FUNCTORS THAT COMMUTE WITH DIRECT AND INVERSE LIMITS

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Abstract. In the first part of this paper, we study additive functors that commute with direct and inverse limits. We prove that, under some conditions, such functors are trivial. In the second part, we state conditions for a covariant half exact functor to commute with direct limits, based on its first left satellite functor.

Introduction

The main examples of funtors that commute with direct and inverse limits are $\otimes_\Lambda$ and Hom_Λ, just as the left derived functors Tor_n^λ for $\otimes_\Lambda$ (see [2], **Prop 9.2**, chapter V, §9 and **Prop. 1.3**, chapter VI, §1). The commutating properties in some canonical functors, provide important information about the module and subcategories arising from the module category. For instance, H. Lenzning proved that a right Λ-module M is finitely presented if and only if the functor $\mathrm{Hom}_\Lambda(M,\cdot)$, preserves direct limits (see [3]). Similarly, in [1], is proved $\mathrm{Ext}^1_\Lambda(M,\cdot)$ commutes with inverse limits if and only if M is projective. Follow this ideas, in this work we prove that if a functor acts trivially in any Λ-quotient (resp. Λ-ideal) and commutes with direct limits (resp. inverse limits), then the functor is trivial (see **Thm. 2.1**).

On the other hand, it is a well known result that for a right extact functor T, its left satellite S_1T coincides with the left derived funtor LT of T. Further, if the functor T is left balanced, *i.e.*, when any of the covariant (resp. contravariant) variables of T is replaced for projective module (injective modules), T becomes an exact functor of the remaining variables, then the left derived functor can be determined as a the left satellite of any of the variables. It is a classical result that for a direct system of modules $\{A^\alpha,\varphi^\alpha_\beta\}$, over a partially directed set Ω, there exists direct system of modules $\{P^\alpha,\phi^\alpha_\beta\}$, such that P^α is a

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projective module of A^α and $\varinjlim P^\alpha$ is a projective module over $\varinjlim A^\alpha$ (see **Lemma 9.5**, [2], Chapter V §9). Inspired on this result, in **Thm. 2.2**, we state necessary and sufficient conditions on a functor T, to commute with direct limits based on its first left satellite S_1T.

1. Direct and inverse limits of modules

Definition 1.1. Let Λ be a ring and let $\{A_\alpha \mid \alpha \in \Omega\}$ be an indexed family of Λ-modules by a set Ω. The **direct product** $\prod_{\boldsymbol{\alpha\in\Omega}} \boldsymbol{A_\alpha}$, is the set defined by:

$$\prod_{\alpha\in\Omega} A_\alpha = \left\{ f : \Omega \to \bigcup_{\alpha\in\Omega} A_\alpha \mid f(\alpha) \in A_\alpha,\ \forall \alpha \in \Omega \right\}.$$

The direct product $\prod_{\alpha\in\Omega} A_\alpha$ is a left Λ-module with coordinatewise addition and left scalar multiplication. The **direct sum** $\bigoplus_{\boldsymbol\alpha} \boldsymbol{A_\alpha}$ is the Λ-submodule of $\prod_{\alpha\in\Omega} A_\alpha$, given by:

$$\bigoplus_{\alpha\in\Omega} A_\alpha = \left\{ f \in \prod_{\alpha\in\Omega} A_\alpha \mid f(\alpha) = 0, \text{ except in a finit number } \right\}.$$

Any element $f \in \prod_{\gamma\in\Omega} A_\gamma$ can be written as $f = (x_\gamma)_{\gamma\in\Omega}$, where $f(\gamma) = x_\gamma \in A_\gamma$, for all $\gamma \in \Omega$. In addition, for each $\alpha \in \Omega$, there is a map $p_\alpha : \prod_{\gamma\in\Omega} A_\gamma \to A_\alpha$, given by

$$p_\alpha(f) = p_\alpha((x_\gamma)_{\gamma\in\Omega}) = x_\alpha = f(\alpha),$$

for all $f = (x_\gamma)_{\gamma\in\Omega} \in \prod_{\alpha\in\Omega} A_\gamma$. Note that p_α is a surjective Λ-homomorphism.

Similarly, for each $\alpha \in \Omega$, we define the map $c^\alpha : A_\alpha \to \bigoplus_{\gamma\in\Omega} A_\gamma$, by $c^\alpha(x)(\beta) = \delta_{\alpha,\beta}\, x$, for all $\beta \in \Omega$. It is clear that c^α is an injective Λ-homomorphism. Observe that any $f \in \bigoplus_{\alpha\in\Omega} A_\alpha$, can be written as:

$$f = \sum_{\alpha\in\Omega} c^\alpha(f(\alpha)).$$

Definition 1.2. A partially ordered set Ω is a **directed set** if for any $\alpha, \beta \in \Omega$, there exists $\gamma \in \Omega$ such that $\gamma \geq \alpha$ and $\gamma \geq \beta$.

Definition 1.3. Let Ω be an ordered partially set. A **direct system** of Λ-modules over Ω is an ordered pair $\{A^\alpha; \varphi_\alpha^\beta\}$, consisting on an index family of Λ-modules $\{A^\alpha \mid \alpha \in \Omega\}$, together with a family of homomorphisms $\{\varphi_\alpha^\beta : A^\alpha \to A^\beta\}$, with $\alpha \leq \beta$, such that $\varphi_\alpha^\alpha = \mathrm{Id}_{A_\alpha}$ and $\varphi_\alpha^\gamma = \varphi_\beta^\gamma \circ \varphi_\alpha^\beta$, for all $\alpha \leq \beta \leq \gamma$, *i.e.*, the following diagram is commutative:

$$\begin{tikzcd} A^\alpha \arrow[rr, "\varphi^\alpha_\gamma"] \arrow[dr, "\varphi^\alpha_\beta"'] & & A^\gamma \\ & A^\beta \arrow[ur, "\varphi^\beta_\gamma"'] & \end{tikzcd}$$

Definition 1.4. Let Ω be a partially ordered set and let $\{A^\alpha, \varphi^\alpha_\beta\}$ be a direct system of Λ-modules over Ω. The **direct limit** (also called **inductive limit** or **colimit**), is a Λ-module $\varinjlim A^\alpha$ and a family of Λ-maps $\{\sigma^\alpha : A^\alpha \to \varinjlim A^\gamma \mid \alpha \in \Omega\}$, such that:

(i) $\sigma^\beta \circ \varphi^\alpha_\beta = \sigma^\alpha$, whenever $\alpha \leq \beta$.
(ii) For every Λ-module X having maps $f^\alpha : A^\alpha \to X$, satisfying $f^\beta \circ \varphi^\alpha_\beta = f^\alpha$, for all $\alpha \leq \beta$, there exists a unique map $\theta : \varinjlim A^\alpha \to X$, such that $\theta \circ \sigma^\alpha = f^\alpha$, for all $\alpha \in \Omega$, *i.e.*, the following diagram is commutative:

(1)
$$\begin{tikzcd} \varinjlim A^\gamma \arrow[rr, "\theta"] & & X \\ & M^\alpha \arrow[ul, "\sigma^\alpha"] \arrow[ur, "f^\alpha"'] & \end{tikzcd}$$

Let Ω be a partially ordered set and let $\{A^\alpha, \varphi^\alpha_\beta\}$ be a direct system of Λ-modules over Ω. We define the Λ-submodule N, of $\oplus_{\alpha\in\Omega} A^\alpha$, as:

$$N = \mathrm{Span}_\Lambda\{c^\beta(\varphi^\alpha_\beta(x)) - c^\alpha(x) \mid x \in A^\alpha,\ \alpha \leq \beta\}.$$

It is not difficult to prove that the quotient $\oplus_{\alpha\in\Omega} A^\alpha/N$ and the maps $\sigma^\alpha : A^\alpha \to \oplus_{\gamma\in\Omega} A^\gamma/N$, defined by $\sigma^\alpha(x) = c^\alpha(x) + N$, for any $\alpha \in \Omega$, are solutions to the diagram (1). Therefore, $\varinjlim A^\alpha = \oplus_{\alpha\in\Omega} A^\alpha/N$. This proves the following basic result (see [4], **Prop. 7.94**):

Proposition 1.5. *The direct limit of any direct system $\{A^\alpha, \varphi^\alpha_\beta\}$ of Λ-modules over a partially ordered index set Ω, exists.*

The following result, gives information about the elements of a direct limit of modules (see [4], **Prop. 7.98**).

Proposition 1.6. *Let $\{A^\alpha, \varphi^\alpha_\beta\}$ be a direct system of left Λ-modules over a directed index set Ω, and let $c^\alpha : A^\alpha \to \oplus_\gamma A^\gamma$ be the α-th injection, so that $\varinjlim A^\alpha = \oplus_{\alpha\in\Omega} A^\alpha/N$, where:*

$$N = \mathrm{Span}_\Lambda\{c^\beta(\varphi^\alpha_\beta(x)) - c^\alpha(x) \mid x \in A^\alpha,\ \alpha \leq \beta\}.$$

(i) *Each element of $\varinjlim A^\alpha$, has a representative of the form $c^\alpha(a) + N$, for some $a \in A^\alpha$(instead of $\sum_\alpha c^\alpha(a^\alpha) + N$).*
(ii) *$c^\alpha(a) + N = 0$ if and only if $\varphi^\alpha_\beta(a) = 0$, for some $\beta \geq \alpha$.*

Definition 1.7. Let $\{A^\alpha, \varphi^\alpha_\beta\}$ be a direct system of Λ-modules. Let $\sigma^\alpha : A^\alpha \to \varinjlim A^\gamma$ be the canonical map defined by $\sigma^\alpha(a) = c^\alpha(a) + N$, for all $a \in A^\alpha$ and $\alpha \in \Omega$. Let $T : \mathrm{Mod}_\Lambda \to \mathrm{Mod}_{\Lambda_1}$ be a one variable covariant functor. Then, $\{T(A^\alpha); T(\varphi^\alpha_\beta)\}$ is a direct system of Λ_1-modules. There exists the Λ_1-homomorphism $\widehat{\sigma} : \varinjlim T(A^\alpha) \to T(\varinjlim A^\alpha)$, given by $\widehat{\sigma}(c^\alpha(x) + N) = T(\sigma^\alpha)(x)$. The functor T is **of the type $\boldsymbol{L\Sigma^*}$** if $\widehat{\sigma}$ is an isomorphism.

Definition 1.8. Let Ω be an ordered partially set. An **inverse system** of Λ-modules over Ω is an ordered pair $\{A_\alpha; \psi^\beta_\alpha\}$, consisting on an index family of Λ-modules $\{A_\alpha \mid \alpha \in \Omega\}$, together with a family of homomorphisms $\{\psi^\beta_\alpha : A_\beta \to A_\alpha\}$, with $\alpha \leq \beta$, such that $\psi^\alpha_\alpha = \mathrm{Id}_{A_\alpha}$ and $\psi^\gamma_\alpha = \psi^\beta_\alpha \circ \psi^\gamma_\beta$, for all $\alpha \leq \beta \leq \gamma$., *i.e.*, the following diagram commutes:

$$\begin{tikzcd}
A_\gamma \arrow[rr, "\psi^\gamma_\alpha"] \arrow[dr, "\psi^\gamma_\beta"'] & & A_\alpha \\
& A_\beta \arrow[ur, "\psi^\beta_\alpha"'] &
\end{tikzcd}$$

Definition 1.9. Let Ω be a partially ordered set and let $\{A_\alpha, \psi^\beta_\alpha\}$ be an inverse system of Λ-modules over Ω. The **inverse limit** (also called **projective limit** or **limit**) is a Λ-module $\varprojlim A_\alpha$ and a family of Λ of Λ-maps $\{\xi_\alpha : \varprojlim A_\gamma \to A_\alpha\}$, such that:

(i) $\psi^\beta_\alpha \circ \xi_\beta = \xi_\alpha$, whenever $\alpha \leq \beta$.
(ii) For every module X having maps $f_\alpha : X \to A_\alpha$, satisfying $\psi^\beta_\alpha \circ f_\beta = f_\alpha$, for all $\alpha \leq \beta$, there exists a unique map $\theta : X \to \varprojlim A_\alpha$, such that $\xi_\alpha \circ \theta = f_\alpha$, for all $x \in \Omega$, *i.e.*, the following diagram commutes:

$$\begin{tikzcd}
X \arrow[rr, "\theta"] \arrow[dr, "f_\alpha"'] & & \varprojlim A_\gamma \arrow[dl, "\xi_\alpha"'] \\
& A_\alpha &
\end{tikzcd} \tag{2}$$

We define the submodule L of the direct product $\Pi_{\alpha\in\Omega} A_\alpha$, as follows:

$$L = \{(a_\alpha)\Pi_{\alpha\in\Omega} A_\alpha \mid a_\alpha = \psi^\beta_\alpha(a_\beta),\ \alpha \leq \beta\}.$$

Let $\xi_\alpha = p_\alpha|_L$, where $p_\alpha : \Pi_{\gamma\in\Omega} A_\gamma \to A_\alpha$ is the projection onto A_α. Then, L and the maps ξ_α, with $\alpha \in \Omega$, are solutions to the diagram (2). This proves the following basic result (see [4], **Prop. 7.90**):

Proposition 1.10. *The inverse limit of any inverse system $\{A_\alpha, \psi^\beta_\alpha\}$ of Λ-modules over a partially ordered set Ω, exists.*

Definition 1.11. Let $\{A_\alpha, \psi^\alpha_\beta\}$ be an inverse system of Λ-modules over a partially ordered set Ω. Let $\xi_\alpha : \varprojlim A_\gamma \to A_\alpha$ be the canonical map given by $\xi_\alpha((a_\gamma)_{\gamma\in\Omega}) = a_\alpha$, for all $(a_\gamma)_{\gamma\in\Omega} \in \varprojlim A_\gamma$. Let $T : \mathrm{Mod}_\Lambda \to \mathrm{Mod}_{\Lambda_1}$ be a one variable covariant funtor. Then, $\{T(A_\alpha); T(\psi^\alpha_\beta)\}$ is an inverse system of Λ_1-modules. There exists the Λ_1-homomorphism $\widehat{\xi} : T(\varprojlim A_\alpha) \to \varprojlim T(A_\alpha)$, given by $\widehat{\xi}(x) = (T(\xi_\alpha(x))_{\alpha\in\Omega})$, for all $x \in T(A)$. The functor T is **of the type $\boldsymbol{R\Sigma^*}$** if $\widehat{\xi}$ is an isomorphism.

Remark 1.12. The corresponding definitions for contravariant functors of the type $L\Sigma^*$ and $R\Sigma^*$, only reverse the arrows in $\widehat{\sigma}$ and $\widehat{\xi}$, respectively.

2. Funtors that commute with direct and inverse limits

Theorem 2.1. *Let T be a one variable functor.*

(i) *If T is a covariant half exact funtor of the type $L\Sigma^*$,* i.e. *commutes with direct limits, and $T(\Lambda/I) = 0$ for each left ideal I of Λ, then $T = 0$.*

(ii) *If T is a contravariant left exact funtor of the type $R\Sigma^*$,* i.e. *commutes with inverse limits and $T(I) = 0$ for each left ideal I of Λ, then $T(M) = 0$, for any left Λ-module M.*

Proof. (**i**) Let A be a left Λ-module and $a \in A$. The Λ-homomorphism $\Lambda \to \langle a\rangle$, $\lambda \to \lambda a$, for all $\lambda \in \Lambda$, is surjective and $\mathrm{Ker}(\Lambda \to \langle a\rangle) = \mathrm{ann}_\Lambda(a)$. Then, we have the following short exact sequence:

$$0 \longrightarrow \mathrm{ann}_\Lambda(a) \longrightarrow \Lambda \longrightarrow \langle a\rangle \longrightarrow 0.$$

It is clear that $\mathrm{ann}_\Lambda(a) = \mathrm{Ker}(\Lambda \to \langle a\rangle)$ is a left ideal of Λ such that $\Lambda/\mathrm{ann}_\Lambda(a)$ is isomorphic to $\langle a\rangle$. Applying the functor T we obtain $T(\langle a\rangle) \simeq T(\Lambda/\mathrm{ann}_\Lambda(a)) = 0$ for all $a \in A$.

Let $\Omega \in \mathrm{Ord}$ be an ordinal such that there exists a bijection $\nu : \Omega \to A$. For an element $a \in A$, we write $a = a_\alpha$ if $\nu(\alpha) = a$ with $\alpha \in \Omega$. For any $\alpha \in \Omega$, we define the following Λ-submodule of M:

$$A^0 = 0,$$

$$A^\alpha = \left\{\sum_{k=1}^{\ell} \lambda_k a_{\alpha_k} \mid \lambda_k \in \Lambda, \alpha_k < \alpha,\ \ell \in \mathbb{Z}_+\right\},\ \alpha > 0.$$

It is clear that if $\alpha \le \beta$, then $A^\alpha \subseteq A^\beta$. Observe that,

$$A^{\alpha+1}/A^\alpha = \{\lambda a_\alpha + A^\alpha \mid \lambda \in \Lambda\},\ \forall \alpha \in \Omega.$$

The assignment $\Lambda \to A^{\alpha+1}/A^\alpha$, $\lambda \mapsto \lambda a_\alpha + A^\alpha$, is a surjective Λ-homomorphism. Then, there exists an ideal I of Λ, such that $\Lambda/I \simeq$

$A^{\alpha+1}/A^\alpha$. Applying the functor T and considering the hypothesis, we have $T(A^{\alpha+1}/A^\alpha) = 0$, for all $\alpha \in \Omega$. It is clear to see that $A^1 = \langle a_0 \rangle$, and $T(A^1) = 0$. Applying the functor T to the short exact sequence:

$$0 \longrightarrow A^1 \longrightarrow A^2 \longrightarrow A^2/A^1 \longrightarrow 0,$$

one obtain that the sequence $0 \to T(A^2) \to 0$, is exact, from we conclude that $T(A^2) = 0$. In general, if $T(A^\alpha) = 0$, then, from the short exact sequence:

$$0 \longrightarrow A^\alpha \longrightarrow A^{\alpha+1} \longrightarrow A^{\alpha+1}/A^\alpha \longrightarrow 0,$$

by applying the functor T, we obtain $T(A^{\alpha+1}) = 0$. Suppose there exists an element $\alpha \in \Omega$ for which $T(A^\alpha) \neq \{0\}$. Let $\gamma \in \Omega$ be the minimal element satisfying $T(A^\gamma) \neq 0$, then $\gamma > 1$. If γ has an immediately predecessor β, by applying the functor T to the following short exact sequence:

$$0 \longrightarrow A^\beta \longrightarrow A^\gamma \longrightarrow A^\gamma/A^\beta \longrightarrow 0,$$

and considering the hypothesis, we obtain $T(A^\gamma) = 0$, a contradiction. This lead us to consider γ as an ordinal limit, *i.e.*,

$$\gamma = \bigcup_{\alpha<\gamma} \alpha.$$

It is clear that γ can be considered as a directed set. Let $\alpha \leq \beta \leq \gamma$, we define $\varphi^\alpha_\beta : A^\alpha \to A^\beta$ as the inclusion. Therefore, the pair $\{A^\alpha, \varphi^\alpha_\beta \,|\, \alpha \leq \beta < \gamma\}$, is a direct system, which guarantees the existence of the direct limit $\varinjlim A^\alpha$. We claim that $\varinjlim A^\alpha$ is isomorphic, as a Λ-module, to A^γ. Indeed, let us consider the direct sum $\bigoplus_{\alpha<\gamma} A^\alpha$:

$$\bigoplus_{\alpha<\gamma} A^\alpha = \left\{ f : \gamma \to \bigcup_{\alpha<\gamma} A^\alpha \;\middle|\; f(\alpha) \in A^\alpha, f(\alpha) = 0, \text{ except in a finit number.} \right\}$$

Let consider the coordinate functions $c^\alpha : A^\alpha \to \bigoplus_{\beta\in\gamma} A^\beta$, defined by

$$x \in A^\alpha \Rightarrow \; c^\alpha(x) : \gamma \to \bigcup_{\beta<\gamma} A^\beta$$

$$c^\alpha(x)(\beta) = \begin{cases} 0, & \text{if } \alpha \neq \beta, \\ x, & \text{if } \alpha = \beta. \end{cases} \qquad \forall x \in A^\alpha, \quad \forall \beta < \gamma.$$

Then, any element $f \in \bigoplus_{\alpha<\gamma} A^\alpha$, can be written as a finite sum of the form

$$f = \sum_{\alpha<\gamma} c^\alpha(f(\alpha)).$$

We define a map $\Phi_\gamma : \bigoplus_{\alpha<\gamma} A^\alpha \to A^\gamma$, by the assignment:

$$\Phi(f) = \sum_{\alpha<\gamma} \varphi^\alpha_\gamma(f(\alpha)), \quad \forall f \in \bigoplus_{\alpha<\gamma} M^\alpha.$$

Let $x_\gamma \in A^\gamma$. There are $\lambda_j \in \Lambda$, $a_{\alpha_j} \in A^{\alpha_j}$, with $\alpha_j < \gamma$ and $j = 1, \ldots, \ell$, such that $x_\gamma = \lambda_1 a_{\alpha_1} + \ldots + \lambda_\ell a_{\alpha_\ell}$. We define the map $f \in \bigoplus_{\alpha<\gamma} A^\alpha$ by $f = c^{\alpha_1}(\lambda_1 a_{\alpha_1}) + \ldots + c^{\alpha_\ell}(\lambda_\ell a_{\alpha_\ell})$. Then, $\Phi(f) = x_\gamma$, which shows that Φ is surjective.

Let us define the following Λ-submodule of $\bigoplus_{\alpha<\gamma} A^\alpha$:

$$N_\gamma = \langle c^\beta(\varphi^\alpha_\beta(x_\alpha)) - c^\alpha(x_\alpha) \,|\, x_\alpha \in A^\alpha,\, \alpha \leq \beta \rangle.$$

Then

$$\begin{aligned}\Phi(c_\beta(\varphi^\alpha_\beta(x_\alpha)) - c^\alpha(x_\alpha)) &= \Phi(c^\beta(\varphi^\alpha_\beta(x_\alpha))) - \Phi(c^\alpha(x_\alpha))) \\ &= \varphi^\beta_\gamma(\varphi^\alpha_\beta(x_\alpha)) - \varphi^\alpha_\gamma(x_\alpha) = \varphi^\alpha_\gamma(x_\alpha) - \varphi^\alpha_\gamma(x_\alpha) \\ &= 0.\end{aligned}$$

This shows that $N_\gamma \subset \mathrm{Ker}(\Phi)$. This induce a Λ-homomorphism

$$\widetilde{\Phi} : \varinjlim A^\alpha = \bigoplus_{\alpha<\gamma} A^\alpha / N_\gamma \to A^\gamma.$$

We already know that $\widetilde{\Phi}$ is surjective. We shall to prove that $\widetilde{\Phi}$ is injective. Let $\overline{f} \in \mathrm{Ker}(\widetilde{\Phi})$. There exists $x_\alpha \in A^\alpha$, such that $\overline{f} = \delta_\alpha(x_\alpha) + N_\gamma$. Then, $0 = \widetilde{\Phi}(\overline{f}) = \varphi^\alpha_\gamma(x_\alpha)$. Therefore, $\overline{f} = 0$ and $\widetilde{\Phi}$ is injective. This proves that $\varinjlim A^\alpha$ is isomorphic to A^γ. Applying the functor T and considering that T commutes with direct limits, we obtain the following:

$$T(A^\gamma) \simeq T(\varinjlim A^\alpha) \simeq \varinjlim T(A^\alpha) = 0.$$

Because $T(A^\alpha) = 0$ for all $\alpha < \gamma$. This contradicts our assumption that γ is the minimal element for which $T(A^\gamma) \neq 0$. Therefore, $T(A^\alpha) = 0$ for all α. Now we claim that $T(A) = 0$. Observe that the correspondence $c^\alpha(a_\alpha) + N \mapsto a_\alpha$, gives an isomorphism between $\varinjlim A^\alpha$ and A. Thus, $T(\varinjlim A^\alpha)$ and $T(A)$ are isomorphic Λ_1-modules. Since $\varinjlim T(A^\alpha)$

is isomorphic to $T(\varinjlim A^\alpha)$, it follows that $\varinjlim T(A^\alpha)$ is isomorphic to $T(A)$. But $T(A^\alpha) = 0$ for all $\alpha \in \Omega$, then $T(A) = 0$.

(**ii**) Let A be a left Λ-module. Let Ω be the ordinal for which there exists a bijection $\nu : \Omega \to A$. In according with the convention stated in (**i**), we write $\nu(\alpha) = a_\alpha \in A$, for all $\alpha \in \Omega$. For $\alpha \in \Omega$, we define the following left Λ-submodules of A:

$$\begin{aligned} A_\alpha &= \left\{ \sum_{k=1}^{\ell} \lambda_k a_{\alpha_k} \mid \lambda_k \in \Lambda,\, \alpha \leq \alpha_k,\, \ell \in \mathbb{Z}_+ \right\},\, \alpha > 0, \\ A_0 &= A. \end{aligned}$$

For $\beta \geq \alpha$, we have $A_\beta \subseteq A_\alpha$. We consider the Λ-homomorphism $\psi_\alpha^\beta : A_\beta \to A_\alpha$ as the inclusion map. It is clear that $\{A_\alpha, \psi_\alpha^\beta\}$ is an inverse system of Λ-modules. We claim that $\varprojlim A_\alpha$ is isomorphic to the intersection $\bigcap\limits_{\alpha\in\Omega} A_\alpha$. Observe that

$$\varprojlim A_\gamma = \{(x_\gamma)_{\gamma\in\Omega} \in \Pi_{\gamma\in\Omega} A_\gamma \mid \psi_\alpha^\beta(x_\beta) = x_\alpha,\, \forall \alpha \leq \beta\}.$$

For any $(x_\gamma)_{\gamma\in\Omega} \in \varprojlim A_\alpha$, we have $x_\alpha = x_\beta$, for all $\alpha, \beta \in \Omega$. Indeed, if $\alpha \leq \beta$, this is clear for the definition of inverse limit. If $\beta \leq \alpha$, then $A_\alpha \subset A_\beta$ and $x_\alpha = \psi_\beta^\alpha(x_\alpha) = x_\beta$. From this, it follows that $x_\alpha \in \bigcap\limits_{\beta\in\Omega} A_\beta$, for all $\alpha \in \Omega$. Let us consider the map $(x_\gamma)_{\gamma\in\Omega} \mapsto x_\alpha$, between $\varprojlim A_\gamma$ and $\bigcap\limits_{\gamma\in\Omega} A_\gamma$. Let $x \in \bigcap\limits_{\gamma\in\Omega} A_\gamma$ and consider the element $(x_\gamma)_{\gamma\in\Omega} \in \varprojlim A_\gamma$, given by $x_\gamma = x$, for all $\gamma \in \Omega$. Then $(x_\gamma)_{\gamma\in\Omega} \mapsto x$. This proves that the map $(x_\gamma)_{\gamma\in\Omega} \mapsto x_\alpha$ is surjective, from it we obtain our claim.

For each $\alpha \in \Omega$, we define the Λ-module P_α as the free module generated by A_α. Therefore, any element $f \in P_\alpha$ can be identified as a function $f : [\alpha, \Omega[\to \Lambda$, such that $f(\beta) \neq 0$, for a finite number of elements β's in the interval $[\alpha, \Omega[$. Let $\epsilon_\alpha : [\alpha, \Omega[\to \Lambda$ be the function defined by $\epsilon_\alpha(\beta) = \delta_{\alpha,\beta}$, for all $\beta \geq \alpha$. Then, any $f \in P_\alpha$ can be written as:

$$f = \sum_{\beta\geq\alpha} f(\beta)\epsilon_\beta.$$

The map $\Phi_\alpha : P_\alpha \to A_\alpha$, given by $\Phi(f) = \sum_{\beta\geq\alpha} f(\beta) a_\beta$, is surjective. For each $\alpha \in \Omega$, let $\iota_\alpha : \mathrm{Ker}(\Phi_\alpha) \to P_\alpha$ be the inclusion map. Then, we have the following short exact sequence:

$$0 \longrightarrow \mathrm{Ker}(\Phi_\alpha) \longrightarrow P_\alpha \longrightarrow A_\alpha \longrightarrow 0.$$

Let $\alpha \leq \beta$ and consider the map $\eta_\alpha^\beta : P_\beta \to P_\alpha$, defined by

$$f \in P_\beta \Rightarrow \eta_\alpha^\beta(f) : [\alpha, \Omega[\to \Lambda,$$
$$\eta_\alpha^\beta(f)(\gamma) = \begin{cases} 0, \text{ if } \alpha \leq \gamma < \beta \\ f(\gamma) \text{ if } \beta \leq \gamma \end{cases}$$

Then, $\{P_\alpha; \eta_\alpha^\beta\}$ is an inverse system of Λ-modules. Observe that the following diagram is commutative:

$$\begin{tikzcd} P_\beta \arrow[r, "\eta_\beta^\alpha"] \arrow[d, "\Phi_\beta"'] & A^\beta \arrow[d, "\psi_\alpha^\beta"'] \\ P^\alpha \arrow[r, "\Phi_\alpha"] & A^\alpha \end{tikzcd} , \quad \psi_\alpha^\beta \circ \Phi_\beta = \Phi_\alpha \circ \eta_\beta^\alpha. \tag{3}$$

Now, we claim that $\varprojlim A_\alpha = 0$. Let $x \in \bigcap_{\gamma\in\Omega} A_\gamma$. Then, for each $\alpha \in \Omega$, there exists $f_\alpha \in P_\alpha$, such that $\Phi_\alpha(f_\alpha) = x$. Let P' be the following Λ-submodule of $\bigoplus_{\gamma\in\Omega} P_\gamma$:

$$P' = \mathrm{Span}_\Lambda\{c^\alpha(f_\alpha) \in \bigoplus_{\gamma\in\Omega} P_\gamma \mid \Phi_\alpha(f_\alpha) = \Phi_\beta(f_\beta), \forall \alpha, \beta \in \Omega\}.$$

It is straightforward to verify that the map $P' \to \varprojlim A_\alpha$, $c^\alpha(f^\alpha) \mapsto \Phi_\alpha(f_\alpha)$ is a surjective Λ-homomorphism. On the other hand, from (3), it follows that $\psi_\alpha^\beta(\Phi_\beta(f_\beta)) = \Phi_\alpha(\eta_\alpha^\beta(f_\beta)) = \Phi_\alpha(f_\alpha)$, for all $\alpha, \beta \in \Omega$. Then, there exists $f' \in \mathrm{Ker}(\Phi_\alpha)$, such that $\eta_\alpha^\beta(f_\beta) = f_\alpha + f'$. Take $\alpha \leq \gamma < \beta$, then $f_\alpha(\gamma) + f'(\gamma) = 0$. Since β and γ are arbitrary, it follows that $f_\alpha = -f' \in \mathrm{Ker}(\Phi_\alpha)$. This proves that if f_α satisfies $c^\alpha(f_\alpha) \in P_\alpha$, then $f_\alpha \in \mathrm{Ker}(\Phi_\alpha)$. Therefore, $\bigcap_{\gamma\in\Omega} A_\gamma = 0$, consequently $\varprojlim A_\alpha = 0$.

Let consider the modules $A_\alpha/A_{\alpha+1}$ and the maps $\tau_\alpha^\beta : A_\beta/A_{\beta+1} \to A_\alpha/A_{\alpha+1}$, defined by $\tau_\alpha^\beta(x + A_{\beta+1}) = x + A_{\alpha+1}$, for all $x \in A_\beta$ and for all $\alpha \leq \beta$. We claim that $\varprojlim A_\alpha/A_{\alpha+1} = 0$. Let $(x_\alpha + A_{\alpha+1})_{\alpha\in\Omega} \in \Pi_{\alpha\in\Omega} A_\alpha/A_{\alpha+1}$, then $\tau_\alpha^\beta(x_\beta + A_{\beta+1}) = x_\alpha + A_{\alpha+1}$, for all $\alpha \leq \beta$. In particular, for $\beta > \alpha$, we have $\beta \geq \alpha + 1$ and $A_\beta \subseteq A_{\alpha+1}$. Then, $x_\beta + A_{\alpha+1} = \tau_\alpha^\beta(x_\beta + A_{\beta+1}) = 0$. Therefore, $(x_\alpha + A_{\alpha+1})_{\alpha\in\Omega} = 0$ and $\varprojlim A_\alpha/A_{\alpha+1} = 0$.

On the other hand, observe that $A_\alpha/A_{\alpha+1} = \{\lambda a_\alpha + A_{\alpha+1} \mid \lambda \in \Lambda\}$. Then, we have the surjective map $\sigma_\alpha : \Lambda \to A_\alpha/A_{\alpha+1}$, given by $\sigma_\alpha(\lambda) = \lambda a_\alpha + A_{\alpha+1}$, for all $\lambda \in \Lambda$. Thus, we obtain the following short exact sequence:

$$0 \longrightarrow \mathrm{Ker}(\sigma_\alpha) \longrightarrow \Lambda \longrightarrow A_\alpha/A_{\alpha+1} \longrightarrow 0, \ \forall \alpha \in \Omega\ ,$$

where $\mathrm{Ker}(\sigma_\alpha) = \{\lambda \in \Lambda \mid \lambda a_\alpha \in A_{\alpha+1}\}$. We apply the functor T to the short exact sequence (2) and taking into account the hypothesis that T is trivial in any left ideal of Λ, (in particular with Λ itself), we deduce that $T(A_\alpha/A_{\alpha+1}) = 0$, for all $\alpha \in \Omega$. Now, consider the following short exact sequence:

$$0 \longrightarrow A_{\alpha+1} \xrightarrow{\iota_\alpha} A_\alpha \longrightarrow A_\alpha/A_{\alpha+1} \longrightarrow 0, \quad \forall \alpha \in \Omega.$$

Applying the funtor T to the above sequence, we obtain the following short exact sequence:

$$0 \longrightarrow T(A_\alpha) \xrightarrow{T(\iota_\alpha)} T(A_{\alpha+1}),$$

for all $\alpha \in \Omega$. In particular, for $\alpha = 0$ we get that $0 \to T(A) \to T(A_1)$. Therefore, for each α, we have an injective map $\rho_\alpha : T(A) \to T(A_{\alpha+1})$, defined recursively by $\rho_{\alpha+1} = T(\iota_{\alpha+1}) \circ \rho_\alpha$ and $\rho_0 = T(\iota_0)$. Let $\rho : T(A) \to \varprojlim T(A_{\alpha+1})$ be the map defined by $\rho(x) = (\rho_\alpha(x))_{\alpha\in\Omega}$, for all $x \in T(A)$. It is clear that ρ is injective. Observe that for each $\alpha \in \Omega$, there is an injective Λ-homomorphism $\psi_\alpha : A_{\alpha+1} \to A_\alpha$, defined by $\psi_\alpha(x) = \psi_\alpha^{\alpha+1}(x)$, for all $x \in A_{\alpha+1}$. This yields an injective Λ-homomorphism $\psi : \varprojlim T(A_{\alpha+1}) \to \varprojlim T(A_\alpha)$, given by $\psi((x_\alpha)_{\alpha\in\Omega}) = (\psi_\alpha(x)_\alpha)_{\alpha\in\Omega}$. Then, $\varprojlim T(A_{\alpha+1}) = 0$. Since T commutes with inverse limits, we finally have $T(A) = 0$. $\square$

Before to state the following result, we give a brief review of the concepts and notation used in the sequel. It is a well-known result that any Λ-module can be embedded into a exact sequence of the form:

$$0 \longrightarrow M \longrightarrow P \longrightarrow A \longrightarrow 0$$

with P a projective module (see [2], **Thm. 2.3**, Chapter I §2). Let T be a one variable covariant functor. We define $S_1T(A) = \mathrm{Ker}(T(M) \to T(P))$. Now, let A' and A modules and let $g : A' \to A$ be a homomorphism. Consider the following diagram with exact rows:

(4)
$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r] \arrow[d, "f'"] & P' \arrow[r] \arrow[d, "f"] & A' \arrow[r, "\zeta"] \arrow[d, "g"] & 0 \\
0 \arrow[r] & M \arrow[r] & P \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

The map g induces the maps f and f', such that (4) is commutative. Furthermore, the map $S_1T(A') \to S_1T(A)$, induced by $T(f')$, is independent of the choice of f (see [2], **Prop. 1.1**, Chapter III §1). We denote this map by $S_1T(g) : S_1T(A') \to S_1T(A)$. This conclusions yield a covariant functor S_1T, called the **left satellite** of ***T***. This definition of satellite may be iterated by setting:

$$S_{n+1}T = S_1(S_nT), \quad S_0T = T.$$

Theorem 2.2.

(i) *Let T be a one variable covariant half exact funtor. If T is of the type $L\Sigma^*$ then S_1T is of the type $L\Sigma^*$.*
(ii) *Let T be a one variable covariant right exact funtor. If S_1T is of the type $L\Sigma^*$ and T commutes with direct limits of projective modules, then T is of the type $L\Sigma^*$.*

Proof. Let $\{A^\alpha, \varphi^\alpha_\beta\}$ be a direct system of left Λ-modules over a directed set Ω and let $A = \varinjlim A^\alpha$. For each $\alpha \in \Omega$, there is a canonical map $\sigma^\alpha : A^\alpha \to A$, defined by $\sigma^\alpha(a) = c^\alpha(a) + N$. In addition, $A = \{c^\alpha(a) + N \mid a \in A^\alpha, \alpha \in \Omega\}$.

For each $\alpha \in \Omega$, let P^α be the free Λ-module generated by A^α, that is:

$$P^\alpha = \{f : A^\alpha \to \Lambda \mid f(a) = 0, \text{ except for a finit number of } a \in A^\alpha\}.$$

For each $a \in A^\alpha$, we define an element $e^\alpha(a) \in P^\alpha$, by

$$e^\alpha(a)(b) = \begin{cases} 0, & \text{if } a \neq b, \\ 1, & \text{if } a = b. \end{cases}$$

Therefore, any $f \in P^\alpha$ can be written as: $f = \sum_{a\in A^\alpha} f(a)e^\alpha(a)$. This shows that P^α is a free Λ-module generated by $e^\alpha(a)$, with $a \in A^\alpha$, that is $P^\alpha = \mathrm{Span}_\Lambda\{e^\alpha(a) \mid a \in A^\alpha\}$. We extend the map φ^α_β to P^α and P^β by through:

$$\begin{array}{rccl} \phi^\alpha_\beta : & P^\alpha & \longrightarrow & P^\beta \\ & f & \mapsto & \displaystyle\sum_{a\in A^\alpha} f(a)e^\beta(\varphi^\alpha_\beta(a)). \end{array}$$

It is straightforward to verify that $\phi^\beta_\gamma \circ \phi^\alpha_\beta = \phi^\alpha_\gamma$, for all $\alpha \le \beta \le \gamma$. Then, $\{P^\alpha; \phi^\beta_\alpha\}$ is a direct system of free Λ-modules. Let $q^\alpha : P^\alpha \to A^\alpha$ be the Λ-homomorphism defined by $q^\alpha(e^\alpha(a)) = a$, for all $a \in A$. Let $M^\alpha = \mathrm{Ker}(q^\alpha)$ and let $\iota^\alpha : M^\alpha \to P^\alpha$ be the inclusion map. Then, we have the following short exact sequence:

$$0 \longrightarrow M^\alpha \xrightarrow{\iota^\alpha} P^\alpha \xrightarrow{q^\alpha} A^\alpha \longrightarrow 0$$

It is straightforward to prove that the following diagram is commutative:

$$\begin{tikzcd} P^\alpha \arrow[r, "\phi^\alpha_\beta"] \arrow[d, "q^\alpha"'] & P^\beta \arrow[d, "q^\beta"'] \\ A^\alpha \arrow[r, "\varphi^\alpha_\beta"] & A^\beta \end{tikzcd}, \quad q^\beta \circ \phi^\alpha_\beta = \varphi^\alpha_\beta \circ q^\alpha.$$

This implies that $\phi^\alpha_\beta(M^\alpha) \subset M^\beta$. Let $\chi^\alpha_\beta = \phi^\alpha_\beta|_{M^\alpha} : M^\alpha \to M^\beta$. Then, $\{M^\alpha; \chi^\alpha_\beta\}$ is a direct system of Λ-modules. In addition, we obtain the following commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & M^\alpha \arrow[r, "\iota^\alpha"] \arrow[d, "\chi^\alpha_\beta"'] & P^\alpha \arrow[r, "q^\alpha"] \arrow[d, "\phi^\alpha_\beta"'] & A^\alpha \arrow[r] \arrow[d, "\varphi^\alpha_\beta"'] & 0 \\ 0 \arrow[r] & M^\beta \arrow[r, "\iota^\beta"] & P^\beta \arrow[r, "t^\beta"] & A^\beta \arrow[r] & 0 \end{tikzcd}$$

Let P be the free Λ-module generated by A, that is:

$$P = \{f : A \to \Lambda \mid f(x) = 0, \text{except for a finit number of } x \in A\}.$$

For any $x \in A$, let $e(x) \in P$ be the element defined by $e(x)(y) = \delta_{x,y}$, for all $y \in A$. Then, any $f \in P$ can be written as $f = \sum_{x \in A} f(x)e(x)$, which shows that P is a free Λ-module generated by $e(x)$, for $x \in A$; that is $P = \mathrm{Span}_\Lambda\{e(x) \mid x \in A\}$. Let $q : P \to A$, be the Λ-homomorphism defined by $q(e(x)) = x$, for all $x \in A$. Let $M = \mathrm{Ker}(q)$ and let $\iota : M \to P$ be the inclusion map. Then, we have the following short exact sequence of Λ-modules:

$$0 \longrightarrow M \xrightarrow{\iota} P \xrightarrow{q} A \longrightarrow 0$$

For each $\alpha \in \Omega$, let $\tau^\alpha : P^\alpha \to P$ be the Λ-homomorphism defined by $\tau^\alpha(e^\alpha(a)) = e(\sigma^\alpha(a))$, for all $a \in A^\alpha$. It is straightforward to prove that $\tau^\beta \circ \phi^\alpha_\beta = \tau^\alpha$, for all $\alpha \leq \beta$. Consider the canonical injective Λ-homomorphism $\sigma^\alpha_P : P^\alpha \to \varinjlim P^\gamma$. Then, there exists a Λ-homomorphism $\tau : \varinjlim P^\alpha \to P$, given by the assignment:

$$\tau(\sigma^\alpha_P(f)) = \tau(c^\alpha_P(f) + N_P) = \tau^\alpha(f), \quad \forall f \in P^\alpha, \quad \forall \alpha \in \Omega.$$

It is not difficult to prove that τ is an isomorphism. Observe that the following diagram is commutative:

$$\begin{tikzcd} P^\alpha \arrow[r, "q^\alpha"] \arrow[d, "\tau^\alpha"'] & A^\alpha \arrow[d, "\sigma^\alpha_A"'] \\ P \arrow[r, "q"] & A \end{tikzcd} \quad \sigma^\alpha_A \circ q^\alpha = t \circ \tau^\alpha, \quad \alpha \in \Omega.$$

This proves that if $f \in M^\alpha = \mathrm{Ker}(q^\alpha)$, then $\tau^\alpha(f) \in M = \mathrm{Ker}(q)$. Let $\theta^\alpha = \tau^\alpha|_{M^\alpha} : M^\alpha \to M$. Then, $\theta^\beta \circ \chi^\alpha_\beta = \theta^\alpha$. Thus, there exists a Λ-homomorphism $\theta : \varinjlim M^\alpha \to M$, given by:

$$\theta(\sigma^\alpha_M(f)) = \theta(c^\alpha_M(f) + N_M) = \theta^\alpha(f), \quad \forall f \in M^\alpha, \quad \forall \alpha \in \Omega.$$

It is not difficult to prove that θ is an isomorphism. Therefore, we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & M^\alpha \arrow[r, "\iota^\alpha"] \arrow[d, "\theta^\alpha"] & P^\alpha \arrow[r, "q^\alpha"] \arrow[d, "\tau^\alpha"] & A^\alpha \arrow[r] \arrow[d, "\sigma^\alpha_A"] & 0 \\
0 \arrow[r] & M \arrow[r, "\iota"] & P \arrow[r, "q"] & A \arrow[r] & 0
\end{tikzcd}$$

Consider the Λ_1-modules $S_1T(A^\alpha) = \mathrm{Ker}(T(\iota^\alpha))$ and $S_1T(A) = \mathrm{Ker}(T(\iota))$, arising from the left satellite of T, S_1T. Then, $\{S_1T(A^\alpha); S_1T(\varphi^\alpha_\beta)\}$ is a direct system of Λ_1-modules, where $S_1T(\varphi^\alpha_\beta) : S_1T(A^\alpha) \to S_1T(A^\beta)$, is defined by $S_1T(\varphi^\alpha_\beta)(x) = T(\chi^\alpha_\beta)(x)$, for all $x \in S_1T(A^\alpha)$. Observe that there exists a Λ_1-homomorphism $\overline{\theta}^\alpha : S_1T(A^\alpha) \to S_1T(A)$, given by $\overline{\theta}^\alpha(x) = T(\theta^\alpha)(x)$, for all $x \in S_1T(A^\alpha)$. Since $S_1T(P) = 0$, for any Λ-projective module, then we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & S_1T(A^\alpha) \arrow[r, "\overline{\iota}^\alpha"] \arrow[d, "\overline{\theta}^\alpha"] & T(M^\alpha) \arrow[r, "T(\iota^\alpha)"] \arrow[d, "T(\theta^\alpha)"] & T(P^\alpha) \arrow[r, "T(q^\alpha)"] \arrow[d, "T(\tau^\alpha)"] & T(A^\alpha) \arrow[d, "T({\sigma_A}^\alpha)"] \\
0 \arrow[r] & S_1T(A) \arrow[r, "\overline{\iota}"] & T(M) \arrow[r, "T(\iota)"] & T(P) \arrow[r, "T(q)"] & T(A)
\end{tikzcd}$$

where $\overline{\iota}^\alpha : S_1T(A^\alpha) \to T(M^\alpha)$ and $\overline{\iota} : S_1T(A) \to T(M)$ are the inclusion maps. Observe that we have the following direct systems of Λ_1-modules:

$$\{T(M^\alpha); T(\chi^\alpha_\beta)\}, \quad \{T(P^\alpha), T(\phi^\alpha_\beta)\}, \quad \{T(A^\alpha); T(\varphi^\alpha_\beta).$$

The Λ_1-homomorphism $T(\theta^\alpha)$, $T(\tau^\alpha)$ and $T(\sigma^\alpha)$, give place to the following Λ_1-homomorphisms,:

$$\begin{array}{rccc} \widehat{\theta}: & \varinjlim T(M^\alpha) & \longrightarrow & T(M) \\ & c^\alpha(x) + N & \mapsto & T(\theta^\alpha)(x) \end{array} \qquad \begin{array}{rccc} \widehat{\tau}: & \varinjlim T(P^\alpha) & \longrightarrow & T(P) \\ & c^\alpha(x) + N & \mapsto & T(\tau^\alpha)(x) \end{array}$$

and

$$\begin{array}{rccc} \widehat{\sigma}: & \varinjlim T(A^\alpha) & \longrightarrow & T(A) \\ & c^\alpha(x) + N & \mapsto & T(\sigma^\alpha)(x). \end{array}$$

In such a way that we obtain the following commutative diagram with exact rows:

(5)

$$\begin{tikzcd}
0 \arrow[r] & \varinjlim S_1T(A^\alpha) \arrow[r, "\rho"] \arrow[d, "\overline{\theta}"] & \varinjlim T(M^\alpha) \arrow[r, "\eta"] \arrow[d, "\widehat{\theta}"] & \varinjlim T(P^\alpha) \arrow[r, "\zeta"] \arrow[d, "\widehat{\tau}"] & \varinjlim T(A^\alpha) \arrow[d, "\widehat{\sigma}"] \\
0 \arrow[r] & S_1T(A) \arrow[r, "\overline{\iota}"] & T(M) \arrow[r, "T(\iota)"] & T(P) \arrow[r, "T(q)"] & T(A)
\end{tikzcd}$$

where ρ, η and ζ, are given by:

$$\begin{array}{rccc} \rho: & \varinjlim S_1T(A^\alpha) & \longrightarrow & \varinjlim T(M^\alpha) \\ & c^\alpha(x)+N & \mapsto & c^\alpha(\iota^\alpha(x))+N \end{array}$$

$$\begin{array}{rccc} \eta: & \varinjlim T(M^\alpha) & \longrightarrow & \varinjlim T(P^\alpha) \\ & c^\alpha(x)+N & \mapsto & c^\alpha(T(\iota^\alpha)(x))+N \end{array}$$

and

$$\begin{array}{rccc} \zeta: & \varinjlim T(P^\alpha) & \longrightarrow & \varinjlim T(A^\alpha) \\ & c^\alpha(x)+N & \mapsto & c^\alpha(T(q^\alpha)(x))+N. \end{array}$$

Now, we shall prove (**i**).

(**i**) By hipothesis, the vertical arrows $\widehat{\theta}$, $\widehat{\tau}$ and $\widehat{\sigma}$ are isomorphism, then $\overline{\theta}$ is an isomorphism, which proves our claim.

(**ii**) Since T is right exact, the diagram (5), takes the form:

$$\begin{tikzcd}
0 \arrow[r] & \varinjlim S_1T(A^\alpha) \arrow[r, "\rho"] \arrow[d, "\overline{\theta}"] & \varinjlim T(M^\alpha) \arrow[r, "\eta"] \arrow[d, "\widehat{\theta}"] & \varinjlim T(P^\alpha) \arrow[r, "\zeta"] \arrow[d, "\widehat{\tau}"] & \varinjlim T(A^\alpha) \arrow[r] \arrow[d, "\widehat{\sigma}"] & 0 \\
0 \arrow[r] & S_1T(A) \arrow[r, "\overline{\iota}"] & T(M) \arrow[r, "T(\iota)"] & T(P) \arrow[r, "T(q)"] & T(A) \arrow[r] & 0
\end{tikzcd}$$

By hypothesis, the vertical arrows $\overline{\theta}$ and $\widehat{\tau}$ are isomorphism, this implies immediately that $\widehat{\sigma}$ is surjective and $\widehat{\theta}$ is injective. Since $\{A^\alpha; \varphi^\alpha_\beta\}$ is an arbitrary direct system of Λ-modules, it follows that $\widehat{\theta}$ is also surjective, therefore $\widehat{\theta}$ is an isomorphism, from it finally we get that $\widehat{\sigma}$ is an isomorphism.

□

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References

[1] Breaz S. *Modules M such that $\mathrm{Ext}^1_R(M, -)$ commutes with direct limits.* https://arxiv.org/pdf/1107.0557.pdf

[2] Cartan H., Eilenberg S., *Homological algebra*, Princeton, New Jersey, Princeton University Press (1956).

[3] Lenzing H., *Endlich präsentierbare Moduln* (Basel) 20 (1969), 262-266

[4] Rotman Joseph J., *Advanced Modern Algebra*, Prentice Hall, First edition, ISBN: 0130878685.

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