot [B \otimes D]$$

with $A, B, C, D \in Obj(\mathcal{C})$*, specifying a lax middle-four interchange law between the product structures. All of this data must satisfy the properties that* $(\mathcal{D}, \otimes, I)$ *and* $(\mathcal{D}, \odot, U)$ *are both monoidal category structures on* $\mathcal{D}$*,* U *is a monoid object in* $(\mathcal{D}, \otimes, I)$*,* I *is a comonoid object in* $(\mathcal{D}, \odot, U)$*, and for all* $A, B, C, D, E, F \in Obj(\mathcal{D})$ *the following diagrams commute.*

$$\begin{tikzcd}
((A \odot B) \otimes (C \odot D)) \otimes (E \odot F) \arrow[r, "\alpha^{\otimes}_{A\odot B, C\odot D, E\odot F}"] \arrow[d, "\boxplus_{A,B,C,D} \otimes \mathbb{1}_{E\odot F}"'] & (A \odot B) \otimes ((C \odot D) \otimes (E \odot F)) \arrow[d, "\mathbb{1}_{A\odot B} \otimes \boxplus_{C,D,E,F}"] \\
((A \otimes C) \odot (B \otimes D)) \otimes (E \odot F) \arrow[d, "\boxplus_{A\otimes C, B\otimes D, E, F}"'] & (A \odot B) \otimes ((C \otimes E) \odot (D \otimes F)) \arrow[d, "\boxplus_{A,B,C\otimes E, D\otimes F}"] \\
((A \otimes C) \otimes E) \odot ((B \otimes D) \otimes F) \arrow[r, "\alpha^{\otimes}_{A,C,E} \odot \alpha^{\otimes}_{B,D,F}"'] & (A \otimes (C \otimes E)) \odot (B \otimes (D \otimes F))
\end{tikzcd}$$

$$\begin{tikzcd}
((A \odot B) \odot C) \otimes ((D \odot E) \odot F) \arrow[r, "\alpha^{\odot}_{A,B,C} \otimes \alpha^{\odot}_{D,E,F}"] \arrow[d, "\boxplus_{A \odot B, C, D \odot E, F}"'] & (A \odot (B \odot C)) \otimes (D \odot (E \odot F)) \arrow[d, "\boxplus_{A, B \odot C, D, E \odot F}"] \\
((A \odot B) \otimes (D \odot E)) \odot (C \otimes F) \arrow[d, "\boxplus_{A,B,D,E} \odot \mathbb{1}_{C \otimes F}"'] & (A \otimes D) \odot ((B \odot C) \otimes (E \odot F)) \arrow[d, "\mathbb{1}_{A \otimes D} \odot \boxplus_{B,C,E,F}"] \\
((A \otimes D) \odot (B \otimes E)) \odot (C \otimes F) \arrow[r, "\alpha^{\odot}_{A \otimes D, B \otimes E, C \otimes F}"'] & (A \otimes D) \odot ((B \otimes E) \odot (C \otimes F))
\end{tikzcd}$$

$$\begin{tikzcd}
A \odot B \arrow[r, "\overline{\lambda}^{\otimes}_{A} \odot \overline{\lambda}^{\otimes}_{B}"] \arrow[d, "\overline{\lambda}^{\otimes}_{A \odot B}"'] & (I \otimes A) \odot (I \otimes B) \\
I \otimes (A \odot B) \arrow[r, "\delta \otimes \mathbb{1}_{A \odot B}"'] & (I \odot I) \otimes (A \odot B) \arrow[u, "\boxplus_{I,I,A,B}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
A \odot B \arrow[r, "\overline{\rho}^{\otimes}_{A} \odot \overline{\rho}^{\otimes}_{B}"] \arrow[d, "\overline{\rho}^{\otimes}_{A \odot B}"'] & (A \otimes I) \odot (B \otimes I) \\
(A \odot B) \otimes I \arrow[r, "\mathbb{1}_{A \odot B} \otimes \delta"'] & (A \odot B) \otimes (I \odot I) \arrow[u, "\boxplus_{A,B,I,I}"']
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes B \arrow[r, "\overline{\lambda}^{\odot}_{A} \otimes \overline{\lambda}^{\odot}_{B}"] \arrow[d, "\overline{\lambda}^{\odot}_{A \otimes B}"'] & (U \odot A) \otimes (U \odot B) \arrow[d, "\boxplus_{U,A,U,B}"] \\
U \odot (A \otimes B) & (U \otimes U) \odot (A \otimes B) \arrow[l, "\phi \odot \mathbb{1}_{A \otimes B}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes B \arrow[r, "\overline{\rho}^{\odot}_{A} \otimes \overline{\rho}^{\odot}_{B}"] \arrow[d, "\overline{\rho}^{\odot}_{A \otimes B}"'] & (A \odot U) \otimes (B \odot U) \arrow[d, "\boxplus_{A,U,B,U}"] \\
(A \otimes B) \odot U & (A \otimes B) \odot (U \otimes U) \arrow[l, "\mathbb{1}_{A \otimes B} \odot \phi"]
\end{tikzcd}$$

The above definition can be stated more succinctly by noting that duoidal categories are precisely the pseudomonoid objects in the category $\mathbf{MonCat}_{lax}$ of monoidal categories and lax-monoidal functors. From this point of view, we think of $(\mathcal{D}, \otimes, I, \odot, U, \delta, \phi, \theta, \boxplus)$ as a monoidal category $(\mathcal{D}, \otimes, I)$ equipped with two lax-monoidal functors $\odot : \mathcal{D} \times \mathcal{D} \to \mathcal{D}$ and $U : \mathbf{1}_{\mathbf{MonCat}_{lax}} \to \mathcal{D}$ over the monoidal product $\otimes$, where $\mathbf{1}_{\mathbf{MonCat}_{lax}}$ is the trivial monoidal category with one object, and $(\mathcal{D}, \odot, U)$ is a pseudomonoid with respect to the cartesian product in $\mathbf{MonCat}_{lax}$, the category of monoidal categories and lax-monoidal functors. The laxivity of the functor $\otimes$ induces the interchange transformation $\boxplus$ and the morphism δ. Similarly, the laxivity of U induces the morphisms ϕ and θ. The six commutative diagrams above follow from the associativity and unity coherence conditions that make $\odot$ a lax-monoidal functor over $\otimes$ and those which make $\otimes$ an oplax monoidal functor over $\odot$. Moreover, note that all of this data makes I a comonoid object with respect to $\odot$. And similarly, it follows that U (thought of as an object in $\mathcal{D}$) is a monoid object with respect to $\otimes$.

13.2. Definition. *A lax-duoidal functor between duoidal categories* $(\mathcal{D}, \otimes, I, \odot, U, \delta, \phi, \theta, \boxplus)$ *and* $(\mathcal{D}', \otimes', I', \odot', U', \delta', \phi', \theta', \boxplus')$ *is given by a functor* $F : \mathcal{D} \to \mathcal{D}'$*, two natural transformations*

$$\beta : \otimes'(F(-), F(-)) \Rightarrow F(\otimes(-,-))$$
$$\gamma : \odot'(F(-), F(-)) \Rightarrow F(\odot(-,-))$$

given by components

$$\beta_{A,B} : F(A) \otimes' F(B) \to F(A \otimes B)$$
$$\gamma_{A,B} : F(A) \odot' F(B) \to F(A \odot B),$$

and two morphism

$$i : I' \to F(I)$$
$$u : U' \to F(U)$$

such that (F, β, i) *is a lax-monoidal functor between* $(\mathcal{D}, \otimes, I)$ *and* $(\mathcal{D}', \otimes', I')$*,* (F, γ, u) *is a lax-monoidal functor between* $(\mathcal{D}, \odot, U)$ *and* $(\mathcal{D}', \odot', U')$*, and for all* $A, B, C, D \in Obj(\mathcal{D})$ *the following diagrams commute.*

$$\begin{tikzcd}
(F(A) \odot' F(B)) \otimes' (F(C) \odot' F(D)) \arrow[r, "\gamma_{A,B} \otimes' \gamma_{C,D}"] \arrow[d, "\boxplus'_{F(A),F(B),F(C),F(D)}"'] & F(A \odot B) \otimes' F(C \odot D) \arrow[d, "\beta_{A \odot B, C \odot D}"] \\
(F(A) \otimes' F(C)) \odot' (F(B) \otimes' F(D)) \arrow[d, "\beta_{F(A),F(C)} \odot' \beta_{F(B),F(D)}"'] & F((A \odot B) \otimes (C \odot D)) \arrow[d, "F(\boxplus_{A,B,C,D})"] \\
F(A \otimes C) \odot' F(B \otimes D) \arrow[r, "\gamma_{A \otimes C, B \otimes D}"'] & F((A \otimes C) \odot (B \otimes D))
\end{tikzcd}$$

$$\begin{tikzcd}
I' \arrow[r, "i"] \arrow[d, "\delta'"'] & F(I) \arrow[r, "F(\delta)"] & F(I \odot I) \\
I' \odot' I' \arrow[rr, "i \odot' i"'] & & F(I) \odot' F(I) \arrow[u, "\gamma_{I,I}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
U' \otimes' U' \arrow[r, "\phi'"] \arrow[d, "u \otimes' u"'] & U' \arrow[r, "u"] & F(U) \\
F(U) \otimes' F(U) \arrow[rr, "\beta_{U,U}"'] & & F(U \otimes U) \arrow[u, "F(\phi)"']
\end{tikzcd}$$

$$\begin{tikzcd}
I' \arrow[r, "i"] \arrow[d, "\theta'"'] & F(I) \arrow[d, "F(\theta)"] \\
U' \arrow[r, "u"'] & F(U)
\end{tikzcd}$$

Duoidal categories, together with all of the duoidal functors between, them form a category which we shall here denote **Duoidal**. It is possible to enrich over objects in this category via the following construction.

13.3. DEFINITION. *A category enriched over a duoidal category* $(\mathcal{D}, \otimes, I, \odot, U, \delta, \phi, \theta, \boxplus)$, *or simply a* $\mathcal{D}$*-category, is an enriched category with respect to the monoidal structure* $(\mathcal{D}, \otimes, I)$*. A* $\mathcal{D}$*-functor is an enriched functor between two* $\mathcal{D}$*-categories which is enriched with respect to the same monoidal structure* $(\mathcal{D}, \otimes, I)$*. A* $\mathcal{D}$*-transformation is an enriched natural transformation between two* $\mathcal{D}$*-functors.*

Note that these enriched categories do not initially appear to use the second monoidal structure. This second product will however become manifest when looking at the category of categories enriched over a fixed duoidal category. Let us first explicitly unpack the definition of a duoidally enriched category.

A $\mathcal{D}$-category $\mathcal{E}$ enriched over the duoidal category $(\mathcal{D}, \otimes, I, \odot, U, \delta, \phi, \theta, \boxplus)$ has, first of all, a collection of objects $Obj(\mathcal{E})$. For each pair $X, Y \in Obj(\mathcal{E})$ we have an object $E(X, Y) \in Obj(\mathcal{D})$ called the *hom-object* between X and Y. For each triple $X, Y, Z \in Obj(\mathcal{E})$ we have a morphism

$$\circ_{X,Y,Z} : E(Y, Z) \otimes E(X, Y) \to E(X, Z)$$

from $\mathcal{D}$ called *composition at* (X, Y, Z). We also have for each $X \in Obj(\mathcal{E})$ a morphism

$$j_X : I \to E(X, X)$$

from $\mathcal{D}$ called the *identity identification at* X. All of this data must satisfy the obvious commutative diagrams ensuring that the composition operation in $\mathcal{E}$ is associative and unital.

As previously mentioned, all such $\mathcal{D}$-categories and $\mathcal{D}$-functors between them form a category, which we shall here denote $\mathcal{D}\mathbf{Cat}$. Here the second monoidal product from the duoidal structure on $\mathcal{D}$ induces a monoidal structure on $\mathcal{D}\mathbf{Cat}$. The tensor product

$$\oplus : \mathcal{D}\mathbf{Cat} \times \mathcal{D}\mathbf{Cat} \to \mathcal{D}\mathbf{Cat}$$

of $\mathcal{D}$-categories $\mathcal{E}$ and $\mathcal{F}$ is given as the cartesian product on objects and for $A, B \in Obj(\mathcal{E})$ and $X, Y \in Obj(\mathcal{F})$ we have

$$E \oplus F((A, X), (B, Y) := E(A, B) \odot F(X, Y)$$

as the hom-objects in $\mathcal{E} \oplus \mathcal{F}$. The unit $\mathbf{1}_\oplus$ with respect to this tensor product is the trivial $\mathcal{D}$-category consisting of a single object $*$ and a single hom-object $\mathbf{1}_\oplus(*, *) := U$, which is precisely the monoidal unit for the second monoidal structure in the underlying duoidal category $\mathcal{D}$ over which the enrichment structure is defined. This allows us to define the following type of $\mathcal{D}$-category.

13.4. Definition. *A monoidal $\mathcal{D}$-category $(\mathcal{M}, \diamond, \iota)$ is a pseudomonoid in the monoidal category $(\mathcal{D}\mathbf{Cat}, \oplus, \mathbf{1}_{\oplus})$, where $\mathcal{M} \in Obj(\mathcal{D}\mathbf{Cat})$, the $\mathcal{D}$-functor $\diamond : \mathcal{M} \oplus \mathcal{M} \to \mathcal{M}$ is the monoidal product, and $\iota : \mathbf{1}_{\oplus} \to \mathcal{M}$ is the unit $\mathcal{D}$-functor such that $\diamond$ is associative and unital, with respect to ι, up to $\mathcal{D}$-transformations.*

Having the structure of a pseudomonoid implies the existence of a morphism

$$\boxdot_{X,Y,Z,W} : M(X,Y) \odot M(Z,W) \to M(X \diamond Z, Y \diamond W)$$

for every $X, Y, Z, W \in Obj(\mathcal{M})$, which encodes how $\diamond$ acts on morphisms. Moreover, these satisfy the usual pentagon and triangle coherence conditions to ensure that $(\mathcal{M}, \diamond, \iota)$ is a pseudomonoid. This structure will play a role below when defining lax-monoidal functors between $\mathcal{D}$ categories. But before we define these functors, it is important to note one final fact. Notice that $\mathcal{D}$-categories come equipped with an underlying category. The underlying category has the same objects as the $\mathcal{D}$-category. Morphisms in the underlying category are given by

$$U(M)(X,Y) := D(I, M(X,Y))$$

for $X, Y \in \mathcal{M}$. As shown by Batanin and Markl[2], this gives a lax-monoidal 2-functor between $\mathcal{D}\mathbf{Cat}$ and $\mathbf{Cat}$. This fact will also play a role in the following definition.

13.5. Definition. *A lax-monoidal $\mathcal{D}$-functor between monoidal $\mathcal{D}$-categories $(\mathcal{M}, \diamond, \iota)$ and $(\mathcal{M}', \diamond', \iota')$ is a triple $(F, \widehat{F}, e)$ consisting of an underlying $\mathcal{D}$-functor $F : \mathcal{M} \to \mathcal{M}'$ together with a $\mathcal{D}$-transformation*

$$\widehat{F} : \diamond'(F(-), F(-)) \Rightarrow F(\diamond(-,-))$$

given by components

$$\widehat{F}_{X,Y} : I \to M'(F(X) \diamond' F(Y), F(X \diamond Y))$$

for $X, Y \in Obj(\mathcal{M})$, and a morphism

$$e : \iota'(*) \to F(\iota(*))$$

in $(\mathcal{M}', \diamond', \iota')$ such that the underlying functor F is a lax-monoidal functor between the underlying monoidal categories, not thought of as $\mathcal{D}$-categories. Moreover, this data must satisfy the following coherence condition for all $X, Y, Z, W \in Obj(\mathcal{M})$

$$\begin{tikzcd}
 & M(X \diamond Z, Y \diamond W) \otimes I \arrow[dr, "F \otimes \widehat{F}_{X,Z}"] & \\
M(X \diamond Z, Y \diamond W) \arrow[ur, "\overline{\rho}^{\mathcal{D}}_{M(X \diamond Z, Y \diamond W)}"] & & M'(F(X \diamond Z), F(Y \diamond W)) \otimes M'(F(X) \diamond' F(Z), F(X \diamond Z)) \arrow[d, "\circ^{\mathcal{M}'}_{F(X)\diamond' F(Z), F(X \diamond Z), F(Y \diamond W)}"] \\
M(X,Y) \odot M(Z,W) \arrow[u, "\boxdot^{\mathcal{M}}_{X,Y,Z,W}"] \arrow[d, "F \odot F"'] & & M'(F(X) \diamond' F(Z), F(Y \diamond W)) \\
M'(F(X), F(Y)) \odot M'(F(Z), F(W)) \arrow[dr, "\overline{\lambda}^{\mathcal{D}}_{M'(F(X),F(Y)) \odot M'(F(Z),F(W))}"'] & & M'(F(Y) \diamond' F(W), F(Y \diamond W)) \otimes M'(F(X) \diamond' F(Z), F(Y) \diamond' F(W)) \arrow[u, "\circ^{\mathcal{M}'}_{F(X)\diamond' F(Z), F(Y) \diamond' F(W), F(Y \diamond W)}"'] \\
 & I \otimes [M'(F(X), F(Y)) \odot M'(F(Z), F(W))] \arrow[ur, "\widehat{F}_{Y,W} \otimes \boxdot^{\mathcal{M}'}_{F(X),F(Y),F(Z),F(W)}"'] &
\end{tikzcd}$$

ensuring that the two pseudomonoid structures are compatible.

We will need the following special type of duoidal category.

13.6. Definition. *A cartesian-duoidal category is a duoidal category* $(\mathcal{D}, \otimes, I, \times, \mathbb{1}, \delta, \phi, \theta, \boxplus)$ *such that the second monoidal structure* $(\mathcal{D}, \times, \mathbb{1})$ *is a cartesian monoidal category.*

Cartesian-duoidal categories form a subcategory of **Duoidal**, which we shall here denote by **CartDuoidal**. We have the following well known result.

13.7. Proposition. *A monoidal category with finite products is cartesian-duoidal.*

Proof. Let $(\mathcal{D}, \otimes, I)$ be a monoidal category with finite products. We wish to show that this category admits a duoidal structure $(\mathcal{D}, \otimes, I, \times, \mathbb{1}, \delta, \phi, \theta, \boxplus)$. The two monoidal products $\otimes, \times : \mathcal{D} \times \mathcal{D} \to \mathcal{D}$ are present by assumption. Similarly, the unit I for $\otimes$ is given. The unit $\mathbb{1}$ for $\times$ follows from the fact that $\mathcal{D}$ has all finite products, as $\mathbb{1}$ is the terminal empty product. The morphism $\delta : I \to I \times I$ is the the universal morphism induced by the universal property of the product $I \times I$. The morphisms $\phi : \mathbb{1} \otimes \mathbb{1} \to \mathbb{1}$ and $\theta : I \to \mathbb{1}$ are the unique maps induced by the fact that $\mathbb{1}$ is terminal.

We shall prove the existence of the map $\boxplus$ by building it in pieces. Consider the product object $(X \times Y) \otimes (Z \times W)$ for $X, Y, Z, W \in Obj(\mathcal{C})$. It suffices to construct a map from this object to the $(X \otimes Z) \times (Y \otimes W)$. Using projection and diagonal maps given by the cartesian structure, we get the following map:

$$(X \times Y) \otimes (Z \times W) \xrightarrow{((\pi_1 \otimes \mathbb{1}) \times (\pi_2 \otimes \mathbb{1})) \circ \Delta} [X \otimes (Z \times W)] \times [Y \otimes (Z \times W)]$$

We can similarly construct the following two maps:

$$[X \otimes (Z \times W)] \times [Y \otimes (Z \times W)] \xrightarrow{([(\mathbb{1} \otimes \pi_1) \times (\mathbb{1} \otimes \pi_2)] \times \mathbb{1}) \circ (\Delta \times \mathbb{1})} [(X \otimes Z) \times (X \otimes W)] \times [Y \otimes (Z \times W)]$$

$$[(X \otimes Z) \times (X \otimes W)] \times [Y \otimes (Z \times W)] \xrightarrow{(\mathbb{1} \times [(\mathbb{1} \otimes \pi_1) \times (\mathbb{1} \otimes \pi_2)]) \circ (\mathbb{1} \times \Delta)} [(X \otimes Z) \times (X \otimes W)] \times [(Y \otimes Z) \times (Y \otimes W)]$$

We then have the following two projection maps:

$$[(X \otimes Z) \times (X \otimes W)] \times [(Y \otimes Z) \times (Y \otimes W)] \to [X \otimes Z] \times [(Y \otimes Z) \times (Y \otimes W)]$$

$$[X \otimes Z] \times [(Y \otimes Z) \times (Y \otimes W)] \to [X \otimes Z] \times [Y \otimes W].$$

By composing each of these five maps together we get the desired map

$$(X \times Y) \otimes (Z \times W) \to (X \otimes Z) \times (Y \otimes W).$$

Lastly we must check that various coherence conditions are satisfied. We first need to check that $(\mathbb{1}, \phi, \theta)$ is a monoid in $(\mathcal{D}, \otimes, I)$. This follows immediately as the commutativity of the associativity and unital commutitive diagrams which ensure that $(\mathbb{1}, \phi, \theta)$ has a monoid structure in $(\mathcal{D}, \otimes, I)$ follows from the fact that $\mathbb{1}$ is terminal in $\mathcal{D}$. We then need to check that (I, δ, θ) is a comonoid in $(\mathcal{D}, \times, \mathbb{1})$. In this case, the commutativity of the coassociativity and counital commutative diagrams ensuring that (I, δ, θ) has a comonoid structure in $(\mathcal{D}, \times, \mathbb{1})$ follows immediately from the cartesian structure. The final coherence conditions which must be checked are the commutativity of the six commutative diagrams required of the data in a duoidal category. Checking that these diagrams commute is routine, following from the fact that the second monoidal structure is cartesian.

■

For simplicity we shall call a category enriched over a cartesian-duoidal category $\mathcal{C}$ a *$\mathcal{C}$-category*. For a fixed $\mathcal{C}$ we shall denote the category of all such enriched categories $\mathcal{C}\mathbf{Cat}$. We can finally state succinctly the key definition of this section.

13.8. Definition. *A cartesian-duoidal enriched monoidal category is a pseudomonoid in* $(\mathcal{C}\mathbf{Cat}, \oplus, \mathbf{1}_\oplus)$.

Note that a strict cartesian-duoidal enriched monoidal category would simply be a monoid in $(\mathcal{C}\mathbf{Cat}, \oplus, \mathbf{1}_\oplus)$. Our present interest in these monoidal categories is that they allow us to generalize the classical definition of a PRO. We conclude this section with the following definition.

13.9. Definition. *An enriched cartesian PRO is a strict cartesian-duoidal enriched monoidal category enriched over a cartesian-duoidal category $\mathcal{C}$ such that the object set can be identified with $\mathbb{N}$ and the monoidal product on objects is identified with addition of natural numbers.*

14. Defining Globular PROs

Just as classical PROs may be presented as a specific type of monoidal category, in what follows we will see that a globular PRO is simply a specific type of cartesian-duoidal enriched category. Before formally defining globular PROs we first need to ensure that **Col** is cartesian-duoidal. But since **Col** is a slice topos it has a natural cartesian product. It then follows immediately from the proposition above that since **Col** has finite products it is moreover cartesian-duoidal. This then ensures us that the category **Col** has the appropriate structure for us to define globular PROs via the following construction.

14.1. Definition. *A globular PRO is a strict cartesian-duoidal enriched monoidal category $(\mathcal{P}, +, O)$ enriched over the cartesian-duoidal category* **Col** *such that the object set of $\mathcal{P}$ is isomorphic to $\mathbb{N}$, the bifunctor $+ : \mathcal{P} \times \mathcal{P} \to \mathcal{P}$ acts as addition of natural numbers on objects, and the unit* **Col***-functor $O : \mathbf{1}_\oplus \to \mathcal{P}$ maps $*$ to the additive identity $0 \in \mathbb{N}$.*

Note that a globular PRO is precisely an enriched cartesian PRO enriched over **Col**. More explicitly, a globular PRO $\mathcal{P}$ has the following structure. $\mathcal{P}$ has as its object set $Obj(\mathcal{P}) \cong \mathbb{N}$. For each pair $n, m \in \mathbb{N}$ we have a *hom-object* $h_{n,m} : \mathcal{P}(n, m) \to \mathcal{T}(\mathbf{1})$ from **Col**, which we will often simply write as $\mathcal{P}(n, m)$. For each triple $n, m, l \in \mathbb{N}$ we have a collection homomorphism $\circ_{n,m,l} : \mathcal{P}(m,l)\Box\mathcal{P}(n,m) \to \mathcal{P}(n,l)$ called *composition at* (n,m,l). We also have for each $n \in \mathbb{N}$ a collection homomorphism $j_n : \mathbf{1} \to \mathcal{P}(n,n)$ called the *identity identification at* n.

Carefully note that the monoidal unit in **Col** is $I : \mathbf{1} \hookrightarrow \mathcal{T}(\mathbf{1})$. Hence the boldface $\mathbf{1}$, which is the source of each identity identification, is not the (unbolded) number $1 \in \mathbb{N}$, but rather the globular set with exactly one cell of every dimension.

All of this data must satisfy, for all $n, m, l, k \in \mathbb{N}$, the following two commutative diagrams ensuring that composition in $\mathcal{P}$ is associative and unital.

$$\begin{tikzcd}
{[\mathcal{P}(l,k)\Box\mathcal{P}(m,l)]\Box\mathcal{P}(n,m)} \arrow[rr, "\alpha^{\mathbf{Col}_\Box}_{\mathcal{P}(l,k),\mathcal{P}(m,l),\mathcal{P}(n,m)}"] \arrow[d, "\circ_{m,l,k}\Box\mathbb{1}_{\mathcal{P}(n,m)}"'] & & \mathcal{P}(l,k)\Box[\mathcal{P}(m,l)\Box\mathcal{P}(n,m)] \arrow[d, "\mathbb{1}_{\mathcal{P}(l,k)}\Box\circ_{n,m,l}"] \\
\mathcal{P}(m,k)\Box\mathcal{P}(n,m) \arrow[dr, "\circ_{n,m,k}"'] & & \mathcal{P}(l,k)\Box\mathcal{P}(n,l) \arrow[dl, "\circ_{n,l,k}"] \\
& \mathcal{P}(n,k) &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{P}(m,m)\Box\mathcal{P}(n,m) \arrow[r, "\circ_{n,m,m}"] & \mathcal{P}(n,m) & \mathcal{P}(n,m)\Box\mathcal{P}(n,n) \arrow[l, "\circ_{n,n,m}"'] \\
\mathbf{1}\Box\mathcal{P}(n,m) \arrow[u, "j_m\Box\mathbb{1}_{\mathcal{P}(n,m)}"] \arrow[ur, "\lambda^{\mathbf{Col}_\Box}_{\mathcal{P}(n,m)}"'] & & \mathcal{P}(n,m)\Box\mathbf{1} \arrow[u, "\mathbb{1}_{\mathcal{P}(n,m)}\Box j_n"'] \arrow[ul, "\rho^{\mathbf{Col}_\Box}_{\mathcal{P}(n,m)}"]
\end{tikzcd}$$

Here $\mathbf{Col}_\Box$ is used to denote that these structure maps are those for the $\Box$ monoidal product as opposed to that of the cartesian product in **Col**.

The globular PRO $\mathcal{P}$ must also come equipped with a monoidal structure encoded in the 2-variable functor $+ : \mathcal{P} \times \mathcal{P} \to \mathcal{P}$. Since $\mathcal{P}$ is an enriched category, the functor $+$ must moreover be an enriched functor of 2-variables. More precisely, this means that $+$ is given on objects by the addition map $+ : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ together with, for each $n, m, l, k \in \mathbb{N}$, collection homomorphisms $+_{n,m,l,k} : \mathcal{P}(n,m) \times \mathcal{P}(l,k) \to \mathcal{P}(n+l, m+k)$, all of which must, for all $n, m, l, k, r, s \in \mathbb{N}$, make the following diagrams commute.

$$\begin{tikzcd}
{[\mathcal{P}(m,r)\times\mathcal{P}(k,s)]\Box[\mathcal{P}(n,m)\times\mathcal{P}(l,k)]} \arrow[rr, "\boxtimes_{\mathcal{P}(m,r),\mathcal{P}(k,s),\mathcal{P}(n,m),\mathcal{P}(l,k)}"] \arrow[d, "+_{m,r,k,s}\Box+_{n,m,l,k}"'] & & {[\mathcal{P}(m,r)\Box\mathcal{P}(n,m)]\times[\mathcal{P}(k,s)\Box\mathcal{P}(l,k)]} \arrow[d, "\circ_{n,m,r}\times\circ_{l,k,s}"] \\
\mathcal{P}(m+k,r+s)\Box\mathcal{P}(n+l,m+k) \arrow[dr, "\circ_{n+l,m+k,r+s}"] & & \mathcal{P}(n,r)\times\mathcal{P}(l,s) \arrow[dl, "+_{n,r,l,s}"'] \\
& \mathcal{P}(n+l,r+s) &
\end{tikzcd}$$

$$\begin{tikzcd}
& \mathbf{1}\times\mathbf{1}\cong\mathbf{1} \arrow[dl, "j_n\times j_m"'] \arrow[dr, "j_{n+m}"] & \\
\mathcal{P}(n,n)\times\mathcal{P}(m,m) \arrow[rr, "+_{n,n,m,m}"] & & \mathcal{P}(n+m,n+m)
\end{tikzcd}$$

$$\begin{tikzcd}
{[\mathcal{P}(n,m)\times\mathcal{P}(l,k)]\times\mathcal{P}(r,s)} \arrow[rr, "\alpha^{\mathbf{Col}_\times}_{\mathcal{P}(n,m),\mathcal{P}(l,k),\mathcal{P}(r,s)}"] \arrow[d, "+_{n,m,l,k}\times\mathbb{1}_{\mathcal{P}(r,s)}"'] & & {\mathcal{P}(n,m)\times[\mathcal{P}(l,k)\times\mathcal{P}(r,s)]} \arrow[d, "\mathbb{1}_{\mathcal{P}(n,m)}\times+_{l,k,r,s}"] \\
\mathcal{P}(n+l,m+k)\times\mathcal{P}(r,s) \arrow[dr, "+_{n+l,m+k,r,s}"] & & \mathcal{P}(n,m)\times\mathcal{P}(l+r,k+s) \arrow[dl, "+_{n,m,l+r,k+s}"'] \\
& \mathcal{P}(n+l+r,m+k+s) &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{T}(\mathbf{1})\times\mathcal{P}(n,m) \arrow[r, "O\times\mathbb{1}_{\mathcal{P}(n,m)}"] \arrow[dr, "\lambda^{\mathbf{Col}_\times}_{\mathcal{P}(n,m)}"'] & \mathcal{P}(0,0)\times\mathcal{P}(n,m)\cong\mathcal{P}(n,m)\times\mathcal{P}(0,0) \arrow[d, "+_{0,0,n,m}"', "+_{n,m,0,0}"] & \mathcal{P}(n,m)\times\mathcal{T}(\mathbf{1}) \arrow[l, "\mathbb{1}_{\mathcal{P}(n,m)}\times O"'] \arrow[dl, "\rho^{\mathbf{Col}_\times}_{\mathcal{P}(n,m)}"] \\
& \mathcal{P}(n,m) &
\end{tikzcd}$$

The first two diagrams ensure that $+$ is a **Col**-functor. The second two ensure that $\mathcal{P}$ is a monoid object **ColCat** with respect to the product $\oplus$, which in this context is simply the cartesian product on homsets. We again adopt the notation $\mathbf{Col}_\times$ to distinguish the structure maps from the cartesian structure on **Col** from the $\Box$ monoidal product.

14.2. DEFINITION. *A morphism of globular PROs between globular PROs $\mathcal{P}$ and $\mathcal{P}'$ is a strict monoidal* **Col**-*functor* $(F, \widehat{F}, e) : \mathcal{P} \to \mathcal{P}'$. *More precisely, such a morphism consists of an underlying* **Col**-*functor*

$$F : \mathcal{P} \to \mathcal{P}'$$

that is the identity on objects, a **Col**-*enriched natural transformation*

$$\widehat{F} : +(F(-), F(-)) \Rightarrow F(+(-,-))$$

with each component

$$\widehat{F}_{n,m} : I \to \mathcal{P}'(F(n) + F(m), F(n+m))$$

for $n, m \in \mathbb{N}$ *having* $\widehat{F}_{n,m} = j^{\mathcal{P}'}_{n+m}$, *and a morphism*

$$e : I \to \mathcal{P}'(0, F(0))$$

such that $e = j^{\mathcal{P}'}_0$, *all of which makes F a strict monoidal functor between the underlying categories $\mathcal{P}$ and $\mathcal{P}'$ not thought of as* **Col**-*categories. Moreover, the diagram*

$$\begin{tikzcd}
 & P(n+l, m+k) \Box I \arrow[dr, "F \Box \widehat{F}_{n,l}"] & \\
P(n+l, m+k) \arrow[ur, "\overline{\rho}^{\mathbf{Col}}_{P(n+l,m+k)}"] & & P'(F(n+l), F(m+k)) \Box P'(F(n) +' F(l), F(n+l)) \arrow[d, "\circ^{\mathcal{P}'}_{F(n)+'F(l), F(n+l), F(m+k)}"] \\
P(n,m) \times P(l,k) \arrow[u, "\boxdot^{\mathcal{P}}_{n,m,l,k}"] \arrow[d, "F \times F"'] & & P'(F(n) +' F(l), F(m+k)) \\
P'(F(n), F(m)) \times P'(F(l), F(k)) \arrow[dr, "\overline{\lambda}^{\mathbf{Col}}_{P'(F(n),F(m)) \times P'(F(l),F(k))}"'] & & P'(F(m) +' F(k), F(m+k)) \Box P'(F(n) +' F(l), F(m) +' F(k)) \arrow[u, "\circ^{\mathcal{P}'}_{F(n)+'F(l), F(m)+'F(k), F(m+k)}"'] \\
 & I \Box [P'(F(n), F(m)) \times P'(F(l), F(k))] \arrow[ur, "\widehat{F}_{m,k} \Box \boxdot^{\mathcal{P}'}_{F(n),F(m),F(l),F(k)}"'] &
\end{tikzcd}$$

must commute for all $n, m, l, k \in \mathbb{N}$.

Together with the morphisms between them, Globular PROs form a category which we shall here denote by **GlobPRO**.

15. The Tautological Globular PRO

Just as with ordinary PROs, before formalizing the notion of an algebra for a globular PRO we will first construct the *tautological globular PRO*, which we shall denote by $GTaut(A)$ given a degenerate collection $a : A \to \mathcal{T}(\mathbf{1})$. Note that in the construction that follows it is not strictly necessary that the collection $a : A \to \mathcal{T}(\mathbf{1})$ be degenerate in order to define a tautological globular PRO. We however make this assumption for the purpose of defining algebras for globular PROs. If $a : A \to \mathcal{T}(\mathbf{1})$ is not degenerate, the final result of this construction gives instead the structure of a module.

We first construct the PRO $GTaut(A)$ by specifying its objects. $GTaut(A)$ has as its set of objects all successive powers cartesian powers A^n in **Col** for $n \in \mathbb{N}$. These can, as in the non-globular case, be naturally identified with $\mathbb{N}$. Under this identification the hom-objects $GTaut(A)(n,m)$ in $GTaut(A)$ are exactly the internal hom $[A^n, A^m]$ of the closed structure corresponding to the product $\square$ in **Col**. To understand composition in $GTaut(A)$ we first need to consider again the hom-tensor adjunction $-\square B \dashv [B,-] : \mathbf{Col} \to \mathbf{Col}$. Let

$$\epsilon^B : [B,-]\square B \Rightarrow \mathbb{1}_{\mathbf{Col}}$$

be the counit of this adjunction, which has components

$$\epsilon^B_X : [B,X]\square B \to X$$

for each collection $x : X \to \mathcal{T}(\mathbf{1})$. We will call each of these components *evaluation*. We shall also use

$$\Psi^B_{X,Y} : \mathrm{Hom}_{\mathbf{Col}}(X\square B, Y) \to \mathrm{Hom}_{\mathbf{Col}}(X, [B,Y])$$

to denote the X, Y component of the natural isomorphism of hom-sets which defines this adjunction. Applying this morphism (or its inverse) is precisely the currying of a morphism in either of these hom-sets. Now consider the composition

$$\theta_{X,Y,Z} : ([Y,Z]\square[X,Y])\square X \xrightarrow{\alpha^{\mathbf{Col}_\square}_{[Y,Z],[X,Z],X}} [Y,Z]\square([X,Y]\square X) \xrightarrow{\mathbb{1}_{[Y,Z]}\square\epsilon^X_Y} [Y,Z]\square Y \xrightarrow{\epsilon^Y_Z} Z$$

in **Col**. Since $\theta_{X,Y,Z} \in \mathrm{Hom}_{\mathbf{Col}}(([Y,Z]\square[X,Y])\square X, Z)$ we can curry it by applying $\Psi^X_{[Y,Z]\square[X,Y],Z}$. This gives the morphism

$$\circ_{X,Y,Z} := \Psi^X_{[Y,Z]\square[X,Y],Z}(\theta_{X,Y,Z}) : [Y,Z]\square[X,Y] \to [X,Z]$$

which we define to be the (X,Y,Z) component of the composition in $GTaut(A)$. In order to get identities for $GTaut(A)$ we must then consider the left unitor

$$\lambda^{\mathbf{Col}_\square}_X : \mathbf{1}\square X \to X$$

with respect to the monoidal product $\square$ in **Col**. The identity identification at X is then defined to be

$$j_X := \Psi^X_{\mathbf{1},X}\left(\lambda^{\mathbf{Col}_\square}_X\right) : \mathbf{1} \to [X,X]$$

in a similar way to that of composition. To define the monoidal product $+$, we first consider the morphism κ_{A^n,A^m,A^l,A^k} which we define via the diagram below.

$$\begin{tikzcd}
([A^n,A^m]\times[A^l,A^k])\square A^{n+l} \cong ([A^n,A^m]\times[A^l,A^k])\square(A^n\times A^l) \arrow[rr, dashed, "\kappa_{A^n,A^m,A^l,A^k}"] \arrow[dr, "\boxtimes_{A^n,A^m,A^l,A^k}"'] & & A^m\times A^k \cong A^{m+k} \\
& ([A^n,A^m]\square A^n)\times([A^l,A^k]\square A^l) \arrow[ur, "\epsilon^{A^n}_{A^m}\times\epsilon^{A^l}_{A^k}"'] &
\end{tikzcd}$$

This allows us to then define

$$+_{A^n,A^m,A^l,A^k} := \Psi^{A^{n+l}}_{[A^n,A^m]\times[A^l,A^k],A^{m+k}}(\kappa_{A^n,A^m,A^l,A^k}) : [A^n, A^m] \times [A^l, A^k] \to [A^{n+l}, A^{m+k}]$$

which is the monoidal product in $GTaut(A)$. It then remains to show that all of this data satisfies the following commutative diagrams:

$$\begin{tikzcd}
([A^l, A^k]\Box[A^m, A^l])\Box[A^n, A^m] \arrow[rr, "\alpha^{\mathbf{Col}_\Box}_{[A^l,A^k],[A^m,A^l],[A^n,A^m]}"] \arrow[d, "\circ_{A^m,A^l,A^k}\Box\mathbb{1}_{[A^n,A^m]}"'] & & {[A^l, A^k]\Box([A^m, A^l]\Box[A^n, A^m])} \arrow[d, "\mathbb{1}_{[A^l,A^k]}\Box\circ_{A^n,A^m,A^l}"] \\
{[A^m, A^k]\Box[A^n, A^m]} \arrow[dr, "\circ_{A^n,A^m,A^k}"'] & & {[A^l, A^k]\Box[A^n, A^l]} \arrow[dl, "\circ_{A^n,A^l,A^k}"] \\
& {[A^n, A^k]} &
\end{tikzcd}$$

$$\begin{tikzcd}
{[A^m, A^m]\Box[A^n, A^m]} \arrow[r, "\circ_{A^n,A^m,A^m}"] & {[A^n, A^m]} & {[A^n, A^m]\Box[A^n, A^n]} \arrow[l, "\circ_{A^n,A^n,A^m}"'] \\
\mathbf{1}\Box[A^n, A^m] \arrow[u, "j_{A^m}\Box\mathbb{1}_{[A^n,A^m]}"] \arrow[ur, "\lambda^{\mathbf{Col}_\Box}_{[A^n,A^m]}"'] & & {[A^n, A^m]\Box\mathbf{1}} \arrow[u, "\mathbb{1}_{[A^n,A^m]}\Box j_{A^n}"'] \arrow[ul, "\rho^{\mathbf{Col}_\Box}_{[A^n,A^m]}"]
\end{tikzcd}$$

$$\begin{tikzcd}
([A^m, A^r] \times [A^k, A^s])\Box([A^n, A^m] \times [A^l, A^k]) \arrow[rr, "\boxtimes_{[A^m,A^r],[A^k,A^s],[A^n,A^m],[A^l,A^k]}"] \arrow[d, "+_{A^m,A^r,A^k,A^s}\Box+_{A^n,A^m,A^l,A^k}"'] & & ([A^m, A^r]\Box[A^n, A^m]) \times ([A^k, A^s]\Box[A^l, A^k]) \arrow[d, "\circ_{A^n,A^m,A^r}\times\circ_{A^l,A^k,A^s}"] \\
{[A^{m+k}, A^{r+s}]\Box[A^{n+l}, A^{m+k}]} \arrow[dr, "\circ_{A^{n+l},A^{m+k},A^{r+s}}"] & & {[A^n, A^r] \times [A^l, A^s]} \arrow[dl, "+_{A^n,A^r,A^l,A^s}"'] \\
& {[A^{n+l}, A^{r+s}]} &
\end{tikzcd}$$

$$\begin{tikzcd}
& \mathbf{1} \times \mathbf{1} \cong \mathbf{1} \arrow[dl, "j_{A^n}\times j_{A^m}"'] \arrow[dr, "j_{A^{n+m}}"] & \\
{[A^n, A^n] \times [A^m, A^m]} \arrow[rr, "+_{A^n,A^n,A^m,A^m}"] & & {[A^{n+m}, A^{n+m}]}
\end{tikzcd}$$

$$\begin{tikzcd}
([A^n, A^m] \times [A^l, A^k]) \times [A^r, A^s] \arrow[rr, "\alpha^{\mathbf{Col}_\times}_{[A^n,A^m],[A^l,A^k],[A^r,A^s]}"] \arrow[d, "+_{A^n,A^m,A^l,A^k} \times \mathbb{1}_{[A^r,A^s]}"'] & & {[A^n, A^m] \times ([A^l, A^k] \times [A^r, A^s])} \arrow[d, "\mathbb{1}_{[A^n,A^m]} \times +_{A^l,A^k,A^r,A^s}"] \\
{[A^{n+l}, A^{m+k}] \times [A^r, A^s]} \arrow[dr, "+_{A^{n+l},A^{m+k},A^r,A^s}"] & & {[A^n, A^m] \times [A^{l+r}, A^{k+s}]} \arrow[dl, "+_{A^n,A^m,A^{l+r},A^{k+s}}"'] \\
& {[A^{n+l+r}, A^{m+k+s}]} &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{T}(\mathbf{1}) \times [A^n, A^m] \arrow[r, "O^{GTaut(A)} \times \mathbb{1}_{[A^n,A^m]}"] \arrow[dr, "\lambda^{\mathbf{Col}_\times}_{[A^n,A^m]}"'] & {[\mathcal{T}(\mathbf{1}), \mathcal{T}(\mathbf{1})] \times [A^n, A^m] \cong [A^n, A^m] \times [\mathcal{T}(\mathbf{1}), \mathcal{T}(\mathbf{1})]} \arrow[d, "+_{\mathbf{1},\mathbf{1},A^n,A^m}"', "+_{A^n,A^m,\mathbf{1},\mathbf{1}}"] & {[A^n, A^m] \times \mathcal{T}(\mathbf{1})} \arrow[l, "\mathbb{1}_{[A^n,A^m]} \times O^{GTaut(A)}"'] \arrow[dl, "\rho^{\mathbf{Col}_\times}_{[A^n,A^m]}"] \\
& {[A^n, A^m]} &
\end{tikzcd}$$

Note that the final diagram maps $\mathcal{T}(\mathbf{1})$ to $[\mathcal{T}(\mathbf{1}), \mathcal{T}(\mathbf{1})]$ rather than to $[A^0, A^0]$. This is because $a^0 : A^0 \to \mathcal{T}(\mathbf{1})$ is the empty cartesian product and is hence the terminal collection $\mathbb{1} : \mathcal{T}(\mathbf{1}) \to \mathcal{T}(\mathbf{1})$. But moreover, the collection $[\mathcal{T}(\mathbf{1}), \mathcal{T}(\mathbf{1})]$ is also $\mathbb{1}$. To see this, consider all possible collection homomorphisms from $a\square\mathbb{1}$ to $\mathbb{1}$. Because $\mathbb{1}$ is terminal, there is only one. And since the functor $-\square\mathbb{1}$ is adjoint to $[\mathbb{1}, -]$, the natural isomorphism of homsets implies that there is a single unique map from A to $[\mathbb{1}, \mathbb{1}]$ (i.e. $[\mathcal{T}(\mathbf{1}), \mathcal{T}(\mathbf{1})]$). Hence $[\mathcal{T}(\mathbf{1}), \mathcal{T}(\mathbf{1})]$ is the terminal collection $\mathbb{1}$.

In showing the commutativity of these diagrams we will often suppress associators by MacLane's coherence theorem. For the first diagram, the one asserting associativity of composition in $GTaut(A)$, we consider the diagram in Figure 1 whose boundary is obtained by currying the boundary of the original diagram. The commutativity of this second diagram then implies the commutativity of the original. The commutativity of the top leftmost square follows by the functoriality of $\square$. The commutativity of the middle, bottom left, and top right squares follows from the fact that composing and then evaluating is equivalent by definition to two consecutive evaluations. Finally, the bottom right square commutes trivially. Therefore composition in $GTaut(A)$ is associative.

$$
\begin{tikzcd}[column sep=small]
 & & {[A^l, A^k]\square[A^m, A^l]\square[A^n, A^m]\square A^n} \arrow[dll, "\circ_{A^m,A^l,A^k}\square\mathbb{1}_{[A^n,A^m]}\square\mathbb{1}_{A^n}"'] \arrow[ddl, "\mathbb{1}_{[A^l,A^k]}\square\mathbb{1}_{[A^m,A^l]}\square\epsilon^{A^m}_{A^l}"] \arrow[ddr, "\mathbb{1}_{[A^l,A^k]}\square\circ_{A^n,A^m,A^l}\mathbb{1}_{A^n}"] \arrow[drr, "\mathbb{1}_{[A^l,A^k]}\square\mathbb{1}_{[A^m,A^l]}\square\epsilon^{A^n}_{A^m}"] & & \\
{[A^m, A^k]\square[A^n, A^m]\square A^n} \arrow[dd, "\mathbb{1}_{[A^m,A^k]}\square\epsilon^{A^n}_{A^m}"'] & & & & {[A^l, A^k]\square[A^m, A^l]\square A^m} \arrow[dd, "\mathbb{1}_{[A^l,A^k]}\square\epsilon^{A^m}_{A^l}"] \\
 & {[A^l, A^k]\square[A^m, A^l]\square A^m} \arrow[dl, "\circ_{A^m,A^l,A^k}\square\mathbb{1}_{A^m}"'] \arrow[dr, "\mathbb{1}_{[A^l,A^k]}\square\epsilon^{A^m}_{A^l}"] & & {[A^l, A^k]\square[A^n, A^l]\square A^n} \arrow[dl, "\mathbb{1}_{[A^l,A^k]}\square\epsilon^{A^n}_{A^l}"'] \arrow[dr, "\mathbb{1}_{[A^l,A^k]}\square\epsilon^{A^n}_{A^l}"] & \\
{[A^m, A^k]\square A^m} \arrow[drr, "\epsilon^{A^m}_{A^k}"] & & {[A^l, A^k]\square A^l} \arrow[d, "\epsilon^{A^l}_{A^k}"'] & & {[A^l, A^k]\square A^l} \arrow[dll, "\epsilon^{A^l}_{A^k}"'] \\
 & & A^k & &
\end{tikzcd}
$$

Figure 1

We then consider the following two diagrams

$$\begin{tikzcd}
 & {[A^m,A^m]\Box[A^n,A^m]\Box A^n} \arrow[rr, "\mathbb{1}_{[A^m,A^m]}\Box\epsilon^{A^n}_{A^m}"] & & {[A^m,A^m]\Box A^m} \arrow[dr, "\epsilon^{A^m}_{A^m}"] & \\
\mathbf{1}\Box[A^n,A^m]\Box A^n \arrow[ur, "j_{A^m}\Box\mathbb{1}_{[A^n,A^m]}\Box\mathbb{1}_{A^n}"] \arrow[drr, "\mathbb{1}_{\mathbf{1}}\Box\epsilon^{A^n}_{A^m}"'] & & & & A^m \\
 & & \mathbf{1}\Box A^m \arrow[uur, "j_{A^m}\Box\mathbb{1}_{A^m}"] \arrow[urr, "\lambda^{\mathbf{Col}_\Box}_{A^m}"'] & &
\end{tikzcd}$$

$$\begin{tikzcd}
{[A^n,A^m]\Box(\mathbf{1}\Box A^n)} \arrow[r, "\mathbb{1}_{[A^n,A^m]}\Box j_{A^n}\Box\mathbb{1}_{A^n}"] \arrow[d, "\overline{\alpha}^{\mathbf{Col}_\Box}_{[A^n,A^m],\mathbf{1},A^n}"'] \arrow[dr, "\mathbb{1}_{[A^n,A^m]}\Box\lambda^{\mathbf{Col}_\Box}_{A^n}"] & {[A^n,A^m]\Box([A^n,A^n]\Box A^n)} \arrow[r, "\mathbb{1}_{[A^n,A^m]}\Box\epsilon^{A^n}_{A^n}"] & {[A^n,A^m]\Box A^n} \arrow[d, "\epsilon^{A^n}_{A^m}"] \\
{([A^n,A^m]\Box\mathbf{1})\Box A^n} \arrow[r, "\rho^{\mathbf{Col}_\Box}_{[A^n,A^m]}\Box\mathbb{1}_{A^n}"] & {[A^n,A^m]\Box A^n} \arrow[r, "\epsilon^{A^n}_{A^m}"] \arrow[ur, "="] & A^m
\end{tikzcd}$$

whose boundaries are obtained by currying the boundaries of the left and right unit axiom diagrams, respectively, for $GTaut(A)$ to be a **Col**-cat. Note then that the left square in the first curried diagram commutes by the naturality of ϵ while the right triangle commutes due to the fact that j_X was defined to be the currying of $\lambda^{\mathbf{Col}_\Box}_X$, the X component of the left unitor from **Col** with respect to the product $\Box$. For the second diagram we have that the leftmost triangle commutes as an instance of the triangle coherence condition with respect to the monoidal product $\Box$ in **Col**. The middle triangle commutes by the definition of j, just as we saw for the rightmost triangle in the previous diagram. The final triangle in second diagram commutes trivially. Thus we have that composition in $GTaut(A)$ is also unital with respect to the same unitors in **Col**.

We then consider the diagram in Figure 2 whose boundary is obtained by currying the diagram asserting that the product $+$ respects composition in $GTaut(A)$. The top left square commutes by the functoriality of the $\Box$ product. The square to the right of this functoriality square commutes by the adjunction used to define $+$. The square to the right of these first two commutative squares also commutes by the functoriality of $\Box$. The bottom left square commutes by the definition of ϵ implying that composition followed by evaluation is the same as double evaluation. The square to its right commutes by the adjunction used to define $+$. The top right square commutes from the fact that the product $\boxtimes$ involves a projection. Hence the two sides of this square must commute as they differ only in the order in which those projections occur. The square below and to the left as well as the square below and to the right of the previous square commute by the naturality of the $\boxtimes$ product. The bottom left square commutes by the fact that composition followed by evaluation is the same as double evaluation.

$$\begin{tikzcd}
 & & & ([A^m,A^r]\times[A^k,A^s])\square([A^n,A^m]\times[A^l,A^k])\square(A^n\times A^l) \arrow[dlll, "+_{A^m,A^r,A^k,A^s}\square+_{A^n,A^l,A^m,A^k}\square\mathbb{1}_{A^{n+l}}"'] \arrow[ddl, "\mathbb{1}_{[A^m,A^r]\times[A^k,A^s]}\square+_{A^n,A^m,A^l,A^k}\square\mathbb{1}_{A^{n+l}}"'] \arrow[d, "\mathbb{1}_{[A^m,A^r]\times[A^k,A^s]}\square\boxtimes_{[A^n,A^m],[A^l,A^k],A^n,A^l}"] \arrow[drr, "\boxtimes_{[A^m,A^r],[A^k,A^s],[A^n,A^m],[A^l,A^k]}\square\mathbb{1}_{A^n\times A^l}"] & & \\
{[A^{m+k},A^{r+s}]\square[A^{n+l},A^{m+k}]\square A^{n+l}} \arrow[ddd, "\circ_{A^{n+l},A^{m+k},A^{r+s}}\mathbb{1}_{A^{n+l}}"'] \arrow[ddr, "\mathbb{1}_{[A^{m+k},A^{r+s}]}\square\epsilon^{A^{n+l}}_{A^{m+k}}"] & & & ([A^m,A^r]\times[A^k,A^s])\square[([A^n,A^m]\square A^n)\times([A^l,A^k]\square A^l)] \arrow[dd, "\mathbb{1}_{[A^m,A^r]\times[A^k,A^s]}\square\epsilon^{A^n}_{A^m}\times\epsilon^{A^l}_{A^k}"] \arrow[dr, "\boxtimes_{[A^m,A^r],[A^k,A^s],[A^n,A^m]\square A^n,[A^l,A^k]\square A^l}"] & & {[([A^m,A^r]\square[A^n,A^m])\times([A^k,A^s]\square[A^l,A^k])]\square(A^n\times A^l)} \arrow[dl, "\boxtimes_{[A^m,A^r]\square[A^n,A^m],[A^k,A^s]\square[A^l,A^k],A^n,A^l)}"'] \arrow[dd, "(\circ_{A^n,A^m,A^r}\times\circ_{A^l,A^k,A^s})\square(\mathbb{1}_{A^n}\times\mathbb{1}_{A^l})"] \\
 & & ([A^m,A^r]\times[A^k,A^s])\square[A^{n+l},A^{m+k}]\square A^{n+l} \arrow[ull, "+_{A^m,A^r,A^k,A^s}\square\mathbb{1}_{[A^{n+l},A^{m+k}]}\square\mathbb{1}_{A^{n+l}}"'] \arrow[dr, "\mathbb{1}_{[A^m,A^r]\times[A^k,A^s]}\square\epsilon^{A^{n+l}}_{A^{m+k}}"] & & ([A^m,A^r]\square[A^n,A^m]\square A^n)\times([A^k,A^s]\square[A^l,A^k]\square A^l) \arrow[ddl, "(\mathbb{1}_{[A^m,A^r]}\square\epsilon^{A^n}_{A^m})\times(\mathbb{1}_{[A^k,A^s]}\square\epsilon^{A^l}_{A^k})"] \arrow[ddr, "(\circ_{A^n,A^m,A^r}\square\mathbb{1}_{A^n})\times(\circ_{A^l,A^k,A^s}\square\mathbb{1}_{A^l})"'] & \\
 & {[A^{m+k},A^{r+s}]\square A^{m+k}} \arrow[ddrr, "\epsilon^{A^{m+k}}_{A^{r+s}}"] & & ([A^m,A^r]\times[A^k,A^s])\square(A^m\times A^k) \arrow[ll, "+_{A^m,A^r,A^k,A^s}\square\mathbb{1}_{A^{m+k}}"'] \arrow[d, "\boxtimes_{[A^m,A^r],[A^k,A^s],A^m,A^k}"'] & & ([A^n,A^r]\times[A^l,A^s])\square(A^n\times A^l) \arrow[d, "\boxtimes_{[A^n,A^r],[A^l,A^s],A^n,A^l}"] \\
{[A^{n+l},A^{r+s}]\square A^{n+l}} \arrow[drrr, "\epsilon^{A^{n+l}}_{A^{r+s}}"'] & & & ([A^m,A^r]\square A^m)\times([A^k,A^s]\square A^k) \arrow[d, "\epsilon^{A^m}_{A^r}\times\epsilon^{A^k}_{A^s}"] & & ([A^n,A^r]\square A^n)\times([A^l,A^s]\square A^l) \arrow[dll, "\epsilon^{A^n}_{A^r}\times\epsilon^{A^l}_{A^s}"] \\
 & & & A^{r+s}=A^r\times A^s & &
\end{tikzcd}$$

Figure 2

We then have the diagram

$$\begin{tikzcd}
 & ([A^n, A^n] \times [A^m, A^m]) \square (A^n \times A^m) \arrow[rr, "\boxtimes_{[A^n,A^n],[A^m,A^m],A^n,A^m}"] & & ([A^n, A^n] \square A^n) \times ([A^m, A^m] \square A^m) \arrow[dr, "\epsilon^{A^n}_{A^n} \times \epsilon^{A^m}_{A^m}"] & \\
(\mathbf{1} \times \mathbf{1}) \square (A^n \times A^m) \arrow[ur, "(j_{A^n} \times j_{A^m}) \square (\mathbb{1}_{A^n} \times \mathbb{1}_{A^m})"] \arrow[rr, "\boxtimes_{\mathbf{1},\mathbf{1},A^n,A^m}"] \arrow[drr, "p_1 \square \mathbb{1}_{A^n \times A^m}"'] & & (\mathbf{1} \square A^n) \times (\mathbf{1} \square A^m) \arrow[ur, "(j_{A^n} \square \mathbb{1}_{A^n}) \times (j_{A^m} \square \mathbb{1}_{A^m})"] \arrow[rr, "\lambda^{\mathbf{Col}_\square}_{A^n} \times \lambda^{\mathbf{Col}_\square}_{A^m}"] & & A^n \times A^m \\
 & & \mathbf{1} \square (A^n \times A^m) \arrow[urr, "\lambda^{\mathbf{Col}_\square}_{A^n \times A^m}"'] & &
\end{tikzcd}$$

whose boundary comes from the currying of the diagram which asserts that + preserves identities. For this diagram, the top left square commutes by the naturality of the interchange morphism $\boxtimes$. The top right triangle commutes by the definition of ϵ. The bottom square commutes as it is the inverse of the unit coherence diagram for $\overline{\lambda}$ following from the duoidal structure for **Col**. Note that although the coherence condition in the definition of a duoidal category is presented with respect to $\overline{\lambda}$ and the morphism $\delta : I \odot I \to I$, the inverse diagram shown here also follows from the fact λ is an isomorphism and the collections $\mathbf{1} \times \mathbf{1} \to \mathcal{T}(\mathbf{1})$ and $\mathbf{1} \hookrightarrow \mathcal{T}(\mathbf{1})$ are isomorphic. Hence this square, and therefore the outer diagram, must commute.

Next we consider the diagram in Figure 3 whose boundary is obtained by currying the associativity diagram required of $GTaut(A)$ to be a monoid object in **ColCat**. The top pentagon commutes from the fact that $\boxtimes$ is defined via a projection and hence the order in which we project does not change the result. In the bottom square, the top triangle commutes by the definition of the associator. The remaining three triangles in this square commute by the functoriality of the cartesian product.

We next consider the diagrams in Figures 4 and 5 whose boundaries are obtained by currying the left and right unit diagrams required of $GTaut(A)$ to be a monoid object in **ColCat**, respectively.

We first consider Figure 4. The leftmost square commutes by the functoriality of $\square$. We shall now, following clockwise from the top of the diagram, check the commutativity of the five regions incident with this leftmost square. The first square commutes by the functoriality of $\square$. The pentagon commutes by the definition of +. The next square commutes by the naturality of $\boxtimes$. The square incident only on an edge is an instance of the sixth commutativity axiom for **Col** to be a duoidal category. The bottom adjacent triangle commutes by the definition of $\overline{\rho}^{\mathbf{Col}_\times}$ and functoriality of $\square$. We shall now look at the right end of the diagram. Starting with the top right-most triangle, we see that this region commutes by the functoriality of the internal hom $[-,-]$. The adjacent square to its right commutes by the naturality of ϵ. The adjacent square directly below this one, along the bottom of the diagram (we shall look at the square to its left last), commutes by the naturality of ρ. The square to the left of this one, which shares an edge with it, commutes by the functoriality of $\times$. The adjacent square above this one also commutes by the functoriality of $\times$. The triangle to the right of this one commutes by the definition of $\mathcal{O}$ and functoriality of $\times$. The final region, the square which was previously skipped, commutes by the fact that the two composites which bound it are two factorizations of

$$
\begin{tikzcd}
 & ([A^n, A^m] \times [A^l, A^k] \times [A^r, A^s])\square(A^n \times A^l \times A^r) \arrow[dl, "\boxtimes_{[A^n,A^m]\times[A^l,A^k],[A^r,A^s],A^n\times A^l,A^r}"] \arrow[dr, "\boxtimes_{[A^n,A^m],[A^l,A^k]\times[A^r,A^s],A^n,A^l\times A^r}"] & \\
(([A^n, A^m] \times [A^l, A^k])\square(A^n \times A^l)) \times ([A^r, A^s]\square A^r) \arrow[d, "\boxtimes_{[A^n,A^m],[A^l,A^k],A^n,A^l} \times \mathbb{1}_{[A^r,A^s]\square A^r}"] & & ([A^n, A^m]\square A^n) \times (([A^l, A^k] \times [A^r, A^s])\square(A^l \times A^r)) \arrow[d, "\mathbb{1}_{[A^n,A^m]\square A^n} \times \boxtimes_{[A^l,A^k],[A^r,A^s],A^l,A^r}"'] \\
(([A^n, A^m]\square A^n) \times ([A^l, A^k]\square A^l)) \times ([A^r, A^s]\square A^r) \arrow[rr, "\alpha^{\mathbf{Col}_\times}_{[A^n,A^m]\square A^n,[A^l,A^k]\square A^l,[A^r,A^s]\square A^r}"] \arrow[dr, "(\mathbb{1}_{[A^n,A^m]\square A^n} \times \epsilon^{A^l}_{A^k}) \times \mathbb{1}_{[A^r,A^s]\square A^r}"] \arrow[dd, "(\epsilon^{A^n}_{A^m} \times \epsilon^{A^l}_{A^k}) \times \mathbb{1}_{[A^r,A^s]\square A^r}"] & & ([A^n, A^m]\square A^n) \times (([A^l, A^k]\square A^l) \times ([A^r, A^s]\square A^r)) \arrow[dl, "\mathbb{1}_{[A^n,A^m]\square A^n} \times (\epsilon^{A^l}_{A^k} \times \mathbb{1}_{[A^r,A^s]\square A^r})"'] \arrow[dd, "\mathbb{1}_{[A^n,A^m]\square A^n} \times (\epsilon^{A^l}_{A^k} \times \epsilon^{A^r}_{A^s})"'] \\
 & ([A^n, A^m]\square A^n) \times A^k \times ([A^r, A^s]\square A^r) \arrow[dl, "\epsilon^{A^n}_{A^m} \times \mathbb{1}_{A^k} \times \mathbb{1}_{[A^r,A^s]\square A^r}"'] \arrow[dr, "\mathbb{1}_{[A^n,A^m]\square A^n} \times \mathbb{1}_{A^k} \times \epsilon^{A^r}_{A^s}"] & \\
A^m \times A^k \times ([A^r, A^s]\square A^r) \arrow[r, "\mathbb{1}_{A^m} \times \mathbb{1}_{A^k} \times \epsilon^{A^r}_{A^s}"] & A^m \times A^k \times A^s & ([A^n, A^m]\square A^n) \times A^k \times A^s \arrow[l, "\epsilon^{A^n}_{A^m} \times \mathbb{1}_{A^k} \times \mathbb{1}_{A^s}"']
\end{tikzcd}
$$

Figure 3

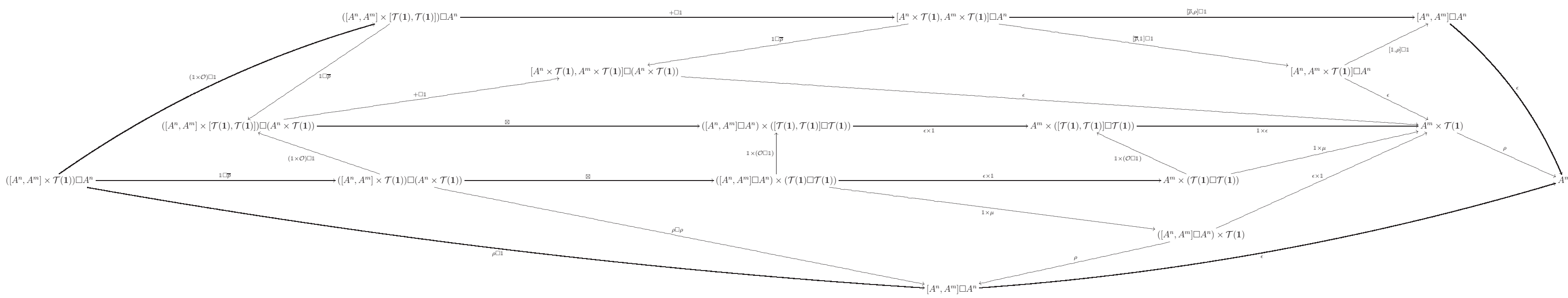

Figure 4

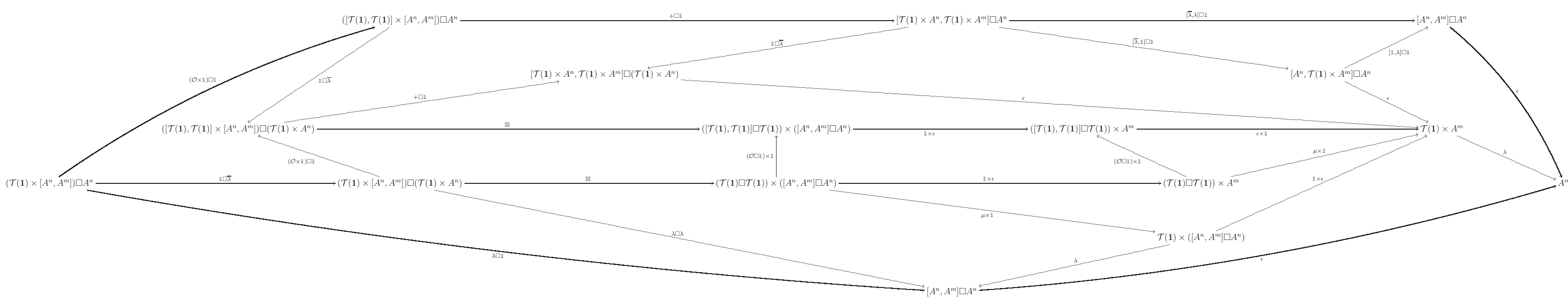

Figure 5

the currying of the following map:

$$[\rho^{\mathbf{Col}_\times}_{A^n}, \mathbb{1}_{A^m \times \mathcal{T}(\mathbf{1})}] : [A^n \times \mathcal{T}(\mathbf{1}), A^m \times \mathcal{T}(\mathbf{1})] \to [A^n, A^m \times \mathcal{T}(\mathbf{1})]$$

We now consider Figure 5. The explanations of why each of these regions commutes are completely analogous to those for Figure 4. The only essential differences are either that the content of certain maps lies in a different cartesian factor (i.e. on the left side of an identity map rather than the right) or that some regions are given in terms of the left unitor transformation instead of the right.

We can hence conclude that the tautological globular PRO is in fact a globular PRO.

16. Algebras for a Globular PRO

Just as in the classical case, we can define algebras for a globular PRO, without use of $GTaut(A)$, via a sequence of action maps as follows.

16.1. Definition. *An algebra for a globular PRO $\mathcal{P}$ is given by a degenerate collection $a : A \to \mathcal{T}(\mathbf{1})$ together with, for all $n, m \in \mathbb{N}$, a series of collection homomorphisms $\Omega_{n,m} : \mathcal{P}(n,m)\Box A^n \to A^m$ which each make the following diagrams commute.*

$$\begin{tikzcd}
{[\mathcal{P}(m,r)\Box\mathcal{P}(n,m)]\Box A^n} \arrow[r, "\alpha^{\mathbf{Col}_\Box}_{\mathcal{P}(m,r),\mathcal{P}(n,m),A^n}"] \arrow[d, "\circ_{n,m,r}\Box\mathbb{1}_{A^n}"'] & \mathcal{P}(m,r)\Box[\mathcal{P}(n,m)\Box A^n] \arrow[r, "\mathbb{1}_{\mathcal{P}(m,r)}\Box\Omega_{n,m}"] & \mathcal{P}(m,r)\Box A^m \arrow[d, "\Omega_{m,r}"] \\
\mathcal{P}(n,r)\Box A^n \arrow[rr, "\Omega_{n,r}"'] & & A^r
\end{tikzcd}$$

$$\begin{tikzcd}
{[\mathcal{P}(n,m)\times\mathcal{P}(r,s)]\Box[A^n\times A^r]} \arrow[r, "+_{n,m,r,s}\times\mathbb{1}_{n+r}"] \arrow[d, "\boxtimes_{\mathcal{P}(n,m),\mathcal{P}(r,s),A^n,A^r}"'] & \mathcal{P}(n+r,m+s)\Box A^{n+r} \arrow[d, "\Omega_{n+r,m+s}"] \\
{[\mathcal{P}(n,m)\Box A^n]\times[\mathcal{P}(r,s)\Box A^r]} \arrow[r, "\Omega_{n,m}\times\Omega_{r,s}"'] & A^m\times A^s = A^{m+s}
\end{tikzcd}$$

$$\begin{tikzcd}
& \mathcal{P}(n,n)\Box A^n \arrow[dr, "\Omega_{n,n}"] & \\
\mathbf{1}\Box A^n \arrow[ur, "j_n\Box\mathbb{1}_{A^n}"] \arrow[rr, "\lambda^{\mathbf{Col}_\Box}_{A^n}"'] & & A^n
\end{tikzcd}$$

We can again express the previous notion of algebras instead as representations of our PRO via currying.

16.2. Definition. *A P-module for a globular PRO $\mathcal{P}$ is a globular PRO homomorphism $f : \mathcal{P} \to GTaut(A)$ for some collection $a : A \to \mathcal{T}(\mathbf{1})$. An algebra is a $\mathcal{P}$-module such that the collection $a : A \to \mathcal{T}(\mathbf{1})$ is degenerate. In other words, the arity map factors as $a = [id] \circ !_A$.*

We once more get an immediate result, here in the context of globular PROs, regarding induced algebras.

16.3. Theorem. *An algebra for a globular PRO $\mathcal{P}$ is an algebra for every globular PRO $\mathcal{Q}$ which maps to $\mathcal{P}$. In particular, an algebra for $\mathcal{P}$ is an algebra for every globular sub-PRO.*

17. The Free Monoidal and Path Category PROs on a **Col**-graph

Every category has an underlying graph. It is obtained by forgetting the composition and identity structure. Analogously, for every enriched category there is an underlying enriched graph which is obtained by the same process. In general, these special graphs are defined as follows.

17.1. Definition. *Given a duoidal category $\mathcal{D}$, a $\mathcal{D}$-graph $\mathbf{G} = (V, E)$ consists of a set of objects V, the elements of which are called vertices, and a family of objects E, which we shall call edge objects, consisting of, for all $X, Y \in V$, an object $G(X, Y)$ in $\mathcal{D}$.*

17.2. Definition. *A $\mathcal{D}$-graph morphism $H : \mathcal{A} \to \mathcal{B}$ consists of a function $H : Obj(\mathcal{A}) \to Obj(\mathcal{B})$ together with a family of morphisms $\{H_{X,Y} : A(X, Y) \to B(H(X), H(Y))\}$ from $\mathcal{D}$ with $X, Y \in Obj(\mathcal{A})$.*

Let $\mathbb{N}\mathcal{D}\mathbf{Graph}$ be the full subcategory of $\mathcal{D}\mathbf{Graph}$ consisting of the $\mathcal{D}$-graphs whose object set is $\mathbb{N}$. If $\mathcal{D}$ has all countable coproducts, a $\mathcal{D}$-graph $\mathbf{G} = (V, E)$ in $\mathbb{N}\mathcal{D}\mathbf{Graph}$ can be seen as a bi-graded object in $\mathcal{D}$ since every object $G(n, m) \in E$ is indexed by a pair of natural numbers. This fact induces a bi-grading on E. But since $\mathcal{D}$ has all countable coproducts, the coproduct over the objects of V gives a single object in $\mathcal{D}$ doubly graded over $\mathbb{N}$. This allows us to canonically identify $\mathbb{N}\mathcal{D}\mathbf{Graph}$ with the category consisting of objects in $\mathcal{D}$ equipped with a bi-grading over $\mathbb{N}$ together with the maps which preserve the bi-grading. We shall here denote this category $\mathbf{BiGrd}\mathcal{D}$. Note that, given a pair of objects $X, Y \in obj(\mathcal{D})$, these are precisely the morphisms $f : X \to Y$ which may be written as a two parameter family of $\mathcal{D}$-morphism $\{f_{i,j} : X(i, j) \to Y(i, j)\}$.

The category $\mathbf{BiGrd}\mathcal{D}$ has a natural monoidal structure given by the functor

$$\oplus : \mathbf{BiGrd}\mathcal{D} \times \mathbf{BiGrd}\mathcal{D} \to \mathbf{BiGrd}\mathcal{D}$$

which maps a pair of bi-graded objects X and Y from $\mathcal{D}$ to the object $X \oplus Y$, which has the following induced grading

$$(X \oplus Y)(n, m) := \coprod_{\substack{n=i+j \\ m=l+k}} X(i, l) \odot Y(j, k)$$

where $\odot$ is the second monoidal product in the duoidal category $\mathcal{D}$. Given two morphisms $f = \{f_{i,j} : X(i,j) \to Z(i,j)\}$ and $g = \{g_{l,k} : Y(l,k) \to W(l,k)\}$ in $\mathbf{BiGrd}\mathcal{D}$ we get

$$f \oplus g = \{(f \oplus g)_{n,m} : (X \oplus Y)(l,k) \to (Z \oplus W)(l,k)\}$$

with components given by

$$(f \oplus g)_{n,m} := \coprod_{\substack{n=i+j \\ m=l+k}} f_{i,l} \odot g_{j,k}$$

for each $n, m \in \mathbb{N}$. The monoidal unit for the product $\oplus$ is the object $\mathfrak{I}$ which is given as $\mathfrak{I}(0,0) = U$, where U is the monoidal unit for the second monoidal structure $\odot$ on $\mathcal{D}$, and $\mathfrak{I}(i,j) = E$, where E is the initial object defined by the empty coproduct in $\mathcal{D}$, for all other $i, j \in \mathbb{N}$. Note that E must exist by the requirement that $\mathcal{D}$ have all countable coproducts. Moreover, the coproduct structure in $\mathcal{D}$ induces a coproduct $X \coprod Y$ in $\mathbf{BiGrd}\mathcal{D}$. It is defined to be the identity on objects and has the coproduct in $\mathcal{D}$ of edge objects $X(i,j) \coprod Y(i,j)$ as its edge object $(X \coprod Y)(i,j)$.

17.3. Definition. *A monoidal $\mathbb{N}\mathcal{D}$-graph $(\mathbf{M}, \diamond, \iota)$ is a monoid in the category $\mathbf{BiGrd}\mathcal{D}$, where $\mathbf{M} \in Obj(\mathbf{BiGrd}\mathcal{D})$, the bi-graded $\mathcal{D}$-morphism $\diamond : \mathbf{M} \oplus \mathbf{M} \to \mathbf{M}$ is the monoidal product, and $\iota : \mathfrak{I} \to \mathbf{M}$ is the unit bi-graded $\mathbf{D}$-morphism such that $\diamond$ is associative and unital with respect to $\oplus$.*

Now that we have a notion of monoidal $\mathcal{D}$-graph, it's then natural to ask the following: given a $\mathcal{D}$-graph $\mathbf{G} \in \mathbb{N}\mathcal{D}\mathbf{Graph}$, can we construct a free monoidal $\mathcal{D}$-graph $M(\mathbf{G})$ on $\mathbf{G}$? Fortunately we can.

17.4. Definition. *Given a $\mathcal{D}$-graph $\mathbf{G} \in \mathbb{N}\mathcal{D}\mathbf{Graph}$, the free monoidal $\mathbb{N}\mathcal{D}$-graph on $\mathbf{G}$ is the $\mathcal{D}$-graph*

$$M(\mathbf{G}) := \coprod_{n \in \mathbb{N}} \bigoplus_{k=1}^{n} \mathbf{G}$$

where both $\mathbf{G}$ and $M(\mathbf{G})$ are thought of as objects in $\mathbf{BiGrd}\mathcal{D}$. The monoidal product for $M(\mathbf{G})$ is given by the canonical functor $\oplus : M(\mathbf{G}) \oplus M(\mathbf{G}) \to M(\mathbf{G})$ which is closed by construction. Note that when $n = 0$ the product $\bigoplus_{k=1}^{n} \mathbf{G}$ is the monoidal unit in $\mathbf{BiGrd}\mathcal{D}$. Hence, the unit morphism $\iota_{M(\mathbf{G})} : \mathfrak{I} \to M(\mathbf{G})$ is the canonical functor which sends the only non-empty summand of $\mathfrak{I}$, $\mathfrak{I}(0,0) = U$, identically to the only non-empty summand of the empty $\oplus$-product $M(\mathbf{G})_0(0,0) = U$. Moreover, $M : \mathbb{N}\mathcal{D}\mathbf{Graph} \to \mathbf{Mon}\mathbb{N}\mathcal{D}\mathbf{Graph}$ gives a functor by sending a given bi-graded $\mathcal{D}$-morphism $f = \{f_{i,j} : X(i,j) \to Y(i,j)\}$ to the morphism

$$M(f) = \{M(f)_{i,j} : X(i,j) \to Y(i,j)\}$$

whose components are given by

$$M(f)_{i,j} = \coprod_{n \in \mathbb{N}} \bigoplus_{k=0}^{n} f_{i,j}$$

for $i, j \in \mathbb{N}$.

It is furthermore clear that the functor $M : \mathbb{N}\mathcal{D}\mathbf{Graph} \to \mathbf{Mon}\mathbb{N}\mathcal{D}\mathbf{Graph}$ has a right adjoint $W : \mathbf{Mon}\mathbb{N}\mathcal{D}\mathbf{Graph} \to \mathbb{N}\mathcal{D}\mathbf{Graph}$ that forgets the monoidal product and unit morphisms with which our $\mathcal{D}$-graph $\mathbf{G}$ is equipped. In the special case where $\mathcal{D}$ is the category $\mathbf{Col}$, we have the following result.

17.5. Theorem. *The functor* $\mathcal{W} : \mathbf{Mon}\mathbb{N}\mathbf{ColGraph} \to \mathbb{N}\mathbf{ColGraph}$ *which forgets both the monoidal product and unit structures for a given monoidal* $\mathbb{N}\mathbf{Col}$*-graph is finitary and monadic over* $\mathbb{N}\mathbf{ColGraph}$.

Proof. It is immediately clear from construction that the functor $\mathcal{M} : \mathbb{N}\mathbf{ColGraph} \to \mathbf{Mon}\mathbb{N}\mathbf{ColGraph}$ is left adjoint to the forgetful functor $\mathcal{W}$. It is furthermore clear from construction that $\mathcal{M}(\mathbb{N}\mathbf{ColGraph})$ is the category of algebras for the monad $\mathcal{W}(\mathcal{M})$. Here $\mathcal{M}$ is precisely the free functor dual to the structure forgotten by $\mathcal{W}$. Hence the comparison functor $K^{\mathcal{W}(\mathcal{M})} : \mathbf{Mon}\mathbb{N}\mathbf{ColGraph} \to (\mathbb{N}\mathbf{ColGraph})^{\mathcal{W}(\mathcal{M})}$ is an equivalence of categories. It remains then to show that $\mathcal{W}$ preserves filtered colimits and is hence finitary. But this is clear from the fact that $\mathcal{W}$ simply forgets the monoidal concatenation operation structure and that the special summand $M(\mathbf{G})(0,0) = E$ has unit structure with respect to this product. This implies that given a filtered diagram in $\mathbb{N}\mathbf{ColCat}$, any objects or morphisms that become equal in a colimit on that diagram were already made equal at some level in the filtered diagram. Moreover, given any filtered diagram in $\mathbb{N}\mathbf{ColCat}$ in which any new elements are generated, the components of that object already existed at some level in the diagram on which the colimit is taken. Hence the preservation of the $\mathbb{N}\mathbf{ColGraph}$ structure in the filtered diagram ensures the preservation of the structure in the colimit. And thus $\mathcal{W}$ preserves filtered colimits and is therefore finitary. ■

In a similar way we can both create and forget the category structure on a given $\mathcal{D}$-graph as well. We will here follow the construction as presented by Wolff[16]. First of all, the general process of forgetting the composition and identity structure for a generic $\mathcal{D}$-category to get a corresponding $\mathcal{D}$-graph gives a forgetful functor $U : \mathcal{D}\mathbf{Cat} \to \mathcal{D}\mathbf{Graph}$ which we shall use in the following definition.

17.6. Definition. *Given a* $\mathcal{D}$*-graph* $\boldsymbol{G}$ *and a* $\mathcal{D}$*-category* $\mathcal{C}$*, a* $\mathcal{D}$*-diagram of type* $\boldsymbol{G}$ *in* $\mathcal{C}$ *is a* $\mathcal{D}$*-graph morphism* $\varphi_{\boldsymbol{G}} : \boldsymbol{G} \to U(\mathcal{C})$ *to the underlying* $\mathcal{D}$*-graph of* $\mathcal{C}$.

We are specifically interested in $\mathbf{Col}$-graphs whose vertex set is the natural numbers. Given such a graph $\mathbf{G}$ we can construct the free globular PRO $P(\mathbf{G})$ on $\mathbf{G}$. But before describing this construction in detail, we first mention the following alternative free construction on a $\mathbf{Col}$-graph.

17.7. Definition. *Given a* $\mathbf{Col}$*-graph* $\mathbf{G}$*, the free* $\mathbf{Col}$*-category* $F(\mathbf{G})$ *generated by* $\mathbf{G}$ *is constructed as follows. First set* $Obj(F(\mathbf{G})) = Obj(\mathbf{G})$*. Then take* $X, Y \in Obj(\mathbf{G})$*. If* $X \neq Y$ *we define the hom-object*

$$F(G)(X,Y) := \coprod G(E_0, E_1)\Box G(E_1, E_2)\Box ...\Box G(E_{n-1}, E_n)$$

where the coproduct is taken over all finite sequences $(E_0 = X, E_1, E_2, ..., E_{n-1}, E_n = Y)$ *with* $E_i \in Obj(\mathbf{G})$ *for* $i \in \{0, 1, 2, ..., n-1, n\}$ *and* $n \geq 1$. *If* $X = Y$ *then we define the hom-object*

$$F(G)(X, X) := [\coprod G(X, E_1)\square G(E_1, E_2)\square ...\square G(E_{n-1}, X)] \coprod I$$

to account for the fact that this hom-object should have enough structure to include identities. The composition map

$$\circ_{X,Y,Z} : F(G)(Y, Z)\square F(G)(X, Y) \to F(G)(X, Z)$$

is then defined in each coproduct summand by concatenating (via the operation $\square$*) strings of hom-objects from the corresponding summands. More explicitly, if we have that both* $\tau_1 = (A, E_1, ..., E_{n-1}, B)$ *and* $\tau_2 = (B, D_1, ..., D_{n-1}, C)$ *are strings of objects, then if we define*

$$\tau_1 \bullet \tau_2 := (A, E_1, ..., E_{n-1}, B, D_1, ..., D_{n-1}, C)$$

and suppose that both $A \neq B$ *and* $B \neq C$*, then we can define* $\circ_{A,B,C}$ *to be the collection homomorphism which satisfies the equation*

$$\circ_{A,B,C}(\iota_{\tau_1}\square\iota_{\tau_2}) = \iota_{\tau_1\bullet\tau_2}(\alpha^k)$$

where α^k *is enough copies of the associator so that the source is completely left parenthesized and* ι_{τ_i} *is the canonical inclusion into the coproduct summand corresponding to the sequence* τ_i. *If* $A = B$ *then we define* $\circ_{A,B,C}$ *so that it satisfies*

$$\circ_{A,B,C}(\iota_{\tau_1}\square\iota_{\tau_2}) = \iota_{\tau_2}(\lambda_{\tau_2})$$

where λ_{τ_2} *is the* τ_2 *component of the left unitor from* **Col**. *If* $B = C$ *then* $\circ_{A,B,C}$ *is defined to satisfy*

$$\circ_{A,B,C}(\iota_{\tau_1}\square\iota_{\tau_2}) = \iota_{\tau_1}(\rho_{\tau_1})$$

with ρ_{τ_1} *being the* τ_1 *component of the right unitor from* **Col**. *And we define* $\circ_{A,B,C}$ *to satisfy*

$$\circ_{A,B,C}(\iota_{\tau_1}\square\iota_{\tau_2}) = \rho_I = \lambda_I$$

if $A = B = C$. *The identity identifications* $j_A : I \to F(G)(X, X)$ *are defined to be the canonical inclusion map into the* I *summand of the corresponding coproduct.*

This construction extends to maps of **Col**-graphs in the obvious way to give a functor F : **ColGraph** $\to$ **ColCat**. We can now generate free **Col**-categories by listing certain generating hom-objects at the graph level.

17.8. THEOREM. *The functor* $U : \mathbb{N}$**ColCat** $\to \mathbb{N}$**ColGraph** *which sends any* $\mathbb{N}$**ColCat** *to its underlying* $\mathbb{N}$**ColGraph** *is finitary and monadic over* $\mathbb{N}$**ColGraph**.

Proof. It is clear from construction both that the functor $F : \mathbb{N}\mathbf{ColGraph} \to \mathbb{N}\mathbf{ColCat}$ is left adjoint to $U : \mathbb{N}\mathbf{ColCat} \to \mathbb{N}\mathbf{ColGraph}$ and that $F(\mathbb{N}\mathbf{ColGraph})$ is precisely the category of algebras for the monad $U(F) : \mathbb{N}\mathbf{ColGraph} \to \mathbb{N}\mathbf{ColGraph}$. In other words, F is precisely the free functor dual to the structure forgotten by U. Hence the comparison functor $K^{U(F)} : \mathbb{N}\mathbf{ColCat} \to (\mathbb{N}\mathbf{ColGraph})^{U(F)}$ is an equivalence of categories. It remains then to show that U preserves filtered colimits and is hence finitary. But this is clear from the fact that U simply forgets the concatenation operation and that certain hom-objects have unit structures with respect to this concatenation. This implies that given a filtered diagram in $\mathbb{N}\mathbf{ColCat}$, any objects or morphisms that become equal in a colimit on that diagram were already made equal at some level in the filtered diagram. Moreover, given any filtered diagram in $\mathbb{N}\mathbf{ColCat}$ in which any new elements are generated, the components of that element already existed in some previous object at some level in the diagram over which the colimit is taken. Hence the preservation of the $\mathbb{N}\mathbf{ColGraph}$ structure in the filtered diagram ensures the preservation of the structure in the colimit. And thus U preserves filtered colimits and is therefore finitary. ■

18. The Globular PRO Monad

In Leinster's presentation of weak ω-categories in *Higher Operads, Higher Categories*[11], he proves the existence of an initial globular operad with contraction using a theorem of Kelly[8] which asserts that the strict pullback in $\mathbf{Cat}$ of two finitary and monadic functors, both of whose target is locally finitely presentable, is monadic. In his construction, the two finitary and monad functors are the underlying functors for the monads on $\mathbf{Col}$ which have as algebras collections with contraction and globular operads respectively. Hence, his pullback monad, which we shall denote $\mathfrak{O}$, has as algebras globular operads with contraction. Applying $\mathfrak{O}$ to the initial object $\{\} \in \mathbf{Col}$ constructs a collection $\mathfrak{O}(\{\})$ that, when thought of as an algebra for $\mathfrak{O}$ when equipped with the structure map $\mu^{\mathfrak{O}}_{\{\}} : \mathfrak{O}^2(\{\}) \to \mathfrak{O}(\{\})$ induced by the component at $\{\}$ of the multiplication transformation for $\mathfrak{O}$, is the initial free globular operad with contraction. Algebras for the operad $\mathfrak{O}(\{\})$ are then by construction weak ω-categories.

We shall eventually use this same trick to construct a globular PRO whose algebras are by construction weak ω-categorifications of a particular equational algebraic theory. We do not yet have the machinery to construct such a PRO. We do however have the machinery to construct a similar kind of monad whose algebras are globular PROs, given the following lemma.

18.1. Lemma. *The category* $\mathbb{N}\mathbf{ColGraph}$ *is locally finitely presentable.*

Proof. Recall that $\mathbb{N}\mathbf{ColGraph}$ is equivalent to the category $\mathbf{BiGrdCol}$. Moreover, we can think of each bi-graded collection as a countable product of ordinary collections over $\mathbb{N} \times \mathbb{N}$. Hence we can express $\mathbf{BiGrdCol}$ as

$$\mathbf{BiGrdCol} \cong \prod_{\mathbb{N}\times\mathbb{N}} \mathbf{Col} \cong \prod_{\mathbb{N}\times\mathbb{N}} {}^{[\mathbb{G}^{op},\mathbf{Set}]}/_{\mathcal{T}(\mathbf{1})} \cong \prod_{\mathbb{N}\times\mathbb{N}} \mathbf{Set}^{Elt(\mathcal{T}(\mathbf{1}))^{op}} \cong$$

$$\left(\mathbf{Set}^{Elt(\mathcal{T}(\mathbf{1}))^{op}}\right)^{\mathbb{N}\times\mathbb{N}} \cong \mathbf{Set}^{(Elt(\mathcal{T}(\mathbf{1}))^{op})\times(\mathbb{N}\times\mathbb{N})} \cong \mathbf{Set}^{(Elt(\mathcal{T}(\mathbf{1}))\times\mathbb{N}\times\mathbb{N})^{op}}$$

where $Elt(\mathcal{T}(\mathbf{1}))$ is the category of elements for the (covariant) presheaf functor $\mathcal{T}(\mathbf{1}) : \mathbb{G}^{op} \to \mathbf{Set}$. We are here using it in order to perform the standard construction for writing a slice presheaf category as a presheaf category. Also note that in the functor category $\left(\mathbf{Set}^{Elt(\mathcal{T}(\mathbf{1}))^{op}}\right)^{\mathbb{N}\times\mathbb{N}}$, the object $\mathbb{N}$ is being thought of as the descrete category with object set $\mathbb{N}$. This shows that **BiGrdCol**, and hence $\mathbb{N}$**ColGraph**, is a presheaf category. The conclusion then follows from the fact that presheaf categories are locally finitely presentable [3]. ∎

18.2. Theorem. *The category* **GlobPRO** *is monadic over* $\mathbb{N}$**ColGraph**.

Proof. Theorems 17.5 and 17.8 show that we have two finitary and monadic underlying functors into $\mathbb{N}$**ColGraph**. Lemma 18.1 allows us to conclude that their strict pullback in **Cat** is monadic via Kelly's theorem. ∎

We now have a monad on $\mathbb{N}$**ColGraph** whose algebras are globular PROs. Let us denote this monad as $\mathfrak{M} : \mathbb{N}\mathbf{ColGraph} \to \mathbb{N}\mathbf{ColGraph}$. Furthermore, applying this monad to the initial $\mathbb{N}$**Col**-graph constructs an $\mathbb{N}$**Col**-graph that, when viewed as an algebra for our pullback monad, is the initial free globular PRO. More generally, we get the following definition from this construction.

18.3. Definition. *The free globular PRO on a* $\mathbb{N}$**Col***-graph* **G** *is the algebra*

$$(\mathfrak{M}(\mathbf{G}), \mu^{\mathfrak{M}}_{\mathbf{G}} : \mathfrak{M}^2(\mathbf{G}) \to \mathfrak{M}(\mathbf{G}))$$

for the globular PRO monad $\mathfrak{M}$.

19. PRO Globularization

It is well know [9] that the functor which takes an enriched category to its underlying ordinary category has a left adjoint which generates the enriched structure. This is done by taking copowers of the monoidal unit from the category over which the enrichment is taking place. We will here perform a similar construction which constructs, from a classical PRO P, a globular PRO $\mathcal{P}$ whose algebras are precisely the algebras for P in **Glob** which have an ω-category structure with operations given by strict ω-functors. This is done by taking copowers not of the unit collection $I : \mathbf{1} \hookrightarrow \mathcal{T}(\mathbf{1})$, but rather the terminal collection $\mathbb{1} : \mathcal{T}(\mathbf{1}) \to \mathcal{T}(\mathbf{1})$. Let P be any ordinary set PRO and consider the following functor $G_P : P \to \mathcal{P}$ which maps P to its globularization.

$$n \mapsto n$$

$$P(n,m) \mapsto \mathcal{P}(n,m) := P(n,m)\cdot \mathbb{1} = \coprod_{P(n,m)} \mathbb{1}$$

Furthermore, the operations $\circ$ and $+$ are induced by the structure in P.

Note first that, for all $n,m,p \in \mathbb{N}$ the hom-object $\mathcal{P}(m,p)\Box\mathcal{P}(n,m)$ can be written

$$\mathcal{P}(m,p)\Box\mathcal{P}(n,m) = \left(\coprod_{P(m,p)} \mathbb{1}\right) \Box \left(\coprod_{P(n,m)} \mathbb{1}\right) = \coprod_{P(m,p)} \left(\mathbb{1}\Box \left(\coprod_{P(n,m)} \mathbb{1}\right)\right) =$$

$$\coprod_{P(m,p)} \left(\coprod_{P(n,m)} (\mathbb{1}\Box\mathbb{1})\right) \cong \coprod_{P(m,p)\times P(n,m)} (\mathbb{1}\Box\mathbb{1})$$

where the final isomorphism is simply a reindexing of the double coproduct by a single coproduct of pairs. The induced composition operations on $\mathcal{P}$, for all $n,m,p \in \mathbb{N}$ are then given by

$$\circ^{\mathcal{P}}_{n,m,p} := \circ^{P}_{n,m,p} \cdot \phi : (P(m,p) \times P(n,m)) \cdot (\mathbb{1}\Box\mathbb{1}) \to P(n,p) \cdot \mathbb{1}$$

where $\circ^{P}_{n,m,p}$ is composition in P and $\phi : \mathbb{1}\Box\mathbb{1} \to \mathbb{1}$ is the morphism in **Col** ensuring that $\mathbb{1}$ is a monoid object with respect to the product $\Box$.

We can similarly write $\mathcal{P}(n,m) \times \mathcal{P}(l,k)$

$$\mathcal{P}(n,m) \times \mathcal{P}(l,k) = \left(\coprod_{P(n,m)} \mathbb{1}\right) \times \left(\coprod_{P(l,k)} \mathbb{1}\right) = \coprod_{P(n,m)} \left(\mathbb{1} \times \left(\coprod_{P(l,k)} \mathbb{1}\right)\right) =$$

$$\coprod_{P(n,m)} \left(\coprod_{P(l,k)} (\mathbb{1} \times \mathbb{1})\right) \cong \coprod_{P(n,m)\times P(l,k)} (\mathbb{1} \times \mathbb{1})$$

for all $n,m,l,k \in \mathbb{N}$. The induced addition operations on $\mathcal{P}$, for all $n,m,l,k \in \mathbb{N}$ are then given by

$$+^{\mathcal{P}}_{n,m,l,k} := +^{P}_{n,m,l,k} \cdot \Phi : (P(n,m) \times P(l,k)) \cdot (\mathbb{1} \times \mathbb{1}) \to P(n+l, m+k) \cdot \mathbb{1}$$

where $+^{P}_{n,m,l,k}$ is addition in P and Φ is the canonical isomorphism which is described by the left (equivalently right) cartesian unitor.

The identity identifications $j_n : \mathbf{1} \to \mathcal{P}(n,n)$ are induced by the composition

$$I \xrightarrow{\xi_n} P(n,n) \cdot I \hookrightarrow P(n,n) \cdot \mathbb{1}$$

$$\sigma_n \mapsto (\iota(*), \sigma_n) \hookrightarrow (\iota_n(*), \sigma_n)$$

for each $n \in \mathbb{N}$, where $\iota_n : \{*\} \to P(n,n)$ is the identity identification from the underlying set PRO P.

19.1. Theorem. *The globularization $\mathcal{P}$ of a PRO P is a globular PRO.*

PROOF. It is clear from construction that $\mathcal{P}$ is a cartesian-duoidal enriched category enriched over the cartesian-duoidal category **Col** with object set $\mathbb{N}$. It is furthermore clear from construction that $+^{\mathcal{P}}$ is simply addition at the level of objects. We need then that $\mathcal{P}$ is a monoid $(\mathcal{P}, +^{\mathcal{P}}, \mathcal{I})$ in $(\mathbf{ColCat}, \times, \mathbf{1}_*)$, where $\mathbf{1}_*$ is the terminal **Col**-cat and $\mathcal{I} : \mathbf{1}_* \to \mathcal{P}$ is the **Col**-functor which maps the single object $* \in \mathbb{1}_*$ to $0 \in \mathbb{N}$ and the unique hom-object $\mathbf{1}_*(*, *)$ to $\mathcal{P}(0, 0) = \mathbb{1}$, the unit for $\times$ in **ColCat**. But this follows immediately from the fact that each of the relevant commutative diagrams was satisfied in the original non-globularized PRO. As this structure is faithfully preserved by the indexing on each hom-object, the induced operations on the globularlized PRO satisfy the analogous commutativity conditions which ensure that $\mathcal{P}$ is a globular PRO as well. Finally, the commutativity of the appropriate diagrams required of $\mathcal{P}$ in order for it to be a globular PRO follow immediately from construction. Therefore $\mathcal{P}$ is a globular PRO. ∎

19.2. THEOREM. *Let $\mathcal{P}$ be the globularization of the ordinary PRO P. The algebras for $\mathcal{P}$ are exactly the strict ω-categories which are algebras for P whose operations in P are given by strict ω-functors.*

PROOF. Let A be an algebra for the globular PRO $\mathcal{P}$. Consider the hom-object $\mathcal{P}(1, 1)$ which acts on A via the action map $\omega : P(1, 1) \cdot \mathcal{T}(\mathbf{1}) \Box A \to A$. Note that the component of ω corresponding to the identity in P gives a map of globular sets $\omega_{\mathbb{1}_P} : \mathcal{T}(\mathbf{1}) \Box A \to A$ which encodes the structure of a strict ω-category on the globular set A (as an algebra for the terminal collection). To see that A is moreover an algebra for the set PRO P, consider that the action map $\Omega_{n,m} : \mathcal{P}(n, m) \Box A^n \to A^m$ may be restricted so that the globular pasting portion of the action only acts by the image of the inclusion of generators $\mathbf{1} \hookrightarrow \mathcal{T}(\mathbf{1})$. This restricted map is precisely an action of the indexing set for the globular operations (i.e. an induced set $P(n, m)$) on the set A. Furthermore, collectively these maps, for all $n, m \in \mathbb{N}$, satisfy the appropriate diagrams to induce the structure of a P-algebra on A. It remains to show that the action of operations in $\mathcal{P}$ act on A by strict ω-functors. This means that two components of an action (the cartesian portion taking place in the indexing set PRO and the globular pasting portion) can be applied in either order. But this follows immediately from the fact that the action map can be factored so that either operation may be performed first together with the fact that each pair in the source $\mathcal{P}(n, m) \Box A^n$ maps to a particular cell in A^m under $\Omega_{n,m}$. Hence both of these factorization show that the operations in $\mathcal{P}$ act on A by strict ω-functors.

Conversely, assume that A is an algebra in **Glob** for the set PRO P which has the structure of a strict ω-category and whose operations in P are given by strict ω-functors. We wish to show that it is also an algebra for $\mathcal{P}$. Since A is a strict ω-category it admits the structure of an algebra for the terminal collection $\mathbb{1}$. Hence there exists an action map $\omega : \mathcal{T}(\mathbf{1}) \Box A \to A$ where A is here the collection equipped with arity map $[id] \circ !_A : A \to \mathcal{T}(\mathbf{1})$. Since A is an algebra for P it also admits a map to the ordinary tautological PRO on A. This means that for each $n, m \in \mathbb{N}$ we have a map $P(n, m) \to \mathbf{Glob}(A^n, A^m)$, each of which can be curried to get maps $\nu_{n,m} : P(n, m) \cdot A^n \to A^m$. Note then that using the identities discussed above we can construct an induced action map

$\Omega_{n,m} : \mathcal{P}(n,m)\square A^n \to A^m$ by first rewriting the domain as

$$\mathcal{P}(n,m)\square A^n = P(n,m)\cdot\mathbb{1} = \left(\coprod_{P(n,m)} \mathcal{T}(\mathbf{1})\right)\square A^n \cong \coprod_{P(n,m)} (\mathcal{T}(\mathbf{1})\square A^n) \cong \coprod_{P(n,m)} (\mathcal{T}(\mathbf{1})\square A)^n$$

and letting $\Omega_{n,m}$ be defined as the composition

$$\Omega_{n,m} := \nu_{n,m} \circ (\mathbb{1}_{P(n,m)} \cdot \omega^n)$$

where $\omega^n : (\mathcal{T}(\mathbf{1})\square A)^n \to A^n$ is simply the nth cartesian power of ω. All that remains to be shown is that the diagrams

$$\begin{tikzcd}
{[\mathcal{P}(m,l)\square\mathcal{P}(n,m)]\square A^n} \arrow[r, "\alpha^{\mathbf{Col}_\square}_{\mathcal{P}(m,l),\mathcal{P}(n,m),A^n}"] \arrow[d, "\circ_{n,m,l}\square\mathbb{1}_{A^n}"'] & \mathcal{P}(m,l)\square[\mathcal{P}(n,m)\square A^n] \arrow[r, "\mathbb{1}_{\mathcal{P}(m,l)}\square\Omega_{n,m}"] & \mathcal{P}(m,l)\square A^m \arrow[d, "\Omega_{m,l}"] \\
\mathcal{P}(n,l)\square A^n \arrow[rr, "\Omega_{n,l}"'] & & A^l
\end{tikzcd}$$

$$\begin{tikzcd}
{[\mathcal{P}(n,m)\times\mathcal{P}(l,k)]\square[A^n\times A^l]} \arrow[r, "\boxtimes_{\mathcal{P}(n,m),\mathcal{P}(l,k),A^n,A^l}"] \arrow[d, "+_{n,m,l,k}\times\mathbb{1}_{n+l}"'] & {[\mathcal{P}(n,m)\square A^n]\times[\mathcal{P}(l,k)\square A^l]} \arrow[d, "\Omega_{n,m}\times\Omega_{l,k}"] \\
\mathcal{P}(n+l,m+k)\square A^{n+l} \arrow[r, "\Omega_{n+l,m+k}"'] & A^{m+k} = A^m\times A^k
\end{tikzcd}$$

$$\begin{tikzcd}
& \mathcal{P}(n,n)\square A^n \arrow[dr, "\Omega_{n,n}"] & \\
\mathbf{1}\square A^n \arrow[ur, "j_n\square\mathbb{1}_{A^n}"] \arrow[rr, "\lambda^{\mathbf{Col}_\square}_{A^n}"'] & & A^n
\end{tikzcd}$$

commute for all $n, m, l, k \in \mathbb{N}$. When unpacking these diagrams explicitly via the definitions provided above for the relevant maps, the first two unfortunately become quite large. This makes it impractical to attempt typesetting the complete diagrams all at once. Instead, in order to show that these three diagrams commute, a schematic has been provided below for the complete diagrams with subsections of the center faces cut out and labeled. Explicit versions of each of these subsections can then be found below, together with an explanation of why this subsection commutes. The third diagram is small enough to be shown explicitly in a single diagram and follows the first two. Note also that all unlabeled edges correspond to either a reindexing operation or a sequence of instances of unitors and interchange morphisms (here used to include a $\square$ product with a cartesian power of a collection in the second variable into a cartesian power of $\square$ products).

COMPOSITION PRESERVES ACTION

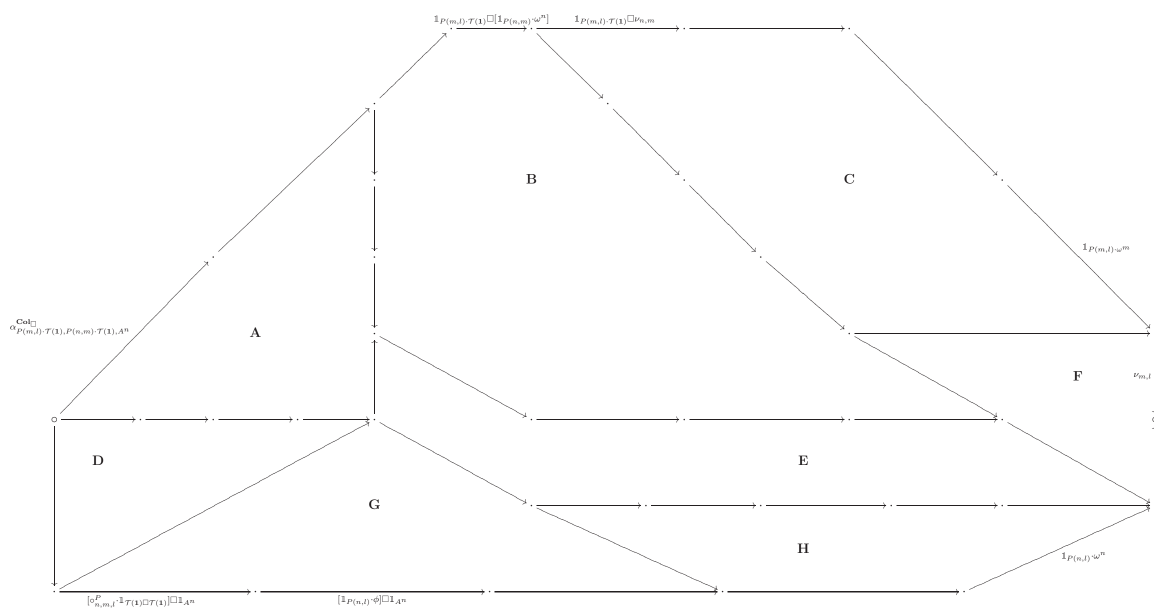

DIAGRAM A

$$
\begin{tikzcd}[column sep=tiny]
 & {[(P(m,l)\cdot\mathcal{T}(\mathbf{1}))\square(P(n,m)\cdot\mathcal{T}(\mathbf{1}))]\square A^n} \arrow[dl] \arrow[dr, "\alpha^{\mathbf{Col}_\square}_{P(m,l)\cdot\mathcal{T}(\mathbf{1}),P(n,m)\cdot\mathcal{T}(\mathbf{1}),A^n}"] & \\
{[P(m,l)\cdot[\mathcal{T}(\mathbf{1})\square(P(n,m)\cdot\mathcal{T}(\mathbf{1}))]]\square A^n} \arrow[d] & & {(P(m,l)\cdot\mathcal{T}(\mathbf{1}))\square[(P(n,m)\cdot\mathcal{T}(\mathbf{1}))\square A^n]} \arrow[d] \\
{[P(m,l)\cdot[P(n,m)\cdot(\mathcal{T}(\mathbf{1})\square\mathcal{T}(\mathbf{1}))]]\square A^n} \arrow[d] & & {(P(m,l)\cdot\mathcal{T}(\mathbf{1}))\square[P(n,m)\cdot(\mathcal{T}(\mathbf{1})\square A^n)]} \arrow[d] \\
{[(P(m,l)\times P(n,m))\cdot(\mathcal{T}(\mathbf{1})\square\mathcal{T}(\mathbf{1}))]\square A^n} \arrow[d] & & {P(m,l)\cdot[\mathcal{T}(\mathbf{1})\square[P(n,m)\cdot(\mathcal{T}(\mathbf{1})\square A^n)]]} \arrow[d] \\
{(P(m,l)\times P(n,m))\cdot[(\mathcal{T}(\mathbf{1})\square\mathcal{T}(\mathbf{1}))\square A^n]} \arrow[dr, "\mathbb{1}_{P(m,l)\times P(n,m)}\cdot\alpha^{\mathbf{Col}_\square}_{\mathcal{T}(\mathbf{1}),\mathcal{T}(\mathbf{1}),A^n}"] & & {P(m,l)\cdot[P(n,m)\cdot[\mathcal{T}(\mathbf{1})\square(\mathcal{T}(\mathbf{1})\square A^n)]]} \arrow[dl] \\
 & {(P(m,l)\times P(n,m))\cdot[\mathcal{T}(\mathbf{1})\square(\mathcal{T}(\mathbf{1})\square A^n)]} &
\end{tikzcd}
$$

We shall show the commutativity of this diagram by describing how each edge of this diagram acts on a generic element. Let $\sigma_\phi \in P(m,l)\cdot\mathcal{T}(\mathbf{1})$ be a cell of shape $\sigma \in \mathcal{T}(\mathbf{1})$ indexed by an operation $\phi \in P(m,l)$. Then let $\Sigma \in \mathcal{T}(P(n,m)\cdot\mathcal{T}(\mathbf{1}))$ be a coloring of σ_ϕ by composite cells ${}^\tau\Sigma_\psi \in P(n,m)\cdot\mathcal{T}(\mathbf{1})$, one for each sub-cell $\tau \in \sigma_\phi$. Note that each composite cell may be indexed by the same ψ because of the connectedness of $\mathcal{T}(\mathbf{1})$. Moreover, let κ_n be a coloring of the shape of (σ_ϕ, Σ). Thus we start both compositions with a cell $((\sigma_\phi,\Sigma),\kappa_n) \in [(P(m,l)\cdot\mathcal{T}(\mathbf{1}))\square(P(n,m)\cdot\mathcal{T}(\mathbf{1}))]\square A^n$.

We begin the first composition by applying the associator for $\square$ to $((\sigma_\phi,\Sigma),\kappa_n)$ to get a coloring $\int$ of σ_ϕ, where $\int$ is induced by the coloring of Σ by κ_n. Hence, for each sub-cell $\tau \in \sigma_\phi$, the composite cell of $\int$ which colors it is $({}^\tau\Sigma_\psi,\kappa_n) \in [P(n,m)\cdot\mathcal{T}(\mathbf{1})]\square A^n$. Since the colored cells of $\int$ all come from the same summand, we may send $({}^\tau\Sigma_\psi,\kappa_n)$ to $({}^\tau\Sigma,\kappa_n)_\psi \in P(n,m)\cdot[\mathcal{T}(\mathbf{1})\square A^n]$. Similarly, the sub-cells of σ_ϕ being colored do not rely on the summand denoted by ϕ to be colored. Hence, this and the previous step together send $(\sigma_\phi,(\Sigma_\psi,\kappa_n))$ to $(\sigma,(\Sigma,\kappa_n)_\psi)_\phi$. Citing this independence from the summand index a third time gives $((\sigma,(\Sigma,\kappa_n))_\psi)_\phi$ which can be re-indexed by a single operation $(\psi,\phi) \in P(m,l)\times P(n,m)$ to get $(\sigma,(\Sigma,\kappa_n))_{(\psi,\phi)} \in [P(m,l)\times P(n,m)]\cdot[\mathcal{T}(\mathbf{1})\square(\mathcal{T}(\mathbf{1})\square A^n)]$.

Along the other composition, we first re-index $((\sigma_\phi,\Sigma),\kappa_n)$ to get $((\sigma,\Sigma)_\phi,\kappa_n)$. Again by the connectedness of $\mathcal{T}(\mathbf{1})$ we can re-index to get $(((\sigma,\Sigma)_\psi)_\phi,\kappa_n)$. Reindexing further gives $((\sigma,\Sigma)_{(\psi,\phi)},\kappa_n)$; hence $((\sigma,\Sigma),\kappa_n)_{(\psi,\phi)} \in [P(m,l)\times P(n,m)]\cdot[(\mathcal{T}(\mathbf{1})\square\mathcal{T}(\mathbf{1}))\square A^n]$. Then applying the associator within this single summand corresponding to $(\psi,\phi) \in P(m,l)\times P(n,m)$ must give the same cell $(\sigma,(\Sigma,\kappa_n))_{(\psi,\phi)}$ in

$[P(m,l) \times P(n,m)] \cdot [\mathcal{T}(\mathbf{1}) \Box (\mathcal{T}(\mathbf{1}) \Box A^n)]$ from above.

DIAGRAM B

$$
\begin{tikzcd}[column sep=tiny]
& {[P(m,l) \cdot \mathcal{T}(\mathbf{1})] \Box [P(n,m) \cdot (\mathcal{T}(\mathbf{1}) \Box A^n)]} \arrow[dl] \arrow[ddr] & \\
{P(m,l) \cdot [\mathcal{T}(\mathbf{1}) \Box [P(n,m) \cdot (\mathcal{T}(\mathbf{1}) \Box A^n)]]} \arrow[dd] & & \\
& & {[P(m,l) \cdot \mathcal{T}(\mathbf{1})] \Box [P(n,m) \cdot (\mathcal{T}(\mathbf{1}) \Box A)^n]} \arrow[dl] \arrow[dd, "\mathbb{1}_{P(m,l) \cdot \mathcal{T}(\mathbf{1})} \Box [\mathbb{1}_{P(n,m)} \cdot \omega^n]"] \\
{P(m,l) \cdot [P(n,m) \cdot [\mathcal{T}(\mathbf{1}) \Box (\mathcal{T}(\mathbf{1}) \Box A^n)]]} \arrow[dd] & {P(m,l) \cdot [\mathcal{T}(\mathbf{1}) \Box [P(n,m) \cdot (\mathcal{T}(\mathbf{1}) \Box A)^n]]} \arrow[dd] & \\
& & {[P(m,l) \cdot \mathcal{T}(\mathbf{1})] \Box [P(n,m) \cdot A^n]} \arrow[dd] \\
{[P(m,l) \times P(n,m)] \cdot [\mathcal{T}(\mathbf{1}) \Box (\mathcal{T}(\mathbf{1}) \Box A^n)]} \arrow[dd] & {P(m,l) \cdot [P(n,m) \cdot [\mathcal{T}(\mathbf{1}) \Box (\mathcal{T}(\mathbf{1}) \Box A)^n]} \arrow[ddl] & \\
& & {P(m,l) \cdot [\mathcal{T}(\mathbf{1}) \Box (P(n,m) \cdot A^n)]} \arrow[dd] \\
{[P(m,l) \times P(n,m)] \cdot [\mathcal{T}(\mathbf{1}) \Box (\mathcal{T}(\mathbf{1}) \Box A)^n]} \arrow[dd, "\mathbb{1}_{P(m,l) \times P(n,m)} \cdot [\mathbb{1}_{\mathcal{T}(\mathbf{1})} \Box \omega^n]"'] & & \\
& & {P(m,l) \cdot [P(n,m) \cdot (\mathcal{T}(\mathbf{1}) \Box A^n)]} \arrow[dll] \arrow[dd] \\
{[P(m,l) \times P(n,m)] \cdot [\mathcal{T}(\mathbf{1}) \Box A^n]} \arrow[dd] & & \\
& & {P(m,l) \cdot [P(n,m) \cdot (\mathcal{T}(\mathbf{1}) \Box A)^n]} \arrow[dll] \arrow[dd, "\mathbb{1}_{P(m,l)} \cdot [\mathbb{1}_{P(n,m)} \cdot \omega^n]"] \\
{[P(m,l) \times P(n,m)] \cdot [\mathcal{T}(\mathbf{1}) \Box A]^n} \arrow[ddr, "\mathbb{1}_{P(m,l) \times P(n,m)} \cdot \omega^n"] & & \\
& & {P(m,l) \cdot [P(n,m) \cdot A^n]} \arrow[dl] \\
& {[P(m,l) \times P(n,m)] \cdot A^n} &
\end{tikzcd}
$$

Each of the top two regions of this diagram commute by the fact that $\Box$ preserves coproduct, and hence $\cdot$ is preserved. The bottom two squares commute by the naturality of the operation of reindexing copowers.

DIAGRAM C

$$\begin{tikzcd}
& {[P(m,l)\cdot\mathcal{T}(\mathbf{1})]\square[P(n,m)\cdot A^n]} \arrow[dl] \arrow[dr, "\mathbb{1}_{P(m,l)\cdot\mathcal{T}(\mathbf{1})}\square\nu_{n,m}"] & \\
P(m,l)\cdot[\mathcal{T}(\mathbf{1})\square(P(n,m)\cdot A^n)] \arrow[d] & & {[P(m,l)\cdot\mathcal{T}(\mathbf{1})]\square A^m} \arrow[d] \\
P(m,l)\cdot[P(n,m)\cdot(\mathcal{T}(\mathbf{1})\square A^n)] \arrow[d] & & P(m,l)\cdot[\mathcal{T}(\mathbf{1})\square A^m] \arrow[d] \\
P(m,l)\cdot[P(n,m)\cdot(\mathcal{T}(\mathbf{1})\square A)^n] \arrow[d, "\mathbb{1}_{P(m,l)\cdot[\mathbb{1}_{P(n,m)}\cdot\omega^n]}"'] & & P(m,l)\cdot[\mathcal{T}(\mathbf{1})\square A]^m \arrow[ddl, "\mathbb{1}_{P(m,l)\cdot\omega^m}"'] \\
P(m,l)\cdot[P(n,m)\cdot A^n] \arrow[dr, "\mathbb{1}_{P(m,l)\cdot\nu_{n,m}}"] & & \\
& P(m,l)\cdot A^m &
\end{tikzcd}$$

This diagram commutes by the fact that the operations in $\mathcal{P}$ act on A as strict ω-functors. Hence, the ν and ω portion of an operation in $\mathcal{P}$ may be performed in either order.

DIAGRAM D

$$\begin{tikzcd}
& {[(P(m,l)\cdot\mathcal{T}(\mathbf{1}))\square(P(n,m)\cdot\mathcal{T}(\mathbf{1}))]\square A^n} \arrow[ddl] \arrow[dr] & \\
& & {[P(m,l)\cdot[\mathcal{T}(\mathbf{1})\square(P(n,m)\cdot\mathcal{T}(\mathbf{1}))]]\square A^n} \arrow[d] \\
{[(P(m,l)\times P(n,m))\cdot(\mathcal{T}(\mathbf{1})\square\mathcal{T}(\mathbf{1}))]\square A^n} \arrow[ddr] & & {[P(m,l)\cdot[P(n,m)\cdot(\mathcal{T}(\mathbf{1})\square\mathcal{T}(\mathbf{1}))]]\square A^n} \arrow[d] \\
& & {[(P(m,l)\times P(n,m))\cdot(\mathcal{T}(\mathbf{1})\square\mathcal{T}(\mathbf{1}))]\square A^n} \arrow[dl] \\
& {[P(m,l)\times P(n,m)]\cdot[(\mathcal{T}(\mathbf{1})\square\mathcal{T}(\mathbf{1}))\square A^n]} &
\end{tikzcd}$$

This diagram commutes by the fact that each of the two sides of the diagram two ways of performing the same copower reindexing.

DIAGRAM E

$$
\begin{tikzcd}[column sep=small]
 & {[P(m,l) \times P(n,m)] \cdot [(\mathcal{T}(\mathbf{1}) \Box \mathcal{T}(\mathbf{1})) \Box A^n]} \arrow[dl, "\circ^P_{n,m,l} \cdot \mathbb{1}_{(\mathcal{T}(\mathbf{1}) \Box \mathcal{T}(\mathbf{1})) \Box A^n}"'] \arrow[ddr, "\mathbb{1}_{P(m,l) \times P(n,m)} \cdot \alpha^{\mathbf{Col}_\Box}_{\mathcal{T}(\mathbf{1}), \mathcal{T}(\mathbf{1}), A^n}"] & \\
P(n,l) \cdot [(\mathcal{T}(\mathbf{1}) \Box \mathcal{T}(\mathbf{1})) \Box A^n] \arrow[dd, "\mathbb{1}_{P(n,l)} \cdot \alpha^{\mathbf{Col}_\Box}_{\mathcal{T}(\mathbf{1}), \mathcal{T}(\mathbf{1}), A^n}"'] & & \\
 & & {[P(m,l) \times P(n,m)] \cdot [\mathcal{T}(\mathbf{1}) \Box (\mathcal{T}(\mathbf{1}) \Box A^n)]} \arrow[dll, "\circ^P_{n,m,l} \cdot \mathbb{1}_{\mathcal{T}(\mathbf{1}) \Box (\mathcal{T}(\mathbf{1}) \Box A^n)}"'] \arrow[dd] \\
P(n,l) \cdot [\mathcal{T}(\mathbf{1}) \Box (\mathcal{T}(\mathbf{1}) \Box A^n)] \arrow[dd] & & \\
 & & {[P(m,l) \times P(n,m)] \cdot [\mathcal{T}(\mathbf{1}) \Box (\mathcal{T}(\mathbf{1}) \Box A)^n]} \arrow[dll, "\circ^P_{n,m,l} \cdot \mathbb{1}_{\mathcal{T}(\mathbf{1}) \Box (\mathcal{T}(\mathbf{1}) \Box A)^n}"'] \arrow[dd, "\mathbb{1}_{P(m,l) \times P(n,m)} \cdot [\mathbb{1}_{\mathcal{T}(\mathbf{1})} \Box \omega^n]"] \\
P(n,l) \cdot [\mathcal{T}(\mathbf{1}) \Box (\mathcal{T}(\mathbf{1}) \Box A)^n] \arrow[dd, "\mathbb{1}_{P(n,l)} \cdot [\mathbb{1}_{\mathcal{T}(\mathbf{1})} \Box \omega^n]"'] & & \\
 & & {[P(m,l) \times P(n,m)] \cdot [\mathcal{T}(\mathbf{1}) \Box A^n]} \arrow[dll, "\circ^P_{n,m,l} \cdot \mathbb{1}_{\mathcal{T}(\mathbf{1}) \Box A^n}"'] \arrow[dd] \\
P(n,l) \cdot [\mathcal{T}(\mathbf{1}) \Box A^n] \arrow[dd] & & \\
 & & {[P(m,l) \times P(n,m)] \cdot [\mathcal{T}(\mathbf{1}) \Box A]^n} \arrow[dll, "\circ^P_{n,m,l} \cdot \mathbb{1}_{[\mathcal{T}(\mathbf{1}) \Box A]^n}"'] \arrow[dd, "\mathbb{1}_{P(m,l) \times P(n,m)} \cdot \omega^n"] \\
P(n,l) \cdot [\mathcal{T}(\mathbf{1}) \Box A]^n \arrow[ddr, "\mathbb{1}_{P(n,l)} \cdot \omega^n"] & & \\
 & & {[P(m,l) \times P(n,m)] \cdot A^n} \arrow[dl, "\circ^P_{n,m,l} \cdot \mathbb{1}_{A^n}"'] \\
 & P(n,l) \cdot A^n &
\end{tikzcd}
$$

Each of these squares commute by the functoriality of the $\cdot$ operation.

DIAGRAM F

$$\begin{tikzcd}
& P(m,l)\cdot[P(n,m)\cdot A^n] \arrow[dl] \arrow[ddr, "\mathbb{1}_{P(m,l)}\cdot\nu_{n,m}"] & \\
{[P(m,l)\times P(n,m)]\cdot A^n} \arrow[dd, "\circ^P_{n,m,l}\cdot\mathbb{1}_{A^n}"'] & & \\
& & P(m,l)\cdot A^m \arrow[ddl, "\nu_{m,l}"'] \\
P(n,l)\cdot A^n \arrow[dr, "\nu_{n,l}"] & & \\
& A^l &
\end{tikzcd}$$

This diagram commutes by the fact that A is an algebra for the underlying set PRO P.

DIAGRAM G

$$\begin{tikzcd}
& {[(P(m,l)\times P(n,m))\cdot(\mathcal{T}(\mathbf{1})\square\mathcal{T}(\mathbf{1}))]\square A^n} \arrow[dr] \arrow[ddl, "{[\circ^P_{n,m,l}\cdot\mathbb{1}_{\mathcal{T}(\mathbf{1})\square\mathcal{T}(\mathbf{1})}]\square\mathbb{1}_{A^n}}"'] & \\
& & {[P(m,l)\times P(n,m)]\cdot[(\mathcal{T}(\mathbf{1})\square\mathcal{T}(\mathbf{1}))\square A^n]} \arrow[dd, "\circ^P_{n,m,l}\cdot\mathbb{1}_{(\mathcal{T}(\mathbf{1})\square\mathcal{T}(\mathbf{1}))\square A^n}"] \\
{[P(n,l)\cdot(\mathcal{T}(\mathbf{1})\square\mathcal{T}(\mathbf{1}))]\square A^n} \arrow[drr] \arrow[dd, "{[\mathbb{1}_{P(n,l)}\cdot\phi]\square\mathbb{1}_{A^n}}"'] & & \\
& & P(n,l)\cdot[(\mathcal{T}(\mathbf{1})\square\mathcal{T}(\mathbf{1}))\square A^n] \arrow[ddl, "{\mathbb{1}_{P(n,l)}\cdot[\phi\square\mathbb{1}_{A^n}]}"] \\
{[P(n,l)\cdot\mathcal{T}(\mathbf{1})]\square A^n} \arrow[dr] & & \\
& P(n,l)\cdot[\mathcal{T}(\mathbf{1})\square A^n] &
\end{tikzcd}$$

Both of these squares commute by the naturality of the reindexing operation.

DIAGRAM H

$$
\begin{tikzcd}
 & & P(n,l)\cdot[(\mathcal{T}(\mathbf{1})\square\mathcal{T}(\mathbf{1}))\square A^n] \arrow[dll, "\mathbb{1}_{P(n,l)}\cdot[\phi\square\mathbb{1}_{A^n}]"'] \arrow[drr, "\mathbb{1}_{P(n,l)}\cdot\alpha^{\mathbf{Col}_\square}_{\mathcal{T}(\mathbf{1}),\mathcal{T}(\mathbf{1}),A^n}"] & & \\
P(n,l)\cdot[\mathcal{T}(\mathbf{1})\square A^n] \arrow[ddd] & & & & P(n,l)\cdot[\mathcal{T}(\mathbf{1})\square(\mathcal{T}(\mathbf{1})\square A^n)] \arrow[d] \\
 & & & & P(n,l)\cdot[\mathcal{T}(\mathbf{1})\square(\mathcal{T}(\mathbf{1})\square A)^n] \arrow[d, "\mathbb{1}_{P(n,l)}\cdot[\mathbb{1}_{\mathcal{T}(\mathbf{1})}\square\omega^n]"] \\
 & & & & P(n,l)\cdot[\mathcal{T}(\mathbf{1})\square A^n] \arrow[d] \\
P(n,l)\cdot[\mathcal{T}(\mathbf{1})\square A]^n \arrow[drr, "\mathbb{1}_{P(n,l)}\cdot\omega^n"'] & & & & P(n,l)\cdot[\mathcal{T}(\mathbf{1})\square A]^n \arrow[dll, "\mathbb{1}_{P(n,l)}\cdot\omega^n"] \\
 & & P(n,l)\cdot A^n & &
\end{tikzcd}
$$

This diagram commutes by the fact that A has the structure of a strict ω-category by assumption.

MONOIDAL SUM PRESERVES ACTION

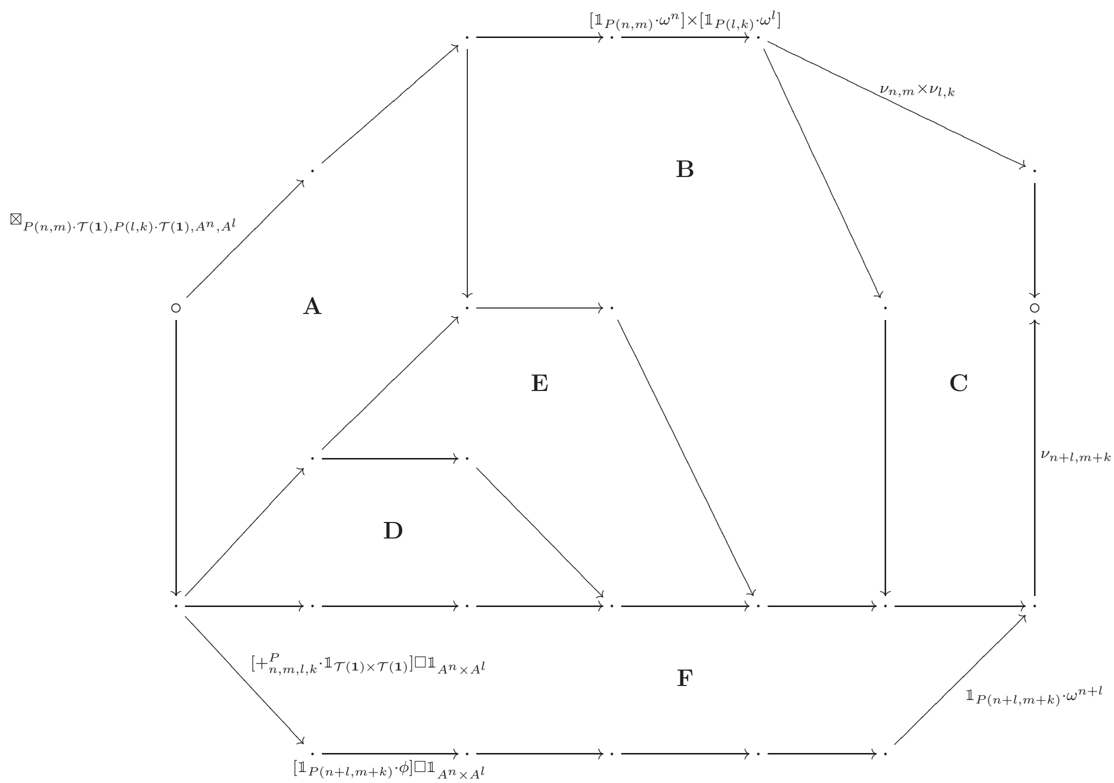

DIAGRAM A

$$
\begin{tikzcd}[column sep=tiny]
& {[(P(n,m)\cdot\mathcal{T}(\mathbf{1}))\times(P(l,k)\cdot\mathcal{T}(\mathbf{1}))]\square[A^n\times A^l]} \arrow[dl] \arrow[dr, "\boxtimes_{P(n,m)\cdot\mathcal{T}(\mathbf{1}),P(l,k)\cdot\mathcal{T}(\mathbf{1}),A^n,A^l}"] & \\
{[(P(n,m)\times P(l,k))\cdot(\mathcal{T}(\mathbf{1})\times\mathcal{T}(\mathbf{1}))]\square[A^n\times A^l]} \arrow[d] & & {[(P(n,m)\cdot\mathcal{T}(\mathbf{1}))\square A^n]\times[(P(l,k)\cdot\mathcal{T}(\mathbf{1}))\square A^l]} \arrow[d] \\
{[P(n,m)\times P(l,k)]\cdot[(\mathcal{T}(\mathbf{1})\times\mathcal{T}(\mathbf{1}))\square(A^n\times A^l)]} \arrow[dr, "\mathbb{1}_{P(n,m)\times P(l,k)}\cdot\boxtimes_{\mathcal{T}(\mathbf{1}),\mathcal{T}(\mathbf{1}),A^n,A^l}"] & & {[P(n,m)\cdot(\mathcal{T}(\mathbf{1})\square A^n)]\times[P(l,k)\cdot(\mathcal{T}(\mathbf{1})\square A^l)]} \arrow[dl] \\
& {[P(n,m)\times P(l,k)]\cdot[(\mathcal{T}(\mathbf{1})\square A^n)\times(\mathcal{T}(\mathbf{1})\square A^l)]} &
\end{tikzcd}
$$

We shall once again justify commutativity by describing how each of the two sides of this diagram act on a generic element. We start with a pair of globular cells (h,k), both with arity shape σ, such that the first is indexed by an operation $\phi \in P(n,m)$ and the second is indexed by on operation $\psi \in P(l,k)$. Hence we may write h as σ_ϕ and k as σ_ψ to get $(h,k) = (\sigma_\phi, \sigma_\psi) \in (P(n,m)\cdot\mathcal{T}(\mathbf{1}))\times(P(l,k)\cdot\mathcal{T}(\mathbf{1}))$. Moreover, $(\sigma_\phi,\sigma_\psi)$ is equipped with a coloring of its arity by cells in $A^n \times A^l$. We now wish to look at two different compositions of maps to see that the corresponding diagram of morphisms commutes.

We begin the first string by applying the middle four interchange to the cells described above. This gives a pair whose first entry is σ_ϕ equipped with the coloring of its arity by the first n cells in the coloring by cells in $A^n \times A^l$. We shall denote this 'word' coloring σ_ϕ by $\kappa_n \in \mathcal{T}(A^n)$. The second entry of the pair is then the cell σ_ψ equipped with the coloring of its arity by the last l cells in the coloring, which we shall denote κ_l. Hence, the interchange transformation sends $((\sigma_\phi,\sigma_\psi),(\kappa_n,\kappa_l))$ to $((\sigma_\phi,\kappa_n),(\sigma_\psi,\kappa_l))$. Since a cell of the cartesian product is a tuple of cells all having the same arity shape, we may re-index $((\sigma_\phi,\kappa_n),(\sigma_\psi,\kappa_l))$ as $((\sigma,\kappa_n)_\phi,(\sigma,\kappa_l)_\psi)$ without any loss of information. Similarly, we can re-index this tuple as $((\sigma,\kappa_n),(\sigma,\kappa_l))_{(\phi,\psi)}$.

If we instead follow the other composition, we first take $((\sigma_\phi,\sigma_\psi),(\kappa_n,\kappa_l))$ and, instead of applying the interchange transformation, re-index it as $((\sigma,\sigma)_{(\phi,\psi)},(\kappa_n,\kappa_l))$. This can then also be re-indexed as $((\sigma,\sigma),(\kappa_n,\kappa_l))_{(\phi,\psi)}$. We can then apply the interchange morphism in just the (ϕ,ψ) summand to get $((\sigma,\kappa_n),(\sigma,\kappa_l))_{(\phi,\psi)}$ as we had before.

DIAGRAM B

$$
\begin{tikzcd}
& {[P(n,m)\cdot(\mathcal{T}(\mathbf{1})\Box A^n)]\times[P(l,k)\cdot(\mathcal{T}(\mathbf{1})\Box A^l)]} \arrow[ddl] \arrow[dr] & \\
& & {[P(n,m)\cdot(\mathcal{T}(\mathbf{1})\Box A)^n]\times[P(l,k)\cdot(\mathcal{T}(\mathbf{1})\Box A)^l]} \arrow[d, "{[\mathbb{1}_{P(n,m)}\cdot\omega^n]\times[\mathbb{1}_{P(l,k)}\cdot\omega^l]}"] \arrow[ddll] \\
{[P(n,m)\times P(l,k)]\cdot[(\mathcal{T}(\mathbf{1})\Box A^n)\times(\mathcal{T}(\mathbf{1})\Box A^l)]} \arrow[d] & & {[P(n,m)\cdot A^n]\times[P(l,k)\cdot A^l]} \arrow[d] \\
{[P(n,m)\times P(l,k)]\cdot[(\mathcal{T}(\mathbf{1})\Box A)^n\times(\mathcal{T}(\mathbf{1})\Box A)^l]} \arrow[d] \arrow[rr, "{\mathbb{1}_{P(n,m)\times P(l,k)}\cdot[\omega^n\times\omega^l]}"] & & {[P(n,m)\times P(l,k)]\cdot[A^n\times A^l]} \arrow[ddl] \\
{[P(n,m)\times P(l,k)]\cdot[\mathcal{T}(\mathbf{1})\Box A]^{n+l}} \arrow[dr, "{\mathbb{1}_{P(n,m)\times P(l,k)}\cdot\omega^{n+l}}"] & & \\
& {[P(n,m)\times P(l,k)]\cdot A^{n+l}} &
\end{tikzcd}
$$

The two topmost squares commute by the naturality of the operation of reindexing copwers. The bottom square commutes by the naturality of the associator for $\times$ in **Col**.

DIAGRAM C

$$
\begin{tikzcd}
& & & {[P(n,m)\cdot A^n]\times[P(l,k)\cdot A^l]} \arrow[dll] \arrow[dd, "\nu_{n,m}\times\nu_{l,k}"] \\
& {[P(n,m)\times P(l,k)]\cdot[A^n\times A^l]} \arrow[dl] & & \\
{[P(n,m)\times P(l,k)]\cdot A^{n+l}} \arrow[dr, "{+^P_{n,m,l,k}\cdot\mathbb{1}_{A^{n+l}}}"] & & & A^m\times A^k \arrow[dd] \\
& P(n+l,m+k)\cdot A^{n+l} \arrow[drr, "\nu_{n+l,m+k}"] & & \\
& & & A^{m+k}
\end{tikzcd}
$$

This diagram commutes by the fact that $+$ is the monoidal product for the underlying set PRO P.

DIAGRAM D

$$\begin{tikzcd}
 & {[(P(n,m)\times P(l,k))\cdot(\mathcal{T}(\mathbf{1})\times\mathcal{T}(\mathbf{1}))]\square[A^n\times A^l]} \arrow[ddl, "{[\mathbb{1}_{P(n,m)\times P(l,k)}\cdot\phi]\square\mathbb{1}_{A^n\times A^l}}"'] \arrow[dr] & \\
 & & {[P(n,m)\times P(l,k)]\cdot[(\mathcal{T}(\mathbf{1})\times\mathcal{T}(\mathbf{1}))\square(A^n\times A^l)]} \arrow[d, "{\mathbb{1}_{P(n,m)\times P(l,k)}\cdot[\phi\square\mathbb{1}_{A^n\times A^l}]}"] \\
{[(P(n,m)\times P(l,k))\cdot\mathcal{T}(\mathbf{1})]\square[A^n\times A^l]} \arrow[rr] \arrow[d] & & {[P(n,m)\times P(l,k)]\cdot[\mathcal{T}(\mathbf{1})\square(A^n\times A^l)]} \arrow[ddl] \\
{[(P(n,m)\times P(l,k))\cdot\mathcal{T}(\mathbf{1})]\square A^{n+l}} \arrow[dr] & & \\
 & {[P(n,m)\times P(l,k)]\cdot[\mathcal{T}(\mathbf{1})\square A^{n+l}]} &
\end{tikzcd}$$

Both of these squares commute by the naturality of the operation of reindexing copowers.

DIAGRAM E

$$\begin{tikzcd}
 & {[P(n,m)\times P(l,k)]\cdot[(\mathcal{T}(\mathbf{1})\times\mathcal{T}(\mathbf{1}))\square(A^n\times A^l)]} \arrow[dl] \arrow[dr] & \\
{[P(n,m)\times P(l,k)]\cdot[\mathcal{T}(\mathbf{1})\square(A^n\times A^l)]} \arrow[d] & & {[P(n,m)\times P(l,k)]\cdot[(\mathcal{T}(\mathbf{1})\square A^n)\times(\mathcal{T}(\mathbf{1})\square A^l)]} \arrow[d] \\
{[P(n,m)\times P(l,k)]\cdot[\mathcal{T}(\mathbf{1})\square A^{n+l}]} \arrow[dr] & & {[P(n,m)\times P(l,k)]\cdot[(\mathcal{T}(\mathbf{1})\square A)^n\times(\mathcal{T}(\mathbf{1})\square A)^l]} \arrow[dl] \\
 & {[P(n,m)\times P(l,k)]\cdot[\mathcal{T}(\mathbf{1})\square A]^{n+l}} &
\end{tikzcd}$$

Note that each object and arrow of this diagram is a copower indexed by $P(n,m)\times P(l,k)$. By the functoriality of $\cdot$, it is hence enough to show that this diagram commutes prior to taking copowers. To see this, note initially that the first map along the left hand side (which is a $\square$-product of the unitor for $\times$ in **Col** with an identity map) is invertible. Note then that the operation of preserving cartesian power in the second variable, which is seen in this diagram in both the last map along the left hand side as well as the second to last along the right hand side, factors as series of inverse unitor maps for $\times$ followed by a series of applications of $\boxtimes$. We hence see, after inverting the first map along the left hand side, that both sides of this diagram are simply a different choice in the order in which these iterated unitors and copies of $\boxtimes$ are applied. Hence the diagram commutes by the coherence theorem for lax-monoidal functors.

DIAGRAM F

$$
\begin{tikzcd}[column sep=tiny]
 & {[(P(n,m)\times P(l,k))\cdot(\mathcal{T}(\mathbf{1})\times\mathcal{T}(\mathbf{1}))]\Box[A^n\times A^l]} \arrow[ddl, "{[+^P_{n,m,l,k}\cdot\mathbb{1}_{\mathcal{T}(\mathbf{1})\times\mathcal{T}(\mathbf{1})}]\Box\mathbb{1}_{A^n\times A^l}}"'] \arrow[dr, "{[\mathbb{1}_{P(n,m)\times P(l,k)}\cdot\phi]\Box\mathbb{1}_{A^n\times A^l}}"] & \\
 & & {[(P(n,m)\times P(l,k))\cdot\mathcal{T}(\mathbf{1})]\Box[A^n\times A^l]} \arrow[dddll, "{[+^P_{n,m,l,k}\cdot\mathbb{1}_{\mathcal{T}(\mathbf{1})}]\Box\mathbb{1}_{A^n\times A^l}}"'] \arrow[dd] \\
{[P(n+l,m+k)\cdot(\mathcal{T}(\mathbf{1})\times\mathcal{T}(\mathbf{1}))]\Box[A^n\times A^l]} \arrow[dd, "{[\mathbb{1}_{P(n+l,m+k)}\cdot\phi]\Box\mathbb{1}_{A^n\times A^l}}"'] & & \\
 & & {[(P(n,m)\times P(l,k))\cdot\mathcal{T}(\mathbf{1})]\Box A^{n+l}} \arrow[dddll, "{[+^P_{n,m,l,k}\cdot\mathbb{1}_{\mathcal{T}(\mathbf{1})}]\Box\mathbb{1}_{A^{n+l}}}"'] \arrow[dd] \\
{[P(n+l,m+k)\cdot\mathcal{T}(\mathbf{1})]\Box[A^n\times A^l]} \arrow[dd] & & \\
 & & {[P(n,m)\times P(l,k)]\cdot[\mathcal{T}(\mathbf{1})\Box A^{n+l}]} \arrow[dddll, "{+^P_{n,m,l,k}\cdot\mathbb{1}_{\mathcal{T}(\mathbf{1})\Box A^{n+l}}}"'] \arrow[dd] \\
{[P(n+l,m+k)\cdot\mathcal{T}(\mathbf{1})]\Box A^{n+l}} \arrow[dd] & & \\
 & & {[P(n,m)\times P(l,k)]\cdot[\mathcal{T}(\mathbf{1})\Box A]^{n+l}} \arrow[dddll, "{+^P_{n,m,l,k}\cdot\mathbb{1}_{[\mathcal{T}(\mathbf{1})\Box A]^{n+l}}}"'] \arrow[dd, "{\mathbb{1}_{P(n,m)\times P(l,k)}\cdot\omega^{n+l}}"] \\
P(n+l,m+k)\cdot[\mathcal{T}(\mathbf{1})\Box A^{n+l}] \arrow[dd] & & \\
 & & {[P(n,m)\times P(l,k)]\cdot A^{n+l}} \arrow[ddl, "{+^P_{n,m,l,k}\cdot\mathbb{1}_{A^{n+l}}}"] \\
P(n+l,m+k)\cdot[\mathcal{T}(\mathbf{1})\Box A]^{n+l} \arrow[dr, "{\mathbb{1}_{P(n+l,m+k)}\cdot\omega^{n+l}}"'] & & \\
 & P(n+l,m+k)\cdot A^{n+l} &
\end{tikzcd}
$$

The topmost square commutes by the functoriality of $\cdot$. The second square from the top commutes by the naturality of the associator for $\times$ in **Col**. The middle square commutes by the naturality of the operation of reindexing copowers. The bottom two squares also commute by the functoriality of $\cdot$.

UNIT IS REPRESENTED BY THE ACTION

$$\begin{tikzcd}
(P(n,n)\cdot\mathcal{T}(\mathbf{1}))\Box A^n \arrow[rr] \arrow[dr] & & P(n,n)\cdot(\mathcal{T}(\mathbf{1})\Box A^n) \arrow[r] & P(n,n)\cdot(\mathcal{T}(\mathbf{1})\Box A)^n \arrow[r, "\mathbb{1}_{P(n,n)}\cdot\omega^n"] & P(n,n)\cdot A^n \arrow[dddd, "\nu_{n,n}"] \\
 & \mathcal{T}(\mathbf{1})\Box(P(n,n)\cdot A^n) \arrow[rr, "\mathbb{1}_{\mathcal{T}(\mathbf{1})}\Box\nu_{n,n}"] & & \mathcal{T}(\mathbf{1})\Box A^n \arrow[dd] & \\
(P(n,n)\cdot\mathbf{1})\Box A^n \arrow[uu, hook] & & & & \\
 & & (\mathbf{1}\Box A)^n \arrow[r, hook] & (\mathcal{T}(\mathbf{1})\Box A)^n \arrow[dr, "\omega^n"] & \\
\mathbf{1}\Box A^n \arrow[uu, "\xi_n\Box\mathbb{1}_{A^n}"] \arrow[uuurrr, hook] \arrow[urr] \arrow[rrrr, "\lambda^{\mathbf{Col}_\Box}_{A^n}"] & & & & A^n
\end{tikzcd}$$

The top right region commutes by the fact that operations in $\mathcal{P}$ act on A as strict ω-functors. The upper of the two middle regions commutes by the fact that the action in this region is by identities from the set PRO P. The lower of the middle two regions commutes by the naturality of the iterated middle four interchange which sends each $\Box$ product with a cartesian power of a collection in the second variable to a canonical cartesian power of a $\Box$ product. To see why the bottom region commutes, consider first the upper path of this region. After an initial reindexing, this composition amounts to an action on an element of $\mathcal{T}(A^n)$ by a generating globular cell from $\mathbf{1} \subset \mathcal{T}(\mathbf{1})$. But since these cells act as identities with respect to the $\Box$ product, this is the same as simply applying $\lambda^{\mathbf{Col}_\Box}_{A^n}$ to $\mathbf{1}\Box A^n$. ∎

20. Contractions and Leinster Fibrations

Although we now have enough structure to encode strict higher order algebraic structures as algebras for our globular PROs, we need one more piece of structure to give us the weak versions. This last bit of structure is a special lifting property. We begin by recalling Leinster's notion of a contraction structure on a collection[11].

20.1. Definition. *Given a globular set (X, s_X, t_X), two n-cells $\nu^-, \nu^+ \in X$ are parallel if $s_X(\nu^-) = s_X(\nu^+)$ and $t_X(\nu^-) = t_X(\nu^+)$. All zero dimensional cells in X are parallel.*

Now, given a map $f : X \to Y$ of globular sets, for each nonzero n-cell $\nu \in Y_n$ we may consider the set

$$\mathrm{Par}_f(\nu) := \{(\rho^-, \rho^+) \in X_{n-1}\times X_{n-1} | \rho^- \text{ and } \rho^+ \text{ are parallel}, f(\rho^-) = s^n_Y(\nu), f(\rho^+) = t^n_Y(\nu)\}$$

of pairs of parallel $(n-1)$-cells in X that map via f to the boundary of ν in Y.

20.2. Definition. *Given a map $f : X \to Y$ of globular sets, a contraction $(f : X \to Y, \kappa^f)$ on f is a sequence of maps $\kappa^f = \{\kappa_\nu : Par_f(\nu) \to X_n\}$, indexed by the nonzero n-cell $\nu \in Y_n$, such that for each nonzero $\nu \in Y$*

$$s^n_X(\kappa_\nu(\rho^-, \rho^+)) = \rho^-$$

$$t^n_X(\kappa_\nu(\rho^-, \rho^+)) = \rho^+$$

$$f(\kappa_\nu(\rho^-, \rho^+)) = \nu$$

for every pair $(\rho^-, \rho^+) \in Par_f(\nu)$.

This definition may be weakened so that for any nonzero n-cell $\nu \in Y_n$ and any pair $(\rho^-, \rho^+) \in \mathrm{Par}_f(\nu)$ we require only that there exists an n-cell $\kappa \in X_n$ such that ρ^- and ρ^+ bound κ in X and $f(\kappa) = \nu$. In other words, whenever we can lift the boundary of an n-cell in Y we are furthermore able to lift the entire cell.

20.3. Definition. *A Leinster fibration is a globular set map $f : X \to Y$ which satisfies the property that for all $n \in \mathbb{N}$ and $\nu \in Y_n$, for each pair $(\rho^-, \rho^+) \in Par_f(\nu)$ there exists a cell $\gamma_\nu^{(\rho^-,\rho^+)} \in X_n$ such that*

$$s^n_X(\gamma_\nu^{(\rho^-,\rho^+)}) = \rho^-$$

$$t^n_X(\gamma_\nu^{(\rho^-,\rho^+)}) = \rho^+$$

$$f(\gamma_\nu^{(\rho^-,\rho^+)}) = \nu$$

Note that in the presence of the axiom of choice being a Leinster fibration is equivalent to the existence of a contraction structure on a globular set map. A contraction structure is a choice of lifts for a Leinster fibration. In this way, we may think of a contraction structure as a *split Leinster fibration*.

20.4. Definition. *A contraction structure on a globular operad $a : A \to \mathcal{T}(\mathbf{1})$ is a contraction on the unique map from a to the terminal collection $\mathcal{T}(\mathbf{1}) \to \mathcal{T}(\mathbf{1})$. In particular, it is a contraction on the arity map a.*

We shall now extend this construction to the theory of globular PROs.

20.5. Definition. *A contraction structure on a $\mathbb{N}$**Col**-graph homomorphism is a map of $\mathbb{N}$**Col**-graphs $F : \mathbf{G} \to \mathbf{H}$ such that each component $F_{n,m} : \mathbf{G}(n, m) \to \mathbf{H}(n, m)$, all of which are maps of globular sets, comes equipped with a specified contraction.*

Most often when working with contractions we will fix the target. Collectively, $\mathbb{N}$**Col**-graphs with contraction over the $\mathbb{N}$**Col**-graph $\mathbf{G}$ form a category $\mathbf{Cont}({}^{\mathbb{N}\mathbf{ColGraph}}/_{\mathbf{G}})$ whose morphisms are those in ${}^{\mathbb{N}\mathbf{ColGraph}}/_{\mathbf{G}}$ which preserve the contraction structure on each hom-object component.

Given any object in ${}^{\mathbb{N}\mathbf{ColGraph}}/_{\mathbf{G}}$ we can expand it to have a canonical contraction structure. In appendix G of Leinster's book[11], he describes a functorial construction for giving a generic collection $p : P \to \mathcal{T}(\mathbf{1})$ a canonical contraction structure

$$(C(p) : CP \to \mathcal{T}(\mathbf{1}), \kappa^{C(p)})$$

by inductively adjoining the requisite cells to P at each dimension to get the new collection $C(p)$ which has a natural induced contraction $\kappa^{C(p)}$. Moreover, Leinster's construction does not require that the object upon which we are adjoining a contraction structure be a collection. There is an analogous construction which sends any globular set map to a globular set map with contraction. This allows us to naturally extend this construction to $\mathbb{N}\mathbf{Col}$-graphs.

20.6. Definition. *Given a $\mathbb{N}\mathbf{Col}$-graph $\mathbf{G}$, for any object $H : \mathbf{H} \to \mathbf{G}$ in ${}^{\mathbb{N}\mathbf{ColGraph}}/_{\mathbf{G}}$, the $\mathbf{Col}$-graph with freely generated contraction structure over $\mathbf{G}$ on H, denoted $C_{\mathbf{G}}(H) : C_{\mathbf{G}}\mathbf{H} \to \mathbf{G}$, is constructed by applying Leinster's free contraction construction to each of the globular set map components*

$$H_{n,m} : \mathbf{H}(n,m) \to \mathbf{G}(n,m)$$

to make them globular sets with contraction

$$\left(C_{\mathbf{G}}(H_{n,m}) : C_{\mathbf{G}}\mathbf{H}(n,m) \to \mathbf{G}(n,m), \kappa^{C_{\mathbf{G}}(H_{n,m})}\right)$$

which collectively induce upon $C_{\mathbf{G}}(H) : C_{\mathbf{G}}\mathbf{H} \to \mathbf{G}$ the structure of a $\mathbb{N}\mathbf{Col}$-graph with contraction over $\mathbf{G}$.

20.7. Theorem. *Given a $\mathbb{N}\mathbf{Col}$-graph $\mathbf{G}$, $C_{\mathbf{G}} : {}^{\mathbb{N}\mathbf{ColGraph}}/_{\mathbf{G}} \to \mathbf{Cont}({}^{\mathbb{N}\mathbf{ColGraph}}/_{\mathbf{G}})$ has a right adjoint $R_{\mathbf{G}} : \mathbf{Cont}({}^{\mathbb{N}\mathbf{ColGraph}}/_{\mathbf{G}}) \to {}^{\mathbb{N}\mathbf{ColGraph}}/_{\mathbf{G}}$ which is finitary and monadic.*

Proof. It is immediately clear that such a right adjoint exists. Given any $\mathbb{N}\mathbf{Col}$-graph with contraction in $\mathbf{Cont}({}^{\mathbb{N}\mathbf{ColGraph}}/_{\mathbf{G}})$, $R_{\mathbf{G}}$ simply forgets the contraction with which each hom-object is equipped. As Leinster shows in his book[11], the right adjoint for his construction is both finitary and monadic over $\mathbf{Col}$. It is then clear by construction that these properties are preserved at the level of $R_{\mathbf{G}}$. ∎

We can now use this construction on $\mathbb{N}\mathbf{Col}$-graphs to create globular PROs whose algebras are weak versions of the algebras for the original globular PRO. First we fix a globular PRO P whose algebras we wish to weaken. We then consider the category ${}^{\mathbb{N}\mathbf{ColGraph}}/_{U(P)}$ of $\mathbf{Col}$-graphs with object set $\mathbb{N}$ sliced over the underlying $\mathbf{Col}$-graph of the PRO whose algebras are the strict models of the theory we wish to weaken. We then repeat the previous construction with $\mathbf{G} = U(P)$.

20.8. Definition. *A contraction structure on a globular PRO homomorphism is a map of globular PROs $F : \mathcal{P} \to \mathcal{P}'$ such that each component $F_{n,m} : \mathcal{P}(n,m) \to \mathcal{P}'(n,m)$ of its underlying $\mathbf{Col}$-functor, each of which is a map of globular sets, comes equipped with a specified contraction.*

20.9. Definition. *Given a globular PRO P, for any object $G : \boldsymbol{G} \to U(P)$ in ${}^{\mathbb{N}\mathbf{ColGraph}}/_{U(P)}$, the $\mathbf{Col}$-graph with freely generated contraction structure over $U(P)$ on G, denoted $C_P(G) : C_P\mathbf{G} \to U(P)$, is constructed by applying Leinster's free contraction construction to each of the globular set map components*

$$G_{n,m} : \mathbf{G}(n,m) \to U(P)(n,m)$$

to make them globular sets with contraction

$$\left(C_P(G_{n,m}) : C_P\mathbf{G}(n,m) \to U(P)(n,m), \kappa^{C_P(G_{n,m})}\right)$$

which collectively induce upon $C_P(G) : C_P\mathbf{G} \to U(P)$ the structure of a globular PRO with contraction over $U(P)$.

21. The Globular PRO with Contraction Monad

With the functor C_P we are almost able to construct the monad for globular PROs with contraction over $U(P)$. However, C_P only allows us to construct free contraction structures on **Col**-graphs over $U(P)$. We will need to extend the corresponding monads for the underlying functors $\mathcal{W} : \mathbf{Mon}\mathbb{N}\mathcal{D}\mathbf{Graph} \to \mathbb{N}\mathcal{D}\mathbf{Graph}$ and $U : \mathbf{ColCat} \to \mathbf{ColGraph}$ from above which allowed us to construct the monad for globular PROs. Then we shall be able to give objects in ${}^{\mathbb{N}\mathbf{ColGraph}}/_{U(P)}$ the full structure of a globular PRO with contraction over $U(P)$. First we will need the following theorem and corollary.

21.1. Theorem. *Let $T : \mathcal{C} \to \mathcal{C}$ be a monad over $\mathcal{C}$ whose associated adjunction is $F \dashv U$, such that $T = UF$, with unit and counit $\eta : \mathbb{1}_{\mathcal{C}} \Rightarrow UF$ and $\epsilon : FU \Rightarrow \mathbb{1}_{\mathbf{T\text{-}Alg}}$ respectively, where $\mathbf{T\text{-}Alg}$ is the category of T-algebras over $\mathcal{C}$. Then, given any T-algebra A, the induced functor $\overline{U} : {}^{\mathbf{T\text{-}Alg}}/_{A} \to {}^{\mathcal{C}}/_{U(A)}$ is monadic over ${}^{\mathcal{C}}/_{U(A)}$.*

Proof. Consider the functor $\mathcal{F} : {}^{\mathcal{C}}/_{U(A)} \to {}^{\mathbf{T\text{-}Alg}}/_{A}$ which is defined on objects as

$$\mathcal{F}(x : X \to U(A)) := \epsilon_A(F(x)) : F(X) \to FU(A) \to A$$

and sends a morphism

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "x"'] & & Y \arrow[dl, "y"] \\ & U(A) & \end{tikzcd}$$

in ${}^{\mathcal{C}}/_{U(A)}$ to the morphism

$$\begin{tikzcd} F(X) \arrow[rr, "\mathcal{F}(f)"] \arrow[d, "F(x)"'] & & F(Y) \arrow[d, "F(y)"] \\ FU(A) \arrow[dr, "\epsilon_A"'] & & FU(A) \arrow[dl, "\epsilon_A"] \\ & A & \end{tikzcd}$$

in ${}^{\mathbf{T\text{-}Alg}}/_{A}$. We shall show that $\mathcal{F} \dashv \overline{U}$ by checking that there is a natural isomorphism Φ between the appropriate hom-sets. Let $n : N \to U(A)$ be any object in ${}^{\mathcal{C}}/_{U(A)}$ and $m : M \to A$ be any object in ${}^{\mathbf{T\text{-}Alg}}/_{A}$. We first need to show that any morphism $f : \mathcal{F}(n) \to m$

can be identified with a morphism $\Phi(f) : n \to \overline{U}(m)$. Consider the following diagram obtained by precomposing $\overline{U}(f)$ with η_N.

$$\begin{tikzcd}
N \arrow[r, "\eta_N"] \arrow[ddr, "n"'] & UF(N) \arrow[r, "\overline{U}(f)"] \arrow[d, "UF(n)"] & U(M) \arrow[ddl, "U(m)"] \\
 & UFU(A) \arrow[d, "U(\epsilon_A)"] & \\
 & U(A) &
\end{tikzcd}$$

Note that this diagram represents a morphism $N \to U(A)$ in ${}^{\mathcal{C}}\!/_{U(A)}$ which is our desired candidate for $\Phi(f)$. It remains then to show that this diagram commutes. Immediately the right square commutes by construction as the image of a morphism under a functor. To see why the left square commutes consider the refinement of the previous diagram

$$\begin{tikzcd}
N \arrow[r, "\eta_N"] \arrow[dd, "n"'] & UF(N) \arrow[r, "\overline{U}(f)"] \arrow[d, "UF(n)"] & U(M) \arrow[ddl, "U(m)"] \\
 & UFU(A) \arrow[d, "U(\epsilon_A)"] & \\
U(A) \arrow[ur, "\eta_{U(A)}"] \arrow[r, "\mathbb{1}_{U(A)}"] & U(A) &
\end{tikzcd}$$

where the top square commutes by the naturality of η and the bottom triangle commutes by the unit/counit relations satisfied by F and U as adjoint functors. Dually, given a morphism $g : n \to \overline{U}(m)$ we wish to find a morphism $\Phi^{-1}(g) : \mathcal{F}(n) \to m$. Consider the diagram obtained by composing $\mathcal{F}$ with ϵ_M.

$$\begin{tikzcd}
F(N) \arrow[rr, "\mathcal{F}(g)"] \arrow[dr, "F(n)"'] & & FU(M) \arrow[r, "\epsilon_M"] \arrow[d, "FU(m)"] & M \arrow[ddl, "m"] \\
& FU(A) \arrow[dr, "\epsilon_A"'] & FU(A) \arrow[d, "\epsilon_A"] & \\
& & A &
\end{tikzcd}$$

Here the commutativity of the left pentagon follows by construction as the image of a functor and the right square follows by the naturality of ϵ. Hence this diagram represents the desired $\Phi^{-1}(g) : \mathcal{F}(n) \to m$ showing that the two homsets are isomorphic. Moreover, this isomorphism is natural in both variables by construction, as the relevant commutative naturality squares can each be seen as restrictions of the corresponding naturality square of the original adjunction.

Now let $\mathfrak{T} := \overline{U}\mathcal{F}$ and consider the comparison functor $K^{\mathfrak{T}} : {}^{\mathbf{T\text{-}Alg}}/_{A} \to ({}^{\mathcal{C}}/_{U(A)})^{\mathfrak{T}}$. We must show that it is an equivalence of categories. To see this we shall here think of T-algebras as a pair $(A, T(A) \to A)$ consisting of the underlying object of the algebra A together with the structure map $T(A) \to A$. From this perspective the functor U simply forgets the associated structure map for the pair. We can then consider a generic element $x^T : (X, T(X) \to X) \to (A, T(A) \to A)$ from ${}^{\mathbf{T\text{-}Alg}}/_{A}$ whose image under $K^{\mathfrak{T}}$ is given by the pair $(\overline{U}(x^T), \overline{U}(\epsilon^{\mathfrak{T}}_{x^T}))$ where $\epsilon^{\mathfrak{T}}_{x^T} : \mathcal{F}\overline{U}(x^T) \to x^T$ is the counit for the $\mathcal{F} \vdash \overline{U}$ adjunction. Hence we have the following:

$$K^{\mathfrak{T}}(x^T) = \left(\begin{tikzcd} X \arrow[d] \\ A \end{tikzcd} , \begin{tikzcd} T(X) \arrow[r] \arrow[d] & X \arrow[dl] \\ T(A) \arrow[r] & A \end{tikzcd} \right)$$

Consider now the following $\mathfrak{T}$-algebra in $({}^{\mathcal{C}}/_{U(A)})^{\mathfrak{T}}$:

$$\left(\begin{tikzcd} Y \arrow[d] \\ A \end{tikzcd} , \mathfrak{T}\left(\begin{tikzcd} Y \arrow[d] \\ A \end{tikzcd} \right) \to \begin{tikzcd} Y \arrow[d] \\ A \end{tikzcd} \right) = \left(\begin{tikzcd} Y \arrow[d] \\ A \end{tikzcd} , \begin{tikzcd} T(Y) \arrow[r] \arrow[d] & Y \arrow[dl] \\ T(A) \arrow[r] & A \end{tikzcd} \right)$$

Together these show that any $\mathfrak{T}$-algebra is the image of a T-algebra under $K^{\mathfrak{T}}$ and moreover this functor is in fact the identity functor on $^{\textbf{T-Alg}}/_A$. Hence $K^{\mathfrak{T}}$ is an equivalence of categories. Therefore $\mathcal{F} \dashv \overline{U}$ is a monadic adjunction. ∎

21.2. Corollary. *The induced functors* $\overline{\mathcal{W}}_P : {^{\mathbf{Mon}\mathbb{N}\mathbf{ColGraph}}}/_{U(P)} \to {^{\mathbb{N}\mathbf{ColGraph}}}/_{U(P)}$ *and* $\overline{U}_P : {^{\mathbb{N}\mathbf{ColCat}}}/_P \to {^{\mathbb{N}\mathbf{ColGraph}}}/_{U(P)}$ *for some globular PRO P, where $\mathcal{W}$ and U are the underlying functors for the free monoid and free* **Col**-*category monads defined above, are finitary and monadic over* ${^{\mathbb{N}\mathbf{ColGraph}}}/_{U(P)}$.

Proof. That these two functors are monadic follows immediately from the previous theorem. Moreover, since the forgetful functors from slice categories to the original category preserves and creates colimits, it follows that if $\mathcal{W}$ and U are finitary, then so are $\overline{\mathcal{W}}_P$ and $\overline{U}_P$. ∎

To complete our construction we will also need that the target category ${^{\mathbb{N}\mathbf{ColGraph}}}/_{U(P)}$ for each of our three finitary and monadic underlying functors is locally finitely presentable.

21.3. Corollary. *Given a globular PRO P, the category* ${^{\mathbb{N}\mathbf{ColGraph}}}/_{U(P)}$*, where the functor* $U : \mathbb{N}\mathbf{ColCat} \to \mathbb{N}\mathbf{ColGraph}$ *is the underlying functor for the free* **Col**-*category adjunction above, is locally finitely presentable.*

Proof. In the proof of Lemma 18.1 we showed that $\mathbb{N}\mathbf{ColGraph}$ is a presheaf category. Since ${^{\mathbb{N}\mathbf{ColGraph}}}/_{U(P)}$ is the slice of a presheaf category and slices of presheaf categories are themselves presheaf categories, it follows that ${^{\mathbb{N}\mathbf{ColGraph}}}/_{U(P)}$ is locally finitely presentable. ∎

21.4. Theorem. *The category of globular PROs with contraction over a fixed PRO P is monadic over* ${^{\mathbb{N}\mathbf{ColGraph}}}/_{U(P)}$.

Proof. By Theorem 20.7 and corollary 21.2 above, we know that $R_{U(\mathcal{P}_T)}, \overline{\mathcal{W}}_{U(\mathcal{P}_T)}$, and $\overline{U}_{\mathcal{P}_T}$ are finitary and monadic. By corollary 21.3 we know that ${^{\mathbb{N}\mathbf{ColGraph}}}/_{U(\mathcal{P}_T)}$ is locally finitely presentable. By applying the same theorem of Kelly[8] as before, we can construct the monadic pullback functor we desire by forming a pullback cube, each face of which is a pullback square. ∎

We shall here use

$$\mathfrak{G}_P : {^{\mathbb{N}\mathbf{ColGraph}}}/_{U(P)} \to {^{\mathbb{N}\mathbf{ColGraph}}}/_{U(P)}$$

to denote the monad constructed by this pullback. Its algebras are by definition globular PROs with contraction over P. Once again, we can apply this monad to the initial object in ${^{\mathbb{N}\mathbf{ColGraph}}}/_{U(P)}$ to construct an $\mathbb{N}$**Col**-graph with contraction over $U(P)$ that, when viewed as an algebra for our pullback monad, is the initial free globular PRO with contraction over P. More generally, we again get the following definition from this construction.

21.5. Definition. *The free globular PRO with contraction over a fixed PRO P generated by a $\mathbb{N}$**Col**-graph homomorphism $H : \mathbf{G} \to U(P)$ is the algebra*

$$(\mathfrak{G}_P(H), \mu_H^{\mathfrak{G}_P} : \mathfrak{G}_P^2(H) \to \mathfrak{G}_P(H))$$

for the globular PRO with contraction over P monad $\mathfrak{G}_P$.

22. Algebras: Fully Weakened

We now have all the requisite structure to construct fully weakened ω-categorified versions of any equational algebraic theory. The process proceeds as follows: We fix an algebraic theory T and consider the PRO P_T whose algebras are models of T. We then consider the globularization $\mathcal{P}_T$ of the classical PRO P_T. As previously shown, algebras for $\mathcal{P}_T$ are strict ω-categorifications of the theory T in the sense that they are precisely those algebras for P_T which carry the structure of a strict ω-category. To further construct the weakened ω-categorifications, we then take $U(\mathcal{P}_T)$, the underlying $\mathbb{N}$**ColGraph** of the globularized PRO $\mathcal{P}_T$, and consider the functor

$$C_{U(\mathcal{P}_T)} : {}^{\mathbb{N}\mathbf{ColGraph}}/_{U(\mathcal{P}_T)} \to \mathbf{Cont}({}^{\mathbb{N}\mathbf{ColGraph}}/_{U(\mathcal{P}_T)})$$

which freely generates, for each $\mathbb{N}$**Col**-graph sliced over $U(\mathcal{P}_T)$, a canonical contraction structure. We consider its right adjoint, the forgetful functor

$$R_{U(\mathcal{P}_T)} : \mathbf{Cont}({}^{\mathbb{N}\mathbf{ColGraph}}/_{U(\mathcal{P}_T)}) \to {}^{\mathbb{N}\mathbf{ColGraph}}/_{U(\mathcal{P}_T)}$$

(which forgets which cells are designated lifts) as well as the forgetful functors

$$\overline{\mathcal{W}}_{U(\mathcal{P}_T)} : {}^{\mathbf{Mon}\mathbb{N}\mathbf{ColGraph}}/_{U(\mathcal{P}_T)} \to {}^{\mathbb{N}\mathbf{ColGraph}}/_{U(\mathcal{P}_T)}$$

and

$$\overline{U}_{\mathcal{P}_T} : {}^{\mathbb{N}\mathbf{ColCat}}/_{\mathcal{P}_T} \to {}^{\mathbb{N}\mathbf{ColGraph}}/_{U(\mathcal{P}_T)}$$

for the free monoid and free path category constructions, respectively, induced by slicing over $U(\mathcal{P}_T)$. By Theorem 23.4, we can hence take what Kelly refers to as the algebraic colimit of the three monads corresponding to our monadic functors (which is technically a limit in the category $\mathbf{Mon}({}^{\mathbb{N}\mathbf{ColGraph}}/_{U(\mathcal{P}_T)})$ of monads over ${}^{\mathbb{N}\mathbf{ColGraph}}/_{U(\mathcal{P}_T)}$) to construct the monad

$$\mathfrak{G}_{\mathcal{P}_T} : {}^{\mathbb{N}\mathbf{ColGraph}}/_{U(\mathcal{P}_T)} \to {}^{\mathbb{N}\mathbf{ColGraph}}/_{U(\mathcal{P}_T)}$$

whose algebras are globular PROs with contraction over $U(\mathcal{P}_T)$. We then note that the category ${}^{\mathbb{N}\mathbf{ColGraph}}/_{U(\mathcal{P}_T)}$ has an initial object $\xi_{U(\mathcal{P}_T)}$, which is the empty $\mathbb{N}$**Col**-graph over $U(\mathcal{P}_T)$. By applying $\mathfrak{G}_{\mathcal{P}_T}$ to $\xi_{U(\mathcal{P}_T)}$ we get the initial free $\mathfrak{G}_{\mathcal{P}_T}$-algebra

$$(\mathfrak{G}_{\mathcal{P}_T}(\xi_{U(\mathcal{P}_T)}), \mu_{\xi_{U(\mathcal{P}_T)}}^{\mathfrak{G}_{\mathcal{P}_T}} : \mathfrak{G}_{\mathcal{P}_T}^2(\xi_{U(\mathcal{P}_T)}) \to \mathfrak{G}_{\mathcal{P}_T}(\xi_{U(\mathcal{P}_T)}))$$

where $\mu^{\mathfrak{G}_{\mathcal{P}_T}}_{\xi_{U(\mathcal{P}_T)}}$ is the component at $\xi_{U(\mathcal{P}_T)}$ of the monad multiplication transformation for $\mathfrak{G}_{\mathcal{P}_T}$. Algebras for $\mathfrak{G}_{\mathcal{P}_T}(\xi_{U(\mathcal{P}_T)})$, when considering just the underlying globular PRO sliced over $U(\mathcal{P}_T)$, are by construction the fully weakened ω-categorified T-algebras. Moreover, the other algebras for our pullback monad $\mathfrak{G}_{\mathcal{P}_T}$ are each globular PROs with contraction over $U(\mathcal{P}_T)$ whose algebras are various partial weakenings of the strict ω-categorified T-algebras.

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