Elements of ∞-Category Theory

Emily Riehl and Dominic Verity

Department of Mathematics, Johns Hopkins University, Baltimore, MD 21218, USA
Email address: `eriehl@math.jhu.edu`

Centre of Australian Category Theory, Macquarie University, NSW 2109, Australia
Email address: `dominic.verity@mq.edu.au`

This text is a rapidly-evolving work in progress — use at your own risk. The most recent version can always be found here:

`www.math.jhu.edu/~eriehl/elements.pdf`

We would be delighted to hear about any comments, corrections, or confusions readers might have. Please send to:

`eriehl@math.jhu.edu`

Commenced on January 14, 2018.

Last modified on April 30, 2020.

We would like to dedicate this book to our supervisors: J.M.E. Hyland and J.P. May.

Peter —

My debt to you is evident in everything that I write. I found it astounding that, after each of three or four consecutive close-readings of my thesis in which you suggested complete structural revisions, your advice remained entirely correct every time. You taught me to derive immense pleasure from each comprehensive reappraisal that is demanded whenever deepened mathematical understanding allows a more fundamental narrative to emerge.

By that standard, this text isn't ready yet. Nevertheless, in honor of your 80th birthday, I wanted to dedicate an early draft to you. I'm hoping you'll like the final version.

— Emily

Contents

Preface

Mathematical objects of a certain sophistication are frequently accompanied by higher homotopical structures: maps between them might be connected by homotopies, which might then be connected by higher homotopies, which might then be connected by even higher homotopies ad infinitum. The natural habitat for such mathematical objects is not an ordinary 1-category but instead an ∞-*category* or, more precisely, an $(\infty, 1)$-*category*, with the index "1" referring to the fact that its morphisms above the lowest dimension 1, the homotopies just discussed, are invertible.

Here the homotopies defining the higher morphisms of an ∞-category are to be regarded as *data* rather than as mere witnesses to an equivalence relation borne by the 1-dimensional morphisms. This has the consequence that all of the categorical structures in an ∞-category are weak. Even at the base level of 1-morphisms, composition is not necessarily uniquely defined but instead witnessed by a 2-morphism and associative up to a 3-morphism whose boundary data involves specified 2-morphism witnesses. Thus, diagrams valued in an ∞-category cannot be said to *commute* on the nose but are instead interpreted as *homotopy coherent*.

A fundamental challenge in defining ∞-categories has to do with giving a precise mathematical meaning of this notion of a weak composition law, not just for the 1-morphisms but also for the morphisms in higher dimensions. Indeed, there is a sense in which our traditional set-based foundations for mathematics are not really suitable for reasoning about ∞-categories; sets don't feature prominently in ∞-categorical data, especially when ∞-categories are considered in a morally correct fashion as objects that are only well-defined up to equivalence. The morphisms between a fixed pair of objects in an ∞-category assemble into an ∞-groupoid, which describes a well-defined homotopy type, though not a well-defined space. When considered up to equivalence, ∞-categories, like ordinary categories, also don't have a well-defined set of objects.

Precision is achieved through a variety of *models* of $(\infty, 1)$-categories, which are Bourbaki-style mathematical structures that encode infinite-dimensional categories with a weak composition law in which all morphisms above dimension 1 are weakly invertible. In order of appearance, these include *simplicial categories*, *quasi-categories* (nee. *weak Kan complexes*), *relative categories*, *Segal categories*, *complete Segal spaces*, and *1-complicial sets* (nee. *saturated 1-trivial weak complicial sets*), each of which comes with an associated array of naturally-occurring examples. The proliferation of models of $(\infty, 1)$-categories begs the question of how they might be compared. In the first decades of the 21st century, Bergner, Joyal–Tierney, Verity, Lurie, and Barwick–Kan built various bridges that prove that each of the models listed above "has the same homotopy theory" in the sense of defining the fibrant objects in Quillen equivalent model categories.[1]

In parallel with the development of models of $(\infty, 1)$-categories and the construction of comparisons between them, Joyal pioneered and Lurie and many others expanded a wildly successful project

[1]A recent book by Bergner surveys all but the last of these models and their interrelationships [**12**]. For a more whirlwind tour, see [**21**].

to extend basic category theory from ordinary 1-categories to $(\infty, 1)$-categories modeled as quasi-categories in such a way that the new quasi-categorical notions restrict along the standard embedding $\mathcal{C}at \hookrightarrow \mathcal{QC}at$ to the standard 1-categorical notions. A natural question is then: does this work extend to other models of $(\infty, 1)$-categories? And to what extent are basic ∞-categorical notions invariant under change of model? For instance, $(\infty, 1)$-categories of manifolds are most naturally constructed as complete Segal spaces, so Kazhdan–Varshavsky [**49**], Boavida de Brito [**29**], and Rasekh [**70, 69**] have recently endeavored to redevelop some of the category theory of quasi-categories using complete Segal space instead so as to have direct access to constructions and definitions that had previously been introduced only in the quasi-categorical model.

For practical, aesthetic, and moral reasons, the ultimate desire of practitioners is to work "model independently," meaning that theorems proven with any of the models of $(\infty, 1)$-categories would apply to them all, with the technical details inherent to any particular model never entering the discussion. Since all models of $(\infty, 1)$-categories "have the same homotopy theory" the general consensus is that the choice of model should not matter greatly, but one obstacle to proving results of this kind is that, to a large extent, precise versions of the categorical definitions that have been established for quasi-categories had not been given for the other models. In cases where comparable definitions do exist in different models, an ad-hoc heuristic proof of model-invariance of the categorical notion in question can typically be supplied, with details to be filled in by experts fluent in the combinatorics of each model, but it would be more reassuring to have a systematic method of comparing the category theory of $(\infty, 1)$-categories in different models via arguments that are somewhat closer to the ground.

Aims of this text

In this text we develop the theory of ∞-categories from first principles in a model-independent fashion using a common axiomatic framework that is satisfied by a variety of models. In contrast with prior "analytic" treatments of the theory of ∞-categories — in which the central categorical notions are defined in reference to the coordinates of a particular model — our approach is "synthetic," proceeding from definitions that can be interpreted simultaneously in many models to which our proofs then apply. While synthetic, our work is not schematic or hand-wavy, with the details of how to make things fully precise left to "the experts" and turtles all the way down.[2] Rather, we prove our theorems starting from a short list of clearly-enumerated axioms, and our conclusions are valid in any model of ∞-categories satisfying these axioms.

The synthetic theory is developed in any *∞-cosmos*, which axiomatizes the universe in which ∞-categories live as objects. So that our theorem statements suggest their natural interpretation, we recast *∞-category* as a technical term, to mean an object in some (typically fixed) ∞-cosmos. Several models of $(\infty, 1)$-categories[3] are ∞-categories in this sense, but our ∞-categories also include certain models of (∞, n)-categories[4] as well as fibered versions of all of the above. Thus each of these

[2]A less rigorous "model-independent" presentation of ∞-category theory might confront a problem of infinite regress, since infinite-dimensional categories are themselves the objects of an ambient infinite-dimensional category, and in developing the theory of the former one is tempted to use the theory of the latter. We avoid this problem by using a very concrete model for the ambient $(\infty, 2)$-category of ∞-categories that arises frequently in practice and is designed to facilitate relatively simple proofs. While the theory of $(\infty, 2)$-categories remains in its infancy, we are content to cut the Gordian knot in this way.

[3]Quasi-categories, complete Segal spaces, Segal categories, and 1-complicial sets (naturally marked quasi-categories) all define the ∞-categories in an ∞-cosmos.

[4]Θ_n-spaces, iterated complete Segal spaces, and n-complicial sets also define the ∞-categories in an ∞-cosmos, as do (nee. weak) complicial sets, a model for (∞, ∞)-categories. We hope to add other models of (∞, n)-categories to this list.

objects are ∞-categories in our sense and our theorems apply to all of them.[5] This usage of the term "∞-categories" is meant to interpolate between the classical one, which refers to any variety of weak infinite-dimensional categories, and the common one, which is often taken to mean quasi-categories or complete Segal spaces.

Much of the development of the theory of ∞-categories takes place not in the full ∞-cosmos but in a quotient that we call the *homotopy 2-category*, the name chosen because an ∞-cosmos is something like a category of fibrant objects in an enriched model category and the homotopy 2-category is then a categorification of its homotopy category. The homotopy 2-category is a strict 2-category — like the 2-category of categories, functors, and natural transformations[6] — and in this way the foundational proofs in the theory of ∞-categories closely resemble the classical foundations of ordinary category theory except that the universal properties they characterize, e.g. when a functor between ∞-categories defines a cartesian fibration, are slightly weaker than in the familiar case of strict 1-categories.

In Part I, we define and develop the notions of equivalence and adjunction between ∞-categories, limits and colimits in ∞-categories, cartesian and cocartesian fibrations and their discrete variants, and prove an external version of the Yoneda lemma all from the comfort of the homotopy 2-category. In an interlude, we digress into abstract ∞-cosmology to give a more careful account of the full class of limit constructions present in any ∞-cosmos. This analysis is used to develop further examples of ∞-cosmoi, whose objects are pointed ∞-categories, or stable ∞-categories, or cartesian or cocartesian fibrations in a given ∞-cosmos.

What's missing from this basic account of the category theory of ∞-categories is a satisfactory treatment of the "hom" bifunctor associated to an ∞-category, which is the prototypical example of what we call a *module*. In Part II, we develop the calculus of modules between ∞-categories and apply this to define and study pointwise Kan extensions. This will give us an opportunity to repackage universal properties proven in Part I as parts of the "formal category theory" of ∞-categories.

This work is all "model-agnostic" in the sense of being blind to details about the specifications of any particular ∞-cosmos. In Part III we prove that the category theory of ∞-categories is also "model-independent" in a precise sense: all categorical notions are preserved, reflected, and created by any "change-of-model" functor that defines what we call a *biequivalence*. This model-independence theorem is stronger than our axiomatic framework might initially suggest in that it also allows us to transfer theorems proven using "analytic" techniques to all biequivalent ∞-cosmoi. For instance, the four ∞-cosmoi whose objects model $(\infty, 1)$-categories are all biequivalent. It follows that the analytically-proven theorems about quasi-categories from **[59]** transfer to complete Segal spaces, and vice versa.

The ideal reader might already have some acquaintance with enriched category theory, 2-category theory, and abstract homotopy theory so that the constructions and proofs with antecedents in these traditions will be familiar. Because ∞-categories are of interest to mathematicians with a wide variety of backgrounds, we review all of the material we need on each of these topics in Appendices A, B, and C, respectively. Some basic facts about quasi-categories first proven by Joyal are needed to establish the corresponding features of general ∞-cosmoi in Chapter 1. We state all of these results in §1.1 but defer

[5]This may seem like sorcery but in some sense is really just the Yoneda lemma. To a close approximation, an ∞-cosmos is a "category of fibrant objects enriched over quasi-categories." When the Joyal–Lurie theory of quasi-categories is expressed in a sufficiently categorical way, it extends to encompass analogous results for the corresponding "representably defined" notions in a general ∞-cosmos.

[6]In fact this is another special case: there is an ∞-cosmos whose objects are ordinary categories and its homotopy 2-category is the usual category of categories, functors, and natural transformations.

the proofs that require a lengthy combinatorial digression to Appendix D, where we also introduce *n-complicial sets*, a model of (∞, n)-categories for any $0 \leq n \leq \infty$. The proofs that many examples of ∞-cosmoi appear "in the wild" can be found in Appendix E, where we also present general techniques that the reader might use to find even more examples. The final appendix addresses a crucial bit of unfinished business. Importantly, the synthetic theory developed in the ∞-cosmos of quasi-categories is fully compatible with the analytic theory developed by Joyal, Lurie, and many others. This is the subject of Appendix F.

Acknowledgments

The first draft of much of this material was written over the course of a semester-long topics course taught at Johns Hopkins in the Spring of 2018 and benefitted considerably from the perspicuous questions asked during lecture by Qingci An, Thomas Brazelton, tslil clingman, Daniel Fuentes-Keuthan, Aurel Malapani-Scala, Mona Merling, David Myers, Apurv Nakade, Martina Rovelli, and Xiyuan Wang.

Further revisions were made during the 2018 MIT Talbot Workshop on the model-independent theory of ∞-categories, organized by Eva Belmont, Calista Bernard, Inbar Klang, Morgan Opie, and Sean Pohorence, with Haynes Miller serving as the faculty sponsor. We were inspired by compelling lectures given at that workshop by Timothy Campion, Kevin Carlson, Kyle Ferendo, Daniel Fuentes-Keuthan, Joseph Helfer, Paul Lessard, Lyne Moser, Emma Phillips, Nima Rasekh, Martina Rovelli, Maru Sarazola, Matthew Weatherley, Jonathan Weinberger, Laura Wells, and Liang Ze Wong as well as by myriad discussions with Elena Dimitriadis Bermejo, Luciana Basualdo Bonatto, Olivia Borghi, Tai-Danae Bradley, Tejas Devanur, Aras Ergus, Matthew Feller, Sina Hazratpour, Peter James, Zhulin Li, David Myers, Maximilien Péroux, Mitchell Riley, Luis Scoccola, Brandon Shapiro, Pelle Steffens, Raffael Stenzel, Paula Verdugo, and Marco Vergura.

Gabriel Drummond-Cole encouraged us to do some much needed restructuring of Chapter 5. The presentation and organization in Chapter 9 was greatly improved by observations of Kevin Carlson and David Myers, who first stated the result appearing as Proposition 9.1.7. The material on the groupoid core of an $(\infty, 1)$-category was informed by discussions with Alexander Campbell and Yuri Sulyma, the latter of whom developed much of this material independently (and first) while working on his PhD thesis. Alexander Campbell was consulted several times during the writing of Appendix A, in particular regarding the subtle interaction between change-of-enrichment functors and underlying categories, and also alerted us to the ∞-cosmos of n-quasi-categories appearing in Proposition E.3.3 and supplied its proof. Naruki Masuda caught several typos and inconsistencies in Appendix D.

tslil clingman suggested a better way to typeset proarrows in the virtual equipment, greatly improving the displayed diagrams in Chapter 8. Doug Ravenel has propogated the use of the character "よ" for the Yoneda embedding. Anna Marie Bohmann, John Bourke, Alexander Campbell, Antoine Chambert-Loir, Gabriel Drummond-Cole, Harry Gindi, Dodam Ih, Chen-wei (Milton) Lin, Naruki Masuda, Emma Phillips, Maru Sarazola, Yuri Sulyma, Paula Verdugo, Jonathan Weinberger, and Hu Xiao wrote to point out typos.

Finally, the authors are grateful to financial support provided by the National Science Foundation for support via grants DMS-1551129 and DMS-1652600, the Australian Research Council via the grant DP160101519, and the Johns Hopkins Catalyst Award program. We also wish to thank the Department of Mathematics at Johns Hopkins University and the Centre of Australian Category Theory at Macquarie University for hosting our respective visits.

Part I

Basic ∞-category theory

CHAPTER 1

∞-Cosmoi and their Homotopy 2-Categories

In this chapter, we introduce the framework within which we will develop the formal category theory of ∞-categories, which goes by the name of an *∞-cosmos*. Informally, an ∞-cosmos is an $(\infty, 2)$-category—that is, a category enriched over $(\infty, 1)$-categories—that is equipped with $(\infty, 2)$-categorical limits, whose objects are ∞-categories in some model. To focus this abstract theory on its intended interpretation, we recast "∞-category" as a technical term, reserved to mean an object of some ∞-cosmos.

Rather surprisingly, we are able to use a very strict notion of "$(\infty, 2)$-category with $(\infty, 2)$-categorical limits," defining an ∞-cosmos to be a particular type of simplicially-enriched category and modeling the $(\infty, 2)$-categorical limits by strictly-defined simplicially-enriched limits, because so many examples satisfy these axioms. More precisely, an ∞-cosmos is a category enriched over quasi-categories, these being a model of $(\infty, 1)$-categories defined as a particular class of simplicial sets. The $(\infty, 2)$-categorical limits are defined as limits of diagrams involving a specified class of maps called *isofibrations*, which have no intrinsic homotopical meaning—in the sense that any functor between ∞-categories is equivalent to an isofibration—but allow us to consider strictly commuting diagrams.

In §1.1, we introduce quasi-categories, reviewing the classical results that will be needed to show that quasi-categories themselves assemble into an ∞-cosmos, providing the prototypical example. General ∞-cosmoi are defined in §1.2, where several examples are given and a number of basic results are proven. In §1.3, we turn our attention to *cosmological functors* between ∞-cosmoi, which preserve all of the defining structure. Cosmological functors serve dual purposes, on the one hand providing technical simplifications in many proofs, and then later on serving as the "change of model" functors that establish the model independence of ∞-category theory.

Finally, in 1.4, we introduce a strictly defined 2-category whose objects are ∞-categories, whose 1-cells are the ∞-functors between them, and whose 2-cells define ∞-natural transformations between these. Any ∞-cosmos has a 2-category of this sort, which we refer to as the *homotopy 2-category* of the ∞-cosmos. In fact, the reader who is particularly impatient to get on to the development of the category theory of ∞-categories can skip this chapter in its entirety, taking the existence of the homotopy 2-category for granted, and proceed straight to Chapter 2.

1.1. Quasi-categories

Before introducing an axiomatic framework that will allow us to develop ∞-category theory in general, we first consider one model in particular: namely, *quasi-categories*, which were introduced in 1973 by Boardman and Vogt **[16]** in their study of homotopy coherent diagrams. Ordinary 1-categories give examples of quasi-categories via the construction of Definition 1.1.4. Joyal first undertook the task of extending 1-category theory to quasi-categories in **[46]** and **[47]** and in several unpublished draft book manuscripts. The majority of the results in this section are due to him.

1.1.1. Notation (the simplex category). Let $\mathbf{\Delta}$ denote the **simplex category** of finite non-empty ordinals $[n] = \{0 < 1 < \cdots < n\}$ and order-preserving maps. These include in particular the

$$\text{elementary face operators} \qquad [n-1] \overset{\delta^i}{\rightarrowtail} [n] \qquad 0 \le i \le n \qquad \text{and the}$$

$$\text{elementary degeneracy operators} \qquad [n+1] \overset{\sigma^i}{\twoheadrightarrow} [n] \qquad 0 \le i \le n$$

whose images respectively omit and double up on the element $i \in [n]$. Every morphism in $\mathbf{\Delta}$ factors uniquely as an epimorphism followed by a monomorphism; these epimorphisms, the **degeneracy operators**, decompose as composites of elementary degeneracy operators, while the monomorphisms, the **face operators**, decompose as composites of elementary face operators.

The category of **simplicial sets** is the category $s\mathcal{S}et := \mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}}$ of presheaves on the simplex category. We write $\Delta[n]$ for the **standard n-simplex**, the simplicial set represented by $[n] \in \mathbf{\Delta}$, and $\Lambda^k[n] \subset \partial\Delta[n] \subset \Delta[n]$ for its **k-horn** and **boundary sphere** respectively. The sphere $\partial\Delta[n]$ is the simplicial subset generated by the codimension-one faces of the n-simplex, while the horn $\Lambda^k[n]$ is the further simplicial subset that omits the face opposite the vertex k.

Given a simplicial set X, it is conventional to write X_n for the set of **n-simplices**, defined by evaluating at $[n] \in \mathbf{\Delta}$. By the Yoneda lemma, each n-simplex $x \in X_n$ corresponds to a map of simplicial sets $x \colon \Delta[n] \to X$. Accordingly, we write $x \cdot \delta^i$ for the ith face of the n-simplex, an $(n-1)$-simplex classified by the composite map

$$\Delta[n-1] \xrightarrow{\delta^i} \Delta[n] \xrightarrow{x} X.$$

Geometrically, $x \cdot \delta^i$ is the "face opposite the vertex i" in the n-simplex x.

Since the morphisms of $\mathbf{\Delta}$ are generated by the elementary face and degeneracy operators, the data of a simplicial set[1] X is often presented by a diagram

$$\cdots \qquad X_3 \;\begin{smallmatrix} \xrightarrow{\delta_3} \\ \xleftarrow{\sigma_2} \\ \xrightarrow{\delta_2} \\ \xleftarrow{\sigma_1} \\ \xrightarrow{\delta_1} \\ \xleftarrow{\sigma_0} \\ \xrightarrow{\delta_0} \end{smallmatrix}\; X_2 \;\begin{smallmatrix} \xrightarrow{\delta_2} \\ \xleftarrow{\sigma_1} \\ \xrightarrow{\delta_1} \\ \xleftarrow{\sigma_0} \\ \xrightarrow{\delta_0} \end{smallmatrix}\; X_1 \;\begin{smallmatrix} \xrightarrow{\delta_1} \\ \xleftarrow{\sigma_0} \\ \xrightarrow{\delta_0} \end{smallmatrix}\; X_0,$$

identifying the sets of n-simplices for each $[n] \in \mathbf{\Delta}$ as well as the (contravariant) actions of the elementary operators, conventionally denoted using subscripts rather than superscripts.

1.1.2. Definition (quasi-category). A **quasi-category** is a simplicial set A in which any **inner horn** can be extended to a simplex, solving the displayed lifting problem:

$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d, hook] & A \\ \Delta[n] \arrow[ur, dashed] \end{tikzcd} \qquad \text{for} \qquad n \ge 2,\ 0 < k < n. \tag{1.1.3}$$

Quasi-categories were first introduced by Boardman and Vogt [**16**] under the name "weak Kan complexes," a **Kan complex** being a simplicial set admitting extensions as in (1.1.3) along all horn

[1] This presentation is also used for more general simplicial *objects* valued in any category.

inclusions $n \geq 1, 0 \leq k \leq n$. Since any topological space can be encoded as a Kan complex,[2] in this way spaces provide examples of quasi-categories.

Categories also provide examples of quasi-categories via the nerve construction.

1.1.4. Definition (nerve). The category $\mathcal{C}at$ of 1-categories embeds fully faithfully into the category of simplicial sets via the **nerve** functor. An n-simplex in the nerve of a 1-category C is a sequence of n composable arrows in C, or equally a functor $\mathbb{n}+\mathbb{1} \to C$ from the ordinal category $\mathbb{n}+\mathbb{1} := [n]$ with objects $0, \ldots, n$ and a unique arrow $i \to j$ just when $i \leq j$.

The map $[n] \mapsto \mathbb{n}+\mathbb{1}$ defines a fully faithful embedding $\mathbf{\Delta} \hookrightarrow \mathcal{C}at$. From this point of view, the nerve functor can be described as a "restricted Yoneda embedding" which carries a category C to the restriction of the representable functor $\hom(-, C)$ to the image of this inclusion. More general "nerve-type constructions" are described in Exercise 1.1.i.

1.1.5. Remark. The nerve of a category C is 2-coskeletal as a simplicial set, meaning that every sphere $\partial\Delta[n] \to C$ with $n \geq 3$ is filled uniquely by an n-simplex in C (see Definition C.5.2). Note a sphere $\partial\Delta[2] \to C$ extends to a 2-simplex if and only if that arrow along its diagonal edge is the composite of the arrows along the edges in the inner horn $\Lambda^1[2] \subset \partial\Delta[2] \to C$. The simplices in dimension 3 and above witness the associativity of the composition of the path of composable arrows found along their **spine**, the 1-skeletal simplicial subset formed by the edges connecting adjacent vertices. In fact, as suggested by the proof of Proposition 1.1.6, any simplicial set in which inner horns admit *unique* fillers is isomorphic to the nerve of a 1-category; see Exercise 1.1.iv.

We decline to introduce explicit notation for the nerve functor, preferring instead to identify 1-categories with their nerves. As we shall discover the theory of 1-categories extends to ∞-categories modeled as quasi-categories in such a way that the restriction of each ∞-categorical concept along the nerve embedding recovers the corresponding 1-categorical concept. For instance, the standard simplex $\Delta[n]$ is isomorphic to the nerve of the ordinal category $\mathbb{n}+\mathbb{1}$, and we frequently adopt the latter notation — writing $\mathbb{1} := \Delta[0]$, $\mathbb{2} := \Delta[1]$, $\mathbb{3} := \Delta[2]$, and so on — to suggest the correct categorical intuition.

To begin down this path, we must first verify the assertion that has implicitly just been made:

1.1.6. Proposition (nerves are quasi-categories). *Nerves of categories are quasi-categories.*

Proof. Via the isomorphism $C \cong \operatorname{cosk}_2 C$ and the adjunction $\operatorname{sk}_2 \dashv \operatorname{cosk}_2$ of C.5.2, the required lifting problem displayed below-left transposes to the one displayed below-right

$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d, hook] & C \cong \operatorname{cosk}_2 C \\ \Delta[n] \arrow[ur, dashed] \end{tikzcd} \qquad \leftrightsquigarrow \qquad \begin{tikzcd} \operatorname{sk}_2 \Lambda^k[n] \arrow[r] \arrow[d, hook] & C \\ \operatorname{sk}_2 \Delta[n] \arrow[ur, dashed] \end{tikzcd}$$

The functor sk_2 replaces a simplicial set by its **2-skeleton**, the simplicial subset generated by the simplices of dimension at most two. For $n \geq 4$, the inclusion $\operatorname{sk}_2 \Lambda^k[n] \hookrightarrow \operatorname{sk}_2 \Delta[n]$ is an isomorphism,

[2]The total singular complex construction defines a functor from topological spaces to simplicial sets that is an equivalence on their respective homotopy categories — weak homotopy types of spaces correspond to homotopy equivalence classes of Kan complexes [**68**, §II.2]. The left adjoint constructed by Exercise 1.1.i "geometrically realizes" a simplicial set as a topological space.

in which case the lifting problems on the right admit (unique) solutions. So it remains only to solve the lifting problems on the left in the cases $n = 2$ and $n = 3$.

To that end consider

$$\begin{tikzcd} \Lambda^1[2] \arrow[r] \arrow[d, hook] & C \\ \Delta[2] \arrow[ur, dashed] \end{tikzcd} \qquad \begin{tikzcd} \Lambda^1[3] \arrow[r] \arrow[d, hook] & C \\ \Delta[3] \arrow[ur, dashed] \end{tikzcd} \qquad \begin{tikzcd} \Lambda^2[3] \arrow[r] \arrow[d, hook] & C \\ \Delta[3] \arrow[ur, dashed] \end{tikzcd}$$

An inner horn $\Lambda^1[2] \to C$ defines a composable pair of arrows in C; an extension to a 2-simplex exists precisely because any composable pair of arrows admits a (unique) composite.

An inner horn $\Lambda^1[3] \to C$ specifies the data of three composable arrows in C, as displayed in the diagram below, together with the composites gf, hg, and $(hg)f$.

$$\begin{tikzcd} & c_1 \arrow[dr, "hg"] \arrow[dd, "g" near end] & \\ c_0 \arrow[ur, "f"] \arrow[rr, "(hg)f" near end] \arrow[dr, "gf"'] & & c_3 \\ & c_2 \arrow[ur, "h"'] & \end{tikzcd}$$

Because composition is associative, the arrow $(hg)f$ is also the composite of gf followed by h, which proves that the 2-simplex opposite the vertex c_1 is present in C; by 2-coskeletality, the 3-simplex filling this boundary sphere is also present in C. The filler for a horn $\Lambda^2[3] \to C$ is constructed similarly. ☐

1.1.7. Definition (homotopy relation on 1-simplices). A parallel pair of 1-simplices f, g in a simplicial set X are **homotopic** if there exists a 2-simplex whose boundary takes either of the following forms

$$\begin{tikzcd} & x_1 \arrow[dr, equal] & \\ x_0 \arrow[ur, "f"] \arrow[rr, "g"'] & & x_1 \end{tikzcd} \qquad \begin{tikzcd} & x_0 \arrow[dr, "f"] & \\ x_0 \arrow[ur, equal] \arrow[rr, "g"'] & & x_1 \end{tikzcd} \tag{1.1.8}$$

or if f and g are in the same equivalence class generated by this relation.

In a quasi-category, the relation witnessed by either of the types of 2-simplex on display in (1.1.8) is an equivalence relation and these equivalence relations coincide:

1.1.9. Lemma (homotopic 1-simplices in a quasi-category). *Parallel 1-simplices f and g in a quasi-category are homotopic if and only if there exists a 2-simplex of any or equivalently all of the forms displayed in* (1.1.8).

Proof. Exercise 1.1.ii. ☐

1.1.10. Definition (the homotopy category [**36**, §2.4]). By 1-truncating, any simplicial set X has an underlying reflexive directed graph

$$\begin{tikzcd} X_1 \arrow[r, shift left=2, "\delta_1"] \arrow[r, shift right=2, "\delta_0"'] & X_0 \arrow[l, "\sigma_0" description] \end{tikzcd},$$

the 0-simplices of X defining the objects and the 1-simplices defining the arrows. By convention, the source of an arrow $f \in X_1$ is its 0th face $f \cdot \delta^1$ (the face opposite 1) while the target is its 1st face $f \cdot \delta^0$ (the face opposite 0). The **free category** on this reflexive directed graph has X_0 as its object set, degenerate 1-simplices serving as identity morphisms, and non-identity morphisms defined to be finite directed paths of non-degenerate 1-simplices. The **homotopy category** $\mathsf{h}X$ of X is the quotient of the free category on its underlying reflexive directed graph by the congruence[3] generated by imposing a composition relation $h = g \circ f$ witnessed by 2-simplices

$$\begin{tikzcd} & x_1 \arrow[dr, "g"] & \\ x_0 \arrow[ur, "f"] \arrow[rr, "h"'] & & x_2 \end{tikzcd}$$

This relation implies in particular that homotopic 1-simplices represent the same arrow in the homotopy category.

The homotopy category of the nerve of a 1-category is isomorphic to the original category, as the 2-simplices in the nerve witness all of the composition relations satisfied by the arrows in the underlying reflexive directed graph. Indeed, the natural isomorphism $\mathsf{h}C \cong C$ forms the counit of an adjunction, embedding $\mathcal{C}at$ as a reflective subcategory of $s\mathcal{S}et$.

1.1.11. Proposition. *The nerve embedding admits a left adjoint, namely the functor which sends a simplicial set to its homotopy category:*

$$\begin{tikzcd} \mathcal{C}at \arrow[r, bend right, hook] \arrow[r, phantom, "\perp"] & s\mathcal{S}et \arrow[l, bend right, "\mathsf{h}"'] \end{tikzcd}$$

In fact, the adjunction of Proposition 1.1.11 exists for formal reasons; see Exercise 1.1.i. Nevertheless, a direct proof can be enlightening:

Proof. For any simplicial set X, there is a natural map $X \to \mathsf{h}X$ from X to the nerve of its homotopy category; since nerves are 2-coskeletal, it suffices to define the map $\mathrm{sk}_2 X \to \mathsf{h}X$, and this is given immediately by the construction of Definition 1.1.10. Note that the quotient map $X \to \mathsf{h}X$ becomes an isomorphism upon applying the homotopy category functor and is already an isomorphism whenever X is the nerve of a category. Thus the adjointness follows from Lemma B.4.2 or by direct verification of the triangle equalities. □

The homotopy category of a quasi-category admits a simplified description.

1.1.12. Lemma (the homotopy category of a quasi-category). *If A is a quasi-category then its **homotopy category** $\mathsf{h}A$ has*

- *the set of 0-simplices A_0 as its objects*
- *the set of homotopy classes of 1-simplices A_1 as its arrows*
- *the identity arrow at $a \in A_0$ represented by the degenerate 1-simplex $a \cdot \sigma^0 \in A_1$*
- *a composition relation $h = g \circ f$ in $\mathsf{h}A$ if and only if, for any choices of 1-simplices representing these arrows, there exists a 2-simplex with boundary*

$$\begin{tikzcd} & a_1 \arrow[dr, "g"] & \\ a_0 \arrow[ur, "f"] \arrow[rr, "h"'] & & a_2 \end{tikzcd}$$

Proof. Exercise 1.1.iii. □

[3] A relation on parallel pairs of arrows of a 1-category is a **congruence** if it is an equivalence relation that is closed under pre- and post-composition: if $f \sim g$ then $hfk \sim hgk$.

1.1.13. Definition (isomorphisms in a quasi-category). A 1-simplex in a quasi-category is an **isomorphism**[4] just when it represents an isomorphism in the homotopy category. By Lemma 1.1.12 this means that $f : a \to b$ is an isomorphism if and only if there exists a 1-simplex $f^{-1} : b \to a$ together with a pair of 2-simplices

$$\begin{tikzcd} & b \arrow[dr, dashed, "f^{-1}"] & \\ a \arrow[ur, "f"] \arrow[rr, equal] & & a \end{tikzcd} \qquad \begin{tikzcd} & a \arrow[dr, "f"] & \\ b \arrow[ur, dashed, "f^{-1}"] \arrow[rr, equal] & & b \end{tikzcd}$$

The properties of the isomorphisms in a quasi-category are most easily proved by arguing in a slightly different category where simplicial sets have the additional structure of a "marking" on a specified subset of the 1-simplices containing the degenerate 1-simplices; maps of these so-called **marked simplicial sets** must then preserve the markings. For instance, each quasi-category has a **natural marking**, where the marked 1-simplices are exactly the isomorphisms. Since the property of being an isomorphism in a quasi-category is witnessed by the presence of 2-simplices with a particular boundary, every map between quasi-categories preserves isomorphisms, inducing a map of the corresponding naturally marked quasi-categories. Because marked simplicial sets will seldom appear outside of the proofs of certain combinatorial lemmas about the isomorphisms in quasi-categories, we save the details for Appendix D.

Let us now motivate the first of several results proven using marked techniques. A quasi-category A is defined to have extensions along all *inner horns*. But if in an outer horn $\Lambda^0[2] \to A$ or $\Lambda^2[2] \to A$, the initial or final edges, respectively, map to isomorphisms in A, then intuitively a filler should exist

$$\begin{tikzcd} & a_1 \arrow[dr, dashed, "hf^{-1}"] & \\ a_0 \arrow[ur, "f", "\sim"'] \arrow[rr, "h"'] & & a_2 \end{tikzcd} \qquad \begin{tikzcd} & a_1 \arrow[dr, "g", "\sim"'] & \\ a_0 \arrow[ur, dashed, "g^{-1}h"] \arrow[rr, "h"'] & & a_2 \end{tikzcd}$$

and similarly for the higher-dimensional outer horns.

1.1.14. Proposition (special outer horn lifting).

(i) Let A be a quasi-category. Then for $n \geq 2$ any outer horns

$$\begin{tikzcd} \Lambda^0[n] \arrow[r, "g"] \arrow[d, hook] & A \\ \Delta[n] \arrow[ur, dashed] & \end{tikzcd} \qquad \begin{tikzcd} \Lambda^n[n] \arrow[r, "h"] \arrow[d, hook] & A \\ \Delta[n] \arrow[ur, dashed] & \end{tikzcd}$$

in which the edges $g|_{\{0,1\}}$ and $h|_{\{n-1,n\}}$ are isomorphisms admit fillers.[5]

(ii) Let A and B be quasi-categories and $f : A \to B$ a map that lifts against the inner horn inclusions. Then for $n \geq 2$ any outer horns

$$\begin{tikzcd} \Lambda^0[n] \arrow[r, "g"] \arrow[d, hook] & A \arrow[d, "f"] \\ \Delta[n] \arrow[ur, dashed] \arrow[r] & B \end{tikzcd} \qquad \begin{tikzcd} \Lambda^n[n] \arrow[r, "h"] \arrow[d, hook] & A \arrow[d, "f"] \\ \Delta[n] \arrow[ur, dashed] \arrow[r] & B \end{tikzcd}$$

[4] Joyal refers to these maps as "isomorphisms" while Lurie refers to them as "equivalences." We prefer, wherever possible, to use the same term for ∞-categorical concepts as for the analogous 1-categorical ones.

[5] Fillers also exist in the case $n = 1$ in the absence of any hypothesis, as required 1-simplices in A can be chosen to be degenerate.

in which the edges $g|_{\{0,1\}}$ and $h|_{\{n-1,n\}}$ are isomorphisms admit fillers.

The proof of Proposition 1.1.14 requires clever combinatorics, due to Joyal, and is deferred to Proposition D.4.5 and Theorem D.4.19 in Appendix D.[6] Here, we enjoy its myriad consequences. Immediately:

1.1.15. COROLLARY. *A quasi-category is a Kan complex if and only if its homotopy category is a groupoid.*

PROOF. If the homotopy category of a quasi-category is a groupoid, then all of its 1-simplices are isomorphisms, and Proposition 1.1.14 then implies that all inner and outer horns have fillers. Thus, the quasi-category is a Kan complex. Conversely, in a Kan complex, all outer horns can be filled and in particular fillers for the horns displayed in Definition 1.1.13 can be used to construct left and right inverses for any 1-simplex.[7] ☐

A quasi-category contains A a canonical **maximal sub Kan complex** $A^{\simeq}$, the simplicial subset spanned by those 1-simplices that are isomorphisms. Just as the arrows in a quasi-category A are represented by simplicial maps $\mathbb{2} \to A$ whose domain is the nerve of the free-living arrow, the isomorphisms in a quasi-category can be represented by diagrams $\mathbb{I} \to A$ whose domain is the free-living isomorphism:

1.1.16. COROLLARY. *An arrow f in a quasi-category A is an isomorphism if and only if it extends to a* ***homotopy coherent isomorphism***

$$\begin{tikzcd} \mathbb{2} \arrow[r, "f"] \arrow[d, hook] & A \\ \mathbb{I} \arrow[ur, dashed] & \end{tikzcd}$$

PROOF. If f is an isomorphism, the map $f\colon \mathbb{2} \to A$ lands in the maximal sub Kan complex contained in A:

$$\begin{tikzcd} \mathbb{2} \arrow[r, "f"] \arrow[d, hook] & A^{\simeq} \subset A \\ \mathbb{I} \arrow[ur, dashed] & \end{tikzcd}$$

By Exercise 1.1.v, the inclusion $\mathbb{2} \hookrightarrow \mathbb{I}$ can be expressed as a sequential composite of outer horn inclusions. Since $A^{\simeq}$, this shows that the required extension exists, and in fact lands in $A^{\simeq} \subset A$. ☐

The category of simplicial sets, like any category of presheaves, is cartesian closed. By the Yoneda lemma and the defining adjunction, an n-simplex in the exponential Y^X corresponds to a simplicial map $X \times \Delta[n] \to Y$, and its faces and degeneracies are computed by precomposing in the simplex variable. Our aim is now to show that the quasi-categories define an exponential ideal in the simplicially enriched category of simplicial sets: if X is a simplicial set and A is a quasi-category, then A^X is a quasi-category. We will deduce this as a corollary of the "relative" version of this result involving a class of maps called isofibrations that we now introduce.

[6]The second statement subsumes the first, but the first is typically used to prove the second.

[7]In a quasi-category, any left and right inverses to a common 1-simplex are homotopic by Lemma 1.1.12. Applying Lemma 1.1.12 it follows that any isomorphism in fact has a single two-sided inverse. In fact, considerably more is true: see Corollary 1.1.16.

1.1.17. Definition (isofibrations between quasi-categories). A simplicial map $f\colon A \to B$ between quasi-categories is a **isofibration** if it lifts against the inner horn inclusions, as displayed below left, and also against the inclusion of either vertex into the free-standing isomorphism $\mathbb{I}$.

$$
\begin{tikzcd}
\Lambda^k[n] \arrow[r] \arrow[d, hook] & A \arrow[d, two heads, "f"] \\
\Delta[n] \arrow[r] \arrow[ur, dashed] & B
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathbb{1} \arrow[r] \arrow[d, hook] & A \arrow[d, two heads, "f"] \\
\mathbb{I} \arrow[r] \arrow[ur, dashed] & B
\end{tikzcd}
$$

To notationally distinguish the isofibrations, we depict them as arrows "$\twoheadrightarrow$" with two heads.

By Theorem D.4.19, the isofibrations between quasi-categories can be understood as those maps that admit fillers for all inner horns as well as special outer horns in dimension $n \geq 1$, as opposed to only those horns with $n \geq 2$ appearing in the statement of Proposition 1.1.14.

1.1.18. Observation.

(i) For any simplicial set X, the unique map $X \to *$ whose codomain is the terminal simplicial set is an isofibration if and only if X is a quasi-category.

(ii) Any class of maps, such as the isofibrations, that is characterized by a right lifting property is automatically closed under composition, product, pullback, retract, and limits of towers; see Lemma C.2.3.

(iii) Combining (i) and (ii), if $A \twoheadrightarrow B$ is an isofibration, and B is a quasi-category, then so is A.

(iv) The isofibrations generalize the eponymous categorical notion. The nerve of any functor $f\colon A \to B$ between categories defines a map of simplicial sets that lifts against the inner horn inclusions. This map then defines an isofibration if and only if given any isomorphism in B and specified object in A lifting either its domain or codomain, there exists an isomorphism in A with that domain or codomain lifting the isomorphism in B.

Much harder to establish is the stability of the class of isofibrations under the formation of "Leibniz[8] exponentials" as displayed in (1.1.20). The proof of this result appears as Proposition D.5.1.

1.1.19. Proposition. *If $i\colon X \hookrightarrow Y$ is a monomorphism and $f\colon A \twoheadrightarrow B$ is an isofibration, then the induced Leibniz exponential map $i \mathbin{\widehat{\pitchfork}} f$*

$$
\begin{tikzcd}
A^Y \arrow[drr, bend left, two heads, "A^i"] \arrow[dr, dashed, two heads, "i\widehat{\pitchfork}f"] \arrow[ddr, bend right, two heads, "f^Y"'] & & \\
& \bullet \arrow[r, two heads] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner" very near start] & A^X \arrow[d, two heads, "f^X"] \\
& B^Y \arrow[r, two heads, "B^i"'] & B^X
\end{tikzcd}
\tag{1.1.20}
$$

is again an isofibration.[9]

1.1.21. Corollary. *If X is a simplicial set and A is a quasi-category, then A^X is a quasi-category. Moreover, a 1-simplex in A^X is an isomorphism if and only if its components at each vertex of X are isomorphisms in A.*

[8]The name alludes to the Leibniz rule in differential calculus, or more specifically to the identification of the domain of the Leibniz product of Lemma D.3.1 with the boundary of the prism; see Remark D.3.2.

[9]Degenerate cases of this result, taking $X = \varnothing$ or $B = *$, imply that the other six maps in this diagram are also isofibrations; see Exercise 1.1.vii.

Proof. The first statement is a special case of Proposition 1.1.19; see Exercise 1.1.vii. The second statement is proven similarly by arguing with marked simplicial sets. See Corollary D.4.18. ☐

1.1.22. Definition (equivalences of quasi-categories). A map $f\colon A \to B$ between quasi-categories is an **equivalence** if it extends to the data of a "homotopy equivalence" with the free-living isomorphism $\mathbb{I}$ serving as the interval: that is, if there exist maps $g\colon B \to A$ and

$$\begin{tikzcd} & A \\ A \arrow[ur, equal] \arrow[r, "\alpha"] \arrow[dr, "gf"'] & A^{\mathbb{I}} \arrow[u, "\mathrm{ev}_0"'] \arrow[d, "\mathrm{ev}_1"] \\ & A \end{tikzcd} \qquad \begin{tikzcd} & B \\ B \arrow[ur, "fg"] \arrow[r, "\beta"] \arrow[dr, equal] & B^{\mathbb{I}} \arrow[u, "\mathrm{ev}_0"'] \arrow[d, "\mathrm{ev}_1"] \\ & B \end{tikzcd}$$

We write "$\xrightarrow{\sim}$" to decorate equivalences and $A \simeq B$ to indicate the presence of an equivalence $A \xrightarrow{\sim} B$.

1.1.23. Remark. If $f\colon A \to B$ is an equivalence of quasi-categories equipped with the homotopy equivalence data displayed above, then the functor $\mathsf{h}f\colon \mathsf{h}A \to \mathsf{h}B$ is an equivalence of categories, with equivalence inverse $\mathsf{h}g\colon \mathsf{h}B \to \mathsf{h}A$ and natural isomorphisms encoded by the composite[10] functors

$$\mathsf{h}A \xrightarrow{\mathsf{h}\alpha} \mathsf{h}(A^{\mathbb{I}}) \longrightarrow (\mathsf{h}A)^{\mathbb{I}} \qquad \mathsf{h}B \xrightarrow{\mathsf{h}\beta} \mathsf{h}(B^{\mathbb{I}}) \longrightarrow (\mathsf{h}B)^{\mathbb{I}}$$

1.1.24. Definition. A map $f\colon X \to Y$ between simplicial sets is a **trivial fibration** if it admits lifts against the boundary inclusions for all simplices

$$\begin{tikzcd} \partial\Delta[n] \arrow[r] \arrow[d, hook] & X \arrow[d, two heads, "\wr", "f"'] \\ \Delta[n] \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \qquad n \geq 0 \tag{1.1.25}$$

We write "$\xrightarrow{\sim}\!\!\!\!\rightarrow$" to decorate trivial fibrations.

1.1.26. Remark. The simplex boundary inclusions $\partial\Delta[n] \hookrightarrow \Delta[n]$ "cellularly generate" the monomorphisms of simplicial sets — see Definition C.2.4 and Lemma C.5.9. Hence the dual of Lemma C.2.3 implies that trivial fibrations lift against any monomorphism between simplicial sets. In particular, applying this to the map $\varnothing \to Y$, it follows that any trivial fibration $X \xrightarrow{\sim}\!\!\!\!\rightarrow Y$ is a split epimorphism.

The notation "$\xrightarrow{\sim}\!\!\!\!\rightarrow$" is suggestive: the trivial fibrations between quasi-categories are exactly those maps that are both isofibrations and equivalences. This can be proven by a relatively standard although rather technical argument in simplicial homotopy theory, appearing as Proposition D.5.2.

1.1.27. Proposition. *For a map $f\colon A \to B$ between quasi-categories the following are equivalent:*

(i) f is a trivial fibration
(ii) f is both an isofibration and an equivalence

[10] Note that $\mathsf{h}(A^{\mathbb{I}}) \not\cong (\mathsf{h}A)^{\mathbb{I}}$ in general. Objects in the latter are homotopy classes of isomorphisms in A, while objects in the former are homotopy coherent isomorphisms, given by a specified 1-simplex in A, a specified inverse 1-simplex, together with an infinite tower of coherence data indexed by the non-degenerate simplices in $\mathbb{I}$.

(iii) *f is a **split fiber homotopy equivalence**: an isofibration admitting a section s that is also an equivalence inverse via a homotopy α from 1_A to sf that composes with f to the constant homotopy from f to f.*

$$\begin{tikzcd} A+A \arrow[r, "(\mathrm{id}_A, sf)"] \arrow[d, hook] & & A \arrow[d, two heads, "f", "\wr"'] \\ A \times \mathbb{I} \arrow[urr, dashed, "\alpha"] \arrow[r, "\pi"'] & A \arrow[r, two heads, "f", "\sim"'] & B \end{tikzcd}$$

As a class characterized by a right lifting property, the trivial fibrations are also closed under composition, product, pullback, limits of towers, and contain the isomorphisms. The stability of these maps under Leibniz exponentiation is proven in Proposition D.5.1, the same result that proves Proposition 1.1.19.

1.1.28. Proposition. *If $i\colon X \to Y$ is a monomorphism and $f\colon A \to B$ is an isofibration, then if either f is a trivial fibration or if i is in the class cellularly generated by the inner horn inclusions and the map $\mathbb{1} \hookrightarrow \mathbb{I}$ then the induced Leibniz exponential map*

$$A^Y \xrightarrow{\widehat{i \pitchfork} f} B^Y \times_{B^X} A^X$$

a trivial fibration.

To illustrate the utility of these Leibniz stability results, we give an "internal" or "synthetic" characterization of the Kan complexes.

1.1.29. Lemma. *A quasi-category A is a Kan complex if and only if the map $A^{\mathbb{I}} \twoheadrightarrow A^{\mathbb{2}}$ induced by the inclusion $\mathbb{2} \hookrightarrow \mathbb{I}$ is a trivial fibration.*

Note that Proposition 1.1.19 implies that $A^{\mathbb{I}} \twoheadrightarrow A^{\mathbb{2}}$ is an isofibration.

Proof. The lifting property that characterizes trivial fibrations transposes to another lifting property, displayed below-right

$$\begin{tikzcd} \partial\Delta[n] \arrow[r] \arrow[d, hook] & A^{\mathbb{I}} \arrow[d, two heads] \\ \Delta[n] \arrow[r] \arrow[ur, dashed] & A^{\mathbb{2}} \end{tikzcd} \qquad \leftrightsquigarrow \qquad \begin{tikzcd} \partial\Delta[n]\times\mathbb{I} \underset{\partial\Delta[n]\times\mathbb{2}}{\cup} \Delta[n]\times\mathbb{2} \arrow[r] \arrow[d, hook] & A \\ \Delta[n]\times\mathbb{I} \arrow[ur, dashed] \end{tikzcd}$$

that asserts that A admits extensions along maps formed by taking the *Leibniz product*—also known as the pushout product—of a simplex boundary inclusion $\partial\Delta[n] \hookrightarrow \Delta[n]$ with the inclusion $\mathbb{2} \hookrightarrow \mathbb{I}$. By Exercise 1.1.v(ii) the inclusion $\mathbb{2} \hookrightarrow \mathbb{I}$ is a sequential composite of left outer horn inclusions. By Corollary D.3.12, a key step along the way to the proofs of Propositions 1.1.19 and 1.1.28, it follows that the Leibniz product is also a sequential composite of left and inner horn inclusions. If A is a Kan complex, then the extensions displayed above right exist, and, by transposing, the map $A^{\mathbb{I}} \twoheadrightarrow A^{\mathbb{2}}$ is a trivial fibration.

Conversely, if $A^{\mathbb{I}} \overset{\sim}{\twoheadrightarrow} A^{\mathbb{2}}$ is a trivial fibration then in particular it is surjective on vertices. Thus every arrow in A is an isomorphism, and Corollary 1.1.15 tells us that A must be a Kan complex. ☐

1.1.30. Digression (the Joyal model structure). The category of simplicial sets bears a Quillen model structure, in the sense of Definition C.3.1, whose fibrant objects are exactly the quasi-categories; all objects are cofibrant. Between fibrant objects, the fibrations, weak equivalences, and trivial fibrations are precisely the classes of isofibrations, equivalences, and trivial fibrations defined above. Proposition 1.1.27 proves that the trivial fibrations are the intersection of the classes of fibrations and weak equivalences. Propositions 1.1.19 and 1.1.28 reflect the fact that the Joyal model structure is a *cartesian closed model category*, satisfying the additional axioms of Definition C.3.9.

We have declined to elaborate on the Joyal model structure for quasi-categories alluded to in Digression 1.1.30 because the only aspects of it that we will need are those described above. The results enumerated here suffice to show that the category of quasi-categories defines an ∞-*cosmos*, a concept to which we now turn.

Exercises.

1.1.i. Exercise. Given any cosimplicial object $C\colon \mathbf{\Delta} \to \mathcal{E}$ valued in any category $\mathcal{E}$, there is an associated **nerve functor** N_C defined by:

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{N_C} & s\mathcal{S}et \\ E & \longmapsto & \hom(C^-, E) \end{array} \qquad \begin{tikzcd} & \mathbf{\Delta} \arrow[dl, "C"'] \arrow[dr, hook, "\yo"] & \\ \mathcal{E} \arrow[rr, bend right, "N_C"'] \arrow[rr, phantom, "\perp"] & & s\mathcal{S}et \arrow[ll, bend right, "\mathrm{lan}_{\yo} C"'] \end{tikzcd}$$

By construction n-simplices in $N_C E$ correspond to maps $C^n \to E$ in $\mathcal{E}$. Show that if $\mathcal{E}$ is cocomplete, then N_C has a left adjoint defined as the left Kan extension of the functor C along the Yoneda embedding $\yo\colon \mathbf{\Delta} \hookrightarrow s\mathcal{S}et$. This gives a second proof of Proposition 1.1.11.

1.1.ii. Exercise (Boardman-Vogt [16]). Consider the set of 1-simplices in a quasi-category with initial vertex a and final vertex b.

(i) Prove that the relation defined by $f \sim g$ if and only if there exists a 2-simplex with boundary

$$\begin{tikzcd} & b \arrow[dr, equal] & \\ a \arrow[ur, "f"] \arrow[rr, "g"'] & & b \end{tikzcd}$$ is an equivalence relation.

(ii) Prove that the relation defined by $f \sim g$ if and only if there exists a 2-simplex with boundary

$$\begin{tikzcd} & a \arrow[dr, "f"] & \\ a \arrow[ur, equal] \arrow[rr, "g"'] & & b \end{tikzcd}$$ is an equivalence relation.

(iii) Prove that the equivalence relations defined by (i) and (ii) are the same.

This proves Lemma 1.1.9.

1.1.iii. Exercise. Consider the free category on the reflexive directed graph

$$\begin{tikzcd} A_1 \arrow[r, shift left=2, "\delta_1"] \arrow[r, shift right=2, "\delta_0"'] & A_0, \arrow[l, "\sigma_0" description] \end{tikzcd}$$

underlying a quasi-category A.

(i) Consider the relation that identifies a pair of sequences of composable 1-simplices with common source and common target whenever there exists a simplex of A in which the sequences of 1-simplices define two paths from its initial vertex to its final vertex. Prove that this relation is stable under pre- and post-composition with 1-simplices and conclude that its transitive closure is a **congruence**: an equivalence relation that is closed under pre- and post-composition.[11]

(ii) Consider the congruence relation generated by imposing a composition relation $h = g \circ f$ witnessed by 2-simplices

$$\begin{tikzcd} & a_1 \arrow[dr, "g"] & \\ a_0 \arrow[ur, "f"] \arrow[rr, "h"'] & & a_2 \end{tikzcd}$$

and prove that this coincides with the relation considered in (i).

(iii) In the congruence relations of (i) and (ii), prove that every sequence of composable 1-simplices in A is equivalent to a single 1-simplex. Conclude that every morphism in the quotient of the free category by this congruence relation is represented by a 1-simplex in A.

(iv) Prove that for any triple of 1-simplices f, g, h in A, $h = g \circ f$ in the homotopy category $\mathsf{h}A$ of Definition 1.1.10 if and only if there exists a 2-simplex with boundary

$$\begin{tikzcd} & a_1 \arrow[dr, "g"] & \\ a_0 \arrow[ur, "f"] \arrow[rr, "h"'] & & a_2 \end{tikzcd}$$

This proves Lemma 1.1.12.

1.1.iv. Exercise. Show that any quasi-category in which inner horns admit unique fillers is isomorphic to the nerve of its homotopy category.

1.1.v. Exercise. Let $\mathbb{I}$ be the nerve of the free-living isomorphism.

(i) Prove that $\mathbb{I}$ contains exactly two non-degenerate simplices in each dimension.

(ii) Inductively build $\mathbb{I}$ from $\mathbb{2}$ by expressing the inclusion $\mathbb{2} \hookrightarrow \mathbb{I}$ as a sequential composite of pushouts of left outer horn inclusions[12] $\Lambda^0[n] \hookrightarrow \Delta[n]$, one in each dimension starting with $n = 2$.[13]

1.1.vi. Exercise. Prove the relative version of Corollary 1.1.16: for any isofibration $p\colon A \twoheadrightarrow B$ between quasi-categories and any $f\colon \mathbb{2} \to A$ that defines an isomorphism in A any homotopy coherent isomorphism in B extending pf lifts to a homotopy coherent isomorphism in A extending f.

$$\begin{tikzcd} \mathbb{2} \arrow[r, "f"] \arrow[d, hook] & A \arrow[d, two heads, "p"] \\ \mathbb{I} \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

1.1.vii. Exercise. Specialize Proposition 1.1.19 to prove the following:

(i) If A is a quasi-category and X is a simplicial set then A^X is a quasi-category.

(ii) If A is a quasi-category and $X \hookrightarrow Y$ is a monomorphism then $A^Y \twoheadrightarrow A^X$ is an isofibration.

[11] Given a congruence relation on the hom-sets of a 1-category, the quotient category can be formed by quotienting each hom-set; see [**61**, §II.8].

[12] By the duality described in Definition 1.2.25, the right outer horn inclusions $\Lambda^n[n] \hookrightarrow \Delta[n]$ can be used instead.

[13] This decomposition of the inclusion $\mathbb{2} \hookrightarrow \mathbb{I}$ reveals which data can always be extended to a homotopy coherent isomorphism: for instance, the 1- and 2-simplices of Definition 1.1.13 together with a single 3-simplex that has these as its outer faces with its inner faces degenerate. Homotopy type theorists refer to this data as a **half adjoint equivalence** [**95**, §4.2].

(iii) If $A \twoheadrightarrow B$ is an isofibration and X is a simplicial set then $A^X \twoheadrightarrow B^X$ is an isofibration.

1.1.viii. Exercise. Anticipating Lemma 1.2.20:

(i) Prove that the equivalences defined in Definition 1.1.22 are closed under retracts.
(ii) Prove that the equivalences defined in Definition 1.1.22 satisfy the 2-of-3 property.

1.1.ix. Exercise. Prove that if $f\colon X \xrightarrow{\sim}\!\!\!\!\twoheadrightarrow Y$ is a trivial fibration between quasi-categories then the functor $\mathsf{h}f\colon \mathsf{h}X \xrightarrow{\sim}\!\!\!\!\twoheadrightarrow \mathsf{h}Y$ is a surjective equivalence of categories.

1.2. ∞-Cosmoi

In §1.1, we presented "analytic" proofs of a few of the basic facts about quasi-categories. The category theory of quasi-categories can be developed in a similar style, but we aim instead to develop the "synthetic" theory of infinite-dimensional categories, so that our results will apply to many models at once. To achieve this, our strategy is not to axiomatize what infinite-dimensional categories *are*, but rather to axiomatize the categorical "universe" in which they *live*.

The following definition introduces the notion of an ∞-*cosmos*, which abstracts the properties of the category of quasi-categories together with the classes of isofibrations, equivalences, and trivial fibrations introduced in §1.1.[14] Firstly, the category of quasi-categories is *enriched* over the category of simplicial sets — the set of morphisms from A to B coincides with the set of vertices of the simplicial set B^A — and moreover these hom-spaces are all quasi-categories. Secondly, certain limit constructions that can be defined in the underlying unenriched category of quasi-categories satisfy universal properties relative to this simplicial enrichment, with the usual isomorphism of sets extending to an isomorphism of simplicial sets. And finally, the classes of isofibrations, equivalences, and trivial fibrations satisfy properties that are familiar from abstract homotopy theory, forming a *category of fibrant objects* à la Brown [**20**] and reviewed in Appendix C. In particular, the use of isofibrations in diagrams guarantees that their strict limits are equivalence invariant, so we can take advantage of up-to-isomorphism universal properties and strict functoriality of these constructions while still working "homotopically."

As will be explained in Digression 1.2.13, there are a variety of models of infinite-dimensional categories for which the category of "∞-categories," as we will call them, and "∞-functors" between them is enriched over quasi-categories and admits classes of isofibrations, equivalences, and trivial fibrations satisfying analogous properties. This motivates the following axiomatization:

1.2.1. Definition (∞-cosmoi). An **∞-cosmos** $\mathcal{K}$ is a category that is enriched over quasi-categories,[15] meaning in particular that

- its morphisms $f\colon A \to B$ define the vertices of a quasi-category denoted $\mathsf{Fun}(A,B)$ and referred to as a **functor-space**

that is also equipped with a specified class of maps that we call **isofibrations** and denote by "$\twoheadrightarrow$" and satisfies the following two axioms:

[14] Metaphorical allusions aside, our ∞-cosmoi resemble the fibrational cosmoi of Street [**89**].

[15] This is to say $\mathcal{K}$ is a simplicially enriched category whose hom-spaces are all quasi-categories; this will be unpacked in 1.2.4.

(i) (completeness) The quasi-categorically enriched category $\mathcal{K}$ possesses a terminal object, small products, pullbacks of isofibrations, limits of countable towers of isofibrations, and cotensors with all simplicial sets, each of these limit notions satisfying a universal property that is enriched over simplicial sets.[16]

(ii) (isofibrations) The class of isofibrations contains all isomorphisms and any map whose codomain is the terminal object; is closed under composition, product, pullback, forming inverse limits of towers, and Leibniz cotensors with monomorphisms of simplicial sets; and has the property that if $f\colon A \twoheadrightarrow B$ is an isofibration and X is any object then $\mathsf{Fun}(X, A) \twoheadrightarrow \mathsf{Fun}(X, B)$ is an isofibration of quasi-categories.

1.2.2. Definition. In an ∞-cosmos $\mathcal{K}$, we define a morphism $f\colon A \to B$ to be

- an **equivalence** if and only if the induced map $f_*\colon \mathsf{Fun}(X, A) \xrightarrow{\sim} \mathsf{Fun}(X, B)$ on functor-spaces is an equivalence of quasi-categories for all $X \in \mathcal{K}$, and
- a **trivial fibration** just when f is both an isofibration and an equivalence.

These classes are denoted by "$\xrightarrow{\sim}$" and "$\overset{\sim}{\twoheadrightarrow}$" respectively.

Put more concisely, one might say that an ∞-cosmos is a "quasi-categorically enriched category of fibrant objects." See Definition C.1.1 and Lemma C.1.3.

1.2.3. Convention (∞-category, as a technical term). Henceforth, we recast ∞-**category** as a technical term to refer to an object in an arbitrary ambient ∞-cosmos. Similarly, we use the term ∞-**functor**—or more commonly the abbreviation "**functor**"—to refer to a morphism $f\colon A \to B$ in an ∞-cosmos. This explains why we refer to the quasi-category $\mathsf{Fun}(A, B)$ between two ∞-categories in an ∞-cosmos as a "functor space": its vertices are the (∞-)functors from A to B.

1.2.4. Digression (simplicial categories). A **simplicial category** $\mathcal{A}$ is given by categories $\mathcal{A}_n$, with a common set of objects and whose arrows are called n-**arrows**, that assemble into a diagram $\mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}at$ of identity-on-objects functors

$$\begin{tikzcd}[column sep=large]
\cdots & \mathcal{A}_3 \arrow[r, shift left=9, "\delta_3"] \arrow[r, leftarrow, shift left=6, "\sigma_2"] \arrow[r, shift left=3, "\delta_2"] \arrow[r, leftarrow, "\sigma_1"] \arrow[r, shift right=3, "\delta_1"] \arrow[r, leftarrow, shift right=6, "\sigma_0"] \arrow[r, shift right=9, "\delta_0"] & \mathcal{A}_2 \arrow[r, shift left=6, "\delta_2"] \arrow[r, leftarrow, shift left=3, "\sigma_1"] \arrow[r, "\delta_1"] \arrow[r, leftarrow, shift right=3, "\sigma_0"] \arrow[r, shift right=6, "\delta_0"] & \mathcal{A}_1 \arrow[r, shift left=3, "\delta_1"] \arrow[r, leftarrow, "\sigma_0"] \arrow[r, shift right=3, "\delta_0"] & \mathcal{A}_0 & =: \mathcal{A}
\end{tikzcd} \tag{1.2.5}$$

The data of a simplicial category can equivalently be encoded by a **simplicially enriched category** with a set of objects and a simplicial set $\mathcal{A}(x, y)$ of morphisms between each ordered pair of objects: an n-arrow in $\mathcal{A}_n$ from x to y corresponds to an n-simplex in $\mathcal{A}(x, y)$; see Exercise 1.2.i. Each endo-hom-space contains a distinguished identity 0-arrow (the degenerate images of which define the corresponding identity n-arrows) and composition is required to define a simplicial map

$$\mathcal{A}(y, z) \times \mathcal{A}(x, y) \xrightarrow{\circ} \mathcal{A}(x, z)$$

the single map encoding the compositions in each of the categories $\mathcal{A}_n$ and also the functoriality of the diagram (1.2.5). The composition is required to be associative and unital, in a sense expressed by

[16] This will be elaborated upon in 1.2.6.

the commutative diagrams

$$\begin{tikzcd} \mathcal{A}(y,z)\times\mathcal{A}(x,y)\times\mathcal{A}(w,x) \arrow[r, "\circ\times 1"] \arrow[d, "1\times\circ"'] & \mathcal{A}(x,z)\times\mathcal{A}(w,x) \arrow[d, "\circ"] \\ \mathcal{A}(y,z)\times\mathcal{A}(w,y) \arrow[r, "\circ"'] & \mathcal{A}(w,z) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A}(x,y) \arrow[r, "\mathrm{id}_y\times 1"] \arrow[d, "1\times\mathrm{id}_x"'] \arrow[dr, equal] & \mathcal{A}(y,y)\times\mathcal{A}(x,y) \arrow[d, "\circ"] \\ \mathcal{A}(x,y)\times\mathcal{A}(x,x) \arrow[r, "\circ"'] & \mathcal{A}(x,y) \end{tikzcd}$$

the latter making use of natural isomorphisms $\mathcal{A}(x,y)\times 1 \cong \mathcal{A}(x,y) \cong 1\times\mathcal{A}(x,y)$ in the domain vertex.

On account of the equivalence between these two presentations, the terms "simplicial category" and "simplicially-enriched category" are generally taken to be synonyms.[17] The category $\mathcal{A}_0$ of 0-arrows is the **underlying category** of the simplicial category $\mathcal{A}$, which forgets the higher dimensional simplicial structure.

In particular, the underlying category $\mathcal{K}_0$ of an ∞-cosmos $\mathcal{K}$ is the category whose objects are the ∞-categories in $\mathcal{K}$ and whose morphisms are the ∞-functors, i.e., the 0-arrows in the functor spaces. In all of the examples to appear below, this recovers the expected category of ∞-categories in a particular model and functors between them.

1.2.6. Digression (simplicially enriched limits). Let $\mathcal{A}$ be a simplicial category. The **cotensor** of an object $A \in \mathcal{A}$ by a simplicial set U is characterized by an isomorphism of simplicial sets

$$\mathcal{A}(X, A^U) \cong \mathcal{A}(X,A)^U \tag{1.2.7}$$

natural in $X \in \mathcal{A}$. Assuming such objects exist, the simplicial cotensor defines a bifunctor

$$\begin{aligned} s\mathcal{S}et^{\mathrm{op}}\times\mathcal{A} &\longrightarrow \mathcal{A} \\ (U,A) &\longmapsto A^U \end{aligned}$$

in a unique way making the isomorphism (1.2.7) natural in U and A as well.

The other simplicial limit notions postulated by axiom 1.2.1(i) are **conical**, which is the term used for ordinary 1-categorical limit shapes that satisfy an enriched analog of the usual universal property; see Definition A.6.14. When these limits exist they correspond to the usual limits in the underlying category, but the usual universal property is strengthened. Applying the covariant representable functor $\mathcal{A}(X,-)\colon \mathcal{A}_0 \to s\mathcal{S}et$ to a limit cone $(\lim_{j\in J} A_j \to A_j)_{j\in J}$ in $\mathcal{A}_0$ yields a natural comparison map

$$\mathcal{A}(X, \lim_{j\in J} A_j) \to \lim_{j\in J} \mathcal{A}(X, A_j). \tag{1.2.8}$$

We say that $\lim_{j\in J} A_j$ defines a **simplicially enriched limit** if and only if (1.2.8) is an isomorphism of simplicial sets for all $X \in \mathcal{A}$.

Considerably more details on the general theory of enriched categories can be found in **[54]** and in Appendix A. Enriched limits are the subjects of §A.4 and §A.5.

1.2.9. Preview (flexible weighted limits in ∞-cosmoi). The axiom 1.2.1(i) implies that any ∞-cosmos $\mathcal{K}$ admits all *flexible limits* (see Definition 6.2.1 and Corollary 6.3.3), a much larger class of simplicially enriched "weighted" limits that will be introduced in §6.2.

[17] The phrase "simplicial object in $\mathcal{C}at$" is reserved for the more general yet less common notion of a diagram $\mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}at$ that is not necessarily comprised of identity-on-objects functors.

We quickly introduce the three examples of ∞-cosmoi that are most easily absorbed, deferring more sophisticated examples to the end of this section. The first of these is the prototypical ∞-cosmos. Using the results of Joyal discussed in §1.1 and the classes of isofibrations, equivalences, and trivial fibrations introduced in Definitions 1.1.17, 1.1.22, and 1.1.24:

1.2.10. Proposition (the ∞-cosmos of quasi-categories). *The full subcategory $\mathcal{QC}at \subset s\mathcal{S}et$ of quasi-categories defines an ∞-cosmos in which the isofibrations, equivalences, and trivial fibrations coincide with the classes already bearing these names.*

Proof. The subcategory $\mathcal{QC}at \subset s\mathcal{S}et$ inherits its simplicial enrichment from the cartesian closed category of simplicial sets: by Proposition 1.1.19, whenever A and B are quasi-categories, $\mathsf{Fun}(A, B) := B^A$ is again a quasi-category.

The limits postulated in 1.2.1(i) exist in the ambient category of simplicial sets.[18] For instance, the defining universal property of the simplicial cotensor (1.2.7) is satisfied by the exponentials of simplicial sets. Moreover, since simplicial sets is a cartesian closed category, each of the conical limits is simplicially enriched in the sense discussed in Digression 1.2.6; the reader is guided through this argument in Exercise 1.2.ii and a general proof appears in Proposition A.5.5.

We now argue that the full subcategory of quasi-categories inherits all these limit notions and at the same time establish the stability of the class of isofibrations under the formation of these limits. In fact, this latter property will help to prove the former. To see this, note that a simplicial set is a quasi-category if and only if the map from it to the point is an isofibration. More generally, if the codomain of any isofibration is a quasi-category then its domain must be as well. So if any of the maps in a limit cone over a diagram of quasi-categories are isofibrations, then it follows that the limit is itself a quasi-category.

Since the class of isofibrations is characterized by a right lifting property, Lemma C.2.3 implies that the isofibrations contains all isomorphism and are closed under composition, product, pullback, and forming inverse limits of towers. In particular, the full subcategory of quasi-categories possesses these limits. This verifies all of the axioms of 1.2.1(i) and 1.2.1(ii) except for the last two: Leibniz closure and closure under exponentiation $(-)^X$. These last closure properties are established in Proposition 1.1.19, and in fact by Exercise 1.1.vii, the former subsumes the latter . This completes the verification of the ∞-cosmos axioms.

It remains to check that the classes of equivalences and of trivial fibrations coincide with those defined by 1.1.24 and 1.1.22. By Proposition 1.1.27 the latter coincidence follows from the former, so it remains only to show that the equivalences of 1.1.22 coincide with the **representably-defined equivalences**: those maps of quasi-categories $f : A \to B$ for which $A^X \to B^X$ is an equivalence of quasi-categories in the sense of Definition 1.1.22. Taking $X = \Delta[0]$, we see immediately that representably-defined equivalences are equivalences, and the converse holds since the exponential $(-)^X$ preserves the data defining a simplicial homotopy. $\square$

Two full examples fit into a common paradigm: both arise as full subcategories of the ∞-cosmos $\mathcal{QC}at$ and inherit their ∞-cosmos structure from this inclusion; see Lemma 6.1.4. But it is also instructive—and ultimately takes less work than went into the proof of Proposition 1.2.10—to describe the resulting ∞-cosmos structures directly.

[18]Any category of presheaves is cartesian closed, complete, and cocomplete — a "cosmos" in the sense of Bénabou.

1.2.11. Proposition (the ∞-cosmos of categories). *The category $\mathcal{C}at$ of 1-categories defines an ∞-cosmos whose isofibrations are the **isofibrations**: functors satisfying the displayed right lifting property:*

$$\begin{tikzcd} \mathbb{1} \arrow[r] \arrow[d, hook] & A \arrow[d, two heads, "f"] \\ \mathbb{I} \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

*The equivalences are the equivalences of categories and the trivial fibrations are **surjective equivalences**: equivalences of categories that are also surjective on objects.*

Proof. It is well-known that the 2-category of categories is complete (and in fact also cocomplete) as a $\mathcal{C}at$-enriched category; see Definition A.6.17 or [**53**]. The categorically enriched category of categories becomes a quasi-categorically enriched category by applying the nerve functor to the hom-categories. It follows formally that these 2-categorical limits become simplicially enriched limits; see §A.7. In particular, as proscribed there and explored in Exercise 1.2.iii, the cotensor of a category A by a simplicial set U is defined to be the functor category $A^{\mathsf{h}U}$. This completes the verification of axiom (i).

Since the class of isofibrations is characterized by a right lifting property, Lemma C.2.3 implies that the isofibrations are closed under all of the limit constructions of 1.2.1(ii) except for the last two, and by Exercise 1.1.vii, the Leibniz closure subsumes the closure under exponentiation.

To verify that isofibrations of categories $f\colon A \twoheadrightarrow B$ are stable under forming Leibniz cotensors with monomorphisms of simplicial sets $i\colon U \hookrightarrow V$, we must solve the lifting problem below-left

$$\begin{tikzcd} \mathbb{1} \arrow[r, "s"] \arrow[d, hook, "\mathrm{dom}"'] & A^{\mathsf{h}V} \arrow[d, "i \widehat{\pitchfork} f"] \\ \mathbb{I} \arrow[r, "\langle \beta, \alpha \rangle"'] \arrow[ur, dashed, "\gamma"] & B^{\mathsf{h}V} \times_{B^{\mathsf{h}U}} A^{\mathsf{h}U} \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} \mathsf{h}U \times \mathbb{I} \cup_{\mathsf{h}U} \mathsf{h}V \arrow[r, "\langle \alpha, s \rangle"] \arrow[d, hook, "\mathsf{h}i \widehat{\times} \mathrm{dom}"'] & A \arrow[d, two heads, "f"] \\ \mathsf{h}V \times \mathbb{I} \arrow[r, "\beta"'] \arrow[ur, dashed, "\gamma"] & B \end{tikzcd}$$

which transposes to the lifting problem above-right, which we can solve by hand. Here the map β defines a natural isomorphism between $fs\colon \mathsf{h}V \to B$ and a second functor. Our task is to lift this to a natural isomorphism γ from s to another functor that extends the natural isomorphism α along $\mathsf{h}i\colon \mathsf{h}U \to \mathsf{h}V$. Note this functor $\mathsf{h}i$ need not be an inclusion, but it is injective on objects, which will be enough.

We define the components of γ by cases. If an object $v \in \mathsf{h}V$ is equal to $i(u)$ for some $u \in \mathsf{h}U$ define $\gamma_{i(u)} \coloneqq \alpha_u$; otherwise, use the fact that f is an isofibration to define γ_v to be any lift of the isomorphism β_v to an isomorphism in A with domain $s(v)$. The data of the map $\gamma\colon \mathsf{h}V \times \mathbb{I} \to A$ also entails the specification of the functor $\mathsf{h}V \to A$ that is the codomain of the natural transformation γ. On objects, this functor is given by $v \mapsto \mathrm{cod}(\gamma_v)$. On morphisms, this functor defined in the unique way that makes γ into a natural transformation:

$$(k\colon v \to v') \mapsto \gamma_{v'} \circ s(k) \circ \gamma_v^{-1}.$$

This completes the proof that $\mathcal{C}at$ defines an ∞-cosmos. We leave the characterization of the classes of equivalences and trivial fibrations to the reader. ∎

1.2.12. Proposition (the ∞-cosmos of Kan complexes). *The category $\mathcal{K}an$ of Kan complexes defines an ∞-cosmos whose isofibrations are the **Kan fibrations**: maps that lift against all horn inclusions $\Lambda^k[n] \hookrightarrow \Delta[n]$ for $n \geq 1$ and $0 \leq k \leq n$.*

Here the proof will proceed along the lines of Lemma 6.1.4. We will show that the subcategory of Kan complexes inherits an ∞-cosmos structure by restricting structure from the ∞-cosmos of quasi-categories.

Proof. By Proposition 1.1.14, an isofibration between Kan complexes is a Kan fibration. Conversely, since the homotopy coherent isomorphism $\mathbb{I}$ can be built from the point $\mathbb{1}$ by attaching fillers to a sequence of outer horns, all Kan fibrations define isofibrations. This shows that between Kan fibrations, the classes of isofibrations and Kan fibrations coincide. So to show that the category of Kan complexes inherits an ∞-cosmos structure by restriction from the ∞-cosmos of quasi-categories, we need only verify that the full subcategory $\mathcal{K}an \hookrightarrow \mathcal{Q}Cat$ is closed under all of the limit constructions of axiom 1.2.1(i). For the conical limits, the argument mirrors the one given in the proof of Proposition 1.2.10, while the closure under cotensors is a consequence of Corollary D.3.12, which says that the Kan complexes also define an exponential ideal in the category of simplicial sets. The remaining axiom 1.2.1(ii) is inherited from the analogous properties established for quasi-categories in Proposition 1.2.10. □

We mention a common source of ∞-cosmoi found in nature to help ground the intuition for readers familiar with Quillen's model categories, a popular framework for "abstract homotopy theory," but reassure newcomers that model categories are not needed outside of Appendix E where these results are proven.

1.2.13. Digression (a source of ∞-cosmoi in nature). As explained in §E.1, certain easily described properties of a model category imply that the full subcategory of fibrant objects defines an ∞-cosmos whose isofibrations, equivalences, and trivial fibrations are the fibrations, weak equivalences, and trivial fibrations between fibrant objects. Namely, any model category that is enriched as such over the Joyal model structure on simplicial sets and with the property that all fibrant objects are cofibrant has this property; see Proposition E.1.1. This model-categorical enrichment over quasi-categories can be defined when the model category is cartesian closed and equipped with a right Quillen adjoint to the Joyal model structure on simplicial sets whose left adjoint preserves finite products; see Corollary E.1.3. In this case, the right adjoint becomes the underlying quasi-category functor (see Proposition 1.3.3(ii)) and the ∞-cosmoi so-produced will then be cartesian closed (see Definition 1.2.23). The ∞-cosmoi listed in Example 1.2.24 all arise in this way.

The following results are consequences of the axioms of Definition 1.2.1. The first of these results tells us that the trivial fibrations enjoy all of the same stability properties satisfied by the isofibrations.

1.2.14. Lemma (stability of trivial fibrations). *The trivial fibrations in an ∞-cosmos define a subcategory containing the isomorphisms and are stable under product, pullback, and forming inverse limits of towers. Moreover, the Leibniz cotensors of any trivial fibration with a monomorphism of simplicial sets is a trivial fibration as is the Leibniz cotensor of an isofibration with a map in the class cellularly generated by the inner horn inclusions and the map $\mathbb{1} \hookrightarrow \mathbb{I}$, and if $E \overset{\sim}{\twoheadrightarrow} B$ is a trivial fibration then so is $\mathsf{Fun}(X, E) \overset{\sim}{\twoheadrightarrow} \mathsf{Fun}(X, B)$.*

Proof. We prove these statements in the reverse order. By axiom 1.2.1(ii) and the definition of the trivial fibrations in an ∞-cosmos, we know that if $E \overset{\sim}{\twoheadrightarrow} B$ is a trivial fibration then is $\mathsf{Fun}(X, E) \overset{\sim}{\twoheadrightarrow} \mathsf{Fun}(X, B)$ is both an isofibration and an equivalence, and hence by Proposition 1.1.27 a trivial fibration. For stability under the remaining constructions, we know in each case that the maps in question are isofibrations in the ∞-cosmos; it remains to show only that the maps are also equivalences. The equivalences in an ∞-cosmos are defined to be the maps that $\mathsf{Fun}(X, -)$ carries to equivalences of quasi-categories, so it suffices to verify that trivial fibrations of quasi-categories satisfy the corresponding stability properties. This is established in Proposition 1.1.28 for the Leibniz stability properties and by the fact that that class is characterized by a right lifting property for the remaining ones. □

Additionally, every trivial fibration is "split" by a section.

1.2.15. Lemma (trivial fibrations split). *Every trivial fibration admits a section*

$$\begin{tikzcd} & E \arrow[d, two heads, "p", "\wr"'] \\ B \arrow[ur, dashed, "s"] \arrow[r, equal] & B \end{tikzcd}$$

that defines a split fiber homotopy equivalence

$$\begin{tikzcd} E + E \arrow[d, hook] \arrow[rr, "{(\mathrm{id}_E, sp)}"] & & E \arrow[d, two heads, "p", "\wr"'] \\ E \times \mathbb{I} \arrow[urr, dashed, "\alpha"] \arrow[r, "\pi"'] & E \arrow[r, two heads, "p", "\sim"'] & B \end{tikzcd}$$

Proof. If $p\colon E \overset{\sim}{\twoheadrightarrow} B$ is a trivial fibration, then by the final stability property of Lemma 1.2.14, so is $p_*\colon \mathrm{Fun}(X,E) \twoheadrightarrow Fun(X,B)$ for any ∞-category X. By Definition 1.1.24, we may solve the lifting problem below-left

$$\begin{tikzcd} \emptyset = \partial\Delta[0] \arrow[d, hook] \arrow[r] & \mathrm{Fun}(B,E) \arrow[d, two heads, "p_*", "\wr"'] \\ \mathbb{1} = \Delta[0] \arrow[ur, dashed, "s"] \arrow[r, "\mathrm{id}_B"'] & \mathrm{Fun}(B,B) \end{tikzcd} \qquad \begin{tikzcd} \mathbb{1} + \mathbb{1} \arrow[d, hook] \arrow[rr, "{(\mathrm{id}_E, sp)}"] & & \mathrm{Fun}(E,E) \arrow[d, two heads, "p_*", "\wr"'] \\ \mathbb{I} \arrow[urr, dashed, "\alpha"] \arrow[r, "!"'] & \mathbb{1} \arrow[r, "p"'] & \mathrm{Fun}(E,B) \end{tikzcd}$$

to find a map $s\colon B \to E$ so that $ps = \mathrm{id}_B$, and then solve the lifting problem above-right to construct the desired fibered homotopy. □

A classical construction in abstract homotopy theory proves the following:

1.2.16. Lemma (Brown factorization lemma). *Any functor $f\colon A \to B$ in an ∞-cosmos may be factored as an equivalence followed by an isofibration, where this equivalence is constructed as a section of a trivial fibration.*

$$\begin{tikzcd} & Pf \arrow[dl, two heads, bend right, "q"', "\sim"] \arrow[dr, two heads, "p"] & \\ A \arrow[ur, "s"'] \arrow[rr, "f"'] & & B \end{tikzcd} \tag{1.2.17}$$

Proof. The displayed factorization is constructed by the pullback of an isofibration formed by the simplicial cotensor of the inclusion $\mathbb{1} + \mathbb{1} \hookrightarrow \mathbb{I}$ into the ∞-category B.

$$\begin{tikzcd} & & A^{\mathbb{I}} \arrow[drr, "f^{\mathbb{I}}"] & & \\ A \arrow[urr, "\Delta"] \arrow[r, dashed, "\sim"', "s"] \arrow[dr, "(A,f)"'] & Pf \arrow[rrr] \arrow[d, two heads, "(q,p)"'] \arrow[drrr, phantom, "\lrcorner"] & & & B^{\mathbb{I}} \arrow[d, two heads, "(\mathrm{ev}_0,\mathrm{ev}_1)"] \\ & A \times B \arrow[rrr, "f \times B"'] & & & B \times B \end{tikzcd}$$

Note the map q is a pullback of the trivial fibration $\mathrm{ev}_0 \colon B^{\mathbb{I}} \xrightarrow{\sim} B$ and is hence a trivial fibration. Its section s, constructed by applying the universal property of the pullback to the displayed cone with summit A, is thus an equivalence. □

By a Yoneda-style argument, the "homotopy equivalence" characterization of the equivalences in the ∞-cosmos of quasi-categories of Definition 1.1.22 extends to an analogous characterization of the equivalences in any ∞-cosmos:

1.2.18. Lemma (equivalences are homotopy equivalences). *A map $f \colon A \to B$ between ∞-categories in an ∞-cosmos $\mathcal{K}$ is an equivalence if and only if it extends to the data of a "homotopy equivalence" with the free-living isomorphism $\mathbb{I}$ serving as the interval: that is, if there exist maps $g \colon B \to A$ and*

$$\begin{tikzcd} & A \\ A \arrow[ur, equal] \arrow[r, "\alpha"] \arrow[dr, "gf"'] & A^{\mathbb{I}} \arrow[u, "\mathrm{ev}_0"'] \arrow[d, "\mathrm{ev}_1"] \\ & A \end{tikzcd} \qquad \begin{tikzcd} & B \\ B \arrow[ur, "fg"] \arrow[r, "\beta"] \arrow[dr, equal] & B^{\mathbb{I}} \arrow[u, "\mathrm{ev}_0"'] \arrow[d, "\mathrm{ev}_1"] \\ & B \end{tikzcd} \tag{1.2.19}$$

in the ∞-cosmos.

Proof. By hypothesis, if $f \colon A \to B$ defines an equivalence in the ∞-cosmos $\mathcal{K}$ then the induced map on post-composition $f_* \colon \mathsf{Fun}(B, A) \xrightarrow{\sim} \mathsf{Fun}(B, B)$ is an equivalence of quasi-categories in the sense of Definition 1.1.22. Evaluating the inverse equivalence $\tilde{g} \colon \mathsf{Fun}(B, B) \xrightarrow{\sim} \mathsf{Fun}(B, A)$ and homotopy $\tilde{\beta} \colon \mathsf{Fun}(B, B) \to \mathsf{Fun}(B, B)^{\mathbb{I}}$ at the 0-arrow $1_B \in \mathsf{Fun}(B, B)$, we obtain a 0-arrow $g \colon B \to A$ together with an isomorphism $\beta \colon \mathbb{I} \to \mathsf{Fun}(B, B)$ from the composite fg to 1_B. By the defining universal property of the cotensor (1.2.7), this isomorphism internalizes to define the map $\beta \colon B \to B^{\mathbb{I}}$ in $\mathcal{K}$ displayed on the right of (1.2.19).

Now the hypothesis that f is an equivalence also provides an equivalence of quasi-categories $f_* \colon \mathsf{Fun}(A, A) \xrightarrow{\sim} \mathsf{Fun}(A, B)$, and the map $\beta f \colon A \to B^{\mathbb{I}}$ represents an isomorphism in $\mathsf{Fun}(A, B)$ from fgf to f. Since f_* is an equivalence, we conclude from Remark 1.1.23 that 1_A and gf are isomorphic in the quasi-category $\mathsf{Fun}(A, A)$: explicitly, such an isomorphism may be defined by applying the inverse equivalence $\tilde{h} \colon \mathsf{Fun}(A, B) \to \mathsf{Fun}(A, A)$ and composing with the components at $1_A, gf \in \mathsf{Fun}(A, A)$ of the isomorphism $\tilde{\alpha} \colon \mathsf{Fun}(A, A) \to \mathsf{Fun}(A, A)^{\mathbb{I}}$ from $1_{\mathsf{Fun}(A,A)}$ to $\tilde{h}f_*$. Now by Corollary 1.1.16 this isomorphism is represented by a map $\mathbb{I} \to \mathsf{Fun}(A, A)$ from 1_A to gf, which internalizes to a map $\alpha \colon A \to A^{\mathbb{I}}$ in $\mathcal{K}$ displayed on the left of (1.2.19).

The converse is easy: the simplicial cotensor construction commutes with $\mathsf{Fun}(X,-)$, so a homotopy equivalence (1.2.19) induces a homotopy equivalences of quasi-categories as in Definition 1.1.22. ☐

1.2.20. Lemma. *The class of equivalences in an ∞-cosmos are closed under retracts and satisfy the* ***2-of-3 property****: given a composable pair of functors and their composite, if any two of these are equivalences so is the third.*

For the reader who solved Exercise 1.1.viii, demonstrating the equivalences between quasi-categories are closed under retracts and satisfy the 2-of-3 property, Lemma 1.2.20 follows easily from the representable definition of equivalences and functoriality. But for sake of completeness, we give an alternate proof of the general cosmological result that makes use of Lemma 1.2.18 and subsumes Exercise 1.1.viii.

Proof. Let $f\colon A \xrightarrow{\sim} B$ be an equivalence equipped with the data of (1.2.19) and consider a retract diagram

$$\begin{tikzcd} C \arrow[r, "u"'] \arrow[d, "h"'] \arrow[rr, bend left, equal] & A \arrow[r, "v"'] \arrow[d, "f", "\wr"'] & C \arrow[d, "h"] \\ D \arrow[r, "s"] \arrow[rr, bend right, equal] & B \arrow[r, "t"] & D \end{tikzcd}$$

By Lemma 1.2.18, to prove that $h\colon C \to D$ is an equivalence, it suffices to construct the data of an inverse homotopy equivalence. To that end define $k\colon D \to C$ to be the composite vgs and then observe from the commutative diagrams

$$\begin{tikzcd} & & A \arrow[r, "v"'] & C \\ C \arrow[r, "u"] \arrow[d, "h"'] \arrow[urrr, bend left, equal] & A \arrow[r, "\alpha"] \arrow[d, "f"'] \arrow[ur, equal] & A^{\mathbb{I}} \arrow[r, "v^{\mathbb{I}}"] \arrow[u, "\mathrm{ev}_0"'] \arrow[d, "\mathrm{ev}_1"] & C^{\mathbb{I}} \arrow[u, "\mathrm{ev}_0"'] \arrow[d, "\mathrm{ev}_1"] \\ D \arrow[r, "s"] \arrow[rrr, bend right, "k"'] & B \arrow[r, "g"] & A \arrow[r, "v"] & C \end{tikzcd}
\qquad
\begin{tikzcd} & & C \arrow[dr, "h"] & \\ & A \arrow[r, "f"'] \arrow[ur, "v"] & B \arrow[r, "t"] & D \\ D \arrow[r, "s"] \arrow[uurr, bend left, "k"] \arrow[drrr, bend right, equal] & B \arrow[r, "\beta"] \arrow[u, "g"'] \arrow[dr, equal] & B^{\mathbb{I}} \arrow[r, "t^{\mathbb{I}}"] \arrow[u, "\mathrm{ev}_0"'] \arrow[d, "\mathrm{ev}_1"] & D^{\mathbb{I}} \arrow[u, "\mathrm{ev}_0"'] \arrow[d, "\mathrm{ev}_1"] \\ & & B \arrow[r, "t"] & D \end{tikzcd}$$

that $v^{\mathbb{I}}\alpha u\colon C \to C^{\mathbb{I}}$ and $t^{\mathbb{I}}\beta s\colon D \to D^{\mathbb{I}}$ define the required "homotopies."

Via Lemma 1.2.18, the 2-of-3 property for equivalences follows from the fact that the set of (homotopy coherent) isomorphisms in a quasi-category is closed under composition. Homotopy coherent isomorphisms in a quasi-category represent isomorphisms in the homotopy category. The composite of a pair of isomorphisms in the homotopy category is then an isomorphism, which by Corollary 1.1.16 can be lifted to a representing homotopy coherent isomorphism.[19] We will apply this to the homotopy coherent isomorphisms in the functor spaces of an ∞-cosmos that form part of the data of an equivalence of ∞-categories.

[19]In fact, by Example D.4.23, homotopy coherent isomorphisms can be composed directly, but we do not need this here.

To prove that equivalences are closed under composition, consider a composable pair of equivalences with their inverse equivalences

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, phantom, "\simeq"] & B \arrow[l, shift left, "k"] \arrow[r, shift left, "g"] \arrow[r, phantom, "\simeq"] & C \arrow[l, shift left, "h"] \end{tikzcd}$$

The homotopies of Lemma 1.2.18 define isomorphisms $\alpha\colon \mathrm{id}_A \cong kf \in \mathsf{Fun}(A, A)$ and $\gamma\colon \mathrm{id}_B \cong hg \in \mathsf{Fun}(B, B)$, the latter of which whiskers to define $k\gamma f\colon kf \cong khgf \in \mathsf{Fun}(B, B)$. Composing these, we obtain an isomorphism $\mathrm{id}_A \cong khgf \in \mathsf{Fun}(A, A)$ defining one of the homotopies that witnesses that kh defines an equivalence inverse of gf. The construction of the other homotopy is dual.

To prove that the equivalences are closed under right cancelation, consider a diagram

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, phantom, "\simeq"] & B \arrow[l, shift left, "k"] \arrow[r, "g"] & C \arrow[ll, bend right, "\ell"', "\sim"] \end{tikzcd}$$

with k an inverse equivalence to f and ℓ and inverse equivalence to gf. We will demonstrate that $f\ell$ defines an inverse equivalence to g. One of the required homotopies $\mathrm{id}_C \cong gf\ell$ is given already. The other is obtained by composing three isomorphisms in $\mathsf{Fun}(B, B)$

$$\mathrm{id}_B \xrightarrow[\cong]{\beta^{-1}} fk \xrightarrow[\cong]{f\delta k} f\ell gfk \xrightarrow[\cong]{f\ell g\beta} f\ell g.$$

The proof of stability of equivalence under left cancelation is dual. $\square$

1.2.21. Remark (equivalences satisfy the 2-of-6 property). In fact the class of equivalences in any ∞-cosmos satisfy the stronger **2-of-6 property**: for any composable triple of functors

$$\begin{tikzcd} & B \arrow[dd, "g"] \arrow[dr, "hg", "\sim"'] & \\ A \arrow[ur, "f"] \arrow[rr, "hgf"] \arrow[dr, "gf"', "\sim"] & & D \\ & C \arrow[ur, "h"'] & \end{tikzcd}$$

if gf and hg are equivalences then f, g, h, and hgf are too. The proof uses Lemma 1.2.20 together with the observation that in the case where $f\colon A \to B$ is an equivalence, the map p of (1.2.17) is also a trivial fibration, and in particular has a section by Lemma 1.2.15. Combining these facts, a result of Blumberg and Mandell [**15**, 6.4] reproduced in Proposition C.1.8 applies to prove that the equivalences have the 2-of-6 property. See Corollary C.1.9.

One of the key advantages of the ∞-cosmological approach to abstract category theory is that there are a myriad varieties of "fibered" ∞-cosmoi that can be built from a given ∞-cosmos, which means that any theorem proven in this axiomatic framework specializes and generalizes to those contexts. The most basic of these derived ∞-cosmoi is the ∞-cosmos of isofibrations over a fixed base, which we introduce now. Other examples of ∞-cosmoi will be introduced in Chapter 6, once we have developed a greater facility with the simplicial limits of axiom 1.2.1(i).

1.2.22. Proposition (sliced ∞-cosmoi). *For any ∞-cosmos $\mathcal{K}$ and any ∞-category $B \in \mathcal{K}$ there is an* ***∞-cosmos $\mathcal{K}_{/B}$ of isofibrations over** B whose*

(i) objects are isofibrations $p\colon E \twoheadrightarrow B$ with codomain B

(ii) functor-spaces, say from $p\colon E \twoheadrightarrow B$ to $q\colon F \twoheadrightarrow B$, are defined by pullback

$$\begin{tikzcd} \mathsf{Fun}_B(p\colon E \twoheadrightarrow B, q\colon F \twoheadrightarrow B) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner" very near start] & \mathsf{Fun}(E,F) \arrow[d, two heads, "q_*"] \\ \mathbb{1} \arrow[r, "p"] & \mathsf{Fun}(E,B) \end{tikzcd}$$

and abbreviated to $\mathsf{Fun}_B(E,F)$ when the specified isofibrations are clear from context

(iii) isofibrations are commutative triangles of isofibrations over B

$$\begin{tikzcd} E \arrow[rr, two heads, "r"] \arrow[dr, two heads, "p"'] & & F \arrow[dl, two heads, "q"] \\ & B & \end{tikzcd}$$

(iv) terminal object is $1\colon B \twoheadrightarrow B$ and products are defined by the pullback along the diagonal

$$\begin{tikzcd} \times_i^B E_i \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner" very near start] & \prod_i E_i \arrow[d, two heads, "\prod_i p_i"] \\ B \arrow[r, "\Delta"] & \prod_i B \end{tikzcd}$$

(v) pullbacks and limits of towers of isofibrations are created by the forgetful functor $\mathcal{K}_{/B} \to \mathcal{K}$

(vi) simplicial cotensors $p\colon E \twoheadrightarrow B$ by $U \in s\mathcal{S}et$ are constructed by the pullback

$$\begin{tikzcd} U \pitchfork_B p \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner" very near start] & E^U \arrow[d, two heads, "p^U"] \\ B \arrow[r, "\Delta"] & B^U \end{tikzcd}$$

(vii) and in which a map over B

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, two heads, "p"'] & & F \arrow[dl, two heads, "q"] \\ & B & \end{tikzcd}$$

is an equivalence in the ∞-cosmos $\mathcal{K}_{/B}$ if and only if f is an equivalence in $\mathcal{K}$.

Proof. Note first that the functor spaces are quasi-categories since axiom 1.2.1(ii) asserts that for any isofibration $q\colon F \twoheadrightarrow B$ in $\mathcal{K}$ the map $q_*\colon \mathsf{Fun}(E,F) \twoheadrightarrow \mathsf{Fun}(E,B)$ is an isofibration of quasi-categories. Other parts of this axiom imply that that each of the limit constructions—such as the products and cotensors constructed in (iv) and (vi)—define isofibrations over B. The closure properties of the isofibrations in $\mathcal{K}_{/B}$ follow from the corresponding ones in $\mathcal{K}$. The most complicated of these is the Leibniz cotensor stability of the isofibrations in $\mathcal{K}_{/B}$, which follows from the corresponding property in $\mathcal{K}$, since for a monomorphism of simplicial sets $i\colon X \hookrightarrow Y$ and an isofibration r over B as in (iii) above, the map $i \,\widehat{\pitchfork}_B\, r$ is constructed by pulling back $i \,\widehat{\pitchfork}\, r$ along $\Delta\colon B \to B^Y$.

The fact that the above constructions define simplicially enriched limits in a simplicially enriched slice category are standard from enriched category theory. It remains only to verify that the equivalences in the ∞-cosmos of isofibrations are created by the forgetful functor $\mathcal{K}_{/B} \to \mathcal{K}$. Suppose first that the map f displayed in (vii) defines an equivalence in $\mathcal{K}$. Then for any isofibration $s\colon A \twoheadrightarrow B$ the induced map on functor-spaces in $\mathcal{K}_{/B}$ is defined by the pullback:

$$\begin{tikzcd} \mathsf{Fun}_B(A,E) \arrow[rr] \arrow[dd, two heads] \arrow[dr, dashed, "f_*", "\sim"'] \arrow[ddrr, phantom, "\lrcorner" very near start] & & \mathsf{Fun}(A,E) \arrow[dr, "f_*", "\sim"'] \arrow[dd, two heads, "p_*"' near start] & \\ & \mathsf{Fun}_B(A,F) \arrow[rr, crossing over] \arrow[dl, two heads] & & \mathsf{Fun}(A,F) \arrow[dl, two heads, "q_*"] \\ \mathbb{1} \arrow[rr, "s"'] & & \mathsf{Fun}(A,B) & \end{tikzcd}$$

Since f is an equivalence in $\mathcal{K}$, the map $f_*\colon \mathsf{Fun}(A,E) \to \mathsf{Fun}(A,F)$ is an equivalence, and so it follows that the induced map on fibers over s is an equivalence as well. This can be verified either by appealing to Lemmas 1.2.14 and 1.2.16 and using standard techniques from simplicial homotopy theory—as done in Lemma C.1.11—or by appealing to Lemma 1.2.18 and using the fact that pullback along s defines a simplicial functor.

For the converse implication, we appeal to Lemma 1.2.18. If $f\colon E \to F$ is an equivalence in $\mathcal{K}_{/B}$ then it admits a homotopy inverse in $\mathcal{K}_{/B}$. The inverse equivalence $g\colon F \to E$ also defines an inverse equivalence in $\mathcal{K}$ and the required simplicial homotopies in $\mathcal{K}$

$$E \xrightarrow{\alpha} \mathbb{I} \pitchfork_B p \longrightarrow E^{\mathbb{I}} \qquad\qquad F \xrightarrow{\beta} \mathbb{I} \pitchfork_B q \to F^{\mathbb{I}}$$

are defined by composing with the top horizontal leg of the pullback defining the cotensor in $\mathcal{K}_{/B}$. □

As mentioned in Digression 1.2.13, many of the ∞-cosmoi we encounter "in the wild" satisfy an additional axiom. Note, however, that this axiom is not inherited by the sliced ∞-cosmoi of Proposition 1.2.22, which is one of the reasons it was not included in Definition 1.2.1.

1.2.23. Definition (cartesian closed ∞-cosmoi). An ∞-cosmos $\mathcal{K}$ is **cartesian closed** if the product bifunctor $- \times -\colon \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ extends to a simplicially enriched two-variable adjunction

$$\mathsf{Fun}(A \times B, C) \cong \mathsf{Fun}(A, C^B) \cong \mathsf{Fun}(B, C^A)$$

in which the right adjoints $(-)^A\colon \mathcal{K} \to \mathcal{K}$ preserve the class of isofibrations for all $A \in \mathcal{K}$.

For instance, the ∞-cosmos of quasi-categories is cartesian closed, with the exponentials defined as (special cases of) simplicial cotensors. This is one of the reasons that we use the same notation for cotensor and for exponential.[20] Note in this case the functor spaces and the exponentials coincide. The same is true for the cartesian closed ∞-cosmoi of categories and of Kan complexes; see Exercise 1.2.iii. In general, the functor space from A to B is the "underlying quasi-category" of the exponential B^A whenever it exists; see Remark 1.3.8.

1.2.24. Example (∞-cosmoi of $(\infty,1)$-categories; §E.2). The following models of $(\infty,1)$-categories define cartesian closed ∞-cosmoi:

[20]Other reasons for this convenient notational conflation will be explained in §2.3 and in Proposition 10.3.4.

(i) Rezk's **complete Segal spaces** define the objects of an ∞-cosmos $\mathcal{CSS}$, in which the isofibrations, equivalences, and trivial fibrations are the corresponding classes of the model structure of [**72**].[21]
(ii) The **Segal categories** defined by Dwyer, Kan, and Smith [**32**] and developed by Hirschowitz and Simpson [**42**] define the objects of an ∞-cosmos $\mathcal{S}egal$, in which the isofibrations, equivalences and trivial fibrations are the corresponding classes of the model structure of [**66**] and [**9**].[22]
(iii) The **1-complicial sets** of [**98**], equivalently the "naturally marked quasi-categories" of [**59**], define the objects of an ∞-cosmos $1\text{-}\mathcal{C}omp$ in which the isofibrations, equivalences and trivial fibrations are the corresponding classes of the model structure from either of these sources.

In §E.3, we prove that certain models of (∞, n)-categories or even (∞, ∞)-categories define ∞-cosmoi: these including n-quasi-categories, Θ_n-spaces, iterated complete Segal spaces, and n-complicial sets, among others.

1.2.25. Definition (co-dual ∞-cosmoi). There is an identity-on-objects functor $(-)^\circ \colon \mathbf{\Delta} \to \mathbf{\Delta}$ that reverses the ordering of the elements in each ordinal $[n] \in \mathbf{\Delta}$. In the notation of 1.1.1, the functor $(-)^\circ$ sends a face map $\delta^i \colon [n-1] \rightarrowtail [n]$ to the face map $\delta^{n-i} \colon [n-1] \rightarrowtail [n]$ and sends the degeneracy map $\sigma^i \colon [n+1] \twoheadrightarrow [n]$ to the degeneracy map $\sigma^{n-i} \colon [n+1] \twoheadrightarrow [n]$. Precomposition with this involutive automorphism induces an involution $(-)^{\mathrm{op}} \colon s\mathcal{S}et \to s\mathcal{S}et$ that sends a simplicial set X to its **opposite simplicial set** X^{op}, with the orientation of the vertices in each simplex reversed. This construction preserves all conical limits and colimits and induces an isomorphism $(Y^X)^{\mathrm{op}} \cong (Y^{\mathrm{op}})^{X^{\mathrm{op}}}$ on exponentials.

For any ∞-cosmos $\mathcal{K}$, there is a **dual ∞-cosmos** $\mathcal{K}^{\mathrm{co}}$ with the same objects but with functor spaces defined by:

$$\mathsf{Fun}_{\mathcal{K}^{\mathrm{co}}}(A, B) \coloneqq \mathsf{Fun}_{\mathcal{K}}(A, B)^{\mathrm{op}}.$$

The isofibrations, equivalences, and trivial fibrations in $\mathcal{K}^{\mathrm{co}}$ coincide with those of $\mathcal{K}$.

Conical limits in $\mathcal{K}^{\mathrm{co}}$ coincide with those in $\mathcal{K}$, while the cotensor of $A \in \mathcal{K}$ with $U \in s\mathcal{S}et$ is defined to be $A^{U^{\mathrm{op}}}$.

A 2-categorical justification for this notation is given in Exercise 1.4.iii.

1.2.26. Definition (discrete ∞-categories). An ∞-category E in an ∞-cosmos $\mathcal{K}$ is **discrete** just when for all $X \in \mathcal{K}$ the functor-space $\mathsf{Fun}(X, E)$ is a Kan complex.

In the ∞-cosmos of quasi-categories, the discrete ∞-categories are exactly the Kan complexes. Similarly, in the ∞-cosmoi of Example 1.2.24 whose ∞-categories are $(\infty, 1)$-categories in some model, the discrete ∞-categories are the ∞-groupoids. These examples suggest that "discrete ∞-categories" might also be called "groupoidal ∞-categories." Importantly for what follows, the discrete ∞-categories can be characterized "internally" to the ∞-cosmos as follows:

[21]Warning: the model category of complete Segal spaces is enriched over simplicial sets in two distinct "directions" — one enrichment makes the simplicial set of maps between two complete Segal spaces into a Kan complex that probes the "spacial" structure while another enrichment makes the simplicial set of maps into a quasi-category that probes the "categorical" structure [**48**]. It is this latter enrichment that we want.

[22]Here we reserve the term "Segal category" for those simplicial objects with a discrete set of objects that are Reedy fibrant and satisfy the Segal condition. The traditional definition does not include the Reedy fibrancy condition because it is not satisfied by the simplicial object defined as the nerve of a Kan complex enriched category. Since Kan complex enriched categories are not among our preferred models of $(\infty, 1)$-categories this does not bother us.

1.2.27. LEMMA. *An ∞-category A is discrete if and only if $A^{\mathbb{I}} \xrightarrow{\sim}\!\!\!\!\twoheadrightarrow A^{\mathbb{2}}$ is a trivial fibration.*

PROOF. By Definition 1.2.2, the isofibration $A^{\mathbb{I}} \twoheadrightarrow A^{\mathbb{2}}$ is a trivial fibration if and only if for all ∞-categories X the induced map on functor spaces

$$\begin{array}{ccc} \mathsf{Fun}(X, A^{\mathbb{I}}) & \longrightarrow\!\!\!\!\!\twoheadrightarrow & \mathsf{Fun}(X, A^{\mathbb{2}}) \\ \wr\| & & \wr\| \\ \mathsf{Fun}(X, A)^{\mathbb{I}} & \longrightarrow\!\!\!\!\!\twoheadrightarrow & \mathsf{Fun}(X, A)^{\mathbb{2}} \end{array}$$

is a trivial fibration of quasi-categories. Via the universal property of the simplicial cotensor, Lemma 1.1.29 tells us that this map is a trivial fibration if and only if $\mathsf{Fun}(X, A)$ is a Kan complex. ☐

We leave it to the reader to show that the discrete ∞-categories in any ∞-cosmos assemble into an ∞-cosmos $\mathcal{K}^{\simeq}$. A proof appears in Proposition 6.1.6 where general techniques for discussing new ∞-cosmoi from given ones are discussed.

Exercises.

1.2.i. EXERCISE. Prove that the following are equivalent:

(i) a simplicial category, as in 1.2.4,

(ii) a category enriched over simplicial sets.

1.2.ii. EXERCISE. Elaborate on the proof of Proposition 1.2.10 by proving that the simplicially enriched category $\mathcal{QC}at$ admits conical products satisfying the universal property of Digression 1.2.6. That is:

(i) Define the cartesian product $A \times B$ and the projection maps $\pi_A \colon A \times B \to A$ and $\pi_B \colon A \times B \to B$ for a pair of quasi-categories A and B and prove that this data satisfies the usual (unenriched) universal property.

(ii) Given another quasi-category X, use (i) and the Yoneda lemma to show that the projection maps induce an isomorphism of quasi-categories

$$(A \times B)^X \xrightarrow{\cong} A^X \times B^X.$$

(iii) Explain how this relates to the universal property of Digression 1.2.6.

(iv) Express the usual 1-categorical universal property of (i) as the “0-dimensional aspect” of the universal property of (ii).

1.2.iii. EXERCISE. To complete the proof that categories inherit a cartesian closed ∞-cosmos structure from quasi-categories via the nerve embedding $\mathcal{C}at \hookrightarrow \mathcal{QC}at$:

(i) Show that the homotopy category functor $\mathsf{h} \colon s\mathcal{S}et \to \mathcal{C}at$ of Definition 1.1.10 preserves finite products. For later use, show also that the restricted homotopy category functor $\mathsf{h} \colon \mathcal{QC}at \to \mathcal{C}at$ preserves all products.

(ii) Conclude, by adjunction, that if A and B are categories then the functor space between their nerves is isomorphic to the nerve of the functor category B^A. This proves that the inclusion $\mathcal{C}at \hookrightarrow \mathcal{QC}at$ preserves exponentials.

(iii) Show, again by adjunction, that if A is a category and U is a simplicial set that the nerve of the functor category $A^{\mathsf{h}U}$ is isomorphic to the simplicial cotensor of the nerve of A with U. This proves that the inclusion $\mathcal{C}at \hookrightarrow \mathcal{QC}at$ preserves simplicial cotensors.

1.2.iv. Exercise. Prove that any object in an ∞-cosmos has a **path object**

$$\begin{tikzcd} & B^{\mathbb{I}} \arrow[dl, bend right] \arrow[dr, "{(\mathrm{ev}_0,\mathrm{ev}_1)}"] & \\ B \arrow[ur, "\sim"] \arrow[rr, "\Delta"'] & & B \times B \end{tikzcd}$$

constructed by cotensoring with the free-living isomorphism.

1.2.v. Exercise. Show that if $\mathcal{K}$ is a cartesian closed ∞-cosmos then $\mathcal{K}^{\mathrm{co}}$ is as well.

1.2.vi. Exercise (Proposition 6.1.6). Use Proposition 1.2.12 to show that the discrete ∞-categories in any ∞-cosmos define an ∞-cosmos whose functor spaces are all Kan complexes.

1.3. Cosmological Functors

Certain "right adjoint type" constructions define maps between ∞-cosmoi that preserve all of the structures axiomatized in Definition 1.2.1. The simple observation that such constructions define *cosmological functors* between ∞-cosmoi will streamline many proofs.

1.3.1. Definition (cosmological functor). A **cosmological functor** is a simplicial functor[23] $F\colon \mathcal{K} \to \mathcal{L}$ between ∞-cosmoi that preserves the specified classes of isofibrations and all of the simplicial limits enumerated in 1.2.1(i).

In general, cosmological functors preserve any ∞-categorical notion that can be characterized internally to the ∞-cosmos — for instance, as additional structure that can be given to a certain map — as opposed to externally — for instance, in a statement that involves a universal quantifier. For example, the equivalences in an ∞-cosmos are characterized externally in Definition 1.2.2, which might lead one to suspect that a non-surjective cosmological functor could fail to preserve them. However, Lemma 1.2.18 characterizes equivalences in terms of the presence of structures definable internally to an ∞-cosmos, so as a result:

1.3.2. Lemma. *Any cosmological functor also preserves the equivalences and the trivial fibrations.*

Proof. By Lemma 1.2.18 the equivalences in an ∞-cosmos coincide with the homotopy equivalences defined relative to cotensoring with the free-living isomorphism. Since a cosmological functor preserves simplicial cotensors, it preserves the data displayed in (1.2.19) and hence carries equivalences to equivalences. The statement about trivial fibrations follows. □

Similarly, arguing from Definition 1.2.26 it would not be clear whether cosmological functors preserve discrete ∞-categories , but using the internal characterization of Lemma 1.2.27 — an ∞-category A is discrete if and only if $A^{\mathbb{I}} \xrightarrow{\sim} A^{\mathbb{2}}$ is a trivial fibration — this follows easily: cosmological functors preserve simplicial cotensors and trivial fibrations.

We now demonstrate that cosmological functors are abundant:

1.3.3. Proposition.

(i) For any ∞-category A in an ∞-cosmos $\mathcal{K}$, $\mathrm{Fun}(A,-)\colon \mathcal{K} \to \mathcal{QC}at$ defines a cosmological functor.
*(ii) Specializing (i) to the terminal ∞-category, each ∞-cosmos has an **underlying quasi-category** functor*

$$(-)_0 \coloneqq \mathrm{Fun}(1,-)\colon \mathcal{K} \to \mathcal{QC}at.$$

[23]See Definition A.2.5.

(iii) For any ∞-cosmos $\mathcal{K}$ and any simplicial set U, the simplicial cotensor defines a cosmological functor $(-)^U : \mathcal{K} \to \mathcal{K}$.
(iv) For any ∞-category A in a cartesian closed ∞-cosmos $\mathcal{K}$, exponentiation defines a cosmological functor $(-)^A : \mathcal{K} \to \mathcal{K}$.
(v) For any functor $f : A \to B$ in an ∞-cosmos $\mathcal{K}$, pullback along f defines a cosmological functor $f^* : \mathcal{K}_{/B} \to \mathcal{K}_{/A}$.
(vi) For any cosmological functor $F : \mathcal{K} \to \mathcal{L}$ and any $A \in \mathcal{K}$, the induced map on slices $F : \mathcal{K}_{/A} \to \mathcal{L}_{/FA}$ defines a cosmological functor.

Proof. The first four of these statements are nearly immediate, the preservation of isofibrations being asserted explicitly as a hypothesis in each case and the preservation of limits following from standard categorical arguments.

For (v), pullback in an ∞-cosmos $\mathcal{K}$ is a simplicially enriched limit construction; one consequence of this is that $f^* : \mathcal{K}_{/B} \to \mathcal{K}_{/A}$ defines a simplicial functor. The action of the functor f^* on a 0-arrow g in $\mathcal{K}_{/B}$ is also defined by a pullback square: since the front and back squares in the displayed diagram are pullbacks the top square is as well

$$\begin{tikzcd}
f^*E \arrow[rr] \arrow[dr, dashed, "f^*(g)"'] \arrow[dd, two heads] \arrow[ddrr, phantom, "\lrcorner", very near start] & & E \arrow[dr, "g"] \arrow[dd, "p"' near start] & \\
& f^*F \arrow[rr] \arrow[dl] \arrow[dr, phantom, "\lrcorner", very near start] & & F \arrow[dl, "q"] \\
A \arrow[rr, "f"'] & & B &
\end{tikzcd}$$

Since isofibrations are stable under pullback, it follows that $f^* : \mathcal{K}_{/B} \to \mathcal{K}_{/A}$ preserves isofibrations. It remains to prove that this functor preserves the simplicial limits constructed in Proposition 1.2.22, which is fundamentally a consequence of the commutativity of limit constructions. In each case, this can be verified explicitly. We illustrate this computation for products and simplicial cotensors, by constructing the commutative cubes:

$$\begin{tikzcd}
& \times_i^A f^*E_i \arrow[rr] \arrow[dl, dashed] \arrow[dd] & & \prod_i f^*E_i \arrow[dl] \arrow[dd] \\
\times_i^B E_i \arrow[rr] \arrow[dd] & & \prod_i E_i \arrow[dd, "\prod_i p_i" near start] & \\
& A \arrow[rr, "\Delta"] \arrow[dl, "f"] & & \prod_i A \arrow[dl, "\prod_i f"] \\
B \arrow[rr, "\Delta"'] & & \prod_i B &
\end{tikzcd}
\qquad
\begin{tikzcd}
& U \pitchfork_A f^*(p) \arrow[rr] \arrow[dl, dashed] \arrow[dd] & & (f^*E)^U \arrow[dl] \arrow[dd, "(f^*p)^U"] \\
U \pitchfork_B p \arrow[rr] \arrow[dd] & & E^U \arrow[dd, "p^U" near start] & \\
& A \arrow[rr, "\Delta"] \arrow[dl, "f"] & & A^U \arrow[dl, "f^U"] \\
B \arrow[rr, "\Delta"] & & B^U &
\end{tikzcd}$$

Since the front, back, and right faces are pullbacks, the left is as well, which is what we wanted to show.

The final statement (vi) is left as Exercise 1.3.i. □

By Propositions 1.2.11 and 1.2.12, the subcategory inclusions $\mathcal{C}at \hookrightarrow \mathcal{QC}at$ and $\mathcal{K}an \hookrightarrow \mathcal{QC}at$ both define cosmological functors; see also Lemma 6.1.4. These cosmological embeddings will make precise the intuition that the formal category theory of 1-categories or of ∞-groupoids can be recovered as a special case of the formal category theory of $(\infty, 1)$-categories.

Non-examples of cosmological functors are also instructive:

1.3.4. Non-Example. The forgetful functor $\mathcal{K}_{/B} \to \mathcal{K}$ is simplicial and preserves the class of isofibrations but does *not* define a cosmological functor, failing to preserve cotensors and products. However, by Proposition 1.3.3(v), its right adjoint $- \times B \colon \mathcal{K} \to \mathcal{K}_{/B}$ does define a cosmological functor.

1.3.5. Non-Example. The cosmological embedding $\mathcal{K}an \hookrightarrow \mathcal{Q}Cat$ has a right adjoint $(-)^{\simeq} \colon \mathcal{Q}Cat \to \mathcal{K}an$ that carries each quasi-category to its "groupoid core" or maximal sub Kan complex, the simplicial subset containing those n-simplices whose edges are all isomorphisms. This "core" functor preserves isofibrations and 1-categorical limits but is *not* cosmological since it is not simplicially enriched: any functor $K \to Q$ whose domain is a Kan complex and whose codomain is a quasi-category factors through the inclusion $Q^{\simeq} \hookrightarrow Q$ via a unique map $K \to Q^{\simeq}$ but in general $\mathsf{Fun}(K, Q) \not\cong \mathsf{Fun}(K, Q^{\simeq})$, since a natural transformation $K \times \Delta[1] \to Q$ will only factor through $Q^{\simeq} \hookrightarrow Q$ in the case where its components are invertible. See Lemma 12.1.14 however.

Certain cosmological functors are especially well-behaved:

1.3.6. Definition (biequivalences). A cosmological functor defines a **biequivalence** $F \colon \mathcal{K} \xrightarrow{\sim} \mathcal{L}$ if additionally

(i) it is **essentially surjective on objects up to equivalence**: for all $C \in \mathcal{L}$ there exists $A \in \mathcal{K}$ so that $FA \simeq C$ and
(ii) it defines a **local equivalence**: for all $A, B \in \mathcal{K}$, the action of F on functor spaces defines an equivalence
$$\mathsf{Fun}(A, B) \xrightarrow{\sim} \mathsf{Fun}(FA, FB).$$

Cosmological biequivalences will be studied more systematically in Chapter 10, where we think of them as "change-of-model" functors. Crucially for our proof of the "model-independence" of $(\infty,1)$-category theory in Chapter 11, we have a variety of cosmological biequivalences between the ∞-cosmoi of $(\infty,1)$-categories:

1.3.7. Example (biequivalences between ∞-cosmoi of $(\infty,1)$-categories; §E.2).

(i) The underlying quasi-category functors defined on the ∞-cosmoi of complete Segal spaces, Segal categories, and 1-complicial sets
$$\mathcal{CSS} \xrightarrow[\sim]{(-)_0} \mathcal{Q}Cat \qquad \mathcal{S}egal \xrightarrow[\sim]{(-)_0} \mathcal{Q}Cat \qquad 1\text{-}\mathcal{C}omp \xrightarrow[\sim]{(-)_0} \mathcal{Q}Cat$$
are all biequivalences. In the first two cases these are defined by "evaluating at the 0th row" and in the last case this is defined by "forgetting the markings."
(ii) There is also a cosmological biequivalence $\mathcal{Q}Cat \xrightarrow{\sim} \mathcal{CSS}$ defined by Joyal and Tierney [**48**].
(iii) The functor $\mathcal{CSS} \xrightarrow{\sim} \mathcal{S}egal$ defined by Bergner [**11**] that "discretizes" a complete Segal spaces also defines a cosmological biequivalence.
(iv) There is a further cosmological biequivalence $(-)^{\natural} \colon \mathcal{Q}Cat \xrightarrow{\sim} 1\text{-}\mathcal{C}omp$ that gives each quasi-category its "natural marking," with all invertible 1-simplices and all simplices in dimension greater than 1 marked.

1.3.8. Remark. The underlying quasi-category functor $(-)_0 \colon \mathcal{K} \to \mathcal{Q}Cat$ carries the internal homs of a cartesian closed ∞-cosmos $\mathcal{K}$ to the corresponding functor spaces: for any ∞-categories A and B in $\mathcal{K}$, we have
$$(B^A)_0 \coloneqq \mathsf{Fun}(1, B^A) \cong \mathsf{Fun}(A, B).$$

In the case where the ∞-cosmos $\mathcal{K}$ is biequivalent to $\mathcal{QC}at$, we will see in Chapters 10 and 11 that this entails no essential loss of categorical information.

Cosmological biequivalences not only preserve equivalences but also reflect and create them:

1.3.9. Lemma. *Let $F\colon \mathcal{K} \xrightarrow{\sim} \mathcal{L}$ be a cosmological biequivalence. Then:*

(i) A functor $f\colon A \to B$ between ∞-categories in $\mathcal{K}$ is an equivalence if and only if $Ff\colon FA \to FB$ is an equivalence in $\mathcal{L}$.
(ii) A pair of ∞-categories in $\mathcal{K}$ are equivalent if and only if their images in $\mathcal{L}$ are equivalent.

Proof. Lemma 1.3.2 implies that cosmological functors preserve equivalences and thus also the condition of there being an equivalence between a pair of ∞-categories in $\mathcal{K}$. To see that equivalences are also reflected, suppose $f\colon A \to B$ is a functor in $\mathcal{K}$ with the property that $Ff\colon FA \xrightarrow{\sim} FB$ is an equivalence in $\mathcal{L}$. Now for any ∞-category X, simplicial functoriality provides a commutative diagram

$$\begin{tikzcd} \mathsf{Fun}(X,A) \arrow[r, "f_*"] \arrow[d, "\wr"] & \mathsf{Fun}(X,B) \arrow[d, "\wr"] \\ \mathsf{Fun}(FX,FA) \arrow[r, "Ff_*", "\sim"'] & \mathsf{Fun}(FX,FB) \end{tikzcd}$$

so from the 2-of-3 property we conclude that $f_*\colon \mathsf{Fun}(X,A) \to \mathsf{Fun}(X,B)$ is an equivalence, proving that f is an equivalence in $\mathcal{K}$.

To see that equivalences are created,c suppose now that A and B are ∞-categories in $\mathcal{K}$ equipped with an equivalence:

$$\begin{tikzcd} FA \arrow[r, shift left, "\tilde{f}"] \arrow[r, phantom, "\simeq"] & FB \arrow[l, shift left, "\tilde{g}"] \end{tikzcd} \qquad \begin{tikzcd} & FA \\ FA \arrow[ur, equal] \arrow[r, "\tilde{\alpha}"] \arrow[dr, "\tilde{g}\tilde{f}"'] & FA^{\mathbb{I}} \arrow[u, "\wr", "\mathrm{ev}_0"'] \arrow[d, "\wr"', "\mathrm{ev}_1"] \\ & FA \end{tikzcd} \qquad \begin{tikzcd} & FB \\ FB \arrow[ur, "\tilde{f}\tilde{g}"] \arrow[r, "\tilde{\beta}"] \arrow[dr, equal] & FB^{\mathbb{I}} \arrow[u, "\wr", "\mathrm{ev}_0"'] \arrow[d, "\wr"', "\mathrm{ev}_1"] \\ & FB \end{tikzcd}$$

in $\mathcal{L}$. Since $\mathsf{Fun}(A,B) \xrightarrow{\sim} \mathsf{Fun}(FA,FB)$ and $\mathsf{Fun}(B,A) \xrightarrow{\sim} \mathsf{Fun}(FB,FA)$ are equivalences of quasi-categories, by Remark 1.1.23 the induced functors between their homotopy categories $\mathsf{h}(\mathsf{Fun}(A,B)) \xrightarrow{\sim} \mathsf{h}(\mathsf{Fun}(FA,FB))$ and $\mathsf{h}(\mathsf{Fun}(B,A)) \xrightarrow{\sim} \mathsf{h}(\mathsf{Fun}(FB,FA))$ are equivalences of categories, and in particular essentially surjective. So we may lift $\tilde{f}$ and $\tilde{g}$ to functors $f\colon A \to B$ and $g\colon B \to A$ so that $Ff \cong \tilde{f}$ and $Fg \cong \tilde{g}$ in $\mathsf{h}(\mathsf{Fun}(A,B))$ and $\mathsf{h}(\mathsf{Fun}(B,A))$ respectively. The commutative diagram

$$\begin{tikzcd} \mathsf{Fun}(B,A) \times \mathsf{Fun}(A,B) \arrow[r, "\circ"] \arrow[d, "\wr"] & \mathsf{Fun}(A,A) \arrow[d, "\wr"] \\ \mathsf{Fun}(FB,FA) \times \mathsf{Fun}(FA,FB) \arrow[r, "\circ"] & \mathsf{Fun}(FA,FA) \end{tikzcd}$$

induces a commutative diagram of functors between homotopy categories. In particular, by composing isomorphisms we see that

$$F(\mathrm{id}_A) = \mathrm{id}_{FA} \cong \tilde{g} \circ \tilde{f} \cong Fg \circ Ff = F(g \circ f)$$

in $\mathsf{h}(\mathsf{Fun}(A,A))$. By fully faithfulness of $\mathsf{h}(\mathsf{Fun}(A,A)) \xrightarrow{\sim} \mathsf{h}(\mathsf{Fun}(FA,FA))$, this isomorphism lifts to an isomorphism $\mathrm{id}_A \cong g \circ f$ in $\mathsf{h}(\mathsf{Fun}(A,A))$. By Corollary 1.1.16, this isomorphism can be represented by a homotopy coherent isomorphism $\mathbb{I} \to \mathsf{Fun}(A, A)$, which internalizes to define a map $\alpha\colon A \to A^{\mathbb{I}}$ as required. The construction of the homotopy coherent isomorphism $\beta\colon B \to B^{\mathbb{I}}$ from $f \circ g$ to id_B proceeds similarly. $\square$

The proof of the creation of equivalences in Lemma 1.3.9 is surprisingly delicate, passing to the homotopy categories of the functor spaces to avoid lifting and composing homotopy coherent isomorphisms; an argument along those lines is also possible, and left to the reader as Exercise 1.3.ii. The next section will give context for the argument just given, introducing the *homotopy 2-category of an ∞-cosmos*. The reader will then have an opportunity to revisit the creation of equivalences in Exercise 1.4.v.

Exercises.

1.3.i. Exercise. Prove that for any cosmological functor $F\colon \mathcal{K} \to \mathcal{L}$ and any $A \in \mathcal{K}$, the induced map $F\colon \mathcal{K}_{/A} \to \mathcal{L}_{/FA}$ defines a cosmological functor.

1.3.ii. Exercise. Sketch a proof that cosmological biequivalences create equivalences between ∞-categor ies by lifting and composing the homotopy coherent isomorphisms given as part of the data of the hypothesized equivalences, without passing to homotopy categories.

1.3.iii. Exercise. Suppose $F\colon \mathcal{K} \to \mathcal{L}$, $G\colon \mathcal{L} \to \mathcal{M}$, and $H\colon \mathcal{M} \to \mathcal{N}$ are cosmological functors, and assume that GF and HG are cosmological biequivalences. Show that F, G, H, and HGF are cosmological biequivalences.

1.4. The Homotopy 2-Category

Small 1-categories define the objects of a strict 2-category[24] $\mathcal{C}at$ of categories, functors, and natural transformations. Many basic categorical notions — those defined in terms of categories, functors, and natural transformations — can be defined internally to the 2-category $\mathcal{C}at$. This suggests a natural avenue for generalization: reinterpreting these same definitions in a generic 2-category using its objects in place of small categories, its 1-cells in place of functors, and its 2-cells in place of natural transformations.

In Chapter 2, we will develop a non-trivial portion of the theory of ∞-categories in any fixed ∞-cosmos following exactly this outline, working internally to a strict 2-category that we refer to as the *homotopy 2-category* that we associate to any ∞-cosmos. The homotopy 2-category of an ∞-cosmos is a quotient of the full ∞-cosmos, replacing each quasi-categorical functor-space by its homotopy category. Surprisingly, this rather destructive quotienting operation preserves quite a lot of information. Indeed, essentially all of the development of the theory of ∞-categories in Part I takes place in the homotopy 2-category of an ∞-cosmos. This said, we caution the reader against becoming overly seduced by homotopy 2-categories. This structure is more of a technical convenience for reducing the complexity of our arguments than a fundamental notion of ∞-category theory.

[24]A comprehensive introduction to strict 2-categories appears as Appendix B. Succinctly, in parallel with Digression 1.2.4, 2-categories can be understood equally as

- "two-dimensional" categories, with objects; **1-cells**, whose boundary are given by a pair of objects; and **2-cells**, whose boundary are given by a parallel pair of 1-cells between a pair of objects—together with partially-defined composition operations governed by this boundary data
- or as categories enriched over $\mathcal{C}at$.

See Definition B.1.1 and Exercise 1.4.i.

The homotopy 2-category for the ∞-cosmos of quasi-categories was first introduced by Joyal in his work on the foundations of quasi-category theory [**47**].

1.4.1. Definition (homotopy 2-category). Let $\mathcal{K}$ be an ∞-cosmos. Its **homotopy 2-category** is the strict 2-category $\mathfrak{h}\mathcal{K}$ whose

- objects are the the objects A, B of $\mathcal{K}$, i.e., the ∞-categories;
- 1-cells $f \colon A \to B$ are the 0-arrows in the functor space $\mathsf{Fun}(A, B)$, i.e., the ∞-functors; and
- 2-cells $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd}$ are homotopy classes of 1-simplices in $\mathsf{Fun}(A, B)$, which we call **∞-natural transformations**.

Put another way $\mathfrak{h}\mathcal{K}$ is the 2-category with the same objects as $\mathcal{K}$ and with hom-categories defined by

$$\mathsf{hFun}(A, B) \coloneqq \mathsf{h}(\mathsf{Fun}(A, B)),$$

that is, as the homotopy category of the quasi-category $\mathsf{Fun}(A, B)$.

The **underlying category** of a 2-category is defined by simply forgetting its 2-cells. Note that an ∞-cosmos $\mathcal{K}$ and its homotopy 2-category $\mathfrak{h}\mathcal{K}$ share the same underlying category $\mathcal{K}_0$ of ∞-categories and ∞-functors in $\mathcal{K}$.

1.4.2. Digression (change of base §A.7). The homotopy category functor $\mathsf{h} \colon s\mathcal{S}et \to \mathcal{C}at$ preserves finite products, as of course does its right adjoint. It follows that the adjunction of Proposition 1.1.11 induces a change-of-base adjunction

$$\begin{tikzcd} 2\text{-}\mathcal{C}at \arrow[r, bend right] \arrow[r, phantom, "\perp"] & s\mathcal{S}et\text{-}\mathcal{C}at \arrow[l, bend right, "\mathsf{h}_*"'] \end{tikzcd}$$

whose left and right adjoints change the enrichment by applying the homotopy category functor or the nerve functor to the hom objects of the enriched category. Here $2\text{-}\mathcal{C}at$ and $s\mathcal{S}et\text{-}\mathcal{C}at$ can each be understood as 2-categories — of enriched categories, enriched functors, and enriched natural transformations — and both change of base constructions define 2-functors; see Proposition A.7.4 or [**18**, 6.4.3]. Since the nerve embedding is fully faithful, 2-categories can be identified as a full subcategory comprised of those simplicial categories whose hom-spaces are nerves of categories; see Exercise 1.4.i.

The proof of Lemma 1.3.9 made use of the following observation.

1.4.3. Lemma.

(i) Every 2-cell in the homotopy 2-category of an ∞-cosmos

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd}$$

is represented by a map $\ulcorner\alpha\urcorner \colon \mathbb{2} \to \mathsf{Fun}(A, B)$ or equivalently by a functor $\ulcorner\alpha\urcorner \colon A \to B^{\mathbb{2}}$, and two such maps represent the same 2-cell if and only if their images are homotopic as 1-simplices in $\mathsf{Fun}(A, B)$.

(ii) Every invertible 2-cell in the homotopy 2-category of an ∞-cosmos

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\cong\Downarrow\alpha"] & B \end{tikzcd}$$

is represented by a map $\ulcorner\alpha\urcorner \colon \mathbb{I} \to \mathsf{Fun}(A, B)$ *or equivalently by a functor* $\ulcorner\alpha\urcorner \colon A \to B^{\mathbb{I}}$*, and two such maps represent the same invertible 2-cell if and only if their common restrictions along* $\mathbb{2} \hookrightarrow \mathbb{I}$ *are homotopic as 1-simplices in* $\mathsf{Fun}(A, B)$*.*

The notion of homotopic 1-simplices referenced here is defined in Lemma 1.1.9.

Proof. The first statement is immediately from the definition of the 2-cells in the homotopy 2-category and the universal property (1.2.7) of the simplicial cotensor. Now a 2-cell in a 2-category is **invertible** if and only if it defines an isomorphism in the appropriate hom-category $\mathsf{hFun}(A, B)$. By Definition 1.1.13 and Corollary 1.1.16 it follows that each invertible 2-cell in $\mathfrak{h}\mathcal{K}$ is represented by a homotopy coherent isomorphism $\mathbb{I} \to \mathsf{Fun}(A, B)$, which similarly internalizes to define a functor. □

An upshot of Digression 1.4.2 is that change of base is an operation that applies to enriched functors as well as enriched categories, as can be directly verified in the case of greatest interest:

1.4.4. Lemma. *Any simplicial functor* $F \colon \mathcal{K} \to \mathcal{L}$ *between ∞-cosmoi induces a 2-functor* $F \colon \mathfrak{h}\mathcal{K} \to \mathfrak{h}\mathcal{L}$ *between their homotopy 2-categories.*

Proof. The action of the induced 2-functor $F \colon \mathfrak{h}\mathcal{K} \to \mathfrak{h}\mathcal{L}$ on objects and 1-cells is given by the corresponding action of $F \colon \mathcal{K} \to \mathcal{L}$; recall an ∞-cosmos and its homotopy 2-category have the same underlying 1-category. Each 2-cell in $\mathfrak{h}\mathcal{K}$ is represented by a 1-simplex in $\mathsf{Fun}(A, B)$ which is mapped via

$$\mathsf{Fun}(A, B) \xrightarrow{\quad F \quad} \mathsf{Fun}(FA, FB)$$

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \arrow[r, mapsto] & FA \arrow[r, bend left, "Ff"] \arrow[r, bend right, "Fg"'] \arrow[r, phantom, "\Downarrow F\alpha"] & FB \end{tikzcd}$$

to a 1-simplex representing a 2-cell in $\mathfrak{h}\mathcal{L}$. Since the action $F \colon \mathsf{Fun}(A, B) \to \mathsf{Fun}(FA, FB)$ on functor spaces defines a morphism of simplicial sets, it preserves faces and degeneracies. In particular, homotopic 1-simplices in $\mathsf{Fun}(A, B)$ are carried to homotopic 1-simplices in $\mathsf{Fun}(FA, FB)$ so the action on 2-cells just described is well-defined. The 2-functoriality of these mappings follows from the simplicial functoriality of the original mapping. □

We now begin to relate the simplicially enriched structures of an ∞-cosmos to the 2-categorical structures in its homotopy 2-category. The first result proves that homotopy 2-categories inherit products from their ∞-cosmoi, which satisfy a 2-categorical universal property. To illustrate, recall that the terminal ∞-category $1 \in \mathcal{K}$ has the universal property $\mathsf{Fun}(X, 1) \cong \mathbb{1}$ for all $X \in \mathcal{K}$. Applying the homotopy category functor we see that $1 \in \mathfrak{h}\mathcal{K}$ has the universal property $\mathsf{hFun}(X, 1) \cong \mathbb{1}$ for all $X \in \mathfrak{h}\mathcal{K}$, which is expressed by saying that the ∞-category 1 defines a **2-terminal object** in the homotopy 2-category. This 2-categorical universal property has both a 1-dimensional and a 2-dimensional aspect. Since $\mathsf{hFun}(X, 1) \cong \mathbb{1}$ is a category with a single object, there exists a unique morphism $X \to 1$ in $\mathcal{K}$. And since $\mathsf{hFun}(X, 1) \cong \mathbb{1}$ has only a single morphism, we see that the only 2-cells in $\mathfrak{h}\mathcal{K}$ with codomain 1 are identities.

1.4.5. Proposition (cartesian (closure)).

(i) The homotopy 2-category of any ∞-cosmos has 2-categorical products.

(ii) The homotopy 2-category of a cartesian closed ∞-cosmos is cartesian closed as a 2-category.

Proof. While the functor $\mathsf{h}\colon s\mathcal{S}et \to \mathcal{C}at$ only preserves finite products, the restricted functor $\mathsf{h}\colon \mathcal{QC}at \to \mathcal{C}at$ preserves *all* products on account of the simplified description of the homotopy category of a quasi-category given in Lemma 1.1.12; see Exercise 1.2.iii. Thus for any set I and family of ∞-categories $(A_i)_{i\in I}$ in $\mathcal{K}$, the homotopy category functor carries the isomorphism of quasi-categories displayed below left to an isomorphisms of hom-categories displayed below right

$$\mathsf{Fun}(X, \textstyle\prod_{i\in I} A_i) \xrightarrow{\cong} \prod_{i\in I} \mathsf{Fun}(X, A_i) \qquad \overset{\mathsf{h}}{\mapsto} \qquad \mathsf{hFun}(X, \prod_{i\in I} A_i) \xrightarrow{\cong} \prod_{i\in I} \mathsf{hFun}(X, A_i).$$

This proves that the homotopy 2-category $\mathfrak{h}\mathcal{K}$ has products whose universal properties have both a 1- and 2-dimensional component, as described for terminal objects above.

If $\mathcal{K}$ is a cartesian closed ∞-cosmos, then for any triple of ∞-categories $A, B, C \in \mathcal{K}$ there exist exponential objects $C^A, C^B \in \mathcal{K}$ characterized by natural isomorphisms

$$\mathsf{Fun}(A \times B, C) \cong \mathsf{Fun}(A, C^B) \cong \mathsf{Fun}(B, C^A).$$

Passing to homotopy categories we have natural isomorphisms

$$\mathsf{hFun}(A \times B, C) \cong \mathsf{hFun}(A, C^B) \cong \mathsf{hFun}(B, C^A),$$

which demonstrates that $\mathfrak{h}\mathcal{K}$ is cartesian closed as a 2-category: functors $A \times B \to C$ transpose to define functors $A \to C^B$ and $B \to C^A$, and 2-cells transpose similarly. $\square$

There is a standard definition of *isomorphism* between two objects in any 1-category, preserved by any functor. Similarly, there is a standard definition of *equivalence* between two objects in any 2-category, preserved by any 2-functor:

1.4.6. Definition (equivalence). An **equivalence** in a 2-category is given by

- a pair of objects A and B
- a pair of 1-cells $f\colon A \to B$ and $g\colon B \to A$
- a pair of invertible 2-cells

$$\begin{tikzcd} A \arrow[r, bend left, equal] \arrow[r, bend right, "gf"'] \arrow[r, phantom, "\cong\Downarrow\alpha"] & A \end{tikzcd} \qquad\qquad \begin{tikzcd} B \arrow[r, bend left, "fg"] \arrow[r, bend right, equal] \arrow[r, phantom, "\cong\Downarrow\beta"] & B \end{tikzcd}$$

When A and B are **equivalent**, we write $A \simeq B$ and refer to the 1-cells f and g as **equivalences**, denoted by "$\xrightarrow{\sim}$."

In the case of the homotopy 2-category of an ∞-cosmos we have a competing definition of equivalence from 1.2.1: namely a 1-cell $f\colon A \xrightarrow{\sim} B$ that induces an equivalence $f_*\colon \mathsf{Fun}(X, A) \xrightarrow{\sim} \mathsf{Fun}(X, B)$ on functor-spaces — or equivalently, by Lemma 1.2.18, a homotopy equivalence defined relative to the interval $\mathbb{I}$. Crucially, all three notions of equivalence coincide:

1.4.7. Theorem (equivalences are equivalences). *In any ∞-cosmos $\mathcal{K}$, the following are equivalent and characterize what it means for a functor $f\colon A \to B$ between ∞-categories to define an **equivalence**.*

(i) For all $X \in \mathcal{K}$, the post-composition map $f_*\colon \mathsf{Fun}(X, A) \xrightarrow{\sim} \mathsf{Fun}(X, B)$ defines an equivalence of quasi-categories.

(ii) *There exists a functor $g\colon B \to A$ and natural isomorphisms $\alpha\colon \mathrm{id}_A \cong gf$ and $\beta\colon fg \cong \mathrm{id}_B$ in the homotopy 2-category.*

(iii) *There exists a functor $g\colon B \to A$ and homotopies*

$$\begin{tikzcd} & A & & & B \\ A \arrow[ur, equal] \arrow[r, "\alpha"] \arrow[dr, "gf"'] & A^{\mathbb{I}} \arrow[u, two heads, "\mathrm{ev}_0"', "\wr"] \arrow[d, two heads, "\mathrm{ev}_1", "\wr"'] & & B \arrow[ur, "fg"] \arrow[r, "\beta"] \arrow[dr, equal] & B^{\mathbb{I}} \arrow[u, two heads, "\mathrm{ev}_0"', "\wr"] \arrow[d, two heads, "\mathrm{ev}_1", "\wr"'] \\ & A & & & B \end{tikzcd}$$

in the ∞-cosmos in $\mathcal{K}$.

As an illustrative comparison of 2-categorical and quasi-categorical techniques, rather than appealing to Lemma 1.2.18, we re-prove it.

Proof. For (i)⇒(ii), if the induced map on post-composition $f_*\colon \mathsf{Fun}(X, A) \xrightarrow{\sim} \mathsf{Fun}(X, B)$ defines an equivalence of quasi-categories, then by Remark 1.1.23, $f_*\colon \mathsf{hFun}(X, A) \xrightarrow{\sim} \mathsf{hFun}(X, B)$ defines an equivalence of categories. In particular, $f_*\colon \mathsf{hFun}(B, A) \xrightarrow{\sim} \mathsf{hFun}(B, B)$ is essentially surjective so there exists $g \in \mathsf{hFun}(B, A)$ and an isomorphism $\beta\colon fg \cong \mathrm{id}_B \in \mathsf{hFun}(B, B)$. Now since $f_*\colon \mathsf{hFun}(A, A) \xrightarrow{\sim} \mathsf{hFun}(A, B)$ is fully faithful, the isomorphism $\beta f\colon fgf \cong f \in \mathsf{hFun}(A, B)$ can be lifted to define an isomorphism $\alpha^{-1}\colon gf \cong \mathrm{id}_A \in \mathsf{hFun}(A, A)$. This defines the data of a 2-categorical equivalence in Definition 1.4.6.

To see that (ii)⇒(iii) recall from Lemma 1.4.3 that the natural isomorphisms $\alpha\colon \mathrm{id}_A \cong gf$ and $\beta\colon fg \cong \mathrm{id}_B$ in $\mathfrak{h}\mathcal{K}$ are represented by maps $\alpha\colon A \to A^{\mathbb{I}}$ and $\beta\colon B \to B^{\mathbb{I}}$ in $\mathcal{K}$ as in (1.2.19).

Finally, (iii)⇒(i) since $\mathsf{Fun}(X, -)$ carries the data of (iii) to the data of an equivalence of categories as in Definition 1.1.22. □

1.4.8. Digression (on the significance of Theorem 1.4.7). It is hard to overstate the importance of Theorem 1.4.7 to the work that follows. The categorical constructions that we will introduce for ∞-categories, ∞-functors, and ∞-natural transformations are invariant under 2-categorical equivalence in the homotopy 2-category and the universal properties we develop similarly characterize 2-categorical equivalence classes of ∞-categories. Theorem 1.4.7 then asserts that such constructions are "homotopically correct": both invariant under equivalence in the ∞-cosmos and precisely identifying equivalence classes of objects.

The equivalence invariance of the functor space in the codomain variable is axiomatic, but equivalence invariance in the domain variable is not.[25] Nor is it evident how this could be proven from either (i) or (iii) of Theorem 1.4.7.

But using (ii) and 2-categorical techniques, there is now a short proof:

1.4.9. Corollary. *Equivalences of ∞-categories $A' \xrightarrow{\sim} A$ and $B \xrightarrow{\sim} B'$ induce an equivalence of functor spaces $\mathsf{Fun}(A, B) \xrightarrow{\sim} \mathsf{Fun}(A', B')$.*

Proof. The simplicial functors $\mathsf{Fun}(A, -)\colon \mathcal{K} \to \mathcal{QC}at$ and $\mathsf{Fun}(-, B)\colon \mathcal{K}^{\mathrm{op}} \to \mathcal{QC}at$ induce 2-functors $\mathsf{Fun}(A, -)\colon \mathfrak{h}\mathcal{K} \to \mathfrak{h}\mathcal{QC}at$ and $\mathsf{Fun}(-, B)\colon \mathfrak{h}\mathcal{K}^{\mathrm{op}} \to \mathfrak{h}\mathcal{QC}at$, which preserve the 2-categorical equivalences of Definition 1.4.6. By Theorem 1.4.7 this is what we wanted to show. □

[25] Lemma 1.3.2 does not apply since $\mathsf{Fun}(-, B)$ is not cosmological.

There is also a standard 2-categorical notion of an isofibration, defined in the statement of Proposition 1.4.10 and elaborated upon in Definition B.4.6. We now show that any isofibration in an ∞-cosmos defines an isofibration in its homotopy 2-category.[26]

1.4.10. Proposition (isofibrations define isofibrations). *Any isofibration $p\colon E \twoheadrightarrow B$ in an ∞-cosmos $\mathcal{K}$ also defines an isofibration in the homotopy 2-category $\mathfrak{h}\mathcal{K}$: given any invertible 2-cell as displayed below left abutting to B with a specified lift of one of its boundary 1-cells through p, there exists an invertible 2-cell abutting to E with this boundary 1-cell as displayed below right that whiskers with p to the original 2-cell.*

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[dr, bend right, "b"'] & E \arrow[d, two heads, "p"] \arrow[dl, phantom, "\cong\Downarrow\beta"] \\ & B \end{tikzcd} \quad = \quad \begin{tikzcd} X \arrow[r, bend left, "e"] \arrow[r, bend right, "\bar{e}"'] \arrow[r, phantom, "\cong\Downarrow\gamma"] & E \arrow[d, two heads, "p"] \\ & B \end{tikzcd}$$

Proof. Put another way, the universal property of the statement says that the functor

$$p_*\colon \mathsf{hFun}(X, E) \twoheadrightarrow \mathsf{hFun}(X, B)$$

is an isofibration of categories in the sense defined in Proposition 1.2.11. By axiom 1.2.1(ii), since $p\colon E \twoheadrightarrow B$ is an isofibration in $\mathcal{K}$, the induced map $p_*\colon \mathsf{Fun}(X, E) \twoheadrightarrow \mathsf{Fun}(X, B)$ is an isofibration of quasi-categories. So it suffices to show that the functor $\mathsf{h}\colon \mathcal{QC}at \to \mathcal{C}at$ carries isofibrations of quasi-categories to isofibrations of categories.[27]

So let us now consider an isofibration $p\colon E \twoheadrightarrow B$ between quasi-categories. By Corollary 1.1.16, every isomorphism β in the homotopy category $\mathsf{h}B$ of the quasi-category B is represented by a simplicial map $\beta\colon \mathbb{I} \to B$. By Definition 1.1.17, the lifting problem

$$\begin{tikzcd} \mathbb{1} \arrow[r, "e"] \arrow[d, hook] & E \arrow[d, two heads, "p"] \\ \mathbb{I} \arrow[r, "\beta"'] \arrow[ur, dashed, "\gamma"] & B \end{tikzcd}$$

can be solved, and the map $\gamma\colon \mathbb{I} \to E$ so-produced represents a lift of the isomorphism from $\mathsf{h}B$ to an isomorphism in $\mathsf{h}E$ with domain e. ☐

1.4.11. Convention (on "isofibrations" in homotopy 2-categories). Since the converse to Proposition 1.4.10 does not hold, there is a potential ambiguity when using the term "isofibration" to refer to a map in the homotopy 2-category of an ∞-cosmos. We adopt the convention that when we declare that a map in $\mathfrak{h}\mathcal{K}$ is an isofibration we always mean this is the stronger sense of defining an isofibration in $\mathcal{K}$. This stronger condition gives us access to the 2-categorical lifting property of Proposition 1.4.10 but also to the many homotopical properties axiomatized in Definition 1.2.1, which guarantee that the strictly defined limits of 1.2.1(i) are automatically equivalence-invariant constructions, as shown in §C.1 and Proposition 6.3.1.

We conclude this chapter with a final definition that can be extracted from the homotopy 2-category of an ∞-cosmos. The 1- and 2-cells in the homotopy 2-category from the terminal ∞-category $1 \in \mathcal{K}$ to a generic ∞-category $A \in \mathcal{K}$ define the objects and morphisms in the *homotopy category* of the ∞-category A.

[26] The converse does not hold, however, nor is it the case that a representably-defined isofibration of quasi-categories is necessarily an isofibration in the ∞-cosmos; consider the case of sliced ∞-cosmoi for instance.

[27] Alternately, argue directly using Lemma 1.4.3.

1.4.12. Definition (homotopy category of an ∞-category). The **homotopy category** of an ∞-category A in an ∞-cosmos $\mathcal{K}$ is defined to be the homotopy category of its underlying quasi-category, that is:

$$\mathsf{h}A := \mathsf{hFun}(1, A) := \mathsf{h}(\mathsf{Fun}(1, A)).$$

As we shall discover, homotopy categories generally bear "derived" analogues of structures present at the level of ∞-categories. An early example of this appears as a remark after the statement of Proposition 2.1.7.

Exercises.

1.4.i. Exercise. Consider a diagram of identity-on-objects functors (1.2.5) defining a simplicial category $\mathcal{A}$ as described in Digression 1.2.4 and Exercise 1.2.i. Show that the hom-spaces in this simplicial category are nerves of categories, and thus $\mathcal{A}$ defines a 2-category, if and only if $\mathcal{A}_\bullet \colon \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}at$ satisfies the **Segal condition**: for each $n \geq 2$, the canonical map

$$\mathcal{A}_n \to \mathcal{A}_1 \underset{\mathcal{A}_0}{\times} \cdots \underset{\mathcal{A}_0}{\times} \mathcal{A}_1,$$

whose codomain is the wide pullback of n copies of $\mathcal{A}_1$ along alternating instances of the maps $\delta_0, \delta_1 \colon \mathcal{A}_1 \rightrightarrows \mathcal{A}_0$, is an isomorphism.[28]

1.4.ii. Exercise.

(i) What is the homotopy 2-category of the ∞-cosmos $\mathcal{C}at$ of 1-categories?

(ii) Prove that the nerve defines a 2-functor $\mathcal{C}at \hookrightarrow \mathfrak{h}\mathcal{QC}at$ that is *locally fully faithful*.

1.4.iii. Exercise. Demonstrate that the homotopy 2-category of the dual cosmos $\mathcal{K}^{\mathrm{co}}$ of an ∞-cosmos $\mathcal{K}$ is the co-dual of the homotopy 2-category $\mathfrak{h}\mathcal{K}$, with the domains and codomains of 2-cells but not 1-cells reversed: in symbols $\mathfrak{h}(\mathcal{K}^{\mathrm{co}}) \cong (\mathfrak{h}\mathcal{K})^{\mathrm{co}}$.

1.4.iv. Exercise (10.3.1). Extend Lemma 1.4.4 to show that if $F \colon \mathcal{K} \to \mathcal{L}$ is a cosmological biequivalence then $F \colon \mathfrak{h}\mathcal{K} \to \mathfrak{h}\mathcal{L}$ is a 2-categorical **biequivalence**, a 2-functor that is essentially surjective on objects up to equivalence that locally defines an equivalence of hom-categories.

1.4.v. Exercise. Let $F \colon \mathcal{K} \xrightarrow{\sim} \mathcal{L}$ be a cosmological biequivalence and let $A, B \in \mathcal{K}$. Re-prove part of the statement of Lemma 1.3.9: that if $FA \simeq FB$ in $\mathcal{L}$ then $A \simeq B$ in $\mathcal{K}$.

1.4.vi. Exercise. Let B be an ∞-category in the ∞-cosmos $\mathcal{K}$ and let $\mathfrak{h}\mathcal{K}_{/B}$ denote the 2-category whose

- objects are isofibrations $E \twoheadrightarrow B$ in $\mathcal{K}$ with codomain B
- 1-cells are 1-cells in $\mathfrak{h}\mathcal{K}$ over B

$$\begin{tikzcd} E \arrow[rr] \arrow[dr, two heads] & & F \arrow[dl, two heads] \\ & B & \end{tikzcd}$$

- 2-cells are 2-cells α in $\mathfrak{h}\mathcal{K}$

$$\begin{tikzcd} E \arrow[rr, bend left, "f"] \arrow[rr, bend right, "g"'] \arrow[rr, phantom, "\Downarrow\alpha"] \arrow[dr, two heads, "p"'] & & F \arrow[dl, two heads, "q"] \\ & B & \end{tikzcd}$$

[28] An additional exercise, for readers familiar with the definition of a 2-category as a category internal to $\mathcal{C}at$ whose "category of objects" is discrete, is to compare that definition to this one.

that lie over B in the sense that $q\alpha = \mathrm{id}_p$.

Argue that the homotopy 2-category $\mathfrak{h}(\mathcal{K}_{/B})$ of the sliced ∞-cosmos has the same underlying 1-category but different 2-cells. How do these compare with the 2-cells of $\mathfrak{h}\mathcal{K}_{/B}$?[29]

[29]A more systematic comparison appears in Proposition 3.6.3.

CHAPTER 2

Adjunctions, Limits, and Colimits I

Heuristically, ∞-categories generalize ordinary 1-categories by adding in higher dimensional morphisms and weakening the composition law. One could imagine "∞-tizing" other types of categorical structure similarly, by adding in higher dimension and weakening properties. The naïve hope would then be that proofs establishing the theory of 1-categories might similarly generalize to give proofs for ∞-categories, just by adding a prefix "∞-" everywhere. In this chapter, we make this dream a reality — at least for a library of basic propositions concerning equivalences, adjunctions, limits, and colimits and the relationships between these notions.

Recall that categories, functors, and natural transformations assemble into a 2-category $\mathcal{C}at$. Similarly, the ∞-categories, ∞-functors, and ∞-natural transformations in any ∞-cosmos assemble into a 2-category, namely the *homotopy 2-category* of the ∞-cosmos, introduced in §1.4. In fact, by Exercise 1.4.ii, $\mathcal{C}at$ can be regarded as a special case of a homotopy 2-category. In this chapter, we will use strict 2-categorical techniques to define *adjunctions* between ∞-categories and *limits* and *colimits* of diagrams valued in an ∞-category and prove that these notions interact in the expected ways. In the homotopy 2-category of categories, this recovers classical results from 1-category theory, and in some cases even specializes to the standards proofs. As these proofs are equally valid in any homotopy 2-category, our arguments also establish the desired generalizations by simply appending the prefix "∞-."

In §2.1, we define an adjunction between ∞-categories to be an adjunction in the homotopy 2-category of ∞-categories, ∞-functors, and ∞-natural transformations. While it takes some work, relegated to §F.5, to justify the moral correctness of this simple definition it has the great advantage that proofs of a number of results concerning the calculus of adjunctions and equivalences can be taken "off the shelf" in the sense that anyone who is sufficiently well-acquainted with 2-categories might know them already. In §2.2, we specialize the theory of adjunctions between ∞-categories to define and study initial and terminal elements inside an ∞-category. This section also serves as a warmup for the more subtle general theory of limits and colimits of diagrams valued in an ∞-category, which is the subject of §2.3. Finally, in §2.4, we study the interactions between these notions, proving that right adjoints preserve limits and left adjoints preserve colimits.

Missing from this discussion is an account of the universal properties associated to the unit of an adjunction or to a limit cone. These will be incorporated when we return to these topics in Chapter 4 after introducing an appropriate "hom ∞-category" with which to state them.

2.1. Adjunctions and Equivalences

In §1.4, we encountered the definition of an *equivalence* between a pair of objects in a 2-category. In the case where the ambient 2-category is the homotopy 2-category of an ∞-cosmos, we observed in Theorem 1.4.7 that the 2-categorical notion of equivalence precisely recaptures the notion of equivalence introduced in Definition 1.2.1 between ∞-categories in the full ∞-cosmos. In each of the examples of ∞-cosmoi we have considered, the representably-defined equivalences in the ∞-cosmos coincide with the standard notion of equivalences between ∞-categories as presented in that particular model.[1] Thus, the 2-categorical notion of equivalence is the “correct” notion of equivalence between ∞-categories.

Similarly, there is a standard definition of an *adjunction* between a pair of objects in a 2-category, which, when interpreted in the homotopy 2-category of ∞-categories, functors, and natural transformations in an ∞-cosmos, will define the correct notion of adjunction between ∞-categories.

2.1.1. Definition (adjunction). An **adjunction** between ∞-categories is comprised of:

- a pair of ∞-categories A and B,
- a pair of ∞-functors $u\colon A \to B$ and $f\colon B \to A$,
- and a pair of ∞-natural transformations $\eta\colon 1_B \Rightarrow uf$ and $\epsilon\colon fu \Rightarrow 1_A$, called the **unit** and **counit** respectively,

so that the triangle equalities hold:[2]

$$\begin{tikzcd} & B \arrow[rr, equals] \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow\epsilon"] & & B \arrow[d, phantom, "\Downarrow\eta"'] \\ A \arrow[ur, "u"] \arrow[rr, equals] & & A \arrow[ur, "u"'] & \end{tikzcd} = \begin{tikzcd} B \arrow[d, phantom, "="] \\ A \arrow[u, bend left, "u"] \arrow[u, bend right, "u"'] \end{tikzcd} \qquad \begin{tikzcd} B \arrow[rr, equals] \arrow[dr, "f"'] & & B \arrow[dr, "f"] & \\ & A \arrow[ur, "u"] \arrow[rr, equals] \arrow[u, phantom, "\Downarrow\eta"] & & A \arrow[u, phantom, "\Downarrow\epsilon"] \end{tikzcd} = \begin{tikzcd} B \arrow[d, bend right, "f"'] \arrow[d, bend left, "f"] \arrow[d, phantom, "="] \\ A \end{tikzcd}$$

The functor f is called the **left adjoint** and u is called the **right adjoint**, a relationship that is denoted symbolically in text by writing $f \dashv u$ or in a displayed diagram such as[3]

$$\begin{tikzcd} A \arrow[r, bend right, "u"'] \arrow[r, phantom, "\perp"] & B \arrow[l, bend right, "f"'] \end{tikzcd}$$

In the future, we generally drop the prefix “∞”- from the functors and natural transformations between ∞-categories.

2.1.2. Digression (why this is the correct definition). For readers who find Definition 2.1.1 implausible — perhaps too simple to be trusted — we offer a few words of justification. Joyal was the first to propose the use of the standard 2-categorical definition of an adjunction, defining an adjunction between quasi-categories to be an adjunction in (what we call) the homotopy 2-category $\mathfrak{h}\mathcal{QC}at$ in the preface to [**47**]. However, this definition was not widely adopted, with most practitioners instead using Lurie’s definition of adjunction between quasi-categories [**59**, §5.2], which takes a quite different form. In Appendix F, we prove that in the ∞-cosmos of quasi-categories, Joyal’s 2-categorical definition of adjunction precisely recovers Lurie’s. As explained in Part III, each of the models of $(\infty, 1)$-categories described in Example 1.2.24 “has the same category theory,” so Definition 2.1.1 agrees with the community consensus notion of adjunction between $(\infty, 1)$-categories.

But what about those ∞-cosmoi whose objects model (∞, n)- or (∞, ∞)-categories? For instance in the ∞-cosmos of complicial sets, the adjunctions defined in the homotopy 2-category are the “pseudo-style” adjunctions. While these are not the most general adjunctions that might be considered — for instance, one could have the triangle equality relations satisfied only up to coherent

[1] For instance, as outlined in Digression 1.2.13, the equivalences in the ∞-cosmoi of Example 1.2.24 recapture the weak equivalences between fibrant-cofibrant objects in the usual model structure.

[2] The left-hand equality of pasting diagrams asserts the composition relation $u\epsilon \cdot \eta u = \mathrm{id}_u$ in the hom-category $\mathsf{hFun}(A, B)$, while the right-hand equality asserts that $\epsilon f \cdot f\eta = \mathrm{id}_f$ in $\mathsf{hFun}(B, A)$. Appendix B and in particular §B.1 are included for the reader who is unfamiliar with the calculus of pasting diagrams.

[3] Some authors contort adjunction diagrams so that the left adjoint is always oriented in a particular direction; we instead use the turnstile symbol “$\perp$” to indicate which adjoint is the left adjoint.

non-invertible 3-cells — they are an important class of adjunctions. One reason for the relevance of Definition 2.1.1 in all ∞-cosmoi is its formal properties vis-a-vis the related notion of equivalence, which Theorem 1.4.7 has established is morally "correct," and with the notions of limits and colimits to be introduced.

Finally, a reasonable objection is that Definition 2.1.1 appears too "low dimensional," comprised of data found entirely in the homotopy 2-category and ignoring the higher dimensional morphisms in an ∞-cosmos. In fact, any adjunction between ∞-categories extends to a *homotopy coherent adjunction* involving data in all dimensions, and moreover such extensions are homotopically unique [**80**].

The definition of an adjunction given in Definition 2.1.1 is "equational" in character: stated in terms of the objects, 1-cells, and 2-cells of a 2-category and their composites. Immediately:

2.1.3. Lemma. *Adjunctions in a 2-category are preserved by any 2-functor.* ☐

Lemma 2.1.3 provides an easy source of examples of adjunctions between quasi-categories. The 2-functors underlying the cosmological functors of Example 1.3.7 then transfer adjunctions defined in one model of $(\infty, 1)$-categories to adjunctions defined in each of the other models.

2.1.4. Example (adjunctions between 1-categories). Via the nerve embedding $\mathcal{C}at \hookrightarrow \mathfrak{h}\mathcal{QC}at$, any adjunction between 1-categories induces an adjunction between their nerves regarded as quasi-categories.

2.1.5. Example (adjunctions between topological categories). Cordier's *homotopy coherent nerve* construction [**24, 25**] defines a 2-functor $\mathfrak{N}\colon \mathcal{K}an\text{-}\mathcal{C}at \to \mathfrak{h}\mathcal{QC}at$ from the 2-category of Kan complex enriched categories, simplicially enriched functors, and simplicial natural transformations, to the homotopy 2-category $\mathfrak{h}\mathcal{QC}at$. In this way, topologically enriched adjunctions define adjunctions between quasi-categories.

2.1.6. Example (Quillen adjunctions). Topologically enriched adjunctions are relatively rare. More prevalent are "up-to-homotopy" topologically enriched adjunctions, such as those given by Quillen adjunctions between (simplicial) model categories. These also define adjunctions between quasi-categories; see a proof of Mazel-Gee [**64**] or [**79**, §6.2].

The preservation of adjunctions by 2-functors, such as those given by Lemma 1.4.4, proves:

2.1.7. Proposition. *Given any adjunction* $\begin{tikzcd} A \arrow[r, bend right, "u"'] \arrow[r, phantom, "\perp"] & B \arrow[l, bend right, "f"'] \end{tikzcd}$ *between ∞-categories then:*

(i) for any ∞-category X,

$$\begin{tikzcd} \mathsf{Fun}(X,A) \arrow[r, bend right, "u_*"'] \arrow[r, phantom, "\perp"] & \mathsf{Fun}(X,B) \arrow[l, bend right, "f_*"'] \end{tikzcd}$$

defines an adjunction between quasi-categories;

(ii) for any ∞-category X,

$$\begin{tikzcd} \mathsf{hFun}(X,A) \arrow[r, bend right, "u_*"'] \arrow[r, phantom, "\perp"] & \mathsf{hFun}(X,B) \arrow[l, bend right, "f_*"'] \end{tikzcd}$$

defines an adjunction between categories;

(iii) for any simplicial set U,

$$\begin{tikzcd} A^U \arrow[r, bend right, "u^U"'] \arrow[r, phantom, "\perp"] & B^U \arrow[l, bend right, "f^U"'] \end{tikzcd}$$

defines an adjunction between ∞-categories;

(iv) and if the ambient ∞-cosmos is cartesian closed, then for any ∞-category C,

$$\begin{tikzcd} A^C \arrow[r, bend right, "u^C"'] \arrow[r, phantom, "\perp"] & B^C \arrow[l, bend right, "f^C"'] \end{tikzcd}$$

defines an adjunction between ∞-categories.

For instance, taking $X = 1$ in (ii) yields a "derived" adjunction between the homotopy categories of the ∞-categories A and B introduced in Definition 1.4.12:

$$\begin{tikzcd} \mathsf{h}A \arrow[r, bend right, "u_*"'] \arrow[r, phantom, "\perp"] & \mathsf{h}B \arrow[l, bend right, "f_*"'] \end{tikzcd}$$

Proof. Any adjunction $f \dashv u$ in the homotopy 2-category $\mathfrak{h}\mathcal{K}$ is preserved by the 2-functors $\mathsf{Fun}(X,-)\colon \mathfrak{h}\mathcal{K} \to \mathfrak{h}\mathcal{QC}at$, $\mathsf{hFun}(X,-)\colon \mathfrak{h}\mathcal{K} \to \mathcal{C}at$, $(-)^U\colon \mathfrak{h}\mathcal{K} \to \mathfrak{h}\mathcal{K}$, and $(-)^C\colon \mathfrak{h}\mathcal{K} \to \mathfrak{h}\mathcal{K}$. □

2.1.8. Remark. There are contravariant versions of each of the adjunction-preservation results of Proposition 2.1.7, the first of which we explain in detail; see Exercise 2.1.i for further discussion. Fixing the codomain variable of the functor-space at any ∞-category $C \in \mathcal{K}$ defines a 2-functor

$$\mathsf{Fun}(-,C)\colon \mathfrak{h}\mathcal{K}^{\mathrm{op}} \to \mathfrak{h}\mathcal{QC}at$$

that is contravariant on 1-cells and covariant on 2-cells.[4] Such 2-functors preserve adjunctions, but exchange left and right adjoints: for instance, given $f \dashv u$ in $\mathcal{K}$, we obtain an adjunction

$$\begin{tikzcd} \mathsf{Fun}(A,C) \arrow[r, bend right, "f^*"'] \arrow[r, phantom, "\perp"] & \mathsf{Fun}(B,C) \arrow[l, bend right, "u^*"'] \end{tikzcd}$$

between the functor-spaces.

The next four results have standard proofs that can be taken "off the shelf" by querying any 2-category theorist who may happen to be standing nearby. The only novelty is the observation that these standard arguments can be applied to the theory of adjunctions between ∞-categories.

[4] On a strict 2-category, the superscript "op" is used to signal that the 1-cells should be reversed but not the 2-cells, the superscript "co" is used to signal that the 2-cells should be reversed but not the 1-cells, and the superscript "coop" is used to signal that both the 1- and 2-cells should be reversed; see Definition B.1.7.

2.1.9. Proposition. *Adjunctions compose: given adjoint functors*

$$
\begin{tikzcd}
C \arrow[r, bend left, "f'"] \arrow[r, phantom, "\perp"] & B \arrow[l, bend left, "u'"] \arrow[r, bend left, "f"] \arrow[r, phantom, "\perp"] & A \arrow[l, bend left, "u"]
\end{tikzcd}
\qquad \rightsquigarrow \qquad
\begin{tikzcd}
C \arrow[r, bend left, "ff'"] \arrow[r, phantom, "\perp"] & A \arrow[l, bend left, "u'u"]
\end{tikzcd}
$$

the composite functors are adjoint.

Proof. Writing $\eta\colon \mathrm{id}_B \Rightarrow uf$, $\epsilon\colon fu \Rightarrow \mathrm{id}_A$, $\eta'\colon \mathrm{id}_c \Rightarrow u'f'$, and $\epsilon'\colon f'u' \Rightarrow \mathrm{id}_B$ for the respective units and counits, the pasting diagrams

$$
\begin{tikzcd}
C \arrow[rrrr, equal] \arrow[dr, "f'"'] & & \arrow[d, phantom, "\Downarrow\eta'"] & & C \\
& B \arrow[rr, equal] \arrow[dr, "f"'] & \arrow[d, phantom, "\Downarrow\eta"] & B \arrow[ur, "u'"'] & \\
& & A \arrow[ur, "u"'] & &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
& & C \arrow[dr, "f'"] \arrow[d, phantom, "\Downarrow\epsilon'"] & & \\
& B \arrow[ur, "u'"] \arrow[rr, equal] & \arrow[d, phantom, "\Downarrow\epsilon"] & B \arrow[dr, "f"] & \\
A \arrow[ur, "u"] \arrow[rrrr, equal] & & & & A
\end{tikzcd}
$$

define the unit and counit of $ff' \dashv u'u$ so that the triangle equalities

$$
\begin{tikzcd}[column sep=small]
C \arrow[rrrr, equal] \arrow[dr, "f'"'] & & \arrow[d, phantom, "\Downarrow\eta'"] & & C \arrow[dr, "f'"] \arrow[d, phantom, "\Downarrow\epsilon'"] & & \\
& B \arrow[rr, equal] \arrow[dr, "f"'] & \arrow[d, phantom, "\Downarrow\eta"] & B \arrow[ur, "u'"] \arrow[rr, equal] & \arrow[d, phantom, "\Downarrow\epsilon"] & B \arrow[dr, "f"] & \\
& & A \arrow[ur, "u"'] \arrow[rrrr, equal] & & & & A
\end{tikzcd}
=
\begin{tikzcd}
C \arrow[d, bend right, "f'"'] \arrow[d, bend left, "f'"] \arrow[d, phantom, "="] \\
B \arrow[d, bend right, "f"'] \arrow[d, bend left, "f"] \arrow[d, phantom, "="] \\
A
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
& & C \arrow[rrrr, equal] \arrow[dr, "f'"] \arrow[d, phantom, "\Downarrow\epsilon'"] & & \arrow[d, phantom, "\Downarrow\eta'"] & & C \\
& B \arrow[ur, "u'"] \arrow[rr, equal] & \arrow[d, phantom, "\Downarrow\epsilon"] & B \arrow[rr, equal] \arrow[dr, "f"'] & \arrow[d, phantom, "\Downarrow\eta"] & B \arrow[ur, "u'"'] & \\
A \arrow[ur, "u"] \arrow[rrrr, equal] & & & & A \arrow[ur, "u"'] & &
\end{tikzcd}
=
\begin{tikzcd}
C \\
B \arrow[u, bend left, "u'"] \arrow[u, bend right, "u'"'] \arrow[u, phantom, "="] \\
A \arrow[u, bend left, "u"] \arrow[u, bend right, "u"'] \arrow[u, phantom, "="]
\end{tikzcd}
$$

hold. ☐

An adjoint to a given functor is unique up to natural isomorphism:

2.1.10. Proposition (uniqueness of adjoints).

(i) *If $f \dashv u$ and $f' \dashv u$, then $f \cong f'$.*

(ii) *Conversely, if $f \dashv u$ and $f \cong f'$ then $f' \dashv u$.*

Proof. Writing $\eta\colon \mathrm{id}_B \Rightarrow uf$, $\epsilon\colon fu \Rightarrow \mathrm{id}_A$, $\eta'\colon \mathrm{id}_c \Rightarrow uf'$, and $\epsilon'\colon f'u \Rightarrow \mathrm{id}_B$ for the respective units and counits, the pasting diagrams

$$
\begin{tikzcd}
B \arrow[rr, equal] \arrow[dr, "f'"'] & \arrow[d, phantom, "\Downarrow\eta'"] & B \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow\epsilon"] & \\
& A \arrow[ur, "u"] \arrow[rr, equal] & & A
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[rr, equal] \arrow[dr, "f"'] & \arrow[d, phantom, "\Downarrow\eta"] & B \arrow[dr, "f'"] \arrow[d, phantom, "\Downarrow\epsilon'"] & \\
& A \arrow[ur, "u"] \arrow[rr, equal] & & A
\end{tikzcd}
$$

define 2-cells $f \Rightarrow f'$ and $f' \Rightarrow f$. The composites $f \Rightarrow f' \Rightarrow f$ and $f' \Rightarrow f \Rightarrow f'$ are computed by pasting these diagrams together horizontally on one side or on the other. Applying the triangle equalities for the adjunctions $f \dashv u$ and $f' \dashv u$ both composites are easily seen to be identities. Hence $f \cong f'$ as functors from B to A.

Part (ii) is left as Exercise 2.1.ii. ☐

We will make repeated use of the following standard 2-categorical result, which says that any equivalence in a 2-category can be promoted to an equivalence that also defines an adjunction:

2.1.11. Proposition (adjoint equivalences). *Any equivalence can be promoted to an* ***adjoint equivalence*** *by modifying one of the 2-cells. That is, the invertible 2-cells in an equivalence can be chosen so as to satisfy the triangle equalities. Hence, if f and g are inverse equivalences then $f \dashv g$ and $g \dashv f$.*

Proof. Consider an equivalence comprised of functors $f\colon A \to B$ and $g\colon B \to A$ and invertible 2-cells

$$\begin{tikzcd} A \arrow[r, bend left, equal] \arrow[r, bend right, "gf"'] \arrow[r, phantom, "\cong\Downarrow\alpha"] & A \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} B \arrow[r, bend left, "fg"] \arrow[r, bend right, equal] \arrow[r, phantom, "\cong\Downarrow\beta"] & B \end{tikzcd}$$

We will construct an adjunction $f \dashv g$ with unit $\eta \coloneqq \alpha$ by modifying β to form the counit ϵ.[5] Since the unit and counit produced in this manner are invertible, it follows also that $g \dashv f$ with unit ϵ^{-1} and counit η^{-1}.

Since α and β are invertible, the "triangle equality composite"

$$\phi \coloneqq f \xRightarrow{f\alpha} fgf \xRightarrow{\beta f} f$$

is an isomorphism, though likely not an identity. Define

$$\epsilon \coloneqq (fg \xRightarrow{\phi^{-1}g} fg \xRightarrow{\beta} \mathrm{id}_B) \coloneqq (fg \xRightarrow{\beta^{-1}fg} fgfg \xRightarrow{f\alpha^{-1}g} fg \xRightarrow{\beta} \mathrm{id}_B)$$

This "corrects" the counit so that now the composite $\epsilon f \cdot f\eta$ in the commutative diagram

$$\begin{tikzcd} & fgf \arrow[dr, Rightarrow, "\phi^{-1}gf"'] \arrow[drr, Rightarrow, "\epsilon f"] & & \\ f \arrow[ur, Rightarrow, "f\alpha"] \arrow[dr, Rightarrow, "\phi^{-1}"'] & & fgf \arrow[r, Rightarrow, "=\beta f"] & f \\ & f \arrow[ur, Rightarrow, "f\alpha"] \arrow[urr, Rightarrow, "\phi"'] & & \end{tikzcd}$$

equals the bottom composite $\phi\cdot\phi^{-1} = \mathrm{id}_f$, where the commutativity of the displayed square of natural isomorphisms is by "naturality of whiskering." [6]

[5]By implementing this construction in the co-dual of the homotopy 2-category, we could alternatively take β to be the counit at the cost of modifying α to form the unit; see Exercise 2.1.iii.

[6]Here "naturality of whiskering," as expressed by Lemma B.1.3, asserts that the pasted composite $\begin{tikzcd} A \arrow[r, bend left, equal] \arrow[r, bend right, "gf"'] \arrow[r, phantom, "\cong\Downarrow\alpha"] & A \arrow[r, bend left, "f"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\cong\Downarrow\phi^{-1}"] & B \end{tikzcd}$ can be computed as a vertical composite of two different "whiskerings" of α and ϕ^{-1}, in this case implying that $\phi^{-1}gf \cdot f\alpha = f\alpha \cdot \phi^{-1}$.

Now by another diagram chase, the other triangle composite $g\epsilon \cdot \eta g$ is an idempotent:

$$\begin{tikzcd} g \arrow[r, Rightarrow, "\eta g"] \arrow[d, Rightarrow, "\eta g"'] & gfg \arrow[r, Rightarrow, "g\epsilon"] \arrow[d, Rightarrow, "\eta gfg"] & g \arrow[d, Rightarrow, "\eta g"] \\ gfg \arrow[r, Rightarrow, "gf\eta g"] \arrow[dr, equal] & gfgfg \arrow[r, Rightarrow, "gfg\epsilon"] \arrow[d, Rightarrow, "g\epsilon fg"] & gfg \arrow[d, Rightarrow, "g\epsilon"] \\ & gfg \arrow[r, Rightarrow, "g\epsilon"'] & g \end{tikzcd}$$

By cancelation, any idempotent isomorphism is the identity, proving that $g\epsilon \cdot \eta g = \mathrm{id}_g$. ☐

One use of Proposition 2.1.11 is to show that adjunctions are equivalence-invariant:

2.1.12. Proposition (equivalence-invariance of adjunctions). *A functor $u \colon A \to B$ between ∞-categories admits a left adjoint if and only if, for any pair of equivalent ∞-categories $A' \simeq A$ and $B' \simeq B$, the equivalent functor $u' \colon A' \to B'$ admits a left adjoint.*

Proof. If $u \colon A \to B$ admits a left adjoint then by composing $f \dashv u$ with the adjoint equivalences $A' \simeq A$ and $B \simeq B'$ we obtain an equivalent adjunction:

$$\begin{tikzcd} A' \arrow[r, bend right, "\sim"'] \arrow[r, phantom, "\perp"] & A \arrow[l, bend right, "\sim"'] \arrow[r, bend right, "u"'] \arrow[r, phantom, "\perp"] & B \arrow[l, bend right, "f"'] \arrow[r, bend right, "\sim"'] \arrow[r, phantom, "\perp"] & B' \arrow[l, bend right, "\sim"'] \end{tikzcd}$$

Conversely, if the equivalent functor $u' \colon A' \simeq A \xrightarrow{u} B \simeq B'$ admits a left adjoint f' then again we obtain a composite adjunction:

$$\begin{tikzcd} A \arrow[r, bend right, "\sim"'] \arrow[r, phantom, "\perp"] & A' \arrow[l, bend right, "\sim"'] \arrow[r, bend right, "\sim"'] & A \arrow[r, bend right, "u"'] \arrow[r, phantom, "\perp"] & B \arrow[r, bend right, "\sim"'] & B' \arrow[llll, bend right, "f'"'] \arrow[r, bend right, "\sim"'] \arrow[r, phantom, "\perp"] & B \arrow[l, bend right, "\sim"'] \end{tikzcd}$$

whose right adjoint is naturally isomorphic to the original functor u. By Proposition 2.1.10 the displayed left adjoint is then a left adjoint to u. ☐

As we shall discover, all of ∞-category theory is equivalence-invariant in this way.

For later use, we close with an example of an abstractly-defined adjunction that can be constructed for any ∞-category in any ∞-cosmos via the results proven in this section.

2.1.13. Lemma. *For any ∞-category A, the "composition" functor*

$$\begin{tikzcd} A^2 \underset{A}{\times} A^2 \arrow[r, "\circ" description] & A^2 \arrow[l, bend right=40, "{(-,\mathrm{id}_{\mathrm{dom}(-)})}"', "\perp"] \arrow[l, bend left=40, "{(\mathrm{id}_{\mathrm{cod}(-)},-)}", "\perp"'] \end{tikzcd} \tag{2.1.14}$$

admits left and right adjoints, which, respectively, "extend an arrow into a composable pair" by pairing it with the identities at its domain or its codomain.

PROOF. There is a dual adjunction in $\mathcal{C}at$ whose functors we describe using notation for simplicial operators introduced in 1.1.1; the full subcategory of $\mathcal{C}at$ spanned by the finite non-empty ordinals is isomorphic to $\mathbf{\Delta}$.

$$\begin{tikzcd} \mathbb{3} \arrow[r, bend left, "s^0"] \arrow[r, bend right, "s^1"'] \arrow[r, phantom, shift left=2, "\top"] \arrow[r, phantom, shift right=2, "\top"] & \mathbb{2} \arrow[l, "d^1"] \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} A^{\mathbb{3}} \arrow[r, "A^{d^1}"] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] & A^{\mathbb{2}} \arrow[l, bend right, "A^{s^0}"'] \arrow[l, bend left, "A^{s^1}"] \end{tikzcd}$$

For any ∞-category A in an ∞-cosmos $\mathcal{K}$, Exercise 2.1.i describes a 2-functor $A^{(-)}: \mathcal{C}at^{\mathrm{op}} \to \mathfrak{h}\mathcal{K}$ carrying the adjoint triple displayed above-left to the one displayed above-right.

Now we claim there is a trivial fibration $A^{\mathbb{3}} \xrightarrow{\sim} A^{\mathbb{2}} \times_A A^{\mathbb{2}}$ constructed as follows. The pushout diagram of simplicial sets displayed below-left is carried by the simplicial cotensor $A^{(-)}: s\mathcal{S}et^{\mathrm{op}} \to \mathcal{K}$ to a pullback diagram displayed below-right; since the legs of the pushout square are monomorphisms, the legs of the pullback square are isofibrations

$$\begin{tikzcd} \Lambda^1[2] \arrow[dr, phantom, "\lrcorner" very near start] & \mathbb{2} \arrow[l] \\ \mathbb{2} \arrow[u, hook] & \mathbb{1} \arrow[l, hook', "d^0"] \arrow[u, hook', "d^1"'] \end{tikzcd} \qquad \begin{tikzcd} A^{\Lambda^1[2]} \arrow[r, two heads] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner" very near start] & A^{\mathbb{2}} \arrow[d, two heads, "\mathrm{ev}_0"] \\ A^{\mathbb{2}} \arrow[r, two heads, "\mathrm{ev}_1"'] & A \end{tikzcd}$$

Lemma 1.2.14 tells us that the cotensor of the inner horn inclusion $\Lambda^1[2] \hookrightarrow \Delta[2] \cong \mathbb{3}$ with the ∞-category A defines a trivial fibration $q: A^{\mathbb{3}} \xrightarrow{\sim} A^{\Lambda^1[2]}$ and the pullback square above-left recognizes its codomain as the desired ∞-category of "composable pairs." By Lemma 1.2.15, any section s to $q: A^{\mathbb{3}} \xrightarrow{\sim} A^{\mathbb{2}} \times_A A^{\mathbb{2}}$ can be made into an equivalence inverse. By Proposition 2.1.11, these functors are both left and right adjoints. The desired adjoint triple may then be constructed as the composite adjunction:

$$\begin{tikzcd} A^{\mathbb{2}} \underset{A}{\times} A^{\mathbb{2}} \arrow[r, "s"] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] & A^{\mathbb{3}} \arrow[l, bend right, "q"'] \arrow[l, bend left, "q"] \arrow[r, "A^{d^1}"] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] & A^{\mathbb{2}} \arrow[l, bend right, "A^{s^0}"'] \arrow[l, bend left, "A^{s^1}"] \end{tikzcd}$$ □

Note that the adjoint functors of (2.1.14) commute with the "endpoint evaluation" functors to $A \times A$. In fact, the units and counits can similarly be fibered over $A \times A$; see Example 3.6.13.

Exercises.

2.1.i. EXERCISE. The aim of this exercise is to spell out the most subtle of the dual adjunction-preservation results discussed in Remark 2.1.8.

(i) Let A be an ∞-category is an ∞-cosmos $\mathcal{K}$. Show that the simplicial cotensor restricts to define a 2-functor $A^{(-)}: \mathfrak{h}\mathcal{QC}at^{\mathrm{op}} \to \mathfrak{h}\mathcal{K}$.

(ii) Argue that the 2-functor of (i) restricts further along the nerve embedding to define a 2-functor $A^{(-)}: \mathcal{C}at^{\mathrm{op}} \to \mathfrak{h}\mathcal{K}$.

(iii) Conclude that for any adjunction between 1-categories as below-left there is an induced adjunction between ∞-categories as below-right:

$$\begin{tikzcd} I \arrow[r, bend right, "u"'] \arrow[r, phantom, "\perp"] & J \arrow[l, bend right, "f"'] \end{tikzcd} \qquad \rightsquigarrow \qquad \begin{tikzcd} A^I \arrow[r, bend right, "f^*"'] \arrow[r, phantom, "\perp"] & A^J \arrow[l, bend right, "u^*"'] \end{tikzcd}$$

2.1.ii. Exercise. Prove Proposition 2.1.10(ii).

2.1.iii. Exercise. Dualize the proof of Proposition 2.1.11 to show that any equivalence can be promoted into an adjoint equivalence in which the counit is part of the originally specified data.

2.2. Initial and Terminal Elements

Employing the tactic used in Definition 1.4.12 to define the homotopy category of an ∞-category, we use the terminal ∞-category 1 to probe inside an ∞-category A. An object a in the homotopy category $\mathsf{h}A$ is defined to be a map of ∞-categories $a\colon 1 \to A$. To avoid the proliferation of the term "objects," and in deference to Lawvere's notion of (generalized) elements [**58**], we refer to maps $a\colon 1 \to A$ as **elements**[7] of the ∞-category A henceforth. This terminology will help us keep track of the "category level" under discussion: elements a live inside ∞-categories A, which are the objects of ∞-cosmoi $\mathcal{K}$—which themselves define "infinite-dimensional categories," albeit of a different sort.

Before introducing limits and colimits of general ∞-category valued diagrams, we warm up by defining initial and terminal elements in an ∞-category A.

2.2.1. Definition (initial/terminal element). An **initial element** in an ∞-category A is a left adjoint to the unique functor $!\colon A \to 1$, as displayed below-left, while a **terminal element** in an ∞-category A is a right adjoint, as displayed below-right.

$$\begin{tikzcd} 1 \arrow[r, bend left, "i"] \arrow[r, phantom, "\perp"] & A \arrow[l, bend left, "!"] \end{tikzcd} \qquad\qquad \begin{tikzcd} 1 \arrow[r, bend right, "t"'] \arrow[r, phantom, "\perp"] & A \arrow[l, bend right, "!"'] \end{tikzcd}$$

Let us unpack the definition of an initial element; dual remarks apply to terminal elements.

2.2.2. Lemma (the minimal data required to present an initial element). *To define an initial element in A, it suffices to specify*

- *an element $i\colon 1 \to A$ and*
- *a natural transformation from the constant functor at i to the identity functor*

$$\begin{tikzcd} & 1 \arrow[dr, "i"] \arrow[d, phantom, "\Downarrow\epsilon"] & \\ A \arrow[ur, "!"] \arrow[rr, equal] & {} & A \end{tikzcd}$$

so that the component $\epsilon i\colon i \Rightarrow i$, an arrow from I to i in $\mathsf{h}A$, is the identity.

Proof. Proposition 1.4.5, whose proof starts in the paragraph before its statement, demonstrates that the ∞-category $1 \in \mathcal{K}$ is terminal in the homotopy 2-category $\mathfrak{h}\mathcal{K}$. The 1-dimensional aspect of this universal property implies that i defines a section of the unique map $A \to 1$, while the

[7] A **generalized element** of A is a functor $f\colon X \to A$. By the Yoneda lemma, an ∞-category is determined by its generalized elements.

2-dimensional aspect asserts that there exist no non-identity 2-cells with codomain 1. In particular, the unit of the adjunction $i \dashv !$ is necessarily an identity and one of the triangle equalities comes for free. The data enumerated above is what remains of Definition 2.1.1 in this setting. ☐

Put more concisely, an initial element i defines a **left adjoint right inverse** to the functor $! \colon A \to 1$. Such adjunctions are studied more systematically in §B.4. In fact, it suffices to assume that the counit component ϵi is an isomorphism, for this in fact implies already that ϵi is the identity; see Exercise 2.2.i.

2.2.3. Lemma (uniqueness). *Any two initial elements in an ∞-category A are isomorphic in $\mathsf{h}A$.*

Proof. By Proposition 2.1.10 any two left adjoints i and i' to the functor $! \colon A \to 1$ are naturally isomorphic. A natural isomorphism between a pair of functors $i, i' \colon 1 \rightrightarrows A$ gives exactly the data of an isomorphism $i \cong i'$ between the corresponding elements of the homotopy category $\mathsf{h}A$. ☐

2.2.4. Remark (representable initiality). Applying the 2-functor $\mathsf{Fun}(X, -) \colon \mathfrak{h}\mathcal{K} \to \mathfrak{h}\mathcal{QC}at$ to an initial element $i \colon 1 \to A$ or terminal element $t \colon 1 \to A$ of an ∞-category $A \in \mathcal{K}$ yields adjunctions

$$\begin{tikzcd} \mathbb{1} \cong \mathsf{Fun}(X,1) \arrow[r, bend left, "i_*"] \arrow[r, phantom, "\perp"] & \mathsf{Fun}(X,A) \arrow[l, bend left, "!"] \end{tikzcd} \qquad \begin{tikzcd} \mathbb{1} \cong \mathsf{Fun}(X,1) \arrow[r, bend right, "t_*"'] \arrow[r, phantom, "\perp"] & \mathsf{Fun}(X,A) \arrow[l, bend right, "!"'] \end{tikzcd}$$

Via the isomorphisms $\mathsf{Fun}(X, 1) \cong \mathbb{1}$ that express the universal property of the terminal ∞-category 1, we see that the constant functor at an initial or terminal element

$$X \xrightarrow{!} 1 \xrightarrow{i} A \qquad \text{or} \qquad X \xrightarrow{!} 1 \xrightarrow{t} A$$

defines an initial or terminal element, respectively, of the functor-space $\mathsf{Fun}(X, A)$. This observation can be summarized by saying that initial or terminal elements are representably initial or terminal at the level of the ∞-cosmos.

Conversely, if $i \colon 1 \to A$ is representability initial, then i defines an initial element of A. This is most easily seen by passing to the homotopy 2-category, where we can show that an initial element $i \colon 1 \to A$ is initial among all *generalized elements* $f \colon X \to A$ in the following precise sense.

2.2.5. Lemma. *An element $i \colon 1 \to A$ is initial if and only if for all $f \colon X \to A$ there exists a unique 2-cell with boundary*

$$\begin{tikzcd} & 1 \arrow[dr, "i"] \arrow[d, phantom, "\Downarrow\exists!"] & \\ X \arrow[ur, "!"] \arrow[rr, "f"'] & {} & A \end{tikzcd}$$

Proof. If $i \colon 1 \to A$ is initial, then the adjunction of Definition 2.2.1 is preserved by the 2-functor $\mathsf{hFun}(X, -) \colon \mathfrak{h}\mathcal{K} \to \mathcal{C}at$, defining an adjunction

$$\begin{tikzcd} \mathbb{1} \cong \mathsf{hFun}(X,1) \arrow[r, bend left, "i_*"] \arrow[r, phantom, "\perp"] & \mathsf{hFun}(X,A) \arrow[l, bend left, "!"] \end{tikzcd}$$

Via the isomorphism $\mathsf{hFun}(X, 1) \cong \mathbb{1}$, this adjunction proves that the constant functor $i!: X \to A$ is initial in the category $\mathsf{hFun}(X, A)$ and thus has the universal property of the statement.

Conversely, if $i: 1 \to A$ satisfies the universal property of the statement, applying this to the generic element of A (the identity map $\mathrm{id}_A: A \to A$) easily produces the data of Lemma 2.2.2. $\square$

Lemma 2.2.5 says that initial elements are *representably initial* in the homotopy 2-category. Specializing the generalized elements to ordinary elements, we see that initial and terminal elements in A respectively define initial and terminal elements in its homotopy category:

$$\begin{tikzcd} \mathbb{1} \arrow[r, bend left, "i"] \arrow[r, bend right, "t"'] & \mathsf{h}A \arrow[l, "!"'] \end{tikzcd} \tag{2.2.6}$$

Continuing the theme of the equivalence-invariance of ∞-categorical notions:

2.2.7. Lemma. *If A has an initial element and $A \simeq A'$ then A' has an initial element and these elements are preserved up to isomorphism by the equivalences.*

Proof. By Proposition 2.1.11, the equivalence $A \simeq A'$ can be promoted to an adjoint equivalence, which can immediately be composed with the adjunction characterizing an initial element i of A:

$$\begin{tikzcd} 1 \arrow[r, bend left, "i"] \arrow[r, phantom, "\perp"] & A \arrow[l, bend left, "!"] \arrow[r, bend left, "\sim"] \arrow[r, phantom, "\perp"] & A' \arrow[l, bend left, "\sim"] \end{tikzcd}$$

The composite adjunction provided by Proposition 2.1.9 proves that the image of i defines an initial element of A', which by construction is preserved by the equivalence $A \xrightarrow{\sim} A'$.

To see that the equivalence $A' \xrightarrow{\sim} A$ also preserves initial elements, we can use the invertible 2-cells of the equivalence to see that i is isomorphic to the image of the image of i in A'. In case the initial objects in mind are not the ones being considered here, we can appeal to the uniqueness of initial elements proven in Lemma 2.2.3. $\square$

We now turn to the general theory of limits and colimits of diagrams valued in an ∞-category. The theory of initial elements previews this material well since in fact an initial element can be understood as an example of both notions: an initial element is the colimit of the empty diagram and also the limit of the diagram encoded by the identity functor, as we explain in Example 2.3.9.

Exercises.

2.2.i. Exercise (B.4.2). Show that if given

- an element $i: 1 \to A$ and
- a natural transformation

$$\begin{tikzcd} & 1 \arrow[dr, "i"] & \\ A \arrow[ur, "!"] \arrow[rr, equal] & \arrow[u, phantom, "\Downarrow\epsilon"] & A \end{tikzcd}$$

so that the component $\epsilon i: i \Rightarrow i$ is an isomorphism in $\mathsf{h}A$, then in fact $\epsilon i = \mathrm{id}_i$. This weakens the minimal data required to present an initial element in the statement of Lemma 2.2.2.

2.2.ii. Exercise. Use Lemma 2.2.5 to show that a representably initial element, as described in Remark 2.2.4, necessarily defines an initial element in A.

2.2.iii. Exercise. Prove that initial elements are preserved by left adjoints and terminal elements are preserved by right adjoints.

2.3. Limits and Colimits

Our aim is now to introduce limits and colimits of diagram valued *inside* an ∞-category A in some ∞-cosmos. We will consider two varieties of diagrams:

- In a generic ∞-cosmos $\mathcal{K}$, we shall consider diagrams indexed by a simplicial set J and valued in an ∞-category A.
- In a cartesian closed ∞-cosmos $\mathcal{K}$, we shall also consider diagram indexed by an ∞-category J and valued in an ∞-category A.[8]

2.3.1. Definition (diagram ∞-categories). For a simplicial set J — or possibly, in the case of a cartesian closed ∞-cosmos, an ∞-category J — and an ∞-category A, we refer to A^J as the **∞-category of J-shaped diagrams in A**. A **diagram** of shape J in A is an element $d\colon 1 \to A^J$.[9]

Both constructions of the ∞-category of diagrams define simplicial bifunctors

$$\begin{array}{ccc} s\mathcal{S}et^{\mathrm{op}} \times \mathcal{K} \longrightarrow \mathcal{K} & \qquad & \mathcal{K}^{\mathrm{op}} \times \mathcal{K} \longrightarrow \mathcal{K} \\ (J, A) \longmapsto A^J & & (J, A) \longmapsto A^J \end{array}$$

In either indexing context, there is a terminal object 1 with the property that $A^1 \cong A$ for any ∞-category A. Restriction along the unique map $!\colon J \to 1$, induces the **constant diagram functor** $\Delta\colon A \to A^J$.

We are deliberately conflating the notation for ∞-categories of diagrams indexed by a simplicial set or by another ∞-category because all of the results we will prove in Part I about the former case will also apply to the latter. For economy of language, we refer only to simplicial set indexed diagrams for the remainder of this section.

2.3.2. Definition (limit and colimit functors). An ∞-category A **admits all colimits** of shape J if the constant diagram functor $\Delta\colon A \to A^J$ admits a left adjoint, while A **admits all limits** of shape J if the constant diagram functor admits a right adjoint:

$$\begin{tikzcd} A^J \arrow[r, bend left, "\mathrm{colim}"] \arrow[r, bend right, "\lim"'] & A \arrow[l, "\Delta" description] \end{tikzcd}$$

In the ∞-cosmos of categories, Definition 2.3.2 reduces to the classically-defined limit and colimit functors, but in a general ∞-category limits and colimits should be thought of as analogous to the classical notions of "homotopy limits" and "homotopy colimits." In certain cases, this correspondence can be made precise. Every quasi-category is equivalent to the homotopy coherent nerve of a Kan complex enriched category [**82**, 7.2.2], and in this context it is known that homotopy limit or homotopy

[8]For the ∞-cosmoi of $(\infty,1)$-categories of Example 1.2.24, there is no essential difference between these notions: in $\mathcal{QC}at$ they are tautologically the same, and in all biequivalent ∞-cosmsoi the ∞-category of diagrams indexed by an ∞-category J is equivalent to the ∞-category of diagrams indexed by its underlying quasi-category, regarded as a simplicial set; see Proposition 10.3.4.

[9]When A^J is the exponential of a cartesian closed ∞-cosmos, diagrams stand in bijection with functors $d\colon J \to A$.

colimit cones in the Kan complex enriched category correspond exactly to limit or colimit cones in the homotopy coherent nerve; see Lurie's [**59**, 4.2.4.1] or [**84**, 6.1.4, 6.2.7]. In the ∞-categorical context, no stricter notion of limit or colimit is available, so the "homotopy" qualifier is typically dropped.

2.3.3. Warning. Limits or colimits of set-indexed diagrams — the case where the indexing shape is a coproduct of the terminal object $\mathbb{1}$ indexed by a set J — are called **products** or **coproducts**, respectively. In this case the ∞-category of diagrams itself decomposes as a product $A^J \cong \prod_J A$. Since the functor

$$\begin{array}{ccc} \mathfrak{h}\mathcal{K} & \xrightarrow{\mathrm{hFun}(1,-)} & \mathcal{C}at \\ A & \longmapsto & \mathrm{h}A \end{array}$$

that carries an ∞-category to its homotopy category preserves products, when J is a set there is a chain of isomorphisms

$$\mathrm{h}(A^J) \cong \mathrm{h}(\textstyle\prod_J A) \cong \prod_J \mathrm{h}A \cong (\mathrm{h}A)^J$$

Thus, in this special case the adjunctions of Definition 2.3.2 that define products or coproducts in an ∞-category descend to the adjunctions that define products or coproducts in its homotopy category:

$$\begin{tikzcd} (\mathrm{h}A)^J \cong \mathrm{h}(A^J) \arrow[r, bend left, "\mathrm{colim}", "\perp"'] \arrow[r, bend right, "\mathrm{lim}"', "\perp"] & \mathrm{h}A \arrow[l, "\Delta" description] \end{tikzcd}$$

This remains true in the case $J = \emptyset$, explaining the observation made in (2.2.6).

However, this argument does *not* extend to more general limit or colimit notions, and such ∞-categorical limits or colimits do not typically descend to limits or colimits in the homotopy category.[10] In §3.2, we observe that the homotopy category construction fails to preserve more complicated cotensors, even in the relatively simple case of $J = \mathbb{2}$.

The problem with Definition 2.3.2 is that it is insufficiently general: many ∞-categories will have certain, but not all, limits of diagrams of a particular indexing shape. So it would be desirable to re-express Definition 2.3.2 in a form that allows us to define the limit of a single diagram $d\colon \mathbb{1} \to A^J$ or of a family of diagrams. To achieve this, we make use of the following 2-categorical notion that op-dualizes the more familiar absolute (Kan) extension diagrams.

2.3.4. Definition (absolute lifting diagrams). Given a cospan $C \xrightarrow{g} A \xleftarrow{f} B$ in a 2-category, an **absolute left lifting** of g through f is given by a 1-cell ℓ and 2-cell λ as displayed below-left

$$\begin{tikzcd} & B \arrow[d, "f"] \\ C \arrow[ur, "\ell"] \arrow[r, "g"'] & A \arrow[ul, phantom, "\Uparrow\lambda" near start] \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "b"] \arrow[d, "c"'] \arrow[dr, phantom, "\Uparrow\chi"] & B \arrow[d, "f"] \\ C \arrow[r, "g"'] & A \end{tikzcd} \quad = \quad \begin{tikzcd} X \arrow[r, "b"] \arrow[d, "c"'] \arrow[dr, phantom, "\exists!\Uparrow\zeta" near start] & B \arrow[d, "f"] \\ C \arrow[ur, "\ell" description] \arrow[r, "g"'] & A \arrow[ul, phantom, "\Uparrow\lambda" near start] \end{tikzcd}$$

so that any 2-cell as displayed above-center factors uniquely through (ℓ, λ) as displayed above-right.

[10] This sort of behavior is expected in abstract homotopy theory: homotopy limits and colimits are not generally limits or colimits in the homotopy category.

Dually, an **absolute right lifting** of g through f is given by a 1-cell r and 2-cell ρ as displayed below-left

$$\begin{tikzcd} & B \arrow[d, "f"] \\ C \arrow[ur, "r"] \arrow[r, "g"'] \arrow[ur, phantom, shift right=3, "\Downarrow\rho"] & A \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "b"] \arrow[d, "c"'] \arrow[dr, phantom, "\Downarrow\chi"] & B \arrow[d, "f"] \\ C \arrow[r, "g"'] & A \end{tikzcd} = \begin{tikzcd} X \arrow[r, "b"] \arrow[d, "c"'] \arrow[r, phantom, shift right=3, "\exists!\Downarrow\zeta"] & B \arrow[d, "f"] \\ C \arrow[ur, "r"] \arrow[r, "g"'] \arrow[ur, phantom, shift right=3, "\Downarrow\rho"] & A \end{tikzcd}$$

so that any 2-cell as displayed above-center factors uniquely through (r, ρ) as displayed above-right.

The adjectives "left" and "right" refer to the handedness of the adjointness of these constructions: when these exist, left and right liftings respectively define left and right adjoints to the composition functor $f_*\colon \mathsf{hFun}(C, B) \to \mathsf{hFun}(C, A)$, with the 2-cells defining the components of the unit and counit of these adjunctions, respectively, at the object g. The adjective "absolute" refers to the following stability property.

2.3.5. Lemma. *Absolute left or right lifting diagrams are stable under restriction of their domain object: if (ℓ, λ) defines an absolute left lifting of g through f, then for any $c\colon X \to C$, the restricted diagram $(\ell c, \lambda c)$ defines an absolute left lifting of gc through f.*

$$\begin{tikzcd} & & B \arrow[d, "f"] \\ X \arrow[r, "c"'] & C \arrow[ur, "\ell"] \arrow[r, "g"'] \arrow[ur, phantom, shift right=3, "\Uparrow\lambda"] & A \end{tikzcd}$$

Proof. Exercise 2.3.ii. ☐

Units and counits of adjunctions provide important examples of absolute left and right lifting diagrams respectively:

2.3.6. Lemma. *A 2-cell $\eta\colon \mathrm{id}_B \Rightarrow uf$ defines the unit of an adjunction $f \dashv u$ if and only if (f, η) defines an absolute left lifting diagram, displayed below-left.*

$$\begin{tikzcd} & A \arrow[d, "u"] \\ B \arrow[ur, "f"] \arrow[r, equal] \arrow[ur, phantom, shift right=3, "\Uparrow\eta"] & B \end{tikzcd} \qquad \begin{tikzcd} & B \arrow[d, "f"] \\ A \arrow[ur, "u"] \arrow[r, equal] \arrow[ur, phantom, shift right=3, "\Downarrow\epsilon"] & A \end{tikzcd}$$

Dually a 2-cell $\epsilon\colon fu \Rightarrow \mathrm{id}_A$ defines the counit of an adjunction if and only if (u, ϵ) defines an absolute right lifting diagram, displayed above-right.

Proof. The universal property of the absolute right lifting diagram

$$\begin{tikzcd} X \arrow[r, "b"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\alpha"] & B \arrow[d, "f"] \\ A \arrow[r, equal] & A \end{tikzcd} = \begin{tikzcd} X \arrow[r, "b"] \arrow[d, "a"'] \arrow[r, phantom, shift right=3, "\exists!\Downarrow\beta"] & B \arrow[d, "f"] \\ A \arrow[ur, "u"] \arrow[r, equal] \arrow[ur, phantom, shift right=3, "\Downarrow\epsilon"] & A \end{tikzcd}$$

asserts that every natural transformation $\alpha\colon fb \Rightarrow a$ has a unique transpose $\beta\colon b \Rightarrow ua$ across the adjunction between the hom-categories of the homotopy 2-category:

$$\begin{tikzcd} \mathsf{hFun}(X,B) \arrow[r, bend left, "f_*"] \arrow[r, phantom, "\perp"] & \mathsf{hFun}(X,A) \arrow[l, bend left, "u_*"] \end{tikzcd}$$

Thus if $f \dashv u$ with counit ϵ, Proposition 2.1.7(ii) supplies this induced adjunction and (u, ϵ) defines an absolute right lifting if id_A through f.

Conversely, the unit and triangle equalities of an adjunction can extracted from the universal property of the absolute right lifting diagram. The details are left as Exercise 2.3.iii. ☐

In particular, the unit and counit of the adjunctions colim $\dashv \Delta \dashv$ lim of Definition 2.3.2 define absolute left and right lifting diagrams:

$$\begin{tikzcd} & A \arrow[d, "\Delta"] \arrow[dl, phantom, "\Uparrow\eta"] \\ A^J \arrow[ur, "\mathrm{colim}"] \arrow[r, equal] & A^J \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[d, "\Delta"] \arrow[dl, phantom, "\Downarrow\epsilon"] \\ A^J \arrow[ur, "\lim"] \arrow[r, equal] & A^J \end{tikzcd}$$

By Lemma 2.3.5, these universal properties are retained upon restricting to any subobject of the ∞-category of diagrams. This motivates the following definition:

2.3.7. Definition. A **colimit** of a family of diagrams $d\colon D \to A^J$ indexed by J in an ∞-category A is given by an absolute left lifting diagram

$$\begin{tikzcd} & A \arrow[d, "\Delta"] \arrow[dl, phantom, "\Uparrow\eta"] \\ D \arrow[ur, "\mathrm{colim}"] \arrow[r, "d"'] & A^J \end{tikzcd}$$

comprised of a **colimit functor** colim$\colon D \to A$ and a **colimit cone** $\eta\colon d \Rightarrow \Delta\,\mathrm{colim}$.

Dually, a **limit** of a family of diagrams $d\colon D \to A^J$ indexed by J in an ∞-category A is given by an absolute right lifting diagram

$$\begin{tikzcd} & A \arrow[d, "\Delta"] \arrow[dl, phantom, "\Downarrow\epsilon"] \\ D \arrow[ur, "\lim"] \arrow[r, "d"'] & A^J \end{tikzcd}$$

comprised of a **limit functor** lim$\colon D \to A$ and a **limit cone** $\epsilon\colon \Delta \lim \Rightarrow d$.

2.3.8. Remark. If A has all limits of shape J, then Lemma 2.3.5 implies that any family of diagrams $d\colon D \to A^J$ has a limit, defined by evaluating the limit functor lim$\colon A^J \to A$ at d, i.e., by the composite $\lim d$. In the ∞-cosmoi of $(\infty,1)$-categories, Corollary 12.2.10 shows that if every diagram $d\colon 1 \to A^J$ has a limit, then A has all J-indexed limits. This is because the terminal $(\infty,1)$-category generates such ∞-cosmoi in a suitable sense, though this generation property does not hold for all ∞-cosmoi.

2.3.9. Example. An initial element $i\colon 1 \to A$ can be regarded as both a colimit of the empty diagram and as a limit of the identity functor, as we now explain. The ∞-category $A^{\varnothing} \simeq 1$ of empty diagrams in A is terminal, so in this context the constant diagram functor reduces to $!\colon A \to 1$. To show that initial elements are colimits in the sense of Definition 2.3.7, we must verify that an initial element defines an absolute left lifting diagram, as displayed below-left, whose 2-cell is the identity:

$$
\begin{tikzcd}
 & A \arrow[d, "!"] \\
1 \arrow[ur, "i"] \arrow[r, equals] \arrow[ur, phantom, shift right=3, "\parallel"] & 1
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "!"'] \arrow[dr, phantom, "\Uparrow\chi"] & A \arrow[d, "!"] \\
1 \arrow[r, equals] & 1
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "!"'] \arrow[dr, phantom, shift left=3, "\exists!\Uparrow\zeta"] & A \arrow[d, "!"] \\
1 \arrow[ur, "i" description] \arrow[r, equals] \arrow[ur, phantom, shift right=3, "\parallel"] & 1
\end{tikzcd}
$$

Since the ∞-category 1 is 2-terminal, there is a unique 2-cell χ inhabiting the central square above, namely the identity. Thus, the universal property of the absolute left lifting diagram asserts the existence of a unique 2-cell $\zeta\colon i! \Rightarrow f$ for any $f\colon X \to A$, exactly as provided by Lemma 2.2.5.

In a cartesian closed ∞-cosmos, we may also use Definition 2.3.7 to characterize an initial object as a limit for the identity functor $\mathrm{id}_A\colon A \to A$. The counit $\epsilon\colon i! \Rightarrow \mathrm{id}_A$ of the adjunction $i \dashv !$ transposes across the 2-adjunction $A\times - \dashv (-)^A$ of Proposition 1.4.5 to define the limit cone displayed below-left:

$$
\begin{tikzcd}
 & A \arrow[d, "\Delta"] \\
1 \arrow[ur, "i"] \arrow[r, "\mathrm{id}_A"'] \arrow[ur, phantom, shift right=3, "\Downarrow\check{\epsilon}"] & A^A
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "!"'] \arrow[dr, phantom, "\Downarrow\chi"] & A \arrow[d, "\Delta"] \\
1 \arrow[r, "\mathrm{id}_A"'] & A^A
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "!"'] \arrow[dr, phantom, shift left=3, "\exists!\Downarrow\zeta"] & A \arrow[d, "\Delta"] \\
1 \arrow[ur, "i" description] \arrow[r, "\mathrm{id}_A"'] \arrow[ur, phantom, shift right=3, "\Downarrow\check{\epsilon}"] & A^A
\end{tikzcd}
$$

The universal property displayed above-right is easiest to verify by transposing across $A\times - \dashv (-)^A$, where we must show that

$$
\begin{tikzcd}
 & X \arrow[dd, "f"] \\
X\times A \arrow[ur, "\pi_X"] \arrow[dr, "\pi_A"'] \arrow[r, phantom, "\Downarrow\hat{\chi}"] & {} \\
 & A
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & X \arrow[dr, "!"'] \arrow[drr, bend left, "f"] \arrow[dr, phantom, shift left=4, "\Downarrow\zeta"] & & \\
X\times A \arrow[ur, "\pi_X"] \arrow[dr, "\pi_A"'] & & 1 \arrow[r, "i"] & A \\
 & A \arrow[ur, "!"] \arrow[urr, bend right, equals] \arrow[ur, phantom, shift right=4, "\Downarrow\epsilon"] & &
\end{tikzcd}
\tag{2.3.10}
$$

Observe that when we restrict the right-hand side of (2.3.10) along the functor $\mathrm{id}_X \times i\colon X \cong X\times 1 \to X\times A$ we recover the 2-cell ζ, since $\epsilon i = \mathrm{id}_i$. This tells us that given χ, we must necessarily define the 2-cell $\zeta\colon f \Rightarrow i!$ to be the restriction of $\hat{\chi}$ along the functor $\mathrm{id}_X \times i$.

From this definition of ζ and the 2-functoriality of the cartesian product—which tells us that $\epsilon\pi_A = \pi_A(X\times\epsilon)$—we have

$$
\begin{tikzcd}
 & X \arrow[dr, "!"'] \arrow[drr, bend left, "f"] \arrow[dr, phantom, shift left=4, "\Downarrow\zeta"] & & \\
X\times A \arrow[ur, "\pi_X"] \arrow[dr, "\pi_A"'] & & 1 \arrow[r, "i"] & A \\
 & A \arrow[ur, "!"] \arrow[urr, bend right, equals] \arrow[ur, phantom, shift right=4, "\Downarrow\epsilon"] & &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & X\times 1 \arrow[dr, "\mathrm{id}_X\times i"] \arrow[d, phantom, "\Downarrow X\times\epsilon"] & & X \arrow[dd, "f"] \\
X\times A \arrow[ur, "\mathrm{id}_X\times !"] \arrow[rr, equals] & {} & X\times A \arrow[ur, "\pi_X"] \arrow[dr, "\pi_A"'] \arrow[r, phantom, "\Downarrow\hat{\chi}"] & {} \\
 & & & A
\end{tikzcd}
$$

By "naturality of whiskering," explained in Lemma B.1.3, the right-hand pasted composite can be computed as the vertical composite of $\pi_X(X\times\epsilon)$ followed by $\hat{\chi}$, but $\pi_X(X\times\epsilon)$ is the identity 2-cell, so this composite is just $\hat{\chi}$. This verifies the desired pasting identity (2.3.10). A very different proof of the same result appears in Corollary 9.5.4.

A few limits and colimits in ∞-categories exist for formal reasons. For example, an abstract 2-categorical lemma enables general proof of a classical result from homotopy theory that computes the colimits, typically called *geometric realizations*, of "split" simplicial objects. Before proving this, we introduce the indexing shapes involved.

2.3.11. Definition (split augmented (co)simplicial objects). Recall $\mathbf{\Delta}$ is the simplex category of finite non-empty ordinals and order-preserving maps introduced in 1.1.1. It defines a full subcategory of the category $\mathbf{\Delta}_+$ of finite ordinals and order-preserving maps, which freely appends the empty ordinal "$[-1]$" as an initial object. The category $\mathbf{\Delta}_+$ in turn defines a wide subcategory of a category $\mathbf{\Delta}_\perp$, which adds an "extra" degeneracy $\sigma^{-1}\colon [n+1] \twoheadrightarrow [n]$ between each pair of consecutive ordinals, including $\sigma^{-1}\colon [0] \twoheadrightarrow [-1]$. The category $\mathbf{\Delta}_+$ also defines a wide subcategory of a category $\mathbf{\Delta}_\top$, which adds an "extra" degeneracy $\sigma^{n+1}\colon [n+1] \twoheadrightarrow [n]$ on the other side between each pair of consecutive ordinals, including $\sigma^0\colon [0] \twoheadrightarrow [-1]$. The categories $\mathbf{\Delta}_\perp$ and $\mathbf{\Delta}_\top$ can be described in another way: there are faithful embeddings of these categories into $\mathbf{\Delta}$ that act on objects by $[n] \mapsto [n+1]$ and identify $\mathbf{\Delta}_\perp$ and $\mathbf{\Delta}_\top$ with the subcategories of finite non-empty ordinals and order-preserving maps that preserve the bottom and top elements in each ordinal respectively. This explains our notation.

Diagrams indexed by $\mathbf{\Delta} \subset \mathbf{\Delta}_+ \subset \mathbf{\Delta}_\perp, \mathbf{\Delta}_\top$ are respectively called **cosimplicial objects**, **coaugmented cosimplicial objects**, and **split coaugmented cosimplicial objects** (in the case of either $\mathbf{\Delta}_\perp$ or $\mathbf{\Delta}_\top$), if they are covariant, and **simplicial objects**, **augmented simplicial objects**, and **split augmented simplicial objects**, if they are contravariant. When it is useful to disambiguate between $\mathbf{\Delta}_\perp$ and $\mathbf{\Delta}_\top$ we refer to the former category as a "bottom splitting" and the latter category as a "top splitting," but this terminology is not standard.

A simplicial object $d\colon 1 \to A^{\mathbf{\Delta}^{\mathrm{op}}}$ in an ∞-category A **admits an augmentation** or **admits a splitting**, if it lifts along the restriction functors

$$\begin{tikzcd} & A^{\mathbf{\Delta}_\perp^{\mathrm{op}}} \arrow[d, two heads] \\ & A^{\mathbf{\Delta}_+^{\mathrm{op}}} \arrow[d, two heads] \\ 1 \arrow[r, "d"'] \arrow[uur, dotted] \arrow[ur, dashed] & A^{\mathbf{\Delta}^{\mathrm{op}}} \end{tikzcd}$$

where in the case of a top splitting, $\mathbf{\Delta}_\perp$ is replaced by $\mathbf{\Delta}_\top$. The family of simplicial objects admitting an augmentation and splitting is then represented by the generalized element $A^{\mathbf{\Delta}_\perp^{\mathrm{op}}} \twoheadrightarrow A^{\mathbf{\Delta}^{\mathrm{op}}}$. The following proposition proves that for any simplicial object admitting a splitting, the augmentation defines the colimit cone; dual results apply to limits of split cosimplicial objects. The limit and colimit cones are defined by cotensoring with the unique natural transformation

$$\begin{tikzcd} \mathbf{\Delta} \arrow[rr, hook] \arrow[dr, "!"'] & \Uparrow\nu & \mathbf{\Delta}_+ \\ & \mathbb{1} \arrow[ur, "{[-1]}"'] & \end{tikzcd} \tag{2.3.12}$$

that exists because $[-1]\colon \mathbb{1} \to \mathbf{\Delta}_+$ is initial; see Lemma 2.2.5.

2.3.13. Proposition (totalization/geometric realization). *Let A be any ∞-category. Every cosimplicial object in A that admits a coaugmentation and a splitting has a limit, with the coaugmentation defining the limit cone. Dually, every simplicial object in A that admits an augmentation and a splitting has a colimit, with the augmentation defining the colimit cone. That is, there exist absolute right and left lifting diagrams*

$$\begin{tikzcd} & & A \arrow[d, "\Delta"] \\ A^{\mathbf{\Delta}_\perp} \arrow[r, "\mathrm{res}"'] & A^{\mathbf{\Delta}_+} \arrow[ur, "\mathrm{ev}_{[-1]}"] \arrow[r, "\mathrm{res}"'] \arrow[ur, phantom, shift right=3, "\Downarrow A^\nu"] & A^{\mathbf{\Delta}} \end{tikzcd} \qquad \begin{tikzcd} & & A \arrow[d, "\Delta"] \\ A^{\mathbf{\Delta}_\perp^{\mathrm{op}}} \arrow[r, "\mathrm{res}"'] & A^{\mathbf{\Delta}_+^{\mathrm{op}}} \arrow[ur, "\mathrm{ev}_{[-1]}"] \arrow[r, "\mathrm{res}"'] \arrow[ur, phantom, shift right=3, "\Uparrow A^{\nu^{\mathrm{op}}}"] & A^{\mathbf{\Delta}^{\mathrm{op}}} \end{tikzcd}$$

$$\begin{tikzcd} & & A \arrow[d, "\Delta"] \\ A^{\mathbf{\Delta}_\top} \arrow[r, "\mathrm{res}"'] & A^{\mathbf{\Delta}_+} \arrow[ur, "\mathrm{ev}_{[-1]}"] \arrow[r, "\mathrm{res}"'] \arrow[ur, phantom, shift right=3, "\Downarrow A^\nu"] & A^{\mathbf{\Delta}} \end{tikzcd} \qquad \begin{tikzcd} & & A \arrow[d, "\Delta"] \\ A^{\mathbf{\Delta}_\top^{\mathrm{op}}} \arrow[r, "\mathrm{res}"'] & A^{\mathbf{\Delta}_+^{\mathrm{op}}} \arrow[ur, "\mathrm{ev}_{[-1]}"] \arrow[r, "\mathrm{res}"'] \arrow[ur, phantom, shift right=3, "\Uparrow A^{\nu^{\mathrm{op}}}"] & A^{\mathbf{\Delta}^{\mathrm{op}}} \end{tikzcd}$$

in which the 2-cells are obtained as restrictions of the cotensor of the 2-cell (2.3.12) *with A. Moreover, such limits and colimits are* ***absolute****, preserved by any functor $f\colon A \to B$ of ∞-categories.*

Proof. By Example B.5.2, the inclusion $\mathbf{\Delta} \hookrightarrow \mathbf{\Delta}_\perp$ admits a right adjoint, which can automatically be regarded as an adjunction "over $\mathbb{1}$" since $\mathbb{1}$ is 2-terminal. The initial element $[-1] \in \mathbf{\Delta}_+ \subset \mathbf{\Delta}_\perp$ defines a left adjoint to the constant functor:

$$\begin{tikzcd} \mathbf{\Delta} \arrow[r, hook] \arrow[d, "!"'] & \mathbf{\Delta}_+ \arrow[r, hook] \arrow[d, phantom, "\perp"] & \mathbf{\Delta}_\perp \arrow[ll, dashed, bend right, "\top"'] \arrow[dll, bend left, "!"] \\ \mathbb{1} \arrow[ur, dashed, "{[-1]}"] \end{tikzcd}$$

and the counit of this adjunction restricts along the inclusions $\mathbf{\Delta} \subset \mathbf{\Delta}_+ \subset \mathbf{\Delta}_\perp$ to the 2-cell (2.3.12). These adjunctions are preserved by the 2-functor $A^{(-)}\colon \mathcal{C}at^{\mathrm{op}} \to \mathfrak{h}\mathcal{K}$, yielding a diagram

$$\begin{tikzcd} & & A \arrow[d, "\Delta"] \arrow[dll, dotted, bend right, "!"'] \\ A^{\mathbf{\Delta}_\perp} \arrow[r, "\mathrm{res}"'] & A^{\mathbf{\Delta}_+} \arrow[ur, "\mathrm{ev}_{[-1]}"] \arrow[r, "\mathrm{res}"'] \arrow[ur, phantom, shift right=3, "\Downarrow A^\nu"] \arrow[u, phantom, "\perp"] & A^{\mathbf{\Delta}} \arrow[ll, dotted, bend left, "\top"] \end{tikzcd}$$

By Lemma B.5.1 these adjunctions witness the fact that evaluation at $[-1]$ and the 2-cell from (2.3.12) define an absolute right lifting of the canonical restriction functor $A^{\mathbf{\Delta}_\perp} \to A^{\mathbf{\Delta}}$ through the constant diagram functor, as claimed. The colimit case is proven similarly by applying the composite 2-functor

$$\mathcal{C}at^{\mathrm{coop}} \xrightarrow{(-)^{\mathrm{op}}} \mathcal{C}at^{\mathrm{op}} \xrightarrow{A^{(-)}} \mathfrak{h}\mathcal{K}$$

A similar argument, starting from Example B.5.3, constructs the absolute lifting diagrams from the top splitting.

Finally, by the 2-functoriality of the simplicial cotensor, any $f\colon A \to B$ commutes with the 2-cells defined by cotensoring with ν or its opposite.

$$\begin{tikzcd} & & A \arrow[r, "f"] \arrow[d, "\Delta"] \arrow[dl, phantom, "\Downarrow A^{\nu}" near end] & B \arrow[d, "\Delta"] \\ A^{\mathbf{\Delta}_\perp} \arrow[r, "\mathrm{res}"'] & A^{\mathbf{\Delta}_+} \arrow[ur, "\mathrm{ev}_{[-1]}"] \arrow[r, "\mathrm{res}"'] & A^{\mathbf{\Delta}} \arrow[r, "f^{\mathbf{\Delta}}"'] & B^{\mathbf{\Delta}} \end{tikzcd} = \begin{tikzcd} & & & B \arrow[d, "\Delta"] \\ A^{\mathbf{\Delta}_\perp} \arrow[r, "f^{\mathbf{\Delta}_\perp}"'] & B^{\mathbf{\Delta}_\perp} \arrow[r, "\mathrm{res}"'] & B^{\mathbf{\Delta}_+} \arrow[ur, "\mathrm{ev}_{[-1]}"] \arrow[r, "\mathrm{res}"'] \arrow[ur, phantom, "\Downarrow A^{\nu}"', shift right=3] & B^{\mathbf{\Delta}} \end{tikzcd}$$

Since the right-hand composite is an absolute right lifting diagram, so is the left-hand composite, which says that $f\colon A \to B$ preserves the totalization of any split coaugmented cosimplicial object in A. $\square$

Exercises.

2.3.i. Exercise. Prove that if an ∞-category A has binary products then it also has ternary products (and in fact n-ary products for all $n \geq 1$). Show further that the ternary product functor can be defined from the binary product functor $- \times -\colon A \times A \to A$ either as the composite $(- \times -) \times -$ or as the composite $- \times (- \times -)$; that is, show that these composites are naturally isomorphic and both satisfy the universal property that characterizes ternary products.

2.3.ii. Exercise. Prove Lemma 2.3.5.

2.3.iii. Exercise. Re-prove the forwards implication of Lemma 2.3.6 by following your nose through a pasting diagram calculation to prove the converse similarly.

2.3.iv. Exercise. Show that diagrams that are isomorphic to absolute right lifting diagrams are themselves absolute right lifting. That is, suppose given an absolute right lifting diagram and natural isomorphisms

$$\begin{tikzcd} & B \arrow[d, "f"] \\ C \arrow[ur, "r"] \arrow[r, "g"'] \arrow[ur, phantom, "\Downarrow\rho"', shift right=3] & A \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, bend left, "r'"] \arrow[r, bend right, "r"'] \arrow[r, phantom, "\cong\Downarrow\theta"] & B \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, bend left, "f'"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\cong\Downarrow\phi"] & A \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\cong\Downarrow\psi"] & A \end{tikzcd}$$

Then the pasted composite

$$\begin{tikzcd} & B \arrow[d, "f"'] \arrow[d, bend left=60, "f'"] \arrow[d, phantom, "\overset{\phi}{\underset{\cong}{\Leftarrow}}", shift left=4] \\ C \arrow[ur, bend left=60, "r'"] \arrow[ur, "r"'] \arrow[ur, phantom, "\cong\Downarrow\theta", shift left=3] \arrow[r, "g"] \arrow[r, bend right=40, "g'"'] \arrow[r, phantom, "\cong\Downarrow\psi", shift right=4] \arrow[ur, phantom, "\Downarrow\rho"', shift right=3] & A \end{tikzcd}$$

is an absolute right lifting diagram.

2.4. Preservation of Limits and Colimits

Famously, right adjoint functors preserve limits and left adjoints preserve colimits. Our aim in this section is to prove this in the ∞-categorical context and exhibit the first examples of initial and final functors, in the sense introduced in Definition 2.4.6 below.

The commutativity of right adjoints and limits is very easily established in the case where the ∞-categories in question admit *all* limits of a given shape: under these hypotheses, the limit functor is right adjoint to the constant diagram functor, which commutes with all functors between the base ∞-categories. Since the left adjoints commute, the uniqueness of adjoints (Proposition 2.1.10) implies that the right adjoints do as well. This outline gives a hint for Exercise 2.4.i.

A slightly more delicate argument is needed in the general case, involving, say, the preservation of a single limit diagram without a priori assuming that any other limits exist. This follows easily from a general lemma about composition and cancelation of absolute lifting diagrams:

2.4.1. Lemma (composition and cancelation of absolute lifting diagrams). *Suppose (r, ρ) defines an absolute right lifting of h through f:*

$$\begin{tikzcd} & C \arrow[d, "g"] \\ & B \arrow[d, "f"] \arrow[u, phantom, shift right=4ex, "\Downarrow\sigma"] \\ D \arrow[r, "h"'] \arrow[ur, "r"] \arrow[uur, bend left, "s"] & A \arrow[u, phantom, shift right=4ex, "\Downarrow\rho"] \end{tikzcd}$$

Then (s, σ) defines an absolute right lifting of r through g if and only if $(s, \rho \cdot f\sigma)$ defines an absolute right lifting of h through fg.

Proof. Exercise 2.4.ii. □

2.4.2. Theorem (RAPL/LAPC). *Right adjoints preserve limits and left adjoints preserve colimits.*

The usual argument that right adjoints preserve limits proceeds like this: a cone over a J-shaped diagram in the image of u transposes across the adjunction $f^J \dashv u^J$ to a cone over the original diagram, which factors through the designated limit cone. This factorization transposes across the adjunction $f \dashv u$ to define the sought-for unique factorization through the image of the limit cone. An ∞-categorical proof along these lines can be given as well and is left as Exercise 2.4.iii. Here we present a slicker packaging of the standard argument. The use of absolute lifting diagrams to express the universal properties of limits and colimits (Definition 2.3.7) and adjoint transposition (Lemma 2.3.6) allows us to economize on the usual proof by suppressing consideration of a generic test cone that must be shown to uniquely factor through the limit cone.

Proof. We prove that right adjoints preserve limits. By taking co-duals the same argument demonstrates that left adjoints preserve colimits.

Suppose $u\colon A \to B$ admits a left adjoint $f\colon B \to A$ with unit $\eta\colon \mathrm{id}_B \Rightarrow uf$ and counit $\epsilon\colon fu \Rightarrow \mathrm{id}_A$. Our aim is to show that any absolute right lifting diagram as displayed below-left is carried to an absolute right lifting diagram as displayed below-right:

$$\begin{tikzcd} & A \arrow[d, "\Delta"] \\ D \arrow[r, "d"'] \arrow[ur, "\lim"] \arrow[ur, phantom, shift right=3ex, "\Downarrow\rho"] & A^J \end{tikzcd} \qquad\qquad \begin{tikzcd} & A \arrow[d, "\Delta"] \arrow[r, "u"] & B \arrow[d, "\Delta"] \\ D \arrow[r, "d"'] \arrow[ur, "\lim"] \arrow[ur, phantom, shift right=3ex, "\Downarrow\rho"] & A^J \arrow[r, "u^J"'] & B^J \end{tikzcd} \tag{2.4.3}$$

By Proposition 2.1.7, the cotensor $(-)^J\colon \mathfrak{h}\mathcal{K} \to \mathfrak{h}\mathcal{K}$ carries the adjunction $f \dashv u$ to an adjunction $f^J \dashv u^J$ with unit η^J and counit ϵ^J. In particular, by Lemma 2.3.6, (u^J, ϵ^J) defines an absolute right lifting of the identity through f^J, which is then preserved by restriction along the functor d. Thus, by Lemma 2.4.1, the diagram on the right of (2.4.3) is an absolute right lifting diagram if and only if the pasted composite displayed below-left defines an absolute right lifting diagram:

$$
\begin{tikzcd}
 & A \arrow[r, "u"] \arrow[d, "\Delta"] & B \arrow[d, "\Delta"] \\
D \arrow[ur, "\lim"] \arrow[r, "d"'] \arrow[r, phantom, shift left=3, "\Downarrow\rho"] & A^J \arrow[r, "u^J"] \arrow[dr, equal] & B^J \arrow[d, "f^J"] \arrow[d, phantom, shift right=3, "\Downarrow\epsilon^J"] \\
 & & A^J
\end{tikzcd}
=
\begin{tikzcd}
 & & B \arrow[d, "f"] \\
 & A \arrow[ur, "u"] \arrow[r, equal] \arrow[d, "\Delta"] \arrow[r, phantom, shift left=3, "\Downarrow\epsilon"] & A \arrow[d, "\Delta"] \\
D \arrow[ur, "\lim"] \arrow[r, "d"'] \arrow[r, phantom, shift left=3, "\Downarrow\rho"] & A^J \arrow[r, equal] & A^J
\end{tikzcd}
=
\begin{tikzcd}
 & B \arrow[d, "f"] \\
 & A \arrow[d, "\Delta"] \\
D \arrow[uur, bend left, "u\lim"] \arrow[ur, "\lim"] \arrow[r, "d"'] \arrow[uur, phantom, "\Downarrow\epsilon\lim"] \arrow[r, phantom, shift left=3, "\Downarrow\rho"] & A^J
\end{tikzcd}
$$

As noted in the proof of Lemma 2.3.6, pasting the 2-cell on the right of (2.4.3) with the counit in this way amounts to transposing the cone under $u\lim$ across the adjunction $f^J \dashv u^J$.

We'll now argue that this transposed cone above-left factors through the limit cone $(\lim, \rho)$ in a canonical way. From the 2-functoriality of the simplicial cotensor in its exponent variable, $f^J\Delta = \Delta f$ and $\epsilon^J\Delta = \Delta\epsilon$. Hence, the pasting diagram displayed above-left equals the one displayed above-center, which equals the diagram above-right by naturality of whiskering.[11] This latter diagram is a pasted composite of two absolute right lifting diagrams, and is then an absolute right lifting diagram in its own right by Lemma 2.4.1; this universal property says that any cone over d whose summit factors through f factors uniquely through the limit cone $(\lim, \rho)$ through a map that then transposes along the adjunction $f \dashv u$. Hence the diagram on the right-hand side of (2.4.3) is an absolute right lifting diagram as claimed. □

By combining Theorem 2.4.2 with Proposition 2.1.11, we have immediately that:

2.4.4. Corollary. *Equivalences preserve limits and colimits.* □

We can also prove a more refined result:

2.4.5. Proposition. *If $A \simeq B$, then any family of diagrams in A admitting a limit or colimit in B also admits a limit or colimit in A that is preserved by the equivalence.*

Proof. By Proposition 2.1.11 the equivalence $B \xrightarrow{\sim} A$ is both left and right adjoint to its equivalence inverse, preserving both limits and colimits of the composite family of diagrams $D \to A^J \xrightarrow{\sim} B^J$. Via the invertible 2-cells of the equivalence $A^J \simeq B^J$ constructed by applying $(-)^J : \mathfrak{h}\mathcal{K} \to \mathfrak{h}\mathcal{K}$ to the equivalence $A \simeq B$, the preserved diagram $D \to A^J \xrightarrow{\sim} B^J \xrightarrow{\sim} A^J$ is isomorphic to the original family of diagrams $D \to A^J$. Thus, we conclude from Exercise 2.3.iv that a family of diagrams in A has a limit or colimit if and only if its image in an equivalent ∞-category B does, and such limits and colimits are preserved by the functors of the equivalence. □

We close this chapter with a limit-preservation result of another sort, which can be used to simplify the calculation of limits or colimits of diagrams with particular shapes.

The following definition makes sense for an arbitrary functor in a cartesian closed ∞-cosmos or for a map between simplicial sets serving as indexing shapes in an arbitrary ∞-cosmos. It will be extended to functors between ∞-categories in an arbitrary ∞-cosmos in Definition 9.5.5.

[11]By naturality of whiskering, $\epsilon^J d \cdot f^J u^J \rho = \rho \cdot \epsilon^J \Delta \lim$, and since $\epsilon^J\Delta = \Delta\epsilon$, this composite equals $\rho \cdot \Delta\epsilon\lim$.

2.4.6. Definition (initial and final functor). A functor $k\colon I \to J$ is **final** if J-indexed colimits exist if and only if, and in such cases coincide with, the restricted I-indexed colimits. That is, $k\colon I \to J$ is final if and only if for any ∞-category A, the square

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "\Delta"'] & A \arrow[d, "\Delta"] \\ A^J \arrow[r, "A^k"'] & A^I \end{tikzcd}$$

preserves and reflects all absolute left lifting diagrams.

Dually a functor $k\colon I \to J$ is **initial** if this square preserves and reflects all absolute right lifting diagrams: or informally, if a generalized element defines a limit of a J-indexed diagram if and only if it defines a limit of the restricted I-indexed diagrams.

Historically, final functors were called "cofinal" with no obvious name for the dual notion. Our preferred terminology hinges on the following mnemonic: the inclusion of an initial element defines an initial functor, while the inclusion of a terminal (aka final) element defines a final functor. These facts are special cases of a more general result we now establish, using exactly the same tactics as deployed to prove Theorem 2.4.2.

2.4.7. Proposition. *Left adjoints define initial functors and right adjoints define final functors.*

Proof. If $k \dashv r$ with unit $\eta\colon \mathrm{id}_I \Rightarrow rk$ and counit $\epsilon\colon kr \Rightarrow \mathrm{id}_J$, then cotensoring into A yields an adjunction

$$\begin{tikzcd} A^J \arrow[r, bend right, "A^k"'] \arrow[r, phantom, "\perp"] & A^I \arrow[l, bend right, "A^r"'] \end{tikzcd}$$

with unit $A^\eta\colon \mathrm{id}_{A^I} \Rightarrow A^kA^r$ and counit $A^\epsilon\colon A^rA^k \Rightarrow \mathrm{id}_{A^J}$.

To prove that k is initial we must show that for any $\rho\colon \Delta\lim \to d$ as displayed below-left,

$$\begin{tikzcd} & A \arrow[d, "\Delta"] \\ D \arrow[ur, "\lim"] \arrow[r, "d"'] \arrow[r, phantom, shift left=3, "\Downarrow\rho"] & A^J \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[r, equal] \arrow[d, "\Delta"] & A \arrow[d, "\Delta"] \\ D \arrow[ur, "\lim"] \arrow[r, "d"'] \arrow[r, phantom, shift left=3, "\Downarrow\rho"] & A^J \arrow[r, "A^k"'] & A^I \end{tikzcd}$$

the left-hand diagram is an absolute right lifting diagram if and only if the right-hand diagram is an absolute right lifting diagram.

By Lemmas 2.3.6 and 2.4.1, the right-hand diagram is an absolute right lifting diagram if and only if the pasted composite displayed below-left

$$\begin{tikzcd} & A \arrow[r, equal] \arrow[d, "\Delta"] & A \arrow[d, "\Delta"] \\ D \arrow[ur, "\lim"] \arrow[r, "d"'] \arrow[r, phantom, shift left=3, "\Downarrow\rho"] & A^J \arrow[r, "A^k"] \arrow[dr, equal] \arrow[dr, phantom, shift left=3, "\Downarrow A^\epsilon"] & A^I \arrow[d, "A^r"] \\ & & A^J \end{tikzcd} = \begin{tikzcd} & A \arrow[d, "\Delta"] \\ D \arrow[ur, "\lim"] \arrow[r, "d"'] \arrow[r, phantom, shift left=3, "\Downarrow\rho"] & A^J \end{tikzcd}$$

is also an absolute right lifting diagram. On noting that $A^r\Delta = \Delta$ and $A^\epsilon\Delta = \mathrm{id}_\Delta$, the left-hand side reduces to the right-hand side, which proves the claim. $\square$

Exercises.

2.4.i. Exercise. Show that any left adjoint $f\colon B \to A$ between ∞-categories admitting all J-shaped colimits preserves them in the sense that the square of functors

$$\begin{tikzcd} B^J \arrow[r, "f^J"] \arrow[d, "\mathrm{colim}"'] \arrow[dr, phantom, "\cong"] & A^J \arrow[d, "\mathrm{colim}"] \\ B \arrow[r, "f"'] & A \end{tikzcd}$$

commutes up to isomorphism.

2.4.ii. Exercise. Prove Lemma 2.4.1.

2.4.iii. Exercise. Give a proof of Theorem 2.4.2 that does not appeal to Lemma 2.4.1 by directly verifying that the diagram on the right of (2.4.3) is an absolute right lifting diagram.

2.4.iv. Exercise (composite adjunctions, revisited). Use Lemma 2.4.1 to give a new proof of Proposition 2.1.9.

CHAPTER 3

Comma ∞-Categories

In Chapter 2, we introduced adjunctions between ∞-categories and limits and colimits of diagrams valued within an ∞-category through definitions that are particularly expedient for establishing the expected interrelationships. But neither 2-categorical definition clearly articulates the universal properties of these notions. Definition 2.3.7 does not obviously express the expected universal property of the limit cone: namely, that the limit cone over a diagram d defines the terminal element of the ∞-category of cones over d, yet-to-be-defined. Nor have we understood how an adjunction $f \dashv u$ induces an equivalence on as-yet-to-be-defined hom-spaces $\mathsf{Hom}_A(fb, a) \simeq \mathsf{Hom}_B(b, ua)$ for a pair of generalized elements.[1] In this section, we make use of the completeness axiom in the definition of an ∞-cosmos to exhibit a general construction that will specialize to give a definition of this ∞-category of cones and also specialize to define these hom-spaces. This construction will also permit us to represent a functor between ∞-categories *as* an ∞-category, in dual "left" or "right" fashions. Using this, we can redefine an adjunction to consist of a pair of functors $f \colon B \to A$ and $u \colon A \to B$ so that the left representation of f is equivalent to the right representation of u over $A \times B$.

Our vehicle for all of these new definitions is the *comma ∞-category* associated to a cospan

$$C \xrightarrow{g} A \xleftarrow{f} B \qquad \rightsquigarrow \qquad \begin{array}{c} \mathsf{Hom}_A(f,g) \\ \big\downarrow{\scriptstyle (p_1,p_0)} \\ C \times B \end{array}$$

Our aim in this chapter is to develop the general theory of comma constructions from the point of view of the homotopy 2-category of an ∞-cosmos. Our first payoff for this work will occur in Chapter 4 where we study the universal properties of adjunctions, limits, and colimits in the sense of the ideas just outlined. The comma construction will also provide the essential vehicle for establishing the model-independence of the categorical notions we will introduce throughout this text.

There is a standard definition of a "comma object" that can be stated in any strict 2-category, defined as a particular weighted limit (see Example A.6.18). Comma ∞-categories do *not* satisfy this universal property in the homotopy 2-category, however. Instead, they satisfy a somewhat peculiar "weak" variant of the usual 2-categorical universal property that to our knowledge has not been discovered elsewhere in the categorical or homotopical literature, expressed in terms of something we call a *smothering functor*. To introduce these universal properties in a concrete rather than abstract framework, we start in §3.1 by considering smothering functors involving homotopy categories of quasi-categories. The intrepid and impatient reader may skip the entirety of §3.1 if they wish to instead first encounter these notions in their full generality.

[1] A 2-categorical version of this result — exhibiting a bijection between *sets* of 2-cells — appears as Lemma 2.3.6, but in an ∞-category we'd hope for a similar equivalence of hom-spaces.

3.1. Smothering functors

Let Q be a quasi-category. Recall from Lemma 1.1.12 that its homotopy category $\mathsf{h}Q$ has

- the elements $1 \to Q$ of Q as its objects;
- the set of homotopy classes of 1-simplices of Q as its arrows, where parallel 1-simplices are homotopic just when they bound a 2-simplex with the remaining outer edge degenerate; and
- a composition relation if and only if any chosen 1-simplices representing the three arrows bound a 2-simplex.

For a 1-category J, it is well-known in classical homotopy theory that the homotopy category of diagrams $\mathsf{h}(Q^J)$ is not equivalent to the category $(\mathsf{h}Q)^J$ of diagrams in the homotopy category — except in very special cases, such as when J is a set (see Warning 2.3.3). The objects of $\mathsf{h}(Q^J)$ are *homotopy coherent diagrams* of shape J in Q, while the objects of $(\mathsf{h}Q)^J$ are mere *homotopy commutative diagrams*. There is, however, a canonical comparison functor

$$\mathsf{h}(Q^J) \to (\mathsf{h}Q)^J$$

defined by applying $\mathsf{h}\colon \mathcal{QC}at \to \mathcal{C}at$ to the evaluation functor $Q^J \times J \to Q$ and then transposing; a homotopy coherent diagram is in particular homotopy commutative.

Our first aim in this section is to better understand the relationship between the arrows in the homotopy category $\mathsf{h}Q$ and what we'll refer to as the **arrows of** Q, namely, the 1-simplices in the quasi-category. To study this we'll be interested in the quasi-category in which the arrows of Q live as elements, namely $Q^{\mathbb{2}}$, where $\mathbb{2} = \Delta[1]$ is the nerve of the "walking" arrow. Our notation deliberately imitates the notation commonly used for the **category of arrows**: if C is a 1-category, then $C^{\mathbb{2}}$ is the category whose objects are arrows in C and whose morphisms are commutative squares, regarded as a morphism from the arrow displayed vertically on the left-hand side to the arrow displayed vertically on the right-hand side. This notational conflation suggests our first motivating question: how does the homotopy category of $Q^{\mathbb{2}}$ relate to the category of arrows in the homotopy category of Q?

3.1.1. Lemma. *The canonical functor* $\mathsf{h}(Q^{\mathbb{2}}) \to (\mathsf{h}Q)^{\mathbb{2}}$ *is*

- *surjective on objects,*
- *full, and*
- ***conservative**, i.e., reflects invertibility of morphisms,*

but not injective on objects nor faithful.

Proof. Surjectivity on objects asserts that every arrow in the homotopy category $\mathsf{h}Q$ is represented by a 1-simplex in Q. This is the conclusion of Exercise 1.1.iii(iii) which outlines the proof of Lemma 1.1.12.

To prove fullness, consider a commutative square in $\mathsf{h}Q$ and choose arbitrary 1-simplices representing each morphism and their common composite:

$$\begin{tikzcd} \bullet \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, "\ell"] & \bullet \arrow[d, "g"] \\ \bullet \arrow[r, "k"'] & \bullet \end{tikzcd}$$

By Lemma 1.1.12, every composition relation in $\mathsf{h}Q$ is witnessed by a 2-simplex in Q; choosing a pair of such 2-simplices defines a diagram $\mathbb{2} \to Q^{\mathbb{2}}$, which represents a morphism from f to g in $\mathsf{h}(Q^{\mathbb{2}})$, proving fullness.

Surjectivity on objects and fullness of the functor $\mathsf{h}(Q^{\mathbb{2}}) \to (\mathsf{h}Q)^{\mathbb{2}}$ are special properties having to do with the diagram shape $\mathbb{2}$. Conservativity is much more general as a consequence of the second statement of Corollary 1.1.21. ☐

The properties of the canonical functor $\mathsf{h}(Q^{\mathbb{2}}) \to (\mathsf{h}Q)^{\mathbb{2}}$ will reappear frequently so are worth giving a name:

3.1.2. DEFINITION. A functor $f\colon A \to B$ between 1-categories is **smothering** if it is surjective on objects, full, and conservative. That is, a functor is smothering if and only if it has the right lifting property with respect to the set of functors:

$$\left\{ \begin{array}{c} \emptyset \\ \downarrow \\ \mathbb{1} \end{array} , \begin{array}{c} \mathbb{1}+\mathbb{1} \\ \downarrow \\ \mathbb{2} \end{array} , \begin{array}{c} \mathbb{2} \\ \downarrow \\ \mathbb{I} \end{array} \right\}$$

Some elementary properties of smothering functors are established in Exercise 3.1.i. The most important of these is:

3.1.3. LEMMA. *Each fibre of a smothering functor is a non-empty connected groupoid.*

PROOF. Suppose $f\colon A \to B$ is smothering and consider the fiber

$$\begin{tikzcd} A_b \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\ \mathbb{1} \arrow[r, "b"] & B \end{tikzcd}$$

over an object b of B. By surjectivity on objects, the fiber is non-empty. Its morphisms are defined to be arrows between objects in the fiber of b that map to the identity on b. By fullness, any two objects in the fiber are connected by a morphism, indeed, by morphisms pointing in both directions. By conservativity, all the morphisms in the fiber are necessarily invertible. ☐

The argument used to prove Lemma 3.1.1 generalizes to:

3.1.4. LEMMA. *If J is a 1-category that is free on a reflexive directed graph and Q is a quasi-category, then the canonical functor $\mathsf{h}(Q^J) \to (\mathsf{h}Q)^J$ is smothering.*

PROOF. Exercise 3.1.ii. ☐

Cotensors are one of the simplicial limit constructions enumerated in axiom 1.2.1(i). Other limit constructions listed there also give rise to smothering functors.

3.1.5. LEMMA. *For any pullback diagram of quasi-categories in which p is an isofibration*

$$\begin{tikzcd} A \underset{B}{\times} E \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, two heads, "p"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

the canonical functor $\mathsf{h}(A \underset{B}{\times} E) \to \mathsf{h}A \underset{\mathsf{h}B}{\times} \mathsf{h}E$ is smothering.

Proof. As $\mathsf{h}\colon \mathcal{QC}at \to \mathcal{C}at$ does not preserve pullbacks, the canonical comparison functor of the statement is not an isomorphism. It is however bijective on objects since the composite functor

$$\mathcal{QC}at \xrightarrow{\mathsf{h}} \mathcal{C}at \xrightarrow{\text{obj}} \mathcal{S}et$$

is given by evaluation on the set of vertices of each quasi-category, and this functor *does* preserve pullbacks.

For fullness, note that a morphism in $\mathsf{h}A \underset{\mathsf{h}B}{\times} \mathsf{h}E$ is represented by a pair of 1-simplices $\alpha\colon a \to a'$ and $\epsilon\colon e \to e'$ in A and E whose images in B are homotopic, a condition that implies in particular that $f(a) = p(e)$ and $f(a') = p(e')$. By Lemma 1.1.9, we can arrange this homotopy however we like, and thus we choose a 2-simplex witness β so as to define a lifting problem

$$\begin{tikzcd} \Lambda^1[2] \arrow[r] \arrow[d, hook] & E \arrow[d, two heads, "p"] & \ni & e \arrow[r, equal] & e \arrow[r, "\epsilon"] & e' \\ \Delta[2] \arrow[r, "\beta"'] \arrow[ur, dashed] & B & \ni & f(a) \arrow[r, equal] \arrow[rr, "f(\alpha)"'] & p(e) \arrow[r, "p(\epsilon)"] & f(a') = p(e') \end{tikzcd}$$

Since p is an isofibration, a solution exists, defining an arrow $\tilde{\epsilon}\colon e \to e'$ in E in the same homotopy class as ϵ so that $p(\tilde{\epsilon}) = f(\alpha)$. The pair $(\alpha, \tilde{\epsilon})$ now defines the lifted arrow in $\mathsf{h}(E \times_B A)$.

Finally, consider an arrow $\mathbb{2} \to A \underset{B}{\times} E$ whose image in $\mathsf{h}A \underset{\mathsf{h}B}{\times} \mathsf{h}E$ is an isomorphism, which is the case just when the projections to E and A define isomorphisms. By Corollary 1.1.16, we may choose a homotopy coherent isomorphism $\mathbb{I} \to A$ extending the given isomorphism $\mathbb{2} \to A$. This data presents us with a lifting problem

$$\begin{tikzcd} \mathbb{2} \arrow[r] \arrow[d, hook] & A \underset{B}{\times} E \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner" very near start] & E \arrow[d, two heads, "p"] \\ \mathbb{I} \arrow[r] \arrow[ur, dotted] \arrow[urr, dashed] & A \arrow[r, "f"'] & B \end{tikzcd}$$

which Exercise 1.1.vi tells us we can solve. This proves that $\mathsf{h}(A \underset{B}{\times} E) \to \mathsf{h}A \underset{\mathsf{h}B}{\times} \mathsf{h}E$ is conservative and hence also smothering. □

A similar argument proves:

3.1.6. Lemma. *For any tower of isofibrations between quasi-categories*

$$\cdots \twoheadrightarrow E_n \twoheadrightarrow E_{n-1} \twoheadrightarrow \cdots \twoheadrightarrow E_2 \twoheadrightarrow E_1 \twoheadrightarrow E_0$$

the canonical functor $\mathsf{h}(\lim_n E_n) \to \lim_n \mathsf{h}E_n$ *is smothering.*

Proof. Exercise 3.1.iii. □

3.1.7. LEMMA. *For any cospan between quasi-categories* $C \xrightarrow{g} A \xleftarrow{f} B$ *consider the quasi-category defined by the pullback*

$$\begin{tikzcd} \mathsf{Hom}_A(f,g) \arrow[r] \arrow[d, two heads, "{(p_1,p_0)}"'] \arrow[dr, phantom, "\lrcorner" very near start] & A^2 \arrow[d, two heads, "{(\mathrm{cod},\mathrm{dom})}"] \\ C \times B \arrow[r, "g\times f"'] & A \times A \end{tikzcd}$$

The canonical functor $\mathsf{h}\mathsf{Hom}_A(f,g) \to \mathsf{Hom}_{\mathsf{h}A}(\mathsf{h}f, \mathsf{h}g)$ *is smothering.*

PROOF. Here, the codomain is the category defined by an analogous pullback

$$\begin{tikzcd} \mathsf{Hom}_{\mathsf{h}A}(\mathsf{h}f,\mathsf{h}g) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner" very near start] & (\mathsf{h}A)^2 \arrow[d, two heads, "{(\mathrm{cod},\mathrm{dom})}"] \\ \mathsf{h}C \times \mathsf{h}B \arrow[r, "\mathsf{h}g\times \mathsf{h}f"'] & \mathsf{h}A \times \mathsf{h}A \end{tikzcd}$$

in $\mathcal{C}at$ and the canonical functor factors as

$$\mathsf{h}\mathsf{Hom}_A(f,g) \to \mathsf{h}(A^2) \times_{\mathsf{h}A\times\mathsf{h}A} (\mathsf{h}C \times \mathsf{h}B) \to (\mathsf{h}A)^2 \times_{\mathsf{h}A\times\mathsf{h}A} (\mathsf{h}C \times \mathsf{h}B)$$

By Lemma 3.1.5 the first of these functors is smothering. By Lemma 3.1.1 the second is a pullback of a smothering functor. By Exercise 3.1.i(i) it follows that the composite functor is smothering. ☐

In the sections that follow, we will discover that the smothering functors just constructed express particular "weak" universal properties of arrow, pullback, and comma constructions in the homotopy 2-category of any ∞-cosmos. It is to the first of these that we now turn.

Exercises.

3.1.i. EXERCISE. Prove that:

(i) The class of smothering functors is closed under composition, retract, product, and pullback.
(ii) The class of smothering functors contains all surjective equivalences of categories.
(iii) All smothering functors are isofibrations, that is, maps that have the right lifting property with respect to $\mathbb{1} \hookrightarrow \mathbb{I}$.
(iv) Prove that if f and gf are smothering functors, then g is a smothering functor.[2]

3.1.ii. EXERCISE. Prove Lemma 3.1.4.

3.1.iii. EXERCISE. Prove Lemma 3.1.6.

3.2. ∞-categories of arrows

In this section, we replicate the discussion from the start of the previous section using an arbitrary ∞-category A in place of the quasi-category Q. The analysis of the previous section could have been developed natively in this general setting but at the cost of an extra layer of abstraction and more confusing notation — with a functor space $\mathsf{Fun}(X, A)$ replacing the quasi-category Q.

Recall an **element** of an ∞-category is defined to be a functor $1 \to A$. Tautologically, the elements of A are the vertices of the **underlying quasi-category** $\mathsf{Fun}(1, A)$ of A. In this section, we will define

[2]It suffices, in fact, to merely assume that f is surjective on objects and arrows.

and study an ∞-category $A^{\mathbb{2}}$ whose elements are the 1-simplices in the underlying quasi-category of A. We refer to $A^{\mathbb{2}}$ as the *∞-category of arrows* in A and call its elements simply *arrows* of A.

In fact, we've tacitly introduced this construction already. Recall $\mathbb{2}$ is our preferred notation for the quasi-category $\Delta[1]$, as this coincides with the nerve of the 1-category $\mathbb{2}$ with a single non-identity morphism $0 \to 1$.

3.2.1. Definition (arrow ∞-category). Let A be an ∞-category. The **∞-category of arrows** in A is the simplicial cotensor $A^{\mathbb{2}}$ together with the canonical endpoint-evaluation isofibration

$$A^{\mathbb{2}} := A^{\Delta[1]} \xrightarrow{(p_1,p_0)} A^{\partial\Delta[1]} \cong A \times A$$

induced by the inclusion $\partial\Delta[1] \hookrightarrow \Delta[1]$. For conciseness, we write $p_0 \colon A^{\mathbb{2}} \twoheadrightarrow A$ for the domain-evaluation induced by the inclusion $0 \colon \mathbb{1} \hookrightarrow \mathbb{2}$ and write $p_1 \colon A^{\mathbb{2}} \twoheadrightarrow A$ for the codomain-evaluation induced by $1 \colon \mathbb{1} \hookrightarrow \mathbb{2}$.

As an object of the homotopy 2-category $\mathfrak{h}\mathcal{K}$, the ∞-category of arrows comes equipped with a canonical 2-cell that we now construct.

3.2.2. Lemma. *For any ∞-category A, the ∞-category of arrows comes equipped with a canonical 2-cell*

$$\begin{tikzcd} A^{\mathbb{2}} \arrow[r, bend left, "p_0"{name=U}] \arrow[r, bend right, "p_1"'{name=D}] & A \arrow[from=U, to=D, phantom, "\Downarrow\kappa"] \end{tikzcd} \tag{3.2.3}$$

*that we refer to as the **generic arrow** with codomain A.*

Proof. The simplicial cotensor has a strict universal property described in Digression 1.2.6: namely $A^{\mathbb{2}}$ is characterized by the natural isomorphism

$$\mathsf{Fun}(X, A^{\mathbb{2}}) \cong \mathsf{Fun}(X, A)^{\mathbb{2}}. \tag{3.2.4}$$

By the Yoneda lemma, the data of the natural isomorphism (3.2.4) is encoded by its "universal element", which is defined to be the image of the identity at the representing object. Here the identity functor $\mathrm{id} \colon A^{\mathbb{2}} \to A^{\mathbb{2}}$ is mapped to an element of $\mathsf{Fun}(A^{\mathbb{2}}, A)^{\mathbb{2}}$, a 1-simplex in $\mathsf{Fun}(A^{\mathbb{2}}, A)$ which represents a 2-cell in the homotopy 2-category defining (3.2.3).

To see that its source and target must be the domain-evaluation and codomain-evaluation maps, note that the action of the simplicial cotensor $A^{(-)}$ on morphisms of simplicial sets is defined so that the isomorphism (3.2.4) is natural in the cotensor variable as well. Thus, by restricting along the endpoint inclusion $\mathbb{1}+\mathbb{1} \hookrightarrow \mathbb{2}$, we may regard the isomorphism (3.2.4) as lying over $\mathsf{Fun}(X, A\times A) \cong \mathsf{Fun}(X, A) \times \mathsf{Fun}(X, A)$. □

There is a 2-categorical limit notion that is analogous to Definition 3.2.1, which constructs, for any object A, the universal 2-cell with codomain A: namely the cotensor with the 1-category $\mathbb{2}$. Its universal property is analogous to (3.2.4) but with the hom-categories of the 2-category in place of the functor spaces. In $\mathcal{C}at$ this constructs the arrow category associated to a 1-category.

In the homotopy 2-category $\mathfrak{h}\mathcal{K}$, by the Yoneda lemma again, the data (3.2.3) encodes a natural transformation

$$\mathsf{hFun}(X, A^{\mathbb{2}}) \to \mathsf{hFun}(X, A)^{\mathbb{2}}$$

of categories but this is *not* a natural isomorphism, nor even a natural equivalence of categories but does express the arrow ∞-category as a "weak" arrow object with a universal property of the following form:

3.2.5. Proposition (the weak universal property of the arrow ∞-category). *The generic arrow* (3.2.3) *with codomain A has a weak universal property in the homotopy 2-category given by three operations:*

(i) ***1-cell induction**: Given a 2-cell over A as below-left*

$$\begin{tikzcd} X \arrow[d, bend right=50, "t"'] \arrow[d, bend left=50, "s"] \arrow[d, phantom, "\overset{\alpha}{\Leftarrow}"] \\ A \end{tikzcd} \quad = \quad \begin{tikzcd} X \arrow[d, dashed, "a"] \arrow[dd, bend right=70, "t"'] \arrow[dd, bend left=70, "s"] \\ A^{2} \arrow[d, bend right, "p_1"'] \arrow[d, bend left, "p_0"] \arrow[d, phantom, "\overset{\kappa}{\Leftarrow}"] \\ A \end{tikzcd}$$

there exists a 1-cell $a\colon X \to A^{2}$ so that $s = p_0 a$, $t = p_1 a$, and $\alpha = \kappa a$.

(ii) ***2-cell induction**: Given a pair of functors $a, a'\colon X \rightrightarrows A^{2}$ and a pair of 2-cells τ_0 and τ_1 so that*

$$\begin{tikzcd} & X \arrow[dl, "a'"'] \arrow[dr, "a"] \arrow[d, phantom, "\overset{\tau_1}{\Leftarrow}"] & \\ A^{2} \arrow[dr, "p_1"'] & {} & A^{2} \arrow[dl, bend right, "p_1"'] \arrow[dl, bend left, two heads, "p_0"] \arrow[dl, phantom, "\overset{\kappa}{\Leftarrow}"] \\ & A & \end{tikzcd} \quad = \quad \begin{tikzcd} & X \arrow[dl, "a'"'] \arrow[dr, "a"] \arrow[d, phantom, "\overset{\tau_0}{\Leftarrow}"] & \\ A^{2} \arrow[dr, bend left, "p_0"] \arrow[dr, bend right, two heads, "p_1"'] \arrow[dr, phantom, "\overset{\kappa}{\Leftarrow}"] & {} & A^{2} \arrow[dl, "p_0"] \\ & A & \end{tikzcd}$$

there exists a 2-cell $\tau\colon a \Rightarrow a'$ so that

$$\begin{tikzcd} & X \arrow[dl, "a'"'] \arrow[dr, "a"] \arrow[d, phantom, "\overset{\tau_1}{\Leftarrow}"] & \\ A^{2} \arrow[dr, "p_1"'] & {} & A^{2} \arrow[dl, "p_1"] \\ & A & \end{tikzcd} = \begin{tikzcd} X \arrow[d, bend right, "a'"'] \arrow[d, bend left, "a"] \arrow[d, phantom, "\overset{\tau}{\Leftarrow}"] \\ A^{2} \arrow[d, "p_1"] \\ A \end{tikzcd} \text{ and } \begin{tikzcd} X \arrow[d, bend right, "a'"'] \arrow[d, bend left, "a"] \arrow[d, phantom, "\overset{\tau}{\Leftarrow}"] \\ A^{2} \arrow[d, "p_0"] \\ A \end{tikzcd} = \begin{tikzcd} & X \arrow[dl, "a'"'] \arrow[dr, "a"] \arrow[d, phantom, "\overset{\tau_0}{\Leftarrow}"] & \\ A^{2} \arrow[dr, "p_0"'] & {} & A^{2} \arrow[dl, "p_0"] \\ & A & \end{tikzcd}$$

(iii) ***2-cell conservativity**: Any 2-cell*

$$\begin{tikzcd} X \arrow[d, bend right, "a'"'] \arrow[d, bend left, "a"] \arrow[d, phantom, "\overset{\tau}{\Leftarrow}"] \\ A^{2} \end{tikzcd}$$

with the property that both $p_1\tau$ and $p_0\tau$ are isomorphisms is an isomorphism.

Proof. Let $Q = \mathsf{Fun}(X, A)$ and apply Lemma 3.1.1 to observe that the natural map of hom-categories

$$\begin{tikzcd} \mathsf{hFun}(X, A^{2}) \arrow[rr] \arrow[dr, "{((p_1)_*,(p_0)_*)}"'] & & \mathsf{hFun}(X, A)^{2} \arrow[dl, "{(\mathrm{ev}_1,\mathrm{ev}_0)}"] \\ & \mathsf{hFun}(X, A) \times \mathsf{hFun}(X, A) & \end{tikzcd}$$

over $\mathsf{hFun}(X, A \times A) \cong \mathsf{hFun}(X, A) \times \mathsf{hFun}(X, A)$ is a smothering functor. Surjectivity on objects is expressed by 1-cell induction, fullness by 2-cell induction, and conservativity by 2-cell conservativity. □

Note that the functors $X \to A^{\mathbb{2}}$ that represent a fixed 2-cell with domain X and codomain A are not unique. However, they are unique up to "fibered" isomorphisms that whisker with $(p_1, p_0)\colon A^{\mathbb{2}} \twoheadrightarrow A \times A$ to an identity 2-cell:

3.2.6. Proposition. *Whiskering with* (3.2.3) *induces a bijection between 2-cells with domain X and codomain A as displayed below-left*

$$\left\{ \begin{tikzcd} X \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] \arrow[r, phantom, "\Downarrow\alpha"] & A \end{tikzcd} \right\} \quad \leftrightsquigarrow \quad \left\{ \begin{tikzcd} & X \arrow[dl, "t"'] \arrow[dr, "s"] \arrow[dd, "a"] & \\ A & & A \\ & A^{\mathbb{2}} \arrow[ul, two heads, "p_1"] \arrow[ur, two heads, "p_0"'] & \end{tikzcd} \right\}_{/\cong}$$

and fibered isomorphism classes of functors $X \to A^{\mathbb{2}}$ as displayed above-right, where the fibered isomorphisms are given by invertible 2-cells

$$\begin{tikzcd} & X \arrow[dl, "t"'] \arrow[dr, "s"] \arrow[dd, bend right, "a'"'] \arrow[dd, bend left, "a"] \arrow[dd, phantom, "\overset{\gamma}{\cong}"] & \\ A & & A \\ & A^{\mathbb{2}} \arrow[ul, two heads, "p_1"] \arrow[ur, two heads, "p_0"'] & \end{tikzcd}$$

so that $p_0\gamma = \mathrm{id}_s$ and $p_1\gamma = \mathrm{id}_t$.

Proof. Lemma 3.1.3 proves that the fibers of the smothering functor of Proposition 3.2.5 are connected groupoids. The objects of these fibers are functors $X \to A^{\mathbb{2}}$ and the morphisms are invertible 2-cells that whisker with $(p_1, p_0)\colon A^{\mathbb{2}} \twoheadrightarrow A \times A$ to an identity 2-cell. The action of the smothering functor defines a bijection between the objects of its codomain and their corresponding fibers. □

Our final task is to observe that the universal property of Proposition 3.2.5 is also enjoyed by any object $(e_1, e_0)\colon E \twoheadrightarrow A \times A$ that is equivalent to the arrow ∞-category $(p_1, p_0)\colon A^{\mathbb{2}} \twoheadrightarrow A \times A$ in the slice ∞-cosmos $\mathcal{K}_{/A\times A}$. We have special terminology to allow us to concisely express the type of equivalence we have in mind.

3.2.7. Definition (fibered equivalence). A **fibered equivalence** over an ∞-category B in an ∞-cosmos $\mathcal{K}$ is an equivalence

$$\begin{tikzcd} E \arrow[rr, "\sim"] \arrow[dr, two heads] & & F \arrow[dl, two heads] \\ & B & \end{tikzcd} \tag{3.2.8}$$

in the sliced ∞-cosmos $\mathcal{K}_{/B}$. We write $E \simeq_B F$ to indicate that the specified isofibrations with these domains are equivalent **over** B.

By Proposition 1.2.22(vii), a fibered equivalence is just a map between a pair of isofibrations over a common base that defines an equivalence in the underlying ∞-cosmos: the forgetful functor $\mathcal{K}_{/B} \to \mathcal{K}$ preserves and reflects equivalences. Note, however, that it does not create them: it is possible for two ∞-categories E and F to be equivalent without there existing any equivalence compatible with a pair of specified isofibration $E \twoheadrightarrow B$ and $F \twoheadrightarrow B$.

3.2.9. Remark. At this point, there is some ambiguity about the 2-categorical data that presents a fibered equivalence related to the question posed in Exercise 1.4.vi. But since Proposition 1.2.22(vii) tells us that a mere equivalence in $\mathfrak{h}\mathcal{K}$ involving a functor of the form (3.2.8) is sufficient to guarantee that this as-yet-unspecified 2-categorical data exists, we defer a careful analysis of this issue to §3.6.

3.2.10. Proposition (uniqueness of arrow ∞-categories). *For any isofibration $(e_1, e_0)\colon E \twoheadrightarrow A \times A$ equipped with a fibered equivalence $e\colon E \xrightarrow{\sim} A^{\mathbb{2}}$, the corresponding 2-cell*

$$\begin{tikzcd} E \arrow[r, bend left, "e_0"] \arrow[r, bend right, "e_1"'] \arrow[r, phantom, "\Downarrow\epsilon"] & A \end{tikzcd}$$

satisfies the weak universal property of Proposition 3.2.5. Conversely, if $(d_1, d_0)\colon D \twoheadrightarrow A\times A$ and $(e_1, e_0)\colon E \twoheadrightarrow A \times A$ are equipped with 2-cells

$$\begin{tikzcd} D \arrow[r, bend left, "d_0"] \arrow[r, bend right, "d_1"'] \arrow[r, phantom, "\Downarrow\delta"] & A \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} E \arrow[r, bend left, "e_0"] \arrow[r, bend right, "e_1"'] \arrow[r, phantom, "\Downarrow\epsilon"] & A \end{tikzcd}$$

satisfying the weak universal property of Proposition 3.2.5, then D and E are fibered equivalent over $A \times A$.

Proof. We prove the first statement. By the definition equation of 1-cell induction $\epsilon = \kappa e$, where κ is the canonical 2-cell of (3.2.3). Hence, the functor induced by pasting with ϵ factors as a composite

$$\begin{tikzcd} \mathsf{hFun}(X,E) \arrow[r, "e_*"] \arrow[dr, "((p_1)_*,(p_0)_*)"'] & \mathsf{hFun}(X,A^{\mathbb{2}}) \arrow[r] & \mathsf{hFun}(X,A)^{\mathbb{2}} \arrow[dl, "(\mathrm{ev}_1,\mathrm{ev}_0)"] \\ & \mathsf{hFun}(X,A)\times\mathsf{hFun}(X,A) & \end{tikzcd}$$

and our task is to prove that this composite functor is smothering. We see that the first functor, defined by post-composing with the equivalence $e\colon E \to A^{\mathbb{2}}$, is an equivalence of categories, and the second functor is smothering. Thus, the composite is clearly full and conservative. To see that it is also surjective on objects, note first that by 1-cell induction any 2-cell

$$\begin{tikzcd} X \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] \arrow[r, phantom, "\Downarrow\alpha"] & A \end{tikzcd}$$

is represented by a functor $a\colon X \to A^{\mathbb{2}}$ over $A\times A$. Composing with any fibered inverse equivalence e' to e yields a functor

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[dr, "(t,s)"'] & A^{\mathbb{2}} \arrow[r, "e'", "\sim"'] \arrow[d, two heads, "(p_1,p_0)"'] & E \arrow[dl, two heads, "(e_1,e_0)"] \\ & A\times A & \end{tikzcd}$$

whose image after post-composing with e is isomorphic to a over $A\times A$. Because this isomorphism is fibered (see Proposition 3.2.6), the image of ae' under the functor $\mathsf{hFun}(X,E) \to \mathsf{hFun}(X,A)^{\mathbb{2}}$ returns the 2-cell α. This proves that this mapping is surjective on objects and hence defines a smothering functor as claimed.

The converse is left to Exercise 3.2.ii and proven in a more general context in Proposition 3.4.11. $\square$

3.2.11. Convention. On account of Proposition 3.2.10, we extend the appellation "∞-category of arrows" from the strict model constructed in Definition 3.2.1 to any ∞-category that is fibered equivalent to it.

Via Lemma 3.1.4, the discussion of this section extends to establish corresponding weak universal properties for the cotensors A^J of an ∞-category A with a free category J. We leave the exploration of this to the reader.

Exercises.

3.2.i. Exercise.

(i) Prove that a parallel pair of 1-simplices in a quasi-category Q are homotopic if and only if they are isomorphic as elements of $Q^{\mathbb{2}}$ via an isomorphism that projects to an identity along $(p_1, p_0)\colon Q^{\mathbb{2}} \twoheadrightarrow Q \times Q$.

(ii) Conclude that a parallel pair of 1-arrows in the functor space $\mathsf{Fun}(X, A)$ between two ∞-categories X and A in any ∞-cosmos represent the same natural transformation if and only if they are isomorphic as elements of $\mathsf{Fun}(X, A)^{\mathbb{2}} \cong \mathsf{Fun}(X, A^{\mathbb{2}})$ via an isomorphism whose domain and codomain components are an identity.

(iii) Conclude that a parallel pair of 1-arrows in the functor space $\mathsf{Fun}(X, A)$, which may be encoded as functors $X \rightrightarrows A^{\mathbb{2}}$, represent the same natural transformation if and only if they are connected by a fibered isomorphism:

$$\begin{tikzcd} X \arrow[rr, bend left] \arrow[rr, bend right] \arrow[rr, phantom, "\cong"] \arrow[dr] & & A^{\mathbb{2}} \arrow[dl, "{(p_1,p_0)}"] \\ & A \times A & \end{tikzcd}$$

3.2.ii. Exercise. Prove the second statement of Proposition 3.2.10.

3.3. Pullbacks and limits of towers

Pullbacks in an ∞-cosmos also have a weak 2-dimensional universal property in the homotopy 2-category. For the most part, we won't make heavy use of this, preferring to exploit the strict universal property of the simplicially enriched limit instead. However, the weak 2-dimensional universal property can be used to prove that equivalences pull back along isofibrations to equivalences and generalize our previous results about the equivalence-invariance of pullbacks in an ∞-cosmos.

3.3.1. Proposition (the weak universal property of the pullback). *The pullback of an isofibration along a functor in an ∞-cosmos*

$$\begin{tikzcd} A \times_B E \arrow[r, "g"] \arrow[d, two heads, "q"'] \arrow[dr, phantom, "\lrcorner"] & E \arrow[d, two heads, "p"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

has a weak universal property in the homotopy 2-category given by three operations:

(i) ***1-cell induction**: Commutative squares $pe = fa$ over the cospan underlying a pullback diagram factor uniquely through the pullback square*

$$\begin{tikzcd} X \arrow[dr, dashed, "\exists! x"] \arrow[drr, bend left, "e"] \arrow[ddr, bend right, "a"'] & & \\ & A \times_B E \arrow[r, "g"] \arrow[d, two heads, "q"'] \arrow[dr, phantom, "\lrcorner" very near start] & E \arrow[d, two heads, "p"] \\ & A \arrow[r, "f"'] & B \end{tikzcd}$$

(ii) ***2-cell induction**: Given a pair of functors $x, x' \colon X \rightrightarrows A \times_B E$ and a pair of 2-cells $\alpha \colon qx \Rightarrow qx'$ and $\epsilon \colon gx \Rightarrow gx'$ as below-left so that $p\epsilon = f\alpha$ there exists a 2-cell $\tau \colon x \Rightarrow x'$ as below-right so that $q\tau = \alpha$ and $g\tau = \epsilon$.*

$$\begin{tikzcd} X \arrow[drr, bend left=20, "gx"] \arrow[drr, bend right=10, "gx'"'] \arrow[drr, phantom, "\epsilon \Downarrow"] \arrow[ddr, bend right=30, "qx'"'] \arrow[ddr, bend left=10, "qx"] \arrow[ddr, phantom, "\alpha \Downarrow"] & & \\ & & E \arrow[d, two heads, "p"] \\ & A \arrow[r, "f"'] \arrow[ur, phantom, "="] & B \end{tikzcd} \qquad \begin{tikzcd} X \arrow[dr, bend left, "x"] \arrow[dr, bend right, "x'"'] \arrow[dr, phantom, "\exists\tau \Downarrow"] & & \\ & A \times_B E \arrow[r, "g"] \arrow[d, two heads, "q"'] \arrow[dr, phantom, "\lrcorner" very near start] & E \arrow[d, two heads, "p"] \\ & A \arrow[r, "f"'] & B \end{tikzcd}$$

(iii) ***2-cell conservativity**: Any 2-cell*

$$\begin{tikzcd} X \arrow[r, bend left, "x"] \arrow[r, bend right, "x'"'] \arrow[r, phantom, "\tau \Downarrow"] & A \times_B E \end{tikzcd}$$

with the property that both $q\tau$ and $g\tau$ are isomorphisms is an isomorphism.

Proof. Apply Lemma 3.1.5 to the pullback diagram of quasi-categories

$$\begin{tikzcd} \mathsf{Fun}(X, A \times_B E) \arrow[r, "g_*"] \arrow[d, two heads, "q_*"'] \arrow[dr, phantom, "\lrcorner" very near start] & E \arrow[d, two heads, "p_*"] \\ A \arrow[r, "f_*"'] & B \end{tikzcd}$$

to observe that the natural map of hom-categories

$$\mathsf{hFun}(X, A \times_B E) \longrightarrow \mathsf{hFun}(X, A) \underset{\mathsf{hFun}(X,B)}{\times} \mathsf{hFun}(X, E)$$

is a bijective-on-objects smothering functor. Bijectivity on objects is expressed by 1-cell induction, fullness by 2-cell induction, and conservativity by 2-cell conservativity. □

Using the weak 2-categorical universal property of the pullback, we can show that ∞-cosmoi are **right proper**, meaning that the pullback of any equivalence along an isofibration defines an equivalence.

3.3.2. Lemma. *In any ∞-cosmos, the pullback of an equivalence along an isofibration is an equivalence.*

$$\begin{tikzcd} F \arrow[r, "g", "\sim"'] \arrow[d, two heads, "q"'] \arrow[dr, phantom, "\lrcorner" very near start] & E \arrow[d, two heads, "p"] \\ A \arrow[r, "\sim", "f"'] & B \end{tikzcd}$$

Proof. By Proposition 2.1.11, we may choose an inverse adjoint equivalence to f and pick invertible 2-cells $\alpha\colon \mathrm{id}_A \cong f^{-1}f$ and $\beta\colon ff^{-1} \cong \mathrm{id}_B$ satisfying the triangle equalities. It is for this reason that we work with the 2-categorical universal property of the pullback rather than the simplicially enriched universal property. Now since the map p is an isofibration, we may use Proposition 1.4.10 to lift the isomorphism $\beta p\colon ff^{-1}p \cong p$ along p to define an isomorphism $\epsilon\colon e \cong \mathrm{id}_E$ with codomain $\mathrm{id}_E\colon E \to E$. By construction $pe = ff^{-1}p$, so by 1-cell induction the pair $(f^{-1}p, e)$ induces a map $g^{-1}\colon E \to F$ so that $qg^{-1} = f^{-1}p$ and $gg^{-1} = e$. In this way we obtain an isomorphism $\epsilon\colon gg^{-1} \cong \mathrm{id}_E$ with $p\epsilon = \beta p$.

Now by 2-cell induction and conservativity of Proposition 3.3.1, to define an isomorphism $\mathrm{id}_F \cong g^{-1}g$, it suffices to exhibit a pair of isomorphisms $\alpha q\colon q \cong f^{-1}fq = f^{-1}pg = qg^{-1}g$ and $\epsilon^{-1}g\colon g \cong gg^{-1}g$ so that $f\alpha q = p\epsilon g$. This latter equation holds because $p\epsilon^{-1}g = \beta^{-1}pg = \beta^{-1}fq = f\alpha q$ by the triangle equality $\beta f \cdot f\alpha = \mathrm{id}_f$ for the adjoint equivalence $f \dashv f^{-1}$. Thus, we may lift the data of an inverse equivalence to f to define an inverse equivalence to its pullback g. □

As a consequence of right properness, we can show that pullback is an equivalence invariant construction in any ∞-cosmos.

3.3.3. Proposition. *Given a diagram of isofibrations and equivalences in any ∞-cosmos*

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "r"', "\wr"] & A \arrow[d, "\wr"', "p"] & B \arrow[l, two heads, "f"'] \arrow[d, "\wr"', "q"] \\ \bar{C} \arrow[r, "\bar{g}"'] & \bar{A} & \bar{B} \arrow[l, two heads, "\bar{f}"] \end{tikzcd}$$

the induced map $C \times_A B \to \bar{C} \times_{\bar{A}} \bar{B}$ between the pullbacks of the horizontal rows is again an equivalence.

Proof. By factoring via Lemma 1.2.16, we can replace the map $\bar{g}$ by an isofibration. By the 2-of-3 property and the right properness of Lemma 3.3.2, the pullback of this isofibration along the equivalence p is equivalent to the map g:

$$\begin{tikzcd} C \arrow[r, dashed, "\sim"'] \arrow[rr, bend left, "g"] \arrow[d, "r"', "\wr"] & P \arrow[r, two heads] \arrow[d, "\wr"'] \arrow[dr, phantom, "\lrcorner" very near start] & A \arrow[d, "\wr"', "p"] \\ \bar{C} \arrow[r, "\sim"] \arrow[rr, bend right, "\bar{g}"'] & \bar{P} \arrow[r, two heads] & \bar{A} \end{tikzcd}$$

By right properness again, the pullback of $P \twoheadrightarrow A$ along f is equivalent to the pullback of $C \to A$ along f and similarly for the lower-horizontal maps. So without loss of generality, we may assume that the maps g and $\bar{g}$ of the statement are fibrations and the left-hand square is a pullback.

Under these new hypothesis, the top, bottom, and front faces of the cube are pullback squares:

$$\begin{tikzcd}[row sep=small, column sep=small]
& C \times_A B \arrow[dl, two heads] \arrow[rr, two heads] \arrow[dd, dashed] & & B \arrow[dl, two heads, "f"'] \arrow[dd, "q", "\wr"'] \\
C \arrow[rr, two heads, "g" near end] \arrow[dd, "r"', "\wr"] & & A \arrow[dd, "p" near start, "\wr"' near start] & \\
& \bar{C} \times_{\bar{A}} \bar{B} \arrow[dl, two heads] \arrow[rr, two heads] & & \bar{B} \arrow[dl, two heads, "\bar{f}"] \\
\bar{C} \arrow[rr, two heads, "\bar{g}"'] & & \bar{A} &
\end{tikzcd}$$

so by pullback composition and cancelation, the back face is a pullback square as well. Now the induced map $C \times_A B \to \bar{C} \times_{\bar{A}} \bar{B}$ is the pullback of the equivalence q along an isofibration and hence is an equivalence by Lemma 3.3.2. $\square$

Exercises.

3.3.i. Exercise. State and prove an analogous result to Proposition 3.3.1 that describes the weak 2-categorical universal property of limits of towers of isofibrations.

3.4. The comma construction

The *comma ∞-category* is defined by restricting the domain and codomain of the ∞-category of arrows $A^{\mathbb{2}}$ along specified functors with codomain A.

3.4.1. Definition (comma ∞-category). Let $C \xrightarrow{g} A \xleftarrow{f} B$ be a diagram of ∞-categories. The **comma ∞-category** is constructed as a pullback of the simplicial cotensor $A^{\mathbb{2}}$ along $g \times f$

$$\begin{tikzcd}
\mathrm{Hom}_A(f,g) \arrow[r, "\phi"] \arrow[d, two heads, "(p_1,p_0)"'] \arrow[dr, phantom, "\lrcorner" very near start] & A^{\mathbb{2}} \arrow[d, two heads, "(p_1,p_0)"] \\
C \times B \arrow[r, "g \times f"'] & A \times A
\end{tikzcd} \tag{3.4.2}$$

This construction equips the comma ∞-category with a specified isofibration $(p_1, p_0) \colon \mathrm{Hom}_A(f,g) \twoheadrightarrow C \times B$ and a canonical 2-cell

$$\begin{tikzcd}
& \mathrm{Hom}_A(f,g) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[dd, phantom, "\Leftarrow\phi"] & \\
C \arrow[dr, "g"'] & & B \arrow[dl, "f"] \\
& A &
\end{tikzcd} \tag{3.4.3}$$

in the homotopy 2-category called the **comma cone**.

3.4.4. Example (arrow ∞-categories as comma ∞-categories). The arrow ∞-category arises as a special case of the comma construction applied to the identity span. This provides us with alternate notation for the generic arrow of (3.2.3), which may be regarded as a particular instance of a comma cone.

$$\begin{tikzcd} & \mathsf{Hom}_A \arrow[dl, "p_1"'] \arrow[dr, "p_0"] & \\ A \arrow[dr, equal] & \arrow[phantom, "\overset{\phi}{\Leftarrow}"] & A \arrow[dl, equal] \\ & A & \end{tikzcd}$$

The following proposition encodes the homotopical properties of the comma construction. The first statement is a special case of Proposition 3.3.3. The proof of the remaining statements is by a standard argument in abstract homotopy theory, which appears as Proposition C.1.12. A hint for this proof is given in Exercise 3.4.i.

3.4.5. Proposition (maps between commas). *A commutative diagram*

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "r"'] & A \arrow[d, "p"] & B \arrow[l, "f"'] \arrow[d, "q"] \\ \bar{C} \arrow[r, "\bar{g}"] & \bar{A} & \bar{B} \arrow[l, "\bar{f}"'] \end{tikzcd}$$

induces a map between the comma ∞-categories

$$\begin{tikzcd} \mathsf{Hom}_A(f,g) \arrow[r, dashed, "\mathsf{Hom}_p(q,r)"] \arrow[d, two heads, "(p_1,p_0)"'] & \mathsf{Hom}_{\bar{A}}(\bar{f},\bar{g}) \arrow[d, two heads, "(p_1,p_0)"] \\ C \times B \arrow[r, "r\times q"] & \bar{C} \times \bar{B} \end{tikzcd}$$

Moreover, if p, q, and r are all

(i) *equivalences,*
(ii) *isofibrations, or*
(iii) *trivial fibrations*

then the induced map is again an equivalence, isofibration, or trivial fibration, respectively.

There is a 2-categorical limit notion that is analogous to Definition 3.4.1, which constructs the universal 2-cell inhabiting a square over a specified cospan. In $\mathcal{C}at$ the category so-constructed is referred to as a *comma category*, from whence we borrow the name. As with the case of ∞-categories of arrows, comma ∞-categories do *not* satisfy this 2-universal property strictly. Instead:

3.4.6. Proposition (the weak universal property of the comma ∞-category). *The comma cone* (3.4.3) *has a weak universal property in the homotopy 2-category given by three operations:*

(i) ***1-cell induction***: *Given a 2-cell over* $C \xrightarrow{g} A \xleftarrow{f} B$ *as below-left*

$$\begin{tikzcd}
 & X \arrow[dl, "c"'] \arrow[dr, "b"] \arrow[dd, phantom, "\overset{\alpha}{\Leftarrow}"] & \\
C \arrow[dr, "g"'] & & B \arrow[dl, "f"] \\
 & A &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & X \arrow[d, dashed, "a"] \arrow[ddl, bend right, "c"'] \arrow[ddr, bend left, "b"] & \\
 & \mathsf{Hom}_A(f,g) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[dd, phantom, "\overset{\phi}{\Leftarrow}"] & \\
C \arrow[dr, "g"'] & & B \arrow[dl, "f"] \\
 & A &
\end{tikzcd}$$

there exists a 1-cell $a\colon X \to \mathsf{Hom}_A(f,g)$ *so that* $b = p_0 a$, $c = p_1 a$, *and* $\alpha = \phi a$.

(ii) ***2-cell induction***: *Given a pair of functors* $a, a'\colon X \rightrightarrows \mathsf{Hom}_A(f,g)$ *and a pair of 2-cells* τ_0 *and* τ_1 *so that*

$$\begin{tikzcd}[column sep=small]
 & X \arrow[dl, "a'"'] \arrow[dr, "a"] \arrow[dd, phantom, "\overset{\tau_1}{\Leftarrow}"] & & \\
\mathsf{Hom}_A(f,g) \arrow[dr, "p_1"'] & & \mathsf{Hom}_A(f,g) \arrow[dl, "p_1"] \arrow[dr, "p_0"] \arrow[dd, phantom, "\overset{\phi}{\Leftarrow}"] & \\
 & C \arrow[dr, "g"'] & & B \arrow[dl, "f"] \\
 & & A &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}[column sep=small]
 & & X \arrow[dl, "a'"'] \arrow[dr, "a"] \arrow[d, phantom, "\overset{\tau_0}{\Leftarrow}"] & \\
 & \mathsf{Hom}_A(f,g) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[dd, phantom, "\overset{\phi}{\Leftarrow}"] & & \mathsf{Hom}_A(f,g) \arrow[dl, "p_0"] \\
C \arrow[dr, "g"'] & & B \arrow[dl, "f"] & \\
 & A & &
\end{tikzcd}$$

there exists a 2-cell $\tau\colon a \Rightarrow a'$ *so that*

$$\begin{tikzcd}[column sep=small]
 & X \arrow[dl, "a'"'] \arrow[dr, "a"] \arrow[d, phantom, "\overset{\tau_1}{\Leftarrow}"] & \\
\mathsf{Hom}_A(f,g) \arrow[dr, "p_1"'] & {} & \mathsf{Hom}_A(f,g) \arrow[dl, "p_1"] \\
 & C &
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[d, bend right, "a'"'] \arrow[d, bend left, "a"] \arrow[d, phantom, "\overset{\tau}{\Leftarrow}"] \\
\mathsf{Hom}_A(f,g) \arrow[d, "p_1"'] \\
C
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
X \arrow[d, bend right, "a'"'] \arrow[d, bend left, "a"] \arrow[d, phantom, "\overset{\tau}{\Leftarrow}"] \\
\mathsf{Hom}_A(f,g) \arrow[d, "p_0"] \\
B
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
 & X \arrow[dl, "a'"'] \arrow[dr, "a"] \arrow[d, phantom, "\overset{\tau_0}{\Leftarrow}"] & \\
\mathsf{Hom}_A(f,g) \arrow[dr, "p_0"'] & {} & \mathsf{Hom}_A(f,g) \arrow[dl, "p_0"] \\
 & B &
\end{tikzcd}$$

(iii) ***2-cell conservativity***: *Any 2-cell*

$$\begin{tikzcd}
X \arrow[d, bend right, "a'"'] \arrow[d, bend left, "a"] \arrow[d, phantom, "\overset{\tau}{\Leftarrow}"] \\
\mathsf{Hom}_A(f,g)
\end{tikzcd}$$

with the property that both $p_1\tau$ *and* $p_0\tau$ *are isomorphisms is an isomorphism.*

PROOF. The cosmological functor $\mathsf{Fun}(X,-)\colon \mathcal{K} \to \mathcal{QC}at$ carries the pullback (3.4.2) to a pullback

$$\begin{tikzcd}
\mathsf{Fun}(X,\mathsf{Hom}_A(f,g)) \cong \mathsf{Hom}_{\mathsf{Fun}(X,A)}(\mathsf{Fun}(X,f),\mathsf{Fun}(X,g)) \arrow[r, "\phi"] \arrow[d, two heads, "(p_1,p_0)"'] \arrow[dr, phantom, "\lrcorner" very near start] & \mathsf{Fun}(X,A)^{\mathbb{2}} \arrow[d, two heads, "(p_1,p_0)"] \\
\mathsf{Fun}(X,C)\times\mathsf{Fun}(X,B) \arrow[r, "\mathsf{Fun}(X,g)\times\mathsf{Fun}(X,f)"'] & \mathsf{Fun}(X,A)\times\mathsf{Fun}(X,A)
\end{tikzcd}$$

of quasi-categories. Now Lemma 3.1.7 demonstrates that the canonical 2-cell (3.4.3) induces a natural map of hom-categories

$$\begin{tikzcd}
\mathsf{hFun}(X,\mathsf{Hom}_A(f,g)) \arrow[rr] \arrow[dr, "((p_1)_*,(p_0)_*)"'] & & \mathsf{Hom}_{\mathsf{hFun}(X,A)}(\mathsf{hFun}(X,f),\mathsf{hFun}(X,g)) \arrow[dl, "(\mathrm{ev}_1,\mathrm{ev}_0)"] \\
& \mathsf{hFun}(X,C)\times\mathsf{hFun}(X,B) &
\end{tikzcd}$$

over $\mathsf{hFun}(X,C\times B)\cong\mathsf{hFun}(X,C)\times\mathsf{hFun}(X,B)$ that is a smothering functor. The properties of 1-cell induction, 2-cell induction, and 2-cell conservativity follow from surjectivity on objects, fullness, and conservativity of this smothering functor respectively. ☐

The 1-cells $X \to \mathsf{Hom}_A(f,g)$ that are induced by a fixed 2-cell $\alpha\colon fb \Rightarrow gc$ are unique up to fibered isomorphism over $C\times B$.

3.4.7. PROPOSITION. *Whiskering with the comma cone* (3.4.3) *induces a bijection between 2-cells as displayed below-left*

$$\left\{\begin{tikzcd}
& X \arrow[dl, "c"'] \arrow[dr, "b"] & \\
C \arrow[dr, "g"'] & \overset{\alpha}{\Leftarrow} & B \arrow[dl, "f"] \\
& A &
\end{tikzcd}\right\}
\quad\leftrightsquigarrow\quad
\left\{\begin{tikzcd}
& X \arrow[dl, "c"'] \arrow[dr, "b"] \arrow[d, "a"] & \\
C & \mathsf{Hom}_A(f,g) \arrow[l, two heads, "p_1"] \arrow[r, two heads, "p_0"'] & B
\end{tikzcd}\right\}_{/\cong}$$

and fibered isomorphism classes of maps of spans from C to B as displayed above-right, where the fibered isomorphisms are given by invertible 2-cells

$$\begin{tikzcd}
& X \arrow[dl, "c"'] \arrow[dr, "b"] \arrow[d, bend right, "a'"'] \arrow[d, bend left, "a"] \arrow[d, phantom, "\overset{\gamma}{\cong}"] & \\
C & \mathsf{Hom}_A(f,g) \arrow[l, two heads, "p_1"] \arrow[r, two heads, "p_0"'] & B
\end{tikzcd}$$

so that $p_0\gamma = \mathrm{id}_b$ and $p_1\gamma = \mathrm{id}_c$.

PROOF. Lemma 3.1.3 proves that the fibers of the smothering functor of Proposition 3.4.6 are connected groupoids. The objects of these fibers are functors $X \to \mathsf{Hom}_A(f,g)$ and the morphisms are invertible 2-cells that whisker with

$$(p_1,p_0)\colon \mathsf{Hom}_A(f,g) \twoheadrightarrow C\times B$$

to an identity 2-cell. The action of the smothering functor defines a bijection between the objects of its codomain and their corresponding fibers. ☐

The construction of the comma ∞-category is also pseudo-functorial in lax maps defined in the homotopy 2-category:

3.4.8. Observation. By 1-cell induction a diagram

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "r"'] \arrow[dr, phantom, "\swArrow\gamma"] & A \arrow[d, "p"] \arrow[dr, phantom, "\swArrow\beta"] & B \arrow[l, "f"'] \arrow[d, "q"] \\ \bar{C} \arrow[r, "\bar{g}"'] & \bar{A} & \bar{B} \arrow[l, "\bar{f}"] \end{tikzcd}$$

induces a map between comma ∞-categories as displayed below-right:

$$\begin{tikzcd} & \mathsf{Hom}_A(f,g) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\swArrow\phi"] & \\ C \arrow[d, "r"'] \arrow[dr, "g"] & {} & B \arrow[d, "q"] \arrow[dl, "f"'] \\ \bar{C} \arrow[dr, "\bar{g}"'] \arrow[r, phantom, "\swArrow\gamma"] & A \arrow[d, "p"] \arrow[r, phantom, "\Leftarrow\beta"] & \bar{B} \arrow[dl, "\bar{f}"] \\ & \bar{A} & \end{tikzcd} = \begin{tikzcd} & \mathsf{Hom}_A(f,g) \arrow[d, dashed, "\beta\downarrow\gamma"] \arrow[ddl, bend right, two heads, "rp_1"'] \arrow[ddr, bend left, two heads, "qp_0"] & \\ & \mathsf{Hom}_{\bar{A}}(\bar{f},\bar{g}) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] & \\ \bar{C} \arrow[dr, "\bar{g}"'] & \overset{\phi}{\Leftarrow} & \bar{B} \arrow[dl, "\bar{f}"] \\ & \bar{A} & \end{tikzcd}$$

that is well-defined and functorial up to fibered isomorphism.

One of many uses of comma ∞-categories is to define the internal mapping spaces between two elements of an ∞-category A. This is one motivation for our notation "Hom_A."

3.4.9. Definition. For any two elements $x, y \colon 1 \rightrightarrows A$ of an ∞-category A, their **mapping space** is the comma ∞-category $\mathsf{Hom}_A(x,y)$ defined by the pullback diagram

$$\begin{tikzcd} \mathsf{Hom}_A(x,y) \arrow[r, "\phi"] \arrow[d, two heads, "{(p_1,p_0)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & A^2 \arrow[d, two heads, "{(p_1,p_0)}"] \\ 1 \arrow[r, "{(y,x)}"'] & A \times A \end{tikzcd}$$

The mapping spaces in any ∞-category are discrete in the sense of Definition 1.2.26.

3.4.10. Proposition (internal mapping spaces are discrete). *For any pair of elements $x, y \colon 1 \rightrightarrows A$ of an ∞-category A, the mapping space $\mathsf{Hom}_A(x,y)$ is discrete.*

Proof. Our task is to prove that for any ∞-category X, the functor space $\mathsf{Fun}(X, \mathsf{Hom}_A(x,y))$ is a Kan complex. This is so just when $\mathsf{hFun}(X, \mathsf{Hom}_A(x,y))$ is a groupoid, i.e., when any 2-cell with codomain $\mathsf{Hom}_A(x,y)$ is invertible. By 2-cell conservativity, a 2-cell with codomain $\mathsf{Hom}_A(x,y)$ is invertible just when its whiskered composite with the isofibration $(p_1,p_0) \colon \mathsf{Hom}_A(x,y) \twoheadrightarrow 1 \times 1$ is an invertible 2-cell, but in fact this whiskered composite is an identity since 1 is terminal. ☐

As in our convention for ∞-categories of arrows, it will be convenient to weaken the meaning of "comma ∞-category" to extend this appellation to any object of $\mathcal{K}_{/C\times B}$ that is fibered equivalent (see Definition 3.2.7) to the strict model $(p_1, p_0)\colon \mathsf{Hom}_A(f,g) \twoheadrightarrow C \times B$ defined by 3.4.1. This is justified because such objects satisfy the weak universal property of Proposition 3.4.6 and conversely any two objects satisfying this weak universal property are equivalent over $C \times B$.

3.4.11. Proposition (uniqueness of comma ∞-categories). *For any isofibration $(e_1, e_0)\colon E \twoheadrightarrow C \times B$ that is fibered equivalent to $\mathsf{Hom}_A(f,g) \twoheadrightarrow C \times B$ the 2-cell*

$$\begin{tikzcd} & E \arrow[dl, "e_1"'] \arrow[dr, "e_0"] & \\ C \arrow[dr, "g"'] \arrow[rr, phantom, "\overset{\epsilon}{\Leftarrow}"] & & B \arrow[dl, "f"] \\ & A & \end{tikzcd}$$

encoded by the equivalence $E \xrightarrow{\sim} \mathsf{Hom}_A(f,g)$ satisfies the weak universal property of Proposition 3.4.6. Conversely, if $(d_1, d_0)\colon D \twoheadrightarrow C \times B$ and $(e_1, e_0)\colon E \twoheadrightarrow C \times B$ are equipped with 2-cells

$$\begin{tikzcd} & D \arrow[dl, "d_1"'] \arrow[dr, "d_0"] & \\ C \arrow[dr, "g"'] \arrow[rr, phantom, "\overset{\delta}{\Leftarrow}"] & & B \arrow[dl, "f"] \\ & A & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & E \arrow[dl, "e_1"'] \arrow[dr, "e_0"] & \\ C \arrow[dr, "g"'] \arrow[rr, phantom, "\overset{\epsilon}{\Leftarrow}"] & & B \arrow[dl, "f"] \\ & A & \end{tikzcd} \tag{3.4.12}$$

satisfying the weak universal property of Proposition 3.4.6, then D and E are fibered equivalent over $C \times B$.

Proof. The proof of the first statement proceeds exactly as in the special case of Proposition 3.2.10. We prove the converse, solving Exercise 3.2.ii.

Consider a pair of 2-cells (3.4.12) satisfying the weak universal properties enumerated in Proposition 3.4.6. 1-cell induction supplies maps of spans

$$\begin{tikzcd} & D \arrow[dl, "d_1"'] \arrow[dr, "d_0"] & \\ C \arrow[dr, "g"'] \arrow[rr, phantom, "\overset{\delta}{\Leftarrow}"] & & B \arrow[dl, "f"] \\ & A & \end{tikzcd} = \begin{tikzcd} & D \arrow[d, "d"] \arrow[ddl, bend right, "d_1"'] \arrow[ddr, bend left, "d_0"] & \\ & E \arrow[dl, "e_1"'] \arrow[dr, "e_0"] & \\ C \arrow[dr, "g"'] \arrow[rr, phantom, "\overset{\epsilon}{\Leftarrow}"] & & B \arrow[dl, "f"] \\ & A & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & E \arrow[dl, "e_1"'] \arrow[dr, "e_0"] & \\ C \arrow[dr, "g"'] \arrow[rr, phantom, "\overset{\epsilon}{\Leftarrow}"] & & B \arrow[dl, "f"] \\ & A & \end{tikzcd} = \begin{tikzcd} & E \arrow[d, "e"] \arrow[ddl, bend right, "e_1"'] \arrow[ddr, bend left, "e_0"] & \\ & D \arrow[dl, "d_1"'] \arrow[dr, "d_0"] & \\ C \arrow[dr, "g"'] \arrow[rr, phantom, "\overset{\delta}{\Leftarrow}"] & & B \arrow[dl, "f"] \\ & A & \end{tikzcd}$$

with the property that $\epsilon de = \epsilon$ and $\delta ed = \delta$. By Proposition 3.4.7 it follows that $de \cong \mathrm{id}_E$ over $C \times B$ and $ed \cong \mathrm{id}_D$ over $C \times B$. This defines the data of a fibered equivalence $D \simeq E$.[3] $\square$

3.4.13. Convention. On account of Proposition 3.4.11, we extend the appellation "comma ∞-category" from the strict model constructed in Definition 3.4.1 to any ∞-category that is fibered equivalent to it and refer to its accompanying 2-cell as the "comma cone."

[3]For the reader uncomfortable with Remark 3.2.9, Proposition 3.6.3 and Lemma 3.6.4 provides a small boost to finish the proof.

For example, in §4.3 we define the ∞-category of cones over a fixed diagram as a comma ∞-category. Proposition 3.4.11 gives us the flexibility to use multiple models for this ∞-category, which will be useful in characterizing the universal properties of limits and colimits.

Exercises.

3.4.i. EXERCISE. Prove Proposition 3.4.5 by observing that the map $\mathsf{Hom}_p(q,r)$ factors as a pullback of the Leibniz cotensor of $\partial\Delta[1] \hookrightarrow \Delta[1]$ with p followed by a pullback of $r \times q$.

3.4.ii. EXERCISE. Use Proposition 3.4.7 to justify the pseudofunctoriality of the comma construction in lax morphisms described in Observation 3.4.8.

3.5. Representable comma ∞-categories

Definition 3.4.1 constructs a comma ∞-category for any cospan. Of particular importance, are the special cases of this construction where one of the legs of the cospan is an identity:

3.5.1. DEFINITION (left and right representations). Any functor $f\colon A \to B$ admits a **left representation** and a **right representation** as a comma ∞-category, displayed below-left and below-right respectively:

$$\begin{tikzcd} & \mathsf{Hom}_B(f,B) \arrow[dl, two heads, "p_1"'] \arrow[dr, two heads, "p_0"] & \\ B \arrow[dr, equal] & \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & A \arrow[dl, "f"] \\ & B & \end{tikzcd} \qquad \begin{tikzcd} & \mathsf{Hom}_B(B,f) \arrow[dl, two heads, "p_1"'] \arrow[dr, two heads, "p_0"] & \\ A \arrow[dr, "f"'] & \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & B \arrow[dl, equal] \\ & B & \end{tikzcd}$$

To save space, we typically depict the **left comma cone over** f displayed above-left and the **right comma cone over** f displayed above-right as inhabiting triangles rather than squares.

By Proposition 3.4.11, the weak universal property of the comma cone characterizes the comma span up to fibered equivalence over the product of the codomain objects. Thus:

3.5.2. DEFINITION. Given a cospan $C \xrightarrow{g} A \xleftarrow{f} B$, the comma ∞-category $\mathsf{Hom}_A(f,g) \twoheadrightarrow C \times B$ is

- **left representable** if there exists a functor $\ell\colon B \to C$ so that $\mathsf{Hom}_A(f,g) \simeq \mathsf{Hom}_C(\ell, C)$ over $C \times B$ and
- **right representable** if there exists a functor $r\colon C \to B$ so that $\mathsf{Hom}_A(f,g) \simeq \mathsf{Hom}_B(B,r)$ over $C \times B$.

In this section, we prove the first of many representability theorems: demonstrating that a functor $g\colon C \to A$ admits an absolute right lifting along $f\colon B \to A$ if and only if the comma ∞-category $\mathsf{Hom}_A(f,g)$ is right representable, the representing functor then defining the postulated lifting. We prove this over the course of three theorems, each strengthening the previous statement. The first theorem characterizes 2-cells

$$\begin{tikzcd} & B \arrow[d, "f"] \\ C \arrow[ur, "r"] \arrow[r, "g"'] \arrow[ur, phantom, shift right=3, "\Downarrow\rho"] & A \end{tikzcd}$$

that define absolute right lifting diagrams via an induced equivalence $\mathsf{Hom}_B(B, r) \simeq_{C\times B} \mathsf{Hom}_A(f, g)$ between comma ∞-categories. The second theorem proves that a functor r defines an absolute right lifting of g through f just when $\mathsf{Hom}_A(f, g)$ is right-represented by r; the difference is that no 2-cell $\rho\colon fr \Rightarrow g$ need be postulated a priori to exist. The final theorem gives a general right-representability criterion that can be applied to construct a right representation to $\mathsf{Hom}_A(f, g)$ without a priori specifying the representing functor r.

3.5.3. Theorem. *The triangle below-left defines an absolute right lifting diagram if and only if the induced 1-cell below-right*

$$
\begin{tikzcd}
 & B \arrow[d, "f"] \\
C \arrow[ur, "r"] \arrow[r, "g"'] \arrow[ur, phantom, "\Downarrow\rho", shift right=3] & A
\end{tikzcd}
\quad\rightsquigarrow\quad
\begin{tikzcd}
 & \mathsf{Hom}_B(B,r) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\
C \arrow[rr, "r"] \arrow[dr, "g"'] & {} \arrow[d, phantom, "\overset{\rho}{\Leftarrow}"] & B \arrow[dl, "f"] \\
 & A &
\end{tikzcd}
=
\begin{tikzcd}
 & \mathsf{Hom}_B(B,r) \arrow[d, dashed, "y"] \arrow[ddl, bend right, two heads, "p_1"'] \arrow[ddr, bend left, two heads, "p_0"] & \\
 & \mathsf{Hom}_A(f,g) \arrow[dl, two heads, "p_1"'] \arrow[dr, two heads, "p_0"] \arrow[d, phantom, "\underset{\Leftarrow}{\phi}"] & \\
C \arrow[dr, "g"'] & {} & B \arrow[dl, "f"] \\
 & A &
\end{tikzcd}
\tag{3.5.4}
$$

defines a fibered equivalence $\mathsf{Hom}_B(B, r) \simeq \mathsf{Hom}_A(f, g)$ *over* $C \times B$.

In [**93**], Street and Walters interpret the equivalence $\mathsf{Hom}_B(B, r) \simeq \mathsf{Hom}_A(f, g)$ encoding an absolute right lifting diagram as asserting that "f is left adjoint to r relative to g." This notion of relative adjunction, first studied by Ulmer [**96**], should be compared with the definition of adjunction given in Proposition 4.1.1.

Proof. Suppose that (r, ρ) defines an absolute right lifting of g through f and consider the corresponding unique factorization of the comma cone under $\mathsf{Hom}_A(f, g)$ through ρ as displayed below-center

$$
\begin{tikzcd}
 & \mathsf{Hom}_A(f,g) \arrow[dl, two heads, "p_1"'] \arrow[dr, two heads, "p_0"] \arrow[d, phantom, "\underset{\Leftarrow}{\phi}"] & \\
C \arrow[dr, "g"'] & {} & B \arrow[dl, "f"] \\
 & A &
\end{tikzcd}
=
\begin{tikzcd}
 & \mathsf{Hom}_A(f,g) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\overset{\zeta}{\Leftarrow}"] & \\
C \arrow[rr, "r"] \arrow[dr, "g"'] & {} \arrow[d, phantom, "\overset{\rho}{\Leftarrow}"] & B \arrow[dl, "f"] \\
 & A &
\end{tikzcd}
=
\begin{tikzcd}
 & \mathsf{Hom}_A(f,g) \arrow[d, dashed, "z"] \arrow[ddl, bend right, two heads, "p_1"'] \arrow[ddr, bend left, two heads, "p_0"] & \\
 & \mathsf{Hom}_B(B,r) \arrow[dl, two heads, "p_1"'] \arrow[dr, two heads, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\
C \arrow[rr, "r"] \arrow[dr, "g"'] & {} \arrow[d, phantom, "\overset{\rho}{\Leftarrow}"] & B \arrow[dl, "f"] \\
 & A &
\end{tikzcd}
\tag{3.5.5}
$$

By 1-cell induction, the 2-cell ζ factors through the right comma cone over r as displayed above-right. Substituting the right-hand side of (3.5.4) into the bottom portion of the above-right diagram, we see that $yz\colon \mathsf{Hom}_A(f, g) \to \mathsf{Hom}_A(f, g)$ is a 1-cell that factors the comma cone for $\mathsf{Hom}_A(f, g)$ through itself. Applying the universal property of Proposition 3.4.7, it follows that there is a fibered isomorphism $yz \cong \mathrm{id}_{\mathsf{Hom}_A(f,g)}$ over $C \times B$.

To prove that $zy \cong \mathrm{id}_{\mathsf{Hom}_B(B,r)}$ it suffices to argue similarly that the right comma cone over r restricts along zy to itself. Since ρ is absolute right lifting, it suffices to verify the equality $\phi zy = \phi$ after pasting below with ρ. But now reversing the order of the equalities in (3.5.5) and (3.5.4) we have

$$
\begin{tikzcd}
& \mathsf{Hom}_B(B,r) \arrow[d, dashed, "y"] \arrow[ddl, bend right, two heads, "p_1"'] \arrow[ddr, bend left, two heads, "p_0"] & \\
& \mathsf{Hom}_A(f,g) \arrow[d, dashed, "z"] \arrow[dl, bend right, two heads, "p_1"'] \arrow[dr, bend left, two heads, "p_0"] & \\
& \mathsf{Hom}_B(B,r) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\
C \arrow[rr, "r"] \arrow[dr, "g"'] & {} \arrow[d, phantom, "\overset{\rho}{\Leftarrow}"] & B \arrow[dl, "f"] \\
& A &
\end{tikzcd}
=
\begin{tikzcd}
& \mathsf{Hom}_B(B,r) \arrow[d, dashed, "y"] \arrow[ddl, bend right, two heads, "p_1"'] \arrow[ddr, bend left, two heads, "p_0"] & \\
& \mathsf{Hom}_A(f,g) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\overset{\zeta}{\Leftarrow}"] & \\
C \arrow[rr, "r"] \arrow[dr, "g"'] & {} \arrow[d, phantom, "\overset{\rho}{\Leftarrow}"] & B \arrow[dl, "f"] \\
& A &
\end{tikzcd}
=
\begin{tikzcd}
& \mathsf{Hom}_B(B,r) \arrow[d, dashed, "y"] \arrow[ddl, bend right, two heads, "p_1"'] \arrow[ddr, bend left, two heads, "p_0"] & \\
& \mathsf{Hom}_A(f,g) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[dd, phantom, "\overset{\phi}{\Leftarrow}"] & \\
C \arrow[dr, "g"'] & & B \arrow[dl, "f"] \\
& A &
\end{tikzcd}
=
\begin{tikzcd}
& \mathsf{Hom}_B(B,r) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\
C \arrow[rr, "r"] \arrow[dr, "g"'] & {} \arrow[d, phantom, "\overset{\rho}{\Leftarrow}"] & B \arrow[dl, "f"] \\
& A &
\end{tikzcd}
$$

which is exactly what we wanted to show. Thus, we see that if (r, ρ) is an absolute right lifting of g through f, then the induced map (3.5.4) defines a fibered equivalence $\mathsf{Hom}_B(B,r) \simeq \mathsf{Hom}_A(f,g)$.

Now, conversely, suppose the 1-cell y defined by (3.5.4) is a fibered equivalence and let us argue that (r, ρ) is an absolute right lifting of g through f. By Proposition 3.4.11, via this fibered equivalence the 2-cell displayed on the left-hand side of the equality in (3.5.4) inherits the weak universal property of a comma cone from $\mathsf{Hom}_A(f,g)$. So Proposition 3.4.7 supplies a bijection displayed below-left-center

$$
\left\{
\begin{tikzcd}
& X \arrow[dl, "c"'] \arrow[dr, "b"] \arrow[dd, phantom, "\overset{\alpha}{\Leftarrow}"] & \\
C \arrow[dr, "g"'] & & B \arrow[dl, "f"] \\
& A &
\end{tikzcd}
\right\}
\leftrightsquigarrow
\left\{
\begin{tikzcd}
& X \arrow[dl, "c"'] \arrow[dr, "b"] \arrow[dd, "a"] & \\
C & & B \\
& \mathsf{Hom}_B(B,r) \arrow[ul, two heads, "p_1"] \arrow[ur, two heads, "p_0"'] &
\end{tikzcd}
\right\}_{/\cong}
\leftrightsquigarrow
\left\{
\begin{tikzcd}
& X \arrow[dl, "c"'] \arrow[dr, "b"] \arrow[d, phantom, "\overset{\xi}{\Leftarrow}"] & \\
C \arrow[rr, "r"'] & {} & B
\end{tikzcd}
\right\}
$$

between 2-cells over the cospan and fibered isomorphism classes of maps of spans that is implemented, from center to left, by whiskering with the 2-cell $\rho p_1 \cdot f\phi \colon fp_0 \Rightarrow gp_1$ in the center of (3.5.4). Proposition 3.4.7 also applies to the right comma cone ϕ over $r \colon C \to B$ giving us a second bijection, displayed above center-right between the same fibered isomorphism classes of maps of spans and 2-cells over r. This second bijection is implemented, from center to right, by pasting with the right comma cone $\phi \colon p_0 \Rightarrow rp_1$. Combining these yields a bijection between the 2-cells displayed on the right and the 2-cells displayed on the left implemented by pasting with ρ, which is precisely the universal property that characterizes absolute right lifting diagrams. $\square$

As a special case of this result, we can now present several equivalent characterizations of fully faithful functors between ∞-categories.

3.5.6. COROLLARY. *The following are equivalent, and define what it means for a functor $f \colon A \to B$ between ∞-categories to be* ***fully faithful****:*

(i) The identity defines an absolute right lifting diagram:

$$\begin{tikzcd} & A \arrow[d, "f"] \\ A \arrow[ur, equal] \arrow[r, "f"'] & B \end{tikzcd}$$

(ii) The identity defines an absolute left lifting diagram:

$$\begin{tikzcd} & A \arrow[d, "f"] \\ A \arrow[ur, equal] \arrow[r, "f"'] & B \end{tikzcd}$$

(iii) For all $X \in \mathcal{K}$, the induced functor

$$f_* \colon \mathsf{hFun}(X, A) \to \mathsf{hFun}(X, B)$$

is a fully faithful functor of 1-categories.

(iv) The functor induced by the identity 2-cell id_f is an equivalence

$$\begin{tikzcd} & A^{\mathbb{2}} \arrow[dl, two heads, "p_1"'] \arrow[dr, two heads, "p_0"] \arrow[dd, "\ulcorner \mathrm{id}_f \urcorner", "\wr"'] & \\ A & & A \\ & \mathsf{Hom}_B(f, f) \arrow[ul, two heads, "p_1"] \arrow[ur, two heads, "p_0"'] & \end{tikzcd}$$

Proof. The statement (iii) is an unpacking of the meaning of both (i) and (ii). Theorem 3.5.3 specializes to prove (i)$\Leftrightarrow$(iv) or dually (ii)$\Leftrightarrow$(iv). $\square$

It is not surprising that post-composition with a fully faithful functor of ∞-categories should induce a fully-faithful functor of hom-categories in the homotopy 2-category. What is surprising is that this definition is strong enough. This result, together with the general case of Theorem 3.5.3 should be provide some retroactive justification for our use of absolute lifting diagrams in Chapter 2.

Having proven Theorem 3.5.3 our immediate aim is to strengthen it to show that a fibered equivalence $\mathsf{Hom}_B(B, r) \simeq \mathsf{Hom}_A(f, g)$ over $C \times B$ implies that $r \colon C \to B$ defines an absolute right lifting of g through f without a previously specified 2-cell $\rho \colon fr \Rightarrow g$.

3.5.7. Theorem. *Given a trio of functors $r \colon C \to B$, $f \colon B \to A$, and $g \colon C \to A$ there is a bijection between 2-cells as displayed below-left and fibered isomorphism classes of maps of spans as displayed below-right*

$$\left\{ \begin{tikzcd} & B \arrow[d, "f"] \\ C \arrow[ur, "r"] \arrow[r, "g"'] \arrow[ur, phantom, "\Downarrow\rho", shift right=3] & A \end{tikzcd} \right\} \quad \leftrightsquigarrow \quad \left\{ \begin{tikzcd} & \mathsf{Hom}_B(B, r) \arrow[dl, two heads, "p_1"'] \arrow[dr, two heads, "p_0"] \arrow[dd, "y"] & \\ C & & B \\ & \mathsf{Hom}_A(f, g) \arrow[ul, two heads, "p_1"] \arrow[ur, two heads, "p_0"'] & \end{tikzcd} \right\}_{/\cong}$$

that is constructed by pasting the right comma cone over r and then applying 1-cell induction to factor through the comma cone for $\mathsf{Hom}_A(f,g)$.

$$
\begin{tikzcd}
& \mathsf{Hom}_B(B,r) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\
C \arrow[rr, "r"] \arrow[dr, "g"'] & \arrow[d, phantom, "\overset{\rho}{\Leftarrow}"] & B \arrow[dl, "f"] \\
& A &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
& \mathsf{Hom}_B(B,r) \arrow[d, dashed, "y"] \arrow[ddl, bend right, "p_1"'] \arrow[ddr, bend left, "p_0"] & \\
& \mathsf{Hom}_A(f,g) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] & \\
C \arrow[dr, "g"'] \arrow[rr, phantom, "\overset{\phi}{\Leftarrow}"] & & B \arrow[dl, "f"] \\
& A &
\end{tikzcd}
$$

Moreover, a 2-cell $\rho\colon fr \Rightarrow g$ displays r as an absolute right lifting of g through f if and only if the corresponding map of spans $y\colon \mathsf{Hom}_B(B,r) \to \mathsf{Hom}_A(f,g)$ is an equivalence.

The second clause is the statement of Theorem 3.5.3, so it remains only to prove the first. We show the claimed construction is a bijection by exhibiting its inverse, the construction of which involves a rather mysterious lemma the significance of which will gradually reveal itself. For instance, Lemma 3.5.8 figures prominently in the proof of the external Yoneda lemma in §5.5 and is also the main ingredient in a "cheap" version of the Yoneda lemma appearing as Corollary 3.5.10.

3.5.8. Lemma. *Let $f\colon A \to B$ be any functor and denote the right comma cone over f by*

$$
\begin{tikzcd}
& \mathsf{Hom}_B(B,f) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\
A \arrow[rr, "f"'] & & B
\end{tikzcd}
$$

Then the codomain-projection functor $p_1\colon \mathsf{Hom}_B(B,f) \twoheadrightarrow A$ admits a right adjoint right inverse[4] induced from the identity 2-cell id_f, defining an adjunction

$$
\begin{tikzcd}
A \arrow[rr, dashed, bend right, "i"'] \arrow[dr, equal] \arrow[rr, phantom, "\perp"] & & \mathsf{Hom}_B(B,f) \arrow[ll, bend right, "p_1"'] \arrow[dl, "p_1"] \\
& A &
\end{tikzcd}
$$

over A whose counit is an identity and whose unit $\eta\colon \mathrm{id} \Rightarrow ip_1$ satisfies the conditions $\eta i = \mathrm{id}_i$, $p_1\eta = \mathrm{id}_{p_1}$ and $p_0\eta = \phi$.

[4]A functor admits a **right adjoint right inverse** just when it admits a right adjoint in an adjunction whose counit is the identity. When the original functor is an isofibration, as is the case here, it suffices to merely assume that the counit is invertible; see Lemma 3.6.10 and Appendix B.

Proof. This adjunction will be constructed using the weak universal properties of the right comma cone over f. The identity 2-cell id_f induces a 1-cell over the right comma cone over f:

$$\begin{tikzcd}
& A \arrow[d, dashed, "i"] \arrow[ddl, bend right, equal] \arrow[ddr, bend left, "f"] & \\
& \mathrm{Hom}_B(B,f) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\
A \arrow[rr, "f"'] & {} & B
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
& A \arrow[dl, equal] \arrow[dr, "f"] \arrow[d, phantom, "="] & \\
A \arrow[rr, "f"'] & {} & B
\end{tikzcd}$$

Note that $p_1 i = \mathrm{id}_A$, so we may take the counit to be the identity 2-cell. Since $\phi i = \mathrm{id}_f$, we have a pasting equality:

$$\begin{tikzcd}
& \mathrm{Hom}_B(B,f) \arrow[dl, "ip_1"'] \arrow[dr, equal] & & \\
\mathrm{Hom}_B(B,f) \arrow[dr, "p_1"'] & = & \mathrm{Hom}_B(B,f) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\
& A \arrow[rr, "f"'] & {} & B
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
& & \mathrm{Hom}_B(B,f) \arrow[dl, "p_1"'] \arrow[ddd, bend left, "p_0"] \\
& A \arrow[d, "i"] \arrow[ddl, bend right, equal] \arrow[ddr, dotted, "f"] \arrow[r, phantom, "\overset{\phi}{\Leftarrow}"] & {} \\
& \mathrm{Hom}_B(B,f) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\
A \arrow[rr, "f"'] & {} & B
\end{tikzcd}$$

which allows us to induce a 2-cell $\eta\colon \mathrm{id} \Rightarrow ip_1$ with defining equations $p_1\eta = \mathrm{id}_{p_1}$ and $p_0\eta = \phi$. The first of these conditions ensures one triangle identity; for the other, we must verify that $\eta i = \mathrm{id}_i$. By 2-cell conservativity, ηi is an isomorphism since $p_1\eta i = \mathrm{id}_A$ and $p_0\eta i = \phi i = \mathrm{id}_f$ are both invertible. By naturality of whiskering, we have

$$\begin{tikzcd}
i \arrow[r, Rightarrow, "\eta i"] \arrow[d, Rightarrow, "\eta i"'] & i \arrow[d, Rightarrow, "\eta i"] \\
i \arrow[r, Rightarrow, "ip_1\eta i"'] & i
\end{tikzcd}$$

and since $p_1\eta = \mathrm{id}_{p_1}$ the bottom edge is an identity. So $\eta i \cdot \eta i = \eta i$ and since ηi is an isomorphism cancelation implies that $\eta i = \mathrm{id}_i$ as required. ☐

One interpretation of Lemma 3.5.8 is best revealed though a special case:

3.5.9. Corollary. *For any element $b\colon 1 \to B$, the identity at b defines a terminal element in $\mathsf{Hom}_B(B,b)$.*

Proof. By Lemma 3.5.8, the codomain-projection from the right representation of any functor admits a right adjoint right inverse induced from its identity 2-cell. In this case, the codomain-projection is the unique functor $!\colon \mathsf{Hom}_B(B,b) \to 1$, so by Definition 2.2.1, this right adjoint identifies a terminal element of $\mathsf{Hom}_B(B,b)$ corresponding to the identity morphism id_b in the homotopy category $\mathsf{h}B$. ☐

The general version of Lemma 3.5.8 has a similar interpretation: in the sliced ∞-cosmos $\mathcal{K}_{/A}$, the identity functor at A defines the terminal object, and Lemma 3.5.8 asserts that id_f induces a terminal element of $\mathsf{Hom}_B(B,f)$ "over A."

Proof of Theorem 3.5.7. The inverse to the function that takes a 2-cell $fr \Rightarrow g$ and produces an isomorphism class of maps $\mathsf{Hom}_B(B,r) \to \mathsf{Hom}_A(f,g)$ over $C \times B$ is constructed by applying Lemma 3.5.8 to the functor $r\colon C \to B$: given a map of spans, restrict along the right adjoint $i\colon C \to \mathsf{Hom}_B(B,r)$ and paste with the comma cone for $\mathsf{Hom}_A(f,g)$ to define a 2-cell $fr \Rightarrow g$.

Starting from a 2-cell $\rho\colon fr \Rightarrow g$, the composite of these two functions constructs the 2-cell displayed below-left

$$
\begin{tikzcd}
& C \arrow[d, dashed, "i"] \arrow[dddl, equal, bend right] \arrow[dddr, bend left, "r"] & \\
& \mathsf{Hom}_B(B,r) \arrow[d, dashed, "y"] \arrow[ddl, two heads, bend right, "p_1"'] \arrow[ddr, two heads, bend left, "p_0"] & \\
& \mathsf{Hom}_A(f,g) \arrow[dl, two heads, "p_1"'] \arrow[dr, two heads, "p_0"] \arrow[dd, phantom, "\overset{\phi}{\Leftarrow}"] & \\
C \arrow[dr, "g"'] & & B \arrow[dl, "f"] \\
& A &
\end{tikzcd}
=
\begin{tikzcd}
& C \arrow[d, dashed, "i"] \arrow[ddl, equal, bend right] \arrow[ddr, bend left, "r"] & \\
& \mathsf{Hom}_B(B,r) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\
C \arrow[rr, "r"] \arrow[dr, "g"'] & \arrow[d, phantom, "\overset{\rho}{\Leftarrow}"] & B \arrow[dl, "f"] \\
& A &
\end{tikzcd}
=
\begin{tikzcd}
& C \arrow[dl, equal, bend right] \arrow[dr, bend left, "r"] \arrow[d, phantom, "="] & \\
C \arrow[rr, "r"] \arrow[dr, "g"'] & \arrow[d, phantom, "\overset{\rho}{\Leftarrow}"] & B \arrow[dl, "f"] \\
& A &
\end{tikzcd}
$$

which equals the above-center pasted composite by the definition of y from ρ, and equals the above-right composite since $\phi i = \mathrm{id}_r$. Thus, when a 2-cell $\rho\colon fr \Rightarrow g$ is encoded as a map $y\colon \mathsf{Hom}_B(B,r) \to \mathsf{Hom}_A(f,g)$ over $C \times B$, and then re-converted into a 2-cell, the original 2-cell ρ is recovered.

For the converse, starting with a map $z\colon \mathsf{Hom}_B(B,r) \to \mathsf{Hom}_A(f,g)$ over $C \times B$, the composite of these two functions constructs an isomorphism class of maps of spans w displayed below-left by applying 1-cell induction for the comma cone $\mathsf{Hom}_A(f,g)$ to the composite 2-cell pasted below-center-left:

$$
\begin{tikzcd}
& \mathsf{Hom}_B(B,r) \arrow[d, dashed, "w"] \arrow[ddl, two heads, bend right, "p_1"'] \arrow[ddr, two heads, bend left, "p_0"] & \\
& \mathsf{Hom}_A(f,g) \arrow[dl, two heads, "p_1"'] \arrow[dr, two heads, "p_0"] \arrow[dd, phantom, "\overset{\phi}{\Leftarrow}"] & \\
C \arrow[dr, "g"'] & & B \arrow[dl, "f"] \\
& A &
\end{tikzcd}
=
\begin{tikzcd}
& \mathsf{Hom}_B(B,r) \arrow[d, two heads, "p_1"'] \arrow[ddddr, two heads, bend left, "p_0"] \arrow[dr, phantom, "\overset{\phi}{\Leftarrow}"] & \\
& C \arrow[d, "i"] \arrow[dddl, equal, bend right] \arrow[dddr, bend left, "r"] & \\
& \mathsf{Hom}_B(B,r) \arrow[d, "z"] \arrow[ddl, two heads, bend right, "p_1"'] \arrow[ddr, two heads, bend left, "p_0"] & \\
& \mathsf{Hom}_A(f,g) \arrow[dl, two heads, "p_1"'] \arrow[dr, two heads, "p_0"] \arrow[dd, phantom, "\overset{\phi}{\Leftarrow}"] & \\
C \arrow[dr, "g"'] & & B \arrow[dl, "f"] \\
& A &
\end{tikzcd}
=
\begin{tikzcd}
& \mathsf{Hom}_B(B,r) \arrow[d, two heads, "p_1"'] \arrow[dd, equal, bend left=60] & \\
& C \arrow[d, "i"] \arrow[dddl, equal, bend right] \arrow[r, phantom, "\overset{\eta}{\Leftarrow}"] & {} \\
& \mathsf{Hom}_B(B,r) \arrow[d, "z"] \arrow[ddl, two heads, bend right, "p_1"'] \arrow[ddr, two heads, bend left, "p_0"] & \\
& \mathsf{Hom}_A(f,g) \arrow[dl, two heads, "p_1"'] \arrow[dr, two heads, "p_0"] \arrow[dd, phantom, "\overset{\phi}{\Leftarrow}"] & \\
C \arrow[dr, "g"'] & & B \arrow[dl, "f"] \\
& A &
\end{tikzcd}
=
\begin{tikzcd}
& \mathsf{Hom}_B(B,r) \arrow[d, "z"] \arrow[ddl, two heads, bend right, "p_1"'] \arrow[ddr, two heads, bend left, "p_0"] & \\
& \mathsf{Hom}_A(f,g) \arrow[dl, two heads, "p_1"'] \arrow[dr, two heads, "p_0"] \arrow[dd, phantom, "\overset{\phi}{\Leftarrow}"] & \\
C \arrow[dr, "g"'] & & B \arrow[dl, "f"] \\
& A &
\end{tikzcd}
$$

Applying Lemma 3.5.8, there exists a 2-cell $\eta\colon \mathrm{id} \Rightarrow ip_1$ so that $p_0\eta = \phi$ — this gives the pasting equality above center — and $p_1\eta = \mathrm{id}$ — which gives the pasting equality above right. Proposition 3.4.7 now implies that $w \cong z$ over $C \times B$. ☐

A dual version of Theorem 3.5.7 represents 2-cells $g \Rightarrow f\ell$ as fibered isomorphism classes of maps $\mathsf{Hom}_B(\ell, B) \to \mathsf{Hom}_A(g, f)$ over $B \times C$. Specializing these results to the case where one of f or g is the identity, we immediately recover a "cheap" form of the Yoneda lemma:

3.5.10. Corollary. *Given a parallel pair of functors, $f, g\colon A \rightrightarrows B$, there are bijections between 2-cells as displayed below-center and fibered isomorphism classes of maps between their left and right representations as comma ∞-categories, as displayed below-left and below-right, respectively:*

$$\left\{\begin{tikzcd} & \mathsf{Hom}_B(g,B) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[dd, "a"] & \\ B & & A \\ & \mathsf{Hom}_B(f,B) \arrow[ul, "p_1"] \arrow[ur, "p_0"'] & \end{tikzcd}\right\}_{/\cong} \quad \leftrightsquigarrow \quad \left\{\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd}\right\} \quad \leftrightsquigarrow \quad \left\{\begin{tikzcd} & \mathsf{Hom}_B(B,f) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[dd, "a"] & \\ A & & B \\ & \mathsf{Hom}_B(B,g) \arrow[ul, "p_1"] \arrow[ur, "p_0"'] & \end{tikzcd}\right\}_{/\cong}$$

that are constructed by pasting with the left comma cone over g and right comma cone over f, respectively:

$$\begin{tikzcd} & \mathsf{Hom}_B(g,B) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\ B & {} & A \arrow[ll, "g"'] \arrow[ll, bend left, "f"] \arrow[ll, phantom, bend left=20, "\Uparrow\alpha"] \end{tikzcd} = \begin{tikzcd} & \mathsf{Hom}_B(g,B) \arrow[d, dashed, "a"] \arrow[ddl, bend right, "p_1"'] \arrow[ddr, bend left, "p_0"] & \\ & \mathsf{Hom}_B(f,B) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\ B & {} & A \arrow[ll, "f"] \end{tikzcd} \qquad \begin{tikzcd} & \mathsf{Hom}_B(B,f) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\ A \arrow[rr, "f"] \arrow[rr, bend right, "g"'] \arrow[rr, phantom, bend right=20, "\Downarrow\alpha"] & {} & B \end{tikzcd} = \begin{tikzcd} & \mathsf{Hom}_B(B,f) \arrow[d, dashed, "a"] \arrow[ddl, bend right, "p_1"'] \arrow[ddr, bend left, "p_0"] & \\ & \mathsf{Hom}_B(B,g) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\ A \arrow[rr, "g"'] & {} & B \end{tikzcd}$$

and then applying 1-cell induction to factor through the left comma cone over f in the former case or the right comma cone over g in the latter. ☐

Combining the results of this section, we prove one final representability theorem that allows us to recognize when a comma ∞-category is right representable in the absence of a predetermined representing functor. This result specializes to give existence theorems for adjoint functors and limits and colimits in the next chapter.

3.5.11. Theorem. *The comma ∞-category $\mathsf{Hom}_A(f, g)$ associated to a cospan $C \xrightarrow{g} A \xleftarrow{f} B$ is right representable if and only if its codomain-projection functor admits a right adjoint right inverse*

$$\begin{tikzcd} & \mathsf{Hom}_A(f,g) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] & \\ C \arrow[ur, dashed, bend right, "i"'] \arrow[ur, phantom, "\perp"] & & B \end{tikzcd}$$

in which case the composite $p_0 i\colon C \to B$ defines the representing functor and the 2-cell represented by the functor $i\colon C \to \mathsf{Hom}_A(f, g)$ defines an absolute right lifting of g through f.

Proof. If $\mathsf{Hom}_A(f,g)$ is represented on the right by a functor $r\colon C \to B$, then $\mathsf{Hom}_A(f,g) \simeq \mathsf{Hom}_B(B,r)$ over $C \times B$ and the codomain-projection functor is equivalent to $p_1\colon \mathsf{Hom}_B(B,r) \twoheadrightarrow C$, which admits a right adjoint right inverse i by Lemma 3.5.8. The proof of Theorem 3.5.3 then shows that i represents an absolute right lifting diagram. Thus, it remains only to prove the converse.

To that end, suppose we are given a right adjoint right inverse adjunction $p_1 \dashv i$. Unpacking the definition, this provides an adjunction

$$\begin{tikzcd}
C \arrow[rr, dashed, "i"'] \arrow[dr, equal] & \perp & \mathsf{Hom}_A(f,g) \arrow[ll, bend right, two heads, "p_1"'] \arrow[dl, two heads, "p_1"] \\
& C &
\end{tikzcd}$$

over C whose counit is an identity and whose unit $\eta\colon \mathrm{id} \Rightarrow ip_1$ satisfies the conditions $\eta i = \mathrm{id}_i$ and $p_1\eta = \mathrm{id}_{p_1}$. By Theorem 3.5.7, to construct the fibered equivalence $\mathsf{Hom}_B(B,r) \simeq_{C\times B} \mathsf{Hom}_A(f,g)$ with $r \coloneqq p_0 i$, it suffices to demonstrate that the 2-cell defined by restricting the comma cone for $\mathsf{Hom}_A(f,g)$ along i

$$\begin{tikzcd}
& C \arrow[d, dashed, "i"] \arrow[ddl, equal, bend right] \arrow[ddr, bend left, "r"] & \\
& \mathsf{Hom}_A(f,g) \arrow[dl, two heads, "p_1"'] \arrow[dr, two heads, "p_0"] & \\
C \arrow[dr, "g"'] \arrow[rr, phantom, "\overset{\phi}{\Leftarrow}"] & & B \arrow[dl, "f"] \\
& A &
\end{tikzcd}$$

defines an absolute right lifting diagram.

By 1-cell induction any 2-cell as displayed below-left induces a 1-cell m as displayed below-center:

$$\begin{tikzcd}
X \arrow[r, "b"] \arrow[d, "c"'] \arrow[dr, phantom, "\Downarrow\chi"] & B \arrow[d, "f"] \\
C \arrow[r, "g"'] & A
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, dashed, "m"'] \arrow[rr, bend left, "b"] \arrow[dr, bend right, "c"'] & \mathsf{Hom}_A(f,g) \arrow[r, two heads, "p_0"'] \arrow[d, two heads, "p_1"'] \arrow[dr, phantom, "\Downarrow\phi"] & B \arrow[d, "f"] \\
& C \arrow[r, "g"'] & A
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, dashed, "m"'] \arrow[rrr, bend left, "b"] \arrow[dr, bend right, "c"'] & \mathsf{Hom}_A(f,g) \arrow[r, equal] \arrow[d, two heads, "p_1"] \arrow[dr, phantom, "\Downarrow\eta"] & \mathsf{Hom}_A(f,g) \arrow[r, two heads, "p_0"'] \arrow[d, two heads, "p_1"'] \arrow[dr, phantom, "\Downarrow\phi"] & B \arrow[d, "f"] \\
& C \arrow[r, equal] \arrow[ur, "i"'] & C \arrow[r, "g"'] & A
\end{tikzcd}$$

Inserting the triangle equality $p_1\eta = \mathrm{id}_{p_1}$ as displayed above-right constructs the desired factorization $p_0\eta m\colon b \Rightarrow rc$ of χ through ϕi.

In fact, by 2-cell induction for the comma cone ϕ, any 2-cell $\tau_0\colon b \Rightarrow rc$ defining a factorization of $\chi\colon fb \Rightarrow gc$ through ϕi must have the form $\tau_0 = p_0\tau$ for some 2-cell $\tau\colon m \Rightarrow ic$ so that $p_1\tau = \mathrm{id}_c$: put another way, the pair (τ_0, id_c) satisfies the compatibility condition of Proposition 3.4.6(ii) to induce a 2-cell $\tau\colon m \Rightarrow ic$. We'll argue that the 2-cell τ is unique, proving that the factorization $p_0\tau\colon b \Rightarrow rc$ is also unique.

To see this, note that the adjunction $p_1 \dashv i$ over C exhibits the right adjoint as a terminal element of the object $p_1\colon \mathsf{Hom}_A(f,g) \twoheadrightarrow C$ in the slice 2-category $(\mathfrak{h}\mathcal{K})_{/C}$. It follows, as in Lemma 2.2.5, that for any object $c\colon X \to C$ and any morphism $m\colon X \to \mathsf{Hom}_A(f,g)$ over C, there exists a unique 2-cell

$m \Rightarrow ic$ over C. Thus, there is a unique 2-cell $\tau\colon m \Rightarrow ic$ with the property that $p_1\tau = \mathrm{id}_c$, and so the factorization $p_0\tau\colon b \Rightarrow rc$ of χ through ϕi must also be unique. □

More concisely, Theorem 3.5.11 shows that a comma ∞-category $\mathsf{Hom}_A(f,g)$ is right representable just when its codomain-projection functor $p_1\colon \mathsf{Hom}_A(f,g) \twoheadrightarrow C$ admits a terminal element as an object of the sliced ∞-cosmos $\mathcal{K}_{/C}$; dually, $\mathsf{Hom}_A(f,g)$ is left representable just when its domain-projection functor admits an initial element as an object of the sliced ∞-cosmos $\mathcal{K}_{/B}$; see Corollary 3.6.11. There is a small gap between this statement and the version proven in Theorem 3.5.11 having to do with the discrepancy between the homotopy 2-category of $\mathcal{K}_{/C}$ and the slice of the homotopy 2-category $\mathfrak{h}\mathcal{K}$ over C. This is the subject to which we now turn.

Exercises.

3.5.i. Exercise. How might one encode the existence of an adjunction $f \dashv u$ between a given opposing pair of functors using comma ∞-categories?

3.6. Sliced homotopy 2-categories and fibered equivalences

The ∞-category $A^{\mathbb{2}}$ of arrows in A together with its domain- and codomain-evaluation functors $(p_0, p_1)\colon A^{\mathbb{2}} \twoheadrightarrow A \times A$ satisfies a weak universal property in the homotopy 2-category that characterizes the ∞-category up to equivalence *over* $A \times A$; see Proposition 3.2.10. Similarly the comma ∞-category is characterized up to fibered equivalence, as defined in Definition 3.2.7.

As commented upon in Remark 3.2.9 there is some ambiguity regarding the 2-categorical data required to specify a fibered equivalence, that we shall now address head-on. The issue is that, for an ∞-category B in an ∞-cosmos $\mathcal{K}$, the homotopy 2-category $\mathfrak{h}(\mathcal{K}_{/B})$ of the sliced ∞-cosmos of Proposition 1.2.22 is not isomorphic to the 2-category $(\mathfrak{h}\mathcal{K})_{/B}$ of isofibrations, functors, and 2-cells over B in the homotopy 2-category $\mathfrak{h}\mathcal{K}$ of $\mathcal{K}$; see Exercise 1.4.vi.

However, there is a canonical comparison functor relating this pair of 2-categories that satisfies a property we now introduce:

3.6.1. Definition (smothering 2-functor). A 2-functor $F\colon \mathcal{A} \to \mathcal{B}$ is **smothering** if it is

- surjective on 0-cells;
- full on 1-cells: for any pair of objects A, A' in $\mathcal{A}$ and 1-cell $k\colon FA \to FA'$ in $\mathcal{B}$, there exists $f\colon A \to A'$ in $\mathcal{A}$ with $Ff = k$;
- full on 2-cells: for any parallel pair $f, g\colon A \rightrightarrows A'$ in $\mathcal{A}$ and 2-cell $\begin{tikzcd} FA \arrow[r, bend left, "Ff"] \arrow[r, bend right, "Fg"'] \arrow[r, phantom, "\Downarrow\beta"] & FA' \end{tikzcd}$ in $\mathcal{B}$, there exists a 2-cell $\alpha\colon f \Rightarrow g$ in $\mathcal{A}$ with $F\alpha = \beta$; and
- conservative on 2-cells: for any 2-cell α in $\mathcal{A}$ if $F\alpha$ is invertible in $\mathcal{B}$ then α is invertible in $\mathcal{A}$.

3.6.2. Remark. Note that smothering 2-functors are surjective on objects 2-functors that are "locally smothering": meaning that the action on hom-categories is by a smothering functor, as defined in 3.1.2.

The prototypical example of a smothering 2-functor solves Exercise 1.4.vi.

3.6.3. Proposition. *Let B be an ∞-category in an ∞-cosmos $\mathcal{K}$. There is a canonical 2-functor*

$$\mathfrak{h}(\mathcal{K}_{/B}) \to (\mathfrak{h}\mathcal{K})_{/B}$$

from the homotopy 2-category of the sliced ∞-cosmos $\mathcal{K}_{/B}$ to the 2-category of isofibrations, functors, and 2-cells over B in $\mathfrak{h}\mathcal{K}$ and this 2-functor is smothering.

This follows more-or-less immediately from Lemma 3.1.5 but we spell out the details nonetheless.

Proof. The 2-categories $\mathfrak{h}(\mathcal{K}_{/B})$ and $(\mathfrak{h}\mathcal{K})_{/B}$ have the same objects — isofibrations with codomain B — and 1-cells — functors between the "total spaces" that commute with these isofibrations to B — so the canonical mapping may be defined to act as the identity on underlying 1-categories.

By the definition of the sliced ∞-cosmos given in Proposition 1.2.22, a 2-cell between functors $f, g\colon E \rightrightarrows F$ from $p\colon E \twoheadrightarrow B$ to $q\colon F \twoheadrightarrow B$ is a homotopy class of 1-simplices in the quasi-category defined by the pullback of simplicial sets below-left

$$\begin{tikzcd} \mathsf{Fun}_B(E,F) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Fun}(E,F) \arrow[d, two heads, "q_*"] \\ \mathbb{1} \arrow[r, "p"'] & \mathsf{Fun}(E,B) \end{tikzcd} \qquad \begin{tikzcd} (\mathsf{hFun})_{/B}(E,F) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{hFun}(E,F) \arrow[d, two heads, "q_*"] \\ \mathbb{1} \arrow[r, "p"'] & \mathsf{hFun}(E,B) \end{tikzcd}$$

Unpacking, a 2-cell $\alpha\colon f \Rightarrow g$ is represented by a 1-simplex $\alpha\colon f \to g$ in $\mathsf{Fun}(E,F)$ that whiskers with q to the degenerate 1-simplex on the vertex $p \in \mathsf{Fun}(E,B)$, and two such 1-simplices represent the same 2-cell if and only if they bound a 2-simplex of the form displayed in (1.1.8) that also whiskers with q to the degenerate 2-simplex on p.

By contrast, a 2-cell in $(\mathfrak{h}\mathcal{K})_{/B}$ is a morphism in the category defined by the pullback of categories above-right. Such 2-cells are represented by 1-simplices $\alpha\colon f \to g$ in $\mathsf{Fun}(E,F)$ that whisker with q to 1-simplices in $\mathsf{Fun}(E,B)$ that are homotopic to the degenerate 1-simplex on p, and two such 1-simplices represent the same 2-cell if and only if they are homotopic in $\mathsf{Fun}(E,F)$.

Applying the homotopy category functor $\mathsf{h}\colon \mathcal{QC}at \to \mathcal{C}at$ to the above-left pullback produces a cone over the above-right pullback, inducing a canonical map

$$\mathfrak{h}(\mathsf{Fun}_B(E,F)) \to (\mathsf{hFun})_{/B}(E,F),$$

which is the action on homs of the canonical 2-functor $\mathfrak{h}(\mathcal{K}_{/B}) \to (\mathfrak{h}\mathcal{K})_{/B}$.

The 2-functor just constructed is bijective on 0- and 1-cells. To see that it is full on 2-cells we must show that any 1-simplex $\alpha\colon f \to g$ in $\mathsf{Fun}(E,F)$, for which $q\alpha\colon p \to p$ is homotopic to $p \cdot \sigma^0\colon p \to p$ in $\mathsf{Fun}(E,B)$, is homotopic in $\mathsf{Fun}(E,F)$ to a 1-simplex from f to g over $p \cdot \sigma^0$. By Lemma 1.1.9, any such α defines a lifting problem

$$\begin{tikzcd} \Lambda^1[2] \arrow[r] \arrow[d, hook] & \mathsf{Fun}(E,F) \arrow[d, two heads, "q_*"] \\ \Delta[2] \arrow[r] \arrow[ur, dashed] & \mathsf{Fun}(E,B) \end{tikzcd} \qquad \begin{matrix} \ni \\ \\ \\ \ni \end{matrix} \qquad \begin{tikzcd}[column sep=small] & f \arrow[dr, "\alpha"] & \\ f \arrow[ur, equal] & & g \\ & \mapsto & \\ & p \arrow[dr, "q\alpha"] & \\ p \arrow[ur, equal] \arrow[rr, equal] & & p \end{tikzcd}$$

A solution exists since $q_*\colon \mathsf{Fun}(E,F) \twoheadrightarrow \mathsf{Fun}(E,B)$ is an isofibration, proving that $\mathfrak{h}(\mathcal{K}_{/B}) \to (\mathfrak{h}\mathcal{K})_{/B}$ is full on 2-cells.

Now suppose $\alpha\colon f \to g$ represents a 2-cell in $\mathsf{Fun}_B(E,F)$ whose image in $(\mathsf{hFun})_{/B}(E,F)$ is an isomorphism. A map in a 1-category defined by a pullback is invertible if and only if its projections along the legs of the pullback cone are isomorphisms. Thus the image of α is invertible if and only if

$\alpha\colon f \to g$ defines an isomorphism in $\mathsf{hFun}(E,F)$, which by Definition 1.1.13 is the case if and only if $\alpha\colon f \to g$ represents an isomorphism in $\mathsf{Fun}(E,F)$. Since α is fibered over the degenerate 1-simplex at p, this presents us with a lifting problem

$$\begin{tikzcd} \mathbb{2} \arrow[r, "\alpha"] \arrow[d, hook] & \mathsf{Fun}_B(E,F) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner" very near start] & \mathsf{Fun}(E,F) \arrow[d, two heads, "q_*"] \\ \mathbb{I} \arrow[r] \arrow[ur, dotted] \arrow[urr, dashed] & \mathbb{1} \arrow[r, "p"'] & \mathsf{Fun}(E,B) \end{tikzcd}$$

which Exercise 1.1.vi tells us we can solve. This proves that $\mathfrak{h}(\mathcal{K}_{/B}) \to (\mathfrak{h}\mathcal{K})_{/B}$ reflects invertibility of 2-cells and hence defines a smothering 2-functor. ☐

Smothering 2-functors are not strictly speaking invertible, but nevertheless 2-categorical structures from the codomain can be lifted to the domain:

3.6.4. Lemma. *Smothering 2-functors reflect equivalences: for any smothering 2-functor $F\colon \mathcal{A} \to \mathcal{B}$ and 1-cell $f\colon A \to B$ in $\mathcal{A}$, if $Ff\colon FA \xrightarrow{\sim} FB$ is an equivalence in $\mathcal{B}$ then f is an equivalence in $\mathcal{A}$.*

Proof. By fullness on 1-cells, an equivalence inverse $g'\colon FB \xrightarrow{\sim} FA$ to Ff lifts to a 1-cell $g\colon B \to A$ in $\mathcal{A}$. By fullness on 2-cells, the isomorphisms $\mathrm{id}_{FA} \cong g' \circ Ff$ and $Ff \circ g' \cong \mathrm{id}_{FB}$ also lift to $\mathcal{A}$ and by conservativity on 2-cells these lifted 2-cells are also invertible. ☐

Applying Lemma 3.6.4 to the smothering 2-functor

$$\mathfrak{h}(\mathcal{K}_{/B}) \to (\mathfrak{h}\mathcal{K})_{/B}$$

we resolve the ambiguity about the 2-categorical data of a fibered equivalence.

3.6.5. Proposition.

(i) Any equivalence in $(\mathfrak{h}\mathcal{K})_{/B}$ lifts to an equivalence in $\mathfrak{h}(\mathcal{K}_{/B})$. That is, fibered equivalences over B may be specified by defining an opposing pair of 1-cells $f\colon E \to F$ and $g\colon F \to E$ over B together with invertible 2-cells $\mathrm{id}_E \cong gf$ and $fg \cong \mathrm{id}_F$ that lie over B in $\mathfrak{h}\mathcal{K}$.

(ii) Moreover, if $f\colon E \to F$ is a map between isofibrations over B that admits an not-necessarily fibered equivalence inverse $g\colon F \to E$ with not-necessarily fibered 2-cells $\mathrm{id}_E \cong gf$ and $fg \cong \mathrm{id}_F$, then this data is isomorphic to a genuine fibered equivalence.

Proof. The first statement is proven by Lemma 3.6.4 and Proposition 3.6.3. The second statement asserts that the forgetful 2-functor $(\mathfrak{h}\mathcal{K})_{/B} \to \mathfrak{h}\mathcal{K}$ reflects equivalences. Exercise 3.6.i shows that for any map between isofibrations over B that admits an equivalence inverse in the underlying 2-category, the inverse equivalence and invertible 2-cells can be lifted to also lie over B. ☐

This gives a 2-categorical proof of Proposition 1.2.22(vii), that for any ∞-category B in an ∞-cosmos $\mathcal{K}$, the forgetful functor $\mathcal{K}_{/B} \to \mathcal{K}$ preserves and reflects equivalences.

The smothering 2-functor $\mathfrak{h}(\mathcal{K}_{/B}) \to (\mathfrak{h}\mathcal{K})_{/B}$ can also be used to lift adjunctions that are fibered 2-categorically over B to adjunctions in the sliced ∞-cosmos $\mathcal{K}_{/B}$.

3.6.6. Definition (fibered adjunction). A **fibered adjunction** over an ∞-category B in an ∞-cosmos $\mathcal{K}$ is an adjunction

$$\begin{tikzcd} E \arrow[rr, bend left, "f"] \arrow[dr, two heads] & \perp & F \arrow[ll, bend left, "u"] \arrow[dl, two heads] \\ & B & \end{tikzcd}$$

in the sliced ∞-cosmos $\mathcal{K}_{/B}$. We write $f \dashv_B u$ to indicate that specified maps over B are **adjoint over** B.

3.6.7. Lemma (pullback and pushforward of fibered adjunctions).

(i) A fibered adjunction over B can be pulled back along any functor $k\colon A \to B$ to define a fibered adjunction over A.

(ii) A fibered adjunction over A can be pushed forward along any isofibration $p\colon A \twoheadrightarrow B$ to define a fibered adjunction over B.

Proof. By Proposition 1.3.3(v), pullback defines a cosmological functor $k^*\colon \mathcal{K}_{/B} \to \mathcal{K}_{/A}$, which descends to a 2-functor $k^*\colon \mathfrak{h}(\mathcal{K}_{/B}) \to \mathfrak{h}(\mathcal{K}_{/A})$ that carries fibered adjunctions over B to fibered adjunctions over A. This proves (i).

Composition with an isofibration $p\colon A \twoheadrightarrow B$ also defines a 2-functor $p_*\colon \mathfrak{h}(\mathcal{K}_{/A}) \to \mathfrak{h}(\mathcal{K}_{/B})$; the reason we ask p to be an isofibration is due to our convention that the objects in the sliced ∞-cosmoi are isofibrations over a fixed base. Thus, composition with an isofibration carries a fibered adjunction over A to a fibered adjunction over B proving (ii). $\square$

In analogy with Lemma 3.6.4, we have:

3.6.8. Lemma. *If $F\colon \mathcal{A} \to \mathcal{B}$ is a smothering 2-functor, then any adjunction in $\mathcal{B}$ may be lifted to an adjunction in $\mathcal{A}$.*

Proof. Exercise 3.6.ii. $\square$

3.6.9. Remark. A direct proof of Lemma 3.6.8 proceeds as follows: since a smothering 2-functor is surjective on objects and full on both 1- and 2-cells, the data of an adjunction in $\mathcal{B}$ may be lifted to an adjunction in $\mathcal{A}$. Since smothering 2-functors are not in general faithful at the level of 2-cells, there is no reason why the triangle identity composites should be identities, but by 2-cell conservativity they are both invertible. Now either the unit or counit may be modified as in the proof of Proposition 2.1.11 by composing with the inverse of one of these triangle identity composite isomorphisms. Now that triangle equality holds and the other triangle identity composite is an idempotent isomorphism and hence also an identity.

John Bourke pointed out that this proof invokes a recharacterization of adjunctions that makes the conclusion of Lemma 3.6.8 obvious: a pair of 1-cells $f\colon B \to A$ and $u\colon A \to B$ in a 2-category form an adjoint pair $f \dashv u$ if and only if there exist 2-cells $\mathrm{id}_B \Rightarrow uf$ and $fu \Rightarrow \mathrm{id}_A$ so that the composites $f \Rightarrow fuf \Rightarrow f$ and $u \Rightarrow ufu \Rightarrow u$ are both invertible.

Many of the examples of fibered adjunctions we will encounter are right adjoint right inverses or left adjoint right inverses to a given isofibration. The next result shows that whenever an isofibration $p\colon E \twoheadrightarrow B$ admits a left adjoint with unit an isomorphism, then this left adjoint may be modified so as to define a left adjoint right inverse, making the adjunction fibered over B. The dual also holds:

3.6.10. Lemma. *Let $p\colon E \twoheadrightarrow B$ be any isofibration that admits a right adjoint $r'\colon B \to E$ with counit $\epsilon\colon pr' \cong \mathrm{id}_B$ an isomorphism. Then r' is isomorphic to a functor r that lies strictly over B and defines a right adjoint right inverse to p. Thus any such p defines a fibered adjunction*

$$\begin{tikzcd} E \arrow[rr, bend left, two heads, "p"] \arrow[rr, phantom, "\perp"] \arrow[dr, two heads, "p"'] & & B \arrow[ll, bend left, dashed, "r"] \arrow[dl, equals] \\ & B & \end{tikzcd}$$

in $\mathcal{K}_{/B}$ whose right adjoint r lies strictly over B, whose counit is the identity 2-cell, and in which the unit η lies over B in the sense that $p\eta = \mathrm{id}_p$.

Proof. Exercise 3.6.iii. □

Since the identity on B defines the terminal object of the sliced ∞-cosmos $\mathcal{K}_{/B}$, Lemma 3.6.10 can be summarized more compactly as follows:

3.6.11. Corollary. *An isofibration $p\colon E \twoheadrightarrow B$ admits a right adjoint right inverse if and only if it admits a terminal element as an object of $\mathcal{K}_{/B}$. Dually, $p\colon E \twoheadrightarrow B$ admits a left adjoint right inverse if and only if it admits an initial element as an object of $\mathcal{K}_{/B}$*

3.6.12. Example. Lemma 3.5.8 constructs an adjunction in the sliced 2-category $\mathfrak{h}\mathcal{K}_{/A}$. Lemma 3.6.8 now allows us to lift it to a genuine adjunction

$$\begin{tikzcd} A \arrow[rr, bend right, dashed, "i"'] \arrow[rr, phantom, "\perp"] \arrow[dr, equals] & & \mathsf{Hom}_B(B,f) \arrow[ll, bend right, two heads, "p_1"'] \arrow[dl, two heads, "p_1"] \\ & A & \end{tikzcd}$$

in the sliced ∞-cosmos $\mathcal{K}_{/A}$. By Corollary 3.6.11 this situation may be summarized by saying that $p_1\colon \mathsf{Hom}_B(B,f) \twoheadrightarrow A$ admits a terminal element over A.

By Lemma 3.6.7(i), we may pull back the fibered adjunction along any element $a\colon 1 \to A$ to obtain an adjunction

$$\begin{tikzcd} 1 \arrow[rr, bend right, dashed, "i_a"'] \arrow[rr, phantom, "\perp"] & & \mathsf{Hom}_B(B,fa) \arrow[ll, bend right, "!"'] \end{tikzcd}$$

that identifies a terminal element in the fiber $\mathsf{Hom}_B(B,fa)$ of $p_1\colon \mathsf{Hom}_B(B,f) \twoheadrightarrow A$ over a. This generalizes the result of Corollary 3.5.9.

3.6.13. Example (the fibered adjoints to composition). For any ∞-category A, the adjoints to the "composition" functor $\circ \colon A^{\mathbb{2}} \underset{A}{\times} A^{\mathbb{2}} \to A^{\mathbb{2}}$ constructed in Lemma 2.1.13 may be constructed by composing a triple of adjoint functors that are fibered over the endpoint-evaluation functors

$$\begin{tikzcd}[column sep=huge]
A^{\mathbb{3}} \arrow[rr, "A^{d^1}" description] \arrow[dr, "{(\mathrm{ev}_2,\mathrm{ev}_0)}"'] & & A^{\mathbb{2}} \arrow[ll, bend right, "A^{s^0}"'] \arrow[ll, bend left, "A^{s^1}"] \arrow[dl, "{(\mathrm{ev}_1,\mathrm{ev}_0)}"] \arrow[ll, phantom, shift right=3, "\perp"] \arrow[ll, phantom, shift left=3, "\perp"] \\
& A \times A &
\end{tikzcd}$$

with an adjoint equivalence involving a functor $A^{\mathbb{3}} \xrightarrow{\sim} A^{\mathbb{2}} \underset{A}{\times} A^{\mathbb{2}}$, which also lies over $A \times A$. Lemma 3.6.10 and its dual implies that these adjoint equivalences can be lifted to fibered adjoint equivalences over $A \times A$, and now both adjoint triples and hence also the composite adjunctions

$$\begin{tikzcd}[column sep=huge]
A^{\mathbb{2}} \underset{A}{\times} A^{\mathbb{2}} \arrow[rr, "\circ" description] \arrow[dr, "{(\mathrm{ev}_2,\mathrm{ev}_0)}"'] & & A^{\mathbb{2}} \arrow[ll, bend right, "{(-,\mathrm{id}_{\mathrm{dom}(-)})}"'] \arrow[ll, bend left, "{(\mathrm{id}_{\mathrm{cod}(-)},-)}"] \arrow[dl, "{(\mathrm{ev}_1,\mathrm{ev}_0)}"] \arrow[ll, phantom, shift right=3, "\perp"] \arrow[ll, phantom, shift left=3, "\perp"] \\
& A \times A &
\end{tikzcd}$$

lie in $\mathcal{K}_{/A \times A}$.

This fibered adjunction figures in the proof of a result that will allow us to convert limit and colimit diagrams into right and left Kan extension diagrams in the next chapter.

3.6.14. Proposition. *A cospan as displayed below-left admits an absolute right lifting if and only if the cospan displayed below-right admits an absolute right lifting*

$$\begin{tikzcd}
& B \arrow[d, "f"] \\
C \arrow[ur, dashed, "r"] \arrow[r, "g"'] \arrow[ur, phantom, shift right=3, "\Downarrow\rho"] & A
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
& \mathrm{Hom}_A(f, A) \arrow[d, "p_1"] \\
C \arrow[ur, dashed, "i"] \arrow[r, "g"'] \arrow[ur, phantom, shift right=3, "\Downarrow\epsilon"] & A
\end{tikzcd}$$

in which case the 2-cell ϵ is necessarily an isomorphism and can be chosen to be an identity.

Proof. By Theorem 3.5.11, a cospan admits an absolute right lifting if and only if the codomain-projection functor from the associated comma ∞-category admits a right adjoint right inverse. Our task is thus to show that this right adjoint right inverse exists for $\mathrm{Hom}_A(f, g)$ if and only if this right adjoint right inverse exists for $\mathrm{Hom}_A(p_1, g)$.

From the defining pullback (3.4.2) that constructs the comma ∞-category $\mathsf{Hom}_A(p_1, g)$ reproduced below-left, we have the below-right pullback square

$$\begin{tikzcd} \mathsf{Hom}_A(p_1,g) \arrow[r] \arrow[d, two heads, "{(p_1,p_0)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Hom}_A(A,g) \arrow[r] \arrow[d, two heads, "{(p_1,p_0)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & A^{\mathbb{2}} \arrow[d, two heads] \\ C \times \mathsf{Hom}_A(f,A) \arrow[r, "C\times p_1"'] \arrow[d, two heads, "\pi"'] & C \times A \arrow[r, "g\times A"'] \arrow[d, two heads, "\pi"] & A \times A \\ \mathsf{Hom}_A(f,A) \arrow[r, "p_1"'] & A & \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \mathsf{Hom}_A(p_1,g) \arrow[r, two heads] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Hom}_A(A,g) \arrow[d, two heads, "p_0"] \\ \mathsf{Hom}_A(f,A) \arrow[r, two heads, "p_1"'] & A \end{tikzcd} \tag{3.6.15}$$

By Lemma 3.6.7, the composition-identity fibered adjunction of Example 3.6.13 pulls back along $g \times f\colon C \times B \to A \times A$ to define a fibered adjunction

$$\begin{tikzcd} \mathsf{Hom}_A(p_1,g) \cong \mathsf{Hom}_A(A,g) \underset{A}{\times} \mathsf{Hom}_A(f,A) \arrow[rr, "\circ"] \arrow[dr, two heads, "{(p_1,p_0)}"'] & & \mathsf{Hom}_A(f,g) \arrow[ll, bend right, "\perp"'] \arrow[ll, bend left, "\perp"] \arrow[dl, two heads, "{(p_1,p_0)}"] \\ & C \times B & \end{tikzcd}$$

which then pushes forward along the projection $\pi\colon C \times B \twoheadrightarrow C$ to a fibered adjunction over C

$$\begin{tikzcd} \mathsf{Hom}_A(p_1,g) \arrow[rr, "\circ"] \arrow[dr, two heads, "p_1"'] & & \mathsf{Hom}_A(f,g) \arrow[ll, bend right, "\perp"'] \arrow[ll, bend left, "\perp"] \arrow[dl, two heads, "p_1"] \\ & C & \end{tikzcd}$$

between the codomain-projection for $\mathsf{Hom}_A(p_1, g)$ and the codomain projection for $\mathsf{Hom}_A(f, g)$. Now by Corollary 3.6.11, $p_1\colon \mathsf{Hom}_A(f, g) \twoheadrightarrow C$ admits a right adjoint right inverse just when the object on the right admits a terminal element, while $p_1\colon \mathsf{Hom}_A(p_1, g) \twoheadrightarrow C$ similarly admits a right adjoint right inverse just when the object on the left admits a terminal element. By Theorem 2.4.2, a terminal element on either side is carried by the appropriate right adjoint to a terminal element on the other side. This proves the equivalence of these conditions.

It remains only to prove that the 2-cell for the absolute right lifting of g through p_1 is invertible. By Theorem 3.5.11, this 2-cell is constructed as by restricting the comma cone along the terminal element, so it is given by the composite

$$\begin{tikzcd} C \arrow[r] & \mathsf{Hom}_A(p_1,g) \arrow[r] & A^{\mathbb{2}} \arrow[r, two heads, bend left, "p_0"] \arrow[r, two heads, bend right, "p_1"'] \arrow[r, phantom, "\Downarrow\kappa"] & A \end{tikzcd}$$

where the left-hand map is the terminal element just constructed and the middle one comes comes from the defining pullback diagram displayed on the left of (3.6.15). As just argued, that terminal element may be chosen to be in the image of the right adjoint $\mathsf{Hom}_A(f, g) \to \mathsf{Hom}_A(A, g) \times_A \mathsf{Hom}_A(f, A) \cong \mathsf{Hom}_A(p_1, g)$, whose component on the left factor is the identity. Simultaneously, the pullback defining the map $\mathsf{Hom}_A(p_1, g) \to A^{\mathbb{2}}$ factors through the projection onto the left factor, so we see that the 2-cell in the absolute right lifting diagram is represented by the composite

$$\begin{tikzcd} C \arrow[r, "i"] & \mathsf{Hom}_A(f,g) \arrow[r, "(\mathrm{id}_{\mathrm{cod}(-)}, -)"] & \mathsf{Hom}_A(A,g) \underset{A}{\times} \mathsf{Hom}_A(f,A) \arrow[r, "\pi"] & \mathsf{Hom}_A(A,g) \arrow[r] & A^{\mathbb{2}} \arrow[r, bend left, "p_0"] \arrow[r, bend right, "p_1"'] \arrow[r, phantom, "\Downarrow\kappa"] & A \end{tikzcd}$$

and hence that this cell is invertible. □

Exercises.

3.6.i. Exercise. Let B be an object in a 2-category $\mathcal{C}$ and consider a map

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, two heads] & & F \arrow[dl, two heads] \\ & B & \end{tikzcd}$$

between isofibrations over B. Prove that if f is an equivalence in $\mathcal{C}$ then f is also an equivalence in the slice 2-category $\mathcal{C}_{/B}$ of isofibrations over B, 1-cells that form commutative triangles over B, and 2-cells that lie over B in the sense that they whisker with the codomain isofibration to the identity 2-cell on the domain isofibration.

3.6.ii. Exercise. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a smothering 2-functor. Show that any adjunction in $\mathcal{B}$ can be lifted to an adjunction in $\mathcal{A}$. Demonstrate furthermore that if we have previously specified a lift of the objects, 1-cells, and either the unit or counit of the adjunction in $\mathcal{B}$, then there is a lift of the remaining 2-cell that combines with the previously specified data to define an adjunction in $\mathcal{A}$. This proves a more precise version of Lemma 3.6.8.

3.6.iii. Exercise. Prove Lemma 3.6.10.

CHAPTER 4

Adjunctions, limits, and colimits II

Comma ∞-categories provide a vehicle for encoding the universal properties of categorical constructions that restrict to define equivalences between the internal mapping spaces introduced in Definition 3.4.9 between suitable pairs of elements. Using the theory developed in Chapter 3, we quickly prove a variety of results of this type first for adjunctions in §4.1 and then for limits and colimits in §4.3. In an interlude in §4.2, we introduce the ∞-categories of cones over or under a diagram as a comma ∞-category and then give a second model for these ∞-categories of cones in the case of diagrams indexed by simplicial sets built from Joyal's join construction. Then we conclude in §4.4 with an application, constructing the loops ⊢ suspension adjunction for *pointed* ∞-categories, containing an element that is both initial and terminal.

4.1. The universal property of adjunctions

Our first result shows that an adjunction between an opposing pair of functors can equally be encoded by a "transposing equivalence" between their left and right representations as comma ∞-categories.

4.1.1. Proposition. *An opposing pair of functors $u\colon A \to B$ and $f\colon B \to A$ define an adjunction $f \dashv u$ if and only if $\mathsf{Hom}_A(f, A) \simeq \mathsf{Hom}_B(B, u)$ over $A \times B$.*

Proof. This is a special case of Theorem 3.5.7. If $f \dashv u$, then Lemma 2.3.6 tells us that its counit $\epsilon\colon fu \Rightarrow \mathrm{id}_A$ defines an absolute right lifting diagram. Theorem 3.5.7 then tells us that the 1-cell induced by the left-hand pasted composite

$$\begin{tikzcd} & \mathsf{Hom}_B(B,u) \arrow[dl, two heads, "p_1"'] \arrow[dr, two heads, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\ A \arrow[rr, "u"] \arrow[dr, equals] & {} \arrow[d, phantom, "\overset{\epsilon}{\Leftarrow}"] & B \arrow[dl, "f"] \\ & A & \end{tikzcd} \quad = \quad \begin{tikzcd} & \mathsf{Hom}_B(B,u) \arrow[d, dashed, "\ulcorner \epsilon \circ f(-) \urcorner"] \arrow[ddl, bend right, two heads, "p_1"'] \arrow[ddr, bend left, two heads, "p_0"] & \\ & \mathsf{Hom}_A(f,A) \arrow[dl, two heads, "p_1"'] \arrow[dr, two heads, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\ A \arrow[dr, equals] & {} & B \arrow[dl, "f"] \\ & A & \end{tikzcd}$$

defines a fibered equivalence $\mathsf{Hom}_B(B, u) \xrightarrow{\sim} \mathsf{Hom}_A(f, A)$ over $A \times B$. We interpret this result as saying that in the presence of an adjunction $f \dashv u$, the right comma cone over u transposes to define the left comma cone over f.[1]

[1] If desired, an inverse equivalence can be constructed by applying the dual of Theorem 3.5.7 to the absolute left lifting diagram presented by the unit.

Conversely, Theorem 3.5.7 tells us that from a fibered equivalence $\mathsf{Hom}_B(B,u) \xrightarrow{\sim} \mathsf{Hom}_A(f,A)$ over $A \times B$ one can extract a 2-cell that defines an absolute right lifting diagram

$$\begin{tikzcd} & B \arrow[d, "f"] \\ A \arrow[ur, "u"] \arrow[r, equal] \arrow[r, phantom, shift left=3, "\Downarrow\epsilon"] & A \end{tikzcd}$$

Lemma 2.3.6 then tells us that this 2-cell defines the counit of an adjunction $f \dashv u$. ☐

4.1.2. Observation (the transposing equivalence). To justify referring to the induced functor

$$\ulcorner \epsilon \circ f(-) \urcorner \colon \mathsf{Hom}_B(B,u) \xrightarrow{\sim} \mathsf{Hom}_A(f,A)$$

as a **transposing equivalence**, recall that the transpose of a 2-cell $\chi\colon b \Rightarrow ua$ across the adjunction $f \dashv u$ is computed by the left-hand pasting diagram below:

$$\begin{tikzcd} & X \arrow[dl, "a"'] \arrow[dr, "b"] \arrow[d, phantom, "\overset{\chi}{\Leftarrow}"] & \\ A \arrow[rr, "u" description] \arrow[dr, equal] & {} \arrow[d, phantom, "\overset{\epsilon}{\Leftarrow}"] & B \arrow[dl, "f"] \\ & A & \end{tikzcd} = \begin{tikzcd} & X \arrow[d, dashed, "x"] \arrow[ddl, bend right, "a"'] \arrow[ddr, bend left, "b"] & \\ & \mathsf{Hom}_B(B,u) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\ A \arrow[rr, "u" description] \arrow[dr, equal] & {} \arrow[d, phantom, "\overset{\epsilon}{\Leftarrow}"] & B \arrow[dl, "f"] \\ & A & \end{tikzcd} = \begin{tikzcd} & X \arrow[d, dashed, "x"] \arrow[ddl, bend right, "a"'] \arrow[ddr, bend left, "b"] & \\ & \mathsf{Hom}_B(B,u) \arrow[d, dashed, "\wr"] \arrow[dl, "p_1"'] \arrow[dr, "p_0"] & \\ A & \mathsf{Hom}_A(f,A) \arrow[l, "p_1"'] \arrow[r, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & B \arrow[ll, bend left, "f"] \\ & {} & \end{tikzcd}$$

By the weak universal property of the right comma cone over u, the 2-cell χ is represented by the induced functor $X \to \mathsf{Hom}_B(B,u)$, which then composes with the transposing equivalence to define a functor $X \to \mathsf{Hom}_A(f,A)$ that represents the transpose of χ, by the pasting diagram equalities from right to left. This observation also justifies our notation, in which we name the fibered equivalence $\ulcorner \epsilon \circ f(-) \urcorner$ after the formula for adjoint transposition.

4.1.3. Corollary. *An adjunction* $\begin{tikzcd} B \arrow[r, bend left, "f"] \arrow[r, phantom, "\perp"] & A \arrow[l, bend left, "u"] \end{tikzcd}$ *induces an equivalence* $\mathsf{Hom}_A(fb,a) \simeq \mathsf{Hom}_B(b,ua)$ *over* $X \times Y$ *for any pair of generalized elements* $a\colon X \to A$ *and* $b\colon Y \to B$.

Proof. By the pullback construction of comma ∞-categories given in (3.4.2), the equivalence $\mathsf{Hom}_A(f,A) \simeq \mathsf{Hom}_B(B,u)$ in $\mathcal{K}_{/A\times B}$ pulls back along $a \times b\colon X \times Y \to A \times B$ to define an equivalence $\mathsf{Hom}_A(fb,a) \simeq \mathsf{Hom}_B(b,ua)$ in $\mathcal{K}_{/X\times Y}$. ☐

In particular, the equivalence of Proposition 4.1.1 pulls back to define an equivalence of internal mapping spaces, introduced in 3.4.9.

4.1.4. Proposition (the universal property of units and counits). *Consider an adjunction*

$$\begin{tikzcd} B \arrow[r, bend left, "f"] \arrow[r, phantom, "\perp"] & A \arrow[l, bend left, "u"] \end{tikzcd} \qquad \textit{with unit } \eta\colon \mathrm{id}_B \Rightarrow uf \textit{ and counit } \epsilon\colon fu \Rightarrow \mathrm{id}_A\,.$$

Then for each element $a\colon 1 \to A$, the component ϵa defines a terminal element of $\mathrm{Hom}_A(f,a)$, and for each element $b\colon 1 \to B$, the component ηb defines an initial element of $\mathrm{Hom}_B(b,u)$.

Proof. By Corollary 4.1.3, the fibered equivalence $\mathrm{Hom}_A(f,A) \simeq_{A\times B} \mathrm{Hom}_B(B,u)$ of Proposition 4.1.1 pulls back to define equivalences

$$\mathrm{Hom}_A(f,a) \simeq_B \mathrm{Hom}_B(B,ua) \qquad \text{and} \qquad \mathrm{Hom}_A(fb,A) \simeq_A \mathrm{Hom}_B(b,u).$$

By Corollary 3.5.9, id_{ua} induces a terminal element of $\mathrm{Hom}_B(B,ua)$ and by Observation 4.1.2 its image across the equivalence $\mathrm{Hom}_B(B,ua) \xrightarrow{\sim} \mathrm{Hom}_A(f,a)$ is again a terminal element, which represents the transposed 2-cell: the component of the counit ϵ at the element a. The proof that the unit component defines an initial element of $\mathrm{Hom}_B(b,u)$ is dual. □

The universal property of unit and counit components captured in Proposition 4.1.4 gives the main idea behind the adjoint functor theorems: a functor $f\colon B \to A$ admits a right adjoint just when for each element $a\colon 1 \to A$, the ∞-category $\mathrm{Hom}_A(f,a)$ admits a terminal element. The image of this terminal element under the domain-projection functor $p_0\colon \mathrm{Hom}_A(f,a) \twoheadrightarrow B$ then defines the element $ua\colon 1 \to B$ and the comma cone defines the component of the counit at a. The universal property of these unit components is then used to extend the mapping on elements to a functor $u\colon A \to B$.

The result just stated is true in the ∞-cosmos of quasi-categories and in other ∞-cosmoi where universal properties are generated by the terminal ∞-category 1; see Corollary 12.2.7.[2] What is true in all ∞-cosmoi is the version of the result just stated where the quantifier "for each element $a\colon 1 \to A$" is replaced with "for each generalized element $a\colon X \to A$," in which case the meaning of "terminal element" should be enhanced to "terminal element over X"; see the remark after Corollary 3.5.9. Since every generalized element factors through the universal generalized element, namely the identity functor at A, it suffices to prove:

4.1.5. Proposition. *A functor $f\colon B \to A$ admits a right adjoint if and only if $\mathrm{Hom}_A(f,A)$ admits a terminal element over A. Dually, $f\colon B \to A$ admits a left adjoint if and only if $\mathrm{Hom}_A(A,f)$ admits an initial element over A.*

Proof. By Proposition 4.1.1, $f\colon B \to A$ admits a right adjoint if and only if the comma ∞-category $\mathrm{Hom}_A(f,A)$ is right representable. Theorem 3.5.11 specializes to tell us that this is the case if and only if the codomain-projection functor $p_1\colon \mathrm{Hom}_A(f,A) \twoheadrightarrow A$ admits a right adjoint right inverse, which by Corollary 3.6.11 is equivalent to postulating a terminal element over A. □

The same suite of results from §3.5 specialize to theorems that encode the universal properties of limits and colimits. Before proving these, we first construct the ∞-category of cones over a fixed diagram and also construct alternate models for the ∞-categories of cones over varying J-indexed diagrams, in the case where J is a simplicial set.

Exercises.

4.1.i. Exercise. Prove that the transposing equivalence of Proposition 4.1.1, as elaborated upon in Observation 4.1.2, is natural with respect to pre-composing with a 2-cell $\beta\colon b' \Rightarrow b$ or post-composing with a 2-cell $\alpha\colon a \Rightarrow a'$.

[2]We delay the discussion of "analytically-proven" theorems about quasi-categories until we demonstrate in Part III that such results apply also in biequivalent ∞-cosmoi. Various "pointwise-determined" universal properties that hold in ∞-cosmoi whose objects are $(\infty,1)$-categories are established in §12.2.

4.2. ∞-categories of cones

4.2.1. Definition (the ∞-category of cones). Let $d\colon 1 \to A^J$ be a J-shaped diagram in an ∞-category A. The **∞-category of cones over** d is the comma ∞-category $\mathrm{Hom}_{A^J}(\Delta, d)$ with comma cone displayed below-left, while the **∞-category of cones under** d is the comma ∞-category $\mathrm{Hom}_{A^J}(d, \Delta)$ with comma cone displayed below-right:

$$\begin{tikzcd} & \mathrm{Hom}_{A^J}(\Delta, d) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[dd, phantom, "\overset{\phi}{\Leftarrow}"] & \\ 1 \arrow[dr, "d"'] & & A \arrow[dl, "\Delta"] \\ & A^J & \end{tikzcd} \qquad \begin{tikzcd} & \mathrm{Hom}_{A^J}(d, \Delta) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[dd, phantom, "\overset{\phi}{\Leftarrow}"] & \\ A \arrow[dr, "\Delta"'] & & 1 \arrow[dl, "d"] \\ & A^J & \end{tikzcd}$$

By replacing the "d" leg of the cospans, Definition 4.2.1 can be modified to allow $d\colon D \to A^J$ to be a family of diagrams or to define ∞-categories of cones over any diagram of shape J: an element of $\mathrm{Hom}_{A^J}(\Delta, A^J)$ is a cone with any summit over any J-indexed diagram.

In the case where the indexing shape J is a simplicial set (and not an ∞-category in a cartesian closed ∞-cosmos), there is another model of the ∞-categories of cones over or under a diagram that may be constructed using Joyal's join construction. The reason for the equivalence is that joins of simplicial sets are known to be equivalent to so-called "fat joins" of simplicial sets, and a particular instance of the fat join construction gives the shape of the cones appearing in Definition 4.2.1. We now introduce these notions.

4.2.2. Definition (fat join). The **fat join** of simplicial sets I and J is the simplicial set constructed by the following pushout:

$$\begin{tikzcd} (I \times J) \sqcup (I \times J) \arrow[r, "\pi_I \sqcup \pi_J"] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner" very near end] & I \sqcup J \arrow[d, hook] \\ I \times \mathbb{2} \times J \arrow[r] & I \diamond J \end{tikzcd}$$

from which it follows that

$$(I \diamond J)_n \coloneqq I_n \sqcup \Big(\coprod_{[n] \twoheadrightarrow [1]} I_n \times J_n \Big) \sqcup J_n.$$

Note there is a natural map $I \diamond J \twoheadrightarrow \mathbb{2}$ induced by the projection $\pi\colon I \times \mathbb{2} \times J \twoheadrightarrow \mathbb{2}$ so that I is the fiber over 0 and J is the fiber over 1:

$$\begin{tikzcd} I \sqcup J \arrow[r, hook] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner" very near start] & I \diamond J \arrow[d, two heads] \\ \mathbb{1} + \mathbb{1} \arrow[r, hook, "{(0,1)}"] & \mathbb{2} \end{tikzcd}$$

4.2.3. Lemma. *For any simplicial set J and ∞-category A we have natural isomorphisms*

$$\mathrm{Hom}_{A^J}(\Delta, A^J) \cong A^{\mathbb{1} \diamond J} \qquad \textit{and} \qquad \mathrm{Hom}_{A^J}(A^J, \Delta) \cong A^{J \diamond \mathbb{1}}.$$

Proof. The simplicial cotensor $A^{(-)}\colon s\mathcal{S}et^{\mathrm{op}} \to \mathcal{K}$ carries the pushout of Definition 4.2.2 to the pullback squares that define the left and right representations of $\Delta\colon A \to A^J$ as a comma ∞-category:

$$\begin{tikzcd} A^{\mathbb{1} \diamond J} \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner" very near start] & (A^J)^{\mathbb{2}} \arrow[d, two heads, "{(p_1,p_0)}"] \\ A^J \times A \arrow[r, "\mathrm{id}\times\Delta"'] & A^J \times A^J \end{tikzcd} \qquad \begin{tikzcd} A^{J \diamond \mathbb{1}} \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner" very near start] & (A^J)^{\mathbb{2}} \arrow[d, two heads, "{(p_1,p_0)}"] \\ A \times A^J \arrow[r, "\Delta\times\mathrm{id}"'] & A^J \times A^J \end{tikzcd}$$ □

4.2.4. Definition (join). The **join** of simplicial sets I and J is the simplicial set $I \star J$

$$\begin{tikzcd} I \sqcup J \arrow[r, hook] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner" very near start] & I \star J \arrow[d, two heads] \\ \mathbb{1} + \mathbb{1} \arrow[r, hook, "{(0,1)}"] & \mathbb{2} \end{tikzcd}$$

with

$$(I \star J)_n := I_n \sqcup (\coprod_{0 \le k < n} I_{n-k-1} \times J_k) \sqcup J_n$$

and with the vertices of these n-simplices oriented so that there is a canonical map $I \star J \to \mathbb{2}$ so that I is the fiber over 0 and J is the fiber over 1. See Definitions D.2.2 and D.2.3 or the original [**46**, §3] for more details.

The join functor $- \star J\colon s\mathcal{S}et \to s\mathcal{S}et$ preserves connected colimits but not the initial object or other coproducts, but this issue can be rectified by replacing the codomain by the slice category under J: the functor $- \star J\colon s\mathcal{S}et \to {}^{J/}s\mathcal{S}et$ preserves all colimits, as proven in Lemma D.2.7 for proof. Contextualized in this way, the join admits a right adjoint, defined by Joyal's **slice** construction, which carries a simplicial map $f\colon J \to X$ to a simplicial set traditionally denoted by $X_{/f}$.

4.2.5. Proposition. *The join functors admit right adjoints*

$$\begin{tikzcd} s\mathcal{S}et \arrow[r, bend left, "I\star-"] \arrow[r, phantom, "\perp"] & {}^{I/}s\mathcal{S}et \arrow[l, bend left, "{}^{-}/\_"] \end{tikzcd} \qquad \begin{tikzcd} s\mathcal{S}et \arrow[r, bend left, "-\star J"] \arrow[r, phantom, "\perp"] & {}^{J/}s\mathcal{S}et \arrow[l, bend left, "{}^{-}/-"] \end{tikzcd}$$

defined by the natural bijections

$$\left\{ \begin{tikzcd}[row sep=small] & I \arrow[dl] \arrow[dr, "h"] & \\ I \star \Delta[n] \arrow[rr] & & X \end{tikzcd} \right\} \cong \left\{ \Delta[n] \longrightarrow {}^{h/}X \right\} \quad \textit{and} \quad \left\{ \begin{tikzcd}[row sep=small] & J \arrow[dl] \arrow[dr, "k"] & \\ \Delta[n] \star J \arrow[rr] & & X \end{tikzcd} \right\} \cong \left\{ \Delta[n] \longrightarrow X_{/k} \right\}.$$

Proof. As in the statement, the simplicial set $X_{/k}$ is defined to have n-simplices corresponding to maps $\Delta[n] \star J \to X$ under J, with the right action by the simplicial operators $[m] \to [n]$ given by pre-composition with $\Delta[m] \to \Delta[n]$. Since the join functor $- \star J\colon s\mathcal{S}et \to {}^{J/}s\mathcal{S}et$ preserves colimits, this extends to a bijection between maps $I \to X_{/k}$ and maps $I \star J \to X$ under J that is natural in I and in $k\colon J \to X$. □

4.2.6. Notation. For any simplicial set J, we write

$$J^{\triangleleft} := \mathbb{1} \star J \qquad \text{and} \qquad J^{\triangleright} := J \star \mathbb{1}$$

and write $\perp$ for the **cone vertex** of $J^{\triangleleft}$ and $\top$ for the **cone vertex** of $J^{\triangleright}$ contributed by the terminal simplicial set $\mathbb{1}$. These simplicial sets are equipped with canonical inclusions

$$J^{\triangleleft} \hookleftarrow J \hookrightarrow J^{\triangleright}$$

4.2.7. Proposition (join vs fat join). *For any simplicial sets I and J and any ∞-category A, there is a natural equivalence*

$$\begin{tikzcd} A^{I\star J} \arrow[rr, "\sim"] \arrow[dr, "\mathrm{res}"'] & & A^{I\diamond J} \arrow[dl, "\mathrm{res}"] \\ & A^{I\sqcup J} & \end{tikzcd}$$

Proof. There is a canonical map of simplicial sets

$$\begin{tikzcd} (I\times J)\sqcup(I\times J) \arrow[r, two heads, "\pi_I\sqcup\pi_J"] \arrow[d, hook] & I\sqcup J \arrow[d, hook] \arrow[dr, hook] & \\ I\times\mathbb{2}\times J \arrow[r, two heads] \arrow[dr] & I\diamond J \arrow[r, dashed] \arrow[d, two heads] \arrow[ul, phantom, "\ulcorner" very near start] & I\star J \arrow[dl] \\ & \mathbb{2} & \end{tikzcd}$$

that commutes with the inclusions of the fibers $I \sqcup J$ over the endpoints of $\mathbb{2}$. This dashed map displayed above is defined on those n-simplices over $I \diamond J$ that map surjectively onto $\mathbb{2}$ to send a triple $(\alpha\colon [n] \twoheadrightarrow [1], \sigma \in I_n, \tau \in J_n)$ representing an n-simplex of $I \diamond J$ to the pair $(\sigma|_{\{0,\ldots,k\}} \in I_k, \tau|_{\{k+1,\ldots,n\}} \in J_{n-k-1})$ representing an n-simplex of $I \star J$, where $k \in [n]$ is the maximal vertex in $\alpha^{-1}(0)$. Proposition D.6.4 of Appendix D proves that this map induces a natural equivalence $Q^{I\star J} \xrightarrow{\sim} Q^{I\diamond J}$ of quasi-categories over $Q^J \times Q^I$. Taking Q to be the functor space $\mathsf{Fun}(X, A)$ proves the claimed equivalence for general ∞-categories. $\square$

By Lemma 4.2.3 $\mathsf{Hom}_{A^J}(\Delta, A^J) \cong A^{\mathbb{1}\diamond J}$ and $\mathsf{Hom}_{A^J}(A^J, \Delta) \cong A^{J\diamond\mathbb{1}}$. Thus Proposition 4.2.7 specializes to give an alternate model for the ∞-categories of cones over or under a diagram.

4.2.8. Corollary. *In particular, there are comma squares*

$$\begin{tikzcd} & A^{J^{\triangleleft}} \arrow[dl, "\mathrm{res}"'] \arrow[dr, "\mathrm{ev}_\perp"] \arrow[dd, phantom, "\overset{\phi}{\Leftarrow}"] & \\ A^J \arrow[dr, equal] & & A \arrow[dl, "\Delta"] \\ & A^J & \end{tikzcd} \qquad \begin{tikzcd} & A^{J^{\triangleright}} \arrow[dl, "\mathrm{ev}_\top"'] \arrow[dr, "\mathrm{res}"] \arrow[dd, phantom, "\overset{\phi}{\Leftarrow}"] & \\ A \arrow[dr, "\Delta"'] & & A^J \arrow[dl, equal] \\ & A^J & \end{tikzcd} \tag{4.2.9}$$

which pullback over a diagram $d\colon 1 \to A^J$ to define alternate models for the ∞-categories of cones over or under d.

Proof. By Lemma 4.2.3, $A^{\mathbb{1}\diamond J}$ and $A^{J\diamond\mathbb{1}}$ are comma ∞-categories. Thus, Proposition 3.4.11 implies that the fibered equivalence of Proposition 4.2.7 equips $A^{J^{\triangleleft}}$ and $A^{J^{\triangleright}}$ with comma cone squares. The

2-cells in (4.2.9) are represented by the maps

$$\begin{tikzcd} J \sqcup J \arrow[r, two heads] \arrow[d, hook] & \mathbb{1} \sqcup J \arrow[d, hook] \arrow[dr, hook] & \\ J \times \mathbb{2} \arrow[r, two heads] \arrow[ur, phantom, "\ulcorner"] \arrow[dr] & \mathbb{1} \diamond J \arrow[r, dashed] \arrow[d, two heads] & J^{\triangleleft} \arrow[dl] \\ & \mathbb{2} & \end{tikzcd} \qquad \begin{tikzcd} J \sqcup J \arrow[r, two heads] \arrow[d, hook] & J \sqcup \mathbb{1} \arrow[d, hook] \arrow[dr, hook] & \\ J \times \mathbb{2} \arrow[r, two heads] \arrow[ur, phantom, "\ulcorner"] \arrow[dr] & J \diamond \mathbb{1} \arrow[r, dashed] \arrow[d, two heads] & J^{\triangleright} \arrow[dl] \\ & \mathbb{2} & \end{tikzcd}$$

which yield 2-cells

$$\begin{tikzcd} A^{J^{\triangleleft}} \arrow[r] \arrow[rr, bend left=60, "\Delta\,\mathrm{ev}_\perp"] \arrow[rr, bend right=60, "\mathrm{res}"'] & (A^J)^{\mathbb{2}} \arrow[r, bend left, "p_0"] \arrow[r, bend right, "p_1"'] \arrow[r, phantom, "\Downarrow\kappa"] & A^J \end{tikzcd} \qquad \begin{tikzcd} A^{J^{\triangleright}} \arrow[r] \arrow[rr, bend left=60, "\mathrm{res}"] \arrow[rr, bend right=60, "\Delta\,\mathrm{ev}_\top"'] & (A^J)^{\mathbb{2}} \arrow[r, bend left, "p_0"] \arrow[r, bend right, "p_1"'] \arrow[r, phantom, "\Downarrow\kappa"] & A^J \end{tikzcd}$$

upon cotensoring into A. □

4.2.10. Warning. By Lemma 4.2.3, the fibers of $A^{\mathbb{1}\diamond J} \twoheadrightarrow A^J$ and $A^{J\diamond\mathbb{1}} \twoheadrightarrow A^J$ over a diagram $d\colon 1 \to A^J$ are isomorphic to the ∞-categories of cones $\mathsf{Hom}_{A^J}(\Delta, d)$ and $\mathsf{Hom}_{A^J}(d, \Delta)$. In the ∞-cosmos of quasi-categories, it is tempting to believe that the fibers of the equivalent isofibrations $A^{J^{\triangleleft}} \twoheadrightarrow A^J$ and $A^{J^{\triangleright}} \twoheadrightarrow A^J$ over d recover Joyal's slice quasi-categories $A_{/d}$ and ${}^{d/}A$, but from a direct comparison of the defining universal properties left as Exercise 4.2.ii, this can easily be seen not to be the case. However, the fiber of $A^{J^{\triangleleft}} \twoheadrightarrow A^J$ over d is equivalent to $A_{/d}$ and the fiber of $A^{J^{\triangleright}} \twoheadrightarrow A^J$ is equivalent to ${}^{d/}A$. Proposition D.6.5 will prove that in the ∞-cosmos of quasi-categories that $A_{/d} \simeq \mathsf{Hom}_{A^J}(\Delta, d)$ and ${}^{d/}A \simeq \mathsf{Hom}_{A^J}(d, \Delta)$ over A.

Exercises.

4.2.i. Exercise. Compute $\Delta[n] \star \Delta[m]$ and $\Delta[n] \diamond \Delta[m]$ and define a section

$$\Delta[n] \star \Delta[m] \to \Delta[n] \diamond \Delta[m]$$

to the map constructed in the proof of Proposition 4.2.7.

4.2.ii. Exercise. Compute the fiber of $A^{J^{\triangleleft}} \twoheadrightarrow A^J$ over $d\colon 1 \to A^J$ in the ∞-cosmos of quasi-categories and prove that this quasi-category is not isomorphic to $A_{/d}$.

4.3. The universal property of limits and colimits

We now return to the general context of Definition 2.3.1, simultaneously considering diagrams valued in an ∞-category A that are indexed either by a simplicial set or by another ∞-category in the case where the ambient ∞-cosmos is cartesian closed. As was the case for Proposition 4.1.1, Theorem 3.5.7 specializes immediately to the context of Definition 2.3.7 to prove:

4.3.1. Proposition (co/limits represent cones). *A family of diagrams $d\colon D \to A^J$ admits a limit if and only if the ∞-category of cones $\mathsf{Hom}_{A^J}(\Delta, d)$ over d is right representable*

$$\mathsf{Hom}_{A^J}(\Delta, d) \simeq_{D\times A} \mathsf{Hom}_A(A, \ell),$$

in which case the representing functor $\ell\colon D \to A$ defines the limit functor. Dually, $d\colon D \to A^J$ admits a colimit if and only if the ∞-category of cones $\mathrm{Hom}_{A^J}(d,\Delta)$ under d is left representable

$$\mathrm{Hom}_{A^J}(d,\Delta) \simeq_{A\times D} \mathrm{Hom}_A(c,A),$$

in which case the representing functor $c\colon D \to A$ defines the colimit functor. □

Theorem 3.5.11 now specializes to tell us that such representations can be encoded by terminal or initial elements, a result which is easiest to interpret in the case of a single diagram rather than a family of diagrams.

4.3.2. Proposition (limits as terminal elements). *Consider a diagram $d\colon 1 \to A^J$ of shape J in an ∞-category A.*

(i) If d admits a limit, then the 1-cell $1 \to \mathrm{Hom}_{A^J}(\Delta, d)$ induced by the limit cone $\epsilon\colon \Delta\ell \Rightarrow d$ defines a terminal element of the ∞-category of cones.

(ii) Conversely, if the ∞-category of cones $\mathrm{Hom}_{A^J}(\Delta, d)$ admits a terminal element, then the cone represented by this element defines a limit cone.

Dually d admits a colimit if and only if the ∞-category $\mathrm{Hom}_{A^J}(d,\Delta)$ of cones under d admits an initial element, in which case the initial element defines the colimit cone.

Proof. By Definition 2.3.7, a limit cone defines an absolute right lifting diagram, which by Theorem 3.5.3, induces an equivalence $\mathrm{Hom}_A(A,\ell) \xrightarrow{\sim} \mathrm{Hom}_{A^J}(\Delta, d)$ over A. By Corollary 3.5.9, the identity at ℓ induces a terminal element of $\mathrm{Hom}_A(A,\ell)$ which the equivalence carries to a terminal element of the ∞-category of cones, this being the element that represents the limit cone $\epsilon\colon \Delta\ell \Rightarrow d$.

Conversely, if $\mathrm{Hom}_{A^J}(\Delta, d)$ admits a terminal element, this defines a right adjoint right inverse to the codomain-projection functor $\mathrm{Hom}_{A^J}(\Delta, d)$. Theorem 3.5.11 then tells us that the cone represented by this element $1 \to \mathrm{Hom}_{A^J}(\Delta, d)$ defines an absolute right lifting of d through Δ. □

4.3.3. Remark. The proof of Proposition 4.3.2 extends without change to the case of a family of diagrams $d\colon D \to A^J$ in place of a single diagram since Theorem 3.5.11 applies at this level of generality. For a family of diagrams d parametrized by D, the ∞-category of cones defines an object $p_1\colon \mathrm{Hom}_{A^J}(\Delta, d) \twoheadrightarrow D$ of the sliced ∞-cosmos $\mathcal{K}_{/D}$ and the terminal elements referred to in both (i) and (ii) should be interpreted as terminal elements in $\mathcal{K}_{/D}$.

4.3.4. Proposition. *An ∞-category A admits a limit of a family of diagrams $d\colon D \to A^J$ indexed by a simplicial set[3] J if and only if there exists an absolute right lifting of d through the restriction functor*

$$\begin{tikzcd} & A^{J^\triangleleft} \arrow[d, two heads, "\mathrm{res}"] \\ D \arrow[ur, "\mathrm{ran}"] \arrow[r, "d"'] \arrow[ur, phantom, shift right=3ex, "\Downarrow\epsilon"] & A^J \end{tikzcd}$$

When these equivalent conditions hold, ϵ is necessarily an isomorphism and may be chosen to be the identity.

[3]We have stated this result for diagrams indexed by simplicial sets because its means is easiest to interpret, but we actually prove it with the codomain-projection functor $p_1\colon \mathrm{Hom}_{A^J}(\Delta, A^J) \twoheadrightarrow A^J$ in place of the equivalent isofibration $A^{J^\triangleleft} \twoheadrightarrow A^J$, and this proof applies equally in the case of diagrams indexed by ∞-categories J in cartesian closed ∞-cosmoi that may or may not have a join operation available.

Proof. By Definition 2.3.7, the family of diagrams admits a limit if and only if d admits an absolute right lifting through $\Delta\colon A \to A^J$. By Proposition 3.6.14, such an absolute lifting diagram exists if and only if d admits an absolute right lifting through codomain-projection functor $p_1\colon \mathsf{Hom}_{A^J}(\Delta, A^J) \twoheadrightarrow A^J$, in which case the 2-cell of this latter absolute right lifting diagram is invertible. By Corollary 4.2.8, the restriction functor $\mathrm{res}\colon A^{J^{\triangleleft}} \twoheadrightarrow A^J$ is equivalent to this codomain-projection functor, so absolute right liftings of d through p_1 are equivalent to absolute right liftings of d through res. If this absolute lifting diagram is inhabited by an invertible 2-cell, the isomorphism lifting property of the isofibration proven in Proposition 1.4.10 can be used to replace the functor $\mathrm{ran}\colon D \to A^{J^{\triangleleft}}$ with an isomorphic functor making the triangle commute strictly. □

Recall from Lemma 2.3.6 that absolute lifting diagrams can be used to encode the existence of adjoint functors. Combining this with Definition 2.3.2, Proposition 4.3.4 specializes to prove:

4.3.5. Corollary. *An ∞-category A admits all limits indexed by a simplicial set J if and only if the restriction functor*

$$\begin{tikzcd} A^{J^{\triangleleft}} \arrow[r, bend left, two heads, "\mathrm{res}"] \arrow[r, phantom, "\perp"] & A^J \arrow[l, bend left, dashed, "\mathrm{ran}"] \end{tikzcd}$$

admits a right adjoint. Dually, an ∞-category A admits all colimits indexed by a simplicial set J if and only if the restriction functor

$$\begin{tikzcd} A^{J^{\triangleright}} \arrow[r, bend right, two heads, "\mathrm{res}"'] \arrow[r, phantom, "\perp"] & A^J \arrow[l, bend right, dashed, "\mathrm{lan}"'] \end{tikzcd}$$

admits a left adjoint. □

4.3.6. Remark. Since the restriction functor is an isofibration, Lemma 3.6.10 applies and the adjunctions of Corollary 4.3.5 can be defined so as to be fibered over the ∞-category of diagrams A^J.

The adjunctions of Corollary 4.3.5 are particular useful in the case of pullbacks and pushouts.

4.3.7. Definition (pushouts and pullbacks). A **pushout** in an ∞-category A is a colimit indexed by the simplicial set

$$\ulcorner := \Lambda^0[2].$$

Dually, a **pullback** in an ∞-category A is a limit indexed by the simplicial set

$$\lrcorner := \Lambda^2[2].$$

Cones over diagrams of shape $\lrcorner$ or cones under diagrams of shape $\ulcorner$ define **commutative squares**, diagrams of shape

$$\square := \Delta[1] \times \Delta[1] \cong \ulcorner^{\triangleright} \cong \lrcorner^{\triangleleft}\ .$$

A **pullback square** in an ∞-category A is an element of $A^{\square}$ in the essential image of the functor ran of Proposition 4.3.4 for some diagram of shape $\lrcorner$. When A admits all pullbacks, these are exactly those elements of $A^{\square}$ at which the component of the unit of the adjunction res $\dashv$ ran of Corollary 4.3.5 is an isomorphism. Dually, a **pushout square** in A is an element in the essential image of the dual functor lan for some diagram of shape $\ulcorner$, i.e., those elements for which the component of the counit of the adjunction lan $\dashv$ res is an isomorphism.

An X-indexed commutative square in A is a diagram $X \to A^{\boxdot}$, or equivalently, an element of $\mathsf{Fun}(X,A)^{\boxdot}$. We label the 0- and 1-simplex components as follows:

$$\begin{tikzcd} d \arrow[r, "u"] \arrow[d, "v"'] \arrow[dr, "w" description] & b \arrow[d, "f"] \\ c \arrow[r, "g"'] & a \end{tikzcd} \qquad \in \qquad \mathsf{Fun}(X,A)$$

The diagram also determines a pair of 2-simplices that witness commutativity $fu = w = gv$ in $\mathsf{hFun}(X,A)$, but the names of these witnesses won't matter for this discussion.

4.3.8. Lemma. *An X-indexed commutative square valued in an ∞-category A in $\mathcal{K}$ as below-left is a pullback square if and only if the induced 2-cell (id_a, v) below-right is an absolute right lifting diagram in $\mathcal{K}_{/X}$:*

$$\begin{tikzcd} d \arrow[r, "u"] \arrow[d, "v"'] \arrow[dr, "w" description] & b \arrow[d, "f"] \\ c \arrow[r, "g"'] & a \end{tikzcd} \qquad \begin{tikzcd} & & \mathsf{Hom}_A(A,b) \arrow[d, "\ulcorner f_* \urcorner"] \arrow[ddl, dotted, two heads, bend left, "p_1"] \\ X \arrow[urr, bend left, "\ulcorner u \urcorner"] \arrow[rr, "\ulcorner g \urcorner"'] \arrow[dr, dotted, equal] \arrow[urr, phantom, "\Downarrow(\mathrm{id}_a, v)"] & & \mathsf{Hom}_A(A,a) \arrow[dl, dotted, two heads, "p_1"] \\ & X & \end{tikzcd}$$

The statement requires some explanation. The 1-simplex $g\colon c \to a$ in $\mathsf{Fun}(X,A)$ represents a 2-cell $\begin{tikzcd} X \arrow[r, bend left, "c"] \arrow[r, bend right, "a"'] \arrow[r, phantom, "\Downarrow g"] & A \end{tikzcd}$, inducing the map $\ulcorner g \urcorner \colon X \to \mathsf{Hom}_A(A,a)$. The map $\ulcorner u \urcorner \colon X \to \mathsf{Hom}_A(A,b)$ is defined similarly. The map $\ulcorner f_* \urcorner \colon \mathsf{Hom}_A(A,b) \to \mathsf{Hom}_A(A,a)$ is characterized by the pasting diagram

$$\begin{tikzcd} & \mathsf{Hom}_A(A,b) \arrow[dl, two heads, "p_1"'] \arrow[dr, two heads, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\ X \arrow[rr, bend left, "b"] \arrow[rr, bend right, "a"'] \arrow[rr, phantom, "\Downarrow f"] & {} & A \end{tikzcd} \quad = \quad \begin{tikzcd} & \mathsf{Hom}_A(A,b) \arrow[d, dashed, "\ulcorner f_* \urcorner"] \arrow[ddl, two heads, bend right, "p_1"'] \arrow[ddr, two heads, bend left, "p_0"] & \\ & \mathsf{Hom}_A(A,a) \arrow[dl, two heads, "p_1"'] \arrow[dr, two heads, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\ X \arrow[rr, "a"'] & {} & A \end{tikzcd}$$

By Proposition 3.4.7, the composite $\ulcorner f_* \urcorner \ulcorner u \urcorner$ is isomorphic to $\ulcorner fu \urcorner$. By 2-cell induction, the 2-cell may be constructed by specifying its domain and codomain components, the former of which we take to be $\begin{tikzcd} X \arrow[r, bend left, "d"] \arrow[r, bend right, "c"'] \arrow[r, phantom, "\Downarrow v"] & A \end{tikzcd}$ and the latter of which we take to be id_a. Note that the 2-cell just constructed lies in $\mathfrak{h}\mathcal{K}_{/X}$ and so can be lifted to $\mathfrak{h}(\mathcal{K}_{/X})$ by Proposition 3.6.3.

Proof. We prove the result in the case $X = 1$ and then deduce the result for families of pullback diagrams from this case. By Corollary 4.2.8, the pullback

$$\begin{tikzcd} A^{\boxdot}_{g \vee f} \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner" very near start] & A^{\boxdot} \arrow[d, two heads, "\mathrm{res}"] \\ 1 \arrow[r, "g \vee f"'] & A^{\lrcorner} \end{tikzcd}$$

is equivalent to the ∞-category of cones over the cospan diagram $g \vee f$. By Proposition 4.3.2, to show that the commutative square defines a pullback diagram is to show that $(u, v, f, g)\colon 1 \to A^{\boxdot}_{g\vee f}$ defines a terminal element in the pullback. We will show that this pullback $A^{\boxdot}_{g\vee f}$ is also equivalent to the comma ∞-category $\mathsf{Hom}_{\mathsf{Hom}_A(A,a)}(\ulcorner f_* \urcorner, \ulcorner g \urcorner)$. By Theorem 3.5.11, the pair $(\ulcorner u \urcorner, (\mathrm{id}_a, v))$ defines an absolute right lifting if and only if it represents a terminal element in this comma ∞-category, which will prove the claimed equivalence.

To see this first consider the diagram, which induces a map between the two pullbacks

$$\begin{tikzcd}
& A_{/f} \arrow[rr] \arrow[dl, dashed, two heads, "p_f"'] \arrow[dd, two heads] & & A^{\mathbb{3}} \arrow[r, two heads, "p_{01}"] \arrow[dl, "p_{02}"'] \arrow[dd, two heads, "p_{12}"] & A^{\mathbb{2}} \arrow[dd, two heads, "p_1"] \\
\mathsf{Hom}_A(A,a) \arrow[rr] \arrow[dd, two heads] & & A^{\mathbb{2}} \arrow[dd, two heads, "p_1"] & & \\
& 1 \arrow[rr, "f"] \arrow[dl, equal] & & A^{\mathbb{2}} \arrow[r, two heads, "p_0"] \arrow[dl, two heads, "p_1"] & A \\
1 \arrow[rr, "a"'] & & A & &
\end{tikzcd}$$

Since $A^{\mathbb{3}} \simeq A^{\Lambda^1[2]}$, the right-hand back square is equivalent to a pullback. Composing the pullback squares in the back face of the diagram, we obtain an equivalence $A_{/f} \xrightarrow{\sim} \mathsf{Hom}_A(A, b)$ and by inspection see that the map $p_f\colon A_{/f} \twoheadrightarrow \mathsf{Hom}_A(A, a)$ is equivalent to the map $\ulcorner f_* \urcorner\colon \mathsf{Hom}_A(A, b) \to \mathsf{Hom}_A(A, a)$ over $\mathsf{Hom}_A(A, a)$.

By applying $(-)^{\mathbb{2}}$ to the pullback diagram that defines $\mathsf{Hom}_A(A, a)$ we obtain a pullback square that factors as:

$$\begin{tikzcd}
\mathsf{Hom}_A(A,a)^{\mathbb{2}} \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & A^{\mathbb{2}\diamond\mathbb{1}} \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & A^{\mathbb{2}\times\mathbb{2}} \arrow[d, two heads, "p_1^{\mathbb{2}}"] \\
1 \arrow[r, "a"] & A \arrow[r, "\Delta"] & A^{\mathbb{2}}
\end{tikzcd}$$

By the equivalence $A^{\mathbb{2}\diamond\mathbb{1}} \simeq A^{\mathbb{2}\star\mathbb{1}}$ of Corollary 4.2.8, the left-hand pullback square shows that $\mathsf{Hom}_A(A, a)^{\mathbb{2}}$ is equivalent to the pullback of $p_2\colon A^{\mathbb{3}} \twoheadrightarrow A$ along $a\colon 1 \to A$. Modulo this equivalence, the map $p_0\colon \mathsf{Hom}_A(A, a)^{\mathbb{2}} \twoheadrightarrow \mathsf{Hom}_A(A, a)$ is the pullback of the fibered map

$$\begin{tikzcd}
A^{\mathbb{3}} \arrow[rr, two heads, "p_{02}"] \arrow[dr, two heads, "p_2"'] & & A^{\mathbb{2}} \arrow[dl, two heads, "p_1"] \\
& A &
\end{tikzcd}$$

along $a\colon 1 \to A$ and the codomain projection is similarly the pullback of the fibered map $p_{12}\colon A^{\mathbb{3}} \twoheadrightarrow A^{\mathbb{2}}$.

Putting this together, it follows that the pullback

$$\begin{tikzcd}
\mathsf{Hom}_{\mathsf{Hom}_A(A,a)}(\ulcorner f_* \urcorner, \mathsf{Hom}_A(A,a)) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Hom}_A(A,a)^{\mathbb{2}} \arrow[d, two heads, "p_0"] \\
\mathsf{Hom}_A(A,b) \arrow[r, "\ulcorner f_* \urcorner"'] & \mathsf{Hom}_A(A,a)
\end{tikzcd}$$

is equivalently computed by forming the limit

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[dd, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & A^{\boxdot} \arrow[r, two heads] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A^{3} \arrow[r, two heads, "p_{12}"] \arrow[d, two heads, "p_{02}"] & A^{2} \\
& A^{3} \arrow[r, two heads, "p_{02}"'] \arrow[d, two heads, "p_{12}"] & A^{2} & \\
1 \arrow[r, "f"'] & A^{2} & &
\end{tikzcd}$$

The codomain projection $p_1 \colon \mathsf{Hom}_{\mathsf{Hom}_A(A,a)}(\ulcorner f_* \urcorner, \mathsf{Hom}_A(A,a)) \twoheadrightarrow \mathsf{Hom}_A(A,a)$ is the pullback of the top-horizontal composite in the above diagram along $\mathsf{Hom}_A(A,a) \to A^{2}$. So we see that the comma ∞-category $\mathsf{Hom}_{\mathsf{Hom}_A(A,a)}(\ulcorner f_* \urcorner, \ulcorner g \urcorner)$ is equivalently computed by the limit below-left, or equivalently by the limit below right, exactly as we claimed:

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \bullet \arrow[rr, two heads] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & & 1 \arrow[d, "g"] \\
\bullet \arrow[r] \arrow[dd, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & A^{\boxdot} \arrow[r, two heads] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A^{3} \arrow[r, two heads, "p_{12}"] \arrow[d, two heads, "p_{02}"] & A^{2} \\
& A^{3} \arrow[r, two heads, "p_{02}"'] \arrow[d, two heads, "p_{12}"] & A^{2} & \\
1 \arrow[r, "f"'] & A^{2} & &
\end{tikzcd}
\qquad
\begin{tikzcd}
A^{\boxdot}_{g \vee f} \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & A^{\boxdot} \arrow[d, two heads, "\mathrm{res}"] \\
1 \arrow[r, "g \vee f"'] & A^{\lrcorner}
\end{tikzcd}$$

The same computation proves the general case for $X \neq 1$ when the comma ∞-category is constructed in $\mathcal{K}_{/X}$; see Proposition 1.2.22 for a description of the simplicial limits in sliced ∞-cosmoi. Alternatively, a diagram $s \colon X \to A^{\boxdot}$ in $\mathcal{K}$ also defines a X-indexed diagram in the ∞-cosmos $\mathcal{K}_{/X}$ valued in the ∞-category $\pi \colon A \times X \to X$. This takes the form of a functor $(d, \mathrm{id}_X) \colon X \to A^{\boxdot} \times X$ over X. It's easy to verify that a diagram valued in $\pi \colon A \times X \twoheadrightarrow X$ whose component at X is the identity has a limit in $\mathcal{K}_{/X}$ if and only if the A-component of the diagram has a limit in $\mathcal{K}$. Since id_X is the terminal object of $\mathcal{K}_{/X}$, this object is the ∞-category $1 \in \mathcal{K}_{/X}$, so the proof just give applies to prove the general case of X-indexed families of commutative squares. □

There is an automorphism of the simplicial set 2×2 that swaps the "intermediate" vertices $(0,1)$ and $(1,0)$, which induces a "transposition" automorphism of $A^{\boxdot}$. By symmetry, a commutative square in A is a pullback if and only if its transposed square is a pullback. This gives a dual form of Lemma 4.3.8 with the roles of f and g and of u and v interchanged. As a corollary, we can easily prove that pullback squares compose both "vertically" and "horizontally" and can be cancelled from the "right" and "bottom":

4.3.9. Proposition (composition and cancelation of pullback squares). *Given a composable pair of X-indexed commutative squares in A and their composite rectangle defined via the equivalence* $A^{3 \times 2} \simeq$

$A^{\boxdot} \underset{A^2}{\times} A^{\boxdot}$

$$\begin{tikzcd} p \arrow[r, "x"] \arrow[d, "y"'] \arrow[dr, "z"] & d \arrow[r, "u"] \arrow[d, "v"] \arrow[dr, "w"] & b \arrow[d, "f"] \\ e \arrow[r, "h"'] & c \arrow[r, "g"'] & a \end{tikzcd}$$

if the right-hand square is a pullback, then the left-hand square is a pullback if and only if the composite rectangle is a pullback.

Proof. By Lemma 4.3.8, we are given an absolute right lifting diagram in $\mathcal{K}_{/X}$

$$\begin{tikzcd} & \mathsf{Hom}_A(A,c) \arrow[d, "\ulcorner g_* \urcorner"] \\ X \arrow[ur, bend left, "\ulcorner v \urcorner"] \arrow[r, "\ulcorner f \urcorner"'] \arrow[ur, phantom, "\Downarrow(\mathrm{id}_a, u)"] & \mathsf{Hom}_A(A,a) \end{tikzcd}$$

By Lemma 2.4.1, the composite diagram

$$\begin{tikzcd} & \mathsf{Hom}_A(A,e) \arrow[d, "\ulcorner h_* \urcorner"] \\ & \mathsf{Hom}_A(A,c) \arrow[d, "\ulcorner g_* \urcorner"] \\ X \arrow[uur, bend left, "\ulcorner y \urcorner"] \arrow[ur, bend left, "\ulcorner v \urcorner"] \arrow[r, "\ulcorner f \urcorner"'] \arrow[ur, phantom, "\Downarrow(\mathrm{id}_a, u)"] & \mathsf{Hom}_A(A,a) \end{tikzcd}$$

with $\Downarrow(\mathrm{id}_c, x)$ in the top triangle, is an absolute right lifting diagram in $\mathcal{K}_{/X}$ if and only if the top triangle is an absolute right lifting diagram in $\mathcal{K}_{/X}$. By Lemma 4.3.8, this is exactly what we wanted to show. $\square$

Terminal elements are special cases of limits where the diagram shape is empty. For any ∞-category A, the ∞-category of diagrams $A^{\varnothing} \cong 1$, which tells us that there is a unique $\varnothing$-indexed diagram in A. In this context, the ∞-categories of cones over or under the unique diagram constructed in Definition 4.2.1 are isomorphic to A. In the case of cones over an empty diagram, the domain-evaluation functor, carrying a cone to its summit, is the identity on A, while in the case of cones under the empty diagram, the codomain-evaluation functor, carrying a cone to its nadir, is the identity on A. The following characterization of terminal elements can be deduced as a special case of Proposition 4.3.1, though we find it easier to argue from Proposition 4.1.1.

4.3.10. Proposition. *For an element $t\colon 1 \to A$ of an ∞-category A,*

(i) t defines a terminal element of A if and only if the domain-projection functor $p_0\colon \mathsf{Hom}_A(A,t) \twoheadrightarrow A$ is a trivial fibration.

(ii) t defines an initial element of A if and only if the codomain-projection functor $p_1\colon \mathsf{Hom}_A(t,A) \twoheadrightarrow A$ is a trivial fibration.

Proof. Recall from Definition 2.2.1, that an element is terminal if and only if it is right adjoint to the unique functor

$$\begin{tikzcd} 1 \arrow[r, bend right, "t"'] \arrow[r, phantom, "\perp"] & A \arrow[l, bend right, "!"'] \end{tikzcd}$$

By Proposition 4.1.1, $! \dashv t$ if and only if there is an equivalence $\mathsf{Hom}_1(!, 1) \simeq_A \mathsf{Hom}_A(A, t)$. By the defining pullback (3.4.2) for the comma ∞-category, the left representation of $!\colon A \to 1$ is A itself, with domain-projection functor the identity. So the component of the equivalence $\mathsf{Hom}_A(A,t) \xrightarrow{\sim} A$ over A must be the domain projection functor $p_0\colon \mathsf{Hom}_A(A,t) \twoheadrightarrow A$, and we conclude that t is a terminal element if and only if this isofibration is a trivial fibration. ☐

4.3.11. Digression (terminal elements of a quasi-category). In the ∞-cosmos of quasi-categories, the isofibration $p_0\colon \mathsf{Hom}_A(A,t) \twoheadrightarrow A$ is equivalent over A to the *slice quasi-category* $A_{/t}$, defined as a right adjoint to the join construction of Definition 4.2.4. Proposition 4.3.10 proves that t is terminal if and only if the projection $A_{/t} \twoheadrightarrow A$ is a trivial fibration in the sense of Definition 1.1.24, which transposes to Joyal's original definition of a terminal element of a quasi-category. See Appendix F for a full proof.

We conclude with two results that could have been proven in Chapter 2, were it not for one small step of the argument, as we explain. A functor $f\colon A \to B$ *preserves* limits if the image of a limit cone in A also defines a limit cone in B. In the other direction, a functor $f\colon A \to B$ *reflects* limits if a cone in A that defines a limit cone in B is also a limit cone in A.

4.3.12. Proposition. *A fully faithful functor $f\colon A \to B$ reflects any limits or colimits that exist in B.*

Proof. The statement for limits asserts that given any family of diagrams $d\colon D \to A^J$ of shape J in A, any functor $\ell\colon D \to A$ and cone $\rho\colon \Delta\ell \Rightarrow d$ as below-left so that the whiskered composite with $f^J\colon A^J \to B^J$ displayed below is an absolute right lifting diagram

$$\begin{tikzcd} & A \arrow[r, "f"] \arrow[d, "\Delta"] & B \arrow[d, "\Delta"] \\ D \arrow[ur, "\ell"] \arrow[r, "d"'] \arrow[ur, phantom, shift left=3, "\Downarrow\rho"] & A^J \arrow[r, "f^J"'] & B^J \end{tikzcd}$$

then (ℓ, ρ) defines an absolute right lifting of $d\colon D \to A^J$ through $\Delta\colon A \to A^J$. Our proof strategy mirrors the results of §2.4. By Corollary 3.5.6(i), to say that f is fully faithful is to say that $\mathrm{id}_A\colon A \to A$ defines an absolute right lifting of f through itself. So by Lemma 2.4.1 and the hypothesis just stated, the composite diagram below-left is an absolute right lifting diagram, and by 2-functoriality of the simplicial cotensor with J, the diagram below-left coincides with the diagram below-right:

$$\begin{tikzcd} & & A \arrow[d, "f"] \\ & A \arrow[ur, equal] \arrow[r, "f"] \arrow[d, "\Delta"] & B \arrow[d, "\Delta"] \\ D \arrow[ur, "\ell"] \arrow[r, "d"'] \arrow[ur, phantom, shift left=3, "\Downarrow\rho"] & A^J \arrow[r, "f^J"'] & B^J \end{tikzcd} \quad = \quad \begin{tikzcd} & & A \arrow[d, "\Delta"] \\ & & A^J \arrow[d, "f^J"] \\ D \arrow[uurr, "\ell"] \arrow[r, "d"'] & A^J \arrow[ur, equal] \arrow[r, "f^J"'] \arrow[u, phantom, "\Downarrow\rho"] & B^J \end{tikzcd}$$

By Corollary 3.5.6(iv) to say that f is fully faithful is to say that $\ulcorner \mathrm{id}_f \urcorner\colon A^{\mathbb{2}} \xrightarrow{\sim} \mathsf{Hom}_B(f,f)$ is a fibered equivalence over $A \times A$. Applying $(-)^J\colon \mathcal{K} \to \mathcal{K}$, this maps to a fibered equivalence $\ulcorner \mathrm{id}_{f^J} \urcorner\colon (A^J)^{\mathbb{2}} \xrightarrow{\sim} \mathsf{Hom}_{B^J}(f^J, f^J)$ over $A^J \times A^J$, proving that if $f\colon A \to B$ is fully faithful, then $f^J\colon A^J \to B^J$ is also.[4] Hence by Corollary 3.5.6(i), $\mathrm{id}_{A^J}\colon A^J \to A^J$ defines an absolute right lifting of f^J through itself.

[4]This is the statement that we could not yet prove in Chapter 2.

Applying Lemma 2.4.1 again, we now conclude that (ℓ, ρ) is an absolute right lifting of d through Δ as required. $\square$

An alternate approach to proving this result is suggested as Exercise 4.3.iii.

Our final result, proves that, for I and J simplicial sets, whenever we are given a J-indexed diagram valued in the ∞-category A^I of I-indexed diagrams in A, its limit may be computed pointwise in the vertices of I as the limit of the corresponding J-indexed diagram in A. Our argument requires the following representable characterization of absolute lifting diagrams whose proof again makes use of the fact that they are preserved by cosmological functors.

4.3.13. Proposition. *A natural transformation defined in an ∞-cosmos $\mathcal{K}$ as below-left is an absolute right lifting diagram if and only if its "externalization" displayed below-right defines a right lifting diagram in $\mathcal{QC}at$*

$$\begin{tikzcd} & B \arrow[d, "f"] \\ C \arrow[ur, "r"] \arrow[r, "g"'] \arrow[ur, phantom, "\Downarrow\rho" near end, shift right=2] & A \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} & \mathsf{Fun}(C,B) \arrow[d, "f_*"] \\ \mathbb{1} \arrow[ur, "r"] \arrow[r, "g"'] \arrow[ur, phantom, "\Downarrow\rho" near end, shift right=2] & \mathsf{Fun}(C,A) \end{tikzcd}$$

that is preserved by precomposition with any functor $c\colon X \to C$ in $\mathcal{K}$ in the sense that the diagram below is also right lifting:

$$\begin{tikzcd} & \mathsf{Fun}(C,B) \arrow[r, "c^*"] \arrow[d, "f_*"] & \mathsf{Fun}(X,B) \arrow[d, "f_*"] \\ \mathbb{1} \arrow[ur, "r"] \arrow[r, "g"'] \arrow[ur, phantom, "\Downarrow\rho" near end, shift right=2] & \mathsf{Fun}(C,A) \arrow[r, "c^*"'] & \mathsf{Fun}(X,A) \end{tikzcd} = \begin{tikzcd} & \mathsf{Fun}(X,B) \arrow[d, "f_*"] \\ \mathbb{1} \arrow[ur, "rc"] \arrow[r, "gc"'] \arrow[ur, phantom, "\Downarrow\rho c" near end, shift right=2] & \mathsf{Fun}(X,C) \end{tikzcd}$$

Moreover the externalized right lifting diagrams in $\mathcal{QC}at$ are in fact absolute.

Proof. Since $\mathsf{Fun}(\mathbb{1}, -)\colon \mathcal{QC}at \to \mathcal{QC}at$ is naturally isomorphic to the identity functor, for any ∞-categories $X, A \in \mathcal{K}$ we have $\mathsf{Fun}(X,A) \cong \mathsf{Fun}(\mathbb{1}, \mathsf{Fun}(X,A))$ and hence

$$\mathsf{hFun}(X,A) \cong \mathsf{hFun}(\mathbb{1}, \mathsf{Fun}(X,A)). \tag{4.3.14}$$

This justifies our use of the same name ρ for the 2-cell in $\mathfrak{h}\mathcal{K}$ and the 2-cell in $\mathfrak{h}\mathcal{QC}at$ in the statement.

To say that (r, ρ) defines a right lifting of g through f in $\mathfrak{h}\mathcal{K}$ asserts a bijection between 2-cells in $\mathsf{hFun}(C,B)$ with codomain r and 2-cells in $\mathsf{hFun}(C,A)$ with codomain g and domain factoring through f implemented by pasting with ρ. Under the correspondence of (4.3.14), this asserts equally that r defines a right lifting of g through f_* in $\mathfrak{h}\mathcal{QC}at$. To say that (r, ρ) defines an *absolute* right lifting of g through f is to assert the analogous right lifting property for the pair $(rc, \rho c)$ defined by restricting along any $c\colon X \to C$. This is exactly the first claim of the statement.

It remains only to argue that if (r, ρ) is an absolute right lifting diagram in $\mathcal{K}$, then its externalized right lifting diagram of quasi-categories is also absolute. To see this, first note that the cosmological functor $\mathsf{Fun}(C, -)\colon \mathcal{K} \to \mathcal{QC}at$ preserves this absolute right lifting diagram, yielding an absolute right lifting diagram of quasi-categories as below left:

$$\begin{tikzcd} & \mathsf{Fun}(C,B) \arrow[d, "f_*"] \\ \mathsf{Fun}(C,C) \arrow[ur, "r_*"] \arrow[r, "g_*"'] \arrow[ur, phantom, "\Downarrow\mathsf{Fun}(C,\rho)" near end, shift right=2] & \mathsf{Fun}(C,A) \end{tikzcd} \qquad \begin{tikzcd} & \mathsf{Fun}(C,B) \arrow[d, "f_*"] \\ \mathbb{1} \arrow[ur, "r"] \arrow[r, "g"'] \arrow[ur, phantom, "\Downarrow\rho" near end, shift right=2] & \mathsf{Fun}(C,A) \end{tikzcd} = \begin{tikzcd} & & \mathsf{Fun}(C,B) \arrow[d, "f_*"] \\ \mathbb{1} \arrow[r, "\mathrm{id}_C"] & \mathsf{Fun}(C,C) \arrow[ur, "r_*"] \arrow[r, "g_*"'] \arrow[ur, phantom, "\Downarrow\mathsf{Fun}(C,\rho)" near end, shift right=2] & \mathsf{Fun}(C,A) \end{tikzcd}$$

We obtain the desired absolute right lifting diagram by evaluating at the identity. □

4.3.15. Proposition. *Let I and J be simplicial sets and let A be an ∞-category. Then a diagram as below-left is an absolute right lifting diagram*

$$\begin{tikzcd} & A^I \arrow[d, "\Delta^I"] \\ D \arrow[ur, "\lim"] \arrow[r, "d"'] \arrow[ur, phantom, "\Downarrow\rho", shift right=3] & A^{I\times J} \end{tikzcd} \qquad \begin{tikzcd} & A^I \arrow[d, "\Delta^I"] \arrow[r, "\mathrm{ev}_i"] & A \arrow[d, "\Delta"] \\ D \arrow[ur, "\lim"] \arrow[r, "d"'] \arrow[ur, phantom, "\Downarrow\rho", shift right=3] & A^{I\times J} \arrow[r, "\mathrm{ev}_i"'] & A^J \end{tikzcd}$$

if for each vertex $i \in I$, the diagram above right is an absolute right lifting diagram.

Note this statement is not a biconditional. Even in the case of strict 1-categories, there may exist coincidental limits of diagram valued in functor categories that are not defined pointwise [**17**, 2.17.10].

Proof. By Proposition 4.3.13, we may externalize and instead show that the diagram of quasi-categories displayed below left

$$\begin{tikzcd} & \mathsf{Fun}(D,A)^I \arrow[d, "\Delta^I"] \\ \mathbb{1} \arrow[ur, "\lim"] \arrow[r, "d"'] \arrow[ur, phantom, "\Downarrow\rho", shift right=3] & \mathsf{Fun}(D,A)^{I\times J} \end{tikzcd} \qquad \begin{tikzcd} & \mathsf{Fun}(D,A)^I \arrow[d, "\Delta^I"] \arrow[r, "\mathrm{ev}_i"] & \mathsf{Fun}(D,A) \arrow[d, "\Delta"] \\ \mathbb{1} \arrow[ur, "\lim"] \arrow[r, "d"'] \arrow[ur, phantom, "\Downarrow\rho", shift right=3] & \mathsf{Fun}(D,A)^{I\times J} \arrow[r, "\mathrm{ev}_i"'] & \mathsf{Fun}(D,A)^J \end{tikzcd}$$

is absolute right lifting lifting if the diagrams above-right are for each vertex $i \in I$. By naturality of our methods, our proof will show that the absolute right lifting diagrams on the left are preserved by postcomposition with the restriction functors induced by $d\colon X \to D$ if the same is true on the right.

We simplify our notation and write Q for the quasi-category $\mathsf{Fun}(D,A)$ and assume that for each $i \in I$ the diagram

$$\begin{tikzcd} & Q^I \arrow[d, "\Delta^I"] \arrow[r, "\mathrm{ev}_i"] & Q \arrow[d, "\Delta"] \\ \mathbb{1} \arrow[ur, "\lim"] \arrow[r, "d"'] \arrow[ur, phantom, "\Downarrow\rho", shift right=3] & Q^{I\times J} \arrow[r, "\mathrm{ev}_i"'] & Q^J \end{tikzcd}$$

is absolute right lifting. By 2-adjunction $(- \times I) \dashv (-)^I$, a diagram as below-left is absolute right lifting if and only if the transposed diagram below-right is a right lifting diagram and this remains the case upon restricting along functors of the form $\pi\colon X \times I \to I$.[5]

$$\begin{tikzcd} & Q^I \arrow[d, "\Delta^I"] \\ \mathbb{1} \arrow[ur, "\lim"] \arrow[r, "d"'] \arrow[ur, phantom, "\Downarrow\rho", shift right=3] & Q^{I\times J} \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} & Q \arrow[d, "\Delta"] \\ I \arrow[ur, "\lim"] \arrow[r, "d"'] \arrow[ur, phantom, "\Downarrow\rho", shift right=3] & Q^J \end{tikzcd} \tag{4.3.16}$$

We'll argue that this right-hand diagram is in fact absolute right lifting, which implies that the left-hand diagram is absolute right lifting as well, as desired.

[5]If the reader is concerned that I is not a quasi-category, there are two ways to proceed. One is to replace I by a quasi-category I' by inductively attaching fillers for inner horns; note that I and I' will have the same sets of vertices. By Proposition 1.1.28, the diagram quasi-categories $Q^{I'}$ and Q^I are equivalent. The other option is to observe that it doesn't matter if I is a quasi-category or not, because we may define $\mathsf{hFun}(I,Q) := \mathsf{h}(Q^I)$ and by Corollary 1.1.21 Q^I is a quasi-category regardless of whether I is.

To see this we appeal to Theorem 12.2.9, which tells us that universal properties in $\mathcal{QC}at$ are detected pointwise. Specifically, this tells us that the triangle above right is an absolute right lifting diagram if and only if the restricted diagram is absolute right lifting for each vertex $i \in I$

$$\begin{tikzcd} & & Q \arrow[d, "\Delta"] \\ \mathbb{1} \arrow[r, "i"] & I \arrow[ur, "\lim"] \arrow[r, "d"'] \arrow[ur, phantom, "\Downarrow\rho", shift left=3] & Q^J \end{tikzcd}$$

and this is exactly what we have assumed in (4.3.16). ☐

Exercises.

4.3.i. Exercise. Prove that if A has a terminal element t then for any element a the mapping space $\mathsf{Hom}_A(a,t)$ is **contractible**, i.e., is equivalent to the terminal ∞-category 1.

4.3.ii. Exercise. Prove that a square in A is a pullback if and only if its "transposed" square, defined by composing with the involution $A^{\boxdot} \cong A^{\boxdot}$ induced from the automorphism of $\mathbb{2}\times\mathbb{2}$ that swaps the "off-diagonal" elements, is a pulllback square.

4.3.iii. Exercise ([**94**, 3.7]). Use Theorem 3.5.3 and Corollary 3.5.6(iv) to prove that a fully faithful functor $f \colon A \to B$ reflects all limits or colimits that exist in A. Why does this argument not also show that $f \colon A \to B$ preserves them?

4.4. Loops and suspension in pointed ∞-categories

4.4.1. Definition (pointed ∞-categories). An ∞-category A is **pointed** if it admits a **zero element**: an element $* \colon 1 \to A$ that is both initial and terminal.

The counit of the adjunction $* \dashv !$ that witnesses the initiality of the zero element defines a natural transformation $\rho \colon *! \Rightarrow \mathrm{id}_A$ that we refer to as the **family of points** of A. Dually, the unit of the adjunction $! \dashv *$ that witnesses the terminality of the zero element defines a natural transformation $\xi \colon \mathrm{id}_A \Rightarrow *!$ that we refer to as the **family of copoints**.

Cospans and spans in an ∞-category A may be defined by gluing together a pair of arrows along their codomains or domains respectively:

$$\begin{tikzcd} A^{\lrcorner} \arrow[r, two heads] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & A^{\mathbb{2}} \arrow[d, two heads, "p_1"] \\ A^{\mathbb{2}} \arrow[r, two heads, "p_1"'] & A \end{tikzcd} \qquad \begin{tikzcd} A^{\ulcorner} \arrow[r, two heads] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & A^{\mathbb{2}} \arrow[d, two heads, "p_0"] \\ A^{\mathbb{2}} \arrow[r, two heads, "p_0"'] & A \end{tikzcd}$$

For instance, the family of points in a pointed ∞-category A is represented by a functor $\rho \colon A \to A^{\mathbb{2}}$ whose domain-component is constant at $*$ and whose codomain component is id_A. Gluing two copies of this map along their codomain defines a diagram $\check{\rho} \colon A \to A^{\lrcorner}$. Dually, there is a diagram $\hat{\xi} \colon A \to A^{\ulcorner}$ defined by gluing the functor $\xi \colon A \to A^{\mathbb{2}}$ that represents the family of copoints to itself along their domains.

4.4.2. Definition (loops and suspension). A pointed ∞-category A **admits loops** if it admits a limit of the family of diagrams

$$\begin{tikzcd} & A \arrow[d, "\Delta"] \\ A \arrow[ur, dashed, "\Omega"] \arrow[r, "\check{\rho}"'] \arrow[ur, phantom, shift left=2, "\Downarrow"] & A^{\lrcorner} \end{tikzcd}$$

in which case the limit functor $\Omega\colon A \to A$ is called the **loops functor**. Dually, a pointed ∞-category A **admits suspensions** if it admits a colimit of the family of diagrams

$$\begin{tikzcd} & A \arrow[d, "\Delta"] \\ A \arrow[ur, dashed, "\Sigma"] \arrow[r, "\hat{\xi}"'] \arrow[ur, phantom, shift left=2, "\Uparrow"] & A^{\ulcorner} \end{tikzcd}$$

in which case the colimit functor $\Sigma\colon A \to A$ is called the **suspension functor**.

Importantly, if A admits loops and suspensions, then the loops and suspension functors are adjoint:

4.4.3. Proposition (the loops-suspension adjunction). *If A is a pointed ∞-category that admits loops and suspensions, then the loops functor is right adjoint to the suspension functor*

$$\begin{tikzcd} A \arrow[r, bend right, "\Omega"'] \arrow[r, phantom, "\perp"] & A \arrow[l, bend right, "\Sigma"'] \end{tikzcd}$$

The main idea of the proof is easy to describe. If A admits all pullbacks and all pushouts, then Corollary 4.3.5 supplies adjunctions

$$\begin{tikzcd} A^{\lrcorner} \arrow[rr, bend right, "\mathrm{ran}"'] \arrow[rr, phantom, "\perp"] & & A^{\square} \arrow[ll, bend right, "\mathrm{res}"'] \arrow[rr, bend right, "\mathrm{res}"'] \arrow[rr, phantom, "\perp"] & & A^{\ulcorner} \arrow[ll, bend right, "\mathrm{lan}"'] \end{tikzcd}$$

that are fibered over $A \times A$ upon evaluating at the intermediate vertices of the commutative square. By pulling back along $(*,*)\colon 1 \to A \times A$, we can pin these vertices at the zero element. Since the zero element is initial and terminal, the ∞-categories of pullback and pushout diagrams of this form are both equivalent to A and the pulled-back adjoints now coincide with the loops and suspension functors.

The only subtlety in the proof that follows is that we have assumed weaker hypotheses: that A admits only loops and suspensions, but perhaps not all pullbacks and pushouts.

Proof. The diagram $\check{\rho}$ lands in a subobject $A^{\lrcorner}_{*}$ of $A^{\lrcorner}$ defined below-left that is comprised of those pullback diagrams whose source elements are pinned at the zero element $*$ of A.

$$\begin{tikzcd} A \arrow[dr, dashed, "\check{\rho}_*"'] \arrow[drr, bend left, "\rho\vee\rho"] \arrow[ddr, bend right] & & \\ & A^{\lrcorner}_{*} \arrow[r, hook] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner" very near start] & A^{\lrcorner} \arrow[d, two heads] \\ & 1 \arrow[r, "{(*,*)}"'] & A\times A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[dr, dashed, "\check{\rho}_*"'] \arrow[drr, bend left, "\rho"] \arrow[ddr, bend right, "\rho"'] & & \\ & A^{\lrcorner}_{*} \arrow[r, two heads, "\sim"] \arrow[d, two heads, "\wr"'] \arrow[dr, phantom, "\lrcorner" very near start] & \mathsf{Hom}_A(*,A) \arrow[d, two heads, "\wr"', "p_1"] \\ & \mathsf{Hom}_A(*,A) \arrow[r, two heads, "\sim", "p_1"'] & A \end{tikzcd}$$

From a second construction of $A^{\lrcorner}_{*}$ displayed above-right and the characterization of initiality given in Proposition 4.3.10, we may apply the 2-of-3 property of equivalences to conclude first that $\rho\colon A \xrightarrow{\sim} \mathsf{Hom}_A(*,A)$ and then that the induced diagram $\check{\rho}_*\colon A \xrightarrow{\sim} A^{\lrcorner}_{*}$ are equivalences. Dually, the diagram $\hat{\xi}\colon A \to A^{\ulcorner}$ defines an equivalence $\hat{\xi}_*\colon A \xrightarrow{\sim} A^{\ulcorner}_{*}$ when its codomain is restricted to the subobject of pushout diagrams whose target elements are pinned at the zero element $*$.

By Proposition 4.3.4, a pointed ∞-category A admits loops or admits suspensions if and only if there exist absolute lifting diagrams as below-left and below-right respectively:

$$\begin{tikzcd} & A^{\boxdot} \arrow[d, "\mathrm{res}"] \\ A \arrow[ur, dashed, "\mathrm{ran}"] \arrow[r, "\check{\rho}"'] \arrow[ur, phantom, "\cong\Downarrow" near end, shift right=3] & A^{\lrcorner} \end{tikzcd} \qquad \begin{tikzcd} & A^{\boxdot} \arrow[d, "\mathrm{res}"] \\ A \arrow[ur, dashed, "\mathrm{lan}"] \arrow[r, "\hat{\xi}"'] \arrow[ur, phantom, "\cong\Uparrow" near end, shift right=3] & A^{\ulcorner} \end{tikzcd}$$

By Theorem 3.5.3 the absolute right lifting diagram defines a right representation $\mathsf{Hom}_{A^{\boxdot}}(A^{\boxdot},\mathrm{ran}) \simeq \mathsf{Hom}_{A^{\lrcorner}}(\mathrm{res},\check{\rho})$, this being a fibered equivalence over $A\times A^{\boxdot}$. The represented comma ∞-category may be pulled back along the inclusion of the subobject $A^{\boxdot}_{*} \hookrightarrow A^{\boxdot}$ of commutative squares in A whose intermediate vertices are pinned at the zero object:

$$\begin{tikzcd}[row sep=small, column sep=small] & \mathsf{Hom}_{A^{\lrcorner}}(\mathrm{res},\check{\rho}) \arrow[rr] \arrow[dd, two heads] & & (A^{\lrcorner})^{2} \arrow[dd, two heads] \\ \mathsf{Hom}_{A^{\lrcorner}_{*}}(\mathrm{res},\check{\rho}_*) \arrow[ur, hook] \arrow[rr, crossing over] \arrow[dd, two heads] & & (A^{\lrcorner}_{*})^{2} \arrow[ur, hook] & \\ & A^{\boxdot}\times A \arrow[rr, "\mathrm{res}\times\check{\rho}" near start] & & A^{\lrcorner}\times A^{\lrcorner} \\ A^{\boxdot}_{*}\times A \arrow[ur, hook] \arrow[rr, "\mathrm{res}\times\check{\rho}_*"'] & & A^{\lrcorner}_{*}\times A^{\lrcorner}_{*} \arrow[ur, hook] \arrow[from=uu, crossing over, two heads] & \end{tikzcd}$$

The right-hand face of this commutative cube is not strictly a pullback but the universal property of the zero element implies that the induced map from $(A^{\lrcorner}_{*})^{2}$ to the pullback is an equivalence. It follows that $\mathsf{Hom}_{A^{\lrcorner}_{*}}(\mathrm{res},\check{\rho}_*)$ is equivalent over $A^{\boxdot}_{*}\times A$ to the pullback of $\mathsf{Hom}_{A^{\lrcorner}}(\mathrm{res},\check{\rho})$ along this inclusion and so the fibered equivalence pulls back to define a right representation for $\mathsf{Hom}_{A^{\lrcorner}_{*}}(\mathrm{res},\check{\rho}_*)$. Dually, the left representation for $\mathsf{Hom}_{A^{\ulcorner}}(\hat{\xi},\mathrm{res})$ pulls back to a left representation for $\mathsf{Hom}_{A^{\ulcorner}_{*}}(\hat{\xi}_*,\mathrm{res})$. By

Theorem 3.5.7 these unpack to define absolute lifting diagrams:

$$\begin{tikzcd} & A_*^{\boxdot} \arrow[d, two heads, "\mathrm{res}"] \\ A \arrow[ur, dashed, "\mathrm{ran}"] \arrow[r, "\check{\rho}_*"'] & A_*^{\lrcorner} \arrow[ul, phantom, near start, "\cong\Downarrow"] \end{tikzcd} \rightsquigarrow \begin{tikzcd} & A_*^{\boxdot} \arrow[d, two heads, "\mathrm{res}"] \\ A_*^{\lrcorner} \arrow[ur, dashed, "\mathrm{ran}"] \arrow[r, equal] & A_*^{\lrcorner} \arrow[ul, phantom, near start, "\cong\Downarrow"] \end{tikzcd} \qquad \begin{tikzcd} & A_*^{\boxdot} \arrow[d, two heads, "\mathrm{res}"] \\ A \arrow[ur, dashed, "\mathrm{lan}"] \arrow[r, "\hat{\xi}_*"'] & A_*^{\ulcorner} \arrow[ul, phantom, near start, "\cong\Uparrow"] \end{tikzcd} \rightsquigarrow \begin{tikzcd} & A_*^{\boxdot} \arrow[d, two heads, "\mathrm{res}"] \\ A_*^{\ulcorner} \arrow[ur, dashed, "\mathrm{lan}"] \arrow[r, equal] & A_*^{\ulcorner} \arrow[ul, phantom, near start, "\cong\Uparrow"] \end{tikzcd}$$

Restricting along the inverse equivalences $A_*^{\lrcorner} \xrightarrow{\sim} A$ and $A_*^{\ulcorner} \xrightarrow{\sim} A$ and pasting with the invertible 2-cell we obtain absolute lifting diagrams whose bottom edge is the identity.

By Lemma 2.3.6, these lifting diagrams define adjunctions:

$$\begin{tikzcd} A \simeq A_*^{\lrcorner} \arrow[r, bend right, "\mathrm{ran}"'] \arrow[r, phantom, "\perp"] & A_*^{\boxdot} \arrow[l, bend right, "\mathrm{res}"'] \arrow[r, bend right, "\mathrm{res}"'] \arrow[r, phantom, "\perp"] & A_*^{\ulcorner} \simeq A \arrow[l, bend right, "\mathrm{lan}"'] \end{tikzcd}$$

which compose to the desired adjunction $\Sigma \dashv \Omega$. □

4.4.4. Definition. An arrow $f \colon 1 \to A^{2}$ from x to y in a pointed ∞-category A **admits a fiber** if A admits a pullback of the diagram defined by gluing f to the component ρy of the family of points. By Proposition 4.3.4 such pullbacks give rise to a pullback square

$$\begin{tikzcd} & A^{\boxdot} \arrow[d, two heads, "\mathrm{res}"] \\ 1 \arrow[ur, "\mathrm{ran}"] \arrow[r, "\rho y \vee f"'] & A^{\lrcorner} \arrow[ul, phantom, near start, "\cong\Downarrow"] \end{tikzcd}$$

that is referred to as the **fiber sequence** for f. Dually, f **admits a cofiber** if A admits a pushout of the diagram defined by gluing f to the component ξx of the family of copoints, in which case the pushout square

$$\begin{tikzcd} & A^{\boxdot} \arrow[d, two heads, "\mathrm{res}"] \\ 1 \arrow[ur, "\mathrm{lan}"] \arrow[r, "\xi x \vee f"'] & A^{\ulcorner} \arrow[ul, phantom, near start, "\cong\Uparrow"] \end{tikzcd}$$

defines the **cofiber sequence** for f.

Fiber and cofiber sequences define commutative squares in A whose lower-left vertex is the zero element $*$. The data of such squares is given by a commutative triangle in A — an element of A^{3} — together with a **nullhomotopy** of the diagonal edge, a witness that this edge factors through the zero element in $\mathfrak{h}A$. Borrowing a classical term from homological algebra, a commutative square in A whose lower-left vertex is the zero element is referred to as a **triangle** in A.

4.4.5. Definition (stable ∞-category). A **stable ∞-category** is a pointed ∞-category A in which

(i) every morphism admits a fiber and a cofiber: that is, there exist absolute lifting diagrams

$$\begin{tikzcd} & A^{\boxdot} \arrow[d, two heads, "\mathrm{res}"] \\ A^{2} \arrow[ur, "\mathrm{ran}"] \arrow[r, "\rho \vee \mathrm{id}"'] & A^{\lrcorner} \arrow[ul, phantom, near start, "\cong\Downarrow"] \end{tikzcd} \qquad \begin{tikzcd} & A^{\boxdot} \arrow[d, two heads, "\mathrm{res}"] \\ A^{2} \arrow[ur, "\mathrm{lan}"] \arrow[r, "\xi \vee \mathrm{id}"'] & A^{\ulcorner} \arrow[ul, phantom, near start, "\cong\Uparrow"] \end{tikzcd}$$

(ii) and a triangle in A defines a fiber sequence if and only if it also defines a cofiber sequence. Such triangles are called **exact**.

A stable ∞-category admits loops and admits suspensions, formed by taking fibers of the arrows in the family of points and cofibers of arrows in the family of copoints respectively.

4.4.6. Proposition. *If A is a stable ∞-category, then $\Sigma \dashv \Omega$ are inverse equivalences.*

Proof. In the proof of Proposition 4.4.3, the adjunction $\Sigma \dashv \Omega$ is constructed as a composite of adjunctions

$$\begin{tikzcd}
A \simeq A_*^{\lrcorner} \arrow[rr, bend right, "\mathrm{ran}"'] \arrow[rr, phantom, "\perp"] & & A_*^{\boxdot} \arrow[ll, bend right, "\mathrm{res}"'] \arrow[rr, bend right, "\mathrm{res}"'] \arrow[rr, phantom, "\perp"] & & A_*^{\ulcorner} \simeq A \arrow[ll, bend right, "\mathrm{lan}"']
\end{tikzcd}$$

that construct fiber and cofiber sequences. By Proposition 2.1.9, the unit and counit of this composite adjunction are given by

$$\begin{tikzcd}
A \arrow[rrrr, equal] \arrow[dr, "\mathrm{lan}"'] \arrow[ddrr, bend right, "\Sigma"'] & & \Downarrow\cong & & A \\
& A_*^{\boxdot} \arrow[rr, equal] \arrow[dr, "\mathrm{res}"'] & \Downarrow\eta & A_*^{\boxdot} \arrow[ur, "\mathrm{res}"'] & \\
& & A \arrow[ur, "\mathrm{ran}"'] \arrow[uurr, bend right, "\Omega"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
& & A \arrow[dr, "\mathrm{lan}"] \arrow[ddrr, bend left, "\Sigma"] & & \\
& A_*^{\boxdot} \arrow[ur, "\mathrm{res}"] \arrow[rr, equal] & \Downarrow\epsilon & A_*^{\boxdot} \arrow[dr, "\mathrm{res}"] & \\
A \arrow[ur, "\mathrm{ran}"] \arrow[uurr, bend left, "\Omega"] \arrow[rrrr, equal] & & \Downarrow\cong & & A
\end{tikzcd}$$

By Definition 4.3.7, the unit of res $\dashv$ ran restricts to an isomorphism on the subobject of pushout squares. In a stable ∞-category, the cofiber sequences in the image of $\mathrm{lan}\colon A \to A_*^{\boxdot}$ are pullback squares, so this tells us that $\eta\,\mathrm{lan}$ is an isomorphism. Dually, the fiber sequences in the image of $\mathrm{ran}\colon A \to A_*^{\boxdot}$ are pushout squares, which tells us that $\epsilon\,\mathrm{ran}$ is an isomorphism. Hence, the unit and counit of $\Sigma \dashv \Omega$ are invertible, so these functors define an adjoint equivalence. ☐

4.4.7. Proposition (finite limits and colimits in stable ∞-categories). *A stable ∞-category admits all pushouts and all pullbacks, and moreover, a square is pushout if and only if it is a pullback.*

Proof. Given a cospan $g \vee f\colon X \to A^{\lrcorner}$ in A, form the cofiber of f followed by the fiber of the composite map $c \to a \to \mathrm{coker}\, f$:

$$\begin{tikzcd}
\ker(qg) \arrow[r, dashed, "u"] \arrow[d, dashed, "v"'] \arrow[dr, phantom, "\lrcorner", very near start] & b \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d, dashed] \\
c \arrow[r, "g"'] & a \arrow[r, dashed, "q"'] \arrow[ur, phantom, "\ulcorner", very near end] & \mathrm{coker}(f)
\end{tikzcd}$$

By Definition 4.4.5(ii), the cofiber sequence $b \to a \to \mathrm{coker}(f)$ is also a fiber sequence. By the pullback cancelation result of Proposition 4.3.9, we conclude that $\ker(qg)$ computes the pullback of the cospan $g \vee f$.

To see that this pullback square is also a pushout, form the fiber of the map v:

$$\begin{tikzcd}
\ker(v) \arrow[r, dashed] \arrow[d, dashed] \arrow[dr, phantom, "\lrcorner", very near start] & \ker(qg) \arrow[r, "u"] \arrow[d, "v"'] \arrow[dr, phantom, "\lrcorner", very near start] & b \arrow[d, "f"'] \\
* \arrow[r] & c \arrow[r, "g"'] & a
\end{tikzcd}$$

By the pullback composition result of Proposition 4.3.9, $\ker(v)$ is also the fiber of the map f. By Definition 4.4.5(ii), the fiber sequences $\ker(v) \to \ker(qg) \to c$ and $\ker(v) \to b \to a$ are also cofiber sequences. Now by the pushout cancelation result of Proposition 4.3.9, we see that the right-hand pullback square is also a pushout square. A dual argument proves that pushouts coincide with pullbacks. $\square$

Exercises.

4.4.i. EXERCISE. Arguing in the homotopy category, show that if an ∞-category A

- admits an initial element i,
- admits a terminal element t, and
- there exists an arrow $t \to i$

then A is a pointed ∞-category.

CHAPTER 5

Fibrations and Yoneda's lemma

The fibers E_b of an isofibration $p\colon E \twoheadrightarrow B$ over an element $b\colon 1 \to B$ are necessarily ∞-categories, so it is natural to ask where the isofibration also encodes the data of functors between the fibers in this family of ∞-categories. Roughly speaking, an isofibration defines a *cartesian fibration* just when the arrows of B act contravariantly functorially on the fibers and a *cocartesian fibration* when the arrows of B act covariantly functorially on the fibers. This action is not strict but rather *pseudofunctorial* in a sense explored in Exercise 5.1.iii. A complete proof of pseudofunctoriality requires the verification of certain conditions, which in this context extend to higher dimensions.

One of the properties that characterizes cocartesian fibrations is an axiom that says that for any 2-cell with codomain B and specified lift of its source 1-cell, there is a lifted 2-cell with codomain E with that one cell as its source. In particular, this lifting property can be applied in the case where the 2-cell in question is a whiskered composite of an arrow in the homotopy category of B as below-left and the lift of the source 1-cell is the canonical inclusion of its fiber:

$$\begin{tikzcd} E_a \arrow[rr, bend left, "\ell_a"] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner" very near start] & & E \arrow[d, two heads, "p"] \\ 1 \arrow[rr, bend left, "a"] \arrow[rr, bend right, "b"'] \arrow[rr, phantom, "\Downarrow\beta"] & & B \end{tikzcd} \quad = \quad \begin{tikzcd} E_a \arrow[rr, bend left, "\ell_a"] \arrow[rr, phantom, "\Downarrow\chi_\beta"] \arrow[rr, dotted, "\beta_*(\ell_a)" description] \arrow[dr, dashed, "\beta_*"'] \arrow[d, two heads] & & E \arrow[d, two heads, "p"] \\ 1 \arrow[rr, bend right, "b"'] & E_b \arrow[l] \arrow[ur, dashed, "\ell_b"'] \arrow[dr, phantom, "\lrcorner" very near start] & B \end{tikzcd} \tag{5.0.1}$$

In this case the codomain $\beta_*(\ell_a)$ of the lifted cell χ_β displayed above right lies strictly above the codomain of the original 2-cell, and thus factors through the pullback defining its fiber. This defines a functor $\beta_*\colon E_a \to E_b$, the "action" of the arrow β on the fibers of p. The pseudofunctoriality of these action maps arises from a universal property required of the specified lifted 2-cells, namely that they are *cartesian* in a sense we now define.

5.1. The 2-category theory of cartesian fibrations

There is a standard notion of cartesian fibration in a 2-category developed by Street [**87**] that recovers the Grothendieck fibrations when specialized to the 2-category $\mathcal{C}at$. This is *not* the correct notion of cartesian fibration between ∞-categories as the universal property the usual notion demands of lifted 2-cells is too strict. Instead of referring to the notions defined here as "weak," we would prefer to refer to the classical notion of cartesian fibration in a 2-category as "strict" were we to refer to it again, which we largely will not.

To remind the reader of the interpretation of the data in the homotopy 2-category of an ∞-cosmos, we refer to 1- and 2-cells as "functors" and "natural transformations." Before defining the notion of cartesian fibration we describe the weak universal property enjoyed by a certain class of "upstairs" natural transformations.

5.1.1. Definition (p-cartesian transformations). Let $p\colon E \twoheadrightarrow B$ be an isofibration. A natural transformation $\begin{tikzcd} X \arrow[r, bend left, "e'"] \arrow[r, bend right, "e"'] \arrow[r, phantom, "\Downarrow\chi"] & E \end{tikzcd}$ with codomain E is p-**cartesian** if

(i) **induction**: Given any natural transformations $\begin{tikzcd} X \arrow[r, bend left, "e''"] \arrow[r, bend right, "e"'] \arrow[r, phantom, "\Downarrow\tau"] & E \end{tikzcd}$ and $\begin{tikzcd} X \arrow[r, bend left, "pe''"] \arrow[r, bend right, "pe'"'] \arrow[r, phantom, "\Downarrow\gamma"] & B \end{tikzcd}$ so that $p\tau = p\chi \cdot \gamma$, there exists a lift $\begin{tikzcd} X \arrow[r, bend left, "e''"] \arrow[r, bend right, "e'"'] \arrow[r, phantom, "\Downarrow\bar{\gamma}"] & E \end{tikzcd}$ of γ so that $\tau = \chi \cdot \bar{\gamma}$.

$$\begin{tikzcd} e'' \arrow[rr, Rightarrow, "\tau"] \arrow[dr, Rightarrow, dotted, "\bar{\gamma}"'] & & e & \in & \mathsf{hFun}(X,E) \arrow[dd, two heads, "p_*"] \\ & e' \arrow[ur, Rightarrow, "\chi"'] \arrow[d, mapsto] & & & \\ pe'' \arrow[rr, Rightarrow, "p\tau"] \arrow[dr, Rightarrow, "\gamma"'] & & pe & \in & \mathsf{hFun}(X,B) \\ & pe' \arrow[ur, Rightarrow, "p\chi"'] & & & \end{tikzcd}$$

(ii) **conservativity**: Any fibered endomorphism of χ is invertible: if $\begin{tikzcd} X \arrow[r, bend left, "e'"] \arrow[r, bend right, "e'"'] \arrow[r, phantom, "\Downarrow\zeta"] & E \end{tikzcd}$ is any natural transformation so that $\chi \cdot \zeta = \chi$ and $p\zeta = \mathrm{id}_{pe'}$ then ζ is invertible.

5.1.2. Remark (why "cartesian"). The induction property for a p-cartesian natural transformation $\chi\colon e' \Rightarrow e$ says that for any $e''\colon X \to E$, there is a surjective function from the set $\mathsf{hFun}(X,E)(e'',e')$ of natural transformations from e'' to e' to the pullback induced in the commutative square

$$\begin{tikzcd} \mathsf{hFun}(X,E)(e'',e') \arrow[r, "\chi\circ-"] \arrow[d, "p"'] & \mathsf{hFun}(X,E)(e'',e) \arrow[d, "p"] \\ \mathsf{hFun}(X,B)(pe'',pe') \arrow[r, "p\chi\circ-"'] & \mathsf{hFun}(X,B)(pe'',pe) \end{tikzcd}$$

Were the induction property strict and not weak, this square would be a pullback, i.e., a "cartesian" square.

It follows that p-cartesian lifts of a given 2-cell with specified codomain are unique up to fibered isomorphism:

5.1.3. Lemma (uniqueness of cartesian lifts). *If* $\begin{tikzcd} X \arrow[r, bend left, "e'"] \arrow[r, bend right, "e"'] \arrow[r, phantom, "\Downarrow\chi"] & E \end{tikzcd}$ *and* $\begin{tikzcd} X \arrow[r, bend left, "e''"] \arrow[r, bend right, "e"'] \arrow[r, phantom, "\Downarrow\chi'"] & E \end{tikzcd}$ *are p-cartesian lifts of a common 2-cell $p\chi$, then there exists an invertible 2-cell* $\begin{tikzcd} X \arrow[r, bend left, "e''"] \arrow[r, bend right, "e'"'] \arrow[r, phantom, "\cong\Downarrow\zeta"] & E \end{tikzcd}$ *so that $\chi' = \chi \cdot \zeta$ and $p\zeta = \mathrm{id}$.* □

Proof. By induction, there exists 2-cells $\zeta\colon e'' \Rightarrow e'$ and $\zeta'\colon e' \Rightarrow e''$ so that $\chi' = \chi \cdot \zeta$ and $\chi = \chi' \cdot \zeta'$ with $p\zeta = p\zeta' = \mathrm{id}$. The composites $\zeta \cdot \zeta'$ and $\zeta' \cdot \zeta$ are then fibered automorphisms of χ and χ' and thus invertible by conservativity. Now the 2-of-6 property for isomorphisms implies that ζ and ζ' are also isomorphisms (though perhaps not inverses). $\square$

We frequently make use of the isomorphism stability of the p-cartesian transformations given by the following suite of observations:

5.1.4. Lemma. *Let $p\colon E \twoheadrightarrow B$ be an isofibration.*

(i) Isomorphisms define p-cartesian transformations.
(ii) Any p-cartesian lift of an identity is a natural isomorphism.
(iii) The class of p-cartesian transformations is closed under pre- and post-composition with natural isomorphisms.

Proof. Exercise 5.1.i. $\square$

Furthermore:

5.1.5. Lemma (more conservativity). *If* $\begin{tikzcd} X \arrow[r, bend left, "e'"] \arrow[r, bend right, "e"'] \arrow[r, phantom, "\Downarrow\chi"] & E \end{tikzcd}$, $\begin{tikzcd} X \arrow[r, bend left, "e''"] \arrow[r, bend right, "e"'] \arrow[r, phantom, "\Downarrow\chi'"] & E \end{tikzcd}$, *and* $\begin{tikzcd} X \arrow[r, bend left, "e''"] \arrow[r, bend right, "e'"'] \arrow[r, phantom, "\Downarrow\zeta"] & E \end{tikzcd}$ *are 2-cells so that χ and χ' are p-cartesian, $\chi' = \chi \cdot \zeta$, and $p\zeta$ is invertible, then ζ is invertible.*

Proof. Given the data in the statement we can use the induction property for χ' applied to the pair $(\chi, p\zeta^{-1})$ to induce a candidate inverse $\hat{\zeta}$ for ζ and then apply the conservativity property to conclude that $\zeta \cdot \hat{\zeta}$ and $\hat{\zeta} \cdot \zeta$ are both isomorphisms. By the 2-of-6 property, ζ is an isomorphism, as desired. $\square$

We now introduce the class of cartesian fibrations.

5.1.6. Definition (cartesian fibration). An isofibration $p\colon E \twoheadrightarrow B$ is a **cartesian fibration** if

(i) Any natural transformation $\beta\colon b \Rightarrow pe$ as below-left admits a p-cartesian lift[1] $\chi_\beta\colon \beta^*e \Rightarrow e$ as below-right:

$$\begin{tikzcd} X \arrow[r, bend left, "e"] \arrow[dr, bend right, "b"'] \arrow[dr, phantom, "\Uparrow\beta"] & E \arrow[d, two heads, "p"] \\ & B \end{tikzcd} = \begin{tikzcd} X \arrow[r, bend left, "e"] \arrow[r, bend right, "\beta^*e"'] \arrow[r, phantom, "\Uparrow\chi_\beta"] & E \arrow[d, two heads, "p"] \\ & B \end{tikzcd}$$

(ii) The class of p-cartesian transformations is closed under restriction: that is, if $\begin{tikzcd} X \arrow[r, bend left, "e'"] \arrow[r, bend right, "e"'] \arrow[r, phantom, "\Downarrow\chi"] & E \end{tikzcd}$ is p-cartesian and $f\colon Y \to X$ is any functor then $\begin{tikzcd} Y \arrow[r, bend left, "e'f"] \arrow[r, bend right, "ef"'] \arrow[r, phantom, "\Downarrow\chi f"] & E \end{tikzcd}$ is p-cartesian.

The lifting property (i) implies that a p-cartesian transformation $\chi\colon e' \Rightarrow e$ is the "universal natural transformation over $p\chi$ with codomain e" in the following weak sense: any transformation

[1]To ask that $\chi_\beta\colon \beta^*e \Rightarrow e$ is a **lift** of $\beta\colon b \Rightarrow pe$ asserts that $p\chi_\beta = \beta$ and hence $p\beta^*e = b$.

$\psi\colon e' \Rightarrow e$ factors through a p-cartesian lift $\chi_\psi\colon (p\psi)^*e \Rightarrow e$ of $p\psi$ via a 2-cell $\gamma\colon e' \Rightarrow (p\psi)^*e$ over an identity, and moreover ψ is p-cartesian if and only if this factorization γ is invertible.

The reason for condition (ii) will become clearer in §5.3. For now, note that since all p-cartesian lifts of a given 2-cell β are isomorphic and the class of p-cartesian cells is stable under isomorphism, to verify the condition (ii) it suffices to show that for any functor $f\colon Y \to X$ there is *some* p-cartesian lift of β that restricts along f to another p-cartesian transformation.

5.1.7. Lemma (composites of cartesian fibrations). *If $p\colon E \twoheadrightarrow B$ and $q\colon B \twoheadrightarrow A$ are cartesian fibrations, then so is $qp\colon E \twoheadrightarrow A$. Moreover, a natural transformation* $\begin{tikzcd} X \arrow[r, bend left, "e'"] \arrow[r, bend right, "e"'] \arrow[r, phantom, "\Downarrow\chi"] & E \end{tikzcd}$ *is qp-cartesian if and only if χ is p-cartesian and $p\chi$ is q-cartesian.*

Proof. The first claim follows immediately from the second, for the lifts required by Definition 5.1.6(i) can be constructed by first taking a q-cartesian lift χ_β and then taking a p-cartesian lift χ_{χ_β} of this lifted cell.

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[ddr, bend right, "a"'] & E \arrow[d, two heads, "p"] \\ \arrow[ur, phantom, "\Uparrow\beta"] & B \arrow[d, two heads, "q"] \\ & A \end{tikzcd} = \begin{tikzcd} X \arrow[r, "e"] \arrow[dr, bend right, "b"'] & E \arrow[d, two heads, "p"] \\ \arrow[ur, phantom, "\Uparrow\chi_\beta"] & B \arrow[d, two heads, "q"] \\ & A \end{tikzcd} = \begin{tikzcd} X \arrow[r, bend left, "e"] \arrow[r, bend right, "\beta^*e"'] \arrow[r, phantom, "\Uparrow\chi_{\chi_\beta}"] & E \arrow[d, two heads, "p"] \\ & B \arrow[d, two heads, "q"] \\ & A \end{tikzcd}$$

and the stability condition 5.1.6(ii) is then inherited from the stability of p- and q-cartesian transformations.

To prove the second claim, first consider a natural transformation $\begin{tikzcd} X \arrow[r, bend left, "e'"] \arrow[r, bend right, "e"'] \arrow[r, phantom, "\Downarrow\chi"] & E \end{tikzcd}$ that is p-cartesian and so that $p\chi$ is q-cartesian. Given any natural transformations $\begin{tikzcd} X \arrow[r, bend left, "e''"] \arrow[r, bend right, "e"'] \arrow[r, phantom, "\Downarrow\tau"] & E \end{tikzcd}$ and $\begin{tikzcd} X \arrow[r, bend left, "qpe''"] \arrow[r, bend right, "qpe'"'] \arrow[r, phantom, "\Downarrow\gamma"] & A \end{tikzcd}$ so that $qp\tau = qp\chi \cdot \gamma$, q-cartesianness of $p\chi$ induces a lift $\begin{tikzcd} X \arrow[r, bend left, "e''"] \arrow[r, bend right, "e'"'] \arrow[r, phantom, "\Downarrow\hat{\gamma}"] & B \end{tikzcd}$ of γ so that $p\tau = p\chi \cdot \hat{\gamma}$. Now p-cartesianness of χ induces a further lift $\begin{tikzcd} X \arrow[r, bend left, "e''"] \arrow[r, bend right, "e'"'] \arrow[r, phantom, "\Downarrow\bar{\gamma}"] & E \end{tikzcd}$ of $\hat{\gamma}$ so that $\tau = \chi \cdot \bar{\gamma}$. Moreover, if $\begin{tikzcd} X \arrow[r, bend left, "e'"] \arrow[r, bend right, "e'"'] \arrow[r, phantom, "\Downarrow\zeta"] & E \end{tikzcd}$ is any natural transformation so that $\chi \cdot \zeta = \chi$ and $qp\zeta = \mathrm{id}$ then $p\chi \cdot p\zeta = p\chi$ and by conservativity for $p\chi$, $p\zeta$ is invertible. Applying Lemma 5.1.5, we conclude that ζ is invertible. This proves that χ is qp-cartesian.

Conversely, if χ is qp-cartesian, then Lemma 5.1.3 implies it is isomorphic to all other qp-cartesian lifts of $qp\chi$. The construction given above produces a qp-cartesian lift of any 2-cell that is p-cartesian and whose image under p is q-cartesian. By the isomorphism stability of p- and q-cartesian transformations of Lemma 5.1.4, χ must also have these properties. $\square$

The following lemma proves that cartesian fibrations come equipped with a "generic p-cartesian transformation."

5.1.8. Lemma. *An isofibration $p\colon E \twoheadrightarrow B$ is a cartesian fibration if and only if the right comma cone over p displayed below-left admits a lift χ as displayed below-right:*

$$\begin{tikzcd} \mathsf{Hom}_B(B,p) \arrow[r, bend left, "p_1"] \arrow[dr, bend right, "p_0"'] \arrow[r, phantom, shift right=4, "\Uparrow\phi"] & E \arrow[d, two heads, "p"] \\ & B \end{tikzcd} = \begin{tikzcd} \mathsf{Hom}_B(B,p) \arrow[r, bend left, "p_1"] \arrow[r, bend right, "r"'] \arrow[r, phantom, "\Uparrow\chi"] & E \arrow[d, two heads, "p"] \\ & B \end{tikzcd} \tag{5.1.9}$$

with the property that the restriction of χ along any $X \to \mathsf{Hom}_B(B,p)$ is a p-cartesian transformation.

Proof. If $p\colon E \twoheadrightarrow B$ is cartesian, then the right comma cone ϕ admits a p-cartesian lift $\chi\colon r \Rightarrow p_1$ by 5.1.6(i), which by 5.1.6(ii) has the property that the restriction of this p-cartesian transformation along any $X \to \mathsf{Hom}_B(B,p)$ is also p-cartesian.

For the converse, suppose we are given the generic p-cartesian transformation $\chi\colon r \Rightarrow p_1$ of the statement and consider a 2-cell $\beta\colon b \Rightarrow pe$ as below-left.

$$\begin{tikzcd} X \arrow[r, bend left, "e"] \arrow[dr, bend right, "b"'] \arrow[dr, phantom, "\Downarrow\beta"] & E \arrow[d, two heads, "p"] \\ & B \end{tikzcd} = \begin{tikzcd} X \arrow[r, dashed, "\ulcorner\beta\urcorner"] \arrow[rr, bend left, "e"] \arrow[drr, bend right, "b"'] & \mathsf{Hom}_B(B,p) \arrow[r, dashed, "p_1"] \arrow[dr, "p_0"'] \arrow[r, phantom, shift right=4, "\Uparrow\phi"] & E \arrow[d, two heads, "p"] \\ & & B \end{tikzcd} = \begin{tikzcd} X \arrow[r, dashed, "\ulcorner\beta\urcorner"] \arrow[rr, bend left, "e"] \arrow[drr, bend right, "b"'] & \mathsf{Hom}_B(B,p) \arrow[r, "p_1"] \arrow[r, bend right, "r"'] \arrow[r, phantom, shift right=2, "\Uparrow\chi"] & E \arrow[d, two heads, "p"] \\ & & B \end{tikzcd}$$

By 1-cell induction $\beta = \phi\ulcorner\beta\urcorner$ for some functor $\ulcorner\beta\urcorner\colon X \to \mathsf{Hom}_B(B,p)$ as above- center. By substituting the equation (5.1.9) as above-right, we see that $\chi\ulcorner\beta\urcorner$ is a lift of β whose codomain is $p_1\ulcorner\beta\urcorner = e$, as required. The hypothesis that restrictions of χ are p-cartesian implies that this lift is a p-cartesian transformation.

Now Lemma 5.1.3 implies that any p-cartesian natural transformation is isomorphic to a restriction of χ. Thus restrictions of p-cartesian transformations are isomorphic to restrictions of χ, and it follows from Lemma 5.1.4 that the class of p-cartesian transformation is closed under restriction. $\square$

The first major result of this section is an internal characterization of cartesian fibrations inspired by a similar result of Street [**87, 90, 91**], which in turn was inspired by previous work of Gray [**39**]; see also [**100**]. Before stating this result, recall from Lemma 3.5.8 that from a functor $p\colon E \twoheadrightarrow B$, we can build a fibered adjunction $p_1 \dashv_E i$, where the right adjoint is induced from the identity 2-cell id_p:

$$\begin{tikzcd} & E \arrow[d, dashed, "i"] \arrow[ddl, equal, bend right] \arrow[ddr, two heads, bend left, "p"] & \\ & \mathsf{Hom}_B(B,p) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\ E \arrow[rr, "p"'] & & B \end{tikzcd} = \begin{tikzcd} & E \arrow[dl, equal] \arrow[dr, "p"] \arrow[d, phantom, "="] & \\ E \arrow[rr, "p"'] & & B \end{tikzcd} \rightsquigarrow \begin{tikzcd} E \arrow[rr, dashed, bend right, "i"'] \arrow[dr, equal] & \perp & \mathsf{Hom}_B(B,p) \arrow[ll, bend right, "p_1"'] \arrow[dl, two heads, "p_1"] \\ & E & \end{tikzcd}$$

Similarly, 1-cell induction for the right comma cone over p applied to the generic arrow for E induces a functor

$$
\begin{tikzcd}
E^{\mathbb{2}} \arrow[d, bend right, "p_1"'] \arrow[d, bend left, "p_0"] \arrow[d, phantom, "\overset{\kappa}{\Leftarrow}"] \\
E \arrow[d, two heads, "p"] \\
B
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
& E^{\mathbb{2}} \arrow[d, dashed, "k"] \arrow[ddl, bend right, "p_1"'] \arrow[ddr, bend left, "pp_0"] & \\
& \mathsf{Hom}_B(B,p) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\
E \arrow[rr, "p"'] & & B
\end{tikzcd}
\tag{5.1.10}
$$

5.1.11. Theorem (an internal characterization of cartesian fibrations). *For an isofibration $p\colon E \twoheadrightarrow B$ the following are equivalent:*

(i) $p\colon E \twoheadrightarrow B$ *defines a cartesian fibration.*

(ii) The functor $i\colon E \to \mathsf{Hom}_B(B,p)$ *admits a right adjoint over* B*:*

$$
\begin{tikzcd}
E \arrow[rr, bend left, "i"] \arrow[rr, phantom, "\perp"] \arrow[dr, two heads, "p"'] & & \mathsf{Hom}_B(B,p) \arrow[ll, dashed, bend left, "r"] \arrow[dl, two heads, "p_0"] \\
& B &
\end{tikzcd}
$$

(iii) The functor $k\colon E^{\mathbb{2}} \to \mathsf{Hom}_B(B,p)$ *admits a right adjoint with invertible counit:*

$$
\begin{tikzcd}
E^{\mathbb{2}} \arrow[r, bend left, "k"] \arrow[r, phantom, "\perp"] & \mathsf{Hom}_B(B,p) \arrow[l, dashed, bend left, "\bar{r}"]
\end{tikzcd}
$$

When these equivalent conditions hold, then for a natural transformation $\begin{tikzcd} X \arrow[r, bend left, "e'"] \arrow[r, bend right, "e"'] \arrow[r, phantom, "\Downarrow\psi"] & E \end{tikzcd}$ *the following are equivalent:*

(iv) ψ *is* p*-cartesian.*

(v) ψ *factors through a restriction of the 2-cell* $p_1\epsilon$*, where* ϵ *is the counit of the adjunction* $i \dashv r$*, via a natural isomorphism* $\bar{\gamma}$ *so that* $p\bar{\gamma} = \mathrm{id}$*:*

$$
\begin{tikzcd}
X \arrow[r, bend left, "e"] \arrow[r, bend right, "e'"'] \arrow[r, phantom, "\Uparrow\psi"] & E
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
X \arrow[r, dashed, "\ulcorner p\psi \urcorner"] \arrow[drr, dashed, bend right, "e'"'] \arrow[rrrr, bend left, "e"] & \mathsf{Hom}_B(B,p) \arrow[rr, equal] \arrow[dr, "r"] \arrow[d, phantom, "\cong\Uparrow\bar{\gamma}"] & \arrow[d, phantom, "\Uparrow\epsilon"] & \mathsf{Hom}_B(B,p) \arrow[r, two heads, "p_1"] & E \\
& {} & E \arrow[ur, "i"] \arrow[urr, equal, bend right] & &
\end{tikzcd}
$$

(vi) The component

$$
\begin{tikzcd}
X \arrow[r, "\ulcorner \psi \urcorner"] & E^{\mathbb{2}} \arrow[r, equal, bend left] \arrow[r, bend right, "\bar{r}k"'] \arrow[r, phantom, "\Downarrow\bar{\eta}"] & E^{\mathbb{2}}
\end{tikzcd}
$$

of the unit for $k \dashv \bar{r}$ *is invertible; that is,* ψ *is in the essential image of the right adjoint.*

The right adjoint of (ii) is the domain-component of the generic cartesian lift of (5.1.9); that cartesian transformation is then recovered as $p_1\epsilon$, where ϵ is the counit of the fibered adjunction $i \dashv r$. This explains the statement of (v). By 1-cell induction, the generic cartesian lift χ can be represented by a functor $\bar{r}\colon \mathsf{Hom}_B(B,p) \to E^{\mathbb{2}}$ and this defines the right adjoint of (iii) and explains the statement of (vi).

Before proving Theorem 5.1.11 we make two further remarks on these postulated adjunctions.

5.1.12. Remark. By Lemma 3.5.8, the functor $i\colon E \to \mathsf{Hom}_B(B,p)$ is itself a right adjoint over B to the codomain-projection functor. Since the counit of the adjunction $p_1 \dashv i$ is an isomorphism, it follows formally that the unit of the adjunction $i \dashv r$ must also be an isomorphism, whenever the adjunction postulated in (ii) exists; see Lemma B.3.8.

5.1.13. Remark. In the case where the ∞-categories $E^{\mathbb{2}}$ and $\mathsf{Hom}_B(B,p)$ are defined by the strict simplicial limits of Definitions 3.2.1 and 3.4.1, the 1-cell k induced in (5.1.10) can be modeled by an isofibration:

$$
\begin{tikzcd}
E^{\mathbb{2}} \arrow[dr, dashed, two heads, "k"] \arrow[drr, bend left, "p^{\mathbb{2}}"] \arrow[ddr, bend right, "p_1"'] & & \\
& \mathsf{Hom}_B(B,p) \arrow[r, two heads] \arrow[d, two heads, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & B^{\mathbb{2}} \arrow[d, two heads, "p_1"] \\
& E \arrow[r, two heads, "p"'] & B
\end{tikzcd}
$$

namely the Leibniz cotensor of the codomain inclusion $1\colon \mathbb{1} \hookrightarrow \mathbb{2}$ and the isofibration $p\colon E \twoheadrightarrow B$. Now Lemma 3.6.10 can be used to rectify the adjunction $k \dashv r$ of (iii) to a right adjoint right inverse adjunction, that is then fibered over $\mathsf{Hom}_B(B,p)$. So when Theorem 5.1.11(iii) holds, we may model the postulated adjunction by a right adjoint right inverse to the isofibration k.

Proof. We'll prove (i)⇒(iii)⇒(ii)⇒(i) and demonstrate the equivalences (iv)⇔(vi) and (iv)⇔(v) in parallel.

(i)⇒(iii): If $p\colon E \twoheadrightarrow B$ is cartesian, then the right comma cone over p admits a cartesian lift along p,

$$
\begin{tikzcd}
\mathsf{Hom}_B(B,p) \arrow[r, bend left, "p_1"] \arrow[dr, bend right, "p_0"'] & E \arrow[d, two heads, "p"] \arrow[dl, phantom, "\Uparrow\phi"] \\
& B
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathsf{Hom}_B(B,p) \arrow[r, bend left, "p_1"] \arrow[r, bend right, "r"'] \arrow[r, phantom, "\Uparrow\chi"] & E \arrow[d, two heads, "p"] \\
& B
\end{tikzcd}
$$

defining a functor $r\colon \mathsf{Hom}_B(B,p) \to E$ over B together with a p-cartesian transformation $\chi\colon r \Rightarrow p_1$. By 1-cell induction, this generic cartesian transformation is represented by a functor

$$
\begin{tikzcd}
\mathsf{Hom}_B(B,p) \arrow[r, bend left, "r"] \arrow[r, bend right, "p_1"'] \arrow[r, phantom, "\Downarrow\chi"] & E
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathsf{Hom}_B(B,p) \arrow[r, dashed, "\bar{r}"] & E^{\mathbb{2}} \arrow[r, bend left, "p_0"] \arrow[r, bend right, "p_1"'] \arrow[r, phantom, "\Downarrow\kappa"] & E
\end{tikzcd}
$$

which we take as our definition of the putative right adjoint $\bar{r}$. By the definition (5.1.10) of k, $\phi k\bar{r} = p\kappa\bar{r} = p\chi = \phi$, so Proposition 3.4.7 supplies a fibered isomorphism $\bar{\epsilon}\colon k\bar{r} \cong \mathrm{id}$ with $p_0\bar{\epsilon} = \mathrm{id}_{p_0}$ and $p_1\bar{\epsilon} = \mathrm{id}_{p_1}$.

To prove that $k \dashv \bar{r}$, it remains to define the unit 2-cell $\bar{\eta}$, which we do by 2-cell induction from a pair given by an identity 2-cell $p_1\bar{\eta} = \mathrm{id}_{p_1}$ and a 2-cell $p_0\bar{\eta}$ that remains to be specified. The required compatibility condition of Proposition 3.4.6(ii) asserts that this $p_0\bar{\eta}$ must define a factorization of the generic arrow

$$\begin{tikzcd} p_0 \arrow[r, Rightarrow, "\kappa"] \arrow[d, Rightarrow, "p_0\bar{\eta}"'] & p_1 \arrow[d, equal, "p_1\bar{\eta}=\mathrm{id}"] \\ p_0\bar{r}k = rk \arrow[r, Rightarrow, "\chi k"'] & p_1 k \end{tikzcd} \tag{5.1.14}$$

through $\kappa\bar{r}k = \chi k$. Note $p\kappa = \phi k$ has χk as its p-cartesian lift, so we define $p_0\bar{\eta}$ by the induction property for the cartesian transformation χk applied to the generic arrow $\kappa\colon p_0 \Rightarrow p_1$. By construction $pp_0\bar{\eta} = \mathrm{id}$.

By Lemma B.4.2, once we verify that $k\bar{\eta}$ and $\bar{\eta}\bar{r}$ are invertible, then this data this together with $\bar{\epsilon}\colon k\bar{r} \cong \mathrm{id}$ defines an adjunction with invertible counit.[2] We prove $k\bar{\eta}$ is invertible by 2-cell conservativity: $p_1k\bar{\eta} = p_1\bar{\eta} = \mathrm{id}$ and $p_0k\bar{\eta} = pp_0\bar{\eta} = \mathrm{id}$.

Similarly, by 2-cell conservativity, to conclude that $\bar{\eta}\bar{r}$ is invertible, it suffices to prove that $p_0\bar{\eta}\bar{r}$ is an isomorphism. Restricting (5.1.14) along $\bar{r}$, we see that $p_0\bar{\eta}\bar{r}$ defines a fibered isomorphism of p-cartesian transformations

$$\begin{tikzcd} r \arrow[rr, Rightarrow, "\chi"] \arrow[dr, Rightarrow, "p_0\bar{\eta}\bar{r}"'] & & p_1 \\ & p_0\bar{r}k = rk \arrow[ur, Rightarrow, "\chi k\bar{r}"'] & \end{tikzcd}$$

so this follows from Lemma 5.1.5.

Finally note that a transformation $\psi\colon e' \Rightarrow e$ is p-cartesian if and only if its factorization through the generic p-cartesian lift of $p\psi$ is invertible. This factorization may be constructed by restricting the 2-cells of (5.1.14) along $\ulcorner\psi\urcorner\colon X \to E^{\mathbb{2}}$ since $\kappa\ulcorner\psi\urcorner = \psi$, so we see that ψ is p-cartesian if and only if $p_0\bar{\eta}\ulcorner\psi\urcorner$ is invertible. By 2-cell conservativity, $\bar{\eta}\ulcorner\psi\urcorner$ is invertible if and only if its domain component is invertible. This proves that (iv)⇔(vi).

(iii)⇒(ii): By Remark 5.1.13, we can model the left adjoint of (iii) by an isofibration $k\colon E^{\mathbb{2}} \twoheadrightarrow \mathrm{Hom}_B(B,p)$ and use Lemma 3.6.10 to rectify the adjunction $k \dashv \bar{r}$ with invertible counit into a right adjoint right inverse adjunction, that is then fibered over $\mathrm{Hom}_B(B,p)$. Composing with the projection $p_0\colon \mathrm{Hom}_B(B,p) \twoheadrightarrow B$, Lemma 3.6.7(ii) then gives us a fibered adjunction over B

$$\begin{tikzcd} E^{\mathbb{2}} \arrow[rr, bend left, "k"] \arrow[rr, phantom, "\perp"] \arrow[dr, "pp_0"'] & & \mathrm{Hom}_B(B,p) \arrow[ll, bend left, "\bar{r}"] \arrow[dl, "p_0"] \\ & B & \end{tikzcd}$$

By the dual of Lemma 3.5.8, the 1-cell $j\colon E \to E^{\mathbb{2}}$ induced by the identity defines a left adjoint right inverse to the domain projection

$$\begin{tikzcd} E \arrow[r, "j"] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] & E^{\mathbb{2}} \arrow[l, bend right, "p_1"'] \arrow[l, bend left, "p_0"] \end{tikzcd} \tag{5.1.15}$$

[2]By Remark B.4.3, we can make $\bar{\eta}$ the unit of this adjunction at the cost of modifying the counit by an isomorphism.

supplying a fibered adjunction $j \dashv p_0$ over E that we push forward along $p\colon E \twoheadrightarrow B$ to a fibered adjunction over B:

$$\begin{tikzcd}
E \arrow[rr, bend left, "j"] \arrow[rr, phantom, "\perp"] \arrow[dr, two heads, "p"'] & & E^{\mathbb{2}} \arrow[ll, bend left, "p_0"] \arrow[dl, two heads, "pp_0"] \\
& B &
\end{tikzcd}$$

This pair of fibered adjunctions composes to define a fibered adjunction over B with left adjoint kj and right adjoint $r \coloneqq p_0\bar{r}$. Proposition 3.4.7 supplies a fibered isomorphism $i \cong kj$ since both i and kj define functors $E \rightrightarrows \mathrm{Hom}_B(B, p)$ are induced by the identity id_p over the right comma cone over p. Composing with this fibered isomorphism, we can replace the left adjoint of the composite adjunction by i

$$\begin{tikzcd}
E \arrow[r, bend left, "j"] \arrow[r, phantom, "\perp"] \arrow[rr, bend left=50, "i"] \arrow[rr, phantom, bend left=35, "\cong"] \arrow[dr, two heads, bend right, "p"'] & E^{\mathbb{2}} \arrow[l, bend left, "p_0"] \arrow[r, bend left, "k"] \arrow[r, phantom, "\perp"] \arrow[d, two heads, "pp_0"] & \mathrm{Hom}_B(B,p) \arrow[l, bend left, "\bar{r}"] \arrow[dl, two heads, bend left, "p_0"] \\
& B &
\end{tikzcd}
=
\begin{tikzcd}
E \arrow[rr, bend left, "i"] \arrow[rr, phantom, "\perp"] \arrow[dr, two heads, "p"'] & & \mathrm{Hom}_B(B,p) \arrow[ll, bend left, "r"] \arrow[dl, two heads, "p_0"] \\
& B &
\end{tikzcd}$$

proving (ii).

(ii)⇒(i): Now suppose given a fibered adjunction

$$\begin{tikzcd}
E \arrow[rr, bend left, "i"] \arrow[rr, phantom, "\perp"] \arrow[dr, two heads, "p"'] & & \mathrm{Hom}_B(B,p) \arrow[ll, bend left, "r"] \arrow[dl, two heads, "p_0"] \\
& B &
\end{tikzcd}$$

We will show that the codomain component of the counit

$$\begin{tikzcd}
\mathrm{Hom}_B(B,p) \arrow[rr, equal] \arrow[dr, "r"'] & \Uparrow\epsilon & \mathrm{Hom}_B(B,p) \arrow[r, two heads, "p_1"] & E \\
& E \arrow[ur, "i"'] \arrow[urr, equal, bend right] & &
\end{tikzcd}$$

satisfies the conditions of the generic p-cartesian transformation described in Lemma 5.1.8. This will then also demonstrate the equivalence (iv)⇔(v).

The first thing to check is that $p_1\epsilon$ defines a lift of the right comma cone along p. To see this, consider the horizontal composite:

$$\begin{tikzcd}
& & & E \arrow[dd, two heads, "p"] \\
\mathrm{Hom}_B(B,p) \arrow[rr, equal] \arrow[dr, "r"'] & \Uparrow\epsilon & \mathrm{Hom}_B(B,p) \arrow[ur, two heads, "p_1"] \arrow[dr, two heads, "p_0"'] \arrow[r, phantom, "\Uparrow\phi"] & {} \\
& E \arrow[ur, "i"'] & & B
\end{tikzcd}$$

Naturality of whiskering provides a commutative square

$$\begin{tikzcd}
p_0ir \arrow[r, equal, "p_0\epsilon"] \arrow[d, equal, "\phi ir"'] & p_0 \arrow[d, Rightarrow, "\phi"] \\
pp_1ir \arrow[r, Rightarrow, "pp_1\epsilon"'] & pp_1
\end{tikzcd}$$

in which $p_0\epsilon = \mathrm{id}$, as a fibered counit, and $\phi ir = \mathrm{id}$, since $\phi i = \mathrm{id}_p$. Thus $pp_1\epsilon = \phi$ and we see that $p_1\epsilon$ is a lift of ϕ along p.

It remains only to verify that the restriction of $p_1\epsilon$ along any functor defines a p-cartesian transformation. To that end, consider $\ulcorner\beta\urcorner\colon X \to \mathsf{Hom}_B(B,p)$ representing a 2-cell $\beta\colon b \Rightarrow pe$; our task is to verify that $p_1\epsilon\ulcorner\beta\urcorner$ is p-cartesian. Note that $pp_1\epsilon\ulcorner\beta\urcorner = \phi\ulcorner\beta\urcorner = \beta$, so to prove the induction property, consider a 2-cell $\tau\colon e'' \Rightarrow e$ and a factorization $p\tau = \beta \cdot \gamma$ for some $\gamma\colon pe'' \Rightarrow b$. Our task is to define a 2-cell $\bar{\gamma}\colon e'' \Rightarrow r\ulcorner\beta\urcorner$ so that the pasted composite

$$\begin{tikzcd}
X \arrow[r, "\ulcorner\beta\urcorner"'] \arrow[rrrr, bend left, "e"] \arrow[drr, bend right, "e''"'] & \mathsf{Hom}_B(B,p) \arrow[dr, "r"] \arrow[rr, equal, bend left] \arrow[d, phantom, "\Uparrow\bar{\gamma}"] & \Uparrow\epsilon & \mathsf{Hom}_B(B,p) \arrow[r, "p_1"] & E \\
& {} & E \arrow[ur, "i"] \arrow[urr, equal, bend right] & &
\end{tikzcd}$$

is τ. Transposing across the adjunction $i \dashv r$, it suffices instead to define a 2-cell $\hat{\gamma}\colon ie'' \Rightarrow \ulcorner\beta\urcorner$ so that $p_1\hat{\gamma} = \tau$. We define $\hat{\gamma}$ by 2-cell induction from this condition and $p_0\hat{\gamma} = \gamma$, a pair which satisfies the 2-cell induction compatibility condition

$$\begin{tikzcd}
pe'' \arrow[r, equal] \arrow[d, Rightarrow, "\gamma = p_0\hat{\gamma}"'] & pe'' \arrow[d, Rightarrow, "pp_1\hat{\gamma} = p\tau"] \\
b \arrow[r, Rightarrow, "\beta"'] & pe
\end{tikzcd}$$

precisely on account of the postulated factorization $p\tau = \beta \cdot \gamma$. This verifies the induction condition of Definition 5.1.1(i).

Now consider an endomorphism $\zeta\colon r\hat{\beta} \Rightarrow r\hat{\beta}$ so that $p_1\epsilon\ulcorner\beta\urcorner \cdot \zeta = p_1\epsilon\ulcorner\beta\urcorner$ and $p\zeta = \mathrm{id}_b$. Write $\bar{r} :=$ $\ulcorner p_1\epsilon\urcorner\colon \mathsf{Hom}_B(B,p) \to E^{\mathbf{2}}$ for the functor induced by 1-cell induction from the generic p-cartesian transformation. Now the conditions defining ζ allow us to induce a 2-cell $\bar{\zeta}\colon \bar{r}\ulcorner\beta\urcorner \Rightarrow \bar{r}\ulcorner\beta\urcorner$ satisfying $p_0\bar{\zeta} = \zeta$ and $p_1\bar{\zeta} = \mathrm{id}$. To prove that ζ is invertible, will make use of the naturality of whiskering square for the horizontal composite

$$\begin{tikzcd}
& \mathsf{Hom}_B(B,p) \arrow[dr, "\bar{r}"] \arrow[d, phantom, "\Downarrow\bar{\zeta}"] & & & & \\
X \arrow[ur, "\ulcorner\beta\urcorner"] \arrow[dr, "\ulcorner\beta\urcorner"'] & {} & E^{\mathbf{2}} \arrow[rr, bend left, "p_0"] \arrow[dr, "k"'] & \Downarrow\bar{\gamma} & E & \rightsquigarrow \\
& \mathsf{Hom}_B(B,p) \arrow[ur, "\bar{r}"'] & & \mathsf{Hom}_B(B,p) \arrow[ur, "r"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
p_0r\ulcorner\beta\urcorner \arrow[r, dotted, "\bar{\gamma}\bar{r}\ulcorner\beta\urcorner", "\cong"'] \arrow[d, Rightarrow, "\zeta = p_0\bar{\zeta}"'] & rk\bar{r}\ulcorner\beta\urcorner \arrow[d, Rightarrow, "\cong"', "rk\bar{\zeta}"] \\
p_0\bar{r}\ulcorner\beta\urcorner \arrow[r, dotted, "\bar{\gamma}\bar{r}\ulcorner\beta\urcorner", "\cong"'] & rk\bar{r}\ulcorner\beta\urcorner
\end{tikzcd}$$

where $\bar{\gamma}$ is a special case of the 2-cell just given this name to be described momentarily for which we will demonstrate that $\bar{\gamma}\bar{r}$ is an isomorphism. The composite $k\bar{\zeta}$ is a 2-cell induced from $p_0k\bar{\zeta} = pp_0\bar{\zeta} = p\zeta = \mathrm{id}$ and $p_1k\bar{\zeta} = p_1\bar{\zeta} = \mathrm{id}$, so by 2-cell conservativity, this is an isomorphism. Now ζ is a composite of three isomorphisms and hence is invertible.

To complete the proof, we must define $\bar{\gamma}$ and prove that $\bar{\gamma}\bar{r}$ is invertible. Specializing the construction just given, we define $\bar{\gamma}$ to be the induced 2-cell satisfying $p\bar{\gamma} = \mathrm{id}$ and $\kappa = p_1\epsilon k \cdot \bar{\gamma}$. Transposing across the fibered adjunction $i \dashv r$, it suffices to define the transposed 2-cell $\hat{\gamma}\colon ip_0 \Rightarrow k$ so that $p_1\hat{\gamma} = \kappa$ and $p_0\hat{\gamma} = \mathrm{id}$. And we may transpose once more along an adjunction $j \dashv p_0$ of (5.1.15) constructed by the dual of Lemma 3.5.8. The counit $\nu\colon jp_0 \Rightarrow \mathrm{id}$ of this adjunction satisfied the defining conditions that $p_0\nu = \mathrm{id}$ and $p_1\nu = \kappa$, so to construct $\hat{\gamma}$ satisfying the conditions just described, it suffices to define instead a 2-cell $\xi\colon i \Rightarrow kj$ satisfying the conditions $p_1\xi = \mathrm{id}$ and $p_0\xi = \mathrm{id}$. These conditions are satisfied by the fibered isomorphism $\xi\colon i \cong kj$ that arises by Proposition 3.4.7 since both 1-cells $E \rightrightarrows \mathrm{Hom}_B(B,p)$ are induced by the identity id_p. Unpacking these transpositions, $\bar{\gamma}$ is defined to be the composite

$$\bar{\gamma} \coloneqq \; p_0 \xRightarrow[\cong]{\eta} rip_0 \xRightarrow[\cong]{\xi} rkjp_0 \xRightarrow{rk\nu} rk$$

To verify that $\bar{\gamma}\bar{r}$ is invertible, it thus suffices to demonstrate that $rk\nu\bar{r}$ is invertible. To see this, we consider another pasting diagram

$$\begin{tikzcd}
& E \arrow[dr, "i"] \arrow[d, phantom, "\Downarrow\epsilon"] & & & E \arrow[dr, "j"] \arrow[d, phantom, "\Downarrow\nu"] \\
\mathrm{Hom}_B(B,p) \arrow[ur, "r"] \arrow[rr, equal] & & \mathrm{Hom}_B(B,p) \arrow[r, "\bar{r}"] \arrow[urr, bend left, "r"] \arrow[rrrr, equal, bend right, "\cong\Downarrow\bar{\epsilon}"'] & E^2 \arrow[ur, "p_0"] \arrow[rr, equal] & & E^2 \arrow[r, "k"] & \mathrm{Hom}_B(B,p) \arrow[r, "r"] & E
\end{tikzcd}$$

By the definition (5.1.10) of k, $\phi k\bar{r} = p\kappa\bar{r} = p\chi = \phi$, so Proposition 3.4.7 supplies a fibered isomorphism $\bar{\epsilon}\colon k\bar{r} \cong \mathrm{id}$ with $p_0\bar{\epsilon} = \mathrm{id}_{p_0}$ and $p_1\bar{\epsilon} = \mathrm{id}_{p_1}$.

Now naturality of whiskering supplies a commutative diagram of 2-cells

$$\begin{tikzcd}
rkjp_0\bar{r}ir \arrow[r, Rightarrow, "rk\nu\bar{r}ir"] \arrow[d, Rightarrow, "rkjr\epsilon"', "\wr"] & rk\bar{r}ir \arrow[r, Rightarrow, "r\bar{\epsilon}ir", "\cong"'] \arrow[d, Rightarrow, "rk\bar{r}\epsilon"] & rir \arrow[d, Rightarrow, "r\epsilon", "\wr"'] \\
rkjp_0\bar{r} \arrow[r, Rightarrow, "rk\nu\bar{r}"'] & rk\bar{r} \arrow[r, Rightarrow, "\cong", "r\bar{\epsilon}"'] & r
\end{tikzcd}$$

Since ϵ is the counit of an adjunction $i \dashv r$ with invertible unit, $r\epsilon$ is an isomorphism, so we see that $rk\nu\bar{r}$ is an isomorphism if and only if $rk\nu\bar{r}ir$ is. And this is the case since $k\nu\bar{r}i$ is an isomorphism by 2-cell conservativity: $p_0k\nu\bar{r}i = \mathrm{id}_p$, while $p_1k\nu\bar{r}i = p_1\epsilon i$, which is an isomorphism again because ϵ is the counit of an adjunction $i \dashv r$ with invertible unit. $\square$

One of the myriad applications of Theorem 5.1.11 is:

5.1.16. Corollary. *Cosmological functors preserve cartesian fibrations and cartesian natural transformations.*

Proof. By Theorem 5.1.11(i)⇔(iii), an isofibration $p\colon E \twoheadrightarrow B$ in an ∞-cosmos $\mathcal{K}$ is cartesian if and only if the isofibration $k\colon E^2 \twoheadrightarrow \mathrm{Hom}_B(B,p)$ defined in Remark 5.1.13 admits a right adjoint right inverse. A cosmological functor $F\colon \mathcal{K} \to \mathcal{L}$ preserves the class of isofibrations and the simplicial limits that define the domain and codomain of this k. Moreover, cosmological functors preserve adjunctions and natural isomorphisms, so if this adjoint exists in $\mathcal{K}$ it also does in $\mathcal{L}$. Similarly, the internal characterization of p-cartesian natural transformations given by Theorem 5.1.11(iv)⇔(vi) is also preserved by cosmological functors. $\square$

Another application of Theorem 5.1.11 is that it allows us to conclude that cartesianness is an equivalence-invariant property of isofibrations.

5.1.17. Corollary. *Consider a commutative square between isofibrations whose horizontals are equivalences*

$$\begin{tikzcd} F \arrow[r, "g", "\sim"'] \arrow[d, two heads, "q"'] & E \arrow[d, two heads, "p"] \\ A \arrow[r, "\sim", "f"'] & B \end{tikzcd}$$

Then p is a cartesian fibration if and only if q is a cartesian fibration in which case g preserves and reflects cartesian transformations: χ is q-cartesian if and only if $g\chi$ is p-cartesian.

Proof. The commutative square of the statement induces a commutative square up to isomorphism whose horizontals are equivalences

$$\begin{tikzcd} F^{\mathbb{2}} \arrow[r, "g^{\mathbb{2}}", "\sim"'] \arrow[d, "k"'] \arrow[dr, phantom, "\cong"] & E^{\mathbb{2}} \arrow[d, "k"] \\ \mathsf{Hom}_A(A, q) \arrow[r, "\sim", "\mathsf{Hom}_f(f,g)"'] & \mathsf{Hom}_B(B, p) \end{tikzcd}$$

By the equivalence-invariance of adjunctions, the left-hand vertical admits a right adjoint with invertible counit if and only if the right-hand vertical does, these adjunctions being defined in such a way that the mate of the given square is an isomorphism built by composing with the natural isomorphisms of the horizontal equivalences (see Proposition B.3.9). By Theorem 5.1.11(i)⇔(iii), it follows that p is cartesian if and only if q is cartesian.

Supposing the postulated adjoints exist, via their construction, the whiskered composites $g^{\mathbb{2}}\eta$ and $\eta g^{\mathbb{2}}$ of the units of the respective adjunctions are isomorphic. Hence the component at an element $\ulcorner\chi\urcorner : X \to F^{\mathbb{2}}$ of $g^{\mathbb{2}}\eta$ is invertible if and only if the component of $\eta g^{\mathbb{2}}$ is invertible; since $g^{\mathbb{2}}$ is an equivalence, the former is the case if and only if the component of η is invertible. By Theorem 5.1.11(iv)⇔(vi), this proves that χ is p-cartesian if and only if $g\chi$ is q-cartesian. □

In terminology we now introduce, the square defined by the equivalences and also the square defined by their inverses[3], defines a *cartesian functor* from q to p.

5.1.18. Definition (cartesian functor). Let $p : E \twoheadrightarrow B$ and $q : F \twoheadrightarrow A$ be cartesian fibrations. A commutative square

$$\begin{tikzcd} F \arrow[r, "g"] \arrow[d, two heads, "q"'] & E \arrow[d, two heads, "p"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

defines a **cartesian functor** if g preserves cartesian transformations: if χ is q-cartesian then $g\chi$ is p-cartesian.

[3]For any inverse equivalences g' and f' to g and f, there is a natural isomorphism $qg' \cong f'fqg' = f'pgg' \cong f'p$. Using the isofibration property of q of Proposition 1.4.10, g' may be replaced by an isomorphic functor g'', which also defines an inverse equivalence to g and for which the square $qg'' = f'p$ commutes strictly.

The internal characterization of cartesian functors makes use of a map between right representable comma ∞-categories induced by the commutative square $fq = pg$ defined by Proposition 3.4.5.

5.1.19. Theorem (an internal characterization of cartesian functors). *For a commutative square*

$$\begin{tikzcd} F \arrow[r, "g"] \arrow[d, two heads, "q"'] & E \arrow[d, two heads, "p"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

between cartesian fibrations the following are equivalent:

(i) The square (g, f) defines a cartesian functor from q to p.

(ii) The mate of the canonical isomorphism

$$\begin{tikzcd} F \arrow[r, "g"] \arrow[d, "i"'] \arrow[dr, phantom, "\cong\swarrow"] & E \arrow[d, "i"] \\ \mathrm{Hom}_A(A,q) \arrow[r, "\mathrm{Hom}_f(f,g)"'] & \mathrm{Hom}_B(B,p) \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} F \arrow[r, "g"] \arrow[dr, phantom, "\cong\searrow"] & E \\ \mathrm{Hom}_A(A,q) \arrow[u, "r"] \arrow[r, "\mathrm{Hom}_f(f,g)"'] & \mathrm{Hom}_B(B,p) \arrow[u, "r"'] \end{tikzcd}$$

in the diagram of functors over $f\colon A \to B$ is an isomorphism.

(iii) The mate of the canonical isomorphism

$$\begin{tikzcd} F^{\mathbb{2}} \arrow[r, "g^{\mathbb{2}}"] \arrow[d, "k"'] \arrow[dr, phantom, "\cong\swarrow"] & E^{\mathbb{2}} \arrow[d, "k"] \\ \mathrm{Hom}_A(A,q) \arrow[r, "\mathrm{Hom}_f(f,g)"'] & \mathrm{Hom}_B(B,p) \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} F^{\mathbb{2}} \arrow[r, "g^{\mathbb{2}}"] \arrow[dr, phantom, "\cong\searrow"] & E^{\mathbb{2}} \\ \mathrm{Hom}_A(A,q) \arrow[u, "\bar{r}"] \arrow[r, "\mathrm{Hom}_f(f,g)"'] & \mathrm{Hom}_B(B,p) \arrow[u, "\bar{r}"'] \end{tikzcd}$$

in the diagram of functors is an isomorphism.

Proof. We will prove (i)$\Leftrightarrow$(ii) and (i)$\Leftrightarrow$(iii). The idea in each case is similar. Conditions (ii) and (iii) imply that g preserves the explicitly chosen cartesian lifts up to isomorphism, which by Lemma 5.1.3 implies that g preserves all cartesian lifts. Conversely, assuming (i), we need to show that a whiskered copy of the counit of $i \dashv r$ and of the unit of $k \dashv \bar{r}$ are isomorphisms. The counit of $i \dashv r$ and unit of $k \dashv \bar{r}$ each encode the data of the factorizations of a natural transformation through the cartesian lift of its projection. It will follow from (i) that the cells in question are cartesian and the factorizations live over identities, so Lemma 5.1.5 will imply that these natural transformations are invertible.

(i)$\Leftrightarrow$(iii): For convenience, we take the functors k to be the isofibrations of Remark 5.1.13, so the square on the left hand side of (iii) commutes strictly and its mate is the 2-cell $\bar{\eta}g^{\mathbb{2}}\bar{r}$. By Theorem 5.1.11(iv)$\Leftrightarrow$(vi), this component of $\bar{\eta}$ is invertible if and only if $g\chi$ is p-cartesian, where χ is the generic q-cartesian lift of Lemma 5.1.8 for the cartesian fibration $q\colon F \twoheadrightarrow A$; recall $\bar{r} = \ulcorner\chi\urcorner$. By that lemma again, $g\chi$ is p-cartesian if and only if g preserves cartesian transformations, since the other canonical q-cartesian lifts are constructed as restrictions of χ.

(i)$\Leftrightarrow$(ii): Let us write $\bar{g}$ for $\mathrm{Hom}_f(f,g)$ to save space. Since the unit of $i \dashv r$ is an isomorphism by Remark 5.1.12, the mate of the isomorphism on the left hand side of (ii) is isomorphic to $r\bar{g}\epsilon$, so our task is to show that this natural transformation is invertible if and only if g defines a cartesian functor. Recall from the proof of Theorem 5.1.11(ii)$\Rightarrow$(i) that $p_1\epsilon$ defines the generic q-cartesian lift of

Lemma 5.1.8 for the cartesian fibration $q\colon F \twoheadrightarrow A$, whiskering with $\bar{g} \coloneqq \mathsf{Hom}_f(f,g)\colon \mathsf{Hom}_A(A,q) \to \mathsf{Hom}_B(B,p)$ carries this to a 2-cell whose projection with p_0 is an identity, since $p_0\epsilon = \mathrm{id}$, and whose projection along p_1 is $gp_1\epsilon$, by the commutativity of the left-hand portion of the diagram below.

$$\begin{tikzcd}
& F \arrow[r, "g"] \arrow[dr, "i"] \arrow[d, phantom, "\Downarrow\epsilon"] & E \arrow[drr, "i"] & & & E \arrow[dr, "i"] \arrow[d, phantom, "\Downarrow\epsilon"] \\
\mathsf{Hom}_A(A,q) \arrow[ur, "r"] \arrow[rr, equal] & & \mathsf{Hom}_A(A,q) \arrow[r, "\bar{g}"'] \arrow[d, two heads, "p_1"'] & \mathsf{Hom}_B(B,p) \arrow[urr, "r"] \arrow[rrr, equal] \arrow[d, two heads, "p_1"] & & & \mathsf{Hom}_B(B,p) \arrow[d, two heads, "p_1"] \\
& & F \arrow[r, "g"'] & E \arrow[rrr, equal] & & & E
\end{tikzcd}$$

Now naturality of whiskering provides a commutative square of natural transformations:

$$\begin{tikzcd}
p_1irigr \arrow[r, Rightarrow, "r\bar{g}\epsilon"] \arrow[d, Rightarrow, "p_1\epsilon igr"', "\cong"] & r\bar{g} \arrow[d, Rightarrow, "p_1\epsilon\bar{g}"] \\
p_1\bar{g}ir \arrow[r, Rightarrow, "p_1\bar{g}\epsilon"'] & p_1\bar{g}
\end{tikzcd}$$

where we've simplified some of the names since $p_1 i = \mathrm{id}$. Since ϵ is the counit of an adjunction $i \dashv r$ with invertible unit, ϵi is an isomorphism. Note $pr\bar{g}\epsilon = p_0\bar{g}\epsilon = \mathrm{id}$ and the right-hand vertical $p_1\epsilon\bar{g}$ is a p-cartesian lift of the restriction of $\phi\bar{g}$, this ϕ being the right comma cone over p, which equals the whiskered right comma cone $f\phi$, this ϕ being the right comma cone over q, by the definition of $\bar{g}$. The bottom horizontal $p_1\bar{g}\epsilon$ is similarly a lift of $f\phi = \phi\bar{g}$. So if g is a cartesian functor, the right-hand vertical and bottom horizontal are both p-cartesian lifts of a common 2-cell and the conservativity property implies that $r\bar{g}\epsilon$ is invertible. Conversely, if $r\bar{g}\epsilon$ is invertible, the $p_1\bar{g}\epsilon = gp_1\epsilon$ is isomorphic to a p-cartesian transformation and is consequently p-cartesian. Since Lemma 5.1.8 constructs the other canonical q-cartesian lifts as restrictions of $p_1\epsilon$, this is the case if and only if g is a cartesian functor. □

5.1.20. Proposition (pullback stability). *If*

$$\begin{tikzcd}
F \arrow[r, "g"] \arrow[d, two heads, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, two heads, "p"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

is a pullback square and p is a cartesian fibration then q is a cartesian fibration. Moreover, a natural transformation χ with codomain F is q-cartesian if and only if $g\chi$ is p-cartesian, and in particular the pullback square defines a cartesian functor.

Proof. We apply Theorem 5.1.11(i)⇒(iii) in the form described in Remark 5.1.13 to the cartesian fibration p, which yields a fibered adjunction

$$\begin{tikzcd} E^{\mathbb{2}} \arrow[rr, bend left, "k"] \arrow[dr, "k"'] & \perp & \mathrm{Hom}_B(B,p) \arrow[ll, dashed, bend left, "\bar{r}"] \arrow[dl, equal] \\ & \mathrm{Hom}_B(B,p) & \end{tikzcd} \tag{5.1.21}$$

We will now argue that this functor k pulls back to the corresponding functor for q. To that end, first note that the top face of the following cube is a pullback since the front, back, and bottom faces are.

$$\begin{tikzcd} & & F^{\mathbb{2}} \arrow[dll, "g^{\mathbb{2}}"'] \arrow[dd, dashed, "k"] \arrow[ddrr, bend left, "q^{\mathbb{2}}"] & & \\ E^{\mathbb{2}} \arrow[dd, dashed, "k"'] \arrow[ddrrr, bend left, "p^{\mathbb{2}}"] & & & & \\ & & \mathrm{Hom}_A(A,q) \arrow[rr, two heads] \arrow[dll, "\mathrm{Hom}_f(f,g)"'] \arrow[dd, "p_1"] & & A^{\mathbb{2}} \arrow[dl, "f^{\mathbb{2}}"] \arrow[dd, "p_1"] \\ \mathrm{Hom}_B(B,p) \arrow[rrr, two heads] \arrow[dd, "p_1"'] & & & B^{\mathbb{2}} \arrow[dd, "p_1"] & \\ & & F \arrow[rr, two heads, "q"] \arrow[dll, "g"'] & & A \arrow[dl, "f"] \\ E \arrow[rrr, two heads, "p"'] & & & B & \end{tikzcd}$$

The right adjoint $(-)^{\mathbb{2}}$ preserves pullbacks, so $F^{\mathbb{2}}$ is the pullback of $p^{\mathbb{2}}$ along $f^{\mathbb{2}}$, and since this pullback square factors through the top face of the cube along the square inducing the maps k, we conclude that this last square is a pullback, as claimed.

Now pullback defines a cosmological functor $\mathrm{Hom}_f(f,g)^* \colon \mathcal{K}_{/\mathrm{Hom}_B(B,p)} \to \mathcal{K}_{/\mathrm{Hom}_A(A,q)}$ that carries the fibered adjunction (5.1.21) to a fibered adjunction

$$\begin{tikzcd} F^{\mathbb{2}} \arrow[rr, bend left, "k"] \arrow[dr, "k"'] & \perp & \mathrm{Hom}_A(A,q) \arrow[ll, dashed, bend left, "\bar{r}"] \arrow[dl, equal] \\ & \mathrm{Hom}_A(A,q) & \end{tikzcd}$$

which by Theorem 5.1.11(iii)⇒(i) proves that q is a cartesian fibration. Moreover, by construction of the adjunction $k \dashv \bar{r}$ as a pullback of the adjunction $k \dashv \bar{r}$, both of the mates in Theorem 5.1.19(iii) are identities, proving that (g, f) defines a cartesian functor.

To see that (g, f) creates cartesian natural transformations, note that a natural transformation χ with codomain F is represented by an element $\ulcorner\chi\urcorner \colon X \to F^{\mathbb{2}}$ and $g\chi$ is represented by the image of this element under the functor $g^{\mathbb{2}} \colon F^{\mathbb{2}} \to E^{\mathbb{2}}$. By Theorem 5.1.11(iv)⇔(vi), χ is q-cartesian just when the component $\bar{\eta}\ulcorner\chi\urcorner$ of the unit of $k \dashv \bar{r}$ is invertible. This unit component $\eta\ulcorner\chi\urcorner$ is the pullback of the corresponding unit component $\eta g^{\mathbb{2}}\ulcorner\chi\urcorner$ indexed by $g\chi$, and by conservativity of the smothering functor

$$\mathsf{hFun}(X, F^{\mathbb{2}}) \to \mathsf{hFun}(X, \mathsf{Hom}_A(A, q)) \underset{\mathsf{hFun}(X, \mathsf{Hom}_B(B,p))}{\times} \mathsf{hFun}(X, E^{\mathbb{2}})$$

if $\eta g^{\mathbb{2}} \ulcorner \chi \urcorner$ is invertible, then so is $\eta \ulcorner \chi \urcorner$. □

Pullback squares provide a key instance of cartesian functors. Another is given by the following lemma, which can be proven using Theorem 5.1.19.

5.1.22. Lemma. *If*

$$\begin{tikzcd} F \arrow[rr, "g"] \arrow[dr, two heads, "q"'] & & E \arrow[dl, two heads, "p"] \\ & B & \end{tikzcd}$$

is a functor between cartesian fibrations that admits a left adjoint over B, then g defines a cartesian functor.

Proof. If $\ell \dashv_B g$, then the cosmological functor $p_1^* \colon \mathcal{K}_{/B} \to \mathcal{K}_{/B^{\mathbb{2}}}$ carries this fibered adjunction to a fibered adjunction

$$\begin{tikzcd} \mathsf{Hom}_B(B, q) \arrow[rr, bend right, "\mathsf{Hom}_{\mathrm{id}_B}(\mathrm{id}_B, g)"'] \arrow[rr, phantom, "\perp"] & & \mathsf{Hom}_B(B, p) \arrow[ll, bend right, "\mathsf{Hom}_{\mathrm{id}_B}(\mathrm{id}_B, \ell)"'] \end{tikzcd}$$

Now both horizontal functors in the commutative square

$$\begin{tikzcd} F^{\mathbb{2}} \arrow[r, "g^{\mathbb{2}}"] \arrow[d, "k"'] \arrow[dr, phantom, "\cong\swArrow"] & E^{\mathbb{2}} \arrow[d, "k"] \\ \mathsf{Hom}_A(A, q) \arrow[r, "\mathsf{Hom}_f(f,g)"'] & \mathsf{Hom}_B(B, p) \end{tikzcd}$$

admit left adjoints and a standard result from the calculus of mates tells us that the mate with respect to the vertical adjunctions $k \dashv \bar{r}$ is an isomorphism if and only if the mate with respect to the horizontal adjunctions is an isomorphism, the latter natural transformation between left adjoints being the transpose of the former natural transformation between their right adjoints. This is the case because the mate with respect to the left adjoints lies is the fiber of the smothering functor of Proposition 3.2.5 for $F^{\mathbb{2}}$. □

Examples of cartesian fibrations are overdue.

5.1.23. Proposition (domain projection). *For any ∞-category A, the domain-projection functor $p_0 \colon A^{\mathbb{2}} \twoheadrightarrow A$ defines a cartesian fibration. Moreover, a natural transformation χ with codomain $A^{\mathbb{2}}$ is p_0-cartesian if and only if $p_1 \chi$ is invertible.*

Before giving the proof, we explain the idea. A 2-cell

$$\begin{tikzcd} X \arrow[r, bend left, "\ulcorner \beta \urcorner"] \arrow[dr, bend right, "a"'] & A^{\mathbb{2}} \arrow[d, two heads, "p_0"] \arrow[dl, phantom, "\Uparrow\alpha"] \\ & A \end{tikzcd}$$

defines a composable pair of 2-cells $\alpha\colon a \Rightarrow x$ and $\beta\colon x \Rightarrow y$ in $\mathsf{hFun}(X, A)$. Composing these we induce a 2-cell $\begin{tikzcd} X \arrow[r, bend left, "\ulcorner\beta\circ\alpha\urcorner"] \arrow[r, bend right, "\ulcorner\beta\urcorner"'] \arrow[r, phantom, "\Downarrow\chi"] & A^{\mathbb{2}} \end{tikzcd}$ representing the commutative square

$$\begin{tikzcd} a \arrow[r, Rightarrow, "\alpha"] \arrow[d, Rightarrow, "\beta\circ\alpha"'] & x \arrow[d, Rightarrow, "\beta"] \\ y \arrow[r, equal] & y \end{tikzcd}$$

so that $p_0\chi = \alpha$, as required, and $p_1\chi = \mathrm{id}$.[4]

Proof. We use Theorem 5.1.11(i)⇔(iii) and prove that p_0 is cartesian by constructing an appropriate adjoint to the functor

$$\begin{tikzcd} (A^{\mathbb{2}})^{\mathbb{2}} \arrow[dr, dashed, two heads, "k"] \arrow[drr, bend left, two heads, "p_0^{\mathbb{2}}"] \arrow[ddr, bend right, two heads, "p_1"'] & & \\ & \mathsf{Hom}_A(A, p_0) \arrow[r, two heads] \arrow[d, two heads, "p_1"'] \arrow[dr, phantom, "\lrcorner" very near start] & A^{\mathbb{2}} \arrow[d, two heads, "p_1"] \\ & A^{\mathbb{2}} \arrow[r, two heads, "p_0"'] & A \end{tikzcd} \qquad \begin{tikzcd} \mathbb{2}\times\mathbb{2} & & \\ & \mathbb{3} \arrow[ul, dashed] & \mathbb{2} \arrow[l, hook'] \arrow[ull, bend right, hook', "0\times\mathbb{2}"'] \\ & \mathbb{2} \arrow[u, hook] \arrow[uul, bend left, hook, "\mathbb{2}\times 1"] & \mathbb{1} \arrow[l, hook', "0"] \arrow[u, hook, "1"'] \end{tikzcd}$$

defined by cotensoring with the 1-categories displayed above right.[5]

To construct a right adjoint with invertible counit to the map k, it suffices to construct a left adjoint left inverse to the inclusion of 1-categories $\mathbb{3} \hookrightarrow \mathbb{2}\times\mathbb{2}$ with image $(0,0) \to (0,1) \to (1,1)$. The left adjoint $\ell\colon \mathbb{2}\times\mathbb{2} \to \mathbb{3}$ is a left inverse on the image of $\mathbb{3}$ and sends $(1,0)$ to the terminal element of $\mathbb{3}$:

$$\mathbb{2}\times\mathbb{2} \ni \left\{ \begin{tikzcd} (0,0) \arrow[r] \arrow[d] & (0,1) \arrow[d] \\ (1,0) \arrow[r] & (1,1) \end{tikzcd} \right\} \quad \mapsto \quad \left\{ \begin{tikzcd} 0 \arrow[r] \arrow[d] & 1 \arrow[d] \\ 2 \arrow[r, equal] & 2 \end{tikzcd} \right\} \quad \in \mathbb{3}$$

Now

$$\begin{tikzcd} (A^{\mathbb{2}})^{\mathbb{2}} \arrow[r, bend left, "k"] \arrow[r, phantom, "\perp"] & A^{\mathbb{3}} \simeq \mathsf{Hom}_A(A, p_0) \arrow[l, bend left, dashed, "A^{\ell}"] \end{tikzcd}$$

defines the desired right adjoint with invertible counit.

[4]While the weak universal property of the arrow ∞-category can be used to induce 2-cells with codomain $A^{\mathbb{2}}$, it cannot be used to prove equations between the induced 2-cells, as required to demonstrate that induction condition of Definition 5.1.1. Thus, some sort of ∞-cosmos-level argument is necessary to establish this result.

[5]The cotensor $A^{(-)}$ carries pushouts of simplicial sets to pullbacks of ∞-categories, and the pushout of $\mathbb{2} \cup_{\mathbb{1}} \mathbb{2}$ of simplicial sets is $\Lambda^1[2]$, not $\mathbb{3} = \Delta[2]$. However, on account of the equivalence of ∞-categories $A^{\mathbb{3}} \simeq A^{\Lambda^1[2]}$, no harm comes from making the indicated substitution.

The characterization of p_0-cartesian transformations now follows from Theorem 5.1.11(iv)⇔(vi). A cell χ with codomain $A^{\mathbb{2}}$ is p_0-cartesian if and only if it its representing element of $(A^{\mathbb{2}})^{\mathbb{2}}$ is in the essential image of the right adjoint $A^\ell\colon A^{\mathbb{3}} \to A^{\mathbb{2}\times\mathbb{2}}$. Clearly if χ is in the essential image then its codomain component must be invertible.

Conversely, suppose the codomain component

$$X \xrightarrow{\ulcorner\chi\urcorner} A^{\mathbb{2}\times\mathbb{2}} \xrightarrow{\mathrm{ev}(1,-)} A^{\mathbb{2}}$$

represents a natural isomorphism. Applying Lemma 1.1.12 to $\mathsf{Fun}(X,A)$ one can build a diagram in $\mathsf{Fun}(X,A)^{\mathbb{2}\times\mathbb{2}} \cong \mathsf{Fun}(X,A^{\mathbb{2}\times\mathbb{2}})$ whose top edge is the given isomorphism, whose vertical edges are isomorphisms, and whose bottom edge is an identity. This glues onto χ to define a diagram $X \to A^{\Lambda^1[2]\times\mathbb{2}}$ and now the composite

$$X \longrightarrow A^{\Lambda^1[2]\times\mathbb{2}} \xrightarrow{\sim} A^{\mathbb{3}\times\mathbb{2}} \xrightarrow{A^c} A^{\mathbb{2}\times\mathbb{2}\times\mathbb{2}}$$

where $c\colon \mathbb{2}\times\mathbb{2} \to \mathbb{3}$ is the surjective functor that sends $(0,0)$ and $(0,1)$ to 0, witnesses an isomorphism between $\ulcorner\chi\urcorner$ and a diagram in the image of A^ℓ, whose codomain component is the identity. ☐

The same argument proves that for any $g\colon C \to A$, the domain-projection $p_0\colon \mathsf{Hom}_A(A,g) \twoheadrightarrow A$ is a cartesian fibration. For $f\colon B \to A$, the domain-projections $p_0\colon \mathsf{Hom}_A(f,A) \twoheadrightarrow B$ and $p_0\colon \mathsf{Hom}_A(f,g) \twoheadrightarrow B$ are obtained by pullback via Proposition 5.1.20. These cartesian fibrations figure in a final important lemma about the class of p-cartesian transformations.

5.1.24. Lemma. *Let $p\colon E \twoheadrightarrow B$ be a cartesian fibration and consider a composable pair of natural transformations* $\begin{tikzcd} X \arrow[r, bend left, "e'"] \arrow[r, bend right, "e"'] \arrow[r, phantom, "\Downarrow\psi"] & E \end{tikzcd}$ *and* $\begin{tikzcd} X \arrow[r, bend left, "e''"] \arrow[r, bend right, "e'"'] \arrow[r, phantom, "\Downarrow\psi'"] & E \end{tikzcd}$ *with codomain E.*

(i) *If ψ and ψ' are p-cartesian, then so is $\psi \cdot \psi'$.*
(ii) *If ψ and $\psi \cdot \psi'$ are p-cartesian, then so is ψ'.*

Proof. For (i), recall from the proof of Lemma 5.1.8 that a p-cartesian lift of $p\psi$ is given by the composite

$$\begin{tikzcd} X \arrow[r, "\ulcorner p\psi\urcorner"] & \mathsf{Hom}_B(B,p) \arrow[r, bend left, "r"] \arrow[r, bend right, "p_1"'] \arrow[r, phantom, "\Downarrow\chi"] & E \end{tikzcd}$$

with the natural transformation $\chi\colon r \Rightarrow p_1$ of (5.1.9) whose domain r is the right adjoint of Theorem 5.1.11(ii). Since we are assuming that ψ is p-cartesian, ψ is isomorphic to this whiskered natural transformation so we may redefine ψ to equal it and redefine ψ' to absorb the isomorphism. By Lemma 5.1.4, this new ψ' is still p-cartesian since we've assumed ψ' is. This modification does not change the composition transformation $\psi \cdot \psi'$ that we desire to show is p-cartesian.

Now by 2-cell induction, the diagram as below-left defines a 2-cell as below-right:

$$\begin{tikzcd} pe'' \arrow[r, Rightarrow, "p\psi'"] \arrow[d, Rightarrow, "p(\psi\cdot\psi')"'] & pe' \arrow[d, Rightarrow, "p\psi"] \\ pe \arrow[r, equal] & pe \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, bend left, "\ulcorner p(\psi\cdot\psi')\urcorner"] \arrow[r, bend right, "\ulcorner p\psi\urcorner"'] \arrow[r, phantom, "\Downarrow\gamma"] & \mathsf{Hom}_B(B,p) \end{tikzcd}$$

By the generalization of Proposition 5.1.23, γ is a p_0-cartesian cell. By Lemma 5.1.22, the fibered right adjoint r

$$\begin{tikzcd} \mathsf{Hom}_B(B,p) \arrow[rr, bend right, "r"'] \arrow[dr, "p_0"'] \arrow[rr, phantom, "\perp"] & & E \arrow[ll, bend right, "i"'] \arrow[dl, "p"] \\ & B & \end{tikzcd}$$

carries γ to a p-cartesian transformation $r\gamma$.

Now the horizontal composite

$$\begin{tikzcd} X \arrow[r, bend left, "\ulcorner p(\psi\cdot\psi')\urcorner"] \arrow[r, bend right, "\ulcorner p\psi \urcorner"'] \arrow[r, phantom, "\Downarrow\gamma"] & \mathsf{Hom}_B(B,p) \arrow[r, bend left, "r"] \arrow[r, bend right, "p_1"'] \arrow[r, phantom, "\Downarrow\chi"] & E \end{tikzcd}$$

provides a commutative diagram of natural transformations

$$\begin{tikzcd} \bar{e} \arrow[r, Rightarrow, "r\gamma"] \arrow[d, Rightarrow, "\chi_{p(\psi\cdot\psi')}"'] & e' \arrow[d, Rightarrow, "\psi"] \\ e \arrow[r, equal] & e \end{tikzcd}$$

In particular, the composite $\psi \cdot r\gamma$ is a p-cartesian natural transformation. Since $pr\gamma = p_0\gamma = p\psi'$, $r\gamma$ is a p-cartesian lift of the cartesian transformation ψ', so ψ' and $r\gamma$ are isomorphic. But now $\psi \cdot \psi'$ is isomorphic to the p-cartesian transformation $\chi_{p(\psi\cdot\psi')}$, and Lemma 5.1.4 proves that $\psi \cdot \psi'$ is p-cartesian.

Now for (ii) suppose ψ and $\psi \cdot \psi'$ are p-cartesian and let $\chi' \colon \bar{e} \Rightarrow e'$ denote a p-cartesian lift of $p\psi'$. Consider the factorization $\psi' = \chi' \cdot \theta$ of ψ' through its p-cartesian lift with $p\theta = \mathrm{id}$. Now part (i) implies that $\psi \cdot \chi'$ is p-cartesian and $\psi \cdot \psi' = \psi \cdot \chi' \cdot \theta$, so Lemma 5.1.5 implies that θ is an isomorphism. Now ψ' is isomorphic to a p-cartesian cell so Lemma 5.1.4 implies that ψ' must be p-cartesian. ☐

This result allows us to strengthen the conclusion of Lemma 5.1.4(ii).

5.1.25. Corollary. *Let $p \colon E \twoheadrightarrow B$ be a cartesian fibration. Then any p-cartesian lift of an isomorphism is an isomorphism.*

Proof. Let $\chi \colon e' \Rightarrow e$ be a p-cartesian lift of an isomorphism $\beta \colon b' \cong b$. Applying the induction property of χ, we obtain a lift of $\chi' \colon e \Rightarrow e'$ of β^{-1} so that $\chi \cdot \chi' = \mathrm{id}_e$. Since χ and id_e are both p-cartesian, Lemma 5.1.24(ii) tells us that χ' is as well. Now by Lemma 5.1.24(i), $\chi' \cdot \chi$ is a p-cartesian lift of $\mathrm{id}_{b'}$ and hence by Lemma 5.1.4(ii) this composite is also an isomorphism. Therefore, we conclude that χ is invertible. ☐

For ∞-categories admitting pullbacks, the codomain-projection functor also defines a cartesian fibration:

5.1.26. Proposition (codomain projection). *Let A be an ∞-category that admits pullbacks in the sense of Definition 4.3.7. Then the codomain-projection functor $p_1 \colon A^2 \twoheadrightarrow A$ is a cartesian fibration and the p_1-cartesian arrows are just those 2-cells that represent pullback squares.*

PROOF. Via Theorem 5.1.11(i)⇔(iii), we desire a right adjoint right inverse to the functor k defined below-left applying A^- to the diagram of simplicial sets appearing below-right:

$$\begin{tikzcd}
(A^{\mathbb{2}})^{\mathbb{2}} \arrow[dr, dashed, two heads, "k"] \arrow[drr, bend left, two heads, "p_1^{\mathbb{2}}"] \arrow[ddr, bend right, two heads, "p_1"'] & & \\
& \mathrm{Hom}_A(A, p_1) \arrow[r, two heads] \arrow[d, two heads, "p_1"'] \arrow[dr, phantom, "\lrcorner" very near start] & A^{\mathbb{2}} \arrow[d, two heads, "p_1"] \\
& A^{\mathbb{2}} \arrow[r, two heads, "p_0"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{2} \times \mathbb{2} & & \\
& \lrcorner \arrow[ul, dashed] \arrow[dr, phantom, "\lrcorner" very near start] & \mathbb{2} \arrow[l] \arrow[ull, bend right, "1\times\mathbb{2}"'] \\
& \mathbb{2} \arrow[u, hook] \arrow[uul, bend left, "\mathbb{2}\times 1"] & \mathbb{1} \arrow[l, hook', "1"] \arrow[u, hook, "1"']
\end{tikzcd}$$

This is done in Corollary 4.3.5:

$$(A^{\mathbb{2}})^{\mathbb{2}} \cong A^{\boxdot} \begin{tikzcd} {} \arrow[r, bend left, two heads, "\mathrm{res}"] \arrow[r, phantom, "\perp"] & {} \arrow[l, bend left, dashed, two heads, "\mathrm{ran}"] \end{tikzcd} A^{\lrcorner} \cong \mathrm{Hom}_A(A, p_1) \qquad \square$$

Exercises.

5.1.i. EXERCISE. Prove Lemma 5.1.4.

5.1.ii. EXERCISE. Attempt to prove directly from Definition 5.1.6 that cosmological functors preserve cartesian fibrations and explain what goes wrong.

5.1.iii. EXERCISE. Show that a cartesian fibration $p\colon E \twoheadrightarrow B$ defines an "incoherent pseudofunctor" $E\colon \mathsf{h}B^{\mathrm{op}} \rightsquigarrow \mathfrak{h}\mathcal{K}$ given by the data:

- a mapping on objects $b \in \mathsf{h}B \mapsto E_b \in \mathfrak{h}\mathcal{K}$;
- a mapping on 1-cells $\beta\colon a \to b \in \mathsf{h}B \mapsto \beta^*\colon E_b \to E_a \in \mathfrak{h}\mathcal{K}$ defined by

$$\begin{tikzcd}
E_b \arrow[rr, bend left, "\ell_b"] \arrow[d, two heads] \arrow[drr, phantom, "\lrcorner" very near start] & & E \arrow[d, two heads, "p"] \\
1 \arrow[rr, bend left, "b"] \arrow[rr, bend right, "a"'] \arrow[rr, phantom, "\Uparrow\beta"] & & B
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
E_b \arrow[rr, bend left, "\ell_b"] \arrow[rr, phantom, "\Uparrow\chi_\beta"] \arrow[rr, dotted, "\beta^*(\ell_b)"'] \arrow[dr, dotted, "\beta^*"] \arrow[dd, two heads] & & E \arrow[dd, two heads, "p"] \\
& E_a \arrow[ur, dashed, "\ell_a"'] \arrow[dl] \arrow[dr, phantom, "\lrcorner" very near start] & \\
1 \arrow[rr, bend right, "a"'] & & B
\end{tikzcd}
\tag{5.1.27}$$

- an invertible 2-cell $\begin{tikzcd} E_b \arrow[r, bend left, equal] \arrow[r, bend right, "\mathrm{id}_b^*"'] \arrow[r, phantom, "\iota_b\Downarrow\cong"] & E_b \end{tikzcd} \in \mathfrak{h}\mathcal{K}$ for each $b \in \mathsf{h}B$; and
- an invertible 2-cell

$$\begin{tikzcd}
& E_b \arrow[dr, "\beta^*"] & \\
E_c \arrow[ur, "\gamma^*"] \arrow[rr, "(\gamma\circ\beta)^*"'] \arrow[rr, phantom, shift left=3, "\alpha_{\beta,\gamma}\Downarrow\cong"] & & E_a
\end{tikzcd}$$

in $\mathfrak{h}\mathcal{K}$ for each composable pair of morphisms in $\mathsf{h}B$.

The coherence conditions present in the full definition of a **pseudo-functor**, which appears as Definition 10.4.1, are not evidently satisfied here, but do follow from the extension of this construction to a homotopy coherent diagram indexed by the underlying quasi-category of B.

5.1.iv. Exercise. Categorify the intuition that cartesian fibrations $p\colon E \twoheadrightarrow B$ and $q\colon F \twoheadrightarrow B$ define "contravariant B-indexed functors valued in ∞-categories" by proving that a cartesian functor

$$\begin{tikzcd} E \arrow[rr, "g"] \arrow[dr, two heads, "p"'] & & F \arrow[dl, two heads, "q"] \\ & B & \end{tikzcd}$$

defines a "natural transformation": show that there exists a natural isomorphism in the square of fibers

$$\begin{tikzcd} E_b \arrow[r, "g"] \arrow[d, "\beta^*"'] \arrow[dr, phantom, "\exists\cong"] & F_b \arrow[d, "\beta^*"] \\ E_a \arrow[r, "g"'] & F_a \end{tikzcd}$$

where the action of an arrow β in the homotopy category of B on the fibers is defined by factoring the domain of a p- or q-cartesian lift of β as displayed in (5.1.27).

5.1.v. Exercise. Use Proposition 5.1.20 to prove that cartesian functors pull back.

5.2. Cocartesian fibrations and bifibrations

By the dual of Proposition 5.1.23, the codomain-projection functor is also a cocartesian fibration, a notion we now introduce.

5.2.1. Definition (p-cocartesian transformations). Let $p\colon E \twoheadrightarrow B$ be an isofibration. A natural transformation $\begin{tikzcd} X \arrow[r, bend left, "e"] \arrow[r, bend right, "e'"'] \arrow[r, phantom, "\Downarrow\chi"] & E \end{tikzcd}$ with codomain E is p-**cocartesian** if

(i) **induction**: Given any natural transformations $\begin{tikzcd} X \arrow[r, bend left, "e"] \arrow[r, bend right, "e''"'] \arrow[r, phantom, "\Downarrow\tau"] & E \end{tikzcd}$ and $\begin{tikzcd} X \arrow[r, bend left, "pe'"] \arrow[r, bend right, "pe''"'] \arrow[r, phantom, "\Downarrow\gamma"] & B \end{tikzcd}$ so that $p\tau = \gamma \cdot p\chi$, there exists a lift $\begin{tikzcd} X \arrow[r, bend left, "e'"] \arrow[r, bend right, "e''"'] \arrow[r, phantom, "\Downarrow\bar{\gamma}"] & E \end{tikzcd}$ of γ so that $\tau = \bar{\gamma} \cdot \chi$.

$$\begin{tikzcd} e \arrow[rr, Rightarrow, "\tau"] \arrow[dr, Rightarrow, "\chi"'] & & e'' & \in & \mathsf{hFun}(X,E) \arrow[d, two heads, "p_*"] \\ & e' \arrow[ur, Rightarrow, dotted, "\bar{\gamma}"'] & & & \\ & \rotatebox{-90}{$\mapsto$} & & & \\ pe \arrow[rr, Rightarrow, "p\tau"] \arrow[dr, Rightarrow, "p\chi"'] & & pe'' & \in & \mathsf{hFun}(X,B) \\ & pe' \arrow[ur, Rightarrow, "\gamma"'] & & & \end{tikzcd}$$

(ii) **conservativity**: Any fibered endomorphism of χ is invertible: if $\begin{tikzcd} X \arrow[r, bend left, "e'"] \arrow[r, bend right, "e'"'] \arrow[r, phantom, "\Downarrow\zeta"] & E \end{tikzcd}$ is any natural transformation so that $\zeta \cdot \chi = \chi$ and $p\zeta = \mathrm{id}_{pe'}$ then ζ is invertible.

5.2.2. Definition (cocartesian fibration). An isofibration $p\colon E \twoheadrightarrow B$ is a **cocartesian fibration** if

(i) Any natural transformation $\beta\colon pe \Rightarrow b$ as below-left admits a lift $\chi_\beta\colon e \Rightarrow \beta_* e$ as below-right

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[dr, bend right, "b"'] \arrow[dr, phantom, "\Downarrow\beta"] & E \arrow[d, two heads, "p"] \\ & B \end{tikzcd} \quad = \quad \begin{tikzcd} X \arrow[r, bend left, "e"] \arrow[r, bend right, "\beta_* e"'] \arrow[r, phantom, "\Downarrow\chi_\beta"] & E \arrow[d, two heads, "p"] \\ & B \end{tikzcd}$$

that is a p-cartesian transformation so that $p\chi = \beta$.

(ii) The class of p-cocartesian transformations is closed under restriction along any functor: that is, if $\begin{tikzcd} X \arrow[r, bend left, "e"] \arrow[r, bend right, "e'"'] \arrow[r, phantom, "\Downarrow\chi"] & E \end{tikzcd}$ is p-cocartesian and $f\colon Y \to X$ is any functor then $\begin{tikzcd} Y \arrow[r, bend left, "ef"] \arrow[r, bend right, "e'f"'] \arrow[r, phantom, "\Downarrow\chi f"] & E \end{tikzcd}$ is p-cocartesian.

The dual to Theorem 5.1.11 asks for left adjoints to $i\colon E \to \mathsf{Hom}_B(p, B)$ and $k\colon E^{\mathbb{2}} \to \mathsf{Hom}_B(p, B)$ in place of right adjoints. See Exercise 5.2.i. This result can be deduced immediately by considering the isofibration $p\colon E \twoheadrightarrow B$ as a map in the dual ∞-cosmos of Definition 1.2.25. Recall that for any ∞-cosmos $\mathcal{K}$, there is a **dual ∞-cosmos** $\mathcal{K}^{\mathrm{co}}$ with the same objects but with functor spaces defined by:

$$\mathsf{Fun}_{\mathcal{K}^{\mathrm{co}}}(A, B) \coloneqq \mathsf{Fun}_{\mathcal{K}}(A, B)^{\mathrm{op}}.$$

The isofibrations, equivalences, and trivial fibrations in $\mathcal{K}^{\mathrm{co}}$ coincide with those of $\mathcal{K}$. Conical limits in $\mathcal{K}^{\mathrm{co}}$ coincide with those in $\mathcal{K}$, while the cotensor of $A \in \mathcal{K}$ with $U \in s\mathcal{S}et$ is defined to be $A^{U^{\mathrm{op}}}$. In particular, the cotensor of an ∞-category with $\mathbb{2}$ is defined to be $A^{\mathbb{2}^{\mathrm{op}}}$, which exchanges the domain and codomain projections from arrow and comma ∞-categories.

5.2.3. Definition. An isofibration $p\colon E \twoheadrightarrow B$ defines a **bifibration** if p is both a cartesian fibration and a cocartesian fibration.

Projections give trivial examples of bifibrations.

5.2.4. Example. For any ∞-categories A and B the projection functor $\pi\colon A \times B \twoheadrightarrow B$ is a bifibration, in which a 2-cell with codomain $A \times B$ is π-cocartesian or π-cartesian if and only if its composite with the projection $\pi\colon A \times B \twoheadrightarrow A$ is an isomorphism.

5.2.5. Proposition. *Let $p\colon E \twoheadrightarrow B$ be a bifibration. Then any arrow* $\begin{tikzcd} X \arrow[r, bend left, "a"] \arrow[r, bend right, "b"'] \arrow[r, phantom, "\Downarrow\beta"] & B \end{tikzcd}$ *induces a fibered adjunction*

$$\begin{tikzcd} E_a \arrow[rr, bend left, "\beta_*"] \arrow[rr, phantom, "\perp"] \arrow[dr, two heads, "p_a"'] & & E_b \arrow[ll, bend left, "\beta^*"] \arrow[dl, two heads, "p_b"] \\ & X & \end{tikzcd} \qquad \begin{tikzcd} E_a \arrow[r, "\ell_a"] \arrow[d, two heads, "p_a"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, two heads, "p"] & E_b \arrow[l, "\ell_b"'] \arrow[d, two heads, "p_b"] \arrow[dl, phantom, "\llcorner", very near start] \\ X \arrow[r, "a"'] & B & X \arrow[l, "b"] \end{tikzcd}$$

between the fibers of p over a and b.

As will be remarked upon following the proof of this result, the left adjoint β_* is the covariantly pseudofunctorial action of the arrow β on the fibers of p defined in (5.0.1), while the right adjoint β^* is the dual contravariantly pseudofunctorial action.

Proof. Write $\ulcorner\beta\urcorner\colon X \to B^{\mathbb{2}}$ for the functor induced by β. Note that the pullbacks defining the fibers over its domain edge a and codomain edge b factor as:

$$\begin{tikzcd} E_a \arrow[r] \arrow[d, two heads, "p_a"'] \arrow[dr, phantom, "\lrcorner" very near start] \arrow[rr, bend left, "\ell_a"] & \mathrm{Hom}_B(p,B) \arrow[r, two heads, "p_0"'] \arrow[d, two heads, "q"'] \arrow[dr, phantom, "\lrcorner" very near start] & E \arrow[d, two heads, "p"] & \mathrm{Hom}_B(B,p) \arrow[l, two heads, "p_1"] \arrow[d, two heads, "q"] \arrow[dl, phantom, "\llcorner" very near start] & E_b \arrow[l] \arrow[d, two heads, "p_b"] \arrow[dl, phantom, "\llcorner" very near start] \arrow[ll, bend right, "\ell_b"'] \\ X \arrow[r, "\ulcorner\beta\urcorner"] \arrow[rr, bend right, "a"'] & B^{\mathbb{2}} \arrow[r, two heads, "p_0"] & B & B^{\mathbb{2}} \arrow[l, two heads, "p_1"'] & X \arrow[l, "\ulcorner\beta\urcorner"'] \arrow[ll, bend left, "b"] \end{tikzcd}$$

Now via Remark 5.1.13, Theorem 5.1.11(i)⇒(iii) and its dual provide a right adjoint right inverse to $k\colon E^{\mathbb{2}} \to \mathrm{Hom}_B(B,p)$ and a left adjoint right inverse to $k\colon E^{\mathbb{2}} \to \mathrm{Hom}_B(p,B)$. Composing the former fibered adjunction with $q\colon \mathrm{Hom}_B(B,p) \twoheadrightarrow B^{\mathbb{2}}$ and the latter fibered adjunction with $q\colon \mathrm{Hom}_B(p,B) \twoheadrightarrow B^{\mathbb{2}}$ we obtain a composable pair of adjunctions

$$\begin{tikzcd} \mathrm{Hom}_B(p,B) \arrow[r, bend left, "\bar{\ell}"] \arrow[r, phantom, "\perp"] \arrow[dr, two heads, "q"'] & E^{\mathbb{2}} \arrow[l, bend left, "k"] \arrow[r, bend left, "k"] \arrow[r, phantom, "\perp"] \arrow[d, two heads, "p^{\mathbb{2}}"] & \mathrm{Hom}_B(B,p) \arrow[l, bend left, "\bar{r}"] \arrow[dl, two heads, "q"] \\ & B^{\mathbb{2}} & \end{tikzcd}$$

fibered over $B^{\mathbb{2}}$; note in both cases that $qk = p^{\mathbb{2}}$. Pulling back the composite adjunction along $\ulcorner\beta\urcorner\colon X \to B^{\mathbb{2}}$ yields the fibered adjunction of the statement. □

5.2.6. Remark (action of arrows on the fibers of a co/cartesian fibration). If $p\colon E \twoheadrightarrow B$ is a cocartesian fibration but not a cartesian fibration, the construction in the proof of Proposition 5.2.5 still defines the functor $\beta_*\colon E_a \to E_b$. Examining the details of this construction, we see that it produces the functor given this name in (5.0.1). The functor just constructed as the pullback over $\ulcorner\beta\urcorner$ of $k\bar{\ell}$ is induced by the composite commutative square

$$\begin{tikzcd} E_a \arrow[r] \arrow[d, two heads, "p_a"'] \arrow[dr, phantom, "\lrcorner" very near start] & \mathrm{Hom}_B(p,B) \arrow[r, "p_1\bar{\ell}"] \arrow[d, two heads, "q"'] & E \arrow[d, two heads, "p"] \\ X \arrow[r, "\ulcorner\beta\urcorner"'] & B^{\mathbb{2}} \arrow[r, two heads, "p_1"'] & B \end{tikzcd}$$

which defines a cone over the pullback defining E_b. The top-horizontal functor takes the codomain-component of the p-cocartesian lift of β with domain ℓ_a. This recovers the description given at the start of this chapter.

Exercises.

5.2.i. Exercise. Formulate the dual to Theorem 5.1.11, providing an internal characterization of cocartesian fibrations.

5.2.ii. Exercise. Prove that for any ∞-category A, the codomain-projection functor $p_1\colon A^{\mathbb{2}} \twoheadrightarrow A$ defines a cocartesian fibration.

5.3. The quasi-category theory of cartesian fibrations

In this section, we reinterpret the notion of cartesian fibration and cartesian natural transformation from the point of view of the ∞-cosmos $\mathcal{K}$, rather than its quotient homotopy 2-category $\mathfrak{h}\mathcal{K}$. In doing so, we recall that the functors $e\colon X \to E$ between ∞-categories are precisely the vertices, or *0-arrows*, in the quasi-categorical functor space $\mathsf{Fun}(X, E)$. The 1-simplices, or *1-arrows*, $\chi\colon e' \to e$ of $\mathsf{Fun}(X, E)$ represent natural transformations $X \underset{e}{\overset{e'}{\underset{\Downarrow\chi}{\rightrightarrows}}} E$. Every natural transformation from e' to e is represented by a 1-arrow $e' \to e$ and a parallel pair of 1-arrows represent the same natural transformation if and only if they are *homotopic*, bounding a 2-arrow in $\mathsf{Fun}(X, E)$ whose 0th or 2nd edge is degenerate.

Before analyzing the 0- and 1-arrows in the functor spaces in particular, we consider the collection of functor spaces of an ∞-cosmos globally and prove another important corollary of the internal characterization of cartesian fibrations. The notion of cartesian fibrations are representably defined in the following sense:

5.3.1. Proposition. *Let $p\colon E \twoheadrightarrow B$ be an isofibration between ∞-categories in an ∞-cosmos $\mathcal{K}$. Then p is a cartesian fibration if and only if:*

(i) For all $X \in \mathcal{K}$, the isofibration $p_*\colon \mathsf{Fun}(X, E) \twoheadrightarrow \mathsf{Fun}(X, B)$ is a cartesian fibration between quasi-categories.

(ii) For all $f\colon Y \to X \in \mathcal{K}$, the square

$$\begin{tikzcd} \mathsf{Fun}(X,E) \arrow[r, "f^*"] \arrow[d, two heads, "p_*"'] & \mathsf{Fun}(Y,E) \arrow[d, two heads, "p_*"] \\ \mathsf{Fun}(X,B) \arrow[r, "f^*"'] & \mathsf{Fun}(Y,B) \end{tikzcd}$$

is a cartesian functor.

Proof. If $p\colon E \twoheadrightarrow B$ is a cartesian fibration, then Theorem 5.1.11(i)$\Rightarrow$(iii) constructs a right adjoint right inverse to $k\colon E^{\mathbb{2}} \twoheadrightarrow \mathsf{Hom}_B(B, p)$. The simplicial bifunctor $\mathsf{Fun}(-,-)\colon \mathcal{K}^{\mathrm{op}} \times \mathcal{K} \to \mathcal{QC}at$ defines a 2-functor $\mathsf{Fun}(-,-)\colon \mathfrak{h}\mathcal{K}^{\mathrm{op}} \times \mathfrak{h}\mathcal{K} \to \mathfrak{h}\mathcal{QC}at$, which transposes to a Yoneda-type embedding $\mathsf{Fun}(-,-)\colon \mathfrak{h}\mathcal{K} \to \mathfrak{h}\mathcal{QC}at^{\mathfrak{h}\mathcal{K}^{\mathrm{op}}}$ from the homotopy 2-category of $\mathcal{K}$ to the 2-category of 2-functors, 2-natural transformations, and modifications. This 2-functor carries the adjunction $k \dashv \bar{r}$ to an adjunction in the 2-category $\mathfrak{h}\mathcal{QC}at^{\mathfrak{h}\mathcal{K}^{\mathrm{op}}}$. This latter adjunction defines, for each $X \in \mathcal{K}$, a right adjoint right inverse adjunction

$$\begin{tikzcd} \mathsf{Fun}(X,E)^{\mathbb{2}} \cong \mathsf{Fun}(X,E^{\mathbb{2}}) \arrow[r, two heads, bend left, "k_*"] \arrow[r, phantom, "\perp"] & \mathsf{Fun}(X, \mathsf{Hom}_B(B,p)) \cong \mathsf{Hom}_{\mathsf{Fun}(X,B)}(\mathsf{Fun}(X,B), p_*) \arrow[l, bend left, "\bar{r}_*"] \end{tikzcd}$$

and for each $f\colon Y \to X$ in $\mathcal{K}$, a strict adjunction morphism[6] commuting strictly with the left adjoints and with the right adjoints:

$$\begin{tikzcd} \mathsf{Fun}(X,E)^{\mathbb{2}} \cong \mathsf{Fun}(X,E^{\mathbb{2}}) \arrow[rr, "f^*"] \arrow[d, two heads, "k_*"', shift right=1.5] & & \mathsf{Fun}(Y,E^{\mathbb{2}}) \cong \mathsf{Fun}(Y,E)^{\mathbb{2}} \arrow[d, two heads, "k_*"', shift right=1.5] \\ \mathsf{Hom}(\mathsf{Fun}(X,B),p_*) \cong \mathsf{Fun}(X,\mathsf{Hom}_B(B,p)) \arrow[u, "\bar{r}_*"', shift right=1.5] \arrow[u, phantom, "\vdash"] \arrow[rr, two heads, "f^*"'] & & \mathsf{Fun}(Y,\mathsf{Hom}_B(B,p)) \cong \mathsf{Hom}(\mathsf{Fun}(Y,B),p_*) \arrow[u, "\bar{r}_*"', shift right=1.5] \arrow[u, phantom, "\dashv"] \end{tikzcd} \tag{5.3.2}$$

By Theorems 5.1.11(iii)⇒(i) and 5.1.19(iii)⇒(i), this demonstrates the two conditions of the statement.

Conversely, supposing $p\colon E \twoheadrightarrow B$ satisfies conditions (i) and (ii) by Theorems 5.1.11(i)⇒(iii) and 5.1.19(i)⇒(iii) there is a commutative square $k_*f^* = f^*k_*$ where both verticals k_* admit right adjoint right inverses $k_* \dashv \bar{r}$ and the mate of the identity $k_*f^* = f^*k_*$ defines an isomorphism $f^*\bar{r} \cong \bar{r}f^*$. By Proposition B.6.2 in Appendix B, this data suffices to internalize the right adjoints to the representable functors $k_*\colon \mathsf{Fun}(X,E^{\mathbb{2}}) \to \mathsf{Fun}(X,\mathsf{Hom}_B(B,p))$ to a functor $\bar{r}_*\colon \mathsf{Fun}(X,\mathsf{Hom}_B(B,p)) \to \mathsf{Fun}(X,E^{\mathbb{2}})$ arising from post-composition with some $\bar{r}\colon \mathsf{Hom}_B(B,p) \to E^{\mathbb{2}}$. This right adjoint $\bar{r}$ is extracted as the image of the identity element

$$\begin{array}{ccc} \mathsf{Fun}(\mathsf{Hom}_B(B,p),\mathsf{Hom}_B(B,p)) & \xrightarrow{\bar{r}} & \mathsf{Fun}(\mathsf{Hom}_B(B,p),E^{\mathbb{2}}) \\ \mathrm{id} & \longmapsto & \bar{r} \end{array}$$

and the unit and counit are internalized similarly; the condition on mates is used to verify the triangle equalities that demonstrate that $k \dashv \bar{r}$. Now Theorem 5.1.11(iii)⇒(i) proves that $p\colon E \twoheadrightarrow B$ is a cartesian fibration. □

An easier argument along the same lines demonstrates:

5.3.3. Corollary. *A commutative square between cartesian fibrations as displayed below-left*

$$\begin{tikzcd} F \arrow[r, "g"] \arrow[d, two heads, "q"'] & E \arrow[d, two heads, "p"] \\ A \arrow[r, "f"'] & B \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \mathsf{Fun}(X,F) \arrow[r, "g_*"] \arrow[d, two heads, "q_*"'] & \mathsf{Fun}(X,E) \arrow[d, two heads, "p_*"] \\ \mathsf{Fun}(X,A) \arrow[r, "f_*"'] & \mathsf{Fun}(X,B) \end{tikzcd}$$

defines a cartesian functor in an ∞-cosmos $\mathcal{K}$ if and only if for all $X \in \mathcal{K}$, the square displayed above right defines a cartesian functor between cartesian fibrations of quasi-categories.

Proof. Exercise 5.3.i. □

Our aim is now to characterize those 1-arrows in $\mathsf{Fun}(X,E)$ that represent p-cartesian natural transformation for some cartesian fibration $p\colon E \twoheadrightarrow B$. Recall that the 1-arrows in $\mathsf{Fun}(X,E)$ are in bijection with the 0-arrows of $\mathsf{Fun}(X,E)^{\mathbb{2}} \cong \mathsf{Fun}(X,E^{\mathbb{2}})$.

[6]A **strict adjunction morphism** is given by a pair of functors, the horizontals of (5.3.2), that define strictly commutative squares with both the left and with the right adjoints and so that the units of each adjunction whisker along these functors to each other and the counits of each adjunction whisker along these functors to each other. See Proposition B.6.2 for more.

5.3.4. Definition (p-cartesian 1-arrow). Let $p\colon E \twoheadrightarrow B$ be a cartesian fibration and consider a 1-arrow χ in $\mathsf{Fun}(X,E)$, defining an element $\chi \in \mathsf{Fun}(X,E)^{\mathbb{2}} \cong \mathsf{Fun}(X,E^{\mathbb{2}})$. Say χ is a **p-cartesian 1-arrow** if it is isomorphic to some object in the image of the right adjoint right inverse functor

$$\mathsf{Fun}(X,\mathsf{Hom}_B(B,p)) \xrightarrow{\bar{r}_*} \mathsf{Fun}(X,E^{\mathbb{2}})$$

of Theorem 5.1.11(iii).

The new notion of p-cartesian 1-arrow coincides exactly with the previous notion of p-cartesian natural transformation:

5.3.5. Lemma. *Consider a cartesian fibration $p\colon E \twoheadrightarrow B$ between ∞-categories. For a 1-arrow $\chi\colon e' \to e$ in $\mathsf{Fun}(X,E)$, the following are equivalent:*

(i) χ is a p-cartesian 1-arrow.

(ii) χ represents a p-cartesian natural transformation $\begin{tikzcd} X \arrow[r, bend left, "e'"] \arrow[r, bend right, "e"'] \arrow[r, phantom, "\Downarrow\chi"] & E \end{tikzcd}$.

(iii) χ represents a natural transformation

$$\begin{tikzcd} \mathbb{1} \arrow[r, bend left, "e'"] \arrow[r, bend right, "e"'] \arrow[r, phantom, "\Downarrow\chi"] & \mathsf{Fun}(X,E) \end{tikzcd}$$

that is cartesian for $p_*\colon \mathsf{Fun}(X,E) \twoheadrightarrow \mathsf{Fun}(X,B)$.

Conversely, a natural transformation $\begin{tikzcd} X \arrow[r, bend left, "e'"] \arrow[r, bend right, "e"'] \arrow[r, phantom, "\Downarrow\chi"] & E \end{tikzcd}$ *is p-cartesian if and only if any representing 1-arrow $\ulcorner\chi\urcorner\colon X \to E^{\mathbb{2}}$ satisfies all of these equivalent conditions.*

Proof. We start with the final clause. Note from Exercise 3.2.i that homotopic 1-arrows are isomorphic as objects of $\mathsf{Fun}(X,E^{\mathbb{2}})$ so if some 1-arrow representing a p-cartesian transformation is a p-cartesian 1-arrow then all representatives of that natural transformation are.

Now the equivalence (i)$\Leftrightarrow$(ii) follows from Theorem 5.1.11(iv)$\Leftrightarrow$(vi) once we establish that a 1-arrow of $\mathsf{Fun}(X,E)$, when encoded as a functor $\chi\colon X \to E^{\mathbb{2}}$ is in the essential image of $\bar{r}_*$ if and only if the component $\bar{\eta}\chi$ of the unit of $k \dashv \bar{r}$ is invertible.

If $\chi\colon e' \to e$ is a p-cartesian 1-arrow, then by definition there exists some $\beta\colon X \to \mathsf{Hom}_B(B,p)$ and an invertible 2-cell

$$\begin{tikzcd} X \arrow[rr, bend left, "\chi"] \arrow[dr, "\beta"'] & \arrow[d, phantom, "\Downarrow\cong"] & E^{\mathbb{2}} \\ & \mathsf{Hom}_B(B,p) \arrow[ur, "\bar{r}"'] & \end{tikzcd}$$

The unit $\bar{\eta}$ of the adjunction $k \dashv \bar{r}$ of Theorem 5.1.11(iii) has the property that $\bar{\eta}\bar{r}$ is invertible, so the component $\bar{\eta}\bar{r}\beta$ is invertible and so $\bar{\eta}\chi$ is also invertible. By Theorem 5.1.11(vi)$\Rightarrow$(iv), this implies that χ represents a p-cartesian natural transformation.

Conversely, if χ defines a p-cartesian natural transformation then Theorem 5.1.11(iv)$\Rightarrow$(vi) tells us that for any representing 1-arrow $\ulcorner\chi\urcorner\colon X \to E^{\mathbb{2}}$, the component $\bar{\eta}\ulcorner\chi\urcorner$ is an isomorphism. In particular, $\ulcorner\chi\urcorner$ is isomorphic to $\bar{r}k\ulcorner\chi\urcorner$, which proves that $\ulcorner\chi\urcorner$ is in the essential image of $\bar{r}_*$ and thus defines a p-cartesian 1-arrow.

Finally, via the characterization of cartesian transformations given in Theorem (vi) and the fact that the adjunction $k \dashv \bar{r}$ in $\mathfrak{h}\mathcal{K}$ induces an adjunction $k_* \dashv \bar{r}_*$ in $\mathfrak{h}\mathcal{QC}at$, the conditions (ii) and (iii) are tautologically equivalent. The former refers to the adjunction between categories $\mathsf{hFun}(X,E)$ and $\mathsf{hFun}(X,B)$, while the latter refers to the adjunction between categories $\mathsf{hFun}(\mathbb{1},\mathsf{Fun}(X,E))$ and $\mathsf{hFun}(\mathbb{1},\mathsf{Fun}(X,B))$ and these are isomorphic. □

Combining Lemma 5.3.5 with Proposition 5.3.1 we arrive at a new equivalent definition of cartesian fibrations.

5.3.6. Corollary. *An isofibration $p\colon E \twoheadrightarrow B$ is a* ***cartesian fibration*** *if and only if*

(i) Any 1-arrow with codomain B admits a p-cartesian lift with specified codomain 0-arrow.
(ii) p-cartesian 1-arrows are stable under precomposition with 0-arrows. □

5.3.7. Lemma. *Let $p\colon E \twoheadrightarrow B$ be a cartesian fibration and consider a 2-arrow*

$$\begin{tikzcd} & e' \arrow[dr, "\psi"] & \\ e'' \arrow[ur, "\psi'"] \arrow[rr, "\psi''"'] & & e \end{tikzcd}$$

in $\mathsf{Fun}(X,E)$.

(i) If ψ and ψ' are p-cartesian 1-arrows, so is ψ''.
(ii) If ψ and ψ'' are p-cartesian 1-arrows, so is ψ'.
(iii) If ψ and ψ'' are p-cartesian 1-arrows and $p\psi'$ is invertible, then ψ' is invertible.

Proof. Via Lemma 5.3.5, (i) and (ii) are Lemmas 5.1.24(i) and (ii), while (iii) is Lemma 5.1.5. □

5.3.8. Lemma. *A 2-cell as below left*

$$\begin{tikzcd} Q \arrow[r, bend left, "e'"] \arrow[r, bend right, "e"'] \arrow[r, phantom, "\Downarrow\chi"] & \mathsf{Fun}(X,E) \end{tikzcd} \qquad \begin{tikzcd} \mathbb{1} \arrow[r, "q"] & Q \arrow[r, bend left, "e'"] \arrow[r, bend right, "e"'] \arrow[r, phantom, "\Downarrow\chi"] & \mathsf{Fun}(X,E) \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} X \arrow[r, bend left, "e'q"] \arrow[r, bend right, "eq"'] \arrow[r, phantom, "\Downarrow\chi q"] & E \end{tikzcd}$$

is p_*-cartesian if and only if each of its components χq is p-cartesian.

Proof. If χ is p_*-cartesian, then so is the restriction χq along any element $q\colon \mathbb{1} \to Q$. By Lemma 5.3.5 this tells us that χq defines a p-cartesian transformation.

Conversely, if χq is a p-cartesian transformation, then Lemma 5.3.5 tells us that χq is a p_*-cartesian transformation. Now consider the factorization $\chi = \chi_\chi \cdot \theta$ through p_*-cartesian lift χ_χ of $p_*\chi$. Because the components χq of χ are p_*-cartesian, the components θq of θ are isomorphisms. By Lemma 12.2.1, an arrow in an exponetial $\mathsf{Fun}(X,E)^Q$ is an isomorphism if and only if it is a pointwise isomorphism, so this implies that χ. By isomorphism stability of cartesian transformations, we thus conclude that χ is p_*-cartesian. □

Exercises.

5.3.i. Exercise. Prove Corollary 5.3.3.

5.4. Discrete cartesian fibrations

Recall from Definition 1.2.26 that an object E in an ∞-cosmos $\mathcal{K}$ is **discrete** if for all $X \in \mathcal{K}$, the functor-space $\mathsf{Fun}(X, E)$ is a Kan complex. Since a quasi-category is a Kan complex just when its homotopy category is a groupoid (see Corollary 1.1.15), equivalently, E is discrete if and only if every natural transformation with codomain E is invertible.

From this definition it follows that an isofibration $p\colon E \twoheadrightarrow B$, considered as an object of $\mathcal{K}_{/B}$, is discrete if and only if any 2-cell with codomain E that whiskers with p to an identity is invertible. In fact, the discrete objects are exactly those isofibrations that define conservative functors in $\mathfrak{h}\mathcal{K}$.

5.4.1. Lemma. *An isofibration $p\colon E \twoheadrightarrow B$ is a discrete object of $\mathcal{K}_{/B}$ if and only if $p\colon E \twoheadrightarrow B$ is a **conservative functor**: meaning any* $X \underset{b}{\overset{a}{\underset{\Downarrow\gamma}{\rightrightarrows}}} E$ *for which $p\gamma$ is an isomorphism is invertible.*

Proof. Exercise 5.4.i. $\square$

Our aim in this section is to study a special class of cartesian fibrations and cocartesian fibrations:

5.4.2. Definition. An isofibration $p\colon E \twoheadrightarrow B$ is a **discrete cartesian fibration** if it is a cartesian fibration and if it is discrete as an object of $\mathcal{K}_{/B}$. Dually, an isofibration $p\colon E \twoheadrightarrow B$ is a **discrete cocartesian fibration** if it is a cocartesian fibration and if it is discrete as an object of $\mathcal{K}_{/B}$

The fibers of a discrete object $p\colon E \twoheadrightarrow B$ in $\mathcal{K}_{/B}$ are discrete ∞-categories. In fact, in ∞-cosmoi $\mathcal{K}$ whose objects model $(\infty, 1)$-categories, an isofibration $p\colon E \twoheadrightarrow B$ defines a discrete object in $\mathcal{K}_{/B}$ if and only if its fibers are discrete ∞-categories, as will be proven in Proposition 12.2.3. Thus, in such ∞-cosmoi, the discrete cartesian fibrations and discrete cocartesian fibrations are "∞-groupoid-valued pseudofunctors."

There is also a direct 2-categorical characterization of the discrete cartesian fibrations, as those maps $p\colon E \twoheadrightarrow B$ with the property that every 2-cell $\beta\colon b \Rightarrow pe$ has an essentially unique lift with codomain e. To prove this, we begin by observing that for a discrete cartesian or cocartesian fibration p, there are no special classes of p-cartesian or p-cocartesian cells, unlike the case for the indiscrete analogues.

5.4.3. Proposition.

(i) *If $p\colon E \twoheadrightarrow B$ is a discrete cartesian fibration, every natural transformation with codomain E is p-cartesian.*

(ii) *An isofibration $p\colon E \twoheadrightarrow B$ is a discrete cartesian fibration if and only if every 2-cell $\beta\colon b \Rightarrow pe$ has an essentially unique lift: given $\chi\colon e' \Rightarrow e$ and $\psi\colon e'' \Rightarrow e$ so that $p\chi = p\psi = \beta$, then there exists an isomorphism $\gamma\colon e'' \Rightarrow e'$ with $\chi \cdot \gamma = \psi$ and $p\gamma = \mathrm{id}$.*

Proof. By the definition of cartesian fibration, any 2-cell ψ with codomain E factors through a p-cartesian lift of $p\psi$ along a 2-cell γ so that $p\gamma = \mathrm{id}$. The discrete objects of $\mathcal{K}_{/B}$ are exactly those isofibrations with the property that any 2-cell with codomain E that whiskers with p to an identity is invertible. In particular, γ is an isomorphism, and now ψ is isomorphic to a p-cartesian transformation and hence by Lemma 5.1.4 itself p-cartesian.

By (i) and Lemma 5.1.3, it's now clear that if $p\colon E \twoheadrightarrow B$ is a discrete cartesian fibration, then any 2-cell $\beta\colon b \Rightarrow pe$ has an essentially unique lift. For the converse, note first that any $p\colon E \twoheadrightarrow B$ satisfying this hypothesis is a discrete object: if $\psi\colon e' \Rightarrow e$ is so that $p\psi = \mathrm{id}$, then $\mathrm{id}\colon e \Rightarrow e$ is another lift of $p\psi$ and essential uniqueness provides an inverse isomorphism $\psi^{-1}\colon e \Rightarrow e'$.

To complete the proof, we now show that any 2-cell $\chi\colon e' \Rightarrow e$ is cartesian for p and to that end consider a pair $\tau\colon e'' \Rightarrow e$ and $\gamma\colon pe'' \Rightarrow pe'$ so that $p\tau = p\chi \cdot \gamma$. By the hypothesis that every 2-cell admits an essentially unique lift, we can construct a lift $\mu\colon \bar{e} \Rightarrow e'$ so that $p\mu = \gamma$. Now τ and $\chi \cdot \mu$ are two lifts of $p\tau$ with the same codomain, so there exists an isomorphism $\theta\colon e'' \Rightarrow \bar{e}$ with $p\theta = \mathrm{id}$. The composite $\mu \cdot \theta$ then defines the desired lift of γ to a cell so that $\tau = \chi \cdot \mu \cdot \theta$. ☐

As an immediate consequence of (i):

5.4.4. Corollary. *Any commutative square*

$$\begin{tikzcd} F \arrow[r, "g"] \arrow[d, two heads, "q"'] & E \arrow[d, two heads, "p"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

from a cartesian fibration q to a discrete cartesian fibration p defines a cartesian functor. ☐

5.4.5. Example (domain projection from an element). For an element $b\colon 1 \to B$, the domain-projection functor $p_0\colon \mathsf{Hom}_B(B, b) \twoheadrightarrow B$ is a discrete cartesian fibration. Cartesianness was established in Proposition 5.1.23 and discreteness follows immediately from 2-cell conservativity. If $\begin{tikzcd} X \arrow[r, bend left, "a"] \arrow[r, bend right, "b"'] \arrow[r, phantom, "\Downarrow\gamma"] & \mathsf{Hom}_B(B, b) \end{tikzcd}$ is a natural transformation for which $p_0\gamma$ is an identity, then since $p_1\gamma$ is also an identity, γ must be invertible.

Dually, the codomain-projection functor $p_1\colon \mathsf{Hom}_B(b, B) \twoheadrightarrow B$ is a discrete cocartesian fibration.

5.4.6. Lemma (pullback stability). *If*

$$\begin{tikzcd} F \arrow[r, "g"] \arrow[d, two heads, "q"'] \arrow[dr, phantom, "\lrcorner" very near start] & E \arrow[d, two heads, "p"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

is a pullback square and p is a discrete cartesian fibration then q is a discrete cartesian fibration.

Proof. In light of Proposition 5.1.20 it remains only to verify that q is discrete. Consider a 2-cell $\begin{tikzcd} X \arrow[r, bend left, "a"] \arrow[r, bend right, "b"'] \arrow[r, phantom, "\Downarrow\gamma"] & F \end{tikzcd}$ so that $q\gamma$ is invertible. Then $fq\gamma = pg\gamma$ is invertible and conservativity of p implies that $g\gamma$ is invertible.

By Lemma 3.1.5, the pullback square of functor spaces

$$\begin{tikzcd} \mathsf{Fun}(X, F) \arrow[r, "g_*"] \arrow[d, two heads, "q_*"'] \arrow[dr, phantom, "\lrcorner" very near start] & \mathsf{Fun}(X, E) \arrow[d, two heads, "p_*"] \\ \mathsf{Fun}(X, A) \arrow[r, "f_*"'] & \mathsf{Fun}(X, B) \end{tikzcd}$$

induces a smothering functor

$$\mathsf{hFun}(X, F) \to \mathsf{hFun}(X, E) \underset{\mathsf{hFun}(X,B)}{\times} \mathsf{hFun}(X, A)$$

We've just verified that the image of γ is an isomorphism, so conservativity implies that γ is also invertible. $\square$

In analogy with Theorem 5.1.11, there is an internal characterization of discrete cartesian fibrations, which in the discrete case takes a much simpler form. Recall any functor $p\colon E \to B$ induces functors $k\colon E^{\mathbb{2}} \to \mathsf{Hom}_B(B,p)$ and $k\colon E^{\mathbb{2}} \to \mathsf{Hom}_B(p,B)$ as in (5.1.10) by applying p to the generic arrow for E.

5.4.7. Proposition (internal characterization of discrete fibrations). *An isofibration $p\colon E \twoheadrightarrow B$ is a discrete cartesian fibration if and only if the functor $k\colon E^{\mathbb{2}} \to \mathsf{Hom}_B(B,p)$ is an equivalence and a discrete cocartesian fibration if and only if the functor $k\colon E^{\mathbb{2}} \to \mathsf{Hom}_B(p,B)$ is an equivalence.*

Recall from Theorem 5.1.11(iii) that $p\colon E \twoheadrightarrow B$ defines a cartesian fibration if and only if $k\colon E^{\mathbb{2}} \to \mathsf{Hom}_B(p,B)$ admits a right adjoint with invertible counit. Proposition 5.4.7 asserts that p defines a discrete cartesian fibration if and only if the unit of that adjunction, a natural transformation that defines the factorization of any natural transformation with codomain E through the canonical p-cartesian lift of its image under p, is an isomorphism, in which case that adjunction defines an adjoint equivalence and all natural transformations with codomain E are p-cartesian.

Proof. Assume first that $p\colon E \twoheadrightarrow B$ is a discrete cartesian fibration. By Theorem 5.1.11(i)⇒(iii), $k\colon E^{\mathbb{2}} \to \mathsf{Hom}_B(B,p)$ then admits a right adjoint $\bar{r}$ with invertible counit $\bar{\epsilon}\colon k\bar{r} \cong \mathrm{id}$. We will show that in this case the unit $\bar{\eta}\colon \mathrm{id} \Rightarrow \bar{r}k$ is also invertible, proving that $k \dashv \bar{r}$ defines an adjoint equivalence.

Since the counit of $k \dashv \bar{r}$ is invertible, $k\bar{\eta}$ is an isomorphism. Thus $p_1k\bar{\eta} = p_1\bar{\eta}$ and $p_0k\bar{\eta} = pp_0\bar{\eta}$ are both isomorphisms. By conservativity of the discrete fibration $p\colon E \twoheadrightarrow B$ proven in Lemma 5.4.1, this implies that $p_0\bar{\eta}$ is invertible and now 2-cell conservativity for $E^{\mathbb{2}}$ reveals that $\bar{\eta}$ is an isomorphism.

Conversely, if $k\colon E^{\mathbb{2}} \xrightarrow{\sim} \mathsf{Hom}_B(B,p)$ is an equivalence, by Proposition 2.1.11, we may choose a right adjoint equivalence inverse $\bar{r}$. The counit of this adjoint equivalence is necessarily an isomorphism, so by Theorem 5.1.11(iii)⇒(i) we know that $p\colon E \twoheadrightarrow B$ is a cartesian fibration. Since the unit of $k \dashv \bar{r}$ is also an isomorphism, Theorem 5.1.19(vi)⇒(iv) tells us that every natural transformation with codomain E is p-cartesian, and now the conservativity property for cartesian transformations of Lemma 5.1.5 tells us that $p\colon E \twoheadrightarrow B$ defines a conservative functor, and in particular is discrete. $\square$

Since equivalences and simplicial limits in an ∞-cosmos are representably-defined notions, it follows immediately from Proposition 5.4.7 that:

5.4.8. Proposition. *An isofibration $p\colon E \twoheadrightarrow B$ in an ∞-cosmos $\mathcal{K}$ defines a discrete cartesian fibration if and only if for all $X \in \mathcal{K}$, the functor $p_*\colon \mathsf{Fun}(X,E) \twoheadrightarrow \mathsf{Fun}(X,B)$ defines a discrete cartesian fibration of quasi-categories.* $\square$

Using the internal characterization, it is straightforward to verify that discrete cartesian fibrations compose and cancel on the left:

5.4.9. Lemma.

(i) If $p\colon E \twoheadrightarrow B$ and $q\colon B \twoheadrightarrow A$ are discrete cartesian fibrations, so is $qp\colon E \twoheadrightarrow A$.

(ii) If $p\colon E \twoheadrightarrow B$ is an isofibration and $q\colon B \twoheadrightarrow A$ and $qp\colon E \twoheadrightarrow A$ are discrete cartesian fibrations, then so is $p\colon E \twoheadrightarrow B$.

Proof. By considering the defining pullback diagrams, the map $E^{\mathbb{2}} \twoheadrightarrow \mathsf{Hom}_A(A, qp)$ that tests whether $qp\colon E \twoheadrightarrow A$ is a discrete cartesian fibration factors as the map $E^{\mathbb{2}} \twoheadrightarrow \mathsf{Hom}_B(B, p)$ that tests whether $p\colon E \twoheadrightarrow B$ is a discrete cartesian fibration followed by a pullback of the map $B^{\mathbb{2}} \twoheadrightarrow \mathsf{Hom}_A(A, q)$ that tests whether $q\colon B \twoheadrightarrow A$ is a discrete cartesian fibration:

$$\begin{tikzcd}
E^{\mathbb{2}} \arrow[dr, dashed, two heads] \arrow[rrrd, bend left] \arrow[dddr, bend right, two heads, "p_1"'] & & & & & \\
& \mathsf{Hom}_B(B,p) \arrow[rr] \arrow[dd, two heads, "p_1"'] \arrow[dr, dashed, two heads] \arrow[dr, phantom, very near start, "\lrcorner"] & & B^{\mathbb{2}} \arrow[rr] \arrow[dr, dashed, two heads] \arrow[dd, two heads, "p_1"] & & A^{\mathbb{2}} \arrow[dd, two heads, "p_1"] \\
& & \mathsf{Hom}_A(A,qp) \arrow[rrru] \arrow[rr] \arrow[dl] \arrow[dr, phantom, very near start, "\lrcorner"] & & \mathsf{Hom}_A(A,q) \arrow[ru] \arrow[dl] \arrow[dr, phantom, very near start, "\lrcorner"] & \\
& E \arrow[rr, two heads, "p"'] & & B \arrow[rr, two heads, "q"'] & & A
\end{tikzcd}$$

Both parts now follow from the 2-of-3 property. □

The internal characterization of discrete cartesian fibrations is useful for establishing further examples.

5.4.10. Lemma. *A trivial fibration $p\colon E \xrightarrow{\sim}\!\!\!\!\twoheadrightarrow B$ is a discrete bifibration.*

Proof. Recall from Remark 5.1.13, that the canonical functors $k\colon E^{\mathbb{2}} \twoheadrightarrow \mathsf{Hom}_B(B, p)$ and $k\colon E^{\mathbb{2}} \twoheadrightarrow \mathsf{Hom}_B(p, B)$ can be constructed as the Leibniz cotensor of the monomorphism $1\colon \mathbb{1} \hookrightarrow \mathbb{2}$ in the first case and $0\colon \mathbb{1} \hookrightarrow \mathbb{2}$ in the second with the trivial fibration $p\colon E \xrightarrow{\sim}\!\!\!\!\twoheadrightarrow B$. By Lemma 1.2.14, both maps are trivial fibrations and in particular equivalences. Now Proposition 5.4.7 proves that p is a discrete cartesian fibration and also a discrete cocartesian fibration. □

A final important family of examples of discrete cartesian fibrations are worth establishing. Proposition 5.1.23 proves that for any ∞-category A, the domain-projection functor $p_0\colon A^{\mathbb{2}} \twoheadrightarrow A$ defines a cartesian fibration. This functor does *not* define a discrete cartesian fibration in the ∞-cosmos $\mathcal{K}$, but recall that p_0-cartesian lifts can be constructed to project to identity arrows along $p_1\colon A^{\mathbb{2}} \twoheadrightarrow A$. This suggests that we might productively consider the domain-projection functor as a map over A, in which case we have the following result:

5.4.11. Proposition. *The functor*

$$\begin{tikzcd}
A^{\mathbb{2}} \arrow[rr, two heads, "{(p_1,p_0)}"] \arrow[dr, two heads, "p_1"'] & & A \times A \arrow[dl, two heads, "\pi"] \\
& A &
\end{tikzcd} \tag{5.4.12}$$

defines a discrete cartesian fibration in the slice ∞-cosmos $\mathcal{K}_{/A}$.

Proof. Note that 2-cell conservativity implies that (5.4.12) is a discrete object in $(\mathcal{K}_{/A})_{/\pi\colon A\times A \twoheadrightarrow A} \cong \mathcal{K}_{/A\times A}$, so it remains only to prove that this functor defines a cartesian fibration. We prove this using Theorem 5.1.11(i)⇔(ii). The first step is to compute the right representable comma object for the functor (5.4.12) by interpreting the formula (3.4.2) in the slice ∞-cosmos $\mathcal{K}_{/A}$ using Proposition 1.2.22. The $\mathbb{2}$-cotensor of the object $\pi\colon A \times A \twoheadrightarrow A$ is $\pi\colon A \times A^{\mathbb{2}} \twoheadrightarrow A$, so this right representable comma is computed by the left-hand pullback in $\mathcal{K}_{/A}$ below:

$$\begin{tikzcd}
\mathsf{Hom}_A(A,p_0) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, very near start, "\lrcorner"] \arrow[ddr, dotted, two heads, "p_2"] & A\times A^{\mathbb{2}} \arrow[r, "\pi"] \arrow[d, two heads, "\mathrm{id}_A\times p_1"] \arrow[dr, phantom, very near start, "\lrcorner"] \arrow[dd, bend right, two heads] & A^{\mathbb{2}} \arrow[d, two heads, "p_1"] \\
A^{\mathbb{2}} \arrow[r, "(p_1,p_0)"] \arrow[dr, two heads, "p_1"'] & A\times A \arrow[r, "\pi"] \arrow[d, two heads, "\pi"] & A \\
& A &
\end{tikzcd}$$

Pasting with the right-hand pullback in $\mathcal{K}$, we recognize that the ∞-category so-constructed coincides with the right representable comma object for the functor $p_0\colon A^{\mathbb{2}} \twoheadrightarrow A$ considered as a map in $\mathcal{K}$. Under the equivalence $\mathsf{Hom}_A(A,p_0) \simeq A^{\mathbb{3}}$ established in the proof of Proposition 5.1.23, the isofibration $p_2\colon \mathsf{Hom}_A(A,p_0) \twoheadrightarrow A$ is evaluation at the final element $2 \in \mathbb{3}$ in the composable pair of arrows. Similarly, the canonical functor $i\colon A^{\mathbb{2}} \to \mathsf{Hom}_A(A,p_0)$ induced by id_{p_0} in $\mathcal{K}$ coincides with the canonical functor $i\colon A^{\mathbb{2}} \to \mathsf{Hom}_A(A,p_0)$ over A induced by $\mathrm{id}_{(p_1,p_0)}$ in $\mathcal{K}_{/A}$.

Now applying Proposition 5.1.23 and Theorem 5.1.11(i)⇒(ii) in $\mathcal{K}$, this functor i admits a right adjoint r over the *domain*-projection functor

$$\begin{tikzcd}
A^{\mathbb{2}} \arrow[rr, bend left, "i"] \arrow[dr, two heads, "p_0"'] & \perp & \mathsf{Hom}_A(A,p_0) \arrow[ll, dashed, bend left, "r"] \arrow[dl, two heads, "p_0"] \\
& A &
\end{tikzcd}$$

By the proofs of Theorem 5.1.11(iii)⇒(ii) and Proposition 5.1.23 this adjunction can be constructed by cotensoring $A^{(-)}$ the composite adjunction of categories

$$\begin{tikzcd}
\mathbb{2} \arrow[r, bend right, "\mathbb{2}\times 0"'] \arrow[rr, bend right=60, "\delta^1"'] & \mathbb{2}\times\mathbb{2} \arrow[l, bend right, "\mathbb{2}\times !"'] \arrow[r, bend right, "\ell"'] \arrow[l, phantom, "\top"] & \mathbb{3} \arrow[l, bend right, "k"'] \arrow[ll, bend right=60, "\sigma^0"'] \arrow[l, phantom, "\top"] \\
& \mathbb{1}+\mathbb{1} \arrow[ul, hook, "(1{,}0)"] \arrow[ur, hook, "(2{,}0)"'] &
\end{tikzcd}
\quad\rightsquigarrow\quad
\begin{tikzcd}
A^{\mathbb{2}} \arrow[rr, bend left, "i"] \arrow[dr, two heads, "(p_1{,}p_0)"'] & \perp & A^{\mathbb{3}} \arrow[ll, bend left, "r"] \arrow[dl, two heads, "(p_2{,}p_0)"] \\
& A\times A &
\end{tikzcd}$$

where $\ell \dashv k$ is described in the proof of Proposition 5.1.23. The composite right adjoint is the functor $\sigma^0\colon \mathbb{3} \twoheadrightarrow \mathbb{2}$ that sends 0 and 1 to 0 and 2 to 1, while the composite left adjoint is the functor $\delta^1\colon \mathbb{2} \rightarrowtail \mathbb{3}$ that sends 0 to 0 and 1 to 2. In particular, this adjunction lies in the strict slice 2-category under the inclusion of the "endpoints" of $\mathbb{2}$ and $\mathbb{3}$.

It follows that upon cotensoring into A, we obtain a fibered adjunction over $A \times A$, which by Theorem 5.1.11(ii)⇒(i) implies that (5.4.12) is a cartesian fibration in $\mathcal{K}_{/A}$, completing the proof. □

Combining Propositions 5.1.23 and Proposition 5.4.11, we can now generalize both results to arbitrary comma ∞-categories.

5.4.13. Corollary. *For any functors* $C \xrightarrow{g} A \xleftarrow{f} B$ *between ∞-categories in an ∞-cosmos $\mathcal{K}$:*

(i) The domain-projection functor $p_0 \colon \mathsf{Hom}_A(f,g) \twoheadrightarrow B$ is a cartesian fibration. Moreover, a natural transformation χ with codomain $\mathsf{Hom}_A(f,g)$ is p_0-cartesian if and only if $p_1\chi$ is invertible.

(ii) The codomain-projection functor $p_1 \colon \mathsf{Hom}_A(f,g) \twoheadrightarrow C$ is a cocartesian fibration. Moreover, a natural transformation χ with codomain $\mathsf{Hom}_A(f,g)$ is p_1-cartesian if and only if $p_0\chi$ is invertible.

(iii) The functor

$$\begin{tikzcd} \mathsf{Hom}_A(f,g) \arrow[rr, two heads, "{(p_1,p_0)}"] \arrow[dr, two heads, "p_1"'] & & C \times B \arrow[dl, two heads, "\pi"] \\ & C & \end{tikzcd}$$

defines a discrete cartesian fibration in $\mathcal{K}_{/C}$.

(iv) The functor

$$\begin{tikzcd} \mathsf{Hom}_A(f,g) \arrow[rr, two heads, "{(p_1,p_0)}"] \arrow[dr, two heads, "p_0"'] & & C \times B \arrow[dl, two heads, "\pi"] \\ & B & \end{tikzcd}$$

defines a discrete cocartesian fibration in $\mathcal{K}_{/B}$.

Proof. We prove (i) and (iii) and leave the dualizations to the reader. For (iii), we first use the cosmological functor $g^* \colon \mathcal{K}_{/A} \to \mathcal{K}_{/C}$, which preserves discrete cartesian fibrations, to establish that

$$\begin{tikzcd} \mathsf{Hom}_A(A,g) \arrow[rr, two heads, "{(p_1,p_0)}"] \arrow[dr, two heads, "p_1"'] & & C \times A \arrow[dl, two heads, "\pi"] \\ & C & \end{tikzcd}$$

defines a discrete cartesian fibration in $\mathcal{K}_{/C}$; this argument works because $p_1 \colon \mathsf{Hom}_A(A,g) \twoheadrightarrow C$ is the pullback of $p_1 \colon A^{\mathbb{2}} \twoheadrightarrow A$ along g. Now

$$\begin{tikzcd} \mathsf{Hom}_A(f,g) \arrow[rr] \arrow[d, two heads, "{(p_1,p_0)}"'] \arrow[ddr, two heads, bend left, "p_1"] \arrow[dr, phantom, "\lrcorner", very near start] & & \mathsf{Hom}_A(A,g) \arrow[d, two heads, "{(p_1,p_0)}"] \arrow[ddl, two heads, bend right, "p_1"'] \\ C \times B \arrow[rr, "C\times f"] \arrow[dr, two heads, "\pi"'] & & C \times A \arrow[dl, two heads, "\pi"] \\ & C & \end{tikzcd}$$

is a pullback square in $\mathcal{K}_{/C}$, so Lemma 5.4.6 now implies that the pullback is also a discrete cartesian fibration.

Using (iii) we can now prove (i). This follows directly from a general claim that if

$$\begin{tikzcd} E \arrow[rr, two heads, "{(q,p)}"] \arrow[dr, two heads, "q"'] & & C \times B \arrow[dl, two heads, "\pi"] \\ & C & \end{tikzcd}$$

defines a cartesian fibration in $\mathcal{K}_{/C}$ then $p\colon E \twoheadrightarrow B$ defines a cartesian fibration in $\mathcal{K}$. By Theorem 5.1.11(i)⇔(ii), this functor defines a cartesian fibration in $\mathcal{K}_{/C}$ if and only if the functor i

$$\begin{tikzcd} E \arrow[rr, bend left, "i"] \arrow[rr, phantom, "\perp"] \arrow[dr, two heads, "{(q,p)}"'] & & \mathrm{Hom}_B(B,p) \arrow[ll, dashed, bend left, "r"] \arrow[dl, two heads, "{(qp_1,p_0)}"] \\ & C \times B & \end{tikzcd}$$

admits a right adjoint r over $C \times B$. Composing with $\pi\colon C \times B \twoheadrightarrow B$, this fibered adjunction defines an adjunction over B, and Theorem 5.1.11(i)⇔(ii) applied this time in $\mathcal{K}$ allows us to conclude that $p\colon E \twoheadrightarrow B$ is a cartesian fibration. □

Note that the domain projection $p_0\colon \mathrm{Hom}_A(f,A) \twoheadrightarrow B$ is the pullback of $p_0\colon A^{\mathbb{2}} \twoheadrightarrow A$ along $f\colon B \to A$, so Proposition 5.1.20 proves directly from Proposition 5.1.23 that this functor is a cartesian fibration, but $p_0\colon \mathrm{Hom}_A(A,g) \twoheadrightarrow A$ is not similarly a pullback of $p_0\colon A^{\mathbb{2}} \twoheadrightarrow A$. This is why a more circuitous argument to the general result is needed.

Exercises.

5.4.i. Exercise. Prove Lemma 5.4.1.

5.5. The external Yoneda lemma

Let $b\colon 1 \to B$ be an element of an ∞-category B and consider its right representation $\mathrm{Hom}_B(B,b)$ as a comma ∞-category. In this case, there is no additional data given by the codomain-projection functor, but Example 5.4.5 observes that the domain-projection functor $p_0\colon \mathrm{Hom}_B(B,b) \twoheadrightarrow B$ has a special property: it defines a discrete cartesian fibration. The fibers of this map over an element $a\colon 1 \to B$ are the internal mapping spaces $\mathrm{Hom}_B(a,b)$ of Definition 3.4.9. In this way, the right representation of the element b encodes the contravariant functor represented by b, which is why all along we've been referring to the comma ∞-categories $\mathrm{Hom}_B(B,b)$ as "representable."

Our aim in this section is to state and prove the Yoneda lemma in this setting, where contravariant representable functors are encoded as discrete cartesian fibrations. A dual statement applies to covariant representable functors encoded as discrete cocartesian fibration $p_1\colon \mathrm{Hom}_B(b,B) \twoheadrightarrow B$, but for ease of exposition we leave the dualization to the reader. Informally, the Yoneda lemma asserts that "evaluation at the identity defines an equivalence," so the first step towards the statement of the Yoneda lemma is to introduce this identity element, which in fact is something we've already encountered.

The identity arrow id_b induces an element $\ulcorner \mathrm{id}_b \urcorner\colon 1 \to \mathrm{Hom}_B(B,b)$ which Corollary 3.5.9 proves is terminal in the ∞-category $\mathrm{Hom}_B(B,b)$ of arrows in B with codomain b. The identity element inclusion defines a functor over B

$$\begin{tikzcd} 1 \arrow[rr, "\ulcorner \mathrm{id}_b \urcorner"] \arrow[dr, "b"'] & & \mathrm{Hom}_B(B,b) \arrow[dl, two heads, "p_0"] \\ & B & \end{tikzcd} \tag{5.5.1}$$

Technically, this functor does not live in the sliced ∞-cosmos $\mathcal{K}_{/B}$ because the domain object $b\colon 1 \to B$ is not an isofibration but nevertheless for any isofibration $p\colon E \twoheadrightarrow B$, restriction along $\ulcorner \mathrm{id}_b \urcorner$ induces a functor between sliced quasi-categorical functor spaces

$$\mathsf{Fun}_B(\mathsf{Hom}_B(B,b) \xrightarrow{p_0} B, E \xrightarrow{p} B) \xrightarrow{\mathrm{ev}_{\ulcorner \mathrm{id}_b \urcorner}} \mathsf{Fun}_B(1 \xrightarrow{b} B, E \xrightarrow{p} B) \tag{5.5.2}$$

Here the codomain is the quasi-category defined by the pullback

$$\begin{tikzcd} \mathsf{Fun}_B(b,p) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner" very near start] & \mathsf{Fun}(1,E) \arrow[d, two heads, "p_*"] \\ \mathbb{1} \arrow[r, "b"] & \mathsf{Fun}(1,B) \end{tikzcd}$$

which is isomorphic to $\mathsf{Fun}(1, E_b)$, the underlying quasi-category of the fiber E_b of $p\colon E \twoheadrightarrow B$ over b.

If a discrete cartesian fibration over B is thought of as a B-indexed discrete ∞-category valued contravariant pseudofunctor, then maps of discrete cartesian fibrations over B are "natural transformations": the "naturality in B" arises because we only allow functors over B. This leads to our first statement of the fibrational Yoneda lemma:

5.5.3. Theorem (external Yoneda lemma, discrete case). *If $p\colon E \twoheadrightarrow B$ is a discrete cartesian fibration, then*

$$\mathsf{Fun}_B(\mathsf{Hom}_B(B,b) \xrightarrow{p_0} B, E \xrightarrow{p} B) \xrightarrow{\mathrm{ev}_{\ulcorner \mathrm{id}_b \urcorner}} \mathsf{Fun}_B(1 \xrightarrow{b} B, E \xrightarrow{p} B) \cong \mathsf{Fun}(1, E_b)$$

is an equivalence of Kan complexes.

Theorem 5.5.3 is subsumed by a generalization that allows $p\colon E \twoheadrightarrow B$ to be any cartesian fibration, not necessarily discrete. In this case, p encodes an "∞-category-valued contravariant B-indexed functor," as does $p_0\colon \mathsf{Hom}_B(B,b) \twoheadrightarrow B$. The correct notion of "natural transformation" between two such functors is now given by a cartesian functor over B; see Exercise 5.1.iv. To that end, for a pair of cartesian fibration $q\colon F \twoheadrightarrow B$ and $p\colon E \twoheadrightarrow B$, we write

$$\mathsf{Fun}^{\mathrm{cart}}_B(F \xrightarrow{q} B, E \xrightarrow{p} B) \subset \mathsf{Fun}_B(F \xrightarrow{q} B, E \xrightarrow{p} B) \tag{5.5.4}$$

for the sub quasi-category containing all those simplices whose vertices define *cartesian* functors from q to p.[7]

5.5.5. Theorem (external Yoneda lemma). *If $p\colon E \twoheadrightarrow B$ is a cartesian fibration, then*

$$\mathsf{Fun}^{\mathrm{cart}}_B(\mathsf{Hom}_B(B,b) \xrightarrow{p_0} B, E \xrightarrow{p} B) \xrightarrow{\mathrm{ev}_{\ulcorner \mathrm{id}_b \urcorner}} \mathsf{Fun}_B(1 \xrightarrow{b} B, E \xrightarrow{p} B) \cong \mathsf{Fun}(1, E_b)$$

is an equivalence of quasi-categories.

The proofs of these theorems overlap significantly and we develop them in parallel. The basic idea is to use the universal property of $\ulcorner \mathrm{id}_b \urcorner$ as a terminal element of $\mathsf{Hom}_B(B,b)$ to define a right adjoint to $\mathrm{ev}_{\ulcorner \mathrm{id}_b \urcorner}$ and prove that when $p\colon E \twoheadrightarrow B$ is discrete or when the domain is restricted to the sub-quasi-category of cartesian functors, this adjunction defines an adjoint equivalence. Note that the functor $\mathrm{ev}_{\ulcorner \mathrm{id}_b \urcorner}$ is the image of the functor $\ulcorner \mathrm{id}_b \urcorner$ under the 2-functor $\mathsf{Fun}_B(-,p)\colon \mathfrak{h}(\mathcal{K}_{/B})^{\mathrm{op}} \to \mathcal{QC}at$. If the adjunction $! \dashv \ulcorner \mathrm{id}_b \urcorner$ lived in the slice ∞-cosmos $\mathcal{K}_{/B}$, this would directly construct a right adjoint to $\mathrm{ev}_{\ulcorner \mathrm{id}_b \urcorner}$. The main technical difficulty in following the outline just given is that the adjunction that witnesses the terminality of $\ulcorner \mathrm{id}_b \urcorner$ does not live in the slice of the homotopy 2-category $\mathfrak{h}\mathcal{K}_{/B}$ but rather in a lax slice of the homotopy 2-category, that we now introduce.

[7] For any quasi-category Q and any subset S of its vertices, there is a "full" sub-quasi-category $Q_S \subset Q$ containing exactly those vertices and all the simplices of Q that they span.

5.5.6. Definition. Consider a 2-category $\mathfrak{h}\mathcal{K}$ and an object $B \in \mathfrak{h}\mathcal{K}$. The **lax slice 2-category** $\mathfrak{h}\mathcal{K}_{/\!/B}$ is the strict 2-category whose

- objects are maps $f\colon X \to B$ in $\mathfrak{h}\mathcal{K}$ with codomain B;
- 1-cells are diagrams
$$\begin{tikzcd} X \arrow[rr, "k"] \arrow[dr, "f"'] & \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}"] & Y \arrow[dl, "g"] \\ & B & \end{tikzcd} \tag{5.5.7}$$
  in $\mathfrak{h}\mathcal{K}$; and
- 2-cells from the 1-cell displayed above to the 1-cell below-right are 2-cells $\theta\colon k \Rightarrow k'$ so that
$$\begin{tikzcd} X \arrow[rr, bend left, "k'"] \arrow[rr, "k"'] \arrow[rr, phantom, bend left=20, "\Uparrow\theta"] \arrow[dr, "f"'] & \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}"] & Y \arrow[dl, "g"] \\ & B & \end{tikzcd} \quad = \quad \begin{tikzcd} X \arrow[rr, "k'"] \arrow[dr, "f"'] & \arrow[d, phantom, "\overset{\alpha'}{\Rightarrow}"] & Y \arrow[dl, "g"] \\ & B & \end{tikzcd}$$

5.5.8. Lemma. *The identity functor* (5.5.1) *is right adjoint to the right comma cone*
$$\begin{tikzcd} \mathsf{Hom}_B(B,b) \arrow[rr, "!"] \arrow[dr, "p_0"'] & \arrow[d, phantom, "\overset{\phi}{\Rightarrow}"] & 1 \arrow[dl, "b"] \\ & B & \end{tikzcd}$$
in $\mathfrak{h}\mathcal{K}_{/\!/B}$.

Proof. Since 1 is the terminal ∞-category, we take the counit of the postulated adjunction to be the identity. By Definition 5.5.6 to define the unit, we must provide a 2-cell:
$$\begin{tikzcd} \mathsf{Hom}_B(B,b) \arrow[rr, bend left, "\ulcorner \mathrm{id}_b \urcorner !"] \arrow[rr, equal] \arrow[rr, phantom, bend left=20, "\Uparrow\eta"] \arrow[dr, "p_0"'] & & \mathsf{Hom}_B(B,b) \arrow[dl, "p_0"] \\ & B & \end{tikzcd} \quad = \quad \begin{tikzcd} \mathsf{Hom}_B(B,b) \arrow[r, "!"] \arrow[dr, "p_0"'] \arrow[dr, phantom, bend left=10, "\overset{\phi}{\Rightarrow}"] & 1 \arrow[r, "\ulcorner \mathrm{id}_b \urcorner"] \arrow[d, "b"] & \mathsf{Hom}_B(B,b) \arrow[dl, "p_0"] \\ & B & \end{tikzcd}$$
so that $p_0\eta = \phi$. This is the defining property of the unit in Lemma 3.5.8. The forgetful 2-functor $\mathfrak{h}\mathcal{K}_{/\!/B} \to \mathfrak{h}\mathcal{K}$ is faithful on 1- and 2-cells, so the verification of the triangle equalities in Lemma 3.5.8 proves that they also hold in $\mathfrak{h}\mathcal{K}_{/\!/B}$. □

Using somewhat non-standard 2-categorical techniques, we will transfer the adjunction of Lemma 5.5.8 to an adjunction between the quasi-categories $\mathsf{Fun}_B(b,p)$ and $\mathsf{Fun}_B(p_0,p)$; see Proposition 5.5.15. Because our initial adjunction lives in the lax rather than the strict slice, the construction will be somewhat delicate, passing through a pair of auxiliary 2-categories that we now introduce.

5.5.9. Definition. Let $\mathfrak{h}\mathcal{K}$ be the homotopy 2-category of an ∞-cosmos and write $\mathfrak{h}\mathcal{K}^{\lrcorner}$ for the strict 2-category whose

- objects are cospans
$$A \xrightarrow{f} B \overset{p}{\twoheadleftarrow} E$$
  in which p is a cartesian fibration;

- 1-cells are diagrams of the form

$$\begin{tikzcd} A' \arrow[r, "f'"] \arrow[d, "a"'] \arrow[dr, phantom, "\Uparrow\phi"] & B' \arrow[d, "b"] & E' \arrow[l, two heads, "p'"'] \arrow[d, "e"] \\ A \arrow[r, "f"'] & B & E \arrow[l, two heads, "p"] \end{tikzcd} \tag{5.5.10}$$

- and whose 2-cells consist of triples $\alpha \colon a \Rightarrow \bar{a}$, $\beta \colon b \Rightarrow \bar{b}$, and $\epsilon \colon e \Rightarrow \bar{e}$ between the verticals of parallel 1-cell diagrams so that $p\epsilon = \beta p'$ and $\bar{\phi} \cdot f\alpha = \beta f' \cdot \phi$.

5.5.11. Definition. Let $\mathfrak{h}\mathcal{K}$ be the homotopy 2-category of an ∞-cosmos and write $\mathfrak{h}\mathcal{K}^{\boxdot}$ for the strict 2-category whose

- objects are pullback squares

$$\begin{tikzcd} F \arrow[r, "g"] \arrow[d, two heads, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, two heads, "p"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

  whose verticals are cartesian fibrations;
- 1-cells are cubes

$$\begin{tikzcd}[row sep=small, column sep=small] F' \arrow[rr, "g'"] \arrow[dr, "\ell"] \arrow[dd, two heads, "q'"'] & & E' \arrow[dr, "e"] \arrow[dd, two heads, "p'" near start] \arrow[dl, phantom, "\Uparrow\chi"] & \\ & F \arrow[rr, "g" near start, crossing over] & & E \arrow[dd, two heads, "p"] \\ A' \arrow[rr, "f'" near end] \arrow[dr, "a"'] & & B' \arrow[dr, "b"] \arrow[dl, phantom, "\Uparrow\phi"] & \\ & A \arrow[rr, "f"'] \arrow[from=uu, two heads, "q" near start, crossing over] & & B \end{tikzcd} \tag{5.5.12}$$

  whose vertical faces commute and in which $\chi \colon g\ell \Rightarrow eg'$ is a p-cartesian lift of $\phi q'$; and
- whose 2-cells are given by quadruples $\alpha \colon a \Rightarrow \bar{a}$, $\beta \colon b \Rightarrow \bar{b}$, $\epsilon \colon e \Rightarrow \bar{e}$, and $\lambda \colon \ell \Rightarrow \bar{\ell}$ in which ϵ and λ are, respectively, lifts of $\beta p'$ and $\alpha q'$ and so that $\phi \cdot f\alpha = \beta f' \cdot \phi$ and $\bar{\chi} \cdot g\lambda = \epsilon g' \cdot \chi$.

These definitions are arranged so that there is an evident forgetful 2-functor $\mathfrak{h}\mathcal{K}^{\boxdot} \to \mathfrak{h}\mathcal{K}^{\lrcorner}$.

5.5.13. Lemma. *The forgetful 2-functor $\mathfrak{h}\mathcal{K}^{\boxdot} \to \mathfrak{h}\mathcal{K}^{\lrcorner}$ is a smothering 2-functor.*

Proof. Proposition 5.1.20 tells us that $\mathfrak{h}\mathcal{K}^{\boxdot} \to \mathfrak{h}\mathcal{K}^{\lrcorner}$ is surjective on objects. To see that it is full on 1-cells, first form the pullbacks of the cospans in (5.5.10), then define χ to be any p-cartesian lift of $\phi q'$ with codomain eg'. By construction, the domain of χ lies strictly over faq' and so this functor factors uniquely through the pullback leg g defining the map ℓ of (5.5.12).

To prove that $\mathfrak{h}\mathcal{K}^{\boxdot} \to \mathfrak{h}\mathcal{K}^{\lrcorner}$ is full on 2-cells, consider a parallel pair of 1-cells in $\mathfrak{h}\mathcal{K}^{\boxdot}$. For one of these we use the notation of (5.5.12) and for the other we denote the diagonal functors by $\bar{a}, \bar{b}, \bar{e}$, and $\bar{\ell}$ and denote the 2-cells by $\bar{\phi}$ and $\bar{\chi}$; the requirement that these 1-cells be parallel implies that the pullback faces are necessarily the same. Now consider a triple $\alpha \colon a \Rightarrow \bar{a}$, $\beta \colon b \Rightarrow \bar{b}$, and $\epsilon \colon e \Rightarrow \bar{e}$ satisfying the conditions of Definition 5.5.9. Our task is to define a fourth 2-cell $\lambda \colon \ell \Rightarrow \bar{\ell}$ so that $q\lambda = \alpha q'$ and $\bar{\chi} \cdot g\lambda = \epsilon g' \cdot \chi$.

To achieve this, we first define a 2-cell $\gamma\colon g\ell \Rightarrow g\bar{\ell}$ using the induction property of the p-cartesian cell $\bar{\chi}\colon g\bar{\ell} \Rightarrow \bar{e}g'$ applied to the composite 2-cell $\epsilon g' \cdot \chi\colon g\ell \Rightarrow \bar{e}g'$ and the factorization $p\epsilon g' \cdot p\chi = \bar{\phi}q' \cdot f\alpha q'$. By construction $p\gamma = f\alpha q'$ so the pair $\alpha q'$ and γ induces a 2-cell $\lambda\colon \ell \Rightarrow \bar{\ell}$ so that $q\lambda = \alpha q'$ and $g\lambda = \gamma$. The quadruple $(\alpha, \beta, \epsilon, \lambda)$ now defines the required 2-cell in $\mathfrak{h}\mathcal{K}^{\boxdot}$.

Finally, for 2-cell conservativity, suppose α, β, and ϵ as above are isomorphisms. By the conservativity property for pullbacks to show that λ is an isomorphism, it suffices to prove that $q\lambda = \alpha q'$ is, which we know already, and that $g\lambda = \gamma$ is invertible. But γ was constructed as a factorization $\epsilon g' \cdot \chi = \bar{\chi} \cdot \gamma$ with $p\gamma = f\alpha q'$. Since ϵ is an isomorphism, $\epsilon g' \cdot \chi$ is p-cartesian, so Lemma 5.1.5 proves that γ is an isomorphism. ☐

5.5.14. Remark. While we cannot directly define a pullback 2-functor $\mathfrak{h}\mathcal{K}^{\lrcorner} \to \mathfrak{h}\mathcal{K}$ in the homotopy 2-category because the 2-categorical universal property of pullbacks in $\mathfrak{h}\mathcal{K}$ is weak and not strict, the zig zag of 2-functors $\mathfrak{h}\mathcal{K}^{\lrcorner} \leftarrow \mathfrak{h}\mathcal{K}^{\boxdot} \to \mathfrak{h}\mathcal{K}$, in which the backwards map is a smothering 2-functor and the forwards map evaluates at the pullback vertex, defines a reasonable replacement.

Using Lemma 5.5.13, we can now construct the desired right adjoint to the functor (5.5.2):

5.5.15. Proposition. *For any element $b\colon 1 \to B$ and any cartesian fibration $p\colon E \twoheadrightarrow B$, the evaluation at the identity functor admits a right adjoint*

$$\begin{tikzcd} \mathsf{Fun}_B(p_0, p) \arrow[r, bend left, "\mathrm{ev}_{\ulcorner \mathrm{id}_b \urcorner}"] \arrow[r, phantom, "\perp"] & \mathsf{Fun}_B(b, p) \arrow[l, bend left, dashed, "R"] \end{tikzcd}$$

defined by the domain-component of the p_*-cartesian lift of the right comma cone over b:

$$\begin{tikzcd} \mathsf{Fun}_B(b,p) \arrow[rr] \arrow[dd, two heads] \arrow[dr, dashed, "R"'] & & \mathsf{Fun}(1,E) \arrow[dd, two heads, "p_*"] \arrow[dr, "!^*"] \arrow[dl, phantom, "\Uparrow\chi"] & \\ & \mathsf{Fun}_B(p_0,p) \arrow[rr] \arrow[dd, two heads] & & \mathsf{Fun}(\mathrm{Hom}_B(B,b),E) \arrow[dd, two heads, "p_*"] \\ \mathbb{1} \arrow[rr, "b"] \arrow[dr, equals] & & \mathsf{Fun}(1,B) \arrow[dr, "!^*"] \arrow[dl, phantom, "\Uparrow\phi"] & \\ & \mathbb{1} \arrow[rr, "p_0"'] & & \mathsf{Fun}(\mathrm{Hom}_B(B,b),B) \end{tikzcd} \tag{5.5.16}$$

The idea will be to transfer the adjunction of Lemma 5.5.8 through a sequence of 2-functors

$$\begin{tikzcd} & \mathfrak{h}\mathcal{QC}at^{\boxdot} \arrow[r, "\mathrm{ev}_\top"] \arrow[d] & \mathfrak{h}\mathcal{QC}at \\ \mathfrak{h}\mathcal{K}^{\mathrm{op}}_{/\!/B} \arrow[r] & \mathfrak{h}\mathcal{QC}at^{\lrcorner} & \end{tikzcd}$$

using Lemma 3.6.8 to lift along the middle smothering 2-functor.

PROOF. Fixing a cartesian fibration $p\colon E \twoheadrightarrow B$ in $\mathcal{K}$, we define a 2-functor[8] $\mathfrak{h}\mathcal{K}_{/\!\!/B}^{\mathrm{op}} \to \mathfrak{h}\mathcal{QC}at^{\lrcorner}$ that carries a 1-cell (5.5.7) to

$$\begin{tikzcd} \mathbb{1} \arrow[r, "g"] \arrow[d, equal] \arrow[dr, phantom, "\Uparrow\alpha"] & \mathsf{Fun}(Y,B) \arrow[d, "k^*"] & \mathsf{Fun}(Y,E) \arrow[l, two heads, "p_*"'] \arrow[d, "e"] \\ \mathbb{1} \arrow[r, "f"'] & \mathsf{Fun}(X,B) & \mathsf{Fun}(X,E) \arrow[l, two heads, "p_*"] \end{tikzcd}$$

and a 2-cell $\theta\colon k \Rightarrow k'$ to the 2-cell that acts via pre-whiskering with θ in its two non-identity components. By Corollary 5.1.16, the functors p_* are cartesian fibration of quasi-categories.

We now apply the 2-functor $\mathfrak{h}\mathcal{K}_{/\!\!/B}^{\mathrm{op}} \to \mathfrak{h}\mathcal{QC}at^{\lrcorner}$ to the adjunction of Lemma 5.5.8 to obtain an adjunction in $\mathfrak{h}\mathcal{QC}at^{\lrcorner}$ and then use the smothering 2-functor of Lemma 5.5.13 and Lemma 3.6.8 to lift this to an adjunction in $\mathfrak{h}\mathcal{QC}at^{\boxdot}$. As elaborated on in Exercise 3.6.ii, the lifted adjunction in $\mathfrak{h}\mathcal{QC}at^{\boxdot}$ can be constructed using any lifts of the objects, 1-cells, and either the unit or counit of the adjunction in $\mathfrak{h}\mathcal{QC}at^{\lrcorner}$.

In particular, we may take the left adjoint of the lifted adjunction in $\mathfrak{h}\mathcal{QC}at^{\boxdot}$ to be any lift of the image in $\mathfrak{h}\mathcal{QC}at^{\lrcorner}$ of the right adjoint of the adjunction $! \dashv \ulcorner \mathrm{id}_b \urcorner$ in $\mathfrak{h}\mathcal{K}_{/\!\!/B}$, and so our left adjoint is

$$\begin{tikzcd} \mathsf{Fun}_B(p_0,p) \arrow[rr] \arrow[dr, dashed, "\mathrm{ev}_{\ulcorner \mathrm{id}_b \urcorner}"'] \arrow[dd, two heads] & & \mathsf{Fun}(\mathsf{Hom}_B(B,b),E) \arrow[dr, "\ulcorner \mathrm{id}_b \urcorner^*"] \arrow[dd, two heads, "p_*" near start] \arrow[d, phantom, "="] & \\ & \mathsf{Fun}_B(b,p) \arrow[rr, crossing over] \arrow[dd, two heads] & {} & \mathsf{Fun}(1,E) \arrow[dd, two heads, "p_*"] \\ \mathbb{1} \arrow[r, dash, "p_0"] \arrow[dr, equal] & {} \arrow[r] & \mathsf{Fun}(\mathsf{Hom}_B(B,b),B) \arrow[dr, "\ulcorner \mathrm{id}_b \urcorner^*"] \arrow[d, phantom, "="] & \\ & \mathbb{1} \arrow[rr, "b"'] & {} & \mathsf{Fun}(1,B) \end{tikzcd}$$

We may also take the right adjoint to be any lift of the image of the right adjoint of the adjunction. This proves that the right adjoint is defined by (5.5.16). Since the counit of $! \dashv \ulcorner \mathrm{id}_b \urcorner$ is an identity, the counit of the lifted adjunction may also be taken to be an identity.

Finally, we compose with the forgetful 2-functor $\mathfrak{h}\mathcal{QC}at^{\boxdot} \to \mathfrak{h}\mathcal{QC}at$ that evaluates at the pullback vertex to project our adjunction in $\mathfrak{h}\mathcal{QC}at^{\boxdot}$ to the desired adjunction in $\mathfrak{h}\mathcal{QC}at$. $\square$

The proof of the discrete case of the Yoneda lemma is now one line.

PROOF OF THEOREM 5.5.3. If $p\colon E \twoheadrightarrow B$ is discrete, then $\mathsf{Fun}_B(p_0,p)$ and $\mathsf{Fun}_B(b,p)$ are Kan complexes, so the adjunction defined in Proposition 5.5.15 is an adjoint equivalence. $\square$

Specializing to the case of two right representable discrete cartesian fibrations, we conclude that the Kan complex of natural transformations is equivalent to the underlying quasi-category of their internal mapping space.

[8] To explain the variance, recall that the 2-functor $\mathsf{Fun}(-,B)\colon \mathfrak{h}\mathcal{K}^{\mathrm{op}} \to \mathfrak{h}\mathcal{QC}at$ is contravariant on 1-cells but covariant on 2-cells. Such 2-functors, like all 2-functors, preserve adjunctions, though in this case the left and right adjoints are interchanged, while the units and counits retain the same roles.

5.5.17. Corollary (external Yoneda embedding). *For any elements $x, y \colon 1 \rightrightarrows A$ in an ∞-category A, evaluation at the identity of x induces an equivalence of Kan complexes*

$$\mathsf{Fun}_A(\mathsf{Hom}_A(A,x), \mathsf{Hom}_A(A,y)) \xrightarrow[\sim]{\mathrm{ev}_{\ulcorner \mathrm{id}_x \urcorner}} \mathsf{Fun}(1, \mathsf{Hom}_A(x,y)) \qquad \square$$

It remains to prove the general case of Theorem 5.5.5. The next step is to observe that the right adjoint in the adjunction of Proposition 5.5.15 lands in the sub quasi-category of cartesian functors from p_0 to p. Lemma 5.5.18 proves this after which it is short work to complete the proof of Theorem 5.5.5 by arguing that this restricted adjunction defines an adjoint equivalence.

5.5.18. Lemma. *For each vertex in $\mathsf{Fun}_B(b,p)$ below-left*

$$\begin{tikzcd} & \mathsf{Fun}_B(b,p) \arrow[rrr, "R"] & & & \mathsf{Fun}_B(p_0,p) & \\ 1 \arrow[rr, "e"] \arrow[dr, "b"'] & & E \arrow[dl, "p"] & \mathsf{Hom}_B(B,b) \arrow[rr, "Re"] \arrow[dr, two heads, "p_0"'] & & E \arrow[dl, two heads, "p"] \\ & B & \mapsto & & B & \end{tikzcd}$$

the map Re in $\mathsf{Fun}_B(p_0,p)$ above-right defines a cartesian functor in $\mathcal{K}_{/B}$.

Proof. From the definition of the right adjoint in (5.5.16) and Lemma 5.3.5, we see that Re is the domain component of a p-cartesian lift χ of the composite natural transformation below-left

$$\begin{tikzcd} \mathsf{Hom}_B(B,b) \arrow[r, two heads, "!"] \arrow[dr, bend right, two heads, "p_0"'] & 1 \arrow[r, "e"] \arrow[d, "b"] \arrow[dl, phantom, "\Uparrow\phi"] & E \arrow[dl, two heads, "p"] \\ & B & \end{tikzcd} = \begin{tikzcd} \mathsf{Hom}_B(B,b) \arrow[rr, bend left, "e!"] \arrow[rr, bend right, "Re"'] \arrow[rr, phantom, "\Uparrow\chi"] \arrow[dr, two heads, "p_0"'] & & E \arrow[dl, two heads, "p"] \\ & B & \end{tikzcd}$$

Since $p_0 \colon \mathsf{Hom}_B(B,b) \twoheadrightarrow B$ is discrete, every natural transformation ψ with codomain $\mathsf{Hom}_B(B,b)$ is p_0-cartesian, so to prove that Re defines a cartesian functor, we must show that $Re\psi$ is p-cartesian. To that end, consider the horizontal composite

$$\begin{tikzcd} X \arrow[r, bend left, "y"] \arrow[r, bend right, "x"'] \arrow[r, phantom, "\Uparrow\psi"] & \mathsf{Hom}_B(B,b) \arrow[r, bend left, "e!"] \arrow[r, bend right, "Re"'] \arrow[r, phantom, "\Uparrow\chi"] & E \end{tikzcd}$$

By naturality of whiskering, we have $\chi y \cdot Re\psi = e!\psi \cdot \chi x = \chi x$, since 1 is the terminal ∞-category and hence $e!\psi$ is an identity. Now Lemma 5.1.24(ii) implies that $Re\psi$ is p-cartesian. $\square$

Proof of Theorem 5.5.5. By Lemma 5.5.18, the adjunction of Proposition 5.5.15 restricts to define an adjunction

$$\begin{tikzcd} \mathsf{Fun}_B^{\mathrm{cart}}(p_0,p) \arrow[r, bend left, "\mathrm{ev}_{\ulcorner \mathrm{id}_b \urcorner}"] \arrow[r, phantom, "\perp"] & \mathsf{Fun}_B(b,p) \arrow[l, bend left, dashed, "R"] \end{tikzcd}$$

Since the counit of the original adjunction $! \dashv \ulcorner \mathrm{id}_b \urcorner$ is an isomorphism and smothering 2-functors are conservative on 2-cells, the counit of the adjunction of Proposition 5.5.15 and hence also of the restricted adjunction is an isomorphism. As in the proof of Theorem 5.5.3, we will prove that $\mathrm{ev}_{\ulcorner \mathrm{id}_b \urcorner}$

is an equivalence by demonstrating that the unit of the restricted adjunction is also invertible. By Lemma 12.2.1, it suffices to verify this elementwise, proving that the component of the unit indexed by a cartesian functor

$$\begin{tikzcd} \mathsf{Hom}_B(B,b) \arrow[rr, "f"] \arrow[dr, two heads, "p_0"'] & & E \arrow[dl, two heads, "p"] \\ & B & \end{tikzcd}$$

is an isomorphism.

Unpacking the proof of Proposition 5.5.15, the unit $\hat{\eta}$ of $\mathrm{ev}_{\ulcorner \mathrm{id}_b \urcorner} \dashv R$ is defined to be a factorization

$$\begin{tikzcd} \mathsf{Fun}_B(p_0,p) \arrow[r, hook] & \mathsf{Fun}(\mathsf{Hom}_B(B,b),E) \arrow[r, "\ulcorner \mathrm{id}_b \urcorner^*"] \arrow[dr, equal] \arrow[r, phantom, shift right=4, "\Uparrow \mathsf{Fun}(\eta,E)"] & \mathsf{Fun}(1,E) \arrow[d, "!^*"] \\ & & \mathsf{Fun}(\mathsf{Hom}_B(B,b),E) \end{tikzcd}$$

$$=$$

$$\begin{tikzcd} \mathsf{Fun}_B(p_0,p) \arrow[r, "\mathrm{ev}_{\ulcorner \mathrm{id}_b \urcorner}"] \arrow[dr, equal] \arrow[dr, phantom, shift left=3, "\Uparrow \hat{\eta}"] & \mathsf{Fun}_B(b,p) \arrow[r, hook] \arrow[d, "R"] \arrow[dr, phantom, "\Uparrow \chi"] & \mathsf{Fun}(1,E) \arrow[d, "!^*"] \\ & \mathsf{Fun}_B(p_0,p) \arrow[r, hook] & \mathsf{Fun}(\mathsf{Hom}_B(B,b),E) \end{tikzcd}$$

of the pre-whiskering 2-cell $\mathsf{Fun}(\eta,E)$ through the p_*-cartesian lift χ. The component of the pre-whiskering 2-cell $\mathsf{Fun}(\eta,E)$ at the cartesian functor f is $f\eta$. Since $p_0 \colon \mathsf{Hom}_B(B,b) \twoheadrightarrow B$ is a discrete cartesian fibration, any 2-cell, such as η, which has codomain $\mathsf{Hom}_B(B,b)$ is p_0-cartesian, and since f is a cartesian functor, we then see that $f\eta$ is p-cartesian.

By Lemma 5.3.8, the components of the p_*-cartesian cell χ define p-cartesian natural transformations in $\mathcal{K}$. As $\hat{\eta}$ is a natural transformation with codomain $\mathsf{Fun}_B(p_0,p)$ its components project along p to the identity. In this way, we see that $\hat{\eta}f$ is a factorization of the p-cartesian transformation $f\eta$ through a p-cartesian lift of ϕ over an identity, and Lemma 5.1.5 proves that $\hat{\eta}f$ is an isomorphism, as desired. ☐

Theorem 5.5.5 implies the following generalization, replacing the elements $b \colon 1 \to B$ by a generalized element $b \colon X \to B$.

5.5.19. Proposition. *For any cartesian fibration $p \colon E \twoheadrightarrow B$ and map $b \colon X \to B$, restricting along the canonical induced functor*

$$\begin{tikzcd} X \arrow[rr, "\ulcorner \mathrm{id}_b \urcorner"] \arrow[dr, "b"'] & & \mathsf{Hom}_B(B,b) \arrow[dl, two heads, "p_0"] \\ & B & \end{tikzcd}$$

defines an equivalence of quasi-categories:

$$\mathsf{Fun}_B^{\mathrm{cart}}(\mathsf{Hom}_B(B,b) \overset{p_0}{\twoheadrightarrow} B, E \overset{p}{\twoheadrightarrow} B) \xrightarrow{\mathrm{ev}_{\ulcorner \mathrm{id}_b \urcorner}} \mathsf{Fun}_B(X \overset{b}{\to} B, E \overset{p}{\twoheadrightarrow} B).$$

Proof. Theorem 5.5.5 applies in $\mathcal{K}_{/X}$ to the cartesian fibration $p \times X \colon E \times X \twoheadrightarrow B \times X$ and the element $(b,X) \colon X \to B \times X$ to define an equivalence

$$\begin{tikzcd} \mathsf{Fun}_{B\times X}^{\mathrm{cart}}(\mathsf{Hom}_B(B,p) \overset{(p_1,p_0)}{\twoheadrightarrow} B \times X, p \times X) \arrow[r, "\mathrm{ev}_{\ulcorner \mathrm{id}_{(b,X)} \urcorner}"] \arrow[d, phantom, "\wr\Vert"] & \mathsf{Fun}_{B\times X}(X \overset{(b,X)}{\longrightarrow} B \times X, p \times X) \arrow[d, phantom, "\wr\Vert"] \\ \mathsf{Fun}_B^{\mathrm{cart}}(\mathsf{Hom}_B(B,b) \overset{p_0}{\twoheadrightarrow} B, E \overset{p}{\twoheadrightarrow} B) \arrow[r, "\mathrm{ev}_{\ulcorner \mathrm{id}_b \urcorner}"] & \mathsf{Fun}_B(X \overset{b}{\to} B, E \overset{p}{\twoheadrightarrow} B) \end{tikzcd}$$

which transposes under the adjunction

$$\begin{tikzcd} \mathcal{K}_{/X} \arrow[r, bend left, "U"] \arrow[r, phantom, "\perp"] & \mathcal{K} \arrow[l, bend left, "-\times X"] \end{tikzcd}$$

to the equivalence of the statement. $\square$

Later we will interpret the result of Proposition 5.5.19 as defining a left biadjoint to the inclusion $\mathcal{C}art(\mathcal{K})_{/B} \hookrightarrow \mathcal{K}_{/B}$ by the functor $(b\colon X \to B) \mapsto (p_0\colon \mathsf{Hom}_B(B,b) \twoheadrightarrow B)$.

Exercises.

5.5.i. Exercise. Given an element $f\colon 1 \to \mathsf{Hom}_A(x,y)$ in the internal mapping space between a pair of elements in an ∞-category A, use the explicit description of the inverse equivalence to the map of Corollary 5.5.17 to construct a map

$$\begin{tikzcd} \mathsf{Hom}_A(A,x) \arrow[rr, dashed, "f_*"] \arrow[dr, two heads, "p_0"'] & & \mathsf{Hom}_A(A,y) \arrow[dl, two heads, "p_0"] \\ & A & \end{tikzcd}$$

which represents the "natural transformation" defined by post-composing with f.[9]

[9] Hint: this construction is a special case of the construction given in the first half of the proof of Lemma 5.5.18.

An interlude on ∞-cosmology

CHAPTER 6

Exotic ∞-cosmoi

Morally an ∞-cosmos can be described as "an $(\infty, 2)$-category with $(\infty, 2)$-categorical limits." For practical reasons, the precise axiomatization given in Definition 1.2.1 employed a particularly strict interpretation of this phrase, as this strictness is available in so many examples. This allows us to define ∞-cosmoi to be simplicially enriched categories whose functor spaces are quasi-categories, for instance.

Our aim in this chapter is to develop further examples of ∞-cosmoi, which will immediately allow us to apply all of the theorems proven in Part I in more exotic contexts, where the ∞-categories of an ∞-cosmos should not be thought of as "$(\infty, 1)$-categories in some model." Some of these examples, such as the ∞-cosmos of isofibrations introduced in §6.1, can be established easily by directly verifying the axioms of Definition 1.2.1.

A larger family of examples, appearing in §6.4, arise as subcategories of previously-defined ∞-cosmoi. For instance, in Proposition 6.4.12 we prove that the ∞-cosmos of isofibrations $\mathcal{K}^{\twoheadrightarrow}$ in any ∞-cosmos $\mathcal{K}$ has a sub ∞-cosmos $\mathcal{C}art(\mathcal{K})$ of cartesian fibrations and cartesian functors between them. This result, and many others of a similar flavor, follow from a common paradigm appearing as Proposition 6.4.3, which states that a *replete* subcategory (see Definition 6.4.1) of an ∞-cosmos inherits an ∞-cosmos structure provided that it is closed under *flexible weighted limits*.

This leads us to the second main theme of this chapter: an elaboration of the $(\infty, 2)$-categorical limits present in any ∞-cosmos. In §6.3, we discover that the cosmological limit notions enumerated in axiom 1.2.1(i) generate a much larger class of simplicially enriched limits which then exist in any ∞-cosmos, which are precisely those simplicially enriched limits that deserve to be called "$(\infty, 2)$-categorical limits." This class of limits is introduced in §6.2 under the name *flexible weighted limits*, borrowing a term from 2-category theory.

To explain the intuition, we recall the formalism of weighted limits from §A.6. An ordinary limit is the universal object with a cone over a given diagram, each cone leg being an arrow whose source is the limit object. In a simplicially enriched context, each cone leg may take the shape of an arbitrary simplicial set, with the cone-commutativity conditions then specified by a simplicial-set-valued simplicial functor, referred to as the *weight*, which specifies the overall shape of the cone. For example, an ordinary cone over a cospan defines a commutative square, but a different choice of weight can specify a square inhabited by a 1-simplex. In this way the comma ∞-categories of Definition 3.4.1 arise as weighted limits; see Example A.6.18.

Intuitively, the flexible weighted limits are those whose weights whose cone shapes do not impose any strict commutativity conditions: pullbacks are not flexible weighted limits, while comma objects are. Flexible weighted limits are also equivalence invariant, while general weighted limits need not be. These are the senses in which flexible weighted limits correspond to $(\infty, 2)$-categorical limit notions.

We prove these results in §6.3 and also see that the cosmological limits enumerated in axiom 1.2.1(i), such as pullbacks of isofibrations, are really flexible weighted limits in disguise. The requirement that certain arrows in the diagram are isofibrations means that strictly commuting cones correspond to pseudo-commutative cones, providing the required flexibility.

6.1. The ∞-cosmos of isofibrations

Our first example of an "exotic ∞-cosmos" is a special case of a more general result that will appear in Appendix E that we nonetheless spell out in detail to illustrate the ideas involved in this sort of argument. The walking arrow category $\mathbb{2}$ is an *inverse Reedy category*, where the domain of the non-identity arrow is assigned "degree 1" and the codomain is assigned "degree zero." This Reedy structure motivates the definitions in the **∞-cosmos of isofibrations** that we now introduce:

6.1.1. Proposition (∞-cosmoi of isofibrations). *For any ∞-cosmos $\mathcal{K}$ there is an ∞-cosmos $\mathcal{K}^{\twoheaddownarrow}$ whose*

(i) objects are isofibrations $p\colon E \twoheadrightarrow B$ in $\mathcal{K}$

(ii) functor-spaces, say from $q\colon F \twoheadrightarrow A$ to $p\colon E \twoheadrightarrow B$, are defined by pullback

$$\begin{tikzcd}
\mathsf{Fun}(F \overset{q}{\twoheadrightarrow} A, E \overset{p}{\twoheadrightarrow} B) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Fun}(F,E) \arrow[d, "p_*"] \\
\mathsf{Fun}(A,B) \arrow[r, "q^*"'] & \mathsf{Fun}(F,B)
\end{tikzcd}$$

(iii) isofibrations from q to p are commutative squares

$$\begin{tikzcd}
F \arrow[rr, two heads, "g"] \arrow[dd, two heads, "q"'] \arrow[dr, two heads] & & E \arrow[dd, two heads, "p"] \\
& \bullet \arrow[ur, dotted, two heads] \arrow[dl, dotted, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \\
A \arrow[rr, two heads, "f"'] & & B
\end{tikzcd}$$

in which the horizontals and the induced map from the initial vertex to the pullback of the cospan are isofibrations in $\mathcal{K}$

(iv) limits are defined pointwise in $\mathcal{K}$

(v) and in which a map

$$\begin{tikzcd}
F \arrow[r, "g", "\sim"'] \arrow[d, two heads, "q"'] & E \arrow[d, two heads, "p"] \\
A \arrow[r, "\sim", "f"'] & B
\end{tikzcd}$$

is an equivalence in the ∞-cosmos $\mathcal{K}^{\twoheaddownarrow}$ if and only if g and f are equivalences in $\mathcal{K}$.

Relative to these definitions, the domain, codomain, and identity functors

$$\begin{tikzcd}
\mathcal{K}^{\twoheaddownarrow} \arrow[r, shift left=2, "\mathrm{dom}"] \arrow[r, shift right=2, "\mathrm{cod}"'] & \mathcal{K} \arrow[l, "\mathrm{id}" description]
\end{tikzcd}$$

are all cosmological.

Proof. The diagram category $\mathcal{K}^{\twoheaddownarrow}$ inherits its simplicially enriched limits, defined pointwise, from $\mathcal{K}$. The functor-spaces described in (ii) are the usual ones for an enriched category of diagrams. This verifies 1.2.1(i).

For axiom 1.2.1(ii) note that the product and simplicial cotensor functors carry pointwise isofibrations to isofibrations. The pullback of an isofibration as in (iii) along a commutatative square from an isofibration r to p may be formed in $\mathcal{K}$. Our task is to show that the induced map t is an isofibration and also that the square from t to r is an isofibration in the sense of (iii):

$$
\begin{tikzcd}
G \times_E F \arrow[rrr] \arrow[drr, two heads] \arrow[ddd, dashed, "t"'] \arrow[ddr, dotted, two heads] & & & F \arrow[drr, two heads] \arrow[ddr, two heads, "z"] \arrow[ddd, two heads, "q"' near start] & & \\
& & G \arrow[rrr, two heads] \arrow[ddd, two heads, "r" near end] & & & E \arrow[ddd, two heads, "p"] \\
& \bullet \arrow[ur, dotted, two heads] \arrow[dl, dotted, two heads] \arrow[rrr, dotted] \arrow[dr, phantom, "\lrcorner", very near start] & & & \bullet \arrow[ur, dotted, two heads] \arrow[dl, dotted, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \\
C \times_B A \arrow[rrr] \arrow[drr, two heads] & & & A \arrow[drr, two heads] & & \\
& & C \arrow[rrr, two heads] & & & B
\end{tikzcd}
\tag{6.1.2}
$$

The map t factors as a pullback of z followed by a pullback of r as displayed above, and is thus an isofibration, as claimed. This observation also verifies that the square from t to r defines an isofibration. A similar argument verifies the Leibniz stability of the isofibrations and that the limit of a tower of isofibrations is an isofibration. This proves that $\mathcal{K}^{\twoheaddownarrow}$ defines an ∞-cosmos in such a way so that the domain, codomain, and identity functors are cosmological.

Finally, since pullbacks of isofibrations in $\mathcal{QC}at$ are invariant under equivalences, a pair of equivalences (g, f) induces an equivalence between the functor-spaces defined in (ii). The converse, that an equivalence in $\mathcal{K}^{\twoheaddownarrow}$ defines a pair of equivalences in $\mathcal{K}$ follows from the fact that domain and codomain-projection functors are cosmological and Lemma 1.3.2. □

In close analogy with Proposition 3.6.3 we have a smothering 2-functor that relates the homotopy 2-category of $\mathcal{K}^{\twoheaddownarrow}$ to the 2-category of isofibrations, commutative squares, and parallel natural transformations in the homotopy 2-category of $\mathcal{K}$.

6.1.3. Lemma. *There is an identity on objects and 1-cells smothering 2-functor $\mathfrak{h}(\mathcal{K}^{\twoheaddownarrow}) \to (\mathfrak{h}\mathcal{K})^{\twoheaddownarrow}$ whose codomain is the 2-category whose*

- *objects are isofibrations in $\mathcal{K}$,*
- *1-cells are commutative squares between such,*
- *2-cells are pairs of 2-cells in $\mathfrak{h}\mathcal{K}$*

$$
\begin{tikzcd}
F \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow\alpha"] \arrow[d, two heads, "q"'] & E \arrow[d, two heads, "p"] \\
A \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow\beta"] & B
\end{tikzcd}
$$

so that $p\alpha = \beta q$.

Proof. Exercise 6.1.i. □

Similarly, any ∞-cosmos admits an ∞-**cosmos of trivial fibrations**, which can be defined as a full subcategory of the ∞-cosmos of isofibrations. The following general result from abstract ∞-cosmology explains how it inherits its ∞-cosmos structure.

6.1.4. Lemma. *Let $\mathcal{K}$ be an ∞-cosmos and let $\mathcal{L} \subset \mathcal{K}$ be a full subcategory. Then $\mathcal{L}$ inherits an ∞-cosmos structure from $\mathcal{K}$ created from the inclusion $\mathcal{L} \hookrightarrow \mathcal{K}$ if and only if $\mathcal{L}$ is closed in $\mathcal{K}$ under the cosmological limit notions enumerated in axiom 1.2.1(i).*

In practice, the full subcategories we consider have the property that any object of $\mathcal{K}$ that is equivalent to an object in $\mathcal{L}$ in fact lies in $\mathcal{L}$—in other words, these subcategories are "replete" in the sense of Definition 6.4.1. In the proof below, we tacitly assume that $\mathcal{L}$ is at least closed under isomorphism so that if a limit of a diagram lies in $\mathcal{L}$ then all of the limits of that diagram do, but this assumption is only used for linguistic convenience and is inessential.

Proof. As a full subcategory, $\mathcal{L}$ inherits its quasi-categorical enrichment from $\mathcal{K}$, and we define a map to be an isofibration in $\mathcal{L}$ if and only if it is an isofibration in $\mathcal{K}$. Note this definition makes axiom 1.2.1(ii) follow immediately once that we've shown that the limits required by 1.2.1(i) coincide with the corresponding limits in $\mathcal{K}$. But this is exactly what is asserted by the hypothesis that $\mathcal{L}$ is closed in $\mathcal{K}$ under the cosmological limit notions.

We've shown that a closed full subcategory inherits an ∞-cosmos structure defined in such a way that the inclusion $\mathcal{L} \hookrightarrow \mathcal{K}$ is a cosmological functor that reflects isofibrations and cosmological limits. Clearly, this inclusion reflects representably-defined equivalences, since $\mathcal{L}$ is a full subcategory of $\mathcal{K}$. But by Lemma 1.3.2 the cosmological functor $\mathcal{L} \hookrightarrow \mathcal{K}$ preserves them as well, which tells us that equivalences in $\mathcal{L}$ are created from $\mathcal{K}$ along with the isofibrations and cosmological limits. □

With this result in hand, we leave the details of the proof of the ∞-cosmos of trivial fibrations to the reader.

6.1.5. Proposition (∞-cosmoi of trivial fibrations). *Let $\mathcal{K}$ be an ∞-cosmos.*

(i) For any ∞-category B in $\mathcal{K}$, the full subcategory $\mathcal{K}^{\simeq}_{/B} \hookrightarrow \mathcal{K}_{/B}$ spanned by the trivial fibrations with codomain B defines an ∞-cosmos, with limits, isofibrations, equivalences, and trivial fibrations created by the inclusion.

(ii) The full subcategory $\mathcal{K}^{\stackrel{\sim}{\twoheadrightarrow}} \hookrightarrow \mathcal{K}^{\twoheadrightarrow}$ spanned by the trivial fibrations defines an ∞-cosmos, with limits, isofibrations, equivalences, and trivial fibrations created by the inclusion.

Proof. The details, which are similar to Propositions 1.2.22 and 6.1.1, are left to Exercise 6.1.ii. □

Note that the sliced ∞-cosmoi $\mathcal{K}^{\simeq}_{/B}$ of trivial fibrations are weakly contractible in the following sense sense: the functor spaces are contractible Kan complexes, and consequently and each functor in $\mathcal{K}^{\simeq}_{/B}$ is an equivalence. Hence the unique functor $\mathcal{K}^{\simeq}_{/B} \to \mathbb{1}$ to the terminal ∞-cosmos is a *cosmological biequivalence*, a notion whose general properties are explored in Chapter 10.

6.1.6. Proposition (∞-cosmoi of discrete ∞-categories). *The full subcategory $\mathcal{K}^{\simeq} \hookrightarrow \mathcal{K}$ spanned by the discrete ∞-categories in any ∞-cosmos form an ∞-cosmos.*

Proof. Recall from Lemma 1.2.27 that an ∞-category A in an ∞-cosmos $\mathcal{K}$ is discrete if and only if the map $A^{\mathbb{I}} \twoheadrightarrow A^{\mathbb{2}}$ is a trivial fibration. This says that the full subcategory $\mathcal{K}^{\mathbb{I}} \hookrightarrow \mathcal{K}$ of discrete ∞-categories is defined by the pullback

$$\begin{tikzcd} \mathcal{K}^{\simeq} \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{K}^{\stackrel{\sim}{\twoheadrightarrow}} \arrow[d, hook] \\ \mathcal{K} \arrow[r, "E"] & \mathcal{K}^{\twoheadrightarrow} \end{tikzcd}$$

along the cosmological functor E that sends an ∞-category A to the isofibration $A^{\mathbb{I}} \twoheadrightarrow A^{\mathbb{2}}$. By Lemma 6.1.4, to show that $\mathcal{K}^{\simeq}$ admits an ∞-cosmos structure inherited from $\mathcal{K}$, we need only show that the discrete ∞-categories are closed in $\mathcal{K}$ under the limit constructions of 1.2.1(i). For the simplicial cotensors, for instance, this follows easily from the defining universal property and the fact that the Kan complexes form an exponential ideal in the category of simplicial sets. A common argument can be given for each of the conical limits; for sake of concreteness, consider a tower of isofibrations between discrete ∞-categories A_n and form the limit in $\mathcal{K}$

$$A \coloneqq \lim\left(\cdots \longrightarrow\!\!\!\!\!\rightarrow A_n \longrightarrow\!\!\!\!\!\rightarrow A_{n-1} \longrightarrow\!\!\!\!\!\rightarrow \cdots \longrightarrow\!\!\!\!\!\rightarrow A_1 \longrightarrow\!\!\!\!\!\rightarrow A_0 \right)$$

The cosmological functor E carries this to a limit diagram in the ∞-cosmos $\mathcal{K}^{\twoheaddownarrow}$

$$\begin{array}{c} A^{\mathbb{I}} \\ \big\downarrow \\ A^{\mathbb{2}} \end{array} \coloneqq \lim \left(\begin{tikzcd} \cdots \arrow[r, two heads] & A_n^{\mathbb{I}} \arrow[r, two heads] \arrow[d, two heads, "\wr"] & A_{n-1}^{\mathbb{I}} \arrow[r, two heads] \arrow[d, two heads, "\wr"] & \cdots \arrow[r, two heads] & A_1^{\mathbb{I}} \arrow[r, two heads] \arrow[d, two heads, "\wr"] & A_0^{\mathbb{I}} \arrow[d, two heads, "\wr"] \\ \cdots \arrow[r, two heads] & A_n^{\mathbb{2}} \arrow[r, two heads] & A_{n-1}^{\mathbb{2}} \arrow[r, two heads] & \cdots \arrow[r, two heads] & A_1^{\mathbb{2}} \arrow[r, two heads] & A_0^{\mathbb{2}} \end{tikzcd} \right)$$

Since each A_n is a discrete ∞-category, each of the objects in this diagram is a trivial fibration. Hence, by Proposition 6.1.5 the limit $A^{\mathbb{I}} \twoheadrightarrow A^{\mathbb{2}}$ is a trivial fibration as well. This proves that A is discrete. $\square$

The proof of Proposition 6.1.1 revealed that it's tedious to manually verify the limit axiom in the construction of new ∞-cosmoi; indeed, even in that relatively basic example, we omitted some details. In the following sections, we develop machinery that will allow us to attack this problem more systematically.

Exercises.

6.1.i. Exercise. Prove Lemma 6.1.3.

6.1.ii. Exercise. Prove Proposition 6.1.5 using Lemma 6.1.4.

6.2. Flexible weighted limits and the collage construction

Our aim in this section is to introduce a special class of simplicial-set-valued weights whose associated weighted limit notions are homotopically well-behaved. Borrowing a term from 2-category theory, we refer to these weights as *flexible*. All of the limits enumerated in 1.2.1(i) are flexible limits. In fact, we will prove that ∞-cosmoi admit all flexible weighted limits because these can be built out of the limits enumerated in 1.2.1(i). In §6.4, we will use this observation to help us verify the limit axiom for newly constructed ∞-cosmoi in a more systematic way.

In this section, we will characterize the class of flexible weights as precisely those whose associated cone shapes, which we refer to as the *collages* of the weights, define *relative simplicial computads*. Roughly, this means that the simplices appearing in a cone are freely attached, relative to the diagram shape. This characterization, which appears as Theorem 6.2.14, will allow us to readily produce examples.

In §6.3, we will establish the homotopical properties of flexible weighted limits and make precise the relationships between this class of the limits and limits assumed present in any ∞-cosmos by axiom 1.2.1(i).

6.2.1. Definition (flexible weights and projective cell complexes). Let $\mathcal{A}$ be a simplicial category.

- A simplicial natural transformation of the form

$$\partial\Delta[n] \times \mathcal{A}(a,-) \hookrightarrow \Delta[n] \times \mathcal{A}(a,-)$$

is called a **projective n-cell at $a \in \mathcal{A}$**.
- A natural transformation $\alpha\colon V \hookrightarrow W$ in $s\mathcal{S}et^{\mathcal{A}}$ that can be expressed as a countable composite of pushouts of coproducts of projective cells is called a **projective cell complex**.
- A weight $W \in s\mathcal{S}et^{\mathcal{A}}$ is **flexible** just when $\emptyset \hookrightarrow W$ is a projective cell complex.

In a simplicial category $\mathcal{M}$, a **flexible weighted limit** is a weighted limit, in the sense of §A.6, of a diagram $F\colon \mathcal{A} \to \mathcal{M}$ whose weight $W\colon \mathcal{A} \to s\mathcal{S}et$ is flexible.

6.2.2. Remark (on generalized projective cells). Since any monomorphism of simplicial sets $U \hookrightarrow V$ can be decomposed as a sequential composite of pushouts of coproducts of boundary inclusions $\partial\Delta[n] \hookrightarrow \Delta[n]$, the class of projective cell complexes may be also be described as the class of maps in $s\mathcal{S}et^{\mathcal{A}}$ that can be expressed as a countable composite of pushouts of coproducts of monomorphisms $U \times \mathcal{A}(a,-) \hookrightarrow V \times \mathcal{A}(a,-)$ for some $a \in \mathcal{A}$.

6.2.3. Example. Recall from §A.4 that the **cotensor** of an object $M \in \mathcal{M}$ in a simplicial category with a simplicial set U is the object $M^U \in \mathcal{M}$ characterized by the simplicial natural isomorphism

$$\mathcal{M}(X, M^U) \cong \mathcal{M}(X, M)^U.$$

This object can be regarded as the limit of the diagram $M\colon \mathbb{1} \to \mathcal{M}$ weighted by $U\colon \mathbb{1} \to s\mathcal{S}et$. Since any simplicial set can be decomposed as a sequential composite of pushouts of coproducts of boundary inclusions $\partial\Delta[n] \hookrightarrow \Delta[n]$, simplicial cotensors are flexible weights.

Recall from §A.5 that the **conical limit** of a diagram $F\colon \mathcal{A} \to \mathcal{M}$ is the limit weighted by the terminal weight $*\colon \mathcal{A} \to s\mathcal{S}et$.

6.2.4. Example. Conical products also define flexible weighted limits, built by attaching one projective 0-cell for each object in the indexing set.

6.2.5. Non-Example. Conical limits indexed by any 1-category that contains non-identity arrows are *not* flexible because the legs of a conical cone over the domain and codomain of each arrow in the diagram are required to define a strictly commutative triangle of 0-arrows. The specifications for a flexible weight allow us to freely attach n-arrows of any dimension, but do not provide a mechanism for demanding strict commutativity of any diagram of n-arrows — only commutativity up to the presence of a higher cell.

6.2.6. Digression (on flexible limits in 2-category theory). Simplicial limits weighted by flexible weights should be thought of as analogous to **flexible 2-limits**, i.e., 2-limits built out of products, inserters, equifiers, and retracts (splittings of idempotents) [**14**]. More precisely, simplicial limits weighted by flexible weights are analogous to the **PIE limits**, those built just from products, inserters, and equifiers, but we choose to adopt the moniker from the slight larger class of weights because we find it to be more evocative. The PIE limits also include iso-inserters, descent objects, comma objects, and Eilenberg-Moore objects, as well as all pseudo, lax, and oplax limits. Many important 2-categories, such as the 2-category of accessible categories and accessible functors, fail to admit all 2-categorical limits, but do admit all PIE-limits [**62**].

The weights for flexible limits are the cofibrant objects in a model structure on the diagram 2-category $\mathcal{C}at^{\mathcal{A}}$ that is enriched over the folk model structure on $\mathcal{C}at$; the PIE weights are exactly the *cellular* cofibrant objects. Correspondingly, the projective cell complexes of Definition 6.2.1 are exactly the cellular cofibrations in the projective model structure on $s\mathcal{S}et^{\mathcal{A}}$.

This suggests that "PIE limits" would be a more precise name for the flexible weighted limits of Definition 6.2.1 and we do not disagree. With apologies to the Australian 2-category theory diaspora, we cannot resist adopting the more evocative term.

Recall that a weight is intended to describe the "shape" of cones over diagrams indexed by a particular category. In the case a weight $W\colon \mathcal{A} \to s\mathcal{S}et$ valued in simplicial sets, we can describe the shape of W-cones directly as a simplicial category called the *collage* of the weight.

6.2.7. Definition (collage construction). The **collage** of a weight $W\colon \mathcal{A} \to s\mathcal{S}et$ is a simplicial category $\mathrm{coll}W$ which contains $\mathcal{A}$ as a full subcategory together with one additional object $\top$ whose endomorphism space is the point. The mapping spaces from any object $a \in \mathcal{A}$ to $\top$ are empty, while the mapping spaces from $\top$ to the image of $\mathcal{A} \hookrightarrow \mathrm{coll}W$ are defined by:

$$\mathrm{coll}W(\top, a) := W(a)$$

with the action maps $\mathcal{A}(a,b) \times W(a) \to W(b)$ from the simplicial functor W used to define composition. This defines a simplicial category together with a canonical inclusion $\mathbb{1} + \mathcal{A} \hookrightarrow \mathrm{coll}W$ that is bijective on objects and fully faithful on $\mathbb{1}$ and $\mathcal{A}$ separately.

6.2.8. Proposition (collage adjunction).

(i) The collage construction defines a fully faithful functor

$$s\mathcal{S}et^{\mathcal{A}} \xrightarrow{\mathrm{coll}} {}^{\mathbb{1}+\mathcal{A}/}s\mathcal{S}et\text{-}\mathcal{C}at$$

from the category of $\mathcal{A}$-indexed weights to the category of simplicial categories under $\mathbb{1} + \mathcal{A}$ whose essential image is comprised of those $\langle e, F\rangle\colon \mathbb{1} + \mathcal{A} \to \mathcal{E}$ that are bijective on objects, fully faithful when restricted to $\mathbb{1}$ and $\mathcal{A}$, and have the property that there are no arrows in $\mathcal{E}$ from the image of F to e.

(ii) The collage functor admits a right adjoint

$$\begin{tikzcd} s\mathcal{S}et^{\mathcal{A}} \arrow[r, bend left, "\mathrm{coll}"] \arrow[r, phantom, "\top"] & {}^{\mathbb{1}+\mathcal{A}/}s\mathcal{S}et\text{-}\mathcal{C}at \arrow[l, bend left, "\mathrm{wgt}"] \end{tikzcd}$$

which carries a pair $\langle e, F\rangle\colon \mathbb{1} + \mathcal{A} \to \mathcal{E}$ to the weight $\mathcal{E}(e, F-)\colon \mathcal{A} \to s\mathcal{S}et$.

Proof. The construction of the collage functor is straightforward and left to the reader. The characterization of its essential image follows from the observation that to define a simplicial functor $W\colon \mathcal{A} \to s\mathcal{S}et$ requires no more and no less than

- the specification of simplicial sets $W(a)$ for each $a \in \mathcal{A}$ and
- and the specification of simplicial maps $\mathcal{A}(a,b) \times W(a) \to W(b)$ for each $a, b \in \mathcal{A}$

so that this action is associative in sense that the diagram

$$\begin{tikzcd} \mathcal{A}(b,c) \times \mathcal{A}(a,b) \times W(a) \arrow[r, "\circ"] \arrow[d] & \mathcal{A}(a,c) \times W(a) \arrow[d] \\ \mathcal{A}(b,c) \times W(b) \arrow[r] & W(c) \end{tikzcd}$$

commutes. This is the same as what is required to extend $\mathbb{1} + \mathcal{A}$ to a simplicial category in which all of the additional maps start at $\top$ and end at an object in $\mathcal{A}$.

The adjunction asserts that simplicial functors

$$\begin{tikzcd} & \mathbb{1} + \mathcal{A} \arrow[dl, hook'] \arrow[dr, "{\langle e,F\rangle}"] & \\ \mathrm{coll}W \arrow[rr, "G"'] & & \mathcal{E} \end{tikzcd}$$

from $\mathrm{coll}W$ to $\mathcal{E}$ under $\mathbb{1} + \mathcal{A}$ stand in natural bijective correspondence with simplicial natural transformations $\gamma \colon W \Rightarrow \mathcal{E}(e, F-)$. Since the inclusion $1 + \mathcal{A} \hookrightarrow \mathrm{coll}W$ is bijective on objects and full on most homs, the data of the simplicial functor requires only the specification of the maps $\mathrm{coll}W(\top, a) = W(a) \to \mathcal{E}(e, Fa)$. These define the components of the simplicial natural transformation γ and functoriality of G corresponds to naturality of γ. □

The collage adjunction has a useful and important interpretation.

6.2.9. Corollary. *The collage of a weight $W \colon \mathcal{A} \to s\mathcal{S}et$ realizes the shape of W-weighted cones in the sense that simplicial functors $G \colon \mathrm{coll}W \to \mathcal{E}$ with domain $\mathrm{coll}W$ stand in bijection to W-cones over the diagram $G|_{\mathcal{A}}$ with summit $G(\top)$.* □

In analogy with Corollary 4.3.5, we can encode simplicial W-weighted limits as a right Kan extension from the indexing simplicial category to the simplicial category that describes the shape of W-cones.

6.2.10. Lemma. *For any simplicial functor $F \colon \mathcal{A} \to \mathcal{E}$ and any weight $W \colon \mathcal{A} \to s\mathcal{S}et$, the weighted limit $\lim_W F$ exists if and only if the pointwise right Kan extension of F along $\mathcal{A} \hookrightarrow \mathrm{coll}W$ exists, in which case $\mathrm{lan}\, F(\top) \cong \lim_W F$.*

Proof. Since $\mathcal{A} \hookrightarrow \mathrm{coll}W$ is fully faithful, pointwise right Kan extensions may be chosen to define genuine extensions

$$\begin{tikzcd} & \mathcal{A} \arrow[dl, hook'] \arrow[dr, "F"] & \\ \mathrm{coll}W \arrow[rr, dashed, "\mathrm{ran}\, F"'] & & \mathcal{E} \end{tikzcd}$$

By Corollary 6.2.9, this data defines a W-cone over F with summit $\mathrm{ran}\, F(\top)$ that we denote by $\lambda \colon W \Rightarrow \mathcal{E}(\mathrm{ran}\, F(\top), F(-))$.

By Corollary 6.2.9 again, the data of a cone over the right Kan extension diagram displayed below-left

$$\begin{tikzcd} & \mathcal{A} \arrow[dl, hook'] \arrow[dr, "F"] \arrow[d, phantom, "\Uparrow\alpha"] & \\ \mathrm{coll}W \arrow[rr, bend right, "G"'] & {} & \mathcal{E} \end{tikzcd} \quad = \quad \begin{tikzcd} & \mathcal{A} \arrow[dl, hook'] \arrow[dr, "F"] & \\ \mathrm{coll}W \arrow[rr, "\mathrm{ran}\, F"] \arrow[rr, bend right=50, "G"'] \arrow[rr, phantom, bend right=25, "\exists!\Uparrow\eta"] & & \mathcal{E} \end{tikzcd}$$

defines a W-cone

$$W \xRightarrow{\gamma} \mathcal{E}(G(\top), G|_{\mathcal{A}}(-)) \xRightarrow{\alpha_*} \mathcal{E}(G(\top), F(-))$$

over F. The universal property of the right Kan extension depicted above-right says this cone factors uniquely through the W-cone λ along a map $\eta_\top \colon G(\top) \to \mathrm{ran}\, F(\top)$. Thus, the right Kan extension of F along $\mathcal{A} \hookrightarrow \mathrm{coll}W$ equips the resulting W-cone with the universal property of the W-weighted limit. □

A particularly convenient aspect of the collage construction is that it allows us to detect the class of flexible weights vis-à-vis a notion that we now introduce.

6.2.11. Definition (relative simplicial computad). The class of all **relative simplicial computads** is the class of all simplicial functors that can be expressed as a countable composite of pushouts of coproducts of

- the unique simplicial functor $\emptyset \hookrightarrow \mathbb{1}$ and
- the simplicial subcomputad inclusion $\mathbb{2}[\partial\Delta[n]] \hookrightarrow \mathbb{2}[\Delta[n]]$ for $n \geq 0$,

where for any simplicial set U, $\mathbb{2}[U]$ denotes the simplicial category with two objects "$-$" and "$+$" and with hom-spaces defined by

$$\mathbb{2}[U](+,-) \coloneqq \emptyset, \quad \mathbb{2}[U](+,+) \coloneqq \mathbb{1} =: \mathbb{2}[U](-,-), \quad \mathbb{2}[U](-,+) \coloneqq U.$$

A simplicial category $\mathcal{A}$ is a **simplicial computad** just when $\emptyset \hookrightarrow \mathcal{A}$ is a relative simplicial computad. Exercise 6.2.viii shows that this characterization agrees with the following elementary characterization of "free" simplicial category, first introduced by Dwyer and Kan [**33**, 4.5].

6.2.12. Definition (simplicial computad). A simplicial category $\mathcal{A}$ is a **simplicial computad** if and only if

- Each category $\mathcal{A}_n$ of n-arrows is freely generated by the graph of atomic[1] n-arrows.
- If f is an atomic n-arrow in $\mathcal{A}_n$ and $\sigma\colon [m] \twoheadrightarrow [n]$ is an epimorphism in $\mathbf{\Delta}$, then the degenerated m-arrow $f \cdot \sigma$ is atomic in $\mathcal{A}_m$.

6.2.13. Digression. The category of simplicial categories bears a model structure due to Bergner whose fibrant objects are the Kan complex enriched categories. The weak equivalences, called **DK-equivalences** after Dwyer and Kan, are simplicial functors $F\colon \mathcal{C} \to \mathcal{D}$ whose action on homs

$$F_{x,y}\colon \mathcal{C}(x,y) \xrightarrow{\sim} \mathcal{D}(Fx,Fy)$$

is via weak homotopy equivalences of simplicial sets and which induce equivalences of homotopy categories.[2] The fibrations are defined by a similar pair of conditions: these are simplicial functors $F\colon \mathcal{C} \to \mathcal{D}$ whose action on homs

$$F_{x,y}\colon \mathcal{C}(x,y) \twoheadrightarrow \mathcal{D}(Fx,Fy)$$

is via a Kan fibration of simplicial sets and which have the property that a 0-arrow $f\colon Fc \to d$ that defines an isomorphism in the homotopy category of $\mathcal{D}$ may be lifted to a 0-arrow $g\colon c \to c'$ that defines an isomorphism in the homotopy category of $\mathcal{C}$. More details can be found in the original [**10**] or in the survey text [**12**].

The morphisms listed in Definition 6.2.11 are the generating cofibrations in the Bergner model structure. Hence, the relative simplicial computads are precisely the *cellular cofibrations*, those that are built as sequential composites of pushouts of coproducts of generating cofibrations (without closing

[1]An arrow f in a 1-category is **atomic** if it is not an identity and if it admits no non-trivial factorizations: i.e., if whenever $f = g \circ h$, either g or h is an identity. A 1-category is **free** if every arrow may be expressed uniquely as a composite of atomic arrows, with the convention that empty composites correspond to identity arrows. Alternatively, a 1-category is **free** if every non-identity arrow may be expressed uniquely as a non-empty composite of atomic arrows and if identity arrows admit no non-trivial factorizations.

[2]The **homotopy category** of a simplicially enriched category is defined by applying the path components functor $\pi_0\colon s\mathcal{S}et \to \mathcal{S}et$ to the hom-spaces. In fact, by the local homotopically fully faithfulness property, it suffices to assume essential surjectivity on homotopy categories.

under retracts). By Exercise 6.2.iv, every retract of a simplicial computad is a simplicial computad. Hence, the cofibrant objects in the Bergner model structure are exactly the simplicial computads. However, for non-cofibrant domains, the notion of Bergner cofibration is more general than the notion of relative simplicial computad.

6.2.14. Theorem (flexible weights and collages). *A natural transformation* $\alpha\colon V \hookrightarrow W$ *between weights in* $s\mathcal{S}et^{\mathcal{A}}$ *is a projective cell complex if and only if* $\mathrm{coll}(\alpha)\colon \mathrm{coll}V \hookrightarrow \mathrm{coll}W$ *is a relative simplicial computad. In particular,* W *is a flexible weight if and only if* $\mathbb{1} + \mathcal{A} \hookrightarrow \mathrm{coll}W$ *is a relative simplicial computad.*

Proof. If $\alpha\colon V \hookrightarrow W$ is a projective cell complex, then it can be presented as a countable composite of pushouts of coproducts of projective cells of varying dimensions indexed by the objects $a \in \mathcal{A}$. Since the collage construction is a left adjoint, it preserves these colimits, and hence the map $\mathrm{coll}(\alpha)\colon \mathrm{coll}V \hookrightarrow \mathrm{coll}W$ is a transfinite composite of pushouts of coproducts of simplicial functors $\mathrm{coll}(\partial\Delta[n] \times \mathcal{A}(a,-)) \hookrightarrow \mathrm{coll}(\Delta[n] \times \mathcal{A}(a,-))$ in ${}^{\mathbb{1}+\mathcal{A}/}s\mathcal{S}et\text{-}\mathcal{C}at$. This composite colimit diagram is connected — note $\mathrm{coll}\varnothing = \mathbb{1} + \mathcal{A}$, so this cell complex presentation is also preserved by the forgetful functor ${}^{\mathbb{1}+\mathcal{A}/}s\mathcal{S}et\text{-}\mathcal{C}at \to s\mathcal{S}et\text{-}\mathcal{C}at$ and the simplicial functor $\mathrm{coll}(\alpha)\colon \mathrm{coll}V \hookrightarrow \mathrm{coll}W$ can be understood as a transfinite composite of pushouts of coproducts of $\mathrm{coll}(\partial\Delta[n] \times \mathcal{A}(a,-)) \hookrightarrow \mathrm{coll}(\Delta[n] \times \mathcal{A}(a,-))$ in $s\mathcal{S}et\text{-}\mathcal{C}at$.

This is advantageous because there is a pushout square in $s\mathcal{S}et\text{-}\mathcal{C}at$

$$\begin{tikzcd} \mathbb{2}[\partial\Delta[n]] \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner" very near end] & \mathrm{coll}(\partial\Delta[n] \times \mathcal{A}(a,-)) \arrow[d, hook] \\ \mathbb{2}[\Delta[n]] \arrow[r] & \mathrm{coll}(\Delta[n] \times \mathcal{A}(a,-)) \end{tikzcd} \qquad \leftrightsquigarrow \qquad \begin{tikzcd} \partial\Delta[n] \arrow[r, hook, "\mathrm{id}_a"] \arrow[d, hook] & \partial\Delta[n] \times \mathcal{A}(a,a) \arrow[d, hook] \\ \Delta[n] \arrow[r, hook, "\mathrm{id}_a"] & \Delta[n] \times \mathcal{A}(a,a) \end{tikzcd} \tag{6.2.15}$$

whose horizontals send the two objects $-$ and $+$ to $\top$ and a and act on the non-trivial hom-spaces via the inclusions displayed above left whose component in $\mathcal{A}(a,a)$ is constant at the identity element at a. The fact that $\mathrm{coll}(\partial\Delta[n] \times \mathcal{A}(a,-)) \hookrightarrow \mathrm{coll}(\Delta[n] \times \mathcal{A}(a,-))$ is a pushout of $\mathbb{2}[\partial\Delta[n]] \hookrightarrow \mathbb{2}[\Delta[n]]$ can be verified by transposing across the adjunction of Proposition 6.2.8 and applying the Yoneda lemma. Hence, $\mathrm{coll}(\alpha)\colon \mathrm{coll}V \hookrightarrow \mathrm{coll}W$ is a transfinite composite of pushouts of coproducts of simplicial functors $\mathbb{2}[\partial\Delta[n]] \hookrightarrow \mathbb{2}[\Delta[n]]$, which proves that this map is a relative simplicial computad.

Conversely, if $\mathrm{coll}(\alpha)\colon \mathrm{coll}V \hookrightarrow \mathrm{coll}W$ is a relative simplicial computad, then it can be presented as a countable composite of pushouts of coproducts of simplicial functors $\mathbb{2}[\partial\Delta[n]] \hookrightarrow \mathbb{2}[\Delta[n]]$; since this inclusion is bijective on objects the inclusion $\varnothing \hookrightarrow \mathbb{1}$ is not needed. Since the only arrows of $\mathrm{coll}W$ that are not present in $\mathrm{coll}V$ have domain $\top$ and codomain $a \in \mathcal{A}$, the characterization of the essential image of the collage functor of Proposition 6.2.8(i) allows us to identify each stage of the countable composite

$$\mathrm{coll}V \hookrightarrow \mathrm{coll}(W^1) \hookrightarrow \cdots \hookrightarrow \mathrm{coll}(W^i) \hookrightarrow \mathrm{coll}(W^{i+1}) \hookrightarrow \cdots \hookrightarrow \mathrm{coll}W$$

as the collage of some weight $W^i\colon \mathcal{A} \to s\mathcal{S}et$. Each attaching map $\mathbb{2}[\partial\Delta[n]] \to \mathrm{coll}W^i$ in the cell complex presentation acts on objects by mapping $-$ and $+$ to $\top$ and a for some $a \in \mathcal{A}$, and hence factors through the top horizontal of the pushout square (6.2.15). Hence, the inclusion $\mathrm{coll}(W^i) \hookrightarrow \mathrm{coll}(W^{i+1})$ is a pushout of a coproduct of the maps $\mathrm{coll}(\partial\Delta[n] \times \mathcal{A}(a,-)) \hookrightarrow \mathrm{coll}(\Delta[n] \times \mathcal{A}(a,-))$, one for each cell $\mathbb{2}[\partial\Delta[n]] \hookrightarrow \mathbb{2}[\Delta[n]]$ whose attaching map sends $+$ to $a \in \mathrm{coll}(W^i)$. As the collage functor is fully faithful, we have now expressed $\mathrm{coll}(\alpha)\colon \mathrm{coll}V \hookrightarrow \mathrm{coll}W$ as a countable composite of pushouts of coproducts of simplicial functors $\mathrm{coll}(\partial\Delta[n] \times \mathcal{A}(a,-)) \hookrightarrow \mathrm{coll}(\Delta[n] \times \mathcal{A}(a,-))$. A fully faithful functor that preserves colimits also reflects them, so in this way we see that $\alpha\colon V \hookrightarrow W$ is a countable composite of pushouts of coproducts of projective cells, proving that it is a projective cell complex as claimed. □

In this text, we have resisted a lengthy digression through homotopy coherent category theory, as those results are not needed here; see [**80, 82**]. Nonetheless, we cannot resist discussing the following example.

6.2.16. Example. The left adjoint $\mathfrak{C}\colon s\mathcal{S}et \to s\mathcal{S}et\text{-}\mathcal{C}at$ to the homotopy coherent nerve functor of Cordier [**24**], which we call *homotopy coherent realization*, carries monomorphisms of simplicial sets to relative simplicial computads (in fact, to simplicial subcomputad inclusions; see Exercise 6.2.viii). In particular, the coherent realization of the canonical inclusion $\mathbb{1} + X \hookrightarrow X^{\triangleleft}$ defines a collage and relative simplicial computad $\mathbb{1} + \mathfrak{C}X \hookrightarrow \mathfrak{C}(X^{\triangleleft})$ and hence a flexible weight W_X for homotopy coherent diagrams of shape X. Since the left adjoint of the collage adjunction is fully faithful, its unit is an isomorphism, and this permits us to define the weight W_X explicitly: for a vertex $x \in X$,

$$W_X(x) \coloneqq \mathfrak{C}(X^{\triangleleft})(\top, x).$$

For homotopy coherent diagrams valued in Kan-complex enriched categories, we might refer to the weight $W_X\colon \mathfrak{C}X \to s\mathcal{S}et$ as the **weight for the pseudo limit of a homotopy coherent diagram of shape** X. This name is not as appropriate, however, for homotopy coherent diagrams valued in quasi-categorically enriched categories, such as ∞-cosmoi. In such contexts, it would be more appropriate to refer to W_X as the weight for *oplax* limits, since in that context the 1-arrows of $\mathfrak{C}(X^{\triangleleft})$ will likely map to non-invertible morphisms.

Exercises.

6.2.i. Exercise. Compute the collage of the weight $U\colon \mathbb{1} \to s\mathcal{S}et$ and use Theorem 6.2.14 to give a second proof that simplicial cotensors are flexible weighted limits. Compare this argument with that given in Example 6.2.3.

6.2.ii. Exercise. Verify — either directly from Definition 6.2.1 or by applying Theorem 6.2.14 — that conical products are flexible weighted limits.

6.2.iii. Exercise. Show that for any weight $W\colon \mathcal{A} \to s\mathcal{S}et$ and simplicial functor $K\colon \mathcal{A} \to \mathcal{B}$, the pushout of simplicial categories

$$\begin{tikzcd} \mathbb{1} + \mathcal{A} \arrow[r, "\mathbb{1}+K"] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner" very near end] & \mathbb{1} + \mathcal{B} \arrow[d, hook] \\ \mathrm{coll} W \arrow[r] & \mathrm{coll}(\mathrm{lan}_K W) \end{tikzcd}$$

computes the collage of the weight $\mathrm{lan}_K W\colon \mathcal{B} \to s\mathcal{S}et$. See Lemma A.6.21.

6.2.iv. Exercise. Verify directly from Definition 6.2.12 that every retract of a simplicial computad is a simplicial computad.

6.2.v. Exercise. A simplicial functor $\mathcal{A} \to \mathcal{B}$ between simplicial computads defines a **simplicial computad morphism** if it maps every atomic arrow in $\mathcal{A}$ to an arrow which is either atomic or an identity in $\mathcal{B}$. This defines a non-full subcategory $s\mathcal{S}et\text{-}\mathcal{C}ptd \subset s\mathcal{S}et\text{-}\mathcal{C}at$ of simplicial computads and their morphisms. Show that the functor that carries a simplicial computad to its underlying diagram of atomic and identity arrows and the inclusion of simplicial computads into simplicial categories define the legs of a pullback cone:

$$\begin{tikzcd} s\mathcal{S}et\text{-}\mathcal{C}ptd \arrow[r, hook] \arrow[d, "\mathrm{atom}"'] \arrow[dr, phantom, "\lrcorner" very near start] & s\mathcal{S}et\text{-}\mathcal{C}at \arrow[d] \\ \mathcal{G}ph^{\Delta^{\mathrm{op}}_{\mathrm{epi}}} \arrow[r, hook, "F^{\Delta^{\mathrm{op}}_{\mathrm{epi}}}"] & \mathcal{C}at^{\Delta^{\mathrm{op}}_{\mathrm{epi}}} \end{tikzcd}$$

6.2.vi. Exercise. A simplicial computad morphism $\mathcal{A} \hookrightarrow \mathcal{B}$ that is injective on objects and faithful displays $\mathcal{A}$ as a **simplicial subcomputad** of $\mathcal{B}$. Show that a simplicial subcategory $\mathcal{A} \hookrightarrow \mathcal{B}$ of a simplicial computad $\mathcal{B}$ displays $\mathcal{A}$ as a simplicial subcomputad of $\mathcal{B}$ just when $\mathcal{A}$ is closed under factorizations: if f and g are composable arrows of $\mathcal{B}$ and $f \circ g \in \mathcal{A}$ then f and g are in $\mathcal{A}$.

6.2.vii. Exercise. Conclude from Exercise 6.2.v that $s\mathcal{S}et\text{-}\mathcal{C}ptd$ has colimits created by either of the functors to $s\mathcal{S}et\text{-}\mathcal{C}at$ or to $\mathcal{G}ph^{\Delta^{\mathrm{op}}_{\mathrm{epi}}}$. Use this to argue that simplicial subcomputads are closed under coproduct, pushout, and colimit of countable sequences.

6.2.viii. Exercise. Prove that if $\mathcal{A}$ is a simplicial computad, then an inclusion of simplicial categories $\mathcal{A} \hookrightarrow \mathcal{B}$ is a simplicial computad inclusion if and only if it is a relative simplicial computad. Conclude, in particular, a simplicial category $\mathcal{B}$ is a simplicial computad if and only if $\emptyset \hookrightarrow \mathcal{B}$ is a relative simplicial computad.

6.3. Homotopical properties of flexible weighed limits

In a $\mathcal{V}$-model category $\mathcal{M}$, the fibrant objects are closed under weighted limits whose weights are projective cofibrant; see Corollary C.3.13. For instance, the fibrant objects in a $\mathcal{C}at$-enriched model structure are closed under flexible weighted limits [**57**, 5.4] in the sense of [**14**]. Specializing this argument to the case of ∞-cosmoi, we obtain the following result:

6.3.1. Proposition (flexible weights are homotopical). *Let $W \colon \mathcal{A} \to s\mathcal{S}et$ be a flexible weight and let $\mathcal{K}$ be an ∞-cosmos.*

(i) The weighted limit $\lim^{\mathcal{A}}_W F$ of any diagram $F \colon \mathcal{A} \to \mathcal{K}$ may be expressed as a countable inverse limit of pullbacks of products of isofibrations

$$Fa^{\Delta[n]} \twoheadrightarrow Fa^{\partial\Delta[n]} \tag{6.3.2}$$

one for each projective n-cell at a in the given projective cell complex presentation of W.

(ii) If $V \hookrightarrow W \in s\mathcal{S}et^{\mathcal{A}}$ is a projective cell complex between flexible weights, then for any diagram $F \colon \mathcal{A} \to \mathcal{K}$, the induced map between weighted limits

$$\lim_W F \twoheadrightarrow \lim_V F$$

is an isofibration.

(iii) If $\alpha \colon F \Rightarrow G$ is a simplicial natural transformation between a pair of diagrams $F, G \colon \mathcal{A} \rightrightarrows \mathcal{K}$ whose components $\alpha_a \colon Fa \xrightarrow{\sim} Ga$ are equivalences, then the induced map

$$\lim_W F \xrightarrow[\sim]{\alpha} \lim_W G$$

is an equivalence.

Proof. To begin, observe that the axioms of Definition A.6.3 imply that the limit of F weighted by the weight $U \times \mathcal{A}(a, -)$, for $U \in s\mathcal{S}et$ and $a \in \mathcal{A}$, is the cotensor Fa^U. Consequently, the map of weighted limits induced by the projective n-cell at a is the isofibration (6.3.2). By definition, any flexible weight is built as a countable composite of pushouts of coproducts of these projective cells and the weighted limit functor $\lim^{\mathcal{A}}_{-} F$ carries each of these conical colimits to the corresponding limit notion. So it follows that $\lim^{\mathcal{A}}_{W} F$ may be expressed as a countable inverse limit of pullbacks of products of the maps (6.3.2). This proves (i).

The same argument proves (ii). By definition, a relative cell complex $V \hookrightarrow W$ is built as a countable composite of pushouts of coproducts of these projective cells and the weighted limit functor $\lim^{\mathcal{A}}_{-} F$ carries each of these conical colimits to the corresponding limit notion. So it follows that $\lim^{\mathcal{A}}_{W} F$ is the limit of a countable tower of isofibrations whose base is $\lim^{\mathcal{A}}_{V} F$, where each of these isofibrations is the pullback of products of the maps (6.3.2) appearing in the projective cell complex decomposition of $V \hookrightarrow W$. As products, pullbacks, and limits of towers of isofibrations are isofibrations, (ii) follows.

In (i), we have decomposed each weighted limit $\lim^{\mathcal{A}}_{W} F$ as the limit of a tower of isofibrations, in which each of these isofibrations is the pullback of a product of the isofibrations (6.3.2). We argue inductively that if $\alpha \colon F \Rightarrow G$ is a componentwise equivalence, then the map induced between the towers of isofibrations for F and for G by the projective cell complex presentation of W is a levelwise equivalence. It follows from the standard argument in abstract homotopy theory reviewed in Appendix C that the inverse limit is then an equivalence, proving (iii).

The bottom of the tower of isofibrations is $\lim_{\varnothing} F \cong 1 \cong \lim_{\varnothing} G$, which is certainly an equivalence. For the inductive step, observe that upon taking the map of weighted limits induced by each projective n-cell at a in W, we obtain a commutative square

$$\begin{tikzcd} Fa^{\Delta[n]} \arrow[r, "\alpha_a^{\Delta[n]}", "\sim"'] \arrow[d, two heads] & Ga^{\Delta[n]} \arrow[d, two heads] \\ Fa^{\partial\Delta[n]} \arrow[r, "\alpha_a^{\partial\Delta[n]}", "\sim"'] & Ga^{\partial\Delta[n]} \end{tikzcd}$$

defining a pointwise equivalence between the isofibrations; the simplicial cotensor, as cosmological functor, preserves equivalences. Now the product of squares of this form gives a commutative square whose verticals are isofibrations and whose horizontals are equivalences. A pullback of this square forms the next layer in the tower of isofibrations; by the inductive hypothesis, the map between the codomains of the pulled back isofibrations is already known to be an equivalence. Now the equivalence-invariance of pullbacks of isofibrations established in Appendix C completes the proof. ☐

Immediately from the construction of Proposition 6.3.1(i):

6.3.3. Corollary (∞-cosmoi admit all flexible weighted limits). *∞-cosmoi admit all flexible weighted limits and cosmological functors preserve them.* ☐

Our aim is now to describe a converse of sorts to Proposition 6.3.1(i), which proves that the flexible weighted limit of any diagram in an ∞-cosmos can be constructed out of the limits of diagrams of isofibrations axiomatized in 1.2.1(i). Over a series of lemmas, we will construct each of the limits listed there as instances of flexible weighted limits. It will follow that any quasi-categorically enriched category equipped with a class of representably-defined isofibrations that possesses flexible weighted limits will admit all of the simplicial limits of 1.2.1(i). This will help us identify new examples of ∞-cosmoi.

To start, simplicial cotensors are flexible weighted limits, as discussed in Example 6.2.3. This leaves only the conical limits. The weights for products are easily seen to be flexible directly from Definition 6.2.1. However, the weights for conical pullbacks or limits of towers of isofibrations are not flexible because the definition of a cone over either diagram shape imposes composition relations on 0-arrows.

6.3.4. Example (the collage of the conical pullback). Let $\lrcorner$ denote the 1-category $c \to a \leftarrow b$. Its collage is the 1-category with four objects and five non-identity 0-arrows as displayed

$$\begin{tikzcd} \top \arrow[r] \arrow[d] \arrow[dr] & b \arrow[d] \\ c \arrow[r] & a \end{tikzcd}$$

regarded as a constant simplicial category. Because the square commutes, this category is not free and hence does *not* define a simplicial computad, though the subcategory $\mathbb{1} + \lrcorner$ is free and hence is a simplicial computad. Exercise 6.2.viii tells us that the inclusion is not a relative simplicial computad and so by Theorem 6.2.14, the weight for the conical pullback is not flexible.

Our strategy is to modify the weights for pullbacks and for limits of countable towers so that each composition equation involved in defining cones over such diagrams is replaced by the insertion of an "invertible" arrow of one dimension up, where we must also take care to define this "invertibility" without specifying any equations between arrows in the next dimension. We have a device for specifying just this sort of isomorphism: recall from Exercise 1.1.v(i) a diagram $\mathbb{I} \to \mathsf{Fun}(A, B)$ specifies a "homotopy coherent isomorphism" between a pair of 0-arrows f and g from A to B, given by:

- a pair of 1-arrows $\alpha\colon f \to g$ and $\beta\colon g \to f$
- a pair of 2-arrows

$$\begin{tikzcd} & g \arrow[dr, "\beta"] \arrow[d, phantom, "\Phi"] & \\ f \arrow[ur, "\alpha"] \arrow[rr, equal] & {} & f \end{tikzcd} \qquad \begin{tikzcd} & f \arrow[dr, "\alpha"] \arrow[d, phantom, "\Psi"] & \\ g \arrow[ur, "\beta"] \arrow[rr, equal] & {} & g \end{tikzcd}$$

- a pair of 3-arrows whose outer faces are Φ and Ψ and whose inner faces are degenerate
- etc

We now introduce the weight for pullback diagrams whose cone shapes are given by squares inhabited by a homotopy coherent isomorphism.

6.3.5. Definition (iso-commas). The **iso-comma object** $C \underset{A}{\tilde{\times}} B$ of a cospan

$$C \xrightarrow{g} A \xleftarrow{f} B$$

in a simplicially-enriched and cotensored category $\mathcal{M}$ is the limit weighted by a weight $W_{\tilde{\times}}\colon \lrcorner \to s\mathcal{S}et$ defined by the cospan

$$\mathbb{1} \overset{1}{\hookrightarrow} \mathbb{I} \overset{0}{\hookleftarrow} \mathbb{1}$$

Under the simplification of Remark A.6.13, the formula for the weighted limit reduces to the equalizer

$$\mathrm{eq}\left(\begin{tikzcd} & A^{\mathbb{I}} \arrow[dr, "{(q_1,q_0)}"] & \\ C \times A^{\mathbb{I}} \times B \arrow[ur, "\pi"] \arrow[dr, "\pi"'] & & A \times A \\ & C \times B \arrow[ur, "g\times f"'] & \end{tikzcd}\right)$$

where the maps $(q_1, q_0)\colon A^{\mathbb{I}} \to A \times A$ are defined by restricting along the endpoint inclusion $\mathbb{1} + \mathbb{1} = \partial\mathbb{I} \hookrightarrow \mathbb{I}$. In an ∞-cosmos, this map is an isofibration and the equalizer defining the iso-comma object is computed by the pullback

$$\begin{tikzcd} C \underset{A}{\dot{\times}} B \arrow[r] \arrow[d, two heads, "{(q_1,q_0)}"'] \arrow[dr, phantom, "\lrcorner"] & A^{\mathbb{I}} \arrow[d, two heads, "{(q_1,q_0)}"] \\ C \times B \arrow[r, "g\times f"'] & A \times A \end{tikzcd} \tag{6.3.6}$$

6.3.7. Lemma. *Iso-comma objects are flexible weighted limits and in particular exist in any ∞-cosmos.*

Proof. Reprising the notation for the category $\lrcorner$ used in Example 6.3.4, the weight $W_{\dot{\times}}$ is constructed by the pushout

$$\begin{tikzcd} \partial\mathbb{I} \times \lrcorner(a,-) \arrow[r] \arrow[d, hook] & \lrcorner(b,-) \sqcup \lrcorner(c,-) \arrow[d, hook] \\ \mathbb{I} \times \lrcorner(a,-) \arrow[r] \arrow[ur, phantom, "\ulcorner"] & W_{\dot{\times}} \end{tikzcd}$$

where the attaching map picks out the two arrows in the cospan $\lrcorner$. As a projective cell complex, $W_{\dot{\times}}$ is built from a projective 0-cell at b, a projective 0-cell at c, and two projective k-cells at a for each $k > 0$, corresponding to the non-degenerate simplices of $\mathbb{I}$. As described by Remark 6.2.2, these may be attached all at once. In this way, we see that $W_{\dot{\times}}$ is a flexible weight, so Corollary 6.3.3 tells us that iso-comma objects exist in any ∞-cosmos, a fact that is also evident from the pullback (6.3.6). ☐

6.3.8. Remark. In the homotopy 2-category of an ∞-cosmos, there is a canonical invertible 2-cell defining the **iso-comma cone**:

$$\begin{tikzcd} & C \underset{A}{\dot{\times}} B \arrow[dl, "q_1"'] \arrow[dr, "q_0"] \arrow[dd, phantom, "\overset{\phi}{\cong}"] & \\ C \arrow[dr, "g"'] & & B \arrow[dl, "f"] \\ & A & \end{tikzcd}$$

that has a weak universal property analogous to that of the comma cone presented in Proposition 3.4.6. The proof, like the proof of that result, makes use of the fact that $A^{\mathbb{I}}$ is the weak $\mathbb{I}$-cotensor in the homotopy 2-category. The proof of this fact is somewhat delicate, making use of marked simplicial sets as appeared already in the proof of Corollary 1.1.16, which gives 1-cell induction.

Our notation for iso-commas is deliberately similar to the usual notation for pullbacks. In an ∞-cosmos, iso-commas can be used to compute "homotopy pullbacks" of diagrams in which neither map is an isofibration. When at least one map of the cospan is an isofibration, these constructions are equivalent.

6.3.9. Lemma (iso-commas and pullbacks). *In an ∞-cosmos $\mathcal{K}$, pullbacks and iso-commas of cospans in which at least one map is an isofibration are equivalent. More precisely, given a pullback square as below-left and an iso-comma square as below-right*

$$\begin{tikzcd} P \arrow[r, "b"] \arrow[d, two heads, "c"'] \arrow[dr, phantom, very near start, "\lrcorner"] & B \arrow[d, two heads, "f"] \\ C \arrow[r, "g"'] & A \end{tikzcd} \qquad\qquad \begin{tikzcd} C \underset{A}{\dot\times} B \arrow[r, two heads, "q_0"] \arrow[d, two heads, "q_1"'] \arrow[dr, phantom, "\phi_{\cong}"] & B \arrow[d, two heads, "f"] \\ C \arrow[r, "g"'] & A \end{tikzcd}$$

$P \simeq C \underset{A}{\dot\times} B$ over C and up to isomorphism over B.

Proof. Applying Lemma 1.2.16 to the functor $b\colon P \to B$, we can replace the span $(c,b)\colon P \to C\times B$ by a span $(cq,p)\colon Pb \twoheadrightarrow C\times B$ whose legs are both isofibrations that is related via an equivalence $s\colon P \xrightarrow{\sim} Pb$ that lies over C on the nose and over B up to isomorphism. We will show that under the hypothesis that f is an isofibration, this new span is equivalent to the iso-comma span.

To see this, note that the factorization constructed in (1.2.17) is in fact defined using an iso-comma, constructed via the pullback in the top square of the diagram below-left. Since the map b is itself defined by a pullback, the bottom square of the diagram below-left is also a pullback, defining the left-hand pullback rectangle:

$$\begin{tikzcd} Pb \arrow[r] \arrow[d, two heads, "{(q,p)}"'] \arrow[dr, phantom, very near start, "\lrcorner"] & B^{\mathbb{I}} \arrow[d, two heads, "{(q_1,q_0)}"] \\ P\times B \arrow[r, "b\times B"] \arrow[d, two heads, "c\times B"'] \arrow[dr, phantom, very near start, "\lrcorner"] & B\times B \arrow[d, two heads, "f\times B"] \\ C\times B \arrow[r, "g\times B"'] & A\times B \end{tikzcd} \qquad \begin{tikzcd} C \underset{A}{\dot\times} B \arrow[r] \arrow[d, two heads, "{(q_1,q_0)}"'] \arrow[dr, phantom, very near start, "\lrcorner"] & A \underset{A}{\dot\times} B \arrow[r, two heads] \arrow[d, two heads, "{(q_1,q_0)}"'] \arrow[dr, phantom, very near start, "\lrcorner"] & A^{\mathbb{I}} \arrow[d, two heads, "{(q_1,q_0)}"] \\ C\times B \arrow[r, "g\times B"'] & A\times B \arrow[r, two heads, "A\times f"'] & A\times A \end{tikzcd}$$

Now the iso-comma is constructed by a similar pullback rectangle, displayed above-right. And because f is an isofibration, Lemma 1.2.14 tells us that the Leibniz tensor $i_0 \mathbin{\widehat{\pitchfork}} f\colon B^{\mathbb{I}} \xrightarrow{\sim}\!\!\!\!\!\rightarrow A \underset{A}{\dot\times} B$ of $i_0\colon \mathbb{1} \hookrightarrow \mathbb{I}$ with $f\colon B \twoheadrightarrow A$ is a trivial fibration. This equivalence commutes with the projections to $A\times B$ and hence the maps $(cq,p)\colon Pb \twoheadrightarrow C\times B$ and $(q_1,q_0)\colon C \underset{A}{\dot\times} B \twoheadrightarrow C\times B$, defined as pullbacks of an equivalent pair of isofibrations along $g\times B$, are equivalent as claimed. $\square$

We now introduce a flexible weight diagrams given by a countable tower of 0-arrows whose cone shapes will have a homotopy coherent isomorphism in the triangle over each generating arrow in the diagram.

6.3.10. Definition (iso-towers). Recall the category $\boldsymbol{\omega}$ whose objects are natural numbers and whose morphisms are freely generated by maps $\iota_{n,n+1}\colon n \to n+1$ for each n.

The **iso-tower** of a diagram $F\colon \boldsymbol{\omega}^{\mathrm{op}} \to \mathcal{M}$ in a simplicially enriched and cotensored category $\mathcal{M}$ is the limit weighted by the diagram $W_{\leftarrow}\colon \boldsymbol{\omega}^{\mathrm{op}} \to s\mathcal{S}et$ defined by the pushout

$$\begin{tikzcd} \coprod_{n\in\boldsymbol{\omega}} \partial\mathbb{I}\times\boldsymbol{\omega}(-,n) \arrow[r, "(\mathrm{id}_n,\iota_{n,n+1})"] \arrow[d, hook] & \coprod_{n\in\boldsymbol{\omega}} \boldsymbol{\omega}(-,n) \arrow[d, hook] \\ \coprod_{n\in\boldsymbol{\omega}} \mathbb{I}\times\boldsymbol{\omega}(-,n) \arrow[r] \arrow[ur, phantom, "\ulcorner" very near end] & W_{\leftarrow} \end{tikzcd} \tag{6.3.11}$$

in $s\mathcal{S}et^{\boldsymbol{\omega}^{\mathrm{op}}}$.

By Definition A.6.3(ii), in an ∞-cosmos the iso-tower of a diagram

$$F \coloneq \qquad \cdots \xrightarrow{f_{n+2,n+1}} F_{n+1} \xrightarrow{f_{n+1,n}} F_n \xrightarrow{f_{n,n-1}} \cdots \xrightarrow{f_{2,1}} F_1 \xrightarrow{f_{1,0}} F_0$$

is constructed by the pullback

$$\begin{tikzcd} \lim_{W_{\leftarrow}} F \arrow[r, "\phi"] \arrow[d, two heads, "\rho"'] \arrow[dr, phantom, "\lrcorner" very near start] & \prod_{n\in\boldsymbol{\omega}} F_n^{\mathbb{I}} \arrow[d, two heads, "\Pi(q_1,q_0)"] \\ \prod_{n\in\boldsymbol{\omega}} F_n \arrow[r, "(f_{n+1,n},\mathrm{id}_{F_n})"'] & \prod_{n\in\boldsymbol{\omega}} F_n\times F_n \end{tikzcd} \tag{6.3.12}$$

The limit cone is generated by a 0-arrow $\rho_n\colon \lim_{W_{\leftarrow}} F \to F_n$ for each $n\in\boldsymbol{\omega}$ together with a homotopy coherent isomorphism ϕ_n in each triangle over a generating arrow $F_{n+1} \to F_n$ in the $\boldsymbol{\omega}^{\mathrm{op}}$-indexed diagram.

6.3.13. Lemma. *Iso-towers are flexible weighted limits and in particular exist in any ∞-cosmos.*

Proof. The weight $W_{\leftarrow}$ is a projective cell complex built by attaching one projective 0-cell at each $n \in \boldsymbol{\omega}$ — forming the coproduct appearing in the upper right-hand corner of (6.3.11) — and then by attaching a projective k-cell at each $n\in\boldsymbol{\omega}$ for each non-degenerate k-simplex of $\mathbb{I}$. Rather than attach each projective k-cell for fixed $n\in\boldsymbol{\omega}$ in sequence, by Remark 6.2.2 these can all be attached at once by taking a single pushout of the "generalized projective cell at n" defined by the map $\partial\mathbb{I}\times\boldsymbol{\omega}(-,n) \hookrightarrow \mathbb{I}\times\boldsymbol{\omega}(-,n)$. These are the maps appearing as the left-hand vertical of (6.3.11). Now Corollary 6.3.3 or the formula (6.3.12) make it clear that such objects exist in any ∞-cosmos. $\square$

6.3.14. Lemma (iso-towers and inverse limits). *In an ∞-cosmos $\mathcal{K}$, the inverse limit of a countable tower of isofibrations is equivalent to the iso-pullback of that tower.*

Proof. We will rearrange the formula (6.3.12) to construct the iso-tower $\lim_{W_{\leftarrow}} F$ as an inverse limit of a countable tower of isofibrations $P\colon \boldsymbol{\omega}^{\mathrm{op}} \to \mathcal{K}$ that is pointwise equivalent to the diagram $F\colon \boldsymbol{\omega}^{\mathrm{op}} \to \mathcal{K}$. In the case where the diagram F is also given by a tower of isofibrations

$$\begin{tikzcd} \lim P \arrow[d, dashed, "\wr"'] & \cong & \cdots \arrow[r, two heads, "p_{n+2,n+1}"] & P_{n+1} \arrow[r, two heads, "p_{n+1,n}"] \arrow[d, "\wr", "e_n"'] & P_n \arrow[r, two heads, "p_{n,n-1}"] \arrow[d, "\wr", "e_n"'] & \cdots \arrow[r, two heads, "p_{2,1}"] & P_1 \arrow[r, two heads, "p_{1,0}"] \arrow[d, "\wr", "e_1"'] & P_0 \arrow[d, "\wr", "e_0"'] \\ \lim F & \cong & \cdots \arrow[r, two heads, "f_{n+2,n+1}"'] & F_{n+1} \arrow[r, two heads, "f_{n+1,n}"'] & F_n \arrow[r, two heads, "f_{n,n-1}"'] & \cdots \arrow[r, two heads, "f_{2,1}"'] & F_1 \arrow[r, two heads, "f_{1,0}"'] & F_0 \end{tikzcd} \tag{6.3.15}$$

the equivalence invariance of the inverse limit of a diagram of isofibrations will imply that the limits $\lim_{W_{\leftarrow}} F \cong \lim P$ and $\lim F$ are equivalent as claimed.

The ∞-categories P_n will be defined as conical limits of truncated versions of the diagram (6.3.12). To start define $P_0 \coloneqq F_0$ and e_0 to be the identity, then define P_1, $p_{1,0}$, and e_1 via the pullback

$$\begin{tikzcd} P_1 \arrow[r, two heads] \arrow[rr, bend left, two heads, "p_{1,0}"] \arrow[d, two heads, "e_1"', "\wr"] \arrow[dr, phantom, very near start, "\lrcorner"] & F_0^{\mathbb{I}} \arrow[r, two heads, "\sim", "q_0"'] \arrow[d, two heads, "\wr"', "q_1"] & F_0 \\ F_1 \arrow[r, two heads, "f_{1,0}"'] & F_0 \end{tikzcd}$$

Note that $P_1 \cong F_1 \underset{F_0}{\dot{\times}} F_0$ computes the iso-comma objects of the cospan given by id_{F_0} and $f_{1,0}$.

Now define P_2, $p_{2,1}$, and e_2 using the composite pullback

$$\begin{tikzcd} P_2 \arrow[r, two heads] \arrow[rr, bend left, two heads, "p_{2,1}"] \arrow[d, two heads, "\wr"] \arrow[dr, phantom, very near start, "\lrcorner"] \arrow[dd, bend right=50, two heads, "e_2"', "\wr"] & \bullet \arrow[r, two heads, "\sim"] \arrow[d, two heads, "\wr"] \arrow[dr, phantom, very near start, "\lrcorner"] & P_1 \arrow[r, two heads] \arrow[d, two heads, "e_1"', "\wr"] \arrow[dr, phantom, very near start, "\lrcorner"] & F_0^{\mathbb{I}} \arrow[r, two heads, "\sim", "q_0"'] \arrow[d, two heads, "\wr"', "q_1"] & F_0 \\ \bullet \arrow[r, two heads] \arrow[d, two heads, "\wr"] \arrow[dr, phantom, very near start, "\lrcorner"] & F_1^{\mathbb{I}} \arrow[r, two heads, "\sim", "q_0"'] \arrow[d, two heads, "\wr"', "q_1"] & F_1 \arrow[r, two heads, "f_{1,0}"'] & F_0 \\ F_2 \arrow[r, two heads, "f_{2,1}"'] & F_1 \end{tikzcd}$$

Continuing inductively, P_n, $p_{n,n-1}$, and e_n are defined by appending the diagram

$$\begin{tikzcd} & F_{n-1}^{\mathbb{I}} \arrow[r, two heads, "\sim", "q_0"'] \arrow[d, two heads, "\wr"', "q_1"] & F_{n-1} \\ F_n \arrow[r, two heads, "f_{n,n-1}"'] & F_{n-1} \end{tikzcd}$$

to the limit cone defining P_{n-1} and taking the limit of this composite diagram.

There is one small problem with the construction just given: it defines a diagram (6.3.15) in which each square commutes up to isomorphism — the isomorphism encoded by the map $P_n \to F_{n-1}^{\mathbb{I}}$ — not on the nose. But because the maps $f_{n+1,n}$ are isofibrations this is no problem. The isomorphism inhabiting the square $e_0 p_{1,0} \cong f_{1,0} e_1$ can be lifted along $f_{1,0}$ to define a new map $e_1' \colon P_1 \xrightarrow{\sim} F_1$ isomorphic to e_1; as observed in the proof of Theorem 1.4.7 this e_1' is then also an equivalence, so we replace e_1 with e_1', and then continue inductively to lift away the isomorphisms in the square $e_1' p_{2,1} \cong f_{2,1} e_2$.

Since inverse limits of towers of isofibrations are equivalence-invariant, it follows that $\lim P \simeq \lim F$. By construction $\lim P \cong \lim_{W_{\leftarrow}} F$, so it follows that $\lim_{W_{\leftarrow}} F \simeq \lim F$, which is what we wanted to show. $\square$

Exercises.

6.3.i. EXERCISE ([**83**, 2.2.2]). Show that ∞-cosmoi admit **wide pullbacks**: limits of finite or countable diagrams of the following form

$$\begin{tikzcd}[column sep=tiny] \cdots & A_n \arrow[dr, "p_{n-1}"'] & & A_{n-1} \arrow[dl, "f_{n-1}"] \arrow[dr, "p_{n-2}"'] & & \cdots \arrow[dl, "f_{n-2}"] \arrow[dr, "p_1"'] & & A_1 \arrow[dl, "f_1"] \arrow[dr, "p_0"'] & & A_0 \arrow[dl, "f_0"] \\ \cdots & & B_{n-1} & & B_{n-2} & \cdots & B_1 & & B_0 \end{tikzcd}$$

and that their construction is invariant under pointwise equivalence between diagrams.

6.4. Replete sub ∞-cosmoi

In this section we generalize Lemma 6.1.4 to subcategories of ∞-cosmoi that are not full. As we shall discover, there are many interesting examples.

For any ∞-cosmos $\mathcal{K}$ and any subcategory of its underlying 1-category — that is for any subset of its objects and subcategory of its 0-arrows — one can form a quasi-categorically enriched subcategory $\mathcal{L} \subset \mathcal{K}$ that contains exactly those objects and 0-arrows and all higher dimensional arrows that they span. We call such subcategories $\mathcal{L}$ **full on positive-dimensional arrows**; note the functor spaces of $\mathcal{L}$ are quasi-categories because all inner horn inclusions are bijective on vertices. We will take particular interest in subcategories that satisfy a further "repleteness" condition.

6.4.1. DEFINITION. Let $\mathcal{K}$ be an ∞-cosmos. A subcategory $\mathcal{L} \subset \mathcal{K}$ is **replete** in $\mathcal{K}$ if it is full on positive-dimensional arrows and moreover:

(i) Every ∞-category in $\mathcal{K}$ that is equivalent to an object in $\mathcal{L}$ lies in $\mathcal{L}$.
(ii) Any equivalence in $\mathcal{K}$ between objects in $\mathcal{L}$ lies in $\mathcal{L}$.
(iii) Any 0-arrow in $\mathcal{K}$ that is isomorphic in $\mathcal{K}$ to an 0-arrow in $\mathcal{L}$ lies in $\mathcal{L}$.

6.4.2. LEMMA. *Suppose $\mathcal{L} \subset \mathcal{K}$ is a replete subcategory of an ∞-cosmos. Then any map $p\colon E \to B$ in $\mathcal{L}$ that defines an isofibration in $\mathcal{K}$ is a representably-defined isofibration in $\mathcal{L}$: that is for all $X \in \mathcal{L}$, $p_*\colon \mathsf{Fun}_{\mathcal{L}}(X,E) \twoheadrightarrow \mathsf{Fun}_{\mathcal{L}}(X,B)$ is an isofibration of quasi-categories.*

PROOF. Since $\mathcal{K}$ is an ∞-cosmos, axiom 1.2.1(ii) requires that $p_*\colon \mathsf{Fun}_{\mathcal{K}}(X,E) \twoheadrightarrow \mathsf{Fun}_{\mathcal{K}}(X,B)$ is an isofibration of quasi-categories. Because the inner horn inclusions are bijective on vertices and $\mathsf{Fun}_{\mathcal{L}}(X,E) \hookrightarrow \mathsf{Fun}_{\mathcal{K}}(X,E)$ is full on positive-dimensional arrows, it follows immediately that the restricted map $p_*\colon \mathsf{Fun}_{\mathcal{L}}(X,E) \twoheadrightarrow \mathsf{Fun}_{\mathcal{L}}(X,B)$ lifts against the inner horn inclusions. Thus it remains only to solve lifting problems of the form displayed below-left

$$\begin{tikzcd} \mathbb{1} \arrow[r, "e"] \arrow[d, hook] & \mathsf{Fun}_{\mathcal{L}}(X,E) \arrow[r, hook] \arrow[d, two heads, "p_*"'] & \mathsf{Fun}_{\mathcal{K}}(X,E) \arrow[d, two heads, "p_*"] \\ \mathbb{I} \arrow[r, "\beta"'] \arrow[urr, dashed] & \mathsf{Fun}_{\mathcal{L}}(X,B) \arrow[r, hook] & \mathsf{Fun}_{\mathcal{K}}(X,B) \end{tikzcd}$$

The lifting problem defines a 0-arrow $e\colon X \to E$ in $\mathcal{L}$ and an isomorphism $\beta\colon b \cong pe$ in $\mathcal{L}$. Its solution in $\mathcal{K}$ defines a 0-arrow $e'\colon X \to E$ in $\mathcal{K}$ so that $pe' = b$ together with an isomorphism $e \cong e'$ in $\mathcal{K}$. By fullness on positive-dimensional arrows, to show that this lift factors through the inclusion $\mathsf{Fun}_{\mathcal{L}}(X,E) \hookrightarrow \mathsf{Fun}_{\mathcal{K}}(X,E)$, we need only argue that the map e' lies in $\mathcal{L}$, but this is the case by condition (iii) of Definition 6.4.1. $\square$

The following result describes a condition under which a replete subcategory $\mathcal{L} \subset \mathcal{K}$ inherits an ∞-cosmos structure created from $\mathcal{K}$.

6.4.3. Proposition. *Suppose $\mathcal{L} \subset \mathcal{K}$ is a replete subcategory of an ∞-cosmos. If $\mathcal{L}$ is closed under flexible weighted limits in $\mathcal{K}$, then $\mathcal{L}$ defines an ∞-cosmos with isofibrations, equivalences, trivial fibrations and simplicial limits created by the inclusion $\mathcal{L} \hookrightarrow \mathcal{K}$, which then defines a cosmological functor.*

When these conditions hold, we refer to $\mathcal{L}$ as a **replete sub ∞-cosmos** of $\mathcal{K}$ and $\mathcal{L} \hookrightarrow \mathcal{K}$ as a **cosmological embedding**.

Proof. To say that a replete subcategory $\mathcal{L} \hookrightarrow \mathcal{K}$ is closed under flexible weighted limits means that for any diagram in $\mathcal{L}$ and any limit cone in $\mathcal{K}$ over that diagram, then the limit cone lies in $\mathcal{L}$ and satisfies appropriate simplicially-enriched universal property of Definition A.6.7 in the subcategory $\mathcal{L}$. We must verify that each of the limits of axiom 1.2.1(i) exist in $\mathcal{L}$. Immediately, $\mathcal{L}$ has a terminal object, products, and simplicial cotensors, since all of these are flexible weighted limits. By Lemmas 6.3.7 and 6.3.13, $\mathcal{L}$ also admits the construction of iso-comma objects and of iso-towers.

Define the class of isofibrations in $\mathcal{L}$ to be those maps in $\mathcal{L}$ that define isofibrations in $\mathcal{K}$. By Lemmas 6.3.9 and 6.3.14, pullbacks and limits of towers of isofibrations are equivalent in $\mathcal{K}$ to the iso-commas and iso-towers formed over the same diagrams. Since these latter limit cones lie in $\mathcal{L}$ by hypothesis, so do the equivalent former cones by repleteness of $\mathcal{L}$ in $\mathcal{K}$.

There is a little more still to verify: namely that pullbacks and limits of towers of isofibrations satisfy the simplicially-enriched universal property as conical limits in $\mathcal{L}$. In the case of a pullback diagram

$$\begin{tikzcd} P \arrow[r, "b"] \arrow[d, two heads, "c"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, two heads, "f"] \\ C \arrow[r, "g"'] & A \end{tikzcd}$$

in $\mathcal{L}$ we must show that for each $X \in \mathcal{L}$, the functor-space $\mathsf{Fun}_{\mathcal{L}}(X, P)$ is isomorphic to the pullback $\mathsf{Fun}_{\mathcal{L}}(X, C) \times_{\mathsf{Fun}_{\mathcal{L}}(X,A)} \mathsf{Fun}_{\mathcal{L}}(X, B)$ of functor spaces. We have such an isomorphism for functor spaces in $\mathcal{K}$ and on account of the commutative diagram

$$\begin{tikzcd} \mathsf{Fun}_{\mathcal{L}}(X,P) \arrow[r, dashed] \arrow[d, hook] & \mathsf{Fun}_{\mathcal{L}}(X,C) \underset{\mathsf{Fun}_{\mathcal{L}}(X,A)}{\times} \mathsf{Fun}_{\mathcal{L}}(X,B) \arrow[d, hook] \\ \mathsf{Fun}_{\mathcal{K}}(X,P) \arrow[r, "\cong"] & \mathsf{Fun}_{\mathcal{K}}(X,C) \underset{\mathsf{Fun}_{\mathcal{K}}(X,A)}{\times} \mathsf{Fun}_{\mathcal{K}}(X,B) \end{tikzcd}$$

and fullness on positive-dimensional arrows, we need only verify surjectivity of the dotted map on 0-arrows. So consider a cone $(h \colon X \to B, k \colon X \to C)$ over the pullback diagram in $\mathcal{L}$. By the universal property of the isocomma $C \underset{A}{\dot\times} B$, there exists a factorization $y \colon X \to C \underset{A}{\dot\times} B$ in $\mathcal{L}$. Composing with the equivalence $C \underset{A}{\dot\times} B \simeq P$, this map is equivalent to the factorization $z \colon X \to P$ of the cone (h, k) through the limit cone (b, c) in $\mathcal{K}$ that exists on account of the strict universal property of the pullback in there. By repleteness, the isomorphism between z and the composite of y with the equivalence suffices to show that z lies in $\mathcal{L}$. Hence, the functor spaces in $\mathcal{L}$ are isomorphic. A similar argument invoking Lemma 6.3.14 proves that inverse limits of towers of isofibrations define conical limits in $\mathcal{L}$. This completes the proof of the limit axiom 1.2.1(i).

Since the isofibrations in $\mathcal{L}$ are a subset of the isofibrations in $\mathcal{K}$ and the limit constructions in both contexts coincide, most of the closure properties of 1.2.1(ii) are inherited from the closure properties in $\mathcal{K}$. The one exception is the requirement that the isofibrations in $\mathcal{L}$ define isofibrations of quasi-categories representably, which was proven for any replete subcategory in Lemma 6.4.2. This proves that $\mathcal{L}$ defines an ∞-cosmos.

Finally, we argue that the equivalences in $\mathcal{L}$ coincide with those of $\mathcal{K}$, which will imply that the trivial fibrations in $\mathcal{L}$ coincide with those of $\mathcal{K}$ as well. Condition (ii) of Definition 6.4.1 implies that for any arrow in $\mathcal{L}$ that defines an equivalence in $\mathcal{K}$, its equivalence inverse and witnessing homotopies of Lemma 1.2.18 lie in $\mathcal{L}$. Because we have already shown that $\mathcal{L}$ admits cotensors with $\mathbb{I}$ preserved by the inclusion $\mathcal{L} \hookrightarrow \mathcal{K}$, Lemma 1.2.18 implies that this data defines an equivalence in $\mathcal{L}$. Conversely, any equivalence in $\mathcal{L}$ extends to the data of (1.2.19) and since $\mathcal{L} \hookrightarrow \mathcal{K}$ preserves $\mathbb{I}$-cotensors, this data defines an equivalence in $\mathcal{K}$. Thus, by construction, the ∞-cosmos structure of $\mathcal{L}$ is preserved and reflected by the inclusion $\mathcal{L} \hookrightarrow \mathcal{K}$ as claimed. □

In practice the repleteness condition of Definition 6.4.1 is satisfied by any subcategory of objects and 0-arrows that is determined by some ∞-categorical property, so the main task in verifying that a subcategory defines an ∞-cosmos is verifying the closure under flexible weighted limits.

6.4.4. Proposition. *For any ∞-cosmos $\mathcal{K}$, let $\mathcal{K}_\top$ denote the quasi-categorically enriched category whose*

(i) objects are ∞-categories in $\mathcal{K}$ that possess a terminal object and

(ii) functor spaces $\mathsf{Fun}_\top(A,B) \subset \mathsf{Fun}(A,B)$ are the sub-quasi-categories whose 0-arrows preserve terminal objects and containing all n-arrows they span.

Then the inclusion $\mathcal{K}_\top \hookrightarrow \mathcal{K}$ creates an ∞-cosmos structure on $\mathcal{K}_\top$ from $\mathcal{K}$, and moreover for each object of $\mathcal{K}_\top$ defined as a flexible weighted limit of some diagram in $\mathcal{K}_\top$, its terminal element is created by the 0-arrow legs of the limit cone.

Proof. We apply Proposition 6.4.3. Lemma 2.2.7 and Proposition 2.1.10 verify the repleteness condition, so it remains only to prove closure under flexible weighted limits, which we do by induction over the tower of isofibrations constructed in Proposition 6.3.1(i), which expresses a flexible weighted limit $\lim_W F$ as the inverse limit of a tower of isofibrations

$$\lim_W F \longrightarrow \cdots \twoheadrightarrow \lim_{W_{k+1}} F \twoheadrightarrow \lim_{W_k} F \twoheadrightarrow \cdots \twoheadrightarrow \lim_{W_0} F \twoheadrightarrow 1$$

each of which is a pullback of products of maps of the form (6.3.2) indexed by the projective cells of the flexible weight W. We'll argue inductively that each ∞-category in this tower possesses a terminal element that's created by the legs of the tower of isofibrations.

For the base case, note that if $(A_i)_{i\in I}$ is a family of ∞-categories possessing terminal elements $t_i \colon 1 \to A_i$, then the product of the adjunctions $! \dashv t_i$ defines an adjunction

$$\begin{tikzcd} 1 \cong \prod_{i\in I} 1 \arrow[r, bend right, "(t_i)_{i\in I} \cong \prod_i t_i"'] \arrow[r, phantom, "\perp"] & \prod_{i\in I} A_i \arrow[l, bend right, "!"'] \end{tikzcd}$$

exhibiting $(t_i)_{i\in I}$ as a terminal element of $\prod_{i\in I} A_i$. By construction, this terminal element is jointly created by the legs of the limit cone. Note that by construction the product-projection functors preserve this terminal element and the map into the product ∞-category $\prod_{i\in I} A_i$ induced by any family of terminal element preserving functors $(f \colon X \to A_i)_{i\in I}$ will preserve terminal elements. This verifies that the subcategory $\mathcal{K}_\top$ is closed under products.

For the inductive step consider a pullback diagram

$$\begin{tikzcd} \lim_{W_{k+1}} F \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner" very near start] & A^{\Delta[n]} \arrow[d, two heads] \\ \lim_{W_k} F \arrow[r, "\ell"'] & A^{\partial\Delta[n]} \end{tikzcd}$$

that arises from the attaching map for a projective n-cell. The inductive hypothesis tells us that $\lim_{W_k} F$ admits a terminal element t_k and for each vertex of $i \in \partial\Delta[n]$, the corresponding component $\ell_i \colon \lim_{W_k} F \to A$ of the limit cone preserves it. Since F is a diagram valued in $\mathcal{K}_\top$ and A is an ∞-category in its image, we know that A must possess a terminal element $t \colon 1 \to A$. By Proposition 2.1.7(iii), the constant diagram at t then defines a terminal element in $A^{\partial\Delta[n]}$ and $A^{\Delta[n]}$, which we also denote by t. By terminality, there is a 1-arrow $\alpha \colon \ell(t_k) \to t \in A^{\partial\Delta[n]}$ whose components at each $i \in \partial\Delta[n]$ are isomorphisms in A. By Lemma 12.2.1, it follows that α is also an isomorphism, which tells us that $\ell(t_k)$ is also a terminal element of $A^{\partial\Delta[n]}$. The same argument demonstrates that terminal elements in simplicial cotensors, in this case by $\partial\Delta[n]$ are jointly created by the 0-arrow components of the limit cone, namely by evaluation on each of the vertices of the cotensoring simplicial set. The proof is now completed by the following lemma: □

6.4.5. Lemma. *Consider a pullback diagram*

$$\begin{tikzcd} F \arrow[r, "g"] \arrow[d, two heads, "q"'] \arrow[dr, phantom, "\lrcorner" very near start] & E \arrow[d, two heads, "p"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

in which the ∞-categories A, B, and E possess a terminal element and the functors f and p preserve them. Then F possesses a terminal element that is created by the legs of the pullback cone q and g.

Proof. If $e \colon 1 \to E$ and $a \colon 1 \to A$ are terminal, then this implies that $f(a) \cong p(e) \in B$. Using the fact that p is an isofibration, there is a lift $e' \cong e$ of this isomorphism along p that then defines another terminal element of E. The pair (a, e') now induces an element t of F that we claim is terminal.

To see this we'll apply Proposition 4.3.10, which proves that t is a terminal element of F if and only if the domain-projection functor $p_0 \colon \mathrm{Hom}_F(F, t) \twoheadrightarrow F$ is a trivial fibration. By construction of t, we know that the domain-projection functors for the elements gt, qt, and $pgt = fqt$ are all trivial fibrations and moreover the top and bottom faces of the cube

$$\begin{tikzcd} & \mathrm{Hom}_F(F,t) \arrow[rr] \arrow[dl, two heads] \arrow[dd, two heads, "p_0"'] \arrow[drr, phantom, "\lrcorner" very near start] & & \mathrm{Hom}_E(E,gt) \arrow[dl, two heads] \arrow[dd, two heads, "\wr"', "p_0"] \\ \mathrm{Hom}_A(A,qt) \arrow[rr] \arrow[dd, two heads, "p_0"', "\wr"] & & \mathrm{Hom}_B(B,fqt) \arrow[dd, two heads, "\wr"', "p_0" near start] & \\ & F \arrow[rr, "g" near start] \arrow[dl, two heads, "q"'] \arrow[drr, phantom, "\lrcorner" very near start] & & E \arrow[dl, two heads, "p"] \\ A \arrow[rr, "f"'] & & B & \end{tikzcd}$$

are pullbacks. Since homotopy pullbacks are homotopical, the fact that the three maps between the cospans are equivalences implies that the map between their pullbacks is also an equivalence, as required. □

Applying the result of Proposition 6.4.4 to $\mathcal{K}^{\mathrm{co}}$ constructs an ∞-cosmos $\mathcal{K}_\perp$ whose objects are ∞-categories in $\mathcal{K}$ that possess an initial object and 0-arrows are initial-element-preserving functors. Combining these, we get an ∞-cosmos for the pointed ∞-categories of Definition 4.4.1, those that possess a zero element.

6.4.6. Proposition. *For any ∞-cosmos $\mathcal{K}$, let $\mathcal{K}_*$ denote the quasi-categorically enriched category of pointed ∞-categories, i.e., ∞-categories that possess a zero element, and functors that preserve them. Then the inclusion $\mathcal{K}_* \hookrightarrow \mathcal{K}$ creates an ∞-cosmos structure on $\mathcal{K}_*$ from $\mathcal{K}$.*

Proof. This result follows directly from Proposition 6.4.4, since the ∞-cosmos $\mathcal{K}_*$ of ∞-categories in $\mathcal{K}$ possessing a zero object is isomorphic to $(\mathcal{K}_\top)_\perp \cong (\mathcal{K}_\perp)_\top$, the idea being that an initial element $a\colon 1 \to A$ in $\mathcal{K}_\top$ is encoded by a terminal-object preserving functor, which says exactly that a is a zero element. □

Applying the result of Proposition 6.4.4 or its dual to the ∞-cosmos $\mathcal{K}_{/B}$ of isofibrations over $B \in \mathcal{K}$, we obtain two new ∞-cosmoi of interest.

6.4.7. Corollary. *For any ∞-category B in an ∞-cosmos $\mathcal{K}$, the sliced ∞-cosmos $\mathcal{K}_{/B}$ admits sub ∞-cosmoi*

$$\mathcal{R}ari(\mathcal{K})_{/B} \hookrightarrow \mathcal{K}_{/B} \hookleftarrow \mathcal{L}ari(\mathcal{K})_{/B}$$

whose

- *objects are isofibrations over B admitting a right adjoint right inverse or left adjoint right inverse, respectively, and*
- *0-arrows are functors over B that commute with the respective right or left adjoints up to fibered isomorphism*

with the ∞-cosmos structures created by the inclusions.

Proof. These ∞-cosmoi are defined by $\mathcal{R}ari(\mathcal{K})_{/B} := (\mathcal{K}_{/B})_\top$ and $\mathcal{L}ari(\mathcal{K})_{/B} := (\mathcal{K}_{/B})_\perp$. □

Leveraging Corollary 6.4.7, we can establish similar cosmological embeddings

$$\mathcal{R}ari(\mathcal{K}) \hookrightarrow \mathcal{K}^{\downarrow} \hookleftarrow \mathcal{L}ari(\mathcal{K})$$

The quasi-categorically enriched subcategories $\mathcal{R}ari(\mathcal{K})$ and $\mathcal{L}ari(\mathcal{K})$ are replete in $\mathcal{K}^{\downarrow}$, so by Proposition 6.4.3 we need only check closure under flexible weighted limits. We argue separately for cotensors, which are easy, and for the conical limits, which are harder. For this, we make use of a general 1-categorical result making use of the fact that the codomain-projection functor $\mathrm{cod}\colon \mathcal{R}ari(\mathcal{K}) \to \mathcal{K}$ is a Grothendieck fibration of underlying 1-categories, defined by restricting the Grothendieck fibration $\mathrm{cod}\colon \mathcal{K}^{\downarrow} \to \mathcal{K}$.

6.4.8. Lemma. *Let $P\colon \mathcal{E} \to \mathcal{B}$ be a Grothendieck fibration between 1-categories. Suppose that $\mathcal{J}$ is a small category, that $D\colon \mathcal{J} \to \mathcal{E}$ is a diagram, and that*

(i) *the diagram $PD\colon \mathcal{J} \to \mathcal{B}$ has a limit L in $\mathcal{B}$ with limit cone $\lambda\colon \Delta L \Rightarrow PD$,* *(ii) the diagram $\lambda^* D \colon \mathcal{J} \to \mathcal{E}_L$*

$$\begin{tikzcd} \mathcal{J} \arrow[r, "D"] \arrow[dr, "\Delta L"'] & \mathcal{E} \arrow[d, "P"] \arrow[dl, phantom, "\Uparrow\lambda", near start] \\ & \mathcal{B} \end{tikzcd} = \begin{tikzcd} \mathcal{J} \arrow[r, bend left, "D"] \arrow[r, bend right, "\lambda^* D"'] \arrow[r, phantom, "\Uparrow\chi"] & \mathcal{E} \arrow[d, "P"] \\ & \mathcal{B} \end{tikzcd}$$

constructed by lifting the cone λ to a cartesian natural transformation $\chi \colon \lambda^* D \Rightarrow D$ has a limit M in the fibre $\mathcal{E}_L$ with limiting cone $\mu \colon \Delta M \Rightarrow \lambda^* D$, and

(iii) the limit $\mu \colon \Delta M \Rightarrow \lambda^* D$ is preserved by the re-indexing functor $u^* \colon \mathcal{E}_L \to \mathcal{E}_B$ associated with any arrow $u \colon B \to L$ in $\mathcal{B}$.

Then the composite cone

$$\Delta M \xRightarrow{\mu} \lambda^* D \xRightarrow{\chi} D$$

displays M as a limit of the diagram D in $\mathcal{E}$.

Proof. Any arrow $f \colon E \to E'$ in the domain of a Grothendieck fibration $P \colon \mathcal{E} \to \mathcal{B}$ factors uniquely up to isomorphism through a "vertical" arrow in the fiber $\mathcal{E}_{PE}$ followed by a "horizontal" cartesian lift of Pf with codomain E'.

Given a cone $\alpha \colon \Delta E \Rightarrow D$ with summit $E \in \mathcal{E}$ over D, by (i) its image $P\alpha \colon \Delta PE \Rightarrow PD$ factors uniquely through the limit cone $\lambda \colon \Delta L \Rightarrow D$ via a map $b \colon PE \to L \in \mathcal{B}$. By the universal property of the cartesian lift χ of λ constructed in (ii), it follows that α factors uniquely through χ via a natural transformation $\beta \colon \Delta E \Rightarrow \lambda^* D$ so that $P\beta = \Delta b$. This arrow factors uniquely up to isomorphism via "vertical" natural transformation $\gamma \colon \Delta E \to \alpha^* D \cong b^* \gamma^* D$ followed by a "horizontal" cartesian lift of b. By (iii), the limit cone $\mu \colon \Delta M \Rightarrow \lambda^* D$ in $\mathcal{E}_L$ pulls back along b to a limit cone in $\mathcal{E}_{PE}$ through which the pullback of β factors via a map $k \colon E \to b^* M$. Thus, β itself factors uniquely through μ via the composite of this map $k \colon E \to b^* M$ with the cartesian arrow $b^* M \to M$ lifting $b \colon PE \to L$. □

6.4.9. Proposition. *For any ∞-cosmos $\mathcal{K}$, the ∞-cosmos of isofibrations admits sub ∞-cosmoi*

$$\mathscr{R}ari(\mathcal{K}) \hookrightarrow \mathcal{K}^{\twoheaddownarrow} \hookleftarrow \mathscr{L}ari(\mathcal{K})$$

whose

- *objects are isofibrations admitting a right adjoint right inverse or left adjoint right inverse, respectively, and*
- *0-arrows are commutative squares between the right or left adjoints, respectively, whose mates are isomorphisms*

with the ∞-cosmos structures created by the inclusions.

We refer to a commutative square between right adjoints whose mate is an isomorphism as an **exact square**.

Proof. The quasi-categorically enriched subcategories $\mathscr{R}ari(\mathcal{K})$ and $\mathscr{L}ari(\mathcal{K})$ are replete in $\mathcal{K}^{\twoheaddownarrow}$, so by Proposition 6.4.3 we need only check that $\mathscr{R}ari(\mathcal{K}) \hookrightarrow \mathcal{K}^{\twoheaddownarrow}$ is closed under flexible weighted limits; the argument for $\mathscr{L}ari(\mathcal{K}) \hookrightarrow \mathcal{K}^{\twoheaddownarrow}$ is dual. We argue separately for cotensors and for the conical limits.

If $p \colon E \twoheadrightarrow B$ is an isofibration admitting a right adjoint right inverse in $\mathcal{K}$ and U is a simplicial set, then the cosmological functor $(-)^U \colon \mathcal{K} \to \mathcal{K}$ carries this data to a right adjoint right inverse to $p^U \colon E^U \twoheadrightarrow B^U$, which proves that the simplicial cotensor in $\mathcal{K}^{\twoheaddownarrow}$ of an object in $\mathscr{R}ari(\mathcal{K})$ lies in $\mathscr{R}ari(\mathcal{K})$. The limit cone for the cotensor is given by the canonical map of simplicial sets $U \to \mathsf{Fun}(E^U \xrightarrow{p^U} B^U, E \xrightarrow{p} B)$ defined on each vertex $u \colon \mathbb{1} \to U$ by the commutative square

$$\begin{tikzcd} E^U \arrow[r, "u^*"] \arrow[d, two heads, "p^U"'] & E \arrow[d, two heads, "p"] \\ B^U \arrow[r, "u^*"'] & B \end{tikzcd} \tag{6.4.10}$$

The maps u^* define the components of a simplicial natural transformation from $(-)^U$ to the identity functor and thus the mate of this commutative square is an identity, so the limit cone for the U-cotensor lies in $\mathscr{R}ari(\mathcal{K})$. Finally, to verify the universal property of the cotensor in $\mathscr{R}ari(\mathcal{K})$, we must show that for any commutative square whose domain is an isofibration admitting a right adjoint right inverse

$$\begin{tikzcd} F \arrow[r] \arrow[d, two heads, "q"] & E^U \arrow[d, two heads, "p^U"'] \\ A \arrow[r] & B^U \end{tikzcd}$$

that composes with each of the squares (6.4.10) to an exact square is itself exact. To see this take the mate to define a 1-arrow in $\mathsf{Fun}(A, E^U) \cong \mathsf{Fun}(A, E)^U$ and note that the hypothesis says that the components of this 1-arrow are invertible for each vertex of U. Lemma 12.2.1 then tells us that this 1-arrow is invertible as required.

Taking U to be a set, the argument just given proves also that $\mathscr{R}ari(\mathcal{K})$ is closed in $\mathcal{K}^{\twoheaddownarrow}$ under products. It remains only to show that it is closed under the remaining conical limits. By pullback stability of fibered adjunctions, the Grothendieck fibration of 1-categories $\mathrm{cod} \colon \mathcal{K}^{\twoheaddownarrow} \to \mathcal{K}$ restricts to $\mathrm{cod} \colon \mathscr{R}ari(\mathcal{K}) \to \mathcal{K}$, so we may appeal to Lemma 6.4.8 to calculate 1-categorical limit cones in $\mathscr{R}ari(\mathcal{K}) \subset \mathcal{K}^{\twoheaddownarrow}$ as composites of cartesian cells with limit cones of fiberwise diagrams. By Corollary 6.4.7, these fiberwise limits in $\mathcal{K}_{/B}$ of diagrams in $\mathscr{R}ari(\mathcal{K})_{/B}$ lie in $\mathscr{R}ari(\mathcal{K})_{/B} \hookrightarrow \mathscr{R}ari(\mathcal{K})$. Moreover, these 1-categorical limits are preserved by the simplicial cotensor, which by Proposition A.5.6 implies that their universal property enriches to define conical limits. In this way we see that $\mathscr{R}ari(\mathcal{K}) \hookrightarrow \mathcal{K}^{\twoheaddownarrow}$ is closed under flexible weighted limits and thus defines a cosmological embedding, as claimed. ☐

Proposition 6.4.9 allows us to construct further ∞-cosmoi of interest.

6.4.11. Proposition. *For any ∞-cosmos $\mathcal{K}$ and simplicial set J, there exist sub ∞-cosmoi*

$$\mathcal{K}_{\top,J} \hookrightarrow \mathcal{K} \hookleftarrow \mathcal{K}_{\perp,J}$$

whose

- *objects are ∞-categories in $\mathcal{K}$ that admit all limits of shape J or all colimits of shape J, respectively,*
- *0-arrows are the functors that preserve them*

with the ∞-cosmos structures created by the inclusions. Moreover for each object of $\mathcal{K}_{\top,J}$ or $\mathcal{K}_{\perp,J}$ defined as a flexible weighted limit of some diagram in that ∞-cosmos, its J-shaped limits or colimits are created by the 0-arrow legs of the limit or colimit cones respectively.

Proof. First note that the quasi-categorically enriched subcategories $\mathcal{K}_{\top,J}$ and $\mathcal{K}_{\perp,J}$ are replete in $\mathcal{K}$, so by Proposition 6.4.3 we need only confirm that the inclusions are closed under flexible weighted limits. We prove this in the case of colimits, the other case being dual.

For any fixed simplicial set J, there is a cosmological functor $F_J\colon \mathcal{K} \to \mathcal{K}^{\twoheaddownarrow}$ defined on objects by mapping an ∞-category A to the isofibration $A^{J^{\triangleright}} \twoheadrightarrow A^J$ in the notation of 4.2.6 and a functor $f\colon A \to B$ to the commutative square

$$\begin{tikzcd} A^{J^{\triangleright}} \arrow[r, "f^{J^{\triangleright}}"] \arrow[d, two heads] & B^{J^{\triangleright}} \arrow[d, two heads] \\ A^J \arrow[r, "f^J"'] & B^J \end{tikzcd}$$

By Corollary 4.3.5, A admits colimits of shape J if and only if this isofibration admits a left adjoint right inverse, and now it is clear that $f\colon A \to B$ preserves these colimits if and only if the square displayed above is exact. In summary, the quasi-categorically enriched subcategory $\mathcal{K}_{\perp,J}$ is defined by the pullback

$$\begin{tikzcd} \mathcal{K}_{\perp,J} \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{L}ari(\mathcal{K}) \arrow[d, hook] \\ \mathcal{K} \arrow[r, "F_J"] & \mathcal{K}^{\twoheaddownarrow} \end{tikzcd}$$

Proposition 6.4.9 proves that $\mathcal{L}ari(\mathcal{K}) \hookrightarrow \mathcal{K}^{\twoheaddownarrow}$ is closed under flexible weighted limits and $F_J\colon \mathcal{K} \to \mathcal{K}^{\twoheaddownarrow}$ preserves them, so by a standard argument $\mathcal{K}_{\perp,J}$ is closed in $\mathcal{K}$ under flexible weighted limits: consider a diagram $D\colon \mathcal{A} \to \mathcal{K}_{\perp,J}$ and a flexible weight and form its flexible weighted limit in $\mathcal{L}ari(\mathcal{K})$ and in $\mathcal{K}$. The functors to $\mathcal{K}^{\twoheaddownarrow}$ carry these to a pair of equivalent cones over the same diagram and since the inclusion $\mathcal{L}ari(\mathcal{K}) \hookrightarrow \mathcal{K}^{\twoheaddownarrow}$ is replete, there exists a limit cone over D in $\mathcal{L}ari(\mathcal{K})$ whose image in $\mathcal{K}^{\twoheaddownarrow}$ is equal to the image of the limit cone in $\mathcal{K}$ under the functor F_J. Now the universal property of the pullback allows us to lift this cone to $\mathcal{K}_{\perp,J}$, and a similar argument demonstrates that the lifted cone is a flexibly weighted limit cone over the original diagram.

Now Proposition 6.4.3 proves that the inclusion $\mathcal{K}_{\perp,J} \hookrightarrow \mathcal{K}$ creates an ∞-cosmos structure. In particular, the closure of the subcategory $\mathcal{K}_{\perp,J}$ under flexible weighted limits in $\mathcal{K}$ implies that J-shaped colimits in an ∞-category defined as a flexible weighted limit are created by the 0-arrow legs of the limit cone, as we explain. Certainly the colimits in an ∞-category in $\mathcal{K}_{\perp,J}$, formed as a weighted limit of a diagram of ∞-categories in $\mathcal{K}_{\perp,J}$, are preserved by the 0-arrow legs of the weighted limit cone, since the 0-arrows in $\mathcal{K}_{\perp,J}$ are J-shaped-colimit-preserving functors. And since the J-colimit completeness of an ∞-category that is defined as the flexible weighted limit in $\mathcal{K}$ can be deduced whenever that diagram lies in the sub ∞-cosmos $\mathcal{K}_{\perp,J}$, these J-colimits are also created. □

6.4.12. Proposition. *The ∞-cosmos of isofibrations admits sub ∞-cosmoi*

$$\mathcal{C}art(\mathcal{K}) \hookrightarrow \mathcal{K}^{\twoheaddownarrow} \hookleftarrow co\mathcal{C}art(\mathcal{K})$$

whose objects are cartesian or cocartesian fibrations, respectively, and whose 0-arrows are cartesian functors, with the ∞-cosmos structures created by the inclusions. Similarly, for any ∞-category B in an ∞-cosmos $\mathcal{K}$, *the sliced ∞-cosmos $\mathcal{K}_{/B}$ admits sub ∞-cosmoi*

$$\mathcal{C}art(\mathcal{K})_{/B} \hookrightarrow \mathcal{K}_{/B} \hookleftarrow co\mathcal{C}art(\mathcal{K})_{/B}$$

whose objects are cartesian or cocartesian fibrations over B, respectively, and whose 0-arrows are cartesian functors, with the ∞-cosmos structures created by the inclusions.

Proof. The quasi-categorically enriched subcategories $\mathcal{C}art(\mathcal{K})$ and $co\mathcal{C}art(\mathcal{K})$ are replete in $\mathcal{K}^{\downarrow}$ by Corollary 5.1.17. By Theorems 5.1.11 and 5.1.19, the quasi-categorically enriched category $\mathcal{C}art(\mathcal{K})$ is defined by the pullback

$$\begin{tikzcd} \mathcal{C}art(\mathcal{K}) \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner" very near start] & \mathcal{R}ari(\mathcal{K}) \arrow[d, hook] \\ \mathcal{K}^{\downarrow} \arrow[r, "K"] & \mathcal{K}^{\downarrow} \end{tikzcd}$$

along the simplicial functor K that sends an isofibration $p\colon E \twoheadrightarrow B$ to the isofibration $k\colon E^{\mathbb{2}} \twoheadrightarrow \mathrm{Hom}_B(B,p)$ defined by Remark 5.1.13. To see this, note that Proposition 6.4.9 observes that the 0-arrows in the functor spaces of $\mathcal{L}ari(\mathcal{K})$ are commutative squares between isofibrations admitting a left adjoint right inverse so that the mate of the identity 2-cell induces an isomorphism in the corresponding square involving the left adjoints. By Theorem 5.1.19, this condition pulls back along the functor K to tell us that 0-arrows in $co\mathcal{C}art(\mathcal{K})$ are commutative squares between cocartesian fibrations that define cartesian functors in the sense of Definition 5.1.18.

The simplicial functor K is constructed out of weighted limits and thus preserves all weighted limits, and the replete subcategory inclusion $\mathcal{R}ari(\mathcal{K}) \hookrightarrow \mathcal{K}^{\downarrow}$ creates flexible weighted limits by Proposition 6.4.9. Hence, $\mathcal{C}art(\mathcal{K})$ is closed in $\mathcal{K}^{\downarrow}$ under flexible weighted limits[3], and now Proposition 6.4.3 proves that the inclusion $\mathcal{C}art(\mathcal{K}) \hookrightarrow \mathcal{K}^{\downarrow}$ creates an ∞-cosmos structure.

The result for cartesian fibrations with a fixed base can be proven directly by a similar argument or deduced by considering $\mathcal{K}_{/B} \subset \mathcal{K}^{\downarrow}$ as the (non-replete!) subcategory whose n-arrows have id_B as their codomain components. $\square$

Exercises.

6.4.i. Exercise. Suppose A is an ∞-category with pullbacks and pushouts and suppose the pushout functor $\mathrm{lan}\colon A^{\ulcorner} \to A^{\boxdot}$ preserves pullbacks, meaning that the left-hand composite is isomorphic to the right-hand absolute right lifting diagram:

$$\begin{tikzcd} & (A^{\ulcorner})^{\boxdot} \arrow[r, "\mathrm{lan}^{\boxdot}"] \arrow[d, two heads, "\mathrm{res}"] & (A^{\boxdot})^{\boxdot} \arrow[d, two heads, "\mathrm{res}"] \\ (A^{\ulcorner})^{\lrcorner} \arrow[ur, "\mathrm{ran}"] \arrow[r, equal] & (A^{\ulcorner})^{\lrcorner} \arrow[r, "\mathrm{lan}^{\lrcorner}"] & (A^{\boxdot})^{\lrcorner} \end{tikzcd} \cong \begin{tikzcd} & & (A^{\boxdot})^{\boxdot} \arrow[d, two heads, "\mathrm{res}"] \\ (A^{\ulcorner})^{\lrcorner} \arrow[r, "\mathrm{lan}^{\lrcorner}"] & (A^{\boxdot})^{\lrcorner} \arrow[ur, "\mathrm{ran}"] \arrow[r, equal] & (A^{\boxdot})^{\lrcorner} \end{tikzcd}$$

[3] The same argument solves the first part of Exercise 6.2.vii.

Show that any pullback square in A is also a pushout square by considering a diagram, such as depicted below, built from the solid-arrow pullback square $p \cong a \times_c b$:

$$\begin{tikzcd}[column sep=small, row sep=small]
& p \arrow[rrrr] \arrow[ddl, dashed] \arrow[drr, dashed] \arrow[dddd] & & & & b \arrow[drr, dashed] \arrow[ddl, dashed] \arrow[dddd] & & \\
& & & b \arrow[rrrr, dotted] \arrow[ddl, dashed] \arrow[dddd, dotted] & & & & b \arrow[ddl, dashed] \arrow[dddd, dotted] \\
a \arrow[drr, dashed] \arrow[dddd, dotted] & & & & c \arrow[drr, dashed] \arrow[dddd, dotted] & & & \\
& & c \arrow[rrrr, dotted] \arrow[dddd, dotted] & & & & c \arrow[dddd, dotted] & \\
& a \arrow[rrrr] \arrow[drr, dashed] \arrow[ddl, dashed] & & & & c \arrow[drr, dashed] \arrow[ddl, dashed] & & \\
& & & c \arrow[rrrr, dotted] \arrow[ddl, dashed] & & & & c \arrow[ddl, dashed] \\
a \arrow[rrrr, dotted] \arrow[drr, dashed] & & & & c \arrow[drr, dashed] & & & \\
& & c \arrow[rrrr, dotted] & & & & c &
\end{tikzcd}$$

6.4.ii. Exercise. Use the previous exercise and an argument similar to that given in the proof of Proposition 6.4.6 to prove that any ∞-cosmos $\mathcal{K}$ admits a sub ∞-cosmos $\mathcal{S}tab(\mathcal{K}) \hookrightarrow \mathcal{K}$ whose objects are the stable ∞-categories of Definition 4.4.5 and whose morphisms are the **exact functors**, which preserve the zero elements and the exact triangles.

6.4.iii. Exercise. Consider a functor between isofibrations

$$\begin{tikzcd}
E \arrow[rr, "g"] \arrow[dr, two heads, "p"] & & F \arrow[dl, two heads, "q"'] \\
& B \arrow[ul, dashed, bend left, "r"] \arrow[ul, phantom, "\perp"] \arrow[ur, dashed, bend right, "s"'] \arrow[ur, phantom, "\perp"] &
\end{tikzcd}$$

in which p admits a right adjoint right inverse r and q admits a right adjoint right inverse s. Prove that if $gr \cong s$ over B, then the mate of the identity $qg = p$ is an isomorphism. This proves that the 0-arrows in the ∞-cosmos $\mathcal{R}ari(\mathcal{K})_{/B}$ are *exact transformations* between right adjoint right inverse adjunctions.

6.4.iv. Exercise. Prove that $\mathcal{L}ari(\mathcal{K}) \cong \mathcal{R}ari(\mathcal{K}^{co})^{co}$.

6.4.v. Exercise. Use Lemma 1.2.27 to show that if $\mathcal{L} \hookrightarrow \mathcal{K}$ is a replete sub ∞-cosmos, then an object $A \in \mathcal{L}$ is discrete if and only if A is discrete as an object of $\mathcal{K}$.

6.4.vi. Exercise ([**38**, 4.1.5]). Given a diagram

$$\begin{tikzcd}
E' \arrow[r] \arrow[d, two heads] & E \arrow[d, two heads] & E'' \arrow[l] \arrow[d, two heads] \\
B' \arrow[r] & B & B'' \arrow[l]
\end{tikzcd}$$

where the vertical maps are cocartesian fibrations and the squares define cartesian functors, verify that the induced functor

$$E' \times_E E'' \to B' \times_B B''$$

is a cocartesian fibration and the projections define cartesian functors. Show also that if the vertical maps are discrete cocartesian fibrations so is this induced functor.

Part II

The calculus of modules

CHAPTER 7

Two-sided fibrations

Recall from Proposition 6.4.12 that for any ∞-category B in an ∞-cosmos $\mathcal{K}$, the quasi-categorically enriched categories $\mathcal{coCart}(\mathcal{K})_{/B}$ and $\mathcal{Cart}(\mathcal{K})_{/B}$ define sub ∞-cosmoi of $\mathcal{K}_{/B}$. In this section, we introduce another sub ∞-cosmos ${}_{A\backslash}\mathcal{Fib}(\mathcal{K})_{/B} \subset \mathcal{K}_{/A\times B}$ whose objects are *two-sided fibrations* from A to B. Several equivalent definitions of this notion are given in §7.1. Iterating Proposion 6.4.12 reveals that ${}_{A\backslash}\mathcal{Fib}(\mathcal{K})_{/B}$ is again an ∞-cosmos, which we study in §7.2. Importantly, a two-sided fibration from B to $\mathbb{1}$ is simply a cocartesian fibration over B, while a two-sided fibration from $\mathbb{1}$ to B is a cartesian fibration over B, so results about two-sided fibrations simultaneously generalize these one-sided notions. For instance, in §7.3, we introduce two-sided representables and prove the Yoneda lemma, generalizing Theorem 5.5.5 for representable (co)cartesian fibrations.

Another reason for our interest in two-sided fibrations is the fact that the discrete objects in ${}_{A}\mathcal{Fib}_{B}$ are precisely the *modules* from A to B, which we define and study in §7.4. The calculus of modules, developed in Chapter 8, is the main site of the formal category theory of ∞-categories, which is the subject of Chapter 9.

7.1. Four equivalent definitions of two-sided fibrations

By factoring, any span in $\mathcal{K}$ from A to B may be replaced up to equivalence by a **two-sided isofibration**, a span $A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$ for which the functor $(q,p)\colon E \twoheadrightarrow A\times B$ is an isofibration. Two-sided isofibrations from A to B are the objects of the ∞-cosmos $\mathcal{K}_{/A\times B}$. In this section, we describe what it means for a two-sided isofibration to be *cocartesian on the left* or *cartesian on the right*, and then introduce *two-sided fibrations*, which integrate these notions.

7.1.1. LEMMA (cocartesian on the left). *For a two-sided isofibration $A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$ in an ∞-cosmos $\mathcal{K}$, the following are equivalent:*

(i) The functor

$$\begin{tikzcd} E \arrow[rr, two heads, "{(q,p)}"] \arrow[dr, two heads, "p"'] & & A\times B \arrow[dl, two heads, "\pi"] \\ & B & \end{tikzcd}$$

is a cocartesian fibration in the slice ∞-cosmos $\mathcal{K}_{/B}$.

(ii) The functor

$$\begin{tikzcd} E \arrow[rr, "{(q,p)}"] \arrow[dr, two heads, "q"'] & & A\times B \arrow[dl, two heads, "\pi"] \\ & A & \end{tikzcd}$$

in $\mathcal{K}_{/A}$ lies in the sub ∞-cosmos $\mathcal{coCart}(\mathcal{K})_{/A}$.

(iii) The functor induced by id_q

$$\begin{tikzcd} E \arrow[rr, "i"'] \arrow[rr, phantom, "\perp"] \arrow[dr, two heads, "{(q,p)}"'] & & \mathsf{Hom}_A(q,A) \arrow[ll, dashed, bend right, "\ell"'] \arrow[dl, two heads, "{(p_1,pp_0)}"] \\ & A \times B & \end{tikzcd}$$

admits a left adjoint in $\mathcal{K}_{/A\times B}$.

(iv) The isofibration $q\colon E \twoheadrightarrow A$ *is a cocartesian fibration in* $\mathcal{K}$ *and for every* q*-cocartesian transformation*

$$\begin{tikzcd} X \arrow[r, bend left, "e"] \arrow[r, bend right, "e'"'] \arrow[r, phantom, "\Downarrow\chi"] & E \end{tikzcd}, \text{ the composite } \begin{tikzcd} X \arrow[r, bend left, "e"] \arrow[r, bend right, "e'"'] \arrow[r, phantom, "\Downarrow\chi"] & E \arrow[r, two heads, "p"] & B \end{tikzcd} \text{ is an isomorphism.}$$

If any of these equivalent conditions are satisfied, we say that $A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$ is **cocartesian on the left**.

Proof. We first prove that the equivalence (i)⇔(iii) is exactly the interpretation of the dual of the equivalence Theorem 5.1.11(i)⇔(ii) applied to the isofibration $(q,p)\colon E \twoheadrightarrow A\times B$ in the ∞-cosmos $\mathcal{K}_{/B}$. This latter result asserts that the isofibration $(q,p)\colon E \twoheadrightarrow A\times B$ is a cocartesian fibration in $\mathcal{K}_{/B}$ if and only if a certain canonical functor from E to the left representation of the functor $(q,p)\colon E \twoheadrightarrow A \times B$ admits a left adjoint over the codomain $\pi\colon A \times B \twoheadrightarrow B$; since $(\mathcal{K}_{/B})_{/\pi\colon A\times B \twoheadrightarrow B} \cong \mathcal{K}_{/A\times B}$ this is the same as asserting this adjunction over $A \times B$.

The only subtlety in interpreting Theorem 5.1.11 in $\mathcal{K}_{/B}$ has to do with the correct interpretation of the left representable comma ∞-category in $\mathcal{K}_{/B}$ for the functor $(q,p)\colon E \twoheadrightarrow A \times B$. This comma ∞-category is constructed as a subobject of the $\mathbb{2}$-cotensor of the object $\pi\colon A\times B \to B$ in $\mathcal{K}_{/B}$, which Proposition 1.2.22 tells us is formed as the left-hand vertical of the pullback diagram

$$\begin{tikzcd} A^{\mathbb{2}} \times B \arrow[r] \arrow[d, two heads, "\pi"'] \arrow[dr, phantom, "\lrcorner" very near start] & (A \times B)^{\mathbb{2}} \arrow[d, two heads, "\pi^{\mathbb{2}}"] \\ B \arrow[r, "\Delta"] & B^{\mathbb{2}} \end{tikzcd}$$

By (3.4.2) the comma ∞-category is constructed by the pullback in $\mathcal{K}_{/B}$

$$\begin{tikzcd} \mathsf{Hom}_A(q,A) \arrow[rr] \arrow[d, two heads, "p_0"'] \arrow[ddr, dotted, "pp_0"] \arrow[dr, phantom, "\lrcorner" very near start] & & A^{\mathbb{2}} \times B \arrow[r, two heads, "p_1\times\mathrm{id}"] \arrow[d, two heads, "p_0\times\mathrm{id}"] \arrow[ddl, "\pi"'] & A \times B \arrow[ddll, bend left, "\pi"] \\ E \arrow[rr, two heads, "{(q,p)}"] \arrow[dr, two heads, "p"'] & & A \times B \arrow[dl, two heads, "\pi"] & \\ & B & & \end{tikzcd} \tag{7.1.2}$$

which is created by the forgetful functor $\mathcal{K}_{/B} \to \mathcal{K}$, and its codomain-projection functor is the top composite $(p_1, pp_0)\colon \mathsf{Hom}_A(q,A) \twoheadrightarrow A \times B$. Now we see that the interpretation of Theorem 5.1.11(i)⇔(ii) in $(\mathcal{K}_{/B})^{\mathrm{co}}$ is exactly the equivalence (i)⇔(iii).

It remains to demonstrate the equivalence with (ii) and (iv). Assuming (iii) and composing with $\pi\colon A\times B \twoheadrightarrow A$, we are left with a fibered adjunction that demonstrates that q is a cocartesian fibration.

The unit of both fibered adjunctions is the same, and by Theorem 5.1.11(v) the composite

$$\begin{tikzcd} \mathsf{Hom}_A(q,A) \arrow[rr, equal, bend left] \arrow[dr, "\ell"'] & \arrow[d, phantom, "\Downarrow\eta"] & \mathsf{Hom}_A(q,A) \arrow[r, two heads, "p_0"] & E \\ & E \arrow[ur, "i"'] & & \end{tikzcd}$$

is the generic q-cocartesian cell. Since η is fibered over $A \times B$, when we postcompose with p we get an identity, which tells us that $p\colon E \twoheadrightarrow B$ carries q-cocartesian cells to isomorphisms. This proves (iv).

We now argue that (iv) implies (ii). By Example 5.2.4, the cocartesian cells for the cocartesian fibration $\pi\colon A \times B \twoheadrightarrow A$ are precisely those 2-cells whose component with codomain B is an isomorphism, so (iv) says exactly that $(q,p)\colon E \twoheadrightarrow A \times B$ defines a cartesian functor from q to π. Thus (iv) implies (ii).

To complete the argument, we show that (ii) implies (i). Assume (ii) and consider a 2-cell in $\mathcal{K}_{/B}$

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[dr, bend right, "(a,b)"'] & E \arrow[d, two heads, "(q,p)"] \arrow[dl, phantom, "\Downarrow(\alpha,\mathrm{id})"] \\ & A \times B \end{tikzcd}$$

Because q is a cocartesian fibration, $\alpha\colon qe \Rightarrow a$ has a q-cocartesian lift $\chi\colon e \Rightarrow e'\colon X \to E$, and since (q,p) is a cartesian functor, the whiskered 2-cell $p\chi\colon b \Rightarrow pe'$ is an isomorphism. Because $(q,p)\colon E \twoheadrightarrow A \times B$ is an isofibration, we may lift the 2-cell $(\mathrm{id}, p\chi^{-1})\colon e' \Rightarrow (a,b)$ to an invertible 2-cell $\gamma\colon e' \Rightarrow e''$ with $q\gamma = \mathrm{id}_a$. The composite $\gamma \cdot \chi\colon e \Rightarrow e''$ is a lift of (α, id) along (q,p) over B, which is easily verified to define a (q,p)-cocartesian lift of (α, id) in $\mathcal{K}_{/B}$ as required by (i). ☐

If $(q,p)\colon E \twoheadrightarrow A \times B$ is a *discrete* cocartesian fibration in $\mathcal{K}_{/B}$ then the converse to the last statement of (iv) holds: any 2-cell χ for which $p\chi$ is an isomorphism is q-cocartesian. See Exercise 7.1.i.

By Lemma 7.1.1 and its dual:

7.1.3. Corollary. *A two-sided isofibration $A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$ is cocartesian on the left and cartesian on the right if the following equivalent conditions are satisfied:*

(i) The functor

$$\begin{tikzcd} E \arrow[rr, "(q,p)"] \arrow[dr, two heads, "q"'] & & A \times B \arrow[dl, two heads, "\pi"] \\ & A & \end{tikzcd}$$

lies in $\mathcal{C}o\mathcal{C}art(\mathcal{K})_{/A}$ and defines a cartesian fibration in $\mathcal{K}_{/A}$.

(ii) The functor

$$\begin{tikzcd} E \arrow[rr, two heads, "(q,p)"] \arrow[dr, two heads, "p"'] & & A \times B \arrow[dl, two heads, "\pi"] \\ & B & \end{tikzcd}$$

lies in $\mathcal{C}art(\mathcal{K})_{/B}$ and defines a cocartesian fibration in $\mathcal{K}_{/B}$. ☐

A two-sided fibration is a span that is cocartesian on the left, cartesian on the right, and satisfies a further compatibility condition that can be stated in a number of equivalent ways, which boil down to the assertion that the processes of taking q-cocartesian and p-cartesian lifts commute:

7.1.4. Theorem. *For a two-sided isofibration $A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$ in $\mathcal{K}$, the following are equivalent:*

(i) *The functor*

$$\begin{tikzcd} E \arrow[rr, "{(q,p)}"] \arrow[dr, two heads, "q"'] & & A \times B \arrow[dl, two heads, "\pi"] \\ & A & \end{tikzcd}$$

defines a cartesian fibration in $\mathcal{coCart}(\mathcal{K})_{/A}$.

(ii) *The functor*

$$\begin{tikzcd} E \arrow[rr, two heads, "{(q,p)}"] \arrow[dr, two heads, "p"'] & & A \times B \arrow[dl, two heads, "\pi"] \\ & B & \end{tikzcd}$$

defines a cocartesian fibration in $\mathcal{Cart}(\mathcal{K})_{/B}$.

(iii) *The canonical functors admit the displayed adjoints in* $\mathcal{K}_{/A\times B}$

$$\begin{tikzcd} E \arrow[r, "i"', "\top"] \arrow[d, "i"] \arrow[d, phantom, shift right=3, "\dashv"] & \mathrm{Hom}_B(B,p) \arrow[l, dashed, bend right, "r"'] \arrow[d, "{(ip_1,\mathrm{id})}"'] \arrow[d, phantom, shift left=3, "\vdash"] \\ \mathrm{Hom}_A(q,A) \arrow[u, dashed, bend left, "\ell"] \arrow[r, "{(\mathrm{id},ip_0)}", "\bot"'] & \mathrm{Hom}_A(q,A) \underset{E}{\times} \mathrm{Hom}_B(B,p) \arrow[l, dashed, bend left, "r"] \arrow[u, dashed, bend right, "\ell"'] \end{tikzcd}$$

and the mate of the identity 2-cell in this displayed commutative square defines an isomorphism $\ell r \cong r\ell$.

(iv) $A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$ *is cocartesian on the left, cartesian on the right, and for any* $a\colon X \to A$, $b\colon X \to B$, *and* $e\colon X \to E$ *and pair of 2-cells* $\alpha\colon qe \Rightarrow a$ *and* $\beta\colon b \Rightarrow pe$, *the map* $\beta^*\alpha_*(e)\colon X \to E$, *obtained as the domain of a* p*-cartesian lift of the composite of* β *with a* q*-cocartesian lift* $\chi_\alpha\colon e \Rightarrow \alpha_*(e)$ *of* α *over* B, *is isomorphic over* $A \times B$ *to the map* $\alpha_*\beta^*(e)\colon X \to E$ *obtained as the codomain of a* q*-cocartesian lift of the composite of* α *with a* p*-cartesian lift* $\chi_\beta\colon \beta^*(e) \Rightarrow e$ *of* β *over* A.

A span $A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$ defines a **two-sided fibration from A to B** if any of these equivalent conditions are satisfied.

Proof. Our strategy will be to show that condition (i) is equivalent to (iii), an equationally witnessed condition in the slice ∞-cosmos $\mathcal{K}_{/A\times B}$. A dual argument will show that condition (ii) is equivalent to (iii). We then unpack this condition to prove its equivalence with (iv).

We use Theorem 5.1.11(ii), which provides a condition that characterizes the cartesian fibrations in any ∞-cosmos via the presence of a fibered adjunction, to re-express (i) in $\mathcal{K}_{/A\times B}$. To apply this characterization to the map displayed in (i), we must first compute the right representable comma object in $\mathcal{coCart}(\mathcal{K})_{/A} \subset \mathcal{K}_{/A}$ associated to the functor $(q,p)\colon E \to A \times B$ over A. By Proposition 6.4.12, it suffices to compute this comma object in $\mathcal{K}_{/A}$. By the dual of the calculation (7.1.2) we gave in the proof of Lemma 7.1.1, the comma object covariantly representing the functor $(q,p)\colon E \to A\times B$ in $\mathcal{coCart}(\mathcal{K})_{/A} \subset \mathcal{K}_{/A}$ is the cocartesian fibration $qp_1\colon \mathrm{Hom}_B(B,p) \twoheadrightarrow A$.

Now Theorem 5.1.11(ii) applied in $\mathcal{coCart}(\mathcal{K})_{/A}$ tells us that condition (i) holds if and only if the canonical functor i admits a right r over $A \times B$

$$\begin{tikzcd} E \arrow[rr, "i"] \arrow[rr, phantom, shift right=2, "\perp"] \arrow[dr, two heads, "(q,p)"'] & & \mathsf{Hom}_B(B,p) \arrow[ll, dashed, bend left, "r"] \arrow[dl, two heads, "(qp_1,p_0)"] \\ & A \times B & \end{tikzcd} \tag{7.1.5}$$

where the three displayed objects are all cocartesian fibrations over A and each of the four displayed maps are cartesian functors between these cocartesian fibrations.

By Lemma 7.1.1, to say that (q,p) defines a cartesian functor between cocartesian fibrations is to say that the span $A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$ is cocartesian on the left, which is the case if and only if the canonical functor i admits a left adjoint

$$\begin{tikzcd} E \arrow[rr, "i"'] \arrow[rr, phantom, shift left=2, "\perp"] \arrow[dr, two heads, "(q,p)"'] & & \mathsf{Hom}_A(q,A) \arrow[ll, dashed, bend right, "\ell"'] \arrow[dl, two heads, "(p_1,pp_0)"] \\ & A \times B & \end{tikzcd}$$

in $\mathcal{K}_{/A\times B}$. Similarly, to say that (qp_1,p_0) defines a cartesian functor between cocartesian fibrations is to say that the span $A \overset{qp_1}{\twoheadleftarrow} \mathsf{Hom}_B(B,p) \overset{p_0}{\twoheadrightarrow} B$ is cocartesian on the left. By Lemma 7.1.1 again, this is equivalent to the hypothesis that the canonical functor from $\mathsf{Hom}_B(B,p)$ to the comma object contravariantly representing qp_1 admits a left adjoint

$$\begin{tikzcd} \mathsf{Hom}_B(B,p) \arrow[rr, "(ip_1{,}\mathrm{id})"'] \arrow[rr, phantom, shift left=2, "\perp"] \arrow[dr, two heads, "(qp_1,p_0)"'] & & \mathsf{Hom}_A(q,A) \underset{E}{\times} \mathsf{Hom}_B(B,p) \arrow[ll, dashed, bend right, "\ell"'] \arrow[dl, two heads, "(p_1,p_0)"] \\ & A \times B & \end{tikzcd}$$

in $\mathcal{K}_{/A\times B}$.

Finally, by Theorem 5.1.19, to say that the functor i of (7.1.5) is cartesian is to say that in the commutative solid-arrow diagram in $\mathcal{K}_{/A\times B}$,

$$\begin{tikzcd} E \arrow[r, "i"] \arrow[d, "i"] & \mathsf{Hom}_B(B,p) \arrow[d, "(ip_1{,}\mathrm{id})"'] \\ \mathsf{Hom}_A(q,A) \arrow[r, "(\mathrm{id}{,}ip_0)"'] \arrow[u, dashed, bend left, "\ell"] \arrow[u, phantom, "\dashv"] & \mathsf{Hom}_A(q,A) \underset{E}{\times} \mathsf{Hom}_B(B,p) \arrow[u, dashed, bend right, "\ell"'] \arrow[u, phantom, "\vdash"] \end{tikzcd} \qquad \begin{tikzcd} E \arrow[r, "i"'] \arrow[r, phantom, shift left=2, "\top"] \arrow[d, "i"'] & \mathsf{Hom}_B(B,p) \arrow[d, "(ip_1{,}\mathrm{id})"] \arrow[l, dashed, bend right, "r"'] \\ \mathsf{Hom}_A(q,A) \arrow[r, "(\mathrm{id}{,}ip_0)"] \arrow[r, phantom, shift right=2, "\perp"] & \mathsf{Hom}_A(q,A) \underset{E}{\times} \mathsf{Hom}_B(B,p) \arrow[l, dashed, bend left, "r"] \end{tikzcd} \tag{7.1.6}$$

the mate of this identity 2-cell involving the left adjoints ℓ is an isomorphism. This "Beck-Chevalley" condition is equivalent to saying that the other mate, displayed above right, associated to the functor r of (7.1.5) is an isomorphism. Finally, to say that r is also a cartesian functor between cocartesian fibrations is to say that the further mate

$$\begin{tikzcd}
E & & \mathrm{Hom}_B(B,p) \arrow[ll, dashed, "r"'] \\
\mathrm{Hom}_A(q,A) \arrow[u, dashed, "\ell"] & & \mathrm{Hom}_A(q,A) \times_E \mathrm{Hom}_B(B,p) \arrow[ll, dashed, "r"] \arrow[u, dashed, "\ell"'] \arrow[ull, phantom, "\Uparrow\cong"]
\end{tikzcd} \tag{7.1.7}$$

is an isomorphism.

Thus, we have shown that condition (i) is equivalent to (iii) positing the existence of adjunctions (7.1.6) in $\mathcal{K}_{/A\times B}$ so that all of the mates of the solid-arrow diagram are isomorphisms. Dualizing and reversing this argument, we see that this is equivalent to condition (ii).

Finally, (iii) and (iv) are equivalent since the existence of the left adjoints in (7.1.6) is equivalent to the span being cocartesian on the left, the existence of the right adjoints is equivalent to being cartesian on the right, and the compatibility condition for the cartesian and cocartesian lifts is the meaning of the isomorphism (7.1.7). □

7.1.8. Corollary. *Any two-sided isofibration $(a,b)\colon X \twoheadrightarrow A\times B$ that is equivalent over $A\times B$ to a two-sided fibration $(q,p)\colon E \twoheadrightarrow A\times B$ is a two-sided fibration.*

Proof. The assertion of Theorem 7.1.4(iii) is invariant under fibered equivalence. □

Theorem 7.1.4 will help us establish an important family of examples involving the ordinal categories from Definition 1.1.4.

7.1.9. Proposition. *For any ∞-category A and any $n \geq 2$, the two-sided isofibration $A \overset{p_{n-1}}{\twoheadleftarrow} A^{\mathbb{n}} \overset{p_0}{\twoheadrightarrow} A$ defines a two-sided fibration.*

This result is a generalization of Proposition 5.1.23 and its dual and the proof uses similar ideas.

Proof. We use Theorem 7.1.4(iii). The right representable comma ∞-category associated to $p_0\colon A^{\mathbb{n}} \twoheadrightarrow A$ is constructed by the pullback

$$\begin{tikzcd}
A^{\mathbb{2}\vee\mathbb{n}} \arrow[r, two heads] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & A^{\mathbb{2}} \arrow[d, two heads, "p_1"] \\
A^{\mathbb{n}} \arrow[r, two heads, "p_0"'] & A
\end{tikzcd}$$

which is equivalent to $(p_n,p_0)\colon A^{\mathbb{n}+\mathbb{1}} \twoheadrightarrow A\times A$ over the endpoint evaluation maps. The canonical map

$$\begin{tikzcd}
A^{\mathbb{n}} \arrow[rr, "A^{\sigma^0}"'] \arrow[dr, "(p_{n-1},p_0)"'] \arrow[rr, phantom, bend left=15, "\top"] & & A^{\mathbb{n}+\mathbb{1}} \arrow[ll, dashed, bend right, "A^{\delta^1}"'] \arrow[dl, "(p_n,p_0)"] \\
& A\times A &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{n} \arrow[rr, dashed, bend left, "\delta^1"] \arrow[rr, phantom, bend left=15, "\perp"] & & \mathbb{n}+\mathbb{1} \arrow[ll, "\sigma_0"] \\
& \mathbb{1}+\mathbb{1} \arrow[ul, hook, "(n-1,0)"] \arrow[ur, hook, "(n,0)"'] &
\end{tikzcd}$$

that tests whether $(p_{n-1},p_0)\colon A^{\mathbb{n}} \twoheadrightarrow A\times A$ is cartesian on the right is given by restriction along the epimorphism $\sigma^0\colon \mathbb{n}+\mathbb{1} \to \mathbb{n}$ that sends the objects $0,1 \in \mathbb{n}+\mathbb{1}$ to $0 \in \mathbb{n}$. This functor admits a left adjoint under the endpoint inclusions displayed above-right, which provides the desired fibered right adjoint displayed above left.

A dual argument shows that $(p_{n-1}, p_0)\colon A^{\mathbb{n}} \twoheadrightarrow A \times A$ is cocartesian on the left. The final condition asks that the mate of the commutative square defined by the degeneracy maps

$$\begin{tikzcd}
\mathbb{n} \arrow[d, dashed, bend right, "\delta^n"'] \arrow[r, dashed, bend left, "\delta^1"] \arrow[r, phantom, "\perp"] & \mathbb{n}+\mathbb{1} \arrow[l, "\sigma^0"] \arrow[d, dashed, bend left, "\delta^{n+1}"] \\
\mathbb{n}+\mathbb{1} \arrow[u, "\sigma^n"'] \arrow[u, phantom, "\vdash"] \arrow[r, dashed, bend right, "\delta^1"'] \arrow[r, phantom, "\top"] & \mathbb{n}+\mathbb{2} \arrow[l, "\sigma^0"'] \arrow[u, "\sigma^{n+1}"] \arrow[u, phantom, "\dashv"]
\end{tikzcd}$$

is an isomorphism, as is easily verified. The square in Theorem 7.1.4(iii) is obtained by applying $A^{(-)}$. $\square$

Theorem 7.1.4 has a relative analogue, whose proof is left to the reader, which characterizes what we call a **cartesian functor** between two-sided fibrations from A to B.

7.1.10. LEMMA. *A map of spans between a pair of two-sided fibrations from A to B*

$$\begin{tikzcd}
& E \arrow[dl, two heads, "q"'] \arrow[dr, two heads, "p"] \arrow[dd, "g"] & \\
A & & B \\
& F \arrow[ul, two heads, "s"] \arrow[ur, two heads, "r"'] &
\end{tikzcd} \tag{7.1.11}$$

defines a cartesian functor between the cartesian fibrations in $\mathcal{coCart}(\mathcal{K})_{/A}$ if and only if it defines a cartesian functor between the cocartesian fibrations in $\mathcal{Cart}(\mathcal{K})_{/B}$.

PROOF. A similar argument to that given in Theorem 7.1.4 shows that the map $g\colon E \to F$ of (7.1.11) is a cartesian functor between cartesian fibrations in $\mathcal{coCart}(\mathcal{K})_{/A}$ if and only if in two commutative diagrams over $A \times B$, the mates are isomorphisms. This condition also characterizes cartesian functors between cocartesian fibrations in $\mathcal{Cart}(\mathcal{K})_{/B}$. The details are left as Exercise 7.1.ii. $\square$

A cartesian functor is just a map of spans (7.1.11) that defines a cartesian functor between the cocartesian fibrations q and s and also a cartesian functor between the cartesian fibrations p and r. It follows from the internal characterization (iii) of Theorem 7.1.4 and a similar internal characterization of cartesian functors derived from Theorem 5.1.19 that:

7.1.12. COROLLARY. *Any cosmological functor preserves two-sided fibrations and cartesian functors between them.* $\square$

Exercises.

7.1.i. EXERCISE. Suppose $(q,p)\colon E \twoheadrightarrow A \times B$ is a *discrete* cocartesian fibration in $\mathcal{K}_{/B}$. Prove, for any 2-cell χ with codomain E, that if $p\chi$ is an isomorphism, then χ is q-cocartesian.

7.1.ii. EXERCISE. Prove Lemma 7.1.10.

7.1.iii. EXERCISE. Prove that

(i) A two-sided isofibration $1 \overset{!}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$ defines a two-sided fibration from 1 to B if and only if $p\colon E \twoheadrightarrow B$ is a cartesian fibration.

(ii) A map of spans

$$\begin{tikzcd}[row sep=small] & E \arrow[dl, two heads, "!"'] \arrow[dr, two heads, "p"] \arrow[dd, "g"] & \\ 1 & & B \\ & F \arrow[ul, two heads, "!"] \arrow[ur, two heads, "q"'] & \end{tikzcd}$$

defines a cartesian functor of two-sided fibrations if and only if $g\colon E \to F$ defines a cartesian functor from p to q.

7.2. The ∞-cosmos of two-sided fibrations

The equivalent conditions (i) and (ii) of Theorem 7.1.4 provide two equivalent ways to define the ∞-cosmos of two-sided fibrations.

7.2.1. Definition (the ∞-cosmos of two-sided fibrations). By Theorem 7.1.4 and Lemma 7.1.10, the pair of quasi-categorically enriched subcategories

$$\mathcal{C}art(co\mathcal{C}art(\mathcal{K})_{/A})_{/\pi\colon A\times B\twoheadrightarrow A} \qquad \text{and} \qquad co\mathcal{C}art(\mathcal{C}art(\mathcal{K})_{/B})_{/\pi\colon A\times B\twoheadrightarrow B}$$

of $\mathcal{K}_{/A\times B}$ coincide. By Proposition 6.4.12, applied twice, this subcategory is an ∞-cosmos, which we call the *∞-cosmos of two-sided fibrations* from A to B and denote by

$${}_{A\backslash}\mathcal{F}ib(\mathcal{K})_{/B} \subset \mathcal{K}_{/A\times B}.$$

By definition

$$\mathsf{Fun}_{{}_{A\backslash}\mathcal{F}ib(\mathcal{K})_{/B}}(E,F) \coloneqq \mathsf{Fun}^{c}_{A\times B}(E,F) \subset \mathsf{Fun}_{A\times B}(E,F)$$

is the quasi-category of maps of spans that define cartesian functors from E to F in the sense of Lemma 7.1.10.

7.2.2. Proposition. *The simplicial subcategory ${}_{A\backslash}\mathcal{F}ib(\mathcal{K})_{/B} \hookrightarrow \mathcal{K}_{/A\times B}$ creates an ∞-cosmos structure on the ∞-cosmos of two-sided fibrations from the ∞-cosmos of two-sided isofibrations.*

Proof. This inclusion factors as

$${}_{A\backslash}\mathcal{F}ib(\mathcal{K})_{/B} \cong co\mathcal{C}art(\mathcal{C}art(\mathcal{K})_{/B})_{/A\times B\twoheadrightarrow B} \hookrightarrow (\mathcal{C}art(\mathcal{K})_{/B})_{/A\times B\twoheadrightarrow B} \hookrightarrow (\mathcal{K}_{/B})_{/A\times B\twoheadrightarrow B} \cong \mathcal{K}_{/A\times B}$$

Applying Proposition 6.4.12, we see that both inclusions create ∞-cosmos structures. □

7.2.3. Observation (two-sided fibrations generalize (co)cartesian fibrations). By Exercise 7.1.iii, a two-sided fibration from B to 1 is a cocartesian fibration over B, while a two-sided fibration from 1 to B is a cartesian fibration over B. Indeed,

$$co\mathcal{C}art(\mathcal{K})_{/B} \cong {}_{B\backslash}\mathcal{F}ib(\mathcal{K})_{/1} \qquad \text{and} \qquad \mathcal{C}art(\mathcal{K})_{/B} \cong {}_{1\backslash}\mathcal{F}ib(\mathcal{K})_{/B}.$$

In this sense, statements about two-sided fibrations simultaneously generalize statements about cartesian and cocartesian fibrations.

7.2.4. Proposition. *For any pair of functors $a\colon A' \to A$ and $b\colon B' \to B$, the cosmological pullback functor*

$$(a,b)^*\colon \mathcal{K}_{/A\times B} \longrightarrow \mathcal{K}_{/A'\times B'}$$

restricts to define a cosmological functor

$$(a,b)^*\colon {}_{A\backslash}\mathcal{F}ib(\mathcal{K})_{/B} \longrightarrow {}_{A'\backslash}\mathcal{F}ib(\mathcal{K})_{/B'}.$$

In particular, the pullback of a two-sided fibration is again a two-sided fibration.

Proof. By factoring (a,b) as $A' \times B' \xrightarrow{\mathrm{id}\times b} A' \times B \xrightarrow{a\times\mathrm{id}} A \times B$ we see that it suffices to consider pullback along one side at a time. Proposition 5.1.20 proves that pullback along $b\colon B' \to B$ preserves cartesian fibrations and cartesian functors, defining a restricted functor

$$\begin{tikzcd} \mathcal{C}art(\mathcal{K})_{/B} \arrow[r, hook] \arrow[d, dashed, "b^*"'] & \mathcal{K}_{/B} \arrow[d, "b^*"] \\ \mathcal{C}art(\mathcal{K})_{/B'} \arrow[r, hook] & \mathcal{K}_{/B'} \end{tikzcd}$$

Since limits and isofibrations in $\mathcal{C}art(\mathcal{K})_{/B}$ are created in $\mathcal{K}_{/B}$, this restricted functor is a cosmological functor. Applying this result to the map

$$\begin{tikzcd} A \times B' \arrow[rr, "\mathrm{id}\times b"] \arrow[dr, two heads, "\pi"'] & & A \times B \arrow[dl, two heads, "\pi"] \\ & A & \end{tikzcd}$$

in the ∞-cosmos $co\mathcal{C}art(\mathcal{K})_{/A}$, we conclude that pullback restricts to define a cosmological functor

$$\begin{tikzcd} {}_{A\backslash}\mathcal{F}ib(\mathcal{K})_{/B} \cong \mathcal{C}art(co\mathcal{C}art(\mathcal{K})_{/A})_{/\pi\colon A\times B \twoheadrightarrow A} \arrow[r, hook] \arrow[d, dashed, "(\mathrm{id}\times b)^*"'] & (co\mathcal{C}art(\mathcal{K})_{/A})_{/\pi\colon A\times B \twoheadrightarrow A} \arrow[r, hook] \arrow[d, "(\mathrm{id}\times b)^*"'] & \mathcal{K}_{/A\times B} \arrow[d, "(\mathrm{id}\times b)^*"] \\ {}_{A\backslash}\mathcal{F}ib(\mathcal{K})_{/B'} \cong \mathcal{C}art(co\mathcal{C}art(\mathcal{K})_{/A})_{/\pi\colon A\times B' \twoheadrightarrow A} \arrow[r, hook] & (co\mathcal{C}art(\mathcal{K})_{/A})_{/\pi\colon A\times B' \twoheadrightarrow A} \arrow[r, hook] & \mathcal{K}_{/A\times B'} \end{tikzcd}$$

□

7.2.5. Lemma. *If $A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$ is a two-sided fibration from A to B, $v\colon A \twoheadrightarrow C$ is a cocartesian fibration and $u\colon B \twoheadrightarrow D$ is a cartesian fibration, then the composite span*

$$C \xtwoheadleftarrow{v} A \xtwoheadleftarrow{q} E \xtwoheadrightarrow{p} B \xtwoheadrightarrow{u} D$$

defines a two-sided fibration from C to D. Moreover, a cartesian functor between two-sided fibrations from A to B induces a cartesian functor

$$\begin{tikzcd} & & E \arrow[dl, two heads, "q"'] \arrow[dr, two heads, "p"] \arrow[dd, "g"] & & \\ C & A \arrow[l, two heads, "v"'] & & B \arrow[r, two heads, "u"] & D \\ & & F \arrow[ul, two heads, "s"] \arrow[ur, two heads, "r"'] & & \end{tikzcd}$$

between two-sided fibrations from C to D.

Note composition in the sense being described here does not define a cosmological functor from ${}_{A\backslash}\mathcal{F}ib(\mathcal{K})_{/B}$ to ${}_{C\backslash}\mathcal{F}ib(\mathcal{K})_{/D}$ because it does not preserve flexible limits.

Proof. By Theorem 7.1.4, it suffices to consider composition on one side at a time, say with a cocartesian fibration $v\colon A \twoheadrightarrow C$. Working in the ∞-cosmos $\mathcal{C}art(\mathcal{K})_{/B}$, we are given cocartesian fibrations

$$\begin{tikzcd} E \arrow[rr, two heads, "(q,p)"] \arrow[dr, two heads, "p"'] & & A \times B \arrow[dl, two heads, "\pi"] \\ & B & \end{tikzcd} \qquad \begin{tikzcd} A \times B \arrow[rr, two heads, "v\times\mathrm{id}"] \arrow[dr, two heads, "\pi"'] & & C \times B \arrow[dl, two heads, "\pi"] \\ & B & \end{tikzcd}$$

These compose to define a cocartesian fibration

$$\begin{tikzcd} E \arrow[rr, two heads, "(vq{,}p)"] \arrow[dr, "p"'] & & C \times B \arrow[dl, "\pi"] \\ & B & \end{tikzcd}$$

and hence a two-sided fibration from C to B, as desired.

Now the vq-cocartesian cells are the q-cocartesian lifts of the v-cocartesian cells. If g is a cartesian functor from q to s, then these are clearly preserved, proving that g also defines a cartesian functor from vq to vs. □

Proposition 7.2.4 and Lemma 7.2.5 combine to prove that two-sided fibrations can be composed "horizontally."

7.2.6. Proposition. *The pullback of a two-sided fibration from A to B along a two-sided fibration from B to C*

$$\begin{tikzcd} & & E \underset{B}{\times} F \arrow[dl, two heads, "\pi_1"'] \arrow[dr, two heads, "\pi_0"] \arrow[dd, phantom, "\lrcorner" rotate=-45] & & \\ & E \arrow[dl, two heads, "q"'] \arrow[dr, two heads, "p"] & & F \arrow[dl, two heads, "s"'] \arrow[dr, two heads, "r"] & \\ A & & B & & C \end{tikzcd}$$

defines a two-sided fibration from A to C as displayed.

Proof. The composite two-sided fibration is constructed in two stages, first by pulling back

$$\begin{tikzcd} E \underset{B}{\times} F \arrow[r] \arrow[d, two heads, "(\pi_1{,}r\pi_0)"'] \arrow[dr, phantom, "\lrcorner" very near start] & F \arrow[d, two heads, "(s{,}r)"] \\ E \times C \arrow[r, two heads, "p \times C"'] & B \times C \end{tikzcd}$$

and then by composing the left leg with the cocartesian fibration $q\colon E \twoheadrightarrow A$. By Proposition 7.2.4 and Lemma 7.2.5, this results is a two-sided fibration from A to C. Alternatively, the composite can be constructed by pulling back along $A \times s$ and composing with the cartesian fibration $r\colon F \twoheadrightarrow C$, resulting in another two-sided fibration from A to C that is canonically isomorphic to the first. □

7.2.7. Example. If $p\colon E \twoheadrightarrow B$ is a cartesian fibration and $q\colon F \twoheadrightarrow A$ is a cocartesian fibration, then the span

$$A \overset{q}{\twoheadleftarrow} F \overset{\pi}{\twoheadleftarrow} F \times E \overset{\pi}{\twoheadrightarrow} E \overset{p}{\twoheadrightarrow} B$$

defines a two-sided fibration from A to B.

7.2.8. Example. By Proposition 7.1.9 and Proposition 7.2.4, a general comma span

$$C \xleftarrow{p_1} \mathsf{Hom}_A(f,g) \xrightarrow{p_0} B$$

is a two-sided fibration, as a pullback of $A \overset{p_1}{\twoheadleftarrow} A^{\mathbb{2}} \overset{p_0}{\twoheadrightarrow} A$. By Proposition 7.2.6, "horizontal composites" of comma spans are also two-sided fibrations. In certain cases, a horizontal composite again defines a span that is equivalent to a comma span, as is the case for:

$$\begin{tikzcd}[column sep=small]
& & \mathsf{Hom}_A(A,g) \underset{A}{\times} \mathsf{Hom}_A(f,A) \arrow[dl, two heads, "\pi_1"'] \arrow[dr, two heads, "\pi_0"] \arrow[dd, phantom, very near start, "\lrcorner" rotate=-45] & & \\
& \mathsf{Hom}_A(A,g) \arrow[dl, two heads, "p_1"'] \arrow[dr, "p_0"] & & \mathsf{Hom}_A(f,A) \arrow[dl, two heads, "p_1"'] \arrow[dr, two heads, "p_0"] & \\
C & & A & & B
\end{tikzcd}$$

which is equivalent to $(p_1, p_0)\colon \mathsf{Hom}_A(f,g) \twoheadrightarrow C \times B$ over $C \times B$. Certain other horizontal composites are not equivalent to comma spans but nonetheless define two-sided fibrations, as is the case for:

$$\begin{tikzcd}[column sep=small]
& & \mathsf{Hom}_A(a,A) \underset{X}{\times} \mathsf{Hom}_B(B,b) \arrow[dl, two heads, "\pi_1"'] \arrow[dr, two heads, "\pi_0"] \arrow[dd, phantom, very near start, "\lrcorner" rotate=-45] & & \\
& \mathsf{Hom}_A(a,A) \arrow[dl, two heads, "p_1"'] \arrow[dr, "p_0"] & & \mathsf{Hom}_B(B,b) \arrow[dl, two heads, "p_1"'] \arrow[dr, two heads, "p_0"] & \\
A & & X & & B
\end{tikzcd}$$

Exercises.

7.2.i. Exercise. Consider a diagram

$$\begin{tikzcd}
& E \arrow[dl, two heads, "q"'] \arrow[dr, two heads, "p"] \arrow[dd, "g"] & & F \arrow[dl, two heads, "s"'] \arrow[dr, two heads, "r"] \arrow[dd, "h"] & \\
A & & B & & C \\
& E' \arrow[ul, two heads, "q'"] \arrow[ur, two heads, "p'"'] & & B' \arrow[ul, two heads, "s'"] \arrow[ur, two heads, "r'"'] &
\end{tikzcd}$$

in which g and h define cartesian functors between two-sided fibrations, from A to B and from B to C, respectively. Prove that g and h pullback to define a cartesian functor $(g,h)\colon E \underset{B}{\times} F \to E' \underset{B}{\times} F'$ between two-sided fibrations from A to C.

7.2.ii. Exercise. Prove the assertion made in Example 7.2.7.

7.3. Representable two-sided fibrations and the Yoneda lemma

We now introduce representable two-sided fibrations and prove a two-sided version of the external Yoneda lemma.

7.3.1. Definition. Specializing Example 7.2.7, for any pair of elements $a\colon 1 \to A$ and $b\colon 1 \to B$, the span

$$A \xleftarrow{p_1\pi} \mathsf{Hom}_A(a,A) \times \mathsf{Hom}_B(B,b) \xrightarrow{p_0\pi} B$$

defines a two-sided fibration from A to B that we refer to as the **two-sided fibration represented by** a and b. Note there is a canonical element $(\mathrm{id}_a, \mathrm{id}_b)\colon 1 \to \mathsf{Hom}_A(a,A) \times \mathsf{Hom}_B(B,b)$ in the fiber over $(a,b)\colon 1 \to A \times B$.

The terminology of Definition 7.3.1 is justified by the Yoneda lemma for two-sided fibrations.

7.3.2. Theorem (Yoneda lemma). *For any elements $a\colon 1 \to A$ and $b\colon 1 \to B$ and any two-sided fibration $A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$, restriction along $(\ulcorner \mathrm{id}_a \urcorner, \ulcorner \mathrm{id}_b \urcorner)\colon 1 \to \mathsf{Hom}_A(a, A) \times \mathsf{Hom}_B(B, b)$ defines an equivalence of quasi-categories*

$$\mathsf{Fun}^{\mathrm{c}}_{A\times B}\left(\begin{array}{c}\mathsf{Hom}_A(a,A)\times\mathsf{Hom}_B(B,b)\\ \downarrow\!\!\downarrow \\ A\times B\end{array}, \begin{array}{c}E\\ \downarrow\!\!\downarrow\\ A\times B\end{array}\right) \xrightarrow{\sim} \mathsf{Fun}_{A\times B}\left(\begin{array}{c}1\\ {\scriptstyle (a,b)}\downarrow \\ A\times B\end{array}, \begin{array}{c}E\\ \downarrow\!\!\downarrow\\ A\times B\end{array}\right).$$

Interpreting this result in a slice ∞-cosmos will enable us to replace the elements a and b with a pair of generalized elements $a\colon X \to A$ and $b\colon X \to B$; see Corollary 7.3.5.

Proof. The two-sided fibration represented by a and b is defined by a pullback

$$\begin{tikzcd} \mathsf{Hom}_A(a,A)\times\mathsf{Hom}_B(B,b) \arrow[r, two heads, "\mathrm{id}\times p_0"] \arrow[d, two heads, "p_1\times\mathrm{id}"'] \arrow[dr, phantom, "\lrcorner" very near start] & \mathsf{Hom}_A(a,A)\times B \arrow[d, two heads, "p_1\times\mathrm{id}"] \\ A\times\mathsf{Hom}_B(B,b) \arrow[r, two heads, "\mathrm{id}\times p_0"'] & A\times B \end{tikzcd}$$

We will argue by applying the Yoneda lemma of Theorem 5.5.5 twice: first for cocartesian fibrations over the object $\pi\colon A \times \mathsf{Hom}_B(B, b) \twoheadrightarrow \mathsf{Hom}_B(B, b)$ the ∞-cosmos $\mathcal{C}art(\mathcal{K})_{/\mathsf{Hom}_B(B,b)}$, and then for cartesian fibrations over the object B in the $\mathcal{K}$.

To set up the first use of the Yoneda lemma, we begin by pulling back the two-sided fibration $A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$ along $p_0\colon \mathsf{Hom}_B(B, b) \to B$ to obtain a two-sided fibration

$$\begin{tikzcd}[column sep=small] & & \mathsf{Hom}_B(p,b) \arrow[dl, two heads, "p_0"'] \arrow[drr, two heads, "p'"] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & \\ & E \arrow[dl, two heads, "q"'] \arrow[dr, two heads, "p"] & & & \mathsf{Hom}_B(B,b) \arrow[dll, two heads, "p_0"] \\ A & & B & & \end{tikzcd}$$

from A to $\mathsf{Hom}_B(B, b)$. By the adjoint correspondence between cartesian functors established by Lemma 7.3.3(iii) below, there is an isomorphism

$$\mathsf{Fun}^{\mathrm{c}}_{A\times B}\left(\begin{array}{c}\mathsf{Hom}_A(a,A)\times\mathsf{Hom}_B(B,b)\\ {\scriptstyle p_1\times p_0}\downarrow\!\!\downarrow \\ A\times B\end{array}, \begin{array}{c}E\\ \downarrow\!\!\downarrow{\scriptstyle (q,p)}\\ A\times B\end{array}\right)$$

$$\cong \mathsf{Fun}^{\mathrm{c}}_{A\times\mathsf{Hom}_B(B,b)}\left(\begin{array}{c}\mathsf{Hom}_A(a,A)\times\mathsf{Hom}_B(B,b)\\ {\scriptstyle p_1\times \mathrm{id}}\downarrow\!\!\downarrow \\ A\times \mathsf{Hom}_B(B,b)\end{array}, \begin{array}{c}\mathsf{Hom}_B(p,b)\\ \downarrow\!\!\downarrow{\scriptstyle (qp_0,p')}\\ A\times \mathsf{Hom}_B(B,b)\end{array}\right).$$

Consider the object $\pi\colon A\times \mathsf{Hom}_B(B,b) \twoheadrightarrow \mathsf{Hom}_B(B,b)$ in the ∞-cosmos $\mathcal{C}art(\mathcal{K})_{/\mathsf{Hom}_B(B,b)}$. The element

$$\begin{tikzcd} \mathsf{Hom}_B(B,b) \arrow[rr, "(a!{,}\mathrm{id})"] \arrow[dr, equals] & & A\times \mathsf{Hom}_B(B,b) \arrow[dl, two heads, "\pi"] \\ & \mathsf{Hom}_B(B,b) & \end{tikzcd}$$

has $p_1\times \mathrm{id}\colon \mathsf{Hom}_A(a,A)\times \mathsf{Hom}_B(B,b) \to A\times \mathsf{Hom}_B(B,b)$ as its representing cocartesian fibration in $\mathcal{C}art(\mathcal{K})_{/\mathsf{Hom}_B(B,b)}$. Applying the Yoneda lemma to this representable cocartesian fibration in $\mathcal{C}art(\mathcal{K})_{/\mathsf{Hom}_B(B,b)}$, we see that restricting along this map defines an equivalence whose codomain

$$\mathsf{Fun}^{\mathrm{c}}_{A\times\mathsf{Hom}_B(B,b)}(\mathsf{Hom}_B(B,b)\to A\times\mathsf{Hom}_B(B,b), \mathsf{Hom}_B(p,b)\twoheadrightarrow A\times \mathsf{Hom}_B(B,b))$$

is the mapping quasi-category defined by the pullback

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Fun}^{\mathrm{c}}_{\mathsf{Hom}_B(B,b)}(\mathrm{id}_{\mathsf{Hom}_B(B,b)}, \mathsf{Hom}_B(p,b)\twoheadrightarrow \mathsf{Hom}_B(B,b)) \arrow[d, two heads, "(qp_0{,}p')\circ-"] \\ 1 \arrow[r, "(a!{,}\mathrm{id})"] & \mathsf{Fun}^{\mathrm{c}}_{\mathsf{Hom}_B(B,b)}(\mathrm{id}_{\mathsf{Hom}_B(B,b)}, A\times \mathsf{Hom}_B(B,b) \xrightarrow{\pi}\!\!\!\!\twoheadrightarrow \mathsf{Hom}_B(B,b)) \end{tikzcd}$$

By Lemma 7.3.3(iii) applied to the adjunction $(p_0\circ -)\dashv p_0^*$ the right-hand vertical map is isomorphic to the left-hand vertical map displayed below

$$\begin{tikzcd} \mathsf{Fun}^{\mathrm{c}}_B(\mathsf{Hom}_B(B,b)\xrightarrow{p_0}\!\!\!\!\twoheadrightarrow B, E\xrightarrow{p}\!\!\!\!\twoheadrightarrow B) \arrow[r, "\sim"] \arrow[d, two heads, "(q{,}p)\circ-"'] & \mathsf{Fun}_B(1\xrightarrow{b} B, E\xrightarrow{p}\!\!\!\!\twoheadrightarrow B) \arrow[d, two heads, "(q{,}p)\circ-"] \\ \mathsf{Fun}^{\mathrm{c}}_B(\mathsf{Hom}_B(B,b)\xrightarrow{p_0}\!\!\!\!\twoheadrightarrow \mathsf{Hom}_B(B,b), A\times B\xrightarrow{\pi}\!\!\!\!\twoheadrightarrow B) \arrow[r, "\sim"] & \mathsf{Fun}_B(1\xrightarrow{b} B, A\times B\xrightarrow{\pi}\!\!\!\!\twoheadrightarrow B) \end{tikzcd}$$

which is equivalent to the vertical map on the right by the Yoneda lemma applied to the cartesian fibration $p_0\colon \mathsf{Hom}_B(B,b)\twoheadrightarrow B$ corresponding to the element $b\colon 1\to B$. Combining these observations, we obtain an equivalence to the pullback

$$\begin{tikzcd} \mathsf{Fun}_{A\times B}(1\xrightarrow{(a,b)} A\times B, E\xrightarrow{(q,p)}\!\!\!\!\twoheadrightarrow A\times B) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Fun}_B(1\xrightarrow{b} B, E\xrightarrow{p}\!\!\!\!\twoheadrightarrow B) \arrow[d, two heads, "(q{,}p)\circ-"] \\ 1 \arrow[r, "(a{,}b)"] & \mathsf{Fun}_B(1\xrightarrow{b} B, A\times B\xrightarrow{\pi}\!\!\!\!\twoheadrightarrow B) \end{tikzcd}$$

This composite equivalence is the map defined by precomposition with the canonical element

$$(\ulcorner \mathrm{id}_a\urcorner, \ulcorner \mathrm{id}_b\urcorner)\colon 1\to \mathsf{Hom}_A(a,A)\times \mathsf{Hom}_B(B,b),$$

completing the proof. $\square$

7.3.3. Lemma.

(i) Suppose $s\colon B\to A$ is a discrete cocartesian fibration and $p\colon E\to B$ is an isofibration. Then $p\colon E\to B$ is a cocartesian fibration if and only if $sp\colon E\twoheadrightarrow A$ is a cocartesian fibration.

(ii) Suppose that s is a discrete cocartesian fibration and p and q are cocartesian fibrations. Then (g,s) defines a cartesian functor from p to q if and only if g defines a cartesian functor from sp to r.

$$\begin{tikzcd} E \arrow[r, "g"] \arrow[d, two heads, "p"'] & F \arrow[d, two heads, "q"] \\ B \arrow[r, two heads, "s"'] & A \end{tikzcd} \qquad \begin{tikzcd} E \arrow[rr, "g"] \arrow[dr, two heads, "sp"'] & & F \arrow[dl, two heads, "q"] \\ & A & \end{tikzcd} \tag{7.3.4}$$

(iii) If $s\colon B \twoheadrightarrow A$ is a discrete cocartesian fibration, then the adjunction $s \circ - \dashv s^*$

$$\mathsf{Fun}_A(E \overset{p}{\twoheadrightarrow} B \overset{s}{\twoheadrightarrow} A, F \overset{q}{\twoheadrightarrow} A) \cong \mathsf{Fun}_B(E \overset{p}{\twoheadrightarrow} B, s^*F \overset{s^*q}{\twoheadrightarrow} B)$$

between composition with and pullback along s restricts to cartesian functors:

$$\mathsf{Fun}^{\mathrm{c}}_A(E \overset{p}{\twoheadrightarrow} B \overset{s}{\twoheadrightarrow} A, F \overset{q}{\twoheadrightarrow} A) \cong \mathsf{Fun}^{\mathrm{c}}_B(E \overset{p}{\twoheadrightarrow} B, s^*F \overset{s^*q}{\twoheadrightarrow} B).$$

Proof. For (i) recall that Lemma 5.1.7 proves that cocartesian fibrations compose, with sp-cocartesian cells being the p-cocartesian lifts of s-cocartesian cells. This proves that sp is cocartesian if p is, and the converse follows as well by the proof of that result: if s is a discrete cocartesian fibration, then any 2-cell with codomain B is s-cocartesian, so we may take the p-cocartesian cells to be precisely the sp-cocartesian cells.

For (ii), note that in proving (i), we have just argued that sp-cocartesian cells are precisely the same as p-cocartesian cells when p is a cocartesian fibration and s is a discrete cocartesian fibration. This proves that the two notions of cartesian functor coincide.[1]

Finally (iii) follows immediately from (ii) and Proposition 5.1.20, which tells us that a map to the pullback of a cocartesian fibration along $s\colon B \to A$ defines a cartesian functor over B if and only if the square as displayed on the left of (7.3.4) defines a cartesian functor. □

7.3.5. Corollary. *The inclusion*

$$_{A\backslash}\mathcal{F}ib_{/B} \hookrightarrow \mathcal{K}_{/A\times B}$$

admits a left biadjoint defined by sending a two-sided isofibration $A \overset{a}{\leftarrow} X \overset{b}{\rightarrow} B$ to the composite two-sided fibration

$$\begin{tikzcd} & & \mathsf{Hom}_A(a,A) \underset{X}{\times} \mathsf{Hom}_B(B,b) \arrow[dl, two heads] \arrow[dr, two heads] \arrow[d, phantom, "\lrcorner" rotate=-45] & & \\ & \mathsf{Hom}_A(a,A) \arrow[dl, two heads, "p_1"'] \arrow[dr, two heads, "p_0"] & {} & \mathsf{Hom}_B(B,b) \arrow[dl, two heads, "p_1"'] \arrow[dr, two heads, "p_0"] & \\ A & & X & & B \end{tikzcd}$$

and equipped with a natural equivalence

$$\mathsf{Fun}^{\mathrm{c}}_{A\times B}\left(\begin{array}{c} \mathsf{Hom}_A(a,A) \underset{X}{\times} \mathsf{Hom}_B(B,b) \\ \downarrow \\ A\times B \end{array} , \begin{array}{c} E \\ \downarrow \\ A\times B \end{array} \right) \xrightarrow{\sim} \mathsf{Fun}_{A\times B}\left(\begin{array}{c} X \\ \downarrow{\scriptstyle (a,b)} \\ A\times B \end{array} , \begin{array}{c} E \\ \downarrow \\ A\times B \end{array} \right).$$

[1] If s is cocartesian but not discrete, then if the left-hand diagram defines a cartesian functor, then so does the right one, but the converse no longer holds.

of functor spaces.

Proof. Let $(q,p)\colon E \twoheadrightarrow A \times B$ be a two-sided fibration in $\mathcal{K}$. Then since $- \times X\colon \mathcal{K} \to \mathcal{K}_{/X}$ is a cosmological functor, the span

$$A \times X \overset{q\times\mathrm{id}}{\twoheadleftarrow} E \times X \overset{p\times\mathrm{id}}{\twoheadrightarrow} B \times X$$

is a two-sided fibration in $\mathcal{K}_{/X}$. The maps $a\colon X \to A$ and $b\colon X \to B$ induce a pair of elements $(a,\mathrm{id})\colon X \to A \times X$ and $(b,\mathrm{id})\colon X \to B \times X$ in $\mathcal{K}_{/X}$ which are respectively covariantly and contravariantly represented by

$$(p_1,p_0)\colon \mathsf{Hom}_A(a,A) \twoheadrightarrow A \times X \qquad \text{and} \qquad (p_1,p_0)\colon \mathsf{Hom}_B(B,b) \twoheadrightarrow B \times X.$$

Applying Theorem 7.3.2 in $\mathcal{K}_{/X}$ to this two-sided isofibration and pair of elements, we find that restriction along the canonical map

$$i\colon X \to \mathsf{Hom}_A(a,A) \underset{X}{\times} \mathsf{Hom}_B(B,b)$$

defines an equivalence of quasi-categories

$$\mathsf{Fun}^{\mathrm{c}}_{A\times B\times X}\left(\begin{array}{c} \mathsf{Hom}_A(a,A) \underset{X}{\times} \mathsf{Hom}_B(B,b) \\ \downarrow \\ A \times B \times X \end{array}, \begin{array}{c} E \times X \\ \downarrow \\ A \times B \times X \end{array} \right) \xrightarrow{\sim} \mathsf{Fun}_{A\times B\times X}\left(\begin{array}{c} X \\ {\scriptstyle (a,b,\mathrm{id})}\downarrow \\ A \times B \times X \end{array}, \begin{array}{c} E \times X \\ \downarrow \\ A \times B \times X \end{array} \right).$$

Transposing the domain and codomain across the adjunction

$$\begin{tikzcd} \mathcal{K} \arrow[r, bend right, "-\times X"'] \arrow[r, phantom, "\perp"] & \mathcal{K}_{/X} \arrow[l, bend right, "\Sigma_X"'] \end{tikzcd}$$

yields the equivalence of functor spaces

$$\mathsf{Fun}^{\mathrm{c}}_{A\times B}\left(\begin{array}{c} \mathsf{Hom}_A(a,A) \underset{X}{\times} \mathsf{Hom}_B(B,b) \\ \downarrow \\ A \times B \end{array}, \begin{array}{c} E \\ \downarrow \\ A \times B \end{array} \right) \xrightarrow{\sim} \mathsf{Fun}_{A\times B}\left(\begin{array}{c} X \\ {\scriptstyle (a,b)}\downarrow \\ A \times B \end{array}, \begin{array}{c} E \\ \downarrow \\ A \times B \end{array} \right)$$

of the statement. ☐

The argument used to establish the equivalence of functor spaces in Corollary 7.3.5 works for any pair of generalized elements $a\colon X \to A$ and $b\colon X \to B$, whether or not this span defines a two-sided isofibration. In the case of spans represented by a single non-identity functor a "one-sided" version of Corollary 7.3.5, which is much more simply established, may be preferred:

7.3.6. PROPOSITION (one-sided Yoneda for two-sided fibrations). *For any two-sided isofibration $A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$ and functor $f \colon A \to B$, restriction along the canonical functor*

$$\begin{tikzcd} & A \arrow[dl, equals] \arrow[dr, "f"] \arrow[d, "\ulcorner \mathrm{id}_f \urcorner"] & \\ A & \mathsf{Hom}_B(B,f) \arrow[l, two heads, "p_1"] \arrow[r, two heads, "p_0"'] & B \end{tikzcd}$$

induce equivalences of functor spaces

$$\mathsf{Fun}^{\mathrm{c}}_{A\times B}\left(\begin{array}{c} \mathsf{Hom}_B(B,f) \\ \downarrow\!\!\!\downarrow {\scriptstyle (p_1,p_0)} \\ A\times B \end{array}, \begin{array}{c} E \\ \downarrow\!\!\!\downarrow {\scriptstyle (q,p)} \\ A\times B \end{array}\right) \xrightarrow{\sim} \mathsf{Fun}_{A\times B}\left(\begin{array}{c} A \\ \downarrow {\scriptstyle (\mathrm{id}_A,f)} \\ A\times B \end{array}, \begin{array}{c} E \\ \downarrow\!\!\!\downarrow {\scriptstyle (q,p)} \\ A\times B \end{array}\right).$$

PROOF. This follows immediately from the external Yoneda lemma of Theorem 5.5.5 applied to the element $(\mathrm{id}_A, f) \colon A \to A \times B$ and the cartesian fibration $(q,p) \colon E \twoheadrightarrow A \times B$ in the ∞-cosmos $\mathcal{coCart}(\mathcal{K})_{/A}$. □

Exercises.

7.3.i. EXERCISE. State and prove the other one-sided Yoneda lemma for two-sided fibrations, establishing an equivalence of functor spaces below-left induced by restricting along the functor below-right

$$\mathsf{Fun}^{\mathrm{c}}_{A\times B}\left(\begin{array}{c} \mathsf{Hom}_A(f,A) \\ \downarrow\!\!\!\downarrow {\scriptstyle (p_1,p_0)} \\ A\times B \end{array}, \begin{array}{c} E \\ \downarrow\!\!\!\downarrow {\scriptstyle (q,p)} \\ A\times B \end{array}\right) \xrightarrow{\sim} \mathsf{Fun}_{A\times B}\left(\begin{array}{c} B \\ \downarrow {\scriptstyle (f,\mathrm{id}_B)} \\ A\times B \end{array}, \begin{array}{c} E \\ \downarrow\!\!\!\downarrow {\scriptstyle (q,p)} \\ A\times B \end{array}\right) \qquad \begin{tikzcd} & B \arrow[dl, "f"'] \arrow[dr, equals] \arrow[d, "i"] & \\ A & \mathsf{Hom}_A(f,A) \arrow[l, two heads, "p_1"] \arrow[r, two heads, "p_0"'] & B \end{tikzcd}$$

7.4. Modules as discrete two-sided fibrations

We are not so much interested in two-sided fibrations but the special case of those two-sided fibrations that are discrete as objects in $\mathcal{K}_{/A\times B}$.

7.4.1. DEFINITION. A **module from A to B** is a two-sided fibration $A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$ that is a discrete object in ${}_{A\backslash}\mathcal{Fib}(\mathcal{K})_{/B}$.

Note that an object in the replete subcosmos ${}_{A\backslash}\mathcal{Fib}(\mathcal{K})_{/B} \hookrightarrow \mathcal{K}_{/A\times B}$ is discrete in there if and only if it is discrete as an object of $\mathcal{K}_{/A\times B}$; see Exercise 6.4.v. Our work in this chapter enables us to give a direct characterization of modules:

7.4.2. LEMMA. *A two-sided isofibration $A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$ defines a module from A to B if and only if it is*

(i) cocartesian on the left,
(ii) cartesian on the right,
(iii) and discrete as an object of $\mathcal{K}_{/A\times B}$.

PROOF. This follows immediately from the characterization of two-sided fibrations given as Theorem 7.1.4(iv). □

Unpacking the definition, we easily establish the following properties of modules.

7.4.3. LEMMA. *If $A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$ defines a module from A to B, then*

(i) *The functors*

$$\begin{tikzcd} E \arrow[rr, two heads, "{(q,p)}"] \arrow[dr, two heads, "p"'] & & A \times B \arrow[dl, two heads, "\pi"] \\ & B & \end{tikzcd} \qquad \begin{tikzcd} E \arrow[rr, two heads, "{(q,p)}"] \arrow[dr, two heads, "q"'] & & A \times B \arrow[dl, two heads, "\pi"] \\ & A & \end{tikzcd}$$

define a discrete cocartesian fibration and a discrete cartesian fibration, respectively, in the slice ∞-cosmoi $\mathcal{K}_{/B}$ and $\mathcal{K}_{/A}$.

(ii) *The functors $q\colon E \twoheadrightarrow A$ and $p\colon E \twoheadrightarrow B$ respectively define a cocartesian fibration and a cartesian fibration in $\mathcal{K}$.*

(iii) *For any 2-cell χ with codomain E, χ is p-cartesian if and only if $q\chi$ is invertible, and χ is q-cocartesian if and only if $p\chi$ is invertible.*

(iv) *In particular, any 2-cell that is fibered over $A \times B$ is both p- and q-cocartesian and any map of spans from a two-sided fibration*

$$\begin{tikzcd} & F \arrow[dl, two heads, "s"'] \arrow[dr, two heads, "r"] \arrow[dd, "g"] & \\ A & & B \\ & E \arrow[ul, two heads, "q"] \arrow[ur, two heads, "p"'] & \end{tikzcd}$$

to a module defines a cartesian functor of two-sided fibrations in the sense of Lemma 7.1.10, and also a cartesian functor from s to q and from r to p.

PROOF. By Lemma 7.1.1, conditions (i) and (iii) together assert exactly that $(q,p)\colon E \twoheadrightarrow A \times B$ defines a discrete cocartesian fibration in $\mathcal{K}_{/B}$, proving (i). Statement (ii) holds for any two-sided fibration; the point of reasserting it here is that it is not necessarily the case that the legs of a module are themselves discrete fibrations (see Exercise 7.4.i).

One direction of statement (iii) is proven as Lemma 7.1.1(iv) while the converse is proven in Exercise 7.1.i. Statement (iv) follows. $\square$

An important property of modules is that they are stable under pullback:

7.4.4. PROPOSITION. *If $A \twoheadleftarrow E \twoheadrightarrow B$ is a module from A to B and $a\colon A' \to A$ and $b\colon B' \to B$ are any pair of functors, then the pullback defines a module $A' \twoheadleftarrow E(b,a) \twoheadrightarrow B'$ from A' to B'.*

PROOF. The cosmological functor

$$(a,b)^*\colon {}_{A\backslash}\mathcal{F}ib(\mathcal{K})_{/B} \longrightarrow {}_{A'\backslash}\mathcal{F}ib(\mathcal{K})_{/B'}$$

of Proposition 7.2.4 preserves discrete objects in these ∞-cosmoi. $\square$

Applying Proposition 7.4.4 to a pair of elements $a\colon 1 \to A$ and $b\colon 1 \to B$, we see that a module from A to B is a two-sided fibration whose fibers $E(b,a)$ are discrete objects. The converse does not generally hold: being discrete as an object of the sliced ∞-cosmos $\mathcal{K}_{/A\times B}$ is a stronger condition than merely having discrete fibers. However, in the ∞-cosmos $\mathcal{QC}at$ the point 1 is a generator in a suitable sense and we have:

7.4.5. LEMMA. *If $E \twoheadrightarrow A \times B$ is a two-sided fibration of quasi-categories whose fibers are Kan complexes, then E is a module.*

Proof. By definition, $E \twoheadrightarrow A \times B$ defines a discrete object in $\mathcal{QC}at_{/A\times B}$ if any 1-simplex in E^X that lies over a degenerate 1-simplex in $(A \times B)^X$ is an isomorphism. As isomorphisms in functor quasi-categories are determined pointwise (see Lemma 12.2.1), it suffices to consider the case $X = \mathbb{1}$ and now this reduces to the hypothesis that the fibers $E(b, a)$ over any pair of points $(a, b) \in A \times B$ are Kan complexes. $\square$

The prototypical example of a module is given by the arrow ∞-category construction.

7.4.6. Proposition. *For any ∞-category A, the arrow ∞-category $A^{\mathbb{2}}$ defines a module from A to A.*

The fact that $A^{\mathbb{2}} \twoheadrightarrow A \times A$ is discrete is related to but stronger than the fact that each fiber over a pair of elements in A, the **internal hom-space** between those elements of A, is a discrete ∞-category, proven in Proposition 3.4.10.

Proof. Proposition 7.1.9 proves that $(p_1, p_0) \colon A^{\mathbb{2}} \twoheadrightarrow A \times A$ is a two-sided fibration, so it remains only to verify the discreteness condition, which we can do in $\mathcal{K}_{/A\times A}$ Discreteness of $A^{\mathbb{2}} \twoheadrightarrow A \times A$ as an object of $\mathcal{K}_{/A\times A}$ is an immediate consequence of 2-cell conservativity of Proposition 3.2.5: if γ is any 2-cell with codomain $A^{\mathbb{2}}$ so that $p_0\gamma$ and $p_1\gamma$ are invertible, then γ is itself invertible. $\square$

By the construction of (3.4.2) and Proposition 7.4.4:

7.4.7. Corollary. *The comma ∞-category $\mathrm{Hom}_A(f, g) \twoheadrightarrow C \times B$ associated to a pair of functors $f \colon B \to A$ and $g \colon C \to A$ defines a module from C to B.* $\square$

Two special cases of these comma modules, those studied in §3.5, deserve a special name:

7.4.8. Definition. To any functor $f \colon A \to B$ between ∞-categories

(i) the module $\mathrm{Hom}_B(B, f)$ from A to B is **right** or **covariantly represented** by f, while
(ii) the module $\mathrm{Hom}_B(f, B)$ from B to A is **left** or **contravariantly represented** by f.

More generally, a module $A \overset{E}{\nrightarrow} B$ is **covariantly** or **contravariantly represented** by f if E is equivalent to the left or right represented modules over $A \times B$.

As we saw in §5.5, the Yoneda lemma for two-sided fibrations simplifies when mapping into a module on account of the observation in Lemma 7.4.3(iv) that any map of spans from a two-sided fibration to a module defines a cartesian functor.

7.4.9. Theorem (Yoneda lemma for modules). *For any elements $a \colon X \to A$ and $b \colon X \to B$ and any module $A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$, restriction along $(\ulcorner \mathrm{id}_a \urcorner, \ulcorner \mathrm{id}_b \urcorner) \colon X \to \mathrm{Hom}_A(a, A) \underset{X}{\times} \mathrm{Hom}_B(B, b)$ defines an equivalence of Kan complexes*

$$\mathrm{Fun}_{A\times B}\left(\begin{array}{c} \mathrm{Hom}_A(a, A) \underset{X}{\times} \mathrm{Hom}_B(B, b) \\ \big\downarrow\!\!\downarrow \\ A \times B \end{array}, \begin{array}{c} E \\ \big\downarrow\!\!\downarrow \\ A \times B \end{array} \right) \xrightarrow{\sim} \mathrm{Fun}_{A\times B}\left(\begin{array}{c} X \\ \big\downarrow{\scriptstyle (a,b)} \\ A \times B \end{array}, \begin{array}{c} E \\ \big\downarrow\!\!\downarrow \\ A \times B \end{array} \right).$$

Similarly for any functors $f\colon A \to B$ or $g\colon B \to A$, restriction along $\ulcorner \mathrm{id}_f \urcorner\colon A \to \mathsf{Hom}_B(B,f)$ or $\ulcorner \mathrm{id}_g \urcorner\colon B \to \mathsf{Hom}_A(g,A)$ define equivalences of Kan complexes

$$\mathsf{Fun}_{A\times B}\left(\begin{array}{c}\mathsf{Hom}_B(B,f)\\ \downarrow\!\!\!\!\downarrow \\ A\times B\end{array}, \begin{array}{c}E\\ \downarrow\!\!\!\!\downarrow\\ A\times B\end{array}\right) \xrightarrow{\sim} \mathsf{Fun}_{A\times B}\left(\begin{array}{c}A\\ \downarrow{\scriptstyle (\mathrm{id},f)}\\ A\times B\end{array}, \begin{array}{c}E\\ \downarrow\!\!\!\!\downarrow\\ A\times B\end{array}\right)$$

$$\mathsf{Fun}_{A\times B}\left(\begin{array}{c}\mathsf{Hom}_A(g,A)\\ \downarrow\!\!\!\!\downarrow \\ A\times B\end{array}, \begin{array}{c}E\\ \downarrow\!\!\!\!\downarrow\\ A\times B\end{array}\right) \xrightarrow{\sim} \mathsf{Fun}_{A\times B}\left(\begin{array}{c}B\\ \downarrow{\scriptstyle (g,\mathrm{id})}\\ A\times B\end{array}, \begin{array}{c}E\\ \downarrow\!\!\!\!\downarrow\\ A\times B\end{array}\right). \qquad \square$$

Exercises.

7.4.i. Exercise. Demonstrate by means of an example that if $A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$ defines a module from A to B then it is not necessarily the case that $p\colon E \twoheadrightarrow B$ is a discrete cartesian fibration or $q\colon E \twoheadrightarrow A$ is a discrete cocartesian fibration.

7.4.ii. Exercise.

(i) Explain why the two-sided fibration $(p_{n-1}, p_0)\colon A^{\mathbb{n}} \twoheadrightarrow A \times A$ of Proposition 7.1.9 does not define a module for $n > 2$.

(ii) Conclude that the horizontal composite of modules, as defined in Proposition 7.2.6, is not necessarily a module.

CHAPTER 8

The calculus of modules

The calculus of modules between ∞-categories bears a strong resemblance to the calculus of (bi)modules between unital rings. Here ∞-categories take the place of rings, with functors between ∞-categories playing the role of ring homomorphisms, which we display vertically on the table below. A module E from A to B, like the two-sided fibrations considered in Chapter 7, is an ∞-category on which A "acts on the left" and B "acts on the right" and these actions commute; this is analogous to the situation for bimodules in ring theory and explains our choice of terminology.[1] Modules will now be depicted as $A \overset{E}{\nrightarrow} B$ whenever explicit names for the legs of the constituent span are not needed.

unital rings	A	∞-categories
ring homomorphisms	$\begin{array}{c} A' \\ {\scriptstyle a}\big\downarrow \\ A \end{array}$	∞-functors
bimodules between rings	$A \overset{E}{\nrightarrow} B$	modules between ∞-categories
module maps	$\begin{tikzcd} A' \arrow[r, "E'", "\mid"{marking}] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow"] & B' \arrow[d, "b"] \\ A \arrow[r, "E"', "\mid"{marking}] & B \end{tikzcd}$	module maps

Finally, there is a notion of module map, that we shall introduce below, whose boundary in the most general case is a square comprised of two modules and two functors as above. In ring theory, a module map with this boundary is given by an A'–B' module homomorphism $E' \to E(b, a)$, whose codomain is the A'–B' bimodule defined by restricting the scalar multiplication in the A–B module E along the ring homomorphisms a and b.

The analogy extends deeper than this: unital rings, ring homomorphisms, bimodules, and module maps define a *proarrow equipment*, in the sense of Wood [**101**].[2] Our main result in this chapter is Theorem 8.2.6, which asserts that ∞-categories, functors, modules, and module maps in any ∞-cosmos define a *virtual equipment*, in the sense of Cruttwell and Shulman [**27**].

As a first step, in §8.1 we introduce the double category of two-sided fibrations, which restricts to define a *virtual double category* of modules. A *double category* is a sort of 2-dimensional category with objects; two varieties of 1-morphisms, the "horizontal" and the "vertical"; and 2-dimensional cells fitting

[1] In the 1- and ∞-categorical literature, the names "profunctor," "correspondence," and "distributor" are all used as synonyms for "module."

[2] This can be seen as a special case of the prototypical equipment comprised of $\mathcal{V}$-categories, $\mathcal{V}$-functors, $\mathcal{V}$-modules, and $\mathcal{V}$-natural transformations between then, for any closed symmetric monoidal category $\mathcal{V}$. The equipment for rings is obtained from the case where $\mathcal{V}$ is the category of abelian groups by restricting to abelian group enriched categories with a single object.

into “squares” whose boundaries consist of horizontal and vertical 1-morphisms with compatible domains and codomains. A motivating example from abstract algebra is the double category of modules: objects are rings, vertical morphisms are ring homomorphisms, horizontal morphisms are bimodules, and whose squares are bimodule homomorphisms. In the literature, this sort of structure is sometimes called a *pseudo double category* — morphisms and squares compose strictly in the “vertical” direction but only up to isomorphism in the “horizontal” direction — but we'll refer to this simply as a “double category” as it is the only variety that we will consider.

Our aim in §8.1 is to describe a similar structure whose objects and vertical morphisms are the ∞-categories and functors in any fixed ∞-cosmos, whose horizontal morphisms are modules, and whose squares are module maps, as will be defined in 8.1.6. If the horizontal morphisms are replaced by the larger class of two-sided fibrations, this does define a double category with the horizontal composition operation defined by Proposition 7.2.6. However, on account of Exercise 7.4.ii, the horizontal composition of two-sided fibrations does not preserve the class of modules: the arrow ∞-category $A^{\mathbb{2}}$ defines a module from A to A whose horizontal composite with itself is equivalent to the two-sided fibration $(p_2, p_0)\colon A^{\mathbb{3}} \twoheadrightarrow A \times A$ of Proposition 7.1.9, which is not discrete in $\mathcal{K}_{/A\times A}$. To define a genuine “tensor product for modules” operation would require a two-stage construction: first forming the pullback that defines a composite two-sided fibration as in Proposition 7.2.6, and then reflecting this into a two-sided *discrete* fibration by means of some sort of “homotopy coinverter” construction. As colimits that are not within the purview of the axioms of an ∞-cosmos, this presents somewhat of an obstacle.

Rather than leave the comfort of our axiomatic framework in pursuit of a double category of modules, we instead describe the structure that naturally arises within the axiomatization: it turns out to be familiar to category theorists and robust enough for our desired applications. We first demonstrate that ∞-categories, functors, modules, and module maps assemble into a *virtual double category*, a weaker structure than a double category in which cells are permitted to have a multi horizontal source, as a “virtual” replacement for horizontal composition of modules.

Once the definition of a virtual equipment is given in §8.2, these axioms are very easily checked. The final two sections are devoted to exploring the consequences of this structure, which will be put to full use in Chapter 9, which develops the formal category theory of ∞-categories by introducing Kan extensions in the virtual equipment of modules. In §8.3, we explain how certain horizontal composites of modules can be recognized in the virtual equipment, even if the general construction of the tensor product of an arbitrary composable pair of modules is not known. The final §8.4 collects together many special properties of the modules $A \overset{\mathrm{Hom}_B(B,f)}{\nrightarrow} B$ and $B \overset{\mathrm{Hom}_B(f,B)}{\nrightarrow} A$ represented by a functor $f\colon A \to B$ of ∞-categories, revisiting some of the properties first established in §3.5.

8.1. The double category of two-sided fibrations

Our first task is to define the 2-dimensional morphisms in the double categories that we will introduce.

8.1.1. Definition. Let $A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$ and $A \overset{s}{\twoheadleftarrow} F \overset{r}{\twoheadrightarrow} B$ be two-sided isofibrations. A **map of spans** from $A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$ to $A \overset{s}{\twoheadleftarrow} F \overset{r}{\twoheadrightarrow} B$ is a fibered isomorphism class of strictly commuting functors

$$\begin{tikzcd} & E \arrow[dl, two heads, "q"'] \arrow[dr, two heads, "p"] \arrow[dd, "g"] & \\ A & & B \\ & F \arrow[ul, two heads, "s"] \arrow[ur, two heads, "r"'] & \end{tikzcd}$$

where two such functors g and g' are considered equivalent if there exists a natural isomorphism $\alpha\colon g \cong g'$ so that $r\alpha = \mathrm{id}_q$ and $s\alpha = \mathrm{id}_p$.

8.1.2. Definition. Let $A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$ and $A \overset{s}{\twoheadleftarrow} F \overset{r}{\twoheadrightarrow} B$ be two-sided fibrations. A **map of two-sided fibrations** from $A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$ to $A \overset{s}{\twoheadleftarrow} F \overset{r}{\twoheadrightarrow} B$ is a map of spans in which any and hence every representing map

$$\begin{tikzcd} & E \arrow[dl, two heads, "q"'] \arrow[dr, two heads, "p"] \arrow[dd, "g"] & \\ A & & B \\ & F \arrow[ul, two heads, "s"] \arrow[ur, two heads, "r"'] & \end{tikzcd}$$

defines a cartesian functor of two-sided fibrations defined in Lemma 7.1.10.

8.1.3. Definition. Let $A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$ and $A \overset{s}{\twoheadleftarrow} F \overset{r}{\twoheadrightarrow} B$ be modules. A **map of modules** from E to F is just a map of spans from $A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$ to $A \overset{s}{\twoheadleftarrow} F \overset{r}{\twoheadrightarrow} B$, that is a fibered isomorphism class of strictly commuting functors

$$\begin{tikzcd} & E \arrow[dl, two heads, "q"'] \arrow[dr, two heads, "p"] \arrow[dd, "g"] & \\ A & & B \\ & F \arrow[ul, two heads, "s"] \arrow[ur, two heads, "r"'] & \end{tikzcd}$$

8.1.4. Observation (the 1-categories of spans and maps). The 1-category of two-sided isofibrations from A to B and maps of spans may be obtained as a quotient of the quasi-categorically enriched category $\mathcal{K}_{/A\times B}$, or of its homotopy 2-category $\mathfrak{h}(\mathcal{K}_{/A\times B})$, or of the slice homotopy 2-category $\mathfrak{h}\mathcal{K}_{/A\times B}$: it is the 1-category with the same set of objects and in which the morphisms are isomorphism classes of 0-arrows.

Similarly, the 1-category of two-sided fibrations from A to B and maps of two-sided fibrations may be obtained as a quotient of the quasi-categorically enriched category ${}_{A\backslash}\mathcal{F}ib(\mathcal{K})_{/B}$, or of its homotopy 2-category $\mathfrak{h}({}_{A\backslash}\mathcal{F}ib(\mathcal{K})_{/B})$: it is the 1-category with the same set of objects and in which the morphisms are isomorphism classes of 0-arrows.

By Lemma 7.4.3, the 1-category of modules from A to B and modules maps is a full subcategory of either of the two 1-categories just considered.

The 1-categories of Observation 8.1.4 are of interest because they precisely capture the correct notion of equivalence between two-sided isofibrations, two-sided fibrations, or modules first introduced in Definition 3.2.7.

8.1.5. Lemma.

(i) A pair of two-sided isofibrations are equivalent in $\mathcal{K}_{/A\times B}$ if and only if they are isomorphic in the 1-category of spans from A to B.

(ii) A pair of two-sided fibrations are equivalent in ${}_{A\backslash}\mathcal{F}ib(\mathcal{K})_{/B}$ if and only if they are isomorphic in the 1-category of two-sided fibrations from A to B.

(iii) A pair of modules are equivalent over A times B if and only if they are isomorphic in the 1-category of modules from A to B. □

Each of the Definitions 8.1.3 admits a common generalization, which defines the 2-dimensional maps inhabiting squares.

8.1.6. Definition (maps in squares). A **map of modules** or **map of two-sided fibrations** or **map of two-sided isofibration** from $A' \overset{q'}{\twoheadleftarrow} E' \overset{p'}{\twoheadrightarrow} B'$ to $A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$ over $a\colon A' \to A$ and $b\colon B' \to B$ is

$$\begin{tikzcd} A' \arrow[r, "E'", "\shortmid"{marking}] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow"] & B' \arrow[d, "b"] \\ A \arrow[r, "E"', "\shortmid"{marking}] & B \end{tikzcd} \qquad \begin{tikzcd} E' \arrow[r, "g"] \arrow[d, two heads, "{(q',p')}"'] & E \arrow[d, two heads, "{(q,p)}"] \\ A' \times B' \arrow[r, "a\times b"] & A \times B \end{tikzcd}$$

is a fibered isomorphism class of strictly commuting functors g as displayed above-right, which in the case of two-sided fibrations must preserve the cartesian and cocartesian transformations, where two such functors g and g' are considered equivalent if there exists a natural isomorphism $\alpha\colon g \cong g'$ so that $q\alpha = \mathrm{id}_{aq'}$ and $p\alpha = \mathrm{id}_{bp'}$.

In the case of modules or two-sided isofibrations, the functor-space $\mathsf{Fun}_{a\times b}(E', E)$ of maps from E' to E over $a \times b$ is defined by the pullback

$$\begin{tikzcd} \mathsf{Fun}_{a\times b}(E',E) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Fun}(E',E) \arrow[d, two heads, "{(q,p)}"] \\ \mathbb{1} \arrow[r, "{(aq',bp')}"'] & \mathsf{Fun}(E', A\times B) \end{tikzcd}$$

In the case of two-sided fibrations, the functor space is taken to be the full sub quasi-category

$$\mathsf{Fun}^{\mathrm{c}}_{a\times b}(E',E) \subset \mathsf{Fun}_{a\times b}(E',E)$$

of all n-simplices whose vertices define cartesian functors.

We occasionally extend the notion of map to allow the domain to be an identity span $C \xleftarrow{\mathrm{id}_C} C \xrightarrow{\mathrm{id}_C} C$, but unless the domain is the identity span, we always require the codomain to be at least a two-sided isofibration.

We now introduce the double categories of isofibrations and of two-sided fibrations. These structures can be viewed either as a collection of data present in the homotopy 2-category $\mathfrak{h}\mathcal{K}$ of an ∞-cosmos or as a quotient of quasi-categorically enriched structures, presented by a non-unital internal category defined up to natural isomorphism in the category of ∞-cosmoi and cosmological functors; see Exercise 8.1.i. For economy of language, we adopt the former approach.

8.1.7. Proposition (the double category of two-sided (iso)fibrations). *The homotopy 2-category of an ∞-cosmos supports a* ***double category***[3] ***of two-sided isofibrations*** *whose:*

- *objects are ∞-categories,*
- *vertical arrows are functors,*
- *horizontal arrows $A \overset{E}{\twoheadrightarrow} B$ are two-sided isofibrations $A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$, and*
- *2-cells with boundary*

$$\begin{tikzcd} A' \arrow[r, "E'", "|" marking] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow"] & B' \arrow[d, "b"] \\ A \arrow[r, "E"', "|" marking] & B \end{tikzcd}$$

  are maps of two-sided isofibrations as defined in 8.1.6, or equivalently, are isomorphism classes of objects in the quasi-category $\mathsf{Fun}_{a\times b}(E', E)$.

Vertical composition of arrows and 2-cells is by composition in $\mathcal{K}$, while horizontal composition of arrows and 2-cells is by pullback, which is well-defined and associative up to isomorphism. The ***double category of two-sided fibrations*** *is the sub double category that has the same objects and vertical arrows but whose:*

- *horizontal arrows $A \overset{E}{\twoheadrightarrow} B$ are two-sided fibrations $A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$ and*
- *2-cells with boundary*

$$\begin{tikzcd} A' \arrow[r, "E'", "|" marking] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow"] & B' \arrow[d, "b"] \\ A \arrow[r, "E"', "|" marking] & B \end{tikzcd}$$

  are maps of two-sided fibrations as defined in 8.1.6, or equivalently, are isomorphism classes of objects in $\mathsf{Fun}^{\mathrm{c}}_{a\times b}(E', E)$.

Proof. The composition of horizontal arrows is defined in Proposition 7.2.6, while the horizontal composition of 2-cells is defined in Exercise 7.2.i. By simplicial functoriality of pullback and composition in $\mathcal{K}$, both constructions are associative up to canonical natural isomorphism. $\square$

8.1.8. Remark (why we left out the horizontal unit). We could have formally added the identity span $\Delta\colon A \to A \times A$ to serve as a horizontal unit in the double categories of Proposition 8.1.7 but we find it less confusing to leave them out because when we restrict to the structure of greatest interest, the virtual equipment category of modules, we will see that the arrow ∞-category $A \overset{\mathsf{Hom}_A}{\twoheadrightarrow} A$ plays the role of the horizontal unit for composition in a sense to be described in Proposition 8.2.4.

8.1.9. Definition (virtual double category). A **virtual double category** consists of

- a category of **objects** and **vertical arrows**
- for any pair of objects A, B, a class of **horizontal arrows** $A \twoheadrightarrow B$

[3]As discussed previously, our double categories support vertical composition laws that are strictly unital and associative but horizontal composition laws that are only associative and unital up to isomorphism. For reasons to be explained in Remarks 8.1.8 and 8.1.13, we choose not to require a horizontal unit arrow or 2-cell.

- **cells**, with boundary depicted as follows

$$
\begin{tikzcd}
A_0 \arrow[r, "E_1", "\shortmid"{marking}] \arrow[d, "f"'] & A_1 \arrow[r, "E_2", "\shortmid"{marking}] & \cdots \arrow[r, "E_n", "\shortmid"{marking}] & A_n \arrow[d, "g"] \\
B_0 \arrow[rrr, "F"', "\shortmid"{marking}] \arrow[rrr, phantom, shift left=4ex, "\Downarrow"] & & & B_n
\end{tikzcd}
\tag{8.1.10}
$$

including those whose horizontal source has length zero, in the case $A_0 = A_n$.
- a **composite cell** as below-right, for any configuration as below-left

$$
\begin{tikzcd}
A_0 \arrow[r, dotted, "{E_{11},\ldots,E_{1k_1}}", "\shortmid"{marking}] \arrow[d, "f_0"'] \arrow[dr, phantom, "\Downarrow"] & A_1 \arrow[r, dotted, "{E_{21},\ldots,E_{2k_2}}", "\shortmid"{marking}] \arrow[d, "f_1"] \arrow[dr, phantom, "\Downarrow"] & \cdots \arrow[r, dotted, "{E_{n1},\ldots,E_{nk_n}}", "\shortmid"{marking}] \arrow[d, "\cdots"] \arrow[dr, phantom, "\Downarrow"] & A_n \arrow[d, "f_n"] \\
B_0 \arrow[r, "F_1", "\shortmid"{marking}] \arrow[d, "g"'] & B_1 \arrow[r, "F_2", "\shortmid"{marking}] & \cdots \arrow[r, "F_n", "\shortmid"{marking}] & B_n \arrow[d, "h"] \\
C_0 \arrow[rrr, "G"', "\shortmid"{marking}] \arrow[rrr, phantom, shift left=4ex, "\Downarrow"] & & & C_n
\end{tikzcd}
\quad =: \quad
\begin{tikzcd}
A_0 \arrow[r, "E_{11}", "\shortmid"{marking}] \arrow[d, "gf_0"'] & \cdot \arrow[r, dotted, "{E_{12},\ldots,E_{nk_n-1}}", "\shortmid"{marking}] & \cdot \arrow[r, "E_{nk_n}", "\shortmid"{marking}] & A_n \arrow[d, "hf_n"] \\
C_0 \arrow[rrr, "G"', "\shortmid"{marking}] \arrow[rrr, phantom, shift left=4ex, "\Downarrow"] & & & C_n
\end{tikzcd}
$$

- an **identity cell** for every horizontal arrow

$$
\begin{tikzcd}
A \arrow[r, "E", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}_E"] & B \arrow[d, equal] \\
A \arrow[r, "E"', "\shortmid"{marking}] & B
\end{tikzcd}
$$

so that composition of cells is associative and unital in the usual multi-categorical sense.

8.1.11. Lemma. *The double category of two-sided isofibrations and the double category of two-sided fibrations extend to a virtual double category in which the*

- *objects are ∞-categories,*
- *vertical arrows are functors,*
- *horizontal arrows are two-sided isofibrations or two-sided fibrations as appropriate, and*
- *n-ary cells* (8.1.10) *are 2-cells*

$$
\begin{tikzcd}
A_0 \arrow[r, "E_1 \underset{A_1}{\times} \cdots \underset{A_{n-1}}{\times} E_n", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow"] & A_n \arrow[d, "g"] \\
B_0 \arrow[r, "F"', "\shortmid"{marking}] & B_n
\end{tikzcd}
\tag{8.1.12}
$$

whose single vertical source is the $(n-1)$-fold pullback of the sequence of spans comprising the vertical source in (8.1.10).

Proof. The only thing perhaps worth commenting on is the nullary cells which have an empty sequence as their vertical domain

$$\begin{tikzcd} A \arrow[r, equals, "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow"] & A \arrow[d, "g"] \\ B \arrow[r, "|" marking, "F"'] & C \end{tikzcd}$$

which we interpret as a 0-fold pullback, this being the identity span from A to A. So the nullary cells displayed above are fibered isomorphism classes of maps

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd, "h"] & \\ B & & C \\ & F \arrow[ul, two heads, "s"] \arrow[ur, two heads, "r"'] & \end{tikzcd}$$

where h and h' lie in the same equivalence class if there exists a natural isomorphism $\alpha\colon h \cong h'$ so that $s\alpha = \mathrm{id}_f$ and $r\alpha = \mathrm{id}_g$. ☐

8.1.13. Remark. For instance, the map

$$\begin{tikzcd} & A \arrow[dl, equals] \arrow[dr, equals] \arrow[dd, "\Delta"] & \\ A & & A \\ & A^{\mathbb{I}} \arrow[ul, two heads, "q_1"] \arrow[ur, two heads, "q_0"'] & \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} A \arrow[r, equals, "|" marking] \arrow[d, equals] \arrow[dr, phantom, "\Downarrow"] & A \arrow[d, equals] \\ A \arrow[r, "|" marking, "A^{\mathbb{I}}"'] & A \end{tikzcd}$$

defines a nullary morphism with codomain $A \overset{A^{\mathbb{I}}}{\nrightarrow} A$ in the virtual double category of two-sided isofibrations. Note, however, that despite the fact that $\Delta\colon A \xrightarrow{\sim} A^{\mathbb{I}}$ defines an equivalence in $\mathcal{K}$, this cell does not define an isomorphism in the virtual double category of any kind. It is for this sort of reason that we left out the identity horizontal arrows in Proposition 8.1.7.

Our main example of interest is a full sub virtual double category defined by restricting the class of horizontal arrows and taking all cells between them. Since the only operations given in the structure of a virtual double category are vertical sources and targets, vertical identities, and vertical composition, it is clear that this substructure is closed under all of these operations, and thus inherits the structure of a virtual double category:

8.1.14. Proposition. *For any ∞-cosmos $\mathcal{K}$, there is a virtual double category of modules $\mathcal{M}od(\mathfrak{h}\mathcal{K})$ defined as a full subcategory of either the virtual double categories of isofibrations or the virtual double category of two-sided fibrations whose*

- *objects are ∞-categories,*
- *vertical arrows are functors,*
- *horizontal arrows $A \overset{E}{\nrightarrow} B$ are modules E from A to B,*

- *n-ary cells are fibered isomorphism classes of maps of spans*

$$
\begin{tikzcd}
A_0 \arrow[r, "E_1", "|" marking] \arrow[d, "f"'] & A_1 \arrow[r, "E_2", "|" marking] \arrow[d, phantom, "\Downarrow"] & \cdots \arrow[r, "E_n", "|" marking] & A_n \arrow[d, "g"] \\
B_0 \arrow[rrr, "F"', "|" marking] & & & B_n
\end{tikzcd}
\quad \leftrightsquigarrow \quad
\begin{tikzcd}
& E_1 \underset{A_1}{\times} \cdots \underset{A_{n-1}}{\times} E_n \arrow[dl] \arrow[dr] \arrow[ddd, dashed] & \\
A_0 \arrow[d, "f"'] & & A_n \arrow[d, "g"] \\
B_0 & & B_n \\
& F \arrow[ul, two heads] \arrow[ur, two heads] &
\end{tikzcd}
\tag{8.1.15}
$$

These maps, introduced in Definition 8.1.6, can be thought of as special cases of the n-ary cells of Lemma 8.1.11 where $E_1, \ldots, E_n, F$ are all required to be modules: the single horizontal source in the diagram (8.1.12) *is the two-sided fibration defined by the $(n-1)$-fold pullback of the sequence of modules comprising the horizontal source in the left-hand diagram.* □

We refer to (8.1.15) as an n-**ary module map**. Note a 1-ary module map is just a module map as in Definition 8.1.6. We refer to the finite sequence of modules occurring as the horizontal domain of an n-ary module map as a **composable sequence of modules**, which just means that their horizontal sources and targets are compatible in the evident way.

A hint at the relevance of this notion of n-ary module map is given by the following special case.

8.1.16. Lemma. *There is a bijection between n-ary module maps whose codomain module $B \overset{\mathsf{Hom}_D(h,k)}{\nrightarrow} C$ is a comma as displayed below-left and 2-cells in the homotopy 2-category whose boundary is displayed above-right.*

$$
\begin{tikzcd}
A_0 \arrow[r, "E_1", "|" marking] \arrow[d, "f"'] & A_1 \arrow[r, "E_2", "|" marking] \arrow[d, phantom, "\Downarrow"] & \cdots \arrow[r, "E_n", "|" marking] & A_n \arrow[d, "g"] \\
B \arrow[rrr, "\mathsf{Hom}_D(h,k)"', "|" marking] & & & C
\end{tikzcd}
\quad \leftrightsquigarrow \quad
\begin{tikzcd}
& E_1 \underset{A_1}{\times} \cdots \underset{A_{n-1}}{\times} E_n \arrow[dl] \arrow[dr] & \\
A_0 \arrow[d, "f"'] & \Leftarrow & A_n \arrow[d, "g"] \\
B \arrow[dr, "k"'] & & C \arrow[dl, "h"] \\
& D &
\end{tikzcd}
$$

Proof. Combine Definition 8.1.6 with Proposition 3.4.7. □

For any pair of objects A and B in the virtual double category of modules, there is a vertical 1-category of modules from A to B and module maps over a pair of identity functors, which coincides with the 1-category of modules from A to B introduced in Observation 8.1.4.

8.1.17. Lemma. *A parallel pair of modules $A \overset{E}{\nrightarrow} B$ and $A \overset{F}{\nrightarrow} B$ are isomorphic as objects of vertical 1-category of modules in the virtual double category of modules if and only if the modules E and F are equivalent as spans from A to B.*

Proof. This is a restatement of Lemma 8.1.5(iii). □

For consistency with the rest of the text, we write $E \simeq F$ or $E \simeq_{A \times B} F$ whenever the modules $A \overset{E}{\nrightarrow} B$ and $A \overset{F}{\nrightarrow} B$ are isomorphic as objects of the vertical 1-category of modules from A to B. For instance, Proposition 4.1.1 proves that a functor $f: B \to A$ is left adjoint to a functor $u: A \to B$ if and only if $\mathsf{Hom}_A(f, A) \simeq \mathsf{Hom}_B(B, u)$ over $A \times B$, that is, if and only if the modules $B \overset{\mathsf{Hom}_A(f,A)}{\nrightarrow} A$ and $A \overset{\mathsf{Hom}_B(B,u)}{\nrightarrow} B$ are isomorphic as objects of the vertical 1-category of modules from A to B.

Exercises.

8.1.i. Exercise. Generalize the proof of Proposition 6.1.1 to prove that for any ∞-cosmos $\mathcal{K}$, there is an ∞-cosmos $\mathcal{K}^{\swarrow\searrow}$ whose objects are two-sided isofibrations between an arbitrary pair of ∞-categories. Prove that the domain and codomain define cosmological functors $\mathcal{K}^{\swarrow\searrow} \rightrightarrows \mathcal{K}$. Use this to give a second description of the double category of two-sided isofibrations as as quotient of a structure defined at the level of quasi-categorically enriched categories.

8.1.ii. Exercise. Prove that any double category defines a virtual double category.[4]

8.2. The virtual equipment of modules

The virtual double category of modules $\mathcal{M}od(\mathfrak{h}\mathcal{K})$ in an ∞-cosmos $\mathcal{K}$ has two special properties that characterize what Cruttwell and Shulman term a *virtual equipment*. Before stating the definition, we explore each of these in turn.

8.2.1. Proposition (restriction). *Any diagram in $\mathcal{M}od(\mathfrak{h}\mathcal{K})$ as below-left can be completed to a* ***cartesian cell*** *as below-right*

$$\begin{tikzcd} A' & B' \\ A \arrow[r, "E"', "\shortmid" marking] & B \arrow[from=1-1, to=2-1, "a"'] \arrow[from=1-2, to=2-2, "b"] \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} A' \arrow[r, "{E(b,a)}", "\shortmid" marking] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\rho"] & B' \arrow[d, "b"] \\ A \arrow[r, "E"', "\shortmid" marking] & B \end{tikzcd}$$

characterized by the universal property that any cell as displayed below-left factors uniquely through ρ as below-right:

$$\begin{tikzcd} X_0 \arrow[r, "E_1", "\shortmid" marking] \arrow[d, "af"'] & X_1 \arrow[r, "E_2", "\shortmid" marking] \arrow[d, phantom, "\Downarrow"] & \cdots \arrow[r, "E_n", "\shortmid" marking] & X_n \arrow[d, "bg"] \\ A \arrow[rrr, "E"', "\shortmid" marking] & & & B \end{tikzcd} \quad = \quad \begin{tikzcd} X_0 \arrow[r, "E_1", "\shortmid" marking] \arrow[d, "f"'] & X_1 \arrow[r, "E_2", "\shortmid" marking] \arrow[d, phantom, "\exists!\Downarrow"] & \cdots \arrow[r, "E_n", "\shortmid" marking] & X_n \arrow[d, "g"] \\ A' \arrow[rrr, "{E(b,a)}", "\shortmid" marking] \arrow[d, "a"'] & \Downarrow\rho & & B' \arrow[d, "b"] \\ A \arrow[rrr, "E"', "\shortmid" marking] & & & B \end{tikzcd}$$

Proof. The horizontal source of the cartesian cell is defined by restricting the module $A \overset{E}{\nrightarrow} B$ along the functors a and b:

$$\begin{tikzcd} E(b,a) \arrow[r, "\rho"] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner" very near start] & E \arrow[d, two heads] \\ A' \times B' \arrow[r, "{(a,b)}"'] & A \times B \end{tikzcd} \tag{8.2.2}$$

[4]If the double category lacks horizontal identity morphisms, the corresponding virtual double category may lack nullary morphisms — unless these can be defined in some other way as we did in the proof of Lemma 8.1.11. Note that if we added identity spans to our double category of two-sided isofibrations, then the corresponding virtual double category would be the correct one, which contains the virtual double category of modules. See Remark 8.1.13 however.

By Proposition 7.4.4, this left-hand isofibration defines a module from A' to B', while by Definition 8.1.6 the top horizontal functor represents a module map inhabiting the desired square. As in Definition 8.1.6, the simplicial pullback in $\mathcal{K}$ induces an equivalence[5] of functor spaces:

$$\mathsf{Fun}_{f\times g}(E_1 \underset{A_1}{\times} \cdots \underset{A_{n-1}}{\times} E_n, E(b,a)) \xrightarrow[\sim]{\rho\circ-} \mathsf{Fun}_{af\times bg}(E_1 \underset{A_1}{\times} \cdots \underset{A_{n-1}}{\times} E_n, E)$$

which descends to a bijection on isomorphism classes of objects. This defines the unique factorization of cells as displayed above left through the cartesian restriction cell ρ. □

We refer to the module $A' \overset{E(b,a)}{\nrightarrow} B'$ as the **restriction** of $A \overset{E}{\nrightarrow} B$ along the functors a and b, because the pullback (8.2.2) is analogous to the restriction of scalars of a bimodule along a pair of ring homomorphisms.

8.2.3. Example. The module $C \overset{\mathsf{Hom}_A(f,g)}{\nrightarrow} B$ is the restriction of the module $A \overset{A^2}{\nrightarrow} A$ along $g\colon C \to A$ and $f\colon B \to A$. To make this restriction relationship more transparent, we typically write $A \overset{\mathsf{Hom}_A}{\nrightarrow} A$ when regarding the arrow ∞-category as a module. Since the common notation for "homs" places the contravariant variable on the left and the covariant variable on the right, we've adopted a similar notation convention for restrictions in Proposition 8.2.1.

8.2.4. Proposition (units). *Any object A in $\mathcal{M}od(\mathfrak{h}\mathcal{K})$ is equipped with a canonical nullary* ***cocartesian cell*** *as displayed below*

$$\begin{tikzcd}
A \arrow[r, equal, "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\iota"] & A \arrow[d, equal] \\
A \arrow[r, "\shortmid"{marking}, "\mathsf{Hom}_A"'] & A
\end{tikzcd}$$

characterized by the universal property that any cell in $\mathcal{M}od(\mathfrak{h}\mathcal{K})$ whose horizontal source includes the object A factors uniquely through ι as below-right:

$$\begin{tikzcd}
X \arrow[r, "E_1", "\shortmid"{marking}] \arrow[d, "f"'] & \cdots \arrow[r, "E_n", "\shortmid"{marking}] & A \arrow[r, "F_1", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow"] & \cdots \arrow[r, "F_m", "\shortmid"{marking}] & Y \arrow[d, "g"] \\
B \arrow[rrrr, "\shortmid"{marking}, "G"'] & & & & C
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, "E_1", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}_{E_1}"] & \cdots \arrow[r, "E_n", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}_{E_n}"] & A \arrow[r, equal, "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\iota"] & A \arrow[r, "F_1", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}_{F_1}"] & \cdots \arrow[r, "F_m", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}_{F_m}"] & Y \arrow[d, equal] \\
X \arrow[r, "E_1", "\shortmid"{marking}] \arrow[d, "f"'] & \cdots \arrow[r, "E_n", "\shortmid"{marking}] & A \arrow[r, "\mathsf{Hom}_A", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\exists!"] & A \arrow[r, "F_1", "\shortmid"{marking}] & \cdots \arrow[r, "F_m", "\shortmid"{marking}] & Y \arrow[d, "g"] \\
B \arrow[rrrrr, "\shortmid"{marking}, "G"'] & & & & & C
\end{tikzcd}$$

Proof. The canonical nullary cell is represented by the map of spans

$$\begin{tikzcd}
& A \arrow[dl, equal] \arrow[dr, equal] \arrow[dd, "\iota"] & \\
A & & A \\
& \mathsf{Hom}_A \arrow[ul, two heads, "p_1"] \arrow[ur, two heads, "p_0"'] &
\end{tikzcd}$$

[5] If the pullbacks are defined strictly, then in fact pullback induces an *isomorphism* of functor spaces, but even if $E(b,a)$ is replaced by an equivalent module, the functor spaces are still equivalent, which enough to induce a bijection on isomorphism classes of objects.

induced by the generic arrow with codomain A (3.2.3); recall from Example 8.2.3 that we write $A \overset{\mathsf{Hom}_A}{\nrightarrow} A$ for the module encoded by the arrow ∞-category construction.

In the case where both of the sequences E_i and F_j are empty, the one-sided version of the Yoneda lemma for modules given by Theorem 7.4.9 tells us that restriction along the this map induces an equivalence of functor spaces

$$\mathsf{Fun}_{A\times A}(\mathsf{Hom}_A, G(g,f)) \xrightarrow{\sim} \mathsf{Fun}_{A\times A}(A, G(g,f)).$$

Taking isomorphism classes of objects gives the bijection of the statement.

In the case where one or both of the sequences are non-empty, we may form their horizontal composite two-sided fibrations and then form either the horizontal composite $E \underset{A}{\times} \mathsf{Hom}_A$ or the horizontal composite $\mathsf{Hom}_A \underset{A}{\times} F$ of the composable triple below:

$$\begin{tikzcd}
 & & & & E \underset{A}{\times} \mathsf{Hom}_A \underset{A}{\times} F \arrow[dll, two heads] \arrow[drr, two heads] \arrow[dd, phantom, "\vee", very near start] & & & & \\
 & & \mathsf{Hom}_A(A,p) \arrow[dl, two heads] \arrow[drr, two heads] \arrow[dd, phantom, "\vee", very near start] & & & & \mathsf{Hom}_A(s,A) \arrow[dll, two heads] \arrow[dr, two heads] \arrow[dd, phantom, "\vee", very near start] & & \\
 & E \arrow[dl, two heads, "q"'] \arrow[dr, two heads, "p"] & & & \mathsf{Hom}_A \arrow[dll, two heads, "p_1"'] \arrow[drr, two heads, "p_0"] & & & F \arrow[dl, two heads, "s"'] \arrow[dr, two heads, "r"] & \\
X & & A & & & & A & & Y
\end{tikzcd}$$

In the former case, this constructs the two-sided fibration $(qp_1, p_0)\colon \mathsf{Hom}_A(A,p) \twoheadrightarrow X \times A$, and in the latter case this constructs the two-sided fibration $(p_1, rp_0)\colon \mathsf{Hom}_A(s,A) \twoheadrightarrow A \times Y$. In both cases, by Lemma 7.1.1, the map ι pulls back to define a fibered adjoint functor

$$\begin{tikzcd}
E \arrow[rr, "E \underset{A}{\times} \iota"] \arrow[rr, phantom, "\perp"'] \arrow[dr, two heads, "(q,p)"'] & & \mathsf{Hom}_A(A,p) \arrow[ll, dashed, bend left, "r"] \arrow[dl, two heads, "(qp_1,p_0)"] \\
 & X \times A &
\end{tikzcd}
\qquad
\begin{tikzcd}
F \arrow[rr, "\iota \underset{A}{\times} F"'] \arrow[rr, phantom, "\perp"] \arrow[dr, two heads, "(s,r)"'] & & \mathsf{Hom}_A(s,A) \arrow[ll, dashed, bend right, "\ell"'] \arrow[dl, two heads, "(p_1,rp_0)"] \\
 & A \times Y &
\end{tikzcd}$$

We only require one of these adjunctions, so without loss of generality we use the former. This fibered adjunction pulls back along s and pushes forward along r to define a fibered adjunction

$$\begin{tikzcd}
E \underset{A}{\times} F \arrow[rr, "E \underset{A}{\times} \iota \underset{A}{\times} F"] \arrow[rr, phantom, "\perp"'] \arrow[dr, two heads] & & E \underset{A}{\times} \mathsf{Hom}_A \underset{A}{\times} F \arrow[ll, dashed, bend left, "r"] \arrow[dl, two heads] \\
 & X \times Y &
\end{tikzcd}$$

between two-sided fibrations. Upon mapping into the discrete object $G(g,f) \twoheadrightarrow X \times Y$, this adjunction becomes an adjoint equivalence. In particular, restriction along ι induces an equivalence of Kan complexes

$$\mathsf{Fun}_{X\times Y}(E \underset{A}{\times} \mathsf{Hom}_A \underset{A}{\times} F, G(g,f)) \xrightarrow[\sim]{-\circ(E \underset{A}{\times} \iota \underset{A}{\times} F)} \mathsf{Fun}_{X\times Y}(E \underset{A}{\times} F, G(g,f)),$$

and once again taking isomorphism classes of objects gives the bijection of the statement. □

Propositions 8.2.1 and 8.2.4 imply that the virtual double category of modules is a *virtual equipment* in the sense introduced by Cruttwell and Shulman [**27**, §7].

8.2.5. Definition. A **virtual equipment** is a virtual double category so that

(i) For any horizontal arrow $A \overset{E}{\nrightarrow} B$ and pair of vertical arrows $a\colon A' \to A$ and $b\colon B' \to B$, there exists a horizontal arrow $B' \overset{E(b,a)}{\nrightarrow} A'$ and unary cartesian cell ρ satisfying the universal property of Proposition 8.2.1.

(ii) Every object A admits a **unit** horizontal arrow $A \overset{\mathrm{Hom}_A}{\nrightarrow} A$ equipped with a nullary cocartesian cell ι satisfying the universal property of Proposition 8.2.4.

Thus, Propositions 8.2.1 and 8.2.4 combine to prove:

8.2.6. Theorem. *The virtual double category $\mathcal{M}od(\mathfrak{h}\mathcal{K})$ of modules in an ∞-cosmos $\mathcal{K}$ is a virtual equipment.* □

By abstract nonsense, the relatively simple axioms (i) and (ii) established in Theorem 8.2.6 establish a robust "calculus of modules." In an effort to familiarize the reader with this little-known categorical structure and expedite the proofs of the formal category theory of ∞-categories in Chapter 9, we devote the remainder of this chapter to proving a plethora of results that actually follow formally from this axiomatization: namely, Lemmas 8.3.5 and 8.3.11, Proposition 8.4.1, Theorem 8.4.4, Corollary 8.4.6, Proposition 8.4.7, Corollary 8.4.8, and the bijection of Proposition 8.4.10 between unary cells in the virtual equipment of modules. One additional result is left as Exercise 8.4.iii for the reader.

8.2.7. Notation. We adopt the following notational conventions to streamline certain virtual equipment diagrams.

- We adopt the convention that an unlabeled unary cell whose vertical boundaries are identities and whose horizontal sources and targets is an identity cell.

$$\begin{tikzcd} A \arrow[r, "E", "\mid" marking] \arrow[d, equal] & B \arrow[d, equal] \\ A \arrow[r, "E"', "\mid" marking] & B \end{tikzcd} \quad := \quad \begin{tikzcd} A \arrow[r, "E", "\mid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}_E"] & B \arrow[d, equal] \\ A \arrow[r, "E"', "\mid" marking] & B \end{tikzcd}$$

- Cells whose vertical boundary functors are identities and therefore whose source and target spans lie between the same pair of ∞-categories

$$\mu\colon E_1 \mathbin{\dot\times} \cdots \mathbin{\dot\times} E_n \Rightarrow E \qquad := \qquad \begin{tikzcd} A_0 \arrow[r, "E_1", "\mid" marking] \arrow[d, equal] & A_1 \arrow[r, "E_2", "\mid" marking] \arrow[d, phantom, "\Downarrow\mu"] & \cdots \arrow[r, "E_n", "\mid" marking] & A_n \arrow[d, equal] \\ A_0 \arrow[rrr, "E"', "\mid" marking] & & & A_n \end{tikzcd}$$

may be displayed in line using the notation $\mu\colon E_1 \mathbin{\dot\times} \cdots \mathbin{\dot\times} E_n \Rightarrow E$, an expression which implicitly asserts that the modules appearing in the domain define a composable sequence, with the symbol "$\dot\times$" meant to suggest the pullback appearing as the horizontal domain of (8.1.10) rather than a product.

Exercises.

8.2.i. Exercise. Prove that unital rings, ring homomorphisms, bimodules, and bimodule maps also define a virtual equipment.

8.3. Composition of modules

In a virtual equipment one cannot define the composite of a generic pair $A \overset{E}{\nrightarrow} B$ and $B \overset{F}{\nrightarrow} C$ of horizontal morphisms but there is a mechanism by which a particular horizontal composite $A \overset{G}{\nrightarrow} C$ that happens to exist can be recognized, in which case we write $E \otimes F \simeq G$ to reinforce the intuition provided by the analogy with bimodules. In the virtual equipment of modules, a composition relation $E \otimes F \simeq G$ does not mean that the module G is equivalent to the horizontal composite two-sided isofibration $E \times F$. Rather, in the notation of 8.2.7 a composition relation is witnessed by a module map $E \times F \Rightarrow G$ that defines a cocartesian cell in a sense analogous to the universal property stated in Proposition 8.2.4.[6]

8.3.1. Definition. A composable sequence of modules

$$A_0 \overset{E_1}{\nrightarrow} A_1, A_1 \overset{E_2}{\nrightarrow} A_2, \ldots, A_{n-1} \overset{E_n}{\nrightarrow} A_n$$

admits a composite if there exists a module $A_0 \overset{E}{\nrightarrow} A_n$ and a cocartesian cell

$$\mu\colon E_1 \times \cdots \times E_n \Rightarrow E$$

characterized by the universal property that any cell of the form

$$\begin{tikzcd} X \arrow[r, "F_1", "|"'] \arrow[d, "f"'] & \cdots \arrow[r, "F_m", "|"'] & A_0 \arrow[r, "E_1", "|"'] & \cdots \arrow[r, "E_n", "|"'] \arrow[d, phantom, "\Downarrow"] & A_n \arrow[r, "G_1", "|"'] & \cdots \arrow[r, "G_k", "|"'] & Y \arrow[d, "g"] \\ B \arrow[rrrrrr, "H"', "|"] & & & & & & C \end{tikzcd}$$

factors uniquely through μ as follows:

$$\begin{tikzcd} X \arrow[r, "F_1", "|"'] \arrow[d, equal] & \cdots \arrow[r, "F_m", "|"'] & A_0 \arrow[r, "E_1", "|"'] \arrow[d, equal] & \cdots \arrow[r, "E_n", "|"'] \arrow[d, phantom, "\Downarrow\mu"] & A_n \arrow[r, "G_1", "|"'] \arrow[d, equal] & \cdots \arrow[r, "G_k", "|"'] & Y \arrow[d, equal] \\ X \arrow[r, "F_1", "|"'] \arrow[d, "f"'] & \cdots \arrow[r, "F_m", "|"'] & A_0 \arrow[rr, "E", "|"'] & \arrow[d, phantom, "\Downarrow\exists!"] & A_n \arrow[r, "G_1", "|"'] & \cdots \arrow[r, "G_k", "|"'] & Y \arrow[d, "g"] \\ B \arrow[rrrrrr, "H"', "|"] & & & & & & C \end{tikzcd}$$

A composite $\mu\colon E_1 \times \cdots \times E_n \Rightarrow E$ can be used to reduce the domain of a cell by replacing any occurrence of the composable sequence $E_1 \times \cdots \times E_n$ from A_0 to A_n by a single module E. Particularly in the case of binary composites, we write $E_1 \otimes E_2$ to denote the **composite** of the modules

[6] In nearly all cases where $E \otimes F \simeq G$, precomposition with the cocartesian map $\mu\colon E \times_B F \Rightarrow G$, which by Definition 8.1.1 corresponds to a fibered isomorphism class of maps of spans over $A \times C$, induces an equivalence of Kan complexes $- \circ \mu\colon \mathsf{Fun}_{A\times C}(G, H) \xrightarrow{\sim} \mathsf{Fun}_{A\times C}(E \times_B F, H)$ for all modules H, so the module G may be understood as the "reflection" of the two-sided isofibration $E \times_B F$ into the subcategory of modules. Corollary 7.3.5, reappearing as Proposition 8.3.7 below, is one instance of this.

E_1 and E_2, appearing as the codomain of the cocartesian cell $E_1 \times E_2 \Rightarrow E_1 \otimes E_2$. Lemma 8.1.17 easily implies that composites are unique up to vertical isomorphism, i.e., by Lemma 8.1.17 up to equivalence of modules. Moreover:

8.3.2. Lemma. *Suppose the cells $\mu_i \colon E_{i1} \mathbin{\underset{\cdot}{\times}} \cdots \mathbin{\underset{\cdot}{\times}} E_{ik_i} \Rightarrow E_i$ exhibit composites for $i = 1, \ldots, n$.*

(i) If $\mu \colon E_1 \mathbin{\underset{\cdot}{\times}} \cdots \mathbin{\underset{\cdot}{\times}} E_n \Rightarrow E$ exhibits a composite then the composite cell

$$E_{11} \mathbin{\underset{\cdot}{\times}} \cdots \mathbin{\underset{\cdot}{\times}} E_{nk_n} \xRightarrow{\mu_1 \mathbin{\underset{\cdot}{\times}} \cdots \mathbin{\underset{\cdot}{\times}} \mu_n} E_1 \mathbin{\underset{\cdot}{\times}} \cdots \mathbin{\underset{\cdot}{\times}} E_n \xRightarrow{\mu} E$$

exhibits E as a composite of the sequence $E_{11} \mathbin{\underset{\cdot}{\times}} \cdots \mathbin{\underset{\cdot}{\times}} E_{nk_n}$.

(ii) If the composite cell

$$E_{11} \mathbin{\underset{\cdot}{\times}} \cdots \mathbin{\underset{\cdot}{\times}} E_{nk_n} \xRightarrow{\mu_1 \mathbin{\underset{\cdot}{\times}} \cdots \mathbin{\underset{\cdot}{\times}} \mu_n} E_1 \mathbin{\underset{\cdot}{\times}} \cdots \mathbin{\underset{\cdot}{\times}} E_n \xRightarrow{\mu} E$$

exhibits E as a composite of the sequence $E_{11} \mathbin{\underset{\cdot}{\times}} \cdots \mathbin{\underset{\cdot}{\times}} E_{nk_n}$, then $\mu \colon E_1 \mathbin{\underset{\cdot}{\times}} \cdots \mathbin{\underset{\cdot}{\times}} E_n \Rightarrow E$ exhibits E as a composite of the sequence $E_1 \mathbin{\underset{\cdot}{\times}} \cdots \mathbin{\underset{\cdot}{\times}} E_n$.

Proof. For (i), the required bijection factors as a composite of $n + 1$ bijections induced by the maps $\mu_1, \ldots, \mu_n$ and μ. For (ii), the required bijection induced by μ composes with the bijections supplied by the maps $\mu_1, \ldots, \mu_n$ to a bijection and is thus itself a bijection. □

The trivial instances of composites are easily verified:

8.3.3. Lemma.

(i) The units $\iota \colon \varnothing \Rightarrow \mathsf{Hom}_A$ of Proposition 8.2.4 define nullary composites.

(ii) A unary cell $\mu \colon E \Rightarrow F$ is a composite if and only if it is an isomorphism in the vertical category of modules from A to B and module maps over identity functors, that is, if and only if the modules E and F are equivalent as spans.

Proof. Exercise 8.3.ii. □

8.3.4. Remark. On account of the universal property of restrictions established in Proposition 8.2.1, to prove that a cell $\mu \colon E_1 \mathbin{\underset{\cdot}{\times}} \cdots \mathbin{\underset{\cdot}{\times}} E_n \Rightarrow E$ exhibits a composite, it suffices to prove the factorization property of Definition 8.3.1 in the case where the vertical functors are identities.

As one might hope, the unit modules $A \overset{\mathsf{Hom}_A}{\nrightarrow} A$ are units for the composition of Definition 8.3.1: for any module $A \overset{E}{\nrightarrow} B$, $\mathsf{Hom}_A \otimes E \otimes \mathsf{Hom}_B \simeq E$.

8.3.5. Lemma (composites with units). *For any module $A \overset{E}{\nrightarrow} B$ the unique cell $\circ \colon \mathsf{Hom}_A \times E \times \mathsf{Hom}_B \Rightarrow E$ defined using the universal property of the unit cell*

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\iota"] & A \arrow[r, "E", "|" marking] \arrow[d, equal] & B \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\iota"] & B \arrow[d, equal] \\
A \arrow[r, "\mathsf{Hom}_A", "|" marking] \arrow[d, equal] & A \arrow[r, "E", "|" marking] \arrow[d, phantom, "\Downarrow\circ"] & B \arrow[r, "\mathsf{Hom}_B", "|" marking] & B \arrow[d, equal] \\
A \arrow[rrr, "E"', "|" marking] & & & B
\end{tikzcd}
\quad := \quad
\begin{tikzcd}
A \arrow[r, "E", "|" marking] \arrow[d, equal] & B \arrow[d, equal] \\
A \arrow[r, "E"', "|" marking] & B
\end{tikzcd}$$

displays E as the composite $\mathsf{Hom}_A \otimes E \otimes \mathsf{Hom}_B$.

Proof. The result is immediate from Lemma 8.3.2(ii) and Lemma 8.3.3. $\square$

The argument used in the proof of Proposition 8.2.4 to demonstrate that units define nullary composites in the virtual equipment of modules applies also to more general composites.

8.3.6. Lemma. *Consider an n-ary module morphism $\mu\colon E_1 \underset{A_1}{\times} \cdots \underset{A_{n-1}}{\times} E_n \Rightarrow E$ in the virtual equipment of modules whose codomain is a module from A to B. If any representing map of spans*

$$\begin{tikzcd} E_1 \dot{\times} \cdots \dot{\times} E_n \arrow[rr, "\ulcorner\mu\urcorner"] \arrow[dr, two heads] & & E \arrow[dl, two heads] \\ & A \times B & \end{tikzcd}$$

admits a fibered adjoint over $A \times B$, then μ exihibits E as the composite $E_1 \otimes \cdots \otimes E_n$.

Proof. To verify the universal property of Definition 8.3.1, consider a composable sequence of modules $F_1, \ldots, F_m$ from X to A and a composable sequence of modules $G_1, \ldots, G_k$ from B to Y and form the horizontal composite two-sided fibrations

$$\begin{tikzcd} & F \arrow[dl, two heads, "q"'] \arrow[dr, two heads, "p"] & \\ X & & A \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & G \arrow[dl, two heads, "s"'] \arrow[dr, two heads, "r"] & \\ B & & Y \end{tikzcd}$$

The fibered adjunction of the statement pulls back along $p \times s\colon F \times G \twoheadrightarrow A \times B$ and pushes forward along $q \times r\colon F \times G \to X \times Y$ to a fibered adjoint to

$$\begin{tikzcd} F \underset{A}{\times} E_1 \underset{A_1}{\times} \cdots \underset{A_{n-1}}{\times} E_n \underset{B}{\times} G \arrow[rr, "F \underset{A}{\times} \ulcorner\mu\urcorner \underset{B}{\times} G"] \arrow[dr, two heads] & & F \underset{A}{\times} E \underset{B}{\times} G \arrow[dl, two heads] \\ & X \times Y & \end{tikzcd}$$

Via Remark 8.3.4, it suffices to verify the universal property of the composite for modules $X \overset{H}{\rightarrow\!\!\!\!\!+} Y$. Since modules are discrete, applying $\mathsf{Fun}_{X\times Y}(-, H)$ transforms this fibered adjunction into an adjoint equivalence

$$\mathsf{Fun}_{X\times Y}(F \underset{A}{\times} E \underset{B}{\times} G, H) \xrightarrow[\sim]{-\circ(F \underset{A}{\times} \ulcorner\mu\urcorner \underset{B}{\times} G)} \mathsf{Fun}_{X\times Y}(F \underset{A}{\times} E_1 \underset{A_1}{\times} \cdots \underset{A_{n-1}}{\times} E_n \underset{B}{\times} G, H).$$

Passing to isomorphism classes of objects this now gives the universal property of Definition 8.3.1. $\square$

For instance:

8.3.7. Proposition. *Let $A \overset{E}{\rightarrow\!\!\!\!\!+} B$ be a module encoded by the span $A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$. Then the binary module map*

$$\begin{tikzcd} A \arrow[r, "\mathsf{Hom}_A(q,A)", "\shortmid"{marking}] \arrow[d, equal] & E \arrow[r, "\mathsf{Hom}_B(B,p)", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\mu"] & B \arrow[d, equal] \\ A \arrow[rr, "E"', "\shortmid"{marking}] & & B \end{tikzcd}$$

represented by composite left and right adjoints of Theorem 7.1.4(iii) exhibits $E \simeq \mathsf{Hom}_A(q, A) \otimes \mathsf{Hom}_B(B, p)$ as the composite of the modules representing its legs.

PROOF. Immediate from Theorem 7.1.4(iii) and Lemma 8.3.6 applied twice.[7] □

8.3.8. REMARK. Nothing in the proof of Proposition 8.3.7 requires that the span $A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$ is actually a module except for the interpretation that μ is a binary cell in the virtual equipment of modules. For any two-sided fibration $A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$, it is still the case that restriction along μ defines a bijection between n-ary maps whose source includes the span E and whose codomain is a module stand in bijection with $(n+1)$-ary maps whose source includes $\mathsf{Hom}_A(q, A) \times \mathsf{Hom}_B(B, p)$. By Theorem 7.4.9, this result can be extended further to mere spans E, not necessarily two-sided fibrations.

8.3.9. REMARK. We will make use of another description of the composition map $\mu \colon \mathsf{Hom}_A(q, A) \times \mathsf{Hom}_B(B, p) \Rightarrow E$. The comma cones

$$\begin{tikzcd}
& & \mathsf{Hom}_A(q,A) \underset{E}{\times} \mathsf{Hom}_B(B,p) \arrow[dl] \arrow[dr] \arrow[d, phantom, "\lrcorner"] & & \\
& \mathsf{Hom}_A(q,A) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & & \mathsf{Hom}_B(B,p) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\
A & & E \arrow[ll, two heads, "q"] \arrow[rr, two heads, "p"'] & & B
\end{tikzcd}$$

define a pair of 2-cells to which the premises of Theorem 7.1.4(iv) apply. The conclusion asserts that there is a well-defined fibered isomorphism class of functors $\ulcorner\mu\urcorner \colon \mathsf{Hom}_A(q, A) \underset{E}{\times} \mathsf{Hom}_B(B, p) \to E$ defined by taking the q-cocartesian lift of the left comma cone, composing with the right comma cone, and then taking the codomain of a p-cartesian lift of this composite cell — or by first taking the p-cartesian lift, composing, and then taking the domain of a q-cartesian lift of this composite — this being the functor $\ell r \simeq r\ell$ in the notation of Theorem 7.1.4(iii). In the case where $A \overset{E}{\twoheadrightarrow} B$ is itself a comma module, the resulting fibered isomorphism class of functors $\ulcorner\mu\urcorner$ is the one that classifies the pasted composite above with the comma cone for E.

Any virtual double category has an identity cell for each horizontal arrow $A \overset{E}{\nrightarrow} B$ whose vertical boundary arrows are identities. In a virtual equipment, we also have a **unit cell** for each vertical arrow $f \colon A \to B$ whose horizontal boundary is given by the unit modules $A \overset{\mathsf{Hom}_A}{\nrightarrow} A$ and $B \overset{\mathsf{Hom}_B}{\nrightarrow} B$ of Proposition 8.2.4.

8.3.10. DEFINITION. Using the unit modules in $\mathcal{M}od(\mathfrak{h}\mathcal{K})$, for any functor $f \colon A \to B$ we may define a unary **unit cell** as displayed below-left by appealing to the universal property of the nullary unit cell $\iota \colon \emptyset \Rightarrow \mathsf{Hom}_A$ for A in the equation below-right:

$$\begin{tikzcd}
A \arrow[r, "\mathsf{Hom}_A", "\vert" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow \mathsf{Hom}_f"] & A \arrow[d, "f"] \\
B \arrow[r, "\mathsf{Hom}_B"', "\vert" marking] & B
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
A \arrow[r, equal, "\vert" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow \iota"] & A \arrow[d, equal] \\
A \arrow[r, "\mathsf{Hom}_A", "\vert" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow \mathsf{Hom}_f"] & A \arrow[d, "f"] \\
B \arrow[r, "\mathsf{Hom}_B"', "\vert" marking] & B
\end{tikzcd}
\quad := \quad
\begin{tikzcd}
A \arrow[r, equal, "\vert" marking] \arrow[d, "f"'] & A \arrow[d, "f"] \\
B \arrow[r, equal, "\vert" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow \iota"] & B \arrow[d, equal] \\
B \arrow[r, "\mathsf{Hom}_B"', "\vert" marking] & B
\end{tikzcd}$$

[7]Of course the composite of a left and a right adjoint is not an adjoint but here we're effectively composing adjoint equivalences in which case the direction does not matter.

In the characterization of Lemma 8.1.16, both sides of the pasting equality defining the unary unit cell correspond to the identity 2-cell $\begin{tikzcd} A \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "f"'{name=D}] & B \arrow[from=U, to=D, phantom, "\Downarrow \mathrm{id}_f"] \end{tikzcd}$.

As one might hope, the unit cells are units for the vertical composition of cells in the virtual equipment of modules.

8.3.11. Lemma (composition with unit cells). *Any cell α as below-right equals the pasted composite below-left:*

$$\begin{tikzcd}
A_0 \arrow[r, equals, "\mid"{marking}] \arrow[d, equals] \arrow[dr, phantom, "\Downarrow\iota"] & A_0 \arrow[r, "E_1", "\mid"{marking}] \arrow[d, equals] & A_1 \arrow[r, "E_2", "\mid"{marking}] \arrow[d, equals] & \cdots \arrow[r, "E_n", "\mid"{marking}] \arrow[d, equals, "\cdots"] & A_n \arrow[r, equals, "\mid"{marking}] \arrow[d, equals] \arrow[dr, phantom, "\Downarrow\iota"] & A_n \arrow[d, equals] \\
A_0 \arrow[r, "\mathsf{Hom}_{A_0}", "\mid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\mathsf{Hom}_f"] & A_0 \arrow[r, "E_1", "\mid"{marking}] \arrow[d, "f"] & A_1 \arrow[r, "E_2", "\mid"{marking}] \arrow[dr, phantom, "\Downarrow\alpha"] & \cdots \arrow[r, "E_n", "\mid"{marking}] & A_n \arrow[r, "\mathsf{Hom}_{A_n}", "\mid"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\mathsf{Hom}_g"] & A_n \arrow[d, "g"] \\
B \arrow[r, "\mathsf{Hom}_B"', "\mid"{marking}] \arrow[d, equals] & B \arrow[rrr, "E"', "\mid"{marking}] \arrow[dr, phantom, "\Downarrow\circ"] & & & C \arrow[r, "\mathsf{Hom}_C"', "\mid"{marking}] & C \arrow[d, equals] \\
B \arrow[rrrrr, "E"', "\mid"{marking}] & & & & & C
\end{tikzcd}
=
\begin{tikzcd}
A_0 \arrow[r, "E_1", "\mid"{marking}] \arrow[d, "f"'] & A_1 \arrow[r, "E_2", "\mid"{marking}] \arrow[d, phantom, "\Downarrow\alpha"] & \cdots \arrow[r, "E_n", "\mid"{marking}] & A_n \arrow[d, "g"] \\
B \arrow[rrr, "E"', "\mid"{marking}] & & & C
\end{tikzcd}$$

Proof. By Definition 8.3.10 and the laws for composition with identity cells in a virtual double category stated in Definition 8.1.9, the left-hand composite of the statement equals the left-hand composite cell displayed below and the right-hand side of the statement equals the right-hand composite cell displayed below:

$$\begin{tikzcd}
A_0 \arrow[r, equals, "\mid"{marking}] \arrow[d, "f"'] & A_0 \arrow[r, "E_1", "\mid"{marking}] \arrow[d, "f"] & A_1 \arrow[r, "E_2", "\mid"{marking}] \arrow[d, phantom, "\Downarrow\alpha"] & \cdots \arrow[r, "E_n", "\mid"{marking}] & A_n \arrow[r, equals, "\mid"{marking}] \arrow[d, "g"'] & A_n \arrow[d, "g"] \\
B \arrow[r, equals, "\mid"{marking}] \arrow[d, equals] \arrow[dr, phantom, "\Downarrow\iota"] & B \arrow[rrr, "E"', "\mid"{marking}] \arrow[d, equals] & & & C \arrow[r, equals, "\mid"{marking}] \arrow[d, equals] \arrow[dr, phantom, "\Downarrow\iota"] & C \arrow[d, equals] \\
B \arrow[r, "\mathsf{Hom}_B"', "\mid"{marking}] \arrow[d, equals] & B \arrow[rrr, "E"', "\mid"{marking}] \arrow[dr, phantom, "\Downarrow\circ"] & & & C \arrow[r, "\mathsf{Hom}_C"', "\mid"{marking}] & C \arrow[d, equals] \\
B \arrow[rrrrr, "E"', "\mid"{marking}] & & & & & C
\end{tikzcd}
=
\begin{tikzcd}
A_0 \arrow[r, "E_1", "\mid"{marking}] \arrow[d, "f"'] & A_1 \arrow[r, "E_2", "\mid"{marking}] \arrow[d, phantom, "\Downarrow\alpha"] & \cdots \arrow[r, "E_n", "\mid"{marking}] & A_n \arrow[d, "g"] \\
B \arrow[rrr, "E"', "\mid"{marking}] \arrow[d, equals] & & & C \arrow[d, equals] \\
B \arrow[rrr, "E"', "\mid"{marking}] & & & C
\end{tikzcd}$$

By Lemma 8.3.5, the left-hand side equals the right-hand side. □

8.3.12. Definition (horizontal composition of cells). If given a horizontally composable sequence of unary cells

$$\begin{tikzcd}
A_0 \arrow[r, "E_1", "\mid"{marking}] \arrow[d, "f_0"'] \arrow[dr, phantom, "\Downarrow\alpha_1"] & A_1 \arrow[r, "E_2", "\mid"{marking}] \arrow[d, "f_1"] \arrow[dr, phantom, "\Downarrow\alpha_2"] & \cdots \arrow[r, "E_n", "\mid"{marking}] \arrow[d, phantom, "\cdots"] \arrow[dr, phantom, "\Downarrow\alpha_n"] & A_n \arrow[d, "f_n"] \\
B_0 \arrow[r, "F_1"', "\mid"{marking}] & B_1 \arrow[r, "F_2"', "\mid"{marking}] & \cdots \arrow[r, "F_n"', "\mid"{marking}] & B_n
\end{tikzcd}$$

for which the composable sequences $E_1 \mathbin{\dot\times} \cdots \mathbin{\dot\times} E_n$ and $F_1 \mathbin{\dot\times} \cdots \mathbin{\dot\times} F_n$ both admit composites, then there exists a **horizontal composite unary cell** $\alpha_1 * \cdots * \alpha_n$ that is uniquely determined up to the specification of the composites $\circ \colon E_1 \mathbin{\dot\times} \cdots \mathbin{\dot\times} E_n \Rightarrow E$ and $F_1 \mathbin{\dot\times} \cdots \mathbin{\dot\times} F_n \Rightarrow F$ by the pasting identity:

$$\begin{tikzcd}
A_0 \arrow[r, "E_1", "\shortmid"{marking}] \arrow[d, equal] & A_1 \arrow[r, "E_2", "\shortmid"{marking}] & \cdots \arrow[r, "E_n", "\shortmid"{marking}] & A_n \arrow[d, equal] \\
A_0 \arrow[rrr, "E", "\shortmid"{marking}] \arrow[d, "f_0"'] \arrow[u, phantom, ""] & & & A_n \arrow[d, "f_n"] \\
B_0 \arrow[rrr, "F"', "\shortmid"{marking}] & & & B_n
\arrow[from=1-1, to=2-4, phantom, "\Downarrow\circ"]
\arrow[from=2-1, to=3-4, phantom, "\Downarrow\alpha_1*\cdots*\alpha_n"]
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A_0 \arrow[r, "E_1", "\shortmid"{marking}] \arrow[d, "f_0"'] \arrow[dr, phantom, "\Downarrow\alpha_1"] & A_1 \arrow[r, "E_2", "\shortmid"{marking}] \arrow[d, "f_1"] \arrow[dr, phantom, "\Downarrow\alpha_2"] & \cdots \arrow[r, "E_n", "\shortmid"{marking}] \arrow[d] \arrow[dr, phantom, "\Downarrow\alpha_n"] & A_n \arrow[d, "f_n"] \\
B_0 \arrow[r, "F_1"', "\shortmid"{marking}] \arrow[d, equal] & B_1 \arrow[r, "F_2"', "\shortmid"{marking}] & \cdots \arrow[r, "F_n"', "\shortmid"{marking}] & B_n \arrow[d, equal] \\
B_0 \arrow[rrr, "F"', "\shortmid"{marking}] & & & B_n
\arrow[from=2-1, to=3-4, phantom, "\Downarrow\circ"]
\end{tikzcd}$$

By an argument very similar to the proof of Lemma 8.3.11 using the composites of Lemma 8.3.5, the horizontal composite $\mathsf{Hom}_f * \alpha * \mathsf{Hom}_g$ of a unary cell α with the unit cells Hom_f and Hom_g at its vertical boundary functors recovers α. The upshot of Definition 8.3.12 is that $\mathcal{M}od(\mathfrak{h}\mathcal{K})$ can be understood to contain various "vertical" and "horizontal" bicategories.

8.3.13. Proposition (the vertical 2-category in the virtual equipment). *Any virtual equipment contains a **vertical 2-category** whose objects are the objects of the virtual equipment, whose arrows are the vertical arrows, and whose 2-cells are those unary cells*

$$\begin{tikzcd}
A \arrow[r, "\mathsf{Hom}_A", "\shortmid"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow"] & A \arrow[d, "f"] \\
B \arrow[r, "\mathsf{Hom}_B"', "\shortmid"{marking}] & B
\end{tikzcd}$$

whose horizontal boundary arrows are given by the unit modules.

Proof. To prove that these structures define a 2-category we must define "horizontal" composition of 2-cells (composing along a boundary 0-cell) and "vertical" composition of 2-cells (composing along a boundary 1-cell). The "horizontal" composition in the 2-category is defined via the vertical composition in the virtual double category described in Definition 8.1.9. The "vertical" composition in the 2-category is defined by Definition 8.3.12. To see that this yields a 2-category and not a bicategory note that any bicategory in which the composition of 1-cells is strictly associative and unital is a 2-category; in this case, the 1-cells are the vertical arrows of the virtual double category, which do indeed compose strictly. $\square$

Proposition 8.4.10 will prove that the vertical 2-category in the virtual equipment $\mathcal{M}od(\mathfrak{h}\mathcal{K})$ is isomorphic to the homotopy 2-category $\mathfrak{h}\mathcal{K}$.

8.3.14. Remark (horizontal bicategories in the virtual equipment). Via Definition 8.3.12, a virtual equipment can also be understood to contain various "horizontal" bicategories, defined by taking the 1-cells to be composable modules and the 2-cells to be unary module maps whose vertical boundary functors are identities. Particular horizontal bicategories of interest are described in Definition 8.4.11.

Exercises.

8.3.i. Exercise. Extending Exercise 8.1.ii, prove that in a virtual double category arising from an actual double category that every composable sequence of horizontal morphisms admits a composite in the sense of Definition 8.3.1.

8.3.ii. Exercise. Prove Lemma 8.3.3.

8.4. Representable modules

Any vertical arrow $f\colon A \to B$ in a virtual equipment has a pair of associated horizontal arrows $B \overset{\mathrm{Hom}_B(f,B)}{\nrightarrow} A$ and $A \overset{\mathrm{Hom}_B(B,f)}{\nrightarrow} B$ — defined as restrictions of the horizontal unit arrows — that have universal properties similar to companions and conjoints in an ordinary double category. In the virtual equipment of modules, these are sensible referred to as the *left* and *right representations* of a functor as a module and coincide exactly with the left and right representable first introduced in §3.5 and reappearing in Definition 7.4.8.

8.4.1. Proposition (companion and conjoint relations for representables). *To any functor $f\colon A \to B$ in the virtual equipment of modules, there exist canonical restriction cells displayed below-left and application cells displayed below-right*

$$
\begin{tikzcd}
B \arrow[r, "\mathrm{Hom}_B(f,B)", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\rho"] & A \arrow[d, "f"] \\
B \arrow[r, "\mathrm{Hom}_B"', "\shortmid"{marking}] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "\mathrm{Hom}_B(B,f)", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\rho"] & B \arrow[d, equal] \\
B \arrow[r, "\mathrm{Hom}_B"', "\shortmid"{marking}] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "\mathrm{Hom}_A", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\kappa"] & A \arrow[d, equal] \\
B \arrow[r, "\mathrm{Hom}_B(f,B)"', "\shortmid"{marking}] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "\mathrm{Hom}_A", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\kappa"] & A \arrow[d, "f"] \\
A \arrow[r, "\mathrm{Hom}_B(B,f)"', "\shortmid"{marking}] & B
\end{tikzcd}
$$

defining unary module maps between the unit modules $A \overset{\mathrm{Hom}_A}{\nrightarrow} A$ and $B \overset{\mathrm{Hom}_B}{\nrightarrow} B$ and the left and right representable modules $B \overset{\mathrm{Hom}_B(f,B)}{\nrightarrow} A$ and $A \overset{\mathrm{Hom}_B(B,f)}{\nrightarrow} B$. These satisfy the identities:

$$
\begin{tikzcd}
A \arrow[r, "\mathrm{Hom}_A", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\kappa"] & A \arrow[d, equal] \\
B \arrow[r, "\mathrm{Hom}_B(f,B)", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\rho"] & A \arrow[d, "f"] \\
B \arrow[r, "\mathrm{Hom}_B"', "\shortmid"{marking}] & B
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[r, "\mathrm{Hom}_A", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\mathrm{Hom}_f"] & A \arrow[d, "f"] \\
B \arrow[r, "\mathrm{Hom}_B"', "\shortmid"{marking}] & B
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[r, "\mathrm{Hom}_A", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\kappa"] & A \arrow[d, "f"] \\
A \arrow[r, "\mathrm{Hom}_B(B,f)", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\rho"] & B \arrow[d, equal] \\
B \arrow[r, "\mathrm{Hom}_B"', "\shortmid"{marking}] & B
\end{tikzcd}
\tag{8.4.2}
$$

and

$$
\begin{tikzcd}
B \arrow[r, "\mathrm{Hom}_B(f,B)", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\rho"] & A \arrow[r, "\mathrm{Hom}_A", "\shortmid"{marking}] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\kappa"] & A \arrow[d, equal] \\
B \arrow[r, "\mathrm{Hom}_B", "\shortmid"{marking}] \arrow[d, equal] & B \arrow[r, "\mathrm{Hom}_B(f,B)", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\circ"] & A \arrow[d, equal] \\
B \arrow[rr, "\mathrm{Hom}_B(f,B)"', "\shortmid"{marking}] & & A
\end{tikzcd}
=
\begin{tikzcd}
B \arrow[r, "\mathrm{Hom}_B(f,B)", "\shortmid"{marking}] \arrow[d, equal] & A \arrow[r, "\mathrm{Hom}_A", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\circ"] & A \arrow[d, equal] \\
B \arrow[rr, "\mathrm{Hom}_B(f,B)"', "\shortmid"{marking}] & & A
\end{tikzcd}
\quad
\begin{tikzcd}
A \arrow[r, "\mathrm{Hom}_A", "\shortmid"{marking}] \arrow[d, equal] & A \arrow[r, "\mathrm{Hom}_B(B,f)", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\circ"] & B \arrow[d, equal] \\
A \arrow[rr, "\mathrm{Hom}_B(B,f)"', "\shortmid"{marking}] & & B
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "\mathrm{Hom}_A", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\kappa"] & A \arrow[r, "\mathrm{Hom}_B(B,f)", "\shortmid"{marking}] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\rho"] & B \arrow[d, equal] \\
A \arrow[r, "\mathrm{Hom}_B(B,f)", "\shortmid"{marking}] \arrow[d, equal] & B \arrow[r, "\mathrm{Hom}_B", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\circ"] & A \arrow[d, equal] \\
A \arrow[rr, "\mathrm{Hom}_B(B,f)"', "\shortmid"{marking}] & & B
\end{tikzcd}
\tag{8.4.3}
$$

Proof. The unary module maps κ are defined by the equations (8.4.2) by appealing to the universal property of the restriction cells in Proposition 8.2.1. The relations (8.4.3) could also be verified directly from the axioms of Theorem 8.2.6 via Propositions 8.2.4 and Lemmas 8.3.5 and 8.3.5. Instead, we appeal to Lemma 8.1.16 to characterize each of the cells in the virtual equipment as 2-cells in the homotopy 2-category.

We prove this for the right representables; the co-dual then proves this for the left representables. The binary module morphism on the left-hand side of the equality of (8.4.3) represents the 2-cell below-left, while the right-hand composite is below-right:

$$\begin{tikzcd}[column sep=small]
& & A^2 \underset{A}{\times} \mathsf{Hom}_B(B,f) \arrow[dl] \arrow[dr] \arrow[d, phantom, "\lrcorner" rotate=-45] & & \\
& A^2 \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\overset{\kappa}{\Leftarrow}"] & {} & \mathsf{Hom}_B(B,f) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\
A \arrow[rr, equal] & {} & A \arrow[rr, "f"'] & {} & B
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
& & & A^2 \underset{A}{\times} \mathsf{Hom}_B(B,f) \arrow[dll] \arrow[dr] \arrow[d, phantom, "\lrcorner" rotate=-45] & & \\
& A^2 \arrow[ddl, bend right, "p_1"'] \arrow[d, "k"] \arrow[dr, "p_0"] & & {} & \mathsf{Hom}_B(B,f) \arrow[dll, "p_1"'] \arrow[d, "\rho"] \arrow[ddr, bend left, "p_0"] & \\
& \mathsf{Hom}_B(B,f) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & A \arrow[d, "f"] & & B^2 \arrow[dll, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\overset{\kappa}{\Leftarrow}"] & \\
A \arrow[rr, "f"'] & {} & B \arrow[rrr, equal] & & {} & B
\end{tikzcd}$$

By the definition of the induced functor $k\colon A^2 \to \mathsf{Hom}_B(B,f)$, $\phi k = f\kappa$ and by the definition of the comma cone ϕ for $\mathsf{Hom}_B(B,f)$, $\phi = \kappa\rho$. Thus, the left-hand side equals the right-hand side. $\square$

8.4.4. Theorem. *In the virtual equipment of modules there are bijections between cells of the following four forms*

$$\begin{tikzcd}
C \arrow[r, "\mathsf{Hom}_C(f,C)", "\shortmid"{marking}] \arrow[d, equal] & A \arrow[r, "E", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow"] & B \arrow[d, "g"] \\
C \arrow[rr, "F"', "\shortmid"{marking}] & & D
\end{tikzcd}
\leftrightsquigarrow
\begin{tikzcd}
A \arrow[r, "E", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow"] & B \arrow[d, "g"] \\
C \arrow[r, "F"', "\shortmid"{marking}] & D
\end{tikzcd}
\leftrightsquigarrow
\begin{tikzcd}
A \arrow[r, "E", "\shortmid"{marking}] \arrow[d, "f"'] & B \arrow[r, "\mathsf{Hom}_D(D,g)", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow"] & D \arrow[d, equal] \\
C \arrow[rr, "F"', "\shortmid"{marking}] & & D
\end{tikzcd}$$

$$\rotatebox[origin=c]{-45}{$\leftrightsquigarrow$} \qquad\qquad \rotatebox[origin=c]{45}{$\leftrightsquigarrow$}$$

$$\begin{tikzcd}
C \arrow[r, "\mathsf{Hom}_C(f,C)", "\shortmid"{marking}] \arrow[d, equal] & A \arrow[r, "E", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow"] & B \arrow[r, "\mathsf{Hom}_D(D,g)", "\shortmid"{marking}] & D \arrow[d, equal] \\
C \arrow[rrr, "F"', "\shortmid"{marking}] & & & D
\end{tikzcd}$$

implemented by composing with the canonical cells κ and ρ of Proposition 8.4.1 and with the composition and nullary cells associated to the units.

Proof. The composite bijection carries cells α and β to the cells displayed below-left and below-right respectively:

$$\hat{\alpha} := \begin{tikzcd}
C \arrow[r, "\mathrm{Hom}_C(f,C)"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\rho"] & A \arrow[r, "E"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & B \arrow[r, "\mathrm{Hom}_D(D,g)"] \arrow[d, "g"] \arrow[dr, phantom, "\Downarrow\rho"] & D \arrow[d, equal] \\
C \arrow[r, "\mathrm{Hom}_C"] \arrow[d, equal] & C \arrow[r, "F"] \arrow[dr, phantom, "\Downarrow\circ"] & D \arrow[r, "\mathrm{Hom}_D"] & D \arrow[d, equal] \\
C \arrow[rrr, "F"'] & & & D
\end{tikzcd}
\qquad \text{and} \qquad
\check{\beta} := \begin{tikzcd}
A \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\iota"] & A \arrow[r, "E"] \arrow[d, equal] & B \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\iota"] & B \arrow[d, equal] \\
A \arrow[r, "\mathrm{Hom}_A"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\kappa"] & A \arrow[r, "E"] \arrow[d, equal] & B \arrow[r, "\mathrm{Hom}_B"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\kappa"] & B \arrow[d, "g"] \\
C \arrow[r, "\mathrm{Hom}_C(f,C)"] \arrow[d, equal] & A \arrow[r, "E"] \arrow[dr, phantom, "\Downarrow\beta"] & B \arrow[r, "\mathrm{Hom}_D(D,g)"] & D \arrow[d, equal] \\
C \arrow[rrr, "F"'] & & & D
\end{tikzcd}$$

By Proposition 8.4.1 and Lemma 8.3.11

$$\check{\hat{\alpha}} := \begin{tikzcd}
A \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\iota"] & A \arrow[r, "E"] \arrow[d, equal] & B \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\iota"] & B \arrow[d, equal] \\
A \arrow[r, "\mathrm{Hom}_A"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\kappa"] & A \arrow[r, "E"] \arrow[d, equal] & B \arrow[r, "\mathrm{Hom}_B"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\kappa"] & B \arrow[d, "g"] \\
C \arrow[r, "\mathrm{Hom}_C(f,C)"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\rho"] & A \arrow[r, "E"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & B \arrow[r, "\mathrm{Hom}_D(D,g)"] \arrow[d, "g"] \arrow[dr, phantom, "\Downarrow\rho"] & D \arrow[d, equal] \\
C \arrow[r, "\mathrm{Hom}_C"] \arrow[d, equal] & C \arrow[r, "F"] \arrow[dr, phantom, "\Downarrow\circ"] & D \arrow[r, "\mathrm{Hom}_D"] & D \arrow[d, equal] \\
C \arrow[rrr, "F"'] & & & D
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\iota"] & A \arrow[r, "E"] \arrow[d, equal] & B \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\iota"] & B \arrow[d, equal] \\
A \arrow[r, "\mathrm{Hom}_A"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\mathrm{Hom}_f"] & A \arrow[r, "E"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & B \arrow[r, "\mathrm{Hom}_B"] \arrow[d, "g"] \arrow[dr, phantom, "\Downarrow\mathrm{Hom}_g"] & B \arrow[d, "g"] \\
C \arrow[r, "\mathrm{Hom}_C"] \arrow[d, equal] & C \arrow[r, "F"] \arrow[dr, phantom, "\Downarrow\circ"] & D \arrow[r, "\mathrm{Hom}_D"] & D \arrow[d, equal] \\
C \arrow[rrr, "F"'] & & & D
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "E"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & B \arrow[d, "g"] \\
C \arrow[r, "F"'] & D
\end{tikzcd}$$

The other composite is:

$$\hat{\check{\beta}} := \begin{tikzcd}
C \arrow[r, "\mathrm{Hom}_C(f,C)"] \arrow[d, equal] & A \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\iota"] & A \arrow[r, "E"] \arrow[d, equal] & B \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\iota"] & B \arrow[r, "\mathrm{Hom}_D(D,g)"] \arrow[d, equal] & D \arrow[d, equal] \\
C \arrow[r, "\mathrm{Hom}_C(f,C)"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\rho"] & A \arrow[r, "\mathrm{Hom}_A"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\kappa"] & A \arrow[r, "E"] \arrow[d, equal] & B \arrow[r, "\mathrm{Hom}_B"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\kappa"] & B \arrow[r, "\mathrm{Hom}_D(D,g)"] \arrow[d, "g"] \arrow[dr, phantom, "\Downarrow\rho"] & D \arrow[d, equal] \\
C \arrow[r, "\mathrm{Hom}_C"] \arrow[d, equal] & C \arrow[r, "\mathrm{Hom}_C(f,C)"] \arrow[d, equal] & A \arrow[r, "E"] \arrow[d, phantom, "\Downarrow\beta"] & B \arrow[r, "\mathrm{Hom}_D(D,g)"] & D \arrow[r, "\mathrm{Hom}_D"] \arrow[d, equal] & D \arrow[d, equal] \\
C \arrow[r, "\mathrm{Hom}_C"] \arrow[d, equal] & C \arrow[rrr, "F"'] & {} \arrow[d, phantom, "\Downarrow\circ"] & & D \arrow[r, "\mathrm{Hom}_D"] & D \arrow[d, equal] \\
C \arrow[rrrrr, "F"'] & & & & & D
\end{tikzcd} \tag{8.4.5}$$

By Lemma 8.3.5, we have

$$
\begin{tikzcd}
C \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\iota"] & C \arrow[r, "\mathsf{Hom}_C(f,C)"] \arrow[d, equal] & A \arrow[r, "E"] \arrow[d, equal] & B \arrow[r, "\mathsf{Hom}_D(D,g)"] \arrow[d, equal] & D \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\iota"] & D \arrow[d, equal] \\
C \arrow[r, "\mathsf{Hom}_C"] \arrow[d, equal] & C \arrow[r, "\mathsf{Hom}_C(f,C)"] \arrow[d, equal] \arrow[drrr, phantom, "\Downarrow\beta"] & A \arrow[r, "E"] & B \arrow[r, "\mathsf{Hom}_D(D,g)"] & D \arrow[r, "\mathsf{Hom}_D"] \arrow[d, equal] & D \arrow[d, equal] \\
C \arrow[r, "\mathsf{Hom}_C"] \arrow[d, equal] \arrow[drrrrr, phantom, "\Downarrow\circ"] & C \arrow[rrr, "F"'] & & & D \arrow[r, "\mathsf{Hom}_D"] & D \arrow[d, equal] \\
C \arrow[rrrrr, "F"'] & & & & & D
\end{tikzcd}
=
\begin{tikzcd}
C \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\iota"] & C \arrow[r, "\mathsf{Hom}_C(f,C)"] \arrow[d, equal] & A \arrow[r, "E"] \arrow[d, equal] & B \arrow[r, "\mathsf{Hom}_D(D,g)"] \arrow[d, equal] & D \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\iota"] & D \arrow[d, equal] \\
C \arrow[r, "\mathsf{Hom}_C"] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\circ"] & C \arrow[r, "\mathsf{Hom}_C(f,C)"] & A \arrow[r, "E"] \arrow[d, equal] & B \arrow[r, "\mathsf{Hom}_D(D,g)"] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\circ"] & D \arrow[r, "\mathsf{Hom}_D"] & D \arrow[d, equal] \\
C \arrow[rr, "\mathsf{Hom}_C(f,C)"] \arrow[d, equal] \arrow[drrrrr, phantom, "\Downarrow\beta"] & & A \arrow[r, "E"] & B \arrow[rr, "\mathsf{Hom}_D(D,g)"] & & D \arrow[d, equal] \\
C \arrow[rrrrr, "F"'] & & & & & D
\end{tikzcd}
$$

since both composites equal β. By the universal property of the unit cells in Proposition 8.2.4, the bottom two rows of these diagrams are equal, so we may substitute the bottom two rows of the right-hand diagram for the bottom two rows of (8.4.5) to obtain:

$$
\hat{\hat{\beta}} =
\begin{tikzcd}
C \arrow[r, "\mathsf{Hom}_C(f,C)"] \arrow[d, equal] & A \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\iota"] & A \arrow[r, "E"] \arrow[d, equal] & B \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\iota"] & B \arrow[r, "\mathsf{Hom}_D(D,g)"] \arrow[d, equal] & D \arrow[d, equal] \\
C \arrow[r, "\mathsf{Hom}_C(f,C)"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\rho"] & A \arrow[r, "\mathsf{Hom}_A"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\kappa"] & A \arrow[r, "E"] \arrow[d, equal] & B \arrow[r, "\mathsf{Hom}_B"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\kappa"] & B \arrow[r, "\mathsf{Hom}_D(D,g)"] \arrow[d, "g"] \arrow[dr, phantom, "\Downarrow\rho"] & D \arrow[d, equal] \\
C \arrow[r, "\mathsf{Hom}_C"] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\circ"] & C \arrow[r, "\mathsf{Hom}_C(f,C)"] & A \arrow[r, "E"] \arrow[d, equal] & B \arrow[r, "\mathsf{Hom}_D(D,g)"] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\circ"] & D \arrow[r, "\mathsf{Hom}_D"] & D \arrow[d, equal] \\
C \arrow[rr, "\mathsf{Hom}_C(f,C)"'] \arrow[d, equal] \arrow[drrrrr, phantom, "\Downarrow\beta"] & & A \arrow[r, "E"] & B \arrow[rr, "\mathsf{Hom}_D(D,g)"'] & & D \arrow[d, equal] \\
C \arrow[rrrrr, "F"'] & & & & & D
\end{tikzcd}
$$

By Proposition 8.4.1 and Lemma 8.3.5 this reduces to β.

$$
=
\begin{tikzcd}
C \arrow[r, "\mathsf{Hom}_C(f,C)"] \arrow[d, equal] & A \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\iota"] & A \arrow[r, "E"] \arrow[d, equal] & B \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\iota"] & B \arrow[r, "\mathsf{Hom}_D(D,g)"] \arrow[d, equal] & D \arrow[d, equal] \\
C \arrow[r, "\mathsf{Hom}_C(f,C)"] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\circ"] & A \arrow[r, "\mathsf{Hom}_A"] & A \arrow[r, "E"] \arrow[d, equal] & B \arrow[r, "\mathsf{Hom}_B"] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\circ"] & B \arrow[r, "\mathsf{Hom}_D(D,g)"] & D \arrow[d, equal] \\
C \arrow[rr, "\mathsf{Hom}_C(f,C)"'] \arrow[d, equal] \arrow[drrrrr, phantom, "\Downarrow\beta"] & & A \arrow[r, "E"] & B \arrow[rr, "\mathsf{Hom}_D(D,g)"'] & & D \arrow[d, equal] \\
C \arrow[rrrrr, "F"'] & & & & & D
\end{tikzcd}
=
\begin{tikzcd}
C \arrow[r, "\mathsf{Hom}_C(f,C)"] \arrow[d, equal] \arrow[drrr, phantom, "\Downarrow\beta"] & A \arrow[r, "E"] & B \arrow[r, "\mathsf{Hom}_D(D,g)"] & D \arrow[d, equal] \\
C \arrow[rrr, "F"'] & & & D
\end{tikzcd}
$$

By vertically bisecting this construction, one obtains the one-sided bijections of the statement. □

We frequently apply this result in the following form:

8.4.6. Corollary. *Given modules $A \overset{E}{\nrightarrow} B$ and $C \overset{F}{\nrightarrow} D$ and functor $f\colon A \to C$ and $g\colon B \to D$, there is a bijection between ternary module maps $\mathsf{Hom}_C(f,C) \mathbin{\underset{\cdot}{\times}} E \mathbin{\underset{\cdot}{\times}} \mathsf{Hom}_D(D,g) \Rightarrow F$ and unary module maps $E \Rightarrow F(g,f)$, i.e., between cells*

$$\begin{tikzcd}
C \arrow[r, "\mathsf{Hom}_C(f{,}C)", "\shortmid"{marking}] \arrow[d, equal] & A \arrow[r, "E", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow"] & B \arrow[r, "\mathsf{Hom}_D(D{,}g)", "\shortmid"{marking}] & D \arrow[d, equal] \\
C \arrow[rrr, "F"', "\shortmid"{marking}] & & & D
\end{tikzcd}
\quad \leftrightsquigarrow \quad
\begin{tikzcd}
A \arrow[r, "E", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow"] & B \arrow[d, equal] \\
A \arrow[r, "F(g{,}f)"', "\shortmid"{marking}] & B
\end{tikzcd}$$

Proof. Combine Theorem 8.4.4 with Proposition 8.2.1. $\square$

8.4.7. Proposition. *For any module $A \overset{E}{\nrightarrow} B$ and pair of functors $a\colon X \to A$ and $b\colon Y \to B$, the composite $\mathsf{Hom}_A(A,a) \otimes E \otimes \mathsf{Hom}_B(b,B)$ exists and is given by the restriction $E(b,a)$, with the ternary composite map $\mu\colon \mathsf{Hom}_A(A,a) \mathbin{\underset{\cdot}{\times}} E \mathbin{\underset{\cdot}{\times}} \mathsf{Hom}_B(b,B) \Rightarrow E(b,a)$ defined by the universal property of the restriction by the pasting diagram:*

$$\begin{tikzcd}
X \arrow[r, "\mathsf{Hom}_A(A{,}a)", "\shortmid"{marking}] \arrow[d, equal] & A \arrow[r, "E", "\shortmid"{marking}] & B \arrow[r, "\mathsf{Hom}_B(b{,}B)", "\shortmid"{marking}] & Y \arrow[d, equal] \\
X \arrow[rrr, "E(b{,}a)"', "\shortmid"{marking}] \arrow[d, "a"'] & & & Y \arrow[d, "b"] \\
A \arrow[rrr, "E"', "\shortmid"{marking}] & & & B
\arrow[from=1-2, to=2-3, phantom, "\Downarrow\mu"]
\arrow[from=2-2, to=3-3, phantom, "\Downarrow\rho"]
\end{tikzcd}
\quad \coloneqq \quad
\begin{tikzcd}
X \arrow[r, "\mathsf{Hom}_A(A{,}a)", "\shortmid"{marking}] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\rho"] & A \arrow[r, "E", "\shortmid"{marking}] \arrow[d, equal] & B \arrow[r, "\mathsf{Hom}_B(b{,}B)", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\rho"] & Y \arrow[d, "b"] \\
A \arrow[r, "\mathsf{Hom}_A"', "\shortmid"{marking}] \arrow[d, equal] & A \arrow[r, "E", "\shortmid"{marking}] & B \arrow[r, "\mathsf{Hom}_B"', "\shortmid"{marking}] & B \arrow[d, equal] \\
A \arrow[rrr, "E"', "\shortmid"{marking}] & & & B
\arrow[from=2-2, to=3-3, phantom, "\Downarrow\circ"]
\end{tikzcd}$$

Proof. The horizontal composite two-sided fibration of the composable sequence is

$$\begin{tikzcd}[column sep=small]
& & & \mathsf{Hom}_A(A,a) \underset{A}{\times} E \underset{B}{\times} \mathsf{Hom}_B(b,B) \arrow[dl, ->>] \arrow[dr, ->>] \arrow[dd, phantom, "\lrcorner"{rotate=-45}, very near start] & & & \\
& & \mathsf{Hom}_A(q,a) \arrow[dl, ->>] \arrow[dr, ->>] \arrow[dd, phantom, "\lrcorner"{rotate=-45}, very near start] & & \mathsf{Hom}_B(b,p) \arrow[dl, ->>] \arrow[dr, ->>] \arrow[dd, phantom, "\lrcorner"{rotate=-45}, very near start] & & \\
& \mathsf{Hom}_A(A,a) \arrow[dl, ->>, "p_1"'] \arrow[dr, ->>, "p_0"] & & E \arrow[dl, ->>, "q"'] \arrow[dr, ->>, "p"] & & \mathsf{Hom}_B(b,B) \arrow[dl, ->>, "p_1"'] \arrow[dr, ->>, "p_0"] & \\
X & & A & & B & & Y
\end{tikzcd}$$

from which we see that the binary composite cell

$$\begin{tikzcd}
\mathsf{Hom}_A(q,A) \underset{E}{\times} \mathsf{Hom}_B(B,p) \arrow[rr] \arrow[dr, ->>] & & E \arrow[dl, ->>, "(q{,}p)"] \\
& A \times B &
\end{tikzcd}$$

of Proposition 8.3.7 pulls back along $a \times b\colon X \times Y \to A \times B$ to define the map μ. By Lemma 8.3.6, we conclude that $\mu\colon \mathsf{Hom}_A(A,a) \mathbin{\underset{\cdot}{\times}} E \mathbin{\underset{\cdot}{\times}} \mathsf{Hom}_B(b,B) \Rightarrow E(b,a)$ is a composite. $\square$

As a special case, right representable modules can always be composed with each other and dually left representable modules can always be composed with each other:

8.4.8. COROLLARY. *Any composable pair of functors $A \xrightarrow{f} B \xrightarrow{g} C$ defines a composable pair of right-represented modules and a composable pair of left-represented modules*

$$A \overset{\mathsf{Hom}_B(B,f)}{\longrightarrow\!\!\!\!\!\!\!\!\!+\;\;\;} B \overset{\mathsf{Hom}_C(C,g)}{\longrightarrow\!\!\!\!\!\!\!\!\!+\;\;\;} C \qquad\qquad C \overset{\mathsf{Hom}_C(g,C)}{\longrightarrow\!\!\!\!\!\!\!\!\!+\;\;\;} B \overset{\mathsf{Hom}_B(f,B)}{\longrightarrow\!\!\!\!\!\!\!\!\!+\;\;\;} A$$

and moreover:

$$\mathsf{Hom}_B(B,f)\otimes\mathsf{Hom}_C(C,g) \simeq \mathsf{Hom}_C(C,gf) \quad \textit{and} \quad \mathsf{Hom}_C(g,C)\otimes\mathsf{Hom}_B(f,B) \simeq \mathsf{Hom}_C(gf,C). \qquad \square$$

Combining this result with Proposition 8.3.7 provides a generalization of Theorem 3.5.11, which allows us to detect representable modules.

8.4.9. PROPOSITION. *Let $A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$ encode a module.*

(i) *The module $A \overset{E}{\nrightarrow} B$ is right representable if and only if its left leg $q\colon E \twoheadrightarrow A$ has a right adjoint $r\colon A \to E$ in which case $E \simeq \mathsf{Hom}_B(B, pr)$.*

(ii) *The module $A \overset{E}{\nrightarrow} B$ is left representable if and only if its right leg $p\colon E \twoheadrightarrow B$ has a left adjoint $\ell\colon B \to E$ in which case $E \simeq \mathsf{Hom}_A(q\ell, A)$.*

PROOF. Lemma 3.5.8 proves that the claimed adjoints to the legs $p_1\colon \mathsf{Hom}_B(B,f) \twoheadrightarrow A$ and $p_0\colon \mathsf{Hom}_B(f,B) \twoheadrightarrow A$ of left or right representable modules exist, so it remains only to prove the converse. By Proposition 8.3.7, any module $A \overset{E}{\nrightarrow} B$ can be expressed as a composite $E \simeq \mathsf{Hom}_A(q,A)\otimes \mathsf{Hom}_B(B,p)$ of the left representation of its left leg followed by the right right representation of its right leg. If $q \dashv r$, then by Proposition 4.1.1 $\mathsf{Hom}_A(q,A) \simeq \mathsf{Hom}_E(E,r)$, so $E \simeq \mathsf{Hom}_E(E,r) \otimes \mathsf{Hom}_B(B,p)$. By Corollary 8.4.8, $\mathsf{Hom}_E(E,r) \otimes \mathsf{Hom}_B(B,p) \simeq \mathsf{Hom}_B(B,pr)$ so it follows that $E \simeq \mathsf{Hom}_B(B,pr)$. The case of left representability is dual. $\square$

Finally, we revisit the "cheap" version of the Yoneda lemma presented in Corollary 3.5.10, which encodes natural transformations in the homotopy 2-category as maps of represented modules.

8.4.10. PROPOSITION. *For any parallel pair of functors there are natural bijections between 2-cells in the homotopy 2-category*

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow"] & B \end{tikzcd}$$

and cells in the virtual equipment of modules:

$$\begin{tikzcd} A \arrow[r, "\mathsf{Hom}_B(B,f)", "|"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow"] & B \arrow[d, equal] \\ A \arrow[r, "|"{marking}, "\mathsf{Hom}_B(B,g)"'] & B \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} A \arrow[r, "\mathsf{Hom}_A", "|"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow"] & A \arrow[d, "f"] \\ B \arrow[r, "|"{marking}, "\mathsf{Hom}_B"'] & B \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} B \arrow[r, "\mathsf{Hom}_B(g,B)", "|"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow"] & A \arrow[d, equal] \\ B \arrow[r, "|"{marking}, "\mathsf{Hom}_B(f,B)"'] & A \end{tikzcd}$$

PROOF. The bijection between 2-cells in the homotopy 2-category and unary module maps between left or right representables is simply a restatement of Corollary 3.5.10. Alternatively, the bijection between 2-cells in the homotopy 2-category and the cells displayed in the center follows from Lemma 8.1.16 and Proposition 8.2.4. The bijections in $\mathcal{M}od(\mathfrak{h}\mathcal{K})$ can then be derived from Theorem 8.4.4 and Corollary 8.4.6. $\square$

8.4.11. Definition (the covariant and contravariant embeddings). Proposition 8.4.10 defines the action on 2-cells of two identity-on-objects locally fully faithful homomorphisms

$$\begin{array}{ccc} \mathfrak{h}\mathcal{K} \hookrightarrow \mathcal{M}od(\mathfrak{h}\mathcal{K}) & \qquad & \mathfrak{h}\mathcal{K}^{\mathrm{coop}} \hookrightarrow \mathcal{M}od(\mathfrak{h}\mathcal{K}) \\ A \xrightarrow{f} B \longmapsto A \overset{\mathsf{Hom}_B(B,f)}{\nrightarrow} B & \qquad & A \xrightarrow{f} B \longmapsto B \overset{\mathsf{Hom}_B(f,B)}{\nrightarrow} A \end{array}$$

that embed the homotopy 2-category fully faithfully into the sub "bicategory" of $\mathcal{M}od(\mathfrak{h}\mathcal{K})$ containing only those unary cells whose vertical boundaries are identities.

This substructure of $\mathcal{M}od(\mathfrak{h}\mathcal{K})$ isn't quite a bicategory because not all horizontally composable modules can be composed, but if we restrict only to the right representable modules or only to the left representable modules, then by Corollary 8.4.8 the composites do exist and moreover the embeddings are horizontally as well as vertically pseudofunctorial: given $A \xrightarrow{f} B \xrightarrow{g} C$ we have $\mathsf{Hom}_B(B,f) \otimes \mathsf{Hom}_C(C,g) \simeq \mathsf{Hom}_C(C,gf)$ and $\mathsf{Hom}_C(g,C) \otimes \mathsf{Hom}_B(f,B) \simeq \mathsf{Hom}_C(gf,C)$. We refer to these as the **covariant** and **contravariant** embeddings respectively.

As described in Remark 8.3.14, $\mathcal{M}od(\mathfrak{h}\mathcal{K})$ can be understood to contain genuine bicategories whose 2-cells are unary cells between right-represented modules (or between left-represented modules) whose vertical boundaries are identities. In this way, the covariant and contravariant embeddings can be understood to define genuine bicategorical homomorphisms.

There is a third locally fully faithful embedding of the homotopy 2-category $\mathfrak{h}\mathcal{K}$ into $\mathcal{M}od(\mathfrak{h}\mathcal{K})$ that is identity on objects, sends $f \colon A \to B$ to the corresponding vertical 1-cell, and uses the third bijection of Proposition 8.4.10 to define the action on 2-cells. Since $\mathsf{Hom}_A \otimes \mathsf{Hom}_A \simeq \mathsf{Hom}_A$, the unary cells in this image of this embedding can be composed horizontally as well as vertically, and this embedding is functorial in both directions: vertical composites of natural transformations in $\mathfrak{h}\mathcal{K}$ coincide with horizontal composites of unary cells and horizontal composites of natural transformations in $\mathfrak{h}\mathcal{K}$ coincide with vertical composites of unary cells. The image is precisely the vertical 2-category of Proposition 8.3.13. We will make much greater use of the covariant and contravariant embeddings of Definition 8.4.11 however.

Exercises.

8.4.i. Exercise. Prove Proposition 8.4.1 in any virtual equipment, without appealing to Lemma 8.1.16.

8.4.ii. Exercise. Prove Proposition 8.4.7 in any virtual equipment, without appealing to Lemma 8.3.6.

8.4.iii. Exercise. For any functor $f \colon A \to B$, define a unary cell $\eta \colon \mathsf{Hom}_A \Rightarrow \mathsf{Hom}_B(f,f) \cong \mathsf{Hom}_B(B,f) \otimes \mathsf{Hom}_B(f,B)$ and a binary cell $\epsilon \colon \mathsf{Hom}_B(f,B) \times \mathsf{Hom}_B(B,f) \Rightarrow \mathsf{Hom}_B$. Use this data to demonstrate that the modules $A \overset{\mathsf{Hom}_B(B,f)}{\nrightarrow} B$ and $B \overset{\mathsf{Hom}_B(f,B)}{\nrightarrow} A$ are "adjoint" in a suitable sense.

CHAPTER 9

Formal category theory in a virtual equipment

Mac Lane famously asserted that "all concepts are Kan extensions" [**60**, §X.7] — at least in category theory. Kelly later amended this to assert that the *pointwise Kan extensions*, which he calls simply "Kan extensions" are the important ones, writing "Our present choice of nomenclature is based on our failure to find a single instance where a [non-pointwise] Kan extension plays any mathematical role whatsoever" [**54**, §4]. Using the calculus of modules we can now add the theory of pointwise Kan extensions of functors between ∞-categories to the basic ∞-category theory developed in Part I.

Right and left extensions of a functor $f\colon A \to C$ along a functor $k\colon A \to B$ can be defined internally to any 2-category — at this level of generality the eponym "Kan" is typically dropped. We review this notion in Definition 9.1.1. However, in the homotopy 2-category of an ∞-cosmos, the universal property defining left and right extensions is not strong enough, and indeed the correct universal property is associated to the stronger notion of a *pointwise extension*, for which the values of a right or left extension at an element of B can be computed as limits or colimits indexed by the appropriate comma ∞-category; see Corollary 9.5.3 for a precise statement.

Our aim in this chapter is to define and study pointwise extensions for functors between ∞-categories. In fact, we give multiple definitions of pointwise extension. One is fundamentally 2-categorical: a pointwise extension is an ordinary 2-categorical extension in the homotopy 2-category that is stable under pasting with comma squares. Another definition is that a 2-cell defines a pointwise right extension if and only if its image under the covariant embedding into the virtual equipment of modules defines a right extension there. Theorem 9.3.3 proves that these two notions coincide.

In §9.1, we introduce right liftings and right extensions in the virtual equipment of modules and being to familiarize ourselves with these notions. Before turning our attention to pointwise extensions, we first introduce exact squares in §9.2, a class of squares in the homotopy 2-category that include comma squares and which will be used to characterize the pointwise extensions internally to the homotopy 2-category. Pointwise extensions are introduced in a variety of equivalent ways in §9.3 and applied in §9.4 to develop a few aspects of the formal theory of ∞-categories. In §9.5, we conclude with a discussion of pointwise extensions in a cartesian closed ∞-cosmos, in which context these relate to the absolute lifting diagrams and limits and colimits studied in §2.3-2.4.

9.1. Liftings and extensions of modules

In this section we introduce and study liftings and extensions in the virtual equipment of modules. To motivate Definition 9.1.2, we briefly recall the standard 2-categorical definition:

9.1.1. Definition. A **right extension** of a 1-cell $f\colon A \to C$ along a 1-cell $k\colon A \to B$ is given by a pair $(r\colon B \to C, \nu\colon rk \Rightarrow f)$ as below-left

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "k"'] \arrow[dr, phantom, "\Uparrow\nu"] & C \\ B \arrow[ur, dashed, "r"'] & \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "k"'] \arrow[dr, phantom, "\Uparrow\gamma"] & C \\ B \arrow[ur, "g"'] & \end{tikzcd} = \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "k"'] \arrow[dr, phantom, "\Uparrow\nu"] & C \\ B \arrow[ur, "r"] \arrow[ur, bend right, "g"'] \arrow[ur, phantom, "\Uparrow\exists!"] & \end{tikzcd}$$

so that any similar pair as above-center factors uniquely through ν as above right. The co-dual defines a **left extension** of a 1-cell $f\colon A \to C$ along a 1-cell $k\colon A \to B$.

The op-dual of Definition 9.1.1 defines a notion of **right lifting** in any 2-category. Analogous notions of right extension and right lifting can be defined for horizontal morphisms in a virtual double equipment, where in the presence of restrictions of modules it suffices to consider cells whose vertical functors are identities. We specialize our language to the virtual equipment of modules, as this will be the one case of interest:

9.1.2. Definition. A **right extension** of a module $A \overset{F}{\nrightarrow} C$ along a module $A \overset{K}{\nrightarrow} B$ consists of a pair given by a module $B \overset{R}{\nrightarrow} C$ together with a binary cell

$$\begin{tikzcd} A \arrow[r, "K", "\shortmid"{marking}] \arrow[d, equal] & B \arrow[r, "R", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\nu"] & C \arrow[d, equal] \\ A \arrow[rr, "F"', "\shortmid"{marking}] & & C \end{tikzcd}$$

with the property that every $n+1$-ary cell of the form displayed below-left factors uniquely through $\nu\colon K \mathbin{\underset{\cdot}{\times}} R \Rightarrow F$ as below-right:

$$\begin{tikzcd} A \arrow[r, "K", "\shortmid"{marking}] \arrow[d, equal] & B \arrow[r, "E_1", "\shortmid"{marking}] & \cdots \arrow[r, "E_n", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow"] & C \arrow[d, equal] \\ A \arrow[rrr, "F"', "\shortmid"{marking}] & & & C \end{tikzcd} = \begin{tikzcd} A \arrow[r, "K", "\shortmid"{marking}] \arrow[d, equal] & B \arrow[r, "E_1", "\shortmid"{marking}] \arrow[d, equal] & \cdots \arrow[r, "E_n", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\exists!"] & C \arrow[d, equal] \\ A \arrow[r, "K", "\shortmid"{marking}] \arrow[d, equal] & B \arrow[rr, "R", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\nu"] & & C \arrow[d, equal] \\ A \arrow[rrr, "F"', "\shortmid"{marking}] & & & C \end{tikzcd}$$

Dually, a **right lifting** of $A \overset{F}{\nrightarrow} C$ through $B \overset{H}{\nrightarrow} C$ consists of a pair given by a module $A \overset{L}{\nrightarrow} B$ together with a binary cell

$$\begin{tikzcd} A \arrow[r, "L", "\shortmid"{marking}] \arrow[d, equal] & B \arrow[r, "H", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\lambda"] & C \arrow[d, equal] \\ A \arrow[rr, "F"', "\shortmid"{marking}] & & C \end{tikzcd}$$

with the property that every $n+1$-ary cell of the form displayed below-left factors uniquely through $\lambda\colon L \mathbin{\dot\times} H \Rightarrow F$ as below-right:

$$\begin{tikzcd} A \arrow[r, "E_1", "\shortmid"{marking}] \arrow[d, equal] & \cdots \arrow[r, "E_n", "\shortmid"{marking}] & B \arrow[r, "H", "\shortmid"{marking}] & C \arrow[d, equal] \\ A \arrow[rrr, "F"', "\shortmid"{marking}] \arrow[urrr, phantom, "\Downarrow"] & & & C \end{tikzcd} \quad = \quad \begin{tikzcd} A \arrow[r, "E_1", "\shortmid"{marking}] \arrow[d, equal] & \cdots \arrow[r, "E_n", "\shortmid"{marking}] & B \arrow[r, "H", "\shortmid"{marking}] \arrow[d, equal] & C \arrow[d, equal] \\ A \arrow[rr, "L", "\shortmid"{marking}] \arrow[d, equal] \arrow[ur, phantom, "\Downarrow\exists!"] & & B \arrow[r, "H", "\shortmid"{marking}] & C \arrow[d, equal] \\ A \arrow[rrr, "F"', "\shortmid"{marking}] \arrow[urrr, phantom, "\Downarrow\lambda"] & & & C \end{tikzcd}$$

Because of the asymmetry in Definition 8.1.9, there is no corresponding notion of *left* extension or *left* lifting. It follows easily from these definitions that right extensions or right liftings are unique up to vertical isomorphism in $\mathcal{M}od(\mathfrak{h}\mathcal{K})$; see Exercise 9.1.i.

9.1.3. LEMMA. *For any functor $f\colon A \to B$, the binary cell*

$$\begin{tikzcd} B \arrow[r, "\mathsf{Hom}_B(f,B)", "\shortmid"{marking}] \arrow[d, equal] & A \arrow[r, "\mathsf{Hom}_B(B,f)", "\shortmid"{marking}] & B \arrow[d, equal] \\ B \arrow[rr, "\mathsf{Hom}_B"', "\shortmid"{marking}] \arrow[urr, phantom, "\Downarrow\epsilon"] & & B \end{tikzcd}$$

defined in Exercise 8.4.iii defines a right extension of $B \overset{\mathsf{Hom}_B}{\twoheadrightarrow} B$ through $B \overset{\mathsf{Hom}_B(f,B)}{\twoheadrightarrow} A$ and a right lifting of $B \overset{\mathsf{Hom}_B}{\twoheadrightarrow} B$ through $A \overset{\mathsf{Hom}_B(B,f)}{\twoheadrightarrow} B$.

PROOF. Exercise 9.1.ii. $\square$

The result of Lemma 9.1.3 is a special case of a more general result.

9.1.4. LEMMA. *For any module $C \overset{F}{\twoheadrightarrow} D$ and any pair of functors $f\colon A \to C$ and $g\colon B \to D$ the canonical cells*

$$\begin{tikzcd} C \arrow[r, "\mathsf{Hom}_C(f,C)", "\shortmid"{marking}] \arrow[d, equal] & A \arrow[r, "F(1,f)", "\shortmid"{marking}] & D \arrow[d, equal] \\ C \arrow[rr, "F"', "\shortmid"{marking}] \arrow[urr, phantom, "\Downarrow\rho"] & & D \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "F(g,1)", "\shortmid"{marking}] \arrow[d, equal] & B \arrow[r, "\mathsf{Hom}_D(D,g)", "\shortmid"{marking}] & D \arrow[d, equal] \\ C \arrow[rr, "F"', "\shortmid"{marking}] \arrow[urr, phantom, "\Downarrow\rho"] & & D \end{tikzcd}$$

exhibit $F(1,f)$ as the right extension of F through $\mathsf{Hom}_C(f,C)$ and exhibit $F(g,1)$ as the right lifting of F through $\mathsf{Hom}_D(D,g)$.

PROOF. The canonical cells of the statement arise from applying the bijection of Theorem 8.4.4 to the restriction cells of Proposition 8.2.1, which is to say that in the case of the right extension diagram above-left this cell is obtained as the composite

$$\begin{tikzcd}
C \arrow[r, "\mathrm{Hom}_C(f,C)", "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\rho"] & A \arrow[r, "F(1,f)", "\shortmid" marking] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\rho"] & D \arrow[d, equal] \\
C \arrow[r, "\mathrm{Hom}_C"', "\shortmid" marking] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\circ"] & C \arrow[r, "F"', "\shortmid" marking] & D \arrow[d, equal] \\
C \arrow[rr, "F"', "\shortmid" marking] & & D
\end{tikzcd}$$

Now the bijections of Proposition 8.2.1 and Theorem 8.4.4, mildly generalized to allow the module E to be replaced by a composable sequence of modules, combine to exhibit the universal property of the right extension. ☐

Right extensions and right liftings can be understood as right adjoints to horizontal composition with a module on the left or on the right, respectively. This leads to the following "associativity" result, which we formulate for right extensions, leaving the dual result for right liftings to the reader.

9.1.5. Proposition. *Suppose $A \overset{K}{\nrightarrow} B$, $B \overset{H}{\nrightarrow} C$, and $A \overset{F}{\nrightarrow} D$ are modules so that the composite $K \otimes H$ and right extension $B \overset{R_K F}{\nrightarrow} D$ modules exist. Then the right extension of $R_K F$ along H exists if and only if the right extension of F along $K \otimes H$ exists, in which case $R_H(R_K F) \simeq_{C \times D} R_{K \otimes H} F$.*

Proof. The universal property of the binary composite cell $\mu\colon K \times H \Rightarrow K \otimes H$ and right extension cell $\nu\colon K \times R_K F \Rightarrow F$ provide a bijection between cells involving an arbitrary module $C \overset{R}{\nrightarrow} D$

$$\begin{tikzcd}
A \arrow[r, "K", "\shortmid" marking] \arrow[d, equal] & B \arrow[r, "H", "\shortmid" marking] \arrow[d, phantom, "\Downarrow\mu"] & C \arrow[r, "R", "\shortmid" marking] \arrow[d, equal] & D \arrow[d, equal] \\
A \arrow[rr, "K\otimes H"', "\shortmid" marking] \arrow[d, equal] & & C \arrow[r, "R"', "\shortmid" marking] \arrow[d, phantom, "\Downarrow\exists!"] & D \arrow[d, equal] \\
A \arrow[rrr, "F"', "\shortmid" marking] & & & D
\end{tikzcd}
\quad \leftrightsquigarrow \quad
\begin{tikzcd}
A \arrow[r, "K", "\shortmid" marking] \arrow[d, equal] & B \arrow[r, "H", "\shortmid" marking] \arrow[d, equal] & C \arrow[r, "R", "\shortmid" marking] \arrow[d, phantom, "\Downarrow\exists!"] & D \arrow[d, equal] \\
A \arrow[r, "K"', "\shortmid" marking] \arrow[d, equal] & B \arrow[rr, "R_K F"', "\shortmid" marking] \arrow[d, phantom, "\Downarrow\nu"] & & D \arrow[d, equal] \\
A \arrow[rrr, "F"', "\shortmid" marking] & & & D
\end{tikzcd}$$

If the right extension $R_{K\otimes H} F$ exists we take this for R and use the binary cell on the lower-left to induce the binary cell on the upper-right, which we'll show exhibits $R_{K\otimes H} F$ as the right extension $R_H(R_K F)$ of $R_K F$ along H. Conversely, if the right extension $R_H(R_K F)$ exists we use the binary cell on the upper-right to induce the binary cell on the lower-left, which we'll show exhibits $R_H(R_K F)$ as the right extension $R_{K\otimes H} F$ of F along $K \otimes H$. This transference of universal properties is straightforward, following again from the bijection just exhibited, but with a composable sequence of modules in place of R.

Finally, if separately the right extensions $R_H(R_K F)$ and $R_{K\otimes H} F$ are known to exist, then the argument given above combined with the uniqueness of right extensions combines to show that $R_{K\otimes H} F \simeq_{C\times D} R_H(R_K F)$; see Exercise 9.1.i. ☐

We now explain how Definition 9.1.2 relates to Definition 9.1.1 via the covariant and contravariant embeddings of Definition 8.4.11.

9.1.6. Lemma. *If*

$$\begin{tikzcd} A \arrow[r, "\mathrm{Hom}_B(B,k)", "\shortmid"{marking}] \arrow[d, equal] & B \arrow[r, "\mathrm{Hom}_C(C,r)", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\nu"] & C \arrow[d, equal] \\ A \arrow[rr, "\shortmid"{marking}, "\mathrm{Hom}_C(C,f)"'] & & C \end{tikzcd}$$

defines a right extension in the virtual equipment of modules, then $\nu\colon rk \Rightarrow f$ defines a right extension in the homotopy 2-category. Dually if

$$\begin{tikzcd} A \arrow[r, "\mathrm{Hom}_A(\ell,A)", "\shortmid"{marking}] \arrow[d, equal] & B \arrow[r, "\mathrm{Hom}_B(h,B)", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\lambda"] & C \arrow[d, equal] \\ A \arrow[rr, "\shortmid"{marking}, "\mathrm{Hom}_A(g,A)"'] & & C \end{tikzcd}$$

defines a right lifting in the virtual equipment of modules, then $\lambda\colon g \Rightarrow \ell h$ is a left extension in the homotopy 2-category.

Proof. By Corollary 8.4.8 binary cells $\nu\colon \mathrm{Hom}_B(B,k) \times \mathrm{Hom}_C(C,r) \Rightarrow \mathrm{Hom}_C(C,f)$ correspond to unary cells $\nu\colon \mathrm{Hom}_C(C,rk) \Rightarrow \mathrm{Hom}_C(C,f)$, which by Proposition 8.4.10 correspond to natural transformations $\nu\colon rk \Rightarrow f$ in the homotopy 2-category. Via this correspondence the universal property of Definition 9.1.2 clearly subsumes that of Definition 9.1.1. The left extension case is similar, via the contravariant embedding of Definition 8.4.11. $\square$

A sharper characterization of the right extension diagrams of modules in the image of the covariant embedding will have to wait for Theorem 9.3.3, but we can characterize the right lifting diagrams of modules in the image of the covariant embedding now. Duals of these results apply to the right lifting and right extension diagrams of modules in the image of the contravariant embedding.

Recall the notion of absolute right lifting diagram introduced in Definition 2.3.4.

9.1.7. Proposition. *A 2-cell in the homotopy 2-category of an ∞-cosmos as below-left defines an absolute right lifting diagram if and only if the corresponding binary cell displayed below-right defines a right lifting in the virtual equipment of modules:*

$$\begin{tikzcd} & B \arrow[d, "f"] \\ C \arrow[ur, "r"] \arrow[r, "g"'] \arrow[ur, phantom, shift right=3, "\Downarrow\rho"] & A \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} C \arrow[r, "\mathrm{Hom}_B(B,r)", "\shortmid"{marking}] \arrow[d, equal] & B \arrow[r, "\mathrm{Hom}_A(A,f)", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\hat{\rho}"] & A \arrow[d, equal] \\ C \arrow[rr, "\shortmid"{marking}, "\mathrm{Hom}_A(A,g)"'] & & A \end{tikzcd}$$

Dually, a 2-cell in the homotopy 2-category of an ∞-cosmos as below-left defines an absolute left lifting diagram if and only if the corresponding binary cell displayed below-right defines a right extension in the virtual equipment of modules:

$$\begin{tikzcd} & B \arrow[d, "f"] \\ C \arrow[ur, "\ell"] \arrow[r, "g"'] \arrow[ur, phantom, shift right=3, "\Uparrow\lambda"] & A \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} A \arrow[r, "\mathrm{Hom}_A(f,A)", "\shortmid"{marking}] \arrow[d, equal] & B \arrow[r, "\mathrm{Hom}_B(\ell,B)", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\hat{\lambda}"] & A \arrow[d, equal] \\ A \arrow[rr, "\shortmid"{marking}, "\mathrm{Hom}_A(g,A)"'] & & C \end{tikzcd}$$

Proof. By Proposition 8.4.10, natural transformations in the homotopy 2-category of an ∞-cosmos correspond bijectively to unary squares in the virtual equipment of modules of various forms. By this result, Corollary 8.4.6, and Corollary 8.4.8, there are canonical bijections:

$$
\begin{tikzcd}
& X \arrow[dl, "c"'] \arrow[dr, "b"] & \\
C \arrow[dr, "g"'] \arrow[rr, phantom, "\overset{\chi}{\Leftarrow}"] & & B \arrow[dl, "f"] \\
& A &
\end{tikzcd}
\quad \leftrightsquigarrow \quad
\begin{tikzcd}
X \arrow[r, "\mathrm{Hom}_A(A,fb)", "\mid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow"] & A \arrow[d, equal] \\
X \arrow[r, "\mathrm{Hom}_A(A,gc)"', "\mid" marking] & A
\end{tikzcd}
\quad \leftrightsquigarrow \quad
\begin{tikzcd}
C \arrow[r, "\mathrm{Hom}_C(c,C)", "\mid" marking] \arrow[d, equal] & X \arrow[r, "\mathrm{Hom}_B(B,b)", "\mid" marking] \arrow[d, phantom, ""] & B \arrow[r, "\mathrm{Hom}_A(A,f)", "\mid" marking] & A \arrow[d, equal] \\
C \arrow[rrr, "\mathrm{Hom}_A(A,g)"', "\mid" marking] \arrow[urrr, phantom, "\Downarrow\hat{\chi}"] & & & A
\end{tikzcd}
\tag{9.1.8}
$$

If the binary cell $\hat{\rho}\colon \mathrm{Hom}_B(B,r) \mathbin{\dot\times} \mathrm{Hom}_A(A,f) \Rightarrow \mathrm{Hom}_A(A,g)$ defines a right lifting diagram in the virtual equipment of modules, then there is a unique factorization

$$
\begin{tikzcd}
C \arrow[r, "\mathrm{Hom}_C(c,C)", "\mid" marking] \arrow[d, equal] & X \arrow[r, "\mathrm{Hom}_B(B,b)", "\mid" marking] & B \arrow[r, "\mathrm{Hom}_A(A,f)", "\mid" marking] & A \arrow[d, equal] \\
C \arrow[rrr, "\mathrm{Hom}_A(A,g)"', "\mid" marking] \arrow[urrr, phantom, "\Downarrow\hat{\chi}"] & & & A
\end{tikzcd}
=
\begin{tikzcd}
C \arrow[r, "\mathrm{Hom}_C(c,C)", "\mid" marking] \arrow[d, equal] & X \arrow[r, "\mathrm{Hom}_B(B,b)", "\mid" marking] & B \arrow[r, "\mathrm{Hom}_A(A,f)", "\mid" marking] \arrow[d, equal] & A \arrow[d, equal] \\
C \arrow[rr, "\mathrm{Hom}_B(B,r)"', "\mid" marking] \arrow[d, equal] \arrow[urr, phantom, "\exists!\Downarrow\hat{\zeta}"] & & B \arrow[r, "\mathrm{Hom}_A(A,f)"', "\mid" marking] & A \arrow[d, equal] \\
C \arrow[rrr, "\mathrm{Hom}_A(A,g)"', "\mid" marking] \arrow[urrr, phantom, "\Downarrow\hat{\rho}"] & & & A
\end{tikzcd}
$$

Reversing the canonical bijection (9.1.8), this defines the desired unique factorization

$$
\begin{tikzcd}
X \arrow[r, "b"] \arrow[d, "c"'] \arrow[dr, phantom, "\Downarrow\chi"] & B \arrow[d, "f"] \\
C \arrow[r, "g"'] & A
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, "b"] \arrow[d, "c"'] & B \arrow[d, "f"] \\
C \arrow[r, "g"'] \arrow[ur, "r" description] \arrow[u, phantom, "\exists!\Downarrow\zeta", near start] & A \arrow[ul, phantom, "\Downarrow\rho"]
\end{tikzcd}
$$

in the homotopy 2-category. Thus if $\hat{\rho}\colon \mathrm{Hom}_B(B,r) \mathbin{\dot\times} \mathrm{Hom}_A(A,f) \Rightarrow \mathrm{Hom}_A(A,g)$ is a right lifting, then $\rho\colon fr \Rightarrow g$ is an absolute right lifting.

Conversely, suppose $\rho\colon fr \Rightarrow g$ is an absolute right lifting and consider a cell in the virtual equipment of modules of the following form:

$$
\begin{tikzcd}
C \arrow[r, "E_1", "\mid" marking] \arrow[d, equal] & \cdots \arrow[r, "E_n", "\mid" marking] & B \arrow[r, "\mathrm{Hom}_A(A,f)", "\mid" marking] & A \arrow[d, equal] \\
C \arrow[rrr, "\mathrm{Hom}_A(A,g)"', "\mid" marking] \arrow[urrr, phantom, "\Downarrow\bar{\psi}"] & & & A
\end{tikzcd}
$$

Let $C \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$ denote the composite two-sided fibration $E_1 \mathbin{\dot\times} \cdots \mathbin{\dot\times} E_n$. By Remark 8.3.8 applied to $C \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$, module maps $\bar{\psi}\colon E_1 \mathbin{\dot\times} \cdots \mathbin{\dot\times} E_n \mathbin{\dot\times} \mathrm{Hom}_A(A,f) \Rightarrow \mathrm{Hom}_A(A,g)$ stand in bijection with module maps $\hat{\psi}\colon \mathrm{Hom}_C(q,C) \mathbin{\dot\times} \mathrm{Hom}_B(B,p) \mathbin{\dot\times} \mathrm{Hom}_A(A,f) \Rightarrow \mathrm{Hom}_A(A,g)$, as displayed below-left. As argued in (9.1.8), these stand in canonical bijection with natural transformations as below-center:

$$
\begin{tikzcd}
C \arrow[r, "\mathrm{Hom}_C(q{,}C)", "\shortmid" marking] \arrow[d, equal] & E \arrow[r, "\mathrm{Hom}_B(B{,}p)", "\shortmid" marking] & B \arrow[r, "\mathrm{Hom}_A(A{,}f)", "\shortmid" marking] & A \arrow[d, equal] \\
C \arrow[rrr, "\mathrm{Hom}_A(A{,}g)"', "\shortmid" marking] \arrow[urrr, phantom, "\Downarrow\hat{\psi}"] & & & A
\end{tikzcd}
\quad\rightsquigarrow\quad
\begin{tikzcd}
& E \arrow[dl, "q"'] \arrow[dr, "p"] & \\
C \arrow[dr, "g"'] \arrow[rr, phantom, "\overset{\psi}{\Leftarrow}"] & & B \arrow[dl, "f"] \\
& A &
\end{tikzcd}
\quad\rightsquigarrow\quad
\begin{tikzcd}
& E \arrow[dl, "q"'] \arrow[dr, "p"] \arrow[d, phantom, "\overset{\exists!\xi}{\Leftarrow}"] & \\
C \arrow[dr, "g"'] \arrow[rr, "r"] & {} \arrow[d, phantom, "\overset{\rho}{\Leftarrow}"] & B \arrow[dl, "f"] \\
& A &
\end{tikzcd}
$$

Since $\rho\colon fr \Rightarrow g$ is assumed to be an absolute right lifting, ψ factors uniquely through ρ to define a corresponding 2-cell $\xi\colon p \Rightarrow rq$ as above-right. Applying (9.1.8) again, this constructs a unique factorization in the virtual equipment of modules

$$
\begin{tikzcd}
C \arrow[r, "\mathrm{Hom}_C(q{,}C)", "\shortmid" marking] \arrow[d, equal] & E \arrow[r, "\mathrm{Hom}_B(B{,}p)", "\shortmid" marking] & B \arrow[r, "\mathrm{Hom}_A(A{,}f)", "\shortmid" marking] & A \arrow[d, equal] \\
C \arrow[rrr, "\mathrm{Hom}_A(A{,}g)"', "\shortmid" marking] \arrow[urrr, phantom, "\Downarrow\hat{\psi}"] & & & A
\end{tikzcd}
=
\begin{tikzcd}
C \arrow[r, "\mathrm{Hom}_C(q{,}C)", "\shortmid" marking] \arrow[d, equal] & E \arrow[r, "\mathrm{Hom}_B(B{,}p)", "\shortmid" marking] & B \arrow[r, "\mathrm{Hom}_A(A{,}f)", "\shortmid" marking] \arrow[d, equal] & A \arrow[d, equal] \\
C \arrow[rr, "\mathrm{Hom}_B(B{,}r)"', "\shortmid" marking] \arrow[urr, phantom, "\exists!\Downarrow\hat{\xi}"] \arrow[d, equal] & & B \arrow[r, "\mathrm{Hom}_A(A{,}f)"', "\shortmid" marking] & A \arrow[d, equal] \\
C \arrow[rrr, "\mathrm{Hom}_A(A{,}g)"', "\shortmid" marking] \arrow[urrr, phantom, "\Downarrow\hat{\rho}"] & & & A
\end{tikzcd}
$$

By Remark 8.3.8, this defines a bijection

$$
\begin{tikzcd}
C \arrow[r, "E_1", "\shortmid" marking] \arrow[d, equal] & \cdots \arrow[r, "E_n", "\shortmid" marking] & B \arrow[r, "\mathrm{Hom}_A(A{,}f)", "\shortmid" marking] & A \arrow[d, equal] \\
C \arrow[rrr, "\mathrm{Hom}_A(A{,}g)"', "\shortmid" marking] \arrow[urrr, phantom, "\Downarrow\bar{\psi}"] & & & A
\end{tikzcd}
=
\begin{tikzcd}
C \arrow[r, "E_1", "\shortmid" marking] \arrow[d, equal] & \cdots \arrow[r, "E_n", "\shortmid" marking] & B \arrow[r, "\mathrm{Hom}_A(A{,}f)", "\shortmid" marking] \arrow[d, equal] & A \arrow[d, equal] \\
C \arrow[rr, "\mathrm{Hom}_B(B{,}r)"', "\shortmid" marking] \arrow[urr, phantom, "\exists!\Downarrow\bar{\xi}"] \arrow[d, equal] & & B \arrow[r, "\mathrm{Hom}_A(A{,}f)"', "\shortmid" marking] & A \arrow[d, equal] \\
C \arrow[rrr, "\mathrm{Hom}_A(A{,}g)"', "\shortmid" marking] \arrow[urrr, phantom, "\Downarrow\hat{\rho}"] & & & A
\end{tikzcd}
$$

Thus if $\rho\colon fr \Rightarrow g$ is an absolute right lifting, then $\hat{\rho}\colon \mathrm{Hom}_B(B,r) \times \mathrm{Hom}_A(A,f) \Rightarrow \mathrm{Hom}_A(A,g)$ is a right lifting. $\square$

In Theorem 9.3.3 we will discover that right extensions of modules in the image of the covariant embedding are precisely characterized by the sought-for *pointwise* right extensions in the homotopy 2-category; dually pointwise left extensions correspond to right liftings of modules in the image of the contravariant embedding. In the next section, we build towards the 2-categorical definition of this notion.

Exercises.

9.1.i. Exercise. Suppose $B \overset{R}{\nrightarrow} C$ and $B \overset{S}{\nrightarrow} C$ both define right extensions of a module $A \overset{F}{\nrightarrow} C$ along a module $A \overset{K}{\nrightarrow} B$ in the sense of Definition 9.1.2. Prove that $R \simeq_{B\times C} S$.

9.1.ii. Exercise. Prove Lemma 9.1.3.

9.1.iii. Exercise. Dualize Proposition 9.1.5 to characterize right liftings through a composite of two modules.

9.1.iv. Exercise. Verify the dual statement of Proposition 9.1.7, that absolute left lifting diagrams $\lambda\colon g \Rightarrow f\ell$ correspond to right extension diagrams $\hat{\lambda}\colon \mathrm{Hom}_A(f,A) \times \mathrm{Hom}_B(\ell,B) \Rightarrow \mathrm{Hom}_A(g,A)$.

9.2. Exact squares

To motivate the main definition of this section, let us try to guess the 2-categorical universal property of a pointwise right extension by considering a special case that we already understand. If the ambient ∞-cosmos is cartesian closed, then the pointwise right extension of a diagram $f \colon A \to C$ along a functor $k \colon A \to B$ is intended to define the value of a right adjoint, which may or may not exist in toto, to the restriction functor $\mathrm{res}_k \colon C^B \to C^A$ at the element $f \colon 1 \to C^A$. In the case of extensions along a functor $! \colon A \to 1$, the restriction functor is the constant diagram functor $\Delta \colon C \to C^A$ considered in Definition 2.3.2, and so via Definition 2.3.7 we can understand the pointwise right extension as computing the limit of f. The following lemma describes the transposed form of this universal property.

9.2.1. Lemma. *In a cartesian closed ∞-cosmos, the triangle below-left is an absolute right lifting diagram — defining the limit element and limit cone of f — if and only if the transposed triangle below-center has the property that for any X, the composite diagram below-right is a right extension diagram.*

$$
\begin{tikzcd}
 & C \arrow[d, "\Delta"] \\
1 \arrow[ur, dashed, "\lim f"] \arrow[r, "f"'] \arrow[ur, phantom, shift right=3, "\Downarrow\nu"] & C^A
\end{tikzcd}
\quad \leftrightsquigarrow \quad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "!"'] & C \\
1 \arrow[ur, dashed, "\lim f"'] \arrow[ur, phantom, shift left=3, "\Uparrow\nu"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
X \times A \arrow[r, "\pi"] \arrow[d, "\pi"'] & A \arrow[r, "f"] \arrow[d, "!"'] & C \\
X \arrow[r, "!"] & 1 \arrow[ur, dashed, "\lim f"'] \arrow[ur, phantom, shift left=3, "\Uparrow\nu"] &
\end{tikzcd}
$$

Proof. A factorization of a cone with summit X through the absolute right lifting of f along the constant diagram functor

$$
\begin{tikzcd}
X \arrow[r, "c"] \arrow[d, "!"'] \arrow[dr, phantom, "\Downarrow\chi"] & C \arrow[d, "\Delta"] \\
1 \arrow[r, "f"'] & C^A
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, "c"] \arrow[d, "!"'] \arrow[r, phantom, shift right=3, "\exists!\Downarrow\zeta"] & C \arrow[d, "\Delta"] \\
1 \arrow[r, "f"'] \arrow[ur, dashed, "\lim f"] \arrow[r, phantom, shift left=3, "\Downarrow\nu"] & C^A
\end{tikzcd}
$$

transposes to a factorization as below:

$$
\begin{tikzcd}
X \times A \arrow[r, "\pi"] \arrow[d, "\pi"'] & A \arrow[r, "f"] \arrow[d, phantom, "\Uparrow\chi"] & C \\
X \arrow[urr, bend right, "c"'] & &
\end{tikzcd}
=
\begin{tikzcd}
X \times A \arrow[r, "\pi"] \arrow[d, "\pi"'] & A \arrow[r, "f"] \arrow[d, "!"'] & C \\
X \arrow[r, "!"] \arrow[urr, bend right=60, "c"'] & 1 \arrow[ur, dashed, "\lim f"'] \arrow[ur, phantom, shift left=3, "\Uparrow\nu"] \arrow[ur, phantom, shift right=4, "\exists!\Uparrow\zeta"] &
\end{tikzcd}
\qquad \square
$$

Lemma 9.2.1 reveals that to define the limit of $f \colon A \to C$ in an ∞-cosmos that is not necessarily cartesian closed, it is not enough to form the right extension of $! \colon A \to 1$. In terminology we will introduce in Definition 9.3.1 below, we must ask in addition that the right extension diagram is *stable under pasting* with squares of the form:

$$
\begin{tikzcd}
X \times A \arrow[r, "\pi"] \arrow[d, "\pi"'] & A \arrow[d, "!"] \\
X \arrow[r, "!"'] & 1
\end{tikzcd}
$$

How might we characterize such squares? Firstly, they are pullbacks each of whose legs is a bifibration. Secondly, they are comma squares, where the comma cone is an identity 2-cell best regarded as pointing in a direction compatible with ν. By Lemmas 9.2.5 and 9.2.6, we shall see that both of these are instances of *exact squares*, which we now introduce.

By Proposition 8.4.10, natural transformations in the homotopy 2-category of an ∞-cosmos correspond bijectively to unary squares in the virtual equipment of modules of various forms, and in particular, this result, Theorem 8.4.4, Proposition 8.2.4, and Proposition 8.2.1 defines a canonical bijection:

$$\begin{tikzcd}[column sep=small, row sep=small] & D \arrow[dl, "k"'] \arrow[dr, "h"] & \\ C \arrow[dr, "g"'] \arrow[rr, phantom, "\overset{\alpha}{\Leftarrow}"] & & B \arrow[dl, "f"] \\ & A & \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} D \arrow[r, "\mathrm{Hom}_D", "\shortmid"{marking}] \arrow[d, "gk"'] \arrow[dr, phantom, "\Downarrow"] & D \arrow[d, "fh"] \\ A \arrow[r, "\mathrm{Hom}_A"', "\shortmid"{marking}] & A \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} C \arrow[r, "\mathrm{Hom}_C(k{,}C)", "\shortmid"{marking}] \arrow[d, equal] & D \arrow[r, "\mathrm{Hom}_B(B{,}h)", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\hat{\alpha}"] & B \arrow[d, equal] \\ C \arrow[rr, "\mathrm{Hom}_A(f{,}g)"', "\shortmid"{marking}] & & B \end{tikzcd}$$

9.2.2. Definition (exact squares). A square in the homotopy 2-category of an ∞-cosmos

$$\begin{tikzcd}[column sep=small, row sep=small] & D \arrow[dl, "k"'] \arrow[dr, "h"] & \\ C \arrow[dr, "g"'] \arrow[rr, phantom, "\overset{\alpha}{\Leftarrow}"] & & B \arrow[dl, "f"] \\ & A & \end{tikzcd}$$

is **exact** if and only if the corresponding cell below-left, which under the bijection of Lemma 8.1.16 encodes the below-right pasted composite

$$\begin{tikzcd} C \arrow[r, "\mathrm{Hom}_C(k{,}C)", "\shortmid"{marking}] \arrow[d, equal] & D \arrow[r, "\mathrm{Hom}_B(B{,}h)", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\hat{\alpha}"] & B \arrow[d, equal] \\ C \arrow[rr, "\mathrm{Hom}_A(f{,}g)"', "\shortmid"{marking}] & & B \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd}[column sep=small] & \mathrm{Hom}_C(k,C) \underset{D}{\times} \mathrm{Hom}_B(B,h) \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" very near start] & \\ \mathrm{Hom}_C(k,C) \arrow[dr, "p_0"] \arrow[dd, bend right, "p_1"'] \arrow[dr, phantom, "\overset{\phi}{\Leftarrow}"' near end] & & \mathrm{Hom}_B(B,h) \arrow[dl, "p_1"'] \arrow[dd, bend left, "p_0"] \arrow[dl, phantom, "\overset{\phi}{\Leftarrow}" near end] \\ & D \arrow[dl, "k"'] \arrow[dr, "h"] & \\ C \arrow[dr, "g"'] \arrow[rr, phantom, "\overset{\alpha}{\Leftarrow}"] & & B \arrow[dl, "f"] \\ & A & \end{tikzcd} \tag{9.2.3}$$

displays $\mathrm{Hom}_A(f,g)$ as the composite $\mathrm{Hom}_C(k,C) \otimes \mathrm{Hom}_B(B,h)$ as defined in Definition 8.3.1.

When the boundary square is clear from context, for economy of language we may write that "$\alpha\colon fh \Rightarrow gk$ is an exact square" but note that the definition of exactness requires the specification of the four boundary components of the square inhabited by the 2-cell α.[1]

[1]For instance, compare the statements of Exercise 9.2.ii and Lemma 9.4.3.

9.2.4. REMARK (exactness as a Beck-Chevalley condition). By Proposition 8.4.7, the canonical cell $\mu \mathsf{Hom}_A(A,g) \times \mathsf{Hom}_A(f,A) \Rightarrow \mathsf{Hom}_A(f,g)$ encoded by the map of spans defined by 1-cell induction

$$
\begin{tikzcd}[column sep=small]
& & \mathsf{Hom}_A(A,g) \underset{A}{\times} \mathsf{Hom}_A(f,A) \arrow[dl] \arrow[dr] \arrow[d, phantom, "\lrcorner" rotate=-45] & & \\
& \mathsf{Hom}_A(A,g) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & {} & \mathsf{Hom}_A(f,A) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\
C \arrow[rr, "g"'] & {} & A & {} & B \arrow[ll, "f"]
\end{tikzcd}
=
\begin{tikzcd}
& \mathsf{Hom}_A(A,g) \underset{A}{\times} \mathsf{Hom}_A(f,A) \arrow[d, dashed, "\mu"] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\
& \mathsf{Hom}_A(f,g) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\
C \arrow[dr, "g"'] & {} & B \arrow[dl, "f"] \\
& A &
\end{tikzcd}
$$

is a composite. Exactness says that α induces an isomorphism

$$\hat{\alpha}\colon \mathsf{Hom}_C(k,C) \otimes \mathsf{Hom}_B(B,h) \simeq \mathsf{Hom}_A(A,g) \otimes \mathsf{Hom}_A(f,A)$$

of modules from C to B.

The remainder of this section is devoted to examples of exact squares.

9.2.5. LEMMA (comma squares are exact). *For any cospan $C \xrightarrow{g} A \xleftarrow{f} B$, the comma cone defines an exact square:*

$$
\begin{tikzcd}
& \mathsf{Hom}_A(f,g) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\
C \arrow[dr, "g"'] & {} & B \arrow[dl, "f"] \\
& A &
\end{tikzcd}
$$

PROOF. By Proposition 8.3.7, the module $C \overset{\mathsf{Hom}_A(f,g)}{\nrightarrow} B$ is the composite of the left representation of its left leg followed by the right representation of its right leg: $\mathsf{Hom}_C(p_1,C) \otimes \mathsf{Hom}_B(B,p_0) \simeq \mathsf{Hom}_A(f,g)$. By Remark 8.3.9, the composition map $\mathsf{Hom}_C(p_1,C) \dot{\times} \mathsf{Hom}_B(B,p_0) \Rightarrow \mathsf{Hom}_A(f,g)$ classifies the pasted composite

$$
\begin{tikzcd}
& \mathsf{Hom}_C(p_1,C) \dot{\times} \mathsf{Hom}_B(B,p_0) \arrow[dl] \arrow[dr] \arrow[d, phantom, "\lrcorner" rotate=-45] & \\
\mathsf{Hom}_C(p_1,C) \arrow[dr, "p_0"] \arrow[dd, bend right, "p_1"'] \arrow[r, phantom, "\overset{\phi}{\Leftarrow}"] & {} & \mathsf{Hom}_B(B,p_0) \arrow[dl, "p_1"'] \arrow[dd, bend left, "p_0"] \arrow[l, phantom, "\overset{\phi}{\Leftarrow}"] \\
& \mathsf{Hom}_A(f,g) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\
C \arrow[dr, "g"'] & {} & B \arrow[dl, "f"] \\
& A &
\end{tikzcd}
$$

which recovers the cell $\hat{\phi}$ defined by (9.2.3) that tests the exactness of comma square $\phi\colon fp_0 \Rightarrow gp_1$. $\square$

9.2.6. LEMMA. *If $g\colon C \to A$ is a cartesian fibration or $f\colon B \to A$ is a cocartesian fibration, then the pullback square*

$$\begin{tikzcd}[column sep=small] & P \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_0"] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & \\ C \arrow[dr, "g"'] & & B \arrow[dl, "f"] \\ & A & \end{tikzcd}$$

is exact.

PROOF. The two statements are dual though the positions of the cocartesian and cartesian fibrations cannot be interchanged, as the proof will reveal. If $f\colon B \twoheadrightarrow A$ is a cocartesian fibration, observe that the functor $i\colon P \to \mathsf{Hom}_A(f,g)$ induced by the identity 2-cell $f\pi_0 = g\pi_1$ is a pullback of the functor $i\colon B \to \mathsf{Hom}_A(f,A)$ induced by the identity 2-cell id_f.

$$\begin{tikzcd} P \arrow[rr, "\pi_0"] \arrow[dr, bend right, "i"'] \arrow[dr, phantom, bend left, "\perp"] \arrow[dd, "\pi_1"'] & & B \arrow[dr, bend right, "i"'] \arrow[dr, phantom, bend left, "\perp"] \arrow[dd, "f" near end] & \\ & \mathsf{Hom}_A(f,g) \arrow[ul, dashed, bend right, "\ell"'] \arrow[rr] \arrow[dl, "p_1"] \arrow[dr, phantom, "\lrcorner", very near start] & & \mathsf{Hom}_A(f,A) \arrow[ul, dashed, bend right, "\ell"'] \arrow[dl, "p_1"] \\ C \arrow[rr, "g"'] & & A & \end{tikzcd}$$

Since f is a cocartesian fibration, Theorem 5.1.11(ii) tells us that $i\colon B \to \mathsf{Hom}_A(f,A)$ has a fibered left adjoint over A. This fibered adjunction pulls back via the cosmological functor $g^*\colon \mathcal{K}_{/A} \to \mathcal{K}_{/C}$ to define a fibered left adjoint to $i\colon P \to \mathsf{Hom}_A(f,g)$.

Since $\pi_0 = p_0 i$, Corollary 8.4.8 implies that the canonical cell

$$\mathsf{Hom}_{\mathsf{Hom}_A(f,g)}(\mathsf{Hom}_A(f,g), i) \times \mathsf{Hom}_B(B, p_0) \Rightarrow \mathsf{Hom}_B(B, \pi_0)$$

is a composite. Since $\ell \dashv i$, $\mathsf{Hom}_P(\ell, P) \simeq \mathsf{Hom}_{\mathsf{Hom}_A(f,g)}(\mathsf{Hom}_A(f,g), i)$ by Proposition 4.1.1. And since $p_1 = \pi_1 \ell$, Corollary 8.4.8 again implies that the canonical cell

$$\mathsf{Hom}_C(\pi_1, C) \mathbin{\dot{\times}} \mathsf{Hom}_P(\ell, P) \Rightarrow \mathsf{Hom}_C(p_1, C)$$

is a composite. Composing these bijections, we see that cells with domain $\mathsf{Hom}_C(\pi_1, C) \mathbin{\dot{\times}} \mathsf{Hom}_B(B, \pi_0)$ correspond bijectively to cells with domain $\mathsf{Hom}_C(p_1, C) \mathbin{\dot{\times}} \mathsf{Hom}_B(B, p_0)$.

The equation $\mathrm{id} = \phi i\colon f\pi_0 = fp_0 i \Rightarrow gp_1 i = g\pi_1$ asserts that the identity is the transpose along $\ell \dashv i$ of the cell $\phi\colon fp_0 \Rightarrow gp_1 = g\pi_1\ell$, which tells us that the cells

$$\hat{\mathrm{id}}\colon \mathsf{Hom}_C(\pi_1, C) \mathbin{\dot{\times}} \mathsf{Hom}_B(B, \pi_0) \Rightarrow \mathsf{Hom}_A(f,g) \qquad \text{and}$$
$$\hat{\phi}\colon \mathsf{Hom}_C(p_1, C) \mathbin{\dot{\times}} \mathsf{Hom}_B(B, p_0) \Rightarrow \mathsf{Hom}_A(f,g)$$

correspond under the bijection just described. Since Lemma 9.2.5 proves that $\hat{\phi}$ is a composite, by Lemma 8.3.2 so is $\hat{\mathrm{id}}$. □

9.2.7. Lemma. *For any pair of functors $f\colon A \to B$ and $g\colon C \to D$ the square*

$$\begin{tikzcd} A \times C \arrow[r, "f\times C"] \arrow[d, "A\times g"'] & B \times C \arrow[d, "B\times g"] \\ A \times D \arrow[r, "f\times D"'] & B \times D \end{tikzcd}$$

is exact.

Proof. Exercise 9.2.iii. □

9.2.8. Lemma. *The product of a comma square $\phi\colon fp_0 \Rightarrow gp_1$ with any ∞-category K defines an exact square*

$$\begin{tikzcd} & \mathrm{Hom}_A(f,g) \times K \arrow[dl, two heads, "p_1\times K"'] \arrow[dr, two heads, "p_0\times K"] & \\ C \times K \arrow[dr, "g\times K"'] & \overset{\phi\times K}{\Leftarrow} & B \times K \arrow[dl, "f\times K"] \\ & A \times K & \end{tikzcd}$$

Proof. Exercise 9.2.iv. □

Finally, the plethora of examples of exact squares just established can be composed to yield further comma squares:

9.2.9. Lemma (composites of exact squares). *Given a diagram of squares in the homotopy 2-category*

$$\begin{tikzcd} & H \arrow[dl, "s"'] \arrow[dr, "r"] & & G \arrow[dl, "q"'] \arrow[dr, "p"] & \\ F \arrow[dr, "c"'] & \overset{\gamma}{\Leftarrow} & D \arrow[dl, dashed, "k"'] \arrow[dr, dashed, "h"] & \overset{\beta}{\Leftarrow} & E \arrow[dl, "b"] \\ & C \arrow[dr, "g"'] & \overset{\alpha}{\Leftarrow} & B \arrow[dl, "f"] & \\ & & A & & \end{tikzcd}$$

if $\alpha\colon fh \Rightarrow hk$, $\beta\colon bp \Rightarrow hq$, and $\gamma\colon kr \Rightarrow cs$ are all exact squares then so are the composite rectangles $\alpha q \cdot f\beta$ and $g\gamma \cdot \alpha r$. Consequently, arbitrary "double categorical" composites of exact squares define exact squares.

Proof. The two cases are co-duals, so it suffices to prove that the rectangle $\alpha q \cdot f\beta\colon g(kq) \Rightarrow (fb)p$ is exact. The corresponding cell $\widehat{\alpha q \cdot f\beta}$ displayed below-left factors as below-right:

$$\begin{tikzcd} C \arrow[r, "\mathrm{Hom}_C(k,C)", "|" marking] \arrow[d, equal] & D \arrow[r, "\mathrm{Hom}_D(q,D)", "|" marking] & G \arrow[r, "\mathrm{Hom}_E(E,p)", "|" marking] \arrow[d, phantom, "\Downarrow\widehat{\alpha q\cdot f\beta}"] & E \arrow[d, equal] \\ C \arrow[rrr, "|" marking, "\mathrm{Hom}_A(fb,g)"'] & & & E \end{tikzcd} = \begin{tikzcd} C \arrow[r, "\mathrm{Hom}_C(k,C)", "|" marking] \arrow[d, equal] & D \arrow[r, "\mathrm{Hom}_D(q,D)", "|" marking] \arrow[d, equal] & G \arrow[r, "\mathrm{Hom}_E(E,p)", "|" marking] \arrow[d, phantom, "\Downarrow\hat{\beta}"] & E \arrow[d, equal] \\ C \arrow[r, "|" marking, "\mathrm{Hom}_C(k,C)"'] \arrow[d, equal] & D \arrow[rr, "|" marking, "\mathrm{Hom}_B(b,h)"'] \arrow[d, phantom, "\Downarrow\hat{\alpha}"] & & E \arrow[d, equal] \\ C \arrow[rrr, "|" marking, "\mathrm{Hom}_A(fb,g)"'] & & & E \end{tikzcd}$$

through a cell $\bar{\alpha}$ defined by the universal property of the composite $\mathsf{Hom}_B(B,h) \otimes \mathsf{Hom}_B(b,B) \simeq \mathsf{Hom}_B(b,h)$ of Proposition 8.4.7 by the pasting equality:

$$
\begin{tikzcd}
C \arrow[r, "|" marking, "\mathsf{Hom}_C(k,C)"] \arrow[d, equal] & D \arrow[r, "|" marking, "\mathsf{Hom}_B(B,h)"] \arrow[d, equal] & B \arrow[r, "|" marking, "\mathsf{Hom}_B(b,B)"] \arrow[d, phantom, "\Downarrow\circ"] & E \arrow[d, equal] \\
C \arrow[r, "|" marking, "\mathsf{Hom}_C(k,C)"'] \arrow[d, equal] & D \arrow[rr, "|" marking, "\mathsf{Hom}_B(b,h)"'] \arrow[d, phantom, "\Downarrow\bar{\alpha}"] & & E \arrow[d, equal] \\
C \arrow[rrr, "|" marking, "\mathsf{Hom}_A(fb,g)"'] & & & E
\end{tikzcd}
\quad := \quad
\begin{tikzcd}
C \arrow[r, "|" marking, "\mathsf{Hom}_C(k,C)"] \arrow[d, equal] & D \arrow[r, "|" marking, "\mathsf{Hom}_B(B,h)"] \arrow[d, phantom, "\Downarrow\hat{\alpha}"] & B \arrow[r, "|" marking, "\mathsf{Hom}_B(b,B)"] \arrow[d, equal] & E \arrow[d, equal] \\
C \arrow[rr, "|" marking, "\mathsf{Hom}_A(f,g)"'] \arrow[d, equal] & & B \arrow[r, "|" marking, "\mathsf{Hom}_B(b,B)"] \arrow[d, phantom, "\Downarrow\circ"] & E \arrow[d, equal] \\
C \arrow[rrr, "|" marking, "\mathsf{Hom}_A(fb,g)"'] & & & E
\end{tikzcd}
$$

By exactness of α, $\hat{\alpha}$ and both cells named $\circ$ are composites, so by Lemma 8.3.2 $\bar{\alpha}$ is a composite as well. By exactness of β and Lemma 8.3.2 again it now follows that $\widehat{\alpha q \cdot f\beta}$ is also a composite, proving exactness of the rectangle $\alpha q \cdot f\beta \colon g(kq) \Rightarrow (fb)p$. □

Exercises.

9.2.i. Exercise. Characterize the exact squares in the ∞-cosmos $\mathcal{C}at$ of strict 1-categories.

9.2.ii. Exercise. Prove that the identity cells

$$
\begin{tikzcd}
& A \arrow[dl, equal] \arrow[dr, "k"] & \\
A \arrow[dr, "k"'] & & B \arrow[dl, equal] \\
& B &
\end{tikzcd}
\qquad
\begin{tikzcd}
& A \arrow[dl, "k"'] \arrow[dr, equal] & \\
B \arrow[dr, equal] & & A \arrow[dl, "k"] \\
& B &
\end{tikzcd}
$$

define exact squares.

9.2.iii. Exercise. Prove Lemma 9.2.7.

9.2.iv. Exercise. Prove Lemma 9.2.8.

9.3. Pointwise right and left extensions

In this section we give four definitions of pointwise extension and prove they are equivalent. Our proof will reveal that the general 2-categorical notion of Definition 9.1.1 is on its own too weak.

9.3.1. Definition (stability of extensions under pasting). A right extension diagram $\nu \colon rk \Rightarrow f$ in a 2-category is said to be **stable under pasting with a square α**

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "k"'] \arrow[dr, phantom, "\Uparrow\nu"] & C \\
B \arrow[ur, "r"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& D \arrow[dl, "g"'] \arrow[dr, "h"] & \\
A \arrow[dr, "k"'] \arrow[rr, phantom, "\overset{\alpha}{\Leftarrow}"] & & E \arrow[dl, "b"] \\
& B &
\end{tikzcd}
$$

if the pasted composite

$$
\begin{tikzcd}
D \arrow[r, "g"] \arrow[d, "h"'] \arrow[dr, phantom, "\nearrow\alpha"] & A \arrow[r, "f"] \arrow[d, "k"'] \arrow[dr, phantom, "\Uparrow\nu"] & C \\
E \arrow[r, "b"'] & B \arrow[ur, "r"'] &
\end{tikzcd}
\tag{9.3.2}
$$

defines a right extension $rb\colon E \to C$ of $fg\colon D \to C$ along $h\colon D \to E$. The co-dual of this statement defines what it means for a left extension diagram to be stable under pasting with a square.

9.3.3. Theorem (pointwise right extensions). *For a diagram*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "k"'] \arrow[dr, phantom, "\Uparrow\nu"] & C \\ B \arrow[ur, "r"'] & \end{tikzcd}$$

in the homotopy 2-category of an ∞-cosmos $\mathcal{K}$ the following are equivalent:

(i) $\nu\colon rk \Rightarrow f$ *defines a right extension in* $\mathfrak{h}\mathcal{K}$ *that is stable under pasting with exact squares.*
(ii) $\nu\colon rk \Rightarrow f$ *defines a right extension in* $\mathfrak{h}\mathcal{K}$ *that is stable under pasting with comma squares.*
(iii) $\nu\colon \mathsf{Hom}_B(B,k) \times \mathsf{Hom}_C(C,r) \Rightarrow \mathsf{Hom}_C(C,f)$ *defines a right extension in* $\mathcal{M}od(\mathfrak{h}\mathcal{K})$.
(iv) For any exact square $\alpha\colon bh \Rightarrow kg$, $\nu g \cdot r\alpha\colon \mathsf{Hom}_E(E,h) \times \mathsf{Hom}_C(C,rb) \Rightarrow \mathsf{Hom}_C(C,fg)$ *defines a right extension in* $\mathcal{M}od(\mathfrak{h}\mathcal{K})$.

When these conditions hold, we say $r\colon B \to C$ *defines a* ***pointwise right extension*** *of* $f\colon A \to C$ *along* $k\colon A \to B$.

Proof. Lemma 9.2.5 proves (i)⇒(ii).

To show (ii)⇒(iii), suppose $\nu\colon rk \Rightarrow f$ defines a right extension in $\mathfrak{h}\mathcal{K}$ that is stable under pasting with comma squares and consider a cell in $\mathcal{M}od(\mathfrak{h}\mathcal{K})$:

$$\begin{tikzcd} A \arrow[r, "\mathsf{Hom}_B(B,k)", "\shortmid"{marking}] \arrow[d, equal] & B \arrow[r, "E_1", "\shortmid"{marking}] & \cdots \arrow[r, "E_n", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow"] & C \arrow[d, equal] \\ A \arrow[rrr, "\mathsf{Hom}_C(C,f)"', "\shortmid"{marking}] & & & C \end{tikzcd}$$

Let $B \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} C$ denote the composite two-sided fibration $E_1 \times \cdots \times E_n$. By Remark 8.3.8 applied to $B \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} C$, module maps $\mathsf{Hom}_B(B,k) \times E_1 \times \cdots \times E_n \Rightarrow \mathsf{Hom}_C(C,f)$ stand in bijection with module maps $\mathsf{Hom}_B(B,k) \times \mathsf{Hom}_B(q,B) \times \mathsf{Hom}_C(C,p) \Rightarrow \mathsf{Hom}_C(C,f)$. By Corollary 8.4.6 and Proposition 8.4.7, such module maps stand in bijection with module maps $\mathsf{Hom}_B(q,k) \Rightarrow \mathsf{Hom}_C(p,f)$. By Lemma 8.1.16, these module maps correspond bijectively to 2-cells

$$\begin{tikzcd} & \mathsf{Hom}_B(q,k) \arrow[dl, "p_1"', two heads] \arrow[dr, "p_0", two heads] \arrow[dd, phantom, "\overset{\alpha}{\Leftarrow}"] & \\ A \arrow[dr, "f"'] & & E \arrow[dl, "p"] \\ & C & \end{tikzcd}$$

in the homotopy 2-category. By the hypothesis (ii),

$$\begin{tikzcd} \mathsf{Hom}_B(q,k) \arrow[r, "p_1", two heads] \arrow[d, "p_0"', two heads] \arrow[dr, phantom, "\Swarrow\phi"] & A \arrow[r, "f"] \arrow[d, "k"'] \arrow[dr, phantom, "\Uparrow\nu"] & C \\ E \arrow[r, "q"', two heads] & B \arrow[ur, "r"'] & \end{tikzcd}$$

defines a right extension in $\mathfrak{h}\mathcal{K}$, so α factors uniquely through this pasted composite via a map $\gamma\colon p \Rightarrow rq$. By Proposition 8.4.10, this defines a cell $\gamma\colon \mathsf{Hom}_C(C,p) \Rightarrow \mathsf{Hom}_C(C,rq)$, which by Corollary 8.4.6 gives rise to a canonical cell $\hat{\gamma}\colon \mathsf{Hom}_B(q,B) \times \mathsf{Hom}_C(C,p) \Rightarrow \mathsf{Hom}_C(C,r)$. By Remark 8.3.8 again, this produces the desired unique factorization

$$\begin{tikzcd} A \arrow[r, "\mathsf{Hom}_B(B,k)", "\shortmid"{marking}] \arrow[d, equal] & B \arrow[r, "E_1", "\shortmid"{marking}] & \cdots \arrow[r, "E_n", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow"] & C \arrow[d, equal] \\ A \arrow[rrr, "\mathsf{Hom}_C(C,f)"', "\shortmid"{marking}] & & & C \end{tikzcd} = \begin{tikzcd} A \arrow[r, "\mathsf{Hom}_B(B,k)", "\shortmid"{marking}] \arrow[d, equal] & B \arrow[r, "E_1", "\shortmid"{marking}] \arrow[d, equal] & \cdots \arrow[r, "E_n", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\exists!"] & C \arrow[d, equal] \\ A \arrow[r, "\mathsf{Hom}_B(B,k)", "\shortmid"{marking}] \arrow[d, equal] & B \arrow[rr, "\mathsf{Hom}_C(C,r)", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\nu"] & & C \arrow[d, equal] \\ A \arrow[rrr, "\mathsf{Hom}_C(C,f)"', "\shortmid"{marking}] & & & C \end{tikzcd}$$

To show (iii)⇒(iv), consider a diagram (9.3.2) in which $\alpha\colon bh \Rightarrow kg$ is exact and $\nu\colon \mathsf{Hom}_B(B,k) \times \mathsf{Hom}_C(C,r) \Rightarrow \mathsf{Hom}_C(C,f)$ defines a right extension diagram in $\mathcal{M}od(\mathfrak{h}\mathcal{K})$. Now by Corollary 8.4.6, a cell

$$\mathsf{Hom}_E(E,h) \times E_1 \times \cdots \times E_n \Rightarrow \mathsf{Hom}_C(C,fg)$$

corresponds to a cell

$$\mathsf{Hom}_A(g,A) \times \mathsf{Hom}_E(E,h) \times E_1 \times \cdots \times E_n \Rightarrow \mathsf{Hom}_C(C,f).$$

By exactness of α, this corresponds to a cell

$$\mathsf{Hom}_B(b,k) \times E_1 \times \cdots \times E_n \Rightarrow \mathsf{Hom}_C(C,f),$$

or equivalently, upon restricting along the composite map $\mathsf{Hom}_B(B,k) \times \mathsf{Hom}_B(b,B) \Rightarrow \mathsf{Hom}_B(b,k)$ to a cell

$$\mathsf{Hom}_B(B,k) \times \mathsf{Hom}_B(b,B) \times E_1 \times \cdots \times E_n \Rightarrow \mathsf{Hom}_C(C,f).$$

Since $B \overset{\mathsf{Hom}_C(C,r)}{\nrightarrow} C$ is the right extension of $A \overset{\mathsf{Hom}_C(C,f)}{\nrightarrow} C$ along $A \overset{\mathsf{Hom}_B(B,k)}{\nrightarrow} B$, this corresponds to a cell

$$\mathsf{Hom}_B(b,B) \times E_1 \times \cdots \times E_n \Rightarrow \mathsf{Hom}_C(C,r),$$

which transposes via Corollary 8.4.6 to a cell

$$E_1 \times \cdots \times E_n \Rightarrow \mathsf{Hom}_C(C,rb),$$

which gives the factorization required to prove that $E \overset{\mathsf{Hom}_C(C,rb)}{\nrightarrow} C$ is the right extension of $D \overset{\mathsf{Hom}_C(C,fg)}{\nrightarrow} C$ along $D \overset{\mathsf{Hom}_E(E,h)}{\nrightarrow} E$. A slightly more delicate argument is required to see that this bijection is implemented by composing with the map of modules $\mathsf{Hom}_E(E,h) \times \mathsf{Hom}_C(C,rb) \Rightarrow \mathsf{Hom}_C(C,fg)$ corresponding to the pasted composite (9.3.2), but for this it suffices by the Yoneda lemma to start with the identity cell $\mathrm{id}_{\mathsf{Hom}_C(C,rb)}$ and trace back up through the bijection just described. By Lemma 8.1.16 this is straightforward.

Finally Lemma 9.1.6 and the trivial example of Exercise 9.2.ii prove that (iv)⇒(i). □

9.3.4. Corollary. *The pasted composite* (9.3.2) *of a pointwise right extension with an exact square is a pointwise right extension.*

Proof. Lemma 9.2.9, the pasted composite of two exact squares remains an exact square, so by Theorem 9.3.3(i), the pasted composite of a pointwise right extension remains stable under pasting with exact squares. □

Exercises.

9.3.i. Exercise. Using Lemma 9.1.6 as a hint, state and prove a dual version of Theorem 9.3.3 defining pointwise left extensions in the homotopy 2-category of an ∞-cosmos.

9.4. Formal category theory in a virtual equipment

9.4.1. Proposition. *For any pair of functors $u\colon A \to B$ the following are equivalent:*

(i) The natural transformation $\epsilon\colon fu \Rightarrow \mathrm{id}_A$ is the counit of an adjunction $f \dashv u$.

(ii) There is a pointwise right extension diagram in $\mathfrak{h}\mathcal{K}$

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "u"'] \arrow[r, phantom, shift right=3ex, "\Uparrow\epsilon"] & A \\ B \arrow[ur, "f"'] \end{tikzcd} \tag{9.4.2}$$

*that is **absolute**, preserved by any functor $h\colon A \to C$.*

(iii) The square

$$\begin{tikzcd} & A \arrow[dl, equal] \arrow[dr, "u"] \\ A \arrow[dr, equal] \arrow[rr, phantom, "\overset{\epsilon}{\Leftarrow}"] & & B \arrow[dl, "f"] \\ & A \end{tikzcd}$$

is exact.

Proof. Without the adjective "pointwise" the equivalence (i)⇔(ii) is the op-dual of Lemma 2.3.6, whose proof works in any 2-category and in particular in $\mathfrak{h}\mathcal{K}^{\mathrm{op}}$. It remains only to demonstrate that the right extension produced by (i)⇒(ii) is pointwise. To see this, recall that for any $f\colon B \to A$, Lemma 9.1.3 provides a right lifting in $\mathcal{M}od(\mathfrak{h}\mathcal{K})$ as below left

$$\begin{tikzcd} A \arrow[r, "\mathrm{Hom}_A(f{,}A)", "\shortmid"{marking}] \arrow[d, equal] & B \arrow[r, "\mathrm{Hom}_A(A{,}f)", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\epsilon"] & A \arrow[d, equal] & & A \arrow[r, "\mathrm{Hom}_B(B{,}u)", "\shortmid"{marking}] \arrow[d, equal] & B \arrow[r, "\mathrm{Hom}_A(A{,}f)", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\epsilon"] & A \arrow[d, equal] \\ A \arrow[rr, "\shortmid"{marking}, "\mathrm{Hom}_A"'] & & A \arrow[r, phantom, shift left=4ex, "\cong"] & & A \arrow[rr, "\shortmid"{marking}, "\mathrm{Hom}_A"'] & & A \end{tikzcd}$$

If $f \dashv u$, then by Proposition 4.1.1, the counit induces an equivalence $\mathrm{Hom}_B(B,u) \simeq \mathrm{Hom}_A(f,A)$, so we have an isomorphic right lifting diagram above right. By Proposition 8.4.10 and Theorem 9.3.3 this supplies a natural transformation $\epsilon\colon fu \Rightarrow \mathrm{id}_A$ that defines a pointwise right extension (9.4.2).

Proposition 4.1.1, which demonstrates that $\epsilon\colon fu \Rightarrow \mathrm{id}_A$ is a counit if and only if it induces an equivalence $\mathrm{Hom}_B(B,u) \simeq \mathrm{Hom}_A(f,A)$ also proves (i)⇔(iii), via Lemma 8.3.5 which tells us that $\mathrm{Hom}_A \otimes \mathrm{Hom}_B(B,u) \simeq \mathrm{Hom}_B(B,u)$. □

9.4.3. Lemma. *A functor $k\colon A \to B$ is **fully faithful** when any of the following equivalent conditions hold:*

(i) The square

$$\begin{tikzcd} & A \arrow[dl, equal] \arrow[dr, equal] \\ A \arrow[dr, "k"'] & & A \arrow[dl, "k"] \\ & B \end{tikzcd}$$

is exact.

(ii) The unary cell $\widehat{\mathrm{id}}_k \colon \mathsf{Hom}_A \Rightarrow \mathsf{Hom}_B(k,k)$ *is a composite.*

(iii) The module map $\widehat{\mathrm{id}}_k$ *defines an equivalence of modules* $\mathsf{Hom}_A \simeq \mathsf{Hom}_B(k,k)$ *from* A *to* A.

Proof. The equivalence follows from Definition 9.2.2 and Lemma 8.3.3. $\square$

9.4.4. Proposition (extensions along fully faithful functors). *If*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "k"'] \arrow[dr, phantom, "\Uparrow\nu" near start] & C \\ B \arrow[ur, "r"'] & \end{tikzcd}$$

is a pointwise right extension and k *is fully faithful, then* ν *is an isomorphism.*

Proof. Pasting the pointwise right extension with the exact square of Lemma 9.4.3 yields a pointwise right extension diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\Uparrow\nu" near start] & C \\ A \arrow[ur, "rk"'] & \end{tikzcd}$$

By Lemma 8.3.5 and Theorem 9.3.3(iii), $f \colon A \to C$ also defines a pointwise right extension of itself along the identity. The universal property of these extension diagrams in $\mathfrak{h}\mathcal{K}$ described in Definition 9.1.1 now suffices to construct an inverse isomorphism to ν. $\square$

The dual result, that the 2-cell in a pointwise left extension along a fully faithful functor is invertible, is left to Exercise 9.4.i.

9.4.5. Proposition. *A right adjoint* $u \colon A \to B$ *is fully faithful if and only if the counit* $\epsilon \colon fu \Rightarrow \mathrm{id}_A$ *is an isomorphism.*

Proof. If $f \dashv u$ with counit $\epsilon \colon fu \Rightarrow \mathrm{id}_A$, then Proposition 4.1.1 reveals that pasting with ϵ induces an equivalence of modules $\mathsf{Hom}_B(B,u) \simeq \mathsf{Hom}_A(f,A)$. Equivalently, by Proposition 9.4.1, the counit $\epsilon \colon fu \Rightarrow \mathrm{id}_A$ is exact. If u is fully faithful, then by Lemma 9.2.9 the composite rectangle is also exact.

$$\begin{tikzcd} & & A \arrow[dl, equal] \arrow[dr, equal] & \\ & A \arrow[dl, equal] \arrow[dr, "u"] & & A \arrow[dl, "u"] \\ A \arrow[dr, equal] \arrow[rr, phantom, "\overset{\epsilon}{\Leftarrow}"] & & B \arrow[dl, "f"] & \\ & A & & \end{tikzcd}$$

which by Lemma 8.3.3 is to say that ϵ induces an equivalence of modules $\mathsf{Hom}_A(A,A) \simeq \mathsf{Hom}_A(fu,A)$ under the contravariant embedding of Definition 8.4.11. By Proposition 8.4.10, which says this embedding is fully faithful, it follows that $\epsilon \colon fu \Rightarrow \mathrm{id}_A$ is an isomorphism.

Conversely, assume $f \dashv u$ with invertible counit $\epsilon \colon fu \cong \mathrm{id}_A$. By Proposition 8.4.7 we have a composite $\mathsf{Hom}_B(B,u) \otimes \mathsf{Hom}_B(u,B) \simeq \mathsf{Hom}_B(u,u)$. Substituting the equivalence $\mathsf{Hom}_A(f,A) \simeq \mathsf{Hom}_B(B,u)$ gives another composite $\mathsf{Hom}_A(f,A) \otimes \mathsf{Hom}_B(u,B) \simeq \mathsf{Hom}_B(u,u)$, but now Corollary

8.4.8 gives a third composite $\mathsf{Hom}_A(f, A) \otimes \mathsf{Hom}_B(u, B) \simeq \mathsf{Hom}_A(fu, A)$. Factoring one composite through the other we obtain an equivalence $\mathsf{Hom}_A(fu, A) \simeq \mathsf{Hom}_B(u, u)$ which composes with the equivalence $\mathsf{Hom}_A \simeq \mathsf{Hom}_A(fu, A)$ induced by the invertible counit cell to define a composite $\mathsf{Hom}_A \simeq \mathsf{Hom}_A(fu, A) \simeq \mathsf{Hom}_B(u, u)$, proving, by Lemma 9.4.3, that $u\colon A \to B$ is fully faithful. $\square$

Exercises.

9.4.i. Exercise. Prove that if

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow\lambda", near start] & C \\ B \arrow[ur, "\ell"'] & \end{tikzcd}$$

is a pointwise left extension and k is fully faithful, then λ is an isomorphism.

9.5. Limits and colimits in cartesian closed ∞-cosmoi

In this section we work in an ∞-cosmos $\mathcal{K}$ that is *cartesian closed*, satisfying the extra assumption of Definition 1.2.23. In a cartesian closed ∞-cosmos a functor $f\colon A \to C$ transposes to define an element $f\colon 1 \to C^A$ of the ∞-category of functors from A to C and a family of diagrams $f\colon A \times D \to C$ of shape A parametrized by an ∞-category D transposes to a generalized element $f\colon D \to A^C$. The aim is to reinterpret the limits and colimits of §2.3 as pointwise right and left extensions.

9.5.1. Proposition. *Suppose the diagram below-left is a pointwise right extension diagram in a cartesian closed ∞-cosmos.*

$$\begin{tikzcd} A \times D \arrow[r, "f"] \arrow[d, "k \times D"'] \arrow[dr, phantom, "\Uparrow\nu", near start] & C \\ B \times D \arrow[ur, "r"'] & \end{tikzcd} \qquad \leftrightsquigarrow \qquad \begin{tikzcd} & C^B \arrow[d, "C^k"] \\ D \arrow[ur, "r"] \arrow[r, "f"'] \arrow[ur, phantom, "\Downarrow\nu", near end, shift right=3] & C^A \end{tikzcd}$$

Then its transpose above-right defines an absolute right lifting diagram in the homotopy 2-category which moreover is stable under pasting with the square C^ϕ induced from any exact square ϕ.

Proof. The universal property that characterizes the absolute right lifting diagram

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[d, "d"'] \arrow[dr, phantom, "\Downarrow\chi"] & C^B \arrow[d, "C^k"] \\ D \arrow[r, "f"'] & C^A \end{tikzcd} \quad = \quad \begin{tikzcd} X \arrow[r, "e"] \arrow[d, "d"'] \arrow[r, phantom, "\Downarrow\exists!\zeta", shift right=4] & C^B \arrow[d, "C^k"] \\ D \arrow[ur, "r"] \arrow[r, "f"'] \arrow[r, phantom, "\Downarrow\nu", shift left=4] & C^A \end{tikzcd}$$

transposes to

$$\begin{tikzcd} A \times X \arrow[r, "A \times d"] \arrow[d, "k \times X"'] & A \times D \arrow[r, "f"] \arrow[d, phantom, "\Uparrow\chi"] & C \\ B \times X \arrow[urr, bend right, "e"'] & & \end{tikzcd} \quad = \quad \begin{tikzcd} A \times X \arrow[r, "A \times d"] \arrow[d, "k \times X"'] & A \times D \arrow[r, "f"] \arrow[d, "k \times D"'] \arrow[r, phantom, "\Uparrow\nu", shift right=4] & C \\ B \times X \arrow[r, "B \times d"'] \arrow[urr, bend right=60, "e"'] & B \times D \arrow[ur, "r"'] \arrow[r, phantom, "\Uparrow\exists!\zeta"] & \end{tikzcd}$$

Similarly, the pasted composite of the absolute right lifting diagram with an exponentiated exact square transposes below-left to the diagram below-right

$$\begin{tikzcd} & C^B \arrow[r, "C^h"] \arrow[d, "C^k"] \arrow[dr, phantom, "\Downarrow C^\phi"] & C^E \arrow[d, "C^q"] \\ D \arrow[ur, "r"] \arrow[r, "f"'] \arrow[u, phantom, "\Downarrow\nu"] & C^A \arrow[r, "C^p"'] & C^F \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} F \times D \arrow[r, "p\times D"] \arrow[d, "q\times D"'] \arrow[dr, phantom, "\Downarrow\phi\times D"] & A \times D \arrow[r, "f"] \arrow[d, "k\times D"'] & C \arrow[dl, phantom, "\Uparrow\nu"] \\ E \times D \arrow[r, "h\times D"'] & B \times D \arrow[ur, "r"'] & \end{tikzcd}$$

Lemma 9.2.7 and Lemma 9.2.8 prove that the square defining the product $k \times d$ and the square $\phi \times D$ are exact.

Now if ν is a pointwise right extension, then by Lemma 9.2.7 and Corollary 9.3.4 so is $\nu \cdot (A \times d)$. The transposed universal property of this right extension diagram proves that ν defines an absolute right lifting. Since by Corollary 9.3.4, the pasted composite of ν with the exact square $\phi \times D$ gives another pointwise right extension, the same argument shows that the pasted composite of ν with E^ϕ is again an absolute right lifting diagram. $\square$

9.5.2. Proposition. *In a cartesian closed ∞-cosmos any limit as encoded by the absolute right lifting diagram below-right transposes to define a pointwise right extension diagram as below-left:*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "!"'] & C \arrow[dl, phantom, "\Uparrow\nu"] \\ 1 \arrow[ur, "\lim f"'] & \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} & C \arrow[d, "\Delta"] \\ 1 \arrow[ur, "\lim f"] \arrow[r, "f"'] \arrow[ur, phantom, "\Downarrow\nu"] & C^A \end{tikzcd}$$

Conversely, any pointwise right extension diagram of this form transposes to define a limit in C.

Proof. Proposition 9.5.1 proves that pointwise right extension diagrams transpose to define absolute right lifting diagrams. Conversely, Lemma 9.2.1 reveals that if $\nu\colon \Delta \lim f \Rightarrow f$ is absolute right lifting, then $\nu\colon \ell ! \Rightarrow f$ is a right extension diagram stable under pasting with pullback squares. Over the terminal ∞-category 1, all comma squares are pullback squares, so by Theorem 9.3.3 this stability property proves that $\nu\colon \ell ! \Rightarrow f$ is a pointwise right extension. $\square$

9.5.3. Corollary. *For any pointwise right extension in a cartesian closed ∞-cosmos*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "k"'] & C \arrow[dl, phantom, "\Uparrow\nu"] \\ B \arrow[ur, "r"'] & \end{tikzcd}$$

and any element $b\colon 1 \to B$, the element $rb\colon 1 \to C$ is the limit of the diagram

$$\mathsf{Hom}_B(b,k) \xrightarrow{p_1} A \xrightarrow{f} C$$

Proof. By Theorem 9.3.3(ii), the composite

$$\begin{tikzcd} \mathsf{Hom}_B(b,k) \arrow[r, two heads, "p_1"] \arrow[d, "!"'] \arrow[dr, phantom, "\Nearrow\phi"] & A \arrow[r, "f"] \arrow[d, "k"'] & C \arrow[dl, phantom, "\Uparrow\nu"] \\ 1 \arrow[r, "b"'] & B \arrow[ur, "r"'] & \end{tikzcd}$$

is a pointwise right extension. By Proposition 9.5.2 this can be interpreted as saying that $rb\colon 1 \to B$ defines the limit of the restriction of the diagram $f\colon A \to C$ along $p_1\colon \mathsf{Hom}_B(b,k) \twoheadrightarrow A$. $\square$

In Definition 2.2.1, initial and terminal elements in A were defined respectively as left or right adjoints to the unique functor $!\colon A \to 1$

$$\begin{tikzcd} A \arrow[r, "!"] & 1 \arrow[l, bend right, "i"'] \arrow[l, bend left, "t"] \end{tikzcd}$$

or equivalently, by Definition 2.3.2, as colimits or limits for the empty diagram in $A^{\varnothing} \cong 1$. In a cartesian closed ∞-cosmos, we now show that an initial element may also be characterized as a *limit* and a terminal element may be characterized as a *colimit* of the identity diagram $\mathrm{id}_A\colon A \to A$.

9.5.4. Corollary. *For an ∞-category A in a cartesian closed ∞-cosmos:*

- *(i) An element $t\colon 1 \to A$ is terminal if and only if it defines a colimit for the identity functor $\mathrm{id}_A\colon A \to A$ in which case the unit for the adjunction $! \dashv t$ defines the colimit cone.*
- *(ii) An element $i\colon 1 \to A$ is initial if and only if it defines a limit for the identity functor $\mathrm{id}_A\colon A \to A$ in which case the counit for $i \dashv !$ defines the limit cone.*

Proof. We prove (ii). By Proposition 4.1.1 and Lemma 8.1.17, an element $i\colon 1 \to A$ defines a left adjoint to $!\colon A \to 1$ if and only if the modules $\mathsf{Hom}_A(i,A)$ and $\mathsf{Hom}_1(1,!)$ are isomorphic in $\mathcal{M}od(\mathfrak{h}\mathcal{K})$. By Lemma 9.1.3 and Exercise 9.1.i, this is the case if and only if there is a right extension diagram

$$\begin{tikzcd} A \arrow[r, "\mathsf{Hom}_1(1{,}!)", "|" marking] \arrow[d, equal] & A \arrow[r, "\mathsf{Hom}_A(A{,}i)", "|" marking] \arrow[d, phantom, "\Downarrow\epsilon"] & A \arrow[d, equal] \\ A \arrow[rr, "\mathsf{Hom}_A"', "|" marking] & & A \end{tikzcd}$$

in which case the binary cell corresponds to the counit transformation $\epsilon\colon i! \Rightarrow \mathrm{id}_A$. By Theorem 9.3.3, this is the case if and only if the counit defines a pointwise right extension as below-left, which by Proposition 9.5.2 is equivalent to a limit of the identity diagram as below-right:

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "!"'] \arrow[dr, phantom, "\Uparrow\epsilon" near start] & A \\ 1 \arrow[ur, "i"'] & \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} & A \arrow[d, "\Delta"] \\ 1 \arrow[ur, "i"] \arrow[r, "\mathrm{id}_A"'] \arrow[ur, phantom, "\Downarrow\epsilon" near end] & A^A \end{tikzcd} \qquad \square$$

Recall from Definition 2.4.6 that a functor $k\colon I \to J$ is **final** if and only if for any ∞-category A, the square

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "\Delta"'] & A \arrow[d, "\Delta"] \\ A^J \arrow[r, "A^k"'] & A^I \end{tikzcd}$$

preserves and reflects all absolute left lifting diagrams. Dually a functor $k\colon I \to J$ is **initial** if this square preserves and reflects all absolute right lifting diagrams. We can now give a more concise formulation of these notions.

9.5.5. Definition. A functor $k\colon I \to J$ is **final** if and only if the square below-left is exact and **initial** if and only if the square below-right is exact.

$$\begin{tikzcd}[column sep=small, row sep=small] & I \arrow[dl, "!"'] \arrow[dr, "k"] & \\ 1 \arrow[dr, equal] & & J \arrow[dl, "!"] \\ & 1 & \end{tikzcd} \qquad \begin{tikzcd}[column sep=small, row sep=small] & I \arrow[dl, "k"'] \arrow[dr, "!"] & \\ J \arrow[dr, "!"'] & & 1 \arrow[dl, equal] \\ & 1 & \end{tikzcd}$$

Note that the functor $!\colon J \to 1$ is represented on the right and on the left by the modules $J \overset{J}{\nrightarrow} 1$ and $1 \overset{J}{\nrightarrow} J$. So we see that $k\colon I \to J$ is **final** if and only if $p_0\colon \mathsf{Hom}_J(J,k) \overset{\sim}{\twoheadrightarrow} J$ is a trivial fibration, and $k\colon I \to J$ is **initial** if and only if $p_1\colon \mathsf{Hom}_J(k,J) \overset{\sim}{\twoheadrightarrow} J$ is a trivial fibration.

To reconcile Definition 9.5.5 with Definition 2.4.6 we must prove:

9.5.6. Proposition. *In a cartesian closed ∞-cosmos, if $k\colon I \to J$ is initial and $f\colon J \to C$ is any diagram, then a limit of f also defines a limit of $fk\colon I \to C$ and conversely if the limit of $fk\colon I \to C$ exists, then it also defines a limit of $f\colon J \to C$.*

Proof. By Proposition 9.5.2, a limit of f defines a pointwise right extension as below left, which by Corollary 9.3.4 and Definition 9.5.5 gives rise to another pointwise right extension below-right.

$$\begin{tikzcd} J \arrow[r, "f"] \arrow[d, "!"'] \arrow[r, phantom, shift right=3ex, "\Uparrow\nu"] & C \\ 1 \arrow[ur, "\ell"'] & \end{tikzcd} \qquad \begin{tikzcd} I \arrow[r, "k"] \arrow[d, "!"'] & J \arrow[r, "f"] \arrow[d, "!"'] \arrow[r, phantom, shift right=3ex, "\Uparrow\nu"] & C \\ 1 \arrow[r, equal] & 1 \arrow[ur, "\ell"'] & \end{tikzcd}$$

By Proposition 9.5.2 again, this tells us that ℓ is the limit of fk.

For the converse, suppose we are given a pointwise right extension diagram

$$\begin{tikzcd} I \arrow[r, "fk"] \arrow[d, "!"'] \arrow[r, phantom, shift right=3ex, "\Uparrow\mu"] & C \\ 1 \arrow[ur, "\ell"'] & \end{tikzcd}$$

By Theorem 9.3.3(iii), this universal property tells us that for any composable sequence of modules $E_1 \times \cdots \times E_n$ from 1 to C, composing with $\mu\colon \mathsf{Hom}_!(1,!) \times \mathsf{Hom}_C(C,\ell) \Rightarrow \mathsf{Hom}_C(C,fk)$ defines a bijection between cells $E_1 \times \cdots \times E_n \Rightarrow \mathsf{Hom}_C(C,\ell)$ and the cells below-left:

$$\begin{tikzcd} I \arrow[r, "\mathsf{Hom}_1(1,!)", "\mid"{marking}] \arrow[d, equal] & 1 \arrow[r, "E_1", "\mid"{marking}] & \cdots \arrow[r, "E_n", "\mid"{marking}] & C \arrow[d, equal] \\ I \arrow[rrr, "\mathsf{Hom}_C(C,fk)"', "\mid"{marking}] \arrow[rrr, phantom, shift left=5ex, "\Downarrow"] & & & C \end{tikzcd} \qquad \begin{tikzcd} J \arrow[r, "\mathsf{Hom}_J(k,J)", "\mid"{marking}] \arrow[d, equal] & I \arrow[r, "\mathsf{Hom}_1(1,!)", "\mid"{marking}] & 1 \arrow[r, "E_1", "\mid"{marking}] & \cdots \arrow[r, "E_n", "\mid"{marking}] & C \arrow[d, equal] \\ {} \arrow[rrrr, "\mathsf{Hom}_C(C,f)"', "\mid"{marking}] \arrow[rrrr, phantom, shift left=5ex, "\Downarrow"] & & & & C \end{tikzcd}$$

By Corollary 8.4.6, composing with $\mu\colon \mathsf{Hom}_!(1,!) \times \mathsf{Hom}_C(C,\ell) \Rightarrow \mathsf{Hom}_C(C,fk)$ also defines a bijection between cells $E_1 \times \cdots \times E_n \Rightarrow \mathsf{Hom}_C(C,\ell)$ and the cells above-right.

To say that $k\colon I \to J$ is initial means, by Definition 9.5.5, that the map $\mathsf{Hom}_J(k,J) \times \mathsf{Hom}_1(1,!) \Rightarrow \mathsf{Hom}_1(1,!)$ of modules from J to 1 induced by the identity is a composite. Thus composing with

$\mu\colon \mathsf{Hom}_!(1,!) \times \mathsf{Hom}_C(C,\ell) \Rightarrow \mathsf{Hom}_C(C,fk)$ also defines a bijection between cells $E_1 \times \cdots \times E_n \Rightarrow \mathsf{Hom}_C(C,\ell)$ and the cells

$$\begin{tikzcd} J \arrow[r, "\mathsf{Hom}_1(1,!)", "|" marking] \arrow[d, equal] & 1 \arrow[r, "E_1", "|" marking] & \cdots \arrow[r, "E_n", "|" marking] \arrow[d, phantom, "\Downarrow"] & C \arrow[d, equal] \\ J \arrow[rrr, "\mathsf{Hom}_C(C,f)"', "|" marking] & & & C \end{tikzcd}$$

But this says exactly that the cell $\nu\colon \mathsf{Hom}_1(1,!) \times \mathsf{Hom}_C(C,\ell) \Rightarrow \mathsf{Hom}_C(C,f)$ that corresponds to μ under this series of bijections displays $1 \overset{\mathsf{Hom}_C(C,\ell)}{\nrightarrow} C$ as the right extension of $J \overset{\mathsf{Hom}_C(C,f)}{\nrightarrow} C$ along $J \overset{\mathsf{Hom}_1(1,!)}{\nrightarrow} 1$; unpacking this definition, we see that $\nu k = \mu$. Thus $\nu\colon \ell! \Rightarrow f$ is a pointwise right extension and by Proposition 9.5.2 we conclude that $\ell\colon 1 \to C$ also defines a limit of $f\colon J \to C$ as claimed. $\square$

9.5.7. Definition. In a cartesian closed ∞-cosmos $\mathcal{K}$, an ∞-category E **admits functorial pointwise right extensions** along $k\colon A \to B$ if there is a pointwise right extension

$$\begin{tikzcd} A \times E^A \arrow[r, "\mathrm{ev}"] \arrow[d, "k\times E^A"'] & E \\ B \times E^A \arrow[ur, "\mathrm{ran}_k"'] \arrow[u, phantom, shift left=4ex, "\Uparrow\epsilon"] \end{tikzcd}$$

of the evaluation functor along $k \times E^A$.

By Proposition 9.5.1 if E admits functorial pointwise right extensions along $k\colon A \to B$, there is an absolute right lifting diagram

$$\begin{tikzcd} & E^B \arrow[d, "\mathrm{res}"] \\ E^A \arrow[ur, "\mathrm{ran}_k"] \arrow[r, equal] \arrow[ur, phantom, shift right=3ex, "\Downarrow\epsilon"] & E^A \end{tikzcd}$$

that is stable under pasting with E^ϕ for any exact square ϕ.

9.5.8. Proposition (Beck-Chevalley condition). *For any exact square*

$$\begin{tikzcd} & D \arrow[dl, "q"'] \arrow[dr, "p"] \arrow[dd, phantom, "\overset{\phi}{\Leftarrow}"] & \\ C \arrow[dr, "g"'] & & B \arrow[dl, "f"] \\ & A & \end{tikzcd}$$

in a cartesian closed ∞-cosmos and any ∞-category E, whenever E admits functorial pointwise left and right extensions, the Beck-Chevalley condition is satisfied for the induced 2-cell ϕ^* below-left: the mates of ϕ^* below *center and below right are both isomorphisms.*

$$
\begin{tikzcd}[column sep=small]
& E^A \arrow[dl, "g^*"'] \arrow[dr, "f^*"] & \\
E^C \arrow[dr, "q^*"'] \arrow[rr, phantom, "\overset{\phi^*}{\Leftarrow}"] & & E^B \arrow[dl, "p^*"] \\
& E^D &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
& E^A \arrow[dr, "f^*"] & \\
E^C \arrow[ur, "\mathrm{ran}_g"] \arrow[dr, "q^*"'] \arrow[rr, phantom, "\Downarrow\phi^!"] & & E^B \\
& E^D \arrow[ur, "\mathrm{ran}_p"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
& E^A \arrow[dl, "g^*"'] & \\
E^C \arrow[rr, phantom, "\Uparrow\phi_!"] & & E^B \arrow[ul, "\mathrm{lan}_f"'] \arrow[dl, "p^*"] \\
& E^D \arrow[ul, "\mathrm{lan}_q"] &
\end{tikzcd}
$$

Proof. By Proposition 9.5.1, the pointwise right extensions defining ran_g and ran_p transpose to define absolute right lifting diagrams

$$
\begin{tikzcd}
& E^A \arrow[r, "f^*"] \arrow[d, "g^*"] \arrow[dr, phantom, "\Downarrow\phi^*"] & E^B \arrow[d, "p^*"] \\
E^C \arrow[ur, "\mathrm{ran}_g"] \arrow[r, equal] \arrow[ur, phantom, "\Downarrow\epsilon", near end, shift right=2] & E^C \arrow[r, "q^*"'] & E^D
\end{tikzcd}
\qquad
\begin{tikzcd}
& & E^B \arrow[d, "p^*"] \\
E^C \arrow[r, "q^*"'] & E^D \arrow[ur, "\mathrm{ran}_p"] \arrow[r, equal] \arrow[ur, phantom, "\Downarrow\epsilon", near end, shift right=2] & E^D
\end{tikzcd}
$$

Moreover, the mate $\phi^!$ of ϕ^* defines a factorization of the left-hand diagram through the right-hand diagram:

$$
\begin{tikzcd}
& E^A \arrow[r, "f^*"] \arrow[d, "g^*"] \arrow[dr, phantom, "\Downarrow\phi^*"] & E^B \arrow[d, "p^*"] \\
E^C \arrow[ur, "\mathrm{ran}_g"] \arrow[r, equal] \arrow[ur, phantom, "\Downarrow\epsilon", near end, shift right=2] & E^C \arrow[r, "q^*"'] & E^D
\end{tikzcd}
=
\begin{tikzcd}
& E^A \arrow[r, "f^*"] \arrow[d, phantom, "\Downarrow\phi^!"] & E^B \arrow[d, "p^*"] \\
E^C \arrow[ur, "\mathrm{ran}_g"] \arrow[r, "q^*"'] & E^D \arrow[ur, "\mathrm{ran}_p"] \arrow[r, equal] \arrow[ur, phantom, "\Downarrow\epsilon", near end, shift right=2] & E^D
\end{tikzcd}
$$

Immediately from the universal property of the absolute right liftings of q^* through p^* we have that $\phi^!$ is an isomorphism. The proof for $\phi_!$ is similar using the absolute left lifting diagrams arising from the pointwise left extensions defining lan_f and lan_q □

9.6. Weighted limits and colimits of ∞-categories

Proposition 9.5.2 reveals that in cartesian closed ∞-cosmoi, the notion of limit of a diagram of ∞-categories $f\colon A \to C$ first introduced in §2.3 is recaptured by the notion of pointwise right extension of the diagram $f\colon A \to C$ along $!\colon A \to 1$. With this result in hand, we may now define limits and colimits of diagrams of ∞-categories in arbitrary ∞-cosmoi, not necessarily cartesian closed, as pointwise right or left extensions, respectively. We then generalize these definitions further to define a notion of limit or colimit for a diagram weighted by a module $A \overset{E}{\twoheadrightarrow} B$.

9.6.1. Definition. A **limit** of a diagram of ∞-categories $f\colon A \to C$ is given by a pointwise right extension as below-left

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "!"'] \arrow[dr, phantom, "\Uparrow\nu", near start] & C \\
1 \arrow[ur, dashed, "\lim f"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "!"'] \arrow[dr, phantom, "\Downarrow\rho", near start] & C \\
1 \arrow[ur, dashed, "\mathrm{colim}\, f"']
\end{tikzcd}
$$

while a **colimit** of a diagram of ∞-categories $f\colon A \to C$ is given by a pointwise left extension as above-right.

The notions of weighted limit and colimit generalize Definition 9.6.1 — see Exercise 9.6.i — even though they are defined in the virtual equipment of modules, rather than in the ∞-cosmos itself.

9.6.2. Definition. Given a module $A \overset{W}{\nrightarrow} B$ and a functor $f\colon A \to C$, a functor $\lim_W f\colon B \to C$ defines the W-**weighted limit** of f if it covariantly represents the right extension of $\mathsf{Hom}_C(C, f)$ along W:

$$\begin{tikzcd} A \arrow[r, "W", "\shortmid"{marking}] \arrow[d, equal] & B \arrow[r, dashed, "\mathsf{Hom}_C(C{,}\lim_W f)", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow"] & C \arrow[d, equal] \\ A \arrow[rr, "\mathsf{Hom}_C(C{,}f)"', "\shortmid"{marking}] & & C \end{tikzcd}$$

Dually, given a module $B \overset{V}{\nrightarrow} A$ and a functor $f\colon A \to C$, a functor $\operatorname{colim}_V f\colon B \to C$ defines the V-**weighed colimit** of f if it contravariantly represents the right lifting of $\mathsf{Hom}_C(f, C)$ through V:

$$\begin{tikzcd} C \arrow[r, dashed, "\mathsf{Hom}_C(\operatorname{colim}_V f{,}C)", "\shortmid"{marking}] \arrow[d, equal] & B \arrow[r, "V", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow"] & C \arrow[d, equal] \\ C \arrow[rr, "\mathsf{Hom}_C(f{,}C)"', "\shortmid"{marking}] & & A \end{tikzcd}$$

By Proposition 9.1.5 we have immediately:

9.6.3. Proposition. *For any functor $f\colon A \to E$ and weights $A \overset{W}{\nrightarrow} B$ and $B \overset{V}{\nrightarrow} C$, the functors $\lim_{W\otimes V} f$ and $\lim_V(\lim_W f)$ are isomorphic whenever these weighted limits and the composite weight exist.* □

The following lemma is used to compute the values of the weighted limit functors $\lim_W f\colon B \to C$ or $\operatorname{colim}_V f\colon B \to C$ at a generalized element $b\colon X \to B$.

9.6.4. Lemma. *If $f\colon A \to C$ admits a limit $\lim_W f\colon B \to C$ weighted by $A \overset{W}{\nrightarrow} B$ then for any generalized element $b\colon X \to B$, the functor $\lim_W f \circ b\colon X \to C$ is isomorphic to the limit of f weighted by $A \overset{W(b,1)}{\nrightarrow} X$.*

Proof. Recall from Proposition 8.4.7 that the module $W(b, 1)$ is the composite $W \otimes \mathsf{Hom}_B(b, B)$. Thus, by Proposition 9.1.5, the $W(b, 1)$-weighted limit of f, if it exists, is the functor that right-represents the right extension of the module $\mathsf{Hom}_C(C, \lim_W f)$ along the module $\mathsf{Hom}_B(b, B)$. By Lemma 9.1.4, the right extension of $\mathsf{Hom}_C(C, \lim_W f)$ along $\mathsf{Hom}_B(b, B)$ is given by the restriction $\mathsf{Hom}_C(C, \lim_W f \circ b)$. By Definition 9.6.2, this tells us that $\lim_W f \circ b\colon X \to C$ defines the $W(b, 1)$-weighted limit of $f\colon A \to C$, as claimed. □

9.6.5. Remark. We can use weighted limits to re-express stability of pointwise right extensions under pasting with comma squares

$$\begin{tikzcd} \mathsf{Hom}_B(b, k) \arrow[r, two heads, "p_1"] \arrow[d, "p_0"'] \arrow[dr, phantom, "\Nearrow\phi"] & A \arrow[r, "f"] \arrow[d, "k"'] & C \\ X \arrow[r, "b"'] & B \arrow[ur, "\operatorname{ran}_k f"'] \arrow[u, phantom, shift left=4, "\Uparrow\nu"] & \end{tikzcd}$$

By Definition 9.6.1, the pointwise right extension $\operatorname{ran}_k f\colon B \to C$ encodes the weighted limit functor $\lim_{\mathsf{Hom}_B(B,k)} f\colon B \to C$; this spoils Exercise 9.6.ii. Now by Lemma 9.6.4, the composite

$$\operatorname{ran}_k f \circ b \cong \lim_{\mathsf{Hom}_B(B,k)} f \cong \lim_{\mathsf{Hom}_B(b,k)} f,$$

which is to say the value of the pointwise right extension at a generalized element $b\colon X \to B$ is the limit of $f\colon A \to C$ weighted by the module $A \overset{\mathsf{Hom}_B(b,k)}{\nrightarrow} X$. By Proposition 8.3.7, the module $\mathsf{Hom}_B(b,k)$ is the composite $\mathsf{Hom}_A(p_1, A) \otimes \mathsf{Hom}_X(X, p_0)$ of the modules contravariantly and covariantly represented by its legs. Thus

$$\mathrm{ran}_k f \circ b \cong \lim_{\mathsf{Hom}_B(B,k)} f \cong \lim_{\mathsf{Hom}_B(b,k)} f \cong \lim_{\mathsf{Hom}_X(X,p_0)} (\lim_{\mathsf{Hom}_A(p_1,A)} f).$$

Finally Lemma 9.1.4 allows us to compute the limit of $f\colon A \to C$ weighted by $\mathsf{Hom}_A(p_1, A)$; this is given by the composite functor $fp_1\colon \mathsf{Hom}_B(b,k) \to C$. So

$$\mathrm{ran}_k f \circ b \cong \lim_{\mathsf{Hom}_B(B,k)} f \cong \lim_{\mathsf{Hom}_B(b,k)} f \cong \lim_{\mathsf{Hom}_X(X,p_0)} (\lim_{\mathsf{Hom}_A(p_1,A)} f) \cong \lim_{\mathsf{Hom}_X(X,p_0)} (fp_1) \cong \mathrm{ran}_{p_0}(fp_1).$$

Exercises.

9.6.i. Exercise. Determine weights W_{lim} and W_{colim} so that the W_{lim}-weighted limit of a diagram f is the ordinary limit of f, as defined in 9.6.1, and the W_{colim}-weighted colimit is the ordinary colimit.

9.6.ii. Exercise. Express pointwise right extensions and pointwise left extensions as weighted limits and weighted colimits, respectively.

Part III

Change of model and model independence

CHAPTER 10

Cosmological biequivalences

In this chapter we study a certain class of cosmological functors between ∞-cosmoi that do not merely preserve ∞-categorical structure but also reflect and create it. We refer to these functors as *biequivalences* because the 2-functors they induce between homotopy 2-categories are biequivalences: surjective on objects up to equivalence and defining a local equivalence of hom-categories. Informally, we refer to cosmological biequivalences as "change-of-model functors." For example, the four ∞-cosmoi introduced in Example 1.2.24 whose objects model $(\infty, 1)$-categories are connected by the following biequivalences briefly described in Example 1.3.7:

$$\begin{tikzcd} \mathcal{CSS} \arrow[rr, "\text{discretization}"] \arrow[dr, shift left, "\mathrm{row}_0"] & & \mathcal{S}egal \arrow[dl, shift left, "\mathrm{row}_0"] \\ & \mathcal{Q}Cat \arrow[ul, shift left, "\text{cylinder}"] \arrow[ur, shift left, "\text{prism}"] \arrow[d, shift left, "\natural"] & \\ & 1\text{-}\mathcal{C}omp \arrow[u, shift left, "U"] & \end{tikzcd} \tag{10.0.1}$$

In §10.1, we collect together a number of results about cosmological functors that are scattered throughout the text. In §10.2, we introduce the special class of biequivalences and discuss several examples. Then in §10.3 we establish the basic 2-categorical properties of biequivalences, which will provide the essential ingredient in the proof of the model-independence results in Chapter 11.

10.1. Cosmological functors

Recall from Definition 1.3.1 that a **cosmological functor** is a simplicial functor $F: \mathcal{K} \to \mathcal{L}$ between ∞-cosmoi that preserves

- the specified classes of isofibrations and
- all of the simplicial limits enumerated in 1.2.1(i).

By Lemma 1.3.2, it follows that cosmological functors also preserves the equivalences and the trivial fibrations. By Corollary 6.3.3, cosmological functors preserve all flexible weighted limits, since Proposition 6.3.1 reveals that these can be constructed out of simplicial limits of diagrams of isofibrations of the form listed in 1.2.1(i).

Our aim in this section is to show that cosmological functors preserve all of the ∞-categorical structures we have introduced. In many cases this will not be evident from the original 2-categorical definitions (eg of cartesian fibrations in Definition 5.1.6) but can be deduced rather easier from the accompanying "internal characterization" of each categorical notion (such as given in Theorem 5.1.11(iii)).

First, we orient ourselves to the main examples. Recall Proposition 1.3.3:

10.1.1. Example (cosmological functors).

(i) The nerve embedding defines a cosmological functor $\mathcal{C}at \hookrightarrow \mathcal{Q}Cat$.

(ii) For any object A in an ∞-cosmos $\mathcal{K}$,

$$\mathsf{Fun}(A,-)\colon \mathcal{K} \to \mathcal{QC}at$$

defines a cosmological functor.

(iii) In particular, each ∞-cosmos has an **underlying quasi-category** functor

$$(-)_0 \coloneqq \mathsf{Fun}(1,-)\colon \mathcal{K} \to \mathcal{QC}at,$$

defined by mapping out of the terminal ∞-category. We use the notation "$(-)_0$" for this functor because the underlying quasi-categories of a complete Segal space or Segal category displayed in (10.0.1) are defined by evaluating these simplicial objects at 0 and because this notation for an "underlying category" is commonly used in enriched category theory (see Definition A.1.5).

(iv) For any ∞-cosmos $\mathcal{K}$ and any simplicial set U, the simplicial cotensor defines a cosmological functor

$$(-)^U\colon \mathcal{K} \to \mathcal{K}.$$

(v) For any object A in a cartesian closed ∞-cosmos $\mathcal{K}$, exponentiation defines a cosmological functor

$$(-)^A\colon \mathcal{K} \to \mathcal{K}.$$

(vi) For any map $f\colon A \to B$ in an ∞-cosmos $\mathcal{K}$, pullback defines a cosmological functor

$$f^*\colon \mathcal{K}_{/B} \to \mathcal{K}_{/A}.$$

In the special case of the map $!\colon A \to 1$, the pullback cosmological functor has the form

$$-\times A\colon \mathcal{K} \to \mathcal{K}_{/A}.$$

(vii) For any cosmological functor $F\colon \mathcal{K} \to \mathcal{L}$ and any $A \in \mathcal{K}$, the induced map on slices

$$F\colon \mathcal{K}_{/A} \to \mathcal{L}_{/FA}$$

defines a cosmological functor.

10.1.2. Example. The inclusion of a replete sub ∞-cosmos described in Proposition 6.4.3 is a cosmological functor. Examples include:

(i) The full subcategory of discrete objects

$$\mathcal{D}isc(\mathcal{K}) \hookrightarrow \mathcal{K}$$

defines a replete sub ∞-cosmos; see Proposition 6.1.6.

(ii) The inclusions

$$\mathcal{K}_\top \hookrightarrow \mathcal{K} \hookleftarrow \mathcal{K}_\bot$$

of the ∞-cosmoi of ∞-categories possessing and functors preserving a terminal or initial element. More generally, the inclusions

$$\mathcal{K}_{\top,J} \hookrightarrow \mathcal{K} \hookleftarrow \mathcal{K}_{\bot,J}$$

of the ∞-cosmoi of ∞-categories possessing and functors preserving limits or colimits of shape J; see Propositions 6.4.4 and 6.4.11.

(iii) The inclusions
$$\mathscr{R}ari(\mathcal{K})_{/B} \hookrightarrow \mathcal{K}_{/B} \hookleftarrow \mathscr{L}ari(\mathcal{K})_{/B}$$
or
$$\mathscr{R}ari(\mathcal{K}) \hookrightarrow \mathcal{K}^{\twoheaddownarrow} \hookleftarrow \mathscr{L}ari(\mathcal{K})$$
of ∞-cosmoi whose objects are isofibrations admitting a right or left adjoint right inverse; see Corollary 6.4.7 and Proposition 6.4.9.

(iv) The inclusions
$$\mathcal{C}art(\mathcal{K})_{/B} \hookrightarrow \mathcal{K}_{/B} \hookleftarrow co\mathcal{C}art(\mathcal{K})_{/B}$$
or
$$\mathcal{C}art(\mathcal{K}) \hookrightarrow \mathcal{K}^{\twoheaddownarrow} \hookleftarrow co\mathcal{C}art(\mathcal{K})$$
of the ∞-cosmoi of cartesian or cartesian fibrations and cartesian functors between them; see Proposition 6.4.12.

10.1.3. Observation. Inclusions $\mathcal{K}' \hookrightarrow \mathcal{K}$ and $\mathcal{L}' \hookrightarrow \mathcal{L}$ of replete sub ∞-cosmoi create isofibrations and the simplicial limits of axiom 1.2.1(i). As these inclusions are full on positive-dimensional arrows, a simplicial functor
$$\begin{tikzcd} \mathcal{K} \arrow[r, "F"] & \mathcal{L} \\ \mathcal{K}' \arrow[u, hook] \arrow[r, dashed, "F"] & \mathcal{L}' \arrow[u, hook] \end{tikzcd}$$
restricts to a simplicial functor on the subcategories if and only if it carries the objects and 0-arrows of the subcategory $\mathcal{K}'$ into the objects and 0-arrows of $\mathcal{L}'$, as will be the case whenever the replete sub ∞-cosmoi are "naturally defined" by the same ∞-categorical property. Whenever this is the case, the restricted functor is cosmological.

For example, by Proposition 5.1.20 and Exercise 5.1.v, pullback $f^* \colon \mathcal{K}_{/B} \to \mathcal{K}_{/A}$ preserves cartesian fibrations and cartesian functors. Thus, pullback along any functor $f \colon A \to B$ in $\mathcal{K}$ restricts to define a cosmological functor:
$$\begin{tikzcd} \mathcal{K}_{/B} \arrow[r, "f^*"] & \mathcal{K}_{/A} \\ \mathcal{C}art(\mathcal{K})_{/B} \arrow[u, hook] \arrow[r, dashed, "f^*"] & \mathcal{C}art(\mathcal{K})_{/A} \arrow[u, hook] \end{tikzcd}$$

10.1.4. Example. There are also cosmological functors connecting the ∞-cosmoi of "Reedy fibrant diagrams."

(i) By Proposition 6.1.1, the domain, codomain, and identity functors
$$\begin{tikzcd} \mathcal{K}^{\twoheaddownarrow} \arrow[r, shift left=2, "\mathrm{dom}"] \arrow[r, shift right=2, "\mathrm{cod}"'] & \mathcal{K} \arrow[l, "\mathrm{id}"' description] \end{tikzcd}$$
are all cosmological.

(ii) By Exercise 8.1.i, the domain and codomain functors
$$\begin{tikzcd} \mathcal{K}^{\boxtimes} \arrow[r, shift left, "\mathrm{dom}"] \arrow[r, shift right, "\mathrm{cod}"'] & \mathcal{K} \end{tikzcd}$$
from the ∞-cosmos of two-sided isofibrations are cosmological.

Cosmological functors may fail dramatically to be surjective on objects — $\mathrm{id}\colon \mathcal{K} \to \mathcal{K}^{\twoheaddownarrow}$ comes to mind. Thus a 2-categorical property involving a universal quantifier is not obviously preserved. For instance, by Definition 1.2.26 an object $E \in \mathcal{K}$ is **discrete** just when for all $X \in \mathcal{K}$, $\mathsf{Fun}(X, E)$ is a Kan complex. However, by Lemma 1.2.27, which we belatedly solve, discrete objects admit an internal characterization given in (iv) below:

10.1.5. PROPOSITION. *An object E in an ∞-cosmos $\mathcal{K}$ is* discrete *if the following equivalent conditions are satisfied:*

(i) E is a discrete object in the homotopy 2-category $\mathfrak{h}\mathcal{K}$, that is, every 2-cell with codomain E is invertible.
(ii) For each $X \in \mathcal{K}$, the hom-category $\mathsf{hFun}(X, E)$ is a groupoid.
(iii) For each $X \in \mathcal{K}$, the mapping quasi-category $\mathsf{Fun}(X, E)$ is a Kan complex.
(iv) The isofibration $E^{\mathbb{I}} \twoheadrightarrow E^{\mathbb{2}}$, induced by the inclusion of simplicial sets $\mathbb{2} \hookrightarrow \mathbb{I}$, is a trivial fibration.

PROOF. Here (ii) is an unpacking of (i). The equivalence of (ii) and (iii) is a well-known result of Joyal [**46**, 1.4] reproduced in Corollary 1.1.15. Condition (iv) is equivalent to the assertion that $\mathsf{Fun}(X, E)^{\mathbb{I}} \twoheadrightarrow \mathsf{Fun}(X, E)^{\mathbb{2}}$ is a trivial fibration between quasi-categories for all X. If this is a trivial fibration, then surjectivity on vertices implies that every 1-simplex in $\mathsf{Fun}(X, E)$ is an isomorphism, proving (iii). By Exercise 1.1.v, $\mathbb{2} \hookrightarrow \mathbb{I}$ is in the class of maps cellularly generated by the outer horn inclusions, so (iii) implies (iv). ☐

Note that if E is discrete, then any equivalent object E' is also discrete. Now the result we want is easy to establish.

10.1.6. COROLLARY. *Cosmological functors preserve discrete objects.*

PROOF. If $E \in \mathcal{K}$ is discrete and $F\colon \mathcal{K} \to \mathcal{L}$ is cosmological, then

$$F(E^{\mathbb{I}} \overset{\sim}{\twoheadrightarrow} E^{\mathbb{2}}) \cong (FE)^{\mathbb{I}} \overset{\sim}{\twoheadrightarrow} (FE)^{\mathbb{2}}$$

is a trivial fibration in $\mathcal{L}$, proving that FE is discrete by Proposition 10.1.5(iv). ☐

Importantly:

10.1.7. PROPOSITION. *Cosmological functors preserve comma ∞-categories: if $F\colon \mathcal{K} \to \mathcal{L}$ is a cosmological functor and the diagram below-left*

$$\begin{tikzcd} & E \arrow[dl, "e_1"'] \arrow[dr, "e_0"] & & & & FE \arrow[dl, "Fe_1"'] \arrow[dr, "Fe_0"] & \\ C \arrow[dr, "g"'] \arrow[rr, phantom, "\overset{\epsilon}{\Leftarrow}"] & & B \arrow[dl, "f"] \arrow[r, phantom, "\overset{F}{\rightsquigarrow}"] & {} & FC \arrow[dr, "Fg"'] \arrow[rr, phantom, "\overset{F\epsilon}{\Leftarrow}"] & & FB \arrow[dl, "Ff"] \\ & A & & & & FA & \end{tikzcd} \tag{10.1.8}$$

is a comma cone in $\mathcal{K}$, then the diagram above-right is a comma cone in $\mathcal{L}$.

PROOF. The simplicial pullback (3.4.2) that constructs the comma cone is preserved by any cosmological functor. By Proposition 3.4.11, any comma cone as above left arises from a fibered equivalence $e\colon E \xrightarrow{\sim} \mathsf{Hom}_A(f, g)$ where $\epsilon = \phi e$, and any fibered equivalence of this form defines a comma cone. Since $Fe\colon FE \xrightarrow{\sim} F(\mathsf{Hom}_A(f, g)) \cong \mathsf{Hom}_{FA}(Ff, Fg)$, we conclude that the right-hand data again defines a comma cone. ☐

Using Proposition 10.1.7, we can quickly establish the following preservation properties of cosmological functors by arguing along similar lines to Corollary 10.1.6. For ease of reference, we include on this list the preservation properties that have been previously established.

10.1.9. Proposition. *Cosmological functors preserve:*

(i) Equivalences between ∞-categories.
(ii) Adjunctions between ∞-categories, including right adjoint right inverse adjunctions and left adjoint right inverse adjunctions.
(iii) Invertible 2-cells and mates.
(iv) Homotopy coherent adjunctions and monads.
(v) Trivial fibrations and discrete objects.
(vi) Flexible weighted limits.
(vii) Comma spans and comma cones.
(viii) Absolute right and left lifting diagrams.
(ix) Limits or colimits of diagrams indexed by a simplicial set J inside an ∞-category.
(x) Cartesian and cocartesian fibrations and cartesian functors between them.
(xi) Discrete cartesian fibrations and discrete cocartesian fibrations.
(xii) Two-sided fibrations and cartesian functors between them and also modules.
(xiii) Modules and represented modules.

Proof. Lemma 1.3.2 proves that cosmological functors preserve equivalences between ∞-categories since these can be characterized as 2-categorical equivalences, which are preserved by any 2-functor. Similarly, as observed in Lemma 2.1.3, adjunctions are preserved by any 2-functor, as are invertible 2-cells and mates. In the same way, homotopy coherent adjunctions are preserved by any simplicial functor. So the preservation properties (ii), (iii) and (iv) hold under much weaker hypotheses.

The preservation of trivial fibrations was also established in Lemma 1.3.2 and Corollary 10.1.6 uses this to prove that discrete objects are also preserved, proving (v). Corollary 6.3.3 proves that cosmological functors preserve all flexible weighted limits as stated in (vi). Proposition 10.1.7 proves that cosmological functors preserve comma spans in the ∞-cosmos and comma cones in the homotopy 2-category as stated in (vii).

The preservation property (viii) follows from Theorem 3.5.3, which characterizes absolute lifting diagrams as fibered equivalences of comma ∞-categories, Proposition 10.1.7, which says that cosmological functors preserve commas, and the fact that cosmological functors preserve equivalences. Now (ix) follows from this by Definition 2.3.7 and the fact that cosmological functors preserve simplicial tensors.

The preservation properties (x) and (xi) also follow from Proposition 10.1.7 and the fact that cosmological functors preserve right or left adjoint right inverse adjunctions, mates, and trivial fibrations via the characterizations of Theorem 5.1.11(iii), Theorem 5.1.19(iii), and Proposition 5.4.7. More details are given in the proof of Corollary 5.1.16, which also observes that cartesian natural transformations are preserved by cosmological functors.

Directly from the internal characterization of Theorem 7.1.4 and the preservation of adjunctions and invertible 2-cells, cartesian functors preserve two-sided fibrations; preservation of cartesian functors between them is left to Exercise 10.1.i. This establishes (xii). By Proposition 1.3.3(vi), a cosmological functor induces a direct image cosmological functor between sliced ∞-cosmoi, which then preserves discrete objects by Corollary 10.1.6. Thus, modules are also preserved. By specializing Proposition 10.1.7 to cospans involving identities, it becomes clear that left and right representable commas are preserved. Since a module is representable if and only if it is fibered equivalent to one of these, representable modules are preserved as well, completing the proof of (xiii). □

10.1.10. Non-Example. The composite of the underlying quasi-category functor with the homotopy category functor

$$\mathcal{K} \xrightarrow{(-)_0} \mathcal{QC}at \xrightarrow{\mathsf{h}} \mathcal{C}at$$

defines a simplicial functor that carries an ∞-category A to its homotopy category $\mathsf{h}A$ first introduced in Definition 1.4.12. This functor does preserve isofibrations but does not preserve simplicial cotensors and so is not cosmological. This is a good thing because if it were Proposition 10.1.9(ix) would imply that limits in an ∞-category would also define limits in its homotopy category, which is not generally true.

Exercises.

10.1.i. Exercise. Prove that any cosmological functor preserves cartesian functors between two-sided fibrations. Conclude that any cosmological functor $F\colon \mathcal{K} \to \mathcal{L}$ induces a cosmological functor $F\colon {}_{A\backslash}\mathcal{F}ib(\mathcal{K})_{/B} \to {}_{FA\backslash}\mathcal{F}ib(\mathcal{L})_{/FB}$.

10.2. Cosmological biequivalences as change-of-model functors

Special cosmological functors, called *biequivalences*, preserve and also reflect the category theory developed in this text. Recall Definition 1.3.6:

10.2.1. Definition (cosmological biequivalences). A cosmological functor $F\colon \mathcal{K} \to \mathcal{L}$ is a **biequivalence** when it is:

(i) surjective on objects up to equivalence: i.e., if for all $C \in \mathcal{L}$ there exists $A \in \mathcal{K}$ so that $FA \simeq C$; and
(ii) a local equivalence of quasi-categories: i.e., if for every pair $A, B \in \mathcal{K}$, the map

$$\mathsf{Fun}(A, B) \xrightarrow{\sim} \mathsf{Fun}(FA, FB)$$

is an equivalence of quasi-categories.

More generally, two ∞-cosmoi are **biequivalent** if there exists a finite zig-zag of biequivalences connecting them.

10.2.2. Example (biequivalences between ∞-cosmoi of $(\infty, 1)$-categories). In Appendix E, we will show that the functors (10.0.1) are all biequivalences. Except for the functor $\natural\colon \mathcal{QC}at \to 1\text{-}\mathcal{C}omp$, each of the functors (10.0.1) arises from a simplicially enriched right Quillen equivalence between model categories enriched over the Joyal model structure with all objects cofibrant. Functors of this form are easily seen to encode cosmological biequivalences.

Recall that any ∞-cosmos has an **underlying quasi-category functor**

$$\mathcal{K} \xrightarrow{(-)_0 := \mathsf{Fun}(1,-)} \mathcal{QC}at$$

defined by mapping out of the terminal ∞-category. We now show that for every ∞-cosmos that is biequivalent to $\mathcal{QC}at$, its underlying quasi-category functor is a biequivalence.

10.2.3. Proposition. *If an ∞-cosmos $\mathcal{K}$ is biequivalent to $\mathcal{QC}at$, then the underlying quasi-category functor $(-)_0\colon \mathcal{K} \to \mathcal{QC}at$ is a cosmological biequivalence.*

Proof. We will show that in the presence of cosmological biequivalences

$$\mathcal{K} \xleftarrow[\sim]{G} \mathcal{L} \xrightarrow[\sim]{F} \mathcal{QCat}$$

the underlying quasi-category functor $(-)_0 \colon \mathcal{K} \to \mathcal{QCat}$ is a cosmological biequivalence. By induction, perhaps taking one of these functors to be the identity, the same conclusion holds for any ∞-cosmos connected by a finite zig-zag of biequivalences to $\mathcal{QCat}$.

As F is a biequivalence, for each quasi-category Q, there exists an ∞-category $B \in \mathcal{L}$ so that $FB \simeq Q$. Because F and G are both local equivalences preserving the terminal ∞-category 1 (for which we adopt the same notation in each of $\mathcal{K}$, $\mathcal{L}$, and $\mathcal{QCat}$), there is then a zig-zag of equivalences of quasi-categories

$$(GB)_0 = \mathsf{Fun}(1, GB) \xleftarrow{\sim} \mathsf{Fun}(1, B) \xrightarrow{\sim} \mathsf{Fun}(1, FB) \cong FB \simeq Q.$$

This proves that there exists an ∞-category $GB \in \mathcal{K}$ whose underlying quasi-category is equivalent to Q.

To show that the underlying quasi-category functor $(-)_0 \colon \mathcal{K} \to \mathcal{QCat}$ is a local equivalence, consider a pair of ∞-categories $A, B \in \mathcal{K}$. By essential surjectivity of G, there exist ∞-categories $X, Y \in \mathcal{L}$ so that $GX \simeq A$ and $B \simeq GY$. By pre- and post-composing with these equivalences, Corollary 1.4.9 implies that $\mathsf{Fun}(A, B) \to \mathsf{Fun}(A_0, B_0)$ is equivalent to $\mathsf{Fun}(GX, GY) \to \mathsf{Fun}((GX)_0, (GY)_0)$, so it suffices to prove that the latter map is an equivalence of quasi-categories.

By simplicial functoriality of the local equivalences F and G, there is a commutative diagram whose vertical maps are equivalences

$$\begin{tikzcd}
\mathsf{Fun}(GX, GY) \arrow[r] & \mathsf{Fun}(\mathsf{Fun}(1, GX), \mathsf{Fun}(1, GY)) \arrow[d, "\wr"] \\
& \mathsf{Fun}(\mathsf{Fun}(1, X), \mathsf{Fun}(1, GY)) \\
\mathsf{Fun}(X, Y) \arrow[uu, "\wr"] \arrow[dd, "\wr"'] \arrow[r] & \mathsf{Fun}(\mathsf{Fun}(1, X), \mathsf{Fun}(1, Y)) \arrow[u, "\wr"'] \arrow[d, "\wr"] \\
& \mathsf{Fun}(\mathsf{Fun}(1, X), \mathsf{Fun}(1, FY)) \\
\mathsf{Fun}(FX, FY) \arrow[r, "\cong"] & \mathsf{Fun}(\mathsf{Fun}(1, FX), \mathsf{Fun}(1, FY)) \arrow[u, "\wr"']
\end{tikzcd}$$

Any quasi-category is isomorphic to its underlying quasi-category, so the bottom horizontal map is an isomorphism. By the 2-of-3 property, it follows that the top horizontal map is an equivalence, which is what we wanted to show. □

Recall from Proposition 1.3.3(vi) that a cosmological functor $F \colon \mathcal{K} \to \mathcal{L}$ induces a cosmological functor $F \colon \mathcal{K}_{/B} \to \mathcal{L}_{/FB}$ for any $B \in \mathcal{K}$.

10.2.4. Proposition. *If $F \colon \mathcal{K} \to \mathcal{L}$ is a cosmological biequivalence, then for any $B \in \mathcal{K}$ the induced functor $F \colon \mathcal{K}_{/B} \to \mathcal{L}_{/FB}$ is also a cosmological biequivalence.*

Proof. We first argue that the F defines a local equivalence of sliced mapping quasi-categories, as defined in Proposition 1.2.22(ii). Given a pair of isofibration $p \colon E \twoheadrightarrow B$ and $p' \colon E' \twoheadrightarrow B$ in $\mathcal{K}$, the induced functor on mapping quasi-categories is defined by

$$\begin{tikzcd}
\mathrm{Fun}_B(p,p') \arrow[rr] \arrow[dd, two heads] \arrow[dr, dashed, "\sim"] \arrow[ddr, phantom, "\lrcorner" very near start] & & \mathrm{Fun}(E,E') \arrow[dd, two heads] \arrow[dr, "\sim"] & \\
& \mathrm{Fun}_{FB}(Fp,Fp') \arrow[rr, crossing over] \arrow[ddr, phantom, "\lrcorner" very near start] & & \mathrm{Fun}(FE,FE') \arrow[dd, two heads] \\
1 \arrow[rr] \arrow[dr, equal] & & \mathrm{Fun}(E,B) \arrow[dr, "\sim"] & \\
& 1 \arrow[rr] \arrow[from=uu, two heads, crossing over] & & \mathrm{Fun}(FE,FB)
\end{tikzcd}$$

As the maps between the cospans in $\mathcal{QC}at$ are equivalences, so is the induced map between the pullbacks.

For surjectivity up to equivalence, consider an isofibration $q\colon L \twoheadrightarrow FB$ in $\mathcal{L}$. As F is surjective on objects up to equivalence, there exists some $A \in \mathcal{K}$ together with an equivalence $i\colon FA \xrightarrow{\sim} L \in \mathcal{L}$. As F defines a local equivalence of mapping quasi-categories, there is moreover a functor $f\colon A \to B$ in $\mathcal{L}$ so that $Ff\colon FA \to FB$ is isomorphic to qi in $\mathfrak{h}\mathcal{L}$. The map f need not be an isofibration, but Lemma 1.2.16 allows us to factor f as $A \xrightarrow{\sim} K \overset{p}{\twoheadrightarrow} B$. Choosing an equivalence inverse $j\colon K \xrightarrow{\sim} A$, this data defines a diagram in $\mathfrak{h}\mathcal{L}$ that commutes up to isomorphism:

$$\begin{tikzcd}
FK \arrow[r, "Fj", "\sim"'] \arrow[dr, two heads, "Fp"'] \arrow[dr, phantom, "\cong", shift left=3] & FA \arrow[r, "i", "\sim"'] \arrow[d, dashed, "Ff"] & L \arrow[dl, two heads, "q"] \arrow[dl, phantom, "\cong"', shift right=3] \\
& FB &
\end{tikzcd}$$

Proposition 1.4.10 tells us that isofibrations in ∞-cosmoi define isofibrations in the homotopy 2-category. In particular, we may lift the displayed isomorphism along the isofibration $q\colon L \twoheadrightarrow FB$ to define a commutative triangle:

$$\begin{tikzcd}
FK \arrow[r, "Fj", "\sim"'] \arrow[dr, two heads, "Fp"'] \arrow[dr, phantom, "\cong", shift left=3] & FA \arrow[r, "i", "\sim"'] \arrow[d, dashed, "Ff"] & L \arrow[dl, two heads, "q"] \arrow[dl, phantom, "\cong"', shift right=3] \\
& FB &
\end{tikzcd}
=
\begin{tikzcd}
FK \arrow[rr, bend left, "i\cdot Fj", "\sim"'] \arrow[rr, bend right=15, "\sim", "e"'] \arrow[rr, phantom, "\cong"] \arrow[dr, two heads, "Fp"'] & & L \arrow[dl, two heads, "q"] \\
& FB &
\end{tikzcd}$$

As noted in the proof of Theorem 1.4.7, since e is isomorphic to an equivalence $i\cdot Fj$, it must also define an equivalence. Thus, by Proposition 1.2.22(vii), we have shown that the isofibration $p\colon K \twoheadrightarrow B$ maps under F to an isofibration that is equivalent to our chosen $q\colon L \twoheadrightarrow FB$. $\square$

A similar argument proves that a cosmological biequivalence induces a biequivalence between the corresponding ∞-cosmoi of isofibrations of Proposition 6.1.1.

10.2.5. Proposition. *If $F\colon \mathcal{K} \to \mathcal{L}$ is a cosmological biequivalence then the induced functor $F\colon \mathcal{K}^{\twoheaddownarrow} \to \mathcal{L}^{\twoheaddownarrow}$ is a biequivalence.*

Proof. Exercise 10.2.i. $\square$

We establish another family of biequivalences of sliced ∞-cosmoi:

10.2.6. Proposition. *If $f\colon A \xrightarrow{\sim} B$ is an equivalence in $\mathcal{K}$, then the pullback functor $f^*\colon \mathcal{K}_{/B} \to \mathcal{K}_{/A}$ is a cosmological biequivalence.*

Proof. To see that $f^*\colon \mathcal{K}_{/B} \to \mathcal{K}_{/A}$ is essentially surjective, consider an object $r\colon D \twoheadrightarrow A$, factor the composite map $fr\colon D \to B$ as an equivalence followed by an isofibration, and pull the result back along f.

$$\begin{tikzcd}
D \arrow[dr, dashed] \arrow[drr, bend left, "\sim"] \arrow[ddr, bend right, two heads, "r"'] & & \\
& P \arrow[r, "\sim"] \arrow[d, two heads, "q"'] \arrow[dr, phantom, very near start, "\lrcorner"] & E \arrow[d, two heads, "p"] \\
& A \arrow[r, "\sim", "f"'] & B
\end{tikzcd}$$

By Lemma 3.3.2, the pullback of f is an equivalence, so by the 2-of-3 property, r is equivalent to the isofibration $q\colon P \twoheadrightarrow A$, which is in the image of $f^*\colon \mathcal{K}_{/B} \to \mathcal{K}_{/A}$.

To show that this simplicial functor is a local equivalence, consider a pair of isofibrations $p\colon E \twoheadrightarrow B$ and $q\colon F \twoheadrightarrow B$. We will show that the quasi-category of functors over B is equivalent to the quasi-category of functors over A between their pullbacks

$$\begin{tikzcd}
f^*E \arrow[rr, "h", "\sim"'] \arrow[dd, two heads, "r"'] \arrow[ddrr, phantom, very near start, "\lrcorner"] & & E \arrow[dd, two heads, "p"' near start] & \\
& f^*F \arrow[rr, "k", "\sim"' near start] \arrow[dl, two heads, "s"'] \arrow[drr, phantom, very near start, "\lrcorner"] & & F \arrow[dl, two heads, "q"] \\
A \arrow[rr, "\sim"', "f"'] & & B &
\end{tikzcd}$$

To define the comparison map $\mathsf{Fun}_B(E,F) \to \mathsf{Fun}_A(f^*E, f^*F)$ consider the following commutative prism

$$\begin{tikzcd}
& \mathsf{Fun}_B(E,F) \arrow[rrr] \arrow[dl, dashed, "\sim"'] \arrow[dd, two heads] \arrow[drrr, phantom, very near start, "\lrcorner"] & & & \mathsf{Fun}(E,F) \arrow[dl, "h^*"', "\sim"] \arrow[dd, two heads, "q_*"] \\
\mathsf{Fun}_A(f^*E,f^*F) \arrow[rr] \arrow[dd, two heads] \arrow[ddrr, phantom, very near start, "\lrcorner"] & & \mathsf{Fun}(f^*E,f^*F) \arrow[r, "k_*", "\sim"'] \arrow[dd, two heads, "s_*"] \arrow[ddr, phantom, very near start, "\lrcorner"] & \mathsf{Fun}(f^*E,F) \arrow[dd, two heads, "q_*"] & \\
& \mathbb{1} \arrow[rrr, "p"] \arrow[dl, equal] & & & \mathsf{Fun}(E,B) \arrow[dl, "\sim", "h^*"'] \\
\mathbb{1} \arrow[rr, "r"'] & & \mathsf{Fun}(f^*E,A) \arrow[r, "\sim", "f_*"'] & \mathsf{Fun}(f^*E,B) &
\end{tikzcd}$$

The front-right square is a pullback by the simplicial universal property of f^*F, while the front-left square and back face are the pullbacks that define $\mathsf{Fun}_A(f^*E, f^*F)$ and $\mathsf{Fun}_B(E,F)$. The universal property of the composite front pullback rectangle induces the map $\mathsf{Fun}_B(E,F) \to \mathsf{Fun}_A(f^*E, f^*F)$. As this functor is the pullback of the equivalences h^* of Corollary 1.4.9, the induced map defines an equivalence of quasi-categories. □

Exercises.

10.2.i. Exercise. Prove Proposition 10.2.5.

10.2.ii. Exercise. If $F\colon \mathcal{K} \xrightarrow{\sim} \mathcal{L}$ is a biequivalence and $A \in \mathcal{K}$ and $B \in \mathcal{L}$ are so that $FA \simeq B$ prove that the slice ∞-cosmoi $\mathcal{K}_{/A}$ and $\mathcal{L}_{/B}$ are biequivalent.

10.3. Properties of change-of-model functors

We refer to biequivalence between ∞-cosmoi as *change-of-model functors*. In this section, we enumerate their basic properties. First, we observe that cosmological biequivalences descend to biequivalences between homotopy 2-categories, hence the name:

10.3.1. Proposition. *A cosmological biequivalence $F\colon \mathcal{K} \to \mathcal{L}$ induces a biequivalence $F\colon \mathfrak{h}\mathcal{K} \to \mathfrak{h}\mathcal{L}$ of homotopy 2-categories: i.e., the 2-functor F is*

(i) surjective on objects up to equivalence and
(ii) defines a local equivalence of categories $\mathsf{hFun}(A, B) \xrightarrow{\sim} \mathsf{hFun}(FA, FB)$ for all $A, B \in \mathcal{K}$.

Proof. This is immediate from the definition: Theorem 1.4.7 observed that ∞-cosmos-level equivalences coincide with 2-categorical equivalences, and by Lemma 1.2.18 the homotopy category functor $\mathsf{h}\colon \mathcal{QC}at \to \mathcal{C}at$ carries equivalences of quasi-categories to equivalences of categories. ☐

In particular, recalling Definition 1.4.12, it follows immediately that for every ∞-category $A \in \mathcal{K}$, that the biequivalence F induces an equivalence of homotopy categories $\mathsf{h}A \xrightarrow{\sim} \mathsf{h}FA$; see Exercise 10.3.i.

Any biequivalence between 2-categories induces a variety of local and global bijections, as enumerated below, which can be put to use to solve Exercise 10.3.ii.

10.3.2. Corollary. *Consider any cosmological biequivalence $F\colon \mathcal{K} \to \mathcal{L}$.*

(i) The biequivalence F preserves, reflects, and creates equivalences between ∞-categories, and induces a bijection between equivalence classes of objects.
(ii) The biequivalence F induces local bijections between isomorphism classes of functors extending the bijection of (i): choosing any pairs of objects $A, B \in \mathcal{K}$ and $A', B' \in \mathcal{L}$ and equivalences $FA \simeq A'$ and $FB \simeq B'$, the map

$$\mathsf{hFun}(A, B) \xrightarrow{\sim} \mathsf{hFun}(FA, FB) \xrightarrow{\sim} \mathsf{hFun}(A', B') \tag{10.3.3}$$

defines a bijection between isomorphism classes of functors $A \to B$ in $\mathcal{K}$ and isomorphism classes of functors $A' \to B'$ in $\mathcal{L}$.
(iii) The biequivalence F induces local bijections between 2-cells with specified boundary extending the bijections of (i) and (ii): choosing any pairs of objects $A, B \in \mathcal{K}$ and $A', B' \in \mathcal{L}$, equivalences $FA \simeq A'$ and $FB \simeq B'$, functors $f, g\colon A \rightrightarrows B$ and $f', g'\colon A' \rightrightarrows B'$, and natural isomorphisms

$$\begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "\wr"'] \arrow[dr, phantom, "\cong\alpha"] & FB \arrow[d, "\wr"] \\ A' \arrow[r, "f'"'] & B' \end{tikzcd} \qquad \begin{tikzcd} FA \arrow[r, "Fg"] \arrow[d, "\wr"'] \arrow[dr, phantom, "\cong\beta"] & FB \arrow[d, "\wr"] \\ A' \arrow[r, "g'"'] & B' \end{tikzcd}$$

the map (10.3.3) *induces a bijection between 2-cells $f \Rightarrow g$ in $\mathcal{K}$ and 2-cells $f' \Rightarrow g'$ in $\mathcal{L}$.*

Proof. For (i), any 2-functor preserves equivalences, so our first take is to show that equivalences are also reflected. If $f\colon A \to B$ is a morphism so that $Ff\colon FA \xrightarrow{\sim} FB$ is an equivalence then by the

2-of-3 property and the commutative diagram

$$\begin{tikzcd} \mathsf{Fun}(X,A) \arrow[r, "f_*"] \arrow[d, "\wr"'] & \mathsf{Fun}(X,B) \arrow[d, "\wr"] \\ \mathsf{Fun}(FX,FA) \arrow[r, "Ff_*", "\sim"'] & \mathsf{Fun}(FX,FB) \end{tikzcd}$$

f_* is an equivalence for all X, proving that $f\colon A \to B$ is an equivalence.

To prove that equivalences are also created — meaning that if $FA \simeq FB$ then $A \simeq B$ — note that the equivalence $\mathsf{Fun}(A,B) \xrightarrow{\sim} \mathsf{Fun}(FA,FB)$ implies that any morphism $FA \to FB$ has a lift $A \to B$ up to isomorphism. In this way, the morphisms of an equivalence $f\colon FA \xrightarrow{\sim} FB$ and $g\colon FB \xrightarrow{\sim} FA$ can be lifted to morphisms $\tilde{f}\colon A \to B$ and $\tilde{g}\colon B \to A$ so that $\tilde{g}\tilde{f}$ and id_A have isomorphic images under the equivalence $\mathsf{Fun}(A,A) \xrightarrow{\sim} \mathsf{Fun}(FA,FA)$. Since an equivalence of quasi-categories induces a bijection on isomorphism classes of vertices, we conclude that $\tilde{g}\tilde{f} \cong \mathrm{id}_A \in \mathsf{Fun}(A,A)$. Similarly, $\tilde{f}\tilde{g}$ and id_B have isomorphic images under $\mathsf{Fun}(B,B) \xrightarrow{\sim} \mathsf{Fun}(FB,FB)$ and thus $\tilde{f}\tilde{g} \cong \mathrm{id}_B$, proving that $A \simeq B$.

Finally, the preservation, reflection, and creation of equivalences together with essential surjectivity of a cosmological biequivalence implies that such functors induces a bijection on equivalence classes of objects.

For (ii), by Corollary 1.4.9, the chosen equivalences $FA \simeq A'$ and $FB \simeq B'$ induce an equivalence of quasi-categories

$$\mathsf{Fun}(A,B) \xrightarrow{\sim} \mathsf{Fun}(FA,FB) \xrightarrow{\sim} \mathsf{Fun}(A',B')$$

which descends to the equivalence of homotopy categories (10.3.3). Since equivalences of quasi-categories induce bijections between isomorphism classes of vertices, this yields in particular a bijection between isomorphism classes of functors.

For (iii), the equivalence (10.3.3) is full and faithful, inducing a bijection between cells $f \Rightarrow g$ and $Ff \Rightarrow Fg$. This bijection can be transported along any chosen isomorphisms α and β to yield a bijection between natural transformations $f \Rightarrow g$ in $\mathsf{hFun}(A,B)$ in $\mathcal{K}$ and natural transformations $f' \Rightarrow g'$ in $\mathsf{hFun}(A',B')$ in $\mathcal{L}$. ☐

As an application of Corollary 10.3.2, we now show that the internal hom B^A between ∞-categories A and B in a cartesian closed ∞-cosmos $\mathcal{K}$ that is biequivalent to $\mathcal{QC}at$ is equivalent to the simplicial cotensor B^{A_0} of B with the underlying quasi-category of A. Even if $\mathcal{K}$ is not cartesian closed as an ∞-cosmos, on account of the biequivalence $(-)_0\colon \mathcal{K} \to \mathcal{QC}at$ of Proposition 10.2.3, its homotopy 2-category $\mathfrak{h}\mathcal{K}$ is cartesian closed in the bicategorical sense, replacing the natural isomorphisms of Proposition 1.4.5(ii) with natural equivalences: we define $B^A \in \mathcal{K}$ to be any ∞-category whose underlying quasi-category is $B_0^{A_0}$. By the 2-of-3 property for equivalences

$$\begin{tikzcd} \mathsf{Fun}(X,B^A) \arrow[r, "\sim"] \arrow[d, dashed, "\wr"'] & \mathsf{Fun}(X_0,B_0^{A_0}) \arrow[d, "\cong"] \\ \mathsf{Fun}(X\times A,B) \arrow[r, "\sim"] & \mathsf{Fun}(X_0\times A_0,B_0) \end{tikzcd}$$

we see that $\mathsf{Fun}(X, B^A) \simeq \mathsf{Fun}(X \times A, B)$ for any X. For this reason, the statement of Proposition 10.3.4 does not require that $\mathcal{K}$ is cartesian closed as an ∞-cosmos; the exponentials can be inferred to exist a posteriori.

10.3.4. Proposition. *Suppose $\mathcal{K}$ is an ∞-cosmos that is biequivalent to $\mathcal{QC}at$. Then for any ∞-categories $A, B \in \mathcal{K}$, the exponential $B^A \simeq B^{A_0}$ is equivalent to the cotensor of B with the underlying quasi-category of A.*

Proof. By Corollary 10.3.2(i), cosmological biequivalences reflect equivalences of ∞-categories. Thus, to prove $B^A \simeq B^{A_0}$, it suffices by Proposition 10.2.3 and Corollary 10.3.2 to prove that B^A and B^{A_0} have equivalent underlying quasi-categories. The defining universal properties of the exponential and cotensor provide equivalences

$$\mathsf{Fun}(1, B^A) \simeq \mathsf{Fun}(A, B) \xrightarrow{\sim} \mathsf{Fun}(A_0, B_0) \cong \mathsf{Fun}(1, B^{A_0})$$

which compose with the local equivalence of the underlying quasi-category functor to provide the desired equivalence. □

We now prove that biequivalences reflect and create, as well as preserve, the ∞-categorical structures considered in Section 10.1.

10.3.5. Proposition. *A biequivalence $F: \mathcal{K} \to \mathcal{L}$ between ∞-cosmoi:*

(i) Preserves and reflects invertibility of 2-cells.
(ii) Preserves, reflects, and creates adjunctions between ∞-categories, including right adjoint right inverse adjunctions and left adjoint right inverse adjunctions.
(iii) Preserves and reflects discreteness.
(iv) Preserves, reflects, and creates fibered equivalences.
(v) Preserves and reflects comma and arrow ∞-categories: a cell as on the left of (10.1.8) *is a comma cone in $\mathcal{K}$ if and only if its image is a comma cone in $\mathcal{L}$.*
(vi) Preserves and reflects cartesian and cocartesian fibrations and cartesian functors between them.
(vii) Preserves and reflects discrete cartesian fibrations and discrete cocartesian fibrations.
(viii) Preserves and reflects two-sided fibrations and cartesian functors between them.
(ix) Preserves and reflects modules and represented modules and induces a bijection on equivalence classes of modules between a fixed pair of ∞-categories.
(x) Preserves, reflects, and creates absolute right and left lifting diagrams over a given cospan.
(xi) Preserves and reflects limits or colimits of diagrams indexed by a simplicial set J inside an ∞-category and creates the property of an ∞-category in $\mathcal{K}$ admitting a limit or colimit of a given diagram.

Proof. Properties (i) and (ii) hold for any biequivalence between 2-categories, such as $F: \mathfrak{h}\mathcal{K} \to \mathfrak{h}\mathcal{L}$ and are left to Exercise 10.3.ii as a useful exercise to familiarize oneself with the 2-categorical notion of biequivalence.

Property (iii) follows from (i) and Proposition 10.1.5(i): if FE is discrete, then the image under F of any 2-cell in $\mathcal{K}$ with codomain E is invertible, which implies that that 2-cell is invertible in E.

Property (iv) is a special case of Corollary 10.3.2(i) applied to the induced biequivalence of Proposition 10.2.4. Since both $\mathcal{K}$ and $\mathcal{L}$ admit comma ∞-categories and Proposition 3.4.11 shows that comma spans are characterized by a fibered equivalence class of two-sided isofibrations, (v) follows from (iv).

Proposition (vi) follows from (ii) and (i) and (vii) follows from Corollary 10.3.2(ii) which shows that invertibility of the morphism considered in Proposition 5.4.7 is reflected. Property (viii) follows similarly from Theorem 7.1.4(iii) and (ii) and (i). Preservation and reflection of modules now follows from (iii) and the bijection between equivalence classes follows from (iv); we defer full details until Proposition 11.3.1. The final statement of (ix), on representability, combines (iv) with (v), as will be elaborated upon in Proposition 11.3.2.

The reflection properties of (x) and (xi) follow from Theorem 3.5.3 and the creation properties follow from Theorem 3.5.11 and (ix). The precise steps are spelled out further in Theorem 11.1.1. ☐

10.3.6. COROLLARY. *If $F\colon \mathcal{K} \to \mathcal{L}$ is a cosmological biequivalence then the following induced cosmological functors are all biequivalences:*

(i) $F\colon \mathcal{D}isc(\mathcal{K}) \to \mathcal{D}isc(\mathcal{L})$
(ii) $F\colon \mathcal{K}_{\top,J} \to \mathcal{L}_{\top,J}$ *and* $F\colon \mathcal{K}_{\perp,J} \to \mathcal{L}_{\perp,J}$
(iii) $F\colon \mathscr{R}ari(\mathcal{K})_{/B} \to \mathscr{R}ari(\mathcal{L})_{/FB}$ *and* $F\colon \mathcal{L}ari(\mathcal{K})_{/B} \to \mathcal{L}ari(\mathcal{L})_{/FB}$
(iv) $F\colon \mathscr{R}ari(\mathcal{K}) \to \mathscr{R}ari(\mathcal{L})$ *and* $F\colon \mathcal{L}ari(\mathcal{K}) \to \mathcal{L}ari(\mathcal{L})$
(v) $F\colon \mathcal{C}art(\mathcal{K})_{/B} \to \mathcal{C}art(\mathcal{L})_{/FB}$ *and* $F\colon co\mathcal{C}art(\mathcal{K})_{/B} \to co\mathcal{C}art(\mathcal{L})_{/FB}$
(vi) $F\colon \mathcal{C}art(\mathcal{K}) \to \mathcal{C}art(\mathcal{L})$ *and* $F\colon co\mathcal{C}art(\mathcal{K}) \to co\mathcal{C}art(\mathcal{L})$
(vii) $F\colon {}_{A\backslash}\mathcal{F}ib(\mathcal{K})_{/B} \to {}_{FA\backslash}\mathcal{F}ib(\mathcal{L})_{/FB}$
(viii) $F\colon {}_{A\backslash}\mathcal{M}od(\mathcal{K})_{/B} \to {}_{FA\backslash}\mathcal{M}od(\mathcal{L})_{/FB}$

PROOF. In each case we start with a cosmological biequivalence — perhaps $\mathcal{K}_{/B} \to \mathcal{L}_{/FB}$ or $\mathcal{K}^{\natural} \to \mathcal{L}^{\natural}$ — and must show that the restricted cosmological functor of Observation 10.1.3 is again a biequivalence between the replete sub ∞-cosmoi. Each of the arguments is similar: we know that $F\colon \mathcal{K} \to \mathcal{L}$ induces a bijection between equivalence classes of objects. Since the sub ∞-cosmoi are replete, the equivalence classes of objects of the replete sub ∞-cosmos are given by a subset of the equivalence classes of objects of $\mathcal{K}$. Since, moreover, Proposition 10.3.5 proves that the property that characterizes the objects of the sub ∞-cosmos is preserved, reflected, and created by any biequivalence, it is clear that the bijection between equivalence classes of objects in $\mathcal{K}$ and $\mathcal{L}$ restricts to define a bijection between equivalence classes of objects in the sub ∞-cosmoi.

Now suppose $A, B \in \mathcal{K}$ are objects of its replete sub ∞-cosmos. We must show the local equivalence

$$\mathsf{Fun}(A,B) \xrightarrow{\sim} \mathsf{Fun}(FA,FB)$$

restricts to a local equivalence of the functor spaces of the sub ∞-cosmoi. Since we know that the biequivalence defines a bijection between isomorphism classes of 0-arrows in the sub ∞-cosmoi, the proof is completed by the following lemma. ☐

10.3.7. LEMMA. *Let $f\colon K \xrightarrow{\sim} L$ be an equivalence of quasi-categories. Let $K' \subset K$ and $L' \subset L$ be* ***replete sub quasi-categories*** *— that is, full and replete up to isomorphism on some collection of vertices — and suppose further that f restricts to define a simplicial functor*

$$\begin{tikzcd} K \arrow[r, "f", "\sim"'] & L \\ K' \arrow[u, hook] \arrow[r, dashed, "f"] & L' \arrow[u, hook] \end{tikzcd}$$

that is bijective on isomorphism classes of vertices. Then $f\colon K' \xrightarrow{\sim} L'$ is an equivalence of quasi-categories.

PROOF. Choose any inverse equivalence $g\colon L \xrightarrow{\sim} K$. Since $K' \subset K$ is replete and full on some subset of vertices and $f\colon K \to L$ is bijective on isomorphism classes of vertices, g restricts to define a functor $g\colon L' \to K'$: for each $y \in L'$, $fg(y') \cong y' \in L$ so $g(y')$ must lie in K'. By repleteness and fullness again, the simplicial natural isomorphisms $K \to K^{\mathbb{I}}$ and $L \to L^{\mathbb{I}}$ that witness $gf \cong \mathrm{id}_K$ and $fg \cong \mathrm{id}_L$ restrict to L' and K' to define the desired equivalence. □

Exercise.

10.3.i. EXERCISE. Let $F\colon \mathcal{K} \xrightarrow{\sim} \mathcal{L}$ be a cosmological biequivalence. Show, for any ∞-category $A \in \mathcal{K}$, that F induces an equivalence of homotopy categories $\mathsf{h}A \xrightarrow{\sim} \mathsf{h}FA$.

10.3.ii. EXERCISE. Consider a 2-functor $F\colon \mathcal{C} \to \mathcal{D}$ that defines a biequivalence in the sense of Proposition 10.3.1. Prove that:

(i) A 2-cell $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd}$ in $\mathcal{C}$ is invertible if and only if $F\alpha$ is invertible in $\mathcal{D}$.

(ii) $u\colon A \to B$ admits a left adjoint in $\mathcal{C}$ if and only if $Fu\colon FA \to FB$ admits a left adjoint in $\mathcal{D}$ in which case F preserve the adjunction.

10.3.iii. EXERCISE. Prove that cosmological biequivalences between cartesian closed ∞-cosmoi preserve exponential objects up to equivalence.

10.4. Inverse cosmological biequivalences

Definition 10.2.1 declares two ∞-cosmoi to be **biequivalent** just when they are connected by a finite zig-zag of cosmological biequivalences. In this section, we establish a few useful properties of the "composite" of such a zig-zag, the analysis of which immediately reduces to the base case: describing the inverse $G\colon \mathcal{L} \rightsquigarrow \mathcal{K}$ to a cosmological biequivalence $F\colon \mathcal{K} \xrightarrow{\sim} \mathcal{L}$. This material will be applied in Chapter 12 to streamline proofs that ∞-categorical structures transfer to biequivalent ∞-cosmoi. The reader might consider skipping this for now and referring back to it with the applications of Chapter 12 in mind.

To explain what to expect, consider the analogous 2-categorical case. By Proposition 10.3.1, a cosmological biequivalence $F\colon \mathcal{K} \xrightarrow{\sim} \mathcal{L}$ induces a biequivalence $F\colon \mathfrak{h}\mathcal{K} \xrightarrow{\sim} \mathfrak{h}\mathcal{L}$ of homotopy 2-categories, this being a 2-functor that is

- surjective on objects up to equivalence and
- defines a local equivalence of categories $\mathsf{hFun}(A, B) \xrightarrow{\sim} \mathsf{hFun}(FA, FB)$ for all $A, B \in \mathfrak{h}\mathcal{K}$.

From these properties we may attempt to define an inverse biequivalence G as follows:

- For each $C \in \mathfrak{h}\mathcal{L}$, we choose an $A \in \mathfrak{h}\mathcal{K}$ together with a specified equivalence $\epsilon\colon FA \simeq C$ and define $GC := A$.
- For each pair $C, D \in \mathfrak{h}\mathcal{L}$, we define the action of G on hom-categories to be the composite

$$G_{C,D} := \mathsf{hFun}(C, D) \xrightarrow[\sim]{(-\circ\epsilon, \epsilon^{-1}\circ-)} \mathsf{hFun}(FGC, FGD) \xrightarrow[\sim]{{F_{GC,GD}}^{-1}} \mathsf{hFun}(GC, GD)$$

of the equivalence defined by pre- and post-composing with the maps of the specified equivalences $FGC \simeq C$ and $FGD \simeq D$ together with an inverse of the equivalence defined by the action of F.

These choices are suitably unique: the action on objects is well-defined up to equivalence and the action on hom-categories is well-defined up to natural isomorphism. However, these mappings do not assemble into a 2-functor (see Definition B.2.1): for instance, while the triangle on the left commutes on the nose, the composite triangle on the right only commutes up to isomorphism:

$$\begin{tikzcd} & \mathbb{1} \arrow[dl, "\mathrm{id}_A"'] \arrow[dr, "\mathrm{id}_{FA}"] & \\ \mathsf{hFun}(A,A) \arrow[rr, "F_{A,A}"'] & & \mathsf{hFun}(FA,FA) \end{tikzcd} \qquad \begin{tikzcd} & \mathbb{1} \arrow[dl, "\mathrm{id}_C"'] \arrow[d, "\mathrm{id}_{FGC}"] \arrow[dr, "\mathrm{id}_{GC}"] & \\ \mathsf{hFun}(C,C) \arrow[r, "(-\circ\epsilon,\epsilon^{-1}\circ-)"'] \arrow[ur, phantom, "\cong"] & \mathsf{hFun}(FGC,FGC) \arrow[r, "F^{-1}_{GC,GC}"'] \arrow[ur, phantom, "\cong"] & \mathsf{hFun}(GC,GC) \end{tikzcd}$$

Instead, the mapping $G\colon \mathfrak{h}\mathcal{L} \rightsquigarrow \mathfrak{h}\mathcal{K}$ defines a *pseudofunctor* between the homotopy 2-categories, which we now define. Importantly, the inverse cannot in general be chosen to be a 2-functor; see [**56**, 3.1].

10.4.1. Definition. A **pseudofunctor** $G\colon \mathcal{C} \rightsquigarrow \mathcal{D}$ between 2-categories is given by:

- a mapping on objects $\mathcal{C} \ni x \mapsto Gx \in \mathcal{D}$;
- a mapping on hom-categories $G_{x,y}\colon \mathcal{C}(x,y) \to \mathcal{D}(Gx,Gy)$ for each $x,y \in \mathcal{C}$;
- an invertible 2-cell

$$\begin{tikzcd} & \mathbb{1} \arrow[dl, "\mathrm{id}_x"'] \arrow[dr, "\mathrm{id}_{Gx}"] & \\ \mathcal{C}(x,x) \arrow[rr, "G_{x,x}"'] \arrow[rr, phantom, bend left=20, "\iota_x\Downarrow\cong"] & & \mathcal{D}(Gx,Gx) \end{tikzcd}$$

  for each $x \in \mathcal{C}$, defining an isomorphism $\iota_x\colon \mathrm{id}_{Gx} \cong G\,\mathrm{id}_X$ in $\mathcal{D}(Gx,Gx)$; and
- an natural isomorphism

$$\begin{tikzcd} \mathcal{C}(y,z)\times\mathcal{C}(x,y) \arrow[r, "G\times G"] \arrow[d, "\circ"'] \arrow[dr, phantom, "\alpha^{x,y,z}\Downarrow\cong"] & \mathcal{D}(Gy,Gz)\times\mathcal{D}(Gx,Gy) \arrow[d, "\circ"] \\ \mathcal{C}(x,z) \arrow[r, "G"'] & \mathcal{D}(Gx,Gz) \end{tikzcd}$$

  for each triple of objects $x,y,z \in \mathcal{C}$

so that three pasting diagrams commute:

$$\begin{tikzcd} \mathcal{C}(y,z)\times\mathcal{C}(x,y)\times\mathcal{C}(w,x) \arrow[r, "G\times G\times G"] \arrow[d, "\circ"'] \arrow[dr, phantom, "\alpha^{x,y,z}\times\mathrm{id}\Downarrow\cong"] & \mathcal{D}(Gy,Gz)\times\mathcal{D}(Gx,Gy)\times\mathcal{D}(Gw,Gx) \arrow[d, "\circ"] \\ \mathcal{C}(x,z)\times\mathcal{C}(w,x) \arrow[r, "G\times G"] \arrow[d, "\circ"'] \arrow[dr, phantom, "\alpha^{w,x,z}\Downarrow\cong"] & \mathcal{D}(Gx,Gz)\times\mathcal{D}(Gw,Gx) \arrow[d, "\circ"] \\ \mathcal{C}(w,z) \arrow[r, "G"'] & \mathcal{D}(Gw,Gz) \end{tikzcd}$$

$$=$$

$$\begin{tikzcd} \mathcal{C}(y,z)\times\mathcal{C}(x,y)\times\mathcal{C}(w,x) \arrow[r, "G\times G\times G"] \arrow[d, "\circ"'] \arrow[dr, phantom, "\mathrm{id}\times\alpha^{w,x,y}\Downarrow\cong"] & \mathcal{D}(Gy,Gz)\times\mathcal{D}(Gx,Gy)\times\mathcal{D}(Gw,Gx) \arrow[d, "\circ"] \\ \mathcal{C}(y,z)\times\mathcal{C}(w,y) \arrow[r, "G\times G"] \arrow[d, "\circ"'] \arrow[dr, phantom, "\alpha^{w,y,z}\Downarrow\cong"] & \mathcal{D}(Gy,Gz)\times\mathcal{D}(Gw,Gy) \arrow[d, "\circ"] \\ \mathcal{C}(w,z) \arrow[r, "G"'] & \mathcal{D}(Gw,Gz) \end{tikzcd}$$

$$
\begin{tikzcd}
\mathcal{C}(x,y) \arrow[r, "G"] \arrow[d, "\mathrm{id}_y \times \mathrm{id}"'] \arrow[dr, phantom, "\iota_y\times\mathrm{id}\Downarrow\cong"] & \mathcal{D}(Gx,Gy) \arrow[d, "\mathrm{id}_{Gy}\times\mathrm{id}"] \\
\mathcal{C}(x,y)\times\mathcal{C}(x,x) \arrow[r, "G\times G"] \arrow[d, "\circ"'] \arrow[dr, phantom, "\alpha^{x,x,y}\Downarrow\cong"] & \mathcal{D}(Gx,Gy)\times\mathcal{D}(Gx,Gx) \arrow[d, "\circ"] \\
\mathcal{C}(x,y) \arrow[r, "G"'] & \mathcal{D}(Gx,Gy)
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{C}(x,y) \arrow[r, "G"] \arrow[d, "\mathrm{id}\times\mathrm{id}_x"'] \arrow[dr, phantom, "\mathrm{id}\times\iota_x\Downarrow\cong"] & \mathcal{D}(Gx,Gy) \arrow[d, "\mathrm{id}\times\mathrm{id}_{Gx}"] \\
\mathcal{C}(y,y)\times\mathcal{C}(x,y) \arrow[r, "G\times G"] \arrow[d, "\circ"'] \arrow[dr, phantom, "\alpha^{x,y,y}\Downarrow\cong"] & \mathcal{D}(Gy,Gy)\times\mathcal{D}(Gx,Gy) \arrow[d, "\circ"] \\
\mathcal{C}(x,y) \arrow[r, "G"'] & \mathcal{D}(Gx,Gy)
\end{tikzcd}
$$

where both of these latter composites equal the unit 2-cell $\mathrm{id}_{G_{x,y}}$.

There is a closely related notion of pseudonatural transformation between pseudofunctors.

10.4.2. Definition. For any 2-categories $\mathcal{C}$ and $\mathcal{D}$ and pseudofunctors $F, G\colon \mathcal{C} \rightsquigarrow \mathcal{D}$, a **pseudonatural transformation** $\phi\colon F \Rightarrow G$ is given by:

- a 1-cell $\phi_x\colon Fx \to Gx \in \mathcal{D}$ for every object $x \in \mathcal{C}$ and
- an invertible 2-cell in $\mathcal{D}$

$$
\begin{tikzcd}
Fx \arrow[r, "Ff"] \arrow[d, "\phi_x"'] \arrow[dr, phantom, "\phi_f\Downarrow\cong"] & Fy \arrow[d, "\phi_y"] \\
Gx \arrow[r, "Gf"'] & Gy
\end{tikzcd}
$$

  for each 1-cell $f\colon x \to y \in \mathcal{C}$

so that this data

- is natural, in the sense that for each 2-cell $\begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\gamma"] & y \end{tikzcd}$ in $\mathcal{C}$ the pasted composites are equal

$$
\begin{tikzcd}
Fx \arrow[r, bend left, "Ff"] \arrow[r, bend right, "Fg"'] \arrow[r, phantom, "\Downarrow F\gamma"] \arrow[d, "\phi_x"'] \arrow[dr, phantom, "\phi_g\Downarrow\cong"] & Fy \arrow[d, "\phi_y"] \\
Gx \arrow[r, bend right, "Gg"'] & Gy
\end{tikzcd}
=
\begin{tikzcd}
Fx \arrow[r, bend left, "Ff"] \arrow[d, "\phi_x"'] \arrow[dr, phantom, "\phi_f\Downarrow\cong"] & Fy \arrow[d, "\phi_y"] \\
Gx \arrow[r, bend left, "Gf"] \arrow[r, bend right, "Gg"'] \arrow[r, phantom, "\Downarrow G\gamma"] & Gy
\end{tikzcd}
$$

  and
- respects the composition and unit constraints specified by the pseudofunctors

$$
\begin{tikzcd}
& Fy \arrow[dr, "Fk"] \arrow[d, phantom, "\alpha^{x,y,z}_{f,g}\Downarrow\cong"] & \\
Fx \arrow[ur, "Ff"] \arrow[rr, bend right, "F(kf)"'] \arrow[d, "\phi_x"'] & \arrow[d, phantom, "\phi_{kf}\Downarrow\cong"] & Fz \arrow[d, "\phi_z"] \\
Gx \arrow[rr, bend right, "G(kf)"'] & & Gz
\end{tikzcd}
=
\begin{tikzcd}
& Fy \arrow[dr, "Fk"] \arrow[d, "\phi_y"] & \\
Fx \arrow[ur, "Ff"] \arrow[dd, "\phi_x"'] \arrow[r, phantom, "\phi_f\Downarrow\cong"] & Gy \arrow[dr, "Gk"] \arrow[d, phantom, "\alpha^{x,y,z}_{f,g}\Downarrow\cong"] \arrow[r, phantom, "\phi_k\Downarrow\cong"] & Fz \arrow[d, "\phi_z"] \\
& & Gz \\
Gx \arrow[uur, "Gf"] \arrow[urr, bend right, "G(kf)"'] & &
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
Fx \arrow[r, bend left, "\mathrm{id}_{Fx}"] \arrow[r, bend right, "F\,\mathrm{id}_x"'] \arrow[r, phantom, "\iota_x\Downarrow\cong"] \arrow[d, "\phi_x"'] \arrow[dr, phantom, "\phi_{\mathrm{id}_x}\Downarrow\cong"] & Fx \arrow[d, "\phi_x"] \\
Gx \arrow[r, bend right, "G\,\mathrm{id}_x"'] & Gx
\end{tikzcd}
=
\begin{tikzcd}
Fx \arrow[r, bend left, "\mathrm{id}_{Fx}"] \arrow[d, "\phi_x"'] & Fx \arrow[d, "\phi_x"] \\
Gx \arrow[r, bend left, "\mathrm{id}_{Gx}"] \arrow[r, bend right, "G\,\mathrm{id}_x"'] \arrow[r, phantom, "\iota_y\Downarrow\cong"] & Gx
\end{tikzcd}
$$

As suggested above, one instance in which pseudofunctors naturally arise is as inverses to 2-functors that define biequivalences. The pseudofunctors that arise in this manner are themselves **biequivalences**: surjective on objects up to equivalence and defining local equivalences on hom-categories. These functors are inverses in the sense that there exist **pseudonatural equivalences** between the composites and the identities, these being pseudonatural transformations that are componentwise equivalences; see Exercise 10.4.ii for an alternate characterization. Collectively, this data defines an equivalence of 2-categories, not in the sense of enriched category theory—cf. Definition A.3.15—but in the sense appropriate to bicategory theory:

10.4.3. Proposition. *If $F\colon \mathcal{C} \to \mathcal{D}$ is a 2-functor between 2-categories $\mathcal{C}$ and $\mathcal{D}$ and a biequivalence then there exists a pseudofunctor $G\colon \mathcal{D} \rightsquigarrow \mathcal{C}$, that is also a biequivalence, and is a pseudoinverse to F in the sense that there exist pseudonatural equivalences $\mathrm{id}_{\mathcal{C}} \Rightarrow GF$ and $FG \Rightarrow \mathrm{id}_{\mathcal{D}}$.*

Proof. Exercise 10.4.iii. □

Proposition 10.4.3 describes a classical result in bicategory theory, so we feel content to leave its proof to the exercises; a proof in a slightly more general context also appears in [**44**, 7.4.1], where this result is called "the bicategorical Whitehead theorem." We shall also shortly prove a generalization subsuming this classical statement. Recall from Definition 1.2.1 that an ∞-cosmos is, among other things, a category enriched in the cartesian closed category of quasi-categories, in the sense explicated in §A.1–A.2. By Definition 1.4.1 and Proposition 1.4.5, the cartesian closed category of quasi-categories underlies a cartesian closed *2-category* of quasi-categories $\mathfrak{h}\mathcal{QC}at$. While we have been referring to $\mathfrak{h}\mathcal{QC}at$ as the *homotopy 2-category of quasi-categories*, it really is just the natural 2-categorical enrichment: its objects are quasi-categories, its 1-cells are simplicial functors between them, and its 2-cells deserve to be called "natural transformations" between quasi-categories.

The extra dimension in the 2-category $\mathfrak{h}\mathcal{QC}at$ enables us to define *quasi-categorically enriched pseudofunctors* as follows:

10.4.4. Definition. For quasi-categorically enriched categories $\mathcal{K}$ and $\mathcal{L}$, a **quasi-categorically enriched pseudofunctor** — a **quasi-pseudofunctor** for short — $G\colon \mathcal{K} \rightsquigarrow \mathcal{L}$ is given by:

- a mapping on objects $\mathcal{K} \ni x \mapsto Gx \in \mathcal{L}$;
- a simplicial map of hom quasi-categories $G_{x,y}\colon \mathcal{K}(x,y) \to \mathcal{L}(Gx,Gy)$ for each $x,y \in \mathcal{K}$;
- an invertible 2-cell

$$
\begin{tikzcd}
& \mathbb{1} \arrow[dl, "\mathrm{id}_x"'] \arrow[dr, "\mathrm{id}_{Gx}"] \arrow[d, phantom, "\iota_x\Downarrow\cong"] & \\
\mathcal{K}(x,x) \arrow[rr, "G_{x,x}"'] & & \mathcal{L}(Gx,Gx)
\end{tikzcd}
$$

in the homotopy 2-category of quasi-categories for each $x \in \mathcal{K}$;

- an invertible 2-cell

$$\begin{tikzcd}
\mathcal{K}(y,z)\times\mathcal{K}(x,y) \arrow[r, "G\times G"] \arrow[d, "\circ"'] \arrow[dr, phantom, "\alpha^{x,y,z}\Downarrow\cong"] & \mathcal{L}(Gy,Gz)\times\mathcal{L}(Gx,Gy) \arrow[d, "\circ"] \\
\mathcal{K}(x,z) \arrow[r, "G"'] & \mathcal{L}(Gx,Gz)
\end{tikzcd}$$

in the homotopy 2-category of quasi-categories for each triple of objects $x,y,z\in\mathcal{K}$ so that three pasting diagrams commute:

$$\begin{tikzcd}
\mathcal{K}(y,z)\times\mathcal{K}(x,y)\times\mathcal{K}(w,x) \arrow[r, "G\times G\times G"] \arrow[d, "\circ"'] \arrow[dr, phantom, "\alpha^{x,y,z}\times\mathrm{id}\Downarrow\cong"] & \mathcal{L}(Gy,Gz)\times\mathcal{L}(Gx,Gy)\times\mathcal{L}(Gw,Gx) \arrow[d, "\circ"] \\
\mathcal{K}(x,z)\times\mathcal{K}(w,x) \arrow[r, "G\times G"] \arrow[d, "\circ"'] \arrow[dr, phantom, "\alpha^{w,x,z}\Downarrow\cong"] & \mathcal{L}(Gx,Gz)\times\mathcal{L}(Gw,Gx) \arrow[d, "\circ"] \\
\mathcal{K}(w,z) \arrow[r, "G"'] & \mathcal{L}(Gw,Gz)
\end{tikzcd}$$

$$= \tag{10.4.5}$$

$$\begin{tikzcd}
\mathcal{K}(y,z)\times\mathcal{K}(x,y)\times\mathcal{K}(w,x) \arrow[r, "G\times G\times G"] \arrow[d, "\circ"'] \arrow[dr, phantom, "\mathrm{id}\times\alpha^{w,x,y}\Downarrow\cong"] & \mathcal{L}(Gy,Gz)\times\mathcal{L}(Gx,Gy)\times\mathcal{L}(Gw,Gx) \arrow[d, "\circ"] \\
\mathcal{K}(y,z)\times\mathcal{K}(w,y) \arrow[r, "G\times G"] \arrow[d, "\circ"'] \arrow[dr, phantom, "\alpha^{w,y,z}\Downarrow\cong"] & \mathcal{L}(Gy,Gz)\times\mathcal{L}(Gw,Gy) \arrow[d, "\circ"] \\
\mathcal{K}(w,z) \arrow[r, "G"'] & \mathcal{L}(Gw,Gz)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{K}(x,y) \arrow[r, "G"] \arrow[d, "\mathrm{id}_y\times\mathrm{id}"'] \arrow[dr, phantom, "\iota_y\times\mathrm{id}\Downarrow\cong"] & \mathcal{L}(Gx,Gy) \arrow[d, "\mathrm{id}_{Gy}\times\mathrm{id}"] \\
\mathcal{K}(x,y)\times\mathcal{K}(x,x) \arrow[r, "G\times G"] \arrow[d, "\circ"'] \arrow[dr, phantom, "\alpha^{x,x,y}\Downarrow\cong"] & \mathcal{L}(Gx,Gy)\times\mathcal{L}(Gx,Gx) \arrow[d, "\circ"] \\
\mathcal{K}(x,y) \arrow[r, "G"'] & \mathcal{L}(Gx,Gy)
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{K}(x,y) \arrow[r, "G"] \arrow[d, "\mathrm{id}\times\mathrm{id}_x"'] \arrow[dr, phantom, "\mathrm{id}\times\iota_x\Downarrow\cong"] & \mathcal{L}(Gx,Gy) \arrow[d, "\mathrm{id}\times\mathrm{id}_{Gx}"] \\
\mathcal{K}(y,y)\times\mathcal{K}(x,y) \arrow[r, "G\times G"] \arrow[d, "\circ"'] \arrow[dr, phantom, "\alpha^{x,y,y}\Downarrow\cong"] & \mathcal{L}(Gy,Gy)\times\mathcal{L}(Gx,Gy) \arrow[d, "\circ"] \\
\mathcal{K}(x,y) \arrow[r, "G"'] & \mathcal{L}(Gx,Gy)
\end{tikzcd}$$

where both of these latter composites equal the unit 2-cell $\mathrm{id}_{G_{x,y}}$.

10.4.6. Remark. For any pair of objects $a,b\in\mathcal{L}$ in a quasi-categorically enriched category,

$$\mathsf{Fun}(\mathbb{1},\mathcal{L}(a,b))\cong\mathcal{L}(a,b),$$

and so $\mathsf{hFun}(\mathbb{1},\mathcal{L}(a,b))\cong\mathsf{h}\mathcal{L}(a,b)$. Thus 2-cells in the homotopy 2-category of quasi-categories $\mathfrak{h}\mathcal{QC}at$ with domain $\mathbb{1}$ and codomain $\mathcal{L}(a,b)$ correspond to 2-cells in the homotopy 2-category of $\mathcal{L}$ — defined exactly as in Definition 1.4.1 — from a to b.

In particular, the data of the invertible 2-cell ι_x is no more and no less than an invertible 2-cell $\iota_x\colon \mathrm{id}_{Gx}\cong G\,\mathrm{id}_X$ in the homotopy 2-category of $\mathcal{L}$.

10.4.7. Definition. A quasi-pseudofunctor $G\colon\mathcal{K}\rightsquigarrow\mathcal{L}$ whose codomain $\mathcal{L}$ is an ∞-cosmos is a **biequivalence** when it is:

(i) surjective on objects up to equivalence: if for all $a \in \mathcal{L}$ there exists $x \in \mathcal{K}$ so that $Fx \simeq a$; and
(ii) a local equivalence of quasi-categories: if for every pair $x, y \in \mathcal{K}$, the map

$$\mathcal{K}(x,y) \xrightarrow[\sim]{G_{x,y}} \mathcal{L}(Gx, Gy)$$

is an equivalence of quasi-categories.

10.4.8. Remark. We find it convenient to assume that $\mathcal{L}$ is an ∞-cosmos in Definition 10.4.7 because that provides us access to the equivalent characterizations of the equivalences in $\mathcal{L}$ given by Theorem 1.4.7. In what follows we will ask that an equivalence $a \simeq b$ in $\mathcal{L}$

- defines an equivalence in the homotopy 2-category of $\mathcal{L}$ and
- induces an equivalence of quasi-categories $\mathcal{L}(x,a) \xrightarrow{\sim} \mathcal{L}(x,b)$ in the homotopy 2-category $\mathfrak{h}\mathcal{QC}at$ that is 2-natural in x.

The latter of these properties implies the former, so if we required a notion of quasi-pseudofunctorial biequivalence between general quasi-categorically enriched categories, we could use this notion of equivalence in Definition 10.4.7. But we will make no use of the concept outside of the context provided by ∞-cosmoi and so prefer the simpler terminology. Note that we permit the domain $\mathcal{K}$ to be merely quasi-categorically enriched.

Similarly:

10.4.9. Definition. For quasi-categorically enriched categories $\mathcal{K}$ and $\mathcal{L}$ and quasi-pseudofunctors $F, G \colon \mathcal{K} \rightsquigarrow \mathcal{L}$, a **quasi-categorically enriched pseudonatural transformation** — a **quasi-pseudonatural transformation** for short — $\phi \colon F \Rightarrow G$ is given by:

- a 0-arrow $\phi_x \colon Fx \to Gx \in \mathcal{L}$ for every object $x \in \mathcal{K}$ and
- an invertible 2-cell

$$\begin{tikzcd} \mathcal{K}(x,y) \arrow[r, "F_{x,y}"] \arrow[d, "G_{x,y}"'] \arrow[dr, phantom, "\phi^{x,y}\Downarrow\cong"] & \mathcal{L}(Fx,Fy) \arrow[d, "\phi_y\circ-"] \\ \mathcal{L}(Gx,Gy) \arrow[r, "-\circ\phi_x"'] & \mathcal{L}(Fx,Gy) \end{tikzcd}$$

in the homotopy 2-category of quasi-categories, for each pair of objects $x, y \in \mathcal{K}$

so that this data respects the composition and unit constraints specified by the quasi-pseudofunctors, as expressed by the following two pasting diagrams

$$\begin{tikzcd} \mathcal{K}(y,z)\times\mathcal{K}(x,y) \arrow[r, "F\times F"] \arrow[d, "\circ"'] \arrow[dr, phantom, "\alpha^{x,y,z}\Downarrow\cong"] & \mathcal{L}(Fy,Fz)\times\mathcal{L}(Fx,Fy) \arrow[d, "\circ"] \\ \mathcal{K}(x,z) \arrow[r, "F"] \arrow[d, "G"'] \arrow[dr, phantom, "\phi^{x,z}\Downarrow\cong"] & \mathcal{L}(Fx,Fz) \arrow[d, "\phi_z\circ-"] \\ \mathcal{L}(Gx,Gz) \arrow[r, "-\circ\phi_x"'] & \mathcal{L}(Fx,Gz) \end{tikzcd} \quad =$$

$$\begin{tikzcd}
 & \mathcal{L}(Fy,Fz)\times\mathcal{L}(Fx,Fy) \arrow[dr, "(\phi_z\circ -)\times\mathrm{id}"] \arrow[d, phantom, "\phi^{y,z}\times\mathrm{id}\Downarrow\cong"] & \\
\mathcal{K}(y,z)\times\mathcal{K}(x,y) \arrow[ur, "F\times F"] \arrow[r, dashed, "G\times F"] \arrow[dr, "G\times G"] \arrow[d, "\circ"'] & \mathcal{L}(Gy,Gz)\times\mathcal{L}(Fx,Fy) \arrow[r, "(-\circ\phi_y)\times\mathrm{id}"] \arrow[d, phantom, "\mathrm{id}\times\phi^{x,y}\Downarrow\cong"] & \mathcal{L}(Fy,Gz)\times\mathcal{L}(Fx,Fy) \arrow[d, "\mathrm{id}\times(\phi_y\circ -)"'] \arrow[dd, bend left=60, "\circ"] \\
\mathcal{K}(x,z) \arrow[r, phantom, "\alpha^{x,y,z}\Downarrow\cong"] \arrow[dr, "G"'] & \mathcal{L}(Gy,Gz)\times\mathcal{L}(Gx,Gy) \arrow[r, "\mathrm{id}\times(-\circ\phi_x)"'] \arrow[d, "\circ"] & \mathcal{L}(Gy,Gz)\times\mathcal{L}(Fx,Gy) \arrow[d, "\circ"] \\
 & \mathcal{L}(Gx,Gz) \arrow[r, "-\circ\phi_x"'] & \mathcal{L}(Fx,Gz)
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & \mathbb{1} \arrow[dl, "\mathrm{id}_x"'] \arrow[dr, "\mathrm{id}_{Fx}"] \arrow[d, phantom, "\iota_x\Downarrow\cong"] & \\
\mathcal{K}(x,x) \arrow[rr, "F_{x,x}"'] \arrow[d, "G_{x,x}"'] & \arrow[d, phantom, "\phi^{x,y}\Downarrow\cong"] & \mathcal{L}(Fx,Fx) \arrow[d, "\phi_x\circ -"] \\
\mathcal{L}(Gx,Gx) \arrow[rr, "-\circ\phi_x"'] & & \mathcal{L}(Fx,Gx)
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & \mathbb{1} \arrow[dl, "\mathrm{id}_x"'] \arrow[dr, "\mathrm{id}_{Gx}"] \arrow[d, phantom, "\iota_x\Downarrow\cong"] & \\
\mathcal{K}(x,x) \arrow[rr, "G_{x,x}"'] \arrow[d, "G_{x,x}"'] & & \mathcal{L}(Gx,Gx) \arrow[d, "-\circ\phi_x"] \\
\mathcal{L}(Gx,Gx) \arrow[rr, "-\circ\phi_x"'] & & \mathcal{L}(Fx,Gx)
\end{tikzcd}$$

10.4.10. Remark. Recall the 0-arrows in $\mathcal{K}(x,y)$ correspond to simplicial maps $f\colon \mathbb{1}\to\mathcal{K}(x,y)$. By the argument of Remark 10.4.6, the restriction $\phi^{x,y}f$ defines the component

$$\begin{tikzcd}
Fx \arrow[r, "Ff"] \arrow[d, "\phi_x"'] \arrow[dr, phantom, "\phi_f\Downarrow\cong"] & Fy \arrow[d, "\phi_y"] \\
Gx \arrow[r, "Gf"'] & Gy
\end{tikzcd}$$

of an invertible 2-cell in the homotopy 2-category of $\mathfrak{h}\mathcal{L}$. This 2-cell is automatically "natural," in the sense that for each 2-cell $\begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\gamma"] & y \end{tikzcd}$ in the homotopy 2-category of $\mathcal{K}$ the pasted composites are equal

$$\begin{tikzcd}
Fx \arrow[r, bend left, "Ff"] \arrow[r, bend right, "Fg"'] \arrow[r, phantom, "\Downarrow F\gamma"] \arrow[d, "\phi_x"'] \arrow[dr, phantom, "\phi_g\Downarrow\cong"] & Fy \arrow[d, "\phi_y"] \\
Gx \arrow[r, bend right, "Gg"'] & Gy
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
Fx \arrow[r, bend left, "Ff"] \arrow[d, "\phi_x"'] \arrow[dr, phantom, "\phi_f\Downarrow\cong"] & Fy \arrow[d, "\phi_y"] \\
Gx \arrow[r, bend left, "Gf"] \arrow[r, bend right, "Gg"'] \arrow[r, phantom, "\Downarrow G\gamma"] & Gy
\end{tikzcd}$$

in the homotopy 2-category of $\mathcal{L}$, simply because both pasting composites are represented by the horizontal composite

$$\begin{tikzcd}
\mathbb{1} \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\gamma"] & \mathcal{K}(x,y) \arrow[r, bend left, "(\phi_y\circ -)\circ F_{x,y}"] \arrow[r, bend right, "(-\circ\phi_x)\circ G_{x,y}"'] \arrow[r, phantom, "\phi^{x,y}\Downarrow\cong"] & \mathcal{L}(Fx,Gy)
\end{tikzcd}$$

in the homotopy 2-category of quasi-categories.

Similarly, the composition and unit diagrams of Definition 10.4.9 imply that the corresponding diagrams displayed in Definition 10.4.2 commute in the homotopy 2-category of $\mathcal{L}$; see Exercise 10.4.iv.

10.4.11. LEMMA. *The action on homs of a quasi-pseudofunctor $F\colon \mathcal{K} \rightsquigarrow \mathcal{L}$ between quasi-categorically enriched categories defines a quasi-pseudonatural transformation*

$$\mathcal{K}(-,-) \xrightarrow{F_{-,-}} \mathcal{L}(F-,F-)$$

between simplicial functors $\mathcal{K}^{\mathrm{op}} \times \mathcal{K} \to \mathcal{QC}at$.

PROOF. The 2-cell component of the quasi-pseudonatural transformation associated to a pair of objects (x,y) and (w,z) in $\mathcal{K}^{\mathrm{op}} \times \mathcal{K}$ is given by

$$\begin{tikzcd} \mathcal{K}(y,z) \times \mathcal{K}(w,x) \arrow[r, "\circ"] \arrow[d, "F"'] \arrow[dr, phantom, "\alpha^{w,x,y,z}\Downarrow\cong"] & \mathcal{K}(w,z)^{\mathcal{K}(x,y)} \arrow[d, "F_{w,z}\circ-"] \\ \mathcal{L}(Fw,Fz)^{\mathcal{L}(Fx,Fy)} \arrow[r, "-\circ F_{x,y}"'] & \mathcal{L}(Fw,Fz)^{\mathcal{K}(x,y)} \end{tikzcd}$$

where $\alpha^{w,x,y,z}$ is a transpose of the pasted composite of (10.4.5). We leave the verification of the composition and unit axioms to Exercise 10.4.v. ☐

10.4.12. DEFINITION. A quasi-pseudonatural transformation $\phi\colon F \Rightarrow G$ between quasi-pseudofunctors $F,G\colon \mathcal{K} \rightsquigarrow \mathcal{L}$ whose codomain is an ∞-cosmos is a **quasi-pseudonatural equivalence** if each of its components $\phi_x\colon Fx \to Gx$ defines an equivalence in the homotopy 2-category of $\mathcal{L}$.

Again, as noted in Remark 10.4.8, we find it convenient to assume that $\mathcal{L}$ is an ∞-cosmos so we need not be more explicit about the appropriate notion of equivalence in the target category. Our interest in the class of quasi-pseudonatural equivalences stems from the following result, which can be understood as a version of the bicategorical Yoneda lemma in the context of quasi-categorically enriched categories, quasi-pseudofunctors, quasi-pseudonatural transformations, and the "modifications" between them.

10.4.13. LEMMA. *If there exists a quasi-pseudonatural equivalence*

$$\phi_x\colon \mathcal{K}(x,a) \xrightarrow{\sim} \mathcal{K}(x,b)$$

between the simplicial functors $\mathcal{K}^{\mathrm{op}} \to \mathcal{QC}at$ represented by a pair of objects a,b in an ∞-cosmos $\mathcal{K}$, then a and b are equivalent in $\mathcal{K}$.

PROOF. We will show that the 0-arrow $\phi_a(\mathrm{id}_a)\colon a \to b$ is an equivalence in $\mathcal{K}$. First observe from the diagram

$$\begin{tikzcd} \mathcal{K}(x,a) \arrow[r, "\circ"] \arrow[d, "\circ"'] \arrow[dr, phantom, "\phi^{x,a}\Downarrow\cong"] & \mathcal{K}(x,a)^{\mathcal{K}(a,a)} \arrow[d, "\phi_x\circ-"] \arrow[dr, "\mathrm{ev}_{\mathrm{id}_a}"] & \\ & & \mathcal{K}(x,a) \arrow[dd, "\phi_x"] \\ \mathcal{K}(x,b)^{\mathcal{K}(a,b)} \arrow[r, "-\circ\phi_a"'] \arrow[dr, "\mathrm{ev}_\phi"'] & \mathcal{K}(x,b)^{\mathcal{K}(a,a)} \arrow[dr, "\mathrm{ev}_{\mathrm{id}_a}"] & \\ & \mathcal{K}(x,b) \arrow[r, equal] & \mathcal{K}(x,b) \end{tikzcd}$$

that for every $x \in \mathcal{K}$ the component $\phi_x \colon \mathcal{K}(x,a) \xrightarrow{\sim} \mathcal{K}(x,b)$ — the top-right composite in the above diagram — is isomorphic in $\mathcal{QC}at_2$ to the postcomposition map $\phi \circ -$ — the lower-left composite in the above diagram. In particular, the map $\phi \circ - \colon \mathcal{K}(b,a) \xrightarrow{\sim} \mathcal{K}(b,b)$ is an equivalence, so we may choose a 0-arrow $\psi \colon b \to a$ so that $\beta \colon \phi\psi \cong \mathrm{id}_b$ in the homotopy 2-category $\mathfrak{h}\mathcal{K}$. Now id_a and $\psi\phi$ define a pair of 0-arrows in $\mathcal{K}(a,a)$ whose images under the equivalences $\phi \circ - \colon \mathcal{K}(a,a) \xrightarrow{\sim} \mathcal{K}(a,b)$ are isomorphic via $\beta\phi \colon \phi\psi\phi \cong \phi$. Hence, there exists an isomorphism $\alpha \colon \mathrm{id}_a \cong \psi\phi$ in the homotopy 2-category $\mathfrak{h}\mathcal{K}$, completing the proof that $a \simeq b$ in $\mathcal{K}$. □

Quasi-pseudonatural equivalences may be constructed as adjoint equivalence inverses of simplicial natural transformations that define componentwise equivalences.

10.4.14. Lemma. *Consider a simplicial natural transformation* $\begin{tikzcd} \mathcal{K} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\phi"] & \mathcal{L} \end{tikzcd}$ *between ∞-cosmoi whose 0-arrow components $\phi_x \colon Fx \xrightarrow{\sim} Gx$ all define equivalences in $\mathcal{L}$. Then any choice of adjoint equivalence inverses $\psi_x \colon Gx \xrightarrow{\sim} Fx$ assemble into the components of a quasi-pseudonatural transformation $\psi \colon G \Rightarrow F$.*

Proof. The components of the quasi-pseudonatural transformation ψ are defined by the adjoint equivalence inverse arrows $\psi_x \colon Gx \xrightarrow{\sim} Fx$ and by the pasted composite natural transformation

$$\psi^{x,y} := \quad \begin{tikzcd} \mathcal{K}(x,y) \arrow[r, "G_{x,y}"] \arrow[d, "F_{x,y}"'] & \mathcal{L}(Gx,Gy) \arrow[d, "-\circ\phi_x"'] \arrow[dr, equal] \arrow[d, phantom, shift left=6ex, "\eta_x\cong"] \\ \mathcal{L}(Fx,Fy) \arrow[r, "\phi_y\circ -"] \arrow[dr, equal] \arrow[r, phantom, shift right=3ex, "\epsilon_y\cong"] & \mathcal{L}(Fx,Gy) \arrow[r, "-\circ\psi_x"'] \arrow[d, "\psi_y\circ -"] & \mathcal{L}(Gx,Fy) \arrow[d, "\psi_y\circ -"] \\ & \mathcal{L}(Fx,Fy) \arrow[r, "-\circ\psi_x"'] & \mathcal{L}(Gx,Fy) \end{tikzcd}$$

of the unit and counit isomorphisms of the adjoint equivalence.

Since F and G are simplicial functors, the unit condition simplifies to ask only that the component of this pasted natural transformation at the identity arrow $\mathrm{id}_x \colon \mathbb{1} \to \mathcal{K}(x,x)$ is an identity 2-cell id_{ψ_x}. This component is the pasted composite

$$\begin{tikzcd} Gx \arrow[rr, equal] \arrow[dr, "\psi_x"'] & \arrow[d, phantom, "\eta\cong"] & Gx \arrow[dr, "\psi_x"] & \arrow[d, phantom, "\epsilon\cong"] \\ & Fx \arrow[ur, "\phi_x"] \arrow[rr, equal] & & Fx \end{tikzcd}$$

which is indeed the identity, since the specified data defines an adjoint equivalence.

Similarly, to verify the composition axiom, we must show that the composite

$$\begin{tikzcd}
\mathcal{K}(y,z)\times\mathcal{K}(x,y) \arrow[dr, "\circ"] & & & \\
& \mathcal{K}(x,z) \arrow[r, "G_{x,z}"] \arrow[d, "F_{x,z}"'] & \mathcal{L}(Gx,Gz) \arrow[d, "-\circ\phi_x"'] \arrow[dr, equal] \arrow[dr, phantom, "\eta_x\cong"'] & \\
& \mathcal{L}(Fx,Fz) \arrow[r, "\phi_z\circ-"] \arrow[dr, equal] \arrow[dr, phantom, "\epsilon_z\cong"] & \mathcal{L}(Fx,Gz) \arrow[r, "-\circ\psi_x"'] \arrow[d, "\psi_z\circ-"] & \mathcal{L}(Gx,Fz) \arrow[d, "\psi_z\circ-"] \\
& & \mathcal{L}(Fx,Fz) \arrow[r, "-\circ\psi_x"'] & \mathcal{L}(Gx,Fz)
\end{tikzcd}$$

equals

$$\begin{tikzcd}
\mathcal{K}(y,z)\times\mathcal{K}(x,y) \arrow[r, "G\times G"] \arrow[dr, "F\times G"'] \arrow[dd, "F\times F"'] & \mathcal{L}(Gy,Gz)\times\mathcal{L}(Gx,Gy) \arrow[dr, "-\circ\phi_y"'] \arrow[drr, equal] \arrow[drr, phantom, "\eta_y\cong"] & & \\
& \mathcal{L}(Fy,Fz)\times\mathcal{L}(Gx,Gy) \arrow[r, "\phi_z\circ-"] \arrow[dd, "-\circ\phi_x"] \arrow[ddr, equal] \arrow[ddr, phantom, "\epsilon_z\cong"] & \mathcal{L}(Fy,Gz)\times\mathcal{L}(Gx,Gy) \arrow[r, "-\circ\psi_y"'] \arrow[dd, "\psi_z\circ-"] & \mathcal{L}(Gy,Gz)\times\mathcal{L}(Gx,Gy) \arrow[dd, "\psi_z\circ-"] \\
\mathcal{L}(Fy,Fz)\times\mathcal{L}(Fx,Fy) \arrow[dr, "\phi_y\circ-"'] \arrow[ddr, equal, bend right] & \arrow[r, phantom, "\eta_x\cong"] & & \\
& \mathcal{L}(Fy,Fz)\times\mathcal{L}(Fx,Gy) \arrow[r, "-\circ\psi_x"'] \arrow[d, "\psi_y\circ-"] \arrow[l, phantom, "\epsilon_y\cong"] & \mathcal{L}(Fy,Fz)\times\mathcal{L}(Gx,Gz) \arrow[r, "-\circ\psi_y"] \arrow[d, "\psi_y\circ-"] & \mathcal{L}(Gy,Fz)\times\mathcal{L}(Gx,Gy) \arrow[d, "\circ"] \\
& \mathcal{L}(Fy,Fz)\times\mathcal{L}(Fx,Fy) \arrow[r, "-\circ\psi_x"] & \mathcal{L}(Fy,Fz)\times\mathcal{L}(Gx,Fy) \arrow[r, "\circ"'] & \mathcal{L}(Gx,Fz)
\end{tikzcd}$$

The pre- and post-composition maps appearing in this diagram are 2-natural, so for instance the whiskered composite of ϵ_y and $-\circ\psi_x$ can be formed in either order. Using this commutativity property repeatedly to the second pasting diagram and applying the triangle identity $\phi_y\epsilon_y \cdot \eta_y\phi_y = \mathrm{id}_{\phi_y}$, the second pasting diagram reduces to the first one. ☐

We now show that any biequivalence $F\colon \mathcal{K} \xrightarrow{\sim} \mathcal{L}$ of quasi-categorically enriched categories admits a quasi-pseudofunctorial "inverse" $G\colon \mathcal{L} \rightsquigarrow \mathcal{K}$ equipped with quasi-pseudonatural equivalences $\eta\colon \mathrm{id}_{\mathcal{K}} \Rightarrow GF$ and $\epsilon\colon FG \Rightarrow \mathrm{id}_{\mathcal{L}}$.

10.4.15. Proposition. *If $F\colon \mathcal{K} \to \mathcal{L}$ is a quasi-categorically enriched functor between ∞-cosmoi and a biequivalence then there exists a quasi-pseudofunctor $G\colon \mathcal{L} \rightsquigarrow \mathcal{K}$, that is also a biequivalence. Moreover G is a quasi-pseudoinverse to F in the sense that there exist quasi-pseudonatural equivalences $\mathrm{id}_{\mathcal{K}} \Rightarrow GF$ and $FG \Rightarrow \mathrm{id}_{\mathcal{L}}$.*

Proof. To coherently define an inverse to a biequivalence $F\colon \mathcal{K} \xrightarrow{\sim} \mathcal{L}$, we "fully specify" its data, choosing:

(β) fully specified adjoint equivalences $\epsilon_a\colon Fx \simeq a$ for each $a \in \mathcal{L}$ and
(γ) fully specified inverse adjoint equivalences of quasi-categories

$$\mathcal{K}(x,y) \xrightarrow[\sim]{F_{x,y}} \mathcal{L}(Fx,Fy)$$

for each pair $x, y \in \mathcal{K}$ whose inverse is quasi-pseudonatural in x and y.

In (γ), we apply Lemma 10.4.14 to the simplicial natural transformation $F_{-,-}\colon \mathcal{K}(-,-) \to \mathcal{L}(F-,F-)$ to observe that the pointwise adjoint equivalences to these maps assemble into a quasi-pseudonatural transformation, which is also a pointwise equivalence.

Now, to define $G\colon \mathcal{L} \rightsquigarrow \mathcal{K}$, use (β) to specify for each $a \in \mathcal{L}$ an object $Ga \in \mathcal{K}$ together with an equivalence $\epsilon_a\colon FGa \simeq a$ in $\mathcal{L}$. This defines the mapping of G on objects and the 0-arrow components of the quasi-pseudonatural transformation ϵ. To define the action of G on hom quasi-categories, use this data and (γ) to define

$$G_{a,b} \coloneqq \mathcal{L}(a,b) \xrightarrow[\sim]{(-\circ\epsilon_a,\epsilon_b^{-1}\circ-)} \mathcal{L}(FGa,FGb) \xrightarrow[\sim]{F^{-1}_{Ga,Gb}} \mathcal{K}(Ga,Gb)$$

For each $a \in \mathcal{L}$ define $\iota_a\colon \mathrm{id}_{Ga} \cong G_{a,a}\,\mathrm{id}_a$ to be the composite

$$\begin{tikzcd}
 & \mathbb{1} \arrow[dl, "\mathrm{id}_a"'] \arrow[d, dashed, "\mathrm{id}_{FGa}"] \arrow[dr, "\mathrm{id}_{Ga}"] & \\
\mathcal{L}(a,a) \arrow[r, "(-\circ\epsilon_a,\epsilon_a^{-1}\circ-)"'] \arrow[ur, phantom, "\beta\cong"] & \mathcal{L}(FGa,FGa) \arrow[r, "F^{-1}_{Ga,Ga}"'] \arrow[ur, phantom, "\gamma\cong"] & \mathcal{K}(Ga,Ga)
\end{tikzcd}$$

of the isomorphism $\beta_a\colon \epsilon_a^{-1}\circ\epsilon_a \cong \mathrm{id}_{FGa}$ in the homotopy 2-category of $\mathcal{L}$ with the component of the isomorphism $\gamma\mathrm{id}_{Ga}\colon F^{-1}_{Ga,Ga}\circ F_{Ga,Ga} \cong \mathrm{id}_{\mathcal{K}(Ga,Ga)}$ at id_{Ga}. For each $a,b,c \in \mathcal{L}$, define $\alpha^{a,b,c}$ to be the composite

$$\begin{tikzcd}
\mathcal{L}(b,c)\times\mathcal{L}(a,b) \arrow[r, "(-\circ\epsilon_b,\epsilon_c^{-1}\circ-)\times(-\circ\epsilon_a,\epsilon_b^{-1}\circ-)"] \arrow[d, "\circ"'] \arrow[dr, phantom, "\beta\cong"] & \mathcal{L}(FGb,FGc)\times\mathcal{L}(FGa,FGb) \arrow[r, "F^{-1}_{Gb,Gc}\times F^{-1}_{Ga,Gb}"] \arrow[d, "\circ"] \arrow[dr, phantom, "\gamma\cong"] & \mathcal{K}(Gb,Gc)\times\mathcal{K}(Ga,Gb) \arrow[d, "\circ"] \\
\mathcal{L}(a,c) \arrow[r, "(-\circ\epsilon_a,\epsilon_c^{-1}\circ-)"'] & \mathcal{L}(FGa,FGc) \arrow[r, "F^{-1}_{Ga,Gc}"'] & \mathcal{K}(Ga,Gc)
\end{tikzcd}$$

of the canonical natural transformations built from the data of (β) and (γ).

We next verify that these choices make G into a quasi-pseudofunctor. For the unit condition, we must verify that the composite

$$\begin{tikzcd}
\mathcal{L}(a,b) \arrow[r, "(-\circ\epsilon_a,\epsilon_b^{-1}\circ-)"] \arrow[d, "\mathrm{id}_b\times\mathrm{id}"'] \arrow[dr, phantom, "\beta\cong"] & \mathcal{L}(FGa,FGb) \arrow[r, "F^{-1}_{Ga,Gb}"] \arrow[d, dashed, "\mathrm{id}_{FGb}\times\mathrm{id}"] \arrow[dr, phantom, "\gamma\cong"] & \mathcal{K}(Ga,Gb) \arrow[d, "\mathrm{id}_{Gb}\times\mathrm{id}"] \\
\mathcal{L}(b,b)\times\mathcal{L}(a,b) \arrow[r, "(-\circ\epsilon_b,\epsilon_b^{-1}\circ-)\times(-\circ\epsilon_a,\epsilon_b^{-1}\circ-)"] \arrow[d, "\circ"'] \arrow[dr, phantom, "\beta\cong"] & \mathcal{L}(FGb,FGb)\times\mathcal{L}(FGa,FGb) \arrow[r, "F^{-1}_{Gb,Gb}\times F^{-1}_{Ga,Gb}"] \arrow[d, "\circ"] \arrow[dr, phantom, "\gamma\cong"] & \mathcal{K}(Gb,Gb)\times\mathcal{K}(Ga,Gb) \arrow[d, "\circ"] \\
\mathcal{L}(a,b) \arrow[r, "(-\circ\epsilon_a,\epsilon_b^{-1}\circ-)"'] & \mathcal{L}(FGa,FGb) \arrow[r, "F^{-1}_{Ga,Gb}"'] & \mathcal{K}(Ga,Gb)
\end{tikzcd}$$

is the identity; in fact, each pair of vertical composites is the identity. On the left-hand side, this is on account of one of the triangle equality relations for the adjoint equivalence ϵ_b. On the right-hand side, this is a consequence of quasi-pseudonaturality of the pair $F^{-1}_{Ga,Gb}$ and γ established in Lemma 10.4.14. The right unit constraint and associativity conditions are similar. This completes the proof that $G\colon \mathcal{L} \rightsquigarrow \mathcal{K}$ defines a quasi-pseudofunctor.

By construction, the quasi-pseudofunctor G is a local equivalence, its action on homs defined by composing an equivalence with a map induced by pre- and post-composing with an equivalence in the

∞-cosmos $\mathcal{L}$, which is then an equivalence by Corollary 1.4.9. We use this local equivalence to argue that for each $x \in \mathcal{K}$, there is an equivalence $\eta_x \colon x \xrightarrow{\sim} GFx$, proving essential surjectivity of G. This component is defined by applying the specified inverse adjoint equivalence $F^{-1}_{x,Gfx} \colon \mathcal{L}(Fx, FGFx) \xrightarrow{\sim} \mathcal{K}(x, GFx)$ of (γ) to the inverse of the specified adjoint equivalence $\epsilon^{-1}_{Fx} \colon Fx \to FGFx$ of (β). Since F is a cosmological biequivalence, which caries the map η_x to an equivalence in $\mathcal{L}$, it is easily verified that η_x is itself an equivalence in $\mathcal{K}$. Thus, the quasi-pseudofunctor G is an biequivalence.

It remains only to check quasi-pseudonaturality of η and ϵ. For the latter, we define the component natural isomorphism by the pasting diagram

$$\epsilon^{a,b} := \begin{tikzcd} \mathcal{L}(a,b) \arrow[r, "-\circ\epsilon_a"] \arrow[d, equal] & \mathcal{L}(FGa,b) \arrow[r, "\epsilon_b^{-1}\circ-"] \arrow[drrr, equal, "\cong\beta"] & \mathcal{L}(FGa,FGb) \arrow[r, "F^{-1}_{Ga,Gb}"] \arrow[rr, equal, bend right, "\cong\gamma"'] & \mathcal{K}(Ga,Gb) \arrow[r, "F_{Ga,Gb}"] & \mathcal{L}(FGa,FGb) \arrow[d, "\epsilon_b\circ-"] \\ \mathcal{L}(a,b) \arrow[rrrr, "-\circ\epsilon_a"'] & & & & \mathcal{L}(FGa,b) \end{tikzcd}$$

For the former, using the definition $\eta_x := F^{-1}\epsilon^{-1}_{Fx}$ and the quasi-pseudonaturality of $F^{-1}_{-,-}$, we have a pasting diagram

$$\eta^{x,y} := \begin{tikzcd} \mathcal{K}(x,y) \arrow[r, "F"] \arrow[d, equal] \arrow[dr, phantom, "\gamma\cong"] & \mathcal{L}(Fx,Fy) \arrow[r, "(-\circ\epsilon_{Fx}\text{,}\epsilon^{-1}_{Fy}\circ-)"] \arrow[dr, "\epsilon^{-1}_{Fy}\circ-"'] & \mathcal{L}(FGFx,FGFy) \arrow[r, "F^{-1}_{GFx,GFy}"] \arrow[d, "-\circ\epsilon^{-1}_{Fx}"] \arrow[dl, phantom, "\beta\cong"] \arrow[dr, phantom, "\gamma\cong"] & \mathcal{K}(GFx,GFy) \arrow[d, "-\circ\eta_x"] \\ \mathcal{K}(x,y) \arrow[rrr, bend right, "\eta_y\circ-"'] & & \mathcal{L}(Fx,FGFy) \arrow[r, "F^{-1}_{x,GFy}"] & \mathcal{K}(x,GFy) \end{tikzcd}$$

which defines component natural isomorphism. We leave the verification that these natural transformations satisfy the unit and composition coherence conditions to define quasi-pseudonatural equivalences $\eta \colon \mathrm{id}_{\mathcal{K}} \Rightarrow GF$ and $\epsilon \colon FG \Rightarrow \mathrm{id}_{\mathcal{L}}$ to the reader. ☐

It is easily verified that composites of quasi-pseudofunctors are quasi-pseudofunctors and composites of biequivalences are biequivalences. Hence:

10.4.16. Corollary. *Any zig-zag of cosmological biequivalences composes to define a quasi-pseudofunctor $\mathcal{K} \overset{\sim}{\rightsquigarrow} \mathcal{L}$ between ∞-cosmoi that is also a biequivalence.* ☐

Moreover, the preservation and reflection properties of cosmological biequivalences established in Proposition 10.3.5 extend to their quasi-pseudofunctorial inverses. For future reference, we prove one result in this vein in detail.

10.4.17. Lemma. *A quasi-pseudofunctor $G \colon \mathcal{L} \overset{\sim}{\rightsquigarrow} \mathcal{K}$ defined as an pseudoinverse to a cosmological biequivalence $F \colon \mathcal{K} \xrightarrow{\sim} \mathcal{L}$ preserves and reflects comma ∞-categories: a cell as on the left is a comma cone in $\mathcal{L}$ if and only if its image is a comma cone in $\mathcal{K}$.*

$$\begin{tikzcd} & E \arrow[dl, "e_1"'] \arrow[dr, "e_0"] \\ C \arrow[dr, "g"'] \arrow[rr, phantom, "\overset{\epsilon}{\Leftarrow}"] & & B \arrow[dl, "f"] \\ & A \end{tikzcd} \quad \overset{G}{\rightsquigarrow} \quad \begin{tikzcd} & (GE)' \arrow[dl, "(Ge_1)'"'] \arrow[dr, "(Ge_0)'"] \\ GC \arrow[dr, "Gg"'] \arrow[rr, phantom, "\overset{G\epsilon}{\Leftarrow}"] & & GB \arrow[dl, "Gf"] \\ & GA \end{tikzcd}$$

Note there is no reason why the inverse to a cosmological biequivalence should preserve isofibrations, so the map $(Ge_1, Ge_0)\colon GE \to GC \times GB$ should be replaced by an equivalent isofibration $((Ge_1)', (Ge_0)')\colon GE' \twoheadrightarrow GC \times GB$.

Proof. Proposition 10.3.5(v) proves that the right-hand square defines a comma cone in $\mathcal{K}$ if and only if its image under the cosmological biequivalence $F\colon \mathcal{K} \xrightarrow{\sim} \mathcal{L}$ defines an comma cone in $\mathcal{L}$. On account of the quasi-pseudonatural equivalence $\epsilon\colon FG \Rightarrow \mathrm{id}_{\mathcal{L}}$ of Proposition 10.4.15, the image of the right-hand square is equivalent to the left-hand square. By a mild extension of Propositions 3.4.5 and 3.4.11, the universal property that characterizes the comma square, transfers across this equivalence. □

Exercises.

10.4.i. Exercise. For a fixed pair of 2-categories $\mathcal{C}$ and $\mathcal{D}$, show that the collection of pseudofunctors $\mathcal{C} \rightsquigarrow \mathcal{D}$, pseudonatural transformations, and modifications (see Definition B.2.3) assemble into a 2-category.

10.4.ii. Exercise ([**44**, 6.2.16]). For a pseudonatural transformation $\phi\colon F \Rightarrow G$ between pseudofunctors $F, G\colon \mathcal{C} \rightsquigarrow \mathcal{D}$ between 2-categories $\mathcal{C}$ and $\mathcal{D}$, show that the following are equivalent.

- Each 1-cell component $\phi_x\colon Fx \xrightarrow{\sim} Gx$ is an equivalence in $\mathcal{D}$.
- The 1-cell ϕ defines an equivalence in the 2-category described in Exercise 10.4.i.

10.4.iii. Exercise. Derive a proof of Proposition 10.4.3 from Proposition 10.4.15.

10.4.iv. Exercise. Let $\phi\colon F \Rightarrow G$ be a quasi-pseudonatural transformation between quasi-pseudofunctors $F, G\colon \mathcal{K} \rightsquigarrow \mathcal{L}$. Show that for any pair of 0-arrows $f\colon x \to y$ and $k\colon y \to z$ in the $\mathcal{K}$, the diagrams

$$\begin{tikzcd}
& Fy \arrow[dr, "Fk"] \arrow[d, phantom, "\alpha^{x,y,z}_{f,g}\Downarrow\cong"] & \\
Fx \arrow[ur, "Ff"] \arrow[rr, bend right, "F(kf)"'] \arrow[d, "\phi_x"'] & {} \arrow[d, phantom, "\phi_{kf}\Downarrow\cong"] & Fz \arrow[d, "\phi_z"] \\
Gx \arrow[rr, bend right, "G(kf)"'] & {} & Gz
\end{tikzcd}
=
\begin{tikzcd}
& Fy \arrow[dr, "Fk"] \arrow[dd, "\phi_y"] & \\
Fx \arrow[ur, "Ff"] \arrow[dd, "\phi_x"'] \arrow[r, phantom, "\phi_f\Downarrow\cong"] & {} & Fz \arrow[dd, "\phi_z"] \arrow[l, phantom, "\phi_k\Downarrow\cong"] \\
& Gy \arrow[dr, "Gk"] \arrow[d, phantom, "\alpha^{x,y,z}_{f,g}\Downarrow\cong"] & \\
Gx \arrow[ur, "Gf"] \arrow[rr, bend right, "G(kf)"'] & {} & Gz
\end{tikzcd}$$

and

$$\begin{tikzcd}
Fx \arrow[r, bend left, "\mathrm{id}_{Fx}"] \arrow[r, phantom, "\iota_x\Downarrow\cong"] \arrow[r, bend right, "F\,\mathrm{id}_x"'] \arrow[d, "\phi_x"'] & Fx \arrow[d, "\phi_x"] \\
Gx \arrow[r, bend right, "G\,\mathrm{id}_x"'] \arrow[ur, phantom, "\phi_{\mathrm{id}_x}\Downarrow\cong"] & Gx
\end{tikzcd}
=
\begin{tikzcd}
Fx \arrow[r, bend left, "\mathrm{id}_{Fx}"] \arrow[d, "\phi_x"'] & Fx \arrow[d, "\phi_x"] \\
Gx \arrow[r, bend left, "\mathrm{id}_{Gx}"] \arrow[r, phantom, "\iota_y\Downarrow\cong"] \arrow[r, bend right, "G\,\mathrm{id}_x"'] & Gx
\end{tikzcd}$$

commute in the homotopy 2-category of $\mathcal{L}$.

10.4.v. Exercise. Finish the proof of Lemma 10.4.11.

CHAPTER 11

Proof of model independence

We begin in §11.1 by applying the results of Chapter 10 to give an ad hoc exploration of the model invariance of the fundamental ∞-categorical notions. This discussion previews the line of reasoning that underpins our main theorem in this chapter, which develops a more systematic approach to this model-independence question. Explicitly, in §11.3 we prove that a cosmological biequivalence induces a biequivalence of virtual equipments. In §11.2, we review the construction of the virtual equipment of modules associated to an ∞-cosmos and explain why it describes a suitable context for proving the model-independence of the fundamental ∞-categorical notions.

11.1. Ad hoc model invariance

An ad hoc approach to proving the model-independence of the basic category theory of ∞-categories is given by elaborations of Proposition 10.3.5, as described for instance in the following theorem:

11.1.1. Theorem (model independence of basic category theory I). *The following notions are preserved, reflected, and created by any cosmological biequivalence:*

(i) A left or right adjoint to a functor $u\colon A \to B$.
(ii) A limit or a colimit for a J-indexed diagram $d\colon 1 \to A^J$ in an ∞-category A.

Proof. For (i), if $u\colon A \to B$ admits a left adjoint $f\colon B \to A$, then this adjunction is preserved by any 2-functor, and hence by any cosmological functor $F\colon \mathcal{K} \to \mathcal{L}$. If there merely exists $f\colon B \to A$ in $\mathcal{K}$ so that $Ff \dashv Fu$ in $\mathcal{L}$, then in the case where F is a biequivalence, Corollary 10.3.2(iii) can be used to lift the unit and counit from $\mathcal{L}$ to define 2-cells $\eta\colon \mathrm{id}_B \Rightarrow uf$ and $\epsilon\colon fu \Rightarrow \mathrm{id}_A$ in $\mathcal{K}$. A priori the composites $\epsilon f \cdot f\eta$ and $u\epsilon \cdot \eta u$ need not be identities, but they will be invertible and the standard 2-categorical argument appearing in the proof of Proposition 2.1.11 can be used to modify one of these 2-cells to produce a genuine adjunction $f \dashv u$. A similar line of reasoning can be used in the absence of a candidate left adjoint using Corollary 10.3.2(ii) to lift the left adjoint from $\mathcal{L}$ to a functor $f\colon B \to A$ in $\mathcal{K}$ whose image is isomorphic to the left adjoint to Fu. The rest of the argument proceeds as before.

For (ii), given an absolute lifting diagram in $\mathfrak{h}\mathcal{L}$ as displayed on the right

$$\begin{tikzcd} & FA \arrow[d, "\Delta"] \\ 1 \arrow[ur, "\bar{\ell}"] \arrow[r, "Fd"'] \arrow[ur, phantom, shift right=3, "\Downarrow\bar{\lambda}"] & FA^J \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[d, "\Delta"] \\ 1 \arrow[ur, "\ell"] \arrow[r, "d"'] \arrow[ur, phantom, shift right=3, "\Downarrow\lambda"] & A^J \end{tikzcd}$$

Corollary 10.3.2(ii) and (iii) yield a 1-cell $\ell\colon 1 \to A$ with $F\ell \cong \bar{\ell}$ and a 2-cell $\lambda\colon \Delta\ell \Rightarrow d$, as displayed on the right, so that $F\lambda$ composes with the isomorphism $F\ell \cong \bar{\ell}$ to $\bar{\lambda}$. Our task is to show that that this data defines an absolute right lifting diagram in $\mathcal{K}$, which amounts to showing for all objects X

and for all maps $a\colon X \to A$ that the composition operation with λ induces a bijection

$$\begin{tikzcd}
\mathsf{hFun}(X,A)(a,\ell) \arrow[r, "\lambda\circ-"] \arrow[d, phantom, "\wr\Vert"] & \mathsf{hFun}(X,A^J)(\Delta a, d!) \arrow[d, phantom, "\wr\Vert"] \\
\mathsf{hFun}(FX,FA)(Fa,F\ell) \arrow[r, "F\lambda\circ-"] \arrow[d, phantom, "\wr\Vert"] & \mathsf{hFun}(FX,FA^J)(\Delta Fa, Fd!) \arrow[d, phantom, "\wr\Vert"] \\
\mathsf{hFun}(FX,FA)(Fa,\bar{\ell}) \arrow[r, "\bar{\lambda}\circ-"] & \mathsf{hFun}(FX,FA^J)(\Delta Fa, Fd!)
\end{tikzcd}$$

On account of the isomorphisms of Corollary 10.3.2(ii) and (iii), this follows from the corresponding encoding of the universal property of $\bar{\lambda}$ in $\mathcal{L}$.

This argument also shows that the universal property of absolute lifting diagrams is reflected. To see that absolute lifting diagrams are also preserved, Corollary 10.3.2(i) must be invoked to lift any cone in $\mathcal{L}$ with summit Y over the 2-cell $F\lambda$ to an equivalent one in $\mathcal{K}$ with summit X chosen so that $FX \simeq Y$. As the local bijections Corollary 10.3.2(ii) and (iii) can also be defined relative to specified equivalences of objects, the same style of argument goes through. □

Special features of the homotopy 2-category of a particular ∞-cosmos can also easily be seen to transfer to biequivalent ∞-cosmoi. For instance, Corollary D.4.18 proves that a natural transformation between functors between quasi-categories $\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & A \end{tikzcd}$ is a natural isomorphism if and only if it is a **pointwise isomorphism**,

Exercises.

11.1.i. Exercise. Pick your favorite ∞-categorical notion and give an ad hoc proof of its model independence.

11.2. The context for the model independence theorem

Recall from Chapter 8, that the **virtual double category of modules** $\mathcal{M}od(\mathfrak{h}\mathcal{K})$ in an ∞-cosmos $\mathcal{K}$ consists of

- a category of **objects** and **vertical arrows**, here the ∞-categories and ∞-functors
- for any pair of objects A, B, a class of **horizontal arrows** $A \overset{E}{\nrightarrow} B$, here the modules $(q,p)\colon E \twoheadrightarrow A \times B$ from A to B
- **cells**, with boundary depicted as follows

$$\begin{tikzcd}
A_0 \arrow[r, "E_1", "\mid" marking] \arrow[d, "f"'] & A_1 \arrow[r, "E_2", "\mid" marking] \arrow[d, phantom, "\Downarrow"] & \cdots \arrow[r, "E_n", "\mid" marking] & A_n \arrow[d, "g"] \\
B_0 \arrow[rrr, "F"', "\mid" marking] & & & B_n
\end{tikzcd}$$

including those whose horizontal source has length zero, in the case $A_0 = A_n$. Here, a cell with the displayed boundary is an isomorphism class of objects in the functor space

$$\begin{tikzcd}
\mathsf{Fun}_{f,g}(E_1 \underset{A_1}{\times} \cdots \underset{A_{n-1}}{\times} E_n, F) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Fun}(E_1 \underset{A_1}{\times} \cdots \underset{A_{n-1}}{\times} E_n, F) \arrow[d] \\
\mathbb{1} \arrow[r] & \mathsf{Fun}(E_1 \underset{A_1}{\times} \cdots \underset{A_{n-1}}{\times} E_n, B_0 \times B_n)
\end{tikzcd}$$

i.e., a fibered isomorphism class of maps of spans over $f \times g$

- a **composite cell**, for any configuration

$$\begin{tikzcd}
A_0 \arrow[r, "{E_{11},\ldots,E_{1k_1}}", "\mid"{marking}] \arrow[d, "f_0"'] \arrow[dr, phantom, "\Downarrow"] & A_1 \arrow[r, "{E_{21},\ldots,E_{2k_2}}", "\mid"{marking}] \arrow[d, "f_1"] \arrow[dr, phantom, "\Downarrow"] & \cdots \arrow[r, "{E_{n1},\ldots,E_{nk_n}}", "\mid"{marking}] \arrow[d] \arrow[d, phantom, "\cdots"] \arrow[dr, phantom, "\Downarrow"] & A_n \arrow[d, "f_n"] \\
B_0 \arrow[r, "F_1", "\mid"{marking}] \arrow[d, "g"'] & B_1 \arrow[r, "F_2", "\mid"{marking}] \arrow[d, phantom, "\Downarrow"] & \cdots \arrow[r, "F_n", "\mid"{marking}] & B_n \arrow[d, "h"] \\
C_0 \arrow[rrr, "G"', "\mid"{marking}] & & & C_n
\end{tikzcd}$$

- an **identity cell** for every horizontal arrow

$$\begin{tikzcd}
A \arrow[r, "E", "\mid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow \mathrm{id}_E"] & B \arrow[d, equal] \\
A \arrow[r, "E"', "\mid"{marking}] & B
\end{tikzcd}$$

so that composition of cells is strictly associative and unital in the usual multi-categorical sense.

The virtual double category of modules is a **virtual equipment** in the sense introduced by Cruttwell and Shulman [**27**, §7], which means that it satisfies the two further properties:

(i) For any module and pair of functors as displayed on the left, there exists a module and cartesian cell as displayed on the right

$$\begin{tikzcd}
A' \arrow[d, "a"'] & B' \arrow[d, "b"] \\
A \arrow[r, "E"', "\mid"{marking}] & B
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
A' \arrow[r, "{E(b,a)}", "\mid"{marking}] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow \rho"] & B' \arrow[d, "b"] \\
A \arrow[r, "E"', "\mid"{marking}] & B
\end{tikzcd}$$

characterized by the universal property that any cell as displayed below-left factors uniquely through ρ as below-right:

$$
\begin{tikzcd}
X_0 \arrow[r, "E_1", "\mapsto"'] \arrow[d, "af"'] & X_1 \arrow[r, "E_2"] & \cdots \arrow[r, "E_n"] & X_n \arrow[d, "bg"] \\
A \arrow[rrr, "E"'] \arrow[rrru, phantom, "\Downarrow"] & & & B
\end{tikzcd}
=
\begin{tikzcd}
X_0 \arrow[r, "E_1"] \arrow[d, "f"'] & X_1 \arrow[r, "E_2"] & \cdots \arrow[r, "E_n"] & X_n \arrow[d, "g"] \\
A' \arrow[rrr, "E(b,a)"] \arrow[d, "a"'] \arrow[rrru, phantom, "\exists!\Downarrow"] & & & B' \arrow[d, "b"] \\
A \arrow[rrr, "E"'] \arrow[rrru, phantom, "\Downarrow\rho"] & & & B
\end{tikzcd}
$$

(ii) Every object A admits a *unit* module equipped with a nullary cocartesian cell

$$
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, equal] & A \arrow[d, equal] \\
A \arrow[r, "\mathsf{Hom}_A"'] \arrow[ru, phantom, "\Downarrow\iota"] & A
\end{tikzcd}
$$

satisfying the universal property that any cell in the virtual double category of modules whose horizontal source includes the object A, as displayed on the left

$$
\begin{tikzcd}[column sep=small]
X \arrow[r, "E_1"] \arrow[d, "f"'] & \cdots \arrow[r, "E_n"] & A \arrow[r, "F_1"] & \cdots \arrow[r, "F_m"] & Y \arrow[d, "g"] \\
B \arrow[rrrr, "G"'] \arrow[rrrru, phantom, "\Downarrow"] & & & & C
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
X \arrow[r, "E_1"] \arrow[d, equal] & \cdots \arrow[r, "E_n"] & A \arrow[r, equal] \arrow[d, equal] & A \arrow[r, "F_1"] \arrow[d, equal] & \cdots \arrow[r, "F_m"] & Y \arrow[d, equal] \\
X \arrow[r, "E_1"'] \arrow[d, "f"'] \arrow[ru, phantom, "\Downarrow\mathrm{id}_{E_1}"] & \cdots \arrow[r, "E_n"'] \arrow[ru, phantom, "\Downarrow\mathrm{id}_{E_n}"] & A \arrow[r, "\mathsf{Hom}_A"'] \arrow[ru, phantom, "\Downarrow\iota"] & A \arrow[r, "F_1"'] \arrow[ru, phantom, "\Downarrow\mathrm{id}_{F_1}"] & \cdots \arrow[r, "F_m"'] \arrow[ru, phantom, "\Downarrow\mathrm{id}_{F_m}"] & Y \arrow[d, "g"] \\
B \arrow[rrrrr, "G"'] \arrow[rrrrru, phantom, "\Downarrow\exists!"] & & & & & C
\end{tikzcd}
$$

factors uniquely through ι as displayed on the right.

The module in (i) is defined by pulling back a module $A \overset{E}{\nrightarrow} B$ along functors $a\colon A' \to A$ and $b\colon B' \to B$. The simplicial pullback defining $E(b,a)$ induces an equivalence of functor spaces

$$\mathsf{Fun}_{af,bg}(E_1 \times \cdots \times E_n, E) \simeq \mathsf{Fun}_{f,g}(E_1 \times \cdots \times E_n, E(b,a)),$$

which gives rise to the universal property. See Proposition 8.2.1.

The unit module is the arrow ∞-category, given the notation $A \overset{\mathsf{Hom}_A}{\nrightarrow} A$ when considered as a module from A to A. The universal property of (ii) follows from the Yoneda lemma; see Proposition 8.2.4.

The virtual equipment of modules in $\mathcal{K}$ has a lot of pleasant properties, which follow formally from the axiomatization of a virtual equipment. For instance, certain sequences of composable modules can be said to *have composites*, witnessed by cocartesian cells as in (ii) (see 8.3.5, 8.3.11, 8.4.7, and 8.4.8 and the non-formal composite given in 8.3.7). Also, for any functor $f\colon A \to B$, the modules $A \overset{\mathsf{Hom}_B(B,f)}{\nrightarrow} B$ and $B \overset{\mathsf{Hom}_B(f,B)}{\nrightarrow} A$ behave like adjoints is a sense suitable to a virtual double category; more precisely, the module $A \overset{\mathsf{Hom}_B(B,f)}{\nrightarrow} B$ defines a *companion* and the module $B \overset{\mathsf{Hom}_B(f,B)}{\nrightarrow} A$ defines a *conjoint* to $f\colon A \to B$

(see 8.4.1, 8.4.4, and 8.4.6). Another consequence of Theorem 8.2.6 is the following: for any parallel pair of functors there are natural bijections between 2-cells in the homotopy 2-category

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow"] & B \end{tikzcd}$$

and cells in the virtual equipment of modules:

$$\begin{tikzcd} A \arrow[r, "\mathrm{Hom}_B(B,f)", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow"] & B \arrow[d, equal] \\ A \arrow[r, "\mathrm{Hom}_B(B,g)"', "\shortmid"{marking}] & B \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} A \arrow[r, "\mathrm{Hom}_A", "\shortmid"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow"] & A \arrow[d, "f"] \\ B \arrow[r, "\mathrm{Hom}_B"', "\shortmid"{marking}] & B \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} B \arrow[r, "\mathrm{Hom}_B(g,B)", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow"] & A \arrow[d, equal] \\ B \arrow[r, "\mathrm{Hom}_B(f,B)"', "\shortmid"{marking}] & A \end{tikzcd}$$

See Proposition 8.4.10.

As remarked upon in Definition 8.4.11, as a consequence of these results, there are two locally-fully-faithful homomorphisms $\mathfrak{h}\mathcal{K} \hookrightarrow \mathcal{M}od(\mathfrak{h}\mathcal{K})$ and $\mathfrak{h}\mathcal{K}^{\mathrm{coop}} \hookrightarrow \mathcal{M}od(\mathfrak{h}\mathcal{K})$ — referred to as the *covariant* and *contravariant embeddings*, respectively — embedding the homotopy 2-category into the substructure[1] of $\mathcal{M}od(\mathfrak{h}\mathcal{K})$ comprised only of unary cells whose vertical boundaries are identities. The modules in the image of the covariant embedding are the right representables and the modules in the image of the contravariant embedding are the left representables.

The theme of Chapter 9 could be summarized by saying that the virtual equipment of modules in an ∞-cosmos is a robust setting to develop the category theory of ∞-categories. On the one hand, it contains the homotopy 2-category of the ∞-cosmos, which was the setting for the results of Part I. It is also a very natural home to study ∞-categorical properties that are somewhat awkward to express in the homotopy 2-category. For instance, the weak 2-universal property of comma ∞-categories is now encoded by a bijection in Lemma 8.1.16: cells in the virtual equipment whose codomain is a comma module correspond bijectively to natural transformations of a particular form in the homotopy 2-category. Fibered equivalences of modules, as used to express the universal properties of adjunctions, limits, and colimits in Chapter 4, are now vertical isomorphisms in the virtual equipment between parallel modules. The virtual equipment also cleanly encodes the universal property of pointwise left and right Kan extensions.

In this chapter, we show that a cosmological biequivalence $F\colon \mathcal{K} \xrightarrow{\sim} \mathcal{L}$ induces a *biequivalence of virtual equipments* $F\colon \mathcal{M}od(\mathfrak{h}\mathcal{K}) \xrightarrow{\sim} \mathcal{M}od(\mathfrak{h}\mathcal{L})$ in a suitable sense that we introduce to describe what is true in our setting. We then explain the interpretation of this result: that the category theory of ∞-categories is preserved, reflected, and created by any "change-of-model" functor of this form.

11.3. A biequivalence of virtual equipments

We first elaborate on Proposition 10.3.1(ix).

11.3.1. Proposition. *A cosmological biequivalence $F\colon \mathcal{K} \xrightarrow{\sim} \mathcal{L}$ preserves, reflects, and creates modules:*

(i) An isofibration $E \twoheadrightarrow A \times B$ is a module in $\mathcal{K}$ if and only if $FE \twoheadrightarrow FA \times FB$ is a module in $\mathcal{L}$.

(ii) For every module $G \twoheadrightarrow A' \times B'$ in $\mathcal{L}$ and every pair of ∞-categories A, B in $\mathcal{K}$ with specified equivalences $FA \simeq A'$ and $FB \simeq B'$ there is a module $E \twoheadrightarrow A \times B$ in $\mathcal{K}$ so that FE is equivalent to G over the pair of equivalences.

[1] This substructure is very nearly a bicategory, with horizontal composites of unary cells constructed as in Definition 8.3.12, except that compatible sequences of modules do not always admit a horizontal composite.

PROOF. For (i) consider an isofibration $E \twoheadrightarrow A \times B$ in $\mathcal{K}_{/A\times B}$ and the induced biequivalence $F\colon \mathcal{K}_{/A\times B} \xrightarrow{\sim} \mathcal{L}_{/FA\times FB}$ of Proposition 10.2.4. By Proposition 10.3.5(iii), E defines a discrete object in $\mathcal{K}_{/A\times B}$ if and only if FE defines a discrete object in $\mathcal{L}_{/FA\times FB}$. By Theorem 11.1.1(i), $E \to \mathrm{Hom}_B(B,p)$ admits a right adjoint and $E \to \mathrm{Hom}_A(q,A)$ admits a left adjoint in $\mathcal{K}_{/A\times B}$ if and only if $FE \to F\mathrm{Hom}_B(B,p) \cong \mathrm{Hom}_{FB}(FB,Fp)$ admits a right adjoint and $FE \to F\mathrm{Hom}_A(q,A) \cong \mathrm{Hom}_{FA}(Fq,FA)$ admits a left adjoint in $\mathcal{L}_{/FA\times FB}$. By Lemma 7.4.2 these properties characterize modules.

For (ii), fix a pair of equivalences $FA \simeq A'$ and $FB \simeq B'$, defining an equivalence $e\colon A' \times B' \xrightarrow{\sim} FA \times FB$, and consider the composite biequivalence

$$\mathcal{K}_{/A\times B} \xrightarrow[\sim]{F} \mathcal{L}_{/FA\times FB} \xrightarrow[\sim]{e^*} \mathcal{L}_{/A'\times B'}$$

given by Propositions 10.2.4 and 10.2.6. Consider a module $G \twoheadrightarrow A' \times B'$. By essential surjectivity, there is an isofibration $E \twoheadrightarrow A \times B$ whose image under this cosmological functor—the pullback of $FE \twoheadrightarrow FA \times FB$ along $e\colon A' \times B' \xrightarrow{\sim} FA \times FB$—defines an isofibration $(q,p)\colon E' \twoheadrightarrow A' \times B'$ that is equivalent to G in $\mathcal{L}_{/A'\times B'}$. It remains only to argue that E defines a module from A to B, which will follow, essentially as in the proof of (i), from the fact that $E' \simeq G$ defines a module from A' to B'.

As the image E' of E is equivalent to a discrete object, Proposition 10.3.5(iii) tells us E is discrete in $\mathcal{K}_{/A\times B}$. The final step is to argue that the desired right adjoint to $E \to \mathrm{Hom}_B(B,p)$ is present in the image of the biequivalence $\mathcal{K}_{/A\times B} \xrightarrow{\sim} \mathcal{L}_{/A'\times B'}$, and apply Theorem 11.1.1(i) to deduce its presence in $\mathcal{K}_{/A\times B}$; a similar argument of course applies to the functor $E \to \mathrm{Hom}_A(q,A)$. To see this note that $F\colon \mathcal{K}_{/A\times B} \xrightarrow{\sim} \mathcal{L}_{/FA\times FB}$ carries $\mathrm{Hom}_B(B,p) \twoheadrightarrow A \times B$ to $\mathrm{Hom}_{FB}(FB,Fp) \twoheadrightarrow FA \times FB$. By (i), it suffices to argue that this functor has a right adjoint over $FA \times FB$. Applying Theorem 11.1.1(i) to the biequivalence $e^*\colon \mathcal{L}_{/FA\times FB} \xrightarrow{\sim} \mathcal{L}_{/A'\times B'}$, this follows from the fact that $E' \to \mathrm{Hom}_{B'}(B',p')$ has a right adjoint over $A' \times B'$. □

11.3.2. PROPOSITION. *Let $F\colon \mathcal{K} \xrightarrow{\sim} \mathcal{L}$ be a cosmological biequivalence. Then a module $A \overset{E}{\nrightarrow} B$ in $\mathcal{K}$ is right representable if and only if the module $FA \overset{FE}{\nrightarrow} FB$ is right representable in $\mathcal{L}$, in which case, F carries the representing functor $f\colon A \to B$ in $\mathcal{K}$ to a representing functor $Ff\colon FA \to FB$ in $\mathcal{L}$.*

PROOF. To say that $A \overset{E}{\nrightarrow} B$ is right representable in $\mathcal{K}$ is to say that there exists a functor $f\colon A \to B$ together with an equivalence $E \simeq_{A\times B} \mathrm{Hom}_B(B,f)$ of modules over B. If this is the case then any cosmological functor $F\colon \mathcal{K} \to \mathcal{L}$ carries this to a fibered equivalence $FE \simeq_{FA\times FB} \mathrm{Hom}_{FB}(FB,Ff)$, and hence the module $FA \overset{FE}{\nrightarrow} FB$ is right-represented by $Ff\colon FA \to FB$ in $\mathcal{L}$.

Conversely, if $FA \overset{FE}{\nrightarrow} FB$ is right-represented by some functor $g\colon FA \to FB$, then by Corollary 10.3.2(ii), there exists a functor $f\colon A \to B$ in $\mathcal{K}$ so that $Ff \cong g$ in $\mathcal{L}$. By Proposition 8.4.10, naturally isomorphic functors represent equivalent modules; that is, $\mathrm{Hom}_{FB}(FB,g) \simeq_{FA\times FB} \mathrm{Hom}_{FB}(FB,Ff)$. Thus $FE \simeq_{FA\times FB} \mathrm{Hom}_{FB}(FB,Ff)$. By Corollary 10.3.2(i), this fibered equivalence lifts along the cosmological functor $F\colon \mathcal{K}_{/A\times B} \to \mathcal{L}_{/FA\times FB}$ to a fibered equivalence $E \simeq_{A\times B} \mathrm{Hom}_B(B,f)$, which proves that E is right represented by $f\colon A \to B$ in $\mathcal{K}$. □

Because cosmological functors preserve modules, simplicial pullbacks, and the arrow construction, we see that a functor $F\colon \mathcal{K} \to \mathcal{L}$ induces a functor of virtual equipments $F\colon \mathcal{M}od(\mathfrak{h}\mathcal{K}) \to \mathcal{M}od(\mathfrak{h}\mathcal{L})$ preserving all of the structures described in Theorem 8.2.6.

11.3.3. Theorem (model independence of ∞-category theory). *If $F\colon \mathcal{K} \to \mathcal{L}$ is a cosmological biequivalence, then the induced functor of virtual equipments*

$$F\colon \mathcal{M}od(\mathfrak{h}\mathcal{K}) \to \mathcal{M}od(\mathfrak{h}\mathcal{L})$$

defines a ***biequivalence of virtual equipments****: i.e., it is*

(i) bijective on equivalence classes of objects;
(ii) locally bijective on isomorphism classes of parallel vertical functors extending the bijection of (i);
(iii) locally bijective on equivalence classes of parallel modules extending the bijection of (ii);
(iv) locally bijective on cells extending the bijections of (i), (ii), and (iii).

Consequently, any ∞-categorical notion that can be encoded as an equivalence-invariant proposition in the virtual equipment of modules is model invariant: preserved, reflected, and created by cosmological biequivalences.

Note further that if two ∞-cosmoi are connected by a finite zig-zag of biequivalences, then the bijections described in Theorem 11.3.3 compose.

Proof. Properties (i) and (ii) are restatements of Corollary 10.3.2(i) and (ii).

The local bijection (iii) follows immediately from Proposition 11.3.1 and the fact that for any pair of equivalences $e\colon A' \times B' \xrightarrow{\sim} FA \times FB$, the composite biequivalence

$$\mathcal{K}_{/A\times B} \xrightarrow[\sim]{F} \mathcal{L}_{/FA\times FB} \xrightarrow[\sim]{e^*} \mathcal{L}_{/A'\times B'}$$

preserves, reflects, and creates equivalences between objects (Corollary 10.3.2(i)). Finally (iv) is an application of Corollary 10.3.2(ii) to this cosmological biequivalence. □

For instance, the presence of an adjunctions between ∞-categories and the existence of limits and colimits inside an ∞-category can both be encoded as an equivalence-invariant proposition in the virtual equipment of modules. Using Theorem 11.3.3, we can reprove Theorem 11.1.1.

Module-theoretic proof of Theorem 11.1.1. By Proposition 4.1.1, $u\colon A \to B$ admits a left adjoint if and only if the module $A \overset{\mathrm{Hom}_B(B,u)}{\nrightarrow} B$ is contravariantly represented. Any representing functor in $\mathcal{L}$ lifts up to isomorphism to define a functor $f\colon B \to A$ in $\mathcal{K}$ and now by Theorem 11.3.3(iii) there is an equivalence $\mathrm{Hom}_A(f, A) \simeq \mathrm{Hom}_B(B, u)$ of modules from A to B in $\mathcal{K}$ if and only if there is an equivalence $\mathrm{Hom}_{FA}(Ff, FA) \simeq \mathrm{Hom}_{FB}(FB, fu)$ of modules from FA to FB in $\mathcal{L}$.

Similarly, a diagram $d\colon 1 \to A^J$ admits a limit if and only if the module $1 \overset{\mathrm{Hom}_{A^J}(\Delta,d)}{\nrightarrow} A$ is covariantly represented, and Proposition 11.3.2 or the argument just given completes the proof in this case as well. □

A biequivalence of virtual equipments preserves, reflects, and creates composites of modules.

11.3.4. Lemma. *Let $F\colon \mathcal{K} \xrightarrow{\sim} \mathcal{L}$ be a cosmological biequivalence.*

(i) Then a composable sequence of modules in $\mathcal{K}$ admits a composite in $\mathcal{M}od(\mathfrak{h}\mathcal{K})$ if and only if the image of this sequence admits a composite in $\mathcal{M}od(\mathfrak{h}\mathcal{L})$.
(ii) Hence, cosmological biequivalences preserve and reflect exact squares.

Proof. Via Definition 9.2.2, (ii) follows immediately from (i), so it remains only to show that a biequivalence of virtual equipments $F\colon \mathcal{M}od(\mathfrak{h}\mathcal{K}) \xrightarrow{\sim} \mathcal{M}od(\mathfrak{h}\mathcal{L})$ preserves, reflects, and creates composites of modules. To see that an n-ary composite cell $\mu\colon E_1 \times \cdots \times E_n \Rightarrow E$ in $\mathfrak{h}\mathcal{K}$ is preserved, note that by Theorem 11.3.3(iv), any cell in $\mathcal{M}od(\mathfrak{h}\mathcal{L})$ is isomorphic to a cell in the image of F: first replace the objects by equivalent ones in the image, then replace the vertical functors by naturally isomorphic ones in the image, then replace the modules by equivalent ones in the image over the specified equivalences between their ∞-categorical sources and targets, and then finally apply the local bijection (iv) to replace the cell in $\mathcal{M}od(\mathfrak{h}\mathcal{L})$ by a unique cell in the image of $\mathcal{M}od(\mathfrak{h}\mathcal{K})$ by composing with this data. Now, by local full and faithfulness and essential surjectivity, the universal property of the cocartesian cell $\mu\colon E_1 \mathbin{\dot\times} \cdots \mathbin{\dot\times} E_n \Rightarrow E$ implies that its image $F\mu\colon FE_1 \mathbin{\dot\times} \cdots \mathbin{\dot\times} FE_n \Rightarrow FE$ is again a cocartesian cell. Thus composites $E_1 \otimes \cdots \otimes E_n \simeq E$ are preserved by cosmological biequivalences.

Now if $F\mu\colon FE_1 \mathbin{\dot\times} \cdots \mathbin{\dot\times} FE_n \Rightarrow FE$ is a composite, the fact that $F\colon \mathcal{M}od(\mathfrak{h}\mathcal{K}) \to \mathcal{M}od(\mathfrak{h}\mathcal{L})$ is locally fully faithful implies immediately that $\mu\colon E_1 \mathbin{\dot\times} \cdots \mathbin{\dot\times} E_n \Rightarrow E$ is a composite; thus composites $FE_1 \otimes \cdots \otimes FE_n \simeq FE$ are reflected by cosmological biequivalences.

Finally, suppose the sequence $FE_1 \mathbin{\dot\times} \cdots \mathbin{\dot\times} FE_n$ of modules in $\mathcal{M}od(\mathfrak{h}\mathcal{L})$ admits a composite $FA_0 \overset{G}{\nrightarrow} FA_n$; since the composable sequence of modules is in the image of F the source and target ∞-categories of the composite module G are as well. By Theorem 11.3.3(iii) there exists a module $A_0 \overset{E}{\nrightarrow} A_n$ in $\mathfrak{h}\mathcal{K}$ so that $FE \simeq G$ as modules from FA_0 to FA_n. The cocartesian cell $FE_1 \mathbin{\dot\times} \cdots \mathbin{\dot\times} FE_n \Rightarrow G$ composes with the unary cell of this equivalence to define a cocartesian cell $FE_1 \mathbin{\dot\times} \cdots \mathbin{\dot\times} FE_n \Rightarrow FE$. By Theorem 11.3.3(iv), this lifts to an n-ary cell $E_1 \mathbin{\dot\times} \cdots \mathbin{\dot\times} E_n \Rightarrow E$ in $\mathcal{M}od(\mathfrak{h}\mathcal{K})$. As we've just seen that cocartesianness of cells is reflected by biequivalences $F\colon \mathcal{M}od(\mathfrak{h}\mathcal{K}) \to \mathcal{M}od(\mathfrak{h}\mathcal{L})$, this completes the proof that composites are created by cosmological biequivalences. □

Exercises.

11.3.i. Exercise. Prove that cosmological biequivalences between cartesian closed ∞-cosmoi preserve and reflect initial and final functors.

CHAPTER 12

Applications of model-independence

In this chapter, we establish some elementary properties of a certain class of ∞-cosmoi we might call **∞-cosmoi of $(\infty,1)$-categories**, by which we mean ∞-cosmoi that are biequivalent to $\mathcal{QC}at$. By Proposition 10.2.3 an ∞-cosmos $\mathcal{K}$ is an ∞-cosmos of $(\infty,1)$-categories if and only if its underlying quasi-category functor $(-)_0 := \mathsf{Fun}(1,-)\colon \mathcal{K} \to \mathcal{QC}at$ is a biequivalence — meaning that every quasi-category is equivalent to the underlying quasi-category of an ∞-category in $\mathcal{K}$ and that for any $A, B \in \mathcal{K}$ the map $\mathsf{Fun}(A,B) \to B_0^{A_0}$ is an equivalence of quasi-categories. Several examples of ∞-cosmoi of this form are established in §E.2.

The aim is to illustrate how the model-independence theorem can be used to combine synthetic and analytic techniques to prove results concerning any family of biequivalent ∞-cosmoi. In what follows we appeal to explicit descriptions of $(\infty,1)$-categories as quasi-categories to supply analytic proofs of certain key results — for instance, that a functor defines an equivalence of quasi-categories just when it is fully faithful and essential surjective in a suitable sense. We then explain how the model-independence theorem can be used to transfer these results to biequivalent ∞-cosmoi. In this way, we conclude that any functor defines an equivalence of $(\infty,1)$-categories just when it is fully faithful and essentially surjective, even though we can't translate the specific proof of the quasi-categorical case of this result. We then apply this result to further develop the synthetic theory of ∞-cosmoi of $(\infty,1)$-categories.

Many of the results that follow could have appeared earlier, but in the presence of the results of Chapters 10 and 11 their conclusions apply more broadly, to all ∞-cosmoi of $(\infty,1)$-categories, not just in the quasi-categorical case. In particular, we discuss some special features of the ∞-cosmos of quasi-categories, proving in particular that universal properties in this ∞-cosmos are determined pointwise, again appealing to model-independence to generalize this result to other ∞-cosmoi of $(\infty,1)$-categories.

12.1. Cores and opposites of $(\infty,1)$-categories

The construction of the co-dual of an ∞-cosmos in Definition 1.2.25 makes use of the construction of the opposite of a simplicial set. We start by exploring the role played by this operation in the ∞-cosmos of quasi-categories and then investigate a related operation on other ∞-cosmoi of ∞-categories.

12.1.1. Recall. There is an identity-on-objects involution $(-)^\circ\colon \mathbf{\Delta} \to \mathbf{\Delta}$ that reverses the ordering of the elements in each ordinal $[n] \in \mathbf{\Delta}$, sending a face map $\delta^i\colon [n-1] \rightarrowtail [n]$ to the face map $\delta^{n-i}\colon [n-1] \rightarrowtail [n]$ and sending the degeneracy map $\sigma^i\colon [n+1] \twoheadrightarrow [n]$ to the degeneracy map $\sigma^{n-i}\colon [n+1] \twoheadrightarrow [n]$. Precomposition with $(-)^\circ$ defines a functor $(-)^{\mathrm{op}}\colon s\mathcal{S}et \to s\mathcal{S}et$ which carries a simplicial set X to its **opposite simplicial set** X^{op}.

12.1.2. Lemma. *If X is a quasi-category, then X^{op} is a quasi-category.*

PROOF. The lifting problem below-left is solved by the lifting problem below-right

$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d, hook] & X^{\mathrm{op}} \\ \Delta[n] \arrow[ur, dashed] \end{tikzcd} \qquad \begin{tikzcd} \Lambda^{n-k}[n] \arrow[r] \arrow[d, hook] & X \\ \Delta[n] \arrow[ur, dashed] \end{tikzcd}$$

□

12.1.3. DEFINITION. For a quasi-category A, its **opposite quasi-category** A^{op} is the simplicial set defined by "reversing the ordering of the elements in each ordinal."

12.1.4. LEMMA. *The opposite quasi-category construction defines a cosmological functor* $(-)^{\mathrm{op}}\colon \mathcal{QC}at \to \mathcal{QC}at^{\mathrm{co}}$ *that acts on functor spaces via a natural isomorphism*

$$\mathsf{Fun}(A, B) \xrightarrow{\cong} \mathsf{Fun}(A^{\mathrm{op}}, B^{\mathrm{op}})^{\mathrm{op}}.$$

PROOF. The isomorphism $\mathsf{Fun}(A, B) \cong \mathsf{Fun}(A^{\mathrm{op}}, B^{\mathrm{op}})^{\mathrm{op}}$ is best understood at the level of simplices: the simplicial maps $A \times \Delta[n] \to B$ that define n-simplices in the left-hand functor space correspond bijectively via the isomorphism $(-)^{\mathrm{op}}\colon s\mathcal{S}et \to s\mathcal{S}et$ to simplicial maps $A^{\mathrm{op}} \times \Delta[n]^{\mathrm{op}} \to B^{\mathrm{op}}$, these defining the n-simplices in the right-hand functor space.

By an extension of the proof of Lemma 12.1.2, the opposite of an isofibration is an isofibration. The conical limits in $\mathcal{QC}at$, being defined pointwise in $\mathcal{S}et$, are clearly preserved by restriction along $(-)^{\circ}$. Simplicial cotensors are also preserved: for a quasi-category B and a simplicial set X, $(B^X)^{\mathrm{op}} \cong (B^{\mathrm{op}})^{X^{\mathrm{op}}}$, which accords with the general construction of the cotensor of $B^{\mathrm{op}} \in \mathcal{QC}at^{\mathrm{co}}$ with a simplicial set X as noted in Definition 1.2.25. Alternatively, one can use the natural isomorphism $\mathsf{Fun}(A, B) \cong \mathsf{Fun}(A^{\mathrm{op}}, B^{\mathrm{op}})^{\mathrm{op}}$ to directly verify that the quasi-category $(B^X)^{\mathrm{op}}$ has the universal property that characterizes the cotensor in $\mathcal{QC}at^{\mathrm{co}}$. □

This extends the usual construction of the opposite of a 1-category and the corresponding 2-functor $(-)^{\mathrm{op}}\colon \mathcal{C}at \to \mathcal{C}at^{\mathrm{co}}$. On account of the explicitness of the construction given in Definition 12.1.3, the opposite of a quasi-category is defined up to isomorphism. By contrast, without any additional hypotheses, we'll only be able to define the opposite of an ∞-category in a biequivalent ∞-cosmos up to equivalence. While at first this may seem undesirable, it is arguably morally correct to give the definition in this manner, since from the model-independent point of view, the ∞-category itself ought only be considered up to equivalence.

12.1.5. DEFINITION. Let A be an ∞-category in an ∞-cosmos $\mathcal{K}$ that is biequivalent to $\mathcal{QC}at$. Define the opposite ∞-category A^{op} to be any ∞-category whose underlying quasi-category is A_0^{op}.

By Proposition 10.2.3 and Corollary 10.3.2, the underlying quasi-category functor $(-)_0\colon \mathcal{K} \to \mathcal{QC}at$ is bijective on equivalence classes of objects; hence Definition 12.1.5 is well-defined up to equivalence. This also proves

12.1.6. PROPOSITION. *Let $\mathcal{K}$ be an ∞-cosmos of $(\infty, 1)$-categories.*

(i) Any ∞-category in $\mathcal{K}$ has an opposite, well-defined up to equivalence.
(ii) For any ∞-categories $A, B \in \mathcal{K}$, there is an equivalence $\mathsf{Fun}(A, B) \simeq \mathsf{Fun}(A^{\mathrm{op}}, B^{\mathrm{op}})^{\mathrm{op}}$ that is quasi-pseudonatural in A and in B.

(iii) Moreover, any specified choices in (i) and (ii) assemble into a quasi-pseudofunctor $(-)^{\mathrm{op}} \colon \mathcal{K} \to \mathcal{K}^{\mathrm{co}}$ *defined as the composite of the zig-zag of cosmological biequivalences*

$$\begin{tikzcd} \mathcal{K} \arrow[r, "(-)_0", "\sim"'] \arrow[d, dashed, "(-)^{\mathrm{op}}"', "\wr"] \arrow[dr, phantom, "\cong"] & \mathcal{QC}at \arrow[d, "\wr\vert", "(-)^{\mathrm{op}}"'] \\ \mathcal{K}^{\mathrm{co}} \arrow[r, "\sim", "(-)_0"'] & \mathcal{QC}at^{\mathrm{co}} \end{tikzcd}$$

Proof. The underlying quasi-category functor $(-)_0 \colon \mathcal{K} \to \mathcal{QC}at$ is bijective on equivalence classes of objects by Proposition 10.2.3 and Corollary 10.3.2. Since the underlying quasi-category A_0 of any ∞-category $A \in \mathcal{K}$ has an opposite, this proves that A must have an opposite as well, and thus Definition 12.1.5 is well-defined up to equivalence, with A^{op} characterized by the equivalence of quasi-categories

$$\mathsf{Fun}(1, A^{\mathrm{op}}) \simeq \mathsf{Fun}(1, A)^{\mathrm{op}}.$$

For ∞-categories A and B, by construction $(A^{\mathrm{op}})_0 \simeq (A_0)^{\mathrm{op}}$ and $(B^{\mathrm{op}})_0 \simeq (B_0)^{\mathrm{op}}$. Composing with these equivalences, the biequivalence $(-)_0 \colon \mathcal{K} \to \mathcal{QC}at$ provides local equivalences of quasi-categories:

$$\begin{tikzcd} \mathsf{Fun}(A,B) \arrow[rr, "\sim"] \arrow[d, dashed, "\wr"'] \arrow[drr, phantom, "\cong"] & & \mathsf{Fun}(A_0,B_0) \arrow[d, "\wr\vert"] \\ \mathsf{Fun}(A^{\mathrm{op}},B^{\mathrm{op}})^{\mathrm{op}} \arrow[r, "\sim"] & \mathsf{Fun}((A^{\mathrm{op}})_0,(B^{\mathrm{op}})_0)^{\mathrm{op}} \arrow[r, "\sim"] & \mathsf{Fun}((A_0)^{\mathrm{op}},(B_0)^{\mathrm{op}})^{\mathrm{op}} \end{tikzcd}$$

which compose to define the desired equivalence in such a way that the square commutes up to a homotopy coherent isomorphism $\mathsf{Fun}(A,B) \times \mathbb{I} \to \mathsf{Fun}((A_0)^{\mathrm{op}},(B_0)^{\mathrm{op}})^{\mathrm{op}}$.

The final claim is a special case of Corollary 10.4.16, whose proof specializes to recover the constructions in (i) and (ii). This also proves the quasi-pseudonaturality statement of (ii), via Lemma 10.4.11. □

On account of the equivalence $\mathsf{Fun}(A,B) \simeq \mathsf{Fun}(A^{\mathrm{op}},B^{\mathrm{op}})^{\mathrm{op}}$, a functor between ∞-categories $f \colon A \to B$ has an opposite functor $f^{\mathrm{op}} \colon A^{\mathrm{op}} \to B^{\mathrm{op}}$, well-defined up to isomorphism. Furthermore:

12.1.7. Lemma. *Let $\mathcal{K}$ be an ∞-cosmos of $(\infty,1)$-categories.*

(i) For any ∞-category A and simplicial set U, $(A^U)^{\mathrm{op}} \simeq (A^{\mathrm{op}})^{U^{\mathrm{op}}}$.

(ii) For any functors $f \colon B \to A$ and $g \colon C \to A$, $\mathsf{Hom}_A(f,g)^{\mathrm{op}} \simeq \mathsf{Hom}_{A^{\mathrm{op}}}(g^{\mathrm{op}}, f^{\mathrm{op}})$ over $B^{\mathrm{op}} \times C^{\mathrm{op}}$.

Proof. We have a quasi-pseudonatural equivalence:

$$\begin{aligned} \mathsf{Fun}(X,(A^U)^{\mathrm{op}}) &\simeq \mathsf{Fun}(X^{\mathrm{op}},A^U)^{\mathrm{op}} && \text{by } 12.1.6(ii) \\ &\cong (\mathsf{Fun}(X^{\mathrm{op}},A)^U)^{\mathrm{op}} && \text{by } (1.2.7) \\ &\cong (\mathsf{Fun}(X^{\mathrm{op}},A)^{\mathrm{op}})^{U^{\mathrm{op}}} && \text{by } 12.1.4 \\ &\simeq \mathsf{Fun}(X,A^{\mathrm{op}})^{U^{\mathrm{op}}} && \text{by } 12.1.6(ii) \\ &\cong \mathsf{Fun}(X,(A^{\mathrm{op}})^{U^{\mathrm{op}}}) && \text{by } (1.2.7). \end{aligned}$$

Hence, by Lemma 10.4.13, $(A^U)^{\mathrm{op}} \simeq (A^{\mathrm{op}})^{U^{\mathrm{op}}}$.

The second statement is an application of Lemma 10.4.17 to the composite quasi-pseudofunctorial biequivalence $(-)^{\mathrm{op}} \colon \mathcal{K} \to \mathcal{K}^{\mathrm{co}}$ established in Proposition 12.1.6. □

We now argue that the opposite of an ∞-category behaves like you'd expect. For instance, Exercise 12.1.i reveals that $\mathsf{h}(A^{\mathrm{op}}) \simeq (\mathsf{h}A)^{\mathrm{op}}$. More importantly, the following result provides another perspective on "appeals to duality" where facts about colimits of diagrams in $\mathcal{K}$ were deduced from corresponding proofs about limits in $\mathcal{K}^{\mathrm{co}}$, and similarly results about cartesian fibrations were interpreted in $\mathcal{K}^{\mathrm{co}}$ to conclude the corresponding results about cocartesian fibrations in $\mathcal{K}$.

12.1.8. Proposition. *Let $\mathcal{K}$ be an ∞-cosmos of $(\infty,1)$-categories.*

- *(i) A J-shaped family of diagrams in A has a colimit if and only if the transposed J^{op}-shaped family of diagrams in A^{op} has a limit.*
- *(ii) A functor $p\colon E \twoheadrightarrow B$ defines a cartesian fibration if and only if $p^{\mathrm{op}}\colon E^{\mathrm{op}} \twoheadrightarrow B^{\mathrm{op}}$ defines a cocartesian fibration.*

Note that if $p\colon E \twoheadrightarrow B$ is an isofibration, it is always possible to choose a functor $p^{\mathrm{op}}\colon E^{\mathrm{op}} \twoheadrightarrow B^{\mathrm{op}}$ that is again an isofibration, perhaps by changing the choice of total space E^{op}.

Proof. By Lemma 12.1.7, a J-shaped family of diagrams $d\colon D \to A^J$ transposes to define a J^{op}-shaped family of diagrams $d^{\mathrm{op}}\colon D^{\mathrm{op}} \to (A^{\mathrm{op}})^{J^{\mathrm{op}}}$. By Proposition 4.3.1, d admits a colimit in A if and only if there is an equivalence of comma ∞-categories

$$\mathrm{Hom}_{A^J}(d,\Delta) \simeq_{A\times D} \mathrm{Hom}_A(c,A),$$

in which case the representing functor $c\colon D \to A$ defines the colimit functor.

By Lemma 12.1.7, such an equivalence exists if and only if

$$\mathrm{Hom}_{(A^{\mathrm{op}})^{J^{\mathrm{op}}}}(\Delta,d^{\mathrm{op}}) \simeq_{D^{\mathrm{op}}\times A^{\mathrm{op}}} \mathrm{Hom}_{A^{\mathrm{op}}}(A^{\mathrm{op}},c^{\mathrm{op}}),$$

an equivalence which, by Proposition 4.3.1, characterizes the limit functor $c^{\mathrm{op}}\colon D^{\mathrm{op}} \to A^{\mathrm{op}}$.

The second statement is proven similarly. By Theorem 5.1.11, $p\colon E \twoheadrightarrow B$ defines a cartesian fibration if and only if the induced functor $k\colon E^{\mathbb{2}} \to \mathrm{Hom}_B(B,p)$ admits a right adjoint whose counit is an isomorphism. By applying the quasi-pseudofunctorial biequivalence $(-)^{\mathrm{op}}\colon \mathcal{K} \to \mathcal{K}^{\mathrm{co}}$, this adjunction exists if and only if the opposite functor admits a left adjoint whose unit is an isomorphism, and Lemma 12.1.7 identifies this opposite functor as $k^{\mathrm{op}}\colon (E^{\mathrm{op}})^{\mathbb{2}} \twoheadrightarrow \mathrm{Hom}_{B^{\mathrm{op}}}(p^{\mathrm{op}},B^{\mathrm{op}})$. By the dual of Theorem 5.1.11, such an adjunction exists if and only if $p^{\mathrm{op}}\colon E^{\mathrm{op}} \twoheadrightarrow B^{\mathrm{op}}$ is a cocartesian fibration. ☐

12.1.9. Remark (on dual co/cartesian fibrations). Cartesian fibrations $p\colon E \twoheadrightarrow B$ in an ∞-cosmos of $(\infty,1)$-categories in $\mathcal{K}$, correspond to homotopy coherent diagrams indexed by the underlying quasi-category of B^{op} and valued in the quasi-category of $(\infty,1)$-categories in $\mathcal{K}$. The action on objects carries an element $b\colon 1 \to B$ to ∞-category E_b defined as the fiber of p over b. Similarly, cocartesian fibrations correspond to covariant homotopy coherent diagrams indexed by the underlying quasi-category of the base ∞-category and valued in the quasi-category of $(\infty,1)$-categories in $\mathcal{K}$.

Of course, a homotopy coherent diagram indexed by B_0^{op} could equally be regarded as a *contravariant* diagram indexed by the underlying quasi-category of B or as a *covariant* diagram indexed by the opposite of the underlying quasi-category of B. However, as observed by Barwick, Glasman, and Nardin in [4], the homotopy coherent diagram encoded by the cocartesian fibration $p\colon E^{\mathrm{op}} \twoheadrightarrow B^{\mathrm{op}}$ is *not* the same as the homotopy coherent diagram encoded by the cartesian fibration $p\colon E \twoheadrightarrow B$. Rather, the former is defined by postcomposing the latter with the simplicial functor $(-)^{\mathrm{op}}\colon \mathcal{QC}at \to \mathcal{QC}at^{\mathrm{co}}$.[1]

We now turn our attention to the construction of the Kan complex core of a quasi-category and discuss its analogue in other ∞-cosmoi of $(\infty,1)$-categories.

[1]Or, more accurately, with the $(\infty,1)$-categorical core of this simplicial functor, to be introduced momentarily.

12.1.10. Definition. For a quasi-category A, its **groupoid core** is the largest sub Kan complex $\mathsf{core}(A) \subset A$, which may be constructed as the simplicial subset containing

- all of the vertices of A,
- only those edges that define isomorphisms in A, in the sense of Definition 1.1.13,
- every higher simplex whose edges are all isomorphisms.

12.1.11. Lemma. *The simplicial set defined in the manner described in Definition 12.1.10 is a Kan complex, and indeed is the largest Kan complex contained in the quasi-category A.*

Proof. The inclusion $\mathsf{core}(A) \subset A$ constructed in Definition 12.1.10 is full on simplices of all dimensions except dimension. Thus, to see that $\mathsf{core}(A)$ is a quasi-category, we need only argue that it admits extensions along the horn $\Lambda^1[2] \hookrightarrow \Delta[2]$. By construction, a horn $\Lambda^1[2] \to \mathsf{core}(A)$ picks outs two isomorphisms in A. The filler $\Delta[2] \to A$ witnesses a composition relation in the homotopy category $\mathsf{h}(A)$; thus the composite edge is also an isomorphism, and by fullness this filler lifts to $\Delta[2] \to \mathsf{core}(A)$.

By construction $\mathsf{h}(\mathsf{core}(A))$ is a groupoid; indeed, it is the maximal subgroupoid contained in $\mathsf{h}A$. So by Corollary 1.1.15, $\mathsf{core}(A)$ is a Kan complex.

Finally, a large simplicial subset of $\mathsf{core}(A) \subsetneq K \subset A$ would necessarily contain an additional edge $f\colon x \to y$. If K were a Kan complex, then it would have to admit fillers for $\Lambda^0[2]$- and $\Lambda^2[2]$-horns whose 2nd or 0th faces, respectively, were the 1-simplex f, and whose remaining face is degenerate. The fillers would construct left and right inverses to f in $\mathsf{h}(A)$. Hence, f is an isomorphism in A and already lives in $\mathsf{core}(A)$. $\square$

The inclusion defines a cosmological functor $\mathcal{K}an \hookrightarrow \mathcal{QC}at$, as an instance of Proposition 6.1.6. Functors of quasi-categories preserve isomorphisms, so a functor $f\colon A \to B$ restricts to a functor $f\colon \mathsf{core}(A) \to \mathsf{core}(B)$. In this way the groupoid core construction acts functorially on the underlying category of $\mathcal{QC}at$ and, as an unenriched functor, is right adjoint to the inclusion $\mathcal{K}an \hookrightarrow \mathcal{QC}at$. Note, however, as discussed in Example 1.3.5, that the core construction is not simplicial, at least not with respect to the usual quasi-categorical enrichment of $\mathcal{QC}at$. Indeed, a natural transformation between functors of quasi-categories will only restrict to groupoid cores if each of its components is invertible.

The groupoid core does, however, define a simplicial functor with respect to a new enrichment that we now introduce. An ∞-cosmos is a type of $(\infty,2)$-category since it is a category enriched over a model of $(\infty,1)$-categories. We now introduce the $(\infty,1)$-**categorical core** of an ∞-cosmos. In the following definition, note that since $\mathsf{core}(-)\colon \mathcal{QC}at \to \mathcal{K}an$ is an (unenriched) right adjoint, it preserves products so we may apply it to the functor spaces of a quasi-categorically enriched category to construct a Kan complex enriched subcategory that we now introduce:

12.1.12. Definition ($(\infty,1)$-core of an ∞-cosmos). For any ∞-cosmos $\mathcal{K}$, write $\mathsf{core}_*\mathcal{K} \subset \mathcal{K}$ for the subcategory with the same objects and with homs defined to be the groupoid cores of the functor spaces of $\mathcal{K}$. We refer to $\mathsf{core}_*\mathcal{K}$ as the $(\infty,1)$-**core** of $\mathcal{K}$ and think of it as being the core $(\infty,1)$-category inside this $(\infty,2)$-category.

12.1.13. Remark. The $(\infty,1)$-categorical core is not an ∞-cosmos in the strict sense that we've axiomatized in Definition 1.2.1. It inherits its class of isofibrations and the conical limits from the original ∞-cosmos, but simplicial cotensors exist only weakly: the cotensor of an ∞-category A in $\mathsf{core}_*\mathcal{K}$ by a simplicial set U is constructed by the cotensor in $\mathcal{K}$ of A by a Kan complex replacement $\tilde{U}$ of U, defined by "freely inverting" its edges and adding fillers for horns. This results in an equivalence

$\mathsf{core}(\mathsf{Fun}(X,A))^U \simeq \mathsf{core}(\mathsf{Fun}(X,A^{\tilde{U}}))$ in place of the usual isomorphism. Alternatively, Exercise 12.1.iii suggests an alternate approach to defining the enrichment of an ∞-cosmos in such a way that the $(\infty,1)$-core remains an ∞-cosmos.

12.1.14. Lemma. *The natural inclusion $\mathcal{K}an \hookrightarrow \mathcal{Q}Cat$ factors through the inclusion $\mathsf{core}_*\mathcal{Q}Cat \subset \mathcal{Q}Cat$ and this latter functor admits a simplicially enriched right adjoint left inverse, namely the functor that sends each quasi-category to its groupoid core.*

$$\begin{tikzcd} \mathcal{K}an \arrow[r, hook, bend left] \arrow[r, phantom, "\perp"] & \mathsf{core}_*\mathcal{Q}Cat \arrow[l, dashed, bend left, "\mathsf{core}"] \end{tikzcd}$$

Proof. If K and L are Kan complexes, then so is $\mathsf{Fun}(K,L)$. Hence the natural inclusion $\mathcal{K}an \hookrightarrow \mathcal{Q}Cat$ factors through the $(\infty,1)$-categorical core.

The right adjoint $\mathsf{core}\colon \mathsf{core}_*\mathcal{Q}Cat \to \mathcal{Q}Cat$ acts on objects by the construction of Definition 12.1.10. To define its action on functor spaces, we must supply a canonical map

$$\mathsf{core}(\mathsf{Fun}(A,B)) \to \mathsf{Fun}(\mathsf{core}(A),\mathsf{core}(B)),$$

for any pair of quasi-categories A and B. By Corollary D.4.18, the isomorphism in $\mathsf{Fun}(A,B)$ are simplicial maps $\alpha\colon A \times \Delta[1] \to B$ whose components $\alpha_a\colon \Delta[1] \to B$, indexed by vertices a of A, define isomorphisms in B. Combining this observation with Definition 12.1.10, we see that an n-simplex in $\mathsf{core}(\mathsf{Fun}(A,B))$ is a simplicial map $\phi\colon A \times \Delta[n] \to B$ with the property that upon restriction to any vertex of A and any edge of $\Delta[n]$, the resulting edge in B is an isomorphism. When A is restricted to its Kan complex core, the edges of $\mathsf{core}(A)$ are also isomorphisms. It follows that $\phi\colon \mathsf{core}(A)\times\Delta[n] \to B$ carries every edge of the domain to an isomorphism in B, and hence factors through $\mathsf{core}(B) \hookrightarrow B$, since this inclusion is full on the invertible edges.[2] Thus the n-simplex ϕ restricts to define an n-simplex $\phi\colon \mathsf{core}(A)\times\Delta[n] \to \mathsf{core}(B)$. This defines the canonical map.

Now for a Kan complex K and quasi-category A, the simplicial natural isomorphism

$$\mathsf{core}(\mathsf{Fun}(K,A)) \cong \mathsf{Fun}(K,\mathsf{core}(A))$$

is easily verified. The correspondence on vertices expresses the unenriched adjunction, while the correspondence on higher simplices follows for the reason just discussed and the isomorphism $\mathsf{core}(K) \cong K$. □

12.1.15. Corollary. *If A and B are equivalent quasi-categories, then $\mathsf{core}(A)$ and $\mathsf{core}(B)$ are equivalent Kan complexes.*

Proof. An equivalence of quasi-categories is specified by a pair of 0-arrows together a pair of invertible 1-arrows. As such it is contained in the $(\infty,1)$-categorical core $\mathsf{core}_*\mathcal{Q}Cat \hookrightarrow \mathcal{Q}Cat$ and preserved by the simplicial functor $\mathsf{core}\colon \mathsf{core}_*\mathcal{Q}Cat \to \mathcal{K}an$. □

The core of an ∞-category in a general ∞-cosmos of $(\infty,1)$-categories can be defined in a similar manner to Definition 12.1.5, but this notion also has an up-to-equivalence universal property that we prefer to use as the definition.

[2]In the language of marked simplicial sets, a map in $\mathsf{core}(\mathsf{Fun}(A,B))$ is a marked map $A^\natural \times \Delta[n]^\sharp \to B^\natural$. Upon restriction along $\mathsf{core}(A)^\sharp \hookrightarrow A^\natural$, the domain $\mathsf{core}(A)^\sharp \times \Delta[n]^\sharp$ is maximally marked, and hence factors through the maximally marked core $\mathsf{core}(B)^\sharp \hookrightarrow B^\natural$. See §D.4.

12.1.16. DEFINITION. Let $\mathcal{K}$ be an ∞-cosmos of $(\infty, 1)$-categories and let A be an ∞-category in $\mathcal{K}$. Its **groupoid core** is an ∞-category $\mathsf{core}(A)$ equipped with a map $\iota\colon \mathsf{core}(A) \to A$ so that

- $\mathsf{core}(A)$ is a discrete ∞-category, meaning that $\mathsf{Fun}(X, A)$ is a Kan complex for all X
- if G is a discrete ∞-category, then ι defines an equivalence

$$\mathsf{Fun}(G, \mathsf{core}(A)) \xrightarrow[\sim]{\iota\circ -} \mathsf{core}(\mathsf{Fun}(G, A))$$

In practice, an ∞-cosmos of $(\infty, 1)$-categories frequently comes with an explicit core functor.

12.1.17. PROPOSITION. *Let $\mathcal{K}$ be an ∞-cosmos of $(\infty, 1)$-categories.*

(i) Any ∞-category in $\mathcal{K}$ has a core, well-defined up to equivalence.
(ii) For any ∞-categories $A, B \in \mathcal{K}$, there is an map $\mathsf{core}(\mathsf{Fun}(A, B)) \to \mathsf{Fun}(\mathsf{core}(A), \mathsf{core}(B))$ that is quasi-pseudonatural in A and in B as objects of $\mathsf{core}_*\mathcal{K}$.
(iii) Moreover, any specified choices in (i) and (ii) assemble into a quasi-pseudofunctor $\mathsf{core}\colon \mathsf{core}_*\mathcal{K} \to \mathcal{D}isc(\mathcal{K})$ defined as the composite of the zig-zag of simplicial functors

$$\begin{tikzcd} \mathsf{core}_*\mathcal{K} \arrow[r, "(-)_0", "\sim"'] \arrow[d, dashed, "\mathsf{core}(-)"'] \arrow[dr, phantom, "\cong"] & \mathsf{core}_*\mathcal{Q}Cat \arrow[d, "\mathsf{core}(-)"] \\ \mathcal{D}isc(\mathcal{K}) \arrow[r, "\sim", "(-)_0"'] & \mathcal{D}isc(\mathcal{Q}Cat) \cong \mathcal{K}an \end{tikzcd}$$

Implicitly in the statement of (iii) we have asserted that a cosmological biequivalence $\mathcal{K} \xrightarrow{\sim} \mathcal{L}$ descends to a cosmological biequivalence $\mathcal{D}isc(\mathcal{K}) \xrightarrow{\sim} \mathcal{D}isc(\mathcal{L})$ and a simplicially enriched biequivalence $\mathsf{core}_*\mathcal{K} \xrightarrow{\sim} \mathsf{core}_*\mathcal{L}$. The first statement follows from Proposition 10.3.5(iii) while the second follows from Corollary 12.1.15. We leave the details to Exercise 12.1.iv.

PROOF. By Proposition 10.4.15 the inverse to the cosmological biequivalence $(-)_0\colon \mathcal{D}isc(\mathcal{K}) \xrightarrow{\sim} \mathcal{K}an$, defines a quasi-pseudofunctor and biequivalence $\mathcal{K}an \rightsquigarrow \mathcal{D}isc(\mathcal{K})$, which composes with the simplicial functor $\mathsf{core}((-)_0)\colon \mathsf{core}_*\mathcal{K} \to \mathcal{K}an$ to define the quasi-pseudofunctor $\mathsf{core}\colon \mathsf{core}_*\mathcal{K} \to \mathcal{D}isc(\mathcal{K})$ claimed in (iii).

By Lemma 10.4.11, the action on homs of this quasi-pseudofunctor defines a quasi-pseudonatural transformation

$$\mathsf{core}(\mathsf{Fun}(A, B)) \to \mathsf{Fun}(\mathsf{core}(A), \mathsf{core}(B)),$$

as required in (ii).

It remains only to verify that the action on objects of the quasi-pseudofunctor satisfies the conditions of Definition (12.1.16). By construction, $\mathsf{core}(A)$ is a discrete ∞-category for any $A \in \mathcal{K}$. The map $\iota\colon \mathsf{core}(A) \to A$ is defined by composing whiskering the corresponding inclusion of the Kan complex core of quasi-category with the underlying quasi-category functor and its quasi pseudofunctorial inverse:

$$\begin{tikzcd} \mathsf{core}_*\mathcal{K} \arrow[r, "(-)_0", "\sim"'] & \mathsf{core}_*\mathcal{Q}Cat \arrow[r, bend left, "\mathsf{core}"] \arrow[r, bend right, equal] \arrow[r, phantom, "\Downarrow\iota"] & \mathsf{core}_*\mathcal{Q}Cat \arrow[r, "(-)_0^{-1}", "\sim"'] & \mathsf{core}_*\mathcal{K} \end{tikzcd}$$

Now if G is a discrete ∞-category, then $G_0 = \mathsf{Fun}(1, G)$ is a Kan complex, so by Lemma 12.1.14 $\iota_{A_0} \circ -\colon \mathsf{Fun}(G_0, \mathsf{core}(A_0)) \xrightarrow{\sim} \mathsf{coreFun}(G_0, \mathsf{core}(A_0))$ is an equivalence. By construction $\mathsf{core}(A)$ is defined so that $\mathsf{core}(A)_0 \simeq \mathsf{core}(A_0)$. Note that since $(-)_0\colon \mathcal{K} \xrightarrow{\sim} \mathcal{Q}Cat$ is a biequivalence, this shows that the core of an ∞-category in an ∞-cosmos of $(\infty,1)$-categories is well-defined up to equivalence. Since the simplicial functor $(-)_0 \colon \mathsf{core}_*\mathcal{K} \to \mathsf{core}_*\mathcal{QC}at$ is an equivalence on homs, the functor defined by post-composition with ι_A is equivalent to this functor

$$\begin{tikzcd} \mathsf{Fun}(G,\mathsf{core}(A)) \arrow[r, "\iota_A\circ-"] \arrow[d, "(-)_0"', "\wr"] \arrow[dr, phantom, "\cong"] & \mathsf{coreFun}(G,A) \arrow[d, "\wr"', "(-)_0"] \\ \mathsf{Fun}(G_0,\mathsf{core}(A_0)) \arrow[r, "\iota_{A_0}\circ-", "\sim"'] & \mathsf{coreFun}(G_0,A_0) \end{tikzcd}$$

Thus post-composition with ι_A induces the equivalence $\mathsf{Fun}(G,\mathsf{core}(A)) \simeq \mathsf{coreFun}(G,A)$ required by Definition (12.1.16). This completes the proof of (i). □

The core of an ∞-category can be thought of as an infinite-dimensional extension of the "fundamental groupoid" of the ∞-category A. Recall from Definition 1.4.12 that the **homotopy category** of an ∞-category A is the 1-category defined by $\mathsf{h}A := \mathsf{hFun}(1,A) := \mathsf{h}(\mathsf{Fun}(1,A))$. Similarly:

12.1.18. Definition. The **fundamental groupoid** of an ∞-category A is the 1-groupoid defined by

$$\pi_0 A := \mathsf{hFun}(1,\mathsf{core}A) \cong \mathsf{core}(\mathsf{h}(A)).$$

Exercises.

12.1.i. Exercise. Prove that the homotopy category of the opposite of an ∞-category A is equivalent to the opposite of the homotopy category of A.

12.1.ii. Exercise. Prove that a functor between ∞-categories is an equivalence if and only if its opposite functor is an equivalence.

12.1.iii. Exercise. In consultation with §D.4 and §D.5:

(i) Redefine the notion of an ∞-cosmos from Definition 1.2.1 to be a category enriched over marked simplicial sets, whose functor spaces are naturally marked quasi-categories.
(ii) Describe the construction of the groupoid core of a naturally marked quasi-category and of the $(\infty,1)$-categorical core of an ∞-cosmos with this enrichment.
(iii) Show that $(\infty,1)$-categorical cores are cotensored over simplicial sets, although these cotensors are not preserved by the inclusion $\mathsf{core}_*\mathcal{K} \hookrightarrow \mathcal{K}$.
(iv) Show that the $(\infty,1)$-categorical core of an ∞-cosmos is an ∞-cosmos, although the functor $\mathsf{core}_*\mathcal{K} \hookrightarrow \mathcal{K}$ is not cosmological.

12.1.iv. Exercise. Let $F \colon \mathcal{K} \to \mathcal{L}$ be a cosmological functor. Prove that F induces

(i) a cosmological functor $F \colon \mathcal{D}isc(\mathcal{K}) \to \mathcal{D}isc(\mathcal{L})$ and
(ii) a simplicial functor $F \colon \mathsf{core}_*\mathcal{K} \to \mathsf{core}_*\mathcal{L}$

and show moreover that both functors are biequivalences if the original functor is.

12.2. Pointwise universal properties

In ∞-cosmoi of $(\infty,1)$-categories, the terminal ∞-category 1 plays a special role which can be summarized by the slogan that "universal properties are detected pointwise." In this section, we collect together a number of results that encapsulate this slogan, which are proven through a combination of synthetic and analytic techniques.

For instance, Corollary D.4.18 proves that a natural transformation between functors between quasi-categories $\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & A \end{tikzcd}$ is a natural isomorphism if and only if it is a **pointwise isomorphism**, meaning that each of its components

$$\begin{tikzcd} 1 \arrow[r, "x"] & X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & A \end{tikzcd}$$

is invertible. Consequently:

12.2.1. Lemma. *In an ∞-cosmos of $(\infty,1)$-categories, a natural transformation is a natural isomorphism if and only if it is a pointwise isomorphism.*

Put another way, a natural transformation $\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & A \end{tikzcd}$ between functors between $(\infty,1)$-categories is an isomorphism if and only if each of its components αx defines an isomorphism in the homotopy category of A; see Definition 1.4.12.

Proof. It's clear that any natural isomorphism is a pointwise isomorphism. For the converse, we use the biequivalence $(-)_0 \colon \mathfrak{h}\mathcal{K} \xrightarrow{\sim} \mathfrak{h}\mathcal{QC}at$ of Propositions 10.2.3 and 10.3.1. Suppose $\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & A \end{tikzcd}$ is a pointwise natural isomorphism in $\mathfrak{h}\mathcal{K}$ and consider the underlying natural transformation between underlying quasi-categories $\begin{tikzcd} X_0 \arrow[r, bend left, "f_0"] \arrow[r, bend right, "g_0"'] \arrow[r, phantom, "\Downarrow\alpha_0"] & A_0 \end{tikzcd}$. By construction, vertices of $X_0 \cong \mathsf{Fun}(1, X)$ correspond bijectively to elements of X, so the underlying natural transformation α_0 is a pointwise natural isomorphism in $\mathfrak{h}\mathcal{QC}at$ as well. Thus, Corollary D.4.18 applies to prove that α_0 admits an inverse $\begin{tikzcd} X_0 \arrow[r, bend left, "g_0"] \arrow[r, bend right, "f_0"'] \arrow[r, phantom, "\Downarrow\alpha_0^{-1}"] & A_0 \end{tikzcd}$. By the full and faithfulness of the local equivalence $\mathsf{hFun}(X, A) \xrightarrow{\sim} \mathsf{hFun}(X_0, A_0)$ established in Proposition 10.3.1, this 2-cell lifts to define an inverse natural transformation $\begin{tikzcd} X \arrow[r, bend left, "g"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Downarrow\alpha^{-1}"] & A \end{tikzcd}$ witnessing the invertibility of α. □

We took advantage of a special feature of the cosmological biequivalence $(-)_0 \colon \mathcal{K} \xrightarrow{\sim} \mathcal{QC}at$ to simplify the proof of Lemma 12.2.1 that is worth calling attention to.

12.2.2. Observation (on the elements of the underlying quasi-category). By Corollary 10.3.2(ii), a cosmological biequivalence $F \colon \mathcal{K} \xrightarrow{\sim} \mathcal{L}$ induces a bijection between the isomorphism class of elements of an ∞-category $A \in \mathcal{K}$ and the isomorphism class of elements of $FA \in \mathcal{L}$, and in fact induces an equivalence of homotopy categories $\mathsf{h}A \xrightarrow{\sim} \mathsf{h}FA$, the objects of which are exactly these elements; see Exercise 10.3.i. In particular, any element $x \colon 1 \to FA$ is naturally isomorphic to an element

$Fa\colon 1 \to FA$ that is the image of an element $a\colon 1 \to A$. Since "pointwise" ∞-categorical notions — invertibility of the components of a natural isomorphism, possession of a terminal element in the fibers of a cocartesian fibration — are invariant under isomorphism, if A satisfies some pointwise criterion in $\mathcal{K}$, then FA will satisfy the corresponding pointwise criterion in $\mathcal{L}$.

But in the case of ∞-cosmoi of $(\infty,1)$-categories, Proposition 10.2.3 supplies a cosmological biequivalence $(-)_0\colon \mathcal{K} \to \mathcal{QC}at$ that acts bijectively on elements of ∞-categories. By construction, vertices of $A_0 \cong \mathsf{Fun}(1, A)$ correspond bijectively to elements of A. Thus, every element of the underlying quasi-category A_0 of an ∞-category A comes from some element of A. Consequently, as we saw in the proof of Lemma 12.2.1, "pointwise" properties may be transferred even more readily.

Using Lemma 12.2.1, we can show that an isofibration in an ∞-cosmos of $(\infty,1)$-categories is a discrete object of the slice ∞-cosmos if and only if its fibers are discrete ∞-categories. It follows that cocartesian, cartesian, or two-sided fibrations of $(\infty,1)$-categories are discrete if and only if they have discrete fibers.

12.2.3. Proposition. *Let $p\colon E \twoheadrightarrow B$ be an isofibration in an ∞-cosmos $\mathcal{K}$ of $(\infty,1)$-categories. Then p is discrete as an object of $\mathcal{K}_{/B}$ if and only if the fibers of p are discrete ∞-categories in $\mathcal{K}$.*

Proof. Any element $b\colon 1 \to B$ induces a cosmological functor $b^*\colon \mathcal{K}_{/B} \to \mathcal{K}$ which preserves discrete objects by Corollary 10.1.6. So in fact for any ∞-cosmos $\mathcal{K}$ if $p\colon E \twoheadrightarrow B$ is a discrete object in $\mathcal{K}_{/B}$, then its fibers are discrete ∞-categories.

For the converse we assume that $\mathcal{K}$ is an ∞-cosmos of $(\infty,1)$-categories and appeal to Lemma 12.2.1. To show that $p\colon E \twoheadrightarrow B$ is discrete in $\mathcal{K}_{/B}$ we must argue that the quasi-category defined by the pullback

$$\begin{tikzcd} \mathsf{Fun}_B(f\colon X \to B, p\colon E \twoheadrightarrow B) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Fun}(X,E) \arrow[d, two heads, "p_*"] \\ 1 \arrow[r, "f"] & \mathsf{Fun}(X,B) \end{tikzcd}$$

is a Kan complex. By Corollary 1.1.15, it suffices to show that its homotopy category is a groupoid. To that end, consider a 1-simplex $\alpha\colon e \to e'$ in $\mathsf{Fun}_B(f,p) \hookrightarrow \mathsf{Fun}(X,E)$, representing a natural transformation $\begin{tikzcd} X \arrow[r, bend left, "e"] \arrow[r, bend right, "e'"'] \arrow[r, phantom, "\Downarrow\alpha"] & E \end{tikzcd}$ in the homotopy 2-category. The 1-simplex α has the property that the whiskered composite $p\alpha \in \mathsf{Fun}(X,B)$ is the degenerate 1-simplex at f.[3] By Lemma 3.1.5, there is a smothering functor

$$\mathsf{hFun}_B(f\colon X \to B, p\colon E \twoheadrightarrow B) \to \mathbb{1} \underset{\mathsf{hFun}(X,B)}{\times} \mathsf{hFun}(X,E),$$

which in particular, reflects isomorphisms, so our task is to show that α defines a natural isomorphism. By Lemma 12.2.1 α is invertible if and only if its components αx are invertible for every $x\colon 1 \to X$. Since the 1-simplex $p\alpha$ is degenerate in $\mathsf{Fun}(X,B)$, the 1-simplex $p\alpha x$ is degenerate at the vertex $fx \in \mathsf{Fun}(1,B)$, which says that αx lies in the fiber over $fx\colon 1 \to B$. Since the fibers of $p\colon E \twoheadrightarrow B$ are discrete, this tells us that αx is invertible, so we conclude by Lemma 12.2.1 that α is invertible as claimed. ☐

[3]This implies, but is stronger than, the property that the whiskered composite in the homotopy 2-category equals the 2-cell id_f; see §3.6 for a discussion.

12.2.4. Proposition. *A cocartesian fibration $q\colon E \twoheadrightarrow A$ of quasi-categories admits a right adjoint right inverse $t\colon A \to E$ if and only if for each $a\colon 1 \to A$ the fiber E_a has a terminal element.*

Proof. By Corollary 3.6.11, a right adjoint right inverse to q can be interpreted as defining a terminal element in $q\colon E \twoheadrightarrow A$, considered as an object in the sliced ∞-cosmos $\mathcal{QC}at_{/A}$. By Lemma 3.6.7(i), this fibered adjunction may be pulled back along any element $a\colon 1 \to A$ to define a terminal element in the fiber E_a.

For the converse, let $ta\colon 1 \to E_a$ denote a chosen terminal element in the fiber E_a over $a\colon 1 \to A$. Lemma F.3.1 characterizes those isofibrations between quasi-categories that admit a right adjoint right inverse in terms of a lifting property. In this case, it suffices to show that any lifting problem

$$\begin{tikzcd} 1 \arrow[r, "\{n\}"'] \arrow[rr, bend left, "ta"] & \partial\Delta[n] \arrow[r, "y"] \arrow[d, hook] & E \arrow[d, two heads, "q"] \\ & \Delta[n] \arrow[r, "x"'] \arrow[ur, dashed] & A \end{tikzcd}$$

for $n \geq 1$ has a solution. To that end, consider the simplicial map $k\colon \Delta[n] \times \Delta[1] \to \Delta[n]$ defined on vertices by

$$k(i,0) := i \qquad \text{and} \qquad k(i,1) := n.$$

The composite $xk\colon \Delta[n] \times \Delta[1] \to A$ restricts to define a map $xk\colon \partial\Delta[n] \times \Delta[1] \to A$ which represents a 2-cell whose codomain, defined by evaluating at the vertex $\{1\}$ of $\Delta[1]$, is constant at a and whose domain factors through q along y. This yields a new lifting problem

$$\begin{tikzcd} \partial\Delta[n] \arrow[r, "y"] \arrow[d, hook, "i_0"'] & E \arrow[d, two heads, "q"] \\ \partial\Delta[n] \times \Delta[1] \arrow[r, "xk"'] \arrow[ur, dashed, "z"'] & A \end{tikzcd}$$

which Lemma F.4.9 enables us to solve. By Lemma F.4.9, the lift z represents a q-cocartesian lift ζ of the 2-cell κ represented by the restriction of xk:

$$\begin{tikzcd} \partial\Delta[n] \arrow[r, "y"] \arrow[d, "!"'] \arrow[dr, phantom, "\Downarrow\kappa"] & E \arrow[d, two heads, "q"] \\ 1 \arrow[r, "a"'] & A \end{tikzcd} = \begin{tikzcd} \partial\Delta[n] \arrow[r, bend left, "y"] \arrow[r, phantom, "\Downarrow\zeta"] \arrow[r, dashed, bend right, "u"'] \arrow[d, "!"'] & E \arrow[d, two heads, "q"] \\ 1 \arrow[r, "a"'] & A \end{tikzcd}$$

By construction, the codomain functor of the q-cocartesian lift displayed above right lands in the fiber over a. Now the component of κ at the final vertex $\{n\}\colon 1 \to \partial\Delta[n]$ is id_a, so by Lemma 5.1.4(ii), the component $z\{n\}$ representing the 2-cell $\zeta\{n\}$ is an isomorphism. In particular, the element $u\{n\}\colon 1 \to E$ is isomorphic to the terminal element $y\{n\} = ta$ of E_a, so we may apply the universal property of Proposition F.1.1(v) to extend u to a simplex:

$$\begin{tikzcd} \partial\Delta[n] \arrow[r, "u"] \arrow[d, hook] & E_a \\ \Delta[n] \arrow[ur, dashed, "v"'] \end{tikzcd}$$

This data defines a new lifting problem

$$\begin{tikzcd}
\partial\Delta[n] \arrow[r, hook, "\iota_0"] \arrow[d, hook] \arrow[rr, bend left, "y"] & \partial\Delta[n]\times\Delta[1]\cup\Delta[n]\times\{1\} \arrow[r, "z\cup v"] \arrow[d, hook] & E \arrow[d, two heads, "q"] \\
\Delta[n] \arrow[r, hook, "\iota_0"] \arrow[rr, bend right, "x"'] & \Delta[n]\times\Delta[1] \arrow[r, "xk"] \arrow[ur, dashed] & A
\end{tikzcd}$$

which we solve inductively by choosing lifts of the $n+1$ $(n+1)$-simplices in $\Delta[n]\times\Delta[1]$ not present in $\partial\Delta[n]\times\Delta[1]\cup\Delta[n]\times\{1\}$, starting from the $n+1$-simplex that contains the face $\Delta[n]\times\{1\}$. All but the last of these can be lifted by means of lifting inner horns against the isofibration q. For the final simplex, we must solve an outer horn lifting problem

$$\begin{tikzcd}
\Delta[1] \arrow[r, hook, "\{n,n+1\}"'] \arrow[rr, bend left, "z\{n\}"] & \Lambda^{n+1}[n+1] \arrow[r] \arrow[d, hook] & E \arrow[d, two heads, "q"] \\
& \Delta[n+1] \arrow[r] & A
\end{tikzcd}$$

but in this case the final edge of the outer horn is the isomorphism $z\{n\}$, so Proposition 1.1.14 permits its solution as well. Now this lift restricts to define the sought-for solution to the original lifting problem, proving that $q\colon E \twoheadrightarrow A$ admits a right adjoint right inverse. $\square$

12.2.5. Remark (on duals). Via the cosmological isomorphism $(-)^{\mathrm{op}}\colon \mathcal{QC}at \to \mathcal{QC}at^{\mathrm{co}}$ of Lemma 12.1.4, the proof of Proposition 12.2.4 dualizes to prove that a cartesian fibration of quasi-categories admits a left adjoint right inverse if and only if each fiber has an initial element. See Exercise 12.2.i.

The proof of Proposition 12.2.4 relied heavily on "analytic" techniques. Nonetheless its conclusion transfers to any ∞-cosmos that is biequivalent to the ∞-cosmos of quasi-categories.

12.2.6. Proposition. *In an ∞-cosmos of $(\infty,1)$-categories a cocartesian fibration $q\colon E \twoheadrightarrow A$ of admits a right adjoint right inverse $t\colon A \to E$ if and only if for each $a\colon 1 \to A$ the fiber E_a has a terminal element.*

Proof. The argument that the fibers of an isofibration with right adjoint right inverse admit terminal elements is the same as given in the proof of Proposition 12.2.4. For the converse, suppose $q\colon E \twoheadrightarrow A$ is a cocartesian fibration in an ∞-cosmos $\mathcal{K}$ that is biequivalent to $\mathcal{QC}at$ with the property that for each element $a\colon 1 \to A$ of the base, the fiber E_a has a terminal element. By Proposition 10.2.3, we may use the biequivalence underlying quasi-category functor $(-)_0\colon \mathcal{K} \xrightarrow{\sim} \mathcal{QC}at$ to conclude that q admits a right adjoint right inverse.

Cosmological functors preserve cocartesian fibrations, so the underlying map $q_0\colon E_0 \twoheadrightarrow A_0$ defines a cocartesian fibration of quasi-categories. By construction, vertices of $A_0 \cong \mathsf{Fun}(1,A)$ correspond bijectively to elements of A. Put another way, while Corollary 10.3.2(ii) tells us that a cosmological biequivalence

By Observation 12.2.2, elements of the underlying quasi-category A_0 correspond bijectively to elements of the ∞-category A. By hypothesis, for every $a\colon 1 \to A$ the ∞-category E_a admits a terminal element, so by Proposition 10.1.9 the underlying quasi-category $(E_a)_0$ does as well. In this way, we see that every fiber of the cocartesian fibration of quasi-categories $q_0 \colon E_0 \twoheadrightarrow A_0$ admits a terminal element. By Proposition 12.2.4, q_0 admits a right adjoint right inverse. Now by Proposition 10.3.5(ii), we may conclude that $q \colon E \twoheadrightarrow A$ admits a right adjoint right inverse in $\mathcal{K}$, as desired. □

An important special case of Proposition 12.2.6 proves a result promised in the discussion surrounding Proposition 4.1.5: in an ∞-cosmos of $(\infty, 1)$-categories a functor $f \colon B \to A$ admits a right adjoint just when for each element $a \colon 1 \to A$, the ∞-category $\mathsf{Hom}_A(f, a)$ admits a terminal element. This describes the components of the counit of an adjunction as a "universal arrow" from the functor f to the element a, in the terminology used by Mac Lane [**61**, §III.1].

12.2.7. Corollary. *In an ∞-cosmos of $(\infty, 1)$-categories, a functor $f \colon B \to A$ admits a right adjoint if and only if for each element $a \colon 1 \to A$, the comma ∞-category $\mathsf{Hom}_A(f, a)$ admits a terminal element. Dually, $f \colon B \to A$ admits a left adjoint if and only if for every a, $\mathsf{Hom}_A(a, f)$ admits an initial element.*

Proof. Proposition 4.1.5 demonstrates that in any ∞-cosmos, $f \colon B \to A$ admits a right adjoint if and only if $\mathsf{Hom}_A(f, A)$ "admits a terminal element over A," meaning that $p_1 \colon \mathsf{Hom}_A(f, A) \twoheadrightarrow A$ admits a right adjoint right inverse. By Corollary 5.4.13, this functor is a cocartesian fibration, so Proposition 12.2.6 tells us that p_1 admits a right adjoint right inverse if and only if each fiber $\mathsf{Hom}_A(f, a)$ admits a terminal element.

The dual result follows from the dual of Proposition 12.2.6, whose proof is discussed in Exercises 12.2.i and 12.2.ii. □

Proposition 12.2.6 implies that modules between $(\infty, 1)$-categories admit an analogous "pointwise" representability condition, characterizing those modules that are covariantly or contravariantly represented by a functor in the sense of Definition 7.4.8.

12.2.8. Corollary. *In an ∞-cosmos of $(\infty, 1)$-categories, a module $A \overset{E}{\nrightarrow} B$ is covariantly represented if and only if for each $a \colon 1 \to A$, the module $1 \overset{E(1,a)}{\nrightarrow} B$ is covariantly represented by some element $b \colon 1 \to B$, which is the case if and only if each ∞-category $E(1, a)$ admits a terminal element.*

Proof. By Proposition 8.4.9, a module $A \overset{E}{\nrightarrow} B$ encoded by a span $(q, p) \colon E \twoheadrightarrow A \times B$ is covariantly represented if and only if its left leg $q \colon E \twoheadrightarrow A$ admits a right adjoint right inverse $r \colon A \to E$, in which case $E \simeq \mathsf{Hom}_B(B, pr)$. By Lemma 7.4.3, the left left $q \colon E \twoheadrightarrow A$ defines a cocartesian fibration, so by Proposition 12.2.6, q admits a right adjoint right inverse if and only if the fiber E_a over each element $a \colon 1 \to A$ admits a terminal element. For each $a \colon 1 \to A$, the module $1 \overset{E(1,a)}{\nrightarrow} B$ is given by the pullback along $a \times \mathrm{id} \colon 1 \times B \to A \times B$ and hence is isomorphic to the module $1 \overset{E_a}{\nrightarrow} B$. Applying Proposition 8.4.9 again, this module is covariantly represented by some element $b \colon 1 \to B$ if and only if $! \colon E_a \cong E(1, a) \twoheadrightarrow 1$ admits a right adjoint right inverse, which is the case if and only if this ∞-category admits a terminal element. □

12.2.9. Theorem. *In any ∞-cosmos of $(\infty, 1)$-categories:*

(i) A functor $g \colon C \to A$ admits an absolute right lifting through a functor $f \colon B \to A$ if and only if for all $c \colon 1 \to C$, the comma ∞-category $\mathsf{Hom}_A(f, gc)$ admits a terminal element.

(ii) *A triangle as below-left*

$$\begin{tikzcd} & B \arrow[d, "f"] \\ C \arrow[ur, "r"] \arrow[r, "g"'] \arrow[ur, phantom, shift right=3, "\Downarrow\rho"] & A \end{tikzcd} \qquad \begin{tikzcd} & B \arrow[d, "f"] \\ 1 \arrow[ur, "rc"] \arrow[r, "gc"'] \arrow[ur, phantom, shift right=3, "\Downarrow\rho c"] & A \end{tikzcd}$$

displays r as an absolute right lifting of g through f if and only if for all $c\colon 1 \to C$, the restricted triangle as above-right displays rc as an absolute right lifting of gc through f.

(iii) *A functor $g\colon C \to A$ admits an absolute right lifting through a functor $f\colon B \to A$ if and only if for all $c\colon 1 \to C$ there exists an absolute right lifting of gc through f as below-left*

$$\begin{tikzcd} & B \arrow[d, "f"] \\ 1 \arrow[ur, "r_c"] \arrow[r, "gc"'] \arrow[ur, phantom, shift right=3, "\Downarrow\rho_c"] & A \end{tikzcd} \qquad \begin{tikzcd} & B \arrow[d, "f"] \\ C \arrow[ur, "r"] \arrow[r, "g"'] \arrow[ur, phantom, shift right=3, "\Downarrow\rho"] & A \end{tikzcd}$$

in which case the components of the absolute right lifting of g through f are isomorphic to the corresponding pointwise absolute liftings: $r_c \cong rc$ and hence $\rho_c \cong \rho c$.

In the ∞-cosmos of quasi-categories, Corollary D.6.6 provides an equivalence over A between the comma quasi-category $\mathsf{Hom}_A(A, gc)$ and Joyal's slice quasi-category $A_{/gc}$, which pulls back to an equivalence $\mathsf{Hom}_A(f, gc) \simeq f_{/gc}$.

Proof. Theorems 3.5.7 and 3.5.11 demonstrate that in any ∞-cosmos, a functor $g\colon C \to A$ admits a right lifting through $f\colon B \to A$ if and only if the codomain-projection functor $p_1\colon \mathsf{Hom}_A(f,g) \twoheadrightarrow C$ admits a right adjoint right inverse. By Corollary 5.4.13 this functor is a cocartesian fibration. Now Proposition 12.2.6 proves that in an ∞-cosmos of $(\infty,1)$-categories, $p_1\colon \mathsf{Hom}_A(f,g) \twoheadrightarrow C$ admits a right adjoint right inverse if and only if each fiber $\mathsf{Hom}_A(f,gc)$ over an element $c\colon 1 \to C$ admits a terminal element. This proves the first statement.

Since absolute lifting diagrams restrict, it is immediately clear that any absolute right lifting diagram as above-left, restricts to define a pointwise absolute right lifting diagram as above-right. For the converse, suppose $(rc, \rho c)$ defines an absolute right lifting of gc through f for any $c\colon 1 \to C$. By Theorem 3.5.11, it follows that each comma ∞-category $\mathsf{Hom}_A(f,gc)$ has a terminal element and so by (i), $g\colon C \to A$ must admit an absolute right lifting (s,σ) through $f\colon B \to A$. By its universal property, the pair (r,ρ) factors through (s,σ) via a 2-cell $\tau\colon r \Rightarrow s$ so that $\rho = \sigma \cdot f\tau$. Since $(rc,\rho c)$ and $(sc, \sigma c)$ are both absolute right lifting diagrams, we know that each component τc is an isomorphism. Hence, by Lemma 12.2.1, τ is an isomorphism, and thus (r,ρ) is also an absolute right lifting digram, as desired.

The third statement is a convenient summary of the first two. If gc admits an absolute right lifting through f then $\mathsf{Hom}_A(f,gc)$ has a terminal element and (i) guarantees the existence of an absolute right lifting of g through f. The component of (r,ρ) at c defines a second absolute right lifting of gc through f inducing the claimed isomorphisms. □

As a corollary, we may resolve the question raised in Remark 2.3.8.

12.2.10. Corollary. *In an ∞-cosmos of $(\infty,1)$-categories if A is an ∞-category that admits limits of every diagram $d\colon 1 \to A^J$ of shape J then A admits all limits of shape J: that is, the constant diagram functor admits* *a right adjoint*

$$\begin{tikzcd} A^J \arrow[r, bend left, "\mathrm{colim}"] \arrow[r, bend left, phantom, "\perp"'] \arrow[r, bend right, phantom, "\perp"] \arrow[r, bend right, "\lim"'] & A \arrow[l, "\Delta"'] \end{tikzcd}$$

Proof. By Theorem 12.2.9(iii), the functor $\mathrm{id}\colon A^J \to A^J$ admits an absolute right lifting through $\Delta\colon A \to A^J$ if and only if each $d\colon 1 \to A^J$ admits an absolute right lifting through $\Delta\colon A \to A^J$. By Definition 2.3.7 the latter condition encodes what it means for A to admit a limit of the diagram d, while by Lemma 2.3.6 the former condition encodes what it means for $\Delta\colon A \to A^J$ to admit a right adjoint. $\square$

Another proof of Theorem 12.2.9(ii) is possible. By Theorem 3.5.7, a 2-cell $\rho\colon fr \Rightarrow g$ defines an absolute right lifting if and only if the induced functor $\ulcorner\rho\urcorner\colon \mathrm{Hom}_B(B, r) \to \mathrm{Hom}_A(f, g)$ is an equivalence over $C \times B$. As we shall now discover, equivalences between co/cartesian fibrations or modules can be detected fiberwise in ∞-cosmoi of $(\infty, 1)$-categories.

12.2.11. Proposition. *A cartesian functor*

$$\begin{tikzcd} E \arrow[rr, "g"] \arrow[dr, two heads, "p"'] & & F \arrow[dl, two heads, "q"] \\ & B & \end{tikzcd}$$

between cocartesian fibrations of quasi-categories is a fibered equivalence if and only if it is a fiberwise equivalence, meaning that for each $b\colon 1 \to B$, the induced functor between fibers $g_b\colon E_b \to F_b$ is an equivalence.

Note the subtle difference in terminology between "fibered equivalences" — equivalences over B — and "fiberwise equivalences" — maps inducing equivalences on fibers over elements of B. This result and Proposition 12.2.12 to follow show that in fact these two notions coincide for ∞-cosmoi of $(\infty, 1)$-categories.

Proof. Fibered equivalences are stable under pullback to fibres, so the content is in the converse implication: that any cartesian functor between cocartesian fibrations that induces a fiberwise equivalence is necessarily an equivalence.

The cartesian functor g can be factored in the slice $\mathcal{QC}at_{/B}$ as an equivalence followed by an isofibration. By Corollary 5.1.17, the intermediate object of that factorization is again a cocartesian fibration and the isofibration from it to $q\colon F \twoheadrightarrow B$ is again a cartesian functor. Replacing $p\colon E \twoheadrightarrow B$ by the equivalence cocartesian fibration, it therefore suffices to assume that $g\colon E \twoheadrightarrow B$ is an isofibration and a cartesian functor and postulate that each induced map $g_b\colon E_b \xrightarrow{\sim}\!\!\!\!\rightarrow F_b$ is a trivial fibration. Under these assumptions, we must show that g is itself is a trivial fibration.

To that end, suppose that we are given a lifting problem

$$\begin{tikzcd} \partial\Delta[n] \arrow[r, "e"] \arrow[d, hook] & E \arrow[d, two heads, "g"] \\ \Delta[n] \arrow[r, "f"'] & F \end{tikzcd}$$

over $b\colon \Delta[n] \to B$. Consider the retract diagram

$$\begin{array}{ccccc} \Delta[n] & \xhookrightarrow{\mathrm{id}\times\{0\}} & \Delta[n]\times\Delta[1] & \xrightarrow{\quad r \quad} & \Delta[n] \\ i & \longmapsto & (i,0) & & \\ & & (i,j) & \longmapsto & \begin{cases} i & \text{if } j=0 \\ n & \text{if } j=1 \end{cases} \end{array}$$

and choose a pointwise p-cocartesian lift

$$\begin{tikzcd} \Delta[n]\times\{0\} \arrow[r, "e"] \arrow[d, hook] & E \arrow[d, two heads, "p"] \\ \Delta[n]\times\Delta[1] \arrow[r, "br"'] \arrow[ur, dashed, "\chi"] & B \end{tikzcd}$$

as permitted by Lemma F.4.9. Applying g we obtain a hollow cylinder

$$\partial\Delta[n]\times\Delta[1] \xrightarrow{\chi} E \overset{g}{\twoheadrightarrow} F$$

and since g is a cartesian functor and χ is pointwise p-cocartesian it follows that $g\chi$ is pointwise q-cocartesian. Now by construction the simplex $f\colon \Delta[n]\times\{0\} \to F$ agrees with $g\chi\colon \Delta[n]\times\Delta[1] \to F$ on the subset $\Delta[n]\times\{0\}$ where they are both defined. It follows that they combine to give a well defined simplicial map on the union of their domains and so provide us with a second lifting problem:

$$\begin{tikzcd} \Delta[n]\times\{0\}\cup\partial\Delta[n]\times\Delta[1] \arrow[r, "f\cup g\chi"] \arrow[d, hook] & F \arrow[d, two heads, "q"] \\ \Delta[n]\times\Delta[1] \arrow[r, "br"'] \arrow[ur, dashed, "\rho"] & B \end{tikzcd}$$

which can again be solved to give a pointwise q-cocartesian lift ρ by Lemma F.4.9. Note now that the retraction $r\colon \Delta[n]\times\Delta[1] \to \Delta[n]$ was constructed to map the subset $\Delta[n]\times\{1\}$ onto the vertex $\{n\}$, from which it follows that the n-simplex

$$\Delta[n]\times\{1\} \hookrightarrow \Delta[n]\times\Delta[1] \xrightarrow{br} B$$

is a degenerate image of the final vertex $b_n \coloneqq b\cdot\{n\}$. Now observe that the cylinders χ and ρ were defined to lie over $br\colon \Delta[n]\times\Delta[1] \to B$, so it follows that the restricted maps

$$\Delta[n]\times\{1\} \hookrightarrow \Delta[n]\times\Delta[1] \xrightarrow{\chi} E \qquad \Delta[n]\times\{1\} \hookrightarrow \Delta[n]\times\Delta[1] \xrightarrow{\rho} F$$

land in the fibers E_{b_n} and F_{b_n} of p and q respectively. Consequently we obtain a lifting problem

$$\begin{tikzcd} \partial\Delta[n] \arrow[r, "\chi|_{\{1\}}"] \arrow[d, hook] & E_{b_n} \arrow[d, two heads, "g_{b_n}", "\wr"'] \\ \Delta[n] \arrow[r, "\rho|_{\{1\}}"'] \arrow[ur, dashed, "\gamma"] & F_{b_n} \end{tikzcd}$$

which we may solve since the map of fibers on the right is, by assumption, a trivial fibration. Now the upper left triangle tells us that χ and γ agree on the subset $\Delta[n]\times\{1\}$ where they are both defined.

Thus, these maps combine to give the well defined simplicial map on the union of their domains depicted as the upper-horizontal in the lifting problem on the right of the following diagram:

$$\begin{tikzcd} \partial\Delta[n]\times\{0\} \arrow[r, hook] \arrow[d, hook] \arrow[rr, bend left, "e"] & \Delta[n]\times\{1\}\cup\partial\Delta[n]\times\Delta[1] \arrow[r, dashed, "\gamma\cup\chi"] \arrow[d, hook] & E \arrow[d, two heads, "g"] \\ \Delta[n]\times\{0\} \arrow[r, hook] \arrow[rr, bend right, "f"'] & \Delta[n]\times\Delta[1] \arrow[r, "\rho"] \arrow[ur, dashed] & F \end{tikzcd}$$

A standard argument shows that the lifting problem in the right-hand square can be solved by filling a sequence of inner horns and a single outer horn of shape $\Lambda^{n+1}[n+1]$ whose final edge is the cocartesian lift of the degeneracy at b_n to a simplex with domain e_n. Lemma 5.1.4 observes that cocartesian lifts of degenerate simplices are isomorphisms, so this last horn is actually a "special outer horn" with first edge invertible. Consequently, by Theorem D.4.19 it can therefore be lifted against the isofibration p. This construction fills the sphere at the domain end of the hollow cylinder, solving the original lifting problem. ☐

12.2.12. Proposition (equivalences of co/cartesian fibrations are determined fiberwise). *In an ∞-cosmos of $(\infty,1)$-categories, a cartesian functor*

$$\begin{tikzcd} E \arrow[rr, "g"] \arrow[dr, two heads, "p"'] & & F \arrow[dl, two heads, "q"] \\ & B & \end{tikzcd}$$

between cocartesian fibrations is a fibered equivalence if and only if it is a fiberwise equivalence, meaning that for each $b\colon 1 \to B$, the induced functor between fibers $g_b\colon E_b \to F_b$ is an equivalence.

Proof. Again, it is clear that an equivalence of cocartesian fibrations is necessarily a fiberwise equivalence, so we need only prove the converse. By Propositions 10.2.4 and 10.2.3 if $\mathcal{K}$ is an ∞-cosmos of $(\infty,1)$-categories, then the underlying quasi-category functor induces a cosmological biequivalence $(-)_0\colon \mathcal{K}_{/B} \to \mathcal{QC}at_{/B_0}$. By Corollary 10.3.2, this cosmological biequivalence reflects equivalences, and by Proposition 10.3.5 it preserves cocartesian fibrations and cartesian functors between them. Hence, to show that a cartesian functor and fiberwise equivalence $g\colon E \to F$ is an equivalence over B, it suffices to show that $g_0\colon E_0 \to F_0$ is an equivalence over B_0, which we do by verifying that this functor satisfies the hypotheses of Proposition 12.2.11.

By Observation 12.2.2, elements of the underlying quasi-category B_0 correspond bijectively to elements of the ∞-category B, and the cosmological functor $(-)_0\colon \mathcal{K} \to \mathcal{QC}at$ preserves both fibers and equivalences. In this way we see that $g_0\colon E_0 \to F_0$ is a fiberwise equivalence for all $b\colon 1 \to B_0$. By Proposition 12.2.11 this functor defines a fibered equivalence, and hence g does as well. ☐

Proposition 12.2.12 has the following important corollary:

12.2.13. Corollary (equivalences of modules are determined fiberwise). *In an ∞-cosmos of $(\infty,1)$-categories, a map*

$$\begin{tikzcd} E \arrow[rr, "g"] \arrow[dr, two heads, "{(q,p)}"'] & & F \arrow[dl, two heads, "{(s,r)}"] \\ & A\times B & \end{tikzcd}$$

between modules E and F from A to B is an equivalence if and only if it is a fiberwise equivalence, meaning that for each $a\colon 1 \to A$ and $b\colon 1 \to B$, the induced functor between the fibers $g_{a,b}\colon E(b,a) \to F(b,a)$ is an equivalence.

Proof. Recall from Lemma 7.4.3 that the left-hand legs $q\colon E \twoheadrightarrow A$ and $s\colon F \twoheadrightarrow A$ are cocartesian fibrations and the functor g defines a cartesian functor between them. It follows, by Proposition 12.2.12, that g is an equivalence if and only if for each $a\colon 1 \to A$, the pullback $g_a\colon E(1,a) \to F(1,a)$ is an equivalence. Now each of these ∞-categories defines a module from 1 to B, so by Lemma 7.4.3 again, the pulled back projections $p\colon E(1,a) \twoheadrightarrow B$ and $r\colon F(1,a) \to B$ are cartesian fibrations and g_a defines a cartesian functor between them. By the dual of Proposition 12.2.12, proven in Exercise 12.2.ii, this functor is an equivalence if and only if for each $b\colon 1 \to B$, the induced action $g_{a,b}\colon E(b,a) \to F(a,b)$ on fibers is an equivalence. The stated result follows immediately. $\square$

Corollary 12.2.13 may be applied to characterize fully faithful functors and equivalences between $(\infty,1)$-categories. In **[74]**, Rezk refers to this result as "the fundamental theorem of quasi-category theory."

12.2.14. Proposition. *In an ∞-cosmos of $(\infty,1)$-categories, a functor $f\colon A \to B$ is fully faithful if and only if for all elements $a,a'\colon 1 \to A$, its action on internal mapping spaces $f_{a,a'}\colon \mathsf{Hom}_A(a,a') \to \mathsf{Hom}_B(fa,fa')$ defines an equivalence of discrete ∞-categories.*

Proof. Corollary 3.5.6 defines a functor $f\colon A \to B$ to be fully faithful if and only if the induced functor $A^{\mathbb{2}} \to \mathsf{Hom}_B(f,f)$ between modules from A to A is an equivalence. By Corollary 12.2.13, this is the case if and only if this map defines a fiberwise equivalence, which means exactly that for all elements $a,a'\colon 1 \to A$, its action on internal mapping spaces $f_{a,a'}\colon \mathsf{Hom}_A(a,a') \to \mathsf{Hom}_B(fa,fa')$ defines an equivalence of discrete ∞-categories. $\square$

There is a closely related notion defined on the point-set level: a simplicial functor $F\colon \mathcal{S} \to \mathcal{T}$ between Kan complex enriched categories is **homotopically fully-faithful** if its action $F_{x,y}\colon \mathcal{S}(x,y) \to \mathcal{T}(Fx,Fy)$ on each hom-space is an equivalence of Kan complexes.

12.2.15. Lemma. *A simplicial functor $F\colon \mathcal{S} \to \mathcal{T}$ between Kan complex enriched categories is homotopically fully faithful if and only if $\mathfrak{N}F\colon \mathfrak{N}\mathcal{S} \to \mathfrak{N}\mathcal{T}$ defines a fully faithful functor of quasi-categories.*

Proof. By Corollary **??**, for any fixed pair of objects $x,y \in \mathcal{S}$, there is a natural equivalence of Kan complexes as displayed vertically below,

$$\begin{tikzcd} \mathcal{S}(x,y) \arrow[r, "F_{x,y}"] \arrow[d, tail, "\wr"] & \mathcal{T}(Fx,Fy) \arrow[d, tail, "\wr"] \\ \mathsf{Hom}_{\mathfrak{N}\mathcal{S}}(x,y) \arrow[r, "\ulcorner \mathfrak{N}F \urcorner"] & \mathsf{Hom}_{\mathfrak{N}\mathcal{T}}(Fx,Fy) \end{tikzcd}$$

Thus by Proposition 12.2.14 and the 2-of-3 property, the simplicial functor F is homotopically fully faithful if and only if $\mathfrak{N}F$ is a fully faithful functor of quasi-categories. $\square$

Combining these results, we show that equivalences of $(\infty,1)$-categories are precisely those functors that are pointwise fully faithful and essentially surjective in a suitable sense. Our proof mixes synthetic and analytic techniques:

12.2.16. Theorem. *In an ∞-cosmos of $(\infty,1)$-categories, a functor $f\colon A \to B$ is an equivalence if and only if it is* *(i)* ***fully faithful***: *in the sense that for all elements $a, a' \colon 1 \to A$, the induced map*

$$\mathrm{Hom}_A(a, a') \to \mathrm{Hom}_B(fa, fa')$$

is an equivalence and

(ii) ***essentially surjective*** *in the sense that for all $b \colon 1 \to B$ there exists $a \colon 1 \to A$ and an isomorphism $fa \cong b$ in the homotopy category of B.*

Proof. It is clear that an equivalence of ∞-categories is both fully faithful and essentially surjective. To prove the converse, we start by factoring f as an equivalence followed by an isofibration. Both factors are easily seen to be pointwise fully faithful and essentially surjective, so it suffices to assume that $f \colon A \twoheadrightarrow B$ is an isofibration. Our task is now to show that f is a trivial fibration. In an ∞-cosmos $\mathcal{K}$ of $(\infty, 1)$-categories, the cosmological biequivalence $(-)_0 \colon \mathcal{K} \xrightarrow{\sim} \mathcal{QC}at$ of Proposition 10.2.3 preserves trivial fibration and reflects equivalences. So to show that an isofibration $f \colon A \twoheadrightarrow B$ is a trivial fibration, it suffices to show that the underlying isofibration $f_0 \colon A_0 \twoheadrightarrow B_0$ is a trivial fibration, i.e., that we can solve lifting problems of the form

$$\begin{tikzcd} \partial\Delta[n] \arrow[r] \arrow[d, hook] & \mathrm{Fun}(1, A) \arrow[d, two heads, "f_0"] \\ \Delta[n] \arrow[r] \arrow[ur, dashed] & \mathrm{Fun}(1, B) \end{tikzcd} \tag{12.2.17}$$

for $n \geq 0$.

When $f_0 \colon A_0 \twoheadrightarrow B_0$ is an isofibration, the hypothesis that $f \colon A \twoheadrightarrow B$ is essentially surjective in fact implies that $f_0 \colon A_0 \twoheadrightarrow B_0$ is surjective on vertices. Recall that the homotopy category of B is defined to be $\mathrm{h}B \coloneqq \mathrm{hFun}(1, B) \coloneqq \mathrm{h}(B_0)$. Now Corollary 1.1.16 implies that any isomorphism in the homotopy category of B can be represented by a homotopy coherent isomorphism $\mathbb{I} \to \mathrm{Fun}(1, B)$. By definition, a map $\Delta[0] \to B_0$ is an element $b \colon 1 \to B$. A choice of $a \colon 1 \to A$ and a homotopy coherent isomorphism $\beta \colon \mathbb{I} \to \mathrm{Fun}(1, B)$ representing $fa \cong b$ defines a lifting problem

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner" very near start] & \mathbb{1} \arrow[r, "a"] \arrow[d, hook] & \mathrm{Fun}(1, A) \arrow[d, two heads, "f_0"] \\ \mathbb{1} \arrow[r, hook] \arrow[rr, bend right, "b"'] & \mathbb{I} \arrow[r, "\beta"] \arrow[ur, dashed] & \mathrm{Fun}(1, B) \end{tikzcd}$$

which can be solved by lifting the isomorphism along the isofibration. This solves the lifting problems (12.2.17) in the case $n = 0$.

Applying Proposition 3.4.5 to the commutative diagram below-left, we see that the induced map between modules is an isofibration.

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, equal] & A \arrow[r, equal] \arrow[d, two heads, "f"] & A \arrow[d, equal] \\ A \arrow[r, two heads, "f"'] & B & A \arrow[l, two heads, "f"] \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} & A^2 \arrow[dl, two heads, "p_1"'] \arrow[dr, two heads, "p_0"] \arrow[d, dashed, two heads, "\mathrm{Hom}_f(A,A)"] & \\ A & \mathrm{Hom}_A(f, f) \arrow[l, two heads, "p_1"] \arrow[r, two heads, "p_0"'] & A \end{tikzcd}$$

By Proposition 12.2.14 the hypothesis that f is pointwise fully faithful, inducing equivalences between fibers $f_{a,a'} \colon \mathrm{Hom}_A(a, a') \xrightarrow{\sim} \mathrm{Hom}_B(fa, fa')$, implies that the induced map $\mathrm{Hom}_A(f, f) \colon A^2 \xrightarrow{\sim}\!\!\twoheadrightarrow$

$\mathsf{Hom}_B(f,f)$ is an equivalence and hence under present hypotheses a trivial fibration. The cosmological functor carries this map to a trivial fibration between quasi-categories, which then enjoys the lifting property below-left for $n \geq 0$:

$$\begin{tikzcd} \partial\Delta[n] \arrow[r] \arrow[d, hook] & (A_0)^{\mathbb{2}} \arrow[d, two heads, "\mathsf{Hom}_{f_0}(A_0,A_0)", "\wr"'] \\ \Delta[n] \arrow[r] \arrow[ur, dashed] & \mathsf{Hom}_{B_0}(f_0,f_0) \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} \partial\Delta[n]\times\Delta[1] \underset{\partial\Delta[n]\times\partial\Delta[1]}{\cup} \Delta[n]\times\partial\Delta[1] \arrow[r] \arrow[d, hook] & A_0 \arrow[d, two heads, "f_0"] \\ \Delta[n]\times\Delta[1] \arrow[r] \arrow[ur, dashed] & B_0 \end{tikzcd}$$

Via the description of the comma construction as a weighted limit giving in Example A.6.18, the lifting property above-left transposes across the Leibniz version of the weighted limit two-variable adjunction of Definition A.6.7 to the lifting property displayed above-right, again for $n \geq 0$.

We have already shown that $f_0 \colon A_0 \twoheadrightarrow B_0$ also possesses the right lifting property with respect to the inclusion $\emptyset \hookrightarrow \Delta[0]$. However, this map together with the Leibniz product inclusions $(\partial\Delta[n] \hookrightarrow \Delta[n]) \mathbin{\widehat{\times}} (\partial\Delta[1] \hookrightarrow \Delta[1])$ generate the class of monomorphisms of simplicial sets under transfinite composition, pushout, and retract, it follows now from the fact that the map $\mathsf{Hom}_{f_0}(A_0, A_0)$ is a trivial fibration that $f_0 \colon A_0 \xrightarrow{\sim}\!\!\!\!\!\twoheadrightarrow B_0$ is a trivial fibration. Hence $f \colon A \xrightarrow{\sim}\!\!\!\!\!\twoheadrightarrow B$ is a trivial fibration, which is what we wanted to show. $\square$

Theorem 12.2.16 can be used to give a direct proof of an important result relating quasi-categories to simplicially enriched categories.

12.2.18. Proposition. *Let $F \colon \mathcal{S} \to \mathcal{T}$ be a DK-equivalence of Kan complex enriched categories: a simplicial functor that defines a local equivalence of Kan complexes and is essentially surjective on objects in the homotopy categories of $\mathcal{S}$ and $\mathcal{T}$. Then $\mathfrak{N}F \colon \mathfrak{N}\mathcal{S} \xrightarrow{\sim} \mathfrak{N}\mathcal{T}$ is an equivalence of quasi-categories.*

Proof. Cordier and Porter [**25**] prove that $\mathfrak{N}\mathcal{S}$ and $\mathfrak{N}\mathcal{T}$ are quasi-categories, so by Theorem 12.2.16 it suffices to verify that $\mathfrak{N}F$ is fully faithful and essentially surjective. The homotopy category of the quasi-category $\mathfrak{N}\mathcal{S}$ characterized in Lemma 1.1.12 is equivalent to the homotopy category constructed in Digression 6.2.13. Thus, since F is essentially surjective, $\mathfrak{N}F$ is as well. Since DK-equivalences are homotopically fully faithful, Lemma 12.2.15 shows that $\mathfrak{N}F$ is a fully faithful functor of quasi-categories. The conclusion follows. $\square$

We close with another application of the pointwise universal property results in this section, by describing a common abstract setting which gives rise to adjunctions between quasi-categories defined as homotopy coherent nerves.

12.2.19. Definition. A simplicial functor $U \colon \mathcal{A} \to \mathcal{B}$ between Kan complex enriched categories **admits a pointwise left biadjoint** if for each $b \in \mathcal{B}$ there exists an object $Fb \in \mathcal{A}$ together with a quasi-pseudonatural equivalence

$$\mathcal{A}(Fb, a) \simeq \mathcal{B}(b, Ua)$$

for all $a \in \mathcal{A}$; see Definition 10.4.9.

We now show that the functors considered in Definition 12.2.19 define genuine adjoint functors of quasi-categories upon applying the homotopy coherent nerve.

12.2.20. Proposition. *If $U \colon \mathcal{A} \to \mathcal{B}$ is a simplicial functor between Kan complex enriched categories that admits a pointwise left biadjoint, then its homotopy coherent nerve $\mathfrak{N}U \colon \mathfrak{N}\mathcal{A} \to \mathfrak{N}\mathcal{B}$ admits a left adjoint as a functor between quasi-categories.*

PROOF. By Lemma 2.3.6, $\mathfrak{N}U$ admits a left adjoint if and only if there exists an absolute left lifting of the identity through $\mathfrak{N}U\colon \mathfrak{N}\mathcal{A} \to \mathfrak{N}\mathcal{B}$. By Theorem 12.2.9, this exists if and only if for all objects $b\colon 1 \to \mathfrak{N}\mathcal{B}$ of $\mathcal{B}$ there exists an absolute left lifting of of b through $\mathfrak{N}U$. We define a candidate absolute left lifting diagram

$$\begin{tikzcd} & \mathfrak{N}\mathcal{A} \arrow[d, "\mathfrak{N}U"] \\ 1 \arrow[ur, dashed, "Fb"] \arrow[r, "b"'] & \mathfrak{N}\mathcal{B} \arrow[ul, phantom, "\Uparrow\eta_b"] \end{tikzcd}$$

by choosing the object $Fb \in \mathcal{A}$ for the lift and the 0-arrow $\eta_b\colon b \to UFb$ in $\mathcal{B}$, the image of id_{Fb} under the quasi-pseudonatural equivalence $\mathcal{A}(Fb, a) \simeq \mathcal{B}(b, Ub)$, as a representative for the 2-cell.

To prove that this data defines an absolute right lifting, we appeal to Theorem 3.5.3, which tells us this is the case if and only if η_b induces a fibered equivalence

$$\begin{tikzcd} \mathsf{Hom}_{\mathfrak{N}\mathcal{A}}(Fb, \mathfrak{N}\mathcal{A}) \arrow[rr, "\ulcorner \mathfrak{N}U(-)\cdot\eta_b \urcorner"] \arrow[dr, two heads] & & \mathsf{Hom}_{\mathfrak{N}\mathcal{B}}(b, \mathfrak{N}U) \arrow[dl, two heads] \\ & \mathfrak{N}\mathcal{A} & \end{tikzcd}$$

By Corollary 12.2.13, this map is an equivalence if and only if for all objects $a\colon 1 \to \mathfrak{N}\mathcal{A}$ of $\mathcal{A}$, the component

$$\mathsf{Hom}_{\mathfrak{N}\mathcal{A}}(Fb, a) \xrightarrow{\ulcorner \mathfrak{N}U(-)\cdot\eta_b \urcorner_a} \mathsf{Hom}_{\mathfrak{N}\mathcal{B}}(b, Ua)$$

defines an equivalence of Kan complexes. By Corollary **??**, this map is equivalent to the component

$$\mathcal{A}(Fb, a) \xrightarrow{\sim} \mathcal{B}(b, Ua)$$

of the quasi-pseudonatural equivalence, which proves the claim. □

Exercises.

12.2.i. EXERCISE. Use Lemma 12.1.4 to prove the duals of Propositions 12.2.4 and 12.2.11.

12.2.ii. EXERCISE. State and prove the duals of Proposition 12.2.6 and 12.2.12.

12.2.iii. EXERCISE. Use Corollary 12.2.13 to give a second proof of Theorem 12.2.9(ii).

12.3. Strongly generating functors

In Proposition 4.3.12 we discovered that any fully faithful functor $f\colon A \to B$ between ∞-categories reflects any limits or colimits that exist in B. In this section, we prove a companion result, that shows that any functor that is both fully faithful and strongly generating, a notion we define presently, preserves limits of any diagram in A known to also have a limit in B.

12.3.1. DEFINITION (strongly generating). A functor of ∞-categories $f\colon A \to B$ is **strongly generating** if it satisfies the property that a 2-cell

$$\begin{tikzcd} X \arrow[r, bend left, "h"] \arrow[r, bend right, "k"'] \arrow[r, phantom, "\Downarrow\beta"] & B \end{tikzcd}$$

is invertible whenever the functor $f \downarrow \beta\colon \mathsf{Hom}_B(f, h) \to \mathsf{Hom}_B(f, k)$ induced by the construction of Observation 3.4.8 is an equivalence.

12.3.2. Proposition. *In an ∞-cosmos of $(\infty,1)$-categories, a functor $f\colon A \to B$ is strongly generating if and only if an arrow $\beta\colon b \to b'$ in the homotopy category of B is invertible precisely when its action by post-composition on the intneral mapping spaces $fa \downarrow \beta\colon \mathsf{Hom}_B(fa,b) \to \mathsf{Hom}_B(fa,b')$ is an equivalence of discrete ∞-categories for all elements $a\colon 1 \to A$.*

Proof. Given functors $h,k\colon X \to B$ and a 2-cell $\beta\colon h \Rightarrow k$ as in Definition 12.3.1, the fibers of the induced map $f \downarrow \beta\colon \mathsf{Hom}_B(f,h) \to \mathsf{Hom}_B(f,k)$ over a pair of elements $x\colon 1 \to X$ and $a\colon 1 \to A$ is the action of the component $\beta x\colon hx \Rightarrow kx$ on the internal mapping spaces $fa \downarrow \beta x\colon \mathsf{Hom}_B(fa,hx) \to \mathsf{Hom}_B(fa,kx)$. So we know, by Corollary 12.2.13, that $f \downarrow \beta$ is an equivalence if and only if $fa \downarrow \beta x$ is an equivalence for all elements $a\colon 1 \to A$ and $x\colon 1 \to X$. By Lemma 12.2.1, β is an invertible 2-cell if and only if for each element $x\colon 1 \to X$ its component $\beta x\colon hx \Rightarrow kx$ is an isomorphism in the homotopy category $\mathsf{h}B$. This proves the stated result. $\square$

12.3.3. Observation. The characterization of the Proposition 12.3.2 reveals that a functor $f\colon A \to B$ between $(\infty,1)$-categories is strongly generating if and only if the set of elements

$$\{fa\colon 1 \to B \mid a\colon 1 \to A\}$$

has the property that mapping out of the elements in this set detects isomorphisms with codomain B. In particular, this characterization says nothing about the rest of the structure of the functor f. In this setting, we say that some set G of elements in an $(\infty,1)$-category B is **strongly generating** if it has this isomorphism detection property. It follows from Proposition 12.3.2 that a functor $f\colon A \to B$ is strongly generating if and only if it maps surjectively onto some strongly generating set of elements.

We now show that fully faithful and strongly generating functors of quasi-categories defined via the homotopy coherent nerve construction can be identified by analogous properties at the point-set level.

12.3.4. Definition. A simplicial functor $F\colon \mathcal{S} \to \mathcal{T}$ between Kan complex enriched categories is **homotopically strongly generating** if a 0-arrow $e\colon a \to b$ in $\mathcal{T}$ is an equivalence if and only if for all $x \in \mathcal{S}$ the map $e \circ -\colon \mathcal{T}(Fx,a) \to \mathcal{T}(Fx,b)$ is an equivalence of Kan complexes.

12.3.5. Example. Recall that a 0-arrow $e\colon a \to b$ in a Kan complex enriched category $\mathcal{T}$ is an **equivalence** if there exists a 0-arrow $f\colon b \to a$ and a pair of 1-arrows $\alpha\colon \mathrm{id}_a \to fe$ and $\beta\colon \mathrm{id}_b \to ef$. By the Kan complex enriched analog of Lemma 1.2.18, left as Exercise 12.3.i, $e\colon a \to b$ is an equivalence in $\mathcal{T}$ if and only if for all $z \in \mathcal{T}$, $e \circ -\colon \mathcal{T}(z,a) \to \mathcal{T}(z,b)$ is an equivalence of Kan complexes, which is to say that the identity functor is always homotopically strongly generating.

12.3.6. Lemma. *A simplicial functor $F\colon \mathcal{S} \to \mathcal{T}$ between Kan complex enriched categories is homotopically strongly generating if and only if $\mathfrak{N}F\colon \mathfrak{N}\mathcal{S} \to \mathfrak{N}\mathcal{T}$ is a strongly generating functor of quasi-categories.*

Proof. Any 0-arrow $e\colon a \to b$ in a Kan complex enriched category $\mathcal{T}$ gives rise to an essentially commutative square of Kan complexes whose verticals are the canonical anodyne extensions of Corollary **??**

$$\begin{tikzcd}
\mathcal{T}(Fx,a) \arrow[r, "e\circ-"] \arrow[d, tail, "\wr"'] \arrow[dr, phantom, "\cong"] & \mathcal{T}(Fx,b) \arrow[d, tail, "\wr"] \\
\mathsf{Hom}_{\mathfrak{N}\mathcal{T}}(Fx,a) \arrow[r, "e\circ-"'] & \mathsf{Hom}_{\mathfrak{N}\mathcal{T}}(Fx,b)
\end{tikzcd}$$

where in this case $x \in \mathcal{S}$. It follows that the top horizontal is an equivalence if and only if the bottom horizontal is an equivalence.

By the characterization of the equivalences mentioned in Example 12.3.5 and proven in Exercise 12.3.i, a 0-arrow $e\colon a \to b$ in $\mathcal{T}$ is an equivalence if and only if the corresponding 1-simplex $e\colon a \to b$ defines an isomorphism in the quasi-category $\mathfrak{N}\mathcal{T}$. Hence by Proposition 12.3.2 the condition that characterizes a homotopically strongly generating simplicial functor $F\colon \mathcal{S} \to \mathcal{T}$ is equivalent to the condition that characterizes a strongly generating functor of quasi-categories $\mathfrak{N}F\colon \mathfrak{N}\mathcal{S} \to \mathfrak{N}\mathcal{T}$. ☐

By Observation 12.3.3, in an ∞-cosmos of $(\infty, 1)$-categories we may sensibly refer to a strongly generating set of elements and need not specify the corresponding functor.

12.3.7. Lemma. *For any ∞-category A in an ∞-cosmos of $(\infty, 1)$-categories $\mathcal{K}$, the set of representable modules*

$$\{p_0\colon \mathsf{Hom}_A(A, a) \twoheadrightarrow A \mid a \in A\}$$

is strongly generating in the quasi-category ${}_{1\backslash}\mathsf{Mod}(\mathcal{K})_{/A} := \mathfrak{N}({}_{1\backslash}\mathcal{M}od(\mathcal{K})_{/A})$ of modules from 1 to A.

Proof. By Lemma 12.3.6 it suffices to show that the collection of covariantly represented modules is homotopically strongly generating in the ∞-cosmos ${}_{1\backslash}\mathcal{M}od(\mathcal{K})_{/A} \cong \mathcal{D}isc(\mathcal{C}art(\mathcal{K})_{/A}$. To that end, consider a map

$$\begin{tikzcd} E \arrow[rr, "g"] \arrow[dr, two heads, "p"'] & & F \arrow[dl, two heads, "q"] \\ & A & \end{tikzcd}$$

with the property that for all elements $a\colon 1 \to A$, the map of functor spaces

$$\begin{tikzcd} \mathsf{Fun}_A(\mathsf{Hom}_A(A, a), E) \arrow[r, "g\circ-", "\sim"'] \arrow[d, "\mathrm{ev}_{\ulcorner \mathrm{id}_a \urcorner}"', "\wr"] & \mathsf{Fun}_A(\mathsf{Hom}_A(A, a), F) \arrow[d, "\wr"', "\mathrm{ev}_{\ulcorner \mathrm{id}_a \urcorner}"] \\ E_a \arrow[r, "g"'] & E_b \end{tikzcd}$$

is an equivalence of Kan complexes. By the external Yoneda lemma for discrete cartesian fibrations, this is equivalent to the action $g\colon E_a \to F_a$ of g on the fibers over a, so this tells us that g is a fiberwise equivalence. Now Corollary 12.2.13 tells us that g is an equivalence, as desired. ☐

12.3.8. Proposition. *Suppose $f\colon A \to B$ is a functor between ∞-categories that is both fully faithful and strongly generating. Then f preserves limits of any family of diagrams whose limits exist in both A and B.*

Proof. Consider a family of J-shaped diagrams $d\colon D \to A^J$ so that d admits a limit in A and $f^J d$ admits a limit in B, as presented by absolute right lifting diagrams (ℓ, ρ) and (ℓ', ρ') as displayed below. By the universal property of the latter, we have a factorization

$$\begin{tikzcd} & A \arrow[r, "f"] \arrow[d, "\Delta"] & B \arrow[d, "\Delta"] \\ D \arrow[ur, "\ell"] \arrow[r, "d"'] \arrow[ur, phantom, "\Downarrow\rho", near end, shift right=3] & A^J \arrow[r, "f^J"'] & B^J \end{tikzcd} = \begin{tikzcd} & B \arrow[d, "\Delta"] \\ D \arrow[ur, bend left, "f\ell"] \arrow[ur, dashed, "\ell'"'] \arrow[r, "f^J d"'] \arrow[ur, phantom, "\Downarrow\zeta", shift left=3] \arrow[ur, phantom, "\Downarrow\rho'", near end, shift right=3] & B^J \end{tikzcd}$$

and the claim follows if we can show that ζ is invertible. Rearranging this pasting equality as

$$
\begin{tikzcd}
D \arrow[r, "\ell"] \arrow[d, equal] \arrow[dr, phantom, "\swarrow\rho"] & A \arrow[r, equal] \arrow[d, "\Delta"] & A \arrow[d, equal] \\
D \arrow[r, "d"] \arrow[d, equal] & A^J \arrow[d, "f^J"] & A \arrow[l, "\Delta"'] \arrow[d, equal] \\
D \arrow[r, "f^J d"'] & B^J & A \arrow[l, "f^J\Delta"]
\end{tikzcd}
=
\begin{tikzcd}
D \arrow[r, "\ell"] \arrow[d, equal] & A \arrow[r, equal] \arrow[d, "f"] & A \arrow[d, equal] \\
D \arrow[r, "f\ell"] \arrow[d, equal] \arrow[dr, phantom, "\swarrow\zeta"] & B \arrow[d, equal] & A \arrow[l, "f"'] \arrow[d, equal] \\
D \arrow[r, "\ell'"] \arrow[d, equal] \arrow[dr, phantom, "\swarrow\rho'"] & B \arrow[d, "\Delta"] & A \arrow[l, "f"'] \arrow[d, equal] \\
D \arrow[r, "f^J d"'] & B^J & A \arrow[l, "\Delta f"]
\end{tikzcd}
$$

we see by Observation 3.4.8 that the natural transformations ρ, ρ', and ζ induce functors between the comma ∞-categories constructed from the horizontal cospans, defining a fibered diagram that commutes up to fibered natural isomorphism:

$$
\begin{tikzcd}
& \mathsf{Hom}_A(A,\ell) \arrow[rr, "\mathrm{id}\downarrow\mathrm{id}"] \arrow[dl, "\mathrm{id}\downarrow\rho"'] & & \mathsf{Hom}_B(f,f\ell) \arrow[dr, "\mathrm{id}\downarrow\zeta"] & \\
\mathsf{Hom}_{A^J}(\Delta,d) \arrow[drr, "\mathrm{id}\downarrow\mathrm{id}"'] & & \cong & & \mathsf{Hom}_B(f,\ell') \arrow[dll, "\mathrm{id}\downarrow\rho'"] \\
& & \mathsf{Hom}_{B^J}(f^J\Delta,f^Jd)=\mathsf{Hom}_{B^J}(\Delta f,f^Jd) & &
\end{tikzcd}
$$

Since (ℓ,ρ) is an absolute right lifting diagram, Theorem 3.5.3 tells us that the functor $\mathrm{id}\downarrow\rho$ is an equivalence. Similarly, since (ℓ',ρ') is absolute right lifting, $\mathrm{id}\downarrow\rho'\colon \mathsf{Hom}_B(B,\ell') \to \mathsf{Hom}_{B^J}(\Delta, f^Jd)$ is an equivalence, which pulls back along f to define the equivalence $\mathrm{id}\downarrow\rho'$ in the above pentagon. Since f is fully faithful, Corollary 3.5.6 tells us that $\mathrm{id}\downarrow\mathrm{id}\colon \mathsf{Hom}_A(A,A) \to \mathsf{Hom}_B(f,f)$ is an equivalence, which pulls back to define along ℓ to define the equivalence appearing as the top-horizontal edge of the above pentagon. As observed in the proof of Proposition 4.3.12, since $f\colon A \to B$ is fully faithful, $f^J\colon A^J \to B^J$ is as well, so we conclude similarly that the lower-left functor $\mathrm{id}\downarrow\mathrm{id}\colon \mathsf{Hom}_{A^J}(\Delta,d) \to \mathsf{Hom}_{B^J}(\Delta f^J, f^Jd)$ is an equivalence as well.

Thus, we conclude that the functor $\mathrm{id}\downarrow\zeta\colon \mathsf{Hom}_B(f,f\ell) \to \mathsf{Hom}_B(f,\ell')$ is an equivalence. Since f is strongly generating, we conclude that ζ is invertible as desired. $\square$

Exercises.

12.3.i. Exercise. Prove that in a Kan complex enriched category $\mathcal{S}$ the following are equivalent, characterizing when a 0-arrow $e\colon a \to b$ is an **equivalence** in $\mathcal{S}$:

(i) There exists a 0-arrow $f\colon b \to a$ together with a pair of 1-arrows $\alpha\colon \mathrm{id}_a \to fe$ and $\beta\colon \mathrm{id}_b \to ef$.
(ii) For all $x \in \mathcal{S}$, the map $e \circ -\colon \mathcal{S}(x,a) \xrightarrow{\sim} \mathcal{S}(x,b)$ is an equivalence of Kan complexes.
(iii) The 1-simplex $e\colon a \to b$ in $\mathfrak{N}\mathcal{S}$ corresponding to the 0-arrow e is an isomorphism in the quasi-category $\mathfrak{N}\mathcal{S}$.

Appendix of Abstract Nonsense

APPENDIX A

Basic concepts of enriched category theory

Enriched category theory exists because enriched categories exist in nature. To explain, consider the data of a (small) 1-category C, given by:

- a set of objects
- for each pair of objects $x, y \in C$, a set $C(x,y)$ of arrows in C from x to y
- for each $x \in C$, a specified identity element $\mathrm{id}_x : 1 \to C(x,x)$, and for each $x, y, z \in C$, a specified composition map $\circ : C(y,z) \times C(x,y) \to C(x,z)$ satisfying the associativity and unit conditions:

$$\begin{tikzcd} C(y,z)\times C(x,y)\times C(w,x) \arrow[r, "\circ\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times\circ"'] & C(x,z)\times C(w,x) \arrow[d, "\circ"] \\ C(y,z)\times C(w,y) \arrow[r, "\circ"'] & C(w,z) \end{tikzcd} \qquad \begin{tikzcd} C(x,y) \arrow[r, "\mathrm{id}\times\mathrm{id}_x"] \arrow[d, "\mathrm{id}_y\times\mathrm{id}"'] \arrow[dr, equal] & C(x,y)\times C(x,x) \arrow[d, "\circ"] \\ C(y,y)\times C(x,y) \arrow[r, "\circ"'] & C(x,y) \end{tikzcd} \tag{A.0.1}$$

In many mathematical examples of interest, the set $C(x,y)$ can be given additional structure, in which case it would be strange not to take it account when performing further categorical constructions.

Perhaps there exists a specified zero arrow $0_{x,y} \in C(x,y)$, the set of which defines a two-sided ideal for composition: $g \circ 0 \circ f = 0$. Or extending this, perhaps $C(x,y)$ admits an abelian group structure, for which composition is $\mathbb{Z}$-bilinear. Or in another direction, perhaps the set of arrows from x to y in C form the objects of a 1-category. In this setting, we regard the objects of $C(x,y)$ as "1-dimensional" morphisms from x to y and the arrows of $C(x,y)$ as "2-dimensional" morphisms from x to y in C; now, it's natural to require the composition map to define a functor. Or perhaps the set of arrows from x to y in C form the vertices of a simplicial set, whose higher simplices now provides arrows in each positive dimension; in this setting, it is natural to ask composition to define a simplicial map. In such settings, one says that the 1-category C can be enriched over the category $\mathcal{V}$ in which the objects $C(x,y)$ and diagrams (A.0.1) live[1] — with $\mathcal{V}$ equal to the category of pointed sets, abelian groups, categories, or simplicial sets in the examples just described.

An alternate point of view of enriched category theory is often emphasized — adopted for instance in the classic textbook **[54]** from which we stole the title of this chapter. To borrow a distinction used by Peter May, the term "enriched" can be used as a compound noun — *enriched categories* — or as an adjective — *enriched* categories. In the noun form, an enriched category C has no pre-existing underlying ordinary category, although we shall see below that the underlying unenriched 1-category can always be identified a posteriori. When used as an adjective, an enriched category C is perhaps

[1]To interpret the diagrams (A.0.1) in $\mathcal{V}$ one needs to specify an interpretation for the monoidal product "$\times$" and its unit object 1 (which is not displayed in the diagram). In the examples we will consider, this product is the cartesian product and this unit is the terminal object.

most naturally an ordinary category, whose hom-sets can be given additional structure.[2] While the noun perspective is arguably more elegant when discussing the general theory of enriched categories, the adjective perspective dominates when discussing examples, so we choose to emphasize the adjective form and focus on enriching unenriched categories here.

Before giving a precise definition of enriched category and the enriched functors between them in §A.2, in §A.1 we study that category $\mathcal{V}$ that defines the base for enrichment in which the hom-objects will ultimately live. Here we are interested primarily in two examples $\mathcal{V} = \mathcal{C}at$ and $\mathcal{V} = s\mathcal{S}et$, as well as the unenriched case $\mathcal{V} = \mathcal{S}et$, each of which has the special property of being cartesian closed categories. Since there are some simplifications in enriching over a cartesian closed category, we grant ourselves the luxury of working explicitly with this notion; see **[54]** or **[75**, Chapter 3] for an introduction to categories enriched over a more general symmetric monoidal category.

We continue in §A.3 with an introduction to enriched natural transformations and the enriched Yoneda lemma. These notions allow us to correctly state the universal properties that characterize *tensors* and *cotensors* in §A.4 and *conical limits* and *colimits* in §A.5, both of which are characterized by universal properties involving an enriched natural isomorphism. We conclude in §A.7 with a general theory of change of base — the one part of the theory of enriched categories that is not covered in encyclopedic detail in **[54]**, the original reference instead being **[35]** — which allows us to be more precise about the procedure by which a 2-category may be regarded as a simplicial category or by which a simplicial category may be quotiented to define a 2-category (see e.g., Definition 1.4.1) as alluded to in Digression 1.4.2.

A.1. Cartesian closed categories

Throughout this text, the base category for enrichment will always be taken to be a cartesian closed category:

A.1.1. Definition. A category $\mathcal{V}$ is **cartesian closed** when it

- admits finite products, or equivalently, a terminal object $1 \in \mathcal{V}$ and binary products and
- for each $A \in \mathcal{V}$, the functor $A \times - \colon \mathcal{V} \to \mathcal{V}$ admits a right adjoint $(-)^A \colon \mathcal{V} \to \mathcal{V}$.

A.1.2. Lemma. *In a cartesian closed category $\mathcal{V}$, the product bifunctor is the left adjoint of a two-variable adjunction, this being captured by a commutative triangle of natural isomorphisms*

$$\begin{tikzcd} & \mathcal{V}(A \times B, C) \arrow[dl, phantom, "\cong" sloped] \arrow[dr, phantom, "\cong" sloped] & \\ \mathcal{V}(A, C^B) \arrow[rr, phantom, "\cong"] & & \mathcal{V}(B, C^A) \end{tikzcd} \tag{A.1.3}$$

Proof. The family of functors $(-)^A \colon \mathcal{V} \to \mathcal{V}$ extend to bifunctors

$$(-)^- \colon \mathcal{V}^{\mathrm{op}} \times \mathcal{V} \to \mathcal{V}$$

in a unique way so that the isomorphism defining each adjunction $A \times - \dashv (-)^A$

$$\mathcal{V}(A \times B, C) \cong \mathcal{V}(B, C^A)$$

becomes natural in A (as well as B and C). The details are left as Exercise A.1.i or to **[76**, 4.3.6]. This defines the natural isomorphism on the right-hand side of (A.1.3). The natural isomorphism on the

[2] To quote **[63]** "Thinking from the two points of view simultaneously, it is essential that the constructed ordinary category be isomorphic to the ordinary category that one started out with. Either way, there is a conflict of notation between that preferred by category theorists and that in common use by 'working mathematicians' (to whom **[61]** is addressed).

left-hand is defined by composing with the symmetry isomorphism $A \times B \cong B \times A$. The third natural isomorphism is taken to be the composite of these two. $\square$

A.1.4. Example (cartesian closed categories).

(i) The category of sets is cartesian closed, with B^A defined to be the set of functions from A to B. Transposition across the natural isomorphism (A.1.3) is referred to as "currying."

(ii) The category $\mathcal{Cat}$ of small categories is cartesian closed, with B^A defined to be the category of functors and natural transformations from A to B.[3] The natural isomorphism

$$\mathcal{Cat}(A, C^B) \cong \mathcal{Cat}(A \times B, C) \cong \mathcal{Cat}(B, C^A)$$

identifies natural transformations, which are arrows $\mathbb{2} \to C^A$ in the category of functors, with "homotopies" $A \times \mathbb{2} \to C$.

(iii) For any small category C, the category $\mathcal{Set}^{C^{\mathrm{op}}}$ is cartesian closed. By the Yoneda lemma for $F, G \in \mathcal{Set}^{C^{\mathrm{op}}}$, the value of G^F at $c \in C$ must be defined by

$$G^F(c) \cong \mathcal{Set}^{C^{\mathrm{op}}}(C(-,c), G^F) \cong \mathcal{Set}^{C^{\mathrm{op}}}(F \times C(-,c), G).$$

By the proof of Lemma A.1.2, the action of G^F on a morphism $f : c \to c' \in C$ is defined by precomposition with the corresponding natural transformation $f_* : C(-,c) \to C(-,c')$. This defines the functor G^F. Since any functor $H \in \mathcal{Set}^{C^{\mathrm{op}}}$ is canonically a colimit of representables, this definition extends to the required natural isomorphism $\mathcal{Set}^{C^{\mathrm{op}}}(H, G^F) \cong \mathcal{Set}^{C^{\mathrm{op}}}(F \times H, G)$.

(iv) In particular taking $C = \mathbf{\Delta}^{\mathrm{op}}$, the category of simplicial sets $s\mathcal{Set} := \mathcal{Set}^{\mathbf{\Delta}^{\mathrm{op}}}$ is cartesian closed.

The exponential B^A is frequently referred to as an *internal hom*. As this name suggests, the internal hom B^A can be viewed as a lifting of the hom-set $\mathcal{V}(A, B)$ along a functor that we now introduce.

A.1.5. Definition. For any cartesian closed category $\mathcal{V}$ the **underlying set functor** is the functor

$$\mathcal{V} \xrightarrow{(-)_0 := \mathcal{V}(1,-)} \mathcal{Set}$$

represented by the terminal object $1 \in \mathcal{V}$.

A.1.6. Lemma. *For any pair of objects $A, B \in \mathcal{V}$ in a cartesian closed category, the underlying set of the internal hom B^A is $\mathcal{V}(A, B)$, i.e.:*

$$(B^A)_0 \cong \mathcal{V}(A, B).$$

Proof. Combining Definition A.1.5 with (A.1.3):

$$(B^A)_0 := \mathcal{V}(1, B^A) \cong \mathcal{V}(1 \times A, B) \cong \mathcal{V}(A, B)$$

since there is a natural isomorphism $1 \times A \cong A$. $\square$

It makes sense to ask whether an isomorphism of underlying sets can be "enriched" to lie in $\mathcal{V}$, that is, lifted along the underlying set functor $(-)_0 : \mathcal{V} \to \mathcal{Set}$.

[3] Size matters here: the category of large but locally small categories is not cartesian closed, though it is still possible to define B^A in the case where A is a small category.

A.1.7. LEMMA. *The natural isomorphisms* (A.1.3) *characterizing the defining two-variable adjunction of a cartesian closed category lift to $\mathcal{V}$: for any $A, B, C \in \mathcal{V}$*

$$\begin{tikzcd}[column sep=small] & C^{A\times B} \arrow[dl, phantom, "\rotatebox{45}{$\cong$}"] \arrow[dr, phantom, "\rotatebox{-45}{$\cong$}"] & \\ (C^B)^A & \cong & (C^A)^B \end{tikzcd} \tag{A.1.8}$$

PROOF. This follows from the associativity of finite products and the Yoneda lemma. To prove (A.1.8), it suffices to show that $C^{A\times B}$, $(C^B)^A$, and $(C^A)^B$ represent the same functor. We have a sequence of natural isomorphisms:

$$\begin{aligned}\mathcal{V}(X, (C^B)^A) &\cong \mathcal{V}(X \times A, C^B) \\ &\cong \mathcal{V}((X \times A) \times B, C) \cong \mathcal{V}(X \times (A \times B), C) \\ &\cong \mathcal{V}(X, C^{A\times B}) \\ &\cong \mathcal{V}((A \times B) \times X, C) \cong \mathcal{V}(A \times (B \times X), C) \\ &\cong \mathcal{V}(B \times X, C^A) \\ &\cong \mathcal{V}(X, (C^A)^B),\end{aligned}$$

and hence

$$\mathcal{V}(X, (C^B)^A) \cong \mathcal{V}(X, C^{A\times B}) \cong \mathcal{V}(X, (C^A)^B). \qquad \square$$

A.1.9. REMARK. Note $(-)^1 : \mathcal{V} \to \mathcal{V}$ is naturally isomorphic to the identity functor — i.e., $B^1 \cong B$, — since it is right adjoint to a functor $- \times 1 : \mathcal{V} \to \mathcal{V}$ that is naturally isomorphic to the identity.

A bicomplete cartesian closed category is a special case of a complete and cocomplete closed symmetric monoidal category, this being deemed a **cosmos** by Bénabou, to signify that such bases are an ideal setting for enriched category theory. For obvious reasons, we won't use this term here. There is a competing 2-categorical notion of (fibrational) "cosmos" due to Street [**89**] that is more similar to the notion we consider here, which was the direct inspiration for the terminology we introduce in Definition 1.2.1.

Exercises.

A.1.i. EXERCISE. Prove that in a cartesian closed category $\mathcal{V}$, the family of functors $(-)^A : \mathcal{V} \to \mathcal{V}$ extend to bifunctors

$$(-)^- : \mathcal{V}^{\mathrm{op}} \times \mathcal{V} \to \mathcal{V}$$

in a unique way so that the isomorphism defining each adjunction $A \times - \dashv (-)^A$

$$\mathcal{V}(A \times B, C) \cong \mathcal{V}(B, C^A)$$

becomes natural in A (as well as B and C).

A.1.ii. EXERCISE. The data of a **closed symmetric monoidal category** generalizes Definition A.1.1 by replacing finite products by an arbitrary bifunctor $-\otimes- : \mathcal{V}\times\mathcal{V} \to \mathcal{V}$, replacing the terminal object by an object $I \in \mathcal{V}$, and requiring the additional specification of natural isomorphisms

$$A \otimes (B \otimes C) \cong (A \otimes B) \otimes C \qquad I \otimes A \cong A \cong A \otimes I \qquad A \otimes B \cong B \otimes A$$

satisfying various coherence axioms [**35, 50**]; see also [**51**]. Explain why these isomorphisms do not need to specified in the special case of a cartesian closed category and why the coherence conditions are automatic.

A.2. Enriched categories

We now briefly switch perspectives and explain the meaning of the noun phrase "enriched category" before discussing what is required to "enrich" an ordinary 1-category. Throughout we fix a cartesian closed category $(\mathcal{V}, \times, 1)$ to serve as the base for enrichment.

A.2.1. Definition. A **$\mathcal{V}$-enriched category C** is given by:

- a set of objects
- for each pair of objects $x, y \in C$, an **object of arrows $C(x, y) \in \mathcal{V}$**
- for each $x \in C$, a specified identity element encoded by a map $\mathrm{id}_x \colon 1 \to C(x, x) \in \mathcal{V}$, and for each $x, y, z \in C$, a specified composition map $\circ \colon C(y, z) \times C(x, y) \to C(x, z) \in \mathcal{V}$ satisfying the associativity and unit conditions, both commutative diagrams lying in $\mathcal{V}$:[4]

$$\begin{tikzcd} C(y,z) \times C(x,y) \times C(w,x) \arrow[r, "\circ \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times \circ"'] & C(x,z) \times C(w,x) \arrow[d, "\circ"] \\ C(y,z) \times C(w,y) \arrow[r, "\circ"'] & C(w,z) \end{tikzcd} \qquad \begin{tikzcd} C(x,y) \arrow[r, "\mathrm{id} \times \mathrm{id}_x"] \arrow[d, "\mathrm{id}_y \times \mathrm{id}"'] \arrow[dr, equal] & C(x,y) \times C(x,x) \arrow[d, "\circ"] \\ C(y,y) \times C(x,y) \arrow[r, "\circ"'] & C(x,y) \end{tikzcd}$$

Evidently from the diagrams of (A.0.1), a locally-small 1-category defines a category enriched in $\mathcal{S}et$. The underlying set functor of Definition A.1.5 can be used to define the "underlying category" of an enriched category.

A.2.2. Definition. If C is a $\mathcal{V}$-category, its **underlying category** is the 1-category with the same set of objects and with hom-sets $C(x, y)_0$ defined by applying the underlying set functor $(-)_0 \colon \mathcal{V} \to \mathcal{S}et$ to the hom-objects $C(x, y) \in \mathcal{V}$.

Note the identity arrow $\mathrm{id}_x \colon 1 \to C(x, x)$ of the $\mathcal{V}$-category is by definition an element of the hom-set $C(x, x)_0 := \mathcal{V}(1, C(x, x))$. The composite of two arrows $f \colon 1 \to C(x, y)$ and $g \colon 1 \to C(y, z)$ in the underlying category is defined to be the arrow constructed as the composite

$$1 \xrightarrow{g \times f} C(y, z) \times C(x, y) \xrightarrow{\circ} C(x, z)$$

In analogy with the discussion around Definition A.1.5, when one speaks of "enriching" an a priori unenriched category C over $\mathcal{V}$, the task is to define a $\mathcal{V}$-enriched category as in Definition A.2.1 whose underlying category recovers C. When $\mathcal{V} = \mathcal{C}at$, the task is to define a 2-category whose underlying 1-category is the one given. When $\mathcal{V} = s\mathcal{S}et$, the task is define simplicial hom-sets of n-arrows so that the 0-arrows are the ones given. When a simplicially enriched category C is encoded as a simplicial object $C_\bullet$ in $\mathcal{C}at$ as explained in Digression 1.2.4, its underlying category is the category C_0, further justifying the notion introduced in Definition A.2.2.

For example, a cartesian closed category $\mathcal{V}$ as in Definition A.1.1 can be enriched to define a $\mathcal{V}$-category.

A.2.3. Lemma. *A cartesian closed category $\mathcal{V}$ defines a $\mathcal{V}$-category whose:*

- *objects are the objects of $\mathcal{V}$,*
- *hom-object in $\mathcal{V}$ from A to B is given by the internal hom B^A, and*

[4]These diagrams suppose the associativity and unit natural isomorphisms involving the product bifunctors $\times$ and its unit object 1. In a cartesian closed category these are *canonical* — rather than given by extra data, as is the case in the more general *closed symmetric monoidal category*; see Exercise A.1.ii.

- *the identity map* $\mathrm{id}_A \colon 1 \to A^A$ *and composition map* $\circ \colon C^B \times B^A \to C^A$ *are defined to be the transposes of*
$$1 \times A \xrightarrow{\cong} A \qquad \textit{and} \qquad C^B \times B^A \times A \xrightarrow{\mathrm{id}\times\epsilon} C^B \times B \xrightarrow{\epsilon} C$$
the latter defined using the counit of the cartesian closure adjunction.

PROOF. The task is to verify the commutative diagrams of (A.2.1) in $\mathcal{V}$ and then observe that Lemma A.1.6 reveals that the underlying category of the $\mathcal{V}$-category defined by the statement is the 1-category $\mathcal{V}$. We leave the identity conditions to the reader and verify associativity.

The definition of the composition map as an adjoint transpose implies that *its* adjoint transpose, the top-right composite below, is given by the left-bottom composite:

$$\begin{tikzcd}
C^B \times B^A \times A \arrow[r, "\circ\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times\epsilon"'] & C^A \times A \arrow[d, "\epsilon"] \\
C^B \times B \arrow[r, "\epsilon"'] & C
\end{tikzcd}$$

The associativity diagram below-left commutes if and only if the transposed diagram appearing as the outer boundary composite below-right commutes:

$$\begin{tikzcd}
D^C \times C^B \times B^A \arrow[r, "\circ\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times\circ"'] & D^B \times B^A \arrow[d, "\circ"] \\
D^C \times C^A \arrow[r, "\circ"'] & D^A
\end{tikzcd}
\quad \leftrightsquigarrow \quad
\begin{tikzcd}
D^C \times C^B \times B^A \times A \arrow[rr, "\circ\times\mathrm{id}\times\mathrm{id}"] \arrow[dd, "\mathrm{id}\times\circ\times\mathrm{id}"'] \arrow[dr, "\mathrm{id}\times\mathrm{id}\times\epsilon"] & & D^B \times B^A \times A \arrow[d, "\mathrm{id}\times\epsilon"] \\
& D^C \times C^B \times B \arrow[r, "\circ\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times\epsilon"'] & D^B \times B \arrow[d, "\epsilon"] \\
D^C \times C^A \times A \arrow[r, "\mathrm{id}\times\epsilon"'] & D^C \times C \arrow[r, "\epsilon"'] & D
\end{tikzcd}$$

which follows from bifunctoriality of $\times$ and two instances of the commutative square above. ☐

A.2.4. DEFINITION. The **free $\mathcal{V}$-category** on a 1-category C has

- the same objects as C
- the hom-objects defined to be coproducts $\coprod_{C(x,y)} 1$ of the terminal object 1 indexed by the hom-set $C(x,y)$
- the identity map $\mathrm{id}_x \colon 1 \to \coprod_{C(x,x)} 1$ given by the inclusion of the component indexed by the identity arrow
- the composition map given by acting by the composition function on the indexing sets:

$$\coprod_{C(y,z)} 1 \times \coprod_{C(x,y)} 1 \cong \coprod_{C(y,z)\times C(x,y)} 1 \xrightarrow{\coprod_\circ 1} \coprod_{C(x,z)} 1$$

For example, free $\mathcal{C}at$-enriched categories are those with no non-identity 2-cells and free $s\mathcal{S}et$-enriched categories are those with no non-degenerate arrows in positive dimensions. When context allows, we use the same name for the 1-category C and the free $\mathcal{V}$-category it generates, using language to disambiguate.

A.2.5. DEFINITION. A **$\mathcal{V}$-enriched functor** $F \colon C \to \mathcal{D}$ is given by

- a mapping on objects that carries each $x \in C$ to some $Fx \in \mathcal{D}$

- for each pair of objects $x, y \in \mathcal{C}$, an internal action on the $\mathcal{V}$-objects of arrows given by a morphism $F_{x,y} \colon \mathcal{C}(x,y) \to \mathcal{D}(Fx, Fy) \in \mathcal{V}$ so that the $\mathcal{V}$-functoriality diagrams commute:

$$\begin{tikzcd} \mathcal{C}(y,z) \times \mathcal{C}(x,y) \arrow[r, "\circ"] \arrow[d, "F_{y,z} \times F_{x,y}"'] & \mathcal{C}(x,z) \arrow[d, "F_{x,z}"] \\ \mathcal{D}(Fy,Fz) \times \mathcal{D}(Fx,Fy) \arrow[r, "\circ"] & \mathcal{D}(Fx,Fz) \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r, "\mathrm{id}_x"] \arrow[dr, "\mathrm{id}_{Fx}"'] & \mathcal{C}(x,x) \arrow[d, "F_{x,x}"] \\ & \mathcal{D}(Fx,Fx) \end{tikzcd}$$

A prototypical example is given by the representable functors:

A.2.6. Example. For any $\mathcal{V}$-category $\mathcal{C}$ and object $c \in \mathcal{C}$, the **enriched representable $\mathcal{V}$-functor** $\mathcal{C}(c,-) \colon \mathcal{C} \to \mathcal{V}$ is defined on objects by the assignment $x \in \mathcal{C} \mapsto \mathcal{C}(c,x) \in \mathcal{V}$ and whose internal action on arrows

$$\mathcal{C}(x,y) \xrightarrow{\mathcal{C}(c,-)_{x,y}} \mathcal{C}(c,y)^{\mathcal{C}(c,x)} \qquad \leftrightsquigarrow \qquad \mathcal{C}(x,y) \times \mathcal{C}(c,x) \xrightarrow{\circ} \mathcal{C}(c,y)$$

is given by the adjoint transpose of the internal composition map for $\mathcal{C}$. The $\mathcal{V}$-functoriality diagrams are transposes of associativity and identity diagrams in $\mathcal{C}$.

The contravariant enriched representable functors are defined similarly; see Exercise A.2.ii.

As the next result explains, an enriched representable functor can be thought of as a "two-step" enrichment of the corresponding unenriched representable functor: the first step enriches the hom-sets to hom-objects in $\mathcal{V}$ and the second step enriches the composition function to an internal composition map in $\mathcal{V}$.

A.2.7. Proposition. *To enrich a 1-category $\mathcal{C}$ over $\mathcal{V}$ one must:*

(i) First lift each of the unenriched representable functors through the underlying set functor to define an unenriched functor that encodes the data of the objects of arrows in $\mathcal{V}$:

$$\begin{tikzcd} \mathcal{C} \arrow[rr, dashed, "\mathcal{C}(c,-)"] \arrow[dr, "\mathcal{C}(c,-)"'] & & \mathcal{V} \arrow[dl, "(-)_0"] \\ & \mathcal{S}et & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C} \arrow[r, "\mathcal{C}(c,-)"] & \mathcal{V} \\ x \arrow[r, mapsto] & \mathcal{C}(c,x) \end{tikzcd}$$

(ii) Then enriched each of the unenriched $\mathcal{V}$-valued representable functors defined in (i) to a $\mathcal{V}$-functor $\mathcal{C}(c,-) \colon \mathcal{C} \to \mathcal{V}$ whose internal action on arrows encodes the data of the internal composition map for $\mathcal{C}$.

Proof. The correspondence between the data in (i) and (ii) and the data of a $\mathcal{V}$-enriched category whose underlying category is $\mathcal{C}$ is clear. As discussed in Definition A.2.2, there is no additional data required by the identity arrows in an enriched category. Now the $\mathcal{V}$-functoriality of the enriched representable functor $\mathcal{C}(c,-) \colon \mathcal{C} \to \mathcal{V}$ is expressed by commutative diagrams

$$\begin{tikzcd} \mathcal{C}(y,z) \times \mathcal{C}(x,y) \arrow[r, dashed, "\circ"] \arrow[d, "\mathcal{C}(c,-)_{y,z} \times \mathcal{C}(c,-)_{x,y}"'] & \mathcal{C}(x,z) \arrow[d, "\mathcal{C}(c,-)_{x,z}"] \\ \mathcal{C}(c,z)^{\mathcal{C}(c,y)} \times \mathcal{C}(c,y)^{\mathcal{C}(c,x)} \arrow[r, "\circ"] & \mathcal{C}(c,z)^{\mathcal{C}(c,x)} \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r, "\mathrm{id}_x"] \arrow[dr, "\mathrm{id}_{\mathcal{C}(c,x)}"'] & \mathcal{C}(x,x) \arrow[d, "\mathcal{C}(c,-)_{x,x}"] \\ & \mathcal{C}(c,x)^{\mathcal{C}(c,x)} \end{tikzcd}$$

— where the composition map in $\mathcal{C}$ is the one that has just been defined — and these transpose to the required associativity and identity axioms for the $\mathcal{V}$-category $\mathcal{C}$. □

Both of the constructions of underlying unenriched categories and free categories extend functorially to functors; see Exercise A.2.iii. The relationship between these constructions is summarized by the following proposition, which we also decline to prove, because it will be generalized by a result that we do provide a proof for in §A.7.

A.2.8. Proposition. *The free $\mathcal{V}$-category functor defines a fully faithful left adjoint to the underlying category functor. Consequently, a $\mathcal{V}$-category is free just when it is isomorphic to the free category on its underlying category via the counit of this adjunction.*

Proof. Exercise A.2.iv or see §A.7. ☐

Exercises.

A.2.i. Exercise. Verify the unit condition left to the reader in the proof of Lemma A.2.3.

A.2.ii. Exercise. Define the **opposite** of a $\mathcal{V}$-category and dualize Example A.2.6 to define contravariant enriched representable functors.

A.2.iii. Exercise.

(i) Define the underlying functor of an enriched functor.
(ii) Prove that the passage from enriched functors to underlying unenriched functors is functorial.
(iii) Define the free enriched functor on an unenriched functor.
(iv) Prove the passage from unenriched functors to free enriched functors is functorial.

A.2.iv. Exercise. Prove A.2.8.

A.3. Enriched natural transformations and the enriched Yoneda lemma

Recall an (unenriched) natural transformation $\alpha\colon F \Rightarrow G$ between parallel functors $F, G\colon C \rightrightarrows \mathcal{D}$ is given by:

- the data of an arrow $\alpha_x \in \mathcal{D}(Fx, Gx)$ for each $x \in C$
- subject to the condition that for each morphism $f \in C(x, y)$, the diagram

$$\begin{array}{ccc} Fx & \xrightarrow{\alpha_x} & Gx \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ Fy & \xrightarrow[\alpha_y]{} & Gy \end{array} \tag{A.3.1}$$

  commutes in $\mathcal{D}$.

This enriches to the notion of a $\mathcal{V}$*-natural transformation* whose data is exactly the same — a family of arrows in the underlying category of $\mathcal{D}$ indexed by the objects of C — but with a stronger $\mathcal{V}$-naturality condition expressed by internalizing the naturality condition (A.3.1) given above.

A.3.2. Definition. A $\mathcal{V}$**-enriched natural transformation** $\alpha\colon F \Rightarrow G$ between $\mathcal{V}$-enriched functors $F, G\colon C \rightrightarrows \mathcal{D}$ is given by:

- an arrow $\alpha_x\colon 1 \to \mathcal{D}(Fx, Gx)$ for each $x \in C$

- so that for each pair of objects $x, y \in C$, the follow $\mathcal{V}$-naturality square commutes in $\mathcal{V}$:

$$\begin{tikzcd} C(x,y) \arrow[r, "\alpha_y \times F_{x,y}"] \arrow[d, "G_{x,y}\times\alpha_x"'] & \mathcal{D}(Fy,Gy)\times\mathcal{D}(Fx,Fy) \arrow[d, "\circ"] \\ \mathcal{D}(Gx,Gy)\times\mathcal{D}(Fx,Gx) \arrow[r, "\circ"'] & \mathcal{D}(Fx,Gy) \end{tikzcd} \tag{A.3.3}$$

A.3.4. Example. An arrow $f \colon 1 \to C(x,y)$ in the underlying category of a $\mathcal{V}$-category C defines a $\mathcal{V}$-natural transformation $f^* \colon C(y,-) \Rightarrow C(x,-)$ between the enriched representable functors whose component at $z \in C$ is defined by evaluating the adjoint transpose of the composition map at f:

$$1 \xrightarrow{f} C(x,y) \xrightarrow{C(-,z)_{x,y}} C(x,z)^{C(y,z)}$$

Evaluating one component of the associative diagram for C at f provides the required $\mathcal{V}$-naturality square.

A.3.5. Lemma.

(i) The ***vertical composite*** *of $\mathcal{V}$-enriched natural transformations $\alpha \colon F \Rightarrow G$ and $\beta \colon G \Rightarrow H$, both from C to $\mathcal{D}$ has component $(\beta\cdot\alpha)_x$ at $x \in C$ defined by the composite*

$$1 \xrightarrow{\beta_x\times\alpha_x} \mathcal{D}(Gx,Hx)\times\mathcal{D}(Fx,Gx) \xrightarrow{\circ} \mathcal{D}(Fx,Hx)$$

(ii) The ***horizontal composite*** *of $\alpha \colon F \Rightarrow G$ from C to $\mathcal{D}$ and $\gamma \colon H \Rightarrow K$ from $\mathcal{D}$ to $\mathcal{E}$ has component $(\gamma * \alpha)_x$ at $x \in C$ defined by the composite*

$$\begin{tikzcd} 1 \arrow[r, "\alpha_x"] & \mathcal{D}(Fx,Gx) \arrow[r, "\gamma_{Gx}\times H_{Fx,Gx}"] \arrow[d, "K_{Fx,Gx}\times\gamma_{Fx}"'] & \mathcal{E}(KGx,HGx)\times\mathcal{E}(HFx,HGx) \arrow[d, "\circ"] \\ & \mathcal{E}(KFx,KGx)\times\mathcal{E}(HFx,KFx) \arrow[r, "\circ"'] & \mathcal{D}(HFx,KGx) \end{tikzcd}$$

which is well-defined by $\mathcal{V}$-naturality of γ.

Proof. Exercise A.3.i. □

The data of the **underlying natural transformation** of a $\mathcal{V}$-naturality transformation is given by the same family of arrows. The unenriched naturality condition (A.3.1) is proven by evaluating the enriched naturality condition (A.3.3) at an underlying arrow $f \colon 1 \to C(x,y)$. In particular, the middle four interchange rule for horizontal and vertical composition of $\mathcal{V}$-natural transformations follows from the middle four interchange rule for horizontal and vertical composition of unenriched natural transformations for the data of the latter determines the data of the former. Consequently, Lemma A.3.5 implies that:

A.3.6. Corollary. *For any cartesian closed category $\mathcal{V}$, there is a 2-category $\mathcal{V}$-$\mathcal{C}at$ of $\mathcal{V}$-enriched categories, $\mathcal{V}$-enriched functors, and $\mathcal{V}$-enriched natural transformations. Moreoever, the underlying category functor $\mathcal{V}$-$\mathcal{C}at \to \mathcal{C}at$ and the free $\mathcal{V}$-category functor $\mathcal{C}at \to \mathcal{V}$-$\mathcal{C}at$ are both 2-functors.* □

We now turn our attention to the $\mathcal{V}$-enriched Yoneda lemma, which we present in several forms. One role of the Yoneda lemma is to give a representable characterization of isomorphic objects in C. When C is a $\mathcal{V}$-category, this has several forms involving each of the three notions of "representable" functor appearing in Proposition A.2.7. The notion of $\mathcal{V}$-**natural isomorphism** referred to in the following result is defined to be a $\mathcal{V}$-natural transformation with an inverse for vertical composition.

A.3.7. LEMMA. *For objects x, y in a $\mathcal{V}$-category C the following are equivalent:*

(i) x and y are isomorphic as objects of the underlying category of C.
(ii) The $\mathcal{S}et$-valued unenriched representable functors $C(x,-)_0, C(y,-)_0 \colon C \rightrightarrows \mathcal{S}et$ are naturally isomorphic.
(iii) The $\mathcal{V}$-valued unenriched representable functors $C(x,-), C(y,-)\colon C \rightrightarrows \mathcal{V}$ are naturally isomorphic.
(iv) The $\mathcal{V}$-valued $\mathcal{V}$-functors $C(x,-), C(y,-)\colon C \rightrightarrows \mathcal{V}$ are naturally $\mathcal{V}$-isomorphic.

PROOF. Applying the underlying category functor $(-)_0 \colon \mathcal{V}\text{-}\mathcal{C}at \to \mathcal{C}at$, the fourth statement implies the third. The third statement implies the second by composing the underlying set functor $(-)_0 \colon \mathcal{V} \to \mathcal{S}et$. The second statement implies the first by the unenriched Yoneda lemma; this is still the main point. Finally, the first statement implies the last by a direct construction: if $f\colon 1 \to C(x,y)$ and $g\colon 1 \to C(y,x)$ define an isomorphism in the underlying category of C, the corresponding representable $\mathcal{V}$-natural transformations of Example A.3.4 define a $\mathcal{V}$-natural isomorphism. $\square$

Lemma A.3.7 defines a common notion of **isomorphism** between two objects of an enriched category, which turns out to be no different the usual unenriched notion of isomorphism. This can be thought of as defining a "cheap" form of the enriched Yoneda lemma. The full form of the $\mathcal{V}$-Yoneda lemma enriches the usual statement — a natural isomorphism between the set of natural transformations whose domain is a representable functor to the set defined by evaluating the codomain at the representing object — to an isomorphism in $\mathcal{V}$. The first step to make this precise is to enrich the set of $\mathcal{V}$-enriched natural transformations between a parallel pair of $\mathcal{V}$-functors can to define an object of $\mathcal{V}$.

A.3.8. DEFINITION. Let $\mathcal{V}$ be a complete cartesian closed category and consider a parallel pair of $\mathcal{V}$-functors $F, G\colon C \rightrightarrows \mathcal{D}$, with C a small $\mathcal{V}$-category. Then the **$\mathcal{V}$-object of $\mathcal{V}$-natural transformations** is defined by the equalizer diagram

$$\mathcal{V}^C(F,G) \rightarrowtail \prod_{z\in C} \mathcal{D}(Fz,Gz) \rightrightarrows \prod_{x,y\in C} \mathcal{D}(Fx,Gy)^{C(x,y)}$$

where one map in the equalizer diagram is defined by projecting to $\mathcal{D}(Fx,Gx)$, applying the internal action of G on arrows, and then composing, while the other is defined by projection to $\mathcal{D}(Fy,Gy)$, applying the internal action of F on arrows, and then composing.

A.3.9. LEMMA. *The underlying set of the $\mathcal{V}$-object of $\mathcal{V}$-natural transformations $\mathcal{V}^C(F,G)$ is the set of $\mathcal{V}$-natural transformations from F to G.*

PROOF. By its defining universal property, elements of the underlying set of $\mathcal{V}^C(F,G)$ correspond to maps $\alpha\colon 1 \to \prod_{z\in C}\mathcal{D}(Fz,Gz)$ that equalize the parallel pair of maps described in Definition A.3.8. The map α defines the components of a $\mathcal{V}$-natural transformation $\alpha\colon F \Rightarrow G$ and the commutativity condition transposes to (A.3.3). $\square$

The Yoneda lemma is usually expressed by the slogan "evaluation at the identity is an isomorphism," but since in the enriched context the enriched object of natural transformations is defined via a limit, it is easier to define the map that induces a natural transformation instead. Given an object $a \in \mathcal{A}$

in a small $\mathcal{V}$-category and a $\mathcal{V}$-functor $F\colon \mathcal{A} \to \mathcal{V}$, the internal action of F on arrows transposes to define a map that equalizes the parallel pair

$$Fa \overset{F_{a,-}}{\dashrightarrow} \prod_{z\in\mathcal{A}} Fz^{\mathcal{A}(a,z)} \rightrightarrows \prod_{x,y\in\mathcal{A}} Fy^{Fx\times\mathcal{A}(x,y)} \tag{A.3.10}$$

and thus induces a map $Fa \to \mathcal{V}^{\mathcal{A}}(\mathcal{A}(a,-),F)$ in $\mathcal{V}$.

A.3.11. THEOREM (enriched Yoneda lemma). *For any small $\mathcal{V}$-category $\mathcal{A}$, object $a \in \mathcal{A}$, and $\mathcal{V}$-functor $F\colon \mathcal{A} \to \mathcal{V}$, the canonical map defines an isomorphism in $\mathcal{V}$*

$$Fa \xrightarrow{\cong} \mathcal{V}^{\mathcal{A}}(\mathcal{A}(a,-),F),$$

that is $\mathcal{V}$-natural in both a and F.

PROOF. To prove the isomorphism, it suffices to verify that (A.3.10) is a limit cone. To that end consider another cone over the parallel pair

$$V \xrightarrow{\lambda} \prod_{z\in\mathcal{A}} Fz^{\mathcal{A}(a,z)} \rightrightarrows \prod_{x,y\in\mathcal{A}} Fy^{Fx\times\mathcal{A}(x,y)}$$

and define a candidate factorization by evaluating the component λ_a at id_a:

$$\lambda_a(\mathrm{id}_a) \coloneqq V \xrightarrow{\mathrm{id}_a\times\mathrm{id}} \mathcal{A}(a,a)\times V \xrightarrow{\lambda_a} Fa$$

To see that $\lambda_a(\mathrm{id}_a)\colon V \to Fa$ indeed defines a factorization of λ through the limit cone, it suffices to show commutativity at each component $Fz^{\mathcal{A}}(a,z)$ of the product, which we verify in transposed form:

$$\begin{tikzcd}
\mathcal{A}(a,z)\times V \arrow[d, "\mathrm{id}\times\mathrm{id}_a\times\mathrm{id}"'] \arrow[dr, equal] & \\
\mathcal{A}(a,z)\times\mathcal{A}(a,a)\times V \arrow[r, "\circ\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times\lambda_a"'] & \mathcal{A}(a,z)\times V \arrow[d, "\lambda_z"] \\
\mathcal{A}(a,z)\times Fa \arrow[r, "F_{a,z}"] & Fz
\end{tikzcd}$$

The upper triangle commutes by the identity law for $\mathcal{A}$ while the bottom square commutes because λ defines a cone over the parallel pair. Uniqueness of the factorization $\lambda_a(\mathrm{id}_a)$ follows from the same diagram by taking $z = a$ and evaluating at id_a.

We leave the verification of $\mathcal{V}$-naturality to the reader or to [**54**, §2.4]. ☐

Passing to underlying sets:

A.3.12. COROLLARY. *For any small $\mathcal{V}$-category $\mathcal{A}$, object $a \in \mathcal{A}$, and $\mathcal{V}$-functor $F\colon \mathcal{A} \to \mathcal{V}$, there is a bijection between $\mathcal{V}$-natural transformations $\alpha\colon \mathcal{A}(a,-) \Rightarrow F$ and elements $1 \to Fa$ in the underlying set of Fa implemented by evaluating the component at $a \in \mathcal{A}$ at the identity* id_a. ☐

This gives a criterion for establishing the representability of a $\mathcal{V}$-functor by presenting the minimal data required to establish the defining $\mathcal{V}$-natural isomorphism.

A.3.13. COROLLARY. *For a $\mathcal{V}$-functor $F\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{V}$ and an object $r \in \mathcal{A}$ the following are equivalent and define what it means for r to* ***represent*** *$\mathcal{V}$:*

(i) There exists an isomorphism

$$\mathcal{A}(a,r) \cong Fa$$

in $\mathcal{V}$ that is $\mathcal{V}$-natural in $a \in \mathcal{A}$.

(ii) There exists an element $u\colon 1 \to Fr$ in the underlying set of Fr so that the composite map

$$\mathcal{A}(a,r) \xrightarrow{F_{a,r}} \mathcal{V}(Fr,Fa) \xrightarrow{-\circ u} \mathcal{V}(1,Fa) \cong Fa$$

defines an isomorphism in $\mathcal{V}$ for all $a \in \mathcal{A}$.

Proof. By Corollary A.3.12 the element $u\colon 1 \to Fr$ in the underlying set of Fr determines a unique $\mathcal{V}$-natural transformation $\mathcal{A}(-,r) \Rightarrow F$ whose component at $a \in \mathcal{A}$ is the map of the statement. Thus, the universal element u defines a $\mathcal{V}$-natural isomorphism and not just a $\mathcal{V}$-natural transformation just when the map of the statement is an isomorphism. □

Since we have assumed our bases for enrichment to be cartesian closed, the 2-category $\mathcal{V}$-$\mathcal{C}at$ admits finite products, allowing us to define multivariable $\mathcal{V}$-functors. The following result will imply that the structures characterized by $\mathcal{V}$-natural isomorphisms in §A.4 and §A.5 assemble into $\mathcal{V}$-functors.

A.3.14. Proposition. *Let $M\colon \mathcal{B}^{\mathrm{op}} \times \mathcal{A} \to \mathcal{V}$ be a $\mathcal{V}$-functor so that for each $a \in \mathcal{A}$, the $\mathcal{V}$-functor $M(-,a)\colon \mathcal{B}^{\mathrm{op}} \to \mathcal{V}$ is represented by some $Fa \in \mathcal{B}$, meaning there exists a $\mathcal{V}$-natural isomorphism*

$$\mathcal{B}(b,Fa) \cong M(b,a).$$

Then there is a unique way of extending the mapping $a \in \mathcal{A} \mapsto Fa \in \mathcal{B}$ to a $\mathcal{V}$-functor $F\colon \mathcal{A} \to \mathcal{B}$ so that the isomorphisms are $\mathcal{V}$-natural in $a \in \mathcal{A}$ as well as $b \in \mathcal{B}$.

Proof. By the Yoneda lemma in the form of Corollary A.3.12, to define a family of isomorphisms $\alpha_{b,a}\colon \mathcal{B}(b,Fa) \cong M(b,a)$ for each $a \in \mathcal{A}$ that are $\mathcal{V}$-natural in $b \in \mathcal{B}$ is to define a family of elements $\eta_a\colon 1 \to M(Fa,a)$ for each $a \in \mathcal{A}$. By the $\mathcal{V}$-naturality statement in the Yoneda lemma, for the former isomorphism to be $\mathcal{V}$-natural in a is equivalent to the family of elements $\eta_a\colon 1 \to M(Fa,a)$ being "extraordinarily" $\mathcal{V}$-natural in a. What this means is that for any pair of objects $a,a' \in \mathcal{A}$, the outer square commutes:

$$\begin{tikzcd}
\mathcal{A}(a',a) \arrow[rr, "M(F-,a)_{a,a'}"] \arrow[dr, dashed, "F"] \arrow[dd, "M(Fa',-)_{a',a}"'] & & M(Fa',a)^{M(Fa,a)} \arrow[dd, "M(Fa',a)^{\eta_a}"] \\
& \mathcal{B}(Fa',Fa) \arrow[ur, "M(-,a)"] \arrow[dr, "\alpha_{Fa',a}", "\cong"'] & \\
M(Fa',a)^{M(Fa',a')} \arrow[rr, "M(Fa',a)^{\eta_{a'}}"'] & & M(Fa',a)
\end{tikzcd}$$

The definition of the top-horizontal map $M(F-,a)_{a',a}$ depends on the internal action of F on arrows, which we seek to define, but note that the composite of the other factor with the right vertical map is the natural isomorphism $\alpha_{Fa',a}$. Thus, there is a unique way to define $F_{a',a}$ making the extraordinary $\mathcal{V}$-naturality square commute, which is exactly the claim. $\mathcal{V}$-functoriality of these internal action maps for F follows from $\mathcal{V}$-functoriality of M in the $\mathcal{A}$ variable. □

We close this section with some applications. The correct notions of $\mathcal{V}$-enriched equivalence or $\mathcal{V}$-enriched adjunction are given by interpreting the standard 2-categorical notions of equivalence and adjunction in the 2-category $\mathcal{V}$. For sake of contrast, we present both notions in an alternate form here and leave it to the reader to apply Theorem A.3.11 to relate these to the 2-categorical notions.

A.3.15. Definition. A pair of $\mathcal{V}$-categories $\mathcal{C}$ and $\mathcal{D}$ are **$\mathcal{V}$-equivalent** if there exists a $\mathcal{V}$-functor $F\colon \mathcal{C} \to \mathcal{D}$ that is

- **$\mathcal{V}$-fully faithful**: i.e., each $F_{x,y}\colon \mathcal{C}(x,y) \to \mathcal{D}(Fx,Fy)$ is an isomorphism in $\mathcal{V}$ and
- essentially surjective on objects: i.e., each $d \in \mathcal{D}$ is isomorphic to Fc for some $c \in \mathcal{C}$.

A.3.16. Definition. A **$\mathcal{V}$-enriched adjunction** is given by a pair of $\mathcal{V}$-functors $F\colon \mathcal{B} \to \mathcal{A}$ and $U\colon \mathcal{A} \to \mathcal{B}$ together with isomorphisms

$$\mathcal{A}(Fb,a) \cong \mathcal{B}(b,Ua)$$

that are $\mathcal{V}$-natural in both $a \in \mathcal{A}$ and $b \in \mathcal{B}$.

A.3.17. Remark. By Proposition A.3.14, a $\mathcal{V}$-functor $U\colon \mathcal{A} \to \mathcal{B}$ admits a $\mathcal{V}$-left adjoint if and only if each $\mathcal{B}(b,U-)\colon \mathcal{A} \to \mathcal{V}$ is represented by some object $Fb \in \mathcal{A}$, in which case the data of the $\mathcal{V}$-natural isomorphism $\mathcal{A}(Fb,-) \cong \mathcal{B}(b,U-)$ equips $b \in \mathcal{B} \mapsto Fb \in \mathcal{A}$ with the structure of a $\mathcal{V}$-functor. Dual remarks construct enriched right adjoints.

Exercises.

A.3.i. Exercise. Prove Lemma A.3.5.

A.3.ii. Exercise. Use Corollary A.3.12 to show that the notions of $\mathcal{V}$-equivalence and $\mathcal{V}$-adjunction given in Definitions A.3.15 and A.3.16 are equivalent to the 2-categorical notions in $\mathcal{V}\text{-}\mathcal{C}at$.

A.4. Tensors and cotensors

A $\mathcal{V}$-category $\mathcal{C}$ admits *tensors* just when for all $C \in \mathcal{C}$, the covariant representable functor $\mathcal{C}(C,-)\colon \mathcal{C} \to \mathcal{V}$ admits a left $\mathcal{V}$-adjoint $-\otimes C\colon \mathcal{V} \to \mathcal{C}$. Dually, a $\mathcal{V}$-category $\mathcal{C}$ admits *cotensors* just when the contravariant representable functor $\mathcal{C}(-,C)\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{V}$ admits a mutual right adjoint $C^{-}\colon \mathcal{V}^{\mathrm{op}} \to \mathcal{C}$. The aim in this section is to introduce both constructions formally. In the next section, we establish some useful formal properties of enriched categories that are either *tensored* or *cotensored*.

A.4.1. Definition. A $\mathcal{V}$-category $\mathcal{C}$ is **cotensored**, if for all $V \in \mathcal{V}$ and $C \in \mathcal{C}$, the $\mathcal{V}$-functor $\mathcal{C}(-,C)^V\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{V}$ is represented by an object $C^V \in \mathcal{C}$, i.e., there exists an isomorphism

$$\mathcal{C}(X,C^V) \cong \mathcal{C}(X,C)^V$$

in $\mathcal{V}$ that is $\mathcal{V}$-natural in X. By Proposition A.3.14, the cotensor product defines a unique $\mathcal{V}$-functor

$$\mathcal{C} \times \mathcal{V}^{\mathrm{op}} \xrightarrow{(-)^{-}} \mathcal{C}$$

making the defining isomorphism $\mathcal{V}$-natural in all three variables.

A.4.2. Definition. Dually, a $\mathcal{V}$-category $\mathcal{C}$ is **tensored** if for all $V \in \mathcal{V}$ and $\mathcal{C}$ the $\mathcal{V}$-functor $\mathcal{C}(C,-)^V\colon \mathcal{C} \to \mathcal{V}$ is represented by an object $V \otimes C \in \mathcal{C}$, i.e., there exists an isomorphism

$$\mathcal{C}(V \otimes C,X) \cong \mathcal{C}(C,X)^V$$

in $\mathcal{V}$ that is $\mathcal{V}$-natural in X. By Proposition A.3.14, the tensor product defines a unique $\mathcal{V}$-functor

$$\mathcal{V} \times \mathcal{C} \xrightarrow{-\otimes-} \mathcal{C}$$

making the defining isomorphism $\mathcal{V}$-natural in all three variables.

Immediately from these definitions

A.4.3. Lemma. *A $\mathcal{V}$-category $\mathcal{C}$ is tensored and cotensored if and only if the $\mathcal{V}$-functor $\mathcal{C}(-,-)\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{V}$ is part of a two-variable $\mathcal{V}$-adjunction*

$$\begin{tikzcd}
& \mathcal{C}(V \times A, B) \arrow[dl, phantom, "\cong" sloped] \arrow[dr, phantom, "\cong" sloped] & \\
\mathcal{C}(A, B^V) \arrow[rr, phantom, "\cong"] & & \mathcal{C}(A,B)^V
\end{tikzcd}$$

as expressed by the commutative triangle of $\mathcal{V}$-natural isomorphisms. ☐

A.4.4. Lemma. *In any category $\mathcal{C}$ that is enriched and cotensored over $\mathcal{V}$, there are $\mathcal{V}$-natural isomorphisms*

$$C^1 \cong C \qquad \textit{and} \qquad \begin{tikzcd}
& C^{U\times V} \arrow[dl, phantom, "\cong" sloped] \arrow[dr, phantom, "\cong" sloped] & \\
(C^V)^U \arrow[rr, phantom, "\cong"] & & (C^U)^V
\end{tikzcd}$$

for $U, V \in \mathcal{V}$ and $C \in \mathcal{C}$.

Proof. By Lemma A.3.7, to define the displayed $\mathcal{V}$-natural isomorphisms, it suffices to prove that these objects represent the same $\mathcal{V}$-functors $\mathcal{C}^{\mathrm{op}} \to \mathcal{V}$. By the defining universal property of the cotensor, for any $X \in \mathcal{C}$,

$$\mathcal{C}(X, C^1) \cong \mathcal{C}(X,C)^1 \cong \mathcal{C}(X,C)$$

by Remark A.1.9. Similarly, by the defining universal property of the cotensor, we have $\mathcal{V}$-natural isomorphisms in each of the three vertices of the triangle below

$$\begin{tikzcd}
& \mathcal{C}(X, C^{U\times V}) \cong \mathcal{C}(X,C)^{U\times V} \arrow[dl, phantom, "\cong" sloped] \arrow[dr, phantom, "\cong" sloped] & \\
\mathcal{C}(X, (C^V)^U) \cong (\mathcal{C}(X,C)^V)^U \arrow[rr, phantom, "\cong"] & & \mathcal{C}(X, (C^U)^V) \cong (\mathcal{C}(X,C)^U)^V
\end{tikzcd}$$

whence the connecting $\mathcal{V}$-natural isomorphisms are given by Lemma A.1.7. ☐

Extending Lemma A.2.3:

A.4.5. Lemma. *A cartesian closed category $(\mathcal{V}, \times, 1)$ is enriched, tensored, and cotensored over itself, with tensors defined by the cartesian product and cotensors defined by the internal hom*[5]*:*

$$V \otimes W := V \times W \qquad \textit{and} \qquad W^V := W^V.$$

Proof. Lemma A.1.7 establishes the required isomorphisms (A.1.8) in $\mathcal{V}$. The proof of their $\mathcal{V}$-naturality is left to the reader or [**54**, §1.8]. ☐

The presence of tensors and cotensors provides a convenient mechanism for testing whether an a priori unenriched adjunction may be enriched to a $\mathcal{V}$-adjunction.

A.4.6. Proposition. *Let $U\colon \mathcal{A} \to \mathcal{B}$ and $F\colon \mathcal{B} \to \mathcal{A}$ be unenriched adjoint functors between categories $\mathcal{A}$ and $\mathcal{B}$ that are both enriched over $\mathcal{V}$.*

[5]This justifies the abuse of exponential notation for both internal homs and cotensors.

(i) If $\mathcal{A}$ and $\mathcal{B}$ are cotensored over $\mathcal{V}$ and $U\colon \mathcal{A} \to \mathcal{B}$ is a cotensor-preserving $\mathcal{V}$-functor, then $F \dashv U$ may be enriched to a $\mathcal{V}$-adjunction.

(ii) If $\mathcal{A}$ and $\mathcal{B}$ are tensored over $\mathcal{V}$ and $F\colon \mathcal{B} \to \mathcal{A}$ is a tensor-preserving $\mathcal{V}$-functor, then $F \dashv U$ may be enriched to a $\mathcal{V}$-adjunction.

Conversely, if $F \dashv U$ is a $\mathcal{V}$-enriched adjunction then certainly F and U are $\mathcal{V}$-enriched and respectively tensor- and cotensor-preserving.

Proof. Exercise A.4.i. □

There is a convenient device for turning an a priori unenriched category C into a category that is both enriched and cotensored over simplicial sets that is of utility in constructing ∞-cosmoi. This is a specialization of the notion of *closed $\mathcal{V}$-module* of [**85**, 14.3] to the case $\mathcal{V} = s\mathcal{S}et$, in which case less data is required to prove Proposition A.4.8.

A.4.7. Definition. An unenriched category C is a **right $s\mathcal{S}et$-module** if there exists a bifunctor

$$C \times s\mathcal{S}et^{\mathrm{op}} \xrightarrow{(-)^-} C$$

equipped with natural isomorphisms

$$\begin{tikzcd} C \times s\mathcal{S}et^{\mathrm{op}} \times s\mathcal{S}et^{\mathrm{op}} \arrow[r, "(-)^- \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times (- \times -)"'] \arrow[dr, phantom, "\cong"] & C \times s\mathcal{S}et^{\mathrm{op}} \arrow[d, "(-)^-"] \\ C \times s\mathcal{S}et^{\mathrm{op}} \arrow[r, "(-)^-"'] & C \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "(-{,}1)"] \arrow[dr, "\mathrm{id}"'] \arrow[dr, phantom, bend left, "\cong"] & C \times s\mathcal{S}et^{\mathrm{op}} \arrow[d, "(-)^-"] \\ & C \end{tikzcd}$$

asserting that for $U, V \in s\mathcal{S}et$ and $A \in C$

$$A^{U \times V} \cong (A^U)^V \qquad \text{and} \qquad A \cong A^1$$

with these natural isomorphisms satisfying the manifest pseudo-algebra laws. A right $s\mathcal{S}et$-module is **continuous** if the bifunctor $C \times s\mathcal{S}et^{\mathrm{op}} \to C$ carries limits in its first variable to limits in C and carries colimits in its second variable to limits in C.

A.4.8. Proposition. *A continuous right $s\mathcal{S}et$-module C can be enriched to a $s\mathcal{S}et$-category in a unique way so that there exist isomorphisms in $s\mathcal{S}et$*

$$C(A, B^V) \cong C(A, B)^V$$

for all $A, B \in C$ and $V \in s\mathcal{S}et$ natural[6] in all three variables.

Proof. We must construct an enrichment from the supplied right pseudo-action. This is done straightforwardly by defining the n-arrow in $C(A, B)$ to be arrows $A \to B^{\Delta[n]}$ in C and using the pseudo-action laws and the canonical diagonal co-algebra structure on $\Delta[n]$ to define an associative and unital composition of such. Note that the identity axiom of the right pseudo-action ensures that arrows $A \to B^{\Delta[0]}$ are in bijective correspondence with arrows $A \to B$ and thus that this construction really does give an enrichment of C.

Furthermore, we must show that the pseudo-action supplies cotensors for this derived enrichment: that is that

$$C(A, B^V) \cong C(A, B)^V$$

[6]In fact these isomorphisms are $s\mathcal{S}et$-natural.

for all $A, B \in \mathcal{C}$ and $V \in s\mathcal{S}et$ natural in all three-variables, which fact follows from one of the assumed continuity properties of $(-)^{-} \colon \mathcal{C} \times s\mathcal{S}et^{\mathrm{op}} \to \mathcal{C}$ using the fact that every simplicial set can be expressed as a colimit of representables. □

We have not yet made use of continuity in of the right action bifunctor in its first variable. The reason for this hypothesis will explained in §A.5.

Exercises.

A.4.i. Exercise. Prove Proposition A.4.6.

A.4.ii. Exercise. Let $\mathcal{J}$ be a small unenriched category and let $\mathcal{C}$ be a $\mathcal{V}$-category. Prove that if $\mathcal{C}$ is tensored or cotensored then so is $\mathcal{C}^{\mathcal{J}}$.

A.5. Conical limits

Let $\mathcal{V}$ be a cartesian closed category. The **conical limit** of a $\mathcal{V}$-enriched functor is the limit weighted by the terminal weight, the $\mathcal{V}$-functor that is constant at $1 \in \mathcal{V}$. In what follows, we focus on the slightly less general case of conical limits of unenriched diagrams — those indexed by 1-categories instead of $\mathcal{V}$-categories — as they will be of greatest interest here. Conical limits necessarily define 1-categorical limits, so we pay particular attention to what is required to enriched a 1-categorical limit to a conical limit.

To motivate the universal property that characterizes conical limits, consider a diagram $F \colon \mathcal{J} \to \mathcal{V}$ indexed by a 1-category $\mathcal{J}$ and valued in $\mathcal{V}$ itself. A cone λ over the diagram F with summit suggestively denoted $\lim F \in \mathcal{V}$ is a limit cone if it satisfies the universal property

$$\mathcal{V}(A, \lim F) \cong \lim_{j \in \mathcal{J}} \mathcal{V}(A, Fj),$$

an isomorphism of hom-sets. The next lemma reveals that this isomorphism can always be enriched to lie in $\mathcal{V}$, i.e., lifted along the underlying set functor of Definition A.1.5.

A.5.1. Lemma. *If $\mathcal{V}$ is a cartesian closed category, then any 1-categorical limit cone $\lambda \colon \Delta \lim F \Rightarrow F$ over a diagram $F \colon \mathcal{J} \to \mathcal{V}$ gives rise to an isomorphism in $\mathcal{V}$*

$$(\lim F)^A \cong \lim_{j \in \mathcal{J}} (Fj^A)$$

that is $\mathcal{V}$-natural in $A \in \mathcal{V}$.

Proof. For any $A \in \mathcal{V}$, the exponential $(-)^A \colon \mathcal{V} \to \mathcal{V}$ is right adjoint to the product functor $- \times A \colon \mathcal{V} \to \mathcal{V}$; as such it preserves limits, giving rise to the isomorphism of the statement. □

In terminology we now introduce, Lemma A.5.1 asserts that all 1-categorical limits in a cartesian closed category $\mathcal{V}$ automatically enriched to define *conical limits*.

A.5.2. Definition. Let $\mathcal{C}$ be a $\mathcal{V}$-category and let $\mathcal{J}$ be a 1-category. The **conical limit** of an unenriched diagram $F \colon \mathcal{J} \to \mathcal{C}$ is given by an object $L \in \mathcal{C}$ and a cone $\lambda \colon \Delta L \Rightarrow F$ inducing a $\mathcal{V}$-natural isomorphism of hom-objects in $\mathcal{V}$

$$\mathcal{C}(X, L) \xrightarrow[\cong]{\lambda_*} \lim_{j \in \mathcal{J}} \mathcal{C}(X, Fj) \quad \in \quad \mathcal{V} \tag{A.5.3}$$

for all $X \in \mathcal{C}$.

The isomorphism (A.5.3) looks identical to the usual isomorphism that characterizes 1-categorical limits except for one very important difference: it postulates an isomorphism in $\mathcal{V}$ rather than an isomorphism in $\mathcal{S}et$. In the case where $\mathcal{V} = s\mathcal{S}et$, the isomorphism of vertices that underlies this isomorphism of simplicial sets describes the usual 1-categorical universal property. To say that the limit is *conical* and not merely *1-categorical* is to assert that this universal property extends to all positive dimensions.

Inspecting Definition A.5.2, we see immediately that:

A.5.4. Proposition. *A 1-categorical limit cone is conical just when it is preserved by all $\mathcal{V}$-valued representable functors $\mathcal{C}(X,-)\colon \mathcal{C} \to \mathcal{V}$.* □

The proof of Lemma A.5.1 generalizes to show:

A.5.5. Proposition. *If $\mathcal{C}$ has tensors, then all 1-categorical limits in $\mathcal{C}$ are conical.*

Proof. By Proposition A.5.4, to show that a 1-categorical limit is conical, it suffices to show that it is preserved by the $\mathcal{V}$-valued representable functors $\mathcal{C}(X,-)\colon \mathcal{C} \to \mathcal{V}$ for all $X \in \mathcal{V}$. If $\mathcal{C}$ admits tensors, then each of these functors admits a left adjoint; as right adjoints, they necessarily preserve the 1-categorical limits of the statement. □

A.5.6. Proposition. *Let $\mathcal{C}$ be enriched and cotensored over $\mathcal{V}$. A limit of an unenriched diagram $F\colon \mathcal{J} \to \mathcal{C}$ is a conical limit if and only if it is preserved by cotensors with all objects $V \in \mathcal{V}$.*

Proof. Cotensors are $\mathcal{V}$-enriched right adjoints, which preserve conical limits. The content is that preservation by cotensors suffices to enriched a 1-categorical limit to a $\mathcal{V}$-categorical one.

Recall that a 1-categorical limit cone $(\lambda_j\colon L \to Fj)_{j\in\mathcal{J}}$ is conical just when it is preserved by all $\mathcal{V}$-valued representable functors $\mathcal{C}(X,-)\colon \mathcal{C} \to \mathcal{V}$. To see that the natural map

$$\mathcal{C}(X,L) \to \lim_{j\in\mathcal{J}} \mathcal{C}(X,Fj)$$

is an isomorphism in $\mathcal{V}$ we appeal to the (unenriched) Yoneda lemma and prove that for all $V \in \mathcal{V}$, the map of hom-sets

$$\mathcal{V}(V,\mathcal{C}(X,L))_0 \to \mathcal{V}(V,\lim_{j\in\mathcal{J}} \mathcal{C}(X,Fj))_0 \tag{A.5.7}$$

is an isomorphism. Since unenriched representables preserve 1-categorical limits,

$$\mathcal{V}(V,\lim_{j\in\mathcal{J}} \mathcal{C}(X,Fj))_0 \cong \lim_{j\in\mathcal{J}} \mathcal{V}(V,\mathcal{C}(X,F_j))_0.$$

By the unenriched universal property of cotensors, maps $V \to \mathcal{C}(X,L)$ in $\mathcal{V}$ correspond bijectively to maps $X \to L^V$ in $\mathcal{C}$. So the map (A.5.7) is isomorphic to the map of hom-sets

$$\mathcal{C}(X,L^V)_0 \to \lim_{j\in\mathcal{J}} \mathcal{C}(X,Fj^V)_0 \cong \mathcal{C}(X,\lim_{j\in\mathcal{J}}(Fj^V))_0,$$

where again the unenriched representable commutes with the 1-categorical limit. To say that cotensors preserve the limit $L \cong \lim_{j\in\mathcal{J}} Fj$ means that $\lim_{j\in\mathcal{J}}(Fj^V) \cong (\lim_{j\in\mathcal{J}} Fj)^V \cong L^V$, so by the Yoneda lemma, the map of hom-sets

$$\mathcal{C}(X,L^V)_0 \to \mathcal{C}(X,\lim_{j\in\mathcal{J}}(Fj^V))_0,$$

is an isomorphism, and thus (A.5.7) and hence $\mathcal{C}(X,L) \cong \lim_{j\in\mathcal{J}} \mathcal{C}(X,Fj)$ is an isomorphism as desired. □

A.5.8. PROPOSITION. *Suppose $\mathcal{C}$ is a $\mathcal{V}$-category that admits conical limits indexed by a small category $\mathcal{J}$. Then the limit functor* $\lim \colon \mathcal{C}^{\mathcal{J}} \to \mathcal{C}$ *is a $\mathcal{V}$-functor.*

PROOF. We first prove that the conical limit functor $\lim \colon \mathcal{V}^{\mathcal{J}} \to \mathcal{V}$ is a $\mathcal{V}$-functor. The result for a general $\mathcal{V}$-category $\mathcal{C}$ then follows from the defining $\mathcal{V}$-natural isomorphism (A.5.3) via Proposition A.3.14.

The advantage of this reduction is that we can make use of the fact that $\mathcal{V}$ and $\mathcal{V}^{\mathcal{J}}$ are tensored, the latter following from the former by Exercise A.4.ii. Now by Proposition A.4.6, to show that the conical limit is a $\mathcal{V}$-functor it suffices to show that it's unenriched left adjoint $\Delta \colon \mathcal{V} \to \mathcal{V}^{\mathcal{J}}$ is a $\mathcal{V}$-functor that preserves tensors, and this is straightforward. □

Exercises.

A.5.i. EXERCISE. Specialize the result of Proposition A.5.5 to prove the following: in any 2-category $\mathcal{C}$ that admits tensors with the walking-arrow category $\mathbb{2}$, any 1-categorical limits that $\mathcal{C}$ admits are automatically conical.[7]

A.6. Weighted limits and colimits

The cotensors of §A.4 and conical limits of §A.5 are both instances of a more general notion of *weighted limit* that we now introduce. We continue in the context of a complete and cocomplete cartesian closed symmetric monoidal category $(\mathcal{V}, \times, \mathbb{1})$. The examples we have in mind are $(s\mathcal{S}et, \times, \mathbb{1})$, its cartesian closed subcategory $(\mathcal{C}at, \times, \mathbb{1})$, or its further cartesian closed subcategory $(\mathcal{S}et, \times, \mathbb{1})$.

Ordinary limits and colimits are objects representing the functor of cones with a given apex over or under a fixed diagram. *Weighted* limits and colimits are defined analogously, except that the cones over or under a diagram might have exotic "shapes, " which are allowed to vary with the objects indexing the diagram. More formally, in the $\mathcal{V}$-enriched context, the *weight*, defining the "shape" of a cone over a diagram indexed by $\mathcal{A}$ or under a diagram indexed by $\mathcal{A}^{\mathrm{op}}$, takes the form of a functor in $\mathcal{V}^{\mathcal{A}}$.

Before introducing the general notion of weighted limit and colimit, we first reacquaint ourselves an example that we have seen already in Digression 1.2.6.

A.6.1. EXAMPLE (tensors and cotensors). A diagram indexed by the category $\mathbb{1}$ valued in a $\mathcal{V}$-enriched category $\mathcal{M}$ is just an object A in $\mathcal{M}$. In this case, the weight is just an object U of $\mathcal{V}$. The U-weighted limit of the diagram A is an object of $\mathcal{M}$ denoted A^U — or denoted $\{U, A\}$ whenever superscripts are inconvenient — called the **cotensor** of $A \in \mathcal{M}$ with $U \in \mathcal{V}$ defined by the universal property

$$\mathcal{M}(X, A^U) \cong \mathcal{V}(U, \mathcal{M}(X, A)),$$

this isomorphism between mapping spaces in $\mathcal{V}$. Dually, the U-weighted colimit of A is an object $U \otimes A \in \mathcal{M}$ called the **tensor** of $A \in \mathcal{M}$ with $U \in \mathcal{V}$ defined by the universal property

$$\mathcal{M}(U \otimes A, X) \cong \mathcal{V}(U, \mathcal{M}(A, X)),$$

this isomorphism again between mapping spaces in $\mathcal{V}$. Assuming such objects exist, the cotensor and tensor define $\mathcal{V}$-enriched bifunctors

$$\mathcal{V}^{\mathrm{op}} \times \mathcal{M} \xrightarrow{-^{-}} \mathcal{M} \qquad \mathcal{V} \times \mathcal{M} \xrightarrow{-\otimes-} \mathcal{M}$$

[7]The statement asserts that the presence of tensors with $\mathbb{2}$ implies that the universal property of 1-dimensional limits automatically has an additional 2-dimensional aspect. See the discussion around Proposition 1.4.5 for an illustration of this and see [**53**, pp. 306] for a proof.

in a unique way making the defining isomorphisms natural in U and A as well.

Since $\mathcal{V}$ is cartesian closed, it is tensored and cotensored over itself, with $U \otimes V := U \times V$ and $V^U := \mathcal{V}(U, V)$. In particular, the defining natural isomorphisms characterizing tensors and cotensors in $\mathcal{M}$ can be rewritten as

$$\mathcal{M}(X, A^U) \cong \mathcal{M}(X, A)^U \qquad \text{and} \qquad \mathcal{M}(U \otimes A, X) \cong \mathcal{M}(A, X)^U.$$

The fact that the natural isomorphisms defining tensors and cotensors are required to exist in $\mathcal{V}$ (and not merely in $\mathcal{S}et$) has the following consequence:

A.6.2. Lemma (associativity of tensors and cotensors). *If $\mathcal{M}$ is a $\mathcal{V}$-category with tensors and cotensors then for any $U, V \in \mathcal{V}$ and $A \in \mathcal{M}$, there exist natural isomorphisms*

$$U \otimes (V \otimes A) \cong (U \times V) \otimes A \qquad \text{and} \qquad (A^V)^U \cong A^{U \times V}.$$

Proof. By the defining universal property

$$\begin{aligned}\mathcal{M}(X, (A^V)^U) &\cong \mathcal{V}(U, \mathcal{M}(X, A^V)) \cong \mathcal{V}(U, \mathcal{V}(V, \mathcal{M}(X, A))) \\ &\cong \mathcal{V}(U \times V, \mathcal{M}(X, A)) \cong \mathcal{M}(X, A^{U \times V})\end{aligned}$$

for all $X \in \mathcal{M}$. By the Yoneda lemma $(A^V)^U \cong A^{U \times V}$. The case for tensors is similar. ☐

We now introduce the general notions of weighted limit and weighted colimit from three different viewpoints. We introduce these perspectives in the reverse of the logical order, because we find this route to be the most intuitive. We first describe the axioms that characterize the weighted limit and colimit bifunctors, whenever they exist. We then explain how weighted limits and colimits can be constructed, again assuming these exist. We then finally introduce the general universal property that defines a particular weighted limit or colimit, which tells us when the notions just introduced do in fact exist.

A.6.3. Definition (weighted limits and colimits, axiomatically). For a small $\mathcal{V}$-enriched category $\mathcal{A}$ and a large $\mathcal{V}$-enriched category $\mathcal{M}$, the **weighted limit** and **weighted colimit bifunctors**

$$\lim^{\mathcal{A}}_{-} - \colon (\mathcal{V}^{\mathcal{A}})^{\mathrm{op}} \times \mathcal{M}^{\mathcal{A}} \to \mathcal{M} \qquad \text{and} \qquad \operatorname{colim}^{\mathcal{A}}_{-} - \colon \mathcal{V}^{\mathcal{A}} \times \mathcal{M}^{\mathcal{A}^{\mathrm{op}}} \to \mathcal{M}$$

are characterized by the following pair of axioms whenever they exist:

(i) Weighted (co)limits with representable weights evaluate at the representing object:

$$\lim^{\mathcal{A}}_{\mathcal{A}(a,-)} F \cong F(a) \qquad \text{and} \qquad \operatorname{colim}^{\mathcal{A}}_{\mathcal{A}(-,a)} G \cong G(a).$$

(ii) The weighted (co)limit bifunctors are cocontinuous in the weight: for any diagrams $F \in \mathcal{M}^{\mathcal{A}}$ and $G \in \mathcal{M}^{\mathcal{A}^{\mathrm{op}}}$, the functor $\operatorname{colim}^{\mathcal{A}}_{-} G$ preserves colimits, while the functor $\lim^{\mathcal{A}}_{-} F$ carries colimits to limits.[8]

We interpret axiom (ii) to mean that weights can be "made-to-order": a weight constructed as a colimit of representables — as all $\mathcal{V}$-valued functors are — will stipulate the expected universal property.

[8]More precisely, as will be proven in Proposition A.6.12, the weighted colimit functor $\operatorname{colim}^{\mathcal{A}}_{-} G$ preserves weighted colimits, while the weighted limit functor $\lim^{\mathcal{A}}_{-} F$ carries weighted colimits to weighted limits.

A.6.4. DEFINITION (weighted limits and colimits, constructively). The **limit** of $F \in \mathcal{M}^{\mathcal{A}}$ **weighted by** $W \in \mathcal{V}^{\mathcal{A}}$ is computed by the functor cotensor product:

$$\lim_W^{\mathcal{A}} F \coloneqq \int_{a\in\mathcal{A}} F(a)^{W(a)} \coloneqq \mathrm{eq}\left(\prod_{a\in\mathcal{A}} F(a)^{W(a)} \rightrightarrows \prod_{a,b\in\mathcal{A}} F(b)^{\mathcal{A}(a,b)\times W(a)} \right), \tag{A.6.5}$$

where the product and equalizer should be interpreted as *conical limits*; see Digression 1.2.6 or Definition A.6.14 below. The maps in the equalizer diagram are induced by the actions $\mathcal{A}(a,b)\times W(a) \to W(b)$ and $F(a) \to F(b)^{\mathcal{A}(a,b)}$ of the hom-object $\mathcal{A}(a,b)$ on the $\mathcal{V}$-functors W and F; the latter case makes use of the natural isomorphism $F(b)^{\mathcal{A}(a,b)\times W(a)} \cong (F(b)^{\mathcal{A}(a,b)})^{W(a)}$ of Lemma A.6.2.

Dually, the **colimit** of $G \in \mathcal{M}^{\mathcal{A}^{\mathrm{op}}}$ **weighted by** $W \in \mathcal{V}^{\mathcal{A}}$ is computed by the functor tensor product:

$$\operatorname{colim}_W^{\mathcal{A}} G \coloneqq \int^{a\in\mathcal{A}} W(a)\otimes G(a) \coloneqq \mathrm{coeq}\left(\coprod_{a,b\in\mathcal{A}} (W(a)\times\mathcal{A}(a,b))\otimes G(b) \rightrightarrows \coprod_{a\in\mathcal{A}} W(a)\otimes G(a) \right), \tag{A.6.6}$$

where the coproduct and coequalizer should be interpreted as *conical colimits*. One of the maps in the coequalizer diagram is induced by the action $\mathcal{A}(a,b)\otimes G(b) \to G(a)$ of $\mathcal{A}(a,b)$ on the contravariant $\mathcal{V}$-functor G and the natural isomorphism $(W(a)\times\mathcal{A}(a,b))\otimes G(b) \cong W(a)\otimes(\mathcal{A}(a,b)\otimes G(b))$ of Lemma A.6.2; the other uses the covariant action of $\mathcal{A}(a,b)$ on W as before.

A.6.7. DEFINITION (weighed limits and colimits, the universal property). The **limit** $\lim_W^{\mathcal{A}} F$ of the diagram $F \in \mathcal{M}^{\mathcal{A}}$ weighted by $W \in \mathcal{V}^{\mathcal{A}}$ and the **colimit** $\operatorname{colim}_W^{\mathcal{A}} G$ of $G \in \mathcal{M}^{\mathcal{A}^{\mathrm{op}}}$ weighted by $W \in \mathcal{V}^{\mathcal{A}}$ are characterized by the universal properties:

$$\mathcal{M}(X, \lim_W^{\mathcal{A}} F) \cong \mathcal{V}^{\mathcal{A}}(W, \mathcal{M}(X,F)) \qquad \text{and} \qquad \mathcal{M}(\operatorname{colim}_W^{\mathcal{A}} G, X) \cong \mathcal{V}^{\mathcal{A}}(W, \mathcal{M}(G,X)), \tag{A.6.8}$$

each of these defining an isomorphism between objects of $\mathcal{V}$.

When the indexing category $\mathcal{A}$ is clear from context, as is typically the case, we frequently drop it from the notation for the weighted limit and weighted colimit. We now argue that these three definitions characterize the same objects. Along the way, we obtain results of interest in their own right, that we record separately.

A.6.9. LEMMA. *The category $\mathcal{V}$ admits all weighted limits, as defined by the formula of* (A.6.5) *satisfying the natural isomorphism of* (A.6.8). *Explicitly, for a weight $W\colon \mathcal{A} \to \mathcal{V}$ and a diagram $F\colon \mathcal{A} \to \mathcal{V}$, the weighted limit*

$$\lim_W^{\mathcal{A}} F \coloneqq \mathcal{V}^{\mathcal{A}}(W,F),$$

is the $\mathcal{V}$-object of $\mathcal{V}$-natural transformations from W to V.

PROOF. The $\mathcal{V}$-functor $\mathcal{V}(\mathbb{1}, -)\colon \mathcal{V} \to \mathcal{V}$ represented by the monoidal unit is naturally isomorphic to the identity functor. So taking $X = \mathbb{1}$ in the universal property of (A.6.8) in the case where the diagram $F \in \mathcal{V}^{\mathcal{A}}$ is valued in the $\mathcal{V}$-category $\mathcal{V}$, we have

$$\lim_W^{\mathcal{A}} F \cong \mathcal{V}^{\mathcal{A}}(W,F).$$

Simultaneously, the formula (A.6.5) computes the $\mathcal{V}$-object $\mathcal{V}^{\mathcal{A}}(W,F)$ of $\mathcal{V}$-natural transformations from W to F defined in Definition A.3.8. □

The $\mathcal{V}$-object of $\mathcal{V}$-natural transformations satisfies the natural isomorphism

$$\mathcal{V}(V, \mathcal{V}^{\mathcal{A}}(W, F)) \cong \mathcal{V}^{\mathcal{A}}(W, \mathcal{V}(V, F))$$

for any $V \in \mathcal{V}$. Applying the observation that W-weighted limits of $\mathcal{V}$-valued functors are $\mathcal{V}$-objects of natural transformations to the functor $\mathcal{M}(X, F-)$ and $\mathcal{M}(G-, X)$ in the case of $F \in \mathcal{M}^{\mathcal{A}}$ and $G \in \mathcal{M}^{\mathcal{A}^{\mathrm{op}}}$, we may re-express the natural isomorphism (A.6.8) as:

A.6.10. Corollary. *The weighted limits and weighted colimits of* (A.6.8) *are representably defined as weighted limits in $\mathcal{V}$: for $W \in \mathcal{V}^{\mathcal{A}}$ and $F \in \mathcal{M}^{\mathcal{A}}$ and $G \in \mathcal{M}^{\mathcal{A}^{\mathrm{op}}}$ the weighted limit and colimit are characterized by isomorphisms*

$$\mathcal{M}(X, \lim_W F) \cong \lim_W \mathcal{M}(X, F) \qquad \textit{and} \qquad \mathcal{M}(\operatorname{colim}_W G, X) \cong \lim_W \mathcal{M}(G, X) \tag{A.6.11}$$

natural in X in $\mathcal{V}$. □

We now unify the Definitions A.6.3, A.6.4, and A.6.7.

A.6.12. Proposition. *When the limits and colimits of* (A.6.5) *and* (A.6.6) *exist they define objects satisfying the universal properties* (A.6.8) *or equivalently* (A.6.11) *and bifunctors satisfying the axioms of Definition A.6.3.*

Proof. The proofs are dual, so we confine our attention to the limit case. The general case of the implication Definition A.6.4 ⇒ A.6.7 — for either weighted limits or weighted colimits — is a direct consequence of the special case of this implication for weighted limits valued in $\mathcal{M} = \mathcal{V}$ proven as Lemma A.6.9 and Corollary A.6.10. The limits of (A.6.5) in $\mathcal{M}$ are also defined representably in terms of the analogous limits in $\mathcal{V}$. So the objected defined by (A.6.5) represents the $\mathcal{V}$-functor $\lim_W \mathcal{M}(-, F)$ that defines the weighted limit $\lim_W F$.

It remains to prove that the weighted limits of Definitions A.6.4 and A.6.7 satisfy the axioms of Definition A.6.3. In the case of a $\mathcal{V}$-valued diagram $F \in \mathcal{V}^{\mathcal{A}}$, axiom (i) is the $\mathcal{V}$-Yoneda lemma:

$$\mathcal{V}^{\mathcal{A}}(\mathcal{A}(a, -), F) \cong F(a)$$

proven in Theorem A.3.11. Once again, the general case for $F \in \mathcal{M}^{\mathcal{A}}$ follows from the special case for $\mathcal{V}$-valued diagrams, for to demonstrate an isomorphism $\lim_{\mathcal{A}(a,-)} F \cong F(a)$ in $\mathcal{M}$ it suffices to demonstrate an isomorphism $\mathcal{M}(X, \lim_{\mathcal{A}(a,-)} F) \cong \mathcal{M}(X, F(a))$ in $\mathcal{V}$ for all $X \in \mathcal{M}$ and have such a natural isomorphism by applying (A.6.11) and the observation just made to the functor $\mathcal{M}(X, F-) \in \mathcal{V}^{\mathcal{A}}$.

For the axiom (ii), consider a diagram $K \colon \mathcal{J}^{\mathrm{op}} \to \mathcal{V}^{\mathcal{A}}$ of weights and a weight $V \in \mathcal{V}^{\mathcal{J}}$ so that $\operatorname{colim}^{\mathcal{J}}_V K \cong W$. For any $F \in \mathcal{M}^{\mathcal{A}}$, we will show that the $\mathcal{V}$-functor $\lim^{\mathcal{A}}_{-} F \colon (\mathcal{V}^{\mathcal{A}})^{\mathrm{op}} \to \mathcal{M}$ carries the V-weighted colimit of K to the V-weighted limit of the composite diagram $\lim^{\mathcal{A}}_K F \colon \mathcal{J} \to \mathcal{M}$.

The universal property (A.6.8), applied first to the $\operatorname{colim}^{\mathcal{J}}_V K$-weighted limit of the diagram F and the object X, and then to the V-weighted colimit of the diagram K and the object $\mathcal{M}(X, F)$, supplies isomorphisms:

$$\mathcal{M}(X, \lim^{\mathcal{A}}_{\operatorname{colim}^{\mathcal{J}}_V K} F) \cong \mathcal{V}^{\mathcal{A}}(\operatorname{colim}^{\mathcal{J}}_V K, \mathcal{M}(X, F)) \cong \mathcal{V}^{\mathcal{J}}(V, \mathcal{V}^{\mathcal{A}}(K, \mathcal{M}(X, F))).$$

Applying (A.6.8) twice more, first for the weights Kj for each $j \in \mathcal{J}$ and then for the weight V and the diagram $\lim^{\mathcal{A}}_K F \colon \mathcal{J} \to \mathcal{M}$, we have

$$\cong \mathcal{V}^{\mathcal{J}}(V, \mathcal{M}(X, \lim^{\mathcal{A}}_K F)) \cong \mathcal{M}(X, \lim^{\mathcal{J}}_V \lim^{\mathcal{A}}_K F).$$

By the Yoneda lemma, this proves that

$$\lim^{\mathcal{A}}_{\mathrm{colim}^{\mathcal{J}}_{V} K} F \cong \lim^{\mathcal{J}}_{V} \lim^{\mathcal{A}}_{H} F,$$

i.e., that the weighted limit functor $\lim^{\mathcal{A}}_{-} F$ is carries a weighted colimit of weights to the analogous weighted limit of weights. $\square$

A.6.13. Remark (for unenriched indexing categories). When the indexing category is unenriched, the limit and colimit formulas from Definition A.6.4 simplify

$$\lim^{\mathcal{A}}_{W} F \cong \mathrm{eq}\left(\prod_{a \in \mathcal{A}} F(a)^{W(a)} \rightrightarrows \prod_{\mathcal{A}(a,b)} F(b)^{W(a)} \right)$$

$$\mathrm{colim}^{\mathcal{A}}_{W} G \cong \mathrm{coeq}\left(\coprod_{\mathcal{A}(a,b)} (W(a) \otimes G(b) \rightrightarrows \coprod_{a \in \mathcal{A}} W(a) \otimes G(a) \right)$$

and in fact, it suffices to consider only non-identity arrows or even just atomic arrows.

A.6.14. Definition (conical limits and colimits). The unit for the cartesian product defines a terminal object $\mathbb{1} \in \mathcal{V}$. The constant diagram at the terminal object then defines a terminal object $\mathbb{1} \in \mathcal{V}^{\mathcal{A}}$. A limit weighted by the terminal weight is called a **conical limit** and a colimit weighted by the terminal weight is called a **conical colimit**. It is common to use the simplified notation $\lim F := \lim^{\mathcal{A}}_{\mathbb{1}} F$ and $\mathrm{colim}\, G := \mathrm{colim}^{\mathcal{A}}_{\mathbb{1}} G$.

Conical limits and colimits satisfy the defining universal properties

$$\mathcal{M}(X, \lim F) \cong \mathcal{V}^{\mathcal{A}}(\mathbb{1}, \mathcal{M}(X, F)) \qquad \text{and} \qquad \mathcal{M}(\mathrm{colim}\, G, X) \cong \mathcal{V}^{\mathcal{A}}(\mathbb{1}, \mathcal{M}(G, X)),$$

which say that $\lim F$ and $\mathrm{colim}\, G$ represent the functors of $\mathcal{V}$-enriched conical cones over F or under G, respectively.

We can now properly understand the formulae for weighted limits and colimits given in Definition A.6.4. In particular, these formulae give criteria under which weighted limits or colimits are guaranteed to exist.

A.6.15. Corollary. *If $\mathcal{M}$ is a $\mathcal{V}$-enriched category that admits cotensors and conical limits of all unenriched diagram shapes, then $\mathcal{M}$ admits all weighted limits. Dually, if $\mathcal{M}$ admits tensors and conical colimits of all unenriched diagram shapes, then $\mathcal{M}$ admits all weighted colimits.*

A.6.16. Remark. If $\mathcal{M}$ is a $\mathcal{V}$-category whose underlying unenriched category admits all small limits, then if $\mathcal{M}$ admits cotensors *and tensors* over $\mathcal{V}$, then $\mathcal{M}$ admits all weighted limits. Via the Yoneda lemma, the presence of tensors suffices to internalize the isomorphism of sets expressing the unenriched universal property of limits to an isomorphism in $\mathcal{V}$ that expresses the universal property of conical limits. See Exercise A.6.i.

A.6.17. Definition. A $\mathcal{V}$-category $\mathcal{M}$ is **complete** if it admits all small $\mathcal{V}$-weighted limits, or equivalently, by Corollary A.6.15, if it admits cotensors and all small conical limits. Dually, a $\mathcal{V}$-category $\mathcal{M}$ is **cocomplete** if it admits all small $\mathcal{V}$-weighted colimits, or equivalently, by Corollary A.6.15, if it admits tensors and all small conical colimits.

A.6.18. Example (commas). The comma ∞-category is the limit in the ∞-cosmos $\mathcal{K}$ of the diagram $\lrcorner \to \mathcal{K}$ given by the cospan

$$C \xrightarrow{g} A \xleftarrow{f} B$$

weighted by the diagram $\lrcorner \to s\mathcal{S}et$ given by the cospan

$$\mathbb{1} \xhookrightarrow{1} \mathbb{2} \xhookleftarrow{0} \mathbb{1}$$

Under the simplification of Remark A.6.13, the formula for the weighted limit reduces to the equalizer

$$\mathrm{eq}\left(\begin{tikzcd} & A^{\mathbb{2}} \arrow[dr, "{(p_1,p_0)}"] & \\ C \times A^{\mathbb{2}} \times B \arrow[ur, "\pi"] \arrow[dr, "\pi"'] & & A \times A \\ & C \times B \arrow[ur, "g \times f"'] & \end{tikzcd}\right)$$

which computes the pullback of (3.4.2). The universal property (A.6.8) provides a correspondence between functors $X \to \mathsf{Hom}_A(f,g)$ in $\mathcal{K}$ and simplicial natural transformations, the data of which is given by the three dashed vertical maps that fit into two commutative squares:

$$\begin{tikzcd} \mathbb{1} \arrow[r, hook, "1"] \arrow[d, dashed, "c"'] & \mathbb{2} \arrow[d, dashed, "\alpha"] & \mathbb{1} \arrow[l, hook', "0"'] \arrow[d, dashed, "b"] \\ \mathsf{Fun}(X,C) \arrow[r, "g_*"'] & \mathsf{Fun}(X,A) & \mathsf{Fun}(X,B) \arrow[l, "f_*"] \end{tikzcd}$$

A.6.19. Example (Bousfield-Kan homotopy limits). In their classic book on homotopy limits and colimits [**19**], Bousfield and Kan define the **homotopy limit** of a diagram indexed by a 1-category A and valued in a Kan-complex enriched category $\mathcal{M}$ to be the limit weighted by the functor

$$\begin{aligned} A &\longrightarrow s\mathcal{S}et \\ a &\longmapsto A_{/a} \end{aligned}$$

which carries each object $a \in A$ the nerve of the slice category over a.

A.6.20. Example (Kan extensions as weighted co/limits). The usual colimit or limit formula that computes the value of a pointwise left or right Kan extension of an unenriched functor $F: C \to E$ along $K: C \to D$ at an object $d \in D$ can be succinctly expressed by the weighted colimit or weighted limit

$$\mathrm{lan}_K F(d) := \mathrm{colim}_{D(K-,d)} F \qquad \text{and} \qquad \mathrm{ran}_K F(d) := \lim_{D(d,K-)} F.$$

We conclude with a few results from the general theory of weighted limits and colimits. Immediately from their defining universal properties, it can be verified that:

A.6.21. Lemma (weighted limits of restricted diagrams). *Suppose given a $\mathcal{V}$-functor $K: \mathcal{A} \to \mathcal{B}$, a weight $W: \mathcal{A} \to \mathcal{V}$, and diagrams $F: \mathcal{B} \to \mathcal{M}$ and $G: \mathcal{B}^{\mathrm{op}} \to \mathcal{M}$. Then the W-weighted limit or colimit of the restricted diagram is isomorphic to the $\mathrm{lan}_K W$-weighted limit or colimit of the original diagram:*

$$\lim^{\mathcal{A}}_W (F \circ K) \cong \lim^{\mathcal{B}}_{\mathrm{lan}_K W} F \qquad \textit{and} \qquad \mathrm{colim}^{\mathcal{A}}_W (G \circ K) \cong \mathrm{colim}^{\mathcal{B}}_{\mathrm{lan}_K W} G.$$

Proof. Exercise A.6.iii. $\square$

An enriched adjunction is comprised of a pair of $\mathcal{V}$-functors $F\colon \mathcal{B} \to \mathcal{A}$ and $U\colon \mathcal{A} \to \mathcal{B}$ together with a family of isomoprphisms $\mathcal{A}(Fb, a) \cong \mathcal{B}(b, Ua)$ that are $\mathcal{V}$-natural in both variables; see Definition A.3.16. The usual Yoneda-style argument enriches to show:

A.6.22. Proposition (weighted RAPL/LAPC). *A $\mathcal{V}$-enriched right adjoint functor $U\colon \mathcal{A} \to \mathcal{B}$ preserves all weighted limits that exist in $\mathcal{A}$, while it's $\mathcal{V}$-enriched left adjoint $F\colon \mathcal{B} \to \mathcal{A}$ preserves all weighted colimits that exist in $\mathcal{B}$.*

Proof. Exercise A.6.iv. $\square$

By the axioms of Definition 1.2.1, ∞-cosmoi will admit a large class of simplicially-enriched weighted limits built from the simplicial cotensors and conical simplicial limits enumerated in 1.2.1(i). In practice, ∞-cosmoi often arise as subcategories (of "fibrant objects") in a larger category that is also admits simplicially-enriched weighted colimits, which can then be reflected back into the ∞-cosmos to defined *weighted bicolimits*. This is a story for much later, so we will confine our attention to the case of weighted limits for the rest of this chapter.

Exercises.

A.6.i. Exercise. Suppose $\mathcal{M}$ is a tensored and $\mathcal{V}$-enriched category whose underlying unenriched category admits limits of all unenriched diagram shapes. Show that $\mathcal{M}$ admits conical limits of all unenriched diagram shapes, proving the extension of Corollary A.6.15 described in Remark A.6.16.

A.6.ii. Exercise. Taking the base for enrichment $\mathcal{V}$ to be $\mathcal{S}et$, compute the following weighted limits of a simplicial set X, regarded as a diagram in $\mathcal{S}et^{\mathbb{\Delta}^{\mathrm{op}}}$, weighted by:

(i) the standard n-simplex $\Delta[n] \in \mathcal{S}et^{\mathbb{\Delta}^{\mathrm{op}}}$,

(ii) the **spine** of the n-simplex, the simplicial subset $\Gamma[n] \hookrightarrow \Delta[n]$ obtained by gluing together the n edges from i to $i+1$ into a composable path,

(iii) the n-simplex boundary $\partial\Delta[n] \in \mathcal{S}et^{\mathbb{\Delta}^{\mathrm{op}}}$.[9]

A.6.iii. Exercise. Prove Lemma A.6.21.

A.6.iv. Exercise. Prove Proposition A.6.22.

A.7. Change of base

Change of base for enriched categories was first considered in [**35**]. Their main result, appearing as Proposition A.7.4 below, is that a *lax monoidal functor* $T\colon \mathcal{V} \to \mathcal{W}$ induces a change-of-base 2-functor $T_*\colon \mathcal{V}\text{-}\mathcal{C}at \to \mathcal{W}\text{-}\mathcal{C}at$.

A.7.1. Definition. A **(lax) monoidal functor** between cartesian closed categories $\mathcal{V}$ and $\mathcal{W}$ is a functor $T\colon \mathcal{V} \to \mathcal{W}$ equipped with natural transformations

$$\begin{tikzcd} \mathcal{V}\times\mathcal{V} \arrow[r, "T\times T"] \arrow[d, "\times"'] \arrow[dr, phantom, "\Downarrow\phi"] & \mathcal{W}\times\mathcal{W} \arrow[d, "\times"] \\ \mathcal{V} \arrow[r, "T"'] & \mathcal{W} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{1} \arrow[r, "1"] \arrow[dr, "1"'] & \mathcal{V} \arrow[d, "T"] \arrow[dl, phantom, "\Uparrow\phi_0"] \\ & \mathcal{W} \end{tikzcd}$$

so that the evident associativity and unit diagrams commute.

[9] The limit of a simplicial object weighted by $\partial\Delta[n]$ is called the **nth-matching object**; see Appendix C.

A.7.2. Definition. A **(lax) monoidal natural transformation** between monoidal functors between cartesian closed categories $T, U\colon \mathcal{V} \rightrightarrows \mathcal{W}$ is given by a natural transformation $\theta\colon T \Rightarrow U$ so that the pasting diagrams commute

$$
\begin{tikzcd}
\mathcal{V}\times\mathcal{V} \arrow[r, bend left, "T\times T"] \arrow[r, bend right, "U\times U"'] \arrow[r, phantom, "\Downarrow\theta\times\theta"] \arrow[d, "\times"'] \arrow[dr, phantom, "\Downarrow\phi"] & \mathcal{W}\times\mathcal{W} \arrow[d, "\times"] \\
\mathcal{V} \arrow[r, "U"'] & \mathcal{W}
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{V}\times\mathcal{V} \arrow[r, "T\times T"] \arrow[d, "\times"'] \arrow[dr, phantom, "\Downarrow\phi"] & \mathcal{W}\times\mathcal{W} \arrow[d, "\times"] \\
\mathcal{V} \arrow[r, bend left, "T"] \arrow[r, bend right, "U"'] \arrow[r, phantom, "\Downarrow\theta"] & \mathcal{W}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{1} \arrow[r, "1"] \arrow[dr, bend right, "1"'] \arrow[dr, phantom, "\Uparrow\phi_0"] & \mathcal{V} \arrow[d, bend right, "T"'] \arrow[d, bend left, "U"] \arrow[d, phantom, "\overset{\theta}{\Rightarrow}"] \\
& \mathcal{W}
\end{tikzcd}
=
\begin{tikzcd}
\mathbb{1} \arrow[r, "1"] \arrow[dr, "1"'] \arrow[dr, phantom, "\Uparrow\phi_0"] & \mathcal{V} \arrow[d, "U"] \\
& \mathcal{W}
\end{tikzcd}
$$

A.7.3. Example. A functor $T\colon \mathcal{V} \to \mathcal{W}$ between cartesian closed categories **preserves finite products** just when the natural maps defined for any $A, B \in \mathcal{V}$

$$T(A \times B) \xrightarrow{\cong} TA \times TB \qquad \text{and} \qquad T1 \xrightarrow{\cong} 1$$

are isomorphisms. The inverse isomorphisms then make T into a **strong monoidal functor** between the cartesian closed categories $\mathcal{V}$ and $\mathcal{W}$, with the structure maps of Definition A.7.1 given by natural isomorphisms. Moreover, any natural transformation between product-preserving functors is automatically a monoidal natural transformation.

The following result was first stated as [**35**, II.6.3], with the proof left to the reader. We adopt the same tactic and leave the diagram chases to the reader or to [**26**, 4.2.4] and instead just give the definition.

A.7.4. Proposition. *A monoidal functor $T\colon \mathcal{V} \to \mathcal{W}$ induces a change-of-base 2-functor*

$$T_*\colon \mathcal{V}\text{-}\mathcal{C}at \to \mathcal{W}\text{-}\mathcal{C}at.$$

Proof. The construction of the change-of-base 2-functor is the important thing. Let $\mathcal{C}$ be a $\mathcal{V}$-category and define a $\mathcal{W}$-category $T_*\mathcal{C}$ to have the same objects and to have mapping objects $T_*\mathcal{C}(x, y) \coloneqq T\mathcal{C}(x, y)$. The composition and identity maps are given by the composites

$$T\mathcal{C}(y,z) \times T\mathcal{C}(x,y) \xrightarrow{\phi} T(\mathcal{C}(y,z) \times \mathcal{C}(x,y)) \xrightarrow{T\circ} T\mathcal{C}(x,z) \qquad 1 \xrightarrow{\phi_0} T1 \xrightarrow{T\,\mathrm{id}_x} T\mathcal{C}(x,x)$$

which make use of the structure maps of the monoidal functor. A straightforward diagram chases verifies that $T_*\mathcal{C}$ is a $\mathcal{W}$-category.

If $F\colon \mathcal{C} \to \mathcal{D}$ is a $\mathcal{V}$-functor, then we define a $\mathcal{W}$-functor $T_*F\colon T_*\mathcal{C} \to T_*\mathcal{D}$ to act on objects by $c \in \mathcal{C} \mapsto Fc \in \mathcal{D}$ and with internal action on arrows defined by

$$T\mathcal{C}(x, y) \xrightarrow{TF} \mathcal{D}(Tx, Ty)$$

Again, a straightforward diagram chase verifies that T_*F is $\mathcal{V}$-functorial. It is evident from this definition that $T_*(GF) = T_*G \cdot T_*F$.

Finally, let $\alpha\colon F \Rightarrow G$ be a $\mathcal{V}$-natural transformation between $\mathcal{V}$-functors $F, G\colon \mathcal{C} \rightrightarrows \mathcal{D}$ and define a $\mathcal{W}$-natural transformation $T_*\alpha\colon T_*F \Rightarrow T_*G$ to have components

$$1 \xrightarrow{\phi_0} T1 \xrightarrow{T\alpha_c} T\mathcal{D}(Fc, Gc)$$

Another straightforward diagram chase verifies that $T_*\alpha$ is $\mathcal{W}$-natural.

It remains to verify this assignment is functorial for both horizontal and vertical composition of enriched natural transformations. Consulting Lemma A.3.5, we see that the component of $T_*(\beta \cdot \alpha)$ is defined by the top-horizontal composite while the component of the vertical composite of $T_*\alpha$ with $T_*\beta\colon T_*G \Rightarrow T_*H$ is defined by the bottom composite:

$$\begin{tikzcd} 1 \arrow[r, "\phi_0"] \arrow[dr, "\phi_0\times\phi_0"'] & T1 \arrow[r, "T(\beta_c\times\alpha_c)"] & T(\mathcal{D}(Gc,Hc)\times\mathcal{D}(Fc,Gc)) \arrow[r, "T\circ"] & T\mathcal{D}(Fc,Hc) \\ & T1\times T1 \arrow[u, "\phi"'] \arrow[r, "T\beta_c\times T\alpha_c"'] & T\mathcal{D}(Gc,Hc)\times T\mathcal{D}(Fc,Gc) \arrow[u, "\phi"'] & \end{tikzcd}$$

By the unit axiom for the monoidal functor and naturality of ϕ, these composites agree. The argument for functoriality of horizontal composites is similar. $\square$

For any cartesian closed category $\mathcal{V}$, the underlying set functor $(-)_0\colon \mathcal{V} \to \mathcal{S}et$ is monoidal (Exercise A.7.i). Consequently:

A.7.5. Corollary. *For any cartesian closed category $\mathcal{V}$, the underlying category construction defines a 2-functor*

$$(-)_0\colon \mathcal{V}\text{-}\mathcal{C}at \to \mathcal{C}at. \quad \square$$

A.7.6. Remark. In fact, the change of base procedure is *itself* a 2-functor from the 2-category of monoidal categories, monoidal functors, and monoidal natural transformations to the 2-category of 2-categories, 2-functors, and 2-natural transformations. See [**26**, §4.3] for a discussion and proof.

If the lax monoidal functor $T\colon \mathcal{V} \to \mathcal{W}$ does not commute with the underlying set functors for $\mathcal{V}$ and $\mathcal{W}$ up to natural isomorphism, the change-of-base 2-functor $T_*\colon \mathcal{V}\text{-}\mathcal{C}at \to \mathcal{W}\text{-}\mathcal{C}at$ will not preserve underlying categories, even up to natural isomorphism. Consulting Definition A.2.2, the following condition on T is required to ensure that underlying categories are preserved.

A.7.7. Lemma. *The change-of-base 2-functor induced by a monoidal functor $(T,\phi,\phi_0)\colon \mathcal{V} \to \mathcal{W}$ preserves underlying categories, if and only if, for each $V \in \mathcal{V}$ the composite function on hom-sets*

$$\mathcal{V}(1,V)_0 \xrightarrow{T} \mathcal{W}(T1,TV)_0 \xrightarrow{-\circ\phi_0} \mathcal{W}(1,TV)_0$$

is a bijection.

Proof. The displayed function defines the component at $V \in \mathcal{V}$ of the unique monoidal natural transformation from the underlying set-functor for $\mathcal{V}$ with the composite of T with the underlying set functor for $\mathcal{W}$. By Remark A.7.6, if it defines a monoidal natural isomorphism, then it induces a 2-natural isomorphism between the underlying category 2-functor $(-)_0\colon \mathcal{V}\text{-}\mathcal{C}at \to \mathcal{C}at$ and the composite of the change of base 2-functor $T_*\colon \mathcal{V}\text{-}\mathcal{C}at \to \mathcal{W}\text{-}\mathcal{C}at$ with the underlying category 2-functor $(-)_0\colon \mathcal{W}\text{-}\mathcal{C}at \to \mathcal{C}at$.

Conversely, this condition is necessary for the underlying category of the $\mathcal{W}$-category $T_*\mathcal{V}$ to coincide with the underlying category of the cartesian closed category $\mathcal{V}$. $\square$

One situation in which the condition of Lemma A.7.7 is automatic is when the lax monoidal functor is the right adjoint of a monoidal adjunction, which in this context might be thought of as a *change-of-base adjunction*. The proof, originally given in [**52**], is by a short diagram chase left to Exercise A.7.ii.

Another immediate consequence of Remark A.7.6 is that the monoidal adjunctions of Exercise A.7.ii induce an adjunction between the corresponding change-of-base 2-functors. Between cartesian closed categories $\mathcal{V}$ and $\mathcal{W}$, the statement of this more general result simplifies by applying the ideas of Example A.7.3. The right adjoint functor preserves finite products and so is automatically strong monoidal, and by Exercise A.7.ii(i) the left adjoint is as well. Any natural transformation between product-preserving functors automatically defines a monoidal natural transformation. Consequently:

A.7.8. Proposition. *Any adjunction comprised of finite-product preserving functors between cartesian closed categories induces a change-of-base 2-adjunction*

$$\begin{tikzcd} \mathcal{V} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{W} \arrow[l, bend left, "U"] \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \mathcal{V}\text{-}\mathcal{C}at \arrow[r, bend left, "F_*"] \arrow[r, phantom, "\perp"] & \mathcal{W}\text{-}\mathcal{C}at \arrow[l, bend left, "U_*"] \end{tikzcd} \qquad \square$$

A.7.9. Lemma. *Any adjunction comprised of finite-product preserving functors between cartesian closed categories*

$$\begin{tikzcd} \mathcal{V} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{W} \arrow[l, bend left, "U"] \end{tikzcd}$$

defines a $\mathcal{V}$-enriched adjunction between F and $U_*\mathcal{W}$; i.e., there exists a $\mathcal{V}$-natural isomorphism

$$U\mathcal{W}(Fv, w) \cong \mathcal{V}(v, Uw).$$

Proof. It's easy to verify the $\mathcal{V}$-functoriality of $U\colon U_*\mathcal{W} \to \mathcal{V}$, which implies that for all $v \in \mathcal{V}$, the map

$$U\mathcal{W}(Fv, w) \xrightarrow{U_{Fv,w}} \mathcal{V}(UFv, Uw) \xrightarrow{-\circ\eta_v} \mathcal{V}(v, Uw)$$

is $\mathcal{V}$-natural in $w \in U_*\mathcal{W}$. By Remark A.3.17, to construct a compatible $\mathcal{V}$-enrichment of F, we need only demonstrate that this map in an isomorphism in $\mathcal{V}$.

We do this by constructing an explicit inverse, namely

$$\mathcal{V}(v, Uw) \xrightarrow{\eta} UF\mathcal{V}(v, Uw) \xrightarrow{U(F_{v,Uw})} U\mathcal{W}(Fv, FUw) \xrightarrow{\epsilon_w\circ -} U\mathcal{W}(Fv, w)$$

where the middle map is defined by applying the unenriched functor U to the canonical action map $F_{v,Uw}\colon F\mathcal{V}(v, Uw) \to \mathcal{W}(Fv, FUw)$ from the $\mathcal{W}$-functor $F\colon F_*\mathcal{V} \to \mathcal{W}$.

The proof that this maps are inverses involves a pair of diagram chases, the first of which demonstrates that the top-right composite reduces to the left-bottom composite, which is the identity:

$$\begin{tikzcd} \mathcal{V}(v, Uw) \arrow[r, "\eta"] \arrow[ddrr, "\eta_{Uw}\circ -"'] & UF\mathcal{V}(v, Uw) \arrow[r, "U(F_{v,Uw})"] \arrow[dr, "UF_{v,Uw}"] & U\mathcal{W}(Fv, FUw) \arrow[r, "\epsilon_w\circ -"] \arrow[d, "U_{Fv,FUw}"] & U\mathcal{W}(Fv, w) \arrow[d, "U_{Fv,w}"] \\ & & \mathcal{V}(UFv, UFUw) \arrow[r, "U\epsilon_w\circ -"] \arrow[d, "-\circ\eta_v"] & \mathcal{V}(UFv, Uw) \arrow[d, "-\circ\eta_v"] \\ & & \mathcal{V}(v, UFUw) \arrow[r, "U\epsilon_w\circ -"] & \mathcal{V}(v, Uw) \end{tikzcd}$$

The only subtle point is the commutativity of the trapezoidal region, which expresses the fact that $\eta\colon \mathrm{id}_{\mathcal{V}} \Rightarrow UF$ is a *closed natural transformation* between product-preserving functors between cartesian closed categories. This region commutes because the transposed diagram does:

$$\begin{tikzcd} v \times \mathcal{V}(v,Uw) \arrow[r, equal] \arrow[d, "\eta_v \times \eta"'] & v \times \mathcal{V}(v,Uw) \arrow[r, "\mathrm{ev}"] \arrow[d, "\eta"'] & Uw \arrow[d, "\eta_{Uw}"] \\ UFv \times UF\mathcal{V}(v,Uw) \arrow[r, "\cong"] & UF(v \times \mathcal{V}(v,UW)) \arrow[r, "UF\,\mathrm{ev}"] & UFUw \end{tikzcd}$$

the right-hand square by naturality, and the left-hand square because any naturally transformation between product-preserving functors is automatically a (cartesian) *monoidal* natural transformation. The other diagram chase is similar. $\square$

A.7.10. Proposition. *Given a change of base adjunction comprised of finite-product preserving functors between cartesian closed categories*

$$\begin{tikzcd} \mathcal{V} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{W} \arrow[l, bend left, "U"] \end{tikzcd}$$

then if $\mathcal{M}$ is tensored or cotensored as a $\mathcal{W}$-category, then $U_*\mathcal{M}$ is tensored or cotensored as $\mathcal{V}$-category with the tensors or cotensor of $M \in \mathcal{M}$ by $V \in \mathcal{V}$ defined by

$$V \otimes M := FV \otimes M \qquad \textit{and} \qquad M^V := M^{FV}.$$

Proof. The statements are dual. Suppose $\mathcal{M}$ admits cotensors as a $\mathcal{W}$-category. To verify that $U_*\mathcal{M}$ admits cotensors as a $\mathcal{V}$-category we must supply an isomorphism

$$U\mathcal{M}(X, M^{FV}) \cong (U\mathcal{M}(X,M))^V$$

in $\mathcal{V}$ that is $\mathcal{V}$-natural in X. By the enriched Yoneda lemma, we can extract this isomorphism from an

$$\mathcal{V}(Z, U\mathcal{M}(X, M^{FV})) \cong \mathcal{V}(Z, (U\mathcal{M}(X,M))^V)$$

natural in $Z \in \mathcal{V}$. To that end, we have

$$\begin{aligned} \mathcal{V}(Z, U\mathcal{M}(X, M^{FV})) &\cong U\mathcal{W}(FZ, \mathcal{M}(X, M^{FV})) \cong U\mathcal{W}(FZ, \mathcal{M}(X,M)^{FV}) \\ &\cong U\mathcal{W}(FZ \times FV, \mathcal{M}(X,M)) \cong U\mathcal{W}(F(Z \times V), \mathcal{M}(X,M)) \\ &\cong \mathcal{V}(Z \times V, U\mathcal{M}(X,M)) \cong \mathcal{V}(Z, (U\mathcal{M}(X,M))^V), \end{aligned}$$

by composing the $\mathcal{V}$-natural isomorphisms of Lemma A.7.9 arising from the adjunction $F \dashv U$, the enriched natural isomorphisms arising from the cartesian closed structure on $\mathcal{V}$ and on $U_*\mathcal{W}$, and the fact that F preserves binary products. $\square$

A.7.11. Example. Both adjoints of the adjunction

$$\begin{tikzcd} \mathcal{C}at \arrow[r, bend right] \arrow[r, phantom, "\perp"] & s\mathcal{S}et \arrow[l, bend right, "\mathsf{h}"'] \end{tikzcd}$$

of Proposition 1.1.11 preserve finite products. Hence, by Proposition A.7.8 induces a change-of-base adjunction defined by the 2-functors

$$\begin{tikzcd} 2\text{-}\mathcal{C}at \arrow[r, phantom, "\perp"] \arrow[r, bend right, hook] & s\mathcal{S}et\text{-}\mathcal{C}at \arrow[l, bend right, "\mathsf{h}_*"'] \end{tikzcd}$$

that act identically on objects and act by applying the homotopy category functor or nerve functor, respectively, on homs. Note h also satisfies the condition of Lemma A.7.7 so both adjoints preserve underlying categories, as is evident from direct computation.

Exercises.

A.7.i. Exercise. For any cartesian closed category $\mathcal{V}$, prove that the underlying set functor $(-)_0 \colon \mathcal{V} \to \mathcal{S}et$ is lax monoidal

A.7.ii. Exercise. Consider an adjunction in the 2-category of monoidal categories, monoidal functors, and monoidal natural transformations.

(i) Prove that the left adjoint is strong monoidal.[10]

(ii) Prove that the right adjoint is "pro-normal," satisfying the condition of Lemma A.7.7.

[10]Hint: take the mates of the structure maps of the monoidal right adjoint.

APPENDIX B

An introduction to 2-category theory

2-categories — categories *enriched* in $\mathcal{C}at$ — and double categories — categories defined *internally* to $\mathcal{C}at$ — were first introduced by Charles Ehresmann. A notable early expository account appeared in **[55]**, while a comprehensive modern treatment is given by **[44]**. The basic definitions are given in §B.1, which pays particular attention to the composition of 2-cells in a 2-category by means of *pasting*.

In §B.2, we briefly answer the question: what do 2-categories form? We define three dimensions of morphisms between 2-categories — the 2-functors, 2-natural transformations, and modifications — and observe that these collectively assemble into a *3-category*, this being a category enriched over 2-categories.

In §B.3 and §B.4 we develop the calculus of adjunctions and mates in any 2-category, complementing the results of §2.1, and the special case of right adjoint right inverse adjunctions. Miscellaneous 2-categorical lemmas needed elsewhere appear in §B.5. Finally, in §B.6 we consider the representability of various 2-categorical structures and comment briefly on the bicategorical Yoneda lemma.

B.1. 2-categories and the calculus of pasting diagrams

The category $\mathcal{C}at$ of small categories is cartesian closed with exponentials B^A defined to be the category of functors and natural transformations from A to B and terminal object $\mathbb{1}$. Exploiting the work in Appendix A, we can concisely define a 2-category to be a category enriched over this cartesian closed category. Unpacking this definition, we see it contains a considerable amount of structure:

B.1.1. Definition (2-category). A **2-category** $\mathcal{C}$ is a category enriched in $\mathcal{C}at$. Explicitly it has:

- a collection of objects A, B;
- a collection of arrows $f\colon A \to B$, frequently called **1-cells**, these being the objects of the hom-categories $\mathcal{C}(A, B)$; and
- a collection of arrows between arrows $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd}$, called **2-cells**,[1] these being the morphisms of the hom-categories $\mathcal{C}(A, B)$ from f to g

so that:

(i) For each fixed pair of objects $A, B \in \mathcal{C}$, the 1-cells and 2-cells form a category. In particular:

[1]Implicitly in this graphical representation, is the requirement that a 2-cell α has a domain 1-cell f and a codomain 1-cell g, and these 1-cells have a common domain object A and codomain object B. The objects A and B may be referred to as the 0-cell source and 0-cell target of α.

- A pair of 2-cells as below-left admits a **vertical composite** as below-right:

$$\begin{tikzcd} A \arrow[r, "g" description] \arrow[r, bend left=50, "f"] \arrow[r, bend right=50, "h"'] \arrow[r, phantom, bend left=25, "\Downarrow\alpha"] \arrow[r, phantom, bend right=25, "\Downarrow\beta"] & B \end{tikzcd} \quad =: \quad \begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\Downarrow\beta\cdot\alpha"] & B \end{tikzcd}$$

- Each 1-cell $f\colon A \to B$ has an identity 2-cell $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Downarrow\mathrm{id}_f"] & B \end{tikzcd}$.

(ii) The objects and 1-cells define a category in the ordinary sense; in particular, each object has an identity arrow $\mathrm{id}_A\colon A \to A$.

(iii) The objects and 2-cells form a category. In particular:

- A pair of 2-cells as below-left admits a **horizontal composite** as below-right:

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \arrow[r, bend left, "j"] \arrow[r, bend right, "k"'] \arrow[r, phantom, "\Downarrow\gamma"] & C \end{tikzcd} \quad =: \quad \begin{tikzcd} A \arrow[r, bend left, "jf"] \arrow[r, bend right, "kg"'] \arrow[r, phantom, "\Downarrow\gamma*\alpha"] & C \end{tikzcd}$$

- The identity 2-cells on identity 1-cells

$$\begin{tikzcd} A \arrow[r, bend left, "\mathrm{id}_A"] \arrow[r, bend right, "\mathrm{id}_A"'] \arrow[r, phantom, "\Downarrow\mathrm{id}_{\mathrm{id}_A}"] & A \end{tikzcd}$$

define identities for horizontal composition.

(iv) Finally, the horizontal composition is functorial with respect to the vertical composition:

- The horizontal composite of the identity 2-cells is an identity 2-cell:

$$\begin{tikzcd} A \arrow[r, bend left, "g"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\mathrm{id}_g"] & B \arrow[r, bend left, "k"] \arrow[r, bend right, "k"'] \arrow[r, phantom, "\Downarrow\mathrm{id}_k"] & C \end{tikzcd} \quad = \quad \begin{tikzcd} A \arrow[r, bend left, "kg"] \arrow[r, bend right, "kg"'] \arrow[r, phantom, "\Downarrow\mathrm{id}_{kg}"] & C \end{tikzcd}$$

- In the situation below, the horizontal composite of the vertical composites coincides with the vertical composite of the horizontal composites, a property referred to as **middle-four interchange**

$$\begin{tikzcd} A \arrow[r, "g" description] \arrow[r, bend left=50, "f"] \arrow[r, bend right=50, "h"'] \arrow[r, phantom, bend left=25, "\Downarrow\alpha"] \arrow[r, phantom, bend right=25, "\Downarrow\beta"] & B \arrow[r, "k" description] \arrow[r, bend left=50, "j"] \arrow[r, bend right=50, "\ell"'] \arrow[r, phantom, bend left=25, "\Downarrow\gamma"] \arrow[r, phantom, bend right=25, "\Downarrow\delta"] & C \end{tikzcd}$$

A degenerate special case of horizontal composition, in which all but one of the 2-cells is an identity id_f on its boundary 1-cell f, is called "whiskering."

B.1.2. Definition (whiskering). The **whiskered composite** of a 2-cell $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd}$ with a pair of 1-cells $a \colon X \to A$ and $b \colon B \to Y$ is defined by the horizontal composite:

$$\begin{tikzcd} X \arrow[r, bend left, "bfa"] \arrow[r, bend right, "bga"'] \arrow[r, phantom, "\Downarrow b\cdot\alpha\cdot a"] & Y \end{tikzcd} \quad := \quad \begin{tikzcd} X \arrow[r, "a"] & A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \arrow[r, "b"] & Y \end{tikzcd} \quad := \quad \begin{tikzcd} X \arrow[r, bend left, "a"] \arrow[r, bend right, "a"'] \arrow[r, phantom, "\Downarrow\mathrm{id}_a"] & A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \arrow[r, bend left, "b"] \arrow[r, bend right, "b"'] \arrow[r, phantom, "\Downarrow\mathrm{id}_b"] & Y \end{tikzcd}$$

As the following lemma reveals, horizontal composition can be recovered from vertical composition and whiskering. Our primary interest in this result, however, has to with a rather prosaic consequence appearing as the final part of the statement, which will be applied shockingly often.

B.1.3. Lemma (naturality of whiskering). *Any horizontally-composable pair of 2-cells in a 2-category gives rise to a commutative square in the hom-category from the domain object to the codomain object whose diagonal defines the horizontal composite:*

$$\mathcal{C} \ni \quad \begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \arrow[r, bend left, "h"] \arrow[r, bend right, "k"'] \arrow[r, phantom, "\Downarrow\beta"] & C \end{tikzcd} \qquad \rightsquigarrow \qquad \begin{tikzcd} hf \arrow[r, Rightarrow, "\beta f"] \arrow[d, Rightarrow, "h\alpha"'] \arrow[dr, Rightarrow, "\beta*\alpha" description] & kf \arrow[d, Rightarrow, "k\alpha"] \\ hg \arrow[r, Rightarrow, "\beta g"'] & kg \end{tikzcd} \quad \in \mathcal{C}(A,C)$$

In particular, if any three of the four whiskered 2-cells $h\alpha$, $k\alpha$, βf, and βg are invertible, so is the fourth.

Proof. By middle-four interchange:

$$\beta g \cdot h\alpha = (\beta * \mathrm{id}_g)\cdot(\mathrm{id}_h * \alpha) = (\beta\cdot\mathrm{id}_h)*(\mathrm{id}_g\cdot\alpha) = \beta*\alpha = (\mathrm{id}_k\cdot\beta)*(\alpha\cdot\mathrm{id}_f) = (\mathrm{id}_k*\alpha)\cdot(\beta*\mathrm{id}_f) = k\alpha\cdot\beta f.$$

□

The operations of horizontal and vertical composition are special cases of composition by *pasting*, an operation first introduced by Bénabou [6]. The main result, proven in the 2-categorical context by Power [67] is that a well-formed pasting diagram such as

$$\begin{tikzcd} & B \arrow[rrr, "b"] \arrow[dr, "d"] \arrow[d, phantom, "\Downarrow\gamma"] \arrow[drr, phantom, "\Downarrow\alpha"] & & & F \arrow[dr, "g"] \arrow[drr, bend left, "c"] \arrow[dd, phantom, "\Downarrow\epsilon"] \arrow[drr, phantom, "\Downarrow\beta"] & & \\ A \arrow[ur, "a"] \arrow[rr, "i"] \arrow[drr, "j"'] & & C \arrow[r, "e"] \arrow[d, phantom, "\Downarrow\delta"] & E \arrow[ur, "f"] & & H \arrow[r, "h"] \arrow[d, phantom, "\Downarrow\zeta"] & Z \\ & & D \arrow[ur, "k"] \arrow[rr, "\ell"'] & & G \arrow[ur, "m"] \arrow[urr, bend right, "n"'] & & \end{tikzcd} \tag{B.1.4}$$

has a unique 2-cell composite.[2] We leave the formal statement and proof of this result to the literature and instead describe informally how such pasting composites should be interpreted.

B.1.5. Digression (how to read a pasting diagram). A pasting diagram in a 2-category $\mathcal{C}$ represents a unique composite 2-cell, defining a morphism in one of the hom-categories between a pair of objects. To identify these objects, look at the underlying directed graph of objects and 1-cells in the pasting

[2]This result was generalized to the bicategorical context by Verity [99], in which case the composite 2-cell is well-defined once its source and target 1-cells are specified.

diagram. If the pasting diagram is well-formed, that graph should have a unique source object A and a unique target object Z. This indicates that the pasting diagram defines a 2-cell in the hom-category $\mathcal{C}(A, Z)$. The object A is its **source 0-cell** and the object Z is its **target 0-cell**.

The next step is to identify the source 1-cell and the target 1-cell of the pasting diagram. These should both be objects of $\mathcal{C}(A, Z)$, i.e., 1-cells in the 2-category from A to Z. Again if the pasting diagram is well-formed, the **source 1-cell** should be the unique composable path of 1-cells none of which occur as codomains of any 2-cells in the pasting diagram. In the diagram (B.1.4), these are the 1-cells whose labels appear above the arrow, and their composite is cba. Dually, the **target 1-cell** should be the unique composable path of 1-cells, none of which occur as domains of any 2-cells in the pasting diagram. In (B.1.4), these are the 1-cells whose labels appear below the diagram, and their composite is $n\ell j$.

The final step is to represent the pasting diagram as a vertical composite of 2-cells, each of which is a map between a pair of composite 1-cells from A to Z that trace a composable path through the directed graph of the pasting diagram. Each 2-cell in the pasting diagram will label precisely one of the 2-cells of this composite. The expressions of these vertical 2-cell composites are not necessarily unique and may not necessarily pass through every possible composable path of 1-cells (though there will be some vertical composite of 2-cells that does pass through each path of 1-cells).

To start, pick any 2-cell in the pasting diagram whose 1-cell source can be found as a subsequence of the source 1-cell; in the (B.1.4), either α or β can be chosen first. Whisker it so that it defines a 2-cell from the source 1-cell cba to another path of composable 1-cells from A to Z through the pasting diagram. Then this whiskered composite forms the first step in the sequence of composable 2-cells. Remove this part of the pasting diagram and repeat until you arrive at the target 1-cell. In the example above, there are three possible ways to express the composite pasted cell (B.1.4) as vertical composites of whiskered 2-cell, represented by the three paths through the following commutative diagram in the category $\mathcal{C}(A, Z)$:

$$\begin{tikzcd}
cba \arrow[r, Rightarrow, "c\alpha a"] \arrow[d, Rightarrow, "\beta ba"'] & cfeda \arrow[r, Rightarrow, "cfe\gamma"] \arrow[d, Rightarrow, "\beta feda"'] & cfei \arrow[d, Rightarrow, "\beta fei"] & & & \\
hgba \arrow[r, Rightarrow, "hg\alpha a"'] & hgfeda \arrow[r, Rightarrow, "hgfe\gamma"'] & hgfei \arrow[r, Rightarrow, "hgf\delta"] & hgfkj \arrow[r, Rightarrow, "h\epsilon j"] & hm\ell j \arrow[r, Rightarrow, "\zeta\ell j"] & h\ell j
\end{tikzcd} \in \mathcal{C}(A, Z)$$

B.1.6. Digression (notions of sameness inside a 2-category). From the point of view of 2-category theory, the most natural notion of "sameness" for two objects of a 2-category is *equivalence*: A and B are to be regarded as the same, if there exist 1-cells $f : A \to B$ and $g : B \to A$ together with invertible 2-cells $\alpha : \mathrm{id}_A \cong gf$ and $\beta : fg \cong \mathrm{id}_B$.

From the point of view of 2-category theory, the most natural notion of "sameness" for a parallel pair of morphisms in a 2-category is *isomorphism*: $h, k : A \rightrightarrows B$ are to be regarded as the same if there exists an invertible 2-cell $\gamma : h \cong k$.

From the point of view of 2-category theory, the most natural notion of "sameness" for a pair of 2-cells with common boundary is *equality*. Because a 2-category lacks any higher dimensional morphisms to mediate between 2-cells, there is no weaker notion of sameness available.

A 2-category has four duals, including itself, each of which have the same objects, 1-cells, and 2-cells, but with some domains and codomains swapped.

B.1.7. Definition (op and co duals). Let $\mathcal{C}$ be a 2-category.

- Its **op-dual** $\mathcal{C}^{\mathrm{op}}$ is the 2-category with $\mathcal{C}^{\mathrm{op}}(A,B) \coloneqq \mathcal{C}(B,A)$. This reverses the direction of the 1-cells but not the 2-cells.
- Its **co-dual** $\mathcal{C}^{\mathrm{co}}$ is the 2-category with $\mathcal{C}^{\mathrm{co}}(A,B) \coloneqq \mathcal{C}(A,B)^{\mathrm{op}}$. This reverses the direction of the 2-cells but not the 1-cells.
- Its **coop-dual** $\mathcal{C}^{\mathrm{coop}}$ is the 2-category with $\mathcal{C}^{\mathrm{coop}}(A,B) \coloneqq \mathcal{C}(B,A)^{\mathrm{op}}$. This reverses the direction of both the 2-cells and the 1-cells.

Further details about the specification of $\mathcal{C}^{\mathrm{op}}$ $\mathcal{C}^{\mathrm{co}}$, and $\mathcal{C}^{\mathrm{coop}}$ as $\mathcal{C}at$-enriched categories are left to Exercise B.1.iii.

The data of an equivalence in $\mathcal{C}$ also defines an equivalence in each of its duals.

In Digression 1.2.4, we saw that the data of a simplicial category could be expressed as a diagram of a particular type valued in $\mathcal{C}at$. A small 2-category can be similarly encoded, in fact in two different ways, as a category defined *internally* to the category of categories.

B.1.8. Definition (internal category). Let $\mathcal{E}$ be any category with pullbacks. An **internal category** in $\mathcal{E}$ is given by the data

$$\begin{tikzcd} & C_1 \times_{C_0} C_1 \arrow[dl, "\pi_\ell"'] \arrow[dr, "\pi_r"] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & \\ C_1 \arrow[dr, "d"'] & & C_1 \arrow[dl, "c"] \\ & C_0 & \end{tikzcd} \qquad \begin{tikzcd} C_1 \times_{C_0} C_1 \arrow[r, "\circ"] & C_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & C_0 \arrow[l, "i" description] \end{tikzcd}$$

subject to commutative diagrams that define the domains and codomains of composites and identities and encode the fact that composition is associative and unital. The details are left to Exercise B.1.i or the literature.

For example, a **double category** is a category internal to $\mathcal{C}at$. A 2-category can be realized as a special case of this construction in the following two ways.

B.1.9. Digression (2-categories as category objects). A 2-category may be defined to be an internal category in $\mathcal{C}at$

$$\begin{tikzcd} C_{1,2} \times_{C_0} C_{1,2} \arrow[r, "\circ"] & C_{1,2} \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & C_0 \arrow[l, "i" description] \end{tikzcd}$$

in which the category C_0 is a set, namely the set of objects of the 2-category. The 1- and 2-cells occur as the objects and arrows of the category $C_{1,2}$. The functors $d, c\colon C_{1,2} \to C_0$ send 1- and 2-cells to their domain and codomain 0-cells. The functor $i\colon C_0 \to C_{1,2}$ sends each object to its identity 1-cell. The action of the functor $\circ\colon C_{1,2} \times_{C_0} C_{1,2} \to C_{1,2}$ on objects defines composition of 1-cells and the action on morphisms defines the horizontal composition on 2-cells. Vertical composition on 2-cells is the composition inside the category $C_{1,2}$. Functoriality of this map encodes middle-four interchange. This definition motivates the Segal category model of $(\infty,1)$-categories described in Appendix E.

Dually, a 2-category may be defined to be an internal category in $\mathcal{C}at$

$$\begin{tikzcd} C_{0,2} \times_{C_{0,1}} C_{0,2} \arrow[r, "\circ"] & C_{0,2} \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & C_{0,1} \arrow[l, "i" description] \end{tikzcd}$$

in which the categories $C_{0,1}$ and $C_{0,2}$ have the same set of objects and all four functors are identity-on-objects. Here the common set of objects defines the objects of the 2-category and the arrows of $C_{0,1}$ and $C_{0,2}$ define the 1- and 2-cells, respectively. The functors $d, c\colon C_{1,2} \rightrightarrows C_{0,2}$ define the domain and codomain 1-cells for a 2-cell, which the functor $\circ\colon C_{0,2} \times_{C_{0,1}} C_{0,2} \to C_{0,2}$ encodes vertical composition of 2-cells. The composition inside the category $C_{0,2}$ defines horizontal composition of 2-cells. Functoriality of this map encodes middle-four interchange.

Exercises.

B.1.i. Exercise. Complete the definition of an internal category sketched in Definition B.1.8.

B.1.ii. Exercise. Define a duality involution on double categories that exchanges the two expressions of a 2-category as an internal category appearing in Digression B.1.9.

B.1.iii. Exercise. For any 2-category $\mathcal{C}$, define $\mathcal{C}at$-enriched categories $\mathcal{C}^{\mathrm{op}}$, $\mathcal{C}^{\mathrm{co}}$, and $\mathcal{C}^{\mathrm{coop}}$ along the lines specified by Definition B.1.7.

B.2. The 3-category of 2-categories

Ordinary 1-categories form the objects of a 2-category of categories, functors, and natural transformations. Similarly, 2-categories form the objects of a 3-category of 2-categories, 2-functors, 2-natural transformations, and modifications. In this section, we briefly introduce all of these notions.

Recall from Definition B.1.1, that a 2-category is a category enriched in $\mathcal{C}at$. Similarly, 2-functors and 2-natural transformations are precisely the $\mathcal{C}at$-enriched functors and $\mathcal{C}at$-enriched natural transformations of Appendix A. By Corollary A.3.6, 2-categories, 2-functors, and 2-natural transformations assemble into a 2-category. The 3-dimensional cells between 2-categories — the modifications — are defined using the 2-cells of a 2-category, like the 2-dimensional cells between 1-categories — the natural transformations — are defined using the 1-cells in a 1-category.

B.2.1. Definition. A **2-functor** $F\colon \mathcal{C} \to \mathcal{D}$ between 2-categories is given by:

- a mapping on objects $\mathcal{C} \ni x \mapsto Fx \in \mathcal{D}$;
- a functorial mapping on 1-cells $\mathcal{C} \ni f\colon x \to y \mapsto Ff\colon Fx \to Fy \in \mathcal{D}$ respecting domains and codomains; and
- a mapping on 2-cells

$$\mathcal{C} \ni \quad \begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & y \end{tikzcd} \qquad \mapsto \qquad \begin{tikzcd} Fx \arrow[r, bend left, "Ff"] \arrow[r, bend right, "Fg"'] \arrow[r, phantom, "\Downarrow F\alpha"] & Fy \end{tikzcd} \quad \in \mathcal{D}$$

  that respects 0- and 1-cell sources and targets that is functorial for both horizontal and vertical composition and horizontal and vertical identities.

B.2.2. Definition. A **2-natural transformation** $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\phi"] & \mathcal{D} \end{tikzcd}$ between a parallel pair of 2-functors F and G is given by a family of 1-cells $(\phi_c\colon Fc \to Gc)_{c\in\mathcal{C}}$ in $\mathcal{D}$ indexed by the objects of $\mathcal{C}$ that are natural with respect to the 1-cells $f\colon x \to y$ in $\mathcal{C}$, in the sense that the square

$$\begin{tikzcd} Fx \arrow[r, "Ff"] \arrow[d, "\phi_x"'] & Fy \arrow[d, "\phi_y"] \\ Gx \arrow[r, "Gf"'] & Gy \end{tikzcd}$$

commutes in $\mathcal{D}$, and also natural with respect to the 2-cells $\begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & y \end{tikzcd}$ in $\mathcal{C}$, in the sense that the top-right and bottom-left whiskered composites

$$\begin{tikzcd} Fx \arrow[r, bend left, "Ff"] \arrow[r, bend right, "Fg"'] \arrow[r, phantom, "\Downarrow F\alpha"] \arrow[d, "\phi_x"'] & Fy \arrow[d, "\phi_y"] \\ Gx \arrow[r, bend left, "Gf"] \arrow[r, bend right, "Gg"'] \arrow[r, phantom, "\Downarrow G\alpha"] & Gy \end{tikzcd}$$

are equal: i.e., $\phi_y \cdot F\alpha = G\alpha \cdot \phi_x$.

B.2.3. Definition. A **modification** $\Xi\colon \phi \Rrightarrow \psi$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=60, "F"] \arrow[r, bend right=60, "G"'] & \mathcal{D} \end{tikzcd} \quad \phi \overset{\Xi}{\Rrightarrow} \psi$$

between parallel 2-natural transformations is given by a family of 2-cells in $\mathcal{D}$

$$\begin{tikzcd} Fc \arrow[r, bend left, "\phi_c"] \arrow[r, bend right, "\psi_c"'] \arrow[r, phantom, "\Downarrow\Xi_c"] & Gc \end{tikzcd}$$

indexed by the objects $c \in \mathcal{C}$ with the property that for any 1-cell $f\colon x \to y$ in $\mathcal{C}$, the whiskered composites $\Xi_y \cdot Ff = Gf \cdot \Xi_x$ are equal in $\mathcal{D}$ and for any 2-cell $\alpha\colon f \Rightarrow g$ in $\mathcal{D}$, the horizontal composites in $\mathcal{D}$

$$\begin{tikzcd} Fx \arrow[r, bend left, "Ff"] \arrow[r, bend right, "Fg"'] \arrow[r, phantom, "\Downarrow F\alpha"] & Fy \arrow[r, bend left, "\phi_y"] \arrow[r, bend right, "\psi_y"'] \arrow[r, phantom, "\Downarrow\Xi_y"] & Gy \end{tikzcd} = \begin{tikzcd} Fx \arrow[r, bend left, "\phi_x"] \arrow[r, bend right, "\psi_x"'] \arrow[r, phantom, "\Downarrow\Xi_x"] & Gx \arrow[r, bend left, "Gf"] \arrow[r, bend right, "Gg"'] \arrow[r, phantom, "\Downarrow G\alpha"] & Gy \end{tikzcd}$$

are equal.

Finally, the category of 2-categories is cartesian closed, with internal hom $\mathcal{B}^{\mathcal{A}}$ defined to be the 2-category of 2-functors, 2-natural transformations, and modifications. So now we can define a **3-category** to be a category enriched in 2-categories.

Exercises.

B.2.i. Exercise. For the reader who has a lot of blank paper, unpack the definition of a 3-category just given.

B.3. Adjunctions and mates

As discussed in Chapter 2, any 2-category has an internally-defined notion of adjunction:

B.3.1. Definition (adjunction). An **adjunction** in a 2-category C is comprised of:

- a pair of objects A and B,
- a pair of 1-cells $u\colon A \to B$ and $f\colon B \to A$,
- and a pair of 2-cells $\eta\colon 1_B \Rightarrow uf$ and $\epsilon\colon fu \Rightarrow 1_A$, called the **unit** and **counit** respectively,

so that the triangle equalities hold:

$$\begin{tikzcd} & B \arrow[rr, equal] \arrow[dr, "f"'] \arrow[d, phantom, "\Downarrow\epsilon"] & \arrow[d, phantom, "\Downarrow\eta"] & B \\ A \arrow[ur, "u"] \arrow[rr, equal] & & A \arrow[ur, "u"'] & \end{tikzcd} = \begin{tikzcd} B \\ A \arrow[u, bend left, "u"] \arrow[u, bend right, "u"'] \arrow[u, phantom, "="] \end{tikzcd} \qquad \begin{tikzcd} B \arrow[rr, equal] \arrow[dr, "f"'] & \arrow[d, phantom, "\Downarrow\eta"] & B \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow\epsilon"] & \\ & A \arrow[ur, "u"] \arrow[rr, equal] & & A \end{tikzcd} = \begin{tikzcd} B \arrow[d, bend right, "f"'] \arrow[d, bend left, "f"] \arrow[d, phantom, "="] \\ A \end{tikzcd}$$

The 1-cell f is called the **left adjoint** and u is called the **right adjoint**, a relationship that is denoted symbolically in text by writing $f \dashv u$ or in a displayed diagram such as

$$\begin{tikzcd} A \arrow[r, bend right, "u"'] \arrow[r, phantom, "\perp"] & B \arrow[l, bend right, "f"'] \end{tikzcd} \tag{B.3.2}$$

In the presence of an adjunction as in (B.3.2), certain 2-cells with codomain A "transpose" into 2-cells with codomain B; op-dually, certain 2-cells with domain A "transpose" into 2-cells with domain B:

$$fb \xRightarrow{\alpha} a \quad \leftrightsquigarrow \quad b \xRightarrow{\beta} ua \qquad\qquad df \xRightarrow{\gamma} c \quad \leftrightsquigarrow \quad d \xRightarrow{\delta} cu \tag{B.3.3}$$

Both of these transposition operations admit a common generalization due to [55] referred to as the "mates correspondence" which describes a duality between 2-cells induced by a pair of adjunctions.

B.3.4. Definition (mates). Given any pair of adjunctions and functors

$$\begin{tikzcd} B \arrow[r, "b"] \arrow[d, bend right, "f"'] \arrow[d, phantom, "\dashv"] & B' \arrow[d, bend right, "f'"'] \arrow[d, phantom, "\dashv"] \\ A \arrow[u, bend right, "u"'] \arrow[r, "a"'] & A' \arrow[u, bend right, "u'"'] \end{tikzcd}$$

there is exists a bijection between 2-cells as below-left and 2-cells as below-right

$$\begin{tikzcd} B \arrow[r, "b"] \arrow[d, "f"'] \arrow[dr, phantom, "\swarrow\alpha"] & B' \arrow[d, "f'"] \\ A \arrow[r, "a"'] & A' \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} B \arrow[r, "b"] \arrow[dr, phantom, "\searrow\beta"] & B' \\ A \arrow[u, "u"] \arrow[r, "a"'] & A' \arrow[u, "u'"'] \end{tikzcd} \tag{B.3.5}$$

implemented by pasting with the units and counits of the adjunctions:

$$\begin{tikzcd} B \arrow[r, "b"] \arrow[d, "f"'] \arrow[dr, phantom, "\swarrow\alpha"] & B' \arrow[d, "f'"] \\ A \arrow[r, "a"'] & A' \end{tikzcd} \quad := \quad \begin{tikzcd} B \arrow[r, equal] \arrow[dr, "f"'] \arrow[r, phantom, shift right=3, "\searrow\eta"] & B \arrow[r, "b"] \arrow[dr, phantom, "\searrow\beta"] & B' \arrow[dr, "f'"] \arrow[dr, phantom, shift right=3, "\searrow\epsilon'"] & \\ & A \arrow[u, dashed, "u"] \arrow[r, "a"'] & A' \arrow[u, dashed, "u'"] \arrow[r, equal] & A' \end{tikzcd}$$

$$\begin{tikzcd} B \arrow[r, "b"] \arrow[dr, phantom, "\searrow\beta"] & B' \\ A \arrow[u, "u"] \arrow[r, "a"'] & A' \arrow[u, "u'"'] \end{tikzcd} \quad := \quad \begin{tikzcd} & B \arrow[r, "b"] \arrow[d, dashed, "f"] \arrow[dr, phantom, "\swarrow\alpha"] & B' \arrow[r, equal] \arrow[d, dashed, "f'"] \arrow[dr, phantom, shift left=3, "\swarrow\eta'"] & B' \\ A \arrow[ur, "u"] \arrow[r, equal] \arrow[ur, phantom, shift right=3, "\swarrow\epsilon"] & A \arrow[r, "a"'] & A' \arrow[ur, "u'"'] & \end{tikzcd}$$

Pairs of corresponding 2-cells (B.3.5) under this bijection are referred to as **mates**.

The mates correspondence is respected by horizontal and vertical composition of squares (B.3.5) in the sense made precise by the following result:

B.3.6. Theorem (double-functoriality of the mates correspondence). *For any 2-category C, there is an identity on objects, vertical morphisms, and horizontal morphisms, double isomorphism $\mathbb{L}\mathsf{adj}(C) \cong \mathbb{R}\mathsf{adj}(C)$ between the double categories whose*

- *objects and horizontal morphisms are the objects and 1-cells of C*
- *vertical morphisms are fully-specified adjunctions (f, u, η, ϵ) pointing in the direction of the left adjoint*
- *cells in $\mathbb{L}\mathsf{adj}$ are 2-cells in C of the form displayed below-left, while cells in $\mathbb{R}\mathsf{adj}$ are 2-cells in C of the form displayed below-right:*

$$\begin{tikzcd} B \arrow[r, "b"] \arrow[d, "f"'] \arrow[dr, phantom, "\swarrow\alpha"] & B' \arrow[d, "f'"] \\ A \arrow[r, "a"'] & A' \end{tikzcd} \qquad\qquad \begin{tikzcd} B \arrow[r, "b"] \arrow[dr, phantom, "\searrow\beta"] & B' \\ A \arrow[u, "u"] \arrow[r, "a"'] & A' \arrow[u, "u'"'] \end{tikzcd}$$

that acts on cells by taking mates.

Note that the composition of vertical morphisms in the double categories $\mathbb{L}\mathsf{adj}$ and $\mathbb{R}\mathsf{adj}$ makes use of Proposition 2.1.9.

Proof. The horizontal and vertical functoriality of the mates correspondence of Definition B.3.4 can be verified by an easy diagram chase, or see [**55**, 2.2]. ☐

B.3.7. Warning (mates of isomorphisms need not be isomorphisms). In general it is possible for one of the 2-cells in a mate pair (B.3.5) to be invertible without the other being so, for instance because the adjoint transpose of an isomorphism need not be invertible. However if both horizontal 1-cells a and b are equivalences, or if both adjunctions $f \dashv u$ are adjoint equivalences $f' \dashv u'$, then $\alpha \colon f'b \Rightarrow af$ is invertible if and only if its mate $\beta \colon bu \Rightarrow u'a$ is invertible.

Exercise B.3.ii suggests a new proof that any pair of left adjoints $f' \dashv u$ and $f \dashv u$ to a common 1-cell are isomorphic by applying the double isomorphism $\mathbb{L}\mathsf{adj} \cong \mathbb{R}\mathsf{adj}$. A more complicated argument along the same lines can be used to prove:

B.3.8. Lemma. *Suppose given a triple of adjoint functors $\ell \dashv i \dashv r$. Then the counit of $\ell \dashv i$ is invertible if and only if the unit of $i \dashv r$ is invertible.*

Proof. Let $i\colon A \to B$ and write $\epsilon\colon \ell i \Rightarrow \mathrm{id}_A$ for the counit of $\ell \dashv i$ and $\eta\colon \mathrm{id}_A \Rightarrow ri$ for the unit of $i \dashv r$. If ϵ admits an inverse isomorphism $\epsilon^{-1}\colon \ell i \Rightarrow \mathrm{id}_A$, then the vertical composite in $\mathbb{L}\mathsf{adj}$

$$\begin{tikzcd} A \arrow[r, equals] \arrow[d, equals] \arrow[dr, phantom, "\swarrow \mathrm{id}_i"] & A \arrow[d, "i"] \\ A \arrow[r, "i"] \arrow[d, equals] \arrow[dr, phantom, "\swarrow \epsilon"] & B \arrow[d, "\ell"] \\ A \arrow[r, equals] & A \end{tikzcd}$$

admits an inverse cell for horizontal composition in $\mathbb{L}\mathsf{adj}$:

$$\begin{tikzcd} A \arrow[r, equals] \arrow[d, "\ell i"'] \arrow[dr, phantom, "\swarrow \epsilon^{-1}"] & A \arrow[d, equals] \\ A \arrow[r, equals] & A \end{tikzcd}$$

Applying the horizontal functoriality of the double isomorphism $\mathbb{L}\mathsf{adj} \cong \mathbb{R}\mathsf{adj}$, the mates of these cells must also compose horizontally in $\mathbb{R}\mathsf{adj}$ to identities.[3] Applying the vertical functoriality of the double isomorphism $\mathbb{L}\mathsf{adj} \cong \mathbb{R}\mathsf{adj}$, the mate of the vertical composite equals the composite

$$\begin{tikzcd} A \arrow[r, equals] \arrow[d, equals] \arrow[dr, phantom, "\searrow \eta"] & A \\ A \arrow[r, "i"'] \arrow[d, equals] \arrow[dr, phantom, "\searrow \mathrm{id}_i"] & B \arrow[u, "r"'] \\ A \arrow[r, equals] & A \arrow[u, "i"'] \end{tikzcd}$$

in $\mathbb{R}\mathsf{adj}$. In summary, we conclude that the counit of $\ell \dashv i$ is an isomorphism if and only if the unit of $i \dashv r$ is an isomorphism, in which case its inverse isomorphism $\eta^{-1}\colon ri \Rightarrow \mathrm{id}_A$ is the mate of $\epsilon^{-1}\colon \mathrm{id}_A \Rightarrow \ell i$ via the composite adjunction $\ell i \dashv ri$:

$$\begin{tikzcd} A \arrow[r, equals] \arrow[d, "\ell i"'] \arrow[dr, phantom, "\swarrow \epsilon^{-1}"] & A \arrow[d, equals] \\ A \arrow[r, equals] & A \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} A \arrow[r, equals] \arrow[dr, phantom, "\searrow \eta^{-1}"] & A \arrow[d, equals] \\ A \arrow[r, equals] \arrow[u, "ri"] & A \end{tikzcd}$$ ∎

Elaborating upon Warning B.3.7 we have:

B.3.9. Proposition (equivalence invariance of adjointness). *Suppose given an essentially commutative square whose horizontal arrows are equivalences:*

$$\begin{tikzcd} B \arrow[r, "b", "\sim"'] \arrow[d, "f"] \arrow[dr, phantom, "\cong\swarrow\alpha"] & B' \arrow[d, "f'"'] \\ A \arrow[r, "\sim", "a"'] \arrow[u, dashed, bend left, "u"] \arrow[u, phantom, "\vdash"] & A' \arrow[u, dashed, bend right, "u'"'] \arrow[u, phantom, "\dashv"] \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} B \arrow[r, "b", "\sim"'] \arrow[dr, phantom, "\cong\searrow u'a\epsilon\cdot u'\alpha u\cdot\eta'"] & B' \\ A \arrow[r, "\sim", "a"'] \arrow[u, dashed, bend left, "u"] & A' \arrow[u, dashed, bend right, "u'"'] \end{tikzcd}$$

[3]Since the horizontal morphisms in the cells in question are all id_A, the concern raised in Warning B.3.7 does not apply.

Then f admits a right adjoint u if and only f' admits a right adjoint u', in which case the **mate** *of the isomorphism α is an isomorphism.*

Proof. Proposition 2.1.11 may be used to to choose inverse adjoint equivalences $b' \dashv b$ and $a \dashv a$. If f is a left adjoint, then by Proposition 2.1.9, $f' \cong afb'$ is isomorphic to a left adjoint and so by Proposition 2.1.10 f' is left adjoint to bua'. If $f' \dashv u'$ is defined to be the composite adjunction as in the previous paragraph, the mate of α works out to the whiskered composite of $a'a \cong \mathrm{id}_A$ with bu. By Proposition 2.1.10, any other choice of right adjoint to f' is isomorphic to this one, so the mate of α is still an isomoprhism. □

Exercises.

B.3.i. Exercise. Explain how the bijections (B.3.3) may be realized as special cases of the mates correspondence.

B.3.ii. Exercise. Consider two adjunctions $f \dashv u$ and $f' \dashv u$ as vertical morphisms in $\mathbb{L}\mathsf{adj} \cong \mathbb{R}\mathsf{adj}$ and apply the double functoriality of the mates correspondence to prove that $f \cong f'$.

B.4. Right adjoint right inverse adjunctions

An important class of adjunctions is those whose counit 2-cells are isomorphisms.

B.4.1. Definition. A 1-cell $f\colon B \to A$ in a 2-category admits a **right adjoint right inverse**, abbreviated **RARI**, if it admits a right adjoint $u\colon A \to B$ so that the counit of the adjunction $f \dashv u$ is an isomorphism.

In the situation of Definition B.4.1, f defines a **left adjoint left inverse**, abbreviated **LALI**, to u. The co-dual defines the class **left adjoint right inverse** or **right adjoint left inverse** adjunctions with invertible unit. We leave the dualizations of the results that follow to the reader.

When the counit of $f \dashv u$ is an isomorphism, the whiskered composites $f\eta$ and ηu of the unit must also be isomorphisms. Indeed, to construct an adjunction of this form it suffices to give 2-cells with these properties, as demonstrated by the following 2-categorical lemma.

B.4.2. Lemma. *Suppose we are given a pair of 1-cells $u\colon A \to B$ and $f\colon B \to A$ and a 2-isomorphism $fu \cong \mathrm{id}_A$ in a 2-category. If there exists a 2-cell $\eta'\colon \mathrm{id}_B \Rightarrow uf$ with the property that $f\eta'$ and $\eta' u$ are 2-isomorphisms, then f is left adjoint to u. Furthermore, in the special case where u is a section of f, then f is left adjoint to u with the counit of the adjunction an identity.*

Proof. Let $\epsilon\colon fu \Rightarrow \mathrm{id}_A$ be the isomorphism, taken to be the identity in the case where u is a section of f. We will define an adjunction $f \dashv u$ with counit ϵ by modifying $\eta'\colon \mathrm{id}_B \Rightarrow uf$. The "triangle identity composite" $\theta \coloneqq u\epsilon \cdot \eta' u\colon u \Rightarrow u$ defines an automorphism of u. Define

$$\eta \coloneqq \mathrm{id}_B \xRightarrow{\eta'} uf \xRightarrow{\theta^{-1}f} uf \coloneqq \mathrm{id}_B \xRightarrow{\eta'} uf \xRightarrow{(u\epsilon f)^{-1}} ufuf \xRightarrow{(\eta' uf)^{-1}} uf.$$

Immediately, $u\epsilon \cdot \eta u = \mathrm{id}_u$, as is verified by the calculation:

$$\begin{tikzcd} u \arrow[r, Rightarrow, "\eta' u"'] \arrow[rr, Rightarrow, bend left, "\eta u"] \arrow[dr, Rightarrow, bend right, "\theta"'] & ufu \arrow[r, Rightarrow, "\theta^{-1}fu"'] \arrow[d, Rightarrow, "u\epsilon"] & ufu \arrow[d, Rightarrow, "u\epsilon"] \\ & u \arrow[r, Rightarrow, "\theta^{-1}"'] & u \end{tikzcd}$$

The other triangle identity composite $\phi \coloneqq \epsilon f \cdot f\eta$ is an isomorphism, as a composite of isomorphisms, and also an idempotent:

$$\begin{tikzcd} f \arrow[r, Rightarrow, "f\eta"] \arrow[d, Rightarrow, "f\eta"'] & fuf \arrow[r, Rightarrow, "\epsilon f"] \arrow[d, Rightarrow, "fuf\eta"] & f \arrow[d, Rightarrow, "f\eta"] \\ fuf \arrow[r, Rightarrow, "f\eta uf"] \arrow[dr, equal] & fufuf \arrow[r, Rightarrow, "\epsilon fuf"] \arrow[d, Rightarrow, "fu\epsilon f"] & fuf \arrow[d, Rightarrow, "\epsilon f"] \\ & fuf \arrow[r, Rightarrow, "\epsilon f"'] & f \end{tikzcd}$$

By cancelation, any idempotent isomorphism is the identity, proving that $\epsilon f \cdot f\eta = \mathrm{id}_f$. □

B.4.3. Remark. If desired, the unit of the RARI constructed in the proof of Lemma B.4.2 can be taken to be η', with the counit ϵ modified to absorb the isomorphism θ^{-1}. The details are left as Exercise B.4.i.

By Proposition 9.4.5 to say that an adjunction

$$\begin{tikzcd} A \arrow[r, bend right, "u"'] \arrow[r, phantom, "\perp"] & B \arrow[l, bend right, "f"'] \end{tikzcd}$$

is a RARI is equivalently to say that the right adjoint $u \colon A \to B$ is fully faithful. The presence of a left adjoint to a fully faithful functor provides a convenient characterization of its essential image.

B.4.4. Definition. A generalized element $b \colon X \to B$ is in the **essential image** of the right adjoint $u \colon A \to B$ of a RARI if and only if the unit component $\eta b \colon b \Rightarrow ufb$ is an isomorphism.

B.4.5. Lemma. *A generalized element $b \colon X \to B$ is in the essential image of the right adjoint $u \colon A \to B$ of a RARI if and only if there exists a generalized element $a \colon X \to A$ and invertible 2-cell $\alpha \colon b \cong ua$.*

Proof. The statement generalizes Definition B.4.4, so it remains to argue that if given an invertible 2-cell $\alpha \colon b \cong ua$, the unit component ηb is also an isomorphism. This follows immediately from the banal final statement of Lemma B.1.3. From the horizontally-composable pair

$$\begin{tikzcd} X \arrow[rr, "b"] \arrow[dr, "a"'] & \arrow[d, phantom, "\cong\Downarrow\alpha"] & B \arrow[rr, equal] \arrow[dr, "f"'] & \arrow[d, phantom, "\Downarrow\eta"] & B \\ & A \arrow[ur, "u"'] & & A \arrow[ur, "u"'] & \end{tikzcd}$$

we get a commutative diagram

$$\begin{tikzcd} b \arrow[r, Rightarrow, "\eta b"] \arrow[d, Rightarrow, "\alpha"', "\cong"] & ufb \arrow[d, Rightarrow, "\cong"', "\alpha"] \\ ua \arrow[r, Rightarrow, "\cong", "\eta ua"'] & ufua \end{tikzcd}$$

and $f \dashv u$ is a RARI, ηu is invertible. Thus, if α is invertible so is ηb. □

Sometimes it is more convenient to make use of a stricter notion of RARI, in which the counit ϵ is required to be the identity $fu = \mathrm{id}_A$. In this case it follows from the triangle equalities that $f\eta = \mathrm{id}_f$, so that the unit is fibered over A.

When the left adjoint has a 2-categorical property that we now introduce, we shall see that a right adjoint right inverse up to isomorphism can always be replaced by a right adjoint right inverse up to identity.

B.4.6. Definition. A 1-cell $f\colon B \to A$ in a 2-category defines an **isofibration** — in which case the arrow is typically denoted by "$\twoheadrightarrow$" — if given any invertible 2-cell as displayed below left abutting to A with a specified lift of one of its boundary 1-cells through f, there exists an invertible 2-cell abutting to B with this boundary 1-cell as displayed below right that whiskers with f to the original 2-cell:

$$\begin{tikzcd} X \arrow[r, "b"] \arrow[dr, bend right, "a"'] & B \arrow[d, two heads, "f"] \arrow[dl, phantom, "\cong\Downarrow\alpha"] \\ & A \end{tikzcd} \quad = \quad \begin{tikzcd} X \arrow[r, bend left, "b"] \arrow[r, bend right, "\bar{b}"'] \arrow[r, phantom, "\cong\Downarrow\beta"] & B \arrow[d, two heads, "f"] \\ & A \end{tikzcd}$$

B.4.7. Lemma. *Let $f\colon B \twoheadrightarrow A$ be any isofibration that admits a right adjoint $u'\colon A \to B$ with counit $\epsilon\colon fu' \cong \mathrm{id}_A$ an isomorphism. Then u' is isomorphic to a functor u that lies strictly over A and defines a strict right adjoint right inverse to f. Thus any such f defines a fibered adjunction*

$$\begin{tikzcd} B \arrow[rr, bend left, "f"] \arrow[dr, two heads, "f"'] & \perp & A \arrow[ll, dashed, bend left, "u"] \arrow[dl, equal] \\ & A & \end{tikzcd}$$

Proof. Define the 1-cell $u\colon A \to B$ by lifting the counit isomorphism through the isofibration $f\colon B \twoheadrightarrow A$

$$\begin{tikzcd} A \arrow[r, "u'"] \arrow[dr, bend right, equal] & B \arrow[d, two heads, "f"] \arrow[dl, phantom, "\cong\Downarrow\epsilon"] \\ & A \end{tikzcd} \quad = \quad \begin{tikzcd} A \arrow[r, bend left, "u'"] \arrow[r, bend right, "u"'] \arrow[r, phantom, "\cong\Downarrow\beta"] & B \arrow[d, two heads, "f"] \\ & A \end{tikzcd}$$

Note by construction that $fu = \mathrm{id}_A$. By the triangle equalities for the adjunction $f \dashv u'$, the unit defines a 2-cell $\eta\colon \mathrm{id}_B \Rightarrow u'f$ with $\eta u'$ and $f\eta$ both invertible. The composite 2-cell

$$\eta' := \mathrm{id}_B \xRightarrow{\eta} u'f \xRightarrow[\cong]{\beta f} uf$$

then has the properties that $\eta' u$ and $f\eta$ are both invertible. Applying Lemma B.4.2, this 2-cell may then be modified to define the unit of an adjunction $f \dashv u$ with counit $fu = \mathrm{id}_A$. □

Exercises.

B.4.i. Exercise. Suppose we are given a pair of 1-cells $u\colon A \to B$ and $f\colon B \to A$ and a 2-isomorphism $fu \cong \mathrm{id}_A$ in a 2-category. Modify the proof of Lemma B.4.2 to show that if there exists a 2-cell $\eta\colon \mathrm{id}_B \Rightarrow uf$ with the property that $f\eta'$ and $\eta' u$ are 2-isomorphisms, then f is left adjoint to u with unit η and counit invertible.

B.5. A bestiary of 2-categorical lemmas

B.5.1. Lemma. *Suppose $f \dashv u$ is an adjunction under B in the sense that*

- *the solid-arrow triangles involving both adjoints commute*

$$\begin{tikzcd}
& B \arrow[dl, "c"'] \arrow[dr, "a"] \\
C \arrow[ur, dashed, bend left, "r"] \arrow[ur, phantom, "\dashv"] \arrow[rr, "u"] \arrow[rr, phantom, "\top"'] & & A \arrow[ll, bend left, "f"]
\end{tikzcd}$$

- *and $\eta a = \mathrm{id}_a$ and $\epsilon c = \mathrm{id}_c$.*

Then if c admits a right adjoint r with unit $\iota\colon \mathrm{id}_B \Rightarrow rc$ and counit $\nu\colon cr \Rightarrow \mathrm{id}_C$, then $u\nu$ exhibits r as an absolute right lifting of u through a.

$$\begin{tikzcd}
& B \arrow[d, "a"] \\
C \arrow[ur, dashed, bend left, "r"] \arrow[ur, phantom, "\Downarrow u\nu"] \arrow[r, "u"'] & A
\end{tikzcd}$$

Proof. The argument is purely diagrammatic. Any 2-cell as below-left factors through $u\nu$ as below-right:

$$\begin{tikzcd}
X \arrow[r, "x"] \arrow[d, "y"'] \arrow[dr, phantom, "\Downarrow\chi"] & B \arrow[d, "a"] \\
C \arrow[r, "u"'] & A
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, "x"] \arrow[d, "y"'] \arrow[dr, phantom, "\Downarrow\chi"] & B \arrow[d, "a"] & \\
C \arrow[r, "u"'] \arrow[dr, equal, bend right] \arrow[dr, phantom, "\Downarrow\epsilon"] & A \arrow[d, "f"] \arrow[dr, equal, bend left] \arrow[dr, phantom, "\Downarrow\eta"] & \\
& C \arrow[r, "u"'] & A
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, "x"] \arrow[d, "y"'] \arrow[dr, phantom, "\Downarrow\chi"] & B \arrow[d, "a"] \arrow[dd, bend left, "c"] & \\
C \arrow[r, "u"'] \arrow[dr, equal, bend right] \arrow[dr, phantom, "\Downarrow\epsilon"] & A \arrow[d, "f"] & \\
& C \arrow[r, "u"'] & A
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, "x"] \arrow[d, "y"'] \arrow[dr, phantom, "\Downarrow\chi"] & B \arrow[d, "a"] \arrow[dr, "c"] \arrow[rr, equal] \arrow[rr, phantom, "\Downarrow\iota"'] & & B \arrow[d, "c"] \arrow[dd, bend left, "a"] \\
C \arrow[r, "u"] \arrow[rr, equal, bend right] \arrow[rr, phantom, "\Downarrow\epsilon"'] & A \arrow[r, "f"] & C \arrow[ur, "r"] \arrow[ur, phantom, "\Downarrow\nu"'] \arrow[r, equal] & C \arrow[d, "u"] \\
& & & A
\end{tikzcd}$$

Conversely if ζ defines a factorization of χ through $u\nu$, then

$$\begin{tikzcd}
X \arrow[r, "x"] \arrow[d, "y"'] \arrow[dr, phantom, "\Downarrow\zeta"] & B \\
C \arrow[ur, "r"'] &
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, "x"] \arrow[d, "y"'] \arrow[dr, phantom, "\Downarrow\zeta"] & B \arrow[r, equal] \arrow[d, "c"] \arrow[dr, phantom, "\Downarrow\iota"] & B \\
C \arrow[ur, "r"] \arrow[ur, phantom, "\Downarrow\nu"'] \arrow[r, equal] & C \arrow[ur, "r"'] &
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, "x"] \arrow[d, "y"'] \arrow[dr, phantom, "\Downarrow\zeta"] & B \arrow[rrr, equal] \arrow[d, "c"] \arrow[dr, "a"] \arrow[drrr, "c"] \arrow[drrr, phantom, "\Downarrow\iota"] & & & B \\
C \arrow[ur, "r"] \arrow[ur, phantom, "\Downarrow\nu"'] \arrow[r, equal] & C \arrow[r, "u"] \arrow[rr, equal, bend right] \arrow[rr, phantom, "\Downarrow\epsilon"'] & A \arrow[rr, "f"] & & C \arrow[u, "r"']
\end{tikzcd}$$

$$=
\begin{tikzcd}
X \arrow[r, "x"] \arrow[d, "y"'] \arrow[dr, phantom, "\Downarrow\zeta"] & B \arrow[r, equal] \arrow[d, "a"] \arrow[dr, "c"] \arrow[dr, phantom, "\Downarrow\iota"] & B \\
C \arrow[ur, "r"] \arrow[ur, phantom, "\Downarrow u\nu"'] \arrow[r, "u"] \arrow[rr, equal, bend right] \arrow[rr, phantom, "\Downarrow\epsilon"'] & A \arrow[r, "f"] & C \arrow[u, "r"']
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, "x"] \arrow[d, "y"'] \arrow[dr, phantom, "\Downarrow\chi"] & B \arrow[r, equal] \arrow[d, "a"] \arrow[dr, "c"] \arrow[dr, phantom, "\Downarrow\iota"] & B \\
C \arrow[r, "u"] \arrow[rr, equal, bend right] \arrow[rr, phantom, "\Downarrow\epsilon"'] & A \arrow[r, "f"] & C \arrow[u, "r"']
\end{tikzcd}$$

proving that the factorization constructed above is unique. $\square$

B.5.2. Example. For example, there is a diagram of adjoint functors

$$
\begin{tikzcd}
\mathbf{\Delta} \arrow[r, hook] \arrow[dr, "!"'] & \mathbf{\Delta}_+ \arrow[r, hook] & \mathbf{\Delta}_\perp \arrow[ll, dashed, bend right, "\top"'] \arrow[dl, "!"] \\
& \mathbb{1} \arrow[u, dashed, "{[-1]}"] &
\end{tikzcd}
$$

involving the categories introduced in Definition 2.3.11. The inclusion $\mathbf{\Delta} \hookrightarrow \mathbf{\Delta}_\perp$ freely adjoins a bottom element and its right adjoint is the forgetful functor, also an inclusion. The adjunction $[-1] \dashv\, !$ witnesses the fact that $[-1] \in \mathbf{\Delta}_\perp$ defines an initial object with the counit ν defining the canonical natural transformation from the initial object to the identity functor.

This diagram satisfies the premises of Lemma B.5.1 in $\mathcal{C}at^{\mathrm{op}}$. Let A be an object of any 2-category $\mathcal{C}$ that is cotensored over $\mathcal{C}at$. Then the 2-functor $A^{(-)} \colon \mathcal{C}at^{\mathrm{op}} \to \mathcal{C}$ carries the given data to a diagram of adjoint functors in $\mathcal{C}$ as below-left and hence the triangle below-right is absolute right lifting:

$$
\begin{tikzcd}
& A \arrow[dl, "\Delta"] \arrow[dr, "\Delta"] & \\
A^{\mathbf{\Delta}_\perp} \arrow[ur, dashed, bend left, "\mathrm{ev}_{-1}"] \arrow[rr, "\mathrm{res}", "\top"'] & & A^{\mathbf{\Delta}} \arrow[ll, bend left, "f"]
\end{tikzcd}
\qquad
\begin{tikzcd}
& A \arrow[d, "\Delta"] \arrow[dl, phantom, "\Downarrow A^{\nu}"] \\
A^{\mathbf{\Delta}_\perp} \arrow[ur, dashed, bend left, "\mathrm{ev}_{-1}"] \arrow[r, "\mathrm{res}"'] & A^{\mathbf{\Delta}}
\end{tikzcd}
$$

This proves Proposition 2.3.13.

B.5.3. Example. There is a similar diagram of adjoint functors

$$
\begin{tikzcd}
\mathbf{\Delta} \arrow[r, hook] \arrow[dr, "!"'] & \mathbf{\Delta}_+ \arrow[r, hook] & \mathbf{\Delta}_\top \arrow[ll, dashed, bend right, "\top"'] \arrow[dl, "!"] \\
& \mathbb{1} \arrow[u, dashed, "{[-1]}"] &
\end{tikzcd}
$$

again involving the categories introduced in Definition 2.3.11. The inclusion $\mathbf{\Delta} \hookrightarrow \mathbf{\Delta}_\top$ freely adjoins a top element and its right adjoint is the forgetful functor, also an inclusion. The adjunction $[-1] \dashv\, !$ witnesses the fact that $[-1] \in \mathbf{\Delta}_\top$ defines an initial object with the counit ν defining the canonical natural transformation from the initial object to the identity functor. This proves another version of Proposition 2.3.13 where the "splittings" occur on the other side of the co/simplicial objects.

Exercises.

B.5.i. Exercise. Search for other examples in $\mathcal{C}at$ satisfying the premises of Lemma B.5.1.

B.6. Representable characterizations of 2-categorical notions

Recall from Definition 1.4.6 that an **equivalence** in a 2-category is given by

- a pair of objects A and B
- a pair of 1-cells $f \colon A \to B$ and $g \colon B \to A$

- a pair of invertible 2-cells

$$\begin{tikzcd} A \arrow[r, bend left, equal, ""{name=U, below}] \arrow[r, bend right, "gf"', ""{name=D, above}] \arrow[from=U, to=D, phantom, "\cong\Downarrow\alpha"] & A \end{tikzcd} \qquad\qquad \begin{tikzcd} B \arrow[r, bend left, "fg", ""{name=U, below}] \arrow[r, bend right, equal, ""{name=D, above}] \arrow[from=U, to=D, phantom, "\cong\Downarrow\beta"] & B \end{tikzcd}$$

In analogy with Theorem 1.4.7, we have the following result which tells us that equivalences in a 2-category represent equivalences of categories.

B.6.1. Proposition (equivalences are representably defined). *A 1-cell $f\colon A \to B$ in a 2-category $\mathcal{C}$ defines an equivalence if and only if for all $X \in \mathcal{C}$ the induced functor*

$$\mathcal{C}(X,A) \xrightarrow{f_*} \mathcal{C}(X,B)$$

defines an equivalence of categories.

Proof. Each $X \in \mathcal{C}$ defines a 2-functor $\mathcal{C}(X,-)\colon \mathcal{C} \to \mathcal{C}at$, so if $f\colon A \xrightarrow{\sim} B$ is an equivalence in $\mathcal{C}$, then $f_*\colon \mathcal{C}(X,A) \xrightarrow{\sim} \mathcal{C}(X,B)$ is an equivalence in $\mathcal{C}at$.

Conversely, by essential surjectivity of the equivalence $f_*\colon \mathcal{C}(B,A) \to \mathcal{C}(B,B)$ there exists some $g\colon B \to A$ and isomorphism $\beta\colon fg \cong \mathrm{id}_B$. By fully-faithfulness of $f_*\colon \mathcal{C}(A,A) \to \mathcal{C}(A,B)$ the isomorphism $\beta^{-1}f\colon f \cong fgf$ lifts to an isomorphism $\alpha\colon \mathrm{id}_A \cong gf$. □

Similarly, an adjoint functor in a 2-category induces pointwise-defined adjunctions between the hom-categories, but in this case, a further "exactness" condition is required to convert a representably-defined adjunction into an adjunction in the 2-category.

B.6.2. Proposition (adjunctions are representably defined). *A 1-cell $u\colon A \to B$ in a 2-category $\mathcal{C}$ admits a left adjoint if and only if:*

(i) For all $X \in \mathcal{C}$, the induced functor admits a left adjoint

$$\begin{tikzcd} \mathcal{C}(X,A) \arrow[r, bend right, "u_*"'] \arrow[r, phantom, "\perp"] & \mathcal{C}(X,B) \arrow[l, bend right, dashed, "f^X"'] \end{tikzcd}$$

(ii) For all $k\colon Y \to X \in \mathcal{C}$, the mate of the identity 2-cell is an isomorphism:

$$\begin{tikzcd} \mathcal{C}(X,A) \arrow[r, "k^*"] \arrow[d, "u_*"'] \arrow[dr, phantom, "\Nearrow\mathrm{id}"] & \mathcal{C}(Y,A) \arrow[d, "u_*"] \\ \mathcal{C}(X,B) \arrow[r, "k^*"'] & \mathcal{C}(Y,B) \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \mathcal{C}(X,A) \arrow[r, "k^*"] \arrow[dr, phantom, "\cong\Nwarrow\epsilon^Y k^* f^X \cdot f^Y k^* \eta^X"] & \mathcal{C}(Y,A) \\ \mathcal{C}(X,B) \arrow[r, "k^*"'] \arrow[u, "f^X"] & \mathcal{C}(Y,B) \arrow[u, "f^Y"'] \end{tikzcd}$$

Proof. In the cartesian closed category of 2-categories, the hom 2-functor $\mathcal{C}(-,-)\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}at$ transposes to define a Yoneda embedding 2-functor $\mathcal{C}(-,-)\colon \mathcal{C} \to \mathcal{C}at^{\mathcal{C}^{\mathrm{op}}}$ whose codomain is the 2-category of 2-functors, 2-natural transformations, and modifications. This 2-functor preserves adjunctions, carrying an adjoint pair $f \dashv u$ in $\mathcal{C}$ to an adjunction between the representable 2-functors

$\mathcal{C}(-,A)$ and $\mathcal{C}(-,B)$

$$\begin{tikzcd} \mathcal{C}(-,A) \arrow[r, bend right, "u_*"'] \arrow[r, phantom, "\perp"] & \mathcal{C}(-,B) \arrow[l, bend right, "f_*"'] \end{tikzcd}$$

whose left and right adjoints are the 2-natural transformations $f_* \dashv u_*$ and whose unit and counit are modifications. Evaluating at $X \in \mathcal{C}$, this defines a family of adjunction as in (i) and **strict adjunction morphisms**, i.e., so that any $k\colon Y \to X$ induces a strictly commutative square with respect to the left and right adjoints inhabited by a mate pair of identity 2-cells.

The real content is in the converse. Assuming (i), define a candidate left inverse by $f \coloneqq f^B(\mathrm{id}_B)$. By construction $uf \coloneqq u_*f_B(\mathrm{id}_B)$ so we may define a candidate unit to be the component of the unit η^B of $f^B \dashv u_*$ at id_B:

$$\eta \coloneqq \mathrm{id}_B \xrightarrow{\eta^B_{\mathrm{id}_B}} uf \quad \in \mathcal{C}(B,B).$$

Note that these definitions do not a priori give any information about the other composite $fu \in \mathcal{C}(A,A)$, but condition (ii) defines a natural isomorphism $\alpha\colon f^A u^* \cong u^* f^B$

$$\begin{tikzcd} \mathcal{C}(B,A) \arrow[r, "u^*"] & \mathcal{C}(A,A) \\ \mathcal{C}(B,B) \arrow[u, "f^B"] \arrow[r, "u^*"'] \arrow[ur, phantom, "\cong\Nwarrow \epsilon^A u^* f^B \cdot f^A u^* \eta^B"] & \mathcal{C}(A,B) \arrow[u, "f^A"'] \end{tikzcd}$$

whose component at id_B defines an isomorphism

$$\alpha_{\mathrm{id}_B} \coloneqq f^A(u) \xrightarrow{f^A(\eta u)} f^A(ufu) = f^A u_*(fu) \xrightarrow{\epsilon^A(fu)} fu \quad \in \mathcal{C}(A,A).$$

Using this, we define the counit to be the composite of the inverse of this isomorphism with the component of the counit ϵ^A of $f^A \dashv u_*$ at id_A:

$$\epsilon \coloneqq fu \xrightarrow[\cong]{\alpha^{-1}_{\mathrm{id}_B}} f^A(u) \xrightarrow{\epsilon^A_{\mathrm{id}_A}} \mathrm{id}_A \quad \in \mathcal{C}(A,A).$$

The commutative diagram

$$\begin{tikzcd} u \arrow[r, "\eta u"] \arrow[d, "\eta^A(u)"'] & ufu \arrow[d, "\eta^A(ufu)"] \arrow[dr, equals, bend left] & \\ uf^A(u) \arrow[r, "uf^A(\eta u)"] \arrow[rr, bend right, "u\alpha_{\mathrm{id}_B}"'] & uf^A(ufu) \arrow[r, "u\epsilon^A(fu)"] & ufu \end{tikzcd}$$

reveals that $u\alpha_{\mathrm{id}_B} \cdot \eta^A u = \eta u$, so

$$u\epsilon \cdot \eta u = (u\epsilon^A_{\mathrm{id}_A} \cdot u\alpha^{-1}_{\mathrm{id}_B}) \cdot (u\alpha_{\mathrm{id}_B} \cdot \eta^A u) = u\epsilon^A_{\mathrm{id}_A} \cdot u\alpha_{\mathrm{id}_B} = \mathrm{id}_u,$$

which verifies one of the two triangle equalities.

It is somewhat delicate to prove that the other triangle equality composite

$$f \xrightarrow{f\eta} fuf \xrightarrow{\epsilon f} f \in C(B,A)$$

is the identity because we don't have any way to understand the arrow $f\eta$. Note, however, that this arrow defines an endomorphism of the object $f^B(\mathrm{id}_B) \in C(B,A)$, so if we verify that its transpose under the adjunction $f^B \dashv u_*$ is the unit component $\eta^B_{\mathrm{id}_B}$, then by uniqueness of adjoint transposition, we must have $\epsilon f \cdot f\eta = \mathrm{id}_f$ as desired. This can be verified by direct calculation: the adjoint transpose is computed by applying the functor u_* and then precomposing with $\eta^B_{\mathrm{id}_B} = \eta$, which yields the left-bottom composite below.

$$\begin{tikzcd}
1 \arrow[r, "\eta"] \arrow[d, "\eta"'] & uf \arrow[d, "\eta uf"] \arrow[dr, equal, bend left] & \\
uf \arrow[r, "uf\eta"'] & ufuf \arrow[r, "u\epsilon f"'] & uf
\end{tikzcd}$$

An easy diagram chase making use of the previously-verified triangle equality completes the proof. ☐

Condition (ii) of Proposition B.6.2 is referred to as a "Beck-Chevalley" or *exactness* condition. Another exactness condition appears in a representable characterization of absolute lifting diagrams.

B.6.3. Definition. A trio of functors (u, v, w) between a pair of absolute left lifting diagrams (ℓ, λ) and (ℓ', λ') as below defines a **left exact transformation** if and only if the 2-cell τ induced by the universal property of the absolute left lifting is invertible:

$$\begin{tikzcd}
& B \arrow[d, "f"] \arrow[dr, "v"] & \\
C \arrow[ur, "\ell"] \arrow[r, "g"'] \arrow[dr, "w"'] \arrow[r, phantom, shift left=3, "\Uparrow\lambda"] & A \arrow[dr, "u"] & B' \arrow[d, "f'"] \\
& C' \arrow[r, "g'"'] & A'
\end{tikzcd}
=
\begin{tikzcd}
& B \arrow[dr, "v"] & \\
C \arrow[ur, "\ell"] \arrow[dr, "w"'] & \exists!\Uparrow\tau & B' \arrow[d, "f'"] \\
& C' \arrow[ur, "\ell'"] \arrow[r, "g'"'] \arrow[r, phantom, shift left=3, "\Uparrow\lambda'"] & A'
\end{tikzcd}
\tag{B.6.4}$$

This left exactness condition holds if and only if, in the diagram on the left of (B.6.4), the whiskered 2-cell $u\lambda$ displays $v\ell$ as the absolute left lifting of $g'w$ through f', which is to say that the left exact transformations are those that preserve absolute left lifting diagrams.

B.6.5. Lemma. *The mate of a commutative square between right adjoints as below*

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "u"] & A' \arrow[d, "u'"'] \\
B \arrow[r, "b"'] \arrow[u, bend left, "f"] \arrow[u, phantom, "\dashv"] & B' \arrow[u, bend right, "f'"'] \arrow[u, phantom, "\vdash"]
\end{tikzcd}$$

is invertible if and only if (b, a, b) defines a left exact transformation between the absolute left lifting diagrams (f, η) and (f', η') of id_B *through u and* $\mathrm{id}_{B'}$ *through u'.*

PROOF. The unique 2-cell τ satisfying the pasting diagram below is the mate of $\mathrm{id}\colon bu \Rightarrow u'a$.

$$\begin{tikzcd} & A \arrow[d, "u"] \arrow[dr, "a"] & \\ B \arrow[ur, "f"] \arrow[r, equal] \arrow[dr, "b"'] \arrow[ur, phantom, shift right=3ex, "\Uparrow\eta"] & B \arrow[dr, "b"] & A' \arrow[d, "u'"] \\ & B' \arrow[r, equal] & B' \end{tikzcd} = \begin{tikzcd} & A \arrow[dr, "a"] & \\ B \arrow[ur, "f"] \arrow[dr, "b"'] \arrow[rr, phantom, "\exists!\Uparrow\tau"] & & A' \arrow[d, "u'"] \\ & B' \arrow[ur, "f'"] \arrow[r, equal] \arrow[r, phantom, shift left=3ex, "\Uparrow\eta'"] & B' \end{tikzcd}$$

$\square$

B.6.6. PROPOSITION. *Consider a 2-cell in a 2-category $\mathcal{C}$*

$$\begin{tikzcd} & B \arrow[d, "f"] \\ C \arrow[ur, "\ell"] \arrow[r, "g"'] \arrow[ur, phantom, shift right=2ex, "\Uparrow\lambda"] & A \end{tikzcd}$$

(i) *If (ℓ, λ) defines an absolute left lifting diagram in $\mathcal{C}$, then*

(a) *For all $X \in \mathcal{C}$,*

$$\begin{tikzcd} & \mathcal{C}(X,B) \arrow[d, "f_*"] \\ \mathcal{C}(X,C) \arrow[ur, "\ell_*"] \arrow[r, "g_*"'] \arrow[ur, phantom, shift right=2ex, "\Uparrow\lambda_*"] & \mathcal{C}(X,A) \end{tikzcd}$$

defines an absolute left lifting diagram in $\mathcal{C}at$.

(b) *For all $k\colon Y \to X \in \mathcal{C}$, the induced transformation*

$$\begin{tikzcd} & \mathcal{C}(X,B) \arrow[d, "f_*"] \arrow[dr, "k^*"] & \\ \mathcal{C}(X,C) \arrow[r, "g_*"'] \arrow[dr, "k^*"'] & \mathcal{C}(X,A) \arrow[dr, "k^*"] & \mathcal{C}(Y,B) \arrow[d, "f_*"] \\ & \mathcal{C}(Y,C) \arrow[r, "g_*"'] & \mathcal{C}(Y,A) \end{tikzcd}$$

is left exact.

(ii) *Conversely if* ((i)a) *holds for each $X \in \mathcal{C}$, then (ℓ, λ) defines an absolute left lifting diagram in $\mathcal{C}$.*

(iii) *Moreover, if $g\colon C \to A$ and $f\colon B \to A$ are so that for all $X \in \mathcal{C}$, the functor $g_*\colon \mathcal{C}(X,C) \to \mathcal{C}(X,A)$ admits an absolute left lifting through $f_*\colon \mathcal{C}(X,B) \to \mathcal{C}(X,A)$ for which condition* ((i)b) *holds, then g admits an absolute left lifting through f in $\mathcal{C}$.*

PROOF. We won't make use of the first statement so we leave the details to the reader with the hint that to verify the universal property of an absolute lifting diagram in $\mathcal{C}at$, it suffices to consider cones over the cospan (g_*, f_*) whose summit is the terminal category $\mathbb{1}$.

For the second assertion, consider a cone

$$\begin{tikzcd} X \arrow[r, "b"] \arrow[d, "c"'] \arrow[dr, phantom, "\Uparrow\chi"] & B \arrow[d, "f"] \\ C \arrow[r, "g"'] & A \end{tikzcd}$$

over the cospan (g, f) in $\mathcal{C}$. This data defines a diagram of categories as below-left, which factors uniquely as below-right:

$$\begin{tikzcd} \mathbb{1} \arrow[r, "b"] \arrow[d, "c"'] \arrow[dr, phantom, "\Uparrow\chi"] & \mathcal{C}(X,B) \arrow[d, "f_*"] \\ \mathcal{C}(X,C) \arrow[r, "g_*"'] & \mathcal{C}(X,A) \end{tikzcd} \quad = \quad \begin{tikzcd} \mathbb{1} \arrow[r, "b"] \arrow[d, "c"'] \arrow[r, phantom, shift right=3, "\exists!\Uparrow\zeta"'] & \mathcal{C}(X,B) \arrow[d, "f_*"] \\ \mathcal{C}(X,C) \arrow[r, "g_*"'] \arrow[ur, "\ell_*"] & \mathcal{C}(X,A) \arrow[ul, phantom, shift right=3, "\Uparrow\lambda_*"'] \end{tikzcd}$$

defining the desired unique factorization

$$\begin{tikzcd} X \arrow[r, "b"] \arrow[d, "c"'] \arrow[dr, phantom, "\Uparrow\chi"] & B \arrow[d, "f"] \\ C \arrow[r, "g"'] & A \end{tikzcd} \quad = \quad \begin{tikzcd} X \arrow[r, "b"] \arrow[d, "c"'] \arrow[r, phantom, shift right=3, "\Uparrow\zeta"'] & B \arrow[d, "f"] \\ C \arrow[r, "g"'] \arrow[ur, "\ell"] & A \arrow[ul, phantom, shift right=3, "\Uparrow\lambda"'] \end{tikzcd}$$

For the final statement, we define the pair (ℓ, λ) by evaluating the functor and natural transformation of the postulated absolute left lifting (ℓ^C, λ^C) in the case $X = C$ at $\mathrm{id}_C \in \mathcal{C}(C, C)$. To verify that $\lambda\colon f\ell \Rightarrow g$ defines an absolute left lifting of g through f, consider a functor $c\colon X \to C$. The hypothesis of left-exactness tells us that the composite transformation

$$\begin{tikzcd} & & \mathcal{C}(C,B) \arrow[d, "f_*"] \arrow[dr, "c^*"] & \\ \mathbb{1} \arrow[r, "\mathrm{id}_C"] & \mathcal{C}(C,C) \arrow[ur, "\ell^C"] \arrow[r, "g_*"'] \arrow[dr, "c^*"'] \arrow[ur, phantom, shift right=4, "\Uparrow\lambda^c"'] & \mathcal{C}(C,A) \arrow[dr, "c^*"] & \mathcal{C}(X,B) \arrow[d, "f_*"] \\ & & \mathcal{C}(X,C) \arrow[r, "g_*"'] & \mathcal{C}(X,A) \end{tikzcd}$$

is absolute left lifting. By the proof of the second statement above, this tells us that $(\ell c, \lambda c)$ is an absolute left lifting of gc through f, which proves that (ℓ, λ) is an absolute left lifting as required. $\square$

B.6.7. Remark. The results of Proposition B.6.2 and Proposition B.6.6 can be viewed as applications of the bicategorical Yoneda lemma, which defines a 2-fully faithful embedding of a bicategory $\mathcal{C}$ into the 2-category $[\mathcal{C}^{\mathrm{op}}, \mathcal{C}at]$ of pseudofunctors, pseudonatural transformations, and modifications. If a 1-cell $u\colon A \to B$ in $\mathcal{C}$ satisfies condition (i) of Proposition B.6.2, then by Theorem B.3.6, the left adjoints $f^X\colon \mathcal{C}(X, B) \to \mathcal{C}(X, A)$ define the components of a lax natural transformation. Condition (ii) demands that this lax natural transformation is a pseudo natural transformation. Now 2-fully faithfulness allows us to lift this to an arrow $f\colon B \to A$ in $\mathcal{C}$, which is left adjoint to u.

In the case of Proposition B.6.1, where $u\colon A \to B$ induces equivalences $\mathcal{C}(X, A) \xrightarrow{\sim} \mathcal{C}(X, B)$, the inverse equivalences automatically assemble into a pseudonatural transformation, which is why no additional hypothesis was required.

Exercises.

B.6.i. Exercise. Confirm the assertion made in the proof of Lemma B.6.5.

B.6.ii. Exercise. Verify the forwards implication in Proposition B.6.6.

APPENDIX C

Abstract homotopy theory

The underlying 1-category of an ∞-cosmos, together with its classes of isofibrations, equivalences, and trivial fibrations, defines a *category of fibrant objects*, a classical context for abstract homotopy theory first studied by Brown [**20**]. In §C.1, we develop some of the general theory of categories of fibrant objects with the aim of presenting some classical proofs that were omitted in the main text.

The remainder of this chapter develops material that will be applied in later appendices. In Appendix E we will discover that examples of ∞-cosmoi can be found "in the wild" as model categories that are enriched as such over Joyal's model structure on the category of simplicial sets. For this reason, model categories, enriched model categories, and the functors between them are introduced in §C.3. In the introduction to Chapter I. Axiomatic Homotopy Theory [**68**] where Quillen first introduces the definition of a model category, he highlights the factorization and lifting axioms as being the most important. These are most clearly encapsulated in the categorical notion of a weak factorization system discussed in §C.2, the axioms for which were enumerated later.

Finally, some of the technical combinatorial proofs of Appendix D involve inductive arguments involving the Reedy category $\mathbf{\Delta}$. Thus, we conclude in §C.4 with a brief introduction to Reedy category theory following the presentation of [**78**].

C.1. Abstract homotopy theory in a category of fibrant objects

In this section we work in an (unenriched) category of fibrant objects, a notion first introduced by Brown [**20**]. Examples include the underlying category of an ∞-cosmos or the full subcategory of fibrant objects in a Quillen model category (hence the name).

C.1.1. Definition (category of fibrant objects). A **category of fibrant objects** consists of a category $\mathcal{M}$ together with two subcategories of morphisms $\mathcal{W}$ and $\mathcal{F}$ satisfying the following axioms:

(i) $\mathcal{M}$ has products and in particular a terminal object 1. Moreover, the classes $\mathcal{F}$ and $\mathcal{F} \cap \mathcal{W}$ are each closed under products.
(ii) $\mathcal{W}$ has the **2-of-3 property**: for any composable pair of morphisms, if any two of f, g, and gf is in $\mathcal{W}$ then so is the third.
(iii) Pullbacks of maps in $\mathcal{F}$ exist and lie in $\mathcal{F}$, and pullback also preserves the class $\mathcal{F} \cap \mathcal{W}$.
(iv) Limits of towers of maps in $\mathcal{F}$ exist and also lie in $\mathcal{F}$, and the class $\mathcal{F} \cap \mathcal{W}$ is also closed under forming limits of towers.
(v) For every object B, there exists a **path object** PB together with a factorization of the diagonal into a map in $\mathcal{W}$ followed by a map in $\mathcal{F}$:

$$\begin{tikzcd} & PB \arrow[dr, two heads] & \\ B \arrow[ur, "\sim"] \arrow[rr, "\Delta"'] & & B \times B \end{tikzcd}$$

(vi) All objects are **fibrant**: for every $B \in \mathcal{M}$, the map $B \to 1$ lies in $\mathcal{F}$.

C.1.2. Remark. The original definition only asked for finite products in axiom (i) and omitted axiom (iv). The fact that the classes $\mathcal{F}$ or $\mathcal{F} \cap \mathcal{W}$ are closed under finite products can be proven as a consequence of axiom (iii); see Corollary C.1.14. Here, we ask for these infinite limits to parallel the limit axiom 1.2.1(i) in our definition of an ∞-cosmos. In practice, the classes $\mathcal{F}$ and $\mathcal{F} \cap \mathcal{W}$ are frequently characterized by a right lifting property, in which case all of these closure axioms are automatic; see Lemma C.2.3.

Our primary interest in this notion is on account of the following two examples.

C.1.3. Lemma. *The underlying category of an ∞-cosmos defines a category of fibrant objects with $\mathcal{W}$ the class of equivalences and $\mathcal{F}$ the class of isofibrations.*

Proof. Most of the axioms of Definition C.1.1 are subsumed by the limit and isofibration axioms of Definition 1.2.1. The remaining pieces are established in Lemma 1.2.14, Lemma 1.2.16, and Lemma 1.2.20. □

C.1.4. Lemma. *The full subcategory of fibrant objects in a model category defines a category of fibrant objects with $\mathcal{W}$ the class of weak equivalences and $\mathcal{F}$ the class of fibrations between fibrant objects.*

Proof. Exercise C.3.i. □

In general, it is customary is to refer to the maps in $\mathcal{W}$ as "weak equivalences", the maps in $\mathcal{F}$ as "fibrations," and the maps in $\mathcal{F} \cap \mathcal{W}$ as "trivial fibrations" — unless the specific context dictates alternate names — and depict these classes by the decorated arrows, $\xrightarrow{\sim}$, $\twoheadrightarrow$, and $\overset{\sim}{\twoheadrightarrow}$, respectively.

C.1.5. Remark. Both of the examples just discussed have the additional property of being **right proper**, satisfying an additional axiom:

(vii) Pullbacks of maps in $\mathcal{W}$ along maps in $\mathcal{F}$ define maps in $\mathcal{W}$:

$$\begin{tikzcd} F \arrow[r, "g", "\sim"'] \arrow[d, two heads, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, two heads, "p"] \\ A \arrow[r, "\sim", "f"'] & B \end{tikzcd}$$

For ∞-cosmoi this is proven in Lemma 3.3.2 and for model categories, this was first observed by Reedy in [**71**, Theorem B] (see also [**63**, 15.4.2]).

The factorization axiom in a category of fibrant objects can be generalized to construct factorizations of any map; cf. Lemma 1.2.16

C.1.6. Lemma (Brown factorization lemma). *Any map $f\colon A \to B$ in a category of fibrant objects may be factored as a weak equivalence followed by an fibration, where the weak equivalence is constructed as a section of a trivial fibration.*

$$\begin{tikzcd} & Pf \arrow[dl, two heads, bend right, "q"', "\sim"] \arrow[dr, two heads, "p"] & \\ A \arrow[ur, "s", "\sim"'] \arrow[rr, "f"'] & & B \end{tikzcd}$$

Proof. The displayed factorization is constructed by the pullback of the path object fibration $PB \twoheadrightarrow B \times B$ of (v):

$$\begin{tikzcd}
& & B \arrow[dr, "\sim"] \arrow[ddr, bend left=10, "\Delta"] \\
A \arrow[urr, "f"] \arrow[r, dashed, "\sim", "s"'] \arrow[dr, "{(A,f)}"'] & Pf \arrow[rr] \arrow[d, two heads, "{(q,p)}"'] \arrow[drr, phantom, very near start, "\lrcorner"] & & PB \arrow[d, two heads] \\
& A \times B \arrow[rr, "f \times B"'] & & B \times B
\end{tikzcd}$$

By the 2-of-3 property for the weak equivalences, both projections $PB \overset{\sim}{\twoheadrightarrow} B$ are trivial fibrations. Since the map q is a pullback of one of these projections along $f \colon A \to B$, it follows from axiom (iii) that q is a trivial fibration. Its section s, constructed by applying the universal property of the pullback to the displayed cone with summit A, is thus an equivalence. ☐

C.1.7. Corollary. *If $\mathcal{M}$ is a category of fibrant objects and $B \in \mathcal{M}$, then the category $\mathcal{M}_{/B}$ of fibrations in $\mathcal{M}$ with codomain B and maps over B becomes a category of fibrant objects with weak equivalences and fibrations created by the forgetful functor $\mathcal{M}_{/B} \to \mathcal{M}$.*

Proof. The construction of limits in the slice category $\mathcal{M}_{/B}$ is described in the proof of Proposition 1.2.22; note in particular, that id_B is the terminal object, so all objects in $\mathcal{M}_{/B}$, being fibrations in $\mathcal{M}$, are fibrant. Path objects for a fibration $f \colon A \twoheadrightarrow B$ are constructed by applying Lemma C.1.6 to the "diagonal" map $(f, f) \colon A \to A \times_B A$ from A to the pullback of f along itself. ☐

The dual of a result of Blumberg and Mandell [**15**, 6.4] demonstrates that the equivalences in any ∞-cosmos satisfy the 2-of-6 property. The proof reveals that this holds in any category of fibrant objects in which the class $\mathcal{W}$ is closed under retracts.

C.1.8. Proposition. *Let $\mathcal{M}$ be a category with classes of maps $\mathcal{W}$ and $\mathcal{F}$ so that:*

- *$\mathcal{W}$ satisfies the 2-of-3 property, and is closed under retracts.*
- *The pullback of a map in $\mathcal{F} \cap \mathcal{W}$ is in $\mathcal{F} \cap \mathcal{W}$ and these pullbacks always exist.*
- *Every map in $\mathcal{W}$ factors as a section of a map in $\mathcal{F} \cap \mathcal{W}$ followed by a map in $\mathcal{F}$.*

*Then the class $\mathcal{W}$ satisfies the **2-of-6 property**: for any composable triple of morphisms*

$$\begin{tikzcd}
& B \arrow[dr, "hg", "\sim"'] \arrow[dd, "g"] \\
A \arrow[ur, "f"] \arrow[rr, "hgf"] \arrow[dr, "\sim", "gf"'] & & D \\
& C \arrow[ur, "h"']
\end{tikzcd}$$

if gf and hg are in a class $\mathcal{W}$ then f, g, h, and hgf are too.

Proof. Form the factorization of the weak equivalence hg as displayed below

$$\begin{tikzcd}
& D' \arrow[dl, bend right, two heads, "r"', "\sim"] \arrow[dr, two heads, "p", "\sim"'] \\
B \arrow[ur, "j"', "\sim"] \arrow[rr, "\sim", "hg"'] & & D
\end{tikzcd}$$

and form the pullback of p along h and the induced map t:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[ddrr, bend right, "\sim" sloped, "gf"'] & B \arrow[dr, dashed, "t"] \arrow[drr, bend left, "j", "\sim"' sloped] \arrow[drrr, bend left, equal] \arrow[ddr, bend right, "g"'] & & & \\
& & C' \arrow[r, "u"] \arrow[d, two heads, "q"', "\wr"] \arrow[dr, phantom, "\lrcorner" very near start] & D' \arrow[r, two heads, "\sim", "r"'] \arrow[d, two heads, "p", "\wr"'] & B \\
& & C \arrow[r, "h"'] & D &
\end{tikzcd}$$

By pullback stability of the trivial fibrations, the map q is a weak equivalence, so by the 2-of-3 property and the assumption that gf is a weak equivalence, the composite $tf\colon A \to C'$ must be a weak equivalence. Since the map f is a retract of this composite

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "f"'] & A \arrow[r, equal] \arrow[d, "\wr"', "tf"] & A \arrow[d, "f"] \\
B \arrow[r, "t"'] & C' \arrow[r, "ru"'] & B
\end{tikzcd}$$

so by retract closure of the weak equivalences, $f \in \mathcal{W}$. Now it follows from the 2-of-3 property that g, h, and hgf lie in $\mathcal{W}$ as well. □

C.1.9. Corollary. *The equivalences in an ∞-cosmos satisfy the 2-of-6 property.*

Proof. It remains only to argue that the premises of Proposition C.1.8 hold for the classes of equivalences and trivial fibrations in any ∞-cosmos.

Lemma 1.2.20 proves that the equivalences in $\mathcal{K}$ are also closed under retracts and have the 2-of-3 property. Lemma 1.2.14 proves that the class of trivial fibrations is stable under pullbacks, which exist in any ∞-cosmos. Lemma 1.2.16 constructed the desired factorization, which by the 2-of-3 property factors an equivalence as a section of a trivial fibration followed by a trivial fibration. Now Proposition C.1.8 applies to prove that the equivalences in any ∞-cosmos satisfy the stronger 2-of-6 property. □

The following consequence of Lemma C.1.6, traditionally referred to as "Ken Brown's lemma," is the key to proving the equivalence invariance of many constructions in a category of fibrant objects.

C.1.10. Lemma (Ken Brown's lemma). *Consider a functor $F\colon \mathcal{M} \to \mathcal{K}$ whose domain is a category of fibrant objects and whose codomain is a category with a subcategory of "weak equivalences" satisfying the 2-of-3 property. If F carries trivial fibrations to weak equivalences, then F carries weak equivalences in $\mathcal{M}$ to weak equivalences in $\mathcal{K}$.*

Proof. By the 2-of-3 property of the weak equivalences in $\mathcal{M}$, any weak equivalence in a category of fibrant objects may be factored as a section of a trivial fibration followed by a trivial fibration.

$$\begin{tikzcd}
& Pf \arrow[dl, two heads, bend right, "q"', "\sim" sloped] \arrow[dr, two heads, "p", "\sim"' sloped] & \\
A \arrow[ur, "s"', "\sim" sloped] \arrow[rr, "\sim", "f"'] & & B
\end{tikzcd}$$

By hypothesis, the images of the maps q and p under F are weak equivalences. By the 2-of-3 property of the weak equivalences in $\mathcal{K}$, it follows that the image of s and thus also the image of f are weak equivalences. □

The rest of this section is devoted to applications of Lemma C.1.10 to establish the weak equivalence invariance of limits in a category of fibrant objects. To warm up, as a very easy consequence:

C.1.11. Lemma. *In a category of fibrant objects, a weak equivalence between fibrations pulls back to a weak equivalence between fibrations:*

$$\begin{tikzcd}
P \arrow[rr] \arrow[dr, "u", "\sim"'] \arrow[dd, two heads, "r"'] \arrow[dr, phantom, very near start, "\lrcorner"] & & E \arrow[dr, "e", "\sim"'] \arrow[dd, two heads, "p"'] & \\
& Q \arrow[rr] \arrow[dl, two heads, "s"] \arrow[dr, phantom, very near start, "\lrcorner"] & & F \arrow[dl, two heads, "q"] \\
A \arrow[rr, "f"'] & & B &
\end{tikzcd}$$

Proof. By Corollary C.1.7, slices of a category $\mathcal{M}$ of fibrant objects define categories of fibrant objects and pullback along f defines a functor $f^*\colon \mathcal{M}_{/B} \to \mathcal{M}_{/A}$. Note that the map u in the displayed diagram is the pullback of the map e, so it follows directly from axiom (iii) of Definition C.1.1 that pullback preserves trivial fibrations. Now Lemma C.1.10 implies that it also preserves equivalences. □

Other results in a similar vein require somewhat more delicate arguments. The proofs appearing below are originally due to Reedy in an unpublished manuscript **[71]** that implicitly gave birth to the notion of a "Reedy category" that we introduce in §C.4.

C.1.12. Proposition. *Consider a diagram in a category of fibrant objects:*

$$\begin{tikzcd}
C \arrow[r, "g"] \arrow[d, "r"'] & A \arrow[d, "p"] & B \arrow[l, two heads, "f"'] \arrow[d, "q"] \\
\bar{C} \arrow[r, "\bar{g}"] & \bar{A} & \bar{B} \arrow[l, two heads, "\bar{f}"']
\end{tikzcd}$$

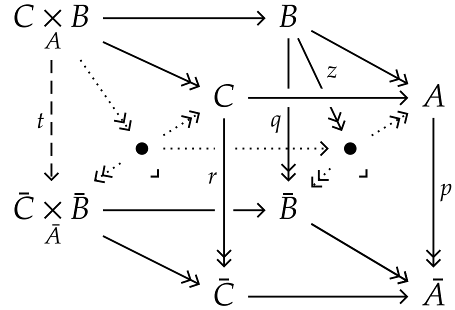
If the map r and the map $z\colon B \to A \times_{\bar{A}} \bar{B}$ are both

(i) fibrations, or

(ii) trivial fibrations

then the induced map from the pullback of f along g to the pullback of $\bar{f}$ along $\bar{g}$ again a fibration or trivial fibration, respectively.

Proof. By considering the commutative diagram and repeatedly applying the pullback composition and cancelation lemma, one discovers that the induced map t factors as a pullback of z followed by a pullback of r as displayed below



and is thus an fibration or trivial fibration if both of these maps are. $\square$

The hypothesis of right properness allows us to prove the following result whose dual form is sometimes called the "gluing lemma."

C.1.13. PROPOSITION. *In a right proper category of fibrant objects, the induced map between the pullbacks of the horizontal rows of a diagram*

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "r"', "\wr"] & A \arrow[d, "\wr"', "p"] & B \arrow[l, two heads, "f"'] \arrow[d, "\wr"', "q"] \\ \bar{C} \arrow[r, "\bar{g}"'] & \bar{A} & \bar{B} \arrow[l, two heads, "\bar{f}"] \end{tikzcd}$$

is again a weak equivalence.

PROOF. Using Lemma C.1.6, the 2-of-3 property from Definition C.1.1, and the right properness of Remark C.1.5, the proof of Proposition 3.3.3 works equally in any right proper category of fibrant objects. $\square$

C.1.14. COROLLARY. *In a category of fibrant objects, finite products of fibrations, trivial fibrations, or weak equivalences are again finite products, trivial fibrations, or weak equivalences.*

PROOF. This can be proven inductively from the case of binary products, which can be constructed as pullbacks over the terminal object. Lemma C.1.10 applies to deduce the result for weak equivalences from the result for trivial fibrations. $\square$

The category of diagrams valued in a category of fibrant objects may itself be equipped with the structure of a category of fibrant objects, at least for certain diagram shapes. A particularly useful family of diagrams includes those indexed by inverse categories.

C.1.15. DEFINITION. A category $\mathcal{I}$ is a **inverse category** if there exists a functor $\deg\colon \mathcal{I} \to \omega^{\mathrm{op}}$ that reflects identities.

The degree functor assigns a natural number degree to each object of $\mathcal{I}$ in such a way that all non-identity morphisms "lower degree," in the sense that the degree of their domain object is strictly greater than the degree of their codomain object. The utility of this axiomatization is it allows us to define the data of an $\mathcal{I}$-indexed diagram or natural transformation by inductively specifying diagrams indexed by the full subcategories

$$\mathcal{I}_{\leq 0} \hookrightarrow \cdots \hookrightarrow \mathcal{I}_{\leq n-1} \hookrightarrow \mathcal{I}_{\leq n} \hookrightarrow \cdots \hookrightarrow \mathcal{I}$$

of objects with bounded degree. To extend $X \in \mathcal{M}^{\mathcal{I}_{\leq n-1}}$ to $\mathcal{M}^{\mathcal{I}_{\leq n}}$ requires the specification, for each object i with degree n of an object $X_i \in \mathcal{M}$ together with a map

$$X_i \to \partial_i X := \lim\left({}^{i/}\mathcal{I}_{\leq n-1} \longrightarrow \mathcal{I}_{\leq n-1} \xrightarrow{X} \mathcal{M} \right) \tag{C.1.16}$$

The mathematics does not change in any substantial way if ω is replaced by the category of ordinals. The reason we restrict to finite degrees is because we've only asked for limits of ω^{op}-indexed towers in Definition C.1.1.

For reasons that will become clear momentarily we define:

C.1.17. DEFINITION. Let $\mathcal{M}$ be a category of fibrant objects and let $\mathcal{I}$ be an inverse category.

- If $X \in \mathcal{M}^{\mathcal{I}}$ is a diagram with the property that for each $i \in \mathcal{I}$, the map $X_i \twoheadrightarrow \partial_i X$ defined by (C.1.16) is a fibration, then call X a **fibrant diagram**.
- If $\alpha\colon X \to Y \in \mathcal{M}^{\mathcal{I}}$ is a natural transformation between fibrant diagrams so that for each $i \in \mathcal{I}$ the map $\hat{m}_i$ defined by

$$
\begin{tikzcd}
X_i \arrow[rr, "\alpha_i"] \arrow[dr, dashed, "\hat{m}_i"] \arrow[dd, two heads] & & Y_i \arrow[dd, two heads] \\
& \bullet \arrow[ur] \arrow[dl] \arrow[dr, phantom, "\lrcorner", very near start] & \\
\partial_i X \arrow[rr, "\partial_i \alpha"'] & & \partial_i Y
\end{tikzcd}
\tag{C.1.18}
$$

is a fibration, then call $\alpha\colon X \to Y$ a **fibrant natural transformation**.

C.1.19. Proposition.

(i) A category of fibrant objects $\mathcal{M}$ admits limits of any fibrant diagram indexed by an inverse category $X \in \mathcal{M}^{\mathcal{I}}$, with $\lim_{\mathcal{I}} F \in \mathcal{M}$ constructed as the limit of a tower

$$\lim_{\mathcal{I}} X \coloneqq \lim_{\omega^{\mathrm{op}}} \left(\cdots \twoheadrightarrow \lim_{\mathcal{I}_{\leq n}} X \twoheadrightarrow \lim_{\mathcal{I}_{\leq n-1}} X \twoheadrightarrow \cdots \twoheadrightarrow \lim_{\mathcal{I}_{\leq 0}} X \right)$$

each layer of which is a pullback

$$
\begin{tikzcd}
\lim_{\mathcal{I}_{\leq n}} X \arrow[r, two heads] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \lim_{\mathcal{I}_{\leq n-1}} X \arrow[d] \\
\prod\limits_{\deg(i)=n} X_i \arrow[r, two heads] & \prod\limits_{\deg(i)=n} \partial_i X
\end{tikzcd}
$$

In particular, each leg of the limit cone $\lim_{\mathcal{I}} X \twoheadrightarrow X_i$ is a fibration as is each map in the image of the fibrant diagram X.

(ii) For any fibrant natural transformation $\alpha\colon X \to Y \in \mathcal{M}^{\mathcal{I}}$, the induced map $\lim_{\mathcal{I}} X \to \lim_{\mathcal{I}} Y$ is the limit composite of a tower whose n-th layer is the map p_n constructed as a pullback in the diagram below:

$$
\begin{tikzcd}
\lim_{\mathcal{I}_{\leq n}} X_i \arrow[rrr] \arrow[drr, two heads] \arrow[ddd, two heads] \arrow[ddr, dotted, "p_n"] & & & \prod\limits_{\deg(i)=n} X_i \arrow[drr, two heads] \arrow[ddd, two heads] \arrow[ddr] & & \\
& & \lim_{\mathcal{I}_{\leq n-1}} X_i \arrow[rrr] \arrow[ddd, two heads] & & & \prod\limits_{\deg(i)=n} \partial_i X \arrow[ddd, two heads] \\
& \bullet \arrow[ur, dotted, two heads] \arrow[dl, dotted] \arrow[rrr, dotted] \arrow[dr, phantom, "\lrcorner", very near start] & & & \bullet \arrow[ur, dotted] \arrow[dl, dotted] \arrow[dr, phantom, "\lrcorner", very near start] & \\
\lim_{\mathcal{I}_{\leq n}} Y_i \arrow[rrr] \arrow[drr, two heads] & & & \prod\limits_{\deg(i)=n} Y_i \arrow[drr, two heads] & & \\
& & \lim_{\mathcal{I}_{\leq n-1}} Y_i \arrow[rrr] & & & \prod\limits_{\deg(i)=n} \partial_i Y
\end{tikzcd}
\tag{C.1.20}
$$

Moreover, each component map $\alpha_i\colon X_i \twoheadrightarrow Y_i$ is a fibration.

Proof. Note that the slice category ${}^{i/}\mathcal{I}_{n-1}$ is again an inverse category in which every object has degree at most $n-1$. In the case where i has degree 1, this category is discrete, so by induction we may assume that the limit $\partial_i X$ defined by (C.1.16) exists. Now the result of (i) follows by direct inspection of the universal property of this construction, or from a more conceptual argument that will prove a generalization of this result in §C.4. The final assertion follows from this construction and is left to Exercise C.1.i.

By (i), it follows that the induced map between the inverse limits is defined as as the limit of an ω^{op}-indexed diagram in the category $\mathcal{M}^{\mathbb{2}}$:

$$\begin{tikzcd} \lim_{\mathcal{I}} X \coloneqq \lim_{n\in\omega^{\mathrm{op}}} \lim_{\mathcal{I}_{\le n}} X \arrow[r, dotted] \arrow[d, dashed, "\lim_{\mathcal{I}} \alpha"'] & \lim_{\mathcal{I}_{\le n}} X \arrow[r] \arrow[d] & \lim_{\mathcal{I}_{\le n-1}} X \arrow[r] \arrow[d] & \cdots \arrow[r] & \lim_{\mathcal{I}_{\le 1}} X \arrow[r] \arrow[d] & \lim_{\mathcal{I}_{\le 0}} X \arrow[d, "p_0"] \\ \lim_{\mathcal{I}} Y \coloneqq \lim_{n\in\omega^{\mathrm{op}}} \lim_{\mathcal{I}_{\le n}} Y \arrow[r, dotted] & \lim_{\mathcal{I}_{\le n}} Y \arrow[r] & \lim_{\mathcal{I}_{\le n-1}} Y \arrow[r] & \cdots \arrow[r] & \lim_{\mathcal{I}_{\le 1}} Y \arrow[r] & \lim_{\mathcal{I}_{\le 0}} Y \end{tikzcd}$$

From this description it is clear that the map $\lim_{\mathcal{I}} \alpha$ factors as the limit composite of a tower whose bottom layer is the pullback of the map p_0 along the lower-horizontal composite above, whose next layer is the pullback of the map p_1 appearing in the right-most square, whose next layer is the pullback of the map p_2 appearing in the second right-most square, and so on, where in each square p_n is the map from the upper left-hand corner to the pullback of the lower-right cospan. Finally, by applying pullback composition and cancelation in the cube (C.1.20), it follows from the fact proven in (i) that the top and bottom faces are pullbacks that the map p_n from the initial vertex to the pullback in the left face is a pullback of the corresponding map from $\prod_{\deg(i)=n} X_i$ to the pullback in the right face. This proves all but the final clause of (ii), which is also left to Exercise C.1.i. □

C.1.21. Proposition. *Let $\mathcal{M}$ be a category of fibrant objects with fibrations $\mathcal{F}$ and weak equivalences $\mathcal{W}$ and let $\mathcal{I}$ be an inverse category. The category $\mathcal{M}^{\mathcal{I}}$ of fibrant diagrams and all natural transformations between them inherits the structure of a category of fibrant objects in which:*

- *the class of weak equivalences is the class of natural transformations whose components lie in $\mathcal{W}$*
- *the class of fibrations is the class of fibrant natural transformations, those $\alpha\colon X \to Y \in \mathcal{M}^{\mathcal{I}}$ so that for each $i \in \mathcal{I}$ the map*

$$X_i \twoheadrightarrow Y_i \times_{\partial_i Y} \partial_i X$$

is in $\mathcal{F}$.

- *the class of trivial fibrations is the class of natural transformations $\alpha\colon X \to Y \in \mathcal{M}^{\mathcal{I}}$ so that for each $i \in \mathcal{I}$ the map*

$$X_i \overset{\sim}{\twoheadrightarrow} Y_i \times_{\partial_i Y} \partial_i X$$

is in $\mathcal{F} \cap \mathcal{W}$.

Proof. The proof is a very lengthy exercise for the reader, which only entails specializing the corresponding arguments from §C.4 to this "one-sided" case. □

The payoff for all this work is that it is now easy to verify the following result.

C.1.22. Proposition. *Let $\mathcal{M}$ be a category of fibrant objects and let $\mathcal{I}$ be an inverse category. Then with the structure of C.1.21 the limit functor* $\lim\colon \mathcal{M}^{\mathcal{I}} \to \mathcal{M}$ *preserves the classes $\mathcal{F}$ and $\mathcal{F} \cap \mathcal{W}$ and hence also $\mathcal{W}$.*

Proof. Consider a map $\alpha\colon X \to Y \in \mathcal{M}^{\mathcal{I}}$ in $\mathcal{F}$ or $\mathcal{F} \cap \mathcal{W}$. By Proposition C.1.19(ii), this map is the limit composite of a tower of maps, each layer of which is the pullback of a product of the maps that we have assumed lies in $\mathcal{F}$ or $\mathcal{F} \cap \mathcal{W}$. Since the classes $\mathcal{F}$ and $\mathcal{F} \cap \mathcal{W}$ are closed under product, pullback, and limits of towers, it is now clear that the limit functor preserves these classes. The fact that it also proves the class $\mathcal{W}$ follows from Lemma C.1.10. □

Consequently:

C.1.23. Corollary. *In a category of fibrant objects:*

(i) A pointwise weak equivalence between cospans of fibrations induces a weak equivalence between their pullbacks.
(ii) A pointwise weak equivalence between towers of fibrations induces a weak equivalence between their inverse limits. □

Exercises.

C.1.i. Exercise.

(i) Verify that each leg of the limit cone constructed in Proposition C.1.19(i) is a fibration.
(ii) Conclude that each morphism in the image of a fibrant diagram is a fibration.
(iii) Arguing along the same lines, verify that each component of a fibrant natural transformation is a fibration.

C.2. Lifting properties, weak factorization systems, and Leibniz closure

Fixing two arrows j and p in a category $\mathcal{M}$, we regard any commutative square of the form

$$\begin{tikzcd} \bullet \arrow[r, "u"] \arrow[d, "j"'] & \bullet \arrow[d, "p"] \\ \bullet \arrow[r, "v"'] \arrow[ur, dashed, "\ell"] & \bullet \end{tikzcd}$$

as presenting a **lifting problem** between j and p, which is solved by constructing a **lift**: a diagonal morphism ℓ making both triangles commute. If every lifting problem between j and p has a solution, we say that j has the **left lifting property** with respect to p and, equivalently, that p has the **right lifting property** with respect to j. When this is the case, we use the suggestive symbol $j \boxslash p$ to assert this lifting property.

Frequently in abstract homotopy theory a class of maps of interest is characterized by a left or right lifting property with respect to another class or set of maps.

C.2.1. Definition. Let $\mathcal{J}$ be a class of maps in a category $\mathcal{M}$.

- Write $\mathcal{J}^{\boxslash}$ for the class of maps that have the **right lifting property** with respect to every morphism in $\mathcal{J}$.
- Write ${}^{\boxslash}\mathcal{J}$ for the class of maps that have the **left lifting property** with respect to every morphism in $\mathcal{J}$.

C.2.2. Example. Definitions 1.1.17 and 1.1.24 charaterize the isofibrations and trivial fibrations between quasi-categories by right lifting properties against the sets of maps

$$\{\Lambda^k[n] \hookrightarrow \Delta[n]\}_{n\geq 2, 0<k<n} \cup \{\mathbb{1} \hookrightarrow \mathbb{I}\} \qquad \text{and} \qquad \{\partial\Delta[n] \hookrightarrow \Delta[n]\}_{n \geq 0}$$

respectively.

Maps characterized by a right lifting property automatically satisfy various closure properties that may now be familiar.

C.2.3. Lemma. *Any class of maps $\mathcal{J}^{\boxslash}$ characterized by a right lifting property contains the isomorphisms and is closed under composition, product, pullback, retract, and limits of towers.*

In the statement, "products" and "retracts" refer to limits formed in the category of arrows, while the "pullbacks" are of a map in $\mathcal{J}^{\boxslash}$ along an arbitrary map. A "tower" refers to a diagram of shape α^{op}, where α is a limit ordinal (most likely ω). Closure under limits of towers asserts that if the images of each of the atomic arrows in the indexing category lie in $\mathcal{J}^{\boxslash}$, then the map from the limit object to the terminal object in the diagram is also in $\mathcal{J}^{\boxslash}$.

Proof. All of the arguments are similar. For instance, suppose q is a pullback of $p \in \mathcal{J}^{\boxslash}$. By juxtaposing a lifting problem as below-left with the pullback square as below-right, we may solve the composite lifting problem of j against p to obtain the dashed diagonal morphism ℓ, and then induce a solution s to the lifting problem of j against q via the cone formed by (v, ℓ) over the pullback diagram

$$\begin{tikzcd}
\bullet \arrow[r, "u"] \arrow[d, "\mathcal{J}\ni j"'] & \bullet \arrow[r, "a"] \arrow[d, "q"] \arrow[dr, phantom, very near start, "\lrcorner"] & \bullet \arrow[d, "p"] \\
\bullet \arrow[r, "v"'] \arrow[ur, dotted, "s"] \arrow[urr, dashed, "\ell"'] & \bullet \arrow[r, "b"'] & \bullet
\end{tikzcd}$$

So q lifts against $\mathcal{J}$ and is therefore in $\mathcal{J}^{\boxslash}$. □

On account of the dual of Lemma C.2.3, any set of maps in a cocomplete category "cellularly generates" a larger class of maps with the same left lifting property.

C.2.4. Definition. Let $\mathcal{J}$ be a class of maps that we think of as "basic cells." A **$\mathcal{J}$-cell complex** is a map built as a transfinite composite of pushouts of coproducts of maps in $\mathcal{J}$:

$$\begin{tikzcd}
\bullet \arrow[r, "\coprod j\in\mathcal{J}"] \arrow[d] & \bullet \arrow[d] & \bullet \arrow[r, "\coprod j\in\mathcal{J}"] \arrow[d] & \bullet \arrow[d] & \\
\bullet \arrow[r] \arrow[rrrr, dashed, bend right, "\in\mathcal{J}\text{-cell}"'] \arrow[ur, phantom, very near end, "\ulcorner"] & \bullet \arrow[r] & \bullet \arrow[r] \arrow[ur, phantom, very near end, "\ulcorner"] & \bullet \arrow[r, dotted] & \bullet \\
& \bullet \arrow[r, "\coprod j\in\mathcal{J}"'] \arrow[u] & \bullet \arrow[u] \arrow[ul, phantom, very near end, "\llcorner"] & &
\end{tikzcd}$$

The class of $\mathcal{J}$-cell complexes $\mathcal{J}$-cell is said to be **cellularly generated** by a set of maps $\mathcal{J}$. The class of maps $\mathcal{J}$-cof **cofibrantly generated** by a set of maps $\mathcal{J}$ is comprised of those maps obtained as retracts of sequential composites of pushouts of coproducts of those maps.

C.2.5. Definition. A **weak factorization system** $(\mathcal{L}, \mathcal{R})$ on a category $\mathcal{M}$ is comprised of two classes of morphisms $\mathcal{L}$ and $\mathcal{R}$ so that

(i) Every morphism in $\mathcal{M}$ may be factored as a morphism in $\mathcal{L}$ followed by a morphism in $\mathcal{R}$.

$$\begin{tikzcd}
\bullet \arrow[rr, "f"] \arrow[dr, "\mathcal{L}\ni\ell"'] & & \bullet \\
& \bullet \arrow[ur, "r\in\mathcal{R}"'] &
\end{tikzcd}$$

(ii) The classes $\mathcal{L}$ and $\mathcal{R}$ respectively have the left and right lifting properties $\mathcal{L} \boxslash \mathcal{R}$ with respect to each other : that is, any commutative square

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "\mathcal{L}\ni\ell"'] & \bullet \arrow[d, "r\in\mathcal{R}"] \\ \bullet \arrow[r] \arrow[ur, dashed] & \bullet \end{tikzcd}$$

admits a diagonal filler as indicated.

(iii) Moreover $\mathcal{L} = {}^{\boxslash}\mathcal{R}$ and $\mathcal{R} = \mathcal{L}^{\boxslash}$.

In the presence of a pair of adjoint functors, lifting properties transpose.

C.2.6. LEMMA. *In the presence of any adjunction*

$$\begin{tikzcd} \mathcal{M} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{N} \arrow[l, bend left, "U"] \end{tikzcd}$$

(i) The lifting problem displayed below left in $\mathcal{N}$ admits a solution if and only if the transposed lifting problem displayed below right admits a solution in $\mathcal{M}$.

$$\begin{tikzcd} FA \arrow[r, "f^\sharp"] \arrow[d, "F\ell"'] & X \arrow[d, "r"] \\ FB \arrow[r, "g^\sharp"'] \arrow[ur, dashed, "k^\sharp"] & Y \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f^\flat"] \arrow[d, "\ell"'] & UX \arrow[d, "Ur"] \\ B \arrow[r, "g^\flat"'] \arrow[ur, dashed, "k^\flat"] & UY \end{tikzcd}$$

(ii) If $\mathcal{M}$ has a weak factorization system $(\mathcal{L}, \mathcal{R})$ while $\mathcal{N}$ has a weak factorization system $(\mathcal{L}', \mathcal{R}')$ then F preserves the left classes if and only if U preserves the right classes:

$$F\mathcal{L} \subset \mathcal{L}' \qquad \leftrightsquigarrow \qquad \mathcal{R} \supset U\mathcal{R}'.$$

The factorizations of Definition C.2.5 are completely irrelevant to the statement of (ii) but we have stated this result for weak factorization systems because that is the context in which it will typically be applied.

PROOF. Exercise C.2.iv. $\square$

Lemma C.2.6(ii) defines the notion of **adjunction of weak factorization systems**, this being an adjoint pair of functors between categories equipped with weak factorization systems so that the left adjoint preserves the left classes and the right adjoint preserves the right classes. Our aim is now to extend this notion to two-variable adjunctions,[1] which are given by a triple of bifunctors, which we write using notation that will suggest the most common examples

$$\mathcal{K} \times \mathcal{L} \xrightarrow{\otimes} \mathcal{M}, \ \ \mathcal{K}^{\mathrm{op}} \times \mathcal{M} \xrightarrow{\{,\}} \mathcal{L}, \ \ \mathcal{L}^{\mathrm{op}} \times \mathcal{M} \xrightarrow{\hom} \mathcal{K} \tag{C.2.7}$$

equipped with a natural isomorphism

$$\mathcal{M}(K \otimes L, M) \cong \mathcal{L}(L, \{K, M\}) \cong \mathcal{K}(K, \hom(L, M)).$$

[1]There is an analogous generalization to n-variable adjunctions that can be found in [**22**, §4].

The "pushout-product" of a bifunctor $-\otimes- \colon \mathcal{K}\times\mathcal{L} \to \mathcal{M}$ defines a bifunctor $-\widehat{\otimes}- \colon \mathcal{K}^{\mathbb{2}}\times\mathcal{L}^{\mathbb{2}} \to \mathcal{M}^{\mathbb{2}}$ that we refer to as the "Leibniz tensor" (when the bifunctor $\otimes$ is called a "tensor"). The "Leibniz cotensor" and "Leibniz hom"

$$\widehat{\{-,-\}} \colon (\mathcal{K}^{\mathbb{2}})^{\mathrm{op}} \times \mathcal{M}^{\mathbb{2}} \to \mathcal{L}^{\mathbb{2}} \qquad \text{and} \qquad \widehat{\hom}(-,-) \colon (\mathcal{L}^{\mathbb{2}})^{\mathrm{op}} \times \mathcal{M}^{\mathbb{2}} \to \mathcal{K}^{\mathbb{2}}$$

are defined dually, using pullbacks in $\mathcal{L}$ and $\mathcal{K}$ respectively.

C.2.8. Definition (Leibniz tensors and cotensors). Given a bifunctor $-\otimes- \colon \mathcal{K}\times\mathcal{L} \to \mathcal{M}$ valued in a category with pushouts, the **Leibniz tensor** of a map $k \colon I \to J$ in $\mathcal{K}$ and a map $\ell \colon A \to B$ in $\mathcal{L}$ is the map $k \widehat{\otimes} \ell$ in $\mathcal{M}$ induced by the pushout diagram below-left:

$$\begin{tikzcd}
I \otimes A \arrow[r, "I\otimes\ell"] \arrow[d, "k\otimes A"'] & I \otimes B \arrow[d] \arrow[ddr, bend left, "k\otimes B"] & \\
J \otimes A \arrow[r] \arrow[drr, bend right, "J\otimes\ell"'] & \bullet \arrow[ul, phantom, "\ulcorner" very near start] \arrow[dr, dashed, "k\widehat{\otimes}\ell"] & \\
& & J \otimes B
\end{tikzcd}
\qquad
\begin{tikzcd}
\{J,X\} \arrow[dr, dashed, "\widehat{\{k,m\}}"] \arrow[drr, bend left, "{\{k,X\}}"] \arrow[ddr, bend right, "{\{J,m\}}"'] & & \\
& \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & \{I,X\} \arrow[d, "{\{I,m\}}"] \\
& \{J,Y\} \arrow[r, "{\{k,Y\}}"'] & \{I,Y\}
\end{tikzcd}$$

In the case of a bifunctor $\{-,-\} \colon \mathcal{K}^{\mathrm{op}} \times \mathcal{M} \to \mathcal{L}$ contravariant in one of its variables valued in a category with pullbacks, the **Leibniz cotensor** of a map $k \colon I \to J$ in $\mathcal{K}$ and a map $m \colon X \to Y$ in $\mathcal{M}$ is the map $\widehat{\{k,m\}}$ induced by the pullback diagram above right.

C.2.9. Proposition. *The Leibniz construction preserves:*

(i) structural isomorphisms: a natural isomorphism

$$X * (Y \otimes Z) \cong (X \times Y) \square Z$$

between suitably composable bifunctors extends to a natural isomorphism

$$f \mathbin{\widehat{*}} (g \mathbin{\widehat{\otimes}} h) \cong (f \mathbin{\widehat{\times}} g) \mathbin{\widehat{\square}} h$$

between the corresponding Leibniz products;

(ii) adjointness: if $(\otimes, \{,\}, \hom)$ *define a two-variable adjunction, then the Leibniz functors* $(\widehat{\otimes}, \widehat{\{,\}}, \widehat{\hom})$ *define a two-variable adjunction between the corresponding arrow categories;*

(iii) colimits in the arrow category: if $\otimes \colon \mathcal{K}\times\mathcal{L} \to \mathcal{M}$ *is cocontinuous in either variable, then so is* $\widehat{\otimes} \colon \mathcal{K}^{\mathbb{2}} \times \mathcal{L}^{\mathbb{2}} \to \mathcal{M}^{\mathbb{2}}$*;*

(iv) pushouts: if $\otimes \colon \mathcal{K}\times\mathcal{L} \to \mathcal{M}$ *is cocontinuous in its second variable, and if* g' *is a pushout of* g*, then* $f \mathbin{\widehat{\otimes}} g'$ *is a pushout of* $f \mathbin{\widehat{\otimes}} g$*;*

(v) *composition, in a sense: the Leibniz tensor $f \widehat{\otimes} (h \cdot g)$ factors as a composite of a pushout of $f \widehat{\otimes} g$ followed by $f \widehat{\otimes} h$*

$$\begin{tikzcd}
I\otimes A \arrow[r, "I\otimes g"] \arrow[d, "f\otimes A"'] & I\otimes B \arrow[r, "I\otimes h"] \arrow[d] & I\otimes C \arrow[d] \arrow[dddrr, bend left, "f\otimes C"] & & \\
J\otimes A \arrow[r] \arrow[drr, "J\otimes g"'] & \bullet \arrow[ul, phantom, "\ulcorner", very near start] \arrow[r] \arrow[dr, dashed, "f\widehat{\otimes} g"] & \bullet \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr] \arrow[ddrr, bend left, "f\widehat{\otimes}(h\cdot g)"] & & \\
 & & J\otimes B \arrow[r] \arrow[drr, "J\otimes h"'] & \bullet \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "f\widehat{\otimes} h"] & \\
 & & & & J\otimes C
\end{tikzcd}$$

(vi) *cell complex structures: if f and g may be presented as cell complexes with cells f_α and g_β, respectively, and if $\otimes$ is cocontinuous in both variables, then $f \widehat{\otimes} g$ may be presented as a cell complex with cells $f_\alpha \widehat{\otimes} g_\beta$.*

PROOF. The components of the induced structural isomorphism between Leibniz products are instances of the given structure isomorphism and hence invertible, proving (i). For (ii), by naturality of the isomorphisms defining a two-variable adjunction $(\otimes, \{,\}, \hom)$, each of the squares below commutes if and only if the other two do, under the hypothesis that the horizontal arrows given the same names in each diagram are transposes:

$$\begin{tikzcd}
J\otimes A \underset{I\otimes A}{\cup} I\otimes B \arrow[r, "{(u,v)}"] \arrow[d, "k\widehat{\otimes}\ell"'] & X \arrow[d, "m"] \\
J\otimes B \arrow[r, "w"'] \arrow[ur, dashed, "x"] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "\ell"'] & \{J,X\} \arrow[d, "\widehat{\{k,m\}}"] \\
B \arrow[r, "{(w,v)}"'] \arrow[ur, dashed, "x"] & \{J,Y\} \underset{\{I,Y\}}{\times} \{I,X\}
\end{tikzcd}
\qquad
\begin{tikzcd}
I \arrow[r, "v"] \arrow[d, "k"'] & \hom(B,X) \arrow[d, "\widehat{\hom}(\ell,m)"] \\
J \arrow[r, "{(u,w)}"'] \arrow[ur, dashed, "x"] & \hom(A,X) \underset{\hom(A,Y)}{\times} \hom(B,Y)
\end{tikzcd}$$

This transposition correspondence extends to solutions to the lifting problems presented by these squares; see Exercise C.2.v.

Property (iii) is immediately since limits or colimits in arrow categories are computed pointwise. For (iv), consider the commutative cube:

$$\begin{tikzcd}
I\otimes A \arrow[rrr] \arrow[drr, "I\otimes g"'] \arrow[ddd, "f\otimes A"'] & & & I\otimes C \arrow[drr, "I\otimes g'"] \arrow[ddd] & & \\
 & & I\otimes B \arrow[rrr] \arrow[dl, dotted] \arrow[ddd] & & & I\otimes D \arrow[dl, dotted] \arrow[ddd] \\
 & \bullet \arrow[rrr, dashed] \arrow[ddr, dashed, "f\widehat{\otimes} g"] & & & \bullet \arrow[ddr, dashed, "f\widehat{\otimes} g'"] & \\
J\otimes A \arrow[ur, dotted] \arrow[rrr] \arrow[drr, "J\otimes g"'] & & & J\otimes C \arrow[ur, dotted] \arrow[drr] & & \\
 & & J\otimes B \arrow[rrr] & & & J\otimes D
\end{tikzcd}$$

Since $I \otimes -$ and $J \otimes -$ preserve the pushout defining g' as a pushout of g, the top and bottom faces of the cube are pushouts. The squares defining the domains of the Leibniz tensors define pushouts inside the left and right-hand faces. It follows by pushout composition and cancelation that $f \widehat{\otimes} g'$ is a pushout of $f \widehat{\otimes} g$ as claimed.

The displayed diagram in (v) proves the assertion made there, so it remains only to prove (vi). First note that pushouts of trasnfinite composites of pushouts are again transfinite composites of pushouts and transfinite composites of transfinite composites are transfinite composites, so it suffices to work one variable at a time and prove that $f \mathbin{\widehat{\otimes}} -$ preserves cell complex presentations for g. To that end, suppose g is a α-composite of maps g_i each of which are pushouts of a coproduct of maps $g'_i = \coprod_j g'_{i,j}$. We may promote this colimit to the arrow category to regard $g = g_{0,\alpha}$ as the colimit of the diagram $\alpha \to \mathcal{L}^2$ with one-step maps

$$\begin{tikzcd} \bullet \arrow[r, "g_i"] \arrow[d, "g_{i,\alpha}"'] & \bullet \arrow[d, "g_{i+1,\alpha}"] \\ \bullet \arrow[r, equal] & \bullet \end{tikzcd}$$

Similarly, the pushout square defining g_i from g'_i can similarly be promoted to a pushout square in the arrow category:

$$\begin{tikzcd} \bullet \arrow[rr, "g'_i"] \arrow[dr, "u_i"] \arrow[dd, "g'_i"'] & & \bullet \arrow[dr, "v_i"] \arrow[dd, equal] & \\ & \bullet \arrow[rr, "g_i", near start] \arrow[dd, "g_{i,\alpha}", near start] & & \bullet \arrow[dd, "g_{i+1,\alpha}"] \\ \bullet \arrow[rr, equal] \arrow[dr, dashed, "g_{i+1,\alpha}v_i"'] & & \bullet \arrow[dr, dashed, "g_{i+1,\alpha}v_i"] & \\ & \bullet \arrow[rr, equal] & & \bullet \end{tikzcd}$$

We interpret this cube as presenting the square in the front face as a pushout of the square in the back face, which decomposes as a coproduct of similar squares, one for each component of $g'_i = \coprod_j g'_{i,j}$. In this way we see that $g = g_{0,\alpha}$ is the domain component of the colimit of a diagram $\alpha \to \mathcal{L}^2$, each step of which is a pushout of a coproduct of maps in the arrow category. Now $f \mathbin{\widehat{\otimes}} - \colon \mathcal{L}^2 \to \mathcal{M}^2$ preserves colimits in the arrow category, and the domain functor $\mathrm{dom}\colon \mathcal{M}^2 \to \mathcal{M}$ preserves colimit as well. Thus, $f \mathbin{\widehat{\otimes}} g$ is a colimit of an α-sequence of pushouts of coproducts of the maps $f \mathbin{\widehat{\otimes}} g_{i,j}$. $\square$

More details establishing these assertions are given in [**78**, §§4-5].

C.2.10. Definition. Let $\mathcal{K}$, $\mathcal{L}$, and $\mathcal{M}$ be cocomplete categories each equipped with weak factorization systems $(\mathcal{M}, \mathcal{E})$, $(\mathcal{C}, \mathcal{F})$, and $(\mathcal{L}, \mathscr{R})$, respectively. A **left Leibniz bifunctor** is a bifunctor

$$\otimes\colon \mathcal{K} \times \mathcal{L} \to \mathcal{M}$$

that is

(i) cocontinuous in each variable separately, and
(ii) has the **Leibniz property**: $\otimes$-pushout products of a map in $\mathcal{M}$ with a map in $\mathcal{C}$ are in $\mathcal{L}$.

Dually, a bifunctor between complete categories equipped with weak factorization systems is a **right Leibniz bifunctor** if it is continuous in each variable separately and if pullback cotensors of maps in the right classes land in the right class. We most frequently apply this definition in the case of a bifunctor

$$\{-,-\}\colon \mathcal{K}^{\mathrm{op}} \times \mathcal{M} \to \mathcal{L}$$

that is contravariant in one of its variables, in which we case the relevant hypothesis is that $\mathcal{K}$ is cocomplete and colimits in the first variable are carried to limits in $\mathcal{L}$. The nature of the duality between left and right Leibniz bifunctors is somewhat subtle to articulate. We leave this as a puzzle for the reader, with the hint to see [**22**].

C.2.11. LEMMA. *If the bifunctors*

$$\mathcal{K}\times\mathcal{L}\xrightarrow{\otimes}\mathcal{M},\quad \mathcal{K}^{\mathrm{op}}\times\mathcal{M}\xrightarrow{\{-,-\}}\mathcal{L},\quad\text{and}\quad \mathcal{L}^{\mathrm{op}}\times\mathcal{M}\xrightarrow{\hom}\mathcal{K}$$

define a two-variable adjunction, and $(\mathcal{M},\mathcal{E})$, $(\mathcal{C},\mathcal{F})$, *and* $(\mathcal{L},\mathcal{R})$ *are three weak factorization systems on* $\mathcal{K}$, $\mathcal{L}$, *and* $\mathcal{M}$ *respectively, then the following are equivalent*

(i) $\otimes\colon\mathcal{K}\times\mathcal{L}\to\mathcal{M}$ *defines a left Leibniz bifunctor.*
(ii) $\{-,-\}\colon\mathcal{K}^{\mathrm{op}}\times\mathcal{M}\to\mathcal{L}$ *defines a right Leibniz bifunctor.*
(iii) $\hom\colon\mathcal{L}^{\mathrm{op}}\times\mathcal{M}\to\mathcal{K}$ *defines a right Leibniz bifunctor.*

When these conditions are satisfied, we say that $(\otimes,\{,\},\hom)$ *defines a* ***Leibniz two-variable adjunction****.*

PROOF. The presence of the adjoints ensures that each of the bifunctors satisfy the required (co)continuity hypotheses. Note that, for instance, $\mathcal{M}\widehat{\otimes}\mathcal{C}\subset\mathcal{L}$ if and only if $\mathcal{M}\widehat{\otimes}\mathcal{C}\boxslash\mathcal{R}$. Now the equivalence of (i), (ii), and (iii) follows from the equivalence of the following three lifting properties:

$$\mathcal{M}\widehat{\otimes}\mathcal{C}\boxslash\mathcal{R}\quad\leftrightsquigarrow\quad\mathcal{C}\boxslash\widehat{\{\mathcal{M},\mathcal{R}\}}\quad\leftrightsquigarrow\quad\mathcal{M}\boxslash\widehat{\hom}(\mathcal{C},\mathcal{R}),$$

the proof of which is left to Exercise C.2.v. $\square$

C.2.12. LEMMA. *For any category* $\mathcal{M}$ *with products and coproducts that is equipped with a weak factorization system* $(\mathcal{L},\mathcal{R})$ *the set-tensor, set-cotensor, and hom*

$$*\colon\mathcal{S}et\times\mathcal{M}\to\mathcal{M},\quad\{-,-\}\colon\mathcal{S}et^{\mathrm{op}}\times\mathcal{M}\to\mathcal{M},\quad\text{and}\quad\hom\colon\mathcal{M}^{\mathrm{op}}\times\mathcal{M}\to\mathcal{S}et$$

respectively define a Leibniz two-variable adjunction relative to the mono-epi weak factorization system $(\mathcal{M},\mathcal{E})$ *on* $\mathcal{S}et$.

PROOF. By Lemma C.2.11, it suffices to prove any one of these bifunctors is Leibniz. When $A\hookrightarrow B$ is a monomorphism in $\mathcal{S}et$, the Leibniz tensor with $f\colon X\to Y$ decomposes as a coproduct of maps that are manifestly in $\mathcal{L}$.

$$\begin{tikzcd}
A*X \arrow[r, hook] \arrow[d, "A*f"'] & B*X\cong A*X\coprod B\backslash A*X \arrow[d, "A*f\coprod\mathrm{id}"] \arrow[ddr, bend left, "B*f"] & \\
A*Y \arrow[r] \arrow[urr, phantom, "\ulcorner", very near end] \arrow[drr, bend right] & A*Y\coprod B\backslash A*X \arrow[dr, "\mathrm{id}\coprod B\backslash A*f"'] & \\
& & B*Y\cong A*Y\coprod B\backslash A*Y
\end{tikzcd}\tag{C.2.13}$$

A slicker proof is also possible. Because every monomorphism may be presented as a cell complex built from a single cell $\varnothing\hookrightarrow 1$, by Proposition C.2.9(vi), it suffices to consider Leibniz tensor with the generating monomorphism $\varnothing\hookrightarrow 1$. But note that the functor

$$\mathcal{M}^2\xrightarrow{(\varnothing\hookrightarrow 1)\hat{*}-}\mathcal{M}^2$$

is naturally isomorphic to the identity, which certainly preserves the left class $\mathcal{L}$. $\square$

C.2.14. REMARK. To prove that $\hom\colon \mathcal{M}^{\mathrm{op}} \times \mathcal{M} \to \mathcal{S}et$ is right Leibniz is to show that for any $\ell \in \mathcal{L}$ and $r \in \mathscr{R}$, the morphism

$$\mathcal{M}(\operatorname{cod}\ell, \operatorname{dom} r) \xrightarrow{r \circ\!-\!\circ \ell} \mathcal{M}(\operatorname{dom}\ell, \operatorname{dom} r) \underset{\mathcal{M}(\operatorname{dom}\ell, \operatorname{cod} r)}{\times} \mathcal{M}(\operatorname{cod}\ell, \operatorname{cod} r)$$

is an epimorphism. The target of this map is the set of commutative squares in $\mathcal{M}$ from ℓ to r, while the fiber over any element is the set of solutions to the lifting problem so-presented. The fact that this is an epimorphism follows from the lifting property $\mathcal{L} \boxslash \mathscr{R}$.

Exercises.

C.2.i. EXERCISE. Finish the proof of Lemma C.2.3.

C.2.ii. EXERCISE.

(i) Prove the "retract argument": Suppose $f = r \circ \ell$ and f has the left lifting property with respect to its right factor r. Then f is a retract of its left factor ℓ.

(ii) Conclude that in the presence of axioms (i) and (ii) of Definition C.2.5, that axiom (iii) may be replaced by the hypothesis that the classes $\mathcal{L}$ and $\mathscr{R}$ are closed under retracts.

C.2.iii. EXERCISE.

(i) Suppose $\mathcal{M}$ is a category with products, pullbacks, and limits of towers equipped with a weak factorization system $(\mathcal{L}, \mathscr{R})$, and let $\mathcal{I}$ be an inverse category. Prove that the category of diagrams $\mathcal{M}^{\mathcal{I}}$ has a weak factorization system whose left class is comprised of those maps whose components are in $\mathcal{L}$ and whose right class is comprised of those maps $\alpha\colon X \to Y$ so that for each $i \in I$, the map $\hat{m}_i\colon X_i \to \partial_i X \times_{\partial_i Y} Y_i$ over (C.1.18) lies in $\mathscr{R}$.

(ii) Give a new proof of Proposition C.1.22 under the additional hypothesis that $\mathcal{F}$ and $\mathcal{F} \cap \mathcal{W}$ are the right classes of weak factorization systems.

C.2.iv. EXERCISE. Prove Lemma C.2.6

C.2.v. EXERCISE. Given a two variable adjunction (C.2.7) and classes of maps $\mathcal{A}, \mathcal{B}, \mathcal{C}$ in $\mathcal{K}, \mathcal{L}, \mathcal{M}$, respectively, prove that the following lifting properties are equivalent

$$\mathcal{A} \hat{\otimes} \mathcal{B} \boxslash \mathcal{C} \quad \Leftrightarrow \quad \mathcal{B} \boxslash \widehat{\{\mathcal{A}, \mathcal{C}\}} \quad \Leftrightarrow \quad \mathcal{A} \boxslash \widehat{\hom}(\mathcal{B}, \mathcal{C}).$$

C.3. Model categories and Quillen functors

The following reformulation of Quillen's definition of "closed model categories" [**68**, I.5.1] was given by Joyal and Tierney [**48**, 7.7], who prove that a category $(\mathcal{M}, \mathcal{W})$ with weak equivalences satisfying the two-of-three property admits a model structure just when there exist classes $\mathcal{C}$ and $\mathcal{F}$ that define a pair of weak factorization systems as follows:

C.3.1. DEFINITION (model category). A **model structure** on a category $\mathcal{M}$ with all small limits and colimits consists of three classes of maps — the **weak equivalences** $\mathcal{W}$ denoted "$\xrightarrow{\sim}$" which must satisfy the two-of-three property, the **cofibrations** $\mathcal{C}$ denoted "$\rightarrowtail$", and the **fibrations** $\mathcal{F}$ denoted "$\twoheadrightarrow$" respectively — so that $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ each define weak factorization systems on $\mathcal{M}$.[2]

[2] There is one axiom in standard definition of a model category — the closure of weak equivalences under retracts — that is not obviously included, but this is a consequence of the axioms given here [**48**, 7.8].

Note that Definitions C.2.5 and C.3.1 are self-dual: if $(\mathcal{L}, \mathscr{R})$ defines a weak factorization system on $\mathcal{M}$ then $(\mathscr{R}, \mathcal{L})$ defines a weak factorization system on $\mathcal{M}^{\mathrm{op}}$. Thus all general theorems about the right classes of fibrations $\mathcal{F}$ and $\mathcal{F} \cap \mathcal{W}$ of **trivial fibrations** "$\overset{\sim}{\twoheadrightarrow}$," imply the dual results involving the left classes $\mathcal{C}$ of cofibrations and $\mathcal{C} \cap \mathcal{W}$ of **trivial cofibrations** "$\overset{\sim}{\rightarrowtail}$." In particular, by Lemma C.1.4, all of the results proven in §C.1 about a category of fibrant objects hold for the fibrations, trivial fibrations, and weak equivalence between the eponymous fibrant objects in a model category that we now define. Thus, the duals of these results hold for the cofibrations, trivial cofibrations and weak equivalences between cofibrant objects.

C.3.2. Definition. In a model category $\mathcal{M}$ an object X is **fibrant** just when the unique map $X \to 1$ to the terminal object is a fibrant and **cofibrant** just when the unique map $\varnothing \to X$ from the initial object is a cofibration. By factoring the unique maps, any object X has a **cofibrant replacment** QX and a **fibrant replacement** RX constructed as follows:

$$\begin{tikzcd} & & & RX \arrow[dr, two heads] & \\ \varnothing \arrow[rr, "!"] \arrow[dr, tail] & & X \arrow[ur, tail, "\sim"] \arrow[rr, "!"'] & & * \\ & QX \arrow[ur, two heads, "\sim"'] & & & \end{tikzcd}$$

Note also that since either class of a weak factorization system determines the other, the trivial cofibrations can be defined without reference to either the cofibrations or weak equivalences as those maps that have the left lifting property with respect to the fibrations, and dually the trivial fibrations are precisely those maps that have the right lifting property with respect to the cofibrations.

C.3.3. Definition. A functor between model categories is

- **left Quillen** if it preserves cofibrations, trivial cofibrations, and cofibrant objects, and
- **right Quillen** if it preserves fibrations, trivial fibrations, and fibrant objects.

Left Quillen functors admit left derived functors while right Quillen functors admit right derived functors. We leave a full account of this to other authors [**75**, ß2.1-2] so as to avoid defining these terms, but an important component of the "derivability" of Quillen functors is captured by the following result:

C.3.4. Lemma. *Any left Quillen functor between model categories preserves weak equivalences between cofibrant objects, while any right Quillen functor preserves weak equivalences between fibrant objects.*

Proof. For right Quillen functors this follows directly from Lemma C.1.10 and C.1.4. The result for left Quillen functors is dual. ☐

Most left Quillen functors are "cocontinuous," preserving all colimits, while most right Quillen functors are "continuous," preserving all limits; when this is the case there is no need to separately assume that cofibrant or fibrant objects are preserved. This is because Quillen functors commonly occur as an adjoint pair:

C.3.5. Definition. Consider an adjunction between a pair of model categories.

$$\begin{tikzcd} \mathcal{M} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{N} \arrow[l, bend left, "U"] \end{tikzcd}$$

By Lemma C.2.6 the following are equivalent, defining a **Quillen adjunction**.

(i) The left adjoint F is left Quillen.
(ii) The right U is right Quillen.
(iii) The left adjoint preserves cofibrations and the right adjoint preserves fibrations.
(iv) The left adjoint preserves trivial fibrations and the right adjoint preserves trivial fibrations.

We now introduce a pair of model structures on diagram categories that are designed to ensure that the diagonal functor $\Delta\colon \mathcal{M} \to \mathcal{M}^{\mathcal{J}}$ is respectively right or left Quillen, so that the colimit and limit functors, respectively, are left or right Quillen. The corresponding left and right derived functors then define the **homotopy colimit** and **homotopy limit** functors.

C.3.6. Definition. Let $\mathcal{M}$ be a model category and let $\mathcal{J}$ be a small category.

(i) The **projective model structure** on $\mathcal{M}^{\mathcal{J}}$ has weak equivalences and fibrations defined pointwise in $\mathcal{M}$.
(ii) The **injective model structure** on $\mathcal{M}^{\mathcal{J}}$ has weak equivalences and cofibrations defined pointwise in $\mathcal{M}$.

When the model category $\mathcal{M}$ is *combinatorial* or more generally *accessible*, the projective and injective model structures always exist. Of course, the projective and injective model structures might happen to exist on $\mathcal{M}^{\mathcal{J}}$, perhaps for particular diagram shapes $\mathcal{J}$, in the absence of these hypotheses.

A Quillen two-variable adjunction is a two-variable adjunction in which the left adjoint is a left Quillen bifunctor while the right adjoints are both right Quillen bifunctors. By Exercise C.2.v, any one of these conditions implying the other two:

C.3.7. Definition. A two-variable adjunction

$$\mathcal{V}\times\mathcal{M} \xrightarrow{\otimes} \mathcal{N}, \quad \mathcal{V}^{\mathrm{op}}\times\mathcal{N} \xrightarrow{\{-,-\}} \mathcal{M}, \quad \mathcal{M}^{\mathrm{op}}\times\mathcal{N} \xrightarrow{\mathrm{hom}} \mathcal{V}$$

between model categories $\mathcal{V}$, $\mathcal{M}$, and $\mathcal{N}$ defines a **Quillen two-variable adjunction** if any, and hence all, of the following equivalent conditions are satisfied:

(i) The functor $\hat{\otimes}\colon \mathcal{V}^2 \times \mathcal{M}^2 \to \mathcal{N}^2$ carries any pair comprised of a cofibration in $\mathcal{V}$ and a cofibration in $\mathcal{M}$ to a cofibration in $\mathcal{N}$ and furthermore this cofibration is a weak equivalence if either of the domain cofibrations are.
(ii) The functor $\widehat{\{-,-\}}\colon (\mathcal{V}^2)^{\mathrm{op}} \times \mathcal{N}^2 \to \mathcal{M}^2$ carries any pair comprised of a cofibration in $\mathcal{V}$ and a fibration in $\mathcal{N}$ to a fibration in $\mathcal{N}$ and furthermore this fibration is a weak equivalence if either of the domain maps are.
(iii) The functor $\widehat{\mathrm{hom}}\colon (\mathcal{M}^2)^{\mathrm{op}} \times \mathcal{N}^2 \to \mathcal{V}^2$ carries any pair comprised of a cofibration in $\mathcal{M}$ and a fibration in $\mathcal{N}$ to a fibration in $\mathcal{V}$ and furthermore this fibration is a weak equivalence if either of the domain maps are.

C.3.8. Remark. Definition C.3.7, asserts that a two-variable adjunction is Quillen if and only if its left adjoint $\otimes\colon \mathcal{V}\times\mathcal{M} \to \mathcal{N}$ is a **left Quillen bifunctor**: a bifunctor that is left Leibniz with respect to all possible choices of constituent weak factorization systems, except the choice of the trivial cofibrations only for $\mathcal{N}$.

Quillen's axiomatization of the additional properties enjoyed by his model structure on the category of simplicial sets has been generalized by Hovey [43, §4.2] to define the notions of monoidal model category and enriched model category. We specialize the former to the cartesian closed categories of §A.1 as those are the only cases we'll need here. If $\mathcal{V}$ has a model structure and also has the structure of a cartesian closed category it is natural to ask that these be compatible in some way.

C.3.9. DEFINITION. A **cartesian closed model category** is a cartesian closed category $(\mathcal{V}, \times, 1)$ with a model structure so that

(i) the cartesian product and internal hom define a Quillen two-variable adjunction and
(ii) furthermore so that the map
$$Q1 \times v \to 1 \times v \cong v$$
are is a weak equivalence whenever v is cofibrant.[3]

Then:

C.3.10. DEFINITION. If $\mathcal{V}$ is a cartesian closed model category a **$\mathcal{V}$-model category** is a model category $\mathcal{M}$ that is

(i) tensored, cotensored, and $\mathcal{V}$-enriched in such a way that $(\otimes, \{,\}, \hom)$ is a Quillen two-variable adjunction and
(ii) the maps
$$Q1 \otimes m \to 1 \otimes m \cong m$$
are weak equivalences if m is cofibrant.[4]

C.3.11. LEMMA. *If $\mathcal{M}$ is a $\mathcal{V}$-model category, then for any cofibrant object M and fibrant object N in $\mathcal{M}$, $\hom(M, N)$ is a fibrant object in $\mathcal{V}$. More generally, for any cofibrant object M and fibration $p\colon N \twoheadrightarrow P$, the induced map $p_*\colon \hom(M, N) \to \hom(M, P)$ is a fibration in $\mathcal{V}$.*

PROOF. By Proposition A.5.5—which implies, for the terminal object $1 \in \mathcal{M}$ and any $M \in \mathcal{V}$, that $\hom(M, 1) \cong 1$ is terminal in $\mathcal{V}$—the second statement subsumes the first. By Exercise C.2.v, the lifting problem below-left for any trivial cofibration i in $\mathcal{V}$ transposes to the lifting problem below-right

$$\begin{tikzcd} U \arrow[r] \arrow[d, tail, "i"', "\wr"] & \hom(M,N) \arrow[d, "p_*"] \\ V \arrow[r] \arrow[ur, dashed] & \hom(M,P) \end{tikzcd} \qquad \begin{tikzcd} U \otimes M \arrow[r] \arrow[d, tail, "i\otimes M"', "\wr"] & N \arrow[d, two heads, "p"] \\ V \otimes M \arrow[r] \arrow[ur, dashed] & P \end{tikzcd}$$

By Exercise C.3.iii, since M is cofibrant, $-\otimes M\colon \mathcal{V} \to \mathcal{M}$ is left Quillen, so $i \otimes M$ is a trivial cofibration in $\mathcal{M}$ and since $p\colon N \twoheadrightarrow P$ is a fibrantion, a solution to the lifting problem exists. ☐

The following result was formulated by Gambino [**37**] in the context of a model category enriched over Quillen's cartesian closed model structure on simplicial sets, but its proof applies in greater generality.

C.3.12. THEOREM. *If $\mathcal{M}$ is a $\mathcal{V}$-model category and $\mathcal{J}$ is a small category, then the weighted colimit functor*
$$\operatorname{colim}^{\mathcal{J}}_{-} -\colon \mathcal{V}^{\mathcal{J}} \times \mathcal{M}^{\mathcal{J}^{\mathrm{op}}} \to \mathcal{M}$$
is left Quillen if the domain has the (injective, projective) or (projective, injective) model structure. Similarly, the weighted limit functor
$$\lim^{\mathcal{J}}_{-} -\colon (\mathcal{V}^{\mathcal{J}})^{\mathrm{op}} \times \mathcal{M}^{\mathcal{J}} \to \mathcal{M}$$
is right Quillen if the domain has the (projective, projective) or (injective, injective) model structure.

[3] If $1 \in \mathcal{V}$ is cofibrant, it suffices to require only the first condition; see Exercise C.3.iii.

[4] Again, by Exercise C.3.iii this condition is unnecessary if $1 \in \mathcal{V}$ is cofibrant.

Proof. By Definition C.3.7 we can prove both statements in adjoint form. The weighted colimit bifunctor of Definition A.6.7 has a right adjoint (used to express the defining universal property of the weighted colimit)

$$\hom(-,-) \colon (\mathcal{M}^{\mathcal{J}^{\mathrm{op}}})^{\mathrm{op}} \times \mathcal{M} \to \mathcal{V}^{\mathcal{J}}$$

which sends $F \in \mathcal{M}^{\mathcal{J}^{\mathrm{op}}}$ and $m \in \mathcal{M}$ to $\hom(F-, m) \in \mathcal{V}^{\mathcal{J}}$.

To prove the statement when $\mathcal{V}^{\mathcal{J}}$ has the projective and $\mathcal{M}^{\mathcal{J}^{\mathrm{op}}}$ has the injective model structure, we must show that this is a right Quillen bifunctor with respect to the pointwise (trivial) cofibrations in $\mathcal{M}^{\mathcal{J}^{\mathrm{op}}}$, (trivial) fibrations in $\mathcal{M}$, and pointwise (trivial) fibrations in $\mathcal{V}^{\mathcal{J}}$. Because the limits involved in the definition of right Quillen bifunctors are also formed pointwise, this follows immediately from the corresponding property of the simplicial hom bifunctor, which was part of the definition of a simplicial model category. The other cases are similar. □

The upshot of Theorem C.3.12 is that there are two approaches to constructing a homotopy colimit: "fattening up the diagram" — for instance, by requiring that its objects are cofibrant and its morphisms are cofibrations — or "fattening up the weight" — typically by taking a cofibrant replacement of the terminal weight. Lemmas 6.3.9 and 6.3.13 can be understood as examples of the general equivalence between these two approaches.

C.3.13. Corollary. *If $\mathcal{M}$ is a $\mathcal{V}$-model category, then for any diagram $F \in \mathcal{M}^{\mathcal{J}}$ whose objects are all fibrant and any projective cofibrant weight $W \in \mathcal{V}^{\mathcal{J}}$, the weighted limit is a fibrant object.*

Proof. By Theorem C.3.12, the weighted limit functor $\lim^{\mathcal{J}}_{-} - \colon (\mathcal{V}^{\mathcal{J}})^{\mathrm{op}} \times \mathcal{M}^{\mathcal{J}} \to \mathcal{M}$ is right Quillen with respect to the projective model structure on the category of weights and the injective model structure on the category of diagrams. Since right Quillen bifunctors preserve fibrant objects, it follows that the limit of a pointwise fibrant diagram weighted by a projective cofibrant weight is fibrant. □

Finally, we will make use of the following theorem which enables the change of base of enrichment for model categories extending the results of §A.7. The premises of Theorem C.3.14 are the obvious extension of the premises of Proposition A.7.8 to the enriched model category context, but the conclusion only allows us to transfer enrichments in the directly of the right adjoint because an enriched model category must also be tensored and cotensored and these properties only transfer in that direction.

The result below is a specialization of a more general theorem proven in [**40**, 3.8] to the cartesian closed bases for enrichment that we have been considering.

C.3.14. Theorem. *Consider a Quillen adjunction between cartesian closed model categories in which the left adjoint preserves finite products:*

$$\begin{tikzcd} \mathcal{V} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{W} \arrow[l, bend left, "U"] \end{tikzcd}$$

Then any $\mathcal{W}$-model category admits the structure of a $\mathcal{V}$-model category with the same underlying unenriched model category with enriched homs, cotensors, and tensors defined by:

$$\hom_{\mathcal{V}}(M,N) := U \hom_{\mathcal{W}}(M,N), \quad V \otimes M := FV \otimes M, \qquad \textit{and} \qquad M^V := M^{FV}.$$

PROOF. By Proposition A.7.10 these definitions make $\mathcal{W}$ into a tensored and cotensored $\mathcal{V}$-enriched category, and Exercise A.7.ii observed that change of base along the right adjoint of a monoidal adjunction preserves underlying 1-categories. It remains only to verify that the functors underlying the $\mathcal{V}$-enriched hom, tensor, and cotensor define a Quillen two-variable adjunction, but this follows easily from the cartesian closure of the model categories $\mathcal{V}$ and $\mathcal{W}$ and the fact that $F \dashv U$ is Quillen. $\square$

Exercises.

C.3.i. EXERCISE. Prove Lemma C.1.4.

C.3.ii. EXERCISE. Verify that a model structure on $\mathcal{M}$, if it exists, is uniquely determined by any of the following data:

(i) The cofibrations and weak equivalences.
(ii) The fibrations and weak equivalences.
(iii) The cofibrations and fibrations.[5]

C.3.iii. EXERCISE.

(i) Prove that if $- \otimes - \colon \mathcal{V} \times \mathcal{M} \to \mathcal{N}$ is a left Quillen bifunctor and $V \in \mathcal{V}$ is cofibrant then $\mathcal{V} \otimes - \colon \mathcal{M} \to \mathcal{N}$ is a left Quillen functor.
(ii) Conclude that the second conditions of Definitions C.3.9 and C.3.10 are unnecessary if $\mathbb{1} \in \mathcal{V}$ is cofibrant.

C.3.iv. EXERCISE. In a locally small category $\mathcal{M}$ with products and coproducts the hom bifunctor is part of a two-variable adjunction:

$$- * - \colon \mathcal{S}et \times \mathcal{M} \to \mathcal{M}, \quad \{-,-\} \colon \mathcal{S}et^{\mathrm{op}} \times \mathcal{M} \to \mathcal{M}, \quad \mathsf{Hom} \colon \mathcal{M}^{\mathrm{op}} \times \mathcal{M} \to \mathcal{S}et.$$

Equipping $\mathcal{S}et$ with the model structure whose weak equivalences are all maps, whose cofibrations are monomorphisms, and whose fibrations are epimorphisms, prove that

(i) $\mathcal{S}et$ is a cartesian monoidal model category.
(ii) Any model category $\mathcal{M}$ is a $\mathcal{S}et$-model category.

C.4. Reedy categories and canonical presentations

In this section we describe a particular structure borne by certain diagram categories $\mathcal{A}$ first exploited by Reedy to prove homotopical results about the category of $\mathcal{A}$-indexed diagrams [**71**]. Our primary examples—$\mathbf{\Delta}$, inverse categories, their opposites, and products of these—are all (strict) *Reedy categories* as defined by Kan, so we confine our attention to this special case. However, we mention for the interested reader, that this theory has been usefully extended by Berger and Moerdijk in such a way as to encompass certain similar categories in which objects are permitted to have non-identity automorphisms [**8**]. Our presentation follows [**78**].

C.4.1. DEFINITION. A **Reedy structure** on a small category $\mathcal{A}$ consists of a **degree function**[6]

$$\deg \colon \operatorname{obj} \mathcal{A} \to \omega$$

[5]By a more delicate observation of Joyal [**47**, E.1.10] a model structure is also uniquely determined by

(iv) The cofibrations and fibrant objects.
(v) The fibrations and cofibrant objects.

[6]The degree function can take values in a different ordinal with no substantial effect on the mathematics.

together with a pair of wide subcategories $\overrightarrow{\mathcal{A}}$ and $\overleftarrow{\mathcal{A}}$ of **degree-increasing** and **degree-decreasing** arrows respectively so that

(i) For each non-identity morphism in $\overrightarrow{\mathcal{A}}$, the degree of its domain is strictly less than the degree of its codomain, and for each non-identity morphism in $\overleftarrow{\mathcal{A}}$, the degree of its domain is strictly greater than the degree of its domain.

(ii) Every $f \in \operatorname{arr}\mathcal{A}$ may be uniquely factored as

$$\begin{tikzcd} \bullet \arrow[rr, "f"] \arrow[dr, "\overleftarrow{\mathcal{A}}\ni\overleftarrow{f}"'] & & \bullet \\ & \bullet \arrow[ur, "\overrightarrow{f}\in\overrightarrow{\mathcal{A}}"'] & \end{tikzcd} \tag{C.4.2}$$

Axiom (i) implies that $\overrightarrow{\mathcal{A}} \cap \overleftarrow{\mathcal{A}} = \operatorname{obj}(\mathcal{A})$, while both conditions together imply that $\mathcal{A}$ contains no non-identity automorphisms; see Exercise C.4.i.

C.4.3. Example. Any inverse category $\mathcal{I}$ as in Definition C.1.15 is a Reedy category, with $\overleftarrow{\mathcal{I}} = \mathcal{I}$ and $\overrightarrow{\mathcal{I}} = \operatorname{obj}\mathcal{I}$. Conversely, any Reedy category $\mathcal{A}$ with $\mathcal{A} = \overleftarrow{\mathcal{A}}$ is an inverse category.

C.4.4. Example. The category $\mathbf{\Delta}_+$ is a Reedy categories with $\overrightarrow{\mathbf{\Delta}}_+$ the monomorphisms and $\overleftarrow{\mathbf{\Delta}}_+$ the epimorphisms. Here it's convenient to take advantage of the order isomorphism $\mathbb{1} + \omega = \omega$ to define $\deg[n] \coloneqq n$. The subcategories $\mathbf{\Delta}$, $\mathbf{\Delta}_\top$ and $\mathbf{\Delta}_\perp$ all inherit analogous Reedy category structures.

C.4.5. Remark. If $\mathcal{A}$ is a Reedy category, then so is $\mathcal{A}^{\mathrm{op}}$: its Reedy structure has the same degree function but has the degree-increasing and degree-decreasing arrows interchanged. In particular, the Reedy structures of Example C.4.3 dualize to define **direct categories**.

C.4.6. Remark. If $\mathcal{A}$ and $\mathcal{B}$ are Reedy categories, so is $\mathcal{A} \times \mathcal{B}$, with $\deg(a, b) \coloneqq \deg(a) + \deg(b)$. See Exercise C.4.ii.

We refer to the unique factorization (C.4.2) as the **Reedy factorization** of the map f and the degree of the object $\operatorname{cod}\overleftarrow{f} = \operatorname{dom}\overrightarrow{f}$ as the **degree** of f. Our next aim is to show that:

(i) It is the minimal degree of an object through which f factors.

(ii) The only factorization of f with this degree is the Reedy factorization.

To prove these assertions, consider the category $\mathcal{F}act\, f$ whose objects are factorizations $a \xrightarrow{g} c \xrightarrow{h} b$ of f and whose morphisms $h \cdot g \to h' \cdot g'$ are maps $k \colon c \to c'$ so that the triangles

$$\begin{tikzcd} & c \arrow[dr, "h"] \arrow[dd, "k"] & \\ a \arrow[ur, "g"] \arrow[dr, "g'"'] & & b \\ & c' \arrow[ur, "h'"'] & \end{tikzcd}$$

commute. Write $\mathcal{F}act_n f \subset \mathcal{F}act\, f$ for the subcategory of factorizations through an object of degree at most n.

C.4.7. Lemma. *The category $\mathcal{F}act\, f$ is connected, and each subcategory $\mathcal{F}act_n f$ is either empty or connected. The minimal n with $\mathcal{F}act_n f$ non-empty is the degree of f, and $\mathcal{F}act_{\deg(f)} f \cong \mathbb{1}$.*

Proof. Consider $h \cdot g \in \mathcal{F}act\, f$ and their Reedy factorizations:

$$\begin{tikzcd}
\bullet \arrow[rr, "g"] \arrow[dr, "\overleftarrow{g}"'] & & \bullet \arrow[rr, "h"] \arrow[dr, "\overleftarrow{h}"] & & \bullet \\
& \bullet \arrow[ur, "\overrightarrow{g}"] \arrow[rr, "k=\overleftarrow{h}\cdot\overrightarrow{g}"'] \arrow[dr, "\overleftarrow{k}"'] & & \bullet \arrow[ur, "\overrightarrow{h}"'] & \\
& & \bullet \arrow[ur, "\overrightarrow{k}"'] & &
\end{tikzcd} \tag{C.4.8}$$

In this way, we define a zig-zag of morphisms in $\mathcal{F}act\, f$ connecting $h \cdot g$ to $\overrightarrow{h\overrightarrow{k}} \cdot \overleftarrow{\overleftarrow{k}g}$, which by axiom (ii) must be the Reedy factorization. This shows that $\mathcal{F}act\, f$ is connected.

Moreover, axiom (i) implies that the degree of $\mathrm{cod}(g) = \mathrm{dom}(h)$ is at least the degree of f. In particular, if $h \cdot g \in \mathcal{F}act_n f$, each of the factorizations in (C.4.8) is as well, proving that $\mathcal{F}act_n f$ is connected if it is non-empty. This diagram also shows that each non-empty category $\mathcal{F}act_n f$ contains the Reedy factorization. Hence, the minimal such n is the degree of f.

Finally, if the degree of $\mathrm{cod}(g) = \mathrm{dom}(h)$ equals the degree of f, then $\overrightarrow{g}$ and $\overleftarrow{h}$ must be identities, from which we deduce that $g \in \overleftarrow{\mathcal{A}}$ and $h \in \overrightarrow{\mathcal{A}}$: i.e., that $h \cdot g$ is the Reedy factorization. Hence $\mathcal{F}act_{\deg(f)} f \cong \mathbb{1}$ is the terminal category as claimed. $\square$

Lemma C.4.7 will be used to establish a "cellular decomposition" for the hom bifunctor $\mathcal{A} \in \mathcal{S}et^{\mathcal{A}^{\mathrm{op}} \times \mathcal{A}}$. That is, we shall use the Reedy structure to present the bifunctor $\mathcal{A}$ as a cell complex in the sense of Definition C.2.4: a sequential composite of pushouts of coproducts of basic "cells" that have a particular form. Lemma C.4.7 implies that the subset of arrows of degree at most n assembles into a sub*functor* of the hom-bifunctor.

C.4.9. Definition (n-skeleton of the hom bifunctor). For any Reedy category $\mathcal{A}$, the n-**skeleton** is the subfunctor

$$\mathrm{sk}_n \mathcal{A} \hookrightarrow \mathcal{A} \in \mathcal{S}et^{\mathcal{A}^{\mathrm{op}} \times \mathcal{A}}$$

of arrows of degree at most n.

There are obvious inclusions $\mathrm{sk}_{n-1} \mathcal{A} \hookrightarrow \mathrm{sk}_n \mathcal{A}$. The colimit of the sequence

$$\varnothing \hookrightarrow \mathrm{sk}_0 \mathcal{A} \hookrightarrow \cdots \hookrightarrow \mathrm{sk}_{n-1} \mathcal{A} \hookrightarrow \mathrm{sk}_n \mathcal{A} \hookrightarrow \cdots \hookrightarrow \mathrm{colim} \cong \mathcal{A}$$

is the hom bifunctor $\mathcal{A}$. The morphisms of degree n first appear in $\mathrm{sk}_n \mathcal{A}$. It remains to express each inclusion $\mathrm{sk}_{n-1} \mathcal{A} \hookrightarrow \mathrm{sk}_n \mathcal{A}$ as a pushout of a coproduct of basic "cells" which our next task is to describe.

The external (pointwise) product defines a bifunctor $\mathcal{S}et^{\mathcal{A}} \times \mathcal{S}et^{\mathcal{A}^{\mathrm{op}}} \xrightarrow{-\Box-} \mathcal{S}et^{\mathcal{A}^{\mathrm{op}} \times \mathcal{A}}$. For any $a \in \mathcal{A}$, there is a natural "composition" map $\circ \colon \mathcal{A}_a \Box \mathcal{A}^a \to \mathcal{A}$ whose domain is the external product of the contravariant $\mathcal{A}^a$ and covariant $\mathcal{A}_a$ representables. By Lemma C.4.7, the composite of any pair of maps that factor through an object a of degree n lies in $\mathrm{sk}_n \mathcal{A}$. Our next task is to describe the subfunctor of the domain of the map

$$\coprod_{\deg(a)=n} \mathcal{A}_a \Box \mathcal{A}^a \xrightarrow{\circ} \mathrm{sk}_n \mathcal{A}$$

that factors through $\mathrm{sk}_{n-1} \mathcal{A} \hookrightarrow \mathrm{sk}_n \mathcal{A}$, for which we require some new notation.

C.4.10. Definition (boundaries of representable functors). If $a \in \mathcal{A}$ has degree n, write

$$\begin{aligned} \partial\mathcal{A}_a &:= \mathrm{sk}_{n-1}\mathcal{A}_a && \in \mathcal{S}et^{\mathcal{A}} && \text{and} \\ \partial\mathcal{A}^a &:= \mathrm{sk}_{n-1}\mathcal{A}^a && \in \mathcal{S}et^{\mathcal{A}^{\mathrm{op}}}. \end{aligned}$$

By Lemma C.4.7, $\partial\mathcal{A}_a \hookrightarrow \mathcal{A}_a$ is the subfunctor of arrows in $\mathcal{A}$ with domain a that do *not* lie in $\overrightarrow{\mathcal{A}}$, while $\partial\mathcal{A}^a \hookrightarrow \mathcal{A}^a$ is the subfunctor of arrows with codomain a that do *not* lie in $\overleftarrow{\mathcal{A}}$.

In particular, the exterior Leibniz product

$$\mathcal{A}_a\square\partial\mathcal{A}^a \cup \partial\mathcal{A}_a\square\mathcal{A}^a \xhookrightarrow{(\partial\mathcal{A}_a\hookrightarrow\mathcal{A}_a)\widehat{\square}(\partial\mathcal{A}^a\hookrightarrow\mathcal{A}^a)} \mathcal{A}_a\square\mathcal{A}^a \tag{C.4.11}$$

defines the subfunctor of pairs of morphisms $h \cdot g$ with $\mathrm{dom}\, h = \mathrm{cod}\, g = a$ in which at least one of the morphisms g and h has degree less than the degree of a.

C.4.12. Proposition. *The square*

$$\begin{tikzcd} \coprod\limits_{\deg(a)=n} \partial\mathcal{A}_a\square\mathcal{A}^a \cup \mathcal{A}_a\square\partial\mathcal{A}^a \arrow[r, hook] \arrow[d, "\circ"'] & \coprod\limits_{\deg(a)=n} \mathcal{A}_a\square\mathcal{A}^a \arrow[d, "\circ"] \\ \mathrm{sk}_{n-1}\mathcal{A} \arrow[r, hook] & \mathrm{sk}_n\mathcal{A} \end{tikzcd} \tag{C.4.13}$$

is both a pullback and a pushout in $\mathcal{S}et^{\mathcal{A}^{\mathrm{op}}\times\mathcal{A}}$.

The fact that (C.4.13) is a pullback is used to facilitate the proof that it is also a pushout.

Proof. An element of the pullback consists of $f \in \mathrm{sk}_{n-1}\mathcal{A}$ together with a factorization $f = h\cdot g$ through an object a of degree n. If both h and g have degree n, then Lemma C.4.7 tells us that $h \cdot g$ is a Reedy factorization, contradicting the fact that f has degree at most $n-1$. So we must have either $h \in \partial\mathcal{A}_a$ or $g \in \partial\mathcal{A}^a$, which tells us that the map from the upper left corner of (C.4.13) surjects onto the pullback. Because the top-horizontal map is monic, the comparison is therefore an isomorphism; i.e., (C.4.13) is a pullback square.

To see that it is a pushout, it suffices now to show that the right-hand vertical is one-to-one on the complement of $\mathrm{sk}_{n-1}\mathcal{A} \hookrightarrow \mathrm{sk}_n\mathcal{A}$. This follows from Lemma C.4.7, which argued that any morphism of degree n has a unique factorization through an object of that degree: namely its Reedy factorization. □

As a corollary of Proposition C.4.12, the two-sided representable $\mathcal{A}$ has a canonical presentation as a cell complex.

C.4.14. Theorem. *The inclusion* $\emptyset \hookrightarrow \mathcal{A}$ *has a canonical presentation as a cell complex:*

$$\begin{tikzcd} & & \coprod\limits_{\deg(a)=n} \partial\mathcal{A}_a\square\mathcal{A}^a \cup \mathcal{A}_a\square\partial\mathcal{A}^a \arrow[r, hook] \arrow[d, "\circ"'] \arrow[dr, phantom, very near end, "\ulcorner"] & \coprod\limits_{\deg(a)=n} \mathcal{A}_a\square\mathcal{A}^a \arrow[d, "\circ"] & \\ \emptyset \arrow[r, hook] & \mathrm{sk}_0\mathcal{A} \arrow[r, dotted] & \mathrm{sk}_{n-1}\mathcal{A} \arrow[r, hook] & \mathrm{sk}_n\mathcal{A} \arrow[r, dotted] & \mathrm{colim}_n\,\mathrm{sk}_n\mathcal{A} \cong \mathcal{A} \end{tikzcd}$$

i.e., a composite of pushouts of coproducts of cells defined as exterior Leibniz products

$$(\partial\mathcal{A}_a \hookrightarrow \mathcal{A}_a)\,\widehat{\square}\,(\partial\mathcal{A}^a \hookrightarrow \mathcal{A}^a),$$

where the cell for each $a \in \mathcal{A}$ of degree n is attached at stage n. □

C.4.15. Remark. One meaning of "canonical" should be "functorial." Indeed, a morphism of Reedy categories—a functor preserving degree and the subcategories of degree-increasing and degree-decreasing maps—induces a morphism of generalized cell complexes: given a morphism $\mathcal{A} \to \mathcal{A}'$ of Reedy categories, there is a natural transformation in $\mathcal{S}et^{\mathcal{A}^{\mathrm{op}} \times \mathcal{A}}$ between the generalized cell complex presentation for $\mathcal{A}$ and the restriction of the generalized cell complex presentation for $\mathcal{A}'$.

As a corollary of Theorem C.4.14, any morphism $f \in \mathcal{M}^{\mathcal{A}}$ is itself a generalized cell complex: the cellular decomposition of $\mathcal{A}$ is translated into a cellular decomposition for f by taking weighted colimits. Taking weighted limits instead transforms the cellular decomposition of $\mathcal{A}$ into a "generalized Postnikov presentation" for f as the limit of a countable tower of pullbacks of ends of a particular form. This sort of result is exemplary of the slogan of **[78]** that "it's all in the weights." Before proving this corollary, let us introduce notation for the maps appearing as the generalized cells.

C.4.16. Definition (latching and matching objects). Let $a \in \mathcal{A}$. The **latching** and **matching objects** of diagram $X \in \mathcal{M}^{\mathcal{A}}$ are defined to be the colimits and limits, respectively, weighted by the boundary representables of appropriate variance:

$$L^a X := \mathrm{colim}_{\partial \mathcal{A}^a} X \qquad M^a X := \lim_{\partial \mathcal{A}_a} X.$$

The boundary inclusions $\partial \mathcal{A}^a \hookrightarrow \mathcal{A}^a$ and $\partial \mathcal{A}_a \hookrightarrow \mathcal{A}_a$ induce the **latching** and **matching maps** $L^a X \to X^a$ and $X^a \to M^a X$, on account of the isomorphisms $\mathrm{colim}_{\mathcal{A}^a} X \cong X^a \cong \lim_{\mathcal{A}_a} X$ of Definition A.6.3(i).

C.4.17. Definition (relative latching and matching maps). The **relative latching** and **relative matching maps** of a natural transformation $f \colon X \to Y \in \mathcal{M}^{\mathcal{A}}$ are defined to be the Leibniz weighted colimits and limits

$$\widehat{\ell}^a f := \widehat{\mathrm{colim}}_{\partial \mathcal{A}^a \hookrightarrow \mathcal{A}^a} f \qquad \widehat{m}^a f := \widehat{\lim}_{\partial \mathcal{A}_a \hookrightarrow \mathcal{A}_a} f,$$

i.e., by the pullbacks and pushouts:

$$\begin{tikzcd}
L^a X \arrow[r] \arrow[d, "L^a f"'] & X^a \arrow[d] \arrow[ddr, bend left, "f^a"] \\
L^a Y \arrow[r] \arrow[drr, bend right] & \ell^a f \arrow[dr, dashed, "\widehat{\ell}^a f"] \arrow[ul, phantom, "\ulcorner", very near start] \\
& & Y^a
\end{tikzcd}
\qquad
\begin{tikzcd}
X^a \arrow[dr, dashed, "\widehat{m}^a f"] \arrow[drr, bend left] \arrow[ddr, bend right, "f^a"'] \\
& m^a f \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & M^a X \arrow[d, "M^a f"] \\
& Y^a \arrow[r] & M^a Y
\end{tikzcd}$$

of the maps $L^a f := \mathrm{colim}_{\partial \mathcal{A}^a} f$ and $M^a f := \lim_{\partial \mathcal{A}_a} f$.

C.4.18. Notation. For any diagram $X \in \mathcal{M}^{\mathcal{A}}$ let

$$\mathrm{sk}_n X := \mathrm{colim}_{\mathrm{sk}_n \mathcal{A}} X \qquad \text{and} \qquad \mathrm{cosk}_n X := \lim_{\mathrm{sk}_n \mathcal{A}} X$$

denote the results of applying the weighted colimit and weighted limit bifunctors

$$\mathrm{colim}_{-} - \colon \mathcal{S}et^{\mathcal{A}^{\mathrm{op}} \times \mathcal{A}} \times \mathcal{M}^{\mathcal{A}} \to \mathcal{M}^{\mathcal{A}} \qquad \text{and} \qquad \lim_{-} - \colon (\mathcal{S}et^{\mathcal{A}^{\mathrm{op}} \times \mathcal{A}})^{\mathrm{op}} \times \mathcal{M}^{\mathcal{A}} \to \mathcal{M}^{\mathcal{A}}$$

to the diagram X with weight $\mathrm{sk}_n \mathcal{A}$.

C.4.19. Corollary. *Let $\mathcal{A}$ be a Reedy category and let $\mathcal{M}$ be bicomplete. Any morphism $f\colon X \to Y \in \mathcal{M}^{\mathcal{A}}$ is a cell complex*

$$X \to X \underset{\mathrm{sk}_0 X}{\cup} \mathrm{sk}_0\, Y \to \cdots \to X \underset{\mathrm{sk}_{n-1} X}{\cup} \mathrm{sk}_{n-1}\, Y \to X \underset{\mathrm{sk}_n X}{\cup} \mathrm{sk}_n\, Y \to \cdots \to \mathrm{colim} \cong Y$$

with the cells

$$(\partial\mathcal{A}_a \hookrightarrow \mathcal{A}_a)\,\widehat{\circledast}\,\widehat{\ell}^a f \tag{C.4.20}$$

indexed by objects a of degree n attached at stage n, and also a "Postnikov tower"

$$X \cong \lim \to \cdots \to \mathrm{cosk}_n X \underset{\mathrm{cosk}_n Y}{\times} Y \to \mathrm{cosk}_{n-1} X \underset{\mathrm{cosk}_{n-1} Y}{\times} Y \to \cdots \to \mathrm{cosk}_0 X \underset{\mathrm{cosk}_0 Y}{\times} Y \to Y$$

whose n-th layer is the product of the maps

$$\widehat{\{\partial\mathcal{A}^a \hookrightarrow \mathcal{A}^a, \widehat{m}^a f\}} \tag{C.4.21}$$

indexed by the objects a of degree n.

Proof. These dual results follow immediately by applying the weighted colimit and weighted limit bifunctors

$$\mathrm{colim}_{-}-\colon \mathcal{S}et^{\mathcal{A}^{\mathrm{op}}\times\mathcal{A}} \times \mathcal{M}^{\mathcal{A}} \to \mathcal{M}^{\mathcal{A}} \qquad \text{and} \qquad \lim_{-}-\colon (\mathcal{S}et^{\mathcal{A}^{\mathrm{op}}\times\mathcal{A}})^{\mathrm{op}} \times \mathcal{M}^{\mathcal{A}} \to \mathcal{M}^{\mathcal{A}}$$

to the cell complex presentations of Theorem C.4.14; recall from Definition A.6.3(ii) that both bifunctors are cocontinuous in the weight.

To see that the generalized cell complex presentation for f has the asserted form, note that for any diagram $X \in \mathcal{M}^{\mathcal{A}}$ and weight defined by an exterior product of $U \in \mathcal{S}et^{\mathcal{A}}$ and $V \in \mathcal{S}et^{\mathcal{A}^{\mathrm{op}}}$, there is a natural isomorphism

$$\mathrm{colim}_{U\square V} X \cong U * \mathrm{colim}_V X,$$

which extends to a natural isomorphism between Leibniz products (Proposition C.2.9(i)).

By the coYoneda lemma, $f \cong \mathrm{colim}_{\mathcal{A}} f \cong \widehat{\mathrm{colim}}_{\varnothing\hookrightarrow\mathcal{A}} f$. By cocontinuity, the Leibniz weighted colimit functor $\widehat{\mathrm{colim}}_{-} f$ preserves generalized cell structures (Proposition C.2.9(vi)). It follows that f admits a canonical presentation as a cell complex with cells

$$\begin{aligned}\widehat{\mathrm{colim}}_{(\partial\mathcal{A}_a\hookrightarrow\mathcal{A}_a)\widehat{\square}(\partial\mathcal{A}^a\hookrightarrow\mathcal{A}^a)} f &\cong (\partial\mathcal{A}_a \hookrightarrow \mathcal{A}_a)\,\widehat{\circledast}\,\widehat{\mathrm{colim}}_{\partial\mathcal{A}^a\hookrightarrow\mathcal{A}^a} f\\ &\cong (\partial\mathcal{A}_a \hookrightarrow \mathcal{A}_a)\,\widehat{\circledast}\,\widehat{\ell}^a f.\end{aligned}$$

□

This presentation is most familiar for the Reedy category $\mathbf{\Delta}^{\mathrm{op}}$, in which case it's conventional to use lower subscripts to designate the contravariant indexing. Here we write $\mathbf{\Delta}^n$ for the standard n-simplex $\mathbf{\Delta}[n]$ and $\partial\mathbf{\Delta}^n$ for its boundary to be consistent with the notation of Definition C.4.10

C.4.22. Example. A simplicial object Y taking values in any cocomplete category admits a skeletal filtration

$$\varnothing \to \mathrm{sk}_0\, Y \to \cdots \to \mathrm{sk}_{n-1}\, Y \to \mathrm{sk}_n\, Y \to \cdots \to Y$$

in which the step from stage $n-1$ to stage n is given by a pushout

$$\begin{tikzcd} \mathbf{\Delta}^n * L_n Y \cup \partial\mathbf{\Delta}^n * Y_n \arrow[r] \arrow[d] & \mathbf{\Delta}^n * Y_n \arrow[d] \\ \mathrm{sk}_{n-1}\, Y \arrow[r] \arrow[ur, phantom, "\ulcorner" very near end] & \mathrm{sk}_n\, Y \end{tikzcd}$$

where $L_nY \to Y_n$ is the object of "degenerate n-simplices."

Considering the Yoneda embedding as a simplicial object $\mathbf{\Delta} \in (\mathcal{S}et^{\mathbf{\Delta}})^{\mathbf{\Delta}^{\mathrm{op}}}$, this specializes to the "canonical cell complex presentation" of the hom bifunctor of Theorem C.4.14

$$\begin{tikzcd} & & \mathbf{\Delta}^n \times \partial\mathbf{\Delta}_n \cup \partial\mathbf{\Delta}^n \times \mathbf{\Delta}_n \arrow[r, hook] \arrow[d] & \mathbf{\Delta}^n \times \mathbf{\Delta}_n \arrow[d] \\ \varnothing \arrow[r, hook] & \cdots \arrow[r, hook] & \mathrm{sk}_{n-1}\,\mathbf{\Delta} \arrow[r, hook] \arrow[ur, phantom, "\ulcorner" very near end] & \mathrm{sk}_n\,\mathbf{\Delta} \arrow[r, hook] & \cdots \arrow[r, hook] & \mathbf{\Delta} \end{tikzcd}$$

In summary, Corollary C.4.19 tells us that we may express a generic natural transformation between diagrams of shape $\mathcal{A}$ as

(i) a cell complex whose cells are Leibniz tensors built from boundary inclusions of covariant representables and relative latching maps,

(ii) and dually as a Postnikov tower whose layers are Leibniz cotensors built from boundary inclusions of contravariant representables and relative matching maps.

This explains the importance of these maps to Reedy category theory, as we shall discover in the next section.

Exercises.

C.4.i. Exercise. Show that any isomorphism in a (strict) Reedy category is an identity.

C.4.ii. Exercise. Show that the product of two Reedy categories is a Reedy category, with the degree of an object defined to be the sum of the degrees.

C.5. The Reedy model structure

Our aim in this section is to explain how any weak factorization system on $\mathcal{M}$ gives rise to a **Reedy weak factorization system** on $\mathcal{M}^{\mathcal{A}}$. We then prove an inductive result that allows us to prove that the Reedy weak factorization systems associated to a model structure on $\mathcal{M}$ define the **Reedy model structure** on $\mathcal{M}^{\mathcal{A}}$. Finally, we prove that the weighted limit and weighted colimit bifunctors define Quillen bifunctors, as a consequence of a more general algebraic result, and discuss the consequences of this result for the theory of homotopy limits and homotopy colimits indexed by strict Reedy categories.

This work requires one preliminary: a discussion of how the skeleta and coskeleta introduced in the previous section feature in the inductive definition of Reedy-shaped diagrams. For a Reedy category $\mathcal{A}$, write

$$\mathcal{A}_{\le 0} \subset \mathcal{A}_{\le 1} \subset \cdots \subset \mathcal{A}_{\le n-1} \subset \mathcal{A}_{\le n} \subset \cdots \subset \mathcal{A}$$

for the full subcategories of objects with degree at most the ordinal appearing in the subscript. These categories give us a new way to understand the skeleton and coskeleton functors introduced in Notation C.4.18.

C.5.1. Lemma. *For any bicomplete category $\mathcal{M}$, restriction and left and right Kan extension define an adjoint triple of functors*

$$\begin{tikzcd} \mathcal{M}^{\mathcal{A}} \arrow[r, "\mathrm{res}_n" description] & \mathcal{M}^{\mathcal{A}_{\le n}} \arrow[l, bend right, "\mathrm{lan}_n"', "\perp"] \arrow[l, bend left, "\mathrm{ran}_n", "\perp"'] \end{tikzcd}$$

with induced comonad $\mathrm{sk}_n := \mathrm{lan}_n \circ \mathrm{res}_n$ *and monad* $\mathrm{cosk}_n := \mathrm{ran}_n \circ \mathrm{res}_n$ *naturally isomorphic to the functors defined by weighted colimit and weighted limit*

$$\mathrm{lan}_n \,\mathrm{res}_n(-) \cong \mathrm{colim}_{\mathrm{sk}_n \mathcal{A}} - \qquad \textit{and} \qquad \mathrm{ran}_n \,\mathrm{res}_n(-) \cong \lim_{\mathrm{sk}_n \mathcal{A}} -.$$

PROOF. Exercise C.5.i. □

Since $\mathrm{sk}_n \mathcal{A} \hookrightarrow \mathcal{A}$ is fully faithful, the functors $\mathrm{lan}_n \dashv \mathrm{res}_n \dashv \mathrm{ran}_n$ of Lemma C.5.1 define a **fully faithful adjoint triple**. For example:

C.5.2. DEFINITION. Specializing the notation above, write $\mathbf{\Delta}_{\leq n} \subset \mathbf{\Delta}$ for the full subcategory of the simplex category of 1.1.1 spanned by the ordinals $[0], \ldots, [n]$. Restriction and left and right Kan extension define adjunctions

$$\begin{tikzcd} \mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}} \arrow[r, "\mathrm{res}_n" description] & \mathcal{S}et^{\mathbf{\Delta}_{\leq n}^{\mathrm{op}}} \arrow[l, bend right, "\mathrm{lan}_n"', "\perp"] \arrow[l, bend left, "\mathrm{ran}_n", "\perp"'] \end{tikzcd}$$

inducing an idempotent comonad $\mathrm{sk}_n := \mathrm{lan}_n \circ \mathrm{res}_n$ and an idempotent monad $\mathrm{cosk}_n := \mathrm{ran}_n \circ \mathrm{res}_n$ on $s\mathcal{S}et$ that are adjoint $\mathrm{sk}_n \dashv \mathrm{cosk}_n$. The counit and unit of this comonad and monad define canonical maps

$$\mathrm{sk}_n X \overset{\epsilon}{\rightarrowtail} X \overset{\eta}{\twoheadrightarrow} \mathrm{cosk}_n X$$

relating a simplicial set X with its n-**skeleton** and n-**coskeleton**. We say X is n-**skeletal** or n-**coskeletal** if the former or latter of these maps, respectively, is an isomorphism.

The canonical map from the n-skeleton of a simplicial set to its n-coskeleton can be defined more generally:

C.5.3. LEMMA. *For any fully faithful inclusion* $\mathcal{B} \hookrightarrow \mathcal{A}$ *and bicomplete category* $\mathcal{M}$, *consider the associated adjoint triple:*

$$\begin{tikzcd} \mathcal{M}^{\mathcal{A}} \arrow[r, "\mathrm{res}" description] & \mathcal{M}^{\mathcal{B}} \arrow[l, bend right, "\mathrm{lan}"', "\perp"] \arrow[l, bend left, "\mathrm{ran}", "\perp"'] \end{tikzcd}$$

(i) The functors $\mathrm{lan}, \mathrm{ran} \colon \mathcal{M}^{\mathcal{B}} \rightrightarrows \mathcal{M}^{\mathcal{A}}$ *are fully faithful; that is, the unit of* $\mathrm{lan} \dashv \mathrm{res}$ *and the counit of* $\mathrm{res} \dashv \mathrm{ran}$ *are isomorphisms.*

(ii) The common composite in the commutative square below defines a canonical natural transformation

$$\begin{tikzcd} \mathrm{lan} \arrow[r, Rightarrow, "\eta \circ \mathrm{lan}"] \arrow[d, Rightarrow, "\mathrm{lan} \circ \epsilon^{-1}"'] \arrow[dr, Rightarrow, dashed, "\tau"] & \mathrm{ran}\,\mathrm{res}\,\mathrm{lan} \arrow[d, Rightarrow, "\mathrm{ran} \circ \eta^{-1}"] \\ \mathrm{lan}\,\mathrm{res}\,\mathrm{ran} \arrow[r, Rightarrow, "\epsilon \circ \mathrm{ran}"'] & \mathrm{ran} \end{tikzcd}$$

While we find the commutative square in (ii) amusing, since the inclusion $\mathcal{B} \hookrightarrow \mathcal{A}$ is fully faithful, that τ can be defined more simply using the universal properties of left and right Kan extensions as the initial and terminal functors that extend a given diagram.

PROOF. It is well-known that a right adjoint functor is fully faithful if and only if the counit is an isomorphism and that the counit of a pointwise right Kan extension along a fully faithful functor is an isomorphism; for proof, specialize the results of Lemma 9.4.3 and Proposition 9.4.4 to the ∞-cosmos $\mathcal{C}at$. These statements and their duals prove (i).

In (ii), τ is defined to be the adjoint transpose of $\eta^{-1}\colon$ res lan $\Rightarrow$ id under res $\dashv$ ran and also to be the adjoint transpose of $\epsilon^{-1}\colon$ id $\Rightarrow$ res ran under lan $\dashv$ res. To see that these definitions agree, observe that the former asserts that the composite of the right two morphisms below is the unique right inverse of the left morphism, while the latter asserts that the composite of the left two morphisms below is the unique left inverse of the right morphism:

$$\mathrm{id} \underset{\cong}{\overset{\eta}{\Longrightarrow}} \mathrm{res}\,\mathrm{lan} \overset{\mathrm{res}\,\tau}{\Longrightarrow} \mathrm{res}\,\mathrm{ran} \underset{\cong}{\overset{\epsilon}{\Longrightarrow}} \mathrm{id}$$

In other words, both definitions assert exactly that the displayed triple composite is the identity. □

In this way, we obtain a natural transformation $\tau_n\colon \mathrm{sk}_n \Rightarrow \mathrm{cosk}_n$ between the comonad and monad introduced in Lemma C.5.1. These structures allow us to inductively define Reedy diagrams:

C.5.4. Proposition (inductive definition of diagrams).

(i) A diagram $X \in \mathcal{M}^{\mathcal{A}_{\leq n-1}}$ together with a family of factorizations

$$\begin{tikzcd} \mathrm{sk}_{n-1} X^a \arrow[rr, "\tau_{n-1}X^a"] \arrow[dr, dashed, "i^a"'] & & \mathrm{cosk}_{n-1} X^a \\ & X^a \arrow[ur, dashed, "p^a"'] & \end{tikzcd}$$

for each object $a \in \mathcal{A}$ of degree n uniquely determines a diagram $X \in \mathcal{M}^{\mathcal{A}_{\leq n}}$ whose restriction to degree $n-1$ coincides with the original diagram.

(ii) A natural transformation $\phi\colon X \to Y \in \mathcal{M}^{\mathcal{A}_{\leq n-1}}$ together with a family of factorizations

$$\begin{tikzcd} \mathrm{sk}_{n-1} X^a \arrow[r, dashed, "i^a"'] \arrow[rr, bend left, "\tau_{n-1}X^a"] \arrow[d, "\mathrm{sk}_{n-1}\phi^a"'] & X^a \arrow[r, dashed, "p^a"'] \arrow[d, dashed, "\phi^a"] & \mathrm{cosk}_{n-1} X^a \arrow[d, "\mathrm{cosk}_{n-1}\phi^a"] \\ \mathrm{sk}_{n-1} Y^a \arrow[r, dashed, "i^a"] \arrow[rr, bend right, "\tau_{n-1}Y^a"'] & Y^a \arrow[r, dashed, "p^a"] & \mathrm{cosk}_{n-1} Y^a \end{tikzcd}$$

for each object $a \in \mathcal{A}$ of degree n uniquely determines a natural transformation $\phi\colon X \to Y \in \mathcal{M}^{\mathcal{A}_{\leq n}}$ whose restriction to degree n coincides with the original natural transformation.

Proof. For (i), it remains to define the action of X on non-identity morphisms whose domain or codomain has degree n. The Reedy factorization of any such morphism $f\colon a \to a'$ is through an object b of degree less than n. By composing the maps in the upper-right or lower-left square, there exist unique dotted-arrow maps making the following diagram commute

$$\begin{tikzcd}
\mathrm{sk}_{n-1} X^a \arrow[r, "i^a"] \arrow[d, "\mathrm{sk}_{n-1} X^{\overleftarrow{f}}"'] & X^a \arrow[r, "p^a"] \arrow[d, dashed, "X^{\overleftarrow{f}}"] & \mathrm{cosk}_{n-1} X^a \arrow[d, "\mathrm{cosk}_{n-1} X^{\overleftarrow{f}}"] \\
\mathrm{sk}_{n-1} X^b \arrow[r, equal, "i^b"] \arrow[d, "\mathrm{sk}_{n-1} X^{\overrightarrow{f}}"'] & X^b \arrow[r, equal, "p^b"] \arrow[d, dashed, "X^{\overrightarrow{f}}"] & \mathrm{cosk}_{n-1} X^b \arrow[d, "\mathrm{cosk}_{n-1} X^{\overrightarrow{f}}"] \\
\mathrm{sk}_{n-1} X^{a'} \arrow[r, "i^{a'}"'] & X^{a'} \arrow[r, "p^{a'}"'] & \mathrm{cosk}_{n-1} X^{a'}
\end{tikzcd}$$

The functoriality of this definition, in a pair of composable maps (f, g) follows from connectedness of the category $\mathcal{F}act_{n-1}(gf)$.

For (ii), apply (i) to the $\mathcal{A}_{\leq n-1}$-shaped diagram $a \mapsto \phi^a$ valued in $\mathcal{M}^{\mathbb{2}}$. $\square$

Now we turn our attention to the main subject of this section. Let $\mathcal{M}$ be a category with a weak factorization system $(\mathcal{L}, \mathcal{R})$ and let $\mathcal{A}$ be a strict Reedy category.

C.5.5. Definition. The **Reedy weak factorization system $(\mathcal{L}[\mathcal{A}], \mathcal{R}[\mathcal{A}])$** on $\mathcal{M}^{\mathcal{A}}$ defined relative to the weak factorization system $(\mathcal{L}, \mathcal{R})$ on $\mathcal{M}$ has:

- as left class $\mathcal{L}[\mathcal{A}]$ those maps $f \colon X \to Y \in \mathcal{M}^{\mathcal{A}}$ whose relative latching maps $\widehat{\ell}^a f \colon \ell^a f \to Y^a \in \mathcal{M}$ are in $\mathcal{L}$, and
- as right class $\mathcal{R}[\mathcal{A}]$ those maps $f \colon X \to Y \in \mathcal{M}^{\mathcal{A}}$ whose relative matching maps $\widehat{m}^a f \colon X^a \to m^a f \in \mathcal{M}$ are in $\mathcal{R}$.

We say a map $f \colon X \to Y \in \mathcal{M}^{\mathcal{A}}$ is **Reedy in $\mathcal{L}$** or **Reedy in $\mathcal{R}$** if its relative latching or relative matching maps are in $\mathcal{L}$ or $\mathcal{R}$, respectively.

The following pair of lemmas, imply that these two classes indeed define a weak factorization system on the category of Reedy diagrams in $\mathcal{M}$.

C.5.6. Lemma. *The maps $i \in \mathcal{L}[\mathcal{A}]$ have the left lifting property with respect to the maps $p \in \mathcal{R}[\mathcal{A}]$.*

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] & K \arrow[d, "p"] \\
B \arrow[r] \arrow[ur, dashed] & L
\end{tikzcd}$$

Proof. By Corollary C.4.19, to show that $i \boxslash p$ for any pair of morphisms $i, p \in \mathcal{M}^{\mathcal{A}}$, it suffices to solve the lifting problems below-left

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d, "(\partial\mathcal{A}_a \hookrightarrow \mathcal{A}_a) \widehat{\circledast} \widehat{\ell}^a i"'] & \bullet \arrow[d, "p"] \\
\bullet \arrow[r] \arrow[ur, dashed] & \bullet
\end{tikzcd}
\quad \leftrightsquigarrow \quad
\begin{tikzcd}
\bullet \arrow[r] \arrow[d, "\widehat{\ell}^a i"'] & \bullet \arrow[d, "\widehat{\lim}_{\partial\mathcal{A}_a \hookrightarrow \mathcal{A}_a} p \cong \widehat{m}^a p"] \\
\bullet \arrow[r] \arrow[ur, dashed] & \bullet
\end{tikzcd}$$

in $\mathcal{M}^{\mathcal{A}}$ for each $a \in \mathcal{A}$. By adjunction, it suffices to solve the transposed lifting problem in $\mathcal{M}$ above-right. If $i \in \mathcal{L}[\mathcal{A}]$ and $p \in \mathcal{R}[\mathcal{A}]$, then by definition $\widehat{\ell}^a i \in \mathcal{L}$ and $\widehat{m}^a p \in \mathcal{R}$, so a solution exists. $\square$

C.5.7. Lemma. *Every map $f \colon X \to Y \in \mathcal{M}^{\mathcal{A}}$ can be factored as a map in $\mathcal{L}[\mathcal{A}]$ followed by a map in $\mathcal{R}[\mathcal{A}]$.*

Proof. We define the components of the factorization of $f^a \colon X^a \to Y^a$ inductively in the degree of a. To start, we use the factorization of $(\mathcal{L}, \mathcal{R})$ to factor all components indexed by objects at degree zero. Since the full subcategory $\mathcal{A}_{\leq 0}$ spanned by these objects is discrete, this defines a factorization of the subdiagram $f \in \mathcal{M}^{\mathcal{A}_{\leq 0}}$.

Continuing inductively, suppose we have factored the restriction $f \in \mathcal{M}^{\mathcal{A}_{<n}}$ as

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\ell"'] & & Y & \in \mathcal{M}^{\mathcal{A}_{<n}} \\ & Z \arrow[ur, "r"'] & & \end{tikzcd}$$

with the relative latching maps $\widehat{\ell}^a \ell \in \mathcal{L}$ and $\widehat{m}^a r \in \mathcal{R}$ for all object a of degree less than n. By Proposition C.5.4, to define the attendant factorization of f^a, it suffices to define an object Z^a of $\mathcal{M}$ together with the dotted arrow maps

$$\begin{tikzcd} L^aX \arrow[rr] \arrow[dd] & & L^aZ \arrow[rr] \arrow[dl] & & L^nY \arrow[dd] \\ & X_n \underset{L^aX}{\times} L^aZ \arrow[dr, dashed] & & & \\ X^a \arrow[ur] \arrow[rr, dashed, "\ell^a"'] \arrow[dd] & & Z^a \arrow[rr, dashed, "r^a"] \arrow[dr, dashed] & & Y^a \arrow[dd] \\ & & & M^aZ \underset{M^aY}{\times} Y^a \arrow[ur] \arrow[dl] & \\ M^aX \arrow[rr] & & M^aZ \arrow[rr] & & M^aY \end{tikzcd}$$

We factor the diagonal map from the pushout to the pullback using $(\mathcal{L}, \mathcal{R})$. The diagonal factors become the a-th relative latching map and matching map of the composite morphisms ℓ^a and r^a so-defined, and in particular lie in the classes $\mathcal{L}$ and $\mathcal{R}$, respectively. It follows from the universal properties of the pushout and the pullback that $f^a = r^a \cdot \ell^a$. By Proposition C.5.4 these definitions extend the natural transformations ℓ and r to degree n. □

It follows from Corollary C.4.19 that if the left class of a weak factorization system $(\mathcal{L}, \mathcal{R})$ on $\mathcal{M}$ is cofibrantly or cellularly generated, as in Definition C.2.4, then the left class of the Reedy weak factorization system is too:

C.5.8. Proposition. *If $(\mathcal{L}, \mathcal{R})$ is a weak factorization system on $\mathcal{M}$ that is cellularly or cofibrantly generated by the class of maps $\mathcal{J}$, then the Reedy weak factorization system $(\mathcal{L}[\mathcal{A}], \mathcal{R}[\mathcal{A}])$ on $\mathcal{M}^{\mathcal{A}}$ is cellularly or cofibrantly generated, respectively, by the class*

$$\{(\partial\mathcal{A}_a \hookrightarrow \mathcal{A}_a) \widehat{\circledast} j\}_{a \in \mathcal{A}, j \in \mathcal{J}}.$$

Proof. By Corollary C.4.19, any morphism $f \colon X \to Y \in \mathcal{M}^{\mathcal{A}}$ may be presented as a cell complex built from cells

$$\{(\partial\mathcal{A}_a \hookrightarrow \mathcal{A}_a) \widehat{\circledast} \widehat{\ell}^a f\}_{a \in \mathcal{A}}.$$

If $f \in \mathcal{L}[\mathcal{A}]$, then $\widehat{\ell}^a f \in \mathcal{L}$ for each a, and by hypothesis these relative latching maps may be presented as cell complexes or retracts of cell complexes built from the maps in the generating class $\mathcal{J}$. By

Proposition C.2.9(vi), the Leibniz tensors $(\partial\mathcal{A}_a \hookrightarrow \mathcal{A}_a) \mathbin{\widehat{\circledast}} \widehat{\ell}^a f$ may then be presented as (retracts of) cell complexes built from the Leibniz tensors of the boundary inclusions and the maps in $\mathcal{J}$, exactly as claimed in the statement. □

For example, the monomorphisms of simplicial sets are cellularly generated by the simplex boundary inclusions $\partial\Delta[n] \hookrightarrow \Delta[n]$ for $n \geq 0$.

C.5.9. Lemma. *The Reedy weak factorization system* $(\mathcal{M}[\boldsymbol{\Delta}^{\mathrm{op}}], \mathcal{E}[\boldsymbol{\Delta}^{\mathrm{op}}])$ *on* $s\mathcal{S}et = \mathcal{S}et^{\boldsymbol{\Delta}^{\mathrm{op}}}$ *defined relative to the (monomorphism, epimorphism) weak factorization system on* $\mathcal{S}et$ *coincides with the (monomorphism, trivial fibration) weak factorization system. Consequently, any monomorphism of simplicial sets decomposes canonically as a sequential composite of pushouts of coproducts of the maps* $\partial\Delta[n] \hookrightarrow \Delta[n]$ *for* $n \geq 0$.

Proof. Monomorphisms of sets are cellularly generated by a single map, the inclusion $! \colon \varnothing \hookrightarrow *$. Consequently, by Proposition C.5.8, the Reedy weak factorization system is cellularly generated as well. In this case, the pushout product functor $- \mathbin{\widehat{\circledast}} ! \colon \mathcal{S}et^{\boldsymbol{\Delta}^{\mathrm{op}}} \to \mathcal{S}et^{\boldsymbol{\Delta}^{\mathrm{op}}}$ is the identity, so the set of generating maps are the familiar simplex boundary inclusions $\{\partial\Delta[n] \hookrightarrow \Delta[n]\}_{[n]\in\boldsymbol{\Delta}}$, the right lifting property against which is used in Definition 1.1.24 to characterize the trivial fibrations. □

C.5.10. Proposition.

(i) If $f \colon X \to Y \in \mathcal{M}^{\mathcal{A}}$ *is Reedy in* $\mathcal{L}$*, that is, if the relative latching maps* $\widehat{\ell}^a f$ *are in* $\mathcal{L}$*, then each of the components* $f^a \colon X^a \to Y^a$ *and each of the latching maps* $L^a f \colon L^a X \to L^a Y$ *are also in* $\mathcal{L}$*.*

(ii) If $f \colon X \to Y \in \mathcal{M}^{\mathcal{A}}$ *is Reedy in* $\mathscr{R}$*, that is, if the relative matching maps* $\widehat{m}^a f$ *are in* $\mathcal{L}$*, then each of the components* $f^a \colon X^a \to Y^a$ *and each of the matching maps* $M^a f \colon M^a X \to M^a Y$ *are also in* $\mathscr{R}$*.*

Proof. We prove the first of these dual statements. The maps f^a and $L^a f$ are the Leibniz weighted colimits of f with the maps $\varnothing \hookrightarrow \mathcal{A}^a$ and $\varnothing \hookrightarrow \partial\mathcal{A}^a$ respectively. Evaluating the covariant variable of the cell complex presentation of Theorem C.4.14 at $a \in \mathcal{A}$, we see that $\varnothing \hookrightarrow \mathcal{A}^a$ is a cell complex whose cells have the form

$$((\partial\mathcal{A}_x)^a \hookrightarrow \mathcal{A}_x^a) \mathbin{\widehat{\square}} (\partial\mathcal{A}^x \hookrightarrow \mathcal{A}^x), \tag{C.5.11}$$

indexed by the objects $x \in \mathcal{A}$. In fact, it suffices to consider those objects with $\deg(x) \leq \deg(a)$; when $\deg(x) > \deg(a)$ the inclusion $(\partial\mathcal{A}_x)^a \hookrightarrow \mathcal{A}_x^a$, and hence the cell (C.5.11), is an isomorphism. Similarly, since $\partial\mathcal{A}^a = \mathrm{sk}_{\deg(a)-1}\mathcal{A}^a$, Theorem C.4.14 implies that $\varnothing \hookrightarrow \partial\mathcal{A}^a$ is a cell complex whose cells have the form (C.5.11) with $\deg(x) < \deg(a)$.

By Proposition C.2.9(vi), the maps f^a and $L^a f$ are then cell complexes whose cells, indexed by the objects $x \in \mathcal{A}$ with the degree bounds just discussed, have the form

$$\widehat{\mathrm{colim}}_{((\partial\mathcal{A}_x)^a \hookrightarrow \mathcal{A}_x^a)\widehat{\square}(\partial\mathcal{A}^x \hookrightarrow \mathcal{A}^x)} f \cong ((\partial\mathcal{A}_x)^a \hookrightarrow \mathcal{A}_x^a) \mathbin{\widehat{\circledast}} \widehat{\ell}^x f, \tag{C.5.12}$$

the isomorphism arising from Proposition C.2.9(i). By Lemma C.2.12, the Leibniz tensor of a monomorphism with a map in the left class of a weak factorization system is again in the left class. Thus, since $(\partial\mathcal{A}_x)^a \hookrightarrow \mathcal{A}_x^a)$ is a monomorphism and $\widehat{\ell}^x f$ is in $\mathcal{L}$, these cells, and thus the maps f^a and $L^a f$ are in $\mathcal{L}$ as well. □

Recall from Definition C.3.1 that a **model structure** on a category $\mathcal{M}$ with a class of weak equivalences $\mathcal{W}$ satisfying the 2-of-3 property is given by two classes of maps $\mathcal{C}$ and $\mathcal{F}$ so that $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ define weak factorization systems. To show that the Reedy weak factorization systems on $\mathcal{M}^{\mathcal{A}}$ relative to a model structure on $\mathcal{M}$ define a model structure on $\mathcal{M}^{\mathcal{A}}$ with the weak equivalences defined pointwise, one lemma is needed.

C.5.13. Lemma. *Let $(\mathcal{W}, \mathcal{C}, \mathcal{F})$ define a model structure on $\mathcal{M}$. Then a map $f\colon X \to Y \in \mathcal{M}^{\mathcal{A}}$*

(i) is Reedy in $\mathcal{C} \cap \mathcal{W}$ if and only if f is Reedy in $\mathcal{C}$ and a pointwise weak equivalence, and

(ii) is Reedy in $\mathcal{F} \cap \mathcal{W}$ if and only if f is Reedy in $\mathcal{F}$ and a pointwise weak equivalence.

Proof. We prove the first of these dual statements. If f is Reedy in $\mathcal{C} \cap \mathcal{W}$, then it is obviously Reedy in $\mathcal{C}$, and Proposition C.5.10 implies that its components f^a are also in $\mathcal{C} \cap \mathcal{W}$. Thus f is a pointwise weak equivalence.

For the converse, we make use of the diagram

$$\begin{tikzcd}
L^aX \arrow[r, "L^a f"] \arrow[d] & L^aY \arrow[d] \arrow[ddr, bend left] & \\
X^a \arrow[r] \arrow[drr, bend right, "f^a"'] & \bullet \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "\widehat{\ell}^a f"] & \\
& & Y^a
\end{tikzcd}$$

which relates the maps $L^a f$, $\widehat{\ell}^a f$, and f^a for any $a \in \mathcal{A}$; this is an instance of Proposition C.2.9(v) applied to $(\varnothing \hookrightarrow \partial\mathcal{A}^a \hookrightarrow \mathcal{A}^a) \mathbin{\widehat{\circledast}_{\mathcal{A}}} f$. Suppose that f is Reedy in $\mathcal{C}$ and a pointwise weak equivalence. By Proposition C.5.10, it follows that $L^a f$ is in $\mathcal{C}$. We will show that $L^a f$ is in fact in $\mathcal{C} \cap \mathcal{W}$ and then apply pushout stability of the left class of a weak factorization system and the 2-of-3 property, to conclude that $\widehat{\ell}^a f \in \mathcal{W}$ and hence that f is Reedy in $\mathcal{C} \cap \mathcal{W}$. We argue by induction. If a has degree zero, then $L^a f$ is the identity at the initial object, which is certainly a weak equivalence, and $\widehat{\ell}^a f = f^a$ is in $\mathcal{C} \cap \mathcal{W}$. If a has degree n, we may now assume that $\widehat{\ell}^x f \in \mathcal{C} \cap \mathcal{W}$ for any x with degree less than the degree of a. By the proof of Proposition C.5.10, $L^a f$ may be presented as a cell complex whose cells (C.5.12) are Leibniz tensors of monomorphisms with maps in $\mathcal{C} \cap \mathcal{W}$, and thus lie in $\mathcal{C} \cap \mathcal{W}$. Thus, we conclude that $L^a f \in \mathcal{C} \cap \mathcal{W}$, completing the proof. ☐

Lemmas C.5.6, C.5.7, and C.5.13 assemble to prove:

C.5.14. Theorem (the Reedy model structure). *If $\mathcal{A}$ is a strict Reedy category and $(\mathcal{W}, \mathcal{C}, \mathcal{F})$ define a model structure on $\mathcal{M}$, then the Reedy weak factorization systems $(\mathcal{C} \cap \mathcal{W}[\mathcal{A}], \mathcal{F}[\mathcal{A}])$ and $(\mathcal{C}[\mathcal{A}], \mathcal{F} \cap \mathcal{W}[\mathcal{A}])$ define a model structure on $\mathcal{M}^{\mathcal{A}}$ with pointwise weak equivalences.* ☐

One reason for our interest in the Reedy model structure is it comes equipped with convenient Quillen bifunctors, which arise from the following result, which produces left Leibniz bifunctors in the sense of Definition C.2.10 in the Reedy diagram context.

C.5.15. Theorem. *Let $\mathcal{A}$ be a Reedy category and let $\otimes\colon \mathcal{K} \times \mathcal{L} \to \mathcal{M}$ be a left Leibniz bifunctor with respect to weak factorization systems $(\mathcal{M}, \mathcal{E})$, $(\mathcal{C}, \mathcal{F})$, and $(\mathcal{L}, \mathcal{R})$. Then the functor tensor product*

$$\otimes_{\mathcal{A}}\colon \mathcal{K}^{\mathcal{A}^{\mathrm{op}}} \times \mathcal{L}^{\mathcal{A}} \to \mathcal{M}$$

is left Leibniz with respect to the Reedy weak factorization systems $(\mathcal{M}[\mathcal{A}^{\mathrm{op}}], \mathcal{E}[\mathcal{A}^{\mathrm{op}}])$ and $(\mathcal{C}[\mathcal{A}], \mathcal{F}[\mathcal{A}])$ and $(\mathcal{L}, \mathcal{R})$.

Proof. The reasons for the cocontinuity of the functor tensor product are well-understood. We argue that $\otimes_{\mathcal{A}}$ has the Leibniz property. Corollary C.4.19 asserts that the maps $f \in \mathcal{K}^{\mathcal{A}^{\mathrm{op}}}$ can be built as cell complexes whose cells are Leibniz products

$$(\partial\mathcal{A}^a \hookrightarrow \mathcal{A}^a) \mathbin{\widehat{*}} \widehat{\ell}_a f,$$

and the maps $g \in \mathcal{L}^{\mathcal{A}}$ can be built as cell complexes whose cells are Leibniz products

$$(\partial\mathcal{A}_b \hookrightarrow \mathcal{A}_b) \mathbin{\widehat{*}} \widehat{\ell}^b g.$$

By Proposition C.2.9(vi), $f \mathbin{\widehat{\otimes}_{\mathcal{A}}} g$ is then a cell complex whose cells have the form

$$\begin{aligned}\left((\partial\mathcal{A}^a \hookrightarrow \mathcal{A}^a) \mathbin{\widehat{*}} \widehat{\ell}_a f\right) &\mathbin{\widehat{\otimes}_{\mathcal{A}}} \left((\partial\mathcal{A}_b \hookrightarrow \mathcal{A}_b) \mathbin{\widehat{*}} \widehat{\ell}^b g\right)\\ &\cong \left((\partial\mathcal{A}^a \hookrightarrow \mathcal{A}^a) \mathbin{\widehat{\times}_{\mathcal{A}}} (\partial\mathcal{A}_b \hookrightarrow \mathcal{A}_b)\right) \mathbin{\widehat{*}} (\widehat{\ell}_a f \mathbin{\widehat{\otimes}} \widehat{\ell}^b g)\end{aligned}$$

To say that f is Reedy in $\mathcal{M}$ and g is Reedy in $\mathcal{C}$ means that $\widehat{\ell}_a f \in \mathcal{M}$ and $\widehat{\ell}^b g \in \mathcal{C}$. Since $\otimes$ is left Leibniz, it follows that $\widehat{\ell}_a f \mathbin{\widehat{\otimes}} \widehat{\ell}^b g \in \mathcal{L}$. The Leibniz functor tensor product

$$(\partial\mathcal{A}^b \hookrightarrow \mathcal{A}^b) \mathbin{\widehat{\times}_{\mathcal{A}}} (\partial\mathcal{A}_a \hookrightarrow \mathcal{A}_a)$$

of the maps in $\mathcal{S}et^{\mathcal{A}^{\mathrm{op}}}$ and in $\mathcal{S}et^{\mathcal{A}}$ amounts to the inclusion into the hom-set $\mathcal{A}^a_b = \mathcal{A}(b,a)$ of the subset of morphisms from b to a that factor through an object of degree strictly less than a or strictly less than b; in particular, this map is a monomorphism. Now Lemma C.2.12 applies to the weak factorization system $(\mathcal{L},\mathcal{R})$ on $\mathcal{M}$ to prove that the Leibniz tensor of this monomorphism with $\widehat{\ell}_a f \mathbin{\widehat{\otimes}} \widehat{\ell}^b g$ remains in $\mathcal{L}$, completing the proof. □

Applying Theorem C.5.15 to Lemma C.2.12, with (monomorphism, epimorphism) taken as the default weak factorization system on $\mathcal{S}et$, we conclude:

C.5.16. Corollary. *For any bicomplete category $\mathcal{M}$ with a weak factorization system $(\mathcal{L},\mathcal{R})$ and any strict Reedy category, the weighted colimit and weighted limit*

$$\mathrm{colim}_{-}-\colon \mathcal{S}et^{\mathcal{A}} \times \mathcal{M}^{\mathcal{A}^{\mathrm{op}}} \to \mathcal{M} \qquad\text{and}\qquad \lim_{-}-\colon (\mathcal{S}et^{\mathcal{A}})^{\mathrm{op}} \times \mathcal{M}^{\mathcal{A}} \to \mathcal{M}$$

define left and right Leibniz bifunctors relative to the Reedy weak factorization systems.

In the setting of a model category, a monoidal model category, or a $\mathcal{V}$-model category (which subsumes the previous two cases by taking $\mathcal{V}$ to be $\mathcal{S}et$ or the model category itself), Corollary C.5.16 specializes to the following result, which helps us understand homotopy limits and colimits of diagrams of Reedy shape.

C.5.17. Corollary. *Let $\mathcal{M}$ be a $\mathcal{V}$-model category and let $\mathcal{A}$ be a strict Reedy category. Then for any weight W in $\mathcal{V}^{\mathcal{A}}$ that is Reedy cofibrant*[7]*, the weighted colimit and weighted limit functors*

$$\mathrm{colim}_W -\colon \mathcal{M}^{\mathcal{A}^{\mathrm{op}}} \to \mathcal{M} \qquad\text{and}\qquad \lim_W -\colon \mathcal{M}^{\mathcal{A}} \to \mathcal{M}$$

are respectively left and right Quillen with respect to the Reedy model structure on $\mathcal{M}^{\mathcal{A}}$.

C.5.18. Example (homotopy limits and colimits). Taking the terminal weight 1 in $\mathcal{S}et^{\mathcal{A}}$, the weighted limit reduces to the ordinary limit functor. The functor $1 \in \mathcal{S}et^{\mathcal{A}}$ is Reedy monomorphic just when, for each $a \in \mathcal{A}$, the category of elements for the weight $\partial\mathcal{A}^a$ is either empty or connected. This is the case if and only if $\mathcal{A}$ has **cofibrant constants**, meaning that the constant $\mathcal{A}$-indexed diagram at any

[7]In the case of $\mathcal{V} = \mathcal{S}et$, "Reedy cofibrant" should be read as "Reedy monomorphic."

cofibrant object in any model category is Reedy cofibrant. Thus, we conclude that if $\mathcal{A}$ has cofibrant constants, then the limit functor $\lim\colon \mathcal{M}^{\mathcal{A}} \to \mathcal{M}$ is right Quillen.

Dually, the colimit functor $\mathrm{colim}\colon \mathcal{M}^{\mathcal{A}} \to \mathcal{M}$ is a special case of the weighted colimit functor with the terminal weight $1 \in \mathcal{S}et^{\mathcal{A}^{\mathrm{op}}}$. This is Reedy monomorphic just when each category of elements for the weights $\partial\mathcal{A}_a$ is either empty or connected, which is the case if and only if $\mathcal{A}$ has **fibrant constants**, meaning that the constant $\mathcal{A}$-indexed diagram at any fibrant object in any model category is Reedy fibrant. Thus, we conclude that if $\mathcal{A}$ has fibrant constants, then the colimit functor $\mathrm{colim}\colon \mathcal{M}^{\mathcal{A}} \to \mathcal{M}$ is left Quillen. See [**78**, §9] for more discussion.

The following lemmas can be used to identify Reedy cofibrant weights for use in applications of Corollary C.5.17.

C.5.19. Lemma. *Any bisimplicial set is Reedy cofibrant as an object of $s\mathcal{S}et^{\Delta^{\mathrm{op}}}$.*

Proof. Exercise C.5.ii. □

Reedy cofibrant cosimplicial objects can also be identified, on account of the following lemma.

C.5.20. Lemma.

(i) Let $X\colon \mathbf{\Delta} \to \mathcal{S}et^{\mathrm{C}^{\mathrm{op}}}$ be a cosimplicial object in a presheaf category. If X is unaugmentable, in the sense that the equalizer of the pair of coface maps $\delta^0, \delta^1\colon X^0 \rightrightarrows X^1$ is empty, then the latching maps of X are all monomorphisms.

(ii) If X is an unaugmentable cosimplicial object in a slice category of a presheaf category, then the latching maps of X are all monomorphisms.

Proof. Since latching objects are defined in terms of certain colimits in $\mathcal{S}et^{\mathrm{C}^{\mathrm{op}}}$ computed pointwise in $\mathcal{S}et$, we may reduce this result to the corresponding one for cosimplicial sets $X\colon \mathbf{\Delta} \to \mathcal{S}et$. A simplex in a cosimplicial set is "non-degenerate" if it is not in the image of a monomorphism from $\mathbf{\Delta}$. The nth latching map $L^nX \to X^n$ is a monomorphism just when each expression of an n-simplex x as the image of a non-degenerate simplex z under a monomorphism $\sigma\colon [k] \rightarrowtail [n]$ is unique.

So suppose we have two such representations $x = \sigma \cdot z = \sigma' \cdot z'$. Any monomorphism $\sigma \in \mathbf{\Delta}$ has a left inverse τ, so we see that $z = \tau\sigma' \cdot z'$. The map $\tau\sigma'$ can be factored as an epimorphism followed by a monomorphism. Because z is non-degenerate, this monomorphism must be the identity, so $\tau\sigma'$ is an epimorphism. Repeating this argument with an left inverse τ' for σ' we see that $\tau'\sigma$ is an epimorphism, so z and z' have the same degree and both epimorphisms are identities. This proves that $z = z'$.

If the set of left inverses for a monomorphism uniquely characterized that monomorphism, then we could conclude that σ and σ' must be equal, and hence that such decompositions would be fully unique. This is true for nearly all monomorphisms in $\mathbf{\Delta}$, the only exceptions being the face maps $\delta^0, \delta^1\colon [0] \rightarrowtail [1]$. However, X is assumed to be unaugmentable, so there is no $z \in Z^0$ with $\delta^0 z = \delta^1 \cdot z$, and thus $L^nX \to X^n$ is a monomorphism for all n. This proves the first statement.

For the second statement, we only need to consider slice categories ${}^{A/}\mathcal{S}et^{\mathrm{C}^{\mathrm{op}}}$ under an object, since the slice category $\mathcal{S}et^{\mathrm{C}^{\mathrm{op}}}{}_{/A}$ is equivalent to the category of presheaves on the category of elements of A. The forgetful functor ${}^{A/}\mathcal{S}et^{\mathrm{C}^{\mathrm{op}}} \to \mathcal{S}et^{\mathrm{C}^{\mathrm{op}}}$ creates monomorphisms and connected colimits, which tells us that the latching maps $L^nX \to X^n$ of a complicial object $X^\bullet$ in ${}^{A/}\mathcal{S}et^{\mathrm{C}^{\mathrm{op}}}$ are calculated in $\mathcal{S}et^{\mathrm{C}^{\mathrm{op}}}$ for all $n > 1$. The direct calculation given above proves that these are monomorphisms. If X is unaugmentable, the latching map $(\delta^0, \delta^1)\colon L^1X \cong X^0 + X^0 \to X^1$ is a monomorphism: arguing in

$\mathcal{S}et^{\mathcal{C}^{\mathrm{op}}}$ and then ultimately in $\mathcal{S}et$, it is easy to see that the pulllback of this map along itself is A. If X is unaugmentable, this says also that that the equalizer of $(\delta^0, \delta^1)\colon X^0 \rightrightarrows X^1$ in $\mathcal{S}et^{\mathcal{C}^{\mathrm{op}}}$ is $A \to X^0$, so this map is a monomorphism. This tells us that the 0th latching map is a monomorphism, completing the proof that $X^\bullet$ is a Reedy monomorphism. ☐

C.5.21. Example (geometric realization and totalization). By Lemma C.5.20, the Yoneda embedding defines a Reedy cofibrant weight $\mathbf{\Delta}^\bullet \in s\mathcal{S}et^{\mathbf{\Delta}}$. The weighted colimit and weighted limit functors

$$\mathrm{colim}_{\mathbf{\Delta}^\bullet} -\colon \mathcal{M}^{\mathbf{\Delta}^{\mathrm{op}}} \to \mathcal{M} \qquad \text{and} \qquad \lim_{\mathbf{\Delta}^\bullet} -\colon \mathcal{M}^{\mathbf{\Delta}} \to \mathcal{M}$$

typically go by the names of **geometric realization** and **totalization**. Corollary C.5.17 proves that if $\mathcal{M}$ is a simplicial model category, then these functors are left and right Quillen.

We frequently invoke Lemma C.5.20 in conjunction with the following result to show that various quasi-categories or Kan complexes are equivalent. The geometric idea is as follows: when $X^\bullet \in \mathcal{M}^{\mathbf{\Delta}}$ is a cosimplicial object in $\mathcal{M}$, then for any object $A \in \mathcal{M}$ we may define a simplicial set $\mathcal{M}(X^\bullet, A)$ in which the n-simplices are maps $X^n \to A$ in $\mathcal{M}$. We think of the cosimplicial object $X^\bullet$ as determining the "shape" of the simplices in $\mathcal{M}(X^\bullet, A)$. The following lemma gives us a tool for proving that the simplicial sets constructed via different shapes are equivalent.

C.5.22. Lemma. *Suppose $\mathcal{M}$ is a model category, $\alpha^\bullet\colon X^\bullet \to Y^\bullet \in \mathcal{M}^{\mathbf{\Delta}}$ is a map of cosimplicial objects, and A is a fibrant object in $\mathcal{M}$. On applying the representable functor $\mathcal{M}(-, A)\colon \mathcal{M}^{\mathrm{op}} \to \mathcal{S}et$ we obtain a map of quasi-categories*

$$\mathcal{M}(\alpha^\bullet, A)\colon \mathcal{M}(Y^\bullet, A) \to \mathcal{M}(X^\bullet, A)$$

which is:

(i) a trivial fibration whenever $\alpha^\bullet$ is a Reedy trivial cofibration in $\mathcal{M}^{\mathbf{\Delta}}$, and

(ii) an equivalence of quasi-categories whenever $\alpha^\bullet$ is a pointwise weak equivalence and $X^\bullet$ and $Y^\bullet$ are Reedy cofibrant.

Proof. The first statement is a special case of Theorem C.5.15 as we now explain. In the terminology of Definition C.2.10, Lemma C.2.12 shows that for any category with products

$$\mathcal{M}(-,-)\colon \mathcal{M}^{\mathrm{op}} \times \mathcal{M} \to \mathcal{S}et$$

is a *right Leibniz bifunctor* relative to the (monomorphism, epimorphism) weak factorization system on $\mathcal{S}et$ and any weak factorization system $(\mathcal{L}, \mathscr{R})$ on $\mathcal{M}$. Essentially what this means is that for any map $\ell\colon K \to L$ in $\mathcal{L}$ and any map $r\colon R \to S$ in $\mathscr{R}$ the function

$$\mathcal{M}(L, R) \xrightarrow{(-\circ\ell, r\circ-)} \mathcal{M}(K, R) \underset{\mathcal{M}(K,S)}{\times} \mathcal{M}(L, S)$$

is an epimorphism; this is another encoding of the lifting property $\ell \boxslash r$ of maps in the left class against maps in the right class.

Now since $\mathbf{\Delta}$ is a strict Reedy category, Theorem C.5.15 tells us that that the corresponding bifunctor

$$\mathcal{M}(-,-)\colon (\mathcal{M}^{\mathbf{\Delta}})^{\mathrm{op}} \times \mathcal{M} \to \mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}}$$

is a right Leibniz bifunctor relative to the (trivial cofibration, fibration) weak factorization system on $\mathcal{M}$, the (Reedy trivial cofibration, Reedy trivial fibration) weak factorization system on $\mathcal{M}^{\mathbf{\Delta}}$, and the (Reedy monomorphism, Reedy epimorphism) weak factorization system on $\mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}}$; by Lemma C.5.9, this latter weak factorization system coincides with the familiar (monomorphism, trivial fibration) weak factorization system. In particular, this bifunctor carries a Reedy trivial cofibration $\alpha^\bullet\colon X^\bullet \to Y^\bullet$ and a fibration $!\colon A \to 1$ to a trivial fibration of simplicial sets, exactly as claimed.

The second statement follows from the first by Ken Brown's Lemma C.1.10. $\square$

C.5.23. Remark. Lemma C.5.22 can be understood as a more primative result in abstract homotopy theory, which does not require a full model structure on $\mathcal{M}$.

In Lemma C.5.22 and the results on which its proof replies, the model category $\mathcal{M}$ may be replaced by a bicomplete category which has:

(i) a class of weak equivalences $\mathcal{W}$ satisfying the 2-of-3 property,
(ii) a weak factorization system $(\mathcal{C}, \mathcal{F}_t)$, with the class $\mathcal{F}_t$ contained among the weak equivalences,
(iii) so that the class $\mathcal{C} \cap \mathcal{W}$ is closed under coproduct, pushout, and sequential composition.

The notion of Reedy cofibrant and Reedy trivial cofibration can be defined relative to the classes $\mathcal{C}$ and $\mathcal{C} \cap \mathcal{W}$. Then for any object A which admits extensions along maps in $\mathcal{C} \cap \mathcal{W}$, the functor $\mathcal{M}(-, A)\colon \mathcal{M}^{\mathrm{op}} \to \mathcal{S}et$ carries maps in $\mathcal{C} \cap \mathcal{W}$ to epimorphisms. Since the Reedy trivial cofibrations in $\mathcal{M}^\Delta$ are built as cell complexes from maps in $\mathcal{C} \cap \mathcal{W}$, axiom (iii) and the construction in the proof of Theorem C.5.15 shows that $\mathcal{M}(-, A)\colon (\mathcal{M}^\Delta)^{\mathrm{op}} \to s\mathcal{S}et$ carries Reedy trivial cofibrations to trivial fibrations of simplicial sets. This proves the first statement of Lemma C.5.22.

For the second statement, we revisit the proof of Lemma C.1.10. By factoring the map $(f, \mathrm{id})\colon X^\bullet + Y^\bullet \to Y^\bullet$ using the Reedy weak factorization system built from (ii) we obtain a factorization of the pointwise weak equivalence $\alpha\colon X^\bullet \to Y^\bullet$ as displayed

$$\begin{tikzcd} & Z^\bullet \arrow[dr, two heads, "q", "\sim"'] & \\ X^\bullet \arrow[ur, tail, "u", "\sim"'] \arrow[rr, "f"', "\sim"] & & Y^\bullet \arrow[ul, tail, bend right, "v"', "\sim"] \end{tikzcd}$$

where the maps u and v are Reedy cofibrations and pointwise weak equivalences and v is a section of q. Axioms (i) and (iii) give us the result proven in Lemma C.5.13, that the maps u and v are Reedy trivial cofibrations. Now the 2-of-3 property assumed in (i) is enough to achieve the conclusion in the second statement of Lemma C.5.22, that $- \circ \alpha\colon \mathcal{M}(Y^\bullet, A) \xrightarrow{\sim} \mathcal{M}(X^\bullet, A)$ is an equivalence of quasi-categories.

Exercises.

C.5.i. Exercise. Prove Lemma C.5.1.

C.5.ii. Exercise ([**75**, 14.3.7]). Prove Lemma C.5.19.

C.5.iii. Exercise ([**75**, 14.3.8]). Prove the relative analog of Lemma C.5.20: if X and Y are both unaugmentable cosimplicial objects in a presheaf category, then any pointwise monomorphism $X \to Y$ is also a Reedy monomorphism, that is, its relative latching maps are monic.

Appendix of Concrete Calculations

APPENDIX D

The combinatorics of (marked) simplicial sets

In this appendix we explore the combinatorics of simplicial sets, proving results alluded to in Chapters 1 and 4. Certain of these results, namely those involving isomorphisms in quasi-categories, are more easily proved in the closely related category of "marked" simplicial sets, where the quasi-categories are identified with those marked simplicial sets that are *1-complicial*. Because the corresponding *n-complicial sets* provide one of the families of examples of ∞-cosmoi appearing in Appendix E, we provide full details of the necessary combinatorial results in that more general context.

D.1. Complicial sets

When a quasi-category is regarded as an $(\infty, 1)$-category, its vertices play the role of the objects and its edges represent the morphisms, with the degenerate edge at a vertex representing its identity. The n-simplices then witness n-ary composition relations. When a complicial set is regarded as an (∞, ∞)-category, its n-simplices must play a dual role: both serving as witnesses for lower-dimensional composition relations and representing a priori non-invertible n-dimensional cells in their own right. To disambiguate between these two interpretations, certain positively-dimensional simplices in a complicial set are *marked* as "thin," indicating that they should be interpreted as "equivalences" witnessing a weak composition relation between their boundary faces. Thus the ambient category in which complicial sets are defined is not the category of ordinary simplicial sets but a closely related category of *marked simplicial sets*[1] that we now introduce.

D.1.1. Definition (marked simplicial sets). A **marked simplicial set** is a simplicial set with a designated subset of **marked** or **thin** positive-dimensional simplices that includes all degenerate simplicies. A map of marked simplicial sets is a simplicial map that preserves marked simplices.

D.1.2. Definition (minimal and maximal marking). The category $s\mathcal{S}et^+$ of marked simplicial sets is equipped with an evident forgetful functor to $s\mathcal{S}et$ admitting both left and right adjoints:

$$\begin{tikzcd}
s\mathcal{S}et^+ \arrow[r, "U" description] \arrow[r, bend right=60, "\mathrm{core}"'] & s\mathcal{S}et \arrow[l, bend right=60, "(-)^\flat"'] \arrow[l, bend left=25, "(-)^\sharp" description]
\end{tikzcd}$$

The left adjoint $(-)^\flat$ defines the **minimal marking** of a simplicial set, in which only the degeneracies are marked, while the right adjoint $(-)^\sharp$ defines the **maximal marking**, with all simplices marked. This

[1]In the original sources **[97, 98]**, marked simplicial sets are called *stratified simplicial sets*. To avoid confusing with the increasingly prominent unrelated notion of stratified spaces, we have elected to change the name. Lurie **[59]** uses the term *marked simplicial sets* for a special case of the more general notion we presently introduce.

functor has a further right adjoint, which takes a marked simplicial set to its **core**, the simplicial set with the same vertices comprised of those marked simplices all of whose faces are also marked.

On various occasions, it is convenient to identify $s\mathcal{S}et$ with either of the fully faithful embeddings into $s\mathcal{S}et^+$ just introduced. Unless otherwise specified, the default convention is to identify simplicial sets with their minimal markings. In particular, with this convention, we are free to regard the standard simplices and their subspaces as minimally marked simplicial sets.

To succinctly introduce other marked simplicial sets, the following terminology will be convenient:

D.1.3. Definition. An inclusion $U \hookrightarrow V$ of marked simplicial sets is:

- **regular**, denoted $U \hookrightarrow_r V$, if thin simplices in U are created in V and
- **entire**, denoted $U \hookrightarrow_e V$, if the map is an isomorphism (or more commonly the identity) on underlying simplicial sets, in which case the only difference between U and V is that V has more marked simplices.

For example, for each $n \geq 1$, we define the **marked simplex** $\mathbf{\Delta}[n]_t$ to be the entire extension of minimally marked simplex $\mathbf{\Delta}[n]$ that also marks the top non-degenerate simplex.

Let $t\mathbf{\Delta} \hookrightarrow s\mathcal{S}et^+$ denote the full subcategory spanned by the minimally marked simplices $\mathbf{\Delta}[n]$, for $n \geq 0$, together with the marked simplices $\mathbf{\Delta}[n]_t$, for $n \geq 1$. It can be built from the simplex category $\mathbf{\Delta}$ of Notation 1.1.1 by:

- adjoining objects $[n]_t$ for $n \geq 1$,
- adjoining maps $\phi\colon [n] \to [n]_t$ for $n \geq 1$ and $\zeta^i\colon [n+1]_t \to [n]$ for $n \geq 0$ and $0 \leq i \leq n$, and
- imposing relations $\zeta^i\phi = \sigma^i$ and $\sigma^i\zeta^{j+1} = \sigma^j\zeta^i$ for $i \leq j$.[2]

For a marked simplicial set X, the maps $\mathbf{\Delta}[n] \to X$ and $\mathbf{\Delta}[n]_t \to X$ respectively parametrize n-simplices in X and marked n-simplices in X. This defines a canonical embedding $s\mathcal{S}et^+ \hookrightarrow \mathcal{S}et^{t\mathbf{\Delta}^{\mathrm{op}}}$, which is easily seen to be fully faithful. Moreover:

D.1.4. Proposition. *There is a reflective fully faithful embedding*

$$\begin{tikzcd} s\mathcal{S}et^+ \arrow[r, hook] \arrow[r, phantom, "\perp"] & \mathcal{S}et^{t\mathbf{\Delta}^{\mathrm{op}}} \arrow[l, bend right, dashed] \end{tikzcd}$$

whose

(i) essential image consists of those presheaves F for which the component maps $- \circ \phi\colon F_{[n]_t} \to F_{[n]}$ are monomorphisms, and

(ii) left adjoint is constructed by replacing the set $F_{[n]_t}$ with the image of the map $- \circ \phi\colon F_{[n]_t} \to F_{[n]}$.

Consequently, $s\mathcal{S}et^+$ is a locally finitely presentable category, and in particular is complete and cocomplete, with limits constructed pointwise as presheaves in $\mathcal{S}et$ and with colimits constructed by applying the reflector to the pointwise colimit of presheaves.

Put in more elementary terms, limits and colimits of marked simplicial sets are created by the underlying simplicial set functor $U\colon s\mathcal{S}et^+ \to s\mathcal{S}et$. A simplex in the limit is marked if and only if *each* of its components, defined by composing with the legs of the limit cone, are marked simplices. A simplex in a colimit is marked if *any* of its lifts along any leg of the colimit cone are marked simplices.

[2] Viktoriya Ozornova and Martina Rovelli pointed out to us that this last family of relations was omitted from the original source [**92**] but should have been included. A corrected definition appears in [**65**, 1.1].

The reflection in (ii) is a sort of "propositional truncation," remembering which simplices should be marked while forgetting the data that indicates why.

PROOF. The right action of the operators in $t\mathbf{\Delta}$ on a presheaf $F \in \mathcal{S}et^{t\mathbf{\Delta}^{\mathrm{op}}}$ gives the sets of elements of F the structure of a marked simplicial set with the exception of one condition: namely that the marked n-simplices form a subset of the n-simplices. This explains the condition appearing in (i) and the construction appearing in (ii). It follows that marked simplicial sets are the category of models for a finite limit sketch, and hence form a locally finitely presentable category. Any reflective full subcategory of a complete and cocomplete category inherits limits in the manner constructed in the statement; see eg [**76**, 4.5.15]. $\square$

Lemma C.5.9 extends to marked simplicial sets as follows:

D.1.5. LEMMA. *The momomorphisms in $s\mathcal{S}et^+$ are cellularly generated by*

$$\{\partial\Delta[n] \hookrightarrow_r \Delta[n]\}_{n\geq 0} \cup \{\Delta[n] \hookrightarrow_e \Delta[n]_t\}_{n\geq 1}.$$

PROOF. Exercise D.1.i. $\square$

A marked simplicial set is a simplicial set with enough structure to talk about composition of simplices in all dimensions. A complicial set is a marked simplicial set in which composites exist and in which thin witnesses to composition compose to define thin simplices, an associativity condition that will ultimately imply that thin simplices are equivalences in a sense that will be made explicit in §D.7. The following form of the definition of a (nee. *weak*) *complicial set*, due to Verity [**98**], modifies an earlier equivalent presentation due to Street [**88**]. Verity's modification focuses on a particular set of *k-admissible n-simplices*, which are thin n-simplices that exhibit their kth face as a composite of their $(k+1)$th and $(k-1)$th faces, in the case where $0 < k < n$. In the case $k = 0$ or $k = n$, a k-admissible n-simplex witnesses an equivalence between the first or last pair of faces, respectively.

D.1.6. DEFINITION (k-admissible n-simplex). For $n \geq 1$ and $0 \leq k \leq n$, the **k-admissible n-simplex** $\Delta^k[n]$ is the entire superset of the standard n-simplex with certain additional faces marked thin: a non-degenerate m-simplex in $\Delta^k[n]$ is thin if and only if it contains all of the vertices $\{k-1, k, k+1\} \cap [n]$. Thin faces include in particular:

- the top dimensional n-simplex
- all codimension-one faces except for the $(k-1)$th, kth, and $(k+1)$th
- the 2-simplex spanned by $[k-1, k, k+1]$ when $0 \leq k \leq n$ or the edge spanned by $[k-1, k, k+1] \cap [n]$ when $k = 0$ or $k = n$.

When drawing pictures of marked simplicial sets, we use the symbol "$\simeq$" to decorate marked simplices and "$\sim$" to decorate marked edges. Our diagrams will also adopt a convention for the direction of the cells inhabiting an unmarked n-simplex. Following the combinatorics introduced by Street in his "Algebra of oriented simplexes" [**88**], we regard an n-simplex as an n-cell from the pasted composite of its odd-numbered faces to the pasted composite of its even-numbered faces.[3] Note this is compatible with the convention already in use for depicting a 1-simplex in a simplicial set as an arrow from its 1st face (the 0th vertex) to its 0th face (the 1st vertex).

D.1.7. EXAMPLE (admissible simplices in low dimensions).

[3]More exactly, Street defines a strict n-category O_n, which he refers to as the n-th *oriental* to be the free strict n-category generated by an n-simplex and its faces, with the orientation conventions described here.

(i) For $n = 1$, both admissible simplices $\Delta^0[1]$ and $\Delta^1[1]$ equal the thin 1-simplex $\Delta[1]_t = \mathbf{\Delta}[1]^\sharp$. A map $\mathbf{\Delta}[1]^\sharp \to A$ is interpreted as defining an equivalence between the two vertices in its image.

(ii) For $n = 2$, the admissible simplex $\Delta^1[2] = \Delta[2]_t$, the thin 2-simplex. A map $\Delta^1[2] \to A$ is interpreted as specifying that the image of the $\{02\}$-edge is a composite of the images of the $\{01\}$- and $\{12\}$-edges.

By contrast, $\Delta^0[2]$ and $\Delta^2[2]$ each have a marked edge, as well as a marked 2-simplex as indicated by the diagrams:

$$\Delta^0[2] \coloneqq \begin{tikzcd} & 1 \arrow[dr] \arrow[d, phantom, "\wr"] & \\ 0 \arrow[ur, "\sim"] \arrow[rr] & {} & 2 \end{tikzcd} \qquad \Delta^1[2] \coloneqq \begin{tikzcd} & 1 \arrow[dr] \arrow[d, phantom, "\wr"] & \\ 0 \arrow[ur] \arrow[rr] & {} & 2 \end{tikzcd} \qquad \Delta^2[2] \coloneqq \begin{tikzcd} & 1 \arrow[dr, "\sim"] \arrow[d, phantom, "\wr"] & \\ 0 \arrow[ur] \arrow[rr] & {} & 2 \end{tikzcd}$$

A map $\Delta^0[2] \to A$ witnesses a homotopy between the image of the $\{12\}$ edge and the image of the $\{02\}$ edge.

(iii) For $n = 3$, the admissible simplices $\Delta^1[3]$ and $\Delta^2[3]$ have their 3rd and 0th faces marked, respectively, as well as the top dimensional 3-simplex, with no other non-degenerate faces marked. We choose to draw admissible 3-simplices in such a way that allows us to see all of their codimension-one faces:

$$\Delta^2[3] \coloneqq \begin{tikzcd} & 1 \arrow[r, "g"] \arrow[d, phantom, "\Uparrow\alpha"] & 2 \arrow[dr, "k"] \arrow[d, phantom, "\beta\Uparrow"] & \\ 0 \arrow[ur, "f"] \arrow[urr, "h"'] \arrow[rrr, "\ell"'] & {} & {} & 3 \end{tikzcd} \simeq \begin{tikzcd} & 1 \arrow[r, "g"] \arrow[drr, "kg"'] \arrow[d, phantom, "\Uparrow\alpha*\beta"] & 2 \arrow[dr, "k"] \arrow[d, phantom, "\wr"] & \\ 0 \arrow[ur, "f"] \arrow[rrr, "\ell"'] & {} & {} & 3 \end{tikzcd}$$

Here we've labelled faces in such a way that allows us to better describe the interpretation of a map $\Delta^2[3] \to A$. Its 0th face, which is itself an admissible simplex $\Delta^1[2]$, witnesses that the edge $\{13\}$ is a composite of the edges $\{12\}$ and $\{23\}$. Note that because the 0th face is thin, its 1st edge is interpreted as a composite kg of g and k, which is needed so that the boundary of the 2-cell appearing in the 2nd face agrees with the boundary of the pasted composite of β and α. On account of this boundary condition and the thin 3-simplex, we interpret the 2nd face as the pasted composite of the 1st and 3rd faces depicted on the right.

The admissible simplex $\Delta^0[3]$ has both its 2nd and 3rd faces marked, as well as the top dimensional 3-simplex, and the edge $\{01\}$. Dually, $\Delta^3[3]$ has its 0th and 1st faces marked, as well as the top dimensional 3-simplex, and the edge $\{23\}$.

$$\Delta^0[3] \coloneqq \begin{tikzcd} & 1 \arrow[r, "fe^{-1}"] \arrow[d, phantom, "\wr"] & 2 \arrow[dr, "g"] \arrow[d, phantom, "\alpha\Uparrow"] & \\ 0 \arrow[ur, "e", "\sim"'] \arrow[urr, "f"'] \arrow[rrr, "h"'] & {} & {} & 3 \end{tikzcd} \simeq \begin{tikzcd} & 1 \arrow[r, "fe^{-1}"] \arrow[drr, "he^{-1}"'] \arrow[d, phantom, "\wr"] & 2 \arrow[dr, "g"] \arrow[d, phantom, "\alpha e^{-1}\Uparrow"] & \\ 0 \arrow[ur, "e", "\sim"'] \arrow[rrr, "h"'] & {} & {} & 3 \end{tikzcd}$$

A map $\Delta^0[3] \to A$ is interpreted as witnessing a homotopy between the pair of non-thin 2-simplices occupying the 0th and 1st faces, respectively.

D.1.8. Remark (the odd dual). Recall that the **opposite** of a simplicial set X is the simplicial set obtained by reindexing along the involution $(-)^{\mathrm{op}} \colon \mathbf{\Delta} \to \mathbf{\Delta}$ that reverses the ordering in each ordinal.

This operation may be extended to marked simplicial sets in a natural way: marking an n-simplex in X^{op} just when the corresponding n-simplex in X is marked. Note, however, that under Street's interpretation of an n-simplex as encoding an n-dimensional morphism from the composite of its odd $(n-1)$-dimensional faces to the composite of its even $(n-1)$-dimensional faces, this operation doesn't simply "reverse the direction of all the cells" in a marked simplicial set. Rather, it reverses the direction of all the simplices in the *odd* dimensional cells, while preserving the direction in all of the even dimensional cells. This is because the odd-dimensional simplices have an even number of codimension-one faces, so reversing their labeling exchanges the groups of "odd" and "even" faces, while even-dimensional simplices have an odd number of codimension-one faces, so the "odd" and "even" groupings are preserved. Thus, we refer to the vertex reordering construction as defining the **odd dual** of a marked simplicial set.

D.1.9. Definition. A **complicial set** is a marked simplicial set that admits extensions along the **elementary anodyne extensions**, which are given by the following two sets of maps:

(i) The **complicial horn extensions**

$$\Lambda^k[n] \hookrightarrow_r \Delta^k[n] \quad \text{for} \quad n \geq 1,\ 0 \leq k \leq n$$

are regular inclusions of k-**admissible** n-**horns**. An inner admissible n-horn parametrizes "admissible composition" of a pair of $(n-1)$-simplices. The extension defines a composite $(n-1)$-simplex together with a thin n-simplex witness.

$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d, hook] & A \\ \Delta^k[n] \arrow[ur, dashed] \end{tikzcd} \tag{D.1.10}$$

(ii) The **complicial thinness extensions**

$$\Delta^k[n]' \hookrightarrow_e \Delta^k[n]'' \quad \text{for} \quad n \geq 2,\ 0 \leq k \leq n,$$

are entire inclusions of two entire supersets of $\Delta^k[n]$. The marked simplicial set $\Delta^k[n]'$ is obtained from $\Delta^k[n]$ by also marking the $(k-1)$th and $(k+1)$th faces, while $\Delta^k[n]''$ has all codimension-one faces marked. This extension problem

$$\begin{tikzcd} \Delta^k[n]' \arrow[r] \arrow[d, hook] & A \\ \Delta^k[n]'' \arrow[ur, dashed] \end{tikzcd} \tag{D.1.11}$$

demands that whenever the composable pair of simplices in an admissible horn are thin, then so is any composite.

D.1.12. Example (complicial horn extensions). For $\Lambda^2[4] \hookrightarrow_r \Delta^2[4]$ the non-thin codimension-one faces in the horn define the two 3-simplices with a common face displayed on the left, while their composite is a 3-simplex as displayed on the right.

$$
\begin{tikzcd}
 & 1 \arrow[d] \arrow[ddr] & \\
 & 2 \arrow[dd] \arrow[dr] & \\
0 \arrow[uur] \arrow[ur] \arrow[rr] \arrow[dr] & & 4 \\
 & 3 \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & 1 \arrow[dd] \arrow[dr] & \\
0 \arrow[ur] \arrow[rr] \arrow[dr] & & 4 \\
 & 3 \arrow[ur] &
\end{tikzcd}
$$

It makes sense to interpret the right hand simplex, the 2nd face of the 2-admissible 4-simplex, as a composite of the 3rd and 1st faces because the 2-simplex

$$
\begin{tikzcd}
 & 2 \arrow[dr] \arrow[d, phantom, "\wr"] & \\
1 \arrow[ur] \arrow[rr] & {} & 3
\end{tikzcd}
$$

is thin.

We refer to the maps that are cellularly generated by the elementary anodyne extensions as **marked anodyne extensions**. For instance, by a mild extension of the argument that solves Exercise 1.1.v:

D.1.13. Lemma. *Either injection $\mathbb{2}^\sharp \hookrightarrow \mathbb{I}^\sharp$ of the marked 1-simplex into the maximally marked isomorphism is an marked anodyne extension, as is the injection $\mathbb{1} \hookrightarrow \mathbb{I}^\sharp$.*

Proof. Exercise D.1.iv. □

D.1.14. Definition. A map of marked simplicial sets is a **complicial isofibration** if it has the right lifting property with respect to the elementary anodyne extensions and if its domain and codomain are complicial sets.

By Exercise D.1.v, a marked map between complicial sets is a complicial isofibration if and only if it lifts against the complicial horn extensions. Of course, complicial isofibrations will then enjoy the right lifting property against all marked anodyne extensions. Among the complicial isofibrations are the **trivial fibrations**, defined to be those maps of marked simplicial sets that lift against the monomorphisms, as characterized by Lemma D.1.5.

The original meaning of "complicial sets" referred to a particular variety that we now call *strict*.

D.1.15. Definition. A **strict complicial set** is a marked simplicial set that admits unique extensions along the elementary anodyne extensions (D.1.10) and (D.1.11).

In the manuscript **[97]**, Verity proves that the strict complicial sets are precisely those marked simplicial sets that are Street nerves of strict ω-categories, resolving a conjecture of Street and Roberts. In this manuscript, we will primarily utilize marked simplicial sets to streamline the proofs of results concerning isomorphisms in quasi-categories and equivalences between quasi-categories. We will discuss this topic more explicitly in §D.4 and §D.7 after developing some combinatorial constructions we will require in the interim. We conclude this section with one final definition.

D.1.16. Definition. A **marked homotopy** between a pair of maps $f, g \colon X \rightrightarrows Y$ is given by a map $\alpha \colon X \times \Delta[1]^\sharp \to Y$ that restricts along the endpoint inclusions $X + X \hookrightarrow_r X \times \Delta[1]^\sharp$ to the maps f and g, respectively. In the case where X and Y are minimally marked simplicial sets, a map $X \times \Delta[1] \to Y$

extends to a map $X \times \Delta[1]^\sharp \to Y$ just when for each 0-simplex $x \in X$, the 1-simplex $(x \cdot \sigma^0, \mathrm{id}_{[1]}) \in X \times \Delta[1]^\sharp$ maps to a degenerate and hence marked 1-simplex of Y.[4]

A **marked homotopy equivalence** consists of:

- a pair of marked maps $f\colon X \to Y$ and $g\colon Y \to X$ and
- a pair of marked homotopies $\alpha\colon X \times \Delta[1]^\sharp \to X$ and $\beta\colon Y \times \Delta[1]^\sharp \to Y$ between id_X and gf and fg and id_Y, respectively.

This definition works best when X and Y are complicial sets, in which case marked homotopies can be reserved and composed. Even when this is not the case, we permit ourselves the reverse the direction of the marked homotopies that comprise a marked homotopy equivalence without comment.

D.1.17. Digression (the Verity model structure for complicial sets). The category of marked simplicial sets bears a cartesian closed, cofibrantly generated model structure whose fibrant objects are exactly the complicial sets and whose cofibrations are the monomorphisms [**98**, §6.2-4]. The fibrations and weak equivalences between fibrant objects are precisely the classes of complicial isofibrations and marked homotopy equivalences defined above. In the following sections, we verify many of these properties for the category of fibrant objects directly, leaving only the verification of the actual model structure, which follows from Jeff Smith's theorem, to the literature.

Exercises.

D.1.i. Exercise. Prove Lemma D.1.5.

D.1.ii. Exercise. Prove that a maximally marked simplicial set defines a complicial set if and only if the underlying simplicial set is a Kan complex.

D.1.iii. Exercise. Prove that the underlying simplicial set of any complicial set in which all simplices of dimension greater than 1 are marked is a quasi-category.[5]

D.1.iv. Exercise. Prove Lemma D.1.13.

D.1.v. Exercise. Let $f\colon A \to B$ be any map of marked simplicial sets whose domain A is a complicial set. Prove that f has the (unique) right lifting property against the complicial thinness extensions.

D.2. The join and slice constructions

In this section, we revisit Joyal's join and slice constructions in considerably more detail than given in Definition 4.2.4 and discuss their extension to marked simplicial sets. We prove that Leibniz joins of monomorphisms and various classes of anodyne maps again define monomorphisms of the same type. The combinatorics are slightly easier if we work with *augmented* simplicial sets in place of ordinary simplicial sets, an approach that follows the original definition of the simplicial join by Ehlers and Porter [**34**].

D.2.1. Definition (ordinal sum). The algebraists' skeletal category $\mathbf{\Delta}_+$ of finite ordinals and order preserving maps — with objects $[n] = \{0 \le 1 \le \cdots \le n\}$ and $[-1] = \varnothing$ — supports a strict (non-symmetric) monoidal structure $(\mathbf{\Delta}_+, \oplus, [-1])$ in which $\oplus$ denotes the **ordinal sum** given

- for objects $[n], [m] \in \mathbf{\Delta}_+$ by $[n] \oplus [m] \coloneqq [n+1+m]$,

[4]The cartesian product of marked simplicial sets is described in more detail in Proposition D.3.4.

[5]A converse of sorts to this result will appear in Theorem D.4.13.

- for arrows $\alpha\colon [n] \to [n']$, $\beta\colon [m] \to [m']$ by $\alpha \oplus \beta\colon [n+1+m] \to [n'+1+m']$ defined by
$$\alpha \oplus \beta(i) = \begin{cases} \alpha(i) & \text{if } i \le n, \\ \beta(i-n-1)+n'+1 & \text{otherwise.} \end{cases}$$

By Day convolution [**28**], the join bifunctor $\oplus\colon \mathbf{\Delta}_+ \times \mathbf{\Delta}_+ \to \mathbf{\Delta}_+$ extends to a (non-symmetric) monoidal closed structure
$$(s\mathcal{S}et_+, \star, \Delta[-1], \mathrm{dec}_l, \mathrm{dec}_r)$$
on the category of augmented simplicial sets $s\mathcal{S}et_+ := \mathcal{S}et^{\mathbf{\Delta}_+^{\mathrm{op}}}$.

D.2.2. Definition (join of augmented simplicial sets). The **join** $X \star Y$ of augmented simplicial sets X and Y may be described explicitly as follows:

- it has simplices pairs $(x,y) \in (X \star Y)_{r+1+s}$ with $x \in X_r$, $y \in Y_s$,
- if (x,y) is a simplex of $X \star Y$ with $x \in X_r$ and $y \in Y_s$ and $\alpha\colon [n] \to [r+1+s]$ is a simplicial operator in $\mathbf{\Delta}_+$, then α may be uniquely decomposed as $\alpha = \alpha_1 \oplus \alpha_2$ with $\alpha_1\colon [n_1] \to [r]$ and $\alpha_2\colon [n_2] \to [s]$, and we define $(x,y)\cdot\alpha := (x\cdot\alpha_1, y\cdot\alpha_2)$.

Note by construction that $\Delta[n] \star \Delta[m] \cong \Delta[n+1+m]$, since $[n] \oplus [m] = [n+1+m]$.[6]

D.2.3. Definition (décalage of augmented simplicial sets). The closures dec_l and dec_r, known as the **left and right décalage constructions**, respectively, are defined as the parametrized right adjoints to the join: to fix handedness, we denote the adjoints, for each augmented simplicial set X, by

$$\begin{tikzcd} s\mathcal{S}et_+ \arrow[r, bend left, "X\star-"] \arrow[r, phantom, "\perp"] & s\mathcal{S}et_+ \arrow[l, bend left, "\mathrm{dec}_l(X,-)"] \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} s\mathcal{S}et_+ \arrow[r, bend left, "-\star X"] \arrow[r, phantom, "\perp"] & s\mathcal{S}et_+ \arrow[l, bend left, "\mathrm{dec}_r(X,-)"] \end{tikzcd}$$

Collectively, the bifunctors

$$s\mathcal{S}et_+ \times s\mathcal{S}et_+ \xrightarrow{-\star-} s\mathcal{S}et_+\,, \quad s\mathcal{S}et_+^{\mathrm{op}} \times s\mathcal{S}et_+ \xrightarrow{\mathrm{dec}_l(-,-)}\colon s\mathcal{S}et_+\,, \quad s\mathcal{S}et_+^{\mathrm{op}} \times s\mathcal{S}et_+ \xrightarrow{\mathrm{dec}_r(-,-)} s\mathcal{S}et_+$$

define a two-variable adjunction and as such preserve limits and colimits in each variable separately.

D.2.4. Observation (simplicial sets vs augmented simplicial sets). The evident functor that forgets the augmentation $U\colon s\mathcal{S}et_+ \to s\mathcal{S}et$ admits both left and right adjoints

$$\begin{tikzcd} s\mathcal{S}et_+ \arrow[r, "U" description] & s\mathcal{S}et \arrow[l, bend right, "\pi_0"'] \arrow[l, bend left, "*"] \arrow[l, phantom, shift right=2, "\perp"] \arrow[l, phantom, shift left=2, "\perp"] \end{tikzcd}$$

where the left adjoint augments a simplicial set X with its set of path components $\pi_0 X$, defined by the coequalizer

$$X_1 \overset{\delta^1}{\underset{\delta^0}{\rightrightarrows}} X_0 \twoheadrightarrow \pi_0 X$$

and the right adjoint augments a simplicial set "trivially" by adding a single -1-simplex. The unit of $\pi_0 \dashv U$ and counit of $U \dashv *$ are both isomorphisms; hence either adjoint defines a fully faithful embedding $s\mathcal{S}et \hookrightarrow s\mathcal{S}et_+$.

[6]A general feature of the Day convolution product is that the Yoneda embedding よ$\colon \mathbf{\Delta}_+ \hookrightarrow \mathcal{S}et^{\mathbf{\Delta}_+^{\mathrm{op}}}$ defines a strong monoidal functor.

Any augmented simplicial set is canonically a coproduct of its terminally-augmented "components":

D.2.5. Lemma. *Let X be an augmented simplicial set and let $x \in X_{-1}$.*

(i) The subset comprised of those simplices in any dimension whose -1-simplex face is x forms a terminally augmented simplicial subset of X.
(ii) The disjoint union of these components is isomorphic to X.

Thus, any augmented simplicial set X admits a canonical decomposition into a coproduct of terminally augmented simplicial sets indexed by the set X_{-1}.

Proof. Exercise D.2.i. □

D.2.6. Definition (join of simplicial sets). By convention, the **join** of a pair of simplicial sets is defined to be the underlying simplicial set of the trivially augmented simplicial sets. Thus, the join bifunctor is the composite

$$\begin{tikzcd} s\mathcal{S}et \times s\mathcal{S}et \arrow[r, dashed, "- \star -"] \arrow[d, hook, "* \times *"'] & s\mathcal{S}et \\ s\mathcal{S}et_+ \times s\mathcal{S}et_+ \arrow[r, "- \star -"] & s\mathcal{S}et_+ \arrow[u, "U"'] \end{tikzcd}$$

Explicitly, n-simplices of $X \star Y$ are pairs comprised of a j-simplex of X and a k-simplex of Y where $j + k = n - 1$, where in the case $j = -1$ such a "pair" consists of a single n-simplex of Y and in the case $k = -1$ such a "pair" consists of a single n-simplex of X. This recovers the definition given in 4.2.4.

As observed in Definition 4.2.4, the join of simplicial sets X and Y admits canonical embeddings

$$X \hookrightarrow X \star Y \hookleftarrow Y$$

which can be understood as the maps obtained by applying $X \star -$ or $- \star Y$ respectively to the maps $\Delta[-1] \to Y$ and $\Delta[-1] \to X$ in $s\mathcal{S}et_+$ that pick out the unique -1-simplices in the trivial augmentations.

D.2.7. Lemma.

(i) The join bifunctor $- \star - \colon s\mathcal{S}et \times s\mathcal{S}et \to s\mathcal{S}et$ preserves connected colimits in each variable separately.
(ii) For any simplicial set X, the join functors

$$s\mathcal{S}et \xrightarrow{X \star -} {}^{X/}s\mathcal{S}et \qquad \text{and} \qquad s\mathcal{S}et \xrightarrow{- \star X} {}^{X/}s\mathcal{S}et$$

preserve all colimits.

Proof. In Definition D.2.6, the join of simplicial sets is defined as the composite of three functors, two of which possess right adjoints and hence preserve all colimits. The third functor $* \colon s\mathcal{S}et \to s\mathcal{S}et_+$ does not possess a right adjoint but nevertheless preserves *connected* colimits as is clear from the following definition: an indexing 1-category J is connected just when the colimit of the constant J-indexed diagram valued at the singleton set is a singleton. This proves (i).

Now the forgetful functor ${}^{X/}s\mathcal{S}et \to s\mathcal{S}et$ strictly creates connected colimits [**76**, 3.3.8], so the join functors of (ii) preserve connected colimits. Arbitrary colimits may be built from connected colimits and coproducts, so to prove (ii) it remains only to argue that these functors preserve coproducts. While $X \star (\coprod_i Y_i) \ncong \coprod_i (X \star Y_i)$ if the latter coproduct is interpreted in $s\mathcal{S}et$, it can be directly verified that

$X \star (\coprod_i Y_i)$ is the quotient of $\coprod_i (X \star Y_i)$ modulo the identification of the image of each inclusion $X \hookrightarrow X \star Y_i$ with a single copy of X, which is exactly the construction of the coproduct in the category ${}^{X/}s\mathcal{S}et$. □

D.2.8. Definition (slice of simplicial sets). The categories $s\mathcal{S}et$ and ${}^{X/}s\mathcal{S}et$ are locally presentable (see [**1**, 1.57]), so the cocontinuous functors of Lemma D.2.7(ii) have right adjoints

$$\begin{tikzcd} s\mathcal{S}et \arrow[r, bend left, "X\star-"] \arrow[r, phantom, "\perp"] & {}^{X/}s\mathcal{S}et \arrow[l, bend left, "{}^{-/}-"] \end{tikzcd} \qquad \begin{tikzcd} s\mathcal{S}et \arrow[r, bend left, "-\star X"] \arrow[r, phantom, "\perp"] & {}^{X/}s\mathcal{S}et \arrow[l, bend left, "-_{/-}"] \end{tikzcd}$$

the values of which at $f\colon X \to A$ define Joyal's sliced simplicial sets ${}^{f/}A$ and $A_{/f}$ characterized by the universal properties

$$\left\{ \begin{tikzcd} & X \arrow[dl, hook'] \arrow[dr, "f"] & \\ X \star Y \arrow[rr] & & A \end{tikzcd} \right\} \cong \left\{ Y \longrightarrow {}^{f/}A \right\} \quad \text{and} \quad \left\{ \begin{tikzcd} & X \arrow[dl, hook'] \arrow[dr, "f"] & \\ Y \star X \arrow[rr] & & A \end{tikzcd} \right\} \cong \left\{ Y \longrightarrow A_{/f} \right\}.$$

See Proposition 4.2.5.

We think of the slice ${}^{f/}A$ as being the simplicial set of **cones under the diagram** f and we think of the dual slice $A_{/f}$ as being the simplicial set of **cones over the diagram** f. This terminology will be reconciled with the terminology of Definition 4.2.1 in Proposition D.6.5. We can also recover these sliced simplicial sets from the décalage construction of Definition D.2.3 via Lemma D.2.5:

D.2.9. Lemma. *For any map $f\colon X \to A$ of simplicial sets, the simplicial sets ${}^{f/}A$ and $A_{/f}$ are the terminally augmented components of* $\mathrm{dec}_l(X, A)$ *and* $\mathrm{dec}_r(X, A)$, *respectively, indexed by the -1-simplex $f\colon X \to A$.*

Proof. We identify simplicial sets X and A with their terminally augmented simplicial sets. Recall that $\Delta[-1]$ is the monoidal unit for the join bifunctor on $s\mathcal{S}et_+$. Consequently, by adjunction, maps $\Delta[-1] \to \mathrm{dec}_l(X, A)$ or $\Delta[-1] \to \mathrm{dec}_r(X, A)$ correspond to maps $X \to A$. For another terminally augmented simplicial set Y, transposing across the adjunction of Definition D.2.3 provides a correspondence:

$$\left\{ \begin{tikzcd} & X \star \Delta[-1] \cong X \arrow[dl, hook', "X\star!"'] \arrow[dr, "f"] & \\ X \star Y \arrow[rr] & & A \end{tikzcd} \right\} \cong \left\{ \begin{tikzcd} & \Delta[-1] \arrow[dl, "!"'] \arrow[dr, "f"] & \\ Y \arrow[rr] & & \mathrm{dec}_l(X, A) \end{tikzcd} \right\}$$

which shows that the simplicial subset of $\mathrm{dec}_l(X, A)$ comprised of those simplices whose -1-simplex face is f has the universal property that defines ${}^{f/}A$. The dual argument proves that the simplicial subset of $\mathrm{dec}_r(X, A)$ comprised of those simplices whose -1-simplex face is f has the universal property that defines $A_{/f}$. In other words, these décalages admit the following canonical decompositions as disjoint unions of (terminally augmented) slices:

$$\mathrm{dec}_r(X, A) = \coprod_{f\colon X \to A} A_{/f} \qquad \mathrm{dec}_l(X, A) = \coprod_{f\colon X \to A} {}^{f/}A$$

□

D.2.10. Definition ((left-/right-/inner-)anodyne extensions).

- The set of **horn inclusions** $\Lambda^k[n] \hookrightarrow \Delta[n]$ for $n \geq 1$ and $0 \leq k \leq n$ cellularly generates the **anodyne extensions**.

- The set of **left horn inclusions** $\Lambda^k[n] \hookrightarrow \Delta[n]$ for $n \geq 1$ and $0 \leq k < n$ cellularly generates the **left anodyne extensions**.
- The set of **right horn inclusions** $\Lambda^k[n] \hookrightarrow \Delta[n]$ for $n \geq 1$ and $0 < k \leq n$ cellularly generates the **right anodyne extensions**.
- The set of **inner horn inclusions** $\Lambda^k[n] \hookrightarrow \Delta[n]$ for $n \geq 2$ and $0 < k < n$ cellularly generates the **inner anodyne extensions**.

We refer to the right classes generated by these maps as **left**, **right**, and **inner fibrations**, respectively. By an easy direct calculation:

D.2.11. Lemma. *The Leibniz join of a horn inclusion and a boundary inclusion is isomorphic to a single horn inclusion:*

$$(\Lambda^k[n] \hookrightarrow \Delta[n]) \mathbin{\widehat{\star}} (\partial\Delta[m] \hookrightarrow \Delta[m]) \cong \Lambda^k[n+1+m] \hookrightarrow \Delta[n+1+m]$$
$$(\partial\Delta[n] \hookrightarrow \Delta[n]) \mathbin{\widehat{\star}} (\Lambda^k[m] \hookrightarrow \Delta[m]) \cong \Lambda^{n+k+1}[n+1+m] \hookrightarrow \Delta[n+1+m]$$

Proof. Since the join bifunctor is the Day convolution of the ordinal sum $[n] \oplus [m] = [n+1+m]$, $\Delta[n] \star \Delta[m] \cong \Delta[n+1+m]$. The domain of the first Leibniz tensor is the simplicial set

$$\Lambda^k[n] \star \Delta[m] \underset{\Lambda^k[n]\star\partial\Delta[m]}{\cup} \Delta[n] \star \partial\Delta[m].$$

From the explicit description of the simplices contained in the join given in Definition D.2.2, is plainly a simplicial subset of $\Delta[n+1+m]$. Since $\partial\Delta[m]$ contains all the codimension-one faces of $\Delta[m]$, the $\Delta[n] \star \partial\Delta[m]$ component contains the jth face of $\Delta[n+1+m]$ for each index $j > n$. Similarly, since $\Lambda^k[n]$ contains all codimension-one faces of $\Delta[n]$-except one, the $\Lambda^k[n] \star \Delta[m]$ component contains the ith face of $\Delta[n+1+m]$ for each index $i \leq n$ except $i = k$. Thus, we see that only the kth face and the $n+1+m$-simplex are missing, which allows us to identify the domain of this Leibniz join with the horn $\Lambda^k[n+1+m]$. The dual argument proves the second claimed isomorphism. □

Lemma D.2.11 reveals that the Leibniz join of an inner horn with a boundary inclusion is an inner horn. Consequently:

D.2.12. Corollary. *If $f: X \to A$ is any simplicial map and A is a quasi-category, then ${}^{f/}A$ and $A_{/f}$ are quasi-categories.*

Proof. By Proposition C.2.9(vi) and Lemma C.5.9, the Leibniz join of an inner horn inclusion with a monomorphism gives a map in the class that is cellularly generated by the inner horn inclusions. By Proposition C.2.9(ii), for any quasi-category A and simplicial set X, the augmented simplicial sets $\mathrm{dec}_l(X, A)$ and $\mathrm{dec}_r(X, A)$ then admit fillers for all inner horns, considered as trivially augmented simplicial sets. By Lemma D.2.9, it follows that for any simplicial map $f: X \to A$, the slices ${}^{f/}A$ and $A_{/f}$ are quasi-categories. □

Our next aim is to prove that the slice quasi-categories are equivalent to the quasi-categories of cones introduced in §4.2. As sketched there, this result hinges on a suitable equivalence between the join construction and the so-called "fat join" construction of Definition 4.2.2, which we now extend to augmented simplicial sets. Recall from Lemma D.2.5 that an augmented simplicial set X canonically decomposes into a coproduct $X \cong \sqcup_{i \in X_{-1}} X^i$ of terminally augmented simplicial sets, indexed by the set of -1-simplices.

D.2.13. Definition (fat join and décalage of augmented simplicial sets). For augmented simplicial sets $X \cong \sqcup_{i \in X_{-1}} X^i$ and $Y \cong \sqcup_{j \in Y_{-1}} Y^j$ their **fat join** is constructed by the pushout:

$$\begin{tikzcd} (X \times Y) \sqcup (X \times Y) \arrow[r, "\pi_X \sqcup \pi_Y"] \arrow[d, hook] & (X \times Y_{-1}) \sqcup (X_{-1} \times Y) \arrow[d, hook] \\ X \times \mathbb{2} \times Y \arrow[r] & X \diamond Y \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \qquad \text{i.e.,} \qquad X \diamond Y := \coprod_{(i,j) \in X_{-1} \times Y_{-1}} X^i \diamond Y^j$$

where $X^i \diamond Y^j$ is the terminally augmented simplicial set corresponding to the fat join of Definition 4.2.2. This construction is arranged so that the bifunctor $- \diamond - \colon s\mathcal{S}et_+ \times s\mathcal{S}et_+ \to s\mathcal{S}et_+$ preserves all colimits in each variable, not simply the connected ones preserved by the bifunctor $- \diamond - \colon s\mathcal{S}et \times s\mathcal{S}et \to s\mathcal{S}et$.

Explicitly, the set of n-simplices $(X \diamond Y)_n$ is the quotient of the set $X_n \times \mathbf{\Delta}([n], [1]) \times Y_n$ of n-simplices of $X \times \mathbb{2} \times Y$ modulo the relation that identifies triples

- $(x, 0, y) \sim (x, 0, y')$ where $0 \colon [n] \to [1]$ is the constant operator and y and y' are in the same component of $Y \cong \sqcup_{j \in Y_{-1}} Y^j$ and
- $(x, 1, y) \sim (x', 1, y)$ where $1 \colon [n] \to [1]$ is the constant operator and x and x' are in the same component of $X \cong \sqcup_{i \in X_{-1}} X^i$.

By cocontinuity and the adjoint functor theorems, the fat join bifunctor on $s\mathcal{S}et_+$ has both left and right closures $\mathrm{fatdec}_l(X, A)$ and $\mathrm{fatdec}_r(X, A)$, called **left and right fat décalage** respectively, which notation we fix by declaring that if X is an augmented simplicial set then $X \diamond - \dashv \mathrm{fatdec}_l(X, -)$ and $- \diamond X \dashv \mathrm{fatdec}_r(X, -)$.

There is a canonical comparison map from the fat join to the join as previewed in the discussion surrounding Proposition 4.2.7.

D.2.14. Lemma. *There exists a canonical map of augmented simplicial sets*

$$s^{X,Y} \colon X \diamond Y \to X \star Y$$

natural in X and Y that in particular defines a natural transformation

$$s^{n,m} \colon \Delta[n] \diamond \Delta[m] \to \Delta[n] \star \Delta[m] \qquad \in s\mathcal{S}et_+^{\mathbf{\Delta}_+ \times \mathbf{\Delta}_+}$$

that is an isomorphism if n or m equals -1 and otherwise arises as a quotient of the map $\bar{s}^{n,m} \colon \Delta[n] \times \Delta[1] \times \Delta[m] \to \Delta[n+1+m]$ defined by its order-preserving action on vertices:

$$\bar{s}^{n,m}(i, j, k) = \begin{cases} i & \text{if } j = 0, \quad \text{and} \\ k + n + 1 & \text{if } j = 1. \end{cases} \tag{D.2.15}$$

Proof. Identifying the set X_{-1} with the augmented simplicial set $\sqcup_{X_{-1}} \Delta[-1]$, the Yoneda lemma supplies a canonical map $X_{-1} \to X$ of augmented simplicial sets, which gives rise to a canonical map

$$\begin{tikzcd} (X \times Y) \sqcup (X \times Y) \arrow[r, two heads, "\pi \sqcup \pi"] \arrow[d, hook] & (X \times Y_{-1}) \sqcup (X_{-1} \times Y) \arrow[d, hook] \arrow[dr] & \\ X \times \mathbb{2} \times Y \arrow[r, two heads] \arrow[dr, two heads, "\pi"'] & X \diamond Y \arrow[ul, phantom, "\ulcorner", very near start] \arrow[r, dashed, "s^{X,Y}"] \arrow[d, two heads] & X \star Y \arrow[dl, two heads] \\ & \mathbb{2} & \end{tikzcd} \tag{D.2.16}$$

Note that the fibers of both $X \diamond Y$ and $X \star Y$ over the endpoints $0, 1$ of $\mathbb{2}$ are $X \times Y_{-1}$ and $X_{-1} \times Y$ respectively, and the map $s^{X,Y}$ commutes with the inclusions of these fibers.

The map $s^{X,Y}$ is defined on those n-simplices over $X \diamond Y$ that map surjectively onto $\mathbb{2}$ by sending a triple $(\sigma \in X_n, \alpha\colon [n] \twoheadrightarrow [1], \tau \in Y_n)$ representing an n-simplex of $X \diamond Y$ to the pair $(\sigma|_{\{0,\ldots,k\}} \in X_k, \tau|_{\{k+1,\ldots,n\}} \in Y_{n-k-1})$ representing an n-simplex of $X \star Y$, where $k \in [n]$ is the maximal vertex in $\alpha^{-1}(0)$.

In the case of a pair of standard simplices, we can be even more explicit. Note that $X \diamond \Delta[-1] \cong X \star \Delta[-1] \cong X \cong \Delta[-1] \diamond X \cong \Delta[-1] \star X$, so it suffices to consider $n, m \geq 0$. In this case, we may describe $s^{n,m}$ as the quotient of a map $\bar{s}^{n,m}\colon \Delta[n] \times \Delta[1] \times \Delta[m] \to \Delta[n+1+m]$ defined by its order-preserving action on vertices, as described in the statement. Note this definition takes simplices related under the congruence described in Definition D.2.13 to the same simplex and thus induces a unique map $s^{n,m}\colon \Delta[n] \diamond \Delta[m] \to \Delta[n] \star \Delta[m]$ on the quotient simplicial set, which can easily be checked to coincide with the definition given in (D.2.16). $\square$

We now prove that the natural comparison between the fat join of simplices and the join of simplices defines a component of a marked homotopy equivalence, in the sense introduced in Definition D.1.16.

D.2.17. Proposition. *For each $n, m \geq -1$, the map of augmented simplicial sets*

$$s^{n,m}\colon \Delta[n] \diamond \Delta[m] \to \Delta[n] \star \Delta[m] \quad \in s\mathcal{S}et_+^{\mathbf{\Delta}_+ \times \mathbf{\Delta}_+}$$

is a marked homotopy retract equivalence which is an isomorphism in the case $n = -1$ or $m = -1$.

Proof. To define a section and left homotopy inverse to $s^{n,m}$, we consider a map $\bar{t}^{n,m}\colon \Delta[n+1+m] \to \Delta[n] \times \Delta[1] \times \Delta[m]$ determined by its order-preserving action on vertices:

$$\bar{t}^{n,m}(i) = \begin{cases} (i, 0, 0) & \text{if } i \leq n, \quad \text{and} \\ (n, 1, i-n-1) & \text{if } i > n \end{cases}$$

and note immediately that that $\bar{s}^{n,m} \circ \bar{t}^{n,m} = \mathrm{id}$. The obverse composite $\bar{t}^{n,m} \circ \bar{s}^{n,m}$ is given by the explicit formula:

$$(\bar{t}^{n,m} \circ \bar{s}^{n,m})(i, j, k) = \begin{cases} (i, 0, 0) & \text{if } j = 0, \quad \text{and} \\ (n, 1, k) & \text{if } j = 1. \end{cases}$$

Now we may define a related order preserving endo-map $\bar{u}^{n,m}$ on $[n] \times [1] \times [m]$ by

$$\bar{u}^{n,m}(i, j, k) = \begin{cases} (i, 0, 0) & \text{if } j = 0, \quad \text{and} \\ (i, 1, k) & \text{if } j = 1 \end{cases}$$

which is of interest because in the pointwise ordering on such maps we have $\bar{u}^{n,m} \leq \bar{t}^{n,m} \circ \bar{s}^{n,m}$ and $\bar{u}^{n,m} \leq \mathrm{id}_{[n]\times[1]\times[m]}$, representing simplicial homotopies:

$$\begin{tikzcd}
\Delta[n]\times\Delta[1]\times\Delta[m] \arrow[dr, "\delta^0"] \arrow[ddrr, bend right, "\bar{t}^{n,m}\circ\bar{s}^{n,m}"'] & & \Delta[n]\times\Delta[1]\times\Delta[m] \arrow[dl, "\delta^1"'] \arrow[dr, "\delta^1"] \arrow[dd, "\bar{u}^{n,m}"] & & \Delta[n]\times\Delta[1]\times\Delta[m] \arrow[dl, "\delta^0"'] \arrow[ddll, bend left, "\mathrm{id}_{\Delta[n]\times\Delta[1]\times\Delta[m]}"] \\
& (\Delta[n]\times\Delta[1]\times\Delta[m])\times\Delta[1] \arrow[dr, "\bar{h}^{n,m}"'] & & (\Delta[n]\times\Delta[1]\times\Delta[m])\times\Delta[1] \arrow[dl, "\bar{k}^{n,m}"] & \\
& & \Delta[n]\times\Delta[1]\times\Delta[m] & &
\end{tikzcd}$$

Passing to quotients under the congruence defined in Definition D.2.13, these maps induce simplicial maps $t^{n,m} \colon \Delta[n] \star \Delta[m] \to \Delta[n] \diamond \Delta[m]$ and $u^{n,m} \colon \Delta[n] \diamond \Delta[m] \to \Delta[n] \diamond \Delta[m]$ so that $s^{n,m} \circ t^{n,m} = \mathrm{id}_{\Delta[n]\star\Delta[m]}$, and simplicial homotopies $h^{n,m}$ and $k^{n,m}$ that assemble into a diagram:

$$\begin{tikzcd}
\Delta[n]\diamond\Delta[m] \arrow[dr, "\delta^0"] \arrow[ddrr, bend right, "t^{n,m}\circ s^{n,m}"'] & & \Delta[n]\diamond\Delta[m] \arrow[dl, "\delta^1"'] \arrow[dr, "\delta^1"] \arrow[dd, "u^{n,m}"] & & \Delta[n]\diamond\Delta[m] \arrow[dl, "\delta^0"'] \arrow[ddll, bend left, "\mathrm{id}_{\Delta[n]\diamond\Delta[m]}"] \\
& (\Delta[n]\diamond\Delta[m])\times\Delta[1] \arrow[dr, "h^{n,m}"'] & & (\Delta[n]\diamond\Delta[m])\times\Delta[1] \arrow[dl, "k^{n,m}"] & \\
& & \Delta[n]\diamond\Delta[m] & &
\end{tikzcd}$$

To see that the maps $h^{n,m}$ and $k^{n,m}$ define marked homotopies, Definition D.1.16 tells us that we must verify that for each 0-simplex $[i,j,k]_\sim$ of $\Delta[n] \diamond \Delta[m]$ the 1-simplex $([i,j,k]_\sim \cdot \sigma^0, \mathrm{id}_{[1]})$ of $(\Delta[n] \diamond \Delta[m]) \times \Delta[1]$ is mapped by $h^{n,m}$ and $k^{n,m}$ to degenerate, and thus marked, simplices in $\Delta[n] \diamond \Delta[m]$. We argue by cases in the index j. If $j = 0$, then $\bar{u}^{n,m}(i,0,k) = (i,0,0) \sim (i,0,k) = \bar{t}^{n,m} \circ \bar{s}^{n,m}(i,0,k)$, so the components of both $h^{n,m}$ and $k^{n,m}$ are degenerate. If $j = 1$, then $\bar{u}^{n,m}(i,1,k) = (i,1,k) \sim (n,i,k) = \bar{t}^{n,m} \circ \bar{s}^{n,m}(i,1,k)$, so again the components of both $h^{n,m}$ and $k^{n,m}$ are degenerate. Thus, $s^{n,m}$ extends to a marked homotopy retract equivalence with equivalence inverse $t^{n,m}$. $\square$

The marked simplicial homotopy equivalence constructed in Proposition D.2.17 witnesses a pointwise weak equivalence in a suitable sense between two diagrams in $s\mathcal{S}et_+^{\mathbb{\Delta}_+\times\mathbb{\Delta}_+}$ considered in Lemma D.2.14. This is the key ingredient in the proof that the canonical map of augmented simplicial sets $s^{X,Y} \colon X \diamond Y \to X \star Y$ is also a weak equivalence in a suitable sense, but this conclusion will require an exploration of the connection between the homotopy theory of marked simplicial sets and the homotopy theory of quasi-categories. We make this connection in §D.4 and then resume this line of reasoning in §D.6.

We close this section with one final result, a marked analogue of Lemma D.2.11. The join construction of Definition D.2.6 is extended to marked simplicial sets in **[97]**.

D.2.18. Definition (join of marked simplicial sets). The simplicial join lifts to a join bifunctor

$$s\mathcal{S}et^+ \times s\mathcal{S}et^+ \xrightarrow{\star} s\mathcal{S}et^+$$

in which a simplex $\Delta[n] \to A \star B$, with components $\Delta[k] \to A$ and $\Delta[n-k-1] \to B$, is marked in $A \star B$ if and only if at least one of the simplices in A or B is marked.

D.2.19. LEMMA.

(i) The Leibniz join of a complicial horn inclusion and a boundary inclusion is isomorphic to a single complicial horn inclusion:

$$(\Lambda^k[n] \hookrightarrow_r \Delta^k[n]) \mathbin{\widehat{\star}} (\partial\Delta[m] \hookrightarrow_r \Delta[m]) \cong \Lambda^k[n+1+m] \hookrightarrow_r \Delta^k[n+1+m]$$

unless $k = n$, in which case the Leibniz join $(\Lambda^n[n] \hookrightarrow_r \Delta^n[n]) \mathbin{\widehat{\star}} (\partial\Delta[m] \hookrightarrow_r \Delta[m])$ is a pushout of the complicial horn inclusion $\Lambda^n[n+1+m] \hookrightarrow_r \Delta^n[n+1+m]$.

(ii) The Leibniz joins

$$(\Lambda^k[n] \hookrightarrow_r \Delta^k[n]) \mathbin{\widehat{\star}} (\Delta[m] \hookrightarrow_e \Delta[m]_t) \cong \Delta^k[n+1+m]' \hookrightarrow_e \Delta^k[n+1+m]''$$
$$(\Delta^k[n]' \hookrightarrow_e \Delta^k[n]'') \mathbin{\widehat{\star}} (\partial\Delta[m] \hookrightarrow_r \Delta[m]) \cong \Delta^k[n+1+m]' \hookrightarrow_e \Delta^k[n+1+m]''$$

unless $k = n$, in which case the Leibniz joins are instead pushout of $\Delta^k[n+1+m]' \hookrightarrow_e \Delta^k[n+1+m]''$, while $(\Delta^k[n]' \hookrightarrow_e \Delta^k[n]'') \mathbin{\widehat{\star}} (\Delta[m] \hookrightarrow_e \Delta[m]_t)$ is the identity map.

The last case of this result is generalized in Exercise D.2.iii.

PROOF. The underlying map of simplicial sets in (i) is identified in Lemma D.2.11, so it remains only to consider the markings. Similarly, in each of the three Leibniz joins considered in (ii), the underlying map of simplicial sets is a Leibniz join of a monomorphism with an identity, and is thus an identity, so it remains only to consider the markings in the resulting entire inclusion. Since a simplex in a join of marked simplicial sets is marked if and only if either of its components are, this description lends itself readily to a case analysis. We leave the details to Exercise D.2.ii or to [**98**, 38]. □

The slice construction of Definition D.2.8 also extends to marked simplicial sets. We adopt new notation for this construction because it is not always the case that the underlying simplicial set of the marked join is the slice of corresponding map of underlying simplicial sets.

D.2.20. LEMMA. *For any map of marked simplicial sets $f\colon X \to A$, there exist marked simplicial sets ${}^{f/\!\!/}A$ and $A_{/\!\!/ f}$ characterized by the universal properties*

$$\left\{ \begin{tikzcd} & X \arrow[dl] \arrow[dr, "f"] & \\ X \star Y \arrow[rr] & & A \end{tikzcd} \right\} \cong \left\{ Y \longrightarrow {}^{f/\!\!/}A \right\} \quad \textit{and} \quad \left\{ \begin{tikzcd} & X \arrow[dl] \arrow[dr, "f"] & \\ Y \star X \arrow[rr] & & A \end{tikzcd} \right\} \cong \left\{ Y \longrightarrow A_{/\!\!/ f} \right\}.$$

PROOF. Exercise D.2.iv. □

Exercises.

D.2.i. EXERCISE. Prove Lemma D.2.5.

D.2.ii. EXERCISE ([**98**, 38]). Finish the proof of Lemma D.2.19.

D.2.iii. EXERCISE. Generalize the last case of Lemma D.2.19(ii) by showing that the Leibniz join of two entire inclusions is an identity.

D.2.iv. EXERCISE. Prove Lemma D.2.20.

D.3. Leibniz stability of cartesian products

We now turn our attention to analogous Leibniz constructions defined with respect to the cartesian product, which in the context of marked simplicial sets is called the *Gray tensor product* in **[98]** for reasons we shall explain. To warm up, let us demonstrate the following basic result about the combinatorics of simplicial sets.

D.3.1. LEMMA. *For any $n, m \geq 0$, the Leibniz product of the simplex boundary inclusions*

$$(\partial\Delta[n] \hookrightarrow \Delta[n]) \mathbin{\widehat{\times}} (\partial\Delta[m] \hookrightarrow \Delta[m]) \cong \partial\Delta[n] \times \Delta[m] \underset{\partial\Delta[n]\times\partial\Delta[m]}{\cup} \Delta[n] \times \partial\Delta[m] \hookrightarrow \Delta[n] \times \Delta[m]$$

is a monomorphism.

PROOF. Products, pushouts, and monomorphisms in $s\mathcal{S}et$ are determined pointwise in the category of sets, so this result follows from the fact that for monomorphisms $S \hookrightarrow T$ and $U \hookrightarrow V$ of sets, the Leibniz product

$$(S \hookrightarrow T) \mathbin{\widehat{\times}} (U \hookrightarrow V) \cong (S \times V \underset{S\times U}{\cup} T \times U \hookrightarrow T \times V)$$

is a monomorphism, which is clear by inspection. □

D.3.2. REMARK (why "Leibniz"). The domain of the inclusion of Lemma D.3.1 defines the boundary of the prism $\partial(\Delta[n] \times \Delta[m])$ and the identification

$$\partial(\Delta[n] \times \Delta[m]) \cong \partial\Delta[n] \times \Delta[m] \underset{\partial\Delta[n]\times\partial\Delta[m]}{\cup} \Delta[n] \times \partial\Delta[m]$$

is formally similar to various identities that are commonly called "the Leibniz rule."

In this section, we prove a number of combinatorial results along the lines of Lemma D.3.1, each of which have corollaries along the following lines:

D.3.3. COROLLARY.

(i) The Leibniz product of any pair of monomorphisms of simplicial sets is again a monomorphism.
(ii) The Leibniz exponential of any trivial fibration of simplicial sets with any monomorphism is again a trivial fibration.

PROOF. By Lemma C.5.9 any monomorphism of simplicial sets can be built as a cell complex from the simplex boundary inclusions. By Proposition C.2.9(vi) it follows that pushout products of monomorphisms can be built as a cell complex out of pushout products of simplex boundary inclusions. Lemma D.3.1 verifies that these maps are monomorphisms, and since the monomorphisms are closed under coproduct, pushout, and sequential composition, the result of (i) follows.

For (ii), recall from Definition 1.1.24 that the trivial fibrations are characterized by the right lifting property against the monomorphisms. To see that the Leibniz exponential $\widehat{\{i, p\}}$ of a trivial fibration p with a monomorphism i is a trivial fibration, it suffices to show that for any other monomorphism j, j has the left lifting property with respect to $\widehat{\{i, p\}}$. By Proposition C.2.9(ii) this lifting problem transposes to one between $i \mathbin{\widehat{\times}} j$ and p, which can be solved by (i). □

D.3.4. PROPOSITION. *The category of marked simplicial sets is cartesian closed with*

- *cartesian product, called the* ***Gray tensor product****, giving by marking every simplex in the cartesian product of the underlying simplicial sets just when both components are marked simplices*

- *internal hom Y^X defined to be the simplicial set whose n-simplices $\sigma \in Y^X$ are marked simplicial maps $\sigma\colon X \times \Delta[n] \to Y$ that are marked just when σ extends to a marked simplicial map*

$$\begin{tikzcd} X \times \Delta[n] \arrow[r, "\sigma"] \arrow[d, hook, "e"'] & Y \\ X \times \Delta[n]_t \arrow[ur, dashed] \end{tikzcd}$$

Proof. It's clear from the universal property of the product and its closure that the cartesian product and internal hom must be defined in this way if these objects exist. To verify the adjunction, recall from Proposition D.1.4 that marked simplicial sets embed as a reflexive full subcategory of a category of presheaves $\mathcal{S}et^{t\mathbf{\Delta}^{\mathrm{op}}}$, and this embedding preserves the products and internal homs as just defined. Now we conclude that these define the functors of a two-variable adjunction on $s\mathcal{S}et^+$ by restricting the corresponding natural isomorphisms from $\mathcal{S}et^{t\mathbf{\Delta}^{\mathrm{op}}}$. ☐

The result of Lemma D.3.1 can easily be extended to the marked context. Our proof will use a simple observation that will also be deployed elsewhere.

D.3.5. Lemma. *The Leibniz product of any map of marked simplicial sets with an entire inclusion is an entire inclusion.*

Proof. By definition the Leibniz product of maps $X \to Y$ and $U \hookrightarrow_e V$ is the induced map of marked simplicial sets

$$\begin{tikzcd} X \times U \arrow[r, hook, "e"'] \arrow[d] & X \times V \arrow[d] \arrow[ddr, bend left] \\ Y \times U \arrow[r, hook, "e"'] \arrow[drr, bend right, hook, "e"'] & \bullet \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed] \\ & & Y \times V \end{tikzcd}$$

Note that the forgetful functor $U\colon s\mathcal{S}et^+ \to s\mathcal{S}et$ preserves products and pushouts, and recall that a map of marked simplicial sets is entire just when the underlying map is an isomorphism. Since the product of a simplicial set with an isomorphism is an isomorphism, the maps $X \times U \hookrightarrow_e X \times V$ and $Y \times U \hookrightarrow_e Y \times V$ are entire. Since pushouts of isomorphisms are isomorphisms, it follows that the remaining horizontal map is also entire. Finally, since isomorphisms obey the 2-of-3 property, the Leibniz product map must also be entire. ☐

D.3.6. Lemma. *The Leibniz product of two regular inclusions is again a regular inclusion.*

Proof. By Corollary D.3.3, the underlying simplicial set of the Leibniz product of two regular inclusions $A \hookrightarrow_r B$ and $C \hookrightarrow_r D$ is the monomorphism

$$A \times D \underset{A \times C}{\cup} B \times C \hookrightarrow B \times D.$$

Note, in particular, that the inclusions of the components $A \times D$ and $B \times C$ jointly surject onto the domain of this map. Our task is to show that any n-simplex in $A \times D \underset{A\times C}{\cup} B \times C$ that is marked in $B \times D$ is marked in $A \times D \underset{A\times C}{\cup} B \times C$. We argue by cases and assume without loss of generality that the n-simplex is in the image of the inclusion from $B \times C$. In this case, the regularity of the map $B \times C \hookrightarrow_r B \times D$ implies that it is marked in $A \times D \underset{A\times C}{\cup} B \times C$ as claimed. ☐

D.3.7. Lemma. *For any $n, m \geq 0$, the Leibniz products*

$$(\partial\Delta[n] \hookrightarrow_r \Delta[n]) \mathbin{\widehat{\times}} (\Delta[m] \hookrightarrow_e \Delta[m]_t) \qquad \textit{and} \qquad (\Delta[n] \hookrightarrow_e \Delta[n]_t) \mathbin{\widehat{\times}} (\Delta[m] \hookrightarrow_e \Delta[m]_t)$$

are monomorphisms of marked simplicial sets.

Proof. By Lemma D.3.5, both Leibniz products are entire maps of marked simplicial sets. Note that every entire map of marked simplicial sets is a monomorphism. ☐

Combining Lemmas D.1.5, D.3.1, and D.3.7, we have:

D.3.8. Corollary.

(i) The Leibniz product of any pair of monomorphisms of marked simplicial sets is again a monomorphism.
(ii) The Leibniz exponential of any trivial fibration of marked simplicial sets with any monomorphism is again a trivial fibration. ☐

Considerably harder is to show the "Leibniz stability" of the class of marked anodyne extensions with the class of marked monomorphisms. We prove a slightly more specific result that also describes the cases of "inner," "left," or "right" marked anodyne extensions, which restrict the inequalities $0 \leq k \leq n$ to $0 < k < n$, $k < n$, or $0 < k$, respectively.

D.3.9. Proposition. *For $n \geq 1$, $m \geq 0$, and $0 < k \leq n$ each of the Leibniz products*

$$(\Lambda^k[n] \hookrightarrow_r \Delta^k[n]) \mathbin{\widehat{\times}} (\partial\Delta[m] \hookrightarrow_r \Delta[m]) \qquad (\Lambda^k[n] \hookrightarrow_r \Delta^k[n]) \mathbin{\widehat{\times}} (\Delta[m] \hookrightarrow_e \Delta[m]_t)$$
$$(\Delta^k[n]' \hookrightarrow_e \Delta^k[n]'') \mathbin{\widehat{\times}} (\partial\Delta[m] \hookrightarrow_r \Delta[m]) \qquad (\Delta^k[n]' \hookrightarrow_e \Delta^k[n]'') \mathbin{\widehat{\times}} (\Delta[m] \hookrightarrow_e \Delta[m]_t)$$

is a right marked anodyne extension and is an inner marked anodyne extension if $k < n$.

Note that, on account of the symmetry of the cartesian product—in contrast to the antisymmetry $(A \star B)^{\mathrm{op}} \cong B^{\mathrm{op}} \star A^{\mathrm{op}}$ of the join—whether the horn inclusion or the simplex boundary inclusion appears on the left or right is immaterial. The proof of this result will require some special notation to describe the cartesian product of simplices.

D.3.10. Digression (on shuffles). By the Yoneda lemma, an r-simplex in $\Delta[n]$ may be represented by a map $i\colon \Delta[r] \to \Delta[n]$. Since $\Delta[n]$ is the nerve of the poset $[n]$, an r-simplex may equally be represented by the ordered sequence of vertices $i_0 \leq \cdots \leq i_r \in [n]$ appearing in its image.

By the universal property of the product, an r-simplex in $\Delta[n] \times \Delta[m]$ is given by a pair $i\colon \Delta[r] \to \Delta[n]$ and $j\colon \Delta[r] \to \Delta[m]$ comprised of an r-simplex in $\Delta[n]$ together with an r-simplex in $\Delta[m]$. Since $\Delta[n] \times \Delta[m]$ is the nerve of the poset $[n] \times [m]$, such simplices correspond bijectively to ordered sequences of pairs

$$(i_0, j_0) \leq (i_1, j_1) \leq \cdots \leq (i_r, j_r) \tag{D.3.11}$$

with each $i_s \in [n]$ and each $j_t \in [m]$.

The non-degenerate $n+m$-simplices of the simplicial set $\Delta[n] \times \Delta[m]$ are called **shuffles**. An $n+m$-simplex (i, j) defines a shuffle just when $i_t + j_t = t$ for all $t \in [n+m]$. If the objects of $[n] \times [m]$ are arranged in a rectangle grid, the shuffles are those maximal-length non-degenerate paths that start from $(0, 0)$ and end with (n, m), by taking steps which change exactly one coordinate at a time.

The first case of the following proof is an adaptation of an argument of Dugger and Spivak [**31**, A.1] to the marked context.

Proof. By Lemma D.3.6, the Leibniz product $(\Lambda^k[n] \hookrightarrow_r \Delta^k[n]) \widehat{\times} (\partial\Delta[m] \hookrightarrow_r \Delta[m])$ is the regular inclusion

$$\Lambda^k[n] \times \Delta[m] \underset{\Lambda^k[n]\times\partial\Delta[m]}{\cup} \Delta^k[n] \times \partial\Delta[m] \hookrightarrow_r \Delta^k[n] \times \Delta[m].$$

A non-degenerate r-simplex (D.3.11) of $\Delta^k[n]\times\Delta[m]$ is missing from the domain of the Leibniz product inclusion just when

- its component $\{i_0, \ldots, i_r\} \supset [n]\backslash\{k\}$ and
- its component $\{j_0, \ldots, j_r\} \supset [m]$.

We will filter this inclusion as a sequence of regular inclusions

$$\Lambda^k[n] \times \Delta[m] \underset{\Lambda^k[n]\times\partial\Delta[m]}{\cup} \Delta^k[n] \times \partial\Delta[m] =: Y^{-1} \hookrightarrow_r Y^0 \hookrightarrow_r \cdots \hookrightarrow_r Y^m = \Delta^k[n] \times \Delta[m]$$

and argue that each $Y^t \hookrightarrow Y^{t+1}$ is left or inner marked anodyne, as appropriate.

Starting from $Y^{-1} \coloneqq \Lambda^k[n] \times \Delta[m] \underset{\Lambda^k[n]\times\partial\Delta[m]}{\cup} \Delta^k[n] \times \partial\Delta[m]$, we define Y^t to be the smallest regular simplicial subset of $\Delta^k[n] \times \Delta[m]$ containing Y^{t-1} together with every simplex (D.3.11) that contains the vertex (k, t). Since every missing simplex is a face of a simplex that contains one of the vertices $(k, 0), \ldots, (k, m)$, it is clear from this description that $Y^m = \Delta^k[n] \times \Delta[m]$.

It remains only to analyze the regular inclusions $Y^{t-1} \hookrightarrow_r Y^t$, which we do by producing another filtration

$$Y^{t-1} = Y^{t,n-1} \hookrightarrow_r Y^{t,n} \hookrightarrow_r \cdots \hookrightarrow_r Y^{t,n+m} = Y^t.$$

Note that every simplex of $\Delta^k[n] \times \Delta[m]$ that contains the vertex (k, t) and has dimension $n - 1$ or less is contained in Y^{-1}, so the simplices containing the vertex (k, t) that are attached to Y^{t-1} to form Y^t have dimensions between n and $n + m$. With this in mind, we define $Y^{t,r}$ to be the smallest regular simplicial subset of $\Delta^k[n] \times \Delta[m]$ containing $Y^{t,r-1}$ and all simplices of dimension r that contain the vertex (k, t). In particular, $Y^{t,n-1} = Y^{t-1}$ and $Y^{t,n+m} = Y^t$.

We now argue that each regular inclusion in this filtration is a pushout of a coproduct of complicial horn inclusions followed by complicial thinness extensions

$$\begin{tikzcd}
\coprod\limits_{\tau\in S^{t,r}} \Lambda^{\ell_\tau}[r] \arrow[r] \arrow[d, hook, "r"'] & Y^{t,r-1} \arrow[d, hook, "r"] & \\
\coprod\limits_{\tau\in S^{t,r}} \Delta^{\ell_\tau}[r] \arrow[r] \arrow[ur, phantom, very near end, "\ulcorner"] & \bullet \arrow[d, hook, "e"'] & \coprod\limits_{\tau\in T^{t,r}} \Delta^{\ell_\tau}[r]' \arrow[l] \arrow[d, hook, "e"] \arrow[dl, phantom, very near end, "\urcorner"] \\
& Y^{t,r} & \coprod\limits_{\tau\in T^{t,r}} \Delta^{\ell_\tau}[r]'' \arrow[l]
\end{tikzcd}$$

indexed by the sets $S^{t,r}$ of r-simplices containing the vertex (k, t) and not already present in $Y^{t,r-1}$ and $T^{t,r} \subset S^{t,r}$ defined to be the subset of those r-simplices τ so that $\tau \cdot \delta^{\ell_\tau}$ is marked in $\Delta^k[n] \times \Delta[m]$. Moreover, for each $\tau \in S^{t,r}$, we will see that that $0 < \ell_\tau$ and if $k < n$ then $\ell_\tau < r$. This will show that the Leibniz product is a right marked anodyne extension, which is an inner marked anodyne extension if $k < n$.

To see this, let $\tau \in S^{t,r}$ be the r-simplex

$$(i_0, j_0) \leq \cdots \leq (i_{\ell_\tau}, j_{\ell_\tau}) = (k, t) \leq \cdots \leq (i_r, j_r)$$

containing (k, t) as its ℓ_τth vertex; for readability we write ℓ for ℓ_τ going forward. Since the set $\{i_0, \ldots, i_r\} \supset [n]$ and $0 < k$ we must also have $0 < \ell$, and if $k < n$, we must also have $\ell < r$. We will argue that:

- Each face of τ except the ℓth is contained in $Y^{t,r-1}$.
- The ℓth face of τ is not in $Y^{t,r-1}$.
- The r-simplex τ is a ℓ-admissible simplex of $\Delta^k[n] \times \Delta[m]$.
- If $\tau \cdot \delta^\ell$ is marked in $\Delta^k[n] \times \Delta[m]$ then so is $\tau \cdot \delta^{\ell-1}$ and $\tau \cdot \delta^{\ell+1}$ (in the case $\ell < r$).

Thus, the union of τ with $Y^{t,r-1}$ may be formed as a pushout of a complicial horn extension $\Lambda^\ell[r] \hookrightarrow_r \Delta^\ell[r]$ as claimed.

For the first item, note that each codimension-one face except for $\tau \cdot \delta^\ell$ has dimension $r - 1$ and contains the vertex (k, t) and thus lies in $Y^{t,r-1}$ as claimed. To see that $Y^{t,r-1}$ does not also contain the face $\tau \cdot \delta^\ell$, we consider the vertex $(i_{\ell-1}, j_{\ell-1})$. If $i_{\ell-1} = k$ then by non-degeneracy, $j_{\ell-1} < t$, in which case we would have $\tau \in Y^{t-1}$, a contradiction. Thus $i_{\ell-1} < k$. Now if $i_{\ell-1} \leq k - 2$, then we would have $\tau \in Y^{-1}$, again a contradiction. So it must be that $i_{\ell-1} = k - 1$. Now if $j_{\ell-1} < t$, then τ would be a face of the $r + 1$-dimensional simplex

$$(i_0, j_0) \leq \cdots \leq (i_{\ell-1}, j_{\ell-1}) \leq (k, t-1) \leq (i_\ell, j_\ell) = (k, t) \leq \cdots \leq (i_r, j_r)$$

which is contained in Y^{t-1}, a contradiction. So we conclude that $(i_{\ell-1}, j_{\ell-1}) = (k - 1, t)$.

From this computation we see that the vertices of $\tau \cdot \delta^\ell$ satisfy $\{i_0, \ldots, i_{\ell-1} = k-1, i_{\ell+1}, \ldots, i_r\} \supset [n] \backslash \{k\}$ and $\{j_0, \ldots, j_{\ell-1}, j_{\ell+1}, \ldots, j_r\} \supset [m]$. Thus, $\tau \cdot \delta^\ell$ is not in Y^{-1}. Furthermore, $\tau \cdot \delta^\ell$ was not added in along the way to $Y^{t,r-1}$, since it is not a face of a simplex containing the vertex (k, s) for any $s < t$. This completes our second task.

We have shown that it is possible to attach τ to $Y^{t,r-1}$ along with its ℓth face by filling a suitable horn. It remains only to argue that the horn $\Lambda^\ell[r] \to Y^{t,r-1}$ along which we are attaching τ is admissible. Since the inclusion $Y^{t,r-1} \hookrightarrow_r \Delta^k[n] \times \Delta[m]$ is regular it suffices to show that each simplex containing the vertices $(i_{\ell-1}, j_{\ell-1})$, (i_ℓ, j_ℓ), and $(i_{\ell+1}, j_{\ell+1})$ — or just the first two of these in the case $\ell = r$ — is marked in $\Delta^k[n] \times \Delta[m]$. We've seen above that $(i_{\ell-1}, j_{\ell-1}) = (k-1, t)$ and $(i_\ell, j_\ell) = (k, t)$. In the case $\ell < r$, since τ is missing from Y^{-1}, $i_{\ell+1}$ must equal k or $k + 1$. But now the component in $\Delta[m]$ of this simplex is degenerate, containing the sequence $t \leq t$, while the component in $\Delta^k[n]$ is either degenerate, contains the sequence $k - 1 \leq k \leq k + 1$, or contains the sequence $k - 1 \leq k$ in the case $\ell = r$ in which case $k = n$. Thus both components are marked simplices, which means that their product is marked in $\Delta^k[n] \times \Delta[m]$ as required.

Finally, we must argue that the simplices attached by the pushout contain all the markings present in the regular subset $Y^{t,r} \hookrightarrow_r \Delta^k[n] \times \Delta[m]$. The only simplices present in $Y^{t,r}$ but not $Y^{t,r-1}$ are in dimensions r and $r - 1$. The newly attached r-simplices are all marked, so we need only concern ourselves with the $(r - 1)$-simplex

$$(i_0, j_0) \leq \cdots \leq (i_{\ell-1}, j_{\ell-1}) = (k - 1, t) \leq (i_{\ell+1}, j_{\ell+1}) \leq \cdots \leq (i_r, j_r)$$

arising as the ℓ-face for each $\tau \in S^{t,r}$ when this simplex is marked in $\Delta^k[n] \times \Delta[m]$.

There are two cases depending on whether $i_{\ell+1} = k + 1$ or $i_{\ell+1} = k$. In the former case, the fact that this simplex is marked tells us that there is a duplication present in the sequence $i_0 \leq \cdots \leq i_{\ell_\tau - 1} \leq i_{\ell_\tau+1} \leq \cdots \leq i_r$ and also in the sequence $j_0 \leq \cdots \leq j_{\ell_\tau-1} \leq j_{\ell_\tau+1} \leq \ldots \leq j_r$. When we substitute $i_\ell = k$ in the first sequence for $i_{\ell-1}$ or $i_{\ell+1}$ either the duplication remains or we now have the subsequence $k-1 \leq k \leq k+1$. Either way, this tells us that the component of the faces $\tau \cdot \delta^{\ell-1}$ and $\tau \cdot \delta^{\ell+1}$ is marked in $\Delta^k[n]$. Similarly, when we substitute $j_\ell = t$ for $j_{\ell-1} = t$, the sequence is unchanged, and when we substitute for $j_{\ell+1}$ our sequence now contains a duplication $t \leq t$. Either way, this tells us that the component of the faces $\tau \cdot \delta^{\ell-1}$ and $\tau \cdot \delta^{\ell+1}$ is marked in $\Delta[m]$. In conclusion, the simplex $\tau \colon \Delta^\ell[r] \to \Delta^k[n] \times \Delta[m]$ extends along $\Delta^k[n] \hookrightarrow_e \Delta^k[n]'$, so we obtain the desired marking of its ℓ-th face by extending along the entire inclusion $\Delta^k[n]' \hookrightarrow_e \Delta^k[n]''$ included in the second pushout.

In the case where $i_{\ell+1} = k$, the sequence of vertices for $\tau\delta^{\ell-1}$ contains $k \leq k$ in its first component and the same sequence of vertices as $\tau\delta^\ell$ in its second component. Thus, $\tau\delta^{\ell-1}$ is marked. The sequence of vertices for $\tau\delta^{\ell+1}$ contains $k-1 \leq k \leq k+1$ in its first component and $t \leq t$ in its second component. Thus, $\tau\delta^{\ell+1}$ is marked, and once more we obtain the desired marking of its ℓ-th face by extending along the entire inclusion $\Delta^k[n]' \hookrightarrow_e \Delta^k[n]''$ included in the second pushout. This completes the proof that the first Leibniz product is a marked anodyne extension.

By Lemma D.3.5, the remaining three Leibniz products are entire inclusions, so all that is required is to verify that the additional markings present in the codomains are the results of complicial thinness extensions. We treat simultaneously the two cases involving Leibniz products

$$(\Delta^k[n]' \hookrightarrow_e \Delta^k[n]'') \mathbin{\widehat{\times}} (A \hookrightarrow B)$$

of a complicial horn inclusion with a marked monomorphism. The only marked simplex of $\Delta^k[n]''$ that is not marked in $\Delta^k[n]'$ is the face $\delta^k \colon \Delta[n-1]_t \to \Delta^k[n]''$, which implies that we have a pullback and pushout square

$$\begin{tikzcd}
\coprod\limits_{\tau \in S} \Delta[n-1] \arrow[r] \arrow[d, hook, "e"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Delta^k[n]' \times B \underset{\Delta^k[n]' \times A}{\cup} \Delta^k[n]'' \times A \arrow[d, hook, "e"] \\
\coprod\limits_{\tau \in S} \Delta[n-1]_t \arrow[r, "{(\delta^k, \tau)}"] & \Delta^k[n]'' \times B \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

where S is the set of marked $(n-1)$-simplices in B that are not present or not marked in A. We argue that for any marked $(n-1)$-simplex $\tau \in B$, the degenerate n-simplex $\tau\sigma^{k-1}$ admits the indicated markings:

$$\begin{tikzcd}
\Delta[n] \arrow[r, hook, "e"'] \arrow[d, "\sigma^{k-1}"'] & \Delta^k[n]'' \arrow[d, dashed] \\
\Delta[n-1]_t \arrow[r, "\tau"] & B
\end{tikzcd}$$

because the $k-1$th and kth faces equal τ, and so are marked, and any face that contains the vertices $k-1$ and k is degenerate and so is also marked; these conditions cover all of the required marked faces.

Now it is clear that the pushout square above factors through the left-hand pushout diagram

$$\begin{tikzcd} \coprod_{\tau\in S} \Delta[n-1] \arrow[r] \arrow[d, hook, "e"'] \arrow[dr, phantom, "\lrcorner"] & \coprod_{\tau\in S} \Delta^k[n]' \arrow[r] \arrow[d, hook, "e"'] & \Delta^k[n]' \times B \underset{\Delta^k[n]'\times A}{\cup} \Delta^k[n]'' \times A \arrow[d, hook, "e"'] \\ \coprod_{\tau\in S} \Delta[n-1]_t \arrow[r, "\delta^k"] & \coprod_{\tau\in S} \Delta^k[n]'' \arrow[r, "{(\mathrm{id},\tau\sigma^{k-1})}"] \arrow[ul, phantom, "\ulcorner"] & \Delta^k[n]'' \times B \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$

demonstrating that the Leibniz product inclusion is a pushout of coproducts of suitable complicial thinness extensions.

The final case of

$$(\Lambda^k[n] \hookrightarrow_r \Delta^k[n]) \widehat{\times} (\Delta[m] \hookrightarrow_e \Delta[m]_t) \cong \Lambda^k[n] \times \Delta[m]_t \underset{\Lambda^k[n]\times\Delta[m]}{\cup} \Delta^k[n] \times \Delta[m] \hookrightarrow_e \Delta^k[n] \times \Delta[m]_t$$

is again an entire inclusion. Since the only simplex that is marked in $\Delta[m]_t$ but not in $\Delta[m]$ is the top-dimensional m-simplex, the only simplices that are marked in the codomain but not in the domain are m-simplices $(\tau, \mathrm{id})\colon \Delta[m]_t \to \Delta^k[n] \times \Delta[m]_t$ in which the image of τ either

- contains $[n]$ or
- contains $[n]\backslash\{k\}$ but not $\{k\}$ and is degenerate.

In particular, this Leibniz product inclusion is an identity if $m < n$. In the case $m = n$, there are $m+1$ simplices that are marked in $\Delta^k[m] \times \Delta[m]_t$ but not in the domain, corresponding to the m m-simplices that are degenerate on the kth face of $\Delta^k[m]$ and the top-dimensional m-simplex.

We will factor the inclusion as a finite composite of pushouts of coproducts of maps $\Delta^s[m+1]' \hookrightarrow_e \Delta^s[m+1]''$ for varying $0 < s \leq m+1$, where each $s < m+1$ if $k < n$. This will prove that this Leibniz product is a complicial thinness extension of the appropriate kind.

We can classify the missing marked simplices in terms of their component $\tau\colon \Delta[m]_t \to \Delta^k[n]$, which we may represent as a sequence $i_0, \ldots, i_m$ of vertices of $[n]$ that either contains $[n]$ or contains $[n]\backslash\{k\}$ and has repetitions. We induct over a partial ordering of these simplices in decreasing order of the sum $\sum_{t=0}^m i_t$.[7]

For a simplex τ with maximal vertex sum $\sum_{t=0}^m i_t$ among those simplices that remain to be marked, let s be minimal so that $i_s \geq k$; when $k = n$ it is possible that all $i_t < k = n$, which gives a second case that we will consider in a moment. Then we consider the $m+1$-simplex:

$$\Delta^s[m+1]'' \xrightarrow{(\tau\cdot\sigma^s, \sigma^{s-1})} \Delta^k[n] \times \Delta[m]_t.$$

By construction, the $s+1$th face is marked in $\Delta^k[n] \times \Delta[m]$, while the $s-1$th face has strictly greater vertex sum, and so is marked by the inductive hypothesis. The faces containing the $s-1$, s, and $s+1$ vertices are all degenerate and thus marked. This proves that the face τ can be marked by forming an extension $\Delta^s[m+1]' \hookrightarrow_e \Delta^s[m+1]''$.

In the case where τ is a simplex where all $i_t < k = n$, then we consider the $m+1$-simplex

$$\Delta^{m+1}[m+1]'' \xrightarrow{(\chi, \sigma^m)} \Delta^n[n] \times \Delta[m]_t.$$

[7] Here the vertex sum of an m-simplex τ is greater than the vertex sum of an m-simplex τ' if and only if τ has greater "depth" in the sense defined in [**98**, 68]. The induction of [**98**, §5.2] involves Leibniz products of inner or left horn inclusions and starts by considering simplices of lowest depth; ours involves an inner or right horn inclusion and starts by considering simplices of highest depth.

where $\chi\colon \Delta[m+1] \to \Delta^n[n]$ is the simplex spanned by the vertices $i_0, \ldots, i_m = n-1, n$. Here the mth face has strictly greater sum, and so is marked by the inductive hypothesis. The faces containing the mth and $m+1$th vertices have a degenerate component in $\Delta[m]$ and have a component in $\Delta^n[n]$ that contains the last two vertices $n-1$ and n. Thus, all such simplices are marked. This proves that the face τ can be marked by forming an extension $\Delta^{m+1}[m+1]' \hookrightarrow_e \Delta^{m+1}[m+1]''$. $\square$

Proposition D.3.9 of course implies its unmarked analogue, refining the result proven in Corollary D.3.3(i):

D.3.12. Corollary. *Let $i\colon A \hookrightarrow B$ and $j\colon K \hookrightarrow L$ be monomorphisms of simplicial sets. If either i or j is also anodyne (or, respectively, left-, right-, or inner-anodyne), then so is the pushout product*

$$A \times L \cup_{A\times K} B \times K \xhookrightarrow{i \hat{\times} j} B \times L$$

Proof. The maximal marking functor $(-)^\sharp\colon s\mathcal{S}et \to s\mathcal{S}et^+$ carries (left/right/inner) horn inclusions to (left/right/inner) marked anodyne extensions, and as a left adjoint, therefore carries all (left/right/inner) anodyne extensions of simplicial sets to (left/right/inner) marked anodyne extensions. As a left and right adjoint, the maximal marking functor also preserves pushout products. Similarly, the forgetful functor $U\colon s\mathcal{S}et^+ \to s\mathcal{S}et$ preserves products and colimits and carries the (left/right/inner) marked anodyne extensions to the classes of anodyne extensions introduced in Definition D.2.10. Thus, this result follows immediately from the first case of Proposition D.3.9. $\square$

Recall from Definition D.1.14 that a marked map between complicial sets is a **complicial isofibration** if it has the right lifting property with respect to the complicial horn inclusions and complicial thinness extensions of Definition D.1.9.

D.3.13. Corollary.

(i) For any quasi-category A and simplicial set X, A^X is again a quasi-category.

(ii) For any complicial set A and marked simplicial set X, A^X is again a complicial set.

(iii) For any complicial isofibration $p\colon E \twoheadrightarrow B$ and any monomorphism of marked simplicial sets $i\colon X \hookrightarrow Y$, the Leibniz exponential $\widehat{\{i,p\}}\colon E^Y \to E^X \underset{B^X}{\times} B^Y$ is a complicial isofibration.

Proof. The second statement is a special case of the third statement, which follows by transposing the result of Proposition D.3.9 across the Leibniz version of the two variable adjunction of Proposition D.3.4. See Proposition C.2.9. The first statement follows similarly by applying Corollary D.3.12 in place of Proposition D.3.9. $\square$

We would like to prove the analogous statement to Corollary D.3.13(iii) for isofibrations between quasi-categories, which requires analogous statements to Proposition D.3.9 and Corollary D.3.12 analyzing Leibniz products of monomorphisms of simplicial sets with the inclusion $\mathbb{1} \hookrightarrow \mathbb{I}$. We shall deduce this by considering the relationship between isomorphisms in quasi-categories and marked edges in complicial sets, which is the subject of the next section.

Exercises.

D.3.i. Exercise. The Leibniz product of a regular inclusion with a non-invertible entire inclusion will be entire but not necessarily regular. Find examples that illustrate both possibilities and state and prove a characterization of those Leibniz products of this form that are necessarily regular. Would this have simplified the proof of Proposition D.3.9?

D.4. Isomorphisms and naturally marked quasi-categories

Our aim in this section is to explain the relevance of Proposition D.3.9 to the theory of quasi-categories. In particular, this will finally enable us to complete the combinatorial work required to supply proofs of the results stated in §1.1, which we dispense with in §D.5.

The connection between the theory of complicial sets and the theory of quasi-categories is established by the two main results proven in this section, Theorems D.4.13 and D.4.19, the first of which explains that any quasi-category can be equipped with a canonically-defined "natural" marking in such a way that it defines a 1-trivial complicial set. The markings on the 1-simplices cannot be arbitrarily assigned: every marked edge must be an equivalence in a sense that we now introduce:

D.4.1. Definition. A 1-simplex f in a marked simplicial set is an **equivalence** if there exist a pair of thin 2-simplices as displayed

$$\begin{tikzcd} & x \arrow[dr, "f"] \arrow[d, phantom, "\wr"] & \\ y \arrow[ur] \arrow[rr, equal] & {} & y \end{tikzcd} \qquad \begin{tikzcd} & y \arrow[dr] \arrow[d, phantom, "\wr"] & \\ x \arrow[ur, "f"] \arrow[rr, equal] & {} & x \end{tikzcd}$$

Note the notion of equivalence is defined relative to the choice of markings on the 2-simplices. A very similar notion is defined for the edges of a quasi-category in Definition 1.1.13 under the name "isomorphism."

D.4.2. Lemma. *Every marked edge in a complicial set is an equivalence.*

Proof. If f is a marked edge in any complicial set A, then the $\Lambda^2[2]$-horn with 0th face f and 1st face degenerate is admissible, so f has a right equivalence inverse. A dual construction involving a $\Lambda^0[2]$-horn shows that f has a left equivalence inverse:

$$\begin{tikzcd} & x \arrow[dr, "f", "\sim"'] \arrow[d, phantom, "\wr"] & \\ y \arrow[ur, dashed, "\sim"] \arrow[rr, equal] & {} & y \end{tikzcd} \qquad \begin{tikzcd} & y \arrow[dr, dashed, "\sim"] \arrow[d, phantom, "\wr"] & \\ x \arrow[ur, "f", "\sim"'] \arrow[rr, equal] & {} & x \end{tikzcd}$$

Note also that the complicial thinness extensions imply further that these one-sided inverses are also marked, so they admit further inverses of their own. □

This result suggests two ways to mark the edges in the nerve of a 1-category.

D.4.3. Lemma. *The nerve of a 1-category defines a complicial set by marking all simplices in dimension greater than one and then either defining:*

(i) the marked edges to be the identity arrows only or
(ii) the marked edges to be all isomorphisms.

Proof. Exercise D.4.i. □

This motivates a definition of the canonical marking of a quasi-category, which is called the "natural marking" in **[59]**.

D.4.4. Definition. For any quasi-category A, its **natural marking** is defined by:

- marking all simplices in dimension greater than one

- marking exactly those edges that are isomorphisms, in the sense of Definition 1.1.13, or equivalently marking all those edges that are equivalences, in the sense of Definition D.4.1.

The natural marking for quasi-categories is convenient for stating and proving an important combinatorial result due to Joyal:

D.4.5. Proposition. *Any naturally marked quasi-category A admits fillers for outer complicial horn inclusions for $n \geq 1$:*

$$\begin{tikzcd} \Lambda^0[n] \arrow[r] \arrow[d, hook, "r"'] & A \\ \Delta^0[n] \arrow[ur, dashed] \end{tikzcd} \qquad \begin{tikzcd} \Lambda^n[n] \arrow[r] \arrow[d, hook, "r"'] & A \\ \Delta^n[n] \arrow[ur, dashed] \end{tikzcd}$$

In the original [**46**], the result is stated without reference to markings as follows: a quasi-category admits fillers for **special outer horns**, left horns $\Lambda^0[n] \to A$ whose initial $\{01\}$-edge is mapped to an isomorphism in A and right horns $\Lambda^n[n] \to A$ whose final $\{n-1n\}$-edge is mapped to an isomorphism in A.

Many proofs of Proposition D.4.5 are possible; see [**31**, §B] or the original [**46**]. We choose to use combinatorial results of Verity [**98**, §4.2] which we present in stages, that use an alternate (a posteriori equivalent) notion of homotopy coherent isomorphism, which a homotopy type theorist would recognize by the name "half adjoint equivalence" [**95**, §4.2].

D.4.6. Digression (subcomplexes of the coherent isomorphism). Recall the **coherent isomorphism** is the simplicial set $\mathbb{I}$ defined as the nerve of the free-living isomorphism. It has exactly two non-degenerate simplices in each dimension. If we label its vertices as "$-$" and "$+$," then its remaining non-degenerate simplices are uniquely determined by their vertices, which are given by alternating sequences of "$-$" and "$+$" starting from either vertex. Following the notation introduced in [**98**, 42], we write $E^+_n, E^-_n \subset \mathbb{I}$ for the simplicial subsets generated by the n-simplices $- + - \cdots \pm$ and $+ - + \cdots \mp$ respectively. Both simplicial subsets include uniquely into both E^+_{n+1} and E^-_{n+1} and these inclusions can be realized as pushouts

$$\begin{tikzcd} \Lambda^0[n+1] \arrow[r, hook] \arrow[d] & \Delta[n+1] \arrow[d] \\ E^{\pm}_n \arrow[r, hook] & E^{\pm}_{n+1} \arrow[ul, phantom, "\ulcorner"] \end{tikzcd} \qquad \begin{tikzcd} \Lambda^{n+1}[n+1] \arrow[r, hook] \arrow[d] & \Delta[n+1] \arrow[d] \\ E^{\pm}_n \arrow[r, hook] & E^{\mp}_{n+1} \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$

as suggested by the hints given to Exercise 1.1.v.

As marked simplicial sets, we give $\mathbb{I}$ and its subcomplexes the maximal marking.

The following result gives a criterion under which an "inner complicial fibration" — a marked map that is only assumed to have the right lifting property against the inner complicial horn inclusions and inner complicial thinness extensions — whose codomain is a complicial set in fact defines a complicial isofibration in the sense of Definition D.1.14.

D.4.7. Proposition. *Let $p\colon A \twoheadrightarrow B$ be an inner complicial fibration whose codomain B is a complicial set. Then p is a complicial isofibration if and only if p admits lifts against the inclusions $E^-_0 \hookrightarrow E^-_1$ and $E^-_1 \hookrightarrow E^-_3$.*

Proof. Clearly any complicial isofibration admits lifts against the complicial anodyne extensions $E^-_0 \hookrightarrow E^-_1$ and $E^-_1 \hookrightarrow E^-_3$. Thus, the heart of this result, and the only part that remains to be proven, is the assertion that any inner complicial fibration whose codomain is a complicial set that admits lifts against this pair of inclusions also admits fillers for outer complicial horn inclusions and outer complicial thinness extensions.

We begin by arguing that an inner complicial fibration $p\colon A \twoheadrightarrow B$ satisfying the conditions of the statement also admits lifts against the dual inclusion $E_0^- \hookrightarrow E_1^+$. Given a lifting problem such as presented by the maps a and b in the square below:

$$\begin{tikzcd}
& E_0^- \arrow[r, "a"] \arrow[dl] \arrow[d, hook] & A \arrow[d, two heads, "p"] \\
E_1^- \arrow[dr] & E_1^+ \arrow[r, "b"] \arrow[d, hook] & B \\
& E_3^- \arrow[ur, dashed] \arrow[uur, dotted] &
\end{tikzcd}$$

there exists the dashed extension of b along $E_1^+ \hookrightarrow E_3^-$, since this inclusion factors as a composite of pushouts of outer complicial horn extensions and B is a complicial set. Now the inclusion $E_0^- \hookrightarrow E_3^-$ factors as indicated $E_0^- \hookrightarrow E_1^- \hookrightarrow E_3^-$ and since p is assumed to lift against both maps, the dotted lift exists as well, which restricts to define a solution to the original lifting problem.

With this result in hand, it follows that the odd dual of $p\colon A \twoheadrightarrow B$ satisfies the same lifting properties as $p\colon A \twoheadrightarrow B$ does, since the odd dual of $E_0^- \hookrightarrow E_1^-$ is $E_0^- \hookrightarrow E_1^+$, while the odd dual of $E_1^- \hookrightarrow E_3^-$ is isomorphic to itself. So it suffices to show that p admits lifts against left complicial horn inclusions $\Lambda^0[n] \hookrightarrow_r \Delta^0[n]$ and left complicial thinness extensions $\Delta^0[n]' \hookrightarrow_e \Delta^0[n]''$ since its odd dual will then share these properties, which implies that p also admits lifts against right complicial horn inclusions and right complicial thinness extensions. The case $\Lambda^0[1] \hookrightarrow \Delta^0[1]$ is the map $E_0^- \hookrightarrow E_1^-$ so it suffices also to assume $n > 1$, in which case we have an isomorphism

$$\Lambda^0[n] \hookrightarrow \Delta^0[n] \cong (\Lambda^0[1] \hookrightarrow \Delta^0[1]) \mathbin{\widehat{\star}} (\partial\Delta[n-2] \hookrightarrow \Delta[n-2])$$

by Lemma D.2.19. Writing $m = n-2$ for concision, consider a lifting problem as presented by the maps a and b:

$$\begin{tikzcd}
E_0^- \star \Delta[m] \cup E_1^- \star \partial\Delta[m] \arrow[rrr, "a"] \arrow[dr, hook, "i"] \arrow[dd, hook] & & & A \arrow[dd, two heads, "p"] \\
& E_2^- \star \Delta[m]_t \cup E_3^- \star \partial\Delta[m] \arrow[r, hook] \arrow[urr, dotted] & E_3^- \star \Delta[m] \arrow[dr, dashed] & \\
E_1^- \star \Delta[m] \arrow[ur, hook] \arrow[urr, hook, bend right, "j"'] \arrow[rrr, "b"'] & & & B
\end{tikzcd}$$

By Lemma D.2.19 and Proposition C.2.9(vi), the map j is a complicial anodyne extension, so since B is a complicial set, the dashed extension exists. To show that the dotted lift exists as well, we argue that the map i is cellularly generated by the inner complicial horn extensions and the map $E_1^- \hookrightarrow E_3^-$. To see this, factor the map i as the composite of the three vertical maps in the middle column of the diagram of pushout squares

$$\begin{tikzcd}
& E_1^- \arrow[r, hook] \arrow[d, hook] & E_0^- \star \Delta[m] \cup E_1^- \star \partial\Delta[m] \arrow[d, hook] & \\
& E_3^- \arrow[r, hook] & \bullet \arrow[d, hook] & E_1^- \star \partial\Delta[m] \cup E_3^- \star \varnothing \arrow[l, hook'] \arrow[d, hook] \\
E_0^- \star \Delta[m] \cup E_2^- \star \partial\Delta[m] \arrow[rr, hook] \arrow[d, hook] & & \bullet \arrow[d] & E_3^- \star \partial\Delta[m] \arrow[l, hook'] \\
E_2^- \star \Delta[m] \arrow[rr, hook] & & E_2^- \star \Delta[m] \cup E_3^- \star \partial\Delta[m] &
\end{tikzcd} \tag{D.4.8}$$

The first attached cell is the map $E_1^- \hookrightarrow E_3^-$ itself. By Digression D.4.6, $E_1^- \hookrightarrow E_3^-$ is a right anodyne extension, so by Lemma D.2.19 the second attached cell $(E_1^- \hookrightarrow E_3^-) \mathbin{\widehat{\star}} (\varnothing \hookrightarrow \partial\Delta[m])$ is an inner anodyne extension. Similarly $E_0^- \hookrightarrow E_2^-$ is a right anodyne extension, so the final attached cell $(E_0^- \hookrightarrow E_2^-) \mathbin{\widehat{\star}} (\partial\Delta[m] \hookrightarrow \Delta[m])$ is also an inner anodyne extension. Thus, p admits lifts against left complicial horn inclusions as claimed.

To see that $p\colon A \twoheadrightarrow B$ also admits outer complicial thinness extensions we make use of the isomorphism

$$(\Lambda^0[1] \hookrightarrow \Delta^0[1]) \mathbin{\widehat{\star}} (\Delta[m] \hookrightarrow \Delta[m]_t) \cong \Delta^0[m+2]' \hookrightarrow \Delta^0[m+2]''$$

of Lemma D.2.19, recalling that $\Lambda^0[1] \hookrightarrow \Delta^0[1]$ is the inclusion $E_0^- \hookrightarrow E_1^-$. So we may consider a lifting problem as presented by the maps a and b:

$$\begin{tikzcd}
E_0^- \star \Delta[m]_t \cup E_1^- \star \Delta[m] \arrow[rrr, "a"] \arrow[dd, hook] \arrow[dr, hook, "i"] & & & A \arrow[dd, two heads, "p"] \\
& E_2^- \star \Delta[m] \cup E_3^- \star \Delta[m] \arrow[r, hook] \arrow[urr, dotted] & E_3^- \star \Delta[m]_t \arrow[dr, dashed] & \\
E_1^- \star \Delta[m]_t \arrow[rrr, "b"'] \arrow[ur, hook] \arrow[urr, hook, "j"'] & & & B
\end{tikzcd}$$

By Lemma D.2.19, the map j is a complicial anodyne extension, so since B is a complicial set, the dashed extension exists. To show that the dotted lift exists as well, we argue that the map i is cellularly generated by the inner complicial horn inclusions, the inner complicial thinness extensions, and the map $E_1^- \hookrightarrow E_3^-$. To see this, factor the map i as the composite of the three vertical maps in the middle column of the diagram of pushout squares

$$\begin{tikzcd}
E_1^- \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner" very near end] & E_0^- \star \Delta[m]_t \cup E_1^- \star \Delta[m] \arrow[d, hook] & \\
E_3^- \arrow[r, hook] & \bullet \arrow[d, hook] & E_1^- \star \Delta[m] \cup E_3^- \star \varnothing \arrow[l, hook'] \arrow[d, hook] \arrow[dl, phantom, "\urcorner" very near end] \\
E_0^- \star \Delta[m]_t \cup E_2^- \star \Delta[m] \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner" very near end] & \bullet \arrow[d] & E_3^- \star \Delta[m] \arrow[l, hook'] \\
E_2^- \star \Delta[m]_t \arrow[r, hook] & E_2^- \star \Delta[m]_t \cup E_3^- \star \Delta[m] &
\end{tikzcd}$$

The first attached cell is the map $E_1^- \hookrightarrow E_3^-$ itself. By Digression D.4.6, $E_1^- \hookrightarrow E_3^-$ is a right anodyne extension, so by Lemma D.2.19 the second attached cell $(E_1^- \hookrightarrow E_3^-) \mathbin{\widehat{\star}} (\varnothing \hookrightarrow \Delta[m])$ is an inner complicial anodyne extension. Similarly $E_0^- \hookrightarrow E_2^-$ is a right anodyne extension, so the final attached cell $(E_0^- \hookrightarrow E_2^-) \mathbin{\widehat{\star}} (\Delta[m] \hookrightarrow \Delta[m]_t)$ is also an inner complicial thinness extension. Thus, p admits lifts against left complicial thinness extensions as claimed. $\square$

Since the inclusions $E_0^- \hookrightarrow E_1^-$ and $E_1^- \hookrightarrow E_3^-$ are left complicial anodyne extensions, Proposition D.4.7 has the following immediate corollary.

D.4.9. Corollary. *Let $p\colon A \twoheadrightarrow B$ be an inner complicial fibration whose codomain B is a complicial set. Then if p is a left complicial fibration or a right complicial fibration, then p is a complicial isofibration.* $\square$

D.4.10. Remark. Note that in the proof of Proposition D.4.7, lifts against the inclusion $E_0^- \hookrightarrow E_1^-$ are only needed to construct lifts for the outer complicial horn extensions $\Lambda^0[1] \hookrightarrow \Delta^0[1]$ and $\Lambda^1[1] \hookrightarrow \Delta^1[1]$. Thus, even if this lifting condition is dropped, the outer complicial horn extensions in higher dimensions can still be constructed.

The argument just given supplies a proof of a special case of "special outer horn filling" that will be useful in proving the general version.

D.4.11. Lemma. *Let $p\colon A \twoheadrightarrow B$ be an inner complicial fibration whose codomain B is a complicial set. Then p admits fillers for left horns*

$$\begin{tikzcd}
\Lambda^0[n] \arrow[r, "a"] \arrow[d, hook] & A \arrow[d, two heads, "p"] \\
\Delta^0[n] \arrow[r, "b"'] \arrow[ur, dashed] & B
\end{tikzcd}$$

with $n > 1$ provided a carries the $\{01\}$ edge of the horn $\Lambda^0[n]$ to a degenerate simplex in A.

Proof. Writing $m = n - 2$ we have an isomorphism

$$\Lambda^0[m+2] \hookrightarrow \Delta^0[m+2] \cong (\Lambda^0[1] \hookrightarrow \Delta^0[1]) \mathbin{\widehat{\star}} (\partial\Delta[m] \hookrightarrow \Delta[m])$$

by Lemma D.2.19, so once more we are asked to consider a lifting problem as presented by the maps a and b:

$$\begin{tikzcd}
E_0^- \star \Delta[m] \cup E_1^- \star \partial\Delta[m] \arrow[rrrr, "a"] \arrow[ddd, hook] \arrow[drr, hook, "k"] \arrow[ddrr, hook, "i"'] & & & & A \arrow[ddd, two heads, "p"] \\
& & E_0^- \star \Delta[m] \cup E_1^- \star \partial\Delta[m] \underset{E_1^-}{\cup} E_3^- \arrow[urr, dashed] \arrow[d, hook, "\ell"'] & & \\
& & E_2^- \star \Delta[m] \cup E_3^- \star \partial\Delta[m] \arrow[r, hook] \arrow[uurr, dotted] & E_3^- \star \Delta[m] \arrow[dr, dashed] & \\
E_1^- \star \Delta[m] \arrow[urr, hook] \arrow[urrr, hook, "j"] \arrow[rrrr, "b"'] & & & & B
\end{tikzcd}$$

The map j is a complicial anodyne extension, so since B is a complicial set, the lower dashed extension exists, and we are left to solve a lifting problem between the map i and the map p. To do so, we factor the map i as a composite of the three middle vertical morphisms displayed in (D.4.8) and let k denote the first of these morphisms while ℓ denotes the composite of the second two. We next solve the lifting problem between k and p by defining the image of the attached E_3^- to be a degenerate 3-simplex; note that k is constructed by attaching this E_3^- to the E_1^- that corresponds to the initial edge of the horn $\Lambda^0[m+2]$, which a maps to a degenerate edge.

Now to construct the dotted lift, it remains only to solve the lifting problem between ℓ and p, and the diagram (D.4.8) reveals that this can be done, as it expresses the map ℓ as the composite of pushouts of inner complicial horn inclusions. ☐

D.4.12. Corollary. *A marked simplicial set A that admits fillers for inner complicial horn extensions and inner complicial thinness extensions is a complicial set if and only if it admits extensions along the map $\Delta[1]^\sharp \hookrightarrow \mathrm{sk}_2\,\mathbb{I}^\sharp$.*

Proof. Applying Proposition D.4.7 to the inner complicial fibration $A \to 1$, to conclude that A is a complicial set, we need only construct extensions along the maps $E_0^- \hookrightarrow E_1^-$ and $E_1^- \hookrightarrow E_3^-$. The former is automatic, since $E_0^- \hookrightarrow E_1^-$ is isomorphic to $\Delta[0] \hookrightarrow \Delta[1]^\sharp$, which admits a retraction, so to complete our proof we will show that if A admits extensions along the map $\Delta[1]^\sharp \hookrightarrow \mathrm{sk}_2\,\mathbb{I}^\sharp$ as well as fillers for inner complicial horns, then A admits extensions along the map $E_1^- \hookrightarrow E_3^-$.

The domain map $\Delta[1]^\sharp \cong E_1^- \to A$ defines a marked 1-simplex $f\colon x \to y$ in A, which by hypothesis we may extend to a map $\mathrm{sk}_2\,\mathbb{I}^\sharp \to A$, which specifies marked 2-simplices

$$\begin{tikzcd}[column sep=small]
& y \arrow[dr, "f^{-1}"] \arrow[d, phantom, "\alpha"] & \\
x \arrow[ur, "f"] \arrow[rr, equal] & {} & x
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
& x \arrow[dr, "f"] \arrow[d, phantom, "\alpha'"] & \\
y \arrow[ur, "f^{-1}"] \arrow[rr, equal] & {} & y
\end{tikzcd}$$

It won't necessarily be the case that the pair of 2-simplices α and α' form the two non-degenerate faces of a map $E_3^- \to A$ but we will construct a replacement β of α' so that α and β form the non-degenerate simplices of $E_3^- \to A$. The 3-simplex E_3^- will be constructed as the 2nd face of the filler to a horn $\Lambda^2[4] \to A$ that we now construct. As orientation for the construction given below, we first summarize the end result:

- the 0th, 2nd, and 3rd vertices will be x, while the 1st and 4th vertex will be y;

- all of the edges will be either id_x, id_y, f, or f^{-1}, with the positioning of these determined uniquely by the vertices;
- the faces $\{0,1,2\}$ and $\{0,1,3\}$ are α, while the face $\{1,2,4\}$ is α';
- the faces $\{0,2,3\}$, $\{0,1,4\}$, $\{0,3,4\}$, and $\{1,2,3\}$ are degenerate; and
- the missing faces $\{0,2,4\}$ called γ, $\{2,3,4\}$ called $\bar{\gamma}$, and $\{1,3,4\}$ called β will be filled in in this order, with the desired 3-simplex E_3^- appearing as the 2nd face of the horn.

$$\begin{tikzcd}
x \arrow[r, "f"] \arrow[rr, bend right, equal, "\alpha"] \arrow[rrrr, bend right=50, "f"'] & y \arrow[r, "f^{-1}"] \arrow[rr, bend left, "f^{-1}"] \arrow[rrr, bend left=50, equal, "\beta"'] & x \arrow[r, equal] \arrow[rr, bend right, "f"', "\bar{\gamma}"] \arrow[r, phantom, shift right=4ex, "\gamma"] & x \arrow[r, "f"] & y
\end{tikzcd}$$

The 4th face is $\alpha\sigma^2$. The 3rd face is constructed by filling the horn $\Lambda^1[3] \to A$ depicted below

$$\begin{tikzcd}
& y \arrow[dr, equal] \arrow[dd, "f^{-1}"] & \\
x \arrow[ur, "f"] \arrow[rr, "f"'] \arrow[dr, equal] \arrow[r, phantom, shift left=2ex, "\alpha"] & \arrow[r, phantom, shift left=2ex, "\alpha'"] & y \\
& x \arrow[ur, "f"'] &
\end{tikzcd}$$

in which the back face is degenerate. By a complicial thinness extension, the face γ defined by filling this horn is marked. Next, the 1st face is constructed by filling a $\Lambda^0[3] \hookrightarrow \Delta^0[3]$ whose 2nd face is γ and 1st and 3rd faces are degenerate, as permitted by Lemma D.4.11.

$$\begin{tikzcd}
& x \arrow[dr, "f"] \arrow[dd, equal] & \\
x \arrow[ur, equal] \arrow[rr, "f"'] \arrow[dr, equal] & \arrow[r, phantom, shift left=2ex, "\gamma"] & y \\
& x \arrow[ur, "f"'] &
\end{tikzcd}$$

This produces another marked 3-simplex $\bar{\gamma}$ defined by filling this horn. The 0th face is constructed by filling the horn $\Lambda^1[3] \to A$ depicted below

$$\begin{tikzcd}
& x \arrow[dr, "f"] \arrow[dd, equal] & \\
y \arrow[ur, "f^{-1}"] \arrow[rr, equal] \arrow[dr, "f^{-1}"'] & \arrow[r, phantom, shift left=2ex, "\bar{\gamma}"] & y \\
& x \arrow[ur, "f"'] &
\end{tikzcd}$$

in which the unlabeled front face is degenerate and the unlabeled back face is α'. The face defined by filling this horn is the replacement 2-simplex β. These four 3-simplices define a map $\Lambda^2[4] \to A$

whose filler defines a 2-simplex face as depicted below:

$$\begin{tikzcd} & y \arrow[dr, equal] \arrow[dd, "f^{-1}" near end] \arrow[dr, phantom, "\beta" near start, shift right=3ex] & \\ x \arrow[ur, "f"] \arrow[rr, "f" near start] \arrow[dr, equal] \arrow[r, phantom, "\alpha", shift left=2ex] & & y \\ & x \arrow[ur, "f"'] & \end{tikzcd}$$

in which both the back and bottom faces are degenerate. This defines the required extension $E_3^- \to A$. $\square$

With Corollary D.4.12 in hand, we can now prove Joyal's special outer horn filling result.

Proof of Proposition D.4.5. Let A be a naturally marked quasi-category. We must show that the unique map $! \colon A \to \mathbb{1}$ satisfies the hypotheses of Corollary D.4.12. For the inner complicial horn extensions, all of the non-degenerate marked simplices are in dimension two and higher, so A admits fillers for these as well, simply because quasi-categories are simplicial sets that admit fillers for all inner horns. Since the composite of a pair of isomorphisms in a quasi-category is again an isomorphism, A admits extensions along $\Delta^1[2]' \hookrightarrow_e \Delta^1[2]''$. The remaining complicial thinness extensions of (D.1.11) are entire inclusions that differ only in markings of simplices in dimension at least two; since all such simplices are thin in the natural marking, A admits these extensions as well.

To conclude, we need only argue that A admits extensions of the form

$$\begin{tikzcd} \Delta[1]^\sharp \arrow[r, "f"] \arrow[d, hook] & A \\ \mathrm{sk}_2\, \mathbb{I}^\sharp \arrow[ur, dashed] & \end{tikzcd}$$

Since A is naturally marked, the attaching map $f \colon x \to y$ defines an isomorphism in A. By Definition 1.1.13, this means there exist 2-simplices

$$\begin{tikzcd} & y \arrow[dr, "f^{-1}"] \arrow[d, phantom, "\alpha"] & \\ x \arrow[ur, "f"] \arrow[rr, equal] & {} & x \end{tikzcd} \qquad \begin{tikzcd} & x \arrow[dr, "f"] \arrow[d, phantom, "\alpha'"] & \\ y \arrow[ur, "f^{-1}"] \arrow[rr, equal] & {} & y \end{tikzcd}$$

which provide the data of the required extension $\mathrm{sk}_2\, \mathbb{I}^\sharp \to A$. $\square$

This proves the hard direction of the following characterization of quasi-categories as complicial sets:

D.4.13. Theorem. *The natural marking of a quasi-category is a complicial set and indeed is the maximal marking that turns a quasi-category into a complicial set. Conversely, the underlying simplicial set of any complicial set with all simplices above dimension one marked is a quasi-category.*

Proof. In the proof of Proposition D.4.5, we have shown that the natural marking of a quasi-category A defines a complicial set: the inner extensions are straightforward, while Proposition D.4.5 proves that A admits the outer complicial horn and thinness extensions of (D.1.10) and (D.1.11). By Lemma D.4.2, it is not possible to mark any additional edges in A and retain the property of being a complicial set.

The converse is elementary, and left to the reader in Exercise D.1.iii. ☐

We now give a few sample applications of Theorem D.4.13, revisiting some results that were proven in §1.1 using Proposition D.4.5. For instance:

D.4.14. Corollary. *A quasi-category A is a Kan complex if and only if its homotopy category is a groupoid.*

Proof. It's clear that the homotopy category of a Kan complex is a groupoid, so we focus our attention on the converse. By Theorem D.4.13, a quasi-category may be regarded as a complicial set, with every simplex above dimension 1 marked, and where the marked edges are exactly the isomorphisms defined to be those 1-simplices that represent isomorphisms in the homotopy category. If the homotopy category of A is a groupoid, then this tells us that A is maximally marked, and Exercise D.1.ii observes that a maximally marked complicial set defines a Kan complex. ☐

For our next result, we revisit Corollary 1.1.16 and eliminate the reference to the maximal Kan complex spanned by the isomorphisms in a quasi-category — the existence of which follows from special outer horn lifting — from the proof given there.

D.4.15. Corollary. *An arrow f in a quasi-category A is an isomorphism if and only if it extends to a **homotopy coherent isomorphism***

$$\begin{tikzcd} \mathbb{2} \arrow[r, "f"] \arrow[d, hook] & A \\ \mathbb{I} \arrow[ur, dashed] \end{tikzcd}$$

Proof. When the quasi-category A is regarded as a naturally marked complicial set, the isomorphism f defines a marked map $f\colon \mathbb{2}^\sharp \to A$. By Lemma D.1.13, the injection $\mathbb{2}^\sharp \hookrightarrow \mathbb{I}^\sharp$ is a marked anodyne extension, and thus a lift

$$\begin{tikzcd} \mathbb{2}^\sharp \arrow[r, "f"] \arrow[d, hook] & A \\ \mathbb{I}^\sharp \arrow[ur, dashed] \end{tikzcd}$$

exists. Forgetting markings, this proves that every isomorphism in A extends to a "homotopy coherent isomorphism." The converse is obvious from Definition 1.1.13. ☐

D.4.16. Lemma. *If A is a naturally marked quasi-category and X is a minimally marked simplicial set, then A^X is a naturally marked quasi-category.*

Proof. By Proposition D.3.4, n-simplices in A^X correspond to maps $X \times \Delta[n] \to A$. Since X is minimally marked, it follows that the underlying simplicial set of A^X coincides with the exponential of the underlying simplicial sets and hence, by Corollary D.3.13(i) defines a quasi-category.

To see that every simplex of dimension greater than one is marked in A^X consider an extension problem

$$\begin{tikzcd} X \times \Delta[n] \arrow[r] \arrow[d, hook, "e"'] & A \\ X \times \Delta[n]_t \arrow[ur, dashed] \end{tikzcd}$$

for $n > 1$. By Proposition D.3.4, the only simplices that are marked in $X \times \Delta[n]_t$ but not in $X \times \Delta[n]$ are n-simplices, and since all n-simplices in A are marked, it is clear that the desired extension exists.

By Corollary D.3.13(ii), A^X is a complicial set, so by Lemma D.4.2 every marked edge in A^X is an isomorphism. It remains only to show that every isomorphism $f\colon \Delta[1] \to A^X$ in the quasi-category A^X is marked, admitting an extension as indicated below-right:

$$\begin{tikzcd} \coprod_{x\in X_0} \Delta[1] \arrow[r] \arrow[d, hook, "e"'] \arrow[rr, bend left, "f_x"] \arrow[dr, phantom, "\lrcorner" near start] & X \times \Delta[1] \arrow[r, "f"] \arrow[d, hook, "e"'] & A \\ \coprod_{x\in X_0} \Delta[1]_t \arrow[r, "{(x\sigma^0, \mathrm{id})}"] \arrow[ur, phantom, "\ulcorner" near end] \arrow[urr, dotted, bend right] & X \times \Delta[1]_t \arrow[ur, dashed] \end{tikzcd} \tag{D.4.17}$$

By Proposition D.3.4, the only simplices that are marked in $X \times \Delta[1]_t$ but not in $X \times \Delta[1]$ are 1-simplices whose component in X is degenerate and whose component in $\Delta[1]_t$ is the non-degenerate 1-simplex, as indicated by the square above-left that is both a pullback and a pushout. The images of such simplices in A define the "components" of f, as indicated by the top composite above.

If f is an isomorphism in A^X then each of its components f_x, the image of f under the evaluation function $\mathrm{ev}_x\colon A^X \to A$, are clearly also isomorphisms, which is the case if and only if each f_x is marked in A. These components f_x are marked in A if and only if the dotted lift exists, and by the universal property of the pushout, this is equivalent to the existence of the dashed lift, as required. □

As a consequence of Lemma D.4.16, we can show:

D.4.18. Corollary. *For any quasi-category A and simplicial set X, an edge in A^X is an isomorphism if and only if each of its components in A, indexed by the vertices of X, are isomorphisms.*

Proof. Regarding A as a naturally marked quasi-category and X as a minimally marked simplicial set, by Lemma D.4.16 a 1-simplex in A^X is marked if and only if it defines an isomorphism in the quasi-category A^X. So the statement asserts that a 1-simplex is marked in A^X if and only if its components are marked in A. By the definition, given in Proposition D.3.4, of the markings in the exponential, a 1-simplex $f\colon \Delta[1] \to A^X$ is marked if and only if the dashed extension of (D.4.17) exists. By the argument given there, this is equivalent to the existence of the dotted extensions, which say exactly that each component f_x is marked in A. □

A similar argument to the one used to prove Proposition D.4.5 that also employs the conclusion of that result, allow us to extend the special outer horn lifting property to isofibrations between naturally marked quasi-categories:

D.4.19. Theorem. *An isofibration between naturally marked quasi-categories admits fillers for outer complicial horn inclusions for $n \geq 1$:*

$$\begin{tikzcd} \Lambda^0[n] \arrow[r] \arrow[d, hook, "r"'] & A \arrow[d, two heads] \\ \Delta^0[n] \arrow[r] \arrow[ur, dashed] & B \end{tikzcd} \qquad \begin{tikzcd} \Lambda^n[n] \arrow[r] \arrow[d, hook, "r"'] & A \arrow[d, two heads] \\ \Delta^n[n] \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

and an inner fibration between naturally marked quasi-categories admits fillers for outer complicial horn inclusions for $n \geq 2$. Consequently, an isofibration between naturally marked quasi-categories is a complicial isofibration.

Proof. By Theorem D.4.13, the codomain B is a complicial set, so to apply Proposition D.4.7 we must argue that an isofibration between naturally marked quasi-categories has the right lifting property against the inner complicial horn extensions, inner complicial thinness extensions, and the two maps $E_0^- \hookrightarrow E_1^-$ and $E_1^- \hookrightarrow E_3^-$. By Remark D.4.10, the first of these maps is only needed to construct the outer complicial horn inclusions for $n = 1$, so the proof given below will establish fillers for the complicial horn inclusions in dimension $n \geq 2$ under the weaker hypothesis where $A \twoheadrightarrow B$ is only assumed to be an inner fibration. We'll concentrate on the case where $A \twoheadrightarrow B$ is an isofibration henceforth.

The inner complicial horn and thinness extensions are straightforward, as in the proof of Proposition D.4.5. To construct a lift

$$\begin{tikzcd}
E_0^- \arrow[r] \arrow[d, hook] & A \arrow[d, two heads, "p"] \\
E_1^- \arrow[r, "b"] \arrow[d, hook] & B \\
\mathbb{I} \arrow[uur, dotted] \arrow[ur, dashed]
\end{tikzcd}$$

we use Lemma D.1.13, recalled as Corollary D.4.15, to extend the codomain to a homotopy coherent isomorphism and then solve the composite lifting problem.

The construction of the lift against $E_1^- \hookrightarrow E_3^-$ is considerably more laborious. To begin, we argue that since A and B are complicial sets and p is an inner complicial fibration, then p admits lifts against the complicial horn inclusion $\Lambda^0[2] \to \Delta^0[2]$. To see this, we identify the codomain $\Delta^0[2]$ with the second face of the 3-simplex $E_2^- \star \Delta[0]$ and consider the lifting problem presented by the exterior diagram

$$\begin{tikzcd}
\Lambda^0[2] \arrow[r, hook, "i_{013}"'] \arrow[d, hook] \arrow[rrr, bend left, "a"] & E_0^- \star \Delta[0] \cup E_2^- \star \varnothing \arrow[rr, dashed] \arrow[d, hook] \arrow[dr, hook, "k"] & & A \arrow[d, two heads, "p"] \\
\Delta^0[2] \arrow[r, hook] \arrow[rr, hook, bend right, "\delta^2"'] \arrow[rrr, bend right=40, "b"'] & \bullet \arrow[r, hook, "j"] \arrow[ul, phantom, "\ulcorner"] \arrow[rr, dotted, bend right] & E_2^- \star \Delta[0] \arrow[r, dashed] \arrow[ur, dotted] & B
\end{tikzcd}$$

The inclusion i_{013} is complicially anodyne, so the top dashed extension exists since A is a complicial set by Theorem D.4.13. This induces the dotted map by the universal property of the pushout. Since the map j is also complicially anodyne, the bottom dashed extension exists since B is a complicial set by Theorem D.4.13. These dashed maps define a new lifting problem between the composite map k and p and since k is an inner complicial anodyne extension, the dotted lift exists, solving the original lifting problem.

Now we can use the fact that p admits lifts along $\Lambda^0[2] \hookrightarrow \Delta^0[2]$ to construct lifts along the horizontal composite map

$$\begin{tikzcd} & E_1^+ \arrow[r, hook] \arrow[d, hook] & E_2^+ \arrow[d, hook] \\ E_1^- \arrow[r, hook] & E_2^- \arrow[r, hook] \arrow[ur, phantom, very near end, "\ulcorner"] & E_2^- \underset{E_1^+}{\cup} E_2^+ \end{tikzcd}$$

since Digression D.4.6 reveals that both maps $E_1^\pm \hookrightarrow E_2^\pm$ are pushouts of $\Lambda^0[2] \hookrightarrow \Delta^0[2]$. Lifts agains the composite $E_1^- \hookrightarrow E_2^- \underset{E_1^+}{\cup} E_2^+$ have the effect of giving the data of a right inverse and also a right inverse to the right inverse to an isomorphism in A, lifting the corresponding data in B. To solve a lifting problem

$$\begin{tikzcd} & E_1^- \arrow[r, "f"] \arrow[dl, hook] \arrow[d, hook, "(\alpha,\alpha')"] & A \arrow[d, two heads, "p"] \\ E_2^- \underset{E_1^+}{\cup} E_2^+ \arrow[r] \arrow[urr, dashed] & E_3^- \arrow[r, "\tau"'] & B \end{tikzcd}$$

we first construct the outer lift. The lift defines a pair of 2-simplices

$$\begin{tikzcd} & y \arrow[dr, "f^{-1}"] \arrow[d, phantom, "\alpha"] & \\ x \arrow[ur, "f"] \arrow[rr, equals] & \, & x \end{tikzcd} \qquad \begin{tikzcd} & x \arrow[dr, "(f^{-1})^{-1}"] \arrow[d, phantom, "\alpha'"] & \\ y \arrow[ur, "f^{-1}"] \arrow[rr, equals] & \, & y \end{tikzcd}$$

in A.

We'll now extend the data of the lifted $E_2^- \underset{E_1^+}{\cup} E_2^+ \to A$ to construct a map $E_3^- \to A$ lifting $E_3^- \to B$ using a mild modification of the construction in the proof of Corollary D.4.12.

As above, we'll define a complicial inner horn extension

$$\begin{tikzcd} \Lambda^2[4] \arrow[r] \arrow[d, hook] & A \arrow[d, two heads, "p"] \\ \Delta^2[4] \arrow[r, "\tau\sigma^2"'] \arrow[ur, dashed] & B \end{tikzcd}$$

so that the lift of the 2nd face defines the simplex $E_3^- \to A$ that we seek. It remains only to define an appropriate horn $\Lambda^2[4] \to A$ over $\tau\sigma^2 \colon \Delta^2[4] \to B$. The 4th face is $\alpha\sigma^2$. The 3rd face is constructed by lifting along a horn $\Lambda^1[3] \hookrightarrow \Delta^1[3]$ whose 3rd face is α, whose 0th face is α', and whose 2nd face is degenerate. Writing γ for the face defined by filling this horn, the 1st face is constructed by filling a $\Lambda^0[3] \hookrightarrow \Delta^0[3]$ whose 2nd face is γ and 1st and 3rd faces are degenerate; the argument required to justify this, as permitted by Lemma D.4.11. The 0th face is constructed by filling the horn $\Lambda^1[3] \to \Delta^1[3]$ whose faces have all already been described. The face defined by filling this horn is the replacement 2-simplex β which witnesses that f is a right inverse to its right inverse f^{-1}. These four 3-simplices define a map $\Lambda^2[4] \to A$ over $\tau\sigma^2$ whose filler defines a 2-simplex face as depicted below:

$$\begin{tikzcd}
& y \arrow[dd, "f^{-1}"] \arrow[dr, equal] & \\
x \arrow[ur, "f"] \arrow[rr, "f"'] \arrow[dr, equal] \arrow[r, phantom, shift left=3, "\alpha"] & \arrow[r, phantom, shift left=3, "\beta"] & y \\
& x \arrow[ur, "f"'] &
\end{tikzcd}$$

in which both the back and bottom faces are degenerate. This defines the required lift along $E_1^- \hookrightarrow E_3^-$. Now Proposition D.4.7 completes the proof that isofibrations lift against outer complicial horn inclusions.

It follows that an isofibration between naturally marked quasi-categories is a complicial isofibration: we've demonstrated that it has the required lifting property against the complicial horn inclusions. By Exercise D.1.v, the fact that A is a complicial set suffices to show that $p\colon A \twoheadrightarrow B$ also lifts against the complicial thinness extensions. $\square$

Theorems D.4.13 and D.4.19 permit the use of complicial techniques to solve lifting problems involving isofibrations between quasi-categories. The results of this section suggest that these techniques are particularly fruitful when isomorphisms are involved. We conclude this section by developing a few specific applications of this principle.

To that end we consider a pair of cosimplicial marked simplicial sets

$$\mathbf{\Delta}[\bullet]^\sharp \hookrightarrow \mathbb{I}[\bullet]^\sharp \quad \in (s\mathcal{S}et^+)^{\mathbf{\Delta}}$$

the former of which is given by the maximally marked simplices $\Delta[n]^\sharp$ and the latter of which is given by the maximally marked contractible groupoids $\mathbb{I}[n]^\sharp$ on objects $0, 1, \ldots, n$.

D.4.20. Lemma. *The natural inclusion $\mathbf{\Delta}[\bullet]^\sharp \hookrightarrow \mathbb{I}[\bullet]^\sharp$ is a Reedy monomorphism in $(s\mathcal{S}et^+)^{\mathbf{\Delta}}$ between Reedy monomorphic cosimplicial objects that is moreover a pointwise weak equivalence.*

Proof. The first half of the statement follows from a general principle: any pointwise monomorphism between "unaugmentable" cosimplicial objects is a Reedy monomorphism between Reedy monomorphic cosimplicial objects, where a cosimplicial object $X^\bullet$ is **unaugmentable** if the equalizer of the face maps $\delta^0, \delta^1\colon X^0 \rightrightarrows X^1$ is empty [**75**, 14.3.8]. The idea is that in the presence of this condition, any simplex in $X^\bullet$ is uniquely expressable as $\alpha \cdot x$, where $\alpha \in \mathbf{\Delta}$ is a monomorphism and $x \in X^\bullet$ is not in the image of any monomorphism. The proof of this relies on the observation that nearly every monomorphism in $\mathbf{\Delta}$ is uniquely determined by its set of left-inverses, the only exceptions being $\delta^0, \delta^1\colon [0] \rightrightarrows [1]$; hence the "unaugmentable" condition. From this definition it is clear that both $\mathbf{\Delta}[\bullet]^\sharp$ and $\mathbb{I}[\bullet]^\sharp$ are unaugmentable, so we conclude that these simplicial objects are Reedy monomorphic and the natural inclusion is a Reedy monomorphism.

Finally to prove that $\Delta[n]^\sharp \to \mathbb{I}[n]^\sharp$ is a pointwise weak equivalence, we appeal to the 2-of-3 property and argue that both $\Delta[n]^\sharp$ and $\mathbb{I}[n]^\sharp$ are contractible in the sense of being marked homotopy equivalent to $\mathbb{1} \in s\mathcal{S}et^+$. The inverse homotopy equivalences are given by $0\colon \mathbb{1} \to \Delta[n]^\sharp$ and $0\colon \mathbb{1} \to \mathbb{I}[n]^\sharp$ and the marked homotopies $\Delta[n]^\sharp \times \Delta[1]^\sharp \to \Delta[n]^\sharp$ and $\mathbb{I}[n]^\sharp \times \Delta[1]^\sharp \to \mathbb{I}[n]^\sharp$ are both defined by the map on objects $(i, 0) \mapsto 0$ and $(i, 1) \mapsto i$. $\square$

Our intent is to use the simplicial objects $\mathbf{\Delta}[\bullet]^\sharp$ and $\mathbb{I}[\bullet]^\sharp$ to "freely invert" the simplices of a simplicial set K. To see how this works, consider also the cosimplicial object $\mathbf{\Delta}[\bullet] \in s\mathcal{S}et^{\mathbf{\Delta}}$ defined by the Yoneda embedding. By the Yoneda lemma, the weighted colimit $\operatorname{colim}_K \mathbf{\Delta}[\bullet] \cong K$ recovers the original simplicial set K. Similarly, since the maximal marking functor $(-)^\sharp \colon s\mathcal{S}et \to s\mathcal{S}et^+$ is a left adjoint, the weighted colimit $\operatorname{colim}_K \mathbf{\Delta}[\bullet]^\sharp \cong K^\sharp$ equips the simplicial set K with the maximal marking. Finally, we define $\tilde{K}^\sharp \coloneqq \operatorname{colim}_K \mathbb{I}[\bullet]^\sharp$ using the weighted colimit bifunctor. The idea of this functor is that it replaces each n-simplex of K by $\mathbb{I}[n]^\sharp$, a "homotopy coherent composite of n-isomorphisms." As the notation suggests, $\tilde{K}^\sharp$ is also maximally marked.

D.4.21. Proposition. *For any simplicial set K, the natural map $K^\sharp \to \tilde{K}^\sharp$ is a trivial cofibration of marked simplicial sets.*

Proof. Any simplicial set K is Reedy monomorphic when considered as an object of $\mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}}$. Hence, by Corollary C.5.17, the weighted colimit functor

$$\operatorname{colim}_K - \colon (s\mathcal{S}et^+)^{\mathbf{\Delta}} \to s\mathcal{S}et^+$$

is left Quillen with respect to the Reedy model structure on $(s\mathcal{S}et^+)^{\mathbf{\Delta}}$. Lemmas C.5.13 and D.4.20 prove that $\mathbf{\Delta}[\bullet]^\sharp \hookrightarrow \mathbb{I}[\bullet]^\sharp$ is a Reedy trivial cofibration, so it follows that $K^\sharp \to \tilde{K}^\sharp$ is a trivial cofibration as claimed. □

We refer to the functor $(\tilde{-}) \colon s\mathcal{S}et \to s\mathcal{S}et$ defined by applying $\operatorname{colim}_{-} \mathbb{I}[\bullet]^\sharp$ and forgetting the markings as the "free-inversion functor." Proposition D.4.21 has the following consequence:

D.4.22. Corollary. *Any diagram $K \to A$ in a quasi-category A whose edges are sent to isomorphisms extends to a diagram indexed by the free inversion of K. More generally, if $K \to A$ sends the edges of K to isomorphisms of A, and $p \colon A \twoheadrightarrow B$ is an isofibration between quasi-categories, then any lifting problem*

$$\begin{tikzcd} K \arrow[r] \arrow[d, hook] & A \arrow[d, two heads, "p"] \\ \tilde{K} \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

has a solution

Proof. The hypothesis is that our given diagram extends to a map $K^\sharp \to A^\natural$ of marked simplicial sets. By Theorem D.4.13, the natural marking of A defines a complicial set and by Theorem D.4.19 $p \colon A^\natural \twoheadrightarrow B^\natural$ is a complicial isofibration, so Proposition D.4.21 implies that the lift

$$\begin{tikzcd} K^\sharp \arrow[r] \arrow[d, hook] & A^\natural \arrow[d, two heads, "p"] \\ \tilde{K}^\sharp \arrow[r] \arrow[ur, dashed] & B^\natural \end{tikzcd}$$

exists. □

D.4.23. Example. Proposition D.4.21 can be applied to the anodyne extension $\Lambda^1[2] \hookrightarrow \Delta[2]$ to prove that composites of homotopy coherent isomorphisms can be lifted along isofibrations of quasi-categories. By definition $\widetilde{\Delta[2]} \cong \mathbb{I}[2]$, the contractible groupoid on three vertices 0, 1, and 2, so we adopt similar notation $\widetilde{\Lambda^1[2]} =: \Lambda^1[\mathbb{I}[2]]$ for the freely-inverted horn. Since $\Lambda^1[2]$ is built by gluing two 1-simplices along a common vertex and the free inversion functor preserves colimits, we see that $\Lambda^1[\mathbb{I}[2]]$ is the union of two homotopy coherent isomorphisms between 0 and 1 and between 1 and

2. Giving these simplicial sets their maximal markings, it follows from the 2-of-3 property applied to the square

$$\begin{tikzcd} \Lambda^1[2]^\sharp \arrow[r, hook] \arrow[d, hook] & \Lambda^1[\mathbb{I}[2]]^\sharp \arrow[d, hook] \\ \Delta[2]^\sharp \arrow[r, hook] & \mathbb{I}[2]^\sharp \end{tikzcd}$$

that the inclusion $\Lambda^1[\mathbb{I}[2]]^\sharp \hookrightarrow \mathbb{I}[2]^\sharp$ is a trivial cofibration of marked simplicial sets. Applying Theorem D.4.19, we conclude that composites of homotopy coherent isomorphisms can be lifted along isofibrations between quasi-categories

$$\begin{tikzcd} \Lambda^1[\mathbb{I}[2]] \arrow[r] \arrow[d, hook] & A \arrow[d, two heads] \\ \mathbb{I}[2] \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

A similar example is left as Exercise D.4.ii. In fact, Joyal proves that $\tilde{(-)}\colon s\mathcal{S}et \to s\mathcal{S}et$ defines a left Quillen functor from Quillen's model structure for Kan complexes to Joyal's model structure for quasi-categories, whose right adjoint is the functor that sends a simplicial set X to the simplicial set whose n-simplices are maps $\mathbb{I}[n] \to X$. His proof, which is replicated in [**75**, 17.6.1], makes use of a lemma that is also of interest.

D.4.24. Lemma. *For any simplicial set K, the natural inclusion $K \hookrightarrow \tilde{K}$ is an anodyne extension.*

Proof. This follows from Proposition D.4.21 and the fact that the functor $s\mathcal{S}et^+ \to s\mathcal{S}et$ that forgets the markings carries marked anodyne extensions to anodyne extensions but a direct proof is also possible. To see that $K \hookrightarrow \tilde{K}$ is an anodyne extension, consider a Kan fibration between Kan complexes $p\colon A \twoheadrightarrow B$. By Corollary D.4.22 a lift

$$\begin{tikzcd} K \arrow[r] \arrow[d, hook] & A \arrow[d, two heads, "p"] \\ \tilde{K} \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

exists from which we conclude that $K \hookrightarrow \tilde{K}$ is an anodyne extension. ☐

D.4.25. Proposition. *The free inversion functor and its left adjoint defines a Quillen adjunction from Quillen's model structure for Kan complexes to Joyal's model structure for quasi-categories.*

Proof. Lemma D.4.20 proves that the functor $\mathbb{I}[\bullet]^\sharp$ is Reedy monomorphic in $(s\mathcal{S}et^+)^\Delta$. Hence, by Theorem C.5.15, the weighted colimit functor

$$\mathrm{colim}_{-}\, \mathbb{I}[\bullet]^\sharp\colon s\mathcal{S}et \cong \mathcal{S}et^{\Delta^{\mathrm{op}}} \to s\mathcal{S}et^+$$

carries Reedy monomorphisms to monomorphisms. Since the Reedy monomorphisms in $\mathcal{S}et^{\Delta^{\mathrm{op}}}$ are just the monomorphisms in $s\mathcal{S}et$, in this way we see that the left adjoint $\tilde{(-)}\colon s\mathcal{S}et \to s\mathcal{S}et$ preserves monomorphisms.

It remains to argue that $\tilde{(-)}$ carries weak equivalences in the Quillen model structure to weak equivalences in the Joyal model structure. If $K \xrightarrow{\sim} L$ is a weak equivalence in the Quillen model structure then by the 2-of-3 property applied to the square

$$\begin{tikzcd} K \arrow[r, "\sim"] \arrow[d, "\wr"] & L \arrow[d, "\wr"] \\ \tilde{K} \arrow[r] & \tilde{L} \end{tikzcd}$$

the map $\tilde{K} \xrightarrow{\sim} \tilde{L}$ is as well. By construction of $\tilde{K}$ and $\tilde{L}$, their homotopy categories of Definition 1.1.10 are groupoids. If we fibrantly replace each object in the Joyal model structure, we obtain quasi-categories $\tilde{K}^\dagger$ and $\tilde{L}^\dagger$ which are weakly equivalent in the Joyal model structure to $\tilde{K}$ and $\tilde{L}$ and whose homotopy categories are thus also groupoids. In particular, by Corollary D.4.14, $\tilde{K}^\dagger$ and $\tilde{L}^\dagger$ are Kan complexes. This gives us a commutative diagram

$$\begin{tikzcd} \tilde{K} \arrow[r] \arrow[d] & \tilde{L} \arrow[d] \\ \tilde{K}^\dagger \arrow[r] & \tilde{L}^\dagger \end{tikzcd}$$

in which the vertical maps are weak equivalences in the Joyal model structure, and hence also the Quillen model structure. Since the top horizontal map is also a weak equivalence is the Quillen model structure, it follows from the 2-of-3 property that $\tilde{K}^\dagger \xrightarrow{\sim} \tilde{L}^\dagger$ is as well, but any weak homotopy equivalence between Kan complexes is also an equivalence of quasi-categories. By the 2-of-3 property again, we conclude that $\tilde{K} \xrightarrow{\sim} \tilde{L}$ is a weak equivalence in the Joyal model structure, as desired. ☐

Exercises.

D.4.i. Exercise. Prove Lemma D.4.3.

D.4.ii. Exercise. Extend the result of Example D.4.23 to show that the maximally marked "$\Lambda^1[3]$-horn of homotopy coherent isomorphisms" $\Lambda^1[\mathbb{I}[3]] \hookrightarrow \mathbb{I}[3]$, whose codomain is the contractible groupoid on four vertices $0, 1, 2, 3$ and whose domain is the union of the three copies of $\mathbb{I}[2]$ spanned by the subsets of three of these four vertices that include the vertex 1, is a complicial anodyne extension.

D.5. Isofibrations between quasi-categories

Our first aim in this section is to integrate the class of isofibrations between quasi-categories into the results proven in §D.3. We start by proving Propositions 1.1.19 and 1.1.28, restated here for convenience.

D.5.1. Proposition.

(i) There is a solution to any lifting problem between the pushout product of a monomorphism $i\colon X \hookrightarrow Y$ and the map $\mathbb{1} \hookrightarrow \mathbb{I}$ and any isofibration $f\colon A \twoheadrightarrow B$.

(ii) If $i\colon X \hookrightarrow Y$ is a monomorphism and $f\colon A \twoheadrightarrow B$ is an isofibration, then the induced Leibniz exponential map

$$A^Y \xrightarrow{\widehat{i \pitchfork f}} B^Y \times_{B^X} A^X$$

is again an isofibration.

(iii) If $i\colon X \hookrightarrow Y$ is a monomorphism and $f\colon A \overset{\sim}{\twoheadrightarrow} B$ is a trivial fibration, then the induced Leibniz exponential map

$$A^Y \xrightarrow{\widehat{i \pitchfork f}} B^Y \times_{B^X} A^X$$

is again an trivial fibration.

(iv) If $i\colon X \hookrightarrow Y$ is in the class generated by the inner horn inclusions and the map $\mathbb{1} \hookrightarrow \mathbb{I}$ and $f\colon A \twoheadrightarrow B$ is an isofibration, then the induced Leibniz exponential map

$$A^Y \xrightarrow{\widehat{i \pitchfork f}} B^Y \times_{B^X} A^X$$

is a trivial fibration.

Proof. It suffices to construct the lift of (i) in marked simplicial sets and then forget the markings. By Lemma D.1.13, $\mathbb{1} \hookrightarrow \mathbb{I}$ is a complicial anodyne extension, when $\mathbb{I}$ is assigned its natural maximal marking. Thus by Proposition D.3.9, the Leibniz product of the minimally marked monomorphism i with this map is again a complicial anodyne extension. By Theorem D.4.19, an isofibration defines a complicial isofibration between naturally marked quasi-categories, so the postulated lift exists.

Parts (ii) and (iv) follow from the conclusion of (i) and a similar result, Corollary D.3.12 that shows that the pushout product of a monomorphism and an inner horn inclusion can be factored as a sequence of pushouts of inner horn inclusions. By transposing across the two-variable adjunction between the pushout product and the Leibniz exponential, these results imply that the map $\widehat{i \pitchfork f}$ lifts against the inner horn inclusions and against the map $\mathbb{1} \hookrightarrow \mathbb{I}$. Part (iii) follows by a similar argument from an easier observation: that the pushout product of two monomorphisms is again a monomorphism. □

D.5.2. Proposition. *For a map $f\colon A \to B$ of quasi-categories the following are equivalent:*

(i) f is at trivial fibration

(ii) f is both an isofibration and an equivalence

(iii) f is a ***split fiber homotopy equivalence****: an isofibration admitting a section s that is also an equivalence inverse via a homotopy from id_A to sf that composes with f to the constant homotopy from f to f.*

Proof. For (i)⇒(ii), observe that the simplex boundary inclusions generate the monomorphisms of simplicial sets under coproduct, pushout, and sequential composition (see Lemma C.5.9), so the lifting property of (1.1.25) implies that the trivial fibrations lift against all monomorphisms of simplicial sets, and in particular against the monomorphisms that detect the class of isofibrations. Thus, trivial fibrations are isofibrations. By the same lifting property, every trivial fibration admits a section

$$\begin{tikzcd} & A \arrow[d, two heads, "f", "\wr"'] \\ B \arrow[ur, dashed, "s"] \arrow[r, equal] & B \end{tikzcd}$$

To show that s defines an inverse homotopy equivalence to f, observe that the other rectangle built from the constant homotopy $\pi\colon A \times \mathbb{I} \to A$

$$\begin{tikzcd} A + A \arrow[rr, "{(\mathrm{id}_A, sf)}"] \arrow[d, hook] & & A \arrow[d, two heads, "f", "\wr"'] \\ A \times \mathbb{I} \arrow[urr, dashed, "\alpha"] \arrow[r, "\pi"'] & A \arrow[r, two heads, "f", "\sim"'] & B \end{tikzcd}$$

commutes since $fsf = f$. The lift defines a homotopy between id_A and sf completing the proof that trivial fibrations are equivalences. And note in fact that the equivalence just constructed is a split fiber homotopy equivalence, proving that (i)⇒(iii).

To prove (ii)⇒(iii), suppose that f is an isofibration with equivalence inverse g. By Lemma D.5.3 below, the homotopies α from id_A to gf and β from fg to id_B may be chosen so as to define a "half-adjoint equivalence," meaning that there exists a map $\Phi\colon A\times\mathbb{I}[2] \to B$, where $\mathbb{I}[2]$ is the contractible groupoid on three objects, whose boundary is formed by $f\alpha$, βf, and the constant homotopy $\mathrm{id}_f := f\pi$.[8]

Applying Proposition D.5.1(i) to the monomorphism $\varnothing \hookrightarrow B$, we find that we can lift the homotopy β between fg and id_B along f

$$\begin{tikzcd}
 & B \arrow[r, "g"] \arrow[d, hook, "i_0"'] & A \arrow[d, two heads, "f"] \\
B \arrow[r, hook, "i_1"'] \arrow[urr, dotted, "s"] \arrow[rr, bend right, equal] & B\times\mathbb{I} \arrow[r, "\beta"] \arrow[ur, dashed, "\gamma"'] & B
\end{tikzcd}$$

The composite map s defines a strict section of f, while the lift defines a homotopy γ from g to s. Applying Proposition D.3.9, Theorem D.4.19, and Example D.4.23, we can solve the lifting problem

$$\begin{tikzcd}
 & A\times\Lambda^1[\mathbb{I}[2]] \arrow[r, "{(\alpha,\gamma f)}"] \arrow[d, hook] & A \arrow[d, two heads, "f"] \\
A\times\mathbb{I} \arrow[r, hook, "\delta^1"'] \arrow[urr, dotted, "\eta"] \arrow[dr, "\pi"'] & A\times\mathbb{I}[2] \arrow[r, "\Phi"'] \arrow[ur, dashed, "\Psi"'] & B \\
 & A \arrow[ur, "f"'] &
\end{tikzcd}$$

The lift defines a composite homotopy η from id_A to sf so that $f\eta = f\pi$ is the constant homotopy. This data exhibits f as a split fiber homotopy equivalence.

Finally, for (iii)⇒(i) to prove that $f\colon A \twoheadrightarrow B$ is a trivial fibration it suffices to show that the Leibniz exponential constructed in the diagram below is surjective on vertices (a vertex in its codomain defining a lifting problem that any lift would solve):

$$\begin{tikzcd}
A^{\Delta[n]} \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right, "f^{\Delta[n]}"'] & & \\
 & \bullet \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & A^{\partial\Delta[n]} \arrow[d, two heads, "f^{\partial\Delta[n]}"] \\
 & B^{\Delta[n]} \arrow[r] & B^{\partial\Delta[n]}
\end{tikzcd}$$

By Proposition 1.1.19, the maps displayed with two heads are all isofibrations. Moreover, equivalences are preserved by exponentiation, so the outer vertical maps are also equivalences.

The point of promoting the equivalence f to a split fiber homotopy equivalence is that surjective equivalences are also stable under pullback, which defines a simplicial functor between slice categories

[8] See Theorem 1.4.7 and Proposition 2.1.11 for an explanation of this terminology.

and surjective equivalences live fully in these slice categories. Since the equivalences satisfy the 2-of-3 property (Exercise 1.1.viii) it now follows that the Leibniz exponential is both an isofibration and equivalence. By (ii)⇒(iii), this map admits a section and in particular is surjective on vertices. Thus, we conclude that split fiber homotopy equivalences are trivial fibrations. □

D.5.3. Lemma. *Any equivalence of quasi-categories*

$$\begin{tikzcd} A \arrow[d, hook, "i_0"'] \arrow[dr, equal] & \\ A \times \mathbb{I} \arrow[r, "\alpha"] & A \\ A \arrow[u, hook', "i_1"] \arrow[r, "f"'] & B \arrow[u, "g"'] \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "g"] \arrow[d, hook, "i_0"'] & A \arrow[d, "f"] \\ B \times \mathbb{I} \arrow[r, "\beta"] & B \\ B \arrow[u, hook', "i_1"] \arrow[ur, equal] & \end{tikzcd}$$

can be extended to a ***half-adjoint equivalence of quasi-categories****, with an additional coherence homotopy* $\Phi\colon A \times \mathbb{I}[2] \to B$ *whose boundary is comprised of the three homotopy coherent isomorphisms:*

$$\begin{tikzcd} & fgf \arrow[dr, dash, "\beta f"] & \\ f \arrow[ur, dash, "f\alpha"] \arrow[rr, dash, "\mathrm{id}_f"'] & \Phi & f \end{tikzcd}$$

at the cost of replacing one of the homotopies α *or* β.

The proof is by a simplicial reinterpretation of the 2-categorical argument that proves Proposition 2.1.11.

Proof. Consider an equivalence of quasi-categories as in the statement. By Example D.4.23, the homotopies $f\alpha$ and βf admit some composite defined by solving the lifting problem

$$\begin{tikzcd} & A \times \Lambda^1[\mathbb{I}[2]] \arrow[r, "(f\alpha, \beta f)"] \arrow[d, hook] & B \\ A \times \mathbb{I} \arrow[r, hook, "\delta^1"'] \arrow[urr, dotted, "\psi"] & A \times \mathbb{I}[2] \arrow[ur, dashed, "\Psi"'] & \end{tikzcd}$$

Restriction along the non-identity involution $\mathbb{I} \to \mathbb{I}$ defines the inverse of any homotopy, denoted by $(-)^{-1}$. We will replace α by the composite $\alpha' := g\psi^{-1} \cdot \alpha$ defined by solving the lifting problem

$$\begin{tikzcd} & A \times \Lambda^1[\mathbb{I}[2]] \arrow[r, "(\alpha, g\psi^{-1})"] \arrow[d, hook] & A \\ A \times \mathbb{I} \arrow[r, hook, "\delta^1"'] \arrow[urr, dotted, "\alpha'"] & A \times \mathbb{I}[2] \arrow[ur, dashed, "\Xi"'] & \end{tikzcd}$$

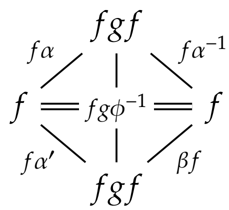
and show that this homotopy defines a half-adjoint equivalence with β.

The witness to the half-adjoint equivalence will be obtained by solving a final lifting problem

$$\begin{tikzcd} & A \times \Lambda^1[\mathbb{I}[3]] \arrow[r, "\Gamma"] \arrow[d, hook] & B \\ A \times \mathbb{I}[2] \arrow[r, hook, "\delta^1"'] \arrow[urr, dotted, "\Phi"] & A \times \mathbb{I}[3] \arrow[ur, dashed] & \end{tikzcd}$$

involving an extension along $\Lambda^1[\mathbb{I}[3]] \hookrightarrow \mathbb{I}[3]$, whose codomain is the contractible groupoid with four objects $0, 1, 2, 3$ and whose domain is the union of the three faces $\mathbb{I}[2]$ which contain the vertex 1. This is permitted by Exercise D.4.ii.

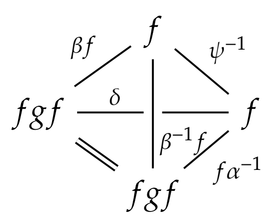
It remains to define the faces of the "horn of homotopy coherent isomorphisms" Γ which is built from the homotopy coherent isomorphisms



The 3rd face is $f\Xi\colon A \times \mathbb{I}[2] \to B$ while the second face is the composite

$$A \times \mathbb{I}[2] \xrightarrow{A \times q} A \times \mathbb{I} \xrightarrow{f\alpha} B$$

where $q\colon \mathbb{I}[2] \to \mathbb{I}$ is the unique map defined by $0, 2 \mapsto 0$ and $1 \mapsto 1$. It remains to define the 0th face. For this, we first extend along another horn $\Lambda^1[\mathbb{I}[3]] \hookrightarrow \mathbb{I}[3]$ of homotopy coherent isomorphisms in B^A as depicted below



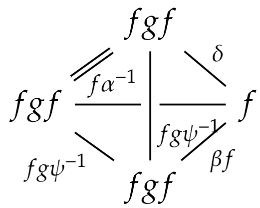
Here the 0th face is Ψ^{-1}, the 3rd face is the composite

$$A \times \mathbb{I}[2] \xrightarrow{A \times q} A \times \mathbb{I} \xrightarrow{\beta f} B$$

and the 2nd face is the composite

$$A \times \mathbb{I}[2] \xrightarrow{A \times d} A \times \mathbb{I}[2] \times \mathbb{I} \xrightarrow{\Psi \times \mathbb{I}} B \times \mathbb{I} \xrightarrow{\beta} B$$

where $d\colon \mathbb{I}[2] \to \mathbb{I}[2] \times \mathbb{I}$ is the unique map defined by $0 \mapsto (2, 0)$, $1 \mapsto (2, 1)$, and $2 \mapsto (0, 1)$. The δ^1-face of this horn can be used to define a final horn $\Lambda^1[\mathbb{I}[3]] \hookrightarrow \mathbb{I}[3]$ of homotopy coherent isomorphisms in B^A as depicted below



whose 3rd face is degenerate, whose 3nd face is an inversion that swaps the first two vertices of the δ^1-face $A \times \mathbb{I}[2] \to B$ just defined, and whose 0th face is the composite

$$A \times \mathbb{I}[2] \xrightarrow{A \times c} A \times \mathbb{I}[2] \times \mathbb{I} \xrightarrow{\Psi \times \mathbb{I}} B \times \mathbb{I} \xrightarrow{\beta} B$$

where $c\colon \mathbb{I}[2] \to \mathbb{I}[2] \times \mathbb{I}$ is the unique map defined by $0 \mapsto (2,0)$, $1 \mapsto (0,0)$, and $2 \mapsto (0,1)$. The filler defines the desired 0th face

$$\begin{tikzcd} & fgf \arrow[dr, dash, "\beta f"] \arrow[d, phantom, "\Sigma"] & \\ fgf \arrow[ur, dash, "fg\psi^{-1}"] \arrow[rr, dash, "f\alpha^{-1}"'] & {} & f \end{tikzcd}$$

that completes the horn $\Gamma\colon A \times \Lambda^1[\mathbb{I}[3]] \to B$ whose filler defines the witness for the half-adjoint equivalence between α' and β. □

D.5.4. Proposition. *A Kan fibration is a trivial fibration if and only if its fibers are contractible.*

Proof. By pullback stability of the class of trivial fibrations, it is clear that the fibers of a trivial fibration are contractible Kan complexes. For the converse, our task is to show that any lifting problem

$$\begin{tikzcd} \partial\Delta[n] \arrow[r, "e"] \arrow[d, hook] & E \arrow[d, two heads, "p"] \\ \Delta[n] \arrow[r, "b"'] & B \end{tikzcd}$$

has a solution. The inclusion of the n-sphere into n-simplex factors as

$$\partial\Delta[n] \xrightarrow{\partial\Delta[n]\times i_1} \partial\Delta[n] \times \Delta[1] \hookrightarrow \Delta[n] \times \Delta[1] \xrightarrow{q} \Delta[n]$$

where q is the quotient map defined on vertices by $(j,i) \mapsto j \cdot i$. The inclusion $\partial\Delta[n]\times i_1$ is an anodyne extension; hence, there is a lift

$$\begin{tikzcd} \partial\Delta[n] \arrow[rrr, "e"] \arrow[d, hook, "\partial\Delta[n]\times i_1"'] & & & E \arrow[d, two heads, "p"] \\ \partial\Delta[n] \times \Delta[1] \arrow[r, hook] \arrow[urrr, dashed, "\ell"] & \Delta[n] \times \Delta[1] \arrow[r, "q"'] & \Delta[n] \arrow[r, "b"'] & B \end{tikzcd}$$

This lift in turn produces a new lifting problem, which we restrict to the sphere sitting over the other endpoint of the cylinder as in the following diagram:

$$\begin{tikzcd} \partial\Delta[n] \arrow[r, hook, "\partial\Delta[n]\times i_0"] \arrow[d, hook] & \partial\Delta[n]\times\Delta[1] \arrow[rr, "\ell"] \arrow[d, hook] & & E \arrow[d, two heads, "p"] \\ \Delta[n] \arrow[r, hook, "\Delta[n]\times i_0"'] \arrow[urrr, dashed, "k"] & \Delta[n]\times\Delta[1] \arrow[r, "q"'] & \Delta[n] \arrow[r, "b"'] & B \end{tikzcd} \tag{D.5.5}$$

By construction, the sphere in this restricted lifting problem lives in the fiber over the vertex $\{0\}$ of the n-simplex b in B; hence, there exists a solution to this lifting as displayed since that fibre is contractible by assumption. Taking the pushout of left-hand square of (D.5.5) we obtain the lifting problem that is the right-hand square in the following diagram:

$$\begin{tikzcd} \partial\Delta[n] \arrow[r, hook, "\partial\Delta[n]\times i_1"] \arrow[d, hook] & \Delta[n]\times\{0\} \underset{\partial\Delta[n]\times\{0\}}{\cup} \partial\Delta[n]\times\Delta[1] \arrow[rr, "m"] \arrow[d, hook] & & E \arrow[d, two heads, "p"] \\ \Delta[n] \arrow[r, hook, "\Delta[n]\times i_1"'] & \Delta[n]\times\Delta[1] \arrow[r, "q"'] \arrow[urr, dashed] & \Delta[n] \arrow[r, "b"'] & B \end{tikzcd}$$

Here the horizontal map m is induced by maps ℓ and k in (D.5.5). The middle vertical is the Leibniz product of a monomorphism and anodyne extension; hence this map is an anodyne extension, and the displayed lift exists. Finally, restriction to the endpoint over the vertex $\{1\}$ of the cylinder, that is composing with the square on the left in the diagram above, defines a solution to the original lifting problem. □

D.6. Equivalence between slices and cones

The work of §D.4 also enables us to supply full proofs of the results sketched in §4.2, which follow easily from the combinatorial work done in §D.2. Recall in particular Proposition D.2.17, which shows that the map of augmented simplicial sets

$$s^{n,m} \colon \Delta[n] \diamond \Delta[m] \to \Delta[n] \star \Delta[m] \quad \in s\mathcal{S}et_+^{\mathbf{\Delta}_+\times\mathbf{\Delta}_+}$$

is a marked homotopy retract equivalence. We argue now that such maps induce equivalences upon mapping into quasi-categories.

D.6.1. Lemma. *Let A be a quasi-category and let $I \to J$ be a map of simplicial sets that extends to a marked homotopy equivalence. Then the induced map $A^J \xrightarrow{\sim} A^I$ is an equivalence of quasi-categories.*

Proof. Equip A with its natural marking so that by Theorem D.4.13 it defines a complicial set. The marked homotopy equivalence defines maps

$$A^J \to A^I, \quad A^I \to A^J, \quad A^I \to (A^I)^{\Delta[1]^\sharp}, \quad A^J \to (A^J)^{\Delta[1]^\sharp}.$$

By Lemma D.4.16, the complicial sets A^I and A^J are also naturally marked quasi-categories. By Exercise 1.1.v, the inclusion $\Delta[1]^\sharp \hookrightarrow \mathbb{I}^\sharp$ is a complicial anodyne extension so by Proposition D.3.9, the restriction maps $(A^I)^{\mathbb{I}^\sharp} \xrightarrow{\sim\!\!\!\!\to} (A^I)^{\Delta[1]^\sharp}$ are trivial fibrations of marked simplicial sets. In particular, there exists lifts

$$\begin{tikzcd} & (A^I)^{\mathbb{I}^\sharp} \arrow[d, two heads, "\wr"] \\ A^I \arrow[r] \arrow[ur, dashed] & (A^I)^{\Delta[1]^\sharp} \end{tikzcd} \qquad \begin{tikzcd} & (A^J)^{\mathbb{I}^\sharp} \arrow[d, two heads, "\wr"] \\ A^J \arrow[r] \arrow[ur, dashed] & (A^J)^{\Delta[1]^\sharp} \end{tikzcd}$$

Forgetting markings, this data defines an inverse equivalence of categories to $A^J \to A^I$. □

In order to use this result to verify that the canonical map of augmented simplicial sets $s^{X,Y} : X \diamond Y \to X \star Y$ is also a weak equivalence in a suitable sense, we first prove that these diagrams are Reedy monomorphic.

D.6.2. Lemma. *The latching maps for the diagrams $F_\diamond, F_\star \in s\mathcal{S}et_+^{\mathbf{\Delta}_+ \times \mathbf{\Delta}_+}$ defined by*

$$F_\diamond^{n,m} := \Delta[n] \diamond \Delta[m] \qquad \text{and} \qquad F_\star^{n,m} := \Delta[n] \star \Delta[m]$$

are the maps

$$(\partial\Delta[n] \hookrightarrow \Delta[n]) \mathbin{\hat{\diamond}} (\partial\Delta[m] \hookrightarrow \Delta[m]) \qquad \text{and} \qquad (\partial\Delta[n] \hookrightarrow \Delta[n]) \mathbin{\widehat{\star}} (\partial\Delta[m] \hookrightarrow \Delta[m]), \tag{D.6.3}$$

which are both monomorphisms. Hence, both $F_\diamond$ and $F_\star$ are Reedy monomorphic.

Proof. By direct calculation,

$$(\partial\Delta[n] \hookrightarrow \Delta[n]) \mathbin{\widehat{\star}} (\partial\Delta[m] \hookrightarrow \Delta[m]) \cong \partial\Delta[n+1+m] \hookrightarrow \Delta[n+1+m],$$

which is clearly a monomorphism. It follows by Proposition C.2.9(vi) and Lemma C.5.9 that Leibniz joins of monomorphisms are monomorphisms.

For the analogous result for the fat join, observe first that $(\partial\Delta[-1] \hookrightarrow \Delta[-1]) \mathbin{\hat{\diamond}} (X \hookrightarrow Y) \cong (X \hookrightarrow Y)$, since $\partial\Delta[-1]$ is the initial object and $\Delta[-1]$ is the unit for the fat join. Thus, it suffices to consider the case of terminally augmented simplicial sets X and Y, where we make use of the fact observed in Definition 4.2.2 that for $n \geq 0$, $(X \diamond Y)_n \cong X_n \sqcup (\bigsqcup_{[n] \twoheadrightarrow [1]} X_n \times Y_n) \sqcup Y_n$ Thus, we see that for any monomorphisms of simplicial sets $X \hookrightarrow Y$ and $U \hookrightarrow V$, the square

$$\begin{tikzcd} (U \diamond X)_n \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & (V \diamond X)_n \arrow[d, hook] \\ (U \diamond Y)_n \arrow[r, hook] & (V \diamond Y)_n \end{tikzcd}$$

is a pullback in the category of sets. For any pullback square comprised of monomorphisms in the category of sets, the pushout inside the square is constructed by the joint image of the lower and right-hand legs: in particular the map $(U \diamond Y)_n \cup_{(U \diamond X)_n} (V \diamond X)_n \hookrightarrow (V \diamond Y)_n$ is a monomorphism. Thus, the Leibniz fat join of two monomorphisms

$$(U \hookrightarrow V) \mathbin{\hat{\diamond}} (X \hookrightarrow Y)$$

is a monomorphism as claimed. By Proposition C.2.9(vi) this argument extends also to non-terminally augmented simplicial sets, since these can be built as cell complexes from the cell boundary inclusions $\partial\Delta[n] \hookrightarrow \Delta[n]$ for $n \geq -1$.

Now by Definition C.4.16, the latching map at the object $([n],[m]) \in \mathbf{\Delta}_+ \times \mathbf{\Delta}_+$ is the map on weighted colimits induced by the maps (D.6.3) of weights. Applying Corollary C.5.16 to the weak factorization system on $s\mathcal{S}et_+$ whose left class is the monomorphisms, it follows that the latching maps for $F_\diamond$ and $F_\star$ are monomorphisms as claimed, proving that these diagrams are Reedy monomorphic. □

Recall the natural comparison map of Lemma D.2.14 from the fat join of a pair of simplicial sets to the join of the pair of simplicial sets.

D.6.4. PROPOSITION. *For all simplicial sets X and Y, the natural map $s\colon X \diamond Y \to X \star Y$ induces an equivalence of quasi-categories $A^{X\star Y} \xrightarrow{\sim} A^{X\diamond Y}$ for all quasi-categories A.*

PROOF. For any pair of terminally augmented simplicial sets X and Y we define their **external product** $X\square Y \in \mathcal{S}et^{\mathbf{\Delta}_+^{\mathrm{op}}\times\mathbf{\Delta}_+^{\mathrm{op}}}$ to be the functor that takes an object $([n],[m])$ to the set $X_n\times Y_m$. We can view this functor as a weight for the diagrams $F_\diamond$ and $F_\star$ introduced in Lemma D.6.2 and use Definition A.6.4 and cocontinuity of the join and fat join bifunctors (in the full subcategory $s\mathcal{S}et \hookrightarrow s\mathcal{S}et_+$ of terminally augmented simplicial sets) to compute the weighted colimits:

$$\begin{aligned}\operatorname{colim}^{\mathbf{\Delta}_+^{\mathrm{op}}\times\mathbf{\Delta}_+^{\mathrm{op}}}_{X\square Y} F_\diamond &\cong \int^{([n],[m])\in\mathbf{\Delta}_+\times\mathbf{\Delta}_+} \coprod_{X_n\times Y_m} \Delta[n]\diamond\Delta[m] \cong \left(\int^{[n]\in\mathbf{\Delta}_+}\coprod_{X_n}\Delta[n]\right)\diamond\left(\int^{[m]\in\mathbf{\Delta}_+}\coprod_{Y_m}\Delta[m]\right)\\ &\cong X\diamond Y\end{aligned}$$

Similarly, $\operatorname{colim}^{\mathbf{\Delta}_+^{\mathrm{op}}\times\mathbf{\Delta}_+^{\mathrm{op}}}_{X\square Y} F_\star \cong X\star Y$.

A direct verification shows that the latching maps of the augmented bisimplicial set $X\square Y \in \mathcal{S}et^{\mathbf{\Delta}_+^{\mathrm{op}}\times\mathbf{\Delta}_+^{\mathrm{op}}}$ are monomorphisms.[9] Hence by Corollary C.5.16, the weighted colimit functor

$$\operatorname{colim}^{\mathbf{\Delta}_+^{\mathrm{op}}\times\mathbf{\Delta}_+^{\mathrm{op}}}_{X\square Y} -\colon s\mathcal{S}et^{\mathbf{\Delta}_+\times\mathbf{\Delta}_+} \to s\mathcal{S}et$$

is a left Leibniz bifunctor with respect to any weak factorization systems of our choosing on $s\mathcal{S}et$.

By Lemma D.6.1, the components $s^{n,m}\colon \Delta[n]\diamond\Delta[m] \to \Delta[n]\star\Delta[m]$ are weak equivalences in the Joyal model structure, as these are characterized as those maps that induce equivalences upon mapping into an arbitrary quasi-category. So Proposition D.2.17 and Lemma D.6.2 establish that the natural transformation $s\colon F_\diamond \to F_\star$ is a pointwise weak equivalence between Reedy monomorphic objects in $s\mathcal{S}et^{\mathbf{\Delta}_+\times\mathbf{\Delta}_+}$. By the dual of Ken Brown's Lemma C.1.10, the induced map on weighted colimits $s^{X,Y}\colon X\diamond Y \to X\star Y$ is then a weak equivalence in the Joyal model structure, which means exactly that it induces an equivalence of quasi-categories $A^{X\star Y} \xrightarrow{\sim} A^{X\diamond Y}$ for all quasi-categories A as claimed. □

In particular, for any quasi-category A, there are natural equivalences $A^{\mathbb{1}\star J} \xrightarrow{\sim} A^{\mathbb{1}\diamond J}$ and $A^{J\star\mathbb{1}} \xrightarrow{\sim} A^{J\diamond\mathbb{1}}$ over $A\times A^J$. By Lemma 4.2.3 the codomains pullback to define the quasi-categories of cones under or over a diagram $d\colon J \to A$, respectively. However, as discussed in Warning 4.2.10, the domains do not pull back to the slice quasi-categories $A_{/d}$ and ${}^{d/}A$ of Definition D.2.8. Nonetheless, we can use the equivalence between the join and fat join constructions to prove that $A_{/d} \simeq \mathsf{Hom}_{A^J}(\Delta, d)$ and ${}^{d/}A \simeq \mathsf{Hom}_{A^J}(d,\Delta)$ over A and do so now.

[9] This follows because $\mathbf{\Delta}_+\times\mathbf{\Delta}_+$ is an **elegant Reedy category**: every element of a presheaf indexed by this category is a degeneracy of some non-degenerate element in a unique way [13].

D.6.5. Proposition. *For any diagram $d\colon J \to A$ indexed by a simplicial set J and valued in a quasi-category A, there are natural equivalences*

$$\begin{tikzcd} A_{/d} \arrow[rr, "\sim"] \arrow[dr, "\mathrm{res}"'] & & \mathsf{Hom}_{A^J}(\Delta, d) \arrow[dl, "\mathrm{res}"] \\ & A & \end{tikzcd} \qquad \begin{tikzcd} {}^{d/}A \arrow[rr, "\sim"] \arrow[dr, "\mathrm{res}"'] & & \mathsf{Hom}_{A^J}(d, \Delta) \arrow[dl, "\mathrm{res}"] \\ & A & \end{tikzcd}$$

between the slice quasi-categories and the quasi-categories of cones.

Proof. We appeal to Lemma C.5.22.

To begin, recall the adjunction of Definition D.2.8

$$\begin{tikzcd} s\mathcal{S}et \arrow[r, bend left, "-\star J"] \arrow[r, phantom, "\perp"] & {}^{J/}s\mathcal{S}et \arrow[l, bend left, "{}^{-/}-"] \end{tikzcd}$$

which gives a correspondence, for a simplicial set I and a map $d\colon J \to A$, between maps $I \star J \to A$ under J and maps of simplicial sets $I \to {}^{A/}d$. A right adjoint to the fat join functor $-\diamond J\colon s\mathcal{S}et \to {}^{J/}s\mathcal{S}et$ can be calculated similarly. By the defining pushout of Definition 4.2.2, the data of a map $I \diamond J \to A$ under J displayed below-left transposes to the data displayed below-right

$$\begin{tikzcd} (I \times J) \sqcup (I \times J) \arrow[r, "\pi_I \sqcup \pi_J"] \arrow[d, hook] & I \sqcup J \arrow[d, hook] \arrow[dr, "- \sqcup d"] \\ I \times \mathbb{2} \times J \arrow[r] \arrow[rr, bend right] & I \diamond J \arrow[ul, phantom, "\ulcorner", very near start] \arrow[r, dashed] & A \end{tikzcd} \qquad \begin{tikzcd} I \arrow[r, dashed] \arrow[rr, bend left] \arrow[dr] & \mathsf{Hom}_{A^J}(\Delta, d) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & A^{\mathbb{2} \times J} \arrow[d, two heads] \\ & 1 \times A \arrow[r, "d \times \Delta"'] & A^J \times A^J \end{tikzcd}$$

whence we see that that the value of the right adjoint to $- \diamond J$

$$\begin{tikzcd} s\mathcal{S}et \arrow[r, bend left, "- \diamond J"] \arrow[r, phantom, "\perp"] & {}^{J/}s\mathcal{S}et \arrow[l, bend left, "\mathsf{Hom}_{\_J}(\Delta, -)"] \end{tikzcd}$$

at $d\colon J \to A$ defines the ∞-category of cones $\mathsf{Hom}_{A^J}(\Delta, d)$ over d.

The natural map $s^{X,Y}\colon X \diamond Y \to X \star Y$ of Lemma D.2.14 defines a natural transformation

$$\begin{tikzcd} \mathbf{\Delta} \arrow[r, bend left, "{\Delta[\bullet] \diamond J}"] \arrow[r, bend right, "{\Delta[\bullet] \star J}"'] \arrow[r, phantom, "\Downarrow s"] & {}^{J/}s\mathcal{S}et \end{tikzcd} \quad = \quad \begin{tikzcd} \mathbf{\Delta} \arrow[r, "\text{よ}"] & s\mathcal{S}et \arrow[r, bend left, "- \diamond J"] \arrow[r, bend right, "- \star J"'] \arrow[r, phantom, "\Downarrow s"] & {}^{J/}s\mathcal{S}et \end{tikzcd}$$

that Proposition D.6.4 is a pointwise weak equivalence in the (sliced) Joyal model structure. Note that by adjunction, $\mathsf{Hom}_{A^J}(\Delta, d)$ is isomorphic to the simplicial set defined by mapping from the cosimplicial object $\Delta[\bullet] \diamond J$ to $d\colon J \to A$ in ${}^{J/}s\mathcal{S}et$, and $A_{/d}$ is isomorphic to the simplicial set defined by mapping from the cosimplicial object $\Delta[\bullet] \star J$ to $d\colon J \to A$ in ${}^{J/}s\mathcal{S}et$. In particular, the mate of the natural transformation s defines a natural comparison map $\hat{s}\colon A_{/d} \to \mathsf{Hom}_{A^J}(\Delta, d)$; since $s^{\Delta[n], \emptyset}$ is the identity, this natural comparison map lies over A.

To apply Lemma C.5.22 to prove that this natural comparison is an equivalence, we need only verify that both cosimplicial objects are Reedy cofibrant, which can be verified by a direct calculation along the lines of that given in the proof of Lemma D.6.2. Lemma C.5.22 now proves that the map $\hat{s}\colon A_{/d} \to \mathrm{Hom}_{A^J}(\Delta, d)$ defines an equivalence of quasi-categories over A, as desired. □

In the case $J = \mathbb{1}$, a diagram $a\colon 1 \to A$ defines an **element** of the quasi-category A, and we have the same result with different notion.

D.6.6. Corollary. *Let $a\colon 1 \to A$ define an element of a quasi-category A. Then there are canonical equivalences $A_{/a} \xrightarrow{\sim} \mathrm{Hom}_A(A, a)$ and ${}^{a/}A \xrightarrow{\sim} \mathrm{Hom}_A(a, A)$ over A.*

Proof. When $J = \mathbb{1}$, the constant diagram functor $\Delta\colon A \to A^J$ appearing in Proposition D.6.5 reduces to the identity functor on A. □

D.7. Equivalences and saturation

Having proven the required results about quasi-categories and isofibrations we indulge in a final section that develops a bit more of the general theory of complicial sets.

In Lemma D.4.2, we observed that the marked edges in a complicial set should be interpreted as equivalences, in a suitable sense. Below we demonstrate that a similar interpretation is appropriate for the higher-dimensional marked simplices as well. Consequently, if a complicial set has a property such as included in Definition D.4.4 of the natural marking that all simplices in dimension $r > n$ are marked—i.e., is *n-trivial* in terminology we will presently introduce—we may interpret that condition as demanding that all simplices in dimension $r > n$ are weakly invertible. Having understood that every marked simplex in a complicial set is an equivalence, we are lead to consider complicial sets that satisfy the converse of this condition, in which every equivalence is marked. Such *saturated* complicial sets are especially important, and we introduce the terminology *n-complicial set* to describe an n-trivial saturated complicial set. The Kan complexes are precisely the *0-complicial sets* while the quasi-categories are precisely the *1-complicial sets*. As this pattern suggests, the *n-complicial sets* define a well-behaved model for (∞, n)-categories in the sense that the full subcategory of such defines a cartesian closed ∞-cosmos, as we prove Proposition E.3.9.

To start our exploration of the higher-dimensional invertible simplices in a complicial set, we introduce an equivalence relation on simplices in a complicial set. To define it, we extend the notation of Definition D.1.6: for $n \geq 1$ and $0 \leq j < k \leq n$ write $\Delta^{j,k}[n]$ for the pushout of $\Delta^j[n]_e \hookleftarrow \Delta[n] \hookrightarrow_e \Delta^k[n]$.

D.7.1. Definition. For every marked simplicial set A, we define an equivalence relation $\sim_{A_n}$ on the set of n-simplices that is generated by the following relation: two n-simplices α, β in a marked simplicial set A are **complicial companions** if there exists an $(n+1)$-simplex τ in A and $0 \leq k \leq n$ so that

- τ is both k-admissible and $(k+1)$-admissible.
- $\alpha = \tau \cdot \delta^k$ and $\beta = \tau \cdot \delta^{k+1}$.

$$\begin{tikzcd} \Delta[n] \arrow[r, hook, "\delta^k"] \arrow[dr, "\alpha"'] & \Delta^{k,k+1}[n+1] \arrow[d, "\tau"] & \Delta[n] \arrow[l, hook', "\delta^{k+1}"'] \arrow[dl, "\beta"] \\ & A & \end{tikzcd}$$

For example, vertices of a marked simplicial set are complicial companions just when they are connected by a finite zig-zag of marked edges.

D.7.2. Lemma. *If A is a complicial set and if $\alpha \simeq_{A_n} \beta$ are n-simplices of A that are complicial companions, then α is marked if and only if β is marked.*

Proof. By induction it suffices to consider the generating relation. Suppose α and β appear, respectively, as the kth and $(k+1)$th faces of a k-admissible and $(k+1)$-admissible $(n+1)$-simplex τ; in particular, these conditions imply apriori that all of the codimension-one faces of τ except perhaps α and β are marked. If β is marked, then the markings of the k-admissible $(n+1)$-simplex τ extend as indicated in the solid arrow diagram

$$\begin{tikzcd}
\Delta^k[n+1] \arrow[r, hook, "e"'] \arrow[rr, bend left, "\tau"] & \Delta^k[n+1]' \arrow[r] \arrow[d, hook, "e"'] & A \\
& \Delta^k[n+1]'' \arrow[ur, dashed] &
\end{tikzcd}$$

and hence the dashed extension exists by Definition D.1.9. Since all codimension faces of $\Delta^k[n+1]''$ are marked, this tells us that α is marked also. If α is assumed to be marked instead, we apply the analogous argument, regarding τ as an $(k+1)$-admissible $(n+1)$-simplex. ☐

D.7.3. Lemma. *In a complicial set A, every n-simplex α is the complicial companion of a pair of n-simplices $\hat{\alpha}$ and $\check{\alpha}$ with the property that $\hat{\alpha}\cdot\delta^0$ is maximally marked and $\check{\alpha}\cdot\delta^n$ is maximally marked.*

In fact, as the proof will show, the face $\hat{\alpha}\cdot\delta^0$ can be taken to be degenerate at the final vertex of α, while the face $\check{\alpha}\cdot\delta^n$ can be arranged to be degenerate at the initial vertex of α.

Proof. We explain the construction of the complicial companion n-simplex $\check{\alpha}$ with $\check{\alpha}\cdot\delta^n$ degenerate at the initial vertex of α; the construction of $\hat{\alpha}$ is the odd dual of Remark D.1.8. To prove that $\alpha \simeq_{A_n} \hat{\alpha}$ we build a diagram starting from the left:

$$\begin{tikzcd}
\Delta[n] \arrow[dr, "\delta^1"] \arrow[ddrrrr, dotted, bend right, "\alpha=\alpha^1"'] & & \Delta[n] \arrow[dl, "\delta^2"'] \arrow[dr, "\delta^2"] \arrow[ddrr, dotted, "\alpha^2"'] & & \Delta[n] \arrow[dl, "\delta^3"'] \arrow[dd, dotted, "\alpha^3"] & \cdots & \Delta[n] \arrow[dr, "\delta^{n-1}"] \arrow[ddll, dotted, "\alpha^{n-1}"] & & \Delta[n] \arrow[dl, "\delta^n"'] \arrow[ddllll, dotted, bend left, "\alpha^n=\check{\alpha}"] \\
& \Delta^{1,2}[n+1] \arrow[drrr, bend right, "\tau^2"] & & \Delta^{2,3}[n+1] \arrow[dr, "\tau^3"] & & \cdots & & \Delta^{n-1,n}[n+1] \arrow[dlll, bend left, "\tau^n"'] & \\
& & & & A & & & &
\end{tikzcd}$$

To explain the idea of the construction of the sequence of complicial companion simplices

$$\alpha = \alpha^1 \simeq_{A_n} \alpha^2 \simeq_{A_n} \alpha^3 \cdots \alpha^{n-1} \simeq_{A_n} \alpha^n = \check{\alpha}$$

consider the sequence of n composable edges $f_{n-1,n}, \ldots, f_{1,2}$ along the spine of $\alpha = \alpha^1$. In α^2, the initial edge is degenerate, the 2nd edge is a composite $f_{1,2}\cdot f_{0,1}$, and the final $n-2$ composable edges along the spine coincide with the corresponding edges in the original sequence. In α^3, the first two edges are degenerate, the 3rd edge is a composite $f_{2,3}\cdot f_{1,2}\cdot f_{0,1}$, and the final $n-3$ edges coincide with the original sequence. In $\alpha^n = \check{\alpha}$, the initial $n-1$-edges are degenerate and the last edge is a composite of the original sequence of n edges.

Inductively starting from the left, we will build the simplex τ^i by filling an admissible horn $\Lambda^i[n+1]$ whose $i-1$th face is the simplex α^{i-1} defined in the previous stage and whose ith face is the simplex α^i used in the next stage. The admissible horn is itself built by extending a "generalized admissible horn" that is defined by gluing α^{i-1} to a degenerate simplex of dimension i. To extend from the generalized admissible horn to the admissible horn $\Lambda^i[n+1]$, a series of admissible simplices are attached to admissible inner horns $\Lambda^k[m]$ starting in dimension $m = 2$. For each of these admissible inner horns, the vertex k is mapped to the vertex i of the simplex τ^i being built, which is to say that all of the composites are being formed at a single vertex.

Writing α^1 for the original n-simplex α, the simplex τ^2 is built by extending a generalized admissible horn defined by the pushout

$$\begin{tikzcd}
& \Delta[1] \arrow[dl, hook', "{\{0,1\}}"'] \arrow[dr, hook, "\delta^1"] \arrow[ddr, equal] & \\
\Delta[n] \arrow[dr] \arrow[ddr, bend right, "\alpha^1"'] & & \Delta[2] \arrow[dl] \arrow[d, two heads, "\sigma^0"] \\
& H^2 \arrow[u, phantom, "\ulcorner" very near start] \arrow[d, dashed] & \Delta[1] \arrow[dl, "\alpha^1\cdot\{0,1\}"] \\
& A &
\end{tikzcd}$$

To extend this generalized admissible horn to a horn $\Lambda^2[n+1]$ whose filler defines τ^2 and α^2, we first attach fillers for each injection $\Lambda^1[2] \to H^2$ that is not already filled in H^2 that sends the inner vertex to the vertex 2 of H^2, which is the final vertex of the $\Delta[1]$ along which the $\Delta[n]$ and $\Delta[2]$ are being glued. We then attach fillers for each injection of a not-already-filled inner k-admissible 3-horn that sends the vertex k to the vertex 2 of H^2. Continuing inductively, we attach fillers for each injection of a not already filled k-admissible m simplex that sends k to the vertex 2. After attaching simplices in dimension $m = n-1$, we will have built a 2-admissible $n+1$ horn extending the original generalized admissible horn H^2. Its filler defines τ^2 and α^2. Note by construction that the initial edge of α^2 is degenerate.

Assume now we've defined the simplex α^{i-1} as the $i-1$th face of $\tau^{i-1} \colon \Delta^{i-2,i-1}[n+1] \to A$. The simplex τ^i is built by extending a generalized complicial horn defined by the pushout

$$\begin{tikzcd}
& \Delta[i-1] \arrow[dl, hook', "\mathrm{first}"'] \arrow[dr, hook, "\delta^1"] \arrow[ddr, equal] & \\
\Delta[n] \arrow[dr] \arrow[ddr, bend right, "\alpha^{i-1}"'] & & \Delta[i] \arrow[dl] \arrow[d, two heads, "\sigma^0"] \\
& H^i \arrow[u, phantom, "\ulcorner" very near start] \arrow[d, dashed] & \Delta[i-1] \arrow[dl, "\alpha^i\cdot\delta^{\mathrm{first}}"] \\
& A &
\end{tikzcd}$$

We extend H^i to an admissible horn $\Lambda^i[n+1]$ by attaching fillers for each injection of a k-admissible m-horn that is not filled already which sends the vertex k to the vertex i. The filler for this admissible horn defines τ^i and α^i. Note that by construction the image of the face of H^i spanned by the vertices $0, \ldots, i-1$ is degenerate on the face spanned by the vertices $0, \ldots, i-2$ of α^{i-1}, which is degenerate at the first vertex of α. Hence, the face of α^i spanned by the vertices $0, \ldots, i-1$ is degenerate at the first vertex of α. This constructs the desired sequence of complicial companion simplices connecting α to an n-simplex $\check{\alpha}$ whose initial codimension-1 face is degenerate, and in particular maximally marked. ☐

D.7.4. DEFINITION. A marked simplicial set X is **n-trivial** if all r-simplices are marked for $r > n$.

The full subcategory of n-trivial marked simplicial sets is reflective and coreflective

$$\begin{tikzcd} s\mathcal{S}et^+_{n\text{-tr}} \arrow[r, hook] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] & s\mathcal{S}et^+ \arrow[l, bend right, "\mathrm{trv}_n"'] \arrow[l, bend left, "\mathrm{core}_n"] \end{tikzcd}$$

in the category of marked simplicial sets. That is **n-trivialization** defines an idempotent monad on $s\mathcal{S}et^+$ with unit the entire inclusion

$$X \hookrightarrow_e \mathrm{trv}_n X$$

of a marked simplicial set X into the marked simplicial set $\mathrm{trv}_n X$ with the same marked simplices in dimensions $1, \ldots, n$, and with all higher simplices "made thin." A complicial set is **n-trivial** if this map is an isomorphism.

The **n-core** $\mathrm{core}_n X$, defined by restricting to those simplices whose faces above dimension n are all thin in X, defines an idempotent comonad with counit the regular inclusion

$$\mathrm{core}_n X \hookrightarrow_r X.$$

Again, a complicial set is n-trivial just when this map is an equivalence. As is always the case for a monad-comonad pair arising in this way, these functors are adjoints: $\mathrm{trv}_n \dashv \mathrm{core}_n$.

The subcategories of n-trivial marked simplicial sets assemble to define a string of inclusions with adjoints

$$\begin{tikzcd} s\mathcal{S}et \arrow[r, hook, "(-)^\sharp", "\cong"'] & s\mathcal{S}et^+_{0\text{- trv}} \arrow[r, hook] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] & s\mathcal{S}et^+_{1\text{- trv}} \arrow[l, bend right, "\mathrm{trv}_1"'] \arrow[l, bend left, "\mathrm{core}_1"] \arrow[r, hook] & \cdots \arrow[r, hook] & s\mathcal{S}et^+_{(n-1)\text{- trv}} \arrow[r, hook] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] & s\mathcal{S}et^+_{n\text{- trv}} \arrow[l, bend right, "\mathrm{trv}_{n-1}"'] \arrow[l, bend left, "\mathrm{core}_{n-1}"] \arrow[r, hook] & \cdots \arrow[r, hook] & s\mathcal{S}et^+ \end{tikzcd}$$

that filter the inclusion of simplicial sets, considered as maximally marked marked simplicial sets, into the category of all marked simplicial sets.

D.7.5. LEMMA. *The n-core of a complicial set is a complicial set.*

PROOF. Exercise D.7.i. ☐

By contrast, the n-trivialization functor does not necessarily preserve complicial sets; see Exercise D.7.ii.

D.7.6. REMARK. By contrast, the left adjoint, which just marks simplices in the appropriate dimension without changing the underlying simplicial set, does not preserve complicial structure; this construction is too naive to define the "freely invert n-arrows" functor from (∞, n)-categories to $(\infty, n -$

1)-categories, whose construction for $n = 1$ is given by Proposition D.4.25. For instance, the minimally marked 1-simplex $\Delta[1]$ defines a 1-trivial complicial set, but $0\text{-}\operatorname{trv}(\Delta[1]) = \Delta[1]^\sharp$ is not a 0-trivial complicial set because its underlying simplicial set is not a Kan complex; see Exercise D.1.ii.

Recall Lemma D.4.3, which described two markings for the nerve of a 1-category which make it into a 1-trivial complicial set: marking only the identity arrows or marking all isomorphisms. Clearly the first of these options defines the minimal 1-trivial marking that makes the nerve of a 1-category into a complicial set. By Lemma D.4.2, the latter option defines the maximal marking that makes the nerve of a 1-category into a complicial set. We now introduce terminology to describe "maximally marked" 1-trivial complicial sets.

D.7.7. Definition. A complicial set is 1-**saturated** if every equivalence it contains among its edges is marked.[10]

If a quasi-category is given a 1-saturated 1-trivial marking, then it is necessarily the natural marking of Definition D.4.4. Conversely, Theorem D.4.13 proves that the underlying simplicial set of any 1-trivial saturated complicial set is a quasi-category and the markings recover the natural markings.

Our aim is now to extend these notions to higher dimensions. To define equivalences and saturation in any dimension, it is helpful to reformulate the notion of 1-equivalence as a lifting property. To state it, we make use of a 3-simplex $\Delta[3]_{\mathrm{eq}}$ in which the edges $\{02\}$ and $\{13\}$ are marked as well as all simplices in dimension greater than 1.

D.7.8. Proposition.

(i) *A 1-simplex in a complicial set A is an equivalence if and only if defines the $\{12\}$-edge of a 3-simplex $\Delta[3]_{\mathrm{eq}} \to A$.*

(ii) *A complicial set is 1-saturated if and only if it admits extensions along the entire inclusion*

$$\begin{tikzcd}
\Delta[3]_{\mathrm{eq}} \arrow[r] \arrow[d, hook, "e"'] & A \\
\Delta[3]^\sharp \arrow[ur, dashed]
\end{tikzcd}$$

Proof. If f is an equivalence, then the witnessing 2-simplices of Definition D.4.1 define the 3rd and 0th faces of an admissible horn $\Lambda^{1,2}[3] \to A$ that fills to define a thin 3-simplex define a thin 3-simplex

$$\begin{tikzcd}
& 1 \arrow[r, "f"] \arrow[dr, phantom, "\wr"] & 2 \arrow[dr, "g"] \arrow[d, phantom, "\wr"] & & & & 1 \arrow[r, "f"] \arrow[drr, equals] & 2 \arrow[dr, "g"] \arrow[d, phantom, "\wr"] \\
0 \arrow[ur, "e"] \arrow[urr, equals] \arrow[rrr, "h"'] & & & 3 & \simeq & 0 \arrow[ur, "e"] \arrow[rrr, "h"'] & \wr & & 3
\end{tikzcd}$$

[10]Since the characterization of equivalences among edges requires prior argreement about which 2-simplices are marked, we typically apply Definition D.7.7 to 1-trivial complicial sets, in which case we say a 1-saturated 1-trivial complicial set is simply "saturated." This terminology will agree with Definition D.7.9.

of the form stipulated in (i). Conversely, the middle edge of any such 3-simplex then admits left and right inverses in the weaker sense of composing to marked edges, as witnessed by the 3rd and 0th faces:

$$\begin{tikzcd} & 1 \arrow[dr, "f"] \arrow[d, phantom, "\alpha \wr"] & \\ 0 \arrow[ur, "e"] \arrow[rr, "\sim", "j"'] & & 2 \end{tikzcd} \qquad \begin{tikzcd} & 2 \arrow[dr, "g"] \arrow[d, phantom, "\beta \wr"] & \\ 1 \arrow[ur, "f"] \arrow[rr, "\sim", "k"'] & & 3 \end{tikzcd}$$

By Lemma D.4.2, the marked edge j admits a right equivalence inverse i, and from this data we may build a 1-admissible 3-horn in which all 2-simplex faces are marked.

$$\begin{tikzcd} & 1 \arrow[r, "e"] \arrow[dl, phantom, "\wr"] & 2 \arrow[dr, "f"] & \\ 0 \arrow[ur, "i", "\sim"'] \arrow[urr, dashed, "e \cdot i"'] \arrow[rrr, equal] & & \wr & 3 \end{tikzcd} \simeq \begin{tikzcd} & 1 \arrow[r, "e"] \arrow[drr, "j"'] & 2 \arrow[dr, "f"] \arrow[d, phantom, "\alpha \wr"] & \\ 0 \arrow[ur, "i", "\sim"'] \arrow[rrr, equal] & \wr & \sim & 3 \end{tikzcd}$$

By a complicial anodyne extension followed by a complicial thinness extension, we produce a thin 2-simplex that witnesses that f admits a right equivalence inverse in the stricter sense of of Definition D.4.1. The dual argument constructs the desired left equivalence inverse.

Now for (ii) recall that a complicial set is 1-saturated if every 1-equivalence is marked. By (i), this implies that for any $\Delta[3]_{\mathrm{eq}} \to A$, the image of the middle edge should be marked. But in this case the complicial thinness extensions $\Delta^2[2]' \hookrightarrow_e \Delta^2[2]''$ and $\Delta^0[2]' \hookrightarrow_e \Delta^0[2]''$ imply that the first and last edges must be marked as well. This proves that a complicial set is 1-saturated if and only if it enjoys the postulated lifting property. $\square$

There are similar extension problems that can be used to define equivalences and detect saturation in any dimension, which are defined by forming the join of the inclusion $\Delta[3]_{\mathrm{eq}} \hookrightarrow_e \Delta[3]^\sharp$ with simplices on one side or the other. It is somewhat subtle to characterize the k-equivalences in a complicial set unless that complicial set is n-trivial for some n because the notion of k-equivalence depends upon composites of k-simplices, which are witnessed by $k+1$-dimensional equivalences. In a general complicial set, the class of equivalences is defined *coinductively* rather than *inductively*. Despite this, there is a surprisingly simple characterization of the saturated complicial sets, in which all equivalences (yet to be defined) are marked:

D.7.9. Definition (saturated complicial set). A complicial set is **saturated** if it admits extensions along the set of entire inclusions

$$\{\Delta[m] \star \Delta[3]_{\mathrm{eq}} \star \Delta[n] \hookrightarrow_e \Delta[m] \star \Delta[3]^\sharp \star \Delta[n] \mid n, m \geq -1\}.$$

In fact, it suffices to require only extensions

$$\begin{tikzcd} \Delta[3]_{\mathrm{eq}} \star \Delta[n] \arrow[r] \arrow[d, hook] & A \\ \Delta[3]^\sharp \star \Delta[n] \arrow[ur, dashed] & \end{tikzcd} \qquad \begin{tikzcd} \Delta[n] \star \Delta[3]_{\mathrm{eq}} \arrow[r] \arrow[d, hook] & A \\ \Delta[n] \star \Delta[3]^\sharp \arrow[ur, dashed] & \end{tikzcd}$$

along inclusions of one-sided joins of the inclusion $\Delta[3]_{\mathrm{eq}} \hookrightarrow_e \Delta[3]^\sharp$ with an n-simplex for each $n \geq -1$, and as it turns out only the left-handed joins or right-handed joins are needed.

By Proposition D.7.8, the $n = -1$ case of Definition D.7.9 asserts that every 1-equivalence in A, defined relative to the marked 2-simplices and marked 3-simplices, is marked. By Proposition D.7.8 again, the general extension property

$$\begin{tikzcd}
\Delta[n] \arrow[r, hook] \arrow[rr, bend left, "\sigma"] & \Delta[3]_{\mathrm{eq}} \star \Delta[n] \arrow[r] \arrow[d, hook] & A & \rightsquigarrow & \Delta[3]_{\mathrm{eq}} \arrow[r] \arrow[d, hook] & A_{/\sigma} \\
 & \Delta[3]^\sharp \star \Delta[n] \arrow[ur, dashed] & & & \Delta[3]^\sharp \arrow[ur, dashed] &
\end{tikzcd}$$

asserts that every 1-equivalence in the slice complicial set $A_{/\sigma}$ is marked.

At first blush, Definition D.7.9 does not seem to be general enough. In the case of a vertex $\sigma\colon \Delta[0] \to A$, 1-equivalences in $A_{/\sigma}$ define 2-simplices in A whose $\{01\}$-edge is a 1-equivalence. In particular, a generic 2-simplex

$$\begin{tikzcd}
 & y \arrow[dr, "g"] \arrow[d, phantom, "\Uparrow \alpha"] & \\
x \arrow[ur, "f"] \arrow[rr, "h"'] & {} & z
\end{tikzcd}$$

with no 1-equivalence edges along its boundary, does not define a 1-equivalence in any slice complicial set. However, there are admissible 3-horns that can be filled to define the pasted composites of α with 1_f and 1_g, respectively:

$$\begin{tikzcd}
 & 1 \arrow[r, "f"] & 2 \arrow[dr, "g"] \arrow[d, phantom, "\Uparrow \alpha"] & & & & 1 \arrow[r, "f"] \arrow[drr, "gf"'] \arrow[d, phantom, "\Uparrow \check{\alpha}"] & 2 \arrow[dr, "g"] \arrow[d, phantom, "\wr"] & \\
0 \arrow[ur, equals] \arrow[urr, "f"'] \arrow[rrr, "h"'] & {} & {} & 3 & \simeq & 0 \arrow[ur, equals] \arrow[rrr, "h"'] & {} & {} & 3
\end{tikzcd}$$

$$\begin{tikzcd}
 & 1 \arrow[r, "g"] & 2 \arrow[dr, equals] \arrow[d, phantom, "\hat{\alpha} \Uparrow"] & & & & 1 \arrow[r, "g"] \arrow[drr, "g"'] \arrow[d, phantom, "\Uparrow \alpha"] & 2 \arrow[dr, equals] \arrow[d, phantom, "="] & \\
0 \arrow[ur, "f"] \arrow[urr, "gf"'] \arrow[rrr, "h"'] & {} & {} & 3 & \simeq & 0 \arrow[ur, "f"] \arrow[rrr, "h"'] & {} & {} & 3
\end{tikzcd}$$

By the complicial thinness extension property, if any of α, $\hat{\alpha}$, or $\check{\alpha}$ are marked, then all of them are. That is, α, $\hat{\alpha}$, and $\check{\alpha}$ are complicial companions in the sense of Definition D.7.1. Lemma D.7.3 generalizes this construction to simplices of arbitrary dimensions.

D.7.10. Definition. In an n-trivial complicial set, an n-simplex $\sigma\colon \Delta[n] \to A$ is an n-**equivalence** if it admits an extension

$$\begin{tikzcd}
\Delta[n] \arrow[r, "\sigma"] \arrow[d, hook] & A \\
\Delta[3]_{\mathrm{eq}} \star \Delta[n-2] \arrow[ur, dashed] &
\end{tikzcd}$$

along the map $\Delta[n] \hookrightarrow \Delta[3]_{\mathrm{eq}} \star \Delta[n-2]$ whose image includes the edge $\{12\}$ of $\Delta[3]_{\mathrm{eq}}$ and all of the vertices of $\Delta[n-2]$, or if it is the complicial companion of an n-simplex which admits such an extension.

The set of n-equivalences identified by Definition D.7.10 depends on the marked $(n+1)$-simplices, which is the reason we have only stated this definition for an n-trivial complicial set. The n-equivalences in a generic complicial set are characterized by an inductive definition, the formulation of which we leave to the reader.

Finally, we introduce special terminology for those complicial sets that most closely represent (∞, n)-categories.

D.7.11. Definition (n-complicial set). A marked simplicial set is an n-**complicial set** if and only if it is a complicial set that is n-trivial and saturated.

D.7.12. Example. For instance, by Exercise D.1.ii, the 0-complicial sets are precisely the Kan complexes, with their maximal marking. By Theorem D.4.13, the 1-complicial sets are precisely the quasi-categories, with their natural marking, which is the largest marking which makes a quasi-category into a complicial set.

D.7.13. Digression (the Verity model structure for n-complicial sets). The category of marked simplicial sets bears a cartesian closed, cofibrantly generated model structure whose fibrant objects are exactly the n-complicial sets and whose cofibrations are the monomorphisms, which is obtained as a left Bousfield localization of the model structure described in Digression D.1.17. In [**98**, §6.2-4] Verity describes a general paradigm for obtaining model structures that localize the model structure for complicial sets. The verification in this particular case is completed by Ozornova and Rovelli [**65**, 1.25].

The closure properties of complicial sets extend to saturated complicial sets and n-complicial sets. The proof, which appears in Corollary D.7.16, depends on two technical lemmas, the first and harder of which was first proven by Ozornova and Rovelli [**65**, B.5-6].

D.7.14. Lemma. *The Leibniz products*

(i) $(\Delta[3]_{\mathrm{eq}} \star \Delta[n] \hookrightarrow \Delta[3]^\sharp \star \Delta[n]) \widehat{\times} (\partial\Delta[m] \hookrightarrow \Delta[m])$ *and*

(ii) $(\Delta[3]_{\mathrm{eq}} \star \Delta[n] \hookrightarrow \Delta[3]^\sharp \star \Delta[n]) \widehat{\times} (\Delta[m] \hookrightarrow \Delta[m]_t)$

are in the class of maps that is cellularly generated by the complicial anodyne extensions together with the saturation extensions of Definition D.7.9.

Proof. When $m = 0$, the map considered in (i) above reduces to the inclusion $\Delta[3]_{\mathrm{eq}} \star \Delta[n] \hookrightarrow \Delta[3]^\sharp \star \Delta[n]$ which is one of the generating saturation extensions, so we may assume that $m > 0$. We begin by observing that the only k-simplices that are marked in the codomain of

$$(\Delta[3]_{\mathrm{eq}} \star \Delta[n]) \times \Delta[m] \underset{(\Delta[3]_{\mathrm{eq}} \star \Delta[n]) \times \partial\Delta[m]}{\cup} (\Delta[3]^\sharp \star \Delta[n]) \times \partial\Delta[m] \hookrightarrow_e (\Delta[3]^\sharp \star \Delta[n]) \times \Delta[m]$$

but not in the domain are k-simplices for $m < k \leq n+2$ of the form

$$\Delta[1] \star \Delta[k-2] \xrightarrow{(\gamma \star \alpha, \beta)} (\Delta[3]_{\mathrm{eq}} \star \Delta[n]) \times \Delta[m]$$

where

- γ is one of the four edges $\{01\}$, $\{12\}$, $\{23\}$, $\{03\}$ of $\Delta[3]_{\mathrm{eq}}$,
- $\alpha\colon \Delta[k-2] \rightarrowtail \Delta[n]$ is injective, and
- $\beta\colon \Delta[k] \twoheadrightarrow \Delta[m]$ is surjective.

These conditions explain the bounds on k given above.

Since $\beta\colon [k] \twoheadrightarrow [m]$ is surjective and order-preserving, $\beta(0) = 0$ and $\beta(1)$ equals either 0 or 1. We factor the inclusion as a sequence of two pushouts

$$\begin{tikzcd}
\coprod \Delta[3]_{\mathrm{eq}} \star \Delta[k-2] \arrow[r, "(\mathrm{id}\star\alpha{,}\beta\sigma^0\sigma^0)"] \arrow[d, hook, "e"'] & (\Delta[3]_{\mathrm{eq}} \star \Delta[n]) \times \Delta[m] \cup (\Delta[3]^\sharp \star \Delta[n]) \times \partial\Delta[m] \arrow[d, hook, "e"] & \\
\coprod \Delta[3]^\sharp \star \Delta[k] \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \bullet \arrow[d, hook, "e"'] & \coprod \Delta^1[k+1]' \arrow[l, "((\gamma\star\alpha)\sigma^1{,}\beta\sigma^0)"'] \arrow[d, hook, "e"] \arrow[dl, phantom, "\urcorner"] \\
& (\Delta[3]^\sharp \star \Delta[n]) \times \Delta[m] & \coprod \Delta^1[k+1]'' \arrow[l]
\end{tikzcd}$$

where the left-hand coproduct is over all pairs (α, β) as above with $\beta(1) = 0$ and the right-hand coproduct is over all triples (γ, α, β) with $\beta(1) = 1$. A detailed argument that the pair of attaching maps preserves the required marked simplices is given in [**65**, B.5]. The top pushout marks all triples (γ, α, β) with $\beta(1) = 0$, while the bottom pushout marks all triples (γ, α, β) with $\beta(1) = 1$.

For the map considered in (i), observe that the only simplices that are marked in the codomain of

$$(\Delta[3]_{\mathrm{eq}} \star \Delta[n]) \times \Delta[m]_t \underset{(\Delta[3]_{\mathrm{eq}}\star\Delta[n])\times\Delta[m]}{\cup} (\Delta[3]^\sharp \star \Delta[n]) \times \Delta[m] \hookrightarrow_e (\Delta[3]^\sharp \star \Delta[n]) \times \Delta[m]_t$$

but not in the domain are m-simplices

$$\Delta[1] \star \Delta[m-2] \xrightarrow{(\gamma\star\alpha,\mathrm{id})} (\Delta[3]_{\mathrm{eq}} \star \Delta[n]) \times \Delta[m]$$

where

- γ is one of the four edges $\{01\}$, $\{12\}$, $\{23\}$, $\{03\}$ of $\Delta[3]_{\mathrm{eq}}$ and
- $\alpha\colon \Delta[m-2] \rightarrowtail \Delta[n]$ is injective,

which implies that we must have $m - 2 \leq n$. To mark the simplices $(\gamma \star \alpha, \mathrm{id})$ we take a pushout along a suitable complicial thinness extension:

$$\begin{tikzcd}
\coprod \Delta^1[m+1]' \arrow[r, "(\gamma\star\alpha)\sigma^1{,}\sigma^0"] \arrow[d, hook, "e"'] & (\Delta[3]_{\mathrm{eq}} \star \Delta[n]) \times \Delta[m]_t \cup (\Delta[3]^\sharp \star \Delta[n]) \times \Delta[m] \arrow[d, hook, "e"] \\
\Delta^1[m+1]'' \arrow[r] \arrow[ur, phantom, "\ulcorner"] & (\Delta[3]^\sharp \star \Delta[n]) \times \Delta[m]_t
\end{tikzcd}$$

where the coproduct is over all unmarked m-simplices $(\gamma \star \alpha, \mathrm{id})$. A detailed argument that the attaching map preserves the required marked simplices is given in [**65**, B.6]. This completes the verification that the two Leibniz products are in the saturated class generated by the complicial anodyne extensions and the saturation maps. $\square$

D.7.15. Lemma. *The Leibniz joins*

(i) $(\Delta[3]_{\mathrm{eq}} \star \Delta[n] \hookrightarrow \Delta[3]^\sharp \star \Delta[n]) \mathbin{\widehat{\star}} (\partial\Delta[m] \hookrightarrow \Delta[m])$,

(ii) $(\partial\Delta[m] \hookrightarrow \Delta[m]) \mathbin{\widehat{\star}} (\Delta[3]_{\mathrm{eq}} \star \Delta[n] \hookrightarrow \Delta[3]^\sharp \star \Delta[n])$,

(iii) $(\Delta[3]_{\mathrm{eq}} \star \Delta[n] \hookrightarrow \Delta[3]^\sharp \star \Delta[n]) \widehat{\star} (\Delta[m] \hookrightarrow \Delta[m]_t)$*, and*

(iv) $(\Delta[m] \hookrightarrow \Delta[m]_t) \widehat{\star} (\Delta[3]_{\mathrm{eq}} \star \Delta[n] \hookrightarrow \Delta[3]^\sharp \star \Delta[n])$

are in the class of maps that is cellularly generated by the complicial anodyne extensions together with the saturation extensions of Definition D.7.9.

Proof. The Leibniz join considered in (i) is evidentally a pushout

$$\begin{tikzcd} \Delta[3]_{\mathrm{eq}} \star \Delta[n] \star \Delta[m] \arrow[r, "\mathrm{id}"] \arrow[d, hook, "e"'] & \Delta[3]_{\mathrm{eq}} \star \Delta[n] \star \Delta[m] \cup \Delta[3]^\sharp \star \Delta[n] \star \partial\Delta[m] \arrow[d, hook, "e"] \\ \Delta[3]^\sharp \star \Delta[n] \star \Delta[m] \arrow[r, "\mathrm{id}"] \arrow[ur, phantom, "\ulcorner"] & \Delta[3]^\sharp \star \Delta[n] \star \Delta[m] \end{tikzcd}$$

of a generating saturation extension. Similarly, the Leibniz join considered in (ii) is evidentally a pushout of the saturation extension

$$\Delta[m] \star \Delta[3]_{\mathrm{eq}} \star \Delta[n] \hookrightarrow_e \Delta[m] \star \Delta[3]^\sharp \star \Delta[n].$$

The Leibniz join

$$\Delta[3]_{\mathrm{eq}} \star \Delta[n] \star \Delta[m]_t \cup \Delta[3]^\sharp \star \Delta[n] \star \Delta[m] \xhookrightarrow[e]{} \Delta[3]^\sharp \star \Delta[n] \star \Delta[m]_t$$

considered in (iii) is the identity (and not just entire), as is the Leibniz join considered in (iv). □

Lemmas D.7.14 and D.7.15 enable us to expeditiously prove the following result, extending Corollary D.3.13.

D.7.16. Corollary.

(i) For any saturated complicial set A and marked simplicial set, A^X is again a saturated complicial set.

(ii) For any n-complicial set A and marked simplicial set, A^X is again an n-complicial set.

(iii) For any complicial set A and map of marked simplicial sets $f\colon X \to A$, the marked slices $A_{/\!\!/ f}$ and ${}^{f/\!\!/}A$ are complicial sets. Moreover, for any complicial isofibration $p\colon A \twoheadrightarrow B$ of complicial sets, any monomorphism of marked simplicial sets $i\colon X \hookrightarrow Y$, and any map $f\colon Y \to A$, the Leibniz marked slices

$$A_{/\!\!/ f} \to A_{/\!\!/ fi} \underset{B_{/\!\!/ pfi}}{\times} B_{/\!\!/ pf} \qquad \text{and} \qquad {}^{f/\!\!/}A \to {}^{fi/\!\!/}A \underset{{}^{pfi/\!\!/}B}{\times} {}^{pf/\!\!/}B$$

are complicial isofibrations.

(iv) For any saturated complicial set A and map of marked simplicial sets $f\colon X \to A$, the marked slices $A_{/\!\!/ f}$ and ${}^{f/\!\!/}A$ are saturated complicial sets.

(v) For any n-complicial set A and map of marked simplicial sets $f\colon X \to A$, the marked slices $A_{/\!\!/ f}$ and ${}^{f/\!\!/}A$ are n-complicial sets.

Proof. The analogue of (i) and (ii) for a complicial set A is proven in Corollary D.3.13, so we pick up where that argument left off. In light of Definition D.7.9, (i) follows immediately from Lemma D.7.14. To extend this result to prove (ii), we need only verify that we may solve lifting problems

$$\begin{tikzcd} \Delta[m] \times X \arrow[r] \arrow[d, hook, "e"'] & A \\ \Delta[m]_t \times X \arrow[ur, dashed] & \end{tikzcd}$$

for $m > n$ under the hypothesis that A is an n-trivial saturated complicial set. On account of the pushout diagram

$$\begin{tikzcd} \coprod\limits_{\sigma \in tX_m} \Delta[m] \arrow[r, "(\mathrm{id},\sigma)"] \arrow[d, hook, "e"'] & \Delta[m] \times X \arrow[r] \arrow[d, hook, "e"'] & A \\ \coprod\limits_{\sigma \in tX_m} \Delta[m]_t \arrow[r] \arrow[urr, dotted] & \Delta[m]_t \times X \arrow[ur, dashed] \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$

where the coproduct is over the set of thin m-simplices in X, this is clear.

The final three statements refer to the marked slices defined by Lemma D.2.20. Lemma D.2.19 proves (iii), while this result combines with Lemma D.7.15 to prove (iv). For the final statement (iv), we need to verify that if A is n-trivial, then the lifting problems

$$\begin{tikzcd} \Delta[m] \star X \arrow[r] \arrow[d, hook, "e"'] & A \\ \Delta[m]_t \star X \arrow[ur, dashed] \end{tikzcd} \qquad \begin{tikzcd} X \star \Delta[m] \arrow[r] \arrow[d, hook, "e"'] & A \\ X \star \Delta[m]_t \arrow[ur, dashed] \end{tikzcd}$$

can be solved for all $m > n$. Since the only simplices that are marked in the codomain of the entire extensions but not in the domain have dimension greater than n, this is clear. □

Exercises.

D.7.i. Exercise ([**98**, 25]). Prove Lemma D.7.5.

D.7.ii. Exercise. Find another example of a complicial set whose n-trivialization is no longer a complicial set.

APPENDIX E

∞-cosmoi found in nature

In this chapter, we establish concrete examples of ∞-cosmoi found in nature. Typically, the objects of these ∞-cosmoi are infinite-dimensional categories as instantiated by some particular non-algebraic model and the functors between them are also morphisms of such. In most cases, there is an accompanying model structure which lends us appropriate classes of isofibrations, equivalences, and trivial fibrations. It would take far too long to re-prove this model structures here, but in each case we provide an appropriate literature citation. The only work that remains for us is to transfer the enrichment found in the model category literature to an enrichment over Joyal's model structure for quasi-categories on simplicial sets.

The general theory of what we might call "quasi-categorically enriched model categories" is discussed in §E.1. In §E.2, we apply these results to establish the familiar ∞-cosmoi of $(\infty, 1)$-categories $\mathcal{CSS}$, $\mathcal{S}egal$, and $1\text{-}\mathcal{C}omp$ together with their accompanying biequivalences to and from $\mathcal{QC}at$. In §E.3, we turn our attention to what might be called *higher ∞-categories*, establishing ∞-cosmoi whose objects are (∞, n)- or even (∞, ∞)-categories in some model. More exotic examples are considered in §E.4. Finally, in §E.5, we discuss a natural generalization of the notion of ∞-cosmos intended to expand the scope of the formal method to develop the theory of ∞-categories even further.

E.1. Quasi-categorically enriched model categories

Many examples of ∞-cosmoi arise as categories of fibrant objects in a model category that is enriched over Joyal's model structure for quasi-categories on simplicial sets described in Digression 1.1.30—at least if all fibrant objects are cofibrant as is surprisingly often the case.[1]

E.1.1. PROPOSITION. *Let $\mathcal{M}$ be any model category that is enriched over the Joyal model structure and in which every fibrant object is cofibrant. Then the full subcategory of fibrant objects $\mathcal{M}_f$ inherits the structure of an ∞-cosmos in which the isofibrations are the fibrations between fibrant objects, the equivalences are the weak equivalences between fibrant objects, and the trivial fibrations are the trivial fibrations between fibrant objects.*

PROOF. Since the fibrant objects in $\mathcal{M}$ are also cofibrant, Lemma C.3.11 implies the simplicially enriched homs between fibrant-cofibrant objects of $\mathcal{M}$ are quasi-categories, which we denote by $\mathsf{Fun}(A, B)$. The same result also implies that any fibration $f: A \twoheadrightarrow B$ between fibrant objects, the induced map $f_*: \mathsf{Fun}(X, A) \twoheadrightarrow \mathsf{Fun}(X, B)$ is an isofibration of quasi-categories.

By Lemma C.1.4, the fibrant objects and fibrations and weak equivalences between them define a category of fibrant objects in the sense of Definition C.1.1. In particular, the unenriched category $\mathcal{M}_f$ possesses a terminal object, small products, pullbacks of isofibrations, and limits of countable towers of isofibrations, with each of these limits created in $\mathcal{M}$. Since $\mathcal{M}$ admits simplicial cotensors,

[1]This hypothesis, that all fibrant objects are cofibrant, is not essential for the development of ∞-category theory. A more general notion of ∞-cosmos that accommodates possibly non-cofibrant objects is introduced in Section E.5.

Proposition A.5.5 implies that these 1-categorical limits are conical, and thus $\mathcal{M}_f$ possesses the conical limits of axiom 1.2.1(i). By hypothesis $\mathcal{M}$ is also cotensored over simplicial sets, and by Definition C.3.7, since all objects in the Joyal model structure are cofibrant, the fibrant objects are closed under simplicial cotensors. Thus $\mathcal{M}_f$ possesses all the limits of 1.2.1(i).

By Lemma C.1.4, the class of fibrations between fibrant objects contains the isomorphisms and all maps to the terminal object and is closed under all the 1-categorical limits of axiom 1.2.1(ii). Leibniz stability a special case of Definition C.3.7, and thus all the required axioms have been verified. □

Furthermore:

E.1.2. Corollary. *Any simplicially enriched right Quillen adjoint between quasi-categorically enriched model categories with all fibrant objects cofibrant defines a cosmological functor that is a cosmological biequivalence whenever the Quillen adjoint defines a Quillen equivalence.*

Proof. Exercise E.1.i. □

With Proposition E.1.1 in hand, the next question is where do model categories enriched over the Joyal model structure come from? Unsurprisingly, this question has not attracted much attention in the literature, but the mathematical community has done us a considerable favor, in many cases, by providing model categories of infinite-dimensional categories that are enriched over some other cartesian closed model category. Our strategy will be to apply Theorem C.3.14 to convert a known enrichment to an enrichment over Joyal's model structure for quasi-categories. Combining that result with Proposition E.1.1, we obtain the following immediate corollary.

E.1.3. Corollary. *Let $\mathcal{V}$ be a cartesian closed model category equipped with a Quillen adjunction*

$$\begin{tikzcd} s\mathcal{S}et \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{V} \arrow[l, bend left, "U"] \end{tikzcd}$$

whose right adjoint is valued in the Joyal model structure and whose left adjoint preserves finite products.

- *(i) Then for any $\mathcal{V}$-model category $\mathcal{M}$ in which every fibrant object is cofibrant, the full subcategory of fibrant objects $\mathcal{M}_f$ defines an ∞-cosmos in which the isofibrations are the fibrations between fibrant objects, the equivalences are the weak equivalences between fibrant objects, the trivial fibrations are the trivial fibrations between fibrant objects, the functor spaces are defined by $\mathsf{Fun}(M,N) := U\mathcal{M}(M,N)$, where $\mathcal{M}(M,N)$ is the hom-object in $\mathcal{V}$, and the simplicial cotensor of $M \in \mathcal{M}_f$ with $S \in s\mathcal{S}et$ are defined by the $\mathcal{V}$-cotensor M^{FS}.*
- *(ii) Moreover, any $\mathcal{V}$-enriched right Quillen adjoint between $\mathcal{V}$-model categories of this form defines a cosmological functor that is a cosmological biequivalence whenever the Quillen adjoint is a Quillen equivalence.* □

E.1.4. Remark. In particular, in the context of the statement of Corollary E.1.3, if the fibrant objects of $\mathcal{V}$ are cofibrant, then $\mathcal{V}$ itself is an ∞-cosmos and U defines a cosmological functor $U\colon \mathcal{V}_f \to \mathcal{QC}at$. Indeed, by Remark A.1.9, the change of base functor U is naturally isomorphic to the underlying quasi-category functor $\mathsf{Fun}(1,-)\colon \mathcal{V}_f \to \mathcal{QC}at$ for the ∞-cosmos structure it induces on $\mathcal{V}_f$.

Corollary E.1.3 inspires the following trivial examples of ∞-cosmoi.

E.1.5. Example (1-categories as ∞-cosmoi). Any complete locally small 1-category C can be made into an ∞-cosmos in which $\mathsf{Fun}(A,B)$ is just the set of morphisms from A to B. By the Yoneda lemma, the equivalences are then the isomorphisms in $\mathcal{C}$ and so by Lemma 1.2.16 all maps must necessarily be isofibrations. The cotensor of an object $A \in \mathcal{C}$ with a simplicial set A is defined by

$$A^S := A^{\pi_0 S} := \prod_{\pi_0 S} A.$$

Ignoring the fact that model categories are typically assumed to have colimits as well as limits, this construction can be seen as a special case of Corollary E.1.3 applied to the adjunction

$$\begin{tikzcd} s\mathcal{S}et \arrow[r, bend left, "\pi_0"] \arrow[r, phantom, "\perp"] & \mathcal{S}et \arrow[l, bend left, "\mathrm{sk}_0"] \end{tikzcd}$$

whose right adjoint embeds $\mathcal{S}et \hookrightarrow s\mathcal{S}et$ as the subcategory of 0-skeletal simplicial sets; see Definition C.5.2. Here the cartesian closed model structure on $\mathcal{S}et$ is not the one considered in Exercise C.3.iv but rather the one in which the weak equivalences are the isomorphisms and all maps are taken to be both cofibrations and fibrations. To see that this adjunction is Quillen, note that π_0 vacuously preserves cofibrations, while sk_0 carries any map to an isofibration of quasi-categories: This latter claim follows by adjunction since the defining lifting-properties below-left transpose to the lifting properties below-right:

$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d, tail, "\wr"'] & \mathrm{sk}_0 A \arrow[d] \\ \Delta[n] \arrow[r] \arrow[ur, dashed] & \mathrm{sk}_0 B \end{tikzcd} \quad \begin{tikzcd} \mathbb{1} \arrow[r] \arrow[d, tail, "\wr"'] & \mathrm{sk}_0 A \arrow[d] \\ \mathbb{I} \arrow[r] \arrow[ur, dashed] & \mathrm{sk}_0 B \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \pi_0\Lambda^k[n] \arrow[r] \arrow[d, equal] & A \arrow[d] \\ \pi_0\Delta[n] \arrow[r] \arrow[ur, dashed, "\exists!"'] & B \end{tikzcd} \quad \begin{tikzcd} \pi_0\mathbb{1} \arrow[r] \arrow[d, equal] & A \arrow[d] \\ \pi_0\mathbb{I} \arrow[r] \arrow[ur, dashed, "\exists!"'] & B \end{tikzcd}$$

Famously, π_0 preserves finite products, so the conditions of change-of-base theorem apply. The homotopy 2-category of an ∞-cosmos arising in this way will have no non-identity 2-cells.

E.1.6. Example (2-categories as ∞-cosmoi). Categorifying the previous example, any 2-category $\mathcal{C}$ with sufficient limits defines an ∞-cosmos where $\mathsf{Fun}(A, B)$ is the nerve of the hom-category of morphisms from A to B in $\mathcal{C}$. By Theorem 1.4.7, the equivalences are necessarily the equivalences in the 2-category. Inspired by Proposition 1.4.10, we take the isofibrations to be the isofibrations in the 2-category.

Interpreting "sufficient limits" to mean the limits of axiom 1.2.1(i), the remaining axiom 1.2.1(ii) can be verified by hand. Alternatively, again ignoring the fact that model categories are typically assumed to have colimits as well as limits, we may apply Corollary E.1.3 to the homotopy category $\dashv$ nerve adjunction

$$\begin{tikzcd} s\mathcal{S}et \arrow[r, bend left, "\mathsf{h}"] \arrow[r, phantom, "\perp"] & \mathcal{C}at \arrow[l, bend left] \end{tikzcd}$$

of Proposition 1.1.11 and make use of the "trivial" $\mathcal{C}at$-enriched model structure of Lack [**57**], whose weak equivalences and fibrations are exactly the equivalences and isofibrations just described.

It remains to unpack the meaning of the weaselly phrase "sufficient limits." By Corollary 6.3.3, the 2-category $\mathcal{C}$ is required to have all PIE-limits, that is 2-categorical products, inserters, and equifiers discussed in Digression 6.2.6. This implies that $\mathcal{C}$ admits pseudopullbacks of all maps, by the construction of Definition 6.3.5, but this doesn't quite imply that $\mathcal{C}$ admits 2-pullbacks of isofibrations. Instead, the proof of Lemma 6.3.9 constructs a *bipullback* of an isofibration, with the usual hom-category isomorphism replaced by a hom-category equivalence. Similar remarks apply to limits of towers of isofibrations. But in practice, the 2-categories that admit PIE limits such as those considered in [14] do seem to admit 2-pullbacks of isofibrations and 2-limits of towers of isofibrations and thus define examples of ∞-cosmoi.

In particular, Example E.1.6 specializes to recover the ∞-cosmos structure on $\mathcal{C}at$ discussed in Proposition 1.2.11. Intriguingly, it also defines an ∞-cosmos structure on $\mathcal{C}at^{\mathrm{op}}$ in which the "isofibrations" are those functors that are injective on objects.[2] Combining these observations with the dual ∞-cosmos construction of Definition 1.2.25, we see that the four 2-categorical duals $\mathcal{C}at$, $\mathcal{C}at^{\mathrm{op}}$, $\mathcal{C}at^{\mathrm{co}}$, and $\mathcal{C}at^{\mathrm{coop}}$ are all ∞-cosmoi.

The ∞-cosmoi of Example E.1.6 admit an abstract characterization as those ∞-cosmoi that are isomorphic (as quasi-categorically enriched categories) to their homotopy 2-categories. In this case, the weak 2-limits of Chapter 3 are actually strict and many of our results specialize to known theorems in the 2-categorical literature.

E.1.7. Example (simplicial model categories as ∞-cosmoi). The identity functor $\mathrm{id}\colon s\mathcal{S}et \to s\mathcal{S}et$ defines a right Quillen adjoint from Quillen's model structure for Kan complexes to Joyal's model structure for quasi-categories; evidently its left adjoint preserves products. Hence, any Kan complex enriched model category—or **simplicial model category** in the usual parliance—may be regarded as a quasi-categorically enriched model category in which each of the mapping-spaces between fibrant-cofibrant objects happens to be a Kan complex. Thus, any simplicial model category whose fibrant objects are cofibrant may be regarded as an ∞-cosmos.

The homotopy 2-categories of ∞-cosmoi arising in this manner are all "$(2,1)$-categories, with every natural transformation defining a natural isomorphism.

Exercises.

E.1.i. Exercise. Extend the proof of Proposition E.1.1 to prove Corollary E.1.2.

E.1.ii. Exercise. Prove the assertions made in Remark E.1.4 if you find them unconvincing.

E.2. ∞-cosmoi of $(\infty, 1)$-categories

The ∞-cosmos for quasi-categories was introduced in Proposition 1.2.10. In this section, we establish three other ∞-cosmoi whose objects define $(\infty,1)$-categories in some model and construct the biequivalences between them displayed in (10.0.1):

$$\begin{tikzcd}
\mathcal{CSS} \arrow[rr, "\mathrm{disc}"] \arrow[dr, shift left, "\mathrm{row}_0"] & & \mathcal{S}egal \arrow[dl, shift left, "\mathrm{row}_0"] \\
& \mathcal{QC}at \arrow[ul, shift left, "\mathrm{cylinder}"] \arrow[ur, shift left, "\mathrm{prism}"] \arrow[d, shift left, "\natural"] & \\
& 1\text{-}\mathcal{C}omp \arrow[u, shift left, "U"] &
\end{tikzcd}$$

A complete Segal space, as defined by Charles Rezk in [72], is firstly a bisimplicial set $X \in \mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}}$, where we regard $X_{m,n}$ as the set of n-simplices in the m^{th} space of a simplicial space

[2]In the "folk" model structure on $\mathcal{C}at$, the fibrations are the isofibrations, the weak equivalences are the equivalences, and the cofibrations are the inijective-on-objects functors. Injective-on-objects functors satisfy an isomorphism extension property dual to the isomorphism lifting property that defines the 2-category notion of isofibration.

$X_\bullet \in s\mathcal{S}et^{\boldsymbol{\Delta}^{\mathrm{op}}}$. It is conventional to regard the simplicial sets $X_m = X_{m,\bullet}$ as the "columns" of the bisimplicial set X, while the simplicial sets $X_{\bullet,n}$ define the "rows."

In a complete Segal space, the diagram

$$\begin{tikzcd} X_0 \arrow[r, shift left=2] & X_1 \arrow[l, shift left=4] \arrow[l, shift right=2] \arrow[r, shift left=4] \arrow[r, shift right=2] & X_2 \arrow[l, shift left=6] \arrow[l] \arrow[l, shift right=6] & \cdots \end{tikzcd}$$

defines a simplicial object in the category of Kan complexes, with each space X_m defining the "total space" of a Kan fibration whose base is the space $M_m X$ of "boundary data" associated with the m-simplex. The spaces X_0 and X_1 are the "spaces of objects and arrows" for the complete Segal space. The so-called "Segal condition" implies that the space X_n may be regarded as the "space of n-composable arrows." A Segal space, satisfying the conditions enumerated thus far, is then something like a "category object up to homotopy." The final "completeness" condition relates the spatial structure of X_0 with the categorical structure just defined, expressing the idea that paths in X_0 should correspond to isomorphisms in X.

A bisimplicial set defines a **complete Segal space** just when it is Reedy fibrant and satisfies the Segal and completeness conditions. All three of these conditions are most easily defined in terms of the weighted limits bifunctor

$$(\mathcal{S}et^{\boldsymbol{\Delta}^{\mathrm{op}}})^{\mathrm{op}} \times s\mathcal{S}et^{\boldsymbol{\Delta}^{\mathrm{op}}} \xrightarrow{\lim_{-} -} s\mathcal{S}et$$

where $s\mathcal{S}et$ is regarded as a $\mathcal{S}et$-enriched category. Note that the weights for $\boldsymbol{\Delta}^{\mathrm{op}}$-indexed diagrams in $s\mathcal{S}et$ are $\boldsymbol{\Delta}^{\mathrm{op}}$-indexed diagrams in $\mathcal{S}et$, i.e., simplicial sets. In more detail:

E.2.1. Definition (complete Segal space).

(i) A simplicial object $X_\bullet \in s\mathcal{S}et^{\boldsymbol{\Delta}^{\mathrm{op}}}$ is **Reedy fibrant** just when the induced map on weighted limits

$$X_m \cong \lim_{\Delta[m]} X \to \lim_{\partial\Delta[m]} X =: M_m X$$

is a Kan fibration of simplicial sets for all $m \geq 0$.

(ii) A Reedy fibrant simplicial object $X_\bullet$ is a **Segal space** just when the induced map on weighted limits

$$X_n \cong \lim_{\Delta[n]} X \to \lim_{\Lambda^k[n]} X$$

is a trivial fibration of simplicial sets for all $n \geq 2$ and $0 < k < n$.[3]

(iii) A Segal space $X_\bullet$ is a **complete Segal space**, just when the induced map on weighted limits

$$\lim_{\mathbb{I}} X \to \lim_{\Delta[0]} X \cong X_0$$

[3]By Reedy fibrancy, the induced map is already a Kan fibration, so to demand that it is a trivial fibration is equivalent to demanding that it is a weak homotopy equivalence. A priori, this definition is stronger than the usual **Segal condition**, which requires that the map induced on weighted limits by the inclusion of the spine of the n-simplex for each $n \geq 2$ is a trivial fibration. The spine inclusions are in the class cellularly generated by the inner horn inclusions, so by Exercise C.2.v applied to the two-variable adjunction involving the weighted limit, our condition clearly implies the classical Segal condition. The proof of the converse is more subtle and can be found as [**48**, 3.4].

is a trivial fibration of simplicial sets, asserting that the "space of isomorphisms in X"[4] is equivalent to the space X_0.[5]

The category of bisimplicial sets, as a presheaf category, is cartesian closed and hence enriched over itself. Among the great supply of product-preserving functors $\mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}} \to \mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}}$ that may be used to convert this to a simplicial enrichment, there are two of particular interest: $\mathrm{column}_0 \colon \mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}} \to \mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}}$, which sends a bisimplicial set X to its space X_0 of 0-simplices and $\mathrm{row}_0 \colon \mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}} \to \mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}}$, which passes to set the set of vertices in each space in the simplicial object. As observed by Joyal and Tierney [**48**], the former construction carries a complete Segal space to a Kan complex, while the latter construction carries a complete Segal space to a quasi-category and will be used to prove:construction that we will use to prove:

E.2.2. Proposition. *The full subcategory $\mathcal{CSS} \hookrightarrow \mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}}$ defines a cartesian closed ∞-cosmos in which the functor space $\mathsf{Fun}(A,B)$ is defined to be the* ***underlying quasi-category****, formed by the vertices in each space of internal hom B^A. With respect to this ∞-cosmos structure:*

(i) The underlying quasi-category functor $(-)_0 \coloneqq \mathrm{row}_0 \colon \mathcal{CSS} \to \mathcal{QC}at$ is a cosmological biequivalence.
(ii) There is a second cosmological biequivalence cylinder$\colon \mathcal{QC}at \to \mathcal{CSS}$, *which carries a quasi-category A to the bisimplicial set whose (m,n)-simplices are simplicial maps $\Delta[m] \times \widetilde{\Delta[n]} \to A$ indexed by the product of the ordinal category with the ordinal groupoid.*

Proof. By a theorem of Rezk, the complete Segal spaces form the fibrant objects in a cartesian closed model structure borne by the category of bisimplicial sets in which all objects are cofibrant [**72**]. Precomposing with the adjoint pair of functors defined by $\pi_1([m]\times[n]) = [m]$ and $\iota_0([m]) = [m]\times[0]$ induces an adjunction as below-right:

$$\begin{tikzcd} \mathbf{\Delta} \arrow[r, bend right, "\iota_0"'] \arrow[r, phantom, "\perp"] & \mathbf{\Delta}\times\mathbf{\Delta} \arrow[l, bend right, "\pi_1"'] \end{tikzcd} \qquad \rightsquigarrow \qquad \begin{tikzcd} \mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}} \arrow[r, bend left, "\pi_1^*"] \arrow[r, phantom, "\perp"] & \mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}} \arrow[l, bend left, "\iota_0^*"] \end{tikzcd} \tag{E.2.3}$$

Joyal and Tierney prove that this pair of functors defines a Quillen equivalence between the model structure for quasi-categories and the model structure for complete Segal spaces [**48**, 4.11]. By inspection, the left adjoint preserves finite products, so Corollary E.1.3 applies to create a cartesian closed ∞-cosmos structure on the full subcategory $\mathcal{CSS}$. By Lemma A.7.9, this makes the adjunction $\pi_* \dashv \iota_0^*$ into a simplicially enriched adjunction. Thus, it follows immediately from Corollary E.1.2 that $(-)_0 \coloneqq \mathrm{row}_0 \coloneqq \iota_0^*$ is a cosmological biequivalence.

[4]Other choices of weight may be used to define the "space of isomorphisms" such as the colimit of the diagram

$$\Delta[2] \xleftarrow{d^0} \Delta[1] \xrightarrow{d^2} \Delta[2]$$

See [**72**, §11] for a discussion.

[5]By the 2-of-3 property, this is equivalent to the arguably more natural condition that the map $\Delta \colon X_0 \to \lim_{\mathbb{I}} X$, induced by $! \colon \mathbb{I} \to \Delta[0]$ is a weak homotopy equivalence.

A second adjunction between simplicial sets and bisimplicial sets pointing in the opposite direction has a left adjoint defined as the left Kan extension of the functor

$$\begin{array}{ccc} \mathbf{\Delta} \times \mathbf{\Delta} & \longrightarrow & \mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}} \\ [m] \times [n] & \longmapsto & \Delta[m] \times \widetilde{\Delta[n]} \end{array}$$

along the Yoneda embedding $\mathbf{\Delta} \times \mathbf{\Delta} \hookrightarrow \mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}}$; here $\widetilde{\Delta[n]}$ is the nerve of the groupoid with $n+1$ objects and one exactly morphism in each hom-set, obtained by freely inverting the morphisms in the ordinal category $\mathbb{n} + \mathbb{1}$. The right adjoint is the corresponding "nerve" functor described in the statement of (ii). Joyal and Tierney also prove that the adjunction

$$\begin{tikzcd} \mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}} \arrow[r, bend left, "\mathrm{lan}"] \arrow[r, phantom, "\perp"] & \mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}} \arrow[l, bend left, "\mathrm{cylinder}"] \end{tikzcd}$$

is a Quillen equivalence with respect to the model structures for complete Segal spaces and quasi-categories [**48**, 4.12]. To conclude from Corollary E.1.2 that cylinder: $\mathcal{QC}at \to \mathcal{CSS}$ is a cosmological biequivalence it remains only to show that this functor is simplicially enriched and preserves simplicial cotensors, or equivalently, by Proposition A.4.6, that the adjunction lan $\dashv$ cylinder is simplicially enriched.

To verify this, we make use of the external product bifunctor:

$$\begin{array}{ccc} \mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}} \times \mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}} & \xrightarrow{\quad \Box \quad} & \mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}} \\ (A, B) & \longmapsto & (A \Box B)_{m,n} := A_m \times B_n \end{array}$$

Since any bisimplicial set X may be recovered as a conical colimit of representables, which the left adjoint of course preserves, it suffices to consider maps from a representable bisimplicial set $\Delta[m]\Box\Delta[n]$ to a simplicial set A. In the simplicial enrichment of $\mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}}$ just defined, the simplicial set of maps from $\Delta[m]\Box\Delta[n]$ to cylinder(A) has k-simplices defined to be the set of $(k, 0)$-simplices in the bisimplicial set cylinder$(A)^{\Delta[m]\Box\Delta[n]}$. Now

$$(\mathrm{cylinder}(A)^{\Delta[m]\Box\Delta[n]})_{k,0} := \mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}}((\Delta[m]\Box\Delta[n]) \times (\Delta[k]\Box\Delta[0]), \mathrm{cylinder}(A))$$

by the definition of the cartesian closed structure on bisimplicial sets

$$\cong \mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}}((\Delta[m] \times \Delta[k])\Box\Delta[n], \mathrm{cylinder}(A))$$

by the definition of the external product

$$\cong \mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}}(\mathrm{lan}((\Delta[m] \times \Delta[k])\Box\Delta[n]), A)$$

by adjunction. Joyal and Tierney prove in [**48**, 2.11] that the left Kan extension acts on the external tensor product by $\mathrm{lan}(B\square\Delta[n]) \cong B \times \widetilde{\Delta[n]}$. So we have

$$\begin{aligned}
&\cong \mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}}((\Delta[m]\times\Delta[k])\times\widetilde{\Delta[n]}, A)\\
&\cong \mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}}((\Delta[m]\times\widetilde{\Delta[n]})\times\Delta[k], A)\\
&\cong \mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}}(\mathrm{lan}(\Delta[m]\square\Delta[n])\times\Delta[k], A)\\
&\cong (A^{\mathrm{lan}(\Delta[m]\square\Delta[n])})_k
\end{aligned}$$

by the definition of the cartesian closed structure on simplicial sets. This proves that the adjunction is compatible with the simplicial enrichments, so it follows from Corollary E.1.2 and [**48**, 4.12] that cylinder: $\mathcal{QC}at \to \mathcal{CSS}$ is a cosmological biequivalence. ☐

A second model of $(\infty,1)$-categories is closely related.

E.2.4. Definition (Segal categories). A **Segal precategory** is a bisimplicial set $X_\bullet \in s\mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}}$ whose space of 0-simplices X_0 is 0-skeletal on the set $X_{0,0}$ of its vertices. A **Segal category** is a Segal category that is Reedy fibrant and satisfying the Segal condition of Definition E.2.1.

Definition E.2.4 is mildly stronger than the usual definition first introduced by Dwyer, Kan, and Smith [**32**] and further developed by Hirschowitz and Simpson [**42**], which states that a Segal precategory $X_\bullet$ is a **Segal category** so that for each $n \geq 2$, the map

$$X_n \cong \lim_{\Delta[n]} X \to X_1 \underset{X_0}{\times} \cdots \underset{X_0}{\times} X_1 \cong \lim_{\Gamma[n]} \Delta[n]$$

induced on weighted limits by the inclusion $\Gamma[n] \hookrightarrow \Delta[n]$ of the spine of the n-simplex is a weak homotopy equivalence of simplicial sets without requiring Reedy fibrancy. We prefer to include Reedy fibrancy in our notion of Segal category so that the Segal categories are precisely the fibrant objects in an appropriate model structure on the category $\mathcal{PC}at$ of Segal precategories, which then gives rise to an ∞-cosmos.

Before we introduce the ∞-cosmos $\mathcal{S}egal$, we explain how to transform a complete Segal space into a Segal category.

E.2.5. Lemma. *There is a functor* disc: $s\mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}} \to s\mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}}$ *defined by the pullback*

$$\begin{tikzcd}
\mathrm{disc}(X) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & X \arrow[d] \\
\mathrm{cosk}_0(X_{0,0}) \arrow[r, hook] & \mathrm{cosk}_0(X_0)
\end{tikzcd}$$

that lands in the subcategory of Segal precategories, and indeed is right adjoint to the inclusion $\mathcal{PC}at \hookrightarrow s\mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}}$. *Moreover, the discretization of a Reedy fibrant Segal space is a Segal category.*

Proof. Since the "vertex evaluation" map $X \to \mathrm{cosk}_0(X_0)$ is bijective on the 0th column, the pullback $\mathrm{disc}(X) \to \mathrm{cosk}_0(X_{0,0})$ must we as well. Hence $\mathrm{disc}(X)_0 \cong X_{0,0}$, which proves that $\mathrm{disc}(X)$ is a Segal precategory. To prove the adjointness, note that for any Segal precategory Y and bisimplicial map $f\colon Y \to X$, the component $f_0\colon Y_0 \to X_0$ factors uniquely through $X_{0,0} \hookrightarrow X_0$ by discreteness of Y. This induces the required unique factorization of f through $\mathrm{disc}(X) \hookrightarrow X$.

Finally, any simplicial space that is 0-coskeletal is automatically Reedy fibrant and a Segal space: the maps of Definition E.2.1(i) and (ii) are both isomorphisms. When X is Reedy fibrant, the map $X_n \to \mathrm{cosk}_0(X_0)_n \cong X_0^n$ is a Kan fibration, so the pullback that defines the simplicial set $\mathrm{disc}(X)_n$ is a homotopy pullback. Applying Lemma C.1.11 to Quillen's model structure for Kan complexes on simplicial sets, the Segal maps (ii) for X pull back to define analogous weak homotopy equivalences for $\mathrm{disc}(X)$. □

E.2.6. Proposition. *The full subcategory $\mathcal{S}egal \hookrightarrow \mathcal{P}Cat$ defines a cartesian closed ∞-cosmos in which the functor space $\mathsf{Fun}(A, B)$ is defined to be the* ***underlying quasi-category****, formed by the vertices in each space of internal hom B^A. With respect to this ∞-cosmos structure:*

(i) The underlying quasi-category functor $(-)_0 \coloneqq \mathrm{row}_0 \colon \mathcal{S}egal \to \mathcal{Q}Cat$ is a cosmological biequivalence.
(ii) There is a cosmological biequivalence $\mathrm{disc}\colon \mathcal{CSS} \to \mathcal{S}egal$ that "discretizes" a complete Segal space into a Segal category.
(iii) There is a second cosmological biequivalence $\mathrm{prism}\colon \mathcal{Q}Cat \to \mathcal{S}egal$, which carries a quasi-category A to the bisimplicial set whose (m, n)-simplices are simplicial maps $\Delta[m] \times \Delta[n] \to A$ whose components at each vertex of $\Delta[m]$ are constant.

Proof. By Pellissier and Bergner, the (Reedy fibrant) Segal category form the fibrant objects in a cartesian closed model structure borne by the category of Segal precategories in which all objects are cofibrant [**66**, **9**, **11**]. The cartesian closed structure on $\mathcal{P}Cat$ can be defined explicitly, or deduced from the observation that $\mathcal{P}Cat$ is a category of presheaves; see Exercise E.2.i.

The adjoint functors of (E.2.3) restrict to an adjunction between simplicial sets and Segal precategories, which Joyal and Tierney again prove define a Quillen equivalence between the model structure for quasi-categories and the model structure for Segal categories [**48**, 5.6]. Arguing as in Proposition E.2.2, Corollary E.1.3 applies to create a cartesian closed ∞-cosmos structure on the full subcategory $\mathcal{S}egal$, and Lemma A.7.9 and Corollary E.1.2 imply that $(-)_0 \coloneqq \mathrm{row}_0 \coloneqq \iota_0^*$ is a cosmological biequivalence.

By a theorem of Bergner, the inclusion $\dashv$ discretization adjunction of Lemma E.2.5 defines a Quillen equivalence between the model structure for complete Segal spaces and the model structure for Segal categories [**11**, §6]. To conclude from Corollary E.1.2 that $\mathrm{disc}\colon \mathcal{CSS} \to \mathcal{S}egal$ is a cosmological biequivalence it remains only to show that this functor is simplicially enriched and preserves simplicial cotensors, or equivalently, by Proposition A.4.6, that the adjunction is simplicially enriched. This follows from the fact that this adjunction commutes with the underlying quasi-category adjunctions for $\mathcal{CSS}$ and $\mathcal{S}egal$; see Remark E.2.7. In particular, since the inclusion $\mathcal{P}Cat \hookrightarrow s\mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}}$ preserves binary products, for any bisimplicial set C and Segal precategory S, $\mathrm{disc}(\mathsf{C}^S) \cong \mathrm{disc}(\mathsf{C})^S$. A similar argument shows that the simplicial cotensors are preserved. Passing to underlying quasi-categories, this induces the desired simplicially enriched adjunction, which makes $\mathrm{disc}\colon \mathcal{CSS} \to \mathcal{S}egal$ simplicial and hence cosmological.

A second adjunction between simplicial sets and Segal precategories pointing in the opposite direction has left adjoint given by restriction along the diagonal functor $\Delta\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}$ and right adjoint, which we call "prism," given by right Kan extension along the same followed by discretization. Joyal and Tierney also prove that this adjunction defines a Quillen equivalence with respect to the model structures for complete Segal spaces and quasi-categories [**48**, 5.7]. As above, to conclude that $\mathrm{prism}\colon \mathcal{Q}Cat \to \mathcal{S}egal$ is a cosmological biequivalence it remains only to argue that this adjunction is simplicially enriched. Since the prism functor is the composite of the right adjoint to the diagonal functor diag: $\mathcal{S}et^{\Delta^{op}\times\Delta^{op}} \to \mathcal{S}et^{\Delta^{op}}$ followed by discretization and we have already argued that the latter adjunction is simplicially enriched, it suffices to show that diag $\dashv$ ran is simplicially enriched.

To that end, consider a bisimplicial set X and a simplicial set A. By definition

$$\begin{aligned}(\mathrm{ran}(A)^X)_k &:= (\mathrm{ran}(A)^X)_{k,0} \\ &:= \mathcal{S}et^{\Delta^{op}\times\Delta^{op}}(X \times (\Delta[k]\square\Delta[0]), \mathrm{ran}(A)) \\ &\cong \mathcal{S}et^{\Delta^{op}\times\Delta^{op}}(\mathrm{diag}(X \times (\Delta[k]\square\Delta[0])), A) \\ &\cong \mathcal{S}et^{\Delta^{op}\times\Delta^{op}}(\mathrm{diag}(X) \times \Delta[k], A) \\ &:= (A^{\mathrm{diag}(X)})_k,\end{aligned}$$

which is what we wanted to show. □

E.2.7. Remark. This discretization functor commutes with the underlying quasi-category functors:

$$\begin{tikzcd} \mathcal{CSS} \arrow[rr, "\mathrm{disc}"] \arrow[dr, "(-)_0"'] & & \mathcal{S}egal \arrow[dl, "(-)_0"] \\ & \mathcal{QC}at & \end{tikzcd}$$

as can most easily be seen by considering the left adjoints to these functors at the level of model categories. However, discretization does not commute with the cylinder and prism constructions on the nose, only up to equivalence. For a quasi-category A, prism(A) is the Segal category with (m, n)-simplices given by the set of simplicial maps $\Delta[m] \times \Delta[n] \to A$ whose components at each vertex of $\Delta[m]$ are constant. By contrast, disc(cylinder(A)) is the Segal category with (m, n)-simplices given by the set of simplicial maps $\Delta[m] \times \widetilde{\Delta[n]} \to A$ whose components at each vertex of $\Delta[m]$ are constant.

E.2.8. Proposition. *The full subcategory* $1\text{-}\mathcal{C}omp \hookrightarrow 1\text{-}\mathcal{S}trat$ *defines a cartesian closed ∞-cosmos in which the functor space* $\mathsf{Fun}(A, B)$ *is defined to be the* ***underlying quasi-category*** *of the internal hom* B^A. *With respect to this ∞-cosmos structure, both the underlying quasi-category functor* $(-)_0 \colon 1\text{-}\mathcal{C}omp \to \mathcal{QC}at$ *and the natural marking functor* $(-)^\natural \colon \mathcal{QC}at \to 1\text{-}\mathcal{C}omp$ *are cosmological.*

Proof. By independent theorems of Lurie [**59**, §3.1.3-4] and Verity [**98**, §6.5], the naturally marked quasi-categories, which we call **1-complicial sets**, form the fibrant objects in a cartesian closed model structure borne by the category of stratified simplicial sets in which all objects are cofibrant. There is an adjunction

$$\begin{tikzcd} s\mathcal{S}et \arrow[r, bend left, "(-)^\flat"] \arrow[r, phantom, "\perp"] & 1\text{-}\mathcal{S}trat \arrow[l, bend left, "U"] \end{tikzcd}$$

in which the right adjoint forgets the marking and the left adjoint assigns each simplicial set the minimal marking, which Lurie proves defines a Quillen equivalence between the model structure for quasi-categories and the model structure for 1-complicial sets [**59**, 3.1.5.1]. By inspection, the left adjoint preserves finite products, so Corollary E.1.3 applies to create a cartesian closed ∞-cosmos structure on the full subcategory $1\text{-}\mathcal{C}omp$; the fact that the model category of 1-complicial sets is enriched over the model structure for quasi-categories via this construction is observed already in [**59**, 3.1.4.5]. As in the proofs of Proposition E.2.2 and E.2.6, it follows that the forgetful functor defines a cosmological biequivalence $U\colon 1\text{-}\mathcal{C}omp \to \mathcal{QC}at$ the coincides with the underlying quasi-category functor; see Remark 1.3.8.

For any quasi-categories A and B, observe that there is a natural isomorphism $\mathsf{Fun}(A,B) \cong \mathsf{Fun}(A^\natural, B^\natural)$ between the functor quasi-category in $\mathcal{QC}at$ and the just-defined functor space in $1\text{-}\mathcal{C}omp$ between their natural markings; the point is that simplicial maps $A \to B$ preserve isomorphisms and hence the natural markings. Verity shows that the natural marking functor $(-)^\sharp\colon \mathcal{QC}at \to 1\text{-}\mathcal{C}omp$ creates the fibrations between fibrant objects [**98**, 114-118], and hence, since limits in $1\text{-}\mathcal{C}omp$ are created in $\mathcal{QC}at$, it follows that the functor $(-)^\natural\colon \mathcal{QC}at \to 1\text{-}\mathcal{C}omp$ is a cosmological biequivalence, and indeed an inverse isomorphism to $(-)_0\colon 1\text{-}\mathcal{C}omp \to \mathcal{QC}at$. □

Exercises.

E.2.i. Exercise. Joyal and Tierney identify the subcategory $\mathcal{PC}at \hookrightarrow \mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}}$ with the category of presheaves indexed by the 1-categorical quotient $\mathbf{\Delta}|_2$ of $\mathbf{\Delta}\times\mathbf{\Delta}$ defined by inverting the maps in the image of the functor $[0]\times\mathbf{\Delta} \hookrightarrow \mathbf{\Delta}\times\mathbf{\Delta}$ [**48**, 5.4]. Redefine the three adjunctions between $\mathcal{PC}at$, $\mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}}$ and $\mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}}$ appearing in the proof of Proposition E.2.6 from this point of view.

E.2.ii. Exercise. Show that each of the cosmological biequivalences cylinder$\colon \mathcal{QC}at \xrightarrow{\sim} \mathcal{CSS}$ and prism$\colon \mathcal{QC}at \xrightarrow{\sim} \mathcal{S}egal$ are, respectively, sections of the underlying quasi-category functors.

E.3. ∞-cosmoi of (∞, n)-categories

In this section, we introduce a variety of ∞-cosmoi whose objects define models of (∞,n)-categories for $1 < n \leq \infty$. In these cases, the ∞-cosmos describes the $(\infty,2)$-category of ∞-categories, ∞-functors, and ∞-natural transformations, omitting higher-dimensional transformations. In individual cases, internal homs or generalized elements may allow access to higher-dimensional non-invertible morphisms.

Because the combinatorics entailed in precisely specifying a model of (∞,n)-categories can be rather involved, to save space, we do not define every one of the higher categorical notions discussed here, instead providing external references to where such definitions can be found.

A few of our models of (∞,n)-categories are defined as presheaves indexed by a 1-category $\mathbf{\Theta}_n$ first introduced by Joyal in an unpublished note [**45**], which we present in an equivalent form due to Berger [**7**].

E.3.1. Definition. For $0 \leq n \leq \infty$, define a family of 1-categories $\mathbf{\Theta}_n$ inductively as follows.

- $\mathbf{\Theta}_0 \coloneqq \mathbb{1}$ is the terminal category and $\mathbf{\Theta}_1 \coloneqq \mathbf{\Delta}$ is the category of finite non-empty ordinals and order-preserving maps.
- $\mathbf{\Theta}_n \coloneqq \mathbf{\Delta}\wr\mathbf{\Theta}_{n-1}$, where $\mathbf{\Delta}\wr - \colon \mathcal{C}at \to \mathcal{C}at$ is the **categorical wreath product** construction.Explicitly, for a 1-category C, $\mathbf{\Delta}\wr C$ is the category whose:
  - objects are tuples $[n](c_1,\ldots,c_n)$ where $[n]\in\mathbf{\Delta}$ and $c_i \in C$.
  - morphisms $(\alpha;\vec{f})\colon [n](c_1,\ldots,c_n) \to [m](c'_1,\ldots,c'_m)$ are given by a simplicial map $\alpha\colon [n] \to [m] \in \mathbf{\Delta}$ together with morphisms $f_{i,j}\colon c_i \to c'_j \in C$ for all $0 < i \leq n$ and $\alpha(i-1) < j \leq \alpha(i)$.

The objects of $\mathbf{\Theta}_n$ define pasting diagrams of k-cells for $0 \leq k \leq n$ while the morphisms define projection, composition, and degeneracy maps. The functor $\mathbf{\Theta}_n \hookrightarrow n\text{-}\mathcal{C}at$ that sends a pasting diagram to the free strict n-category that it generates is full and faithful [**7**, 3.7].

For instance, the morphism in $\mathbf{\Theta}_2$

$$[2]([1],[1]) \xrightarrow{(\delta^2;(\delta^1,!,\mathrm{id}))} [3]([2],[0],[1])$$

corresponds to the 2-functor between the free 2-categories generated by the pasting diagrams

$$\begin{tikzcd} 0 \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & 1 \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & 2 \end{tikzcd} \qquad \mapsto \qquad \begin{tikzcd} 0 \arrow[r, bend left, ""{name=U}] \arrow[r, ""{name=M}] \arrow[r, bend right, ""{name=D}] \arrow[r, phantom, shift left=2, "\Downarrow"] \arrow[r, phantom, shift right=2, "\Downarrow"] & 1 \arrow[r] & 2 \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & 3 \end{tikzcd}$$

that sends 0 to 0, 1 to 1, and 2 to 3, and sends the left 2-cell of the domain to the vertical composite of the leftmost 2-cells of the codomain and the right 2-cell of the domain to the whiskered composite of the rightmost 2-cell of the codomain.

E.3.2. Lemma. *For any 1-category with a terminal element t, the adjunction below-left induces an adjunction below-right:*

$$\begin{tikzcd} \mathbb{1} \arrow[r, bend right, "t"'] \arrow[r, phantom, "\perp"] & C \arrow[l, bend right, "!"'] \end{tikzcd} \qquad \rightsquigarrow \qquad \begin{tikzcd} \mathbf{\Delta} \cong \mathbf{\Delta} \wr \mathbb{1} \arrow[r, bend right, "\mathbf{\Delta}\wr t"'] \arrow[r, phantom, "\perp"] & \mathbf{\Delta} \wr C \arrow[l, bend right, "\mathbf{\Delta}\wr !"'] \end{tikzcd}$$

Proof. The categorical wreath product construction defines a 2-functor $\mathbf{\Delta} \wr - : \mathcal{C}at \to \mathcal{C}at$. □

Ara introduced a model of (∞, n)-categories for each $1 \leq n < \infty$ called **n-quasi-categories** as presheaves on $\mathbf{\Theta}_n$ characterized by a particular right lifting property described in [**2**, §5]. The case of 1-quasi-categories coincides with the usual notion of quasi-categories.

E.3.3. Proposition. *For each $n \geq 1$, the full subcategory $n\text{-}\mathcal{QC}at \hookrightarrow \mathcal{S}et^{\mathbf{\Theta}_n^{\mathrm{op}}}$ defines a cartesian closed ∞-cosmos of n-quasi-categories.*

Proof. Ara constructs a cartesian closed model structure on the category $\mathcal{S}et^{\mathbf{\Theta}_n^{\mathrm{op}}}$ generalizing the Joyal model structure in the case $n = 1$ in which the fibrant objects are exactly the n-quasi-categories and in which the cofibrations are the monomorphisms [**2**]; in particular, all objects are cofibrant. Hence, to induce a cartesian closed ∞-cosmos structure on the full subcategory $n\text{-}\mathcal{QC}at$ it suffices to find a Quillen adjunction between this model structure and the model structure for quasi-categories whose left adjoint $\mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}} \to \mathcal{S}et^{\mathbf{\Theta}_n^{\mathrm{op}}}$ preserves binary products.

To that end, note that $[0] \in \mathbf{\Theta}_{n-1}$ is terminal for all $n > 1$, so Lemma E.3.2 provides an adjunction as below-left and hence an adjunction as below-right

$$\begin{tikzcd} \mathbf{\Delta} \arrow[r, bend right, "\mathbf{\Delta}\wr[0]"'] \arrow[r, phantom, "\perp"] & \mathbf{\Theta}_n \arrow[l, bend right, "\mathbf{\Delta}\wr !"'] \end{tikzcd} \qquad \rightsquigarrow \qquad \begin{tikzcd} \mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}} \arrow[r, bend left, "(\mathbf{\Delta}\wr !)^*"] \arrow[r, phantom, "\perp"] & \mathcal{S}et^{\mathbf{\Theta}_n^{\mathrm{op}}} \arrow[l, bend left, "(\mathbf{\Delta}\wr[0])^*"] \end{tikzcd}$$

The right adjoint $\mathbf{\Delta} \wr [0] \colon \mathbf{\Delta} \hookrightarrow \mathbf{\Theta}_n$ induces $\mathbf{\Delta}$ as the subcategory of "pasting diagrams comprised of only 1-cells"; identifying $\mathbf{\Theta}_n$ with its image in $n\text{-}\mathcal{C}at$, the left adjoint $\mathbf{\Delta}\wr!$ then discards all the cells in dimension greater than 1 in each object of $\mathbf{\Theta}_n$.[6] Hence, the forgetful functor $(\mathbf{\Delta} \wr [0])^* \colon \mathcal{S}et^{\mathbf{\Theta}_n^{\mathrm{op}}} \to \mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}}$ forgets higher-dimensional cells. Its left adjoint, as a restriction functor between categories of presheaves, has its own left adjoint, and so clearly preserves products.

[6]More precisely, the left adjoint to the inclusion $\mathcal{C}at \hookrightarrow n\text{-}\mathcal{C}at$, restricts to define the functor $\mathbf{\Delta}\wr! \colon \mathbf{\Theta}_t \to \mathbf{\Delta}$.

Indeed, for the same reason, the left adjoint preserves all limits and hence also preserves monomorphisms (which can be characterized as those maps whose kernel pair is given by identities). By a result of Joyal and Tierney [**48**, 7.15], to prove that an adjunction is Quillen, it suffices to show that the left adjoint preserves cofibrations, as we've just done, and the right adjoint preserves fibrations between fibrant objects. By Lemma C.2.6, this means that we need only verify that the left adjoint carries the inner horn inclusions $\{\Lambda^k[n] \hookrightarrow \Delta[n]\}_{n\geq 2, 0<k<n}$ and the map $\mathbb{1} \hookrightarrow \mathbb{I}$ to trivial cofibrations in Ara's model structure. In fact, by [**48**, 3.5], it suffices to consider the spine inclusions $\{\Gamma[n] \hookrightarrow \Delta[n]\}_{n\geq 2}$ in place of the inner horn inclusions, which we shall.

To see this, its helpful to note, as observed in [**2**, §6], that that the left adjoint commutes with the nerve embeddings of strict 1-categories and strict n-categories:

$$\begin{tikzcd} \mathcal{C}at \arrow[r, hook] \arrow[d, hook] & n\text{-}\mathcal{C}at \arrow[d] \\ \mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}} \arrow[r, "(\Delta\wr !)^*"] & \mathcal{S}et^{\mathbf{\Theta}_n^{\mathrm{op}}} \end{tikzcd}$$

In particular, the left adjoint carries the 1-categorical nerve of $\mathbb{1} \hookrightarrow \mathbb{I}$ to the strict n-categorical nerve of this map and, since the left adjoint also preserves colimits, it carries the inner horn inclusion $\Gamma[n] \hookrightarrow \Delta[n]$ to the corresponding "spine inclusion" for the object $[n]([0], \ldots, [0]) \in \mathbf{\Theta}_n$. As both types of maps are among Ara's "localizer of n-quasi-categories" of [**2**, 5.17], they are certainly trivial cofibrations. Hence, the adjunction is Quillen, as claimed, and Corollary E.1.3 applies to create a cartesian closed ∞-cosmos structure on n-$\mathcal{QC}at$. □

Another model of (∞, n)-categories, for $0 \leq n < \infty$ is due to Rezk [**73**]. A $\mathbf{\Theta}_n$**-space** is a simplicial presheaf on $\mathbf{\Theta}_n$-satisfying Reedy fibrancy, Segal, and completeness conditions analogous to those of Definition E.2.1. A $\mathbf{\Theta}_1$-space is exactly a complete Segal space, while a $\mathbf{\Theta}_0$-space is just a Kan complex.

E.3.4. Proposition. *For each $n \geq 1$, the full subcategory $\mathbf{\Theta}_n\text{-}\mathcal{S}p \hookrightarrow s\mathcal{S}et^{\mathbf{\Theta}_n^{\mathrm{op}}}$ defines a cartesian closed ∞-cosmos of $\mathbf{\Theta}_n$-spaces for which the underlying complete Segal space functor $U\colon \mathbf{\Theta}_n\text{-}\mathcal{S}p \to \mathbf{\Theta}_1\text{-}\mathcal{S}p \cong \mathcal{CSS}$ is comological.*

Proof. Rezk constructs a cartesian closed model structure on the category $s\mathcal{S}et^{\mathbf{\Theta}_n^{\mathrm{op}}}$ generalizing his model structure for complete Segal spaces in the case $n = 1$, in which the fibrant objects are exactly the $\mathbf{\Theta}_n$-spaces and in which the cofibrations are the monomorphisms [**73**]; in particular, all objects are cofibrant. Hence, to induce a cartesian closed ∞-cosmos structure on the full subcategory $\mathbf{\Theta}_n\text{-}\mathcal{S}p$ it suffices to find a Quillen adjunction between this model structure and the model structure for complete Segal spaces whose left adjoint $s\mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}} \to s\mathcal{S}et^{\mathbf{\Theta}_n^{\mathrm{op}}}$ preserves binary products. We then apply Corollary E.1.3 to the composite of this adjunction with the adjunction (E.2.3).

As in the proof of Proposition E.3.3, we obtain the desired adjunction from Lemma E.3.2 applied to the terminal object $[0] \in \mathbf{\Theta}_{n-1}$.

$$\begin{tikzcd} \mathbf{\Delta} \arrow[r, bend right, "\Delta\wr[0]"'] \arrow[r, phantom, "\perp"] & \mathbf{\Theta}_n \arrow[l, bend right, "\Delta\wr !"'] \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} s\mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}} \arrow[r, bend left, "(\Delta\wr !)^*"] \arrow[r, phantom, "\perp"] & s\mathcal{S}et^{\mathbf{\Theta}_n^{\mathrm{op}}} \arrow[l, bend left, "(\Delta\wr[0])^*"] \end{tikzcd} \tag{E.3.5}$$

The left adjoint has a further left adjoint given by left Kan extension, and so preserves products.

It remains only to argue that this adjunction is Quillen. The model structure for $\mathbf{\Theta}_n$-spaces, and by specialization, also the model structure for complete Segal spaces, are defined as left Bousfield localiations of the injective (or, equivalently, Reedy) model structures on simplicial presheaves. In the injective model structures, the cofibrations and trivial cofibrations are defined objectwise in $s\mathcal{S}et$, so the left adjoint restriction functor is manifestly left Quillen with respect to these model structures. Consequently, the adjunction is Quillen for the localized model structures, if and only if the right adjoint, which Rezk refers to as the "underlying simplicial space" functor, preserves fibrant objects, because in that the case the left adjoint will preserve the new trivial fibrations, which are defined in terms of these. A functor $X \in s\mathcal{S}et^{\mathbf{\Theta}_n^{\mathrm{op}}}$ is fibrant if and only if it satisfies Reedy, Segal, and completeness conditions. Since the adjunction (E.3.5) is Quillen for the injective/Reedy model structure, the Reedy fibrancy condition is preserved, and Rezk proves that the Segal condition is preserved as well [**73**, 7.2]. By definition, the completeness condition for $\mathbf{\Theta}_n$-spaces is created from the completeness condition for underlying simplicial spaces [**73**, §7], so this is preserved as well. Hence, the right adjoint (E.3.5) restricts to a functor $U\colon \mathbf{\Theta}_n\text{-}\mathcal{S}p \to \mathcal{CSS}$, which we call the **underlying complete Segal space functor**.

Corollary E.1.3 applies to create a cartesian closed ∞-cosmos structure on $\mathbf{\Theta}_n\text{-}\mathcal{S}p$. By Lemma A.7.9, the adjunction (E.3.5) is enriched over bisimplicial sets, and so the second part of Corollary E.1.3 applies to prove that the underlying complete Segal space functor is cosmological. ☐

There is another model for (∞, n)-categories that generalizes the complete Segal space model for $(\infty, 1)$-categories, which makes use of the notion of a *Rezk object* valued in a model category.

E.3.6. Definition (Rezk). Let $\mathcal{M}$ be a model category.

(i) A simplicial object $X_\bullet \in \mathcal{M}^{\Delta^{\mathrm{op}}}$ is **Reedy fibrant** just when the induced map on weighted limits

$$X_m \cong \lim_{\Delta[m]} X \to \lim_{\partial\Delta[m]} X =: M_m X$$

is a fibration for all $m \geq 0$.

(ii) A Reedy fibrant simplicial object $X_\bullet$ is a **Segal object** just when the induced map on weighted limits

$$X_n \cong \lim_{\Delta[n]} X \to \lim_{\Lambda^k[n]} X$$

is a trivial fibration for all $n \geq 2$ and $0 < k < n$.

(iii) A Segal object $X_\bullet$ is a **Rezk object**, just when the induced map on weighted limits

$$\lim_{\mathbb{I}} X \to \lim_{\Delta[0]} X \cong X_0$$

is a trivial fibration.

A map $p\colon X_\bullet \to Y_\bullet \in \mathcal{M}^{\Delta^{\mathrm{op}}}$ is a **Rezk isofibration** if the relative analogues of the maps appearing in (i), (ii), and (iii) formed by the Leibniz weighted limit of p with the appropriate maps of weights, are respectively fibrations, trivial fibrations, and a trivial fibration.

Our formulation of Definition E.2.1 was designed to make it clear that the complete Segal spaces are precisely the Rezk objects valued in Quillen's model structure for Kan complexes.

E.3.7. Proposition. *Suppose $\mathcal{M}$ is a Cisinski model category.*[7] *Then the full subcategory $\mathscr{R}ezk_{\mathcal{M}} \hookrightarrow \mathcal{M}^{\Delta^{\mathrm{op}}}$ of Rezk objects defines an ∞-cosmos.*

[7] A **Cisinski model structure** is a combinatorial model structure on a Grothendieck topos in which the cofibrations are exactly the monomorphisms. It follows that the Reedy model structure on $\mathcal{M}^{\Delta^{\mathrm{op}}}$ coincides with the injective model structure, and in particular that all objects are cofibrant [**23, 13**].

Proof. We prove this result directly from Proposition E.1.1 by proving that a left Bousfield localization of the Reedy model structure on $\mathcal{M}^{\Delta^{op}}$ defines a Cisinski model structure in which the fibrant objects are exactly the Rezk objects that is enriched over the model structure for quasi-categories.

To begin, observe that the category $\mathcal{M}^{\Delta^{op}}$ is enriched, tensored, and cotensored over simplicial sets, with homs suggestively denoted by "Fun," in such a way that the Leibniz tensors of monomorphisms of simplicial sets with (trivial) Reedy cofibrations are (trivial) Reedy cofibrations [**30**, 4.4]. We will apply Jeff Smith's theorem [**5**] to prove that $\mathcal{M}^{\Delta^{op}}$ admits a model structure in which

- the cofibrations are the Reedy/injective cofibrations, these being the monomorphisms,
- the fibrant objects are the Rezk objects,
- the fibrations between fibrant objects are the Rezk isofibrations, and
- weak equivalences are the **Rezk weak equivalences**, though maps $U \to V$ that induce equivalences of quasi-categories $\mathsf{Fun}(V, X) \to \mathsf{Fun}(U, X)$ for all Rezk objects X.

Note that by adjunction, a map $p\colon X \to Y \in \mathcal{M}^{\Delta^{op}}$ is a Rezk isofibration if and only if for all monomorphisms $m\colon A \to B \in \mathcal{M}$, the induced map

$$\mathcal{M}(B, X) \xrightarrow{\widehat{\mathcal{M}}(m,p)} \mathcal{M}(A, X) \underset{\mathcal{M}(A,Y)}{\times} \mathcal{M}(B, Y)$$

of simplicial sets is an isofibration of quasi-categories; see Exercise C.2.v. By Corollary D.3.12 and Proposition D.5.1, this is the case if and only if this map has the right lifting property with respect to maps in the set

$$\mathcal{I} \hat{\times} \mathcal{J}, \quad \text{where} \quad \mathcal{I} := \{\partial\Delta[n] \hookrightarrow \Delta[n]\}_{n\geq 0} \quad \text{and} \quad \mathcal{J} := \{\Lambda^k[n] \hookrightarrow \Delta[n]\}_{n\geq 2, 0<k<n} \cup \{\mathbb{1} \hookrightarrow \mathbb{I}\}.$$

By adjunction again, and Proposition C.2.9(i), p is a Rezk isofibration if and only if it has the right lifting property with respect to the sets of maps $(i\hat{\times}j)\hat{*}m \cong j\hat{\otimes}(i\hat{*}m)$ for all $i \in \mathcal{I}$, $j \in \mathcal{J}$, and m among the generating cofibrations in $\mathcal{M}$, where $*$ denotes the pointwise tensor $*\colon s\mathcal{S}et \times \mathcal{M} \to \mathcal{M}^{\Delta^{op}}$ and $\otimes\colon s\mathcal{S}et \times \mathcal{M}^{\Delta^{op}} \to \mathcal{M}^{\Delta^{op}}$ denotes the simplicial tensor. Since the Reedy cofibrations in $\mathcal{M}^{\Delta^{op}}$ are generated by the set of maps $i\hat{*}m$ for $i \in \mathcal{I}$ and as m ranges over the generating cofibrations in $\mathcal{M}$ [**78**, 7.7], we conclude, again by adjunction, that p is a Rezk isofibration between Rezk objects if and only if

$$\mathsf{Fun}(V, X) \xrightarrow{\widehat{\mathsf{Fun}}(c,p)} \mathsf{Fun}(U, X) \underset{\mathsf{Fun}(U,Y)}{\times} \mathsf{Fun}(V, Y)$$

is an isofibration of quasi-categories for all monomorphisms $c\colon U \to V$ in $\mathcal{M}^{\Delta^{op}}$.

Now it's easy to verify the conditions of Jeff Smith's theorem. The Rezk weak equivalences are accessible and satisfy the 2-of-3 property. We argue that the Rezk weak equivalences contain all Reedy weak equivalences and hence the Reedy trivial fibrations, characterized by the right lifting property against the monomorphisms. Transposing the observations already made in [**30**, 4.4] about the Reedy model structure on $\mathcal{M}^{\Delta^{op}}$, we see that for any Reedy trivial cofibration $w\colon U \to V$ and Rezk object X, $w^*\colon \mathsf{Fun}(V, X) \to \mathsf{Fun}(U, X)$ is an equivalence of quasi-categories. By Ken Brown's lemma C.1.10, the same is true when w is a mere Reedy weak equivalence. Note that a map $w\colon U \to V$ is both a Rezk weak equivalence and a cofibration just when $w^*\colon \mathsf{Fun}(V, X) \to \mathsf{Fun}(U, X)$ is a trivial fibration between quasi-categories. This characterization proves that the class of Rezk weak equivalences and cofibrations is stable under pushout and transfinite composition. Smith's theorem now implies that the model structure for Rezk objects exists.

To see that the model structure for Rezk objects is enriched over the model structure for quasi-categories, we must verify the three conditions for a Quillen two variable adjunction

$$(\otimes, \{,\}, \mathsf{Fun})\colon s\mathcal{S}et \times \mathcal{M}^{\Delta^{\mathrm{op}}} \to \mathcal{M}^{\Delta^{\mathrm{op}}}.$$

The fact that Leibniz tensors of cofibrations are cofibrations was verified already for the Reedy model structure on $\mathcal{M}^{\Delta^{\mathrm{op}}}$ and the localized model structure for Rezk objects has the same cofibrations. To verify the remaining 2/3rds of this axiom, we appeal to a result of Dugger [**30**, 3.2], which tells us that in the presence of the first 1/3rd, to verify that Leibniz tensors of monomorphisms of simplicial sets with trivial cofibrations are trivial cofibrations, it suffices to show that the simplicial cotensor $(-)^K\colon \mathcal{M}^{\Delta^{\mathrm{op}}} \to \mathcal{M}^{\Delta^{\mathrm{op}}}$ preserves fibrations between fibrant objects. For left Bousfield localizations, Rezk fibrations between Rezk objects coincide with Reedy fibrations between Rezk objects [**41**, 3.3.16]. It's easy to verify directly that $(-)^K$ preserves Rezk objects, and the preservation of Reedy fibrations is one of the facts we knew already.

For the final 1/3 of the Quillen two-variable adjunction, we use the second part of Dugger's [**30**, 3.2], which tells us that in the presence of the first 2/3rds, we need only verify that for all Rezk objects Z and trivial cofibrations of simplicial sets $j\colon J \to K$, the map $Z^j\colon Z^K \to Z^J$ is a Rezk weak equivalence (assuming $\mathcal{M}^{\Delta^{\mathrm{op}}}$ is left proper, which is the case here since all objects are cofibrant). In fact, we can show that this map is a trivial fibration, by checking the right lifting property against the monomorphisms $c\colon U \to V \in \mathcal{M}^{\Delta^{\mathrm{op}}}$. Transposing, we see that $c \boxslash Z^j$ if and only if $j \boxslash c^*\colon \mathsf{Fun}(V, Z) \to \mathsf{Fun}(U, V)$. But we verified three paragraphs above that c^* is an isofibration between quasi-categories, so the desired lifting property holds. □

Barwick's n-**fold complete Segal space** model of (∞, n)-categories is formed by iterating the Rezk objects construction n times [**3**]. For this to make sense, note that the model structure for Rezk objects on $\mathcal{M}^{\Delta^{\mathrm{op}}}$ remains a Cisinski model structure, so this construction can be iterated. Specialization Proposition E.3.7, we conclude that for all $n \geq 1$, there exist ∞-cosmoi $\mathcal{CSS}_n$ of n-fold complete Segal spaces.

E.3.8. Remark. If $\mathcal{M}$ is a left proper combinatorial model category, the proof just given constructs a model structure on $\mathcal{M}^{\Delta^{\mathrm{op}}}$ whose fibrant objects are the Rezk objects that is enriched as a model category over the model structure for quasi-categories. The only hitch is that without the Cisinski condition, it's possible that not all fibrant objects are cofibrant. Nonetheless, this generalization can be understood as defining an ∞-cosmos of a sort to be discussed in §E.4.

Verity constructs a general family of cartesian model structures on the category of stratified simplicial sets whose fibrant objects are complicial sets of various flavors and whose fibrations are the corresponding notions of complicial fibration [**98**, §9.3]. A restriction of one of these model structures to the case of *1-trivial stratified simplicial sets*, with all simplices above dimension 1 marked, underlies the ∞-cosmos of Proposition E.2.8. Here, we consider model structures whose fibrant objects model (∞, n)-categories for $0 \leq n \leq \infty$, in which case we write **complicial set** to mean ∞-complicial set. The definitions are arranged so that a 0-complicial set is a (maximally marked) Kan complex, a 1-complicial set is a (naturally marked) quasi-category, and a k-complicial set is an n-complicial set whenever $m < n$.

E.3.9. Proposition. *For each $0 \leq n \leq \infty$, the full subcategory in the category of stratified simplicial sets spanned by the complicial sets defines a cartesian closed ∞-cosmos n-$\mathcal{C}omp$. Moreover, whenever $m < n$, the* *functor* $\mathsf{core}\colon n\text{-}\mathcal{C}omp \to m\text{-}\mathcal{C}omp$ *that discards all simplices in dimension* $k > m$ *that are not marked is cosmological.*

Proof. For a suitable class of monomorphisms $\mathcal{K}$, Verity defines a cartesian closed model structure on the category of stratified simplicial sets whose fibrant objects and fibrations between them are the $\mathcal{K}$-complicial sets and $\mathcal{K}$-complicial fibrations, characterized by a right lifting property against $\mathcal{K}$ [**98**, §9.3]. The cofibrations are the monomorphisms so in particular all objects are cofibrant. In more detail, for the m-complicial sets, the class of monomorphisms is defined to be

$$\begin{aligned}\mathcal{K}_m \coloneqq \left\{\Lambda^k[n] \hookrightarrow_r \Delta^k[n]\right\}_{n\geq 1, k\in[n]} &\cup \left\{\Delta^k[n]' \hookrightarrow_e \Delta^k[n]''\right\}_{n\geq 2, k\in[n]} \\ &\cup \left\{\Delta[r] \hookrightarrow_e \Delta[r]_t\right\}_{r>m} \cup \left\{\Delta[j] \star \Delta[3]_{\mathrm{eq}} \star \Delta[k] \hookrightarrow \Delta[j]\star\Delta[3]^\sharp \star \Delta[k]\right\}_{j,k\geq -1}\end{aligned}$$

See [**77**] for an explanation of this notation. The first set of maps are referred to as the **complicial horn extensions** while the second set define the **complicial thinness extensions**. The third set imposes the condition that all simplices in dimension greater than m are marked, while the final condition is **saturation**, which in the presence of the other conditions, implies that all equivalences are marked. To apply Verity's theorem, the sets $\mathcal{K}_m$ must satisfy some technical conditions spelled out in [**98**, 91-92]. In this case, these conditions have been verified in forthcoming work of Viktoriya Ozornova and Martina Rovelli. By construction, the n-complicial sets live in the subcategory $n\text{-}\mathcal{S}trat$ of n-trivial stratified simplicial sets (with all simplices in dimension greater than n marked), and we may restrict the cartesian closed model structures to these subcategories.

The ∞-cosmoi $0\text{-}\mathcal{C}omp$ and $1\text{-}\mathcal{C}omp$ are isomorphic to the ∞-cosmoi $\mathcal{K}an$ and $\mathcal{QC}at$ respectively, so for now we consider $2 \leq n \leq \infty$. To define the ∞-cosmos $n\text{-}\mathcal{C}omp$, we apply Corollary E.1.3 to convert these self enrichments into an enrichment over quasi-categories via a string of Quillen adjunctions whose left adjoints preserve binary products:

$$\begin{tikzcd} s\mathcal{S}et \arrow[r, bend left, "(-)^\flat"] \arrow[r, phantom, "\perp"] & 1\text{-}\mathcal{S}trat \arrow[l, bend left, "U"] \arrow[r, bend left, hook] \arrow[r, phantom, "\perp"] & 2\text{-}\mathcal{S}trat \arrow[l, bend left, "\mathsf{core}_1"] \arrow[r, bend left, hook] \arrow[r, phantom, "\perp"] & \cdots \arrow[l, bend left] \arrow[r, bend left, hook] \arrow[r, phantom, "\perp"] & (n-1)\text{-}\mathcal{S}trat \arrow[l, bend left] \arrow[r, bend left, hook] \arrow[r, phantom, "\perp"] & n\text{-}\mathcal{S}trat \arrow[l, bend left, "\mathsf{core}_{n-1}"] & \cdots \end{tikzcd}$$

In the limiting case, we also consider adjunctions

$$\begin{tikzcd} n\text{-}\mathcal{S}trat \arrow[r, bend left, hook] \arrow[r, phantom, "\perp"] & \mathcal{S}trat \arrow[l, bend left, "\mathsf{core}_n"] \end{tikzcd}$$

where $\mathsf{core}_n X \hookrightarrow X$ is the simplicial subset containing only those simplices in dimension greater than n that are marked. By adjunction it is easy to verify that these functors carry $(n+1)$-complicial sets to n-complicial sets. Since the left adjoints preserve monomorphisms and products, this is enough to verify that the adjunctions are Quillen. Corollary E.1.3 now induces the desired ∞-cosmoi and cosmological core functors. □

Building on past work of Hirschowitz-Simpson [**42**] and Pellissier [**66**], Simpson iterates the construction of the model structure for Segal categories [**86**, §19.2-4]. When the base model category is taken to be Quillen's model structure for Kan complexes, the n-th iteration defines the notion of **Segal n-categories**, the Simpson considers more general model categorical bases. Under suitable hypotheses, satisfied in the case of Segal n-categories, the model structure so produced is cartesian closed and has all objects cofibrant, which strongly suggests that there exists an ∞-cosmos spanned by its fibrant objects: the Reedy fibrant Segal n-categories. We leave the confirmation of this as an exercise for the interested reader.

Exercises.

E.3.i. Exercise. Given an explicit formulation of the "relative analogue" of the conditions (i), (ii), and (iii) used in Definition E.3.6 to define the notion of **Rezk isofibration**.

E.3.ii. Exercise. Investigate potential ∞-cosmos structures on the Segal n-categories of Hirschowitz and Simpson [**42**].

E.3.iii. Exercise. Search for cosmological biequivalences between the ∞-cosmoi constructed in this section (and please share your discoveries with the authors).

E.4. Other examples

In this section, we present a few additional examples of ∞-cosmoi to complement those found in Chapter 6. Left as an exercise for now.

E.5. ∞-cosmoi with non-cofibrant objects

See [**81**, §2] for now.

APPENDIX F

Compatibility with the analytic theory of quasi-categories

The aim in this section is to prove that the synthetic theory of quasi-categories is compatible with the analytic theory pioneered by André Joyal, Jacob Lurie, and many others.

F.1. Initial and terminal elements

In this section, we complete the argument sketched in Digression 4.3.11 and prove that the synthetic definition of a terminal element in a quasi-category coincides with the analytic definition first introduced by Joyal [**46**, 4.1]. In the following result, we prove the equivalence between three synthetic definitions of a terminal element—(i), which appeared first in Definition 2.2.1, (ii), which is essentially contained in Lemma 2.2.2, and (iii), which appeared as Proposition 4.3.10—and two analytic definitions of a terminal element (iv) and (v), which Joyal proves are equivalent [**46**, 4.2].

F.1.1. PROPOSITION. *For a quasi-category A and element $t\colon 1 \to A$ the following are equivalent:*

(i) The element t defines a right adjoint to the unique functor:

$$\begin{tikzcd} 1 \arrow[r, bend right, "t"'] \arrow[r, phantom, "\perp"] & A \arrow[l, bend right, "!"'] \end{tikzcd}$$

(ii) There exists a natural transformation

$$\begin{tikzcd} A \arrow[rr, equal] \arrow[dr, "!"'] & {} \arrow[d, phantom, "\Downarrow\eta"] & A \\ & 1 \arrow[ur, "t"'] & \end{tikzcd}$$

so that the component ηt is an isomorphism.

(iii) The domain-projection functor

$$\mathsf{Hom}_A(A,t) \xrightarrow[\sim]{p_0} A$$

defines a trivial fibration.

(iv) The projection functor

$$A_{/t} \xrightarrow{\sim} A$$

whose domain is the slice of A over t is a trivial fibration.

(v) Any sphere in A whose final vertex is t admits a filler:

$$\begin{tikzcd} 1 \arrow[r, "{[n]}"'] \arrow[rr, bend left, "t"] & \partial\Delta[n] \arrow[r] \arrow[d, hook] & A \\ & \Delta[n] \arrow[ur, dashed] & \end{tikzcd}$$

*When these conditions hold, t defines a **terminal element** of A.*

Proof. Unpacking (i), all that is required to define an adjunction $! \dashv t$ is to define a unit natural transformation $\eta\colon \mathrm{id}_A \Rightarrow t!$ so that the component $\eta t = \mathrm{id}_t$ is an identity; see Lemma 2.2.2. But it suffices, as claimed by (ii), to require only that ηt is an isomorphism. The proof is a specialization of Lemma B.4.2. The vertical composite $(\eta t) \cdot (\eta t)$ is computed by the pasting diagram:

$$\begin{tikzcd} 1 \arrow[r, "t"] & A \arrow[rr, equal] \arrow[dr, "!"'] & \arrow[d, phantom, "\Downarrow\eta"] & A \arrow[rr, equal] \arrow[dr, "!"'] & \arrow[d, phantom, "\Downarrow\eta"] & A \\ & & 1 \arrow[ur, "t"'] & & 1 \arrow[ur, "t"'] & \end{tikzcd}$$

By middle-four interchange, we can evaluating this composite by first whiskering the left-hand η with $!$, which yields $!\eta = \mathrm{id}_!$ since the quasi-category 1 is 2-terminal.[1] Hence $(\eta t) \cdot (\eta t) = (\eta t)$, so ηt is an idempotent isomorphism, and hence, by cancelation, an identity. This proves the equivalence of (i) and (ii).

Proposition 4.3.10 establishes the equivalences of (i) and (iii) in any ∞-cosmos.

By Proposition D.6.4, for any vertex t in a quasi-category, we have an equivalence

$$\begin{tikzcd} \mathsf{Hom}_A(A,t) \arrow[rr, "\sim"] \arrow[dr, two heads, "p_0"'] & & A_{/t} \arrow[dl] \\ & A & \end{tikzcd}$$

between the domain-projection isofibration and the canonical projection from the slice construction of Proposition 4.2.5. Consequently, by the two-of-three property, one isofibration is an equivalence if and only if the other is, proving the equivalence of (iii) and (iv).

By Definition 1.1.24, the projection $A_{/t} \twoheadrightarrow A$ is a trivial fibration if and only if the following right lifting property holds for all $n \geq 0$

$$\begin{tikzcd} \partial\Delta[n] \arrow[r, "u"] \arrow[d, hook] & A_{/t} \arrow[d, two heads] \\ \Delta[n] \arrow[r, "v"'] \arrow[ur, dashed, "w"] & A \end{tikzcd}$$

Via the adjunction

$$\begin{tikzcd} s\mathcal{S}et \arrow[r, bend left, "-\star\Delta[0]"] \arrow[r, phantom, "\perp"] & {}^{\Delta[0]/}s\mathcal{S}et \arrow[l, bend left, "-/-"] \end{tikzcd}$$

[1] See the discussion in the paragraph preceding Proposition 1.4.5.

the sphere $u\colon \partial\mathbf{\Delta}[n] \to A_{/t}$ transposes into a map $\Lambda^{n+1}[n+1] \to A$ with final vertex A, with the simplex $v\colon \mathbf{\Delta}[n] \to A$ providing a filler for the open face of the horn. Thus, together the maps u and v transpose to define a sphere $\partial\Delta[n+1] \to A$ with final vertex t. The desired lift $w\colon \mathbf{\Delta}[n] \to A_{/t}$ exists just when this transposed sphere admits a filler. In this way, we see that the right lifting properties

$$\forall n \geq 0 \quad \begin{tikzcd} \partial\Delta[n] \arrow[r] \arrow[d, hook] & A_{/t} \arrow[d, two heads] \\ \Delta[n] \arrow[r] \arrow[ur, dashed] & A \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} 1 \arrow[r, "{[n]}"'] \arrow[rr, bend left, "t"] & \partial\Delta[n] \arrow[r] \arrow[d, hook] & A \\ & \Delta[n] \arrow[ur, dashed] \end{tikzcd} \quad \forall n \geq 1$$

are transposes, proving the equivalence of (iv) and (v). □

There is a relative extension of Joyal's characterization (v), which is involved with the construction of a homotopy coherent adjunction.

F.1.2. Lemma. *Suppose E and B are quasi-categories which possess a terminal element and $p\colon E \twoheadrightarrow B$ is an isofibration which preserves them: if t is terminal in E then pt is terminal in B. Then any lifting problem of the following form has a solution*

$$\begin{tikzcd} 1 \arrow[r, "{[n]}"'] \arrow[rr, bend left, "t"] & \partial\Delta[n] \arrow[r, "u"] \arrow[d, hook] & E \arrow[d, two heads, "p"] \\ & \Delta[n] \arrow[r, "v"'] \arrow[ur, dashed] & B \end{tikzcd}$$

Proof. Using the universal property of the terminal object t in E and Proposition F.1.1(v), we we may extend the sphere u to a map $w\colon \mathbf{\Delta}[n] \to E$. This defines two maps $pw, v\colon \mathbf{\Delta}[n] \rightrightarrows B$ with a common boundary $pu\colon \partial\mathbf{\Delta}[n] \to B$, which we may use to define a sphere $h\colon \partial\mathbf{\Delta}[n+1] \to B$ with $h\delta^{n+1} = pw$ and $h\delta^n = v$ by starting with the degenerate simplex $pw\sigma^n\colon \mathbf{\Delta}[n+1] \to B$, restricting to its boundary, and then replacing the n^{th} face in this sphere with $v\colon \mathbf{\Delta}[n] \to B$. By construction, h maps the object $[n+1]$ to the object pt which is terminal in B, so it follows that we may fill this sphere to define a simplex $k\colon \mathbf{\Delta}[n+1] \to B$.

We may also construct a map $g\colon \Lambda^n[n+1] \to E$ by restriction from the degenerate simplex $w\sigma^n\colon \mathbf{\Delta}[n+1] \to E$, which then defines a factorization of the commutative square of the statement:

$$\begin{tikzcd} \partial\Delta[n] \arrow[r, "u"] \arrow[d, hook] & E \arrow[d, two heads, "p"] \\ \Delta[n] \arrow[r, "v"'] & B \end{tikzcd} = \begin{tikzcd} \partial\mathbf{\Delta}[n] \arrow[r, hook, "\delta^n"] \arrow[d, hook] & \Lambda^n[n+1] \arrow[r, "g"] \arrow[d, hook] & E \arrow[d, two heads, "p"] \\ \mathbf{\Delta}[n] \arrow[r, hook, "\delta^n"'] & \mathbf{\Delta}[n+1] \arrow[r, "k"'] \arrow[ur, dashed, "\ell"] & B \end{tikzcd}$$

Since the central vertical of this commutative rectangle is an inner horn inclusion and its right hand vertical is an isofibration of quasi-categories, it follows that the lifting problem on the right has a solution $\ell\colon \mathbf{\Delta}[n+1] \to E$ as marked, and now it is clear that the map $\ell\delta^n\colon \mathbf{\Delta}[n] \to E$ provides a solution to the original lifting problem. □

Exercises.

F.1.i. Exercise. Prove that if A and B are quasi-categories which possess a terminal element and $f\colon A \to B$ is a functor which preserves terminal elements, then given any lifting problem as below-left in which t is terminal in A

$$\begin{tikzcd} 1 \arrow[r, "{[n]}"'] \arrow[rr, bend left, "t"] & \partial\Delta[n] \arrow[r] \arrow[d, hook] & A \arrow[d, "f"] \\ & \Delta[n] \arrow[r] & B \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \partial\Delta[n] \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\Downarrow\cong"] & A \arrow[d, "f"] \\ \Delta[n] \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

there exists a lift as above-right so that the upper-left triangle commutes up to natural isomorphism and the bottom-right triangle commutes on the nose.

F.2. Limits and colimits

In this section, we expand Proposition 4.3.2 to prove that the synthetic definition of a limit of a diagram indexed by a simplicial set and taking values in a quasi-category coincides with the analytic definition first introduced by Joyal [**46**, 4.5]. In the following result, we prove the equivalence between four synthetic definitions of a limit cone—(i), the original Definition 2.3.7, (ii), appearing in Proposition 9.5.1, (iii), appearing in Proposition 4.3.4, and (iv), from Proposition 4.3.2—and one analytic one (v), which is Joyal's. In this case, by the results just cited and Proposition F.1.1, there is nothing left to do but state the result and provide references for its components.

F.2.1. Proposition. *Consider a diagram $d\colon J \to A$, where J is a simplicial set and A is a quasi-category. The following are equivalent*

(i) There exists an absolute right lifting diagram

$$\begin{tikzcd} & A \arrow[d, "\Delta"] \\ 1 \arrow[r, "d"'] \arrow[ur, "\ell"] \arrow[r, phantom, shift left=3, "\Downarrow\lambda"] & A^J \end{tikzcd}$$

(ii) There exists an absolute right lifting diagram

$$\begin{tikzcd} & A^{J^{\triangleleft}} \arrow[d, two heads, "\mathrm{res}"] \\ 1 \arrow[r, "d"'] \arrow[ur, "\ulcorner\lambda\urcorner"] \arrow[r, phantom, shift left=3, "\Downarrow\cong"] & A^J \end{tikzcd}$$

(iii) There exists a pointwise right extension diagram

$$\begin{tikzcd} J \arrow[r, "d"] \arrow[d, "!"'] \arrow[r, phantom, shift right=3, "\Uparrow\lambda"] & A \\ 1 \arrow[ur, "\ell"'] \end{tikzcd}$$

(iv) The quasi-category of cones $\mathsf{Hom}_{A^J}(\Delta, d)$ *admits a terminal element* $\ulcorner\lambda\urcorner$*, representing a cone*

$$\begin{tikzcd} & 1 \arrow[d, "\ulcorner\lambda\urcorner"] & \\ & \mathsf{Hom}_{A^J}(\Delta,d) \arrow[dl] \arrow[dr, "p_0"] \arrow[dd, phantom, "\overset{\phi}{\Leftarrow}"] & \\ 1 \arrow[dr, "d"'] & & A \arrow[dl, "\Delta"] \\ & A^J & \end{tikzcd} \quad = \quad \begin{tikzcd} & 1 \arrow[dr, "\ell"] \arrow[dd, phantom, "\overset{\lambda}{\Leftarrow}"] & \\ 1 \arrow[ur, equal] \arrow[dr, "d"'] & & A \arrow[dl, "\Delta"] \\ & A^J & \end{tikzcd}$$

(v) *The quasi-category* $A_{/d}$ *admits a terminal element* $\ulcorner\lambda\urcorner\colon 1 \to A_{/d}$*, transposing to define an extension of* d *to a diagram*

$$\begin{tikzcd} & J \arrow[dl, hook] \arrow[dr, "d"] & \\ J^{\triangleleft} \arrow[rr, "\ulcorner\lambda\urcorner"'] & & A \end{tikzcd}$$

When these conditions hold, the data variously labeled (ℓ, λ) *or* $\ulcorner\lambda\urcorner$ *defines the* ***limit cone*** *over* d*.*

Proof. The equivalence of (i) and (ii) is proven in Proposition 4.3.4 for any ∞-category A and simplicial set J; the simplicial set $J^{\triangleleft} := \Delta[0] \star J$ is the join from Definition 4.2.4. The equivalence of (i) and (iii) is proven in Proposition 4.3.4 for any cartesian closed ∞-cosmos. Proposition 4.3.2 proves the equivalence between (i) and (iv) for any diagram valued in any ∞-cosmos.

Finally, the equivalence between (iv) and (v) is a consequence of the equivalence of quasi-categories $\mathsf{Hom}_{A^J}(\Delta, d) \simeq A_{/d}$ over A of Proposition D.6.5, which provides two models for the quasi-category of cones over d. The final ingredient Lemma 2.2.7, which that if one of these quasi-categories has a terminal element, they both do, as terminal elements are preserved by the equivalence. $\square$

F.3. Right adjoint right inverse adjunctions

To our knowledge, right adjoint right inverse adjunctions between quasi-categories have not been given much attention. Nonetheless, we pause to establish a useful analytic characterization of such adjunctions, which will help us compare various other synthetic and analytic definitions.

F.3.1. Lemma. *An isofibration* $f\colon B \twoheadrightarrow A$ *of quasi-categories admits a right adjoint right inverse if and only if for every element* $a\colon 1 \to A$*, there exists an element* $ua\colon 1 \to B$ *with* $fua = a$ *that has the property that that any lifting problem with* $n \geq 1$

$$\begin{tikzcd} 1 \arrow[r, "{[n]}"'] \arrow[rr, bend left, "ua"] & \partial\Delta[n] \arrow[r] \arrow[d, hook] & B \arrow[d, two heads, "f"] \\ & \Delta[n] \arrow[r] \arrow[ur, dashed] & A \end{tikzcd} \tag{F.3.2}$$

has a solution.

Proof. If u is the right adjoint right inverse, then $\epsilon\colon fu = \mathrm{id}_A$ is the identity, and the induced fibered equivalence $\ulcorner \epsilon \circ f(-) \urcorner\colon \mathsf{Hom}_B(B, u) \xrightarrow{\sim} \mathsf{Hom}_A(f, A)$ of Proposition 4.1.1 is represented by the map induced by f between the comma ∞-categories defined in Proposition 3.4.5.

$$\begin{tikzcd}
& \mathsf{Hom}_B(B,u) \arrow[rr] \arrow[dl, dashed, "\ulcorner \epsilon\circ f(-)\urcorner"'] \arrow[dd, two heads] \arrow[drr, phantom, "\lrcorner" very near start] & & B^{\mathbb{2}} \arrow[dl, two heads, "f^{\mathbb{2}}"] \arrow[dd, two heads] \\
\mathsf{Hom}_A(f,A) \arrow[rr] \arrow[dd, two heads] \arrow[drr, phantom, "\lrcorner" very near start] & & A^{\mathbb{2}} \arrow[dd, two heads] & \\
& A\times B \arrow[rr, "u\times B"] \arrow[dl, equal] & & B\times B \arrow[dl, two heads, "f\times f"] \\
A\times B \arrow[rr, "f\times A"'] & & A\times A &
\end{tikzcd}$$

By that result—or alternatively, by Proposition C.1.12, on which its proof relies—we see that that the induced map between comma ∞-categories is also an isofibration. Combining these facts, we see that $\ulcorner \epsilon \circ f(-) \urcorner\colon \mathsf{Hom}_B(B, u) \mathrel{\overset{\sim}{\twoheadrightarrow}} \mathsf{Hom}_A(f, A)$ is a trivial fibration over $A \times B$. This trivial fibration pulls back over any vertex $a\colon 1 \to A$ to define a trivial fibration $\mathsf{Hom}_B(B, ua) \mathrel{\overset{\sim}{\twoheadrightarrow}} \mathsf{Hom}_A(f, a)$ over B. By Corollary D.6.6, the domain and codomain are equivalent to Joyal's slices, so the isofibration $f\colon B_{/ua} \twoheadrightarrow f_{/a}$ induced by f is also a trivial fibration between quasi-categories. The defining lifting property of Definition 1.1.24

$$\begin{tikzcd}
\partial\Delta[n-1] \arrow[r] \arrow[d, hook] & B_{/ua} \arrow[d, two heads, "\wr", "f"'] \\
\Delta[n-1] \arrow[r] \arrow[ur, dashed] & f_{/a}
\end{tikzcd}$$

for $n \geq 1$ transposes to the lifting property of (F.3.2).

Conversely, the lifting property (F.3.2) can be used to inductively define a section $u\colon A \to B$ of f extending the choices of elements $ua\colon 1 \to B$ lifting each $a\colon 1 \to A$. The inclusion $\mathrm{sk}_0 A \hookrightarrow A$ can be expressed as a countable composite of pushouts of coproducts of maps $\partial\mathbf{\Delta}[n] \hookrightarrow \mathbf{\Delta}[n]$ with $n \geq 1$, and each intermediate lifting problem required to define a lift

$$\begin{tikzcd}
1 \arrow[r, "a"] \arrow[rr, bend left, "ua"] & \mathrm{sk}_0 A \arrow[r, "u"] \arrow[d, hook] & B \arrow[d, two heads, "f"] \\
& A \arrow[r, equal] \arrow[ur, dashed, "u"] & A
\end{tikzcd}$$

will have the form of (F.3.2). To show that u is a right adjoint right inverse to f, it suffices, by Lemma B.4.2 to define a 2-cell $\eta\colon \mathrm{id}_B \Rightarrow uf$ that whiskers with u and with f to isomorphisms. We construct a representative for η by solving the lifting problem

$$\begin{tikzcd}
B \sqcup B \arrow[rr, "\mathrm{id}_B \sqcup uf"] \arrow[d, hook] & & B \arrow[d, two heads, "f"] \\
B \times \Delta[1] \arrow[r, two heads, "\pi"'] \arrow[urr, dashed, "\eta"'] & B \arrow[r, "f"'] & A
\end{tikzcd}$$

By construction $f\eta = \mathrm{id}_f$ is certainly invertible.

To show that ηu is an isomorphism it suffices, by Corollary D.4.18, to check that each of its components $\eta u(a)\colon ua \to ufua = ua$ are isomorphisms in A. Inverse isomorphisms can be found by elementary applications of the lifting property (F.3.2), whose details we leave to the reader. $\square$

F.4. Cartesian and cocartesian fibrations

The aim in this section is to establish the equivalence between synthetic and analytic characterizations of when an isofibration $p\colon E \twoheadrightarrow B$ between quasi-categories defines a cartesian or cocartesian fibration. The following result compares the three synthetic definitions proven equivalent in Theorem 5.1.11 with two analytic definitions due to Lurie [**59**, §2.4.1]. After we give its proof we comment on a few non-obvious aspects of the comparison.

F.4.1. Proposition. *For an isofibration $p\colon E \twoheadrightarrow B$ between quasi-categories, the following are equivalent and define what it means for p to be a* ***cartesisan fibration****:*

(i) Every natural transformation $\beta\colon b \Rightarrow pe$ as below-left admits a lift $\chi\colon e' \Rightarrow e$ as below-right:

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[dr, bend right, "b"'] \arrow[dr, phantom, "\Uparrow\beta" near start] & E \arrow[d, two heads, "p"] \\ & B \end{tikzcd} \quad = \quad \begin{tikzcd} X \arrow[r, bend left, "e"] \arrow[r, bend right, "e'"'] \arrow[r, phantom, "\Uparrow\chi"] & E \arrow[d, two heads, "p"] \\ & B \end{tikzcd}$$

with the property that:

- ***induction****: Given any functor $x\colon Y \to X$ and natural transformations* $\begin{tikzcd} Y \arrow[r, bend left, "e''"] \arrow[r, bend right, "ex"'] \arrow[r, phantom, "\Downarrow\tau"] & E \end{tikzcd}$ *and* $\begin{tikzcd} Y \arrow[r, bend left, "pe''"] \arrow[r, bend right, "pe'x"'] \arrow[r, phantom, "\Downarrow\gamma"] & B \end{tikzcd}$ *so that $p\tau = p\chi x \cdot \gamma$, there exists a lift* $\begin{tikzcd} Y \arrow[r, bend left, "e''"] \arrow[r, bend right, "e'x"'] \arrow[r, phantom, "\Downarrow\bar{\gamma}"] & E \end{tikzcd}$ *of γ so that $\tau = \chi x \cdot \bar{\gamma}$.*
- ***conservativity****: Any fibered endomorphism of a restriction of χ is invertible: if* $\begin{tikzcd} Y \arrow[r, bend left, "e'x"] \arrow[r, bend right, "e'x"'] \arrow[r, phantom, "\Downarrow\zeta"] & E \end{tikzcd}$ *is any natural transformation so that $\chi x \cdot \zeta = \chi x$ and $p\zeta = \mathrm{id}_{pe'}$ then ζ is invertible.*

(ii) The functor $\ulcorner \mathrm{id}_p \urcorner\colon E \to \mathsf{Hom}_B(B, p)$ admits a right adjoint over B.

(iii) The functor $k \coloneqq \widehat{\{\delta^0, p\}}\colon E^{\mathbb{2}} \twoheadrightarrow \mathsf{Hom}_B(B, p)$ admits a right adjoint right inverse.

(iv) Any 1-simplex $\beta\colon b \to pe$ in B admits a lift $\chi\colon e' \to e$ in E so that any lifting problem for $n \geq 1$

$$\begin{tikzcd} \Delta[1] \arrow[r, "{[n]\times\mathrm{id}}"'] \arrow[rrr, bend left, "\chi"] & \partial\Delta[n] \times \Delta[1] \underset{\partial\Delta[n]\times\{1\}}{\cup} \Delta[n] \times \{1\} \arrow[rr, two heads] \arrow[d, hook] & & E \arrow[d, two heads, "p"] \\ & \Delta[n] \times \Delta[1] \arrow[rr] \arrow[urr, dashed] & & B \end{tikzcd} \tag{F.4.2}$$

has a solution.

(v) Any 1-simplex $\beta\colon b \to pe$ in B admits a lift $\chi\colon e' \to e$ in E so that any lifting problem for $n \geq 2$

$$\begin{tikzcd}
\Delta[1] \arrow[r, "\{n-1,n\}"'] \arrow[rr, bend left, "\chi"] & \Lambda^n[n] \arrow[r] \arrow[d] & E \arrow[d, two heads, "p"] \\
& \Delta[n] \arrow[r] \arrow[ur, dashed] & B
\end{tikzcd} \tag{F.4.3}$$

has a solution.

Condition (v) appears to be mildly stronger than [**59**, 2.4.2.1], which only requires that p is an inner fibration with the lifting property (F.4.3), but it follows easily that any such p must be an isofibration; see Exercise F.4.i.

Proof. Condition (i) is a repackaging of the original Definition 5.1.6 with the induction and conservativity properties of the p-cartesian natural transformations of Definition 5.1.1 strengthened to include the observation of Lemma 5.1.5 and the global requirement that these properties be inherited by restrictions of p-cartesian natural transformations. Modulo this translation, the equivalence of (i), (ii), and (iii) is proven in Theorem 5.1.11.

It remains to verify the equivalence between any of these synthetic conditions and the corresponding analytic ones. We'll demonstrate that (iii)⇔(iv) and (iv)⇔(v).

By Lemma F.3.1, the isofibration k admits a right adjoint right inverse if and only if any 1-simplex $\beta\colon b \to pe$ in B admits a lift $\chi\colon e' \to e$ in E with the lifting property

$$\begin{tikzcd}
1 \arrow[r, "{[n]}"'] \arrow[rr, bend left, "\chi"] & \partial\Delta[n] \arrow[r] \arrow[d, hook] & E^{\mathbb{2}} \arrow[d, two heads, "k"] \\
& \Delta[n] \arrow[r] \arrow[ur, dashed] & \mathsf{Hom}_B(B, p)
\end{tikzcd} \tag{F.4.4}$$

for $n \geq 1$. Since the isofibration k is the Leibniz cotensor of p with the inclusion $1\colon \mathbf{\Delta}[0] \hookrightarrow \Delta[1]$, this lifting property is equivalent to the transposed lifting property

$$\begin{tikzcd}
\Delta[1] \arrow[r, "{[n]\times\mathrm{id}}"'] \arrow[rr, bend left, "\chi"] & \partial\Delta[n]\times\Delta[1] \underset{\partial\Delta[n]\times\{1\}}{\cup} \Delta[n]\times\{1\} \arrow[r] \arrow[d, hook] & E \arrow[d, two heads, "p"] \\
& \Delta[n]\times\Delta[1] \arrow[r] \arrow[ur, dashed] & B
\end{tikzcd}$$

again for $n \geq 1$, proving the equivalence between (iii) and (iv).

Now we'll show that the lifting property (F.4.3) assumed in (v) suffices to solve this lifting problem. Our task is to find lifts along p for each of the $n+1$ shuffles of $\Delta[n]\times\Delta[1]$. We number these shuffles $0, \ldots, n$ starting from the closed end of the cylinder. Proceeding inductively for $k < n$, we choose a lift for the kth shuffle by filling a $\Lambda^{k+1}[n+1]$ horn, which can be done since p is an isofibration between quasi-categories. To lift the nth shuffle, we're required to fill a $\Lambda^{n+1}[n+1]$ horn whose final $\{n, n+1\}$ edge is χ, which can be done with the lifting property (F.4.3). This proves that (v)⇒(iv).

The converse implication holds on account of the retract diagram

$$\begin{tikzcd}
\Delta[1] \arrow[r, "\{n,n+1\}"'] \arrow[rr, bend left, "{[n]\times\mathrm{id}}"] & \Lambda^{n+1}[n+1] \arrow[r, hook] \arrow[d, hook] & \partial\Delta[n]\times\Delta[1] \underset{\partial\Delta[n]\times\{1\}}{\cup} \Delta[n]\times\{1\} \arrow[r] \arrow[d, hook] & \Lambda^{n+1}[n+1] \arrow[d, hook] \\
& \Delta[n+1] \arrow[r, hook, "\iota"] & \Delta[n]\times\Delta[1] \arrow[r, "\rho"] & \Delta[n+1]
\end{tikzcd} \tag{F.4.5}$$

in which $\iota\colon \mathbf{\Delta}[n+1] \hookrightarrow \mathbf{\Delta}[n]\times\Delta[1]$ is the inclusion of the last shuffle at the open end of the cylinder and $\rho\colon \mathbf{\Delta}[n]\times\Delta[1] \to \mathbf{\Delta}[n+1]$ is the projection that collapses the closed end of the cylinder to a point. In coordinates:

$$\iota(i) \coloneqq \begin{cases} (i,0) & 0 \le i \le n \\ (n,1) & i = n+1 \end{cases} \qquad \text{and} \qquad \rho(i,j) \coloneqq \begin{cases} i & j = 0 \\ n+1 & j = 1. \end{cases}$$

By Lemma C.2.3 it's now clear that the lifting property (F.4.2) implies the lifting property (F.4.3). ☐

The proof of Proposition F.4.1 demonstrated that the lifting properties of (iv) and (v) define equivalent analytic characterizations of what it means for a 1-simplex in the domain of an isofibration $p\colon E \twoheadrightarrow B$ to be "p-cartesian," see also [**59**, 2.4.1.8]. A priori this analytic definition of what it means for an edge in the domain of an isofibration p between quasi-categories to be p-cartesian appears to be significantly stronger than our synthetic Definition 5.1.1 and indeed this is the case—unless p is a cartesian fibration, in which case this weaker universal property, miraculously, turns out to be strong enough.

F.4.6. Proposition. *Fix $p\colon E \twoheadrightarrow B$ an isofibration of quasi-categories and consider a 1-simplex $\chi\colon e' \to e$ in E. The following are equivalent and characterize when χ is p-**cartesian**:*

(i) The induced map to the pullback in the square

$$\begin{tikzcd}
E_{/\chi} \arrow[r, two heads] \arrow[d, two heads, "p"'] & E_{/e} \arrow[d, two heads, "p"] \\
B_{/p\chi} \arrow[r, two heads] & B_{/pe}
\end{tikzcd}$$

is a trivial fibration.

(ii) Any lifting problem for $n \ge 2$

$$\begin{tikzcd}
\Delta[1] \arrow[r, "\{n-1,n\}"'] \arrow[rr, bend left, "\chi"] & \Lambda^n[n] \arrow[r] \arrow[d] & E \arrow[d, two heads, "p"] \\
& \Delta[n] \arrow[r] \arrow[ur, dashed] & B
\end{tikzcd}$$

has a solution.

(iii) Any lifting problem for $n \geq 1$

$$\begin{tikzcd}
\Delta[1] \arrow[r, "{[n]\times\mathrm{id}}"'] \arrow[rr, bend left, "\chi"] & \partial\Delta[n]\times\Delta[1] \underset{\partial\Delta[n]\times\{1\}}{\cup} \Delta[n]\times\{1\} \arrow[r] \arrow[d, hook] & E \arrow[d, two heads, "p"] \\
& \Delta[n]\times\Delta[1] \arrow[r] \arrow[ur, dashed] & B
\end{tikzcd}$$

has a solution.

If $p\colon E \twoheadrightarrow B$ *is known to be a cartesian fibration, then the following criterian also characterizes the* p*-cartesian edges* $\chi\colon e' \to e$*.*

(iv)
- ***induction**: Given any 1-simplices* $\tau\colon e'' \to e'$ *in E and* $\gamma\colon pe'' \to pe'$ *in B that bound a 2-simplex*

$$\begin{tikzcd}
pe'' \arrow[rr, "p\tau"] \arrow[dr, "\gamma"'] & & pe \\
& pe' \arrow[ur, "p\chi"'] &
\end{tikzcd}$$

in B, there exists a lift $\bar{\gamma}\colon e'' \to e'$ *of* γ *that bounds a 2-simplex*

$$\begin{tikzcd}
e'' \arrow[rr, "\tau"] \arrow[dr, "\bar{\gamma}"'] & & e \\
& e' \arrow[ur, "\chi"'] &
\end{tikzcd}$$

in E.

$$\begin{tikzcd}
e'' \arrow[rr, "\tau"] \arrow[dr, dotted, "\bar{\gamma}"'] & & e & & \\
& e' \arrow[ur, "\chi"'] & & \in & \mathsf{h}E \arrow[dd, two heads, "p_*"] \\
& \arrow[d, mapsto] & & & \\
pe'' \arrow[rr, "p\tau"] \arrow[dr, "\gamma"'] & & pe & \in & \mathsf{h}B \\
& pe' \arrow[ur, "p\chi"'] & & &
\end{tikzcd}$$

- ***conservativity**: Any fibered endomorphism of* χ *is invertible: if* $\zeta\colon e' \to e'$ *is a 1-simplex in E so that there exists a 2-simplex*

$$\begin{tikzcd}
e' \arrow[rr, "\chi"] \arrow[dr, "\zeta"'] & & e \\
& e' \arrow[ur, "\chi"'] &
\end{tikzcd}$$

in E and $p\zeta$ *is an isomorphism, then* ζ *is an isomorphism.*

Proof. By the adjunction of Proposition 4.2.5, the lifting property that characterizes the trivial fibration of (i)

$$\begin{tikzcd}
\partial\Delta[n] \arrow[r] \arrow[d, hook] & E_{/\chi} \arrow[d, two heads] \\
\Delta[n] \arrow[r] \arrow[ur, dashed] & B_{/p\chi} \underset{B_{/pe}}{\times} E_{/e}
\end{tikzcd}$$

for $n \geq 0$ transposes to the lifting property of (ii), as argued in the proof of Proposition F.1.1. The proof of Proposition F.4.1(iv)⇔(v) demonstrates that (ii)⇔(iii).

Finally, in the case where $p\colon E \twoheadrightarrow B$ is known to be a cartesian fibration, Lemma 5.3.5 proves that the synthetically p-cartesian natural transformations, satisfying the weak universal property of (iv), coincide with the class of **p-cartesian 1-arrows** introduced in Definition 5.3.4, namely those vertices in the essential image of the right adjoint right inverse $\bar{r}\colon \mathsf{Hom}_B(B,p) \to E^{\mathbb{2}}$ to $k\colon E^{\mathbb{2}} \twoheadrightarrow \mathsf{Hom}_B(B,p)$. By Lemma F.3.1, these vertices have a lifting property (F.4.4), which we've just seen transposes to the lifting property of (iii). Thus, in the case where p is a cartesian fibration, (iv) is as strong as the a priori higher-dimensional lifting properties (i), (ii), and (iii). □

We now extend the notion of cartesian edge from Definition 5.3.4 to define a **cartesian cylinder** to describe a variant of the lifting properties appearing in Proposition F.4.6.

F.4.7. Definition (cartesian cylinders). Suppose that $p\colon E \twoheadrightarrow B$ is a cartesian fibration of quasi-categories and that X is any simplicial set. We say that a cylinder $e\colon X \times \Delta[1] \to E$ is **pointwise p-cartesian** if and only if for each 0-simplex $x \in X$ it maps the 1-simplex $(x \cdot \sigma^0, \mathrm{id}_{[1]})\colon (x,0) \to (x,1)$ to a p-cartesian arrow in E.

F.4.8. Lemma. *Let $p\colon E \twoheadrightarrow B$ be a cartesian fibration of quasi-categories. A cylinder $e\colon X \times \Delta[1] \to E$ is pointwise p-cartesian if and only if its dual $e\colon \Delta[1] \to E^X$ defines a p^X-cartesian arrow for the cartesian fibration $p^X\colon E^X \twoheadrightarrow B^X$.*

Proof. First note that Corollary 5.1.16 implies that $p^X\colon E^X \twoheadrightarrow B^X$ is a cartesian fibration. To prove the stated equivalence, note that a vertex is in the essential image of $\bar{r}^X\colon \mathsf{Hom}_{B^X}(B^X,p^X) \to E^{X\times\mathbb{2}}$ if and only if it transposes to a diagram $X \to E^{\mathbb{2}}$ whose vertices land in the essential image of $\bar{r}\colon \mathsf{Hom}_B(B,p) \to E$. □

Thus, the exponentiated functor $p^X\colon E^X \twoheadrightarrow B^X$ is a cartesian fibration whose cartesian arrows are the pointwise p-cartesian cylinders. It follows that for all $f\colon X \to Y$ restriction along f defines a cartesian functor of cartesian fibrations:

$$\begin{tikzcd} E^Y \arrow[r, "E^f"] \arrow[d, two heads, "p^Y"'] & E^X \arrow[d, two heads, "p^X"] \\ B^Y \arrow[r, "B^f"'] & B^X \end{tikzcd}$$

We give one further lifting property characterization of the p-cartesian edges for an isofibration $p\colon E \twoheadrightarrow B$ between quasi-categories that will be used in the body of the text.

F.4.9. Lemma. *Let $X \hookrightarrow Y$ be a simplicial subset of a simplicial set Y.*

(i) Any lifting problem for $n \geq 0$

$$\begin{tikzcd} X \times \Delta[1] \cup Y \times \{1\} \arrow[r, "e"] \arrow[d, hook] & E \arrow[d, two heads, "p"] \\ Y \times \Delta[1] \arrow[r, "b"'] \arrow[ur, dashed, "\bar{e}"] & B \end{tikzcd}$$

with the property that the cylinder $X \times \Delta[1] \subseteq X \times \Delta[1] \cup Y \times \{1\} \xrightarrow{e} E$ is pointwise p-cartesian admits a solution $\bar{e}$ which is also pointwise p-cartesian.

(ii) Any lifting problem for $n \geq 1$

$$\begin{tikzcd} X \times \mathbf{\Delta}[n] \cup Y \times \Lambda^n[n] \arrow[r, "e"] \arrow[d, hook] & E \arrow[d, two heads, "p"] \\ Y \times \mathbf{\Delta}[n] \arrow[r, "b"'] \arrow[ur, dashed, "\bar{e}"'] & B \end{tikzcd}$$

in which the cylinder $Y \times \mathbf{\Delta}^{\{n-1,n\}} \subseteq X \times \mathbf{\Delta}[n] \cup Y \times \Lambda^n[n] \xrightarrow{e} E$ is pointwise p-cartesian admits a solution $\bar{e}$.

Proof. The Leibniz tensor with $X \hookrightarrow Y$ defines a functor $s\mathcal{S}et^{\mathbb{2}} \to s\mathcal{S}et^{\mathbb{2}}$ that preserves the retract diagram (F.4.5), so the lifting property (ii) follows from (i). In turn, the lifting property (i) follows inductively from Proposition F.4.6(iii) combined with the fact that any monomorphism $X \hookrightarrow Y$ can be decomposed as a sequential composite of pushouts of coproducts of inclusions $i_n : \partial\Delta[n] \hookrightarrow \Delta[n]$, and by Proposition C.2.9(vi), the pushout product with $\{1\} \hookrightarrow \Delta[1]$ is then similarly a sequential composite of pushouts of coproducts of the pushout products $\partial\mathbf{\Delta}[n] \times \Delta[1] \cup \mathbf{\Delta}[n] \times \{1\} \hookrightarrow \mathbf{\Delta}[n] \times \Delta[1]$. ☐

A similar argument proves the following result.

F.4.10. Lemma. *A cocartesian fibration $q : E \twoheadrightarrow A$ of quasi-categories admits a right adjoint right inverse $t : A \to E$ if and only if for each object $a \in A$ the fiber E_a over that object has a terminal object.*

Proof. To prove necessity, consider the following commutative diagram:

$$\begin{tikzcd} 1 \arrow[r, "{[n]}"'] \arrow[rr, bend left, "ta"] & \partial\Delta[n] \arrow[r] \arrow[d, hook] & E_a \arrow[r, hook] \arrow[d, two heads, "q"] & E \arrow[d, two heads, "q"] \\ & \Delta[n] \arrow[r] \arrow[ur, dashed] \arrow[urr, dotted] & 1 \arrow[r, "a"'] & A \end{tikzcd}$$

Under our assumption that q has a right adjoint right inverse t we may apply the lifting condition depicted in equation F.3.2 of Lemma F.3.1 to show that the outer composite square has a lifting (the dotted arrow) and then apply the pullback property of the right hand square to obtain a lifting for the left hand square (the dashed arrow). This lifting property of the left hand square shows that ta is a terminal object in E_a.

To establish sufficiency, we start by taking the object $ta \in E$ to be the terminal object in the fiber E_a for each object $a \in A$. Now by Lemma F.3.1 our desired result follows if we can show that each lifiting problem

$$\begin{tikzcd} 1 \arrow[r, "{[n]}"'] \arrow[rr, bend left, "ta"] & \partial\Delta[n] \arrow[r, "y"'] \arrow[d, hook] & E \arrow[d, two heads, "q"] \\ & \Delta[n] \arrow[r, "x"'] & A \end{tikzcd} \tag{F.4.11}$$

has a solution. Consider the order preserving function $k : [n] \times [1] \to [n]$ defined by $k(i, 0) := i$ and $k(i, 1) := n$. Taking nerves, this gives rise to a simplicial map $k : \mathbf{\Delta}[n] \times \Delta[1] \to \mathbf{\Delta}[n]$ and it is easy to check that this restricts to a simplicial map $k\colon \mathbf{\Delta}[n] \times \Delta[1] \to \mathbf{\Delta}[n]$ which we may compose with $x\colon \mathbf{\Delta}[n] \to A$ to give (a representative of) a 2-cell

$$\begin{tikzcd}
\partial\Delta[n] \arrow[r, "y"] \arrow[d, "!"'] \arrow[dr, phantom, "\Downarrow\kappa"] & E \arrow[d, two heads, "q"] \\
1 \arrow[r, "a"'] & A
\end{tikzcd}$$

with the property that $\kappa[n]$ is the identity 2-cell on a. Now we may take a cocartesian lift $\chi\colon y \Rightarrow u$ of κ and by construction the image of $u\colon \mathbf{\Delta}[n] \to E$ is contained entirely in the fiber $E_a \subseteq E$. What is more, we know that $\chi[n]$ is an isomorphism since we have, by pre-composition stability of cocartesian 2-cells, that it is a cocartesian lift of the identity 2-cell on a. Consequently, we see that $u\{n\}$ is a terminal object of E_a, since it is isomorphic to ta, and it follows that we may apply its universal property to extend $u\colon \partial\mathbf{\Delta}[n] \to E_a$ to a simplex $v\colon \mathbf{\Delta}[n] \to E_a$.

We may combine (a representative of) the 2-cell χ with v to assemble the upper horizontal map in the following commutative square:

$$\begin{tikzcd}
\partial\mathbf{\Delta}[n] \times \Delta[1] \cup \mathbf{\Delta}[n] \times \{1\} \arrow[r] \arrow[d, hook] & E \arrow[d, two heads, "q"] \\
\mathbf{\Delta}[n] \times \Delta[1] \arrow[r] \arrow[ur, dashed] & A
\end{tikzcd}$$

Now we can construct a solution for this lifting problem by successively picking fillers for each of the non-degenerate $(n+1)$-simplices in $\mathbf{\Delta}[n]\times\Delta[1]$. These are guaranteed to exist for the first $n-1$ of those because q is an isofibrations and they entail the filling of an inner horn. To obtain a filler for the last one we need to fill an outer horn, but observe that its final edge maps to an isomorphism of E, since it is the image of $\chi[n]$, and so it too has a filler. Finally on restricting the resulting map $\ell\colon \mathbf{\Delta}[n]\times\Delta[1] \to E$ to the initial end of the cylinder that is its domain, we obtain an n-simplex $\mathbf{\Delta}[n] \to E$ which is easily seen to be a solution to the original lifting problem in (F.4.11) as required. ☐

A final result demonstrates that discrete cocartesian and discrete cartesian fibrations between quasi-categories coincide with the classes of **left fibrations** and **right fibrations** introduced by Joyal [**46**].

F.4.12. Proposition. *For an isofibration $p\colon E \twoheadrightarrow B$ between quasi-categories, the following are equivalent and define what it means for p to be a* ***discrete cartesian fibration****:*

(i) The map $p\colon E \twoheadrightarrow B$ is a cartesian fibration whose fibers are Kan complexes.
(ii) Every 2-cell $\beta\colon b \Rightarrow pe$ in the homotopy 2-category of quasi-categories has an essentially unique lift: given $\chi\colon e' \Rightarrow e$ and $\psi\colon e'' \Rightarrow e$ so that $p\chi = p\psi = \beta$, then there exists an isomorphism $\gamma\colon e'' \Rightarrow e'$ with $\chi \cdot \gamma = \psi$ and $p\gamma = \mathrm{id}$.
(iii) The induced functor $k\colon E^{\mathbf{2}} \to \mathsf{Hom}_B(B,p)$ is a trivial fibration.
(iv) For $n \geq 0$ and $0 < k \leq n$ any lifting problem

$$\begin{tikzcd}
\Lambda^k[n] \arrow[r] \arrow[d, hook] & E \arrow[d, two heads, "p"] \\
\Delta[n] \arrow[r] \arrow[ur, dashed] & B
\end{tikzcd}$$

has a solution.

Proof. The first characterization (i) is a reinterpretation of the original Definition 5.4.2 using Proposition 12.2.3, which says that an isofibration $p\colon E \twoheadrightarrow B$ defines a discrete object in $\mathcal{QC}at_{/B}$ if and only if its fibers are Kan complexes. The equivalence of (i) with (ii) is proven in Proposition 5.4.3, while the equivalence of (i) with (iii) is proven in Proposition 5.4.7.

We conclude by demonstrating the equivalence of (iii) and (iv). Recall from Remark 5.1.13, that the canonical functors $k\colon E^{\mathbb{2}} \twoheadrightarrow \mathrm{Hom}_B(B,p)$ can be constructed as the Leibniz cotensor of the monomorphism $\mathbf{1}\colon \mathbb{1} \hookrightarrow \mathbb{2}$ with the trivial fibration $p\colon E \xrightarrow{\sim}\!\!\!\!\!\twoheadrightarrow B$. By adjunction, k is a trivial fibration if and only if the lifting problem below-right has a solution

$$\begin{tikzcd} \partial\Delta[n] \arrow[r] \arrow[d, hook] & E^{\mathbb{2}} \arrow[d, two heads, "k", "\wr"'] \\ \Delta[n] \arrow[r] \arrow[ur, dashed] & \mathrm{Hom}_B(B,p) \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} \partial\Delta[n]\times\Delta[1] \cup \Delta[n]\times\{1\} \arrow[r] \arrow[d, hook] & E \arrow[d, two heads, "p"] \\ \Delta[n]\times\Delta[1] \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

By Proposition D.3.9, if $p\colon E \twoheadrightarrow B$ is a right fibration, satisfying condition (iv), then the lifting problem above-right admits a solution and hence $k\colon E^{\mathbb{2}} \twoheadrightarrow \mathrm{Hom}_B(B,p)$ is a trivial fibration proving (iii). Conversely, for $0 < k \leq n$ the horn inclusion $\Lambda^k[n] \hookrightarrow \Delta[n]$ is a retract

$$\begin{tikzcd} \Lambda^k[n] \arrow[r, hook, "\Lambda^k[n]\times\{0\}"] \arrow[d, hook] & \Lambda^k[n]\times\Delta[1] \cup \Delta[n]\times\{1\} \arrow[r, two heads, "r"] \arrow[d, hook] & \Lambda^k[n] \arrow[d, hook] \\ \Delta[n] \arrow[r, hook, "\Delta[n]\times\{0\}"] & \Delta[n]\times\Delta[1] \arrow[r, two heads, "r"] & \Delta[n] \end{tikzcd} \quad \text{where} \quad \begin{aligned} r(i,0) &= i \\ r(i,1) &= \begin{cases} i & i \neq k-1 \\ k & i = k-1 \end{cases} \end{aligned}$$

Thus, to solve the lifting problem postulated by (iv), it suffices to show that p lifts against the pushouts products $(\Lambda^k[n] \hookrightarrow \Delta[n])\hat{\times}(\{1\} \hookrightarrow \Delta[1])$, which transposes to a lifting problem between the monomorphism $\Lambda^k[n] \hookrightarrow \Delta[n]$ and $k\colon E^{\mathbb{2}} \twoheadrightarrow \mathrm{Hom}_B(B,p)$. If (iii) holds and k is a trivial fibration, then this constructs the desired lift. $\square$

Exercises.

F.4.i. Exercise. Suppose $p\colon E \to B$ is an inner fibration between quasi-categories so that any 1-simplex $\beta\colon b \to pe$ in B admits a lift $\chi\colon e' \to e$ in E so that any lifting problem for $n \geq 2$

$$\begin{tikzcd} \Delta[1] \arrow[r, "\{n-1,n\}"'] \arrow[rr, bend left, "\chi"] & \Lambda^n[n] \arrow[r] \arrow[d] & E \arrow[d, two heads, "p"] \\ & \Delta[n] \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

has a solution. Show that p is an isofibration.

F.5. Adjunctions

The comparison between the analytic and synthetic definitions of adjunction between quasi-categories is somewhat more subtle as these are typically presented with different data. The synthetic definition, originally due to Joyal [47], of an adjunction involves two specified functors $f\colon B \to A$ and $u\colon A \to B$, together with specified maps $\eta\colon B\times\Delta[1] \to B$ and $\eta\colon A\times\Delta[1] \to A$ up to homotopy in

B^B or A^A, together with 2-simplices in the quasi-categories B^A and A^B that witness the triangle equalities. By contrast, the analytic notion, due to Lurie [**59**, 5.2.2.1], is defined to be an isofibration[2] that is both a cartesian fibration and a cocartesian fibration equipped with specified equivalences $M_0 \simeq B$ and $M_1 \simeq A$.

Since Proposition 2.1.12 demonstrates that the synthetic notion of adjunction is equivalence-invariant, we simplify our notation somewhat and let B and A denote the fibers over 0 and 1 respectively of the isofibration $M \twoheadrightarrow \Delta[1]$. Our aim in this section is to show that from a cocartesian and cartesian fibration $M \twoheadrightarrow \Delta[1]$, one can extract an adjunction between B and A, with the adjoint functors determined uniquely up to isomorphism, and conversely from a 2-categorical adjunction, one can construct a corresponding **correspondence** $M \twoheadrightarrow \Delta[1]$, which is unique up to fibered equivalence.

F.5.1. Proposition. *Let M be a quasi-category equipped with a map $M \twoheadrightarrow \Delta[1]$ that is both a cocartesian fibration and cartesian fibration. Then*
- *the fibers $B \coloneqq M_0$ and $A \coloneqq M_1$ and*
- *the functors $f \colon B \to A$ and $u \colon A \to B$ defined by the cocartesian and cartesian lift, respectively, of the generic arrow in $\Delta[1]$*

define an adjunction

$$\begin{tikzcd} A \arrow[r, bend right, "u"'] \arrow[r, phantom, "\perp"] & B \arrow[l, bend right, "f"'] \end{tikzcd}$$

Proof. This is a special case of Proposition 5.2.5. We recall the construction of f and u and leave the rest of the details to that result and Remark 5.2.6. Let χ denote a cocartesian lift of the generic arrow in $\Delta[1]$ whose domain is the inclusion $B \hookrightarrow M$ of the fiber over 0. The codomain of this lifted arrow lands in the fiber over 1 and thus factors uniquely through the inclusion $A \hookrightarrow M$ of that fiber:

$$\begin{tikzcd} B \arrow[rr, hook, bend left] \arrow[d, two heads] \arrow[drr, phantom, very near start, "\lrcorner"] & & M \arrow[d, two heads] \\ \mathbb{1} \arrow[rr, bend left, "0"] \arrow[rr, bend right, "1"'] \arrow[rr, phantom, "\Downarrow"] & & \Delta[1] \end{tikzcd} = \begin{tikzcd} B \arrow[rr, hook, bend left] \arrow[rr, phantom, "\Downarrow\chi"] \arrow[dr, dashed, "f"'] \arrow[rr, dotted, bend right=10] \arrow[dd, two heads] & & M \arrow[dd, two heads] \\ & A \arrow[ur, hook, bend right] \arrow[dl] & \\ \mathbb{1} \arrow[rr, bend right, "1"'] & & \Delta[1] \end{tikzcd}$$

This factorization defines the functor $f \colon B \to A$. Since cocartesian lifts of a fixed arrow are unique up to isomorphism, this proves that the left adjoint f is unique up to isomorphism as claimed. The construction of the right adjoint $u \colon A \to B$ is dual, involving a cartesian lift of the generic arrow in $\Delta[1]$. ☐

The converse makes use of something we call the **quasi-categorical collage construction**.

F.5.2. Definition (the quasi-categorical collage construction). For any cospan $f \colon A \to C$ and $g \colon B \to C$ of quasi-categories, define a simplicial set $\mathrm{col}(f,g)$ by declaring that

$$\mathrm{col}(f,g)_n = \left\{ \left(\Delta[i] \xrightarrow{a} A, \Delta[j] \xrightarrow{b} B, \Delta[n] \xrightarrow{c} C \right) \,\middle|\, \begin{array}{lll} c|_{\{0,\ldots,i\}} & = f(a), & i,j \geq -1, \\ c|_{\{n-j,\ldots,n\}} & = g(b), & i+j = n-1. \end{array} \right\}$$

[2]The nitpicker might note that Lurie only requires an inner fibration, but any inner fibration over $\Delta[1]$ is automatically an isofibration. In fact, so long as M is a quasi-category, any simplicial map $M \to \Delta[1]$ is automatically an isofibration.

with the convention that conditions indexed by $\mathbf{\Delta}[-1]$ are empty (or that each simplicial set is terminally augmented). There are simplicial maps

$$\begin{tikzcd} B \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & \mathrm{col}(f,g) \arrow[d, "\rho"] & A \arrow[l, hook'] \arrow[d] \arrow[dl, phantom, "\llcorner" very near start] \\ \{1\} \arrow[r, hook] & \Delta[1] & \{0\} \arrow[l, hook'] \end{tikzcd}$$

the top ones being the evident inclusions. The map ρ is defined to send an n-simplex $(a\colon \mathbf{\Delta}[i] \to A, b\colon \mathbf{\Delta}[j] \to B, c\colon \mathbf{\Delta}[n] \to C)$ to the n-simplex $[n] \to [1]$ that carries $0, \ldots, i$ to 0 and $i+1, \ldots, n$ to 1. Note that the fiber of ρ over 0 is isomorphic to A while the fiber of ρ over 1 is isomorphic to B

As was our custom for two-sided fibrations and modules, we customarily write $B + A \hookrightarrow \mathrm{col}(f,g)$ for the inclusions of the fibers over 1 and 0 — with the fiber over 1 on the left and the fiber over 0 on the right. This positions the covariantly-acting quasi-category on the "left" and the contravariantly-acting quasi-category on the "right."

F.5.3. Lemma. *The map* $\rho\colon \mathrm{col}(f,g) \to \Delta[1]$ *is an inner fibration. In particular, the simplicial set* $\mathrm{col}(f,g)$ *is a quasi-category.*

Proof. Since the fibers of ρ over 0 and 1 are the quasi-categories A and B, it suffices to consider inner horns

$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d] & \mathrm{col}(f,g) \arrow[d, "\rho"] \\ \mathbf{\Delta}[n] \arrow[r, "\alpha"'] \arrow[ur, dashed] & \Delta[1] \end{tikzcd}$$

for which $\alpha\colon [n] \to [1]$ is a surjection. Suppose α carries $0, \ldots, i$ to 0 and $i+1, \ldots, n$ to 1. Note that for any $0 < k < n$, the faces $\{0, \ldots, i\}$ and $\{i+1, \ldots, n\}$ of $\mathbf{\Delta}[n]$ belong to the horn $\Lambda^k[n]$. In particular, the map $\Lambda^k[n] \to \mathrm{col}(f,g)$ identifies simplices $a\colon \mathbf{\Delta}[i] \to A$ and $\mathbf{\Delta}[n-i-1] \to B$ together with a horn $\Lambda^k[n] \to C$ whose initial and final faces are the images of these simplices under $f\colon A \to C$ and $g\colon B \to C$. Since C is a quasi-category this horn admits a filler $c\colon \mathbf{\Delta}[n] \to C$ and the triple (a,b,c) defines an n-simplex in $\mathrm{col}(f,g)$ solving the lifting problem. ☐

We write $\mathrm{col}(f,B)$ for the collage of $f\colon A \to B$ with the identity on B.

F.5.4. Lemma. *For any* $f\colon A \to B$*, the map* $\rho\colon \mathrm{col}(f,B) \to \Delta[1]$ *is a cocartesian fibration.*

Proof. To prove the claim, we need only specify cocartesian lifts of the non-degenerate 1-simplex of $\Delta[1]$ and demonstrate that these edges have the corresponding universal property. To that end, for any vertex $a \in A_0$, let $\chi_a\colon \Delta[1] \to \mathrm{col}(f,B)$ be the 1-simplex

$$\chi_a := (a\colon \mathbf{\Delta}[0] \to A, fa\colon \mathbf{\Delta}[0] \to B, fa \cdot \sigma^0\colon \Delta[1] \to B),$$

defined by the copy degenerate edge at $fa \in B_0$ lying over the 1-simplex in $\Delta[1]$. To show that χ_a is ρ-cocartesian, we must construct fillers for any left horn

$$\begin{tikzcd} \Delta[1] \arrow[r, hook, "{\{0,1\}}"'] \arrow[rr, bend left, "\chi_a"] & \Lambda^0[n] \arrow[r] \arrow[d] & \mathrm{col}(f,B) \arrow[d, "\rho"] \\ & \mathbf{\Delta}[n] \arrow[r, "\beta"'] \arrow[ur, dashed] & \Delta[1] \end{tikzcd}$$

whose initial edge is χ_a. Note that this condition implies that the bottom map $\beta\colon [n] \to [1]$ carries 0 to 0 and the remaining vertices to 1. The map $\Lambda^0[n] \to \mathrm{col}(f,B)$ defines a horn $\Lambda^0[n] \to B$ in the quasi-category B whose first edge is degenerate. By Proposition 1.1.14, this "special outer horn" admits a filler $b\colon \mathbf{\Delta}[n] \to B$ and the triple

$$(a\colon \mathbf{\Delta}[0] \to A, b \cdot \delta^0\colon \mathbf{\Delta}^{n-1} \to B, b\colon \mathbf{\Delta}[n] \to B)$$

defines an n-simplex in $\mathrm{col}(f,B)$ that solves the lifting problem. □

F.5.5. Proposition. *For any $f\colon A \to B$ between quasi-categories, the collage* $\mathrm{col}(f,B)$ *defines the **oplax colimit** of f in $\mathcal{QC}at$. That is* $\mathrm{col}(f,B)$ *defines a cone under the pushout diagram*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook, "\mathrm{id}\times\delta^0"'] & B \arrow[d, hook] \arrow[ddr, bend left] & \\ A\times\Delta[1] \arrow[r] \arrow[drr, bend right, "h"'] & P \arrow[ul, phantom, "\ulcorner" very near start] \arrow[dr, dashed, "k"] & \\ & & \mathrm{col}(f,B) \end{tikzcd}$$

so that the induced map k is inner anodyne, and in particular an equivalence in the Joyal model structure.

Proof. The map k is a quotient of the map h, which has the following explicit description. For each n-simplex $(a,\alpha)\colon \mathbf{\Delta}[n] \to A \times \Delta[1]$ define $i := |\alpha^{-1}(0)| - 1$, so that $-1 \le i \le n$. Then h carries (a,α) to the n-simplex of $\mathrm{col}(f,B)$ corresponding to the triple

$$(a|_{\{0,\ldots,i\}}\colon \mathbf{\Delta}[i] \to A, fa|_{\{i+1,\ldots,n\}}\colon \mathbf{\Delta}[n-i-1] \to B, fa\colon \mathbf{\Delta}[n] \to B).$$

Note that the composite $\rho h\colon A \times \Delta[1] \to \Delta[1]$ is the projection.

It remains to present k as a sequential composite of pushouts of coproducts of inner horn inclusions. To do so, first note that

$$\mathrm{col}(f,B)_n = A_n \coprod A_{n-1} \times_{B_{n-1}} B_n \coprod \cdots \coprod A_0 \times_{B_0} B_n \coprod B_n$$

where each map $B_n \to B_i$ is the initial face map corresponding to $\{0,\ldots,i\} \hookrightarrow \mathbf{\Delta}[n]$. From the perspective of this decomposition, P_n is the subset containing the sets A_n and B_n and the subset of $A_i \times_{B_i} B_n$ whose component in B_n is in the image of f. The n-simplices of $\mathrm{col}(f,B)$ that remain to be attached correspond to elements of $A_i \times_{B_i} B_n$, for $0 \le i < n$, that are not in the image of f in the sense just discussed. Note in particular that $k\colon P_0 \hookrightarrow \mathrm{col}(f,B)_0$ is an isomorphism and $k\colon P_n \hookrightarrow \mathrm{col}(f,B)_n$ is an injection for all $n \ge 1$.

To enumerate our attaching maps, we start with the collection of non-degenerate n-simplices of $\mathrm{col}(f,B)$ for $n \geq 1$ that are not in the image of f and remove also those elements of $A_i \times_{B_i} B_n$ whose components $b \in B_n$ are in the image of the degeneracy map $\sigma_i \colon B_{n-1} \to B_n$. Partially order this set of simplices first in the order of increasing n and the in order of increasing index i; that is we lexicographically order the collection of pairs (n,i) with $n \geq 1$ and $0 \leq i < n$. We will filter the inclusion $P \hookrightarrow \mathrm{col}(f,B)$ as

$$P \hookrightarrow P_{<(1,0)} \hookrightarrow P_{<(2,0)} \hookrightarrow P_{<(2,1)} \hookrightarrow P_{<(3,0)} \hookrightarrow \cdots \hookrightarrow P_{<(n,i)} \hookrightarrow \cdots \hookrightarrow \mathrm{colim} \cong \mathrm{col}(f,B)$$

where the simplicial set $P_{<(n,i)}$ is built from the previous one by a pushout of a coproduct of inner horns indexed by the set of n-simplices $(a,b) \in A_i \times_{B_i} B_n$ with b not in the image of f or σ_i. The filler for the horn indexed by (a,b) will attach this n simplex to B_n as the missing face of the horn and also the $n+1$ simplex $(a, b\sigma^i) \in A_i \times_{B_i} B_{n+1}$.

Consider a simplex $(a,b) \in A_i \times_{B_i} B_n$ with b not in the image of f or σ_i. Define a horn

$$\begin{tikzcd}
\Lambda^{i+1}[n+1] \arrow[r] \arrow[d, hook] & P_{<(n,i)} \arrow[d, hook] \\
\boldsymbol{\Delta}[n+1] \arrow[r, "{(a,b\sigma^i)}"'] & \mathrm{col}(f,B)
\end{tikzcd}$$

For each $0 \leq j < i+1$, the δ^j-face of the $n+1$ simplex $(a, b\sigma^i)$ is the n-simplex $(a\delta^j, b\sigma^i\delta^j)$, which lies in $P_{<(n,i-1)}$ or in $B \hookrightarrow P$ in the case $i = 0$. For each $i+1 < j \leq n+1$, the δ^j-face of the $n+1$ simplex $(a, b\sigma^i)$ is the n-simplex $(a, b\sigma^i\delta^j) = (a, b\delta^{j-1}\sigma^i) \in A_i \times_{B_i} B_n$, which was previously attached to $P_{<(n-1,i)}$. So the $\Lambda^{i+1}[n+1]$ indeed maps to $P_{<(n,i)}$, permitting an inductive construction of the next simplicial set in this sequence as the pushout

$$\begin{tikzcd}
\coprod\limits_{\sim} \Lambda^{i+1}[n+1] \arrow[r] \arrow[d, hook] & P_{<(n,i)} \arrow[d] \\
\coprod\limits_{\sim} \boldsymbol{\Delta}[n+1] \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & \bullet
\end{tikzcd}$$

defining $P_{<(n+1,0)}$ in the case $i = n-1$ and $P_{<(n,i+1)}$ otherwise. $\square$

F.5.6. Theorem. *Consider a pair of functors between quasi-categories $f \colon B \to A$ and $u \colon A \to B$. Then f is left adjoint to u if and only if the collages $\mathrm{col}(f,A)$ and $\mathrm{col}(B,u)$ are equivalent under $A+B$ and over $\Delta[1]$, in which case $\mathrm{col}(f,A) \twoheadrightarrow \Delta[1]$ or equivalently $\mathrm{col}(B,u) \twoheadrightarrow \Delta[1]$ defines both a cocartesian and a cartesian fibration.*

Proof. First suppose that $\mathrm{col}(f,A) \simeq \mathrm{col}(B,u)$ under $A+B$ and over $\Delta[1]$. By Lemma F.5.4 and Corollary 5.1.17 this means that the map $\mathrm{col}(f,A) \to \Delta[1]$ is both a cocartesian and a cartesian fibration. By Proposition F.5.1 it follows that the 1-arrow in $\Delta[1]$ from 0 to 1 induces an adjunction between the fibers B and A. By inspection of that proof, the left adjoint functor so-constructed in the case of the bifibration $\mathrm{col}(f,A) \to \Delta[1]$ is f; substituting the equivalent bifibration $\mathrm{col}(B,u) \to \Delta[1]$, we see that the right adjoint is equivalent to u.

For the converse, we work in the opposite ∞-cosmos $\mathcal{QC}at^{\mathrm{op}}$, an ∞-cosmos in which "not all objects are cofibrant," as described in §E.5. In that context, Proposition F.5.5 proves that $\mathrm{col}(f,A)$ and $\mathrm{col}(B,u)$ construct the contravariant and covariant comma objects associated to the functors f

and u. If $f \dashv u$ in $\mathcal{QCat}$ then these functors are also adjoint in $\mathcal{QCat}^{\mathrm{op}}$ and Proposition 4.1.1 then proves that the commas $\mathrm{col}(f, A)$ and $\mathrm{col}(B, u)$ are equivalent under $A + B$. By construction, this equivalence also lies over $\Delta[1]$. $\square$

Bibliography

[1] J. Adámek and J. Rosický. *Locally Presentable and Accessible Categories*. Cambridge University Press, 1994.

[2] D. Ara. Higher quasi-categories vs higher rezk spaces. *Journal of K-Theory*, 14(3):701–749, 2014.

[3] C. Barwick. (∞, n)-*Cat as a closed model category*. PhD thesis, University of Pennsylvania, 2005.

[4] C. Barwick, S. Glasman, and D. Nardin. Dualizing cartesian and cocartesian fibrations. *Theory Appl. Categ.*, 33(4):67–94, 2018.

[5] T. Beke. Sheafifiable homotopy model categories. *Mathematical Proceedings of the Cambridge Philosophical Society*, 129(03):447–475, 2001.

[6] J. Bénabou. Introduction to bicategories. In *Reports of the Midwest Category Seminar*, number 47 in Lecture Notes in Math., pages 1–77. Springer-Verlag, Berlin, 1967.

[7] C. Berger. Iterated wreath product of the simplex category and iterated loop spaces. *Advances in Mathematics*, 213(1):230–270, 2007.

[8] C. Berger and I. Moerdijk. On an extension of the notion of Reedy category. *Mathematische Zeitschrift*, pages 1–28, 2008.

[9] J. E. Bergner. A characterization of fibrant Segal categories. *Proceedings of the AMS*, 2007.

[10] J. E. Bergner. A model category structure on the category of simplicial categories. *Transactions of the American Mathematical Society*, 359:2043–2058, 2007.

[11] J. E. Bergner. Three models for the homotopy theory of homotopy theories. *Topology*, 46(4):397–436, September 2007.

[12] J. E. Bergner. *An introduction to* $(\infty, 1)$-*categories*. Cambridge University Press, 2018.

[13] J. E. Bergner and C. Rezk. Reedy categories and the θ-construction. *Mathematische Zeitschrift*, 274(1–2):499—514, 2013. http://arxiv.org/abs/1110.1066 [math.AT].

[14] G. J. Bird, G. M. Kelly, A. J. Power, and R. H. Street. Flexible limits for 2–categories. *J. Pure Appl. Algebra*, 61, November 1989.

[15] A. Blumberg and M. Mandell. Algebraic K-theory and abstract homotopy theory. *Advances in Mathematics*, 226:3760–3812, 2011.

[16] J. Boardman and R. Vogt. *Homotopy Invariant Algebraic Structures on Topological Spaces*, volume 347 of *Lecture Notes in Mathematics*. Springer-Verlag, 1973.

[17] F. Borceux. *Handbook of Categorical Algebra 1: Basic Category Theory*. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 1994.

[18] F. Borceux. *Handbook of Categorical Algebra 2: Categories and Structures*. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 1994.

[19] A. K. Bousfield and D. M. Kan. *Homotopy limits, completions and localizations*. Lecture Notes in Mathematics, Vol. 304. Springer-Verlag, Berlin, 1972.

[20] K. S. Brown. Abstract homotopy theory and generalized sheaf cohomology. *Transactions of the American Mathematical Society*, 186:419–458, 1973.

[21] O. A. Camarena. A whirlwind tour of the world of $(\infty, 1)$-categories. arXiv:1303.4669, 2013.

[22] E. Cheng, N. Gurski, and E. Riehl. Cyclic multicategories, multivariable adjunctions and mates. *J. K-Theory*, 13:337–396, 2014.

[23] D. C. Cisinski. *Les préfaisceaux comme modèles des types d'homotopie*, volume 308 of *Astérisque*. Soc. Math. France, 2006.

[24] J.-M. Cordier. Sur la notion de diagramme homotopiquement cohérent. *Cahiers topologie et géométrie différentielle catégoriques*, 23:93–112, 1982.

[25] J.-M. Cordier and T. Porter. Vogt's theorem on categories of homotopy coherent diagrams. *Mathematical Proceedings of the Cambridge Philosophical Society*, 100:65–90, 1986.

[26] G. S. H. Cruttwell. *Normed Spaces and the Change of Base for Enriched Categories*. PhD thesis, Dalhousie University, December 2008.

[27] G. S. H. Cruttwell and M. A. Shulman. A unified framework for generalized multicategories. *Theory Appl. Categ.*, 24(21):580–655, 2010.

[28] B. Day. On closed categories of functors. In *Reports of the Midwest Category Seminar IV*, volume 137 of *Lecture Notes in Mathematics*, pages 1–38. Springer-Verlag, 1970.

[29] P. B. de Brito. Segal objects and the grothendieck construction. arXiv:1605.00706, 2016.

[30] D. Dugger. Replacing model categories with simplicial ones. *Transactions of the American Mathematical Society*, 353(12):5003–5028, 2001.

[31] D. Dugger and D. I. Spivak. Mapping spaces in quasi-categories. *Algebr. Geom. Topol.*, 11:263–325, 2011.

[32] W. Dwyer, D. Kan, and J. Smith. Homotopy commutative diagrams and their realizations. *J. Pure Appl. Algebra*, 57:5–24, 1989.

[33] W. G. Dwyer and D. M. Kan. Simplicial localizations of categories. *J. Pure Appl. Algebra*, 17(267–284), 1980.

[34] P. J. Ehlers and T. Porter. Joins for (augmented) simplicial sets. *Journal of Pure and Applied Algebra*, 145:37–44, 2000.

[35] S. Eilenberg and G. Kelly. Closed categories. In *Proceedings of the Conference on Categorical Algebra (La Jolla 1965)*, pages 421–562. Springer-Verlag, 1966.

[36] P. Gabriel and M. Zisman. *Calculus of Fractions and Homotopy Theory*. Number 35 in Ergebnisse der Math. Springer-Verlag, 1967.

[37] N. Gambino. Weighted limits in simplicial homotopy theory. *J. Pure Appl. Algebra*, 214(7):1193–1199, 2010.

[38] D. Gepner, R. Haugseng, and J. Kock. ∞-operads as analytic monads. arXiv:1712.06469, 2017.

[39] J. Gray. Fibred and cofibred categories. In S. Eilenberg, D. Harrison, H. Röhrl, and S. MacLane, editors, *Proceedings of the Conference on Categorical Algebra (La Jolla 1965)*, pages 21–83. Springer-Verlag Berlin Heidelberg, 1966.

[40] B. Guillou and J. May. Enriched model categories and presheaf categories. arXiv:1110.3567, 2011.

[41] P. Hirschhorn. *Model Categories and their Localizations*. Number 99 in Mathematical Surveys and Monographs. American Mathematical Society, 2003.

[42] A. Hirschowitz and C. Simpson. Descente pour les n-champs. `arXiv:math/9807049`, 2001.

[43] M. Hovey. *Model Categories*, volume 63 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1999.

[44] N. Johnson and D. Yau. 2-dimesional categories. arXiv:2002.06055, 2020.

[45] A. Joyal. Disks, duality and θ-categories. https://ncatlab.org/nlab/files/JoyalThetaCategories.pdf, September 1997.

[46] A. Joyal. Quasi-categories and Kan complexes. *Journal of Pure and Applied Algebra*, 175:207–222, 2002.

[47] A. Joyal. *The theory of quasi-categories and its applications*. Quadern 45 vol II. Centre de Recerca Matemàtica Barcelona, `http://mat.uab.cat/~kock/crm/hocat/advanced-course/Quadern45-2.pdf`, 2008.

[48] A. Joyal and M. Tierney. Quasi-categories vs Segal spaces. In A. D. et al, editor, *Categories in Algebra, Geometry and Mathematical Physics (StreetFest)*, volume 431 of *Contemporary Mathematics*, pages 277–325. American Mathematical Society, 2007.

[49] D. Kazhdan and Y. Varshavsky. Yoneda lemma for complete segal spaces. *Funct Anal Its Appl*, 48(2):81–106, 2014.

[50] G. Kelly. On Mac Lane's conditions for coherence of natural associativities, commutativities, etc. *J. Algebra*, 1:397–402, 1964.

[51] G. Kelly and S. Mac Lane. Coherence in Closed Categories. *J. Pure Appl. Algebra*, 1(1):97–140, 1971.

[52] G. M. Kelly. Doctrinal adjunction. In *Category Seminar (Proc. Sem., Sydney, 1972/1973)*, volume Vol. 420 of *Lecture Notes in Mathematics*, pages 257–280. Springer, Berlin, 1974.

[53] G. M. Kelly. Elementary observations on 2-categorical limits. *Bulletin of the Australian Mathematical Society*, 49:301–317, 1989.

[54] G. M. Kelly. Basic concepts of enriched category theory. *Reprints in Theory and Applications of Categories*, 10, 2005.

[55] G. M. Kelly and R. H. Street. *Review of the Elements of 2-Categories*, volume 420 of *Lecture Notes in Mathematics*, pages 75–103. Springer-Verlag, 1974.

[56] S. Lack. A quillen model structure for 2-categories. *K-Theory*, 26(2):171–205, 2002.

[57] S. Lack. Homotopy-theoretic aspects of 2-monads. *Journal of Homotopy and Related Structures*, 2(2):229–260, 2007.

[58] F. W. Lawvere. An elementary theory of the category of sets (long version) with commentary. *Repr. Theory Appl. Categ.*, 11:1–35, 2005.

[59] J. Lurie. *Higher Topos Theory*, volume 170 of *Annals of Mathematical Studies*. Princeton University Press, Princeton, New Jersey, 2009.

[60] S. Mac Lane. *Categories for the Working Mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, 1971.

[61] S. Mac Lane. *Categories for the Working Mathematician*. Graduate Texts in Mathematics. Springer-Verlag, second edition, 1998.

[62] M. Makkai and R. Paré. *Accessible Categories: The Foundations of Categorical Model Theory*, volume 104 of *Contemporary Mathematics*. American Mathematical Society, 1989.

[63] J. P. May and K. Ponto. *More concise algebraic topology: Localization, completion, and model categories*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 2012.

[64] A. Mazel-Gee. Quillen adjunctions induce adjunctions of quasicategories. *New York J. Math.*, 22:57–93, 2016.

[65] V. Ozornova and M. Rovelli. Model structures for (∞, n)-categories on (pre)stratified simplicial sets and prestratified simplicial spaces. arXiv:1809.10621, 2018.

[66] R. Pellissier. *Catégories enrichies faibles*. PhD thesis, Université de Nice-Sophia Antipolis, 2002.

[67] A. J. Power. A 2-categorical pasting theorem. *Journal of Algebra*, 129:439–445, 1990.

[68] D. G. Quillen. *Homotopical Algebra*, volume 43 of *Lecture Notes in Mathematics*. Springer-Verlag, 1967.

[69] N. Rasekh. Cartesian fibrations and representability. arXiv:1711.03670, 2017.

[70] N. Rasekh. Yoneda lemma for simplicial spaces. arXiv:1711.03160, 2017.

[71] C. Reedy. Homotopy theory of model categories. Unpublished manuscript available from http://www-math.mit.edu/~psh/reedy.pdf, 1974.

[72] C. Rezk. A model for the homotopy theory of homotopy theory. *Transactions of the American Mathematical Society*, 353(3):973–1007, 2001.

[73] C. Rezk. A cartesian presentation of weak n-categories. *Geometry and Topology*, 2010.

[74] C. Rezk. Stuff about quasi-categories. https://faculty.math.illinois.edu/~rezk/quasicats.pdf, March 2019.

[75] E. Riehl. *Categorical Homotopy Theory*, volume 24 of *New Mathematical Monographs*. Cambridge University Press, 2014.

[76] E. Riehl. *Category Theory in Context*. Aurora Modern Math Originals. Dover Publications, 2016.

[77] E. Riehl. Complicial sets, an overture. In *2016 MATRIX Annals*, number 1–28 in MATRIX Book Series. Springer, 2017.

[78] E. Riehl and D. Verity. The theory and practice of Reedy categories. *Theory and Applications of Categories*, 29(9):256–301, 2014.

[79] E. Riehl and D. Verity. The 2-category theory of quasi-categories. *Adv. Math.*, 280:549–642, 2015.

[80] E. Riehl and D. Verity. Homotopy coherent adjunctions and the formal theory of monads. *Adv. Math.*, 286:802–888, 2016.

[81] E. Riehl and D. Verity. Yoneda structures and the calculus of modules for quasi-categories. *In preparation*, 2016.

[82] E. Riehl and D. Verity. The comprehension construction. *Higher Structures*, 2(1):116–190, 2018. arXiv:1706.10023.

[83] E. Riehl and D. Verity. On the construction of limits and colimits in ∞-categories. arXiv:1808.09835, 2018.

[84] E. Riehl and D. Verity. Recognizing quasi-categorical limits and colimits in homotopy coherent nerves. arXiv:1808.09834, 2018.

[85] M. Shulman. Homotopy limits and colimits and enriched category theory. Arxiv preprint arXiv:math/0610194 [math.AT], 2009.

[86] C. Simpson. *Homotopy Theory of Higher Categories*. Number 19 in New Mathematical Monographs. Cambridge University Press, 2012.

[87] R. Street. Fibrations and Yoneda's lemma in a 2-category. In *Category Seminar (Proc. Sem., Sydney, 1972/1973)*, volume Vol. 420 of *Lecture Notes in Math.*, pages 104–133. Springer, Berlin, 1974.

[88] R. Street. The algebra of oriented simplexes. *Journal of Pure and Applied Algebra*, 49:283–335, 1987.

[89] R. H. Street. Elementary cosmoi I. In *Category Seminar (Proc. Sem., Sydney, 1972/1973)*, volume Vol. 420 of *Lecture Notes in Math.*, pages 134–180. Springer, Berlin, 1974.

[90] R. H. Street. Fibrations in bicategories. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, XXI-2:111–159, 1980.

[91] R. H. Street. Correction to "fibrations in bicategories". *Cahiers topologie et géométrie différentielle catégoriques*, 28(1):53–56, 1987.

[92] R. H. Street. Weak omega-categories. In *Diagrammatic Morphisms and Applications*, volume 318 of *Contemporary Mathematics*, pages 207–213. American Mathematical Society, 2003.
[93] R. H. Street and R. Walters. Yoneda structures on 2-categories. *J. Algebra*, 50:350–379, 1978.
[94] Y. J. Sulyma. ∞-categorical monadicity and descent. *New York Journal of Mathematics*, 23:749–777, 2017.
[95] The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics*. `https://homotopytypetheory.org/book`, Institute for Advanced Study, 2013.
[96] F. Ulmer. Properties of dense and relative adjoint functors. *J. Algebra*, 3:77–95, 1968.
[97] D. Verity. Complicial sets, characterising the simplicial nerves of strict ω-categories. *Memoirs of the AMS*, 193(905), May 2008.
[98] D. Verity. Weak complicial sets I, basic homotopy theory. *Advances in Mathematics*, 219:1081–1149, September 2008.
[99] D. Verity. Enriched categories, internal categories and change of base. *Reprints in Theory and Application of Categories*, 20:1–266, June 2011.
[100] M. Weber. Yoneda Structures from 2-toposes. *Applied Categorical Structures*, 15(3):259–323, 2007.
[101] R. J. Wood. Abstract proarrows I. *Cahiers topologie et géométrie différentielle catégoriques*, XXIII-3:279–290, 1982.