

Diagrams, Fibrations, and the Decomposition of Colimits

George Peschke

Dept. of Mathematical and Statistical Sciences, University of Alberta, Edmonton, Alberta, Canada, T6G 2G1

Walter Tholen¹

Department of Mathematics and Statistics, York University, Toronto, Ontario, Canada, M3J 1T6

Abstract

This paper consists of two parts. First, within the framework of Grothendieck’s fibrational category theory, we present a certain web of fundamental adjunctions surrounding the category of all small diagrams in a given category. This extends earlier work of Guitart and the Ehresmann school and promises to be of independent interest. Then we demonstrate the utility of those adjunctions by deriving three formulae for (co-)limits: a ‘twisted’ generalization of the well known Fubini formula—known from a monograph by Chakólski and Scherer, and a ‘colimit decomposition formula’, a special case of which has been found independently and a little earlier by Batanin and Berger.

Extending the formation of the diagram category from a given category to a given functor, we establish a generalized Guitart adjunction, as part of a network of 2-adjunctions which, at its core, includes the equivalence of split Grothendieck (co-)fibrations and strictly (co-)indexed categories.

Keywords: diagram category, colimit decomposition, twisted Fubini formula, (split) fibration, Grothendieck construction, (op)lax colimit, strictification of lax-commutative diagrams, free split cofibration.

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1. Introduction

1.1. Motivation

An initial goal of this work was the development of techniques that may facilitate the computation of “complicated” objects, say in algebra or topology, from smaller and more easily computed “pieces”. To provide a potential tool, the first author showed that the colimit of a diagram $X : \mathcal{K} \rightarrow \mathcal{X}$ in a cocomplete category \mathcal{X} may be undertaken “piecewise”, whenever the small category \mathcal{K} is itself expressed as $\mathcal{K} = \operatorname{colim} \Phi$ in the category \mathbf{Cat} of small categories, for some functor $\Phi : \mathcal{D} \rightarrow \mathbf{Cat}$. If $K_d : \Phi d \rightarrow \mathcal{K}$ ($d \in \mathcal{D}$) denotes the colimit cocone of Φ , we obtain the following *Colimit Decomposition Formula*:

$$\operatorname{colim}^{\mathcal{K}}(X) \cong \operatorname{colim}^{d \in \mathcal{D}}(\operatorname{colim}^{\Phi d}(X K_d)). \quad (\text{CDF})$$

Email addresses: `gepe@ualberta.ca` (George Peschke), `tholen@mathstat.yorku.ca` (Walter Tholen)

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$$\begin{array}{ccc}
\Phi d \xrightarrow{K_d} \operatorname{colim} \Phi = \mathcal{K} \xrightarrow{X} \mathcal{X} & & X(K_d x) \longrightarrow \operatorname{colim} X K_d \longrightarrow \operatorname{colim} X \\
\vdots \nearrow K_e & & \vdots \nearrow \\
\Phi e & & X(K_d y) \longrightarrow \operatorname{colim} X K_e
\end{array}$$

(Note that this visualization of the formula must not be misinterpreted as a kind of preservation of a colimit by X : the arrows on the left live in \mathbf{CAT} , while those on the right are in \mathcal{X} .) The CDF and its limit analogue (4.2) were presented by the first author in a talk presented in May 2017 in the Workshop on Categorical Methods in Non-Abelian Algebra in Louvain-la-Neuve. The argument consisted of brute force constructions in the category $\mathbf{Diag}^\circ(\mathcal{X})$ of small diagrams in \mathcal{X} , see (2.1), with a strong suspicion that fundamental properties of the evident functor $\mathbf{Diag}^\circ(\mathcal{X}) \rightarrow \mathbf{Cat}$ would give a transparent and conceptual explanation. From this suspicion the present collaboration evolved.

After a talk presented by the second author at CT2019, it turned out that the CDF had actually appeared slightly earlier than the first author's 2017 presentation, as Lemma 7.13 in the paper [2] by Michael Batanin and Clemens Berger, who give credit to Steve Lack for the short proof they present.

We found that Grothendieck's *fibrational category theory* provides a good framework for our purposes and that it yields insights on categories of diagrams which are of interest in their own right. As hoped for, we obtain transparent and conceptual proofs for the CDF and its variations, see Sections 3 and 4. Here is an outline.

1.2. The Grothendieck construction and the Guitart adjunction

Consider a Grothendieck fibration $P : \mathcal{E} \rightarrow \mathcal{B}$. By assigning to every object b of the base category \mathcal{B} its fibre P_b in \mathcal{E} one obtains a pseudo-functor $\mathcal{B}^{\text{op}} \rightarrow \mathbf{CAT}$ into the huge 2-category of all categories (see the end of this Introduction for notational conventions with respect to \mathbf{CAT} and \mathbf{Cat}). Conversely, the Grothendieck construction produces for every pseudo-functor $\Phi : \mathcal{B}^{\text{op}} \rightarrow \mathbf{CAT}$ the total category of Φ , here denoted by $\int^\circ \Phi$, which is fibred over \mathcal{B} . In this way, fibred categories over \mathcal{B} are equivalently presented as contravariantly indexed categories, that is: as contravariant \mathbf{CAT} -valued pseudo-functors defined on \mathcal{B} . Dually, *Grothendieck cofibrations* (nowadays more frequently called *opfibrations*²) $\mathcal{E} \rightarrow \mathcal{B}$ correspond equivalently to covariantly indexed categories $\Phi : \mathcal{B} \rightarrow \mathbf{CAT}$, via the dual Grothendieck construction, here denoted by $\int_\circ \Phi$. Under this equivalence, so-called split cofibrations correspond, by definition, to those pseudo-functors that are actually functors, and one may further restrict the equivalence to small-fibred split cofibrations and \mathbf{Cat} -valued (as opposed to \mathbf{CAT} -valued) functors.

These facts are well known and amply documented in the literature, albeit predominantly in a context that leaves the base \mathcal{B} fixed; in historical order, references include [15], [13], [14], [3] [25], [34], [4], [5], [22], [12], [20], [24], [35], [33], [38], [23]. Less known is the fact that the Grothendieck construction was studied very early on by Ehresmann [10] and his school, under the name *produit*

²In this paper we adhere to Grothendieck's original terminology, but note that, in order to prevent confusion with the terminology used in topology, in recent years Grothendieck cofibrations have more commonly been referred to as opfibrations. However, this departure from the standard categorical procedure for dualizing terms causes further deviations from established standards, such as "opcartesian" versus "cocartesian". See also [27] for a brief discussion of this terminological difficulty.

croisé. In particular, Guitart (see [16], [17], [18]) showed that the assignment $\Phi \mapsto \int_{\circ} \Phi$ leads to a left-adjoint functor

$$\int_{\circ} : \mathbf{CAT}/\mathbf{Cat} \longrightarrow \mathbf{CAT}$$

whose right adjoint, here denoted by \mathbf{Diag}° , deserves independent interest. It assigns to a category \mathcal{X} the category $\mathbf{Diag}^{\circ}(\mathcal{X})$ of all small diagrams in \mathcal{X} which, curiously enough, may be thought of as arising via the Grothendieck construction. Indeed, with the functor $[\square, \mathcal{X}] = \mathcal{X}^{\square} : \mathbf{Cat}^{\text{op}} \rightarrow \mathbf{CAT}$ one has

$$\mathbf{Diag}^{\circ}(\mathcal{X}) = \int^{\circ} \mathcal{X}^{\square},$$

to be considered as a (fibred) category over \mathbf{Cat} ; explicitly, a morphism $(F, \varphi) : (\mathcal{I}, X) \rightarrow (\mathcal{J}, Y)$ with small categories \mathcal{I}, \mathcal{J} is given by a lax-commutative diagram

$$\begin{array}{ccc} \mathcal{I} & \xrightarrow{F} & \mathcal{J} \\ & \searrow \varphi \Rightarrow \swarrow & \\ & X & \swarrow Y \\ & & \mathcal{X} \end{array}$$

in \mathbf{CAT} .

Since one actually has an adjunction of 2-functors, the first nice, but immediate, consequence of the Guitart adjunction is the known characterization of the Grothendieck category $\int_{\circ} \Phi$ as a lax colimit of Φ in \mathbf{CAT} . Much less known, or expected, is the second consequence that we draw from the Guitart adjunction, as it gives us the connection with the CDF. Considering the functor Φ again as a diagram in \mathbf{Cat} (and, thus, renaming the category \mathcal{B} as \mathcal{D} from this viewpoint), a diagram $T : \int_{\circ} \Phi \rightarrow \mathcal{X}$ in a category \mathcal{X} corresponds to a \mathcal{D} -shaped diagram in $\mathbf{Diag}^{\circ}(\mathcal{X})$, given by a family $T(d, -) : \Phi d \rightarrow X$ ($d \in \mathcal{D}$) of diagrams in \mathcal{X} . If all these have colimits in \mathcal{X} , then one has the *Twisted Fubini Colimit Formula*:

$$\text{colim}^{(d,x) \in \int_{\circ} \Phi} T(d, x) \cong \text{colim}^{d \in \mathcal{D}} (\text{colim}^{x \in \Phi d} T(d, x)), \quad (\text{TFCF})$$

with the colimit on either side existing if the one on the other side exists. This formula appears in the Appendix of the *Memoir* [9] by Wojciech Chachólski and Jérôme Scherer. In Section 4 we explain how the TFCF implies the CDF.

1.3. A network of global 2-adjunctions

It seems peculiar that, in the Guitart adjunction, the left adjoint \int_{\circ} does not keep track of the fact that, for the \mathbf{Cat} -valued functor Φ , the total category $\int_{\circ} \Phi$ actually lives over \mathbf{Cat} . However, not forgetting this important fact, and still maintaining an adjunction, means that we should extend the functor \mathbf{Diag}° , so that it operates not just on categories, but also on functors, most generally considered as objects of the arrow category \mathbf{CAT}^2 . Accordingly, a major undertaking in this paper is the replacement of \mathbf{CAT} by \mathbf{CAT}^2 as the domain of the Guitart adjunction, as depicted on the right of the diagram

$$\mathbf{CAT} \begin{array}{c} \xleftarrow{!_{\square}} \\ \xrightarrow{\text{Dom}} \end{array} \mathbf{CAT}^2 \begin{array}{c} \xleftarrow{\mathbf{Diag}^{\circ}} \\ \xrightarrow{\int_{\circ}} \end{array} \mathbf{CAT}/\mathbf{Cat},$$

If we compose this extended 2-adjunction on the right with the rather trivial 2-adjunction on the left, one obtains back the original Guitart adjunction. The significance of the extended Guitart adjunction on the right is that it communicates well with the Grothendieck equivalence between split cofibrations with small fibres and \mathbf{Cat} -valued functors. In fact, the 2-functor \int_{\circ} above factors as

$$\mathbf{CAT}^2 \xleftarrow{\text{incl}} \mathbf{SCoFIB}_{\text{sf}} \xleftarrow[\simeq]{\int_{\circ}} \mathbf{CAT} // \mathbf{Cat} \xleftarrow{\text{incl}} \mathbf{CAT} / \mathbf{Cat} ,$$

with the (non-full) subcategory $\mathbf{SCoFIB}_{\text{sf}}$ of small-fibred split cofibrations and their cleavage-preserving morphisms in \mathbf{CAT}^2 , and with the left-adjoint (non-full) inclusion functor of the comma category $\mathbf{CAT} / \mathbf{Cat}$ into the lax comma category $\mathbf{CAT} // \mathbf{Cat}$, all considered as 2-categories. (Objects and morphisms of $\mathbf{CAT} // \mathbf{Cat}$ are defined as in $\mathbf{Diag}^{\circ}(\mathbf{Cat})$ —just specialize \mathcal{X} to \mathbf{Cat} above—, except that there is no smallness requirement for \mathcal{I}, \mathcal{J} .)

Of course, dropping the requirement of small-fibredness, we may factor the 2-functor \int_{\circ} of the extended Guitart adjunction also as

$$\mathbf{CAT}^2 \xleftarrow{\text{incl}} \mathbf{SCoFIB} \xleftarrow[\simeq]{\int_{\circ}} \mathbf{CAT} // \mathbf{CAT} \xleftarrow{\text{incl}} \mathbf{CAT} / \mathbf{Cat} .$$

Now the full inclusion on the left has become a right adjoint, with its left adjoint producing the *free split cofibration* generated by an arbitrary functor. Either way, the extended Guitart adjunction factors through the classical Grothendieck equivalence between (certain) cofibrations and (certain) indexed categories.

1.4. Organization of the paper and terminological conventions

To make the paper self-contained and to introduce notation, we collect background material on fibrational category theory and the Grothendieck construction in the Appendix (Section 9). In doing so, we clarify some essential details which don't seem to be documented explicitly in the literature.

Section 2 introduces diagram categories, shows how to compute colimits and limits in them, and presents the Guitart adjunction, with the characterization of the Grothendieck construction as a lax colimit in \mathbf{CAT} following from it. The Twisted Fubini Colimit Formula appears in Section 3, followed by three independent proofs for the Colimit Decomposition Formula in Section 4. In Sections 5–7 we present successively the extended Guitart adjunction, the Grothendieck equivalence, and finally their global interactions. In doing so, we restrict ourselves to the consideration of split (co)fibrations and functorially indexed categories, but take full account of the 2-categorical nature of the global correspondences. However, in the supplementary Section 8 we briefly show how diagram categories may be considered as 2-(co)fibered categories over \mathbf{Cat} . The expectation is that there are richer or higher-dimensional contexts (such as those recently considered in [37], [30], [30], [28], [27], [32]), in which this network of adjunctions, as well as the the decomposition formulae, may be established.

Throughout the paper, the term *category* refers to an ordinary category, also called 1-category; when a category carries a higher-dimensional structure and is considered with it, we will say so explicitly. Categories may be large (so that their objects may form a proper class), but they are always assumed to be *locally small*, so that their hom-functors take values in \mathbf{Set} . Categories whose object class is a set are called *small*. \mathbf{Cat} denotes the category of small categories, and \mathbf{CAT} is the *huge category* of all (1-)categories, which contains \mathbf{Set} and \mathbf{Cat} as particular objects. The huge

category of classes and their maps is denoted by \mathbf{SET} . These casual conventions may be made more precise and justified through the provision of Grothendieck universes.

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2. Diagram categories and the Guitart adjunction

2.1. Two types of diagram categories of a given category

Let \mathcal{X} be a category. Since every small category \mathcal{I} is, via the formation of the functor category $\mathcal{X}^{\mathcal{I}} = [\mathcal{I}, \mathcal{X}] = \mathbf{CAT}(\mathcal{I}, \mathcal{X})$, exponentiable in \mathbf{CAT} , one has the (internal hom-)functor

$$\mathcal{X}^{\square} = [\square, \mathcal{X}] : \mathbf{Cat}^{\mathrm{op}} \rightarrow \mathbf{CAT}, \quad (F : \mathcal{I} \rightarrow \mathcal{J}) \mapsto (F^* : \mathcal{X}^{\mathcal{J}} \rightarrow \mathcal{X}^{\mathcal{I}}, Y \mapsto YF).$$

Applying both, the Grothendieck construction and the dual Grothendieck construction (see Section 9.5) to \mathcal{X}^{\square} yields the *diagram categories*³

$$\mathbf{Diag}^{\circ}(\mathcal{X}) = \int^{\circ} \mathcal{X}^{\square} \quad \text{and} \quad \mathbf{Diag}_{\circ}(\mathcal{X}) = \int_{\circ} \mathcal{X}^{\square}.$$

For the reader's convenience, here is a detailed description:

Definition 2.1. The objects of the category $\mathbf{Diag}^{\circ}(\mathcal{X})$ are pairs (\mathcal{I}, X) with $X : \mathcal{I} \rightarrow \mathcal{X}$ a functor of a small category \mathcal{I} ; a morphism $(F, \varphi) : (\mathcal{I}, X) \rightarrow (\mathcal{J}, Y)$ is given by a functor $F : \mathcal{I} \rightarrow \mathcal{J}$ of small categories and a natural transformation $\varphi : X \rightarrow YF$, as depicted on the left of the diagram

$$\begin{array}{ccc} \mathcal{I} & \xrightarrow{F} & \mathcal{J} \\ & \searrow \varphi \Rightarrow \swarrow & \\ & X & \swarrow Y \\ & & \mathcal{X} \end{array} \qquad \begin{array}{ccc} \mathcal{I} & \xleftarrow{F} & \mathcal{J} \\ & \searrow \varphi \Rightarrow \swarrow & \\ & X & \swarrow Y \\ & & \mathcal{X} \end{array}$$

The category $\mathbf{Diag}_{\circ}(\mathcal{X})$ has the same objects as $\mathbf{Diag}^{\circ}(\mathcal{X})$, but a morphism $(F, \varphi) : (\mathcal{I}, X) \rightarrow (\mathcal{J}, Y)$ in $\mathbf{Diag}_{\circ}(\mathcal{X})$ is now given by a functor $F : \mathcal{J} \rightarrow \mathcal{I}$ and a natural transformation $\varphi : XF \rightarrow Y$, as depicted on the right of the above diagram.

The composite of (F, φ) followed by $(G, \psi) : (\mathcal{J}, Y) \rightarrow (\mathcal{K}, Z)$ in $\mathbf{Diag}^{\circ}(\mathcal{X})$ and $\mathbf{Diag}_{\circ}(\mathcal{X})$ is respectively given by

$$(GF, \psi F \cdot \varphi) \quad \text{and} \quad (FG, \psi \cdot \varphi G).$$

One has the obvious forgetful functors

$$D^{\mathcal{X}} : \mathbf{Diag}^{\circ}(\mathcal{X}) \rightarrow \mathbf{Cat} \quad \text{and} \quad D_{\mathcal{X}} : \mathbf{Diag}_{\circ}(\mathcal{X}) \rightarrow \mathbf{Cat}^{\mathrm{op}}$$

which remember just the top rows of the above triangles; in the notation of (9.5), they are precisely the functors $\Pi^{\mathcal{X}^{\square}}$ and $\Pi_{\mathcal{X}^{\square}}$, respectively. Consequently one has the following Proposition, which is also easily established “directly”.

³In the literature one finds the notation $\mathbf{Cat} // \mathcal{X}$ for either of these categories. We use it later on in special cases.

Proposition 2.2. $D^{\mathcal{X}}$ is a split fibration, with cleavage

$$\theta_{(\mathcal{J}, Y)}^F = (F, 1_{YF}) : F^*(\mathcal{J}, Y) = (\mathcal{I}, YF) \rightarrow (\mathcal{J}, Y) \text{ for } F : \mathcal{I} \rightarrow \mathcal{J} \text{ and } Y \in \mathcal{X}^{\mathcal{J}}.$$

Dually, $D_{\mathcal{X}}$ is a split cofibration, with cleavage

$$\delta_{(\mathcal{I}, X)}^F = (F, 1_{XF}) : (\mathcal{I}, X) \rightarrow F_!(\mathcal{I}, X) = (\mathcal{J}, XF) \text{ for } F : \mathcal{J} \rightarrow \mathcal{I} \text{ and } X \in \mathcal{X}^{\mathcal{I}}.$$

Since the two types of Grothendieck constructions are dual to each other (as described in 9.5), so are the two types of diagram categories. This fact one may easily see in a direct manner, by an application of the bijective 2-functor

$$\square^{\text{op}} : \text{CAT}^{\text{co}} \longrightarrow \text{CAT}, \quad [\alpha : S \Longrightarrow T : \mathcal{C} \rightarrow \mathcal{D}] \longmapsto [\alpha^{\text{op}} : S^{\text{op}} \Longleftarrow T^{\text{op}} : \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}^{\text{op}}]$$

(which maps morphisms covariantly but 2-cells contravariantly) to the inscribed triangle on the right of the diagram of Definition 2.1. In this way one establishes an isomorphism between the dual of the (ordinary) category $\text{Diag}_{\circ}(\mathcal{X})$ and the category $\text{Diag}^{\circ}(\mathcal{X}^{\text{op}})$, coherently so with respect to the forgetful functors, as shown in the commutative diagram

$$\begin{array}{ccc} (\text{Diag}_{\circ}(\mathcal{X}))^{\text{op}} & \xrightarrow[\cong]{\square^{\text{op}}} & \text{Diag}^{\circ}(\mathcal{X}^{\text{op}}) \\ (D_{\mathcal{X}})^{\text{op}} \downarrow & & \downarrow D(\mathcal{X}^{\text{op}}) \\ \text{Cat} & \xrightarrow[\cong]{\square^{\text{op}}} & \text{Cat} . \end{array}$$

The objects and morphisms of \mathcal{X} may be considered as living in both, $\text{Diag}^{\circ}(\mathcal{X})$ and $\text{Diag}_{\circ}(\mathcal{X})$. Indeed, there are full embeddings

$$E^{\mathcal{X}} : \mathcal{X} \rightarrow \text{Diag}^{\circ}(\mathcal{X}) \quad \text{and} \quad E_{\mathcal{X}} : \mathcal{X} \rightarrow \text{Diag}_{\circ}(\mathcal{X})$$

which interpret every object X of \mathcal{X} as a functor $1 \rightarrow \mathcal{X}$ of the terminal category 1 and every morphism $f : X \rightarrow Y$ in \mathcal{X} as a natural transformation, giving respectively the (hardly distinguishable) morphisms

$$\begin{array}{ccc} 1 & \xrightarrow{\text{Id}_1} & 1 \\ & \searrow f \Longrightarrow \swarrow & \\ & X & Y \\ & \searrow & \swarrow \\ & \mathcal{X} & \end{array} \quad \begin{array}{ccc} 1 & \xleftarrow{\text{Id}_1} & 1 \\ & \swarrow f \Longrightarrow \searrow & \\ & X & Y \\ & \swarrow & \searrow \\ & \mathcal{X} & \end{array}$$

in $\text{Diag}^{\circ}(\mathcal{X})$ and $\text{Diag}_{\circ}(\mathcal{X})$. $E^{\mathcal{X}}$ and $E_{\mathcal{X}}$ cooperate with the dualization isomorphism for the diagram categories, as shown in the commutative diagram

$$\begin{array}{ccc} & \mathcal{X}^{\text{op}} & \\ (E_{\mathcal{X}})^{\text{op}} \swarrow & & \searrow (E^{\mathcal{X}})^{\text{op}} \\ (\text{Diag}_{\circ}(\mathcal{X}))^{\text{op}} & \xrightarrow[\cong]{\square^{\text{op}}} & \text{Diag}^{\circ}(\mathcal{X}^{\text{op}}) \end{array}$$

With (co)completeness to be understood to mean that every small diagram comes with a choice of (co)limit, one has the following “folklore” proposition, whose routine proof we may skip.

- Proposition 2.3.** (1) *The category \mathcal{X} is functorially cocomplete if, and only if, $E^{\mathcal{X}}$ is a reflective embedding (with left adjoint $\text{colim} : \text{Diag}^{\circ}(\mathcal{X}) \rightarrow \mathcal{X}$).*
(2) *The category \mathcal{X} is functorially complete if, and only if, $E_{\mathcal{X}}$ is a coreflective embedding (with right adjoint $\text{lim} : \text{Diag}_{\circ}(\mathcal{X}) \rightarrow \mathcal{X}$).*

Remark 2.4. (1) We emphasize that, here, we are considering $\text{Diag}^{\circ}(\mathcal{X})$ and $\text{Diag}_{\circ}(\mathcal{X})$, just like \mathcal{X} , as 1-categories and, thus, ignore their obvious 2-categorical structures which make $D^{\mathcal{X}}$ and $D_{\mathcal{X}}$ 2-functors. In the case of $\text{Diag}^{\circ}(\mathcal{X})$, a 2-cell $\alpha : (F, \varphi) \Rightarrow (F', \varphi')$ is simply a natural transformation $\alpha : F \rightarrow F'$ with $Y\alpha \cdot \varphi = \varphi'$:

$$\begin{array}{ccc} \mathcal{I} & \begin{array}{c} \xrightarrow{(F, \varphi)} \\ \alpha \Downarrow \\ \xrightarrow{(F', \varphi')} \end{array} & \mathcal{J} \\ & \begin{array}{c} \searrow X \\ \swarrow Y \end{array} & \\ & \mathcal{X} & \end{array}$$

As $D^{\mathcal{X}}$ should preserve the horizontal and vertical compositions of 2-cells, there is no choice of how to define them in $\text{Diag}^{\circ}(\mathcal{X})$. All verifications proceed routinely.

(2) Likewise, at this point we ignore the obvious fact that the functor $\mathcal{X}^{\square} = [-, \mathcal{X}] : \text{Cat}^{\text{op}} \rightarrow \text{CAT}$, as considered at the beginning of this section, is actually a 2-functor: it maps every natural transformation $\alpha : F \Rightarrow F'$ (covariantly) to the transformation $\alpha^* : F^* \Rightarrow F'^*$ with $\alpha_Y^* = Y\alpha : YF \rightarrow YF'$, preserving both the vertical and horizontal composition of natural transformations.

(3) With 1 denoting the terminal category we trivially have

$$\text{Diag}^{\circ}(\emptyset) \cong \text{Diag}_{\circ}(\emptyset) \cong 1 \text{ and } \text{Diag}^{\circ}(1) \cong \text{Cat}, \text{Diag}_{\circ}(1) \cong \text{Cat}^{\text{op}}.$$

Much less obvious is the fact that the (ordinary) category $\text{Diag}^{\circ}(\text{Cat})$ is equivalent to a suitably defined category which has the split cofibrations of small categories as its objects, and that $\text{Diag}_{\circ}(\text{Cat}^{\text{op}})$ is equivalent to the dual of the category of split fibrations of small categories, as one may conclude from the 2-categorical equivalence formulated in Corollary 6.6.

2.2. Limits and colimits in diagram categories

We continue to work with a fixed category \mathcal{X} and now consider the question of how to form limits or colimits in its diagram categories. Without any use of fibrational methods, one confirms the following assertions:

- Proposition 2.5.** (1) *$\text{Diag}^{\circ}(\mathcal{X})$ has coproducts and equalizers, and $D^{\mathcal{X}}$ preserves them. If $\text{Diag}^{\circ}(\mathcal{X})$ has coequalizers, then so does \mathcal{X} , and $E^{\mathcal{X}}$ preserves them.*
(2) *$\text{Diag}_{\circ}(\mathcal{X})$ has products and coequalizers, and $D_{\mathcal{X}}$ preserves them. If $\text{Diag}_{\circ}(\mathcal{X})$ has equalizers, then so does \mathcal{X} , and $E_{\mathcal{X}}$ preserves them.*

The question of how to form coequalizers in $\text{Diag}^{\circ}(\mathcal{X})$ (or, equivalently, equalizers in $\text{Diag}_{\circ}(\mathcal{X})$) is best addressed with fibrational methods, as follows. For every functor $F : \mathcal{I} \rightarrow \mathcal{J}$ of small categories, the functor $F^* : \mathcal{X}^{\mathcal{J}} \rightarrow \mathcal{X}^{\mathcal{I}}$ has, by definition, a left adjoint $F_!$ if, and only if, for every $X \in \mathcal{X}^{\mathcal{I}}$, a (chosen) *left Kan extension* $F_!X = \text{Lan}_F X$ of X along F exists. The existence of this extension is certainly guaranteed when \mathcal{X} is cocomplete; conversely, a colimit of $X : \mathcal{I} \rightarrow \mathcal{X}$ can be obtained as the left Kan extension of X along the functor $\mathcal{I} \rightarrow 1$ to the terminal category. Consequently, with Proposition 2.2 and Theorem 9.1 we obtain:

Proposition 2.6. *The split fibration $D^{\mathcal{X}}$ is a bifibration if, and only if, the category \mathcal{X} is (small-) cocomplete. Dually, the split cofibration $D_{\mathcal{X}}$ is a bifibration if, and only if, \mathcal{X} is (small-)complete.*

With the help of Theorem 9.3, Corollary 9.4, and Proposition 2.6 we can now state:

Theorem 2.7. (1) *If \mathcal{X} is a cocomplete category, then $\text{Diag}^{\circ}(\mathcal{X})$ is also cocomplete, and $D^{\mathcal{X}}$ preserves all colimits. Furthermore, if \mathcal{X} has limits of shape \mathcal{D} , so does $\text{Diag}^{\circ}(\mathcal{X})$, and $D^{\mathcal{X}}$ preserves them.*

(2) *If \mathcal{X} is complete, then $\text{Diag}_{\circ}(\mathcal{X})$ is also complete, and $D_{\mathcal{X}}$ preserves all limits (so that $D_{\mathcal{X}}$ transforms limits in $\text{Diag}_{\circ}(\mathcal{X})$ into colimits in Cat). Furthermore, if \mathcal{X} has colimits of shape \mathcal{D} , so does $\text{Diag}_{\circ}(\mathcal{X})$, and $D_{\mathcal{X}}$ preserves them (i.e., transforms limits into colimits in Cat).*

Proof. (1) Cocompleteness of \mathcal{X} is needed to make $D^{\mathcal{X}}$ a bifibration. Its fibres, i.e., the functor categories of \mathcal{X} , have the types of (co)limits that \mathcal{X} has. Since the base category, Cat , is bicomplete, the assertions follow with the statements referred to above.

(2) With the dualization principle as stated before Proposition 2.3, item (2) follows from an application of (1) to \mathcal{X}^{op} , rather than to \mathcal{X} . \square

2.3. Strictification of morphisms in diagram categories

Other than containing \mathcal{X} as a full subcategory, $\text{Diag}^{\circ}(\mathcal{X})$ contains, of course, also the ordinary comma category Cat/\mathcal{X} as a *non-full* subcategory. In fact, for *small* (!) \mathcal{X} one has the following proposition, embedded in the proof of Lemma 7.13 of [2], with credit given to S. Lack. For arbitrary \mathcal{X} , see Remark 2.9(2); we can still draw the essential conclusion of preservation of colimits (see Corollary 2.10) that is needed in the first proof of the CDF (see Section 4).

Proposition 2.8. *For a small category \mathcal{X} , the inclusion functor $\text{Cat}/\mathcal{X} \rightarrow \text{Diag}^{\circ}(\mathcal{X})$ has a right adjoint.*

Proof. One defines a functor

$$\text{Strict} : \text{Diag}^{\circ}(\mathcal{X}) \rightarrow \text{Cat}/\mathcal{X},$$

which transforms lax commutative triangles into strictly commutative triangles. It assigns to an object X in $\text{Diag}^{\circ}(\mathcal{X})$ the comma category $\mathcal{X} \downarrow X$, equipped with its domain functor which takes an object $(u : a \rightarrow Xi, i)$ in $\mathcal{X} \downarrow X$ with $i \in \mathcal{I}$ to the object $a \in \mathcal{X}$. On morphisms it is defined by

$$\begin{array}{ccc} \begin{array}{ccc} \mathcal{I} & \xrightarrow{F} & \mathcal{J} \\ & \varphi \Rightarrow & \\ X & \searrow & \swarrow Y \\ & \mathcal{X} & \end{array} & \xmapsto{\text{Strict}} & \begin{array}{ccc} \mathcal{X} \downarrow X & \xrightarrow{\text{Strict}(F, \varphi)} & \mathcal{X} \downarrow Y \\ & \text{dom}_X \searrow & \swarrow \text{dom}_Y \\ & \mathcal{X} & \end{array} \end{array}$$

where the functor $\text{Strict}(F, \varphi)$ takes an object $(u : a \rightarrow Xi, i)$ in $\mathcal{X} \downarrow X$ with $i \in \mathcal{I}$ to the object $(\varphi_i \cdot u : a \rightarrow Y(Fi), Fi)$ in $\mathcal{X} \downarrow Y$. It is now a routine exercise to establish a natural isomorphism

$$(\text{Cat}/\mathcal{X})(X, \text{dom}_Y) \cong \text{Diag}^{\circ}(\mathcal{X})(X, Y)$$

in Set . \square

Remark 2.9. (1) With 2-cells $\alpha : F \Rightarrow F'$ given by natural transformations α satisfying $Y\alpha = 1_X$,

$$\begin{array}{ccc}
\mathcal{I} & \begin{array}{c} \xrightarrow{F} \\ \alpha \Downarrow \\ \xrightarrow{F'} \end{array} & \mathcal{J} \\
& \searrow X \quad \swarrow Y & \\
& \mathcal{X} &
\end{array}$$

\mathbf{Cat}/\mathcal{X} becomes a 2-category. In fact, it is the 2-subcategory of $\mathbf{Diag}^\circ(\mathcal{X})$ whose 2-cells are described in Remark 2.4(1). One easily confirms that the isomorphism at the end of the proof of Proposition 2.8 actually lives in \mathbf{Cat} . Consequently, one has in fact a 2-adjunction

$$\text{Inclusion} \dashv \text{Strict} : \mathbf{Diag}^\circ(\mathcal{X}) \rightarrow \mathbf{Cat}/\mathcal{X}.$$

(2) When \mathcal{X} is large, *i.e.*, a \mathbf{CAT} -object, we still have a right adjoint, Strict , to the inclusion 2-functor

$$\mathbf{CAT}/\mathcal{X} \rightarrow \mathbf{DIAG}^\circ(\mathcal{X}),$$

where $\mathbf{DIAG}^\circ(\mathcal{X})$ is defined like $\mathbf{Diag}^\circ(\mathcal{X})$, except that the domain \mathcal{I} of an object $X : \mathcal{I} \rightarrow \mathcal{X}$ is not constrained to be small. We temporarily step into this higher universe to prove the following corollary, which will be used in the proof of Theorem 4.1.

Corollary 2.10. *For any category \mathcal{X} , the inclusion functor $\mathbf{Cat}/\mathcal{X} \rightarrow \mathbf{Diag}^\circ(\mathcal{X})$ preserves all colimits.*

Proof. Trivially, the full embedding $\mathbf{Cat} \rightarrow \mathbf{CAT}$ preserves all colimits and, hence, so does $\mathbf{Cat}/\mathcal{X} \rightarrow \mathbf{CAT}/\mathcal{X}$. By (a large version of) Proposition 2.8, a colimit taken in \mathbf{Cat}/\mathcal{X} is therefore also a colimit in $\mathbf{DIAG}^\circ(\mathcal{X})$, and it trivially maintains that role in its “home” category $\mathbf{Diag}^\circ(\mathcal{X})$. \square

Remark 2.11. We note that, in the notation of 9.2, every small-fibred split fibration $P : \mathcal{E} \rightarrow \mathcal{B}$ comes with a *mate*

$$P^\leftarrow : \mathcal{B} \rightarrow \mathbf{Diag}_\circ(\mathcal{E}), \quad (u : a \rightarrow b) \mapsto ((u^*, \theta^u) : J_a \rightarrow J_b),$$

which, when composed with the split fibration $D_\mathcal{E} : \mathbf{Diag}_\circ(\mathcal{E}) \rightarrow \mathbf{CAT}^{\text{op}}$, reproduces the functor $(\Phi^P)^{\text{op}} : \mathcal{B} \rightarrow \mathbf{CAT}^{\text{op}}$ of 9.3. Although Φ^P maintains sufficient information about P to reproduce P (as in Proposition 9.5), the mate P^\leftarrow may well be regarded as doing so more comprehensively.

2.4. The Guitart adjunction

It is hardly surprising that $\mathbf{Diag}^\circ(\mathcal{X})$, constructed as a Grothendieck category over \mathbf{Cat} , behaves 2-functorially in the variable \mathcal{X} . But it is a nice twist that the assignment $\mathcal{X} \mapsto \mathbf{Diag}^\circ(\mathcal{X})$ (considered as a category over \mathbf{Cat}) has a left adjoint, given again by the Grothendieck construction. This fact was stated by Guitart [17] (see also [18]) in 1-categorical terms. In what follows, we give some details in 2-categorical terms. A generalization is formulated, and proved, as Theorem 5.4.

Considering \mathbf{Cat} as a 1-category and \mathbf{CAT} as a (huge) 2-category, containing \mathbf{Cat} as one of its objects, we form the 2-category $\mathbf{CAT}/\mathbf{Cat}$ as in Remark 2.9(1). Then the 2-functor

$$\mathbf{Diag}^\circ : \mathbf{CAT} \longrightarrow \mathbf{CAT}/\mathbf{Cat}$$

assigns to a category \mathcal{X} the fibration $D^\mathcal{X} : \mathbf{Diag}^\circ(\mathcal{X}) \rightarrow \mathbf{Cat}$; it extends a functor $T : \mathcal{X} \rightarrow \mathcal{Y}$ from ordinary to “variable” objects of \mathcal{X} by post-composition with T , that is: \mathbf{Diag}° assigns to T the $\mathbf{CAT}/\mathbf{Cat}$ -morphism $D^\mathcal{X} \rightarrow D^\mathcal{Y}$, given by the functor

$$T(-) : \mathbf{Diag}^\circ(\mathcal{X}) \longrightarrow \mathbf{Diag}^\circ(\mathcal{Y}), \quad [(F, \varphi) : (\mathcal{I}, X) \rightarrow (\mathcal{J}, Y)] \longmapsto [(F, T\varphi) : (\mathcal{I}, TX) \rightarrow (\mathcal{J}, TY)];$$

and it assigns to a natural transformation $\tau : T \rightarrow T'$ the natural transformation $\tau(-) : T(-) \rightarrow T'(-)$, given by $(\text{Id}_{\mathcal{I}}, \tau X) : (\mathcal{I}, TX) \rightarrow (\mathcal{I}, T'X)$ for all objects (\mathcal{I}, X) in $\text{Diag}^\circ(\mathcal{X})$.

The 2-functor

$$\text{CAT} \longleftarrow \text{CAT}/\text{Cat} : \int_{\circ}$$

assigns to a CAT/Cat -object $\Phi : \mathcal{B} \rightarrow \text{Cat}$ its dual Grothendieck category $\int_{\circ} \Phi$, and to a CAT/Cat -morphism $\Sigma : \Phi \rightarrow \Psi$ with $\Psi : \mathcal{C} \rightarrow \text{Cat}$ the functor

$$(\Sigma -, =) : \int_{\circ} \Phi \longrightarrow \int_{\circ} \Psi, \quad [(u, f) : (a, x) \rightarrow (b, y)] \longmapsto [(\Sigma u, f) : (\Sigma a, x) \rightarrow (\Sigma b, y)].$$

A natural transformation $\sigma : \Sigma \rightarrow \Sigma'$ with $\Psi\sigma = 1_{\Phi}$ is sent by \int_{\circ} to the natural transformation $(\Sigma -, =) \rightarrow (\Sigma' -, =)$ whose component at an object (a, x) in $\int_{\circ} \Phi$ is the morphism $(\sigma_a, 1_{(\Phi a)_x}) : (\Sigma a, x) \rightarrow (\Sigma' a, x)$ in $\int_{\circ} \Psi$.

Theorem 2.12. *The 2-functor \int_{\circ} is left adjoint to the 2-functor Diag° .*

Proof. It suffices to show that, for every category \mathcal{X} and every functor $\Phi : \mathcal{B} \rightarrow \text{Cat}$, one has bijective functors

$$\hat{\square} : \text{CAT}(\int_{\circ} \Phi, \mathcal{X}) \rightleftarrows (\text{CAT}/\text{Cat})(\Phi, D^{\mathcal{X}}) : \check{\square}$$

that are natural in \mathcal{X} and Φ . To this end, for a functor $T : \int_{\circ} \Phi \rightarrow \mathcal{X}$, one lets the functor $\hat{T} : \mathcal{B} \rightarrow \text{Diag}(\mathcal{X})$ map an object $a \in \mathcal{B}$ to the functor

$$T_a : \Phi a \rightarrow \mathcal{X}, \quad (f : x \rightarrow x') \longmapsto [T(1_a, f) : (T(a, x) \rightarrow T(a, x'))].$$

\hat{T} maps a morphism $u : a \rightarrow b$ in \mathcal{B} to the $\text{Diag}^\circ(\mathcal{X})$ -morphism

$$\begin{array}{ccc} \Phi a & \xrightarrow{\Phi u} & \Phi b \\ & \varphi^u : \Longrightarrow & \\ T_a \searrow & & \swarrow T_b \\ & \mathcal{X} & \end{array}$$

where the natural transformation φ^u is defined by $\varphi_x^u = T(u, 1_{\Phi u(x)})$, for all $x \in \Phi a$; clearly then, $D^{\mathcal{X}}\hat{T} = \Phi$. Also, for a natural transformation $\tau : T \rightarrow T'$ one has the 2-cell $\hat{\tau} : \hat{T} \rightarrow \hat{T}'$, the components of which are the $\text{Diag}^\circ(\mathcal{X})$ -morphisms

$$\begin{array}{ccc} \Phi a & \xrightarrow{\text{Id}_{\Phi a}} & \Phi a \\ & \hat{\tau}_a : \Longrightarrow & \\ T_a \searrow & & \swarrow T'_a \\ & \mathcal{X} & \end{array},$$

defined by $(\hat{\tau}_a)_x = \tau_{(a, x)}$, for all $a \in \mathcal{B}, x \in \Phi a$.

Conversely, for a functor $\Sigma : \mathcal{B} \rightarrow \text{Diag}(\mathcal{X})$ with $D^{\mathcal{X}}\Sigma = \Phi$, one defines the functor $\check{\Sigma} : \int_{\circ} \Phi \rightarrow \mathcal{X}$, as follows. For $u : a \rightarrow b$ in \mathcal{B} , writing the $\text{Diag}(\mathcal{X})$ -morphism Σu in the form $\Sigma u = (\Phi u, \varphi^u)$ (as in the triangle on the left of the following diagram), one lets $\check{\Sigma}$ map a morphism $(u, f) : (a, x) \rightarrow (b, y)$ in $\int_{\circ} \Phi$ to the composite morphism of the triangle on the right:

$$\begin{array}{ccc}
\Phi a & \xrightarrow{\Phi u} & \Phi b \\
& \searrow \Sigma a \quad \varphi^u : \Rightarrow & \swarrow \Sigma b \\
& \mathcal{X} &
\end{array}
\qquad
\begin{array}{ccc}
& & (\Sigma b)(\Phi u)x \\
& \nearrow \varphi_x^u & \searrow (\Sigma b)f \\
(\Sigma a)x = \check{\Sigma}(a, x) & \xrightarrow{\check{\Sigma}(u, f)} & \check{\Sigma}(b, y) = (\Sigma b)y .
\end{array}$$

Not surprisingly now, a natural transformation $\sigma : \Sigma \rightarrow \Sigma'$ with $D^{\mathcal{X}}\sigma = 1_{\Phi}$ gives the natural transformation $\check{\sigma} : \check{\Sigma} \rightarrow \check{\Sigma}'$, defined by $\check{\sigma}_{(a, x)} = (\sigma_a)_x$, for all $(a, x) \in \int_{\circ} \Phi$.

All verifications proceed in a standard manner. \square

Let us make explicit how, for a functor $\Phi : \mathcal{B} \rightarrow \mathbf{Cat}$, Theorem 2.12 provides an effective characterization of the category $\int_{\circ} \Phi$ in the 2-category \mathbf{CAT} . A *lax cocone* over Φ with vertex \mathcal{X} is given by a family of functors $\Sigma_a : \Phi a \rightarrow \mathcal{X}$ ($a \in \mathcal{B}$) and a family of natural transformations $\varphi^u : \Sigma_a \rightarrow \Sigma_b(\Phi u)$ ($u : a \rightarrow b$ in \mathcal{B}), satisfying the conditions

$$\varphi^{1_a} = 1_{\Sigma_a} \quad \text{and} \quad \varphi^{v \cdot u} = \varphi^v(\Phi u) \cdot \varphi^u,$$

for all $u : a \rightarrow b, v : b \rightarrow c$ in \mathcal{B} . We recall that the category $\int_{\circ} \Phi$ is the vertex of the lax cocone over Φ , given by the functors

$$J_a : \Phi a \rightarrow \int_{\circ} \Phi, \quad (h : x \rightarrow x') \mapsto (1_a, h) : (a, x) \rightarrow (a, x'),$$

and the natural transformations

$$\delta^u : J_a \rightarrow J_b(\Phi u), \quad \text{with } \delta_x^u = (u, 1_{\Phi u(x)}) : (a, x) \rightarrow (b, (\Phi u)x),$$

for all $u : a \rightarrow b$ in \mathcal{B} and $x \in \Phi a$ (see 9.3). This lax cocone is initial amongst all lax cocones over Φ , in the following sense:

Corollary 2.13 (Lax Colimit Characterization of $\int_{\circ} \Phi$). *For every lax cocone over Φ , given by $(\Sigma_a : \Phi a \rightarrow \mathcal{X})_{a \in \mathcal{B}}, (\varphi^u : \Sigma_a \rightarrow \Sigma_b(\Phi u))_{u : a \rightarrow b}$, there is a uniquely determined functor $T : \int_{\circ} \Phi \rightarrow \mathcal{X}$ with $TJ_a = \Sigma_a$ and $T\delta^u = \varphi^u$, for all $u : a \rightarrow b$ in \mathcal{B} .*

$$\begin{array}{ccc}
\Phi a & \xrightarrow{\Phi u} & \Phi b \\
& \searrow J_a \quad \delta^u : \Rightarrow & \swarrow J_b \\
& \int_{\circ} \Phi & \\
& \searrow \Sigma_a \quad \varphi^u : \Rightarrow & \swarrow \Sigma_b \\
& \mathcal{X} &
\end{array}$$

(Note: A vertical dashed arrow labeled T points from $\int_{\circ} \Phi$ to \mathcal{X} .)

Proof. The lax cocone $(J_a), (\delta^u)$ describes the adjunction unit $J : \mathcal{B} \rightarrow \mathbf{Diag}^{\circ}(\int_{\circ} \Phi)$, and the corollary just paraphrases its universal property, as indicated by the diagram

$$\begin{array}{ccc}
\mathcal{B} & \xrightarrow{J} & \mathbf{Diag}^{\circ}(\int_{\circ} \Phi) \\
& \searrow \Sigma & \downarrow T(-) \\
& & \mathbf{Diag}^{\circ}(\mathcal{X})
\end{array}
\qquad
\begin{array}{c}
\int_{\circ}(\Phi) \\
\downarrow T (= \check{\Sigma}) \\
\mathcal{X} .
\end{array}$$

□

The dualization of Corollary 2.13 for a functor $\Phi : \mathcal{B}^{\text{op}} \rightarrow \text{Cat}$ reads as follows:

Corollary 2.14 (Oplax Colimit Characterization of $\int^{\circ} \Phi$). *For every oplax cocone over Φ , given by $(\Sigma_a : \Phi a \rightarrow \mathcal{X})_{a \in \mathcal{B}}$, $(\varphi^u : \Sigma_a(\Phi u) \rightarrow \Sigma_b)_{u: a \rightarrow b}$, there is a uniquely determined functor $T : \int^{\circ} \Phi \rightarrow \mathcal{X}$ with $TJ_a = \Sigma_a$ and $T\theta^u = \varphi^u$, for all $u : a \rightarrow b$ in \mathcal{B} .*

Remark 2.15. (1) A “direct” proof of Corollary 2.14 makes essential use of the (vertical, Π^{Φ} -cartesian)–factorization $(u, f) = (u, 1_{\Phi u(y)}) \cdot (1_a, f) = \theta_y^u \cdot J_a f$ of every morphism $(u, f) : (a, x) \rightarrow (b, y)$ in $\int^{\circ} \Phi$. Likewise for Corollary 2.13.

(2) Of course, Corollaries 2.13 and 2.14 remain valid *verbatim* if the functor Φ is CAT-valued (rather than Cat-valued).

As another important consequence of Theorem 2.12 we note:

Corollary 2.16. $\text{Diag}^{\circ} : \text{CAT} \rightarrow \text{CAT}/\text{Cat}$ preserves all (weighted) limits, and its left adjoint \int_{\circ} preserves all (weighted) colimits.

2.5. Diag° has the structure of a normal pseudomonad on CAT

That Diag° belongs to a (pseudo-)monad on CAT was already observed in [16]. But since a detailed exposition of this claim, even in a 1-categorical form, does not seem to be readily accessible, we outline the construction of the monad; a more detailed exposition and proof of the claim as given by this section’s header appears in [31].

Our 2-functor

$$\text{Diag}^{\circ} : \text{CAT} \longrightarrow \text{CAT}, \quad \mathcal{X} \longmapsto \text{Diag}^{\circ}(\mathcal{X}),$$

arises by post-composing the right-adjoint of Theorem 2.12 with the forgetful 2-functor $\text{CAT}/\text{Cat} \rightarrow \text{CAT}$. The full embedding $E^{\mathcal{X}} : \mathcal{X} \rightarrow \text{Diag}^{\circ}(\mathcal{X})$ of 2.2 may then be considered as the \mathcal{X} -component of a (strictly) 2-natural transformation

$$E : \text{Id}_{\text{CAT}} \longrightarrow \text{Diag}^{\circ}$$

since, as one easily confirms, every natural transformation $\alpha : F \rightarrow F' : \mathcal{X} \rightarrow \mathcal{Y}$ satisfies

$$\text{Diag}^{\circ}(F)E^{\mathcal{X}} = E^{\mathcal{Y}}F \quad \text{and} \quad \text{Diag}^{\circ}(\alpha)E^{\mathcal{X}} = E^{\mathcal{Y}}\alpha.$$

In order to establish Diag° as the carrier of a pseudo-monad, we now define for every category \mathcal{X} a functor

$$M^{\mathcal{X}} : \text{Diag}^{\circ}(\text{Diag}^{\circ}(\mathcal{X})) \longrightarrow \text{Diag}^{\circ}(\mathcal{X}).$$

An object in $\text{Diag}^{\circ}(\text{Diag}^{\circ}(\mathcal{X}))$ is a functor $\Sigma : \mathcal{B} \rightarrow \text{Diag}^{\circ}(\mathcal{X})$ with \mathcal{B} small so that, with $\Phi := D^{\mathcal{X}}\Sigma : \mathcal{B} \rightarrow \text{Cat}$, for every object a in \mathcal{B} one has a functor $\Sigma a : \Phi a \rightarrow \mathcal{X}$, and for every morphism $u : a \rightarrow b$ in \mathcal{B} a morphism $(\Phi u, \sigma^u) : \Sigma a \rightarrow \Sigma b$ in $\text{Diag}^{\circ}(\mathcal{X})$. Considering Σ as a lax cocone with vertex \mathcal{X} , by Corollary 2.13 we may represent Σ equivalently as a functor

$$M^{\mathcal{X}}(\Sigma) := \tilde{\Sigma} : \int_{\circ}(\Phi) \longrightarrow \mathcal{X}, \quad [(u, f) : (a, x) \rightarrow (b, y)] \longmapsto [\Sigma b(f) \cdot \sigma_x^u : \Sigma a(x) \rightarrow \Sigma b(y)].$$

A $\text{Diag}^{\circ}(\text{Diag}^{\circ}(\mathcal{X}))$ -morphism $(S, \tau) : \Sigma \rightarrow \Xi$ with codomain $\Xi : \mathcal{C} \rightarrow \text{Diag}^{\circ}(\mathcal{X})$ is given by a functor $S : \mathcal{B} \rightarrow \mathcal{C}$ of small categories and a natural transformation $\tau : \Sigma \rightarrow \Xi S$ whose component at $a \in \mathcal{B}$ is, in turn, given by a $\text{Diag}^{\circ}(\mathcal{X})$ -morphism $(R_a, \rho^a) : \Sigma a \rightarrow \Xi a$, as in

$$\begin{array}{ccc}
\Phi a & \xrightarrow{R_a} & \Psi(Sa) \\
& \searrow \Sigma a & \swarrow \Xi a \\
& \mathcal{X} &
\end{array}
\quad \rho^a : \Rightarrow$$

where $\Psi := D^{\mathcal{X}}\Xi$. Now one lets $M^{\mathcal{X}}$ assign to (S, τ) the $\text{Diag}^\circ(\mathcal{X})$ -morphism $(\check{S}, \check{\tau})$, as shown by

$$\begin{array}{ccc}
\mathcal{B} & \xrightarrow{S} & \mathcal{C} \\
& \searrow \Sigma & \swarrow \Xi \\
& \text{Diag}^\circ(\mathcal{X}) &
\end{array}
\quad \tau : \Rightarrow
\quad \mapsto \quad
\begin{array}{ccc}
\int_{\circ} \Phi & \xrightarrow{\check{S} = \text{Gr}_{\circ}(S, D^{\mathcal{X}}\tau)} & \int_{\circ} \Psi \\
& \searrow \check{\Sigma} & \swarrow \check{\Xi} \\
& \mathcal{X} &
\end{array}
\quad \check{\tau} : \Rightarrow$$

where the functor \check{S} and the natural transformation $\check{\tau}$ are defined by

$$\check{S}(u, f) = (Su, R_b f) \quad \text{and} \quad \check{\tau}_{(a, x)} = \rho_x^a,$$

for all morphisms $(u, f) : (a, x) \rightarrow (b, y)$ in $\int_{\circ} \Phi$.

3. The twisted Fubini formulae for limits and colimits

We now exploit the adjunction of Theorem 2.12 for the computation of colimits of those diagrams in a category \mathcal{X} whose shape is the dual Grothendieck category of a functor $\Phi : \mathcal{D} \rightarrow \text{Cat}$. Such a diagram $T : \int_{\circ} \Phi \rightarrow \mathcal{X}$ in \mathcal{X} corresponds equivalently to a diagram $\hat{T} : \mathcal{D} \rightarrow \text{Diag}^\circ(\mathcal{X})$ in $\text{Diag}^\circ(\mathcal{X})$ with $D^{\mathcal{X}}\hat{T} = \Phi$. As we will show in Theorem 3.2, the colimit of \hat{T} facilitates the computation of the colimit of T . The essence of its proof lies in the next lemma, for which we use the following notation. Given $\Phi : \mathcal{D} \rightarrow \text{Cat}$ and \mathcal{X} , every functor $F : \mathcal{D} \rightarrow \mathcal{X}$ gives us trivially the functor

$$\tilde{F} : \mathcal{D} \rightarrow \text{Diag}^\circ(\mathcal{X}), \quad (u : d \rightarrow e) \mapsto \left(\begin{array}{ccc} \Phi d & \xrightarrow{\Phi u} & \Phi e \\ & \searrow \Delta F u & \swarrow \Delta F e \\ & \mathcal{X} & \end{array} \right)$$

with $D^{\mathcal{X}}\tilde{F} = \Phi$. (As usual, we use Δ for constant-value functors and transformations.) For a natural transformation $\alpha : F \rightarrow F'$, one defines a natural transformation $\tilde{\alpha} : \tilde{F} \rightarrow \tilde{F}'$ with $D^{\mathcal{X}}\tilde{\alpha} = 1_\Phi$ whose components are $\tilde{\alpha}_d = (\text{Id}_{\Phi d}, \Delta \alpha_d)$. This defines the functor

$$\tilde{\square} : \text{CAT}(\mathcal{D}, \mathcal{X}) \rightarrow (\text{CAT}/\text{Cat})(\Phi, D^{\mathcal{X}}).$$

Lemma 3.1. *For every functor $\Phi : \mathcal{D} \rightarrow \text{Cat}$ and every category \mathcal{X} , the functor $\tilde{\square}$ makes the diagram*

$$\begin{array}{ccc}
\text{CAT}(\int_{\circ} \Phi, \mathcal{X}) & \xrightarrow[\cong]{\tilde{\square}} & (\text{CAT}/\text{Cat})(\Phi, D^{\mathcal{X}}) \\
\uparrow \Delta & & \uparrow \tilde{\square} \\
\mathcal{X} & \xrightarrow{\Delta} & \text{CAT}(\mathcal{D}, \mathcal{X})
\end{array}$$

commute. If, for all $d \in \mathcal{D}$, the category \mathcal{X} is Φd -cocomplete, then $\tilde{\square}$ has a left adjoint.

Proof. Checking the commutativity of the diagram is a routine matter. In order to construct a left adjoint $\tilde{\square} \dashv \square$, assuming \mathcal{X} to be Φd -cocomplete and using the same notation as in the proof of Theorem 2.12, for a functor $\Sigma : \mathcal{D} \rightarrow \text{Diag}^\circ(\mathcal{X})$ with $D^\mathcal{X}\Sigma = \Phi$, we define $\bar{\Sigma} : \mathcal{D} \rightarrow \mathcal{X}$ on objects by

$$\bar{\Sigma}d = \text{colim}(\Sigma_d : \Phi d \rightarrow \mathcal{X});$$

this definition extends canonically to morphisms. (Of course, for \mathcal{X} cocomplete, $\bar{\Sigma}$ is the composite functor $\text{colim} \circ \Sigma$, with $\text{colim} \dashv E_\mathcal{X} : \mathcal{X} \rightarrow \text{Diag}^\circ(\mathcal{X})$, as in Proposition 2.3.) For every functor $F : \mathcal{D} \rightarrow \mathcal{X}$ one now obtains a natural bijection

$$\text{CAT}(\mathcal{D}, \mathcal{X})(\bar{\Sigma}, F) \rightarrow (\text{CAT}/\text{Cat})(\Phi, D^\mathcal{X})(\Sigma, \tilde{F}),$$

which associates with a natural transformation $\alpha : \bar{\Sigma} \rightarrow F$ its mate $\alpha^\sharp : \Sigma \rightarrow \tilde{F}$, as follows: for every $d \in \mathcal{D}$, the natural transformation $\alpha_d^\sharp : S_d \rightarrow \Delta Fd : \Phi d \rightarrow \mathcal{X}$ is simply the composite transformation

$$\Sigma_d \longrightarrow \Delta(\text{colim}(\Sigma_d)) \xrightarrow{\Delta\alpha_d} \Delta Fd.$$

This confirms the adjunction. \square

With the notation used in the proofs of Theorem 2.12 and of Proposition 3.1 one now obtains a general Fubini-type colimit formula that seems to have been proved first by Chachólski and Scherer [9, 40.2], as follows:

Theorem 3.2 (Twisted Fubini Colimit Formula). *For a functor $\Phi : \mathcal{D} \rightarrow \text{Cat}$, let the category \mathcal{X} be Φd -cocomplete, for all $d \in \mathcal{D}$. Then the colimit of any diagram $T : \int_\circ \Phi \rightarrow \mathcal{X}$ exists in \mathcal{X} if, and only if, the colimit of the diagram $\mathcal{D} \rightarrow \mathcal{X}$, $d \mapsto \text{colim}(\tilde{T}d)$, exists in \mathcal{X} , and in that case the two colimits coincide:*

$$\text{colim}^{(d,x) \in \int_\circ \Phi} T(d, x) \cong \text{colim}^{d \in \mathcal{D}} (\text{colim}^{x \in \Phi d} T(d, x)).$$

Proof. By the commutative diagram and the adjunction established in Proposition 3.1, cocones $T \Longrightarrow \Delta X : \int_\circ \Phi \rightarrow \mathcal{X}$ correspond bijectively to cocones $\tilde{T} \Longrightarrow \Delta X : \mathcal{D} \rightarrow \mathcal{X}$, and naturally so in $X \in \mathcal{X}$. Consequently, the universal representation of either type of cocone exists if the other does, and they then coincide, up to a canonical isomorphism. \square

Remark 3.3. (1) Note that, since all Φd ($d \in \mathcal{D}$) are small, also $\int_\circ \Phi$ is small when \mathcal{D} is small.

(2) It is not hard to prove Theorem 3.2 “directly”, without recourse to Theorem 2.12 and Proposition 3.1: one may simply prove that the composite cocone

$$T(d', -) \longrightarrow \text{colim}_{x \in \Phi d'} T(d', x) \longrightarrow \text{colim}_{d \in \mathcal{D}} (\text{colim}_{x \in \Phi d} T(d, x))$$

(as given by the right-hand side of the formula) is well defined and serves as a colimit cocone for T , and conversely.

(3) As observed in [31], the Twisted Fubini Colimit Formula for a cocomplete category \mathcal{X} is equivalently expressed by the fact that, for the pseudo-algebra $(\mathcal{X}, \text{colim} : \text{Diag}^\circ(\mathcal{X}) \rightarrow \mathcal{X})$ with respect to the pseudo-monad Diag° (see 2.5), the diagram

$$\begin{array}{ccc}
\mathrm{Diag}^\circ(\mathrm{Diag}^\circ(\mathcal{X})) & \xrightarrow{\mathrm{Diag}^\circ(\mathrm{colim})} & \mathrm{Diag}^\circ(\mathcal{X}) \\
\mathrm{colim} \downarrow & & \downarrow \mathrm{colim} \\
\mathrm{Diag}^\circ(\mathcal{X}) & \xrightarrow{\mathrm{colim}} & \mathcal{X}
\end{array}$$

commutes up to isomorphism.

The dualization of Theorem 3.2 reads as follows:

Corollary 3.4 (Twisted Fubini Limit Formula). *For a functor $\Phi : \mathcal{D}^{\mathrm{op}} \rightarrow \mathbf{Cat}$, let the category \mathcal{X} be Φd -complete, for all $d \in \mathcal{D}$. Then the limit of any diagram $T : \int^\circ \Phi \rightarrow \mathcal{X}$ exists in \mathcal{X} if, and only if, the limit of the diagram $\mathcal{D} \rightarrow \mathcal{X}$, $d \mapsto \lim T(d, -)$, exists in \mathcal{X} , and in that case the two limits coincide:*

$$\lim_{(d,x) \in \int^\circ \Phi} T(d, x) \cong \lim_{d \in \mathcal{D}} (\lim_{x \in \Phi d} T(d, x)).$$

Theorem 3.2 implies the “untwisted” Fubini formula that is recorded in Mac Lane’s book [29]:

Corollary 3.5 (Fubini (Co)Limit Formula). *For every functor $T : \mathcal{D} \times \mathcal{E} \rightarrow \mathcal{X}$ into an \mathcal{E} -cocomplete category \mathcal{X} , the colimit of T exists in \mathcal{X} if, and only if, the colimit of the \mathcal{D} -indexed diagram $d \mapsto \mathrm{colim}_{\mathcal{E}} T(d, -)$ exists in \mathcal{X} , and then the two colimits coincide:*

$$\mathrm{colim}^{(d,e) \in \mathcal{D} \times \mathcal{E}} T(d, e) \cong \mathrm{colim}^{d \in \mathcal{D}} (\mathrm{colim}^{e \in \mathcal{E}} T(d, e)).$$

Likewise for limits.

Proof. Let $\Phi : \mathcal{D} \rightarrow \mathbf{Cat}$ be the functor which has constant value \mathcal{E} (formally assumed to be a small category). Then $\int_\circ \Phi = \mathcal{D} \times \mathcal{E}$, and the assertion of the corollary follows from Theorem 3.2. \square

4. The Colimit Decomposition Formula: three proofs

4.1. The basic formula and its short first proof

Given a (small) diagram $\Phi : \mathcal{D} \rightarrow \mathbf{Cat}$, we let

$$K_d : \Phi d \longrightarrow \mathcal{K} = \mathrm{colim} \Phi \quad (d \in \mathcal{D})$$

denote its colimit cocone in the 1-category \mathbf{Cat} . We re-state the formula given in the Introduction and provide (modulo a small correction) a short proof of it, as it was first stated by Batanin and Berger as Lemma 7.13 in [2]:

Theorem 4.1 (Colimit Decomposition Formula). *For every diagram $X : \mathcal{K} \rightarrow \mathcal{X}$ in a cocomplete category \mathcal{X} , the \mathcal{K} -shaped colimit of X may be computed as the \mathcal{D} -shaped colimit of the diagram given by the Φd -shaped colimits of $X K_d$, for every $d \in \mathcal{D}$:*

$$\mathrm{colim}^{\mathcal{K}} X \cong \mathrm{colim}^{d \in \mathcal{D}} (\mathrm{colim}^{\Phi d} X K_d).$$

Proof 1 of the CDF. Since, trivially, the domain functor $\mathbf{Cat}/\mathcal{X} \rightarrow \mathbf{Cat}$ reflects colimits, the given colimit in \mathbf{Cat} gives us the colimit cocone

$$\begin{array}{ccc}
\Phi d & \xrightarrow{K_d} & \mathcal{K} \\
& \searrow X K_d & \swarrow X \\
& & \mathcal{X}
\end{array}$$

in Cat/\mathcal{X} . By Corollary 2.10, the inclusion functor $\text{Cat}/\mathcal{X} \rightarrow \text{Diag}^\circ(\mathcal{X})$ preserves this colimit, and then, by Proposition 2.3, it is again preserved by the left-adjoint functor $\text{colim} : \text{Diag}(\mathcal{X}) \rightarrow \mathcal{X}$. But this is precisely the claim of the theorem. \square

The dualization of the theorem reads as follows:

Corollary 4.2 (Limit Recomposition Formula). *As above, let \mathcal{K} be the colimit of Φ in Cat , with colimit injections K_d . Then, for a diagram $X : \mathcal{K} \rightarrow \mathcal{X}$ in a complete category \mathcal{X} , the limit of X in \mathcal{X} may be computed stepwise, according to the formula*

$$\lim_{\mathcal{K}} X \cong \lim_{d \in \mathcal{D}} (\lim_{\Phi d} X K_d).$$

Proof. Apply Theorem 4.1 to $X^{\text{op}} : \mathcal{K}^{\text{op}} \cong \text{colim}^{d \in \mathcal{D}} (\Phi d)^{\text{op}} \longrightarrow \mathcal{X}^{\text{op}}$. \square

4.2. A generalized colimit decomposition formula and the second proof of the CDF

Our second proof of Theorem 4.1 is based on (what turns out to be) a generalization of the decomposition formula. This generalization follows from the lifting of colimits along a bifibration with cocomplete fibres, as given in Corollary 9.4. By Proposition 2.6, for \mathcal{X} cocomplete, we may apply this corollary to the bifibration $D^{\mathcal{X}} : \text{Diag}^\circ(\mathcal{X}) \rightarrow \text{Cat}$, keeping in mind that cocartesian liftings are given by left Kan extensions in this case. Hence, in order to obtain the colimit of a diagram $T : \mathcal{D} \rightarrow \text{Diag}^\circ(\mathcal{X})$, we follow the dualization of the construction given in the proof of Theorem 9.3 and, with

$$\Phi = D^{\mathcal{X}} T : \mathcal{D} \rightarrow \text{Cat},$$

form the colimit \mathcal{K} of Φ in Cat , as in 4.1. Then, for every $u : d \rightarrow e$ in \mathcal{D} , writing the $\text{Diag}^\circ(\mathcal{X})$ -object Td as $(\Phi d, X_d : \Phi d \rightarrow \mathcal{X})$ and the morphism Tu as $(\Phi u, \varphi^u)$, as in the triangle on the left,

$$\begin{array}{ccc}
\Phi d & \xrightarrow{\Phi u} & \Phi e \\
& \searrow \varphi^u \Rightarrow & \swarrow \\
& & \mathcal{X}
\end{array}
\quad
\begin{array}{ccc}
\Phi d & \xrightarrow{K_d} & \mathcal{K} \\
& \searrow \kappa_d \Rightarrow & \swarrow \\
& & \mathcal{X}
\end{array}$$

$X_d \searrow \quad \swarrow X_e$
 $X_d \searrow \quad \swarrow L_d$

we form the left Kan extensions $L_d := \text{Lan}_{K_d} X_d$. These extensions come with diagram morphisms as in the triangle on the right and form a \mathcal{D} -shaped diagram in $\mathcal{X}^{\mathcal{K}}$ (the fibre of $D^{\mathcal{X}}$ at \mathcal{K}). Its colimit $X := \text{colim}^{d \in \mathcal{D}} L_d$ has colimit injections $\lambda_d : L_d \rightarrow X$. Finally then, the composite $\text{Diag}^\circ(\mathcal{X})$ -morphisms

$$(K_d, \lambda_d K_d \cdot \kappa_d) : (\Phi d, X_d) \rightarrow (\mathcal{K}, X)$$

present (\mathcal{K}, X) as a colimit of T in $\text{Diag}^\circ(\mathcal{X})$.

Remark 4.3. In the construction above, we may think of X as the *joint left Kan extension* of the functors X_d along K_d , characterized by the universal property that, for every functor $Y : \mathcal{K} \rightarrow \mathcal{X}$ and any family $(\mu_d)_{d \in \mathcal{D}}$ of natural transformations $\mu_d : X_d \rightarrow Y K_d$ with $\mu_e(\Phi u) \cdot \varphi^u = \mu_d$ for all $u : d \rightarrow e$ in \mathcal{D} , there is a unique natural transformation $\beta : X \rightarrow Y$ with $\beta K_d \cdot \lambda K_d \cdot \kappa_d = \mu_d$, for all $d \in \mathcal{D}$.

Since the left adjoint functor colim of Proposition 2.3(1) preserves colimits, we obtain:

Theorem 4.4 (General Colimit Decomposition Formula). *For a cocomplete category \mathcal{X} and any diagram $T : \mathcal{D} \rightarrow \text{Diag}^\circ(\mathcal{X})$ with $D^\mathcal{X}T = \Phi$, writing Td as $(\Phi d, X_d : \Phi d \rightarrow \mathcal{X})$ for all $d \in \mathcal{D}$, one has*

$$\text{colim}^\mathcal{K} X \cong \text{colim}^{d \in \mathcal{D}} (\text{colim}^{\Phi d} X_d)$$

in \mathcal{X} , where $\mathcal{K} = \text{colim}^{d \in \mathcal{D}} \Phi d$ with colimit injections K_d in Cat and $X = \text{colim}^{d \in \mathcal{D}} (\text{Lan}_{K_d} X_d)$ is a colimit of left Kan extensions in the functor category $\mathcal{X}^\mathcal{K}$.

Remark 4.5. Although the formula given in Theorem 4.4 may formally look similar to the CDF of Theorem 4.1, there is a crucial difference between the two statements: whereas in Theorem 4.4 X is formed with the help of the given diagrams X_d , in Theorem 4.1 one proceeds the other way around and defines X_d with the help of X as XK_d .

Here is the dualization of Theorem 4.4, obtainable with the dualization procedure given after Proposition 2.2.

Corollary 4.6 (General Limit Recomposition Formula). *For a complete category \mathcal{X} and any diagram $T : \mathcal{D} \rightarrow \text{Diag}_\circ(\mathcal{X})$, $d \mapsto (\Phi d, X_d : \Phi d \rightarrow \mathcal{X})$, with $D_\mathcal{X}T = \Phi^{\text{op}}$ and $\Phi : \mathcal{D}^{\text{op}} \rightarrow \text{Cat}$, one has*

$$\lim_\mathcal{K} X \cong \lim_{d \in \mathcal{D}} (\lim_{\Phi d} X_d)$$

in \mathcal{X} , where $\mathcal{K} = \text{colim}^{d \in \mathcal{D}} \Phi d$ with colimit injections K_d in Cat , and where $X = \lim_{d \in \mathcal{D}} (\text{Ran}_{K_d} X_d)$ is a limit of right Kan extensions in $\mathcal{X}^\mathcal{K}$.

Let us now show how the General Colimit Decomposition Formula may be used to derive the CDF of Theorem 4.1:

Proof 2 of the CDF. We are given the diagrams $\Phi : \mathcal{D} \rightarrow \text{Cat}$ and $X : \mathcal{K} \rightarrow \mathcal{X}$, where \mathcal{K} is the colimit of Φ , with colimit injections $K_d : \Phi d \rightarrow \mathcal{K}$. They allow us to form the diagram $T_X : \mathcal{D} \rightarrow \text{Diag}^\circ(\mathcal{X})$, sending $u : d \rightarrow e$ in \mathcal{D} to the $\text{Diag}^\circ(\mathcal{X})$ -morphism

$$\begin{array}{ccc} \Phi d & \xrightarrow{\Phi u} & \Phi e \\ & \searrow 1 \Rightarrow \swarrow & \\ X_d = XK_d & & XK_e = X_e \end{array} \quad \mathcal{X}$$

which actually lives in Cat/\mathcal{X} ; so, T_X factors through Cat/\mathcal{X} . The assertion of Theorem 4.1 will follow from an application of Theorem 4.4 to T_X , once we have shown the following lemma, formulated in the terminology introduced in Remark 4.3. \square

Lemma 4.7. *X is the joint left Kan extension of the functors X_d along K_d , $d \in \mathcal{D}$.*

Proof. We have to check the relevant universal property, as described in Remark 4.3. To this end we consider a functor $Y : \mathcal{K} \rightarrow \mathcal{X}$ and a family of natural transformations $(\mu_d : XK_d \rightarrow YK_d)_{d \in \mathcal{D}}$ with $\mu_e(\Phi u) = \mu_d$ for all $u : d \rightarrow e$ in \mathcal{D} and must present μ_d as $\mu_d = \beta K_d$ ($d \in \mathcal{D}$), for a unique natural transformation $\beta : X \rightarrow Y$. But this follows immediately from the fact that the functor $\mathcal{X}^{(-)} : \text{Cat}^{\text{op}} \rightarrow \text{Cat}$ transforms the colimit cocone $(K_d : \Phi d \rightarrow \mathcal{K})$ in Cat into a limit cone $(K_d^* : \mathcal{X}^\mathcal{K} \rightarrow \mathcal{X}^{\Phi d})$ in Cat . Indeed, for \mathcal{X} small, as a consequence of the cartesian closedness of Cat , this fact follows from the self-jointness of $\mathcal{X}^{(-)} : \text{Cat}^{\text{op}} \rightarrow \text{Cat}$ (see, for example, Proposition

27.7 in [1]); for \mathcal{X} large, *pro forma* one has to step temporarily into the colossal category \mathbb{CAT} to generate the needed natural bijective correspondence between families of natural transformations μ_d and natural transformations β , in the same way as in the small case. \square

4.3. Obtaining the Colimit Decomposition Formula via Fubini: the third proof of the CDF

Our third proof of Theorem 4.1 takes advantage of the twisted Fubini formula of Theorem 3.2.

Proof 3 of the CDF. Once again, we are given the diagrams $\Phi : \mathcal{D} \rightarrow \mathbf{Cat}$ and $X : \mathcal{K} \rightarrow \mathcal{X}$, where \mathcal{K} is the colimit of Φ , with colimit injections $K_d : \Phi d \rightarrow \mathcal{K}$. Via (4.8) we have a cofinal functor $Q : \int_{\circ} \Phi \rightarrow \mathcal{K}$. With Theorem 3.2, one concludes

$$\mathrm{colim}^{\mathcal{K}} X \cong \mathrm{colim}^{\int_{\circ} \Phi} XQ \cong \mathrm{colim}^{d \in \mathcal{D}} (\mathrm{colim}^{\Phi d} XK_d),$$

for every diagram $X : \mathcal{K} \rightarrow \mathcal{X}$ in a cocomplete category \mathcal{X} . \square

Lemma 4.8. *For a functor $\Phi : \mathcal{D} \rightarrow \mathbf{Cat}$, with \mathcal{D} small, the comparison functor*

$$Q : \int_{\circ} \Phi \rightarrow \mathcal{K} = \mathrm{colim} \Phi, ((u, f) : (d, x) \rightarrow (e, y)) \mapsto (K_e f : K_d x = K_e(\Phi u)x \rightarrow K_e y),$$

from the lax to the strict colimit of Φ (see Corollary 2.13) is a cofinal quotient functor.

Sketch of proof. Let $\Gamma(\Phi)$ denote the coproduct of the categories Φd , $d \in \mathcal{D}$. We have the commutative diagram

$$\begin{array}{ccc} \Gamma(\Phi) & \xrightarrow{J} & \int_{\circ} \Phi \\ \pi \downarrow & \swarrow Q & \downarrow q \\ \mathrm{colim} \Phi & \xrightarrow{I} & (\int_{\circ} \Phi)/\mathcal{Q} \end{array}$$

in \mathbf{Cat} , with π the canonical quotient functor to the colimit, and with the functor J induced by the injections J_d , $d \in \mathcal{D}$ (as considered in 2.13). The functor Q determines relations \mathcal{Q} for a quotient functor q on $\int_{\circ} \Phi$, which turns out to be cofinal. Then qJ factors through π , via the functor I , whose inverse is the factorization of Q through q . Therefore, Q is a cofinal quotient functor as well. \square

5. Extending the Guitart adjunction, making Grothendieck a left adjoint

We return to the Guitart adjunction

$$\mathbf{CAT} \begin{array}{c} \xrightarrow{\mathrm{Diag}^{\circ}} \\ \xleftarrow{\int_{\circ}} \end{array} \mathbf{CAT}/\mathbf{Cat}.$$

of Theorem 2.12. Realizing that $\int_{\circ} \Phi$ is cofibred over the domain of any functor $\Phi : \mathcal{B} \rightarrow \mathbf{Cat}$, so that the 2-functor \int_{\circ} actually takes values in the morphism category \mathbf{CAT}^2 of \mathbf{CAT} , in this section we indicate how to extend the 2-functor Diag° and, in fact, the entire 2-adjunction, from \mathbf{CAT} to \mathbf{CAT}^2 . We also make precise that the 2-functor \int_{\circ} may actually be defined on the “lax slice” $\mathbf{CAT} // \mathbf{Cat}$, rather than on its subcategory $\mathbf{CAT}/\mathbf{Cat}$.

5.1. The diagram category $\text{Diag}^\circ(P)$ of a functor $P : \mathcal{E} \rightarrow \mathcal{B}$.

We start by giving a fibered version of the formation of the category $\text{Diag}^\circ(\mathcal{X})$, replacing \mathcal{X} by a functor P , in such a way that the original construction is described as $\text{Diag}^\circ(\mathcal{X} \rightarrow 1)$, with 1 the terminal category. Hence, for any functor $P : \mathcal{E} \rightarrow \mathcal{B}$ we define the category $\text{Diag}^\circ(P)$, as follows:

- *objects* are triples (a, \mathcal{I}, X) , with a an object in \mathcal{B} and $X : \mathcal{I} \rightarrow \mathcal{E}_a = P^{-1}(a)$ a functor of a small category \mathcal{I} ;
- a *morphism* $(u, F, \varphi) : (a, \mathcal{I}, X) \rightarrow (b, \mathcal{J}, Y)$ is given by a morphism $u : a \rightarrow b$ in \mathcal{B} , a functor $F : \mathcal{I} \rightarrow \mathcal{J}$, and a natural transformation $\varphi : J_a X \rightarrow J_b Y F$ with $P\varphi = \Delta u$ (the constant transformation with value u); J_a, J_b are inclusion functors, as in

$$\begin{array}{ccc}
 \mathcal{I} & \xrightarrow{F} & \mathcal{J} \\
 \downarrow X & & \downarrow Y \\
 \mathcal{E}_a & \xrightarrow{\varphi} & \mathcal{E}_b \\
 \searrow J_a & & \swarrow J_b \\
 & \mathcal{E} & \\
 \begin{array}{ccc} a & \xrightarrow{u} & b \end{array} & & \begin{array}{c} \mathcal{E} \\ \downarrow P \\ \mathcal{B} \end{array}
 \end{array}$$

- *composition*: $(v, G, \psi) \cdot (u, F, \varphi) = (v \cdot u, GF, \psi F \cdot \varphi)$.

The category $\text{Diag}^\circ(P)$ comes equipped with the obvious functors

$$\begin{aligned}
 B^P : \text{Diag}^\circ(P) &\rightarrow \mathcal{B}, & (u, F, \varphi) &\mapsto u, \\
 D^P : \text{Diag}^\circ(P) &\rightarrow \text{Cat}, & (u, F, \varphi) &\mapsto F, \\
 E^P : \mathcal{E} &\rightarrow \text{Diag}^\circ(P), & x &\mapsto (Px, 1, \Delta x : 1 \rightarrow \mathcal{E}_{Px}), \quad (f : x \rightarrow y) \mapsto (Pf, \text{Id}_1, \Delta f).
 \end{aligned}$$

E^P is a full embedding which makes the diagram

$$\begin{array}{ccc}
 \mathcal{E} & \xrightarrow{E^P} & \text{Diag}^\circ(P) \\
 \downarrow P & & \downarrow B^P \\
 & \mathcal{B} &
 \end{array}$$

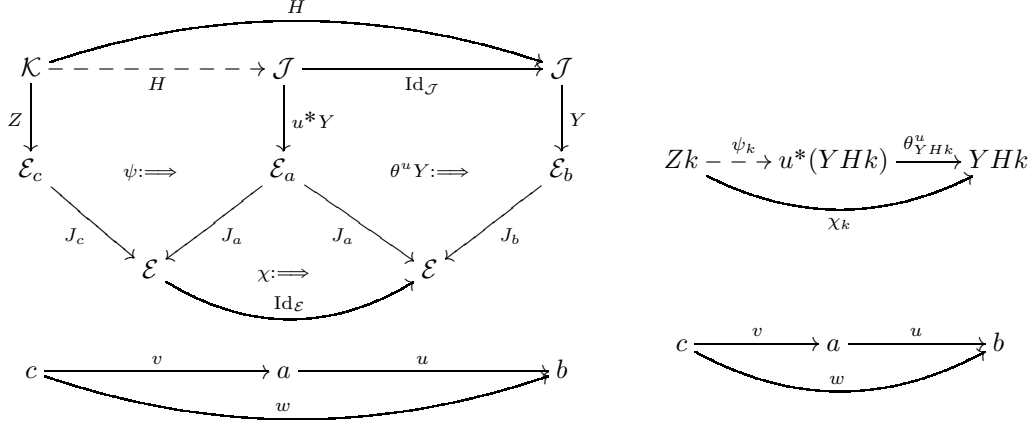
commute. For D^P and B^P one easily proves:

Proposition 5.1. (1) D^P is a split fibration.

(2) B^P is a (split) (co)fibration if, and only if, P has the corresponding property.

Proof. (1) The D^P -cartesian lifting of $F : \mathcal{I} \rightarrow \mathcal{J}$ at (b, \mathcal{J}, Y) may be taken to be $(1_b, F, 1_{J_b Y F}) : (b, \mathcal{I}, Y F) \rightarrow (b, \mathcal{J}, Y)$.

(2) For a (split) fibration P , with P -cartesian liftings denoted by θ , we claim that the B^P -cartesian lift of $u : a \rightarrow b$ at (b, \mathcal{J}, Y) may be taken to be $(u, \text{Id}_{\mathcal{J}}, \theta^u Y)$. Indeed, the required universal property, as depicted by the left side of the diagram below, follows from a pointwise application of the corresponding P -cartesian property, as depicted on the right side:



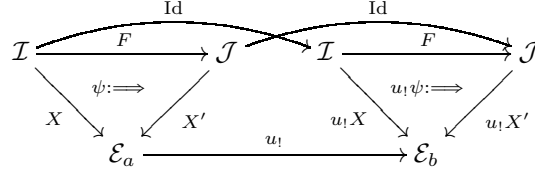
Conversely, if B^P is a (split) fibration, a P -cartesian lifting of $u : a \rightarrow b$ at y may be realized as a B^P -cartesian lifting of $u : a \rightarrow b$ at $(b, 1, \Delta y : 1 \rightarrow \mathcal{E}_b)$.

When P or B^P is a (split) opfibration, the proof proceeds analogously. \square

We should clarify further the interdependency of the diagram constructions for categories and for functors. Trivially, for a category \mathcal{X} , one has $\text{Diag}^\circ(\mathcal{X}) \cong \text{Diag}^\circ(\mathcal{X} \rightarrow 1)$. Less trivially, when $P : \mathcal{E} \rightarrow \mathcal{B}$ is a split cofibration, with the help of the Grothendieck construction we may build $\text{Diag}^\circ(P)$ from the categories $\text{Diag}^\circ(\mathcal{E}_a)$ ($a \in \mathcal{B}$), as follows. Consider the functor

$$\Theta_P : \mathcal{B} \rightarrow \text{CAT}, (u : a \rightarrow b) \mapsto [u_!(-) : \text{Diag}_\circ(\mathcal{E}_a) \rightarrow \text{Diag}_\circ(\mathcal{E}_b)],$$

where the functor $u_!(-)$ maps $(F, \psi) : (\mathcal{I}, X) \rightarrow (\mathcal{J}, X')$ in $\text{Diag}^\circ(\mathcal{E}_a)$ to $(F, u_!\psi)$ in $\text{Diag}^\circ(\mathcal{E}_b)$:



Proposition 5.2. *If $P : \mathcal{B} \rightarrow \mathcal{E}$ is a split cofibration, then, as a cofibred category over \mathcal{B} , the category $\text{Diag}^\circ(P)$ is isomorphic to the dual Grothendieck category $\int_\circ \Theta_P$, by an isomorphism that maps objects identically.*

Proof. For every morphism $u : a \rightarrow b$ in \mathcal{B} , where $a = Px$ and $b = Py$ with $x, y \in \mathcal{E}$, one has the natural bijection

$$\mathcal{E}_b(u!x, y) \rightarrow \mathcal{E}_u(x, y), (f : u!x \rightarrow y) \mapsto f \cdot \delta_x^u,$$

where $\mathcal{E}_u(x, y) = \mathcal{E}(x, y) \cap P^{-1}(u)$ (see 9.2). Given functors $X : \mathcal{I} \rightarrow \mathcal{E}_a$, $Y : \mathcal{J} \rightarrow \mathcal{E}_b$, $F : \mathcal{I} \rightarrow \mathcal{J}$, exploiting the above bijection for $x = Xi$, $y = YFi$ ($i \in \mathcal{I}$), one obtains the natural bijection

$$\{\psi \mid \psi : u_!X \rightarrow YF \text{ nat.tr.}\} \rightarrow \{\varphi \mid \varphi : J_a X \rightarrow J_b YF \text{ nat.tr.}, P\varphi = \Delta u\}, \psi \mapsto J_b \psi \cdot \delta^u X.$$

Equivalently, writing (a, X) instead of $(a, (\mathcal{I}, X))$, we have the natural bijection

$$\{\psi \mid (u, F, \psi) \in (\int_\circ \Theta_P)((a, X), (b, Y))\} \rightarrow \{\varphi \mid (u, F, \varphi) \in \text{Diag}^\circ(P)((a, X), (b, Y))\}, \psi \mapsto J_b \psi \cdot \delta^u X.$$

With objects kept fixed, this defines a bijective functor $\int_\circ \Theta_P \rightarrow \text{Diag}^\circ(P)$ which obviously commutes with the \mathcal{B} -valued split cofibrations:

$$\begin{array}{ccc}
\int_{\circ} \Theta_P & \xrightarrow{\cong} & \text{Diag}^{\circ}(P) \\
\Pi_{\Theta_P} \searrow & & \swarrow B^P \\
& \mathcal{B} &
\end{array}$$

□

In the same way as one arrives at the definition of morphisms of $\text{Diag}_{\circ}(\mathcal{X})$ once those of $\text{Diag}^{\circ}(\mathcal{X})$ have been defined, one may also define the morphisms of the category $\text{Diag}_{\circ}(P)$; that is: keeping the same objects, but inverting the direction of the functor F while keeping the direction of the natural transformation φ in the definition of a morphism (u, F, φ) in $\text{Diag}^{\circ}(P)$, one defines the morphisms of the category $\text{Diag}_{\circ}(P)$. The dualization of Proposition 5.2 then says that, when P is a split fibration, $\text{Diag}_{\circ}(P)$ is isomorphic to $\int^{\circ} \Theta^P$ as a fibred category over \mathcal{B} , with

$$\Theta^P : \mathcal{B}^{\text{op}} \rightarrow \text{CAT}, (u : a \rightarrow b \text{ in } \mathcal{B}) \mapsto [u^*(-) : \text{Diag}_{\circ}(\mathcal{E}_b) \rightarrow \text{Diag}_{\circ}(\mathcal{E}_a)].$$

5.2. Review of the 2-categories CAT^2 , $\text{CAT} // \text{Cat}$ and CAT / Cat .

In order to extend the transitions

$$(P : \mathcal{E} \rightarrow \mathcal{B}) \mapsto (D^P : \text{Diag}^{\circ}(P) \mapsto \text{Cat}), \quad (\Phi : \mathcal{B} \rightarrow \text{Cat}) \mapsto (\Pi_{\Phi} : \int_{\circ} \Phi \rightarrow \mathcal{B}),$$

2-functorially, we form the 2-categories CAT^2 , $\text{CAT} // \text{Cat}$ and CAT / Cat in a standard manner:

- The objects of CAT^2 are functors $P : \mathcal{E} \rightarrow \mathcal{B}$ of 1-categories (= CAT-objects); a morphism $(S, T) : P \rightarrow Q$ is given by functors that make the square on the left of the diagram

$$\begin{array}{ccc}
\mathcal{E} & \xrightarrow{T} & \mathcal{F} \\
P \downarrow & & \downarrow Q \\
\mathcal{B} & \xrightarrow{S} & \mathcal{C}
\end{array}
\qquad
\begin{array}{ccc}
& T & \\
\mathcal{E} & \xrightarrow{\beta \Downarrow} & \mathcal{F} \\
P \downarrow & T' & \downarrow Q \\
& S & \\
\mathcal{B} & \xrightarrow{\alpha \Downarrow} & \mathcal{C} \\
& S' &
\end{array}$$

commutative; and a 2-cell $(\alpha, \beta) : (S, T) \Rightarrow (S', T')$ is a pair of natural transformations $\alpha : S \rightarrow S'$, $\beta : T \rightarrow T'$ with $Q\beta = \alpha P$; their horizontal and vertical compositions are inherited from the 2-category CAT in each of the two components.

- The objects of $\text{CAT} // \text{Cat}$ are functors $\Phi : \mathcal{B} \rightarrow \text{Cat}$ of 1-categories; for $\Psi : \mathcal{C} \rightarrow \text{Cat}$, a morphism $(\Sigma, \tau) : \Phi \rightarrow \Psi$ is given by a functor $\Sigma : \mathcal{B} \rightarrow \mathcal{C}$ and a natural transformation $\tau : \Phi \rightarrow \Psi\Sigma$; a 2-cell $(\sigma, \mu) : (\Sigma, \tau) \Rightarrow (\Sigma', \tau')$ is a natural transformation $\sigma : \Sigma \rightarrow \Sigma'$ together with a modification⁴ $\mu : \Psi\sigma \cdot \tau \rightarrow \tau'$; this means that, for every object $a \in \mathcal{B}$, we have a natural transformation $\mu_a : (\Psi\sigma_a)\tau_a \rightarrow \tau'_a$, such that, for every morphism $u : a \rightarrow b$ in \mathcal{B} , the following two natural transformations coincide:

$$(\Psi\Sigma'u)\mu_a : (\Psi\Sigma'u)(\Psi\sigma_a)\tau_a \rightarrow (\Psi\Sigma'u)\tau'_a \quad \text{and} \quad \mu_b(\Phi u) : (\Psi\sigma_b)\tau_b(\Phi u) \rightarrow \tau'_b(\Phi u).$$

(These two transformations have the same domain and codomain, by the naturality of σ, τ .)

⁴For this term to make sense here, we consider the ordinary category \mathcal{B} as a discrete 2-category (*i.e.*, as having identical 2-cells, so that $\Phi, \Psi\Sigma^{(\cdot)}$ become 2-functors and τ, τ' 2-natural transformations, for free.)



The horizontal and vertical compositions are such that the CAT -valued assignment $(\Sigma, \tau) \mapsto \Sigma$ becomes a 2-functor.

Important Note: CAT//Cat has a richer 2-categorical structure than $\text{DIAG}^\circ(\mathcal{X})$ (as defined in Remark 2.9(2)), which is due to the fact that, when considering 2-cells in CAT//Cat , we are invoking the 2-categorical structure of Cat , in order to form modifications. These are all identities when Cat is replaced by a 1-category \mathcal{X} , considered as a 2-category with identity 2-cells.

- CAT/Cat , as already defined in Remark 2.9(1), is the sub-2-category of CAT//Cat whose morphisms $(\Sigma, \tau) : \Phi \rightarrow \Psi$ satisfy $\tau = 1_\Phi$, so that $\Phi = \Psi\Sigma$; consequently, a 2-cell $\sigma : \Sigma \Longrightarrow \Sigma'$ in CAT/Cat is just a natural transformation satisfying $\Psi\sigma = 1_\Phi$.

Proposition 5.3. *The transitions $P \mapsto D^P$ and $\Phi \mapsto \Pi_\Phi$ are the object assignments of 2-functors*

$$\text{Diag}^\circ : \text{CAT}^2 \rightarrow \text{CAT/Cat} \quad \text{and} \quad \int_\circ : \text{CAT//Cat} \rightarrow \text{CAT}^2.$$

Proof. We just describe the assignments for morphisms and 2-cells and leave all routine verifications to the reader. Diag° assigns to a morphism $(S, T) : P \rightarrow Q$ the functor

$$\Sigma = \text{Diag}^\circ(S, T) : \text{Diag}^\circ(P) \rightarrow \text{Diag}^\circ(Q)$$

which, in turn, is given by the morphism assignment

$$((u, F, \varphi) : (a, \mathcal{I}, X) \rightarrow (b, \mathcal{J}, Y)) \mapsto ((Su, F, T\varphi) : (Sa, \mathcal{I}, T_a X) \rightarrow (Sb, \mathcal{J}, T_b Y));$$

here T_a is the restriction of T that makes the square of the diagram below commute.

$$\begin{array}{ccc} \mathcal{E}_a & \xrightarrow{T_a} & \mathcal{F}_{Sa} \\ J_a \downarrow & & \downarrow J_{Sa} \\ \mathcal{E} & \xrightarrow{T} & \mathcal{F} \end{array} \qquad \begin{array}{ccc} \text{Diag}^\circ(P) & \xrightarrow{\Sigma} & \text{Diag}^\circ(Q) \\ & \searrow D^P & \swarrow D^Q \\ & \text{Cat} & \end{array}$$

Trivially, the triangle on the right commutes as well, so that Σ is indeed a morphism in CAT/Cat . For a 2-cell $(\alpha, \beta) : (S, T) \Longrightarrow (S', T')$, one defines the natural transformation

$$\sigma = \text{Diag}^\circ(\alpha, \beta) : \Sigma = \text{Diag}^\circ(S, T) \longrightarrow \Sigma' = \text{Diag}^\circ(S', T') \quad \text{by}$$

$$\sigma_{(a, \mathcal{I}, X)} = (\alpha_a, \text{Id}_{\mathcal{I}}, \beta J_a X) : \Sigma(a, \mathcal{I}, X) = (Sa, \mathcal{I}, T_a X) \longrightarrow \Sigma'(a, \mathcal{I}, X) = (Sa', \mathcal{I}, T'_a X),$$

for all objects (a, \mathcal{I}, X) in $\text{Diag}^\circ(P)$. Note that $\sigma_{(a, \mathcal{I}, X)}$ is well defined since $Q(\beta J_a X) = \alpha P J_a X = \Delta \alpha_a$.

$$\begin{array}{ccccc}
\mathcal{I} & \xrightarrow{\text{Id}_{\mathcal{I}}} & \mathcal{I} & & \\
\downarrow X & & \downarrow X & & \\
\mathcal{E}_a & \xrightarrow{J_a} & \mathcal{E} & \xleftarrow{J_a} & \mathcal{E}_a \\
\downarrow T_a & & \downarrow T & \begin{array}{c} \beta \Rightarrow \\ \end{array} & \downarrow T'_a \\
\mathcal{F}_{Sa} & \xrightarrow{J_{Sa}} & \mathcal{F} & \xleftarrow{J_{S'a}} & \mathcal{F}_{S'a} \\
\downarrow S_a & & \downarrow \alpha_a & & \downarrow S'a \\
S_a & \xrightarrow{\alpha_a} & S'a & &
\end{array}
\quad \begin{array}{c} \mathcal{F} \\ \downarrow Q \\ \mathcal{C} \end{array}$$

\int_{\circ} assigns to the $(\text{CAT} // \text{Cat})$ -morphism $(\Sigma, \tau) : \Phi \rightarrow \Psi$ the CAT^2 -morphism given by the square

$$\begin{array}{ccc}
\int_{\circ} \Phi & \xrightarrow{T} & \int_{\circ} \Psi \\
\Pi_{\Phi} \downarrow & & \downarrow \Pi_{\Psi} \\
\mathcal{B} & \xrightarrow{\Sigma} & \mathcal{C}
\end{array}$$

where the functor T maps a morphism $(u, f) : (a, x) \rightarrow (b, y)$ to

$$T(u, f) = (\Sigma u, \tau_b(f)) : T(a, x) = (\Sigma a, \tau_a(x)) \longrightarrow T(b, y) = (\Sigma b, \tau_b(y));$$

note that the naturality of τ makes $\tau_b(f)$ have the correct domain, namely $\tau_b(\Phi u(x)) = \Psi(\Sigma u)(\tau_a(x))$.

Given a 2-cell $(\sigma, \mu) : (\Sigma, \tau) \Rightarrow (\Sigma', \tau')$ in $\text{CAT} // \text{Cat}$, we need to define a natural transformation $\beta : T \rightarrow T'$, where $(\Sigma, T) = \int_{\circ}(\Sigma, \tau)$, $(\Sigma', T') = \int_{\circ}(\Sigma', \tau')$, that satisfies $\Pi_{\Psi} \beta = \sigma \Pi_{\Phi}$. To this end, for $(a, x) \in \int_{\circ} \Phi$, we put $\beta_{(a, x)} = (\sigma_a, (\mu_a)_x)$, which is a well-defined morphism $T(a, x) \rightarrow T'(a, x)$ in $\int_{\circ} \Psi$ since $(\mu_a)_x$ is a morphism $\Psi \sigma_a(\tau_a(x)) \rightarrow \tau'_a(x)$ in $\Psi(\Sigma' a)$. \square

5.3. The extended Guitart adjunction

We are now ready to prove that the restriction of the 2-functor $\int_{\circ} : \text{CAT} // \text{Cat} \rightarrow \text{CAT}^2$ to CAT / Cat is left adjoint to Diag° of Proposition 5.3:

Theorem 5.4. $\int_{\circ} \dashv \text{Diag}^{\circ} : \text{CAT}^2 \rightarrow \text{CAT} / \text{Cat}$ is an adjunction of 2-functors.

Proof. In generalization of the adjunction established in the proof of Theorem 2.12, for all functors $\Phi : \mathcal{B} \rightarrow \text{Cat}$, $Q : \mathcal{F} \rightarrow \mathcal{C}$ we must, naturally in Φ and Q , establish functors

$$\text{CAT}^2(\Pi_{\Phi}, Q) \begin{array}{c} \xrightarrow{\square} \\ \xleftarrow{\square} \end{array} (\text{CAT} / \text{Cat})(\Phi, D^Q)$$

that are inverse to each other. In doing so, we follow the notation used in the proof of Proposition 5.3, with slight adjustments. In particular, we write (c, Z) instead of (c, \mathcal{K}, Z) for objects of $\text{Diag}^{\circ}(Q)$.

“ \rightarrow ”: First, given the commutative square on the left, we must define the functor $\Sigma = \widehat{(S, T)}$ of the commutative triangle on the right:

$$\begin{array}{ccc}
\int_{\circ} \Phi & \xrightarrow{T} & \mathcal{F} \\
\Pi_{\Phi} \downarrow & & \downarrow Q \\
\mathcal{B} & \xrightarrow{S} & \mathcal{C}
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{B} & \xrightarrow{\Sigma} & \text{Diag}^{\circ}(Q) \\
\Phi \searrow & & \swarrow D^Q \\
& \text{Cat} &
\end{array}$$

Σ sends an object $a \in \mathcal{B}$ to the $\text{Diag}^{\circ}(Q)$ -object (Sa, T_a) , with the functor

$$T_a : \Phi a \rightarrow \mathcal{F}_{Sa}, \quad (f : x \rightarrow x') \mapsto (T(1_a, f) : T(a, x) \rightarrow T(a, x')),$$

and a morphism $u : a \rightarrow b$ in \mathcal{B} is sent to the $\text{Diag}^{\circ}(Q)$ -morphism

$$\Sigma u = (Su, \Phi u, T\delta^u) : \Sigma a = (Sa, T_a) \longrightarrow \Sigma b = (Sb, T_b),$$

where $\delta_x^u = (u, 1_{\Phi u(x)}) : (a, x) \rightarrow (b, \Phi u(x))$ is the Π_{Φ} -cocartesian lift of u at $x \in \Phi a$. The commutativity of the square above guarantees $Q(T\delta^u) = S\Pi_{\Phi}\delta^u = \Delta Su$, as required.

$$\begin{array}{ccccc}
\Phi a & \xrightarrow{\Phi u} & \Phi b & & \\
T_a \downarrow & \searrow J_a & \delta^u \Rightarrow & \swarrow J_b & \downarrow T_b \\
\mathcal{F}_{Sa} & & \int_{\circ} \Phi & & \mathcal{F}_{Sb} \\
& \searrow J_{Sa} & \downarrow T & \swarrow J_{Sb} & \\
& & \mathcal{F} & &
\end{array}$$

We note that the emerging functor Σ satisfies $D^Q \Sigma = \Phi$, as required. To establish the functoriality of $\widehat{\square}$, for a 2-cell $(\alpha, \beta) : (S, T) \Longrightarrow (S', T')$ one defines the natural transformation

$$\sigma = \widehat{(\alpha, \beta)} : \Sigma = \widehat{(S, T)} \longrightarrow \Sigma' = \widehat{(S', T')}$$

by $\sigma_a = (\alpha_a, \text{Id}_{\Phi a}, \beta_{(a, -)}) : (Sa, T_a) \longrightarrow (S'a, T'_a)$ in $\text{Diag}^{\circ}(Q)$, with $\beta_{(a, x)} : T_a x \rightarrow T'_a x$ ($x \in \Phi a$). Note that one has $D^Q \sigma = 1_{\Phi}$, as required.

“ \leftarrow ”: Conversely now, given the functor Σ of the commutative triangle on the right of the above diagram, we must define the pair of functors $(S, T) = \check{\Sigma}$, making the square on the left commute. With $B^Q : \text{Diag}^{\circ}(Q) \rightarrow \mathcal{C}$ as in 5.1, we put $S = B^Q \Sigma$. For objects $a \in \mathcal{B}$ and $x \in \Phi a$, having the functor $\Sigma a : \Phi a \rightarrow \mathcal{F}_{Sa}$, we put $T(a, x) = \Sigma a(x)$. For a morphism $u : a \rightarrow b$ in \mathcal{B} , we may write the $\text{Diag}^{\circ}(Q)$ -morphism $\Sigma u : \Sigma a \rightarrow \Sigma b$ as

$$\Sigma u = (Su, \Phi u, \varphi^u) : \Sigma a \rightarrow \Sigma b, \quad \text{with } \varphi^u : J_{Sa} \Sigma a \rightarrow J_{Sb} \Sigma b \Phi u \text{ and } Q\varphi^u = \Delta Su.$$

For $(u, f) : (a, x) \rightarrow (b, y)$ in $\int_{\circ} \Phi$, we can now define $T(u, f) : T(a, x) \rightarrow T(b, y)$ as the composite arrow

$$\Sigma a(x) \xrightarrow{\varphi_x^u} \Sigma b(\Phi u(x)) \xrightarrow{\Sigma b(f)} \Sigma b(y).$$

Its second morphism is Q -vertical since the functor Σb takes its values in \mathcal{F}_{Sb} . Consequently,

$$QT(u, f) = Q(\varphi_x^u) = Su = S\Pi_{\Phi}(u, f),$$

so that we indeed have $QT = S\Pi_\Phi$. Clearly, $\tilde{\square}$ becomes a functor since, for a 2-cell $\sigma : \Sigma \Longrightarrow \Sigma'$, we may define $(\alpha, \beta) = \tilde{\sigma} : (S, T) = \tilde{\Sigma} \Longrightarrow (S', T') = \tilde{\Sigma}'$, by putting $\alpha = B^Q\sigma$, then writing σ_a as $(\alpha_a, \text{Id}_{\Phi a}, \beta^a : J_{Sa}\Sigma a \rightarrow J_{S'a}\Sigma' a)$ and finally setting $\beta_{(a,x)} = \beta_x^a$, for all objects $(a, x) \in \text{Gr}_\circ(\Phi)$.

Finally, we must confirm that the functors $\hat{\square}, \check{\square}$ are inverse to each other. First, given $(S, T) : \Pi_\Phi \rightarrow Q$ in CAT^2 , let $\Sigma = \widehat{(S, T)}$ and $(\tilde{S}, \tilde{T}) = \check{\Sigma}$. Then, trivially, $\tilde{S} = B_Q\Sigma = S$, and for all $(a, x) \in \int_\circ \Phi$ one has $\tilde{T}(a, x) = \Sigma a(x) = T_a(x) = T(a, x)$; likewise, $\tilde{T}(1_a, h) = T(1_a, h)$ for every morphism h in Φa . For an arbitrary morphism $(u, f) : (a, x) \rightarrow (b, y)$ in $\int_\circ(\Phi)$, with its (cocartesian, vertical)-factorization $(u, f) = (1_b, f) \cdot (u, 1_{\Phi u(x)}) = J_b(f) \cdot \delta_x^u$, and with φ^u as in “ \leftarrow ”, one obtains

$$\tilde{T}(u, f) = \Sigma b(f) \cdot \varphi_x^u = T_b(f) \cdot T\delta_x^u = T(J_b(f) \cdot \delta_x^u) = T(u, f).$$

This shows $\tilde{T} = T$. Conversely, given $\Sigma : \Phi \rightarrow D^Q$ in CAT/Cat , one argues very similarly that the transitions $\Sigma \mapsto \check{\Sigma} = (S, T) \mapsto \hat{\Sigma} = \widehat{(S, T)}$ actually return Σ . Indeed, having $u : a \rightarrow b$ and writing $\Sigma u = (S, \Phi u, \varphi^u)$ as above, we deduce

$$T\delta_x^u = T(u, 1_{\Phi u(x)}) = \Sigma b(1_{\Phi u(x)}) \cdot \varphi_x^u = \varphi_x^u,$$

for all $x \in \Phi a$ and, hence, $\check{\Sigma}u = (Su, \Phi u, T\delta^u) = (Su, \Phi u, \varphi^u) = \Sigma u$.

This then shows that $\hat{\square}, \check{\square}$ are inverse to each other on the objects of their (co)domains. Showing that the same happens for the morphisms (*i.e.*, the 2-cells in CAT^2 and CAT/Cat) involves only easy routine checks. \square

Remark 5.5. The Guitart adjunction of Theorem 2.12 follows from the extended Guitart adjunction of Theorem 5.4, with the help of (the quite trivial) adjunction

$$\text{Dom} \dashv !_\square : \text{CAT} \rightarrow \text{CAT}^2,$$

where the right adjoint to the domain functor Dom (which exhibits CAT^2 as fibered over CAT) assigns to a category \mathcal{X} the functor $!_\mathcal{X} : \mathcal{X} \rightarrow 1$ (which happens to be a bifibration), considered as an object of CAT^2 . Post-composing this adjunction of 2-functors with the extended Guitart adjunction produces the Guitart adjunction, as the composite adjunction

$$\text{CAT} \begin{array}{c} \xrightarrow{!_\square} \\ \top \\ \xleftarrow{\text{Dom}} \end{array} \text{CAT}^2 \begin{array}{c} \xrightarrow{\text{Diag}^\circ} \\ \top \\ \xleftarrow{\int_\circ} \end{array} \text{CAT}/\text{Cat}.$$

6. The Grothendieck equivalence via the extended Guitart adjunction

In this section we show how the 2-equivalence of split cofibrations $P : \mathcal{E} \rightarrow \mathcal{B}$ and CAT -valued functors $\Phi : \mathcal{B} \rightarrow \text{CAT}$, with functorial and natural changes of the base category \mathcal{B} permitted, may be obtained from the fundamental adjunction of Theorem 5.4. Initially we will restrict ourselves to the consideration of split cofibrations with small fibres. We also formulate the dualized statement for split fibrations.

6.1. Strictification of lax-commutative diagrams

As the Grothendieck equivalence for strict cofibrations with small fibres involves the 2-category CAT//Cat , rather than its subcategory CAT/Cat , our first goal is to map the latter 2-category into the former, with a right adjoint to the inclusion functor. To explain the importance of this step we start with the observation, that the left-adjoint 2-functor $\int_{\circ} : \text{CAT/Cat} \rightarrow \text{CAT}^2$ of Theorem 5.4, assigning to the functor $\Phi : \mathcal{B} \rightarrow \text{Cat}$ the split cofibration $\Pi_{\Phi} : \text{Gr}_{\circ}(\Phi) \rightarrow \mathcal{B}$, actually takes values in the (non-full) sub-2-category

$$\text{SCoFIB}_{\text{sf}}$$

of CAT^2 . Its objects are split cofibrations with small fibres, and its morphisms $(S, T) : P \rightarrow Q$ are morphisms of split cofibrations $P : \mathcal{E} \rightarrow \mathcal{B}$, $Q : \mathcal{F} \rightarrow \mathcal{C}$, *i.e.*, CAT^2 -morphisms that respect the cocleavages:

$$T_b u_! = (S u)_! T_a \quad \text{and} \quad T \delta^u = \delta^{S u} T_a,$$

for all $u : a \rightarrow b$ in \mathcal{B} , where $T_a : \mathcal{E}_a \rightarrow \mathcal{F}_{S a}$ is a restriction of T ; 2-cells are as in CAT^2 (see 5.2). As a consequence, the functor $T : \mathcal{E} \rightarrow \mathcal{F}$ must transform the designated (P -cocartesian, P -vertical)-factorization of a morphism $f : x \rightarrow y$ in \mathcal{E} into the designated (Q -cocartesian, Q -vertical)-factorization of $T f : T x \rightarrow T y$ in \mathcal{F} :

$$T[x \xrightarrow{\delta_x^{P f}} (P f)_!(x) \xrightarrow{\nu_f} y] = [T x \xrightarrow{\delta_{T x}^{S P f}} (S P f)_!(T x) \xrightarrow{\nu_{T f}} T y].$$

Even when we consider the extension of \int_{\circ} to CAT//Cat as in Proposition 5.3, the values still lie in $\text{SCoFIB}_{\text{sf}}$, as one confirms easily. So, we have the commutative diagram

$$\begin{array}{ccc} & \text{CAT//Cat} & \\ \int_{\circ} \swarrow & & \nwarrow \text{Inclusion} \\ \text{SCoFIB}_{\text{sf}} & \xleftarrow{\int_{\circ}} & \text{CAT/Cat} \end{array}$$

in which the bottom 2-functor has a right adjoint, Diag° , as a consequence of Theorem 5.4. As we want to show that the extended Guitart adjunction factors through CAT//Cat , leading to a non-trivial factorization of Diag° as a composite of right-adjoint 2-functors. To this end, we now prove:

Proposition 6.1. *The inclusion 2-functor $\text{CAT/Cat} \rightarrow \text{CAT//Cat}$ has a right adjoint, Strict , given by strictification of lax-commutative diagrams.*

Proof. As an ordinary functor, Strict may be described by a slight adjustment of the 2-functor Strict established in Proposition 2.8, where the ordinary category \mathcal{X} is now taken to be the 2-category Cat . As a result, the strictification needs to account for the greater supply of 2-cells in CAT//Cat than that in $\text{DIAG}^{\circ}(\text{Cat})$. The action of Strict on objects and morphisms is now visualized by

$$\begin{array}{ccc} \begin{array}{ccc} \mathcal{B} & \xrightarrow{S} & \mathcal{C} \\ & \tau : \Rightarrow & \\ \Phi \searrow & & \swarrow \Psi \\ & \text{Cat} & \end{array} & \xrightarrow{\quad \text{Strict} \quad} & \begin{array}{ccc} \text{Cat} \Downarrow \Phi & \xrightarrow{\text{Strict}(S, \tau)} & \text{Cat} \Downarrow \Psi \\ & \text{dom}_{\Phi} \searrow & \swarrow \text{dom}_{\Psi} \\ & \text{Cat} & \end{array} \end{array}$$

with the *lax comma category* $\text{Cat} \Downarrow \Phi$ replacing the ordinary comma category $\text{Cat} \downarrow \Phi$ that (with \mathcal{X} instead of Cat) was considered in Proposition 2.8. We write the objects of $\text{Cat} \Downarrow \Phi$ in the form (a, \mathcal{I}, X) with $a \in \mathcal{B}$ and $X : \mathcal{I} \rightarrow \Phi a$ in Cat and let the functor $\text{Strict}(S, \tau)$ map a morphism $(u, F, \varphi) : (a, \mathcal{I}, X) \rightarrow (b, \mathcal{J}, Y)$ as is indicated by

$$\begin{array}{ccc} \begin{array}{ccc} \mathcal{I} & \xrightarrow{F} & \mathcal{J} \\ X \downarrow & \varphi \rightrightarrows & \downarrow Y \\ \Phi a & \xrightarrow{\Phi u} & \Phi b \end{array} & \longmapsto & \begin{array}{ccc} \mathcal{I} & \xrightarrow{F} & \mathcal{J} \\ \tau_a X \downarrow & \tau_b \varphi \rightrightarrows & \downarrow \tau_b Y \\ \Psi(Sa) & \xrightarrow{\Psi(Su)} & \Psi(Sb) \end{array} \\ & & \text{Strict}(S, \tau) \end{array}$$

Strict maps a 2-cell $(\sigma, \mu) : (S, \tau) \rightarrow (S', \tau')$ to the natural transformation $\text{Strict}(\sigma, \mu) : \text{Strict}(S, \tau) \rightarrow \text{Strict}(S', \tau')$, defined at the object $(a, \mathcal{I}, X) \in \text{Cat} \Downarrow \Phi$ as the morphism

$$(\text{Strict}(\sigma, \mu))_{(a, \mathcal{I}, X)} = (Sa, \text{Id}_{\mathcal{I}}, \mu_a X) : (Sa, \mathcal{I}, \tau_a X) \rightarrow (S'a, \mathcal{I}, \tau'_a X)$$

in $\text{Cat} \Downarrow \Psi$. For any functors Φ, Ψ as above, the needed adjunction isomorphism

$$(\text{CAT} // \text{Cat})(\Phi, \Psi) \cong (\text{CAT} // \text{Cat})(\Phi, \text{dom}_{\Psi})$$

of categories associates with $(S, \tau) : \Phi \rightarrow \Psi$ the functor

$$(S, \tau)^{\sharp} : \mathcal{B} \rightarrow \text{Cat} \Downarrow \Psi, \quad (u : a \rightarrow b) \mapsto ((Su, \Phi u, 1_{\tau_b \Phi u}) : (Sa, \Phi a, \tau_a) \rightarrow (Sb, \Phi b, \tau_b)),$$

and a 2-cell $(\sigma, \mu) : (S, \tau) \rightarrow (S', \tau')$ corresponds to the natural transformation $(\sigma, \mu)^{\sharp}$, defined at every $a \in \mathcal{B}$ by

$$(\sigma, \mu)_a^{\sharp} = (\sigma_a, \text{Id}_{\Phi a}, \mu_a) : (Sa, \Phi a, \tau_a) \rightarrow (S'a, \Phi a, \tau'_a).$$

Note that $(\sigma, \mu)^{\sharp}$ is indeed a 2-cell in $\text{CAT} // \text{Cat}$ since $\text{dom}_{\Psi}(\sigma, \mu)^{\sharp} = 1_{\Phi}$. We omit the details of all the lengthy, but routine verifications. \square

6.2. Replacing diagrams by fibres

When $P : \mathcal{E} \rightarrow \mathcal{B}$ is a split cofibration with small fibres, considered as an object of $\text{SCoFIB}_{\text{sf}}$, rather than mapping it with the 2-functor Diag° , we may now map P to its (covariant) *fibre decomposition functor*

$$\text{Fib}_o(P) = \Phi_P : \mathcal{B} \rightarrow \text{Cat}, \quad u! : \mathcal{E}_a \rightarrow \mathcal{E}_b = P^{-1}(b),$$

considered as an object of $\text{CAT} // \text{Cat}$.

Proposition 6.2. *The assignment $P \mapsto \text{Fib}_o(P)$ extends to a 2-functor $\text{Fib}_o : \text{SCoFIB}_{\text{sf}} \rightarrow \text{CAT} // \text{Cat}$.*

Proof. Keeping the notation of 6.1, we map a morphism $(S, T) : P \rightarrow Q$ in $\text{SCoFIB}_{\text{sf}}$ to the $\text{CAT} // \text{Cat}$ -morphism

$$\text{Fib}_o(S, T) = (S, \tau) : \Phi_P \rightarrow \Phi_Q,$$

where we define the natural transformation $\tau : \Phi_P \rightarrow \Phi_Q$ by restricting the functor T , via $\tau_a = T_a : \mathcal{E}_a \rightarrow \mathcal{F}_{Sa}$; the naturality of τ follows from (S, T) being a morphism of split cofibrations. For a 2-cell $(\alpha, \beta) : (S, T) \rightarrow (S', T')$ in SOFIB_{sf} , we define the 2-cell

$$\text{Fib}_o(\alpha, \beta) = (\alpha, \mu) : (S, \tau) \rightarrow (S', \tau') = \text{Fib}_o(S', T')$$

in CAT/Cat by specifying the modification $\mu : \tau \rightarrow \tau'$, as follows: for every $a \in \mathcal{B}$, we define the natural transformation $\mu_a : (\alpha_a)_! \tau_a \rightarrow \tau'_a$, by letting $(\mu_a)_x$ be the Q -vertical factor of the canonical $(Q$ -cocartesian, Q -vertical)-factorization of β_x , for every $x \in \mathcal{E}_a$:

$$\begin{array}{ccc}
 & (\alpha_a)_!(Tx) & \\
 \delta_{Tx}^{\alpha_a} \nearrow & \downarrow (\mu_a)_x & \\
 Tx & \xrightarrow{\beta_x} T'x & \\
 \alpha_a \nearrow & & \\
 Sa & \xrightarrow{\alpha_a} S'a &
 \end{array}
 \quad
 \begin{array}{c}
 \mathcal{F} \\
 \downarrow Q \\
 \mathcal{C}
 \end{array}$$

Naturality of every μ_a follows easily from the naturality of β and δ^{α_a} ; indeed, for every $f : x \rightarrow x'$ in \mathcal{E}_a one has

$$\begin{aligned}
 T'f \cdot (\mu_a)_x \cdot \delta_{Tx}^{\alpha_a} &= T'f \cdot \beta_x && \text{(definition of } (\mu_a)_x \text{)} \\
 &= \beta_{x'} \cdot Tf && \text{(naturality of } \beta \text{)} \\
 &= (\mu_a)_{x'} \cdot \delta_{Tx'}^{\alpha_a} \cdot Tf && \text{(definition of } (\mu_a)_{x'} \text{)} \\
 &= (\mu_a)_{x'} \cdot (\alpha_a)_!(Tf) \cdot \delta_{Tx}^{\alpha_a} && \text{(naturality of } \delta^{\alpha_a} \text{)},
 \end{aligned}$$

which implies the desired equality $T'_a f \cdot (\mu_a)_x = (\mu_a)_{x'} \cdot (\alpha_a)_!(T_a f)$ in \mathcal{F}_{S_a} .

For μ to qualify as a modification, we must verify that the natural transformations $(S'u)_! \mu_a$ and $\mu_b u_!$ coincide, for all $u : a \rightarrow b$ in \mathcal{B} . Indeed, by the naturality of α and the preservation of cocartesian liftings by T and T' , they have the common domain $(S'u)_!(\alpha_a)_! T_a = (\alpha_b)_!(Su)_! T_a = (\alpha_b)_! T_b u_!$ and the common codomain $(S'u)_! T'_a = T'_b u_!$. Hence, it remains to be shown that, for all $x \in \mathcal{E}_a$, we have the equality $(S'u)_!((\mu_a)_x) = (\mu_b)_{u_!x}$ in \mathcal{F}_{S_a} , which follows from the following sequence of equalities that may be traced by chasing around this diagram:

$$\begin{array}{c}
\begin{array}{ccccc}
Tx & \xrightarrow{\delta_{Tx}^{\alpha_a}} & (\alpha_a)_!(Tx) & \xrightarrow{\delta_{(\alpha_a)_!}^{S'u}} & (S'u)_!(\alpha_a)_!(Tx) = (\alpha_b)_!(T(u_!x)) \\
\downarrow 1_{Tx} & \searrow \delta_{Tx}^{S'u} = T(\delta_x^u) & \downarrow \delta_{Tx}^{S'u \cdot \alpha_a} = \delta_{Tx}^{\alpha_b \cdot Su} & \searrow \delta_{(\alpha_a)_!}^{S'u} & \\
Tx & \xrightarrow{\delta_{Tx}^{S'u} = T(\delta_x^u)} & (Su)_!(Tx) = T(u_!x) & \xrightarrow{\delta_{T(u_!x)}^{\alpha_b}} & (S'u)_!(\alpha_a)_!(Tx) = (\alpha_b)_!(T(u_!x)) \\
& \downarrow 1_{T(u_!x)} & \downarrow (\mu_a)_x & & \downarrow (\mu_b)_{u_!x} \\
Tx & \xrightarrow{\delta_{Tx}^{S'u} = T(\delta_x^u)} & (Su)_!(Tx) = T(u_!x) & \xrightarrow{\beta_x} & T'x \\
& \downarrow \delta_{Tx}^{S'u} = T(\delta_x^u) & \downarrow \beta_{u_!x} & & \downarrow \delta_{T'x}^{S'u} = T'(\delta_x^u) \\
(Su)_!(Tx) = T(u_!x) & \xrightarrow{\beta_{u_!x}} & (S'u)_!(T'x) = T'(u_!x) & &
\end{array} \\
\\
\begin{array}{ccc}
Sa & \xrightarrow{\alpha_a} & S'a \\
\searrow Su & & \searrow S'u \\
Sb & \xrightarrow{\alpha_b} & S'b
\end{array}
\end{array}$$

$$\begin{aligned}
(S'u)_!((\mu_a)_x) \cdot \delta_{Tx}^{S'u \cdot \alpha_a} &= (S'u)_!((\mu_a)_x) \cdot \delta_{(\alpha_a)_!(Tx)}^{S'u} \cdot \delta_{Tx}^{\alpha_a} && \text{(composition of cocleavages)} \\
&= \delta_{T'x}^{S'u} \cdot (\mu_a)_x \cdot \delta_{Tx}^{\alpha_a} && \text{(naturality of } \delta^{S'u}) \\
&= T'(\delta_x^u) \cdot \beta_x && (T' \text{ preserves cocleavages; def. of } (\mu_a)_x) \\
&= \beta_{u_!x} \cdot T(\delta_x^u) && \text{(naturality of } \beta) \\
&= (\mu_b)_{u_!x} \cdot \delta_{T(u_!x)}^{\alpha_b} \cdot \delta_{Tx}^{Su} && \text{(def. of } (\mu_b)_{u_!x}; T \text{ preserves cocleavages)} \\
&= (\mu_b)_{u_!x} \cdot \delta_{Tx}^{\alpha_b \cdot Su} && \text{(composition of cocleavages)} \\
&= (\mu_b)_{u_!x} \cdot \delta_{Tx}^{S'u \cdot \alpha_a} && \text{(naturality of } \alpha).
\end{aligned}$$

The remaining lengthy verifications for the 2-functoriality of Fib_\circ may be left to the reader. \square

6.3. The Grothendieck Equivalence Theorem for split cofibrations

We are now ready to formulate the following “folklore” theorem, a sufficiently elaborate proof of which does not seem to be easily available in the literature, at least not for variable base category:

Theorem 6.3. *The 2-functors $\int_\circ \dashv \text{Fib}_\circ : \text{SCoFIB}_{\text{sf}} \longrightarrow \text{CAT} // \text{Cat}$ are adjoint 2-equivalences.*

Proof. We first establish an invertible 2-natural transformation $\kappa : \int_\circ \circ \text{Fib}_\circ \rightarrow 1_{\text{Id}_{\text{SCoFIB}_{\text{sf}}}}$. For a split cofibration $P : \mathcal{E} \rightarrow \mathcal{B}$ with small fibres, the CAT^2 -morphism $\kappa_P = (K_P, \text{Id}_{\mathcal{B}})$ as depicted by

$$\begin{array}{ccc}
\int_\circ \Phi_P & \xrightarrow{K_P} & \mathcal{E} \\
\Pi_{\Phi_P} \downarrow & & \downarrow P \\
\mathcal{B} & \xrightarrow{\text{Id}_{\mathcal{B}}} & \mathcal{B}
\end{array}$$

is given by the “composition functor”

$$K_P : ((u, f) : (a, x) \rightarrow (b, y)) \mapsto (f \cdot \delta_x^u : x \rightarrow y)$$

which, in the dual situation, is displayed in Theorem 9.5. By design, κ_P is a morphism of split cofibrations and, quite trivially, invertible. To confirm its 2-naturality, we consider, in the notation of Section 5.2, a 2-cell $(\alpha, \beta) : (S, T) \Longrightarrow (S', T') : P \longrightarrow Q$ in $\text{SCoFIB}_{\text{sf}}$ and the following diagram:

$$\begin{array}{ccc} & (S, \tilde{T}) & \\ \Pi_{\Phi_P} \nearrow & & \searrow \Pi_{\Phi_Q} \\ & (\alpha, \tilde{\beta}) \Downarrow & \\ & (S', \tilde{T}') & \\ \kappa_P = (K_P, \text{Id}_{\mathcal{B}}) \downarrow & & \downarrow (K_Q, \text{Id}_{\mathcal{C}}) = \kappa_Q \\ & (S, T) & \\ P \nearrow & & \searrow Q \\ & (\alpha, \beta) \Downarrow & \\ & (S', T') & \end{array}$$

Here, the functors \tilde{T}, \tilde{T}' and the natural transformation $\tilde{\beta} : \tilde{T} \rightarrow \tilde{T}'$ are obtained by applying to the 2-cell (α, β) first Fib_{\circ} and then \int_{\circ} , with both 2-functors leaving the “base” transformation α unchanged. According to the definitions given in the proofs of Propositions 5.3 and 6.2, one has

$$\tilde{T} : \int_{\circ} \Phi_P \rightarrow \int_{\circ} \Phi_Q, \quad [(u, f) : (a, x) \rightarrow (b, y)] \mapsto [(Su, Tf) : (Sa, Tx) \rightarrow (Sb, Ty)],$$

$$\tilde{\beta}_{(a,x)} = (\alpha_a, (\mu_a)_x) : (Sa, Tx) \rightarrow (S'a, T'x), \text{ with } (\mu_a)_x \cdot \delta_{Tx}^{\alpha_a} = \beta_x.$$

Now, from $K_Q(\tilde{\beta}_{(a,x)}) = (\mu_a)_x \cdot \delta_{Tx}^{\alpha_a} = \beta_x = \beta_{K_P(a,x)}$ for all $(a, x) \in \int_{\circ} \Phi_P$ one has $K_Q \tilde{\beta} = \beta K_P$, which is the crucial ingredient to concluding the equality

$$\kappa_Q \cdot \int_{\circ} (\text{Fib}_{\circ}(\alpha, \beta)) = (\alpha, \beta) \cdot \kappa_P,$$

i.e., the 2-naturality of κ .

Next, we establish an invertible 2-natural transformation $\Lambda : 1_{\text{Id}_{\text{CAT//Cat}}} \rightarrow \text{Fib}_{\circ} \circ \int_{\circ}$ which, at the CAT//Cat -object $\Phi : \mathcal{B} \rightarrow \text{Cat}$, is the morphism $\Lambda_{\Phi} = (\text{Id}_{\mathcal{B}}, \lambda^{\Phi}) : \Phi \rightarrow \Phi_{\Pi_{\Phi}}$, where $\lambda_a^{\Phi} : \Phi a \rightarrow (\int_{\circ} \Phi)_a$ is the trivial bijective functor $x \mapsto (a, x)$ (see Theorem 9.5 in the dual situation). We check the 2-naturality of Λ and, in the notation of 5.2, consider a 2-cell $(\sigma, \mu) : (\Sigma, \tau) \Longrightarrow (\Sigma', \tau') : \Phi \longrightarrow \Psi$ in CAT//Cat . An examination of the definitions given in the proofs of Propositions 5.3 and 6.2, show that, up to the identifications $\lambda^{\Phi}, \lambda^{\Psi}$, the composite functor $\text{Fib}_{\circ} \circ \int_{\circ}$ maps (σ, μ) to itself. As a consequence one obtains the needed equality

$$\Lambda_{\Psi} \cdot (\sigma, \mu) = \text{Fib}_{\circ}(\int_{\circ}(\sigma, \mu)) \cdot \Lambda_{\Phi}.$$

This completes the proof. \square

The proof of Theorem 6.3 remains intact if we drop the condition of small-fibredness and consider the 2-category SCoFIB with objects all split cofibrations $P : \mathcal{E} \rightarrow \mathcal{B}$. These then correspond to CAT -valued functors, rather than to Cat -valued functors. Hence, one has to define the 2-category⁵ CAT//CAT just as CAT//Cat has been defined, to obtain the *Grothendieck Equivalence Theorem* [15] for split cofibrations:

⁵The notation CAT//CAT is to be understood as analogous to the standard lax-comma category notation CAT//Cat . While the latter is legitimate (as Cat is an object of CAT), the former is not; rather, CAT//CAT has to be considered as a full subcategory of $\mathbb{C}\text{AT//CAT}$, with some higher-universe $\mathbb{C}\text{AT}$ that contains CAT as an object.

Corollary 6.4. *The 2-functors $\int_{\circ} \dashv \text{Fib}_{\circ} : \text{SCoFIB} \longrightarrow \text{CAT} // \text{CAT}$ are adjoint 2-equivalences.*

We can finally compose the 2-equivalence of Theorem 6.3 with the 2-adjunction of Proposition 6.1, to obtain an alternative proof for the left-adjointness of the restricted functor \int_{\circ} of 6.1, without recourse to the fundamental adjunction of Theorem 5.4. However, the advantage of having established the fundamental adjunction of Theorem 5.4 first is that we may *conclude* that the right adjoints Diag° and $\text{Strict} \circ \text{Fib}_{\circ}$ (with Strict as in Proposition 6.1) coincide, up to 2-natural isomorphism—a fact that is a bit cumbersome to confirm when pursued directly. Either way, we have established the following important fact:

Corollary 6.5. *The diagram*

$$\begin{array}{ccc}
 & \text{CAT} // \text{Cat} & \\
 \text{Fib}_{\circ} \nearrow & & \nwarrow \text{Strict} \\
 \text{SCoFIB}_{\text{sf}} & \xrightarrow{\int_{\circ}} \text{CAT} / \text{Cat} & \\
 & \text{Diag}^{\circ} \quad \text{Incl} & \\
 & \text{---} & \\
 & \int_{\circ} &
 \end{array}$$

of adjunctions of 2-functors commutes. In particular, the 2-functor Diag° factors through the Grothendieck equivalence Fib_{\circ} .

6.4. The Grothendieck Equivalence Theorem for split fibrations

It's time for us to dualize Theorem 6.3 and Corollary 6.4 and to consider the sub-2-category

$$\text{SFIB}$$

of CAT^2 , which is defined just like SCoFIB in 6.1, except that its objects $P : \mathcal{E} \rightarrow \mathcal{B}$ are now split fibrations (rather than split cofibrations), and that its morphisms $(S, T) : P \rightarrow Q$ preserve cleavages, so that

$$T_a u^* = (S_u)^* T_b \quad \text{and} \quad T \vartheta^u = \vartheta^{Su} T_b,$$

for all $u : a \rightarrow b$ in \mathcal{B} , where $T_a : \mathcal{E}_a \rightarrow \mathcal{F}_{S_a}$ is a restriction of T ; 2-cells are as in CAT^2 (see 5.2).

An application of the bijective 2-functor $\square^{\text{op}} : \text{CAT}^{\text{co}} \rightarrow \text{CAT}$ to objects, morphisms and 2-cells of CAT^2 gives rise to the bijective 2-functor $\square^{\text{op}} : (\text{CAT}^2)^{\text{co}} \rightarrow \text{CAT}^2$ with

$$[(\alpha, \beta) : (S, T) \Rightarrow (S', T') : P \rightarrow Q] \longmapsto [(\alpha^{\text{op}}, \beta^{\text{op}}) : (S^{\text{op}}, T^{\text{op}}) \Leftarrow ((S')^{\text{op}}, (T')^{\text{op}}) : P^{\text{op}} \rightarrow Q^{\text{op}}].$$

It maps morphisms covariantly but 2-cells contravariantly, and it restricts to a bijective 2-functor

$$\square^{\text{op}} : \text{SFIB}^{\text{co}} \longrightarrow \text{SCoFIB}.$$

The bijective 2-functor $\square^{\text{op}} : \text{CAT}^{\text{co}} \rightarrow \text{CAT}$ gives also rise to the bijective 2-functor

$$\square^{\text{op}}(-) : \text{CAT} // \text{CAT} \longrightarrow \text{CAT} // \text{CAT},$$

which post-composes every object, morphism and 2-cell with the functor \square^{op} :

$$[(\sigma, \mu) : (\Sigma, \tau) \Rightarrow (\Sigma', \tau') : \Phi \rightarrow \Psi] \longmapsto [(\square^{\text{op}} \sigma, \square^{\text{op}} \mu) : (\square^{\text{op}} \Sigma, \square^{\text{op}} \tau) \Rightarrow (\square^{\text{op}} \Sigma', \square^{\text{op}} \tau') : \square^{\text{op}} \Phi \rightarrow \square^{\text{op}} \Psi].$$

Now we may define the 2-functor Fib° as the dualization of the 2-functor Fib_{\circ} of Corollary 6.4, that is: as the composite 2-functor given by the commutative diagram

$$\begin{array}{ccc}
\mathbf{SFIB}^{\mathrm{co}} & \xrightarrow{\mathrm{Fib}^\circ} & \mathbf{CAT} // \mathbf{CAT} \\
\downarrow \square^{\mathrm{op}} & & \uparrow \square^{\mathrm{op}}(-) \\
\mathbf{SCoFIB} & \xrightarrow{\mathrm{Fib}_\circ} & \mathbf{CAT} // \mathbf{CAT}.
\end{array}$$

Chasing a split fibration $P : \mathcal{E} \rightarrow \mathcal{B}$ around the lower path of the diagram shows that, as expected, Fib° maps P to its fibre representation $\Phi^P : \mathcal{B}^{\mathrm{op}} \rightarrow \mathbf{CAT}$ (as in 9.1), and morphisms and 2-cells of \mathbf{SFIB} get mapped as indicated by

$$\begin{array}{ccc}
\begin{array}{ccc}
\mathcal{E} & \begin{array}{c} \xrightarrow{T} \\ \beta \Downarrow \\ \xrightarrow{T'} \\ S \\ \xrightarrow{\alpha} \end{array} & \mathcal{F} \\
P \downarrow & & \downarrow Q \\
\mathcal{B} & \begin{array}{c} \xrightarrow{\alpha} \\ \alpha \Downarrow \\ \xrightarrow{\alpha} \end{array} & \mathcal{E} \\
& S' &
\end{array} & \mapsto & \begin{array}{ccc}
\mathcal{B}^{\mathrm{op}} & \begin{array}{c} \xrightarrow{(S^{\mathrm{op}}, \tau^{\mathrm{op}})} \\ (\alpha^{\mathrm{op}}, \mu^{\mathrm{op}}) \Uparrow \\ \xrightarrow{((S')^{\mathrm{op}}, (\tau')^{\mathrm{op}})} \end{array} & \mathcal{C}^{\mathrm{op}} \\
\Phi^P \searrow & & \swarrow \Phi^Q \\
& \mathbf{Cat} &
\end{array}
\end{array}$$

Here the natural transformations $\tau : \Phi^P \rightarrow \Phi^Q S^{\mathrm{op}}$, $\tau' : \Phi^P \rightarrow \Phi^Q (S')^{\mathrm{op}}$, analogously to the respective definitions for Fib_\circ in the proof of Proposition 6.2, are determined as the restrictions $\tau_a = T_a : \mathcal{E}_a \rightarrow \mathcal{F}_{Sa}$, $\tau'_a = T'_a : \mathcal{E}_a \rightarrow \mathcal{F}_{S'a}$ of T for all $a \in \mathcal{B}$, while the transformations $\mu_a : \tau_a \rightarrow (\alpha_a)^* \tau'_a$ comprising the modification $\mu : \tau \rightarrow \tau'$ are defined as the Q -vertical factors in the factorization $\beta_x = \theta_{T'_x}^{\alpha_a} \cdot (\mu_a)_x$, for all $x \in \mathcal{E}_a$.

Now, since Fib_\circ is an equivalence of 2-categories, its dualization, Fib° is one as well. Moreover, one obtains its quasi-inverse, \int° , from the 2-functor \int_\circ , by the same dualization procedure that has produced Fib° from Fib_\circ . Indeed, the dualization diagram

$$\begin{array}{ccc}
\mathbf{SFIB}^{\mathrm{co}} & \begin{array}{c} \xleftarrow{\mathrm{Fib}^\circ} \\ \int^\circ \end{array} & \mathbf{CAT} // \mathbf{CAT} \\
\downarrow \square^{\mathrm{op}} & & \uparrow \square^{\mathrm{op}}(-) \\
\mathbf{SCoFIB} & \begin{array}{c} \xleftarrow{\mathrm{Fib}_\circ} \\ \int_\circ \end{array} & \mathbf{CAT} // \mathbf{CAT}.
\end{array}$$

commutes at both, the Fib - and the \int -level. Hence, with Theorem 6.3 and Corollary 6.4 we conclude:

Corollary 6.6. *The 2-functors $\int^\circ \dashv \mathrm{Fib}^\circ : \mathbf{SFIB}^{\mathrm{co}} \rightarrow \mathbf{CAT} // \mathbf{CAT}$ are adjoint 2-equivalences. By restriction to the small-fibred split fibrations they give the 2-equivalences $\int^\circ \dashv \mathrm{Fib}^\circ : (\mathbf{SFIB}_{\mathrm{sf}})^{\mathrm{co}} \rightarrow \mathbf{CAT} // \mathbf{Cat}$.*

7. A left adjoint to the Grothendieck construction: free split cofibrations

In this section we give a novel proof of the essentially known fact⁶ that the composite 2-functor

$$\mathbf{CAT} // \mathbf{CAT} \xrightarrow{\int_\circ} \mathbf{SCoFIB} \xrightarrow{\mathrm{Incl}} \mathbf{CAT}^2$$

⁶See ncatlab.org, “Grothendieck construction”, for a proof of a corresponding statement with fixed base category.

has a left adjoint. Indeed, since \int_{\circ} is a 2-equivalence (by Corollary 6.4), it suffices to show that the inclusion 2-functor has a left adjoint, *i.e.*, that \mathbf{SCoFIB} is 2-reflective in \mathbf{CAT}^2 . This means that we must show *how an arbitrary functor P can be freely “made” into a split cofibration*, compatibly so with the relevant 2-categorical structures.

7.1. Free split cofibrations

We define the 2-functor

$$\mathbf{Free} : \mathbf{CAT}^2 \rightarrow \mathbf{SCoFIB},$$

as follows. For every functor $P : \mathcal{E} \rightarrow \mathcal{B}$, the dual Grothendieck construction applied to the trivial slice functor $P/\square : \mathcal{B} \rightarrow \mathbf{CAT}$ gives us (in generalization of Example 9.6(3)) the split cofibration

$$\mathbf{Free}(P) := \Pi_{P/\square} = \text{cod}_P : \int_{\circ} P/\square = P \downarrow \mathcal{B} \longrightarrow \mathcal{B}$$

of the comma category $P \downarrow \mathcal{B}$. Here we therefore write an object in $P \downarrow \mathcal{B}$ as a pair (h, x) with $x \in \mathcal{E}$ and $h : Px \rightarrow a$ in \mathcal{B} ; a morphism $(u, f) : (h, x) \rightarrow (k, y)$ is given by the commutative square on the left of the diagram

$$\begin{array}{ccc} Px & \xrightarrow{Pf} & Py \\ \downarrow h & & \downarrow k \\ a & \xrightarrow{u} & b \end{array} \quad = \quad \begin{array}{ccccc} Px & \xrightarrow{P1_x} & Px & \xrightarrow{Pf} & Py \\ \downarrow h & & \downarrow u \cdot h & & \downarrow k \\ a & \xrightarrow{u} & b & \xrightarrow{1_b} & b \end{array}$$

The right part of the diagram describes the designated $(\text{cod}_P\text{-cocartesian}, \text{cod}_P\text{-vertical})$ -factorization of (u, f) , so that one has:

$$u_1(h, x) = (u \cdot h, x), \quad \delta_{(h, x)}^u = (u, 1_x) : (h, x) \rightarrow (u \cdot h, x), \quad \nu_{(u, f)} = (1_b, f) : (u \cdot h, x) \rightarrow (k, y).$$

The definition of \mathbf{Free} on morphisms and 2-cells is also straightforward. In the notation of 5.2, the action of \mathbf{Free} is described by

$$[(\alpha, \beta) : (S, T) \Longrightarrow (S', T') : P \longrightarrow Q] \longmapsto [(\alpha, \bar{\beta}) : (S, \bar{T}) \Longrightarrow (S', \bar{T}') : \text{cod}_P \longrightarrow \text{cod}_Q],$$

where

$$\begin{aligned} \bar{T} : P \downarrow \mathcal{B} &\longrightarrow Q \downarrow \mathcal{C}, \quad ((u, f) : (h, x) \rightarrow (k, y)) \longmapsto ((Su, Tf) : (Sh, Tx) \rightarrow (Sk, Ty)), \\ \bar{\beta}_{(h, x)} &= (\alpha_a, \beta_x) : \bar{T}(h, x) = (Sh, Tx) \longrightarrow \bar{T}'(h, x) = (S'h, T'x), \end{aligned}$$

for all objects $(h : Px \rightarrow a, x)$ in $P \downarrow \mathcal{B}$. Trivially, \bar{T} transforms cod_P -cocleavages into cod_Q -cocleavages. We now prove:

Theorem 7.1. *The 2-functor \mathbf{Free} is left adjoint to the inclusion $\mathbf{SCoFIB} \longrightarrow \mathbf{CAT}^2$.*

Proof. Given a \mathbf{CAT}^2 -object $P : \mathcal{E} \rightarrow \mathcal{B}$, we consider the functor

$$H_P : \mathcal{E} \rightarrow P \downarrow \mathcal{B}, \quad (f : x \rightarrow y) \longmapsto (Pf, f) : (1_{Px}, x) \rightarrow (1_{Py}, y)$$

and claim that $(\text{Id}_{\mathcal{B}}, H_P) : P \rightarrow \text{cod}_P$ serves as the unit at P of the 2-adjunction $\mathbf{Free} \dashv \text{Incl}$. Hence, we show that, for every split opfibration $Q : \mathcal{F} \rightarrow \mathcal{C}$, the precomposition with $(\text{Id}_{\mathcal{B}}, H_P)$ provides a bijective functor

$$(-) \cdot (\text{Id}_{\mathcal{B}}, H_P) : \mathbf{SCoFIB}(\text{cod}_P, Q) \longrightarrow \mathbf{CAT}^2(P, Q).$$

To establish its bijectivity on objects, we consider any CAT^2 -morphism $(S, T) : P \rightarrow Q$ and show that there is only one cocleavage-preserving functor $\tilde{T} : P \downarrow \mathcal{B} \rightarrow \mathcal{F}$ with $Q\tilde{T} = S \text{cod}_P$ and $\tilde{T}H_P = T$. First, we observe that, for every morphism $(u, f) : (h, x) \rightarrow (k, y)$ in $P \downarrow \mathcal{B}$ (as in the diagram above), one has the following:

1. $(u, f) = (1_b, f) \cdot (u, 1_x) = \nu_{(u, f)} \cdot \delta_{(h, x)}^u$.
2. With the functor $h_! : (P \downarrow \mathcal{B})_{Px} = P/Px \rightarrow (P \downarrow \mathcal{B})_a = P/a$, the object $(h, x) \in P/a$ may be written as $h_!(1_{Px}, x)$, where $(1_{Px}, x) \in P/Px$.
3. Likewise, with the functor $k_! : P/Py \rightarrow P/b$, the object $(k, y) \in P/b$ may be written as $k_!(1_{Py}, y)$, where $(1_{Py}, y) \in P/Py$, and the morphism $\nu_{(u, f)}$ in P/b may be written as $k_!(1_{Py}, y)$, with the morphism $(1_{Py}, f) : (Pf, x) \rightarrow (1_{Py}, y)$ in P/Py .
4. The morphism $(1_{Py}, f)$ in P/Py as in 3. may be written as $(1_{Py}, f) = \nu_{H_P f}$.

Consequently, for any functor $\tilde{T} : P \downarrow \mathcal{B} \rightarrow \mathcal{F}$ satisfying the above properties, one necessarily has

$$\begin{aligned} \tilde{T}(u, f) &= \tilde{T}(k_!(\nu_{H_P f})) \cdot \tilde{T}(\delta_{h_!(1_{Px}, x)}^u) \\ &= (Sk)_!(\tilde{T}(\nu_{H_P f})) \cdot \delta_{\tilde{T}(h_!(H_P x))}^{Su} \\ &= (Sk)_!(\nu_{\tilde{T}(H_P f)}) \cdot \delta_{(Sh)_!(\tilde{T}(H_P x))}^{Su} \\ &= (Sk)_!(\nu_{Tf}) \cdot \delta_{(Sh)_!(Tx)}^{Su}, \end{aligned}$$

as in

$$\begin{array}{ccccc} (Sh)_!(Tx) & \xrightarrow{\delta_{(Sh)_!(Tx)}^{Su}} & (Su)_!(Sh)_!(Tx) & = & (Sk)_!(Q(Tf))_!(Tx) \xrightarrow{(Sk)_!(\nu_{Tf})} (Sk)_!(Ty) \\ & & & & \downarrow Q \\ Sa & \xrightarrow{Su} & Sb & \xrightarrow{1_{Sb}} & Sb \\ & & & & \downarrow \mathcal{C} \end{array}$$

Therefore, \tilde{T} is unique. Conversely, setting

$$\tilde{T}(u, f) = (Sk)_!(\nu_{Tf}) \cdot \delta_{(Sh)_!(Tx)}^{Su},$$

one has to verify the needed properties for \tilde{T} . Showing that \tilde{T} preserves the composition requires a careful application of the formulae for the $(Q\text{-cocartesian}, Q\text{-vertical})$ -factorization of composite arrows (see Section 9.3). Indeed, using the definition of \tilde{T} for $(u, f) : (h, x) \rightarrow (k, y)$, $(v, g) : (k, y) \rightarrow (\ell, z)$ in $P \downarrow \mathcal{B}$ and the naturality of the transformation δ^{Sv} , we obtain

$$\begin{aligned} \tilde{T}(v \cdot u, g \cdot f) &= (S\ell)_!(\nu_{Tg \cdot Tf}) \cdot \delta_{(Sh)_!(Tx)}^{Sv \cdot Su} \\ &= (S\ell)_![\nu_{Tg} \cdot (S(Pg))_!(\nu_{Tf})] \cdot \delta_{(Su)_!(Sh)_!(Tx)}^{Sv} \cdot \delta_{(Sh)_!(Tx)}^{Su} \\ &= (S\ell)_!(\nu_{Tg}) \cdot (Sv)_!(Sk)_!(\nu_{Tf}) \cdot \delta_{(Sk)_!(S(Pf))_!(Tx)}^{Sv} \cdot \delta_{(Sh)_!(Tx)}^{Su} \\ &= (S\ell)_!(\nu_{Tg}) \cdot \delta_{(Sk)_!(Ty)}^{Sv} \cdot (Sk)_!(\nu_{Tf}) \cdot \delta_{(Sh)_!(Tx)}^{Su} \\ &= \tilde{T}(v, g) \cdot \tilde{T}(u, f). \end{aligned}$$

The verification of the other needed properties of \tilde{T} is straightforward.

It remains to be shown that $(-) \cdot (\text{Id}_{\mathcal{B}}, H_P)$ is fully faithful. Given a 2-cell

$$(\alpha, \beta) : (S, T) \Longrightarrow (S', T') : P \longrightarrow Q$$

in CAT^2 as above, we should find a natural transformation $\tilde{\beta} : \tilde{T} \rightarrow \tilde{T}'$, unique with $Q\tilde{\beta} = \alpha \text{cod}^P$ and $\tilde{\beta}H_P = \beta$. Since any object (h, x) in $P \downarrow \mathcal{B}$ gives rise to the morphism

$$\delta_x^h = (h, 1_x) : H_P x = (1_{Px}, x) \longrightarrow (h, x)$$

in $P \downarrow \mathcal{B}$, the naturality of any such $\tilde{\beta}$ and the preservation of cocleavages by T and T' force

$$\tilde{\beta}_{(h,x)} \cdot \delta_{Tx}^{Sh} = \tilde{\beta}(h, x) \cdot \tilde{T}(\delta_x^h) = \tilde{T}'(\delta_x^h) \cdot \tilde{\beta}_{H_P x} = \delta_{T'x}^{Sh} \cdot \beta_x.$$

Since, with the naturality of α , one has

$$Q(\tilde{\beta}_{(h,x)} \cdot \delta_{Tx}^{Sh}) = \alpha_x \cdot Sh = S'h \cdot \alpha_{Px} = Q(\delta_{T'x}^{S'h} \cdot \beta_x),$$

we see that, by the Q -cocartesianess of δ_{Tx}^{Sh} , the morphism $\tilde{\beta}_{(h,x)}$ is necessarily the only \mathcal{F} -morphism with $\tilde{\beta}_{(h,x)} \cdot \delta_{Tx}^{Sh} = \delta_{T'x}^{S'h} \cdot \beta_x$ and $Q(\tilde{\beta}_{(h,x)}) = \alpha_{Px}$. Conversely, taking this as the definition of $\tilde{\beta}_{(h,x)}$, one routinely shows that $\tilde{\beta}$ has the required properties. \square

We may now compose the 2-adjunction of Theorem 7.1 with the Grothendieck equivalence of Corollary 6.4, as in

$$\begin{array}{ccc} & \text{SCoFIB} & \\ \text{Free} \nearrow & & \nwarrow \text{Fib}_\circ \\ \text{CAT}^2 & \xrightarrow{\text{Incl}} & \text{CAT} // \text{CAT} \\ & \searrow \text{Incl} & \\ & \text{CAT} & \end{array}$$

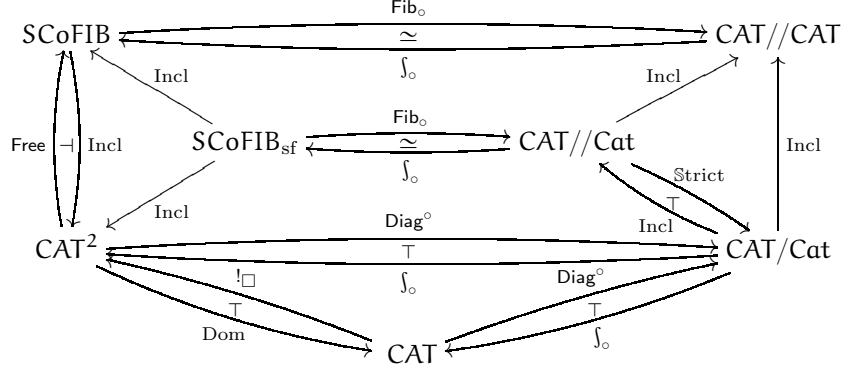
\int_\circ

Since the 2-functor $\text{Fib}_\circ \circ \text{Free}$ assigns to the CAT^2 -object $P : \mathcal{E} \rightarrow \mathcal{B}$ the fibre representation functor of the functor $\text{cod}^P : P \downarrow \mathcal{B} \rightarrow \mathcal{B}$, the fibres of which are the slice categories P/b ($b \in \mathcal{B}$), we conclude:

Corollary 7.2. *The 2-functor $\int_\circ : \text{CAT} // \text{CAT} \longrightarrow \text{CAT}^2$ has a left adjoint which maps the CAT -object $P : \mathcal{E} \rightarrow \mathcal{B}$ to the $\text{CAT} // \text{CAT}$ -object $P/\square : \mathcal{B} \rightarrow \text{CAT}$, $b \mapsto P/b$ (considered in 7.1).*

7.2. A network of global 2-adjunctions

With the help of the following diagram we summarize the 2-adjunctions established in this and the previous sections:



- The top horizontal adjunction displays the Grothendieck 2-equivalence between split cofibrations and CAT -valued functors (Corollary 6.4). It restricts to a 2-equivalence between split cofibrations with small fibres and Cat -valued functors (Theorem 6.3), as shown by the middle horizontal adjunction. The 2-equivalence Fib_o decomposes a split cofibration into the “family” of its fibres, indexed by its base category, while the Grothendieck construction \int_o reassembles such gadgets.
- The “vertical” 2-functor Free modifies a given functor by “freely adding cocartesian liftings” to it, showing that the totality of split cofibrations is 2-reflective amongst all functors (Theorem 7.1). The composition of this 2-adjunction with the top horizontal adjunction is described in Corollary 7.2.
- The bottom horizontal 2-adjunction relates arbitrary functors (rather than split cofibrations) to Cat -valued functors (Theorem 5.4). Its left adjoint, \int_o , trivially factors through the namesakes above it. Not being able to functorially relate the fibres of an arbitrary functor with each other, the right adjoint, Diag° , relates the totality of all small diagrams over the fibres with each other, rather than the fibres themselves. Regarding categories \mathcal{X} as functors $!_{\mathcal{X}} : \mathcal{X} \rightarrow 1$, the lower horizontal 2-adjunction reduces to the lower right diagonal 2-adjunction, as first considered in its ordinary form by Guitart (Theorem 2.12).
- The fibre-representation 2-functor, Fib_o , of the middle horizontal equivalence maps morphisms of its domain to lax-commutative diagrams over Cat , while the right adjoint of the lower horizontal adjunction, Diag° , maps morphisms to strictly commutative diagrams over Cat . In fact, the restriction of the latter 2-functor factors through the former (up to isomorphism), by the strictification 2-functor, Strict , which is right adjoint to a (non-full) inclusion functor (Proposition 6.1, Corollary 6.5)

8. Diagram categories as 2-(co)fibrations over Cat

In this supplementary section we pay tribute to the fact that $D^{\mathcal{X}} : \text{Diag}^\circ(\mathcal{X}) \rightarrow \text{Cat}$ and $D_{\mathcal{X}} : \text{Diag}_o(\mathcal{X}) \rightarrow \text{Cat}^{\text{op}}$ are 2-functors (see Remarks 2.4) and investigate under which conditions on \mathcal{X} (if any), $D^{\mathcal{X}}$ or $D_{\mathcal{X}}$ may be a (co)fibration as such. For that we employ Buckley’s [8] improved version of Hermida’s [19] notion of 2-fibration. We recall the relevant definitions:

Definition 8.1. Let $P : \mathcal{E} \rightarrow \mathcal{B}$ be a 2-functor.

- (1) A 1-cell $f : x \rightarrow y$ in the 2-category \mathcal{E} is *P-2-cartesian* if, for all objects C in \mathcal{E} , the diagram

$$\begin{array}{ccc} \mathcal{E}(z, x) & \xrightarrow{\mathcal{E}(z, f)} & \mathcal{E}(z, y) \\ P_{z, x} \downarrow & & \downarrow P_{z, y} \\ \mathcal{B}(Pz, Px) & \xrightarrow[\mathcal{B}(Pz, Pf)]{} & \mathcal{B}(Pz, Py) \end{array}$$

is a pullback in \mathbf{Cat} .

- (2) A 2-cell $\alpha : f \rightarrow f' : x \rightarrow y$ in \mathcal{E} is *P-2-cartesian* if it is $P_{x, y}$ -cartesian, with respect to the ordinary functor $P_{x, y} : \mathcal{E}(x, y) \rightarrow \mathcal{B}(Px, Py)$.
- (3) P is a (cloven) 2-fibration if
- (a) for all 1-cells $u : a \rightarrow b$ in \mathcal{B} and y objects in \mathcal{E}_b , there is a (chosen) P -2-cartesian lifting $f : x \rightarrow y$ in \mathcal{E} , so that $Px = a$ and $Pf = u$;
 - (b) for all objects $x, y \in \mathcal{E}$, the ordinary functor $P_{x, y} : \mathcal{E}(x, y) \rightarrow \mathcal{B}(Px, Py)$ is a (cloven) fibration;
 - (c) P -2-cartesianness of 2-cells in \mathcal{E} is preserved by horizontal composition.
- (4) P is a 2-cofibration if $P^{\text{coop}} : \mathcal{E}^{\text{coop}} \rightarrow \mathcal{B}^{\text{coop}}$ is a 2-fibration.

Remark 8.2. (1) By definition, the 1-cell $f : x \rightarrow y$ in \mathcal{E} is P -2-cartesian if, and only if, for all objects $z \in \mathcal{E}$, the functor

$$\mathcal{E}(z, x) \rightarrow \mathcal{B}(Pz, Px) \times_{\mathcal{B}(Pz, Py)} \mathcal{E}(z, y), \quad (\tau : t \rightarrow t') \mapsto (P\tau : Pt \rightarrow Pt', f\tau : ft \rightarrow ft'),$$

is an isomorphism of categories. Its bijectivity on objects is equivalent to f being P -cartesian in the ordinary sense, while its full faithfulness adds the following condition to the 1-categorical notion: for all 2-cells $\zeta : w \Rightarrow w' : Pz \rightarrow Px$ in \mathcal{B} and $\rho : h \Rightarrow h' : z \rightarrow y$ in \mathcal{E} with $(Pf)\zeta = P\rho$, there is a unique 2-cell $\tau : t \Rightarrow t' : z \rightarrow x$ in \mathcal{E} with $P\tau = \zeta$ and $f\tau = \rho$.

(2) By definition, P -2-cartesianness of a 2-cell $\alpha : f \Rightarrow f' : x \rightarrow y$ in \mathcal{E} means that, for all 1-cells $k : x \rightarrow y$, the map

$$\mathcal{E}(x, y)(k, f) \rightarrow \mathcal{B}(Px, Py)(Pk, Pf) \times_{\mathcal{B}(Px, Py)(Pk, Pf')} \mathcal{E}(x, y)(k, f'), \quad \mu \mapsto (P\mu, \sigma \cdot \mu),$$

is bijective, that is: for all 2-cells $\gamma : Pk \Rightarrow Pf$ and $\lambda : k \Rightarrow f'$ in, respectively, \mathcal{B} and \mathcal{E} , with $P\alpha \cdot \gamma = P\lambda$, one has $P\mu = \lambda$ and $\alpha \cdot \mu = \lambda$, for a unique 2-cell $\mu : k \rightarrow f$ in \mathcal{E} .

(3) P is a 2-fibration if, and only if,

- (a) for every 1-cell $u : a \rightarrow Py$ in \mathcal{B} with y in \mathcal{E} , there is a P -2-cartesian lifting $f : x \rightarrow y$ in \mathcal{E} with $Pf = u$;
- (b) for every 2-cell $\xi : u \Rightarrow Pf' : Px \rightarrow Py$ in \mathcal{B} with a (P -2-cartesian) 1-cell $f' : x \rightarrow y$ in \mathcal{E} , there is a P -2-cartesian lifting $\alpha : f \Rightarrow f' : x \rightarrow y$ with $P\alpha = \xi$;
- (c) for all 1-cells $t : z \rightarrow x$, $s : y \rightarrow w$ and 2-cells $\alpha : f \Rightarrow f' : x \rightarrow y$ in \mathcal{E} , if α is 2-cartesian, so are $\alpha t : ft \rightarrow f't$ and $s\alpha : sf \rightarrow sf'$. (Of course, since P -cartesianness of 2-cells is closed under vertical composition, closure under (horizontal) pre- and post-composition with 1-cells suffices to make the property closed also under horizontal composition.)

Remark 8.3. The definition of (in a quite obvious sense) split 2-fibration as given above is motivated by the fact that a 2-fibration is, via a 2-categorical Grothendieck construction, 3-equivalently represented by a 2-functor $\mathcal{B}^{\text{coop}} \rightarrow 2\text{Cat}$; see [8]. In fact, Buckley [8] proved a more general result at the bicategorical (rather than the 2-categorical) level.

Theorem 8.4. (1) *The 2-functor $D_{\mathcal{X}}^{\text{op}} : (\text{Diag}_{\circ}(\mathcal{X}))^{\text{op}} \rightarrow \text{Cat}$ is a 2-fibration, for every category \mathcal{X} .*
(2) *If the category \mathcal{X} is cocomplete, then $D^{\mathcal{X}} : \text{Diag}^{\circ}(\mathcal{X}) \rightarrow \text{Cat}$ is a 2-cofibration.*

Proof. (1) Recall that a morphism $(F, \varphi) : (\mathcal{I}, X) \rightarrow (\mathcal{J}, Y)$ in $(\text{Diag}_{\circ}(\mathcal{X}))^{\text{op}}$ is given by small categories \mathcal{I}, \mathcal{J} , functors F, X, Y , and a natural transformation φ , as in the triangle below on the left, and a 2-cell $\alpha : (F, \varphi) \Rightarrow (F', \varphi')$ is given by a natural transformation $\alpha : F \rightarrow F'$ with $\varphi = \varphi' \cdot Y\alpha$:



Now, given $F : \mathcal{I} \rightarrow \mathcal{J}$ in Cat and $(\mathcal{J}, Y) \in (\text{Diag}_{\circ}(\mathcal{X}))^{\text{op}}$, we have the trivial $D_{\mathcal{X}}^{\text{op}}$ -cartesian lifting $(F, 1_{FY}) : (\mathcal{I}, YF) \rightarrow (\mathcal{J}, Y)$ at the 1-category level (Proposition 2.2). To show that $(F, 1_{FY})$ is $D_{\mathcal{X}}^{\text{op}}$ -2-cartesian, it suffices to consider a natural transformation $\zeta : G \Rightarrow G' : \mathcal{K} \rightarrow \mathcal{I}$ and a 2-cell $\rho : (H, \gamma) \Rightarrow (H', \gamma') : (\mathcal{K}, Z) \rightarrow (\mathcal{J}, Y)$ with $FG = H$, $FG' = H'$, $F\zeta = \rho$, and show that $\zeta : (G, \gamma) \Rightarrow (G', \gamma') : (\mathcal{K}, Z) \rightarrow (\mathcal{I}, YF)$ is actually a 2-cell in $(\text{Diag}_{\circ}(\mathcal{X}))^{\text{op}}$. But this is trivial: the given identity $\gamma' \cdot Y\rho = \gamma$ may just be restated as the needed identity $\gamma' \cdot (YF)\zeta = \gamma$.

Next, in order to verify property (b) of Remark 8.2, we consider a 1-cell $(F', \varphi') : (\mathcal{I}, X) \rightarrow (\mathcal{J}, Y)$ in $(\text{Diag}_{\circ}(\mathcal{X}))^{\text{op}}$ and a 2-cell $\alpha : F \Rightarrow F' : \mathcal{I} \rightarrow \mathcal{J}$ in Cat and show that the emerging 2-cell $\alpha : (F, \varphi := \varphi' \cdot Y\alpha) \Rightarrow (F', \varphi')$ is $D_{\mathcal{X}}^{\text{op}}$ -2-cartesian. Indeed, given 2-cells $\lambda : (K, \kappa) \rightarrow (F', \varphi')$ and $\gamma : K \rightarrow F$ in, respectively, $(\text{Diag}_{\circ}(\mathcal{X}))^{\text{op}}$ and Cat with $\alpha \cdot \gamma = \lambda$, the given identity $\varphi' \cdot Y\lambda = \kappa$ translates to $\varphi \cdot Y\gamma = \kappa$, thus making $\gamma : (K, \kappa) \rightarrow (F, \varphi)$ a 2-cell in $\text{Diag}^*(\mathcal{X})$, as desired.

Finally, to verify property (c), for 1-cells $(T, \eta) : (\mathcal{K}, Z) \rightarrow (\mathcal{I}, X)$, $(S, \varepsilon) : (\mathcal{J}, Y) \rightarrow (\mathcal{L}, W)$ and the $D_{\mathcal{X}}^{\text{op}}$ -2-cartesian 2-cell $\alpha : (F, \varphi) \Rightarrow (F', \varphi') : (\mathcal{I}, X) \rightarrow (\mathcal{J}, Y)$ as above, we must show that the horizontal composites

$$(FT, \eta \cdot \varphi T) \xRightarrow{\alpha T} (F'T, \eta \cdot \varphi' T), \quad (SF, \varphi \cdot \varepsilon F) \xRightarrow{S\alpha} (SF', \varphi' \cdot \varepsilon F')$$

are $D_{\mathcal{X}}^{\text{op}}$ -2-cartesian as well. Indeed, from $\varphi' \cdot Y\alpha = \varphi$ one obtains immediately

$$\eta \cdot \varphi' T \cdot Y(\alpha T) = \eta \cdot \varphi T, \quad \varphi' \cdot \varepsilon F' \cdot W(S\alpha) = \varphi' \cdot Y\alpha \cdot \varepsilon F = \varphi \cdot \varepsilon F,$$

as desired.

(2) We now consider $(\mathcal{I}, X) \in \text{Diag}^{\circ}(\mathcal{X})$ and $F : \mathcal{I} \rightarrow \mathcal{J}$ in Cat and form the $D^{\mathcal{X}}$ -cocartesian lifting $(F, \varphi) : (\mathcal{I}, X) \rightarrow (\mathcal{J}, Y)$ at the 1-categorical level, so that $\varphi : X \rightarrow YF$ presents Y as a left Kan extension of X along F (Proposition 2.6). To show that (F, φ) is $D^{\mathcal{X}}$ -2-cocartesian, given any 2-cells $\tau : G \Rightarrow G' : \mathcal{J} \rightarrow \mathcal{K}$ and $\rho : (H, \gamma) \Rightarrow (H', \gamma') : (\mathcal{I}, X) \rightarrow (\mathcal{K}, Z)$ with $GF = H$, $G'F = H'$, $\tau F = \rho$, we let $\beta : Y \rightarrow ZH$, $\beta' : Y \rightarrow ZH'$ be determined by the identities

$\beta F \cdot \varphi = \gamma$, $\beta' F \cdot \varphi = \gamma'$ and must then confirm that $\tau : (G, \beta) \Longrightarrow (G', \beta') : (\mathcal{J}, Y) \rightarrow (\mathcal{K}, Z)$ is a 2-cell in $\text{Diag}^\circ(\mathcal{X})$. But this is straightforward, since from

$$(Z\tau \cdot \beta)F \cdot \varphi = Z\tau F \cdot \beta F \cdot \varphi = Z\tau F \cdot \gamma = Z\rho \cdot \gamma = \gamma' = \beta' F \cdot \varphi$$

one deduces the desired identity $Z\tau \cdot \beta = \beta'$.

Finding a $D^\mathcal{X}$ -2-cocartesian lifting for a 2-cell $\alpha : F \Longrightarrow F'$ in Cat that comes with a 1-cell $(F, \varphi) : (\mathcal{I}, X) \rightarrow (\mathcal{J}, Y)$ in $\text{Diag}^\circ(\mathcal{X})$ proceeds as in (1): one just puts $\varphi' := Y\alpha \cdot \varphi$ and easily shows that, given 2-cells $\lambda : (F, \varphi) \Longrightarrow (K, \kappa)$ in $\text{Diag}^\circ(\mathcal{X})$ and $\chi : F' \rightarrow K$ in Cat with $\chi \cdot \alpha = \lambda$, then $\chi : (F', \varphi') \rightarrow (K, \kappa)$ actually lives in $\text{Diag}^\circ(\mathcal{X})$. Likewise, also the easy proof that pre- and post-composition with 1-cells in $\text{Diag}^\circ(\mathcal{X})$ preserves the $D^\mathcal{X}$ -2-cocartesianess of $\alpha : (F, \varphi) \Longrightarrow (F', \varphi')$ proceeds as in (1). \square

Remark 8.5. We note that the 2-fibration $D_{\mathcal{X}}^{\text{op}}$ is split, in the obvious sense that the induced functor

$$\Pi_{D_{\mathcal{X}}^{\text{op}}} : \text{Cat}^{\text{op}} \rightarrow \text{CAT}, (F : \mathcal{I} \rightarrow \mathcal{J} \text{ in } \text{Cat}) \mapsto (F^* : [\mathcal{J}, \mathcal{X}]^{\text{op}} \rightarrow [\mathcal{I}, \mathcal{X}]^{\text{op}}, Y \mapsto YF),$$

is actually a 2-functor. It assigns to a small category \mathcal{I} its fibre in $(\text{Diag}_\circ(\mathcal{X}))^{\text{op}}$, which is precisely the category $[\mathcal{I}, \mathcal{X}]^{\text{op}}$. Furthermore, for all objects $(\mathcal{I}, X), (\mathcal{J}, Y)$ in $(\text{Diag}_\circ(\mathcal{X}))^{\text{op}}$, the fibration

$$(\text{Diag}_\circ(\mathcal{X}))^{\text{op}}(X, Y) \longrightarrow [\mathcal{I}, \mathcal{J}], (F, \varphi) \mapsto F$$

is actually discrete.

9. Appendix: A primer on Grothendieck fibrations and the Grothendieck construction

9.1. Cartesian morphisms

Given a functor $P : \mathcal{E} \rightarrow \mathcal{B}$, a morphism $f : x \rightarrow y$ in \mathcal{E} is a *lifting* (along P) of a morphism $u : a \rightarrow b$ in \mathcal{B} if $Pf = u$. The lifting f is *P -cartesian* if every diagram of solid arrows below can be filled uniquely, as shown:

$$\begin{array}{ccc} z & \xrightarrow[t]{h} & x \xrightarrow{f} y \\ & \searrow & \downarrow P \\ Pz & \xrightarrow[w]{Ph} & Px \xrightarrow{Pf} Py \end{array} \quad \begin{array}{c} \mathcal{E} \\ \downarrow P \\ \mathcal{B} \end{array}$$

Thus, if $h : z \rightarrow y$ in \mathcal{E} and $w : Pz \rightarrow Px$ in \mathcal{B} satisfy $Pf \cdot w = Pg$, then there is exactly one morphism $t : z \rightarrow x$ in \mathcal{E} with $f \cdot t = g$ and $Pt = w$; equivalently, for every object z in \mathcal{E} , the square

$$\begin{array}{ccc} \mathcal{E}(z, x) & \xrightarrow{\mathcal{E}(z, f)} & \mathcal{E}(z, y) \\ P_{z, x} \downarrow & & \downarrow P_{z, y} \\ \mathcal{B}(Pz, Px) & \xrightarrow{\mathcal{B}(Pz, Pf)} & \mathcal{B}(Pz, Py) \end{array}$$

is a pullback diagram in Set .

We note that, for any functor P , a P -cartesian lifting f of u is an isomorphism if, and only if, u is an isomorphism. The class $\text{Cart}(P)$ of P -cartesian morphisms in \mathcal{E} contains all isomorphisms of \mathcal{E} , is closed under composition, and satisfies the cancellation condition ($g \cdot f \in \text{Cart}(P) \implies f \in \text{Cart}(P)$) whenever g is monic or P -cartesian. Moreover, the class $\text{Cart}(P)$ is stable under those pullbacks in \mathcal{E} which P transforms into monic pairs; in particular, $\text{Cart}(P)$ is stable under the pullbacks that are preserved by P .

9.2. Grothendieck fibrations

For an object b in \mathcal{B} we denote by \mathcal{E}_b the *fibre* of $P : \mathcal{E} \rightarrow \mathcal{B}$ at b ; this is the (non-full) subcategory of \mathcal{E} of all morphisms in \mathcal{E} that are liftings of 1_b . Hence, for the inclusion functor $J_b : \mathcal{E}_b \rightarrow \mathcal{E}$ the functor $PJ_b = \Delta b$ is constant. The morphisms in the fibres of P are also called *P -vertical*. The functor P is a (*Grothendieck*) *fibration* if, for every morphism $u : a \rightarrow b$ in \mathcal{B} and every object y in \mathcal{E}_b , there is a P -cartesian lifting $f : x \rightarrow y$ in \mathcal{E} . Since such a lifting is unique up to isomorphism when considered as an object in the slice category \mathcal{E}/y , one may call f *the* P -cartesian lifting of u at y . In fact, we will assume throughout that our fibrations are *cloven*; this means, that a choice of P -cartesian liftings, also called a *cleavage*, has been made for all u and y . We denote the chosen P -cartesian lifting of $u : a \rightarrow b$ in \mathcal{B} at $y \in \mathcal{E}_b$ by $\theta_y^u : u^*(y) \rightarrow y$. With this notation one sees immediately that a functor P is a fibration if, and only if, for every object y in \mathcal{E} , the induced functor

$$P_y : \mathcal{E}/y \longrightarrow \mathcal{B}/Py$$

of the slice categories has a right adjoint right inverse (*rari*), namely θ_y (see [14]).

For a fibration $P : \mathcal{E} \rightarrow \mathcal{B}$, one also calls \mathcal{E} *fibred over* \mathcal{B} . Every morphism $f : x \rightarrow y$ in \mathcal{E} then has a (P -vertical, P -cartesian)-factorization, as in

$$\begin{array}{ccc} x & & \\ \downarrow \epsilon_f & \searrow f & \\ u^*(y) & \xrightarrow{\theta_y^u} & y \end{array},$$

where $u = Pf$, and where the P -vertical morphism ϵ_f is uniquely determined by f . In fact, there is, for all morphisms $u : a \rightarrow b$ and objects $x \in \mathcal{E}_a$, $y \in \mathcal{E}_b$ a natural bijective correspondence

$$\mathcal{E}_u(x, y) \cong \mathcal{E}_a(x, u^*(y)),$$

where $\mathcal{E}_u(x, y) = \mathcal{E}(x, y) \cap P^{-1}(u)$. The P -vertical morphisms and, more generally, the \mathcal{E} -morphisms which are mapped by P to isomorphisms, are orthogonal to P -cartesian morphisms. As a consequence one obtains that a functor $P : \mathcal{E} \rightarrow \mathcal{B}$ is a fibration if, and only if, P is an *iso-fibration* (that is; if every isomorphism $u : a \rightarrow b$ in \mathcal{B} admits a P -cartesian lifting at every $y \in \mathcal{E}_b$), and if $(P^{-1}(\text{Iso } \mathcal{B}), \text{Cart}(P))$ is an orthogonal factorization system of \mathcal{E} ; the second property means equivalently that P is a *Street fibration* [34].

For every morphism $u : a \rightarrow b$ in \mathcal{B} , the domains of the P -cartesian liftings of u at the objects of \mathcal{E}_b give the object assignment of a functor $u^* : \mathcal{E}_b \rightarrow \mathcal{E}_a$ that makes $\theta^u : J_a u^* \rightarrow J_b$ a natural transformation; for a morphism $j : y \rightarrow y'$ in \mathcal{E}_b one has $u^*(j) = \epsilon_{j \cdot \theta_y^u}$.

$$\begin{array}{ccc}
\mathcal{E}_a & \xleftarrow{u^*} & \mathcal{E}_b \\
& \searrow \theta^u \rightrightarrows & \swarrow J_b \\
& \mathcal{E} &
\end{array}
\qquad
\begin{array}{ccc}
u^*(y) & \xrightarrow{\theta_y^u} & y \\
u^*(j) \downarrow & & \downarrow j \\
u^*(y') & \xrightarrow{\theta_{y'}^u} & y'
\end{array}$$

The commutative diagram below shows that the object assignment $b \mapsto \mathcal{E}_b$ leads to a pseudofunctor

$$\Phi^P : \mathcal{B}^{\text{op}} \longrightarrow \text{CAT}, \quad (u : a \rightarrow b \text{ in } \mathcal{B}) \longmapsto (u^* : \mathcal{E}_b \rightarrow \mathcal{E}_a)$$

and, thus, presents the fibration P as an *indexed category* [25].

$$\begin{array}{c}
x \xrightarrow{f} y \\
\epsilon_f \downarrow \quad \searrow \theta_y^u \\
u^*(y) \xrightarrow{\theta_y^u} y \\
\downarrow u^*(\epsilon_g) \quad \searrow \theta_z^u \\
(v \cdot u)^*(z) \cong u^*(v^*(z)) \xrightarrow{\theta_{v^*(z)}^u} v^*(z) \xrightarrow{\theta_z^v} z
\end{array}
\qquad
\begin{array}{c}
y \xrightarrow{g} z \\
\downarrow \epsilon_g \quad \searrow \theta_z^v \\
v^*(z) \xrightarrow{\theta_z^v} z
\end{array}
\qquad
y \cong 1_b^*(y) \xrightarrow{\theta_y^{1_b}} y$$

$$a \xrightarrow{u} b \xrightarrow{v} c = Pz \qquad b \xrightarrow{1_b} b$$

If Φ^P is actually a functor, with the above canonical isomorphisms becoming identities, so that

$$(v \cdot u)^* = u^* v^*, \quad (1_b)^* = \text{Id}_{\mathcal{E}_b}, \quad \text{and} \quad \theta^{v \cdot u} = \theta^v \cdot \theta^u v^*, \quad \theta^{1_b} = 1_{\text{Id}_{\mathcal{E}_b}}$$

for all composable morphisms u, v and objects b in \mathcal{B} , then P is called a *split fibration*.

A functor $P : \mathcal{E} \rightarrow \mathcal{B}$ is *small-fibred* if all of its fibres are small; in case of a fibration P , this means that Φ^P takes its values in Cat . A fibration P is *discrete* if all of its fibres are discrete, that is: if Φ^P takes its values in SET . Clearly a functor P is a discrete fibration if, and only if there is, for every $u : a \rightarrow b$ in \mathcal{B} and $y \in \mathcal{E}_b$, exactly one lifting with codomain y ; the fibration is necessarily split.

Here is how some elementary properties manifest themselves for a fibration $P : \mathcal{E} \rightarrow \mathcal{B}$: P is faithful (full; essentially surjective on objects) if, and only if, for every $b \in \mathcal{B}$, the fibre \mathcal{E}_b is a preordered class (has all of its homs non-empty; is non-empty, respectively). When \mathcal{E} has a terminal object, a fibration P is an equivalence of categories if, and only if, P preserves the terminal object and reflects isomorphisms. (In the last statement, the preservation of the terminal object is essential: for a monic arrow $f : x \rightarrow y$ in a category \mathcal{C} , the discrete fibration $f \cdot (-) : \mathcal{C}/x \rightarrow \mathcal{C}/y$ is fully faithful, but does not preserve the terminal object 1_x of \mathcal{C}/x , unless f is an isomorphism in \mathcal{C} .)

9.3. Grothendieck cofibrations and bifibrations

For a functor $P : \mathcal{E} \rightarrow \mathcal{B}$, a morphism $f : x \rightarrow y$ in \mathcal{E} is *P-cocartesian* if f is P^{op} -cartesian in \mathcal{E}^{op} , with $P^{\text{op}} : \mathcal{E}^{\text{op}} \rightarrow \mathcal{B}^{\text{op}}$. This means that every solid-arrow diagram below on the left can be filled uniquely as shown.

$$\begin{array}{ccc}
 & h & \\
 x \xrightarrow{f} y & \dashrightarrow_s & z \\
 & Ph & \\
 Px \xrightarrow{Pf} Py & \xrightarrow{v} & Pz
 \end{array}
 \qquad
 \begin{array}{ccc}
 x & \xrightarrow{\delta_x^u} & u_!(x) = y \\
 a = Px & \xrightarrow{u} & b
 \end{array}
 \qquad
 \begin{array}{c}
 \mathcal{E} \\
 \downarrow P \\
 \mathcal{B}
 \end{array}$$

P is a (cloven Grothendieck) cofibration if $P^{\text{op}} : \mathcal{E}^{\text{op}} \rightarrow \mathcal{B}^{\text{op}}$ is a fibration⁷. This means that for every morphism $u : a \rightarrow b$ in \mathcal{B} and every object x in \mathcal{E}_a one has a (chosen) P -cocartesian lifting, which we denote by $\delta_x^u : x \rightarrow u_!(x)$; this fixes the *cocleavage* $\delta^u : J_a \rightarrow J_b u_!$. Every morphism $f : x \rightarrow y$ in \mathcal{E} now admits the (P -cocartesian, P -vertical)-factorization $f = \nu_f \cdot \delta_x^u$, with $u = Pf$.

$$\begin{array}{ccc} \mathcal{E}_a & \xrightarrow{u_!} & \mathcal{E}_b \\ & \delta^u \rightrightarrows & \\ J_a \searrow & & \swarrow J_b \\ & \mathcal{E} & \end{array} \qquad \begin{array}{ccc} & & y \\ & \nearrow f & \uparrow \nu_f \\ x & \xrightarrow{\delta_x^u} & u_!(x) \end{array}$$

One obtains a pseudofunctor

$$\Phi_P : \mathcal{B} \longrightarrow \mathbf{CAT}, \quad (u : a \rightarrow b) \longmapsto (u! : \mathcal{E}_a \rightarrow \mathcal{E}_b),$$

and the cofibration P is *split* if Φ_P is a functor; more precisely, if

$$(v \cdot u)! = v! u!, \quad (1_b)! = \text{Id}_{\mathcal{E}_b}, \quad \text{and} \quad \delta^{v \cdot u} = \delta^v u! \cdot \delta^u, \quad \delta^{1_b} = 1_{\text{Id}_{\mathcal{E}_b}},$$

for all composable morphisms u, v and objects b in \mathcal{B} .

A functor P is a *bifibration* if it is simultaneously a fibration and a cofibration. The following criterion is certainly known but is not easily found and clearly spelled out in the literature:

Theorem 9.1. *The following assertions are equivalent for a functor $P : \mathcal{E} \rightarrow \mathcal{B}$:*

- (i) P is a bifibration;
- (ii) P is a fibration, and the functor u^* has a left adjoint $u_!$, for all $u : a \rightarrow b$ in \mathcal{B} ;
- (iii) P is a cofibration, and the functor $u_!$ has a right adjoint u^* , for all $u : a \rightarrow b$ in \mathcal{B} .

For a bifibration P , the units η^u and counits ε^u of the adjunctions $u_! \dashv u^*$ are determined by the commutative diagrams

$$\begin{array}{ccc}
J_a & & J_b \\
J_a \eta^u \downarrow & \searrow \delta^u & \uparrow \theta^u \\
J_a u^* u! & \xrightarrow{\theta^u u!} & J_b u!
\end{array}$$

Corollary 9.2. *For a bifibration $P : \mathcal{E} \rightarrow \mathcal{B}$ and every morphism $u : A \rightarrow B$ in \mathcal{B} , the functor $u^* : \mathcal{E}_b \rightarrow \mathcal{E}_a$ preserves all limits and $u_! : \mathcal{E}_a \rightarrow \mathcal{E}_b$ preserves all colimits.*

⁷Grothendieck cofibrations are now commonly referred to as *opfibrations*: see Footnote 1 of the Introduction.

9.4. Limits and colimits in a bifibred category

It is certainly known how to form ordinary (co)limits of a specified type in a bifibred category \mathcal{E} from given (co)limits of the same type in the base category \mathcal{B} and the fibres of the fibration. (For notions of, and criteria for, fibrational completeness, see [5, Section 8.5].) We sketch here a detailed but compact proof of this fact, as we trace its steps in our main application in Section 3.

Theorem 9.3. *Let $P: \mathcal{E} \rightarrow \mathcal{B}$ be a bifibration. If limits of shape \mathcal{D} exists in \mathcal{B} and in all fibers of P , then \mathcal{D} -limits exist also in \mathcal{E} and are preserved by P .*

Proof. Construction: Choose a cleavage θ and a cocleavage δ for P . For a diagram $F: \mathcal{D} \rightarrow \mathcal{E}$, let $b := \lim(PF)$ in \mathcal{B} , with limit cone $\beta: \Delta b \rightarrow PF$. For every object d in \mathcal{D} we have the P -cartesian lifting of β_d at Fd ,

$$(\alpha_d: Ld \rightarrow Fd) := (\theta_{Fd}^{\beta_d}: \beta_d^*(Fd) \rightarrow Fd) ,$$

to obtain a functor $L: \mathcal{D} \rightarrow \mathcal{E}_b$, together with a natural transformation $\alpha: J_b L \rightarrow F$. By design, $PJ_b L = \Delta b$ and $P\alpha = \beta$. Now let $z := \lim(L)$, with limit cone $\lambda: \Delta z \rightarrow L$. We claim that the composite transformation

$$\Delta z \xrightarrow{J_b \lambda} J_b L \xrightarrow{\alpha} F$$

is a limit cone in \mathcal{E} .

Weak universal property: Consider any cone $\mu: \Delta x \rightarrow F$ in \mathcal{E} . Its P -image factors as $\beta \cdot \Delta u = P\mu$, for a unique \mathcal{B} -morphism $u: Px \rightarrow b$. As α_d is P -cartesian, for every $d \in \mathcal{D}$ one has a morphism $\gamma_d: x \rightarrow Ld$, unique with $\alpha_d \cdot \gamma_d = \mu_d$ and $P\gamma_d = u$. This gives a cone $\gamma: \Delta x \rightarrow J_b L$ in \mathcal{E} with $\alpha \cdot \gamma = \mu$ and $P\gamma = \Delta u$. With the P -cocartesian lifting of u ,

$$(f: x \rightarrow y) := (\delta_x^u: x \rightarrow u!x) ,$$

the cone γ factors as $J_b \kappa \cdot \Delta f = \gamma$, for a unique cone $\kappa: \Delta y \rightarrow L$ in \mathcal{E}_b . In turn, κ factors as $\lambda \cdot \Delta k = \kappa$, for a unique \mathcal{E}_b -morphism $k: y \rightarrow z$, thus completing the factorization of μ through $\alpha \cdot J_b \lambda$:

$$(\alpha \cdot J_b \lambda) \cdot \Delta(k \cdot f) = \alpha \cdot J_b \kappa \cdot \Delta f = \alpha \cdot \gamma = \mu.$$

Strict universal property: If $g: x \rightarrow z$ is any \mathcal{E} -morphism with $\alpha \cdot J_b \lambda \cdot \Delta g = \mu$, we must confirm $g = k \cdot f$. An application of P to the given identity shows $\beta \cdot \Delta Pg = P\mu = \beta \cdot \Delta u$ and, hence, $Pg = u$. Now g factors through the P -cocartesian morphism f as $\ell \cdot f = g$, for a morphism $\ell: y \rightarrow z$ in \mathcal{E}_b . As f is P -cocartesian,

$$\alpha \cdot J_b(\lambda \cdot \Delta \ell) = \alpha \cdot J_b \lambda \cdot \Delta k = \alpha \cdot J_b \kappa .$$

Thus $\lambda \cdot \Delta \ell = \kappa$, since every component of α is P -cartesian. The limit property in \mathcal{E}_b gives $\ell = k$, so that $g = \ell \cdot f = k \cdot f$ completes the proof. \square

An application of the theorem to P^{op} instead of P produces the dual statement:

Corollary 9.4. *Let $P: \mathcal{E} \rightarrow \mathcal{B}$ be a bifibration. If colimits of shape \mathcal{D} exist in \mathcal{B} and in all fibers of P , then \mathcal{D} -colimits exist in \mathcal{E} and are preserved by P .*

9.5. The Grothendieck construction for indexed categories

The indexed category $\Phi^P : \mathcal{B}^{\text{op}} \rightarrow \text{CAT}$ of a fibration $P : \mathcal{E} \rightarrow \mathcal{B}$ preserves all information about the base category \mathcal{B} and the fibres $\mathcal{E}_b = \Phi^P b$ ($b \in \mathcal{B}$), including their pseudo-functorial interaction. The Grothendieck construction shows how one can rebuild the category \mathcal{E} from that information. Below (on the left) is the definition of the *Grothendieck category* (also *total category*) of Φ , usually denoted by $\int^\circ \Phi$, in the split (=strict) case, that is, for a genuine functor $\Phi : \mathcal{B}^{\text{op}} \rightarrow \text{CAT}$.

On the right we describe the dual construction, *i.e.*, give the definition of the *dual Grothendieck category*, $\int_\circ \Phi$, for a functor $\Phi : \mathcal{B} \rightarrow \text{CAT}$. In the case $\Phi = \Phi_P$ where $P : \mathcal{E} \rightarrow \mathcal{B}$ is a cofibration, it recovers the category \mathcal{E} .

<p>The Grothendieck category $\int^\circ \Phi$ of a functor $\Phi : \mathcal{B}^{\text{op}} \rightarrow \text{CAT}$ is the category with</p> <ul style="list-style-type: none"> • <i>objects</i> pairs (b, y), for $b \in \mathcal{B}$ and $y \in \Phi b$; • <i>morphisms</i> $(u, f) : (a, x) \rightarrow (b, y)$, for $u : a \rightarrow b$ in \mathcal{B} and $f : x \rightarrow (\Phi u)y$ in Φa; <div style="text-align: center; margin: 10px 0;"> $\begin{array}{ccc} (a, x) & & \\ (1_a, f) \downarrow & \searrow (u, f) & \\ (a, (\Phi u)y) & \xrightarrow{(u, 1_{(\Phi u)y})} & (b, y) \\ a \xrightarrow{u} b & & \end{array}$ </div> <ul style="list-style-type: none"> • <i>composition</i> $(v, g) \cdot (u, f) = (v \cdot u, (\Phi u)g \cdot f)$. <p>$\int^\circ \Phi$ is fibred over \mathcal{B}, with split fibration</p> $\Pi^\Phi : \int^\circ \Phi \rightarrow \mathcal{B}, \quad (u, f) \mapsto u,$ $u^*(b, y) = (a, (\Phi u)y),$ $\theta_{(b, y)}^u = (u, 1_{(\Phi u)y}), \quad \epsilon_{(u, f)} = (1_a, f).$	<p>The dual Grothendieck category $\int_\circ \Phi$ of a functor $\Phi : \mathcal{B} \rightarrow \text{CAT}$ is the category with</p> <ul style="list-style-type: none"> • <i>objects</i> pairs (a, x), for $a \in \mathcal{B}$ and $x \in \Phi a$; • <i>morphisms</i> $(u, f) : (a, x) \rightarrow (b, y)$, for $u : a \rightarrow b$ in \mathcal{B} and $f : (\Phi u)x \rightarrow y$ in Φb; <div style="text-align: center; margin: 10px 0;"> $\begin{array}{ccc} & & (b, y) \\ & \nearrow (u, f) & \uparrow (1_b, f) \\ (a, x) & \xrightarrow{(u, 1_{(\Phi u)x})} & (b, (\Phi u)x) \\ a \xrightarrow{u} b & & \end{array}$ </div> <ul style="list-style-type: none"> • <i>composition</i> $(v, g) \cdot (u, f) = (v \cdot u, g \cdot (\Phi v)f)$. <p>$\int_\circ \Phi$ is fibred over \mathcal{B}, with split cofibration</p> $\Pi_\Phi : \int_\circ \Phi \rightarrow \mathcal{B}, \quad (u, f) \mapsto u,$ $u_!(a, x) = (b, (\Phi u)x),$ $\delta_{(a, x)}^u = (u, 1_{(\Phi u)x}), \quad \nu_{(u, f)} = (1_b, f).$
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One can make precise in which sense the construction on the right is dual to the construction on the left, as follows. Given $\Phi : \mathcal{B} \rightarrow \text{CAT}$, dualize the “base” \mathcal{B} and every “fibre” Φb ($b \in \mathcal{B}$), that is: form the indexed category

$$\Phi^\circ := [\mathcal{B} = (\mathcal{B}^{\text{op}})^{\text{op}} \xrightarrow{\Phi} \text{CAT} \xrightarrow{(-)^{\text{op}}} \text{CAT}].$$

Then there is a trivial bijective functor mapping objects and morphisms identically and making

$$\begin{array}{ccc}
 \int^\circ(\Phi^\circ) & \xrightarrow{\cong} & (\int_\circ \Phi)^{\text{op}} \\
 \searrow \Pi^{\Phi^\circ} & & \swarrow (\Pi_\Phi)^{\text{op}} \\
 & \mathcal{B}^{\text{op}} &
 \end{array}$$

commute. (We note that there is also the Borceux-Kock dualization of a fibration which dualizes the fibres but not the base, turning the (vertical, cartesian) factorization for a fibration into a (cartesian, vertical) factorization for a “dual fibration”; see [26] for details).

An elementary (and quite obvious) rendition of the equivalence of split fibrations and strictly functorial indexed categories reads as follows; as a 2-categorical equivalence it is formulated as Corollary 6.6 .

Theorem 9.5. (i) *For every split fibration $P : \mathcal{E} \rightarrow \mathcal{B}$ with cleavage θ , there is a bijective functor K^P , satisfying $PK^P = \Pi^{\Phi^P}$ and preserving the cleavages, given by*

$$\begin{array}{ccc} \int^{\circ} \Phi^P & \xrightarrow{K^P} & \mathcal{E} \\ \Pi^{\Phi^P} \downarrow & & \downarrow P \\ \mathcal{B} & \xlongequal{\quad} & \mathcal{B} \end{array} \quad \begin{array}{ccc} (b, y) & \xrightarrow{\quad} & y \\ [(u, f) : (a, x) \rightarrow (b, y)] & \xrightarrow{\quad} & [\theta_y^u \cdot f : x \rightarrow y] \end{array}$$

(ii) *For every functor $\Phi : \mathcal{B}^{\text{op}} \rightarrow \text{CAT}$, there is a natural isomorphism $\Lambda^{\Phi} : \Phi \rightarrow \Phi^{\Pi^{\Phi}}$ whose component at $b \in \mathcal{B}$ is the bijective functor*

$$\Lambda_b^{\Phi} : \Phi b \longrightarrow (\int^{\circ} \Phi)_b, \quad (y \xrightarrow{f} y') \mapsto [(b, y) \xrightarrow{(1_b, f)} (b, y')].$$

Under the above dualization principle one concludes from the Theorem that split cofibrations correspond equivalently to functors $\Phi : \mathcal{B} \rightarrow \text{CAT}$. Furthermore, Theorems 9.1, 9.3 and Corollaries 9.4, 9.2 may now be formulated in indexed-category form, as follows.

Corollary 9.6. (1) *A functor $\Phi : \mathcal{B}^{\text{op}} \rightarrow \text{CAT}$ has the property that every Φu (with u a morphism in \mathcal{B}) has a left adjoint if, and only if, $\Pi^{\Phi} : \int^{\circ} \Phi \rightarrow \mathcal{B}$ is a bifibration. In that case, if \mathcal{B} and all categories Φb ($b \in \mathcal{B}$) have (co)limits of a specified diagram type \mathcal{D} , so does $\int^{\circ} \Phi$.*

(2) *A functor $\Phi : \mathcal{B} \rightarrow \text{CAT}$ has the property that every Φu (with u a morphism in \mathcal{B}) has a right adjoint if, and only if, $\Pi_{\Phi} : \int_{\circ} \Phi \rightarrow \mathcal{B}$ is a bifibration. In that case, if \mathcal{B} and all categories Φb ($b \in \mathcal{B}$) have (co)limits of a specified diagram type \mathcal{D} , so does $\int_{\circ} \Phi$.*

9.6. Standard examples

- (1) For any category \mathcal{C} , the functors $\text{Id} : \mathcal{C} \rightarrow \mathcal{C}$ and $! : \mathcal{C} \rightarrow 1$ (where 1 is terminal in CAT) are split bifibrations. Every morphism in \mathcal{C} is Id -(co)cartesian and $!$ -vertical; the $!$ -(co)cartesian morphisms are the isomorphisms in \mathcal{C} . The indexed categories induced by Id and $!$ have (up to isomorphism) constant value 1 and \mathcal{C} , respectively.
- (2) For a fixed object A in a category \mathcal{C} , consider its hom-functor $\mathcal{C}(-, A) : \mathcal{C}^{\text{op}} \rightarrow \text{Set}$ as having discrete-category values. Then $\int^{\circ} \mathcal{C}(-, A)$ is the slice category \mathcal{C}/A , presented as a discretely-fibred category over \mathcal{C} .
- (3) The slice categories of (2) define a functor $\mathcal{C}/(-) : \mathcal{C} \rightarrow \text{CAT}$, $A \mapsto \mathcal{C}/A$ whose dual Grothendieck category $\int_{\circ} \mathcal{C}/(-)$ is the arrow category \mathcal{C}^2 (where the only non-identical morphism in the category 2 is $0 \rightarrow 1$), equipped with its codomain functor $\text{cod} = \Pi_{\mathcal{C}/(-)} : \mathcal{C}^2 \rightarrow \text{CAT}$. Hence, cod is a split cofibration, and it is a (cloven) fibration precisely when \mathcal{C} has (chosen) pullbacks. A morphism $(f, u) : x \rightarrow y$ in \mathcal{C}^2 , represented by the commutative square

$$\begin{array}{ccc}
\bullet & \xrightarrow{f} & \bullet \\
x \downarrow & & \downarrow y \\
a & \xrightarrow{u} & b
\end{array}$$

in \mathcal{C} , is cod-cocartesian precisely when f is an isomorphism, and it is cod-cartesian precisely when it is a pullback diagram in \mathcal{C} .

- (4) Bijective functors (isomorphisms in CAT) are split bifibrations. The composite of two (split) fibrations is again a split fibration, and so is any pullback in CAT of a (split) fibration; likewise for (split) cofibrations.
- (5) A left action of a group G on a group N is described by a homomorphism $\phi : G \rightarrow \text{Aut}(N)$ or, equivalently, by a functor $\phi : G \rightarrow \text{Cat}$ which maps the only object of G (seen as a category) to N (seen as a category and, hence, as an object in Cat). The dual Grothendieck category $\int_{\circ} \phi$ is (up to switching coordinates) precisely the semidirect product $N \rtimes G$. A right action of G on N is given by a functor $G^{\text{op}} \rightarrow \text{Cat}$ with value N or, equivalently by its Grothendieck category $\int^{\circ} \phi$.
- (6) There is a functor $\Phi : \text{Rng}^{\text{op}} \rightarrow \text{CAT}$ which assigns to a ring R the category Mod_R of (left) R -modules; every homomorphism $\varphi : R \rightarrow S$ gives the functor $\varphi^* : \text{Mod}_S \rightarrow \text{Mod}_R$ which considers every S -module N as an R -module, via $ra = \varphi(r)a$ for all $r \in R$, $a \in N$. The category $\int^{\circ} \Phi$ is the category Mod of all modules; its objects are pairs (R, M) where R is a ring and M is an R -module, and its morphisms $(\varphi, f) : (R, M) \rightarrow (S, N)$ are given by a morphism $\varphi : R \rightarrow S$ in Rng and an R -linear map $f : M \rightarrow \varphi^*(N)$. The projection $\Pi^{\Phi} : \text{Mod} \rightarrow \text{Rng}$ is a split fibration.
- (7) The functor $\mathcal{O} : \text{Set} \rightarrow \text{Cat}$ assigns to every set X the set of topologies on X , ordered by \supseteq and, as such, considered as a small category; for a map $f : X \rightarrow Y$ one has the monotone map $f^* : \mathcal{O}(Y) \rightarrow \mathcal{O}(X)$, defined by taking inverse images. The Grothendieck category $\int^{\circ} \mathcal{O}$ is the category Top of topological spaces, with underlying Set -functor $\Pi^{\mathcal{O}}$, which is in fact a split bifibration. More generally, one may characterize topological functors with small fibres (see [1]) as those fibrations P for which the indexed category Φ^P takes values in the category of complete lattices and their inf-preserving maps (see [39, 36]), making P in fact a split bifibration.
- (8) Considering the functor $\text{Id}_{\text{Set}} : \text{Set} \rightarrow \text{Set}$ as having discrete-category values, one obtains the category Set_{\bullet} of pointed sets as its dual Grothendieck category $\int_{\circ} \text{Id}_{\text{Set}}$. Writing the domain of Id_{Set} in the form $(\text{Set}^{\text{op}})^{\text{op}}$ we also have the Grothendieck category $\int^{\circ} \text{Id}_{\text{Set}}$, which is precisely the category $\text{Set}_{\bullet}^{\text{op}}$.

The “categorification” of the last (rather trivial) example leads to an important fact, which we describe next.

9.7. The classifying split (co)fibration

The dual Grothendieck category $\int_{\circ} \text{Id}_{\text{Cat}}$ of the functor $\text{Id}_{\text{Cat}} : \text{Cat} \rightarrow \text{Cat}$ (with its codomain to be embedded into CAT) is the category Cat_{\bullet} of *small lax-pointed categories*. Its objects (\mathcal{C}, x) are given by a small category \mathcal{C} equipped with an object $x \in \mathcal{C}$, and a morphism $(F, f) : (\mathcal{C}, x) \rightarrow (\mathcal{D}, y)$ is given by a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ and a morphism $f : Fx \rightarrow y$ in \mathcal{D} . The forgetful functor $\Pi_{\bullet} := \Pi_{\text{Id}_{\text{Cat}}} : \text{Cat}_{\bullet} \rightarrow \text{Cat}$ is a small-fibred split cofibration, called *classifying*, since one has the following rather obvious fact:

Theorem 9.7. *Every small-fibred split cofibration is a pullback (in CAT) of the classifying split cofibration, as shown in the diagram*

$$\begin{array}{ccc} \mathcal{E} & \longrightarrow & \mathbf{Cat}_\bullet \\ P \downarrow & & \downarrow \Pi_\bullet \\ \mathcal{B} & \xrightarrow{\Phi_P} & \mathbf{Cat} \end{array} \quad [f : x \rightarrow y] \longmapsto [((Pf)_!, \nu_f) : (\mathcal{E}_{Px}, x) \rightarrow (\mathcal{E}_{Py}, y)].$$

Writing the domain of $\text{Id}_{\mathbf{Cat}}$ as $(\mathbf{Cat}^{\text{op}})^{\text{op}}$, we can also form the Grothendieck category $\int^\circ \text{Id}_{\mathbf{Cat}}$, which is the category \mathbf{Cat}^\bullet of *small oplax-pointed categories*. Its objects are the same as those of \mathbf{Cat}_\bullet , but its morphisms $(F, f) : (\mathcal{C}, x) \rightarrow (\mathcal{D}, y)$ are now given by functors $F : \mathcal{D} \rightarrow \mathcal{C}$ equipped with a morphism $f : x \rightarrow Fy$ in \mathcal{C} . The forgetful functor $\Pi^\bullet := \Pi^{\text{Id}_{\mathbf{Cat}}} : \mathbf{Cat}^\bullet \rightarrow \mathbf{Cat}^{\text{op}}$ classifies the small-fibred split fibrations:

Corollary 9.8. *Every small-fibred split fibration is a pullback (in CAT) of the classifying split fibration, as shown in the diagram*

$$\begin{array}{ccc} \mathcal{E} & \longrightarrow & \mathbf{Cat}^\bullet \\ P \downarrow & & \downarrow \Pi^\bullet \\ \mathcal{B} & \xrightarrow{(\Phi^P)^{\text{op}}} & \mathbf{Cat}^{\text{op}} \end{array} \quad [f : x \rightarrow y] \longmapsto [((Pf)^*, \nu_f) : (\mathcal{E}_{Px}, x) \rightarrow (\mathcal{E}_{Py}, y)].$$

Of course, nothing prevents us from dropping the restriction of P being small-fibred.: Theorem 9.7 and Corollary 9.8 remain true *verbatim* if we delete “small-fibred” and replace $\mathbf{Cat}, \mathbf{Cat}_\bullet, \mathbf{Cat}^\bullet$ by $\mathbf{CAT}, \mathbf{CAT}_\bullet, \mathbf{CAT}^\bullet$, respectively, and \mathbf{CAT} by the *colossal category* \mathbf{CAT} which contains \mathbf{CAT} as an object, with the last exchange only formally needed for the provision of a legitimate home of the amended pullback diagrams.

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