Injective Hulls of (L, V)-Categories

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Abstract

In this communication some results obtained in [Rum16] are generalized to categories enriched in a commutative quantale V. Using these results is it shown that every (L, V)-category admits an injective hull. A connection between Isbell-duality and the construction of injective hulls made in [Ban73] is made.

1 Introduction

In their article [LBKR12], Lambek et al. studied injective objects and injective hulls in the category of ordered monoids with sub-multiplicative monotones maps. They proved that quantales are the injective objects and that every ordered monoid admits an injective hull. This injective hull is obtained as the fixed points of a quantic nuclei on the quantales of downclosed subsets.

In [Rum16] the problem is studied further by considering the so called *Quantum B-Algebras*, orderded sets equipped with two implications that mimic the residuals in an quantale. It was shown that quantales are injective in the category formed by them by taking as morphisms oplax homomorphisms. Moreover, it was proven that every Quantum B-Algebra admits an injective hull.

In this paper we show how the two results can be put under a common roof and generalized to the realm of enriched categories. This is done by considering (L, V)-categories, a particular example of (T, V)-categories, obtained by taking the list monad L. In this way both ordered monoids with submultiplicative monotone maps and quantum B-algebras with oplax homomorphisms, or more generally their enriched counterparts, become full sub-categories of (L, V)-$\mathtt{Cat}$: the category formed by taking (L, V)-categories along with (L, V)-functor (which is a notion that is able to capture the both the submultiplicativiness and the oplaxness).

For such categories, Clementino, Hofmann and Tholen, have shown how many constructions coming from enriched category theory: [Hof11, CH09b] ((relative) cocompleteness and cocompletion), [Hof14] (completeness), could be developed in the more general context of (T, V)-categories. In particular in [Hof11] is it shown, generalizing results regarding topological spaces ([Esc97, Esc98]), how injective objects could be characterized as algebras for a Koch-Zöberlein monad which is a generalization of the preasheaf monad.

Using this characterization of injectives as algebras for a monad, by pure categorical arguments is possible to generalize Theorem 4.1 of [LBKR12] and proving that quantales (their enriched counterparts) are injective also in (L, V)-$\mathtt{Cat}$. Since quantales can be seen both as ordered monoids and as quantum B-algebras, we can recover the characterization of injectives contained in [LBKR12] and in [Rum16].

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Regarding the problem of the construction of injective hulls, by using ideas coming from [Rum16] we will prove that, for a full-subcategory of (L,V)-Cat a sufficient and necessary condition for its objects to admit an injective hulls is that quantales are objects of such categories.

We would like to stress the fact that results of this paper are not a mere exercise in *fuzzyfication*, that is to say, a rephrase of the ones in [Rum16] for a general V. In order to be able to generalize them, on needs a more general theory of colimits, which is provided by the notion of (L,V)-colimits we are going to introduce. This provides a nice an elegant categorical way to shred light on some universal constructions that in the ordered case are somehow "hidden".

The structure of the paper is the following:

- The first section is devoted to introduce some background material on V-categories and to introduce the notion of (L,V)-categories. We will show how many constructions coming from enriched category theory (distributors, colimits, colimits complections etc.) can be developed in the more general context of (L,V)-categories.
- The second section is, as the title suggests, an *intermezzo*. We recall how injectives and injective hulls are built in the category of ordered sets and monotone maps between them. We show how the same ideas apply to the enriched case, for a general commutative quantale V. The aim of the section is to introduce some ideas that later will be applied in the more general context of (L,V)-categories.
- The third section forms the core of the paper. Promonoidal categories are introduced and their relation with (L,V)-categories is explained. Quantum B-algebras are introduced as representables promonoidal categories, mirroring in a "dual" way the case of monoidal's ones. A characterization of injectives in (L,V)-Cat is given from which will follow the ones for promonoidal categories and quantum B-algebras. Using some ideas from the previous sections we will prove that every quantum B-algebras admits an injective hull, which is constructed by defining a quantic nuclei on the category of (L,V)-presheaves.
- The fourth section is devoted to characterize those sub-categories of (L,L)-Cat for which injective hulls exist. Using ideas from the previous section we will prove that a necessary and sufficient condition is given by having quantales as objects.
- The final section sketch a connection between Isbell-dually in the ordered case and the construction of injective hulls for topological spaces contained in [Ban73].

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2 Prolegomena on (L,V)-Cat and V-Cat

Enriched categories are a generalization of ordinary categories, in which the *hom-sets* take values in a *cosmos*, a symmetric-closed monoidal category V. The standard reference about them is [Kel82].

In this paper we are going to consider only the case in which V is a *commutative quantale*, that is to say, a commutative monoid in the category of sup-lattices and suprema preserving monotone maps. Although this might seen not so useful, Lawvere, in his seminal paper [Law73], showed how taking enrichment in a commutative quantale is not only useful as a toy model in which some general constructions become easier to understand (due to the absence of coherent conditions), but that they are worthy to study as their own, since they are able to capture some important mathematical structures like metric spaces.

(L,V)-categories are a special case of the more generals (T,V)-categories, where the list monad L is considered. They could also be seen as the ordered version of *multicategories* ([Lei04] for an account on them). The basic idea is that, instead of having arrows with just a single domain, we allow them to have as domain a list of objects.

In the following section we are going to recall/introduce some basics notions of such categories. Our point of view is slightly different from the more "standard" one contained in [Kel82]. Our point of view will be more "relational": following [BCSW83, CT03], we will introduce the *quantaloid* of V-relations and defining V-categories starting from there. This might seen as an overkilling, but it will be clear in the section related to (L, V)-categories, how this allows us to smoothly introduce some concepts (as distributors, presheaves and colimits) in the (L, V)-case.

The section is structured as follows:

- The first two subsections are devoted to introduce V-categories, V-distributors and (co)-limits, providing a lot of examples of their use and explaining how they generalize some (possibly) more familiar concepts in order theory.
- In following two subsections we will introduce (L, V)-categories and by mimic what is done for V-categories we will introduce (L, V)-distributors and (L, V)-colimits, explaining how they are able to capture in more abstract and categorical way some constructions in order theory.
- In the last subsection we will introduce the correspondent enriched version of quantales, a concept that plays a central role through the whole paper.

2.1 V-Categories and V-Functors

As we've already remarked before, we are going to consider only categories enriched in a commutative quantale:

Definition 2.1. A commutative quantale $(V, \otimes, k)$ is complete lattice endowed with a (commutative) multiplication preserving suprema in each variable $\otimes : V \times V \to V$ for which $k \in V$ is a neutral element.

Remark 2.2. By the adjoint functor theorem applied to order sets, it follows that $- \otimes =$ admits a right adjoint (in each variable) denoted by: $[-, =]$ and called "internal hom".

Examples 2.3.

1. The two-elements boolean algebra $2 = \{0, 1\}$ with $\wedge$ as multiplication and $\Rightarrow$ as internal hom is a quantale.
2. More generally, every frame F becomes a quantale with the multiplication given by $\wedge$. In particular $k := \top$.
3. $([0, +\infty])^{\mathrm{op}}$ (with the reverse natural order) with $\mathtt{max}$ as multiplication is a quantale. The internal hom is given by the "truncated sum" defined as $v \ominus u := \mathtt{max}(v - u, 0)$.
4. Consider the set:

$$\Delta := \{\psi : [0, +\infty] \to [0, 1] \mid \text{for all } \alpha \in [0, +\infty] : \psi(\alpha) = \bigvee_{\beta < \alpha} \psi(\beta)\},$$

   of distribution functions; with the pointwise order it becomes a complete order set. For all $\psi, \phi \in \Delta$, $\alpha \in [0, +\infty]$, define the following multiplication:

$$\psi \otimes \phi(\alpha) := \bigvee_{\beta + \gamma < \alpha} \psi(\beta) * \phi(\gamma),$$

   Where $*$ is the ordinary multiplication on $[0, 1]$. It can be shown that $(\Delta, \otimes, k)$ is quantale, where $k(\alpha) = 1$ for all $\alpha > 0$ and $k(0) := 0$.

As we stated in the introduction of this section, we are going to present V-categories from a more "relational" point of view. The first step is to define the so called *quantaloid of V-relation*, which is the enriched generalization of the quantaloid $\mathtt{Rel}$ of 2 valued relations. For an account of quantaloids, we refer to [Stu14] for a brief overview and to [Stu05] for a more depth description.

The quantaloid V-$\mathtt{Rel}$ is the quantaloid whose objects are sets, and an arrow $r : X \nrightarrow Y$ is given by a function:

$$r : X \times Y \to Y.$$

The composition of $r : X \to Y$, $s : Y \to Z$ is given as "matrix multiplication" and is defined pointwise as:

$$s \cdot r(x,y) := \bigvee_{y \in Y} r(x,y) \otimes s(y,z);$$

the identity arrow $\mathrm{Id} : X \nrightarrow X$ is defined as:

$$\mathrm{Id}(x_1, x_2) := \begin{cases} \perp & \text{if } x_1 \neq x_2 \\ k & \text{if } x_1 = x_2 \end{cases}$$

The complete order on V-$\mathtt{Rel}(X,Y)$ is the one induced (pointwise) by V, i.e:

$$r \leq r' \text{ in } V\text{-}\mathtt{Rel}(X,Y) \text{ iff } r(x,y) \leq r'(x,y) \text{ in } V \text{ for all } x,y \in X,Y. \tag{1}$$

Remark 2.4. Notice that the order is complete because V is. Since multiplication of V preserves suprema in both variables and because suprema commute with suprema, one has:

$$(\bigvee_i r_i) \cdot (\bigvee_j s_j) = \bigvee_{i,j} r_i \cdot s_j.$$

Proving that V-$\mathtt{Rel}$ is a quantaloid.

Remark 2.5. Notice that in the case in which $V = 2$, 2-$\mathtt{Rel}$ is the quantaloid of relations, and the "matrix multiplication" defined previously becomes the "classical" relational composition.

Remark 2.6. Note that (1) is equivalent to:

$$k \leq \bigwedge_{x,y \in X,Y} [r(x,y), s(x,y)];$$

indeed, fixing $x,y \in X,Y$ we have:

$$r(x,y) \leq r'(x,y) \text{ iff } r(x,y) \otimes k \leq r'(x,y) \text{ iff } k \leq [r(x,y), s(x,y)].$$

Remark 2.7. Notice that every function $f : X \to Y$ could be seen as a V-relation in a straightforward way as follows:

$$f(x,y) := \begin{cases} \perp & \text{if } f(x) \neq y \\ k & \text{if } f(x) = y \end{cases}$$

The identity in V-$\mathtt{Rel}(X,X)$ is an example of this construction.

We have also an involution $(-)^\circ : V\text{-}\mathtt{Rel}^{\mathtt{op}} \to V\text{-}\mathtt{Rel}$ defined as $r^\circ(y,x) := r(x,y)$, which satisfies:

$$(1_X)^\circ = 1_X, \;\; (s \cdot r)^\circ = r^\circ \cdot s^\circ \;\; (r^\circ)^\circ = r.$$

Definition 2.8. A V-category (X,a) is a pair, where X is a set and $a : X \nrightarrow X$ is a V-relation that satisfies:

- $Id \leq a$;

- $a \cdot a \leq a$.

Remark 2.9. In the paper, when the V-structure is clear from the context, we'll denote a V-category (X, a) simply as X.

Definition 2.10. Given two V-categories $(X, a), (Y, b)$, a *V-functor* $f : (X, a) \to (Y, b)$ is a function between the two underlying sets such that:

$$a \leq f^\circ \cdot b \cdot f,$$

which pointwise means, for all $x, y \in X$:

$$a(x, y) \leq b(f(x), f(y)).$$

If the equality holds, we call it *fully-faithful*.

Examples 2.11.

1. In the case in which $V = 2$, as already stated before, a 2-category is an ordered set and a 2-functor is a monotone map.

2. Categories enriched in the quantale $([0, +\infty])^{\mathrm{op}}$, as first recognized by Lawvere in his seminal paper [Law73], are generalized metric spaces and $([0, +\infty])^{\mathrm{op}}$-functors between them are non-expansive maps.

3. Categories enriched in Δ are called probabilistic metric spaces as first recognized in [Fla97].

4. The quantale V could be viewed as a V-category with the V-structure given by its internal hom $[-, =]$.

5. By using the involution in V-`Rel`, for every V-category (X, a) one can define its opposite category $X^{\mathrm{op}} = (X, a^\circ)$.

6. Given two V-categories $(X, a), (Y, b)$ one can define the V-category formed by all V-functors $f : (X, a) \to (Y, b)$, denoted by $([X, Y], [X, Y](-, =))$, by defining the following V-structure:

   $$[X, Y](f, g) := \bigwedge_{x \in X} b(f(x), g(x)).$$

   In particular we have two very important V-categories:

   $$\mathbb{D}(X) := [X^{\mathrm{op}}, V], \text{ the category of presheaves,}$$

   $$\mathbb{U}(X) := [X, V]^{\mathrm{op}}, \text{ the category of co-presheaves.}$$

   Notice that they are the generalization (for a general V) of the classical downward (upward) closet subsets construction, which correspond to the case in which $V = 2$.

7. Given a V-category (X, a), there are two V-functors, called the Yoneda and the co-Yoneda embedding:

   $$y_X : (X, a) \rightarrowtail \mathbb{D}(X), \quad x \mapsto a(-, x),$$

   $$\lambda_X : (X, a) \rightarrowtail \mathbb{U}(X), \quad x \mapsto a(x, =).$$

   The adjective "embedding" is justified by the fact that one can prove that they are both fully-faithful. Moreover, it can be proved that:

   $$\mathbb{U}(X)[\lambda_X(x), g] = g(x), \quad \mathbb{D}(X)[y_X(x), g] = g(x).$$

   The last result is known as the (co)-Yoneda Lemma. This justifies the adjective "embedding", since from (co)-Yoneda Lemma it immediately follows that both are fully-faithful.
   Again, it is worth noticing that the two functors generalize the familiar monotone maps:

   $$x \mapsto \downarrow x, \quad x \mapsto \uparrow x,$$

   which associate to an element the set of elements which are below (above) it.

8. Given two V-categories $(X,a),(Y,b)$ we can define their tensor product:

$$X \boxtimes Y = (X \times Y, a \otimes b),$$

In particular one has: $X \boxtimes K \simeq X$ (where K denotes the one-point V-category $(1,k)$).

In this way we can define V-$\mathtt{Cat}$ as the category whose object are V-categories and whose arrows are V-functors among them; moreover, V-$\mathtt{Cat}$ becomes an order enriched category, by defining, for two V-functor $f,g:(X,a)\to(Y,b)$:

$$f \leq g \text{ iff } k \leq \bigwedge_{x\in X} b(f(x),g(x)).$$

Moreover, one can show that, with the tensor product previously defined, V-$\mathtt{Cat}$ becomes a closed monoidal category, since one can show that for three V-categories $(X,a),(Y,b),(Z,c)$ one has:

$$V\text{-}\mathtt{Cat}(X \boxtimes Y, Z) \simeq V\text{-}\mathtt{Cat}(X,[Y,Z]) \simeq V\text{-}\mathtt{Cat}(Y,[X,Z]).$$

2.2 Distributors and Weighted (co)-Limits

Distributors were introduced by Bénabou in [Bén73] and since them they played an important role in category theory. They could be seen as generalizations of ideal relations from order theory. That is to say, subsets of the cartesian product of two ordered set X,Y which are upward closed in X and downward closed in Y.

Weighted (co)-limits encompass the classical notion of (co)-limits coming from "ordinary" category theory, by admitting a "weight" given by a distributor. In the case of ordered sets, due to the fact that every distributor is a (particular) subset, the weight "disappear" and becomes the "set of condition" over which suprema/infima are taken.

Definition 2.12. Given two V-categories $(X,a),(Y,b)$, a V-*distributor* (or simply a distributor) $j:(X,a)\looparrowright(Y,b)$ is a V-relation between them such that:

- $j \cdot a \leq j$;
- $b \cdot j \leq j$.

Since the composite of two distributors is again a distributor, we can define the quantaloid V-$\mathtt{Dist}$ in the same way as we defined V-$\mathtt{Rel}$. In V-$\mathtt{Dist}(X,X)$ the V-structure a plays the role of the identity, since for every $j:X\looparrowright Y$, one has:

$$b \cdot j = j \cdot a = j.$$

Remark 2.13. By juggling with the definition of distributor, one can show that distributors between two V-categories $(X,a),(Y,b)$ are in bijective correspondence with V-functors between $X^{\mathrm{op}} \boxtimes Y$ and V. if it easy to prove that this correspondence is functorial and that gives and equivalence of order sets:

$$V\text{-}\mathtt{Dist}(X,Y) \simeq V\text{-}\mathtt{Cat}(X^{\mathrm{op}} \boxtimes Y, V) \simeq V\text{-}\mathtt{Cat}(Y, \mathbb{D}(X)).$$

In particular, to every V-distributor $j:X\looparrowright Y$ we can associate its *mate*:

$$\ulcorner j \urcorner : Y \to \mathbb{D}(X), \quad y \mapsto j(-,y).$$

Given a V-functor $f:(X,a)\to(Y,b)$, we can define two arrows in V-$\mathtt{Rel}$:

- $f_* : X \looparrowright Y,\ \ f_*(x,y) := b(f(x),y)$;
- $f^* : Y \looparrowright X,\ \ f^*(y,x) := b(y,f(x))$.

One has:

Lemma 2.14. *The V-relations f_* and f^* are both distributors, moreover $f_* \dashv f^*$ in V-$\mathtt{Dist}$.*

In this way we have two bifunctors:

$$(-)_* : V\text{-}\mathtt{Cat}^{co} \to V\text{-}\mathtt{Dist}, \;\; (-)^* : V\text{-}\mathtt{Cat} \to V\text{-}\mathtt{Dist}.$$

Theorem 2.15. $(-)_* : V\text{-}\mathtt{Cat}^{co} \to V\text{-}\mathtt{Dist}$ *defines a proarrow equipment [Woo82, Woo85] on V-$\mathtt{Cat}$, that is to say:*

- $(-)_*$ *is locally fully-faithful;*
- *For all V-functor $f : (X, a) \to (Y, b)$, there exists a right adjoint to f_* in V-$\mathtt{Dist}$.*

Proof. For the first point we have to show that:

$$f \leq g \text{ in } V\text{-}\mathtt{Cat} \text{ iff } g_* \leq f_* \text{ in } V\text{-}\mathtt{Dist}.$$

$\Rightarrow$). Fix an x, from $k \leq \bigwedge_{x\in X} b(f(x), g(x))$ we get $k \leq b(f(x), g(x))$. By applying composition we get:

$$b(f(x), y) \leq b(f(x), g(x)) \otimes b(g(x), y) \leq b(f(x), y).$$

$\Leftarrow$). For every x we get:

$$k \leq b(g(x), g(x)) \leq b(f(x), g(x)),$$

hence:

$$k \leq \bigwedge_{x\in X} b(f(x), g(x)).$$

The second point follows from the previous lemma.

$\square$

As a corollary we get:

Corollary 2.16. *A function $f : (X, a) \to (Y, b)$ between two V-categories is a V-functor iff $f_* : X \nrightarrow Y$ is a V-distributor; equivalently iff $f^* : Y \nrightarrow X$ is a V-distributor.*

This theorem allows us to "mirror" all the good features V-$\mathtt{Dist}$ has in V-$\mathtt{Cat}$, giving us an "higher" perspective on notions like: weighted (co)-limits, (co)-ends, Kan extensions, and adjoint V-functors.
The key fact is that V-$\mathtt{Dist}$ is "closed", meaning that composition as adjoint in each variable, more precisely, given:

$$\alpha : Z \nrightarrow X, \;\; \beta : X \nrightarrow Y,$$

one has:

- $(-) \cdot \alpha \dashv (-) \rhd \alpha$;
- $\beta \cdot (=) \dashv \beta \lhd (=)$.

Where:

$$\begin{aligned} (-) \cdot \alpha : V\text{-}\mathtt{Dist}(X, Y) &\to V\text{-}\mathtt{Dist}(Z, Y) \\ \beta &\longmapsto \beta \cdot \alpha. \end{aligned}$$

$$\begin{aligned} \beta \cdot (=) : V\text{-}\mathtt{Dist}(Z, X) &\to V\text{-}\mathtt{Dist}(Z, Y) \\ \alpha &\longmapsto \beta \cdot \alpha \end{aligned}$$

and:

$$(\gamma \rhd \alpha)(x, y) := \bigwedge_{z\in Z} [\alpha(z, x), \gamma(z, y)],$$

$$(\beta \lhd \gamma)(z, x) := \bigwedge_{y\in Y} [\beta(x, y), \gamma(z, y)].$$

Definition 2.17. Let $f : (X, a) \to (Y, b)$ be a V-functor, (Z, c) a V-$\mathtt{Cat}$, and $j : X \nrightarrow Z$ be a distributor. We say that a V-functor $g : (Z, c) \to (Y, b)$ is the *colimt of f weighted by j*, and denote it by: $\mathtt{colim}(j, f)$, if $g_* \simeq f_* \rhd j$, that is to say:

$$g_*(z, y) := b(g(z), y) = \bigwedge_{x \in X} [j(x, z), f_*(x, y)], \ (1).$$

Dually, if we have a distributor $l : Z \nrightarrow X$, we say that a V-functor $h : (X, a) \to (Y, b)$ is the *limit of f weighted by l*, and denote it by: $\mathtt{lim}(l, f)$, if $h^* \simeq l \lhd f^*$, that is to say:

$$h^*(y, z) := b(y, h(z)) = \bigwedge_{x \in X} [l(z, x), f^*(y, z)], \ (2).$$

A V-category (Y, b) is called *(co)-complete*, if it has all weighted (co)-limits; meaning that, for all V-functors $f : (X, a) \to (Y, b)$ and for all distributors $(j : X \nrightarrow Z)$ $h : Z \nrightarrow X$, the (co)-limits of f weighted by (j)-h exists.

Remark 2.18. Using the fact that:

$$\mathbb{D}(X)(j, l) := \bigwedge_{x \in X} [j(x), l(x)],$$

(and similarly for $\mathbb{U}(X)$), we can re-write (1) and (2) as:

$$b(\mathtt{colim}(j, f)(z), y) \simeq \mathbb{D}(X)(j(-, z), b(f(-), y)),$$

$$b(y, \mathtt{lim}(l, f)(z)) \simeq \mathbb{U}(X)(b(y, f(=)), l(z, =)).$$

Examples 2.19. Let (X, a) be a V-category and $x : K \to X$ be a point (seen as a V-functor from the one point V-category $(1, k)$). Let $u \in V$ and view it as a distributor $\tilde{u} : K \nrightarrow K$ by defining $u(1, 1) := u$.

1. The colimit of x, weighted by $\tilde{u}$, usually denoted as $x \odot u$, is called *co-power*. Unravelling the definition of colimits, we get (for all $y \in Y$):

$$a(x \odot u, y) = [u, a(x, y)].$$

   A category in which all the co-powers exist is called co-powered.

2. Dually, due to the form of $\tilde{u}$, we can consider also the limit of x weighted by $\tilde{u}$. This is usually denoted as $x \pitchfork u$ and it is called *power*.
   A category in which all the powers exist is called powered.

Let $f : I \to X, \ \ i \mapsto x_i$ be a family of elements of X:

1. Let $\alpha : I \nrightarrow K$ be the constant distributor $\alpha(i) := k$. The colimit of f weighted by α is an element $\tilde{x}$, such that (for all $y \in Y$):

$$a(\tilde{x}, y) = \bigwedge_i a(x_i, y).$$

   Such colimit is called *conical supremum*, and due to its universal property (which coincides with the usual suprema's one in the case in which $V = 2$) is usually denoted as $\tilde{x} := \bigvee_i x_i$.

2. Dually, let $\beta : K \nrightarrow X$ be the same distributor as above, but seen as an arrow from K to X. The limit of f weighted by β is an element $\tilde{x}$, such that (for all $y \in Y$):

$$a(y, \tilde{x}) = \bigwedge_i a(y, x_i).$$

   Such limit is called *conical infumum*; as before, in the case in which $V = 2$, it coincides with the infimum of the family $\{x_i\}_i$, and for this reason is usally denoted as $\tilde{x} := \bigwedge_i x_i$.

All the (co)-limits we described above play a very important role, as they are basic building blocks: every (co)-limits can be written as a suitable composite of them.
Let $f : (X,a) \to (Y,b)$ be a V-functor and $j : X \nrightarrow Z$ be a distributor. If (Y,b) is co-powered and $\mathtt{colim}(j,f)$ exists, then we have:

$$\mathtt{colim}(j,f)(z) \simeq \bigvee_{x\in X} j(x,z) \odot f(x).$$

Dually, if (Y,b) is powered and $\mathtt{lim}(l,f)$ exists (for a distributor $l : Z \nrightarrow X$), one has:

$$\mathtt{lim}(l,f)(z) \simeq \bigwedge_{x\in X} l(z,x) \pitchfork f(x).$$

Remark 2.20. Just as the set of downsets and the set of upsets of an order set X have all suprema and infima, for a V-category (X,a), $\mathbb{D}(X)$ and $\mathbb{D}(X)$ are both complete and (co)-complete V-categories.

Another important concept we are going to use is the concept of *adjunction* between a pair of V-functors

$$\begin{tikzcd} (X,a) \arrow[r, bend left, "f"] \arrow[r, phantom, "\perp"] & (Y,b). \arrow[l, bend left, "g"] \end{tikzcd}$$

This concept generalize the classical notion of Galois connection from order theory.

Definition 2.21. Given two V-functors $f : (X,a) \to (Y,b)$, $g : (Y,b) \to (X,a)$; f is *left adjoint* to g (vice versa: g is *right adjoint* to f) and denoted: $f \dashv g$, if $f_* \simeq g^*$. Pointwise this means that, for all $x, y \in X, Y$:

$$b(f(x), y) = a(x, g(y)).$$

Examples 2.22. Let (X,a) be a V-category.

1. If $x \odot u$ exists for all $u \in V$, then the universal property of the colimits:

$$a(x \odot u, y) = [u, a(x,y)],$$

   can be rephrased by saying that $x \odot u \dashv a(x, =)$.
   Thus, co-powered categories can be equivalently described as those for which $a(x,=)$ admits a left adjoint for all $x \in X$.

2. If $x \pitchfork u$ exists for all $u \in V$, then the universal property of the limits:

$$[u, a(y,x)] = a(y, x \pitchfork u),$$

   can be rephrased by saying that $- \pitchfork u \dashv a(-,x)$.
   Thus, powered categories can be equivalently described as those for which $a(-,x)$ admits a right adjoint for all $x \in X$.

Let $f : (X,a) \to (Y,b)$ and $g : (Y,b) \to (Z,c)$ two V-functors and $j : X \nrightarrow D$ be a distributor. From the definition of weighted colimit it follows that $\mathtt{colim}(j, gf) \leq g(\mathtt{colim}(j,f))$.
Vice versa, if we consider a distributor $l : D \nrightarrow X$, from the definition of weighted limit it follows that $g(\mathtt{lim}(l,f)) \leq \mathtt{lim}(h, gf)$.
If the equality holds, we say that g preserves the (co)-limit of f weighted by (j)-h. If g preserves all (co)-limits, we say that is *(co)-continuous*.

Examples 2.23. 1. $a(x,=) : X \to V$, is continuous.

2. $a(-,x) : X^{\mathrm{op}} \to V$ sends weighted colimits to weighted limits. Meaning that $a(\mathtt{colim}(j,f),x) = \mathtt{lim}(j^\circ, a(-,x))$, for suitable j and f; where j° is the distributor obtained from j by applying the involution $(-)^\circ$.

3. Left adjoints V-functors are continuous; vice versa, right adjoints are (co)-continuous.

Adjunctions allow to give a nice characterization of (co)-complete V-categories:

Theorem 2.24. *Let (X,a) be a V-category. Then (X,a) is complete iff the (co)-Yoneda has a right adjoint; dually, it is cocomplete if the Yoneda has a left adjoint. Moreover, in case they exist, they have the following form:*

$$\mathtt{lim}: \mathbb{U}(X) \to X, \;\; l \mapsto \mathtt{lim}(l, Id),$$

$$\mathtt{colim}: \mathbb{D}(X) \to X, \;\; j \mapsto \mathtt{colim}(j, Id).$$

Remark 2.25. Since both $\mathbb{U}(-)$ and $\mathbb{D}(-)$ define *Kock–Zöberlein* monads (see [Koc95] for a far reaching treatment on them) on V-$\mathtt{Cat}$, in order to prove that the (co)-Yoneda has a (right)-left adjoint is sufficient to provide a left inverse of it.

2.3 (L,V)-Categories and (L,V)-functors

Recall that the list monad is defined as:

$$L: \mathtt{Sets} \to \mathtt{Sets}, \;\; f: X \to Y \mapsto Lf: \amalg_{n\geq 0} X^n \to \amalg_{m \geq 0} Y^m, \;\; \underline{x} := (x_1, ..., x_n) \mapsto (f(x_1), ..., f(x_n)),$$

Whit unity and multiplication given by:

- $e_X : X \to L(X), \;\; x \to (x)$;
- $m_X : L^2(X) \to L(X), \;\; (\underline{x}_1, ..., \underline{x}_n) \mapsto (x_{11}, ..., x_{k1}, ..., x_{n1}, ..., x_{ln})$.

Remark 2.26. Let $\underline{x}$, $\underline{w}$ be two lists; in order to avoid possible confusion with the list of list $\underline{\underline{y}} = (\underline{x}, \underline{w})$, we will denote the list obtained by concatenating $\underline{x}$ and $\underline{w}$ as $(\underline{x}; \underline{w})$. Moreover, in the case in which one of of the two is the single element list, we will use the shortcut $(\underline{x}; w)$, instead of $(\underline{x}; (w))$.

We can extend (in a functorial way) the list monad L to V-relation by defining, for $r: X \nrightarrow Y$:

$$\tilde{L}r : L(X) \nrightarrow L(Y), \;\; (\underline{x}, \underline{y}) \mapsto \begin{cases} \perp \text{ if the two lists haven't the same lenght} \\ r(x_1, y_n) \otimes ... \otimes r(x_n, y_n). \text{ otherwise} \end{cases}$$

It can be easely proven that this particular extension defines a monad on (L,V)-$\mathtt{Rel}$ that preserves the involution.

Remark 2.27. From now on, we will use L for both (the ordinary list monad and its extension).

This allows us to define an order-enriched category (L,V)-$\mathtt{Rel}$ in which a morphism $r: X \multimap Y$ is a V-relation of the form:

$$r: L(X) \nrightarrow Y,$$

and in which composition is given by:

$$s \circ r := s \cdot Lr \cdot m_X^\circ,$$

and in which $e_X^\circ : X \multimap X$ is the identity.

Remark 2.28. Note that, due to the Kleisli-style composition we defined, $- \circ r$ preserves suprema, but (in general) $s \circ (=)$ doesn't. That's the reason why (L,V)-$\mathtt{Rel}$ is not a quantaloid, but just an order-enriched category.

Definition 2.29. An (L,V)-category is a pair (X,a), where X is a set and $a: X \multimap X$ is a (L,V)-$\mathtt{Rel}$ satisfying:

- $e_X^\circ \leq a$;

- $a \circ a \leq a$.

Definition 2.30. Given two (L,V)-categories $(X,a),(Y,b)$, an (L,V)-*functor* $f : (X,a) \to (Y,b)$, is a function between the two underlying sets such that:

$$a \leq f^\circ \cdot b \cdot Lf,$$

which pointwise means, for all $\underline{x} \in LX,\ y \in X$:

$$a(\underline{x}, y) \leq b(Lf(\underline{x}), f(y)).$$

If the equality holds, we call it fully-faithful.

In this way we can define (L,V)-$\mathtt{Cat}$ as the category whose object are (L,V)-categories and whose arrows are (L,V)-functors among them; moreover, (L,V)-$\mathtt{Cat}$ becomes an order-enriched category, by defining, for two (L,V)-functors $f,g : (X,a) \to (Y,b)$:

$$f \leq g \text{ iff } k \leq \bigwedge_{\underline{x} \in LX} b(Lf(x), g(x)).$$

Examples 2.31.

1. Every set X defines an (L,V)-category by taking e_X° as (T,V)-structure. In particular we can define the one-point (L,V)-category $E := (1, e_1^\circ)$.

2. For two (L,V)-categories $(X,a),(Y,b)$, we can form their tensor product $X \boxtimes Y := (X \times Y, a \boxtimes b)$, where:
$$a \boxtimes b(\gamma, (x,y)) := a(L\pi_1(\gamma), x) \otimes b(L\pi_2(\gamma), y);$$
where $\gamma \in L(X \times Y)$ and π_1, π_2 are the obvious projections. Unluckily, in general it is not true that $X \boxtimes E \simeq E$.

3. V itself defines an (L,V)-category with: $[\underline{v}, w] := [v_1 \otimes \ldots \otimes v_n, w]$.

4. In general any monoidal V-category $(X, a, *, u_X)$ defines an (L,V)-category in a similar way by defining $a(\underline{x}, y) := a(x_1 * \ldots * x_n, y)$.[1] (L,V)-categories defined in this way are called *representables* and their (L,V)-structure will be denoted by [2] $\hat{a} := a \cdot \alpha$, where:
$$\alpha : L(X) \to X, \quad \underline{x} \mapsto x_1 * \ldots * x_n.$$
In this way we can define a 2-functor $K : V\text{-}\mathtt{Cat}^L \to (L,V)\text{-}\mathtt{Cat}$, which has a left adjoint $M : (L,V)\text{-}\mathtt{Cat} \to V\text{-}\mathtt{Cat}$ that sends an (L,V)-category (X,a) to $(LX, La \cdot m_X^\circ, m_X)$ and an (L,V)-functor f to Tf. M is also a 2-functor.
Using the aforementioned adjuction, is it possible to extend the monad L to a monad on (L,V)-$\mathtt{Cat}$, denoted L as well. Moreover one can prove (see [CCH15]) that there is an equivalence:
$$V\text{-}\mathtt{Cat}^L \simeq (L,V)\text{-}\mathtt{Cat}^L.$$

5. A priori, due to the non-symmetric form arrows in (L,V)-$\mathtt{Rel}$ have, it is not clear how to define define an (L,V)-category that seems to play the role of a dual. Luckily, one can use the adjunction $K \dashv M$ and the involution in V-$\mathtt{Rel}$ to define, for an (L,V)-category (X,a), its opposite category as: $X^{\mathrm{op}} := (LX, m_X \cdot La^\circ \cdot m_X)$. At first this might seen as an *ad hoc* definition, but if we apply this construction to a V-category (X,a), seen as an (L,V)-category $(LX, e_X^\circ \cdot a)$, we get:
$$X^{\mathrm{op}} := K(LX, La^\circ),$$
where (LX, La°) is the dual, as a V-category, of (LX, La).

[1] In particular $a((-), y) := a(u_X, y)$.

[2] When it is clear from the context that we're dealing with representables (L,V)-categories we will indiscriminately use a to denote both.

6. For any (L,V)-category (X,a), we can form the (L,V)-category, denoted $(\mathbb{D}_L(X), \mathbb{D}_L(X)[-,=])$ whose underlying set consists of all (L,V)-functors of the form: $f : X^{\mathrm{op}} \boxtimes E \to V$ and whose (L,V)-structure is given by:
$$\mathbb{D}_L(X)[\underline{f}, g] := \bigwedge_{\underline{x} \in LX} [(f_1(\underline{x}), \dots, f_n(\underline{x})), g(x))].$$
We have a fully-faithful functor, called the Yoneda embedding, $y_X : X \rightarrowtail \mathbb{D}_L(X)$, defined as:
$$x \mapsto a(-,x).$$
Moreover, it can be proved that:
$$\mathbb{D}_L(X)[Ly_X(\underline{x}), g] = g(\underline{x}).$$
The last result is known as the Yoneda Lemma and justifies the term "embedding".

2.4 Distributors and Weighted Colimits in (L,V)-$\mathtt{Cat}$

The relational point of view used for introducing V-categories allows us introduce the corresponding notion of distributor for (L,V)-categories by considering the composition $\circ$ defined in the previous section. Unfortunately, due to the non-symmetric form $\circ$ has (and due to other technical details) we won't be able to provide a theory of weighted limits, but only one for weighted colimits.

Definition 2.32. [CH09a] Given two (L,V)-categories $(X,a),(Y,b)$ a (L,V)-*distributor* $j : (X,a) \looparrowright (Y,b)$ is a (L,V)-relation between them such that:

- $j \circ a \leq j$;
- $b \circ j \leq j$.

Just as in the V-case, we can define an order-enriched category (L,V)-$\mathtt{Dist}$, where the composition is the one defined in (L,V)-$\mathtt{Rel}$.

Remark 2.33. As in the V-case, one can prove the equivalence between:
$$(L,V)\text{-}\mathtt{Dist}(X,Y) \simeq (L,V)\text{-}\mathtt{Cat}(X^{\mathrm{op}} \boxtimes Y, V) \simeq (L,V)\text{-}\mathtt{Cat}(Y, \mathbb{D}_L(X)).$$
In particular, to every (L,V)-distributor $j : X \looparrowright Y$ we can associate its *mate*:
$$\ulcorner j \urcorner : Y \to \mathbb{D}(X), \quad y \mapsto j(-,y).$$

Just as in the V-case, we have the equivalent of Theorem 2.15. Where now, for an (L,V)-functor $f : (X,a) \to (Y,b)$, the two (adjoint) associated (L,V)-distributors are:

1. $f_\circledast : X \looparrowright Y, \;\; f_\circledast(\underline{x}, y) := b(Lf(\underline{x}), y)$.
2. $f^\circledast : Y \looparrowright X, \;\; f^\circledast(\underline{y}, x) := b(\underline{y}, f(x))$.

As already noticed in Remark 2.28, due to the intrinsic asymmetry Kleisli composition has, contrary to the V-case, we can only prove that:
$$\begin{aligned} (-) \circ \alpha : (L,V)\text{-}\mathtt{Dist}(X,Y) &\to (L,V)\text{-}\mathtt{Dist}(Z,Y) \\ \beta &\longmapsto \beta \circ \alpha. \end{aligned}$$
Has a right adjoint:
$$\begin{aligned} (-) \blacktriangleright \alpha : (L,V)\text{-}\mathtt{Dist}(Z,Y) &\to (L,V)\text{-}\mathtt{Dist}(X,Y) \\ \beta &\longmapsto \beta \blacktriangleright \alpha := \beta \rhd \hat{\alpha}, \end{aligned}$$
where $\hat{\alpha} := L\alpha \cdot m_X^\circ$.

Definition 2.34. Let $f:(X,a)\to(Y,b)$ be a (L,V)-functor and $j:X \looparrowright Z$ be an (L,V)-distributor. We say that a (L,V)-functor $g:(Z,c)\to(Y,b)$ is the *colimt of f weighted by j*, and denote it by: $L\text{-}\mathtt{colim}(j,f)$, if $g_{\circledast}\simeq f_{\circledast}\blacktriangleright j$, that is to say:

$$g_{\circledast}(\underline{z},y):=b(Lg(\underline{z}),y)=\bigwedge_{\underline{x}\in LX}[\hat{j}(\underline{x},z),f_{\circledast}(\underline{x},y)],\quad(1).$$

An (L,V)-category (Y,b) is called *cocomplete*, if it has all weighted colimits; meaning that, for all (L,V)-functors $f:(X,a)\to(Y,b)$and for all (L,V)-distributors $h:Z\looparrowright X$, the colimits of f weighted by h exists.

Contrary to what happens in V-$\mathtt{Cat}$, in (L,V)-$\mathtt{Cat}$ cocompleteness cannot be reduced to distributors into K (a.k.a. presheaves), nevertheless the corresponding of Theorem 2.24 still holds also in the (L,V)-case.

Examples 2.35.
1. Using the Yoneda Lemma, one can prove that every element of $\mathbb{D}_L(X)$ can be written has the colimit of y_X weighted over itself.
2. Consider the V-$\mathtt{Rel}$: $u:E\looparrowright E$, where $u\in V$. Given an (L,V)-category (X,a) and a point $x\in(X,a)$, $L\text{-}\mathtt{colim}(u\cdot a(-,x),Id)(1)$ (if exists) coincides with the co-power $x\odot u$ of the underling V-category $(X,e_X\cdot a)$.

In the case the target category is representable, we have the following:

Proposition 2.36. *Let $f:(X,a)\to(Y,b\cdot\alpha)$ be an (L,V)-functor, with $(Y,b\cdot\alpha)$ be a representable (L,V)-category, and $j:X\looparrowright E$ be an (L,V)-distributor. If $L\text{-}\mathtt{colim}(j,f)$ exists, then:*

$$L\text{-}\mathtt{colim}(j,f)(1)\simeq\mathtt{colim}(e_1^{\circ}\cdot\hat{j},\alpha Lf).$$

Where $\alpha Lf:K(X)\to(Y,b)$.

Before the proof we state a useful Lemma.

Lemma 2.37. *In the hypotesis above, $\alpha Lf:K(X)\to(Y,b)$. is a V-functor and $e_1^{\circ}\cdot\hat{j}$ is a V-distributor.*

Proof. Sine f is an (L,V)-functor, then $f_{\circledast}:=b\cdot\alpha Lf=:(\alpha Lf)_*$ is an (L,V)-distributor, hence (in particular):

$$f_{\circledast}\cdot La\cdot m_X^{\circ}\leq f_{\circledast};$$

moreover, because $b\cdot b\leq b$, then we have also:

$$b\cdot b\cdot\alpha Lf\leq b\cdot\alpha Lf,$$

from which follows that $(\alpha Lf)_*$ is a V-distributor, hence that αLf is a V-functor.
A quick calculation shows that $e_1^{\circ}\cdot\hat{j}=j$; moreover, since j is an (L,V)-distributor we have:

$$e_1^{\circ}\cdot\hat{j}\cdot La\cdot m_X^{\circ}=j\cdot La\cdot m_X^{\circ}=j\circ a\leq j.$$

Which shows that $e_1^{\circ}\cdot\hat{j}:LX\looparrowright E$ is a V-distributor. Hence by Corollary 2.16 the result follows. ∎

Proof. (Of the proposition) We have:

$$\begin{aligned}\tilde{b}(e_Y(L\text{-}\mathtt{colim}(j,f)(1)),c)&=b(L\text{-}\mathtt{colim}(j,f)(1),c)\\&=\bigwedge_{\underline{x}\in LX}[\hat{j}(\underline{x},e_1(1)),b(\alpha Lf(\underline{x}),c)],\\&=\bigwedge_{\underline{x}\in LX}[j(\underline{x}),b(\alpha Lf(\underline{x}),c)],\\&=b(\mathtt{colim}(e_1^{\circ}\cdot\hat{j},\alpha Lf),c).\end{aligned}$$

From which the result follows. ∎

Corollary 2.38. *Assuming the same hypotesis of the previous proposition,* $\tilde{b}(e_Y(L\text{-}\mathtt{colim}(j,f)(1)),c) = \mathtt{lim}(e_1^\circ \cdot \hat{j}, b(\alpha Lf(-),c))$.

Proof. It follows from Example 2 of 2.23. □

Corollary 2.39. *Assuming the same hypotesis of the previous proposition,* $L\text{-}\mathtt{colim}(j,f)(1) \simeq \bigvee_{\underline{x}\in LX} j(\underline{x}) \odot \alpha Lf(\underline{x})$. *Where the conical colimit and the co-power are taken in the underlying* V*-category* (Y,b).

Proof. Follows from the discussion on weighted colimits done in 2.19. □

Definition 2.40. Given two (L,V)-functors $f:(X,a)\to(Y,b)$, $g:(Y,b)\to(X,a)$; f is *left adjoint* to g (vice versa: g is *right adjoint* to f) and denoted: $f\dashv g$, if $f_\circledast \simeq g^\circledast$. Pointwise this means that, for all $\underline{x}, y \in LX, Y$:

$$b(Lf(\underline{x}),y) = a(\underline{x}, g(y)).$$

Examples 2.41.

1. Any (L,V)-functor $f:(X,a)\to(Y,b)$ defines an adjoint pair of (L,V)-functors:

$$\begin{tikzcd} \mathbb{D}_L(X) \arrow[r, bend left, "\mathbb{D}(f):=-\circ f^\circledast"] \arrow[r, phantom, "\perp"] & \mathbb{D}_L(Y) \arrow[l, bend left, "-\circ f_\circledast"] \end{tikzcd}$$

2. Given an (L,V)-functor $f:(X,a)\to(Y,b)$, with (Y,b) cocomplete, we can consider the following functor:

$$\mathtt{Lan}_y(f) := L\text{-}\mathtt{colim}(y_\circledast, f) : \mathbb{D}_L(X) \to Y,$$

which is called the *left Kan extension of* f *along* y_X; it can be shown that $\mathtt{Lan}_y(f)$ has a right adjoint R given by:

$$R(y) := b(Lf(-),y) : Y \to \mathbb{D}_L(X).$$

Let $f:(X,a)\to(Y,b)$ and $g:(Y,b)\to(Z,c)$ two (L,V)-functors and $j: X \nrightarrow D$ be a distributor. From the definition of weighted colimit it follows that $L\text{-}\mathtt{colim}(j,gf)\le g(T\text{-}\mathtt{colim}(j,f))$.
If the equality holds, we say that g preserves the colimit of f weighted by h. If g preserves all colimits, we say that is co-continuous. As in the V-case, we have that left adjoint (L,V)-functors are co-continuous.

2.5 (Enriched)-Quantales and Lax Monoidal Monads

In this subsection we are going to define the V-version of quantales, state some of their properties and introduce lax monoidal monads along with their properties.

It is well known that quantales can be described as monoids in the monoidal category of sup-lattices, where the monoidal structure on Sup could be described as the one induced by the *strong-commutative monad*:

$$P:\mathtt{Sets}\to\mathtt{Sets},\ \ P(X):=2^X.$$

(see [JT84] for details).
Since Sup is equivalent to the category of cocomplete 2-categories (with co-continuous 2-functors as arrows), it is natural to define an enriched quantale as a monoid in the category of cocomplete V-category, where the monoidal structure is the one induced by the strong commutative monad:

$$P:\mathtt{Sets}\to\mathtt{Sets},\ \ P(X):=V^X.$$

This motivates the following definition:

Definition 2.42. An enriched quantale (for simplicity, from now on we will call it: a "quantale") $(Q, *, k)$ is an object of $\mathtt{Mon}(\mathtt{Sets}^{P_V}, \otimes_V, V)$.

Remark 2.43. From the definition of quantale, it follows that $- * x : Q \to Q$ and $x* =: Q \to Q$ are two co-continuous V-functors. Since Q is cocomplete, it follows that both have right adjoints, given by:

$$y \rhd (=) := \mathtt{colim}(Q(- * y, =), Id), \quad x \lhd (=) := \mathtt{colim}(Q(x* =, =); Id),$$

which can be re-written pointwise as:

$$y \rhd z := \bigvee_{x \in X} Q(x * y, z) \odot x, \quad x \lhd z := \bigvee_{y \in Y} Q(x * y, z) \odot y.$$

Note that in the ordered case, the last one correspond to the familiar notion of *left* and *right implication* respectively.

Recall that a V-functor $f : (X, a, *_X, k_X) \to (Y, b, *_Y, k_Y)$ between two monoidal V-categories is called *lax-monoidal* if:

$$k_Y \le f(k_X), \quad f(x) *_Y f(y) \le f(x *_X y).$$

Let $\mathtt{Mon}((V\text{-}\mathtt{Cat}))_{lax}$, be the category formed by monoidal V-category with lax-monoiadal functors between them.

Proposition 2.44. *$K : \mathtt{Mon}((V\text{-}\mathtt{Cat}))_{\mathtt{lax}} \to (L, V)\text{-}\mathtt{Cat}$ is fully-faithful. Here K is the functor defined in Example 4*

Proof. The fact that K is faithful is straightforward.
Let $f : K(X) \to K(N)$, be a (L, V)-functor with $\tilde{a}$, $\tilde{b}$ being their corresponding (L, V)-structure. By definition we have that:

$$\tilde{a}(\underline{x}, y) \le \tilde{b}(Lf(\underline{x}), f(y)), \quad \text{where } Lf(\underline{x}) := (f(x_1), ..., f(x_n)),$$

which by definitions implies:

$$a(x_1 *_X ... *_X x_n, y) \le b(f(x_1) *_Y ... *_Y f(x_n), f(y)).$$

By taking $y = x_1 *_X ... *_X x$, from $k \le a(x, x)$, it follows that:

$$f(x) *_Y f(y) \le f(x *_X y).$$

While, by taking as first argument the empty list:

$$k_Y \le f(k_X).$$

□

By relaxing the definition of quantale we can define another useful category to which ideas we are going to develop in the following sections apply.

Definition 2.45. A monoidal V-category $(X, a, *_X, k_X)$ is called *residuated* if for every $x \in X$ $x *_X (=), (-) *_X x$ have right adjoints: $x \rhd (=), x \lhd (=)$. The category formed by residuated monoidal V-categories with lax monoidal functor between them is denoted by: $\mathtt{ResMon}((V\text{-}\mathtt{Cat}))_{\mathtt{lax}}$

Remark 2.46. From the definitinon is clear that quantales form a full-subcategory of both $\mathtt{ResMon}((V\text{-}\mathtt{Cat}))_{\mathtt{lax}}$ and $\mathtt{Mon}((V\text{-}\mathtt{Cat}))_{\mathtt{lax}}$.

Remark 2.47. Notice that a morphism f in $\mathtt{ResMon}((V\text{-}\mathtt{Cat}))_{\mathtt{lax}}$ satisfies:

$$f(y \rhd z) \leq f(y) \rhd f(z), \quad f(y \lhd z) \leq f(y) \lhd f(z);$$

Indeed, from:

$$f(y \rhd z) * f(y) \leq f((y \rhd z) * y), \text{ and } (y \rhd z) * y \leq z,$$

it follows:

$$f(y \rhd z) \leq f(y) \rhd f(z).$$

In the same way, but starting from:

$$f(y) * f(y \lhd z) \leq f(y * (y \lhd z)),$$

we get:

$$f(y \lhd z) \leq f(y) \lhd f(z).$$

Definition 2.48. Given a quantale Q, a *lax-monoidal monad* $T : Q \to Q$ is a lax-monoidal functor such that:

- $k_Q \leq T(k_Q)$;
- $T^2 = T$;

Remark 2.49. Lax-monoidal monads are generalization, for a general quantale, of the notion of quantic nuclei.

Theorem 2.50. *Let Q be a quantale and $t : Q \to Q$ be lax-monoidal monad, then:*

$$Q^T := \{x \in Q \text{ such that } T(x) = x\},$$

is a quantale; moreover the inclusion $Q^T \rightarrowtail Q$ is a V-functor that has a left adjoint π_T which is a morphism of quantale.

Proof. The proof is a straightforward generalization of the corresponding proof for quantic nuclei. The V-structure on Q^T is the one induced by Q while the right adjoint to the inclusion functor is: $\pi_T(x) := T(x)$. $\square$

3 Intermezzo

In this section we recall how injectives and injective hulls are built in the category of ordered sets and monotone maps between them. We show how the same ideas apply to the enriched case, for a general commutative quantale V.

Recall that in a category C, an object X, is called *injective* (w.r.t a class of moprhims J), if for every diagram of the form:

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & X \\ Z & \end{tikzcd}$$

where $Y \to X$ is in J, there exists an extension:

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & X \\ Z \arrow[ur, dotted] & \end{tikzcd}$$

making the diagram commutes.
Given an object Y, E is called an *injective hull* if there exists a monomorphism $e : A \to E$ such that:

- E is J-injective;
- For every $f : E \to X$, $fe \in J$ implies $f \in J$.

Let $(X, \leq)$ be an ordered set. Recall that we have two embedding:

$$\downarrow : X \rightarrowtail \mathbb{D}(X), \quad x \mapsto \downarrow x := \{w \in X, \quad w \leq x\},$$

$$\uparrow : X \rightarrowtail \mathbb{U}(X), \quad x \mapsto \uparrow x := \{w \in X, \quad x \leq w\}.$$

It is well known that there exists an adjunction:

$$\begin{tikzcd} \mathbb{D}(X) \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathbb{U}(X) \arrow[l, bend left, "R"] \end{tikzcd}$$

Where, for $j \in \mathbb{D}(X)$, $l \in \mathbb{U}(X)$, one has:

$$L(j) := \{x \in X : \forall_{w \in X} w \in j \Rightarrow w \leq x\}, \;\; R(l) := \{x \in X : \forall_{w \in X} w \in l \Rightarrow x \leq w\}.$$

This adjunction induces a monad (closure operator) T on $\mathbb{D}(X)$ defined, for $j \in \mathbb{D}(X)$, as follow:

$$\begin{aligned} T(j) &:= \{x \in X : \forall_{y \in X} y \in L(j) \Rightarrow x \leq y\}, \\ &= \{x \in X : \forall_{y \in X} y (\forall_{w \in X} w \in j \Rightarrow w \leq y) \Rightarrow x \leq y\}, \\ &= \bigwedge_{y \in X} \{y : \forall_{w \in X} w \in j \Rightarrow w \leq y\}, \\ &= \bigwedge_{y \in X} \{y : j \leq \downarrow y\}. \end{aligned}$$

In particular is immediate from the definition that, for all $x \in X$:

$$T(\downarrow x) := \bigwedge_{y \in X} \{y : \downarrow x \leq \downarrow y\} = \downarrow x.$$

Thus we have the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, tail, "\tilde{\downarrow}(-)"] \arrow[dr, tail, bend right, "\downarrow(-)"'] & \mathbb{D}(X)^T \\ & \mathbb{D}(X) \arrow[u, "T"'] \end{tikzcd}$$

The fixed point of this monad for an ordered set $\mathbb{D}(X)^T$ called the *Dedekind-MacNeille completion* of X. The term "completion" refers to the property $\mathbb{D}(X)^T$ has: every element of $\mathbb{D}(X)^T$ is both a supremum of elements of X and an infimum of elements of X, and $\mathbb{D}(X)^T$ is the smallest complete ordered set that has this property. Moreover, in [Mac37], MacNeille proved that w.r.t monotone embedding $\mathbb{D}(X)^T$ is the injective hull of X.

This construction admits an "easy" generalization to the enriched case. First of all, we recall that injectives in V-`Cat` w.r.t fully-faithful functors are cocomplete V-categories; this can be easily showed using the characterization of cocompleteness we mentioned in 2.24.

Remark 3.1. Injectives in V-`Cat` (w.r.t fully-faithful functors) can be equivalently characterized as algebras for the Koch-Zöberlein monad $\mathbb{D}(-)$, as explained in [Esc97, Esc98, EF99] in a more general context.

Let (X,a) be a V-category. As in the ordered case, we have an adjunction, called the *Isbell duality*:

$$\begin{tikzcd} \mathbb{D}(X) \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathbb{U}(X) \arrow[l, bend left, "R"] \end{tikzcd}$$

Where, for $j \in \mathbb{D}(X)$, $l \in \mathbb{U}(X)$, one has:

$$L(j) := \bigwedge_{x\in X}[j(x), a(x,=)] \simeq (y_X)^* \cdot j_*, \quad R(l) := \bigwedge_{x\in X}[j(x), a(-,x)] \simeq l^* \cdot (\lambda_X)_*.$$

The monad induced by this adjunction, T, can be calculated as before:

$$\begin{aligned} T(j) &:= \bigwedge_{x\in X}[L(j)(x), a(-,x)], \\ &= \bigwedge_{x\in X} L(j)(x) \pitchfork y_x(x), \\ &= \mathtt{lim}(L(j), y_X), \\ &= \mathtt{lim}((y_X)^* \cdot j_*, y_X). \end{aligned}$$

In particular is immediate from the definition that, for all $x \in X$:

$$T(y_X(x)) := \mathtt{lim}((y_X)^* \cdot j_*, y_X) \simeq y_X(x).$$

Thus as before, the Yoneda factorizes:

$$\begin{tikzcd} X \arrow[r, tail, "\tilde{y_X}"] \arrow[dr, tail, bend right, "y_X"'] & \mathbb{D}(X)^T \\ & \mathbb{D}(X) \arrow[u, "T"'] \end{tikzcd}$$

The fixed point of this monad form a V-category $\mathbb{D}(X)^T$ in which every element j can be written as:

$$j \simeq \mathtt{colim}(j^* \cdot (y_X)_*, y_X), \quad j \simeq \mathtt{lim}((y_X)^* \cdot j_*, y_X),$$

where the first isomorphism is the well know fact that every presheaves is the weighted colimit over itself of the Yoneda ([Kel82]), and the second one follows from being j a fixed point of T.
This motivates the following definition:

Definition 3.2. A fully-faithful V-functor $i : (X,a) \rightarrowtail (Y,b)$ is called *dense*, if every element $y \in (Y,b)$ is of the form:

$$y \simeq \mathtt{colim}(y^* \cdot i_*, i), \quad y \simeq \mathtt{lim}(i^* \cdot y_*, i),$$

In this way we have a nice characterization of essential morphisms in V-$\mathtt{Cat}$ as contained in the following:

Theorem 3.3. *A fully-faithful V functor $i : (X,a) \rightarrowtail (Y,b)$ is dense iff is essential.*

Before proving it, we state a straightforward lemma:

Lemma 3.4. *Suppose we have a fully faithful V-functor $f : (Y,b) \to (Z,c)$ and a functor $g : (X,a) \to (Y,b)$. If fg is dense, then g is dense too.*

Proof. (Of the theorem) $\Rightarrow$). Let $f : (Y,b) \to (Z,c)$ be a V-functor such that $f\big|_X$ is fully-faithful. We have to show that:

$$c(f(x), f(y)) \le b(x,y), \text{ for all x,y} \in Y.$$

Since i is dense, we can write:

$$x \simeq \mathtt{colim}(x^* \cdot i_*, i(x)), \quad y \simeq \mathtt{lim}(i^* \cdot y_*, i(y)).$$

By playing with the properties of (co)-limits, we have:

$$\begin{aligned}
b(x,y) &= b(\mathtt{colim}(x^* \cdot i_*, i(w)), \mathtt{lim}(i^* \cdot y_*, i(q))),\\
&= \mathtt{lim}(x^* \cdot i_*, \mathtt{lim}, (i^* \cdot y_*), b(i(w), i(q)))),\\
&\text{(since } i \text{ is fully-faithful)},\\
&= \mathtt{lim}(x^* \cdot i_*, \mathtt{lim}, (i^* \cdot y_*), a(w,q)))\\
&\text{(since } f\big|_X \text{ is fully-faithful)},\\
&= \mathtt{lim}(x^* \cdot i_*, \mathtt{lim}, (i^* \cdot y_*), c(f\,i(w), f\,i(q)))),\\
&= c(\mathtt{colim}(x^* \cdot i_*, f\,i(w)), \mathtt{lim}(i^* \cdot y_*, f\,i(q))),\\
&\geq c(f(\mathtt{colim}(x^* \cdot i_*, i(w)), f(\mathtt{lim}(i^* \cdot y_*, i(q))),\\
&\geq c(f(x), f(y)).
\end{aligned}$$

Which shows that i is essential.
$\Leftarrow$). Because $\mathbb{D}(X)^T$ is injective there exists a V-functor $Y \to \mathbb{D}(X)^T$ that makes the following commutative:

$$\begin{tikzcd}
Y \arrow[dr, dashed] & X \arrow[l, tail] \arrow[d, tail] \\
& \mathbb{D}(X)^T
\end{tikzcd}$$

And because i is essential, then we have that the dashed arrow is an embedding. The result follows from the previous lemma.

□

This theorem ends our *intermezzo* section. In the next section we will apply similar ideas in order to construct the injective hulls in (L,V)-$\mathtt{Cat}$. Unfortunately the procedure will not be so smooth as for V-categories.

The first inconvenience is due to the fact that (as for now), since we don't have an equivalent of $\mathbb{U}(X)$ for an (L,V)-category (X,a), we don't have an analogue of the Isbell duality and thus we will have to define the monad T "manually"; Moreover, since we need the category of algebras for this monad to be an (L,V)-category we need T to be a lax-monoidal monad.

The second inconvenience follows directly from the first one. In order to prove that T is lax-monoidal we will need to reduce our attention to a particular sub-category of (L,V)-$\mathtt{Cat}$: the category of *Quantum B-algebras*. For this category we will be able to mimic all the construction done in this section and build injective hulls as algebras for a lax-monoidal monad which resembles the one we introduced before. With a slight modification of definition 3.2, where we will substitute the V-colimit part with an (L,V)-colimit, we will be able to proof the equivalent of 3.3. Luckily the restriction to quantum B-algebras will not prevent us to construct the injective hull for all (L,V)-categories, by embedding every (L,V)-category in a quantum B-algebras we will provide an injective hull for all (L,V)-categories.

4 Injective Hulls of (Enriched) Quantum B-Algebras

Promonoidal categories were introduced by Day in his thesis [Day71]. They originated from the observation that, in order to define a monoidal structure on $[X,V]$, what you need is only a *promonoidal* structure on X. As we are going to describe, a promonoidal category is a monoid in V-$\mathtt{Dist}$, with the monoidal structure inherited from the on in V-Cat we described briefly in the previous section.

Quantum B-Algebras were introduced in [Rum13]. The motivation comes from non-commutative algebraic logic, for which they provide an unified semantic as better explained in [Rum13, RY14]. They were defined as orderded sets equipped with two implications that mimic the residuals in an quantale.

Since ordered monoids (and more general their enriched counterpart) are promonoidal categories for which (part of) the promonoidal structure is representable, is natural to ask if the same holds for quantum B-algebras. Unsurprisingly the answer, as briefly shown in the last section of [Rum16], is positive: quantum B-algebras can be seen as *representables* promonoidal category.
In the following section we'll take this point of view in order to present the enriched version of quantum B-algebras, and we'll show how (L,V)-categories provide a common roof for both of them by showing how some categorical constructions become natural when one consider both as being fully-faithful subcategories of (L,V)-Cat.

The section is structured as follows:

- In the first subsection are devoted to introduce promonoidal categories and quantum B-algebras along with some of their basics properties.
- In the second subsection we show how injectives in both categories can be easily characterize from the characterization of injectives in (L,V)-Cat, and we generalize some constructions of [Rum16] to the enriched case, by showing that every (enriched) quantum B-algebra admits an injective hulls.

We would like to stress the fact that results of 4.2 are not a mere exercise in *fuzzyfication*, that is to say, a rephrase of the ones in [Rum16] for a general V. In order to be able to generalize them, on needs a more general theory of colimits, which is provided by the notion of (L,V)-colimits introduced in the previous section. This shred light on some universal constructions that in the ordered case are somehow "hidden".

4.1 Promonoidal V-Categories and Quantum B-Algebras

In the previous section we briefly explained how V-Cat can be equipped with a monoidal structure, by defining, for two V-categories $(X,a),(Y,b)$, their tensor product as:

$$X \boxtimes Y = (X \boxtimes Y, a \otimes b).$$

This monoidal structure induces a monoidal structure on V-Dist, where for $j : X \to\!\!\!\!\!\to Y,\ \ l : Z \to\!\!\!\!\!\to W$:

$$j \boxtimes l : X \boxtimes Z \to\!\!\!\!\!\to Y \boxtimes W,\ \ j \boxtimes l((x,z),(y,w)) := j(x,y) \otimes l(y,w).$$

With this in mind, is possible to talk about monoids in V-Dist.

Definition 4.1. A *promonoidal V-category* is a monoid in V-Dist. This means it is a V-category (X,a) together with two distributors:

- $P : X \boxtimes X \to\!\!\!\!\!\to X$;
- $J : 1 \to\!\!\!\!\!\to X$.

Such that:

- $P \cdot (P \boxtimes Id) = P \cdot (Id \boxtimes P)$;
- $P \cdot (J \boxtimes Id) = Id,\ \ P \cdot (Id \boxtimes J) = \mathrm{Id}$.

Remark 4.2. The last two conditions pointwise mean:

- $\bigvee_{x\in X} P(a,x,d)\otimes P(b,c,x) = \bigvee_{x\in X} P(x,a,d)\otimes P(a,b,x)$.
- $\bigvee_{z\in X} J(z)\otimes P(x,z,w) = a(x,w)$, $\bigvee_{z\in X} J(z)\otimes P(z,x,w) = a(x,w)$.

Definition 4.3. Following Day [DS95], given two promonoidal categories (X,a,P,J), (Y,b,R,U), a *promonoidal functor* between them is a V-functor $f:(X,a)\to(Y,b)$, such that, for all $x,y,z\in X$:

$$P(x,y,z)\le R(f(x),f(y),f(z)),\quad J(x)\le U(f(x)).$$

If the reverse inequalities hold, we call it *strong*.

Examples 4.4. 1. Any monoidal V-category $(X,a,*_X,u_X)$ defines a promonoidal one by taking:

- $P(x,y,z) := a(x *_X y, z)$.
- $J(x) := a(u_x, x)$.

2. Any quantales $(Q,*,u_Q)$ defines a promonoidal catgory in two (equivalents) ways: the first one is the one described before, while the second one is as follows:

- $P(x,y,z) := Q(x, y\rhd z) = Q(y, x\lhd z)$.
- $J(x) := Q(u_Q, x)$.

The equivalence of the two formulations follows by adjunction.

The last example will be explored further when we will introduce *Quantum B-Algebras*.

Remark 4.5. As mentioned in the introduction of the section, a promonoidal structure induces a monoidal structure on the corresponding category of presheaves. More precisely, given a promonoidal category (X,a,P,J) we can define a monoidal structure on $(\mathbb{D}(X), *_D, J^\circ)$, as follows:

$$j *_D l(x) := \bigvee_{w,z} P^\circ(w,z,x)\otimes j(w)\otimes l(w).$$

In particular, when the promonoidal structure comes from a monoidal one, the previous formula reads as:

$$j *_D l(x) := \bigvee_{w,z} a(x, w*z)\otimes j(w)\otimes l(w).$$

In this case $(\mathbb{D}(X), *_D, J^\circ)$ becomes a quantale, where as an example, the right implication is:

$$l\rhd h(x) := \bigwedge_y [l(y), h(x*y)].$$

Proposition 4.6. *V-*$\mathtt{Pro}$ *is a fully-faithful subcategory of* (L,V)-$\mathtt{Cat}$.

Proof. Given a promonoidal V-category (X,a,P,J) defines an (L,V)-category $(X, \tilde{a} := \amalg_{n\ge 0} a_n)$ in the following inductive way:

- $a_0 := J$;
- $a_1 := a$;
- $a_2 := P$;
- $a_n := P\cdot(Id\boxtimes a_{n-1})$.

Indeed, $e_X^\circ \leq \tilde{a}$ follows directly from the definition, while $\tilde{a} \circ \tilde{a} \leq \tilde{a}$ follows from the inductive definition of $\tilde{a}$, from the properties of P and J, and from the fact that a is a V-structure, in the following way: Let $\underline{\underline{x}} := (\underline{x}_1, ..., \underline{x}_n) \in LLX$, $\underline{y} := (y_1, ..., y_n) \in LX$, and $z \in X$. We want to show:

$$L\tilde{a}(\underline{\underline{x}}, \underline{y}) \otimes \tilde{a}(\underline{y}, z) \leq \tilde{a}(m_X(\underline{\underline{x}}), z).$$

Unravelling the definition this means:

$$a_{l(\underline{x}_1)}(\underline{x}_1, y_1) \otimes ... \otimes a_{l(\underline{x}_n)}(\underline{x}_1, y_1) \otimes a_n(\underline{y}, z) \leq a_{l(\underline{x}_1)+...+l(\underline{x}_n)}(m_X(\underline{\underline{x}}), z), \quad (1)$$

where $l(-)$ denotes the length of a list.
The prove of (1) is a long and frustrating matter of bookkeeping relying on the properties P and a have. Instead of proving (1), we prefer to illustrate the properties used in a simple case which is sufficient general to encompass (1).
Consider $\underline{\underline{x}} := ((x_1, x_2), x_3)$ and $y = (y_1, y_2)$. Then (1) becomes:

$$\begin{aligned}
P(x_1, x_2, y_1) \otimes a(x_3, y_2) \otimes P(y_1, y_2, z) &= \bigvee_d P(x_1, x_2, y_1) \otimes a(y_1, d) \otimes a(x_3, y_2) \otimes P(d, y_2, z), \\
&\text{(from } a \cdot P \leq P) \\
&\leq \bigvee_d P(x_1, x_2, d) \otimes a(x_3, y_2) \otimes P(d, y_2, z), \\
&\text{(from } P \cdot (Id \boxtimes a) \leq P \cdot (a \boxtimes a) \leq P) \\
&\leq \bigvee_d P(x_1, x_2, d) \otimes P(d, x_3, z), \\
&= \tilde{a}((x_1, x_2, x_3), z)
\end{aligned}$$

Suppose that $f : (X, a, P, J) \to (Y, b, R, U)$ is a promonoidal functor. Then, for all $\underline{x} \in LX$, $y \in X$ (with $l(\underline{x}) = n$):

$$\begin{aligned}
\tilde{a}(\underline{x}, y) := a_n(\underline{x}, y) &:= \bigvee_c a_{n-1}((x_1, ..., x_{n-1}), c) \otimes P(c, x_{n+1}, y), \\
&\text{(by induction on } a_n\text{, using the fact that } f \text{ is a promonoidal functor)} \\
&\leq \bigvee_c b_{n-1}((f(x_1), ..., f(x_{n-1})), f(c)) \otimes R(f(c), f(x_{n+1}), f(y)), \\
&:= b_n(Lf(\underline{x}), f(y)) = b(Lf(\underline{x}), f(y)).
\end{aligned}$$

This shows that f is an (L, V)-functor.
If: $\tilde{a}(\underline{x}, y) \leq b(Lf(\underline{x}), f(y))$, take:

$$\underline{x} = (x_1, x_2) \Rightarrow P(x_1, x_2, y) \leq R(f(x_1), f(x_2), f(y)),$$

$$\underline{x} = (x) \Rightarrow a(x, y) \leq b(f(x), f(y)),$$

$$\underline{x} = (-) \Rightarrow J(x) \leq U(f(x)).$$

This shows that V-$\texttt{Pro}$ is a fully-faithful subcategory of (L, V)-$\texttt{Cat}$.

□

Definition 4.7. A *quantum B-algebra* (X_B, a) is a representable promonoidal V-category. This means it is a promonoidal V-category (X, a, P, J) equipped with two binary operations: $\rhd, \lhd \colon X \times X \to X$ and an element $u_X \in X$, such that:

- $P(x, -, y) \simeq a(-, x \lhd y)$;

- $P(-,x,y) \simeq a(-,x \rhd y)$;
- $J(x) = a(u,x)$.

Remark 4.8. From the representability condition it directly follows that $\rhd, \lhd$ are two bi-functors of the form:

$$\rhd, \lhd \colon X^{\mathrm{op}} \boxtimes X \to X.$$

Moreover, from representability, it follows that:

- $a(x, y \rhd z) = a(y, x \lhd z)$;
- $x \rhd (y \lhd z) = y \lhd (x \rhd z)$;
- $u_X \rhd (=)$ $(u_X \lhd (=))$ is the identity.

In this way we can recover the definition of (unital) quantum B-algebras given in [Rum13].

By expressing what it means to be *promonoidal functor* for quantum B-algebras, we have:

Definition 4.9. A morphism between two quantum B-algebras $f : (X_B, a) \to (Y_B, b)$ is V-functor satisfying:

$$f(x \rhd y) \leq f(x) \rhd f(y),\ f(x \lhd y) \leq f(x) \lhd f(y),\ u_Y \leq f(u_X) \quad (1)$$

If also:

$$f(x) \rhd f(y) \leq f(x \rhd y),\ f(x) \lhd f(y) \leq f(x \lhd y)), \quad (2)$$

then f is called strict.

Remark 4.10. It is relative easy to derive these conditions from the promonoidal functor's one using representability. As an example we show how to derive the condition on the unities: $u_Y \leq f(u_X)$.

$$\begin{array}{rc}
 & J(-) \leq U(f(-)) \\ \hline
 & (u_x)_* \leq f^* \cdot (u_Y)_* \\ \hline
\text{(from } f_* \dashv f^*\text{)} & f_* \cdot (u_X)_* \leq (u_Y)_* \\ \hline
 & b(f(u_X),=) \leq b(u_Y,=) \\ \hline
 & u_Y \leq f(u_X)
\end{array}$$

From 4.6, it follows:

Proposition 4.11. *The category of quantum B-algebras is a fully-faithful subcategory of* (L,V)-$\mathtt{Cat}$.

Remark 4.12. Let (X_B, a) be a quantum B-algebra, by unravelling the definition contained in 4.6, the corresponding (L,V)-structure is given as follows. For a pair $(\underline{x}, y)$ defines:

$$\tilde{a}((x_1,\dots,x_n),y) := \tilde{a}((x_1,\dots,x_{n-1}),x_n \rhd y) := \dots := a(x_1, x_2 \rhd \dots x_n \rhd y),$$

$$\tilde{a}((-),y) := a(u_X, y), \quad \text{where } (-) \text{ is the empty list.}$$

Remark 4.13. Let Q be a quantale. We can see it as a quantum B-algebra and as an (L,V)-category. It is easy to see that the two structure (call them Q_B and Q_L) coincide; indeed:

$$Q_B((q_1,\dots,q_n),w) := Q(q_1, q_n \rhd \dots \rhd w) = Q(q_1 * \dots * q_n, w) =: Q_L((q_1,\dots,q_n),w).$$

4.2 Injectives and Injective Hulls

In the previous section we recalled the notion of injective object and of injective hull w.r.t a class of morphism J. As we did for V-$\mathtt{Cat}$ from now on, when we write "injective" we always mean injective w.r.t fully-faithful functors/embedding. Since (L,V)-$\mathtt{Cat}$ serves as a common roof under which all the constructions are performed, when we are dealing with: promonoidal categories, quantum B-algebras, etc. "injective" will always mean w.r.t to the corresponding notion in (L,V)-$\mathtt{Cat}$.

In the context of (T,V)-categories in [Hof11] it was shown that:

Theorem 4.14. *An (L,V)-category (X,a) is injective iff is cocomplete.*

We can go further in our analysis and characterize them as quantales:

Theorem 4.15. *An (L,V)-category (X,a) is injective w.r.t fully-faithful functors iff is a quantale.*

Before giving a direct proof, we state a useful lemma:

Lemma 4.16. *Let Q be a quantale and (Y,b) be an (L,V)-category. Suppose an (L,V)-functor $f : Y \to Q$ is given. Then there exists an (L,V)-functor $g : \mathbb{D}_L(Y) \to Q$ making the following commute:*

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, tail, "y_Y"'] & Q \\ \mathbb{D}_L(Y) \arrow[ur, dotted, "g"'] & \end{tikzcd}$$

Proof. Let $g := \mathtt{Lan}_{y_Y}(f)$. Since $\mathtt{Lan}_{y_Y}(f)(j) := L\text{-}\mathtt{colim}(y_\circledast, f)(1)$, which by Proposition 2.36 is the same as $\mathtt{colim}(e_1^\circ \cdot \hat{y_\circledast}, \alpha L f)$ and Q is a quantale, g is well defined (as a function). Moreover it makes the diagram commute.
If we prove that there exists an (L,V)-functor $R : Q \to \mathbb{D}_L(Y)$ such that $R^\circledast \simeq g_\circledast$ we can conclude that g is an (L,V)-functor too.
By unravelling the definition of colimits, we have that:

$$\begin{aligned} g(j) &\simeq \bigvee_{\underline{y}\in LY} \mathbb{D}_L(Y)[Ly_Y(\underline{y}), j] \odot (f(y_1) * \ldots * f(y_n)) \\ &\quad \text{(by Yoneda lemma)} \\ &\simeq \bigvee_{\underline{y}\in LY} j(\underline{y}) \odot (f(y_1) * \ldots * f(y_n)) \end{aligned}$$

Define $R(w) := Q(Lf(-), w)$. Since R is the mate of $f_\circledast$, by Remark 2.33 it follows that R is an (L,V)-functor. In order to conclude, we have to show that:

$$g_\circledast(\underline{j}, w) = R^\circledast(\underline{j}, w).$$

Without loos of generality we can suppose that $\underline{j} = (j_1, j_2)$.

We can compute:

$$
\begin{aligned}
g_{\circledast}(\underline{j},w) &= Q(g(j_1) * g(j_2), w)\\
&= Q(\bigvee_{\underline{y}\in LY} j_1(\underline{y}) \odot (f(y_1) * \dots * f(y_n)) * \bigvee_{\underline{z}\in LY} j_2(\underline{z}) \odot (f(z_1) * \dots * f(z_n)), w)\\
&\text{(since the multiplication of } Q \text{ preserves colimits being a quantale)}\\
&= \bigwedge_{\underline{y}\in LY} \bigwedge_{\underline{z}\in LY} [j_1(\underline{y}) \otimes j_2(\underline{z}), Q((f(y_1) * \dots * f(y_n)) * (f(z_1) * \dots * f(z_n)), w)]\\
&= \bigwedge_{(\underline{y};\underline{z})\in LY} [Lj(\underline{y};\underline{z}), R(w)(\underline{y};\underline{z})]\\
&= \bigwedge_{\underline{x}\in LY} [Lj(\underline{x}), R(w)(\underline{x})]\\
&\text{(by the definition of the } (L,V)\text{-structure on } \mathbb{D}_L(Y)\text{, as explained in Example 6 of 2.31)}\\
&= \mathbb{D}_L(Y)[\underline{j}, R(w)]\\
&= R^{\circledast}(\underline{j}, w)
\end{aligned}
$$

Thus g is an (L,V)-functor as we wanted.

$\square$

Proof. (Of the theorem).
$\Leftarrow$). Suppose Q is a quantale and Let $g : (X,a) \to Q$ be an (L,V)-functor, with (X,a) be an (L,V)-category. Suppose that $f : (X,a) \rightarrowtail (Y,b)$ is a fully-faithful (L,V)-functor. Consider the Yoneda embedding $y_Y : Y \rightarrowtail \mathbb{D}_L(Y)$, by the previous lemma there exists an extension such that the following diagram commutes:

$$
\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, tail, "f"] & Q \\
Y \arrow[d, tail, "y_Y"] & \\
\mathbb{D}_L(Y) \arrow[uur, dotted] &
\end{tikzcd}
$$

Hence Q is injective.
$\Rightarrow$). Suppose that Q is injective. By 4.14, it follows that Q is a cococomplete (L,V)-category, hence that is representable. Since the 2-functor:

$$(-)_0 : (L,V)\text{-}\mathtt{Cat} \to V\text{-}\mathtt{Cat},$$

sends injectives objects in (L,V)-$\mathtt{Cat}$ to injective objects in V-$\mathtt{Cat}$ and since injectives in V-$\mathtt{Cat}$ are cocomplete V-categories (as explained in the previous section), it follows that Q is a cocomplete V-category. It remains to show that the monoidal structure on Q preserves V-colimits (in each variable).
Since Q is representable $\mathbb{D}(Q)$ becomes a quantale w.r.t Day's convolution product (as explained in 4.5); moreover, the Yoneda embedding:

$$y_Q : Q \rightarrowtail \mathbb{D}(Q),$$

is a strong monoidal V-functor. Thus, since we have:

$$
\begin{aligned}
Q(\underline{x}, w) &:= Q(x_1 * \dots * x_n, w)\\
&= \mathbb{D}(Q)(y_Q(x_1 * \dots * x_n), y_Q(w))\\
&= \mathbb{D}(Q)(y_Q(x_1) * \dots * y_Q(x_n), y_Q(w))\\
&:= \mathbb{D}(Q)(Ly_Q(\underline{x}), y_Q(w))
\end{aligned}
$$

y_Q is also an embedding in (L,V)-$\mathtt{Cat}$.
Because Q is injective, there exists an extension:

$$\begin{tikzcd} Q \arrow[r, equal] \arrow[d, tail, "y_Q"'] & Q \\ \mathbb{D}(Q) \arrow[ur, dotted, "h"] & \end{tikzcd}$$

With $h \cdot y_Q = Id_Q$.
Since the same holds in V-$\mathtt{Cat}$, by Theorem 2.24, h must be of the form:

$$\begin{aligned} h : \mathbb{D}(Q) &\to Q \\ j &\mapsto \mathtt{colim}(j, Id) \end{aligned}$$

In order to show that Q is a quantale, by Theorem 2.24, it is sufficient to show that the monoidal structure preserves colimits of the form $\mathtt{colim}(j, Id)$, for $j \in \mathbb{D}(Q)$. Let $x \in Q$ and $j \in \mathbb{D}(Q)$. We have:

$$\begin{aligned} x * h(j) := {} & h(y_Q(x)) * h(j) \\ & \text{(since } h \text{ is lax-monoidal)} \\ & \leq h(y_Q(x) * j) \\ & \text{(since every presheaves is the colimit of the Yoneda)} \\ & = h(y_Q(x) * \mathtt{colim}(j, Y_Q)) \\ & \text{(since Day's convolution preserves colimits)} \\ & = h(\mathtt{colim}(j, Y_Q(x) * Y_Q(=))) \\ & \text{(since the Yoneda is strong)} \\ & = h(\mathtt{colim}(j, Y_Q(x * =))) \\ & \text{(since h preserves colimits being left adj. to } y_Q \text{ in } V\text{-}\mathtt{Cat}) \\ & \simeq \mathtt{colim}(j, h(Y_Q(x * =))) \\ & \simeq \mathtt{colim}(j, x * (=)) \end{aligned}$$

Hence, because $\mathtt{colim}(j, x * (=)) \leq x * \mathtt{colim}(j, Id)$ follows from the u.p of colimits, we have:

$$x * \mathtt{colim}(j, Id) \simeq \mathtt{colim}(j, x * (=)).$$

Thus Q is a quantale.

□

With the aid of the previous theorem, we can characterize those full sub-categories of (L,V)-$\mathtt{Cat}$ whose injectives are quantales in the following and simple way:

Theorem 4.17. *Let C be a full sub-category of (L,V)-$\mathtt{Cat}$ which contains quantales. Then an object (X,a) is injective iff is a quantale.*

Proof. $\Leftarrow$). Quantales are injectives in (L,V)-$\mathtt{Cat}$ and C is a fully-faithful subcategory of it. Thus, if X is a quantale, then it is injective in C too.

$\Rightarrow$). If X is an injective object in C, then in (L,V)-$\mathtt{Cat}$ we have the following diagram:

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, tail, "y_X"'] & X \\ \mathbb{D}_L(X) \arrow[ur, dashed, "\exists h"] & \end{tikzcd}$$

The commutativity of the diagram implies that $h \cdot y_X = Id$. This means that $h : \mathbb{D}_L(X) \to X_B$ defines an algebra structure for the monad $\mathbb{D}_L(-)$, but because algebras for this monads are quantales (as shown in), this implies that X is a quantale too. $\square$

As corollaries of the previous theorem, we get a characterization of injectives as quantales in each of the categories displayed in the following diagram:

$$\begin{tikzcd}
 & (L,V)\text{-}\mathtt{Cat} & \\
 & V\text{-}\mathtt{Pro} \arrow[u, tail] & \\
 \mathtt{Mon}((V\text{-}\mathtt{Cat}))_{\mathtt{lax}} \arrow[ur, tail] & & \mathtt{QBalg} \arrow[ul, tail] \\
 & \mathtt{ResMon}((V\text{-}\mathtt{Cat}))_{\mathtt{lax}} \arrow[ul, tail] \arrow[ur, tail] &
\end{tikzcd}$$

In order to build injective hulls we mimic what is done for V-$\mathtt{Cat}$ and introduce the notion of *dense morphism*. The difference here is that, due to the nature colimits in our "roof" category, in our definition we consider L-$\mathtt{colim}$ instead of $\mathtt{colim}$.

Definition 4.18. Let (X, a) be in (L, V)-$\mathtt{Cat}$. An embedding $i : (X, a) \rightarrowtail Q$, where Q is a quantale (seen as a (L, V)-$\mathtt{Cat}$) is called *dense*, if every $q \in Q$ can be written as:

$$q \simeq \mathtt{lim}(i^* \cdot q_*, i), \quad q \simeq L\text{-}\mathtt{colim}(i_\circledast \circ q^\circledast, i)(1).$$

If only the latter holds, we call it *pre-dense*.

Remark 4.19. Notice that the limit in the previous definition is a V-enriched limit. We consider the 2-functor:

$$(-)_0 : (L, V)\text{-}\mathtt{Cat} \to V\text{-}\mathtt{Cat},$$

and take the limit accordingly.

Remark 4.20. Using the decomposition of limits (see Example 2.19), we can write the first condition as:

$$\mathtt{lim}(i^* \cdot q_*, i) \simeq \bigwedge_{x \in X} Q(q, i(x)) \pitchfork i(x),$$

which in the ordered case reduce to the more familiar:

$$q = \bigwedge_{\{x \,:\, q \leq i(x)\}} i(x).$$

While, unravelling the second condition and using Corollary 2.39, one has:

$$\begin{aligned} L\text{-}\mathtt{colim}(i_\circledast \circ q^\circledast, i)(1) &\simeq \bigvee_{\underline{x} \in LX} Q(\alpha L i(\underline{x}), q) \odot \alpha L i(\underline{x}), \\ &= \bigvee_{\underline{x} \in LX} Q(i(x_1) * \ldots * i(x_n), q) \odot (i(x_1) * \ldots * i(x_n)), \end{aligned}$$

which in the order case reduce to:

$$q = \bigvee_{\{\underline{x} \,:\, i(x_1) * \ldots * i(x_n) \leq q\}} i(x_1) * \ldots * i(x_n).$$

The last condition shows how, in the ordered case, X generates the quantale Q. This also shows how, in the enriched case, (L, V)-categories (more specifically the theory of (L, V)-colimits) are able to "capture", in an elegant and concise way, what we might call "monoidal colimits"; that is to say to provide a way to capture the idea of "quantale generated by a V-category", an intuitive construction that would be difficult to formalize without the notion of (L, V)-categories.

Definition 4.21. An embedding $i : (X_B, a) \rightarrowtail Q$, where Q is a quantale, of a quantum B-algebra is dense, if it is dense in (L, V)-$\mathtt{Cat}$.

Remark 4.22. We stress the fact that we are considering embedding of (L, V)-categories into quantales, which are cocomplete (L, V)-categories and complete V-categories.

Lemma 4.23. *Let $i : (X, a) \rightarrowtail (Y, b)$ be a fully-faithful V-functor between two V-categories, with (Y, b) complete. Then:*

$$T : (Y, b) \to (Y, b), \quad y \mapsto \mathtt{lim}(i^* \cdot y_\star, i),$$

is a V-functor.

Proof. Notice that T can be written as the composite of the following V-functors:

$$Y \xrightarrow{\lambda_Y} \mathbb{U}(Y) \xrightarrow{i^{-1}} \mathbb{U}(X) \xrightarrow{\mathbb{U}(i)} \mathbb{U}(Y) \xrightarrow{\mathtt{lim}} Y.$$

Where:

- $\lambda_Y : y \mapsto y_* := b(y, =)$;
- $i^{-1} : \psi \mapsto i^* \cdot \psi$;
- $\mathbb{U}(i) : \phi \mapsto i_* \cdot \phi$;
- $\mathtt{lim} : \gamma \mapsto \mathtt{lim}(\gamma, Id)$.

This is because: $\mathtt{lim}(i^* \cdot i_* \cdot y_*, Id) \simeq \mathtt{lim}(i^* \cdot y_\star, i)$

$\square$

Lemma 4.24. *With the same notations as before we have: $T \cdot i \simeq i$, meaning that, for all $x \in (X, a)$, $T(i(x)) = i(x)$.*

Proof. The result follows by noticing that, since i is fully-faithful:

$$i^* \cdot i(x)_* \simeq x_*,$$

and thus that: $x_* \cdot i(x)^* = x_* \cdot x^* \cdot i^* \leq i^*$, in a universal way. This proves our desired result.

$\square$

Proposition 4.25. *Let $i : (X_B, a) \rightarrowtail Q$, where Q is a quantale, be a pre-dense strict morphism in* $\mathtt{QBalg}$*. Then $T : Q \to Q$ defined before is a lax-monoidal monad.*

Proof. We want to prove that, for all $a \in Q$:

- $a \leq T(a)$;
- $k_Q \leq T(k_Q)$;
- $T^2(a) = T(a)$;
- $T(a) * T(b) \leq T(a * b)$.

We have the following chain of adjunctions:

$$\begin{tikzcd} \mathbb{U}(X) \arrow[r, bend left, "\mathbb{U}(i)"] \arrow[r, phantom, "\top"] & \mathbb{U}(Q) \arrow[l, bend left, "i^{-1}"] \arrow[r, bend left, "\mathtt{lim}"] \arrow[r, phantom, "\top"] & Q \arrow[l, bend left, "(-)_\star := \lambda_Q"] \end{tikzcd}$$

where: $i^{-1} : \psi \mapsto i^* \cdot \psi$.
Since $T(a) \simeq \mathtt{lim}(i^* \cdot a_\star, i)$, it follows that:

$$k \leq Q(z, T(a)) = \mathbb{U}(X_B)(i^* \cdot z_*, i^* \cdot a_*) \quad \text{iff for all } x \in X \; Q(a, i(x)) \leq Q(z, i(x)).$$

From this immediately follows: $a \leq T(a)$, hence also $k_Q \leq T(k_Q)$.
From the latter, being T a V-functor, it follows that $T(a) \leq T^2(a)$. For the other direction, we need to prove that:

$$Q(a, i(x)) \leq Q(T^2(a), i(x)).$$

Being T a V-functor, from the previous lemma, it follows that:

$$Q(a, i(x)) \leq Q(T(a), T(i(x))) = Q(T(a), i(x)) \leq Q(T^2(a), i(x)).$$

Let's prove that $T(a) * T(b) \leq T(a * b)$. For $x \in X$, by using the pre-denseness of i, we have:

$$Q(a * b, i(x)) = Q(a * L\text{-}\mathtt{colim}(\phi, Id), i(x)),$$

where, for the sake of notation, $\phi := i_\circledast \circ b^\circledast$.
By unravelling the definition of colimit, we get:

$$\begin{aligned}
Q(a * b, i(x)) = {} & Q(a * L\text{-}\mathtt{colim}(\phi, i(w), i(x)), \\
& \text{(from Proposition 2.36)}, \\
& = Q(a * \mathtt{colim}(e_1^\circ \cdot \phi, \alpha Li(\underline{w})), i(x)), \\
& = \mathtt{lim}(e_1^\circ \cdot \phi, Q(a * \alpha Li(\underline{w}), i(x))), \\
& = \mathtt{lim}(e_1^\circ \cdot \phi, Q(a * (i(w_1) * \ldots * i(w_n)), i(x))), \\
& \text{(by inductively applying adjunction)} \\
& = \mathtt{lim}(e_1^\circ \cdot \phi, Q(a, i(w_1) \rhd (\ldots \rhd i(w_n) \rhd i(x)))), \\
& \text{(by inductively applying } i(x \rhd y) = i(x) \rhd i(y)\text{)} \\
& = \mathtt{lim}(e_1^\circ \cdot \phi, Q(a, i(w_1 \rhd \ldots \rhd (w_n \rhd x)))), \\
& \text{(being } T \text{ a } V\text{-functor and from the previous lemma)} \\
& \leq \mathtt{lim}(e_1^\circ \cdot \phi, Q(T(a), i(w_1 \rhd \ldots \rhd (w_n \rhd x)))), \\
& \text{(by inductively applying adjunction, plus } i(x \rhd y) = i(x) \rhd i(y) \text{ backward)} \\
& \mathtt{lim}(e_1^\circ \cdot \phi, Q(T(a) * (i(w_1) * \ldots * i(w_n)), i(x))), \\
& = Q(T(a) * b, i(x)).
\end{aligned}$$

Now, by repeating the same argument, but now expressing $T(a)$ as a weighted colimit, and by using $\lhd$ instead of $\rhd$, we get:

$$Q(a * b, i(x)) \leq Q(T(a) * b, i(x)) \leq Q(T(a) * T(b), i(x)).$$

□

Corollary 4.26. *In the same hypotesis of the previous proposition, $i' : X_B \to Q^T$ is dense.*

Proof. From T being a left adjoint (as V-functor) and from begin a morphism of quantale, it follows that T is left adjoint also as (L, V)-functor, as:

$$Q^T(T(a_1) *' \ldots *' T(a_n), b) = Q^T(T(a_1 * \ldots * a_n), b) = Q(a_1 * \ldots * a_n, g(b)).$$

This implies that T preserves (L, V)-colimits as well, hence that i' is pre-dense.
Denseness follows by noticing that, by construction, every element of Q^T is such that $T(a) \simeq a$.

□

Proposition 4.27. *Every quantum B-algebra X admits a strict embedding into a quantale.*

Proof. Let (X,a) be the corresponding object in (L,V)-Cat. Consider the Yoneda embedding in (L,V)-Cat:

$$y_X : X \rightarrowtail \mathbb{D}_L(X) \quad x \mapsto a(-,x).$$

We have that:

$$\begin{aligned} a((\underline{x},w),z) &= \mathbb{D}_L(X)[Ly_X((\underline{x};w)), y_X(z)], \\ &= \mathbb{D}_L(X)[(\alpha L y_X(\underline{x})) * y_X(w), y_X(z)], \\ &= \mathbb{D}_L(X)[\alpha L y_X(\underline{x})), y_X(w) \rhd y_X(z))], \\ &= \mathbb{D}_L(X)[L y_X(\underline{x})), y_X(w) \rhd y_X(z))]. \end{aligned}$$

Since, by definition we have that:

$$a((\underline{x};w),z) = a(\underline{x}, w \rhd z) = \mathbb{D}_L(X)[L y_X(\underline{x}), y_X(w \rhd z)];$$

from which it follows that, for all $\underline{x} \in LX$:

$$\mathbb{D}_L(X)[L y_X(\underline{x})), y_X(w) \rhd y_X(z))] = \mathbb{D}_L(X)[L y_X(\underline{x}), y_X(w \rhd z)];$$

hence, by representability:

$$y_X(x_2) \rhd y_X(x) = y_X(x_2 \rhd x).$$

By using the same argument, but starting from: $a((w;\underline{x}),z)$, we can also prove that:

$$y_X(x_2) \lhd y_X(x) = y_X(x_2 \lhd x).$$

□

Remark 4.28. Notice that $\mathbb{D}_L(X)$ is an object of QBalg and when viewed as (L,V)-Cat its (L,V)-structure $\tilde{Q}_{\mathbb{D}_L(X)}$ is the same as $\alpha \cdot Q_{\mathbb{D}_L(X)}$ by adjointness.

Corollary 4.29. *$X \to \mathbb{D}_L(X)$ is pre-dense strict morphism.*

Proof. Every element of $\mathbb{D}_L(X)$ is a weighted colimit of y_X in a canonical way, by Yoneda lemma it follows:

$$\phi \simeq L\text{-}\mathtt{colim}(y_X, \phi) \simeq L\text{-}\mathtt{colim}(y_X, (y_X)_{\circledast} \circ \phi^{\circledast}).$$

□

Corollary 4.30. *For every quantum B-algebra X_B, there exists a quantale Q and a dense embedding $i : X_B \rightarrowtail Q$.*

Proof. Apply 4.25 to the Yoneda embedding:

$$y_X : X_B \rightarrowtail \mathbb{D}_L(X).$$

□

Remark 4.31. As already remarked before, our class J is the class of fully-faithful functors in (L,V)-Cat.

Proposition 4.32. *Every dense morphism in (L,V)-Cat is essential.*

Proof. Suppose that $(X,a) \to Q$ is dense and let $g : Q \to (Z,b)$ such that $g\big|_X$ is fully-faithful
We want to prove that:

$$b(Lg(\underline{a}), g(a)) \leq Q(a_1 * a_2, a), \text{ (where, for simplicity } \underline{a} = (a_1,a_2)).$$

Since the embedding is dense, we can write:

$$a_1 = L\text{-}\mathtt{colim}(i_{\circledast} \circ a_1^{\circledast}, i)(1), \;\; a_2 = L\text{-}\mathtt{colim}(i_{\circledast} \circ a_2^{\circledast}, i)(1), \;\; a = \mathtt{lim}(i^* \cdot a_*, i).$$

In order to increase readability, we denote:

$$j_1 := i_{\circledast} \circ a_1^{\circledast}, \;\; j_2 := i_{\circledast} \circ a_2^{\circledast}, \;\; j_3 := i^* \cdot a_*.$$

We have that:

$$\begin{aligned}
Q(a_1 * a_2, a) = {} & Q(\mathtt{colim}(e_1^\circ \cdot \hat{j}_1, \alpha Li(\underline{x})) * \mathtt{colim}(e_1^\circ \cdot \hat{j}_1, \alpha Li(\underline{w})), \mathtt{lim}(j_3, i(x))) \\
& \text{(since $*$ preserves colimits, being Q a quantale),} \\
& = \mathtt{lim}(e_1^\circ \cdot \hat{j}_1, \mathtt{lim}(e_1^\circ \cdot \hat{j}_2, \mathtt{lim}(j_3, Q(\alpha Li(\underline{x}) * \alpha Li(\underline{w}), i(x))))), \\
& \text{(being i fully faithful),} \\
& = \mathtt{lim}(e_1^\circ \cdot \hat{j}_1, \mathtt{lim}(e_1^\circ \cdot \hat{j}_2, \mathtt{lim}(j_3, a((\underline{x}; \underline{w}), x)))), \\
& \text{(since $g\big|_X$ is fully-faithful),} \\
& = \mathtt{lim}(e_1^\circ \cdot \hat{j}_1, \mathtt{lim}(e_1^\circ \cdot \hat{j}_2, \mathtt{lim}(j_3, b(Lg(\underline{x}; \underline{w}), g(x))))).
\end{aligned}$$

By using the Yoneda embedding $y_Z : (Z, b) \to \mathbb{D}_L(Z)$, we have that $y_Z g$ is an embedding (restricted to X, of course). Moreover we have (denote $y_Z g i := f$):

$$y_Z g(\mathtt{lim}(j_3, i(x))) \leq \mathtt{lim}(j_3, f(x)), \;\; (1)$$

$$\mathtt{colim}(e_1^\circ \cdot \hat{j}_1, \alpha Lf(\underline{x})) \leq y_Z g(\mathtt{colim}(e_1^\circ \cdot \hat{j}_1, \alpha Li(\underline{x}))), \;\; \mathtt{colim}(e_1^\circ \cdot \hat{j}_2, \alpha Lf(\underline{w})) \leq y_Z g(\mathtt{colim}(e_1^\circ \cdot \hat{j}_2, \alpha Li(\underline{w}))),$$

which combined together give:

$$\mathtt{colim}(e_1^\circ \cdot \hat{j}_1, \alpha Lf(\underline{x})) * \mathtt{colim}(e_1^\circ \cdot \hat{j}_2, \alpha Lf(\underline{w})) \leq y_Z g(\mathtt{colim}(e_1^\circ \cdot \hat{j}_1, \alpha Li(\underline{x}))) * y_Z g(\mathtt{colim}(e_1^\circ \cdot \hat{j}_2, \alpha Li(\underline{w}))), \;\; (2).$$

Thus:

$$\begin{aligned}
Q(a_1 * a_2, a) = {} & \mathtt{lim}(e_1^\circ \cdot \hat{j}_1, \mathtt{lim}(e_1^\circ \cdot \hat{j}_2, \mathtt{lim}(j_3, b(Lg(\underline{x}; \underline{w}), g(x_k))))), \\
& \text{(using the Yoneda embedding),} \\
& = \mathtt{lim}(e_1^\circ \cdot \hat{j}_1, \mathtt{lim}(e_1^\circ \cdot \hat{j}_2, \mathtt{lim}(j_3, \mathbb{D}_L(Z)[(\alpha Lf(\underline{x}; \underline{w}), f(x)]))), \\
& = \mathtt{lim}(e_1^\circ \cdot \hat{j}_1, \mathtt{lim}(e_1^\circ \cdot \hat{j}_2, \mathtt{lim}(j_3, \mathbb{D}_L(Z)[(\alpha Lf(\underline{x})) * (\alpha Lf(\underline{w})), f(x)]))), \\
& \text{(since $*$ preserves colimits, being $\mathbb{D}_L(Z)$ a quantale),} \\
& = \mathbb{D}_L(Z)[\mathtt{colim}(e_1^\circ \cdot \hat{j}_1, \alpha Lf(\underline{x})) * \mathtt{colim}(e_1^\circ \cdot \hat{j}_2, \alpha Lf(\underline{w})), \mathtt{lim}(j_3, f(x)]), \\
& \text{(using (1) + (2)),} \\
& \geq \mathbb{D}_L(Z)[y_Z g(a_1) * y_Z g(a_2), y_Z g(a)], \\
& = \mathbb{D}_L(Z)[L(y_Z g)(\underline{a}), y_Z g(a)], \\
& \text{(since the Yoneda is an embedding),} \\
& = b(Lg(\underline{a}), g(a)).
\end{aligned}$$

Which proves that:

$$b(Lg(\underline{a}), g(a)) \leq Q(a_1 * a_2, a).$$

□

Remark 4.33. Since the notion of denseness in $\mathtt{QBalg}$ and (L, V)-$\mathtt{Cat}$ coincide, the same results holds for dense morphisms in $\mathtt{QBalg}$.

Theorem 4.34. *In* QBalg $i:(X_B,a)\rightarrowtail Q$ *is dense iff is esssential.*

Proof. The first part is the previous proposition, while for the converse implication suppose that $i:(X_B,a)\rightarrowtail Q$ is essential. Consider the left Kan extension along the Yoneda embedding of i and take its image factorization, the result will be the quantale generated by X in Q (and in particular will be a strict embedding):

$$\begin{tikzcd}
X \arrow[r, tail, "y_X"] \arrow[dr, tail] & \mathbb{D}_L(X) \arrow[r, tail, "\mathtt{Lan}_y(i)"] \arrow[d, two heads] & Q \\
& \mathtt{Im}(\mathtt{Lan}_y(i)) \arrow[ur, tail] &
\end{tikzcd}$$

Now, consider the following:

$$\begin{tikzcd}
X \arrow[r, tail, "y_X"] \arrow[dr, tail] & \mathtt{Im}(\mathtt{Lan}_y(f)) \arrow[r, tail, "\mathtt{Lan}_y(f)"] \arrow[d, two heads] & Q \arrow[dl, "f"] \\
& (\mathtt{Im}(\mathtt{Lan}_y(f)))^T &
\end{tikzcd}$$

where f is obtained by applying the injective property to j. In particular, being i essential it follows that f is fully-faithful, but since j is full and essentially on objects, it follows f is too, hence f is an equivalence. Hence $Q\simeq W^j$ which implies i is dense. ☐

5 Injective Hulls of (L,V)-Categories

In this section we apply the results proven in the last one to (L,V)-Cat (and more general to particular sub-categories of it). The idea is to use the Yoneda embedding to view any (L,V)-category as a subset of the quantale $\mathbb{D}_L(X)$ and consider the quantum B-algebra generated by it.

Due to the equational definition quantum B-algebras admit, if we have a subset $X\subsetneq Q$ of a quantale Q containing the unit u_Q, the quantum B-algebra generated by X (inside Q) is obtained by inductively add all the "missing" implications until the process reaches a "saturation". If the unit is not included in X we have to add it in the first step of the construction; this might cause a problem, since we are "artificially" adding an element which is "alien" and can not be constructed inductively starting from elements of X.

In order to overcome this problem, we introduce a slight variation of quantum B-algebras: *pre-Quantum B-Algebras*. A pre-quantum B-algebra is a promonoidal category (X,P,J) where only P is representable. In this way we obtain a new sub-category of (L,V)-Cat where all the constructions made in the last section still work, since all it requires for them to work is the presence of an "implicational structure"; moreover, in the process of building the quantum B-algebra generated by a subset, we won't have to "artificially" add the unit of the quantale as we should have if we considered *vanilla* quantum B-algebras.

Definition 5.1. A *pre-quantum B-algebra* (X_B,a,J) is a *pre-representable* promonoidal V-category. This means it is a promonoidal V-category (X,a,P,J) equipped with two binary operations: $\rhd,\lhd\colon X\times X\to X$ such that:

- $P(x,-,y)\simeq a(-,x\lhd y)$;
- $P(-,x,y)\simeq a(-,x\rhd y)$;

By expressing what it means to be *promonoidal functor* for pre-quantum B-algebras, we have:

Definition 5.2. A morphism between two pre-quantum B-algebras $f : (X_B, a, J) \to (Y_B, b, U)$ is V-functor satisfying:

$$f(x \rhd y) \le f(x) \rhd f(y),\ f(x \lhd y) \le f(x) \lhd f(y),\ J(x) \le U(f(x))\ \ (1)$$

If also:

$$f(x) \rhd f(y) \le f(x \rhd y),\ f(x) \lhd f(y) \le f(x \lhd y)),\ \ (2)$$

then f is called strict.

As for *vanilla* quantum B-algebras, we have:

Proposition 5.3. *The category of pre-quantum B-algebras,* PQBalg*, is a fully-faithful subcategory of* (L,V)-Cat.

Remark 5.4. Notice that, since the strictness condition involves only the implications, the key results contained in Proposition 4.25 and Proposition 4.27 remain true when applied to pre-quantum B-algebras. Thus we can construct the injective hull of a pre-quantum B-algebra (X_B, a, J) as done in the previous section.

The last step is to descrive what is the pre-quantum B-algebra generated by a subset of a quantale Q.

Definition 5.5. Let $X \subseteq Q$ be a subset of a quantale $(Q, *, u_Q)$. The pre-quantum B-algebra generated by X, denoted as X_B, is the smallest pre-quantum B-algebra containing X.

Proposition 5.6. $X_B = (\bigcup_i X_i, \tilde{Q}, J)$*, where:*

- $X_0 := X,\ \ X_{i+1} := \{a \rhd b,\ \ c \lhd d,\ \ a,b,c,d \in X_i\} \cup X_i.$
- $\tilde{Q}$ *is the restriction of the* V*-structure on* Q *to* $\bigcup_i X_i$*,.*
- $J(x) := Q(u_Q, x).$

Proof. Is straightforward to verify that $\bigcup_i X_i$ is a pre-quantum B-algebra; hence that $X_B \subseteq \bigcup_i X_i$.
For the other inclusion is sufficient to prove by induction that $X_i \subseteq X_B$. □

Remark 5.7. Note that $X_B \rightarrowtail Q$ is a strict embedding.

Let (X, a) be in (L,V)-Cat. Consider the Yoneda embedding: $y_X : (X, a) \rightarrowtail \mathbb{D}_L(X)$. Define X_B to be the pre-quantum B-algebra generated by $y_X(X)$ in $\mathbb{D}_L(X)$. By definition the Yoneda, factorizes as:

$$X \rightarrowtail X_B \rightarrowtail \mathbb{D}_L(X).$$

Let $\tilde{X}_B$ be its injective hull in PQBalg. We have that: $X_B \rightarrowtail \tilde{X}_B$ is essential (and also dense) in (L,V)-Cat by 4.32. Consider the composite:

$$(X, a) \rightarrowtail X_B \rightarrowtail \tilde{X}_B,$$

We have:

Proposition 5.8. $(X, a) \rightarrowtail X_B \rightarrowtail \tilde{X}_B$ *is essential in* (L,V)-Cat.

Proof. Let $f : \tilde{X} \to (Z, b)$ be such that $f\big|_X$ is an embedding. By composing with y_Z we can suppose that Z is a quantale, and thus that f is a morphism in $(V\text{-}\mathtt{Cat}^L)_{\mathtt{lax}}$.
If we show that $f\big|_{X_B}$ is an embedding, being $\tilde{X}$ the injective hull of X_B, it will follow that f is an embedding too, hence the result.
First we are going to prove that:

$$Z(Lf(\underline{a}), f(w)) \le \tilde{a}_{X_B}((a_1, \ldots, a_n), w),\ \text{with } a_1, \ldots, a_n \in X_B \text{ and } w = y_X(x),\ \text{with } x \in X.$$

With a calculation similar to the one done in Proposition 4.32, by using the properties of f and the fact that $f\big|_X$ is an embedding, we have (where we consider $\underline{a} = (a_1, a_2)$ for a matter of convenience):

$$\begin{aligned}
\tilde{a}_{X_B}((a_1, a_2), w) := {} & a_{X_B}(a_1, a_2 \rhd w), \\
& \text{(because } X_B \text{ is the pre-quantum B-algebra generated by } Y_X(X) \text{ in } \mathbb{D}_L(X)\text{)} \\
& = \mathbb{D}_L(X)[a_1, a_2 \rhd y_X(x)], \\
& = \mathbb{D}_L(X)[a_1 * a_2, y_X(x)], \\
& = \mathtt{lim}(e_1^\circ \cdot j_1, \mathtt{lim}(e_1^\circ \cdot j_1, a((\underline{x}_1; \underline{x}_2), x))), \\
& = \mathtt{lim}(e_1^\circ \cdot j_1, \mathtt{lim}(e_1^\circ \cdot j_1, Z(Lf(\underline{x}_1; \underline{x}_2)), f(w))), \\
& \geq Z(f(a_1) * f(a_2), f(w)).
\end{aligned}$$

Hence:

$$Z(Lf(\underline{a}), f(w)) \leq \tilde{a}_{X_B}(\underline{a}, w).$$

If $w \notin X$, by induction, we can suppose is of the form $w = y \rhd z$ (or $y \lhd z$), for $y, z \in y_X(X)$.
From $f(w) * f(y) \leq f(w * y)$ it follows that $f(w) \leq f(y) \rhd f(z)$; hence that:

$$\begin{aligned}
Z(Lf(\underline{a}), f(w)) & \leq Z(Lf(\underline{a}), f(y) \rhd f(z)), \\
& := Z(f(a_1) * f(a_2), f(y) \rhd f(z), \\
& = Z(f(a_1) * f(a_2) * f(y), f(z)), \\
& =: Z(Lf(\underline{a}; y), f(z)).
\end{aligned}$$

Since we have supposed $z \in y_X(X)$, by the case we've already analyzed, it follows:

$$Z(Lf(\underline{a}; y), f(z)) \leq \tilde{a}_{X_B}(\underline{a}; y, z) := \tilde{a}_{X_B}(\underline{a}, y \rhd z),$$

hence that:

$$Z(Lf(\underline{a}), f(w)) \leq \tilde{a}_{X_B}(\underline{a}, w),$$

for $w \in X_B$.

$\square$

Remark 5.9. Now it should be clear why we have introduced pre-quantum B-algebras. If we had considered *vanilla* quantum B-algebras, in the last proposition we would have faced a problem, since we would not be able to prove that:

$$Z(f(a_1) * f(a_2), f(u)) \leq \tilde{a}_{X_B}((a_1, a_2), u),$$

where u is the unit of $\mathbb{D}_L(X)$.

Remark 5.10. One might ask why we did not consider pre-quantum B-algebras in the first place instead of introducing *vanilla* quantum B-algebras. The reason for our choice is due to the fact that pre-quantum B-algebras doe not look "natural" as quantum B-algebras do; requiring the representability of only the "multiplicative" part of a promonoidal category is an artifice we employed in order to overcome the minor problem one would have with the unit in the previous proposition.

With the aid of this proposition we can prove our main theorem:

Theorem 5.11. *Let C be a full sub-category of (L, V)-$\mathtt{Cat}$ which contains quantales. Then every object (X, a) admits an injective hull.*

Proof. From Theorem 4.17 we get that the injectives in C are quantales. From the previous theorem, for every object of C there exists an essential embedding of it into a quantale. Hence the result.

$\square$

In particular this theorem applies to all the categories displayed in the following:

$$\begin{tikzcd}
 & (L,V)\text{-}\mathtt{Cat} & \\
 & V\text{-}\mathtt{Pro} \arrow[u, tail] & \\
\mathtt{Mon}((V\text{-}\mathtt{Cat}))_{\mathtt{lax}} \arrow[ur, tail] & & \mathtt{QBalg} \arrow[ul, tail] \\
 & \mathtt{ResMon}((V\text{-}\mathtt{Cat}))_{\mathtt{lax}} \arrow[ul, tail] \arrow[ur, tail] &
\end{tikzcd}$$

6 Ending Remaks

Finally, we mention the case of topological spaces which, by [Bar70], can be also seen as examples of "generalised multicategories". Injective topological spaces are characterised in [Sco72] as precisely the continuous lattices, and [Hof11] it is observed that injective topological spaces are those spaces where the (topological analogon of the) Yoneda embedding has a left adjoint in the ordered category of topological spaces and continuous maps. Here one considers the space

$$2^{(UX)^{\mathrm{op}}}$$

where 2 is the Sierpiński space and UX denotes the set of all ultrafilters on X which, with a certain topology, becomes the space $(UX)^{\mathrm{op}}$. The Yoneda map $y : X \to 2^{(UX)^{\mathrm{op}}}$ sends a point $x \in X$ to the set $\{\mathfrak{x} \in UX \mid \mathfrak{x} \to x\}$ of all ultrafilters convergent to x. In [HT10, Example 4.10] it is observed that

$$2^{(UX)^{\mathrm{op}}} \longrightarrow FX, \quad \mathscr{A} \longmapsto \bigcap \mathscr{A}$$

is am isomorphism; here $FX = \{\text{filters of open subsets of } X\}$ with the sets

$$U^{\#} = \{\mathfrak{f} \in FX \mid U \in \mathfrak{f}\}, \qquad (U \text{ open in } X)$$

forming a basis for the topology of FX (for instance, see [Esc97]). On the other hand, the topological analogous to the covariant presheaf category (see [Hof14]) is the lower Vietories space

$$VX = \{A \subseteq X \mid A \text{ closed}\},$$

here the topology is generated by the sets

$$B^{\Diamond} = \{A \in VX \mid A \cap B \neq \varnothing\}, \qquad (B \text{ open in } X).$$

Injective hulls for topological spaces are described in [Ban73], we point out here that, similarly to the situation for V-categories, this description is ultimately linked to the *Isbell adjunction*. For a topological space X, the Isbell adjunction is given by the monotone maps

$$\begin{tikzcd}
FX \arrow[r, bend left, "(-)^{+}"] \arrow[r, phantom, "\perp"] & VX \arrow[l, bend left, "(-)^{-}"]
\end{tikzcd}$$

where

$$A^{-} = \bigcap \{\mathfrak{x} \in UX \mid \mathfrak{x} \to x \in A\} \quad \text{and} \quad \mathfrak{f}^{+} = \lim \mathfrak{f},$$

for all closed subsets $A \subseteq X$ and all filters of open subsets $\mathfrak{f}$. Note that $(-)^{-} : VX \to FX$ is continuous but $(-)^{+} : FX \to VX$ is in general not. We also recall from [Hof13] the following definition.

Definition 6.1. A topological space X is F-core-compact whenever, for all $x \in X$ and all open neighbourhoods U of x, there exists an open neighbourhood V of x so that $V \ll_F U$. Here $V \ll_F U$ whenever, for all filters $\mathfrak{f}$ of opens with $V \in \mathfrak{f}$, $\lim \mathfrak{f} \cap U \neq \varnothing$.

Remark 6.2. For open subsets U, V of X, $V \ll_F U$ if and only if there exists some $x \in X$ with $V \subseteq \downarrow x \subseteq U$. We also note that F-core-compact spaces were introduced in [Ern91] under the name *C-space*. Moreover, in [Ban73, Proposition 3] it is shown that these are exactly the topological spaces which admit an injective hull.

Proposition 6.3. *A topological space X is F-core-compact if and only if the map $(-)^+ \colon FX \to VX$ is continuous.*

Proof. Assume first that X is F-core-compact. Let $B \subseteq X$ be open and $\mathfrak{f} \in FX$ with $\lim \mathfrak{f} \in B^\lozenge$, that is $\lim \mathfrak{f} \cap B \neq \varnothing$. Let $x \in \lim \mathfrak{f} \cap B$. By hypothesis, there is an open neighbourhood U of x with $U \ll_F V$. Then $\mathfrak{f} \in U^\#$ and, for every $\mathfrak{g} \in U^\#$, $\lim \mathfrak{g} \in B^\lozenge$.

Assume now that $(-)^+ \colon FX \to VX$ is continuous. Let $x \in X$ and let B be an open neighbourhood of x. Then, with $\mathfrak{f}$ being the open neighbourhood filter of x, $\mathfrak{f} \to x \in B$, hence $\lim \mathfrak{f} \in B^\lozenge$. Since $(-)^+$ is continuous, there is some open neighbourhood U of x so that, for all $\mathfrak{g} \in U^\#$, $\lim \mathfrak{g} \in B^\lozenge$. □

With this notation, we can reformulate [Ban73, Proposition 3].

Theorem 6.4. *A topological space X has an injective hull if and only if $(-)^+ \colon FX \to VX$ is continuous.*

In other words, X has an injective hull precisely when the Isbell adjuncion exists in the category of topological spaces and continuous maps. In this case, the injective hull of a space X is given by the embedding of X into the subspace λX of FX given by all joins of neighbourhood filters (see [Ban73, Proposition 2]). Similarly to the Dedekind-MacNeille completion, these filters are precisely the fix points of the Isbell adjunction. To see this, consider $2^{(UX)^{\mathrm{op}}} \simeq FX$ and, for $\psi \in 2^{(UX)^{\mathrm{op}}}$, note that

$$(\psi^-)^+ = \bigcap \{ y_X(x) \mid x \in \psi^- \};$$

therefore, for $\mathfrak{f} \in FX$, $\mathfrak{f} = (\mathfrak{f}^-)^+$ precisely when $\mathfrak{f}$ is a join of neighbourhood filters.

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