A HIGHER GROTHENDIECK CONSTRUCTION

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ABSTRACT. In this note we present a *Grothendieck construction* for functors taking values in quasi-categories. We construct a simplicial space from such a functor whose zeroth row is the desired construction. Using our construction we give a new proof of rectification theorem for coCartesian fibrations of simplicial sets.

CONTENTS

1. Introduction

The Grothendieck construction is ubiquitous in category theory. This construction associates to a (pseudo) functors $F : D \to \mathbf{Cat}$, a (op)fibration over the (small) category D. The construction establishes an equivalence and therefore allows us to switch between $\mathbf{Cat}$-valued functors and fibrations. In this note we want to extend the classical Grothendieck construction to quasi-category valued functors with the aim of establishing an equivalence between a category of $\mathcal{S}$-valued functors and appropriately defined (simplicial)-fibrations over the nerve of the domain category of functors. In this note we will be primarily working with the adaptation of Joyal model category structure on $\mathcal{S}$, [Joy08b], [Joy08a], to marked simplicial sets.

It is well known that a *left fibration* of simplicial sets over the nerve of a (small) category $N(D)$ is determined upto equivalence by a homotopy coherent diagram taking values in Kan complexes, see [Cis19, 5.3]. The same holds for *coCartesian fibration* of simplicial sets over $N(D)$ with respect to homotopy coherent diagrams taking values in quasi-categories, see [Lur09, Ch. 3]. The main goal of this note is to show that the aforementioned homotopy coherent diagrams can be rectified *i.e.* upto equivalence they can be replaced by an honest functor. More precisely, we will show that for each coCartesian fibration $p : X \to N(D)$, there exists a (honest) functor $Z : D \to \mathcal{S}$, taking values in quasi-categories whose *Grothendieck construction*, denoted $\int^{d \in D} Z$, is equivalent to the fibration p in a suitably defined model category structure on $\mathcal{S}^+/N(D)$. Such a result first appeared in [Lur09, Ch. 3] where the author defines an extension of the classical Grothendieck construction called *relative nerve* which determines a functor $N_\bullet(D) : [D, \mathcal{S}] \to \mathcal{S}/N(D)$. The author goes further to present another version of the relative nerve for *marked* simplicial sets which is a functor $N_\bullet^+(D) : [D, \mathcal{S}^+] \to \mathcal{S}^+/N(D)$. This functor is shown to be the right Quillen functor of a Quillen adjunction between the *coCartesian* model category structure on $\mathcal{S}^+/N(D)$ and the projective model category structure on $[D, (\mathcal{S}^+, \mathbf{Q})]$. The guiding principle of our Grothendieck construction is that homotopy colimit of a functor $H : D \to \mathcal{S}$ taking values in quasi-categories should be obtained upon inverting the coCartesian edges of the total space of the Grothendieck construction. We recall that a homotopy colimit of a functor $G : D \to \mathbf{Cat}$ is obtained in this way. Our Grothendieck construction is isomorphic to the relative nerve of a functor $H : D \to \mathcal{S}$ but our construction is a part of a larger structure, namely a simplicial space, which we extract out of the functor H. The main objective of this note is to establish a Quillen equivalence whose left Quillen functor has a (total left) derived functor which is isomorphic to a (total right) derived functor of (a marked simplicial sets version of) our Grothendieck construction.

In section 2 of this note we describe a (higher) *Grothendieck construction* for functors taking values in $\mathcal{S}$. In the same section we define a simplicial space (or a bisimplicial set) for each functor $F : D \to \mathcal{S}$ which encodes the information in the functor as a fibration. We define the Grothendieck construction of F to be the zeroth row of the aforementioned simplicial space. This defines a functor $\int^{d \in D} - : [D, \mathcal{S}] \to \mathcal{S}/N(D)$. In section 3 we define a version of our Grothendieck construction functor for marked simplicial sets $\int_+^{d \in D} - : [D, \mathcal{S}^+] \to \mathcal{S}^+/N(D)$. In the same section we establish a Quillen equivalence $(\mathfrak{L}_D^+, \mathfrak{R}_D^+)$ between the projective model category structure on $[D, \mathcal{S}^+]$ and the coCartesian model category structure on $\mathcal{S}^+/N(D)$. This result implies that a (total right) derived functor

of our Grothendieck construction functor is isomorphic to a (total left) drerived functor of $\mathfrak{L}_D^+$.

A version of this result for left fibrations has been proved in [HM15] where the authors establish a Quillen equivalence between the *covariant* model category structure on $\mathcal{S}/N(D)$, see [Joy08b, Ch. 8] and the *projective* model category structure on the functor category $[D, (\mathcal{S}, \mathbf{Kan})]$.

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2. A Grothendieck construction

In this section we will describe a *Grothendieck construction* for quasi-categories. The classical Grothendieck construction defines a functor

$$\int^{d \in D} - : [D; \mathbf{Cat}] \to \mathbf{Cat}/D$$

The construction described in this note also defines a functor, which we denote by $\int^{d \in D} -$, which is a *left Kan extension* of the above functor alpng the Nerve functor $[D; N] : [D; \mathbf{Cat}] \to [D; \mathcal{S}]$.

Let $X : D \to \mathcal{S}$ be a functor. We recursively define a collection of simplicial sets as follows:

$$\mathcal{G}_0^X(d) := X(d).$$

For a map $f : d_1 \to d_2$ in D, we define a simplicial set $\mathcal{G}_1(f)$ by the following pullback square:

$$
\begin{array}{ccc}
\mathcal{G}_1^X(f) & \xrightarrow{\quad p_2(f) \quad} & [\Delta[1]; X(d_2)] \\
{\scriptstyle p_1(f)} \downarrow & & \downarrow {\scriptstyle [d_1; X(d_2)] \times [d_0; X(d_2)]} \\
[\Delta[0]; X(d_1)] \times [\Delta[0]; X(d_2)] & \xrightarrow[{[\Delta[0]; X(f)] \times id}]{} & [\Delta[0]; X(d_2)] \times [\Delta[0]; X(d_2)]
\end{array}
$$

Remark 1. For each object $d \in D$

$$\mathcal{G}_1^X(id_d) = [\Delta[1], X(d)].$$

For a pair of maps $f_1 : d_1 \to d_2$, $f_2 : d_2 \to d_3$ in D, we define a simplicial set $\mathcal{G}_2^X(f_1, f_2)$ by the following pullback square:

$$
\begin{array}{ccc}
\mathcal{G}_2^X(f_1, f_2) & \xrightarrow{\quad p_2((f_1, f_2)) \quad} & [\Delta[2]; X(d_3)] \\
{\scriptstyle p_1((f_1, f_2))} \downarrow & & \downarrow {\scriptstyle ([d_0; X(d_3)], [d_1; X(d_3)], [d_2; X(d_3)])} \\
\mathcal{G}_1^X(f_2) \times \mathcal{G}_1^X(f_2 f_1) \times \mathcal{G}_1^X(f_1) & \xrightarrow[{F_3 \times F_2 \times F_1}]{} & \prod_3 [\Delta[1]; X(d_3)]
\end{array}
$$

where F_1 is the composite map:

$$\mathcal{G}_1^X(f_1) \overset{p_2(f_1)}{\to} [\Delta[1]; X(d_2)] \overset{[\Delta[1]; X(f_2)]}{\to} [\Delta[1]; X(d_3)],$$

$F_3 = p_2(f_2)$, $F_2 = p_2(f_2 f_1)$ and $p_1(f_1, f_2) = (p_1(f_1), p_1(f_2), p_1(f_2 f_1))$.

Remark 2. For each $f \in Mor(D)$, the simplicial sets $\mathcal{G}_2^X((f, id))$ and $\mathcal{G}_2^X((id, f))$ are given by the following two pullback squares respectively:

$$
\begin{array}{ccc}
\mathcal{G}_2^X(f, id) & \xrightarrow{\quad p_2((f, id)) \quad} & [\Delta[2]; X(d_2)] \\
{\scriptstyle p_1((f, id))} \downarrow & & \downarrow {\scriptstyle ([d_1; X(d_3)], [d_2; X(d_2)])} \\
\mathcal{G}_1^X(f) \times \mathcal{G}_1^X(f) & \xrightarrow[{F_2 \times F_1}]{} & [\Delta[1]; X(d_2)] \times [\Delta[1]; X(d_2)]
\end{array}
$$

and

$$\mathcal{G}_2^X(id,f) \xrightarrow{\ p_2((id,f))\ } [\Delta[2]; X(d_2)]$$

$$p_1((id,f)) \downarrow \qquad\qquad\qquad\qquad\qquad \downarrow ([d_0;X(d_2)],[d_1;X(d_2)],[d_2;X(d_2)])$$

$$\mathcal{G}_1^X(f) \times \mathcal{G}_1^X(f) \times [\Delta[1], X(d_1)] \xrightarrow[F_3 \times F_2 \times F_1]{} [\Delta[1]; X(d_2)] \times [\Delta[1]; X(d_2)] \times [\Delta[1]; X(d_2)]$$

For an n-tuple $\sigma = (f_1, f_2, \ldots, f_n) \in (N(D))_n$, we define a simplicial set $\mathcal{G}_n^X(\sigma)$ by the following pullback square:

$$(1) \qquad \mathcal{G}_n^X(\sigma) \xrightarrow{\ p_2(\sigma)\ } [\Delta[n]; X(d_{n+1})]$$

$$p_1(\sigma) \downarrow \qquad\qquad\qquad\qquad \downarrow H$$

$$\prod_{i=0}^{n} \mathcal{G}_{n-1}^X(d_i(\sigma)) \xrightarrow[F_{n+1} \times \cdots F_1]{} \prod_n [\Delta[n-1]; X(d_{n+1})]$$

where $H = ([d_0; X(d_{n+1})], [d_1; X(d_{n+1})], \ldots, [d_n; X(d_{n+1})])$ and for $2 \leq i \leq n+1$ the simplicial map F_i is the following composite:

$$\mathcal{G}_{n-1}^X(d_i(\sigma)) \xrightarrow{p_2(d_i(\sigma))} [\Delta[n-1]; X(d_{n+1})]$$

The map F_1 is the following composite

$$\mathcal{G}_{n-1}^X(d_n(\sigma)) \xrightarrow{p_2(d_n(\sigma))} [\Delta[n-1]; X(d_n)] \xrightarrow{[\Delta[n-1];X(f_n)]} [\Delta[n-1]; X(d_{n+1})]$$

Remark 3. For the canonical simplex $\sigma = \mathbf{id}(d)_n$, see definition **??**, the simplicial set

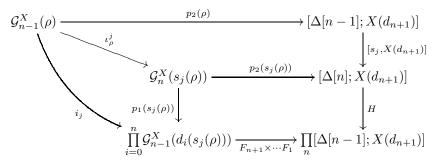$$\mathcal{G}_n^X(\mathbf{id}(d)_n) = [\Delta[n], X(d_{n+1})].$$

Definition 2.1. For a pair consisting of an n-simplex $\sigma \in N(D)_n$ and a functor $X : D \to \mathcal{S}$, we will refer to $\mathcal{G}_n^X(\sigma)$ as the 1-*Gerbe over* σ determined by X.

Proposition 2.2. *For each $(n-1)$-simplex ρ in $N(D)$ there is an inclusion map*

$$\iota_\rho^j : \mathcal{G}_{n-1}^X(\rho) \to \mathcal{G}_n^X(s_j(\rho))$$

where s_j is the jth degeneracy operator of $N(D)$ for $1 \leq j \leq n$.

Proof. The simplicial map ι_ρ^j is the unique map into the pullback shown in the following diagram:

$$\mathcal{G}_{n-1}^X(\rho) \xrightarrow{\ p_2(\rho)\ } [\Delta[n-1]; X(d_{n+1})]$$

$$\iota_\rho^j$$

$$\downarrow [s_j, X(d_{n+1})]$$

$$\mathcal{G}_n^X(s_j(\rho)) \xrightarrow{\ p_2(s_j(\rho))\ } [\Delta[n]; X(d_{n+1})]$$

$$i_j \qquad p_1(s_j(\rho)) \downarrow \qquad\qquad\qquad \downarrow H$$

$$\prod_{i=0}^{n} \mathcal{G}_{n-1}^X(d_i(s_j(\rho))) \xrightarrow[F_{n+1} \times \cdots F_1]{} \prod_n [\Delta[n-1]; X(d_{n+1})]$$

where i_j is the inclusion into the jth component namely $\mathcal{G}_{n-1}^X(d_j s_j(\rho)) = \mathcal{G}_{n-1}^X(\rho)$. $\qquad\square$

Proposition 2.3. *There is a simplicial space i.e. a functor* $\left(\int^{d\in D} X\right)_\bullet : \Delta^{op} \to \mathcal{S}$
whose degree n simplicial-set is defined as follows:

$$\left(\int^{d\in D} X\right)_\bullet([n]) := \underset{\sigma \in (N(D))_n}{\sqcup} \{\sigma\} \times \mathcal{G}_n^X(\sigma)$$

Proof. We will define the degeneracy and face operators. Each $\mathcal{G}_n^X(\sigma)$ is equipped
with a projection map

$$d_i(p_1(\sigma)) : \mathcal{G}_n^X(\sigma) \to \mathcal{G}_{n-1}^X(d_i(\sigma))$$

For $i \in \{0, 1, 2, \ldots, n\}$, this map is given by the following composite:

$$\mathcal{G}_n^X(\sigma) \overset{p_1(\sigma)}{\to} \prod_{i=0}^{n} \mathcal{G}_{n-1}^X(d_i(\sigma)) \overset{pr_i}{\to} \mathcal{G}_{n-1}^X(d_i(\sigma)),$$

where $f_n : d_n \to d_{n+1}$ is the last map in $\sigma = (f_1, \ldots, f_n)$ and pr_i are the obvious
projections from the product. The maps $d_i(p_1(\sigma))$ join together to form a map

$$d_i : \underset{\sigma \in (N(D))_n}{\sqcup} \mathcal{G}_n^X(\sigma) \to \underset{\rho \in (N(D))_{n-1}}{\sqcup} \mathcal{G}_{n-1}^X(\sigma)$$

which is our *ith* face operator for $0 \le i \le n$.

The maps ι_ρ^j from proposition 2.2 gives us the *ith* degeneracy map

$$s_j : \underset{\rho \in (N(D))_{n-1}}{\sqcup} \mathcal{G}_{n-1}^X(\rho) \to \underset{\sigma \in (N(D))_n}{\sqcup} \mathcal{G}_n^X(\rho)$$

$\square$

Notation 2.4. Each pair (K, L) of simplicial sets defines a *bisimplicial sets i.e.* a
functor

$$K \square L : \Delta^{op} \times \Delta^{op} \to \textbf{Sets}$$

as follows:

$$K \square L([m], [n]) := K_m \times L_n$$

Remark 4. The simplicial space $\left(\int^{d\in D} X\right)_\bullet$ is equipped with a map of simplicial
spaces:

$$p_\bullet^X : \left(\int^{d\in D} X\right)_\bullet \to N(D) \square \Delta[0].$$

Notation 2.5. Each simplicial space $Z : \Delta^{op} \to \mathcal{S}$ determines a bisimplicial set,
also denoted by Z

$$Z : \Delta^{op} \times \Delta^{op} \to \textbf{Sets}$$

by $Z([m], [n]) = (Z[m])_n$. Further we denote the following simplicial set by $i_1^*(Z)$:

$$\Delta \overset{(-,[0])}{\to} \Delta \times \Delta \overset{Z}{\to} \textbf{Sets}$$

Now we can define the (total space of) the Grothendieck construction of $X :$
$D \to \mathcal{S}$ as follows:

$$(2) \qquad \int^{d\in D} X = i_1^*\left(\left(\int^{d\in D} X\right)_\bullet\right)$$

Remark 5. The set of n-simplices of $\int^{d \in D} X$ can be represented as follows:

$$\left(\int^{d \in D} X \right)_n = \bigsqcup_{\sigma \in N(D)_n} \{\sigma\} \times \left(\mathcal{G}_n^X(\sigma) \right)_0.$$

Remark 6. An n-simplex δ of $\int^{d \in D} X$ is a pair $\delta = (\sigma, \beta)$ where $\sigma = (f_1, f_2, \ldots, f_n) \in N(D)_n$ and $\beta \in \mathcal{G}_n^X(\sigma)$ *i.e.* $\beta = (\underline{\beta}, \beta)$. This pair consists of $(\underline{\beta_{n-1}}, \beta_{n-1}) = \underline{\beta} \in \mathcal{G}_{n-1}^X(d_n(\sigma))$ and $\beta \in X(d_{n+1})_n$, where $f_n : d_n \to d_{n+1}$. The n-simplex δ satisfies the following two conditions:

(1) $X(f_n)(p_2((\underline{\beta}))) = d_n(\beta)$.

(2) For $0 \le i \le n-2$

$$(d_i(\underline{\beta}), d_i(\beta)) \in \mathcal{G}_{n-1}^X(d_i(\sigma)).$$

Remark 7. Let $\beta = (\underline{\beta}, \beta) \in \mathcal{G}_n^X(\sigma)$, where $\sigma \in N(D)_n$ as in remark (6). We observe that $d_n(\beta) = \underline{\beta} = (\underline{\beta_{n-1}}, \beta_{n-1})$. Further, $d_{n-1}(\underline{\beta}) = \underline{\beta_{n-1}} = (\underline{\beta_{n-2}}, \beta_{n-2}) \in \mathcal{G}_{n-2}^X(d_{n-2}(\sigma))$. Since n is finite, there exists a $\beta_0 \in \mathcal{G}_0^X(d_1)$ such that

(3) $$\beta_0 = d_1 \circ \cdots \circ d_{n-1} \circ d_n(\beta).$$

The notion of *relative nerve* was introduced in [Lur09, 3.2.5.2]. Next we will review this notion:

Definition 2.6. Let D be a category, and $f : D \to \mathcal{S}$ a functor. The nerve of D relative to f is the simplicial set $N_f(D)$ whose n-simplices are sets consisting of:

(i) a functor $d : [n] \to D$; We write $d(i, j)$ for the image of $i \le j$ in $[n]$.

(ii) for every nonempty subposet $J \subseteq [n]$ with maximal element j, a map $\tau^J : \Delta^J \to f(d(j))$,

(iii) such that for nonempty subsets $I \subseteq J \subseteq [n]$ with respective maximal elements $i \le j$, the following diagram commutes:

$$
\begin{array}{ccc}
\Delta^I & \xrightarrow{\tau^I} & f(d(i)) \\
\downarrow & & \downarrow{\scriptstyle f(d(i,j))} \\
\Delta^J & \xrightarrow[\tau^J]{} & f(d(j))
\end{array}
$$

For any f, there is a canonical map $p_f : N_f(D) \to N(D)$ down to the ordinary nerve of D, induced by the unique map to the terminal object $\Delta^0 \in \mathcal{S}$ [Lur09, 3.2.5.4]. When f takes values in quasi-categories, this canonical map is a coCartesian fibration.

Remark 8. A vertex of the simplicial set $N_f(D)$ is a pair (c, g), where $c \in Ob(D)$ and $g \in f(c)_0$. An edge $\underline{e} : (c, g) \to (d, k)$ of the simplicial set $N_f(D)$ consists of a pair (e, h), where $e : c \to d$ is an arrow in D and $h : f(e)_0(g) \to k$ is an edge of $f(d)$.

An immidiate consequence of the above definition is the following proposition:

Proposition 2.7. *Let* $f : D \to \mathcal{S}$ *be a functor, then the fiber of* $p_f : N_f(D) \to N(D)$ *over any* $d \in Ob(D)$ *is isomorphic to the simplicial set* $f(d)$.

The following lemma is a consequence of this definition and the above discussion:

Lemma 2.8. *For each functor $X : D \to \mathcal{S}$, we have the following isomorphism in the category $\mathcal{S}/N(D)$:*

$$\int^{d \in D} X \cong N_X(D).$$

Proof. An n-simplex in $\int^{d \in D} X$ is a pair (σ, β), where $\sigma \in N(D)_n$. This n-simplex σ can be viewed as a functor $\sigma : [n] \to D$. The inclusion of each non-empty subposet $i_J : J \subseteq [n]$ gives a map

$$\left(\int^{d \in D} X \right)(i_J) : \left(\int^{d \in D} X \right)_n \to \left(\int^{d \in D} X \right)_J.$$

We are using the fact that J is isomorphic to an object of Δ which we also denote by J. The inclusion map can now be seen as a map in Δ. This map gives us a J-simplex $\left(\int^{d \in D} X \right)(i_J)((\sigma, \beta))$. Now the second projection map $p_2(\left(\int^{d \in D} X \right)(i_J)((\sigma, \beta)))$ gives us a simplicial map:

$$\Delta[J] \to X(\sigma(j'))$$

where j' is the maximal element of J. For an inclusion $J' \subseteq J$, condition (iii) of definition 2.6 is satisfied because the composite $J' \subseteq J \subseteq [n]$ determines a composite map in Δ. This defines a map $f : \left(\int^{d \in D} X \right)_n \to N_X(D)_n$. Now we define the inverse map. An n-simplex γ in $N_X(D)$ contains a functor $d : [n] \to D$ which uniquely determines an n-simplex σ of $N(D)$. We recall that an n-simplex in $\int^{d \in D} X$ is a pair (σ, β) whose second component β is a pair $(\underline{\beta}, \beta)$, where $\beta \in X(d(n))$. The n-simplex γ contains a simplicial map

$$\Delta[n] \to X(d(n)).$$

We define β to be the n-simplex of $X(d(n))$ which represents the above map. We have an inclusion $[n-1] \hookrightarrow [n]$ in Δ. The n-simplex γ contains another simplicial map

$$\Delta[n-1] \to X(d(n-1)).$$

We define $\underline{\beta}$ to be the pair $(\alpha, (\underline{\beta_{n-1}}, \beta_{n-1}))$ consisting of an $(n-1)$-simplex β_{n-1} of $X(d(n-1))$ which represents the above map and $\alpha = d_n(\sigma)$. The first condition of remark 6 is equivalent to the commutativity of the following diagram:

$$
\begin{array}{ccc}
\Delta[n-1] & \xrightarrow{\beta_{n-1}} & X(d(n-1)) \\
\downarrow & & \downarrow{\scriptstyle X(d(n-1,n))} \\
\Delta[n] & \xrightarrow{\beta} & X(d(n))
\end{array}
$$

The second condition of remark 6 follows from definition 2.6 (iii). $\square$

Next we will define a function object for the category $\mathcal{S}/D$. We shall denote by $[X, Y]_D$ the simplicial set of maps from X to Y in $\mathcal{S}/D$. An n-simplex in $[X, Y]_D$ is a map $\Delta[n] \times X \to Y$ in $\mathcal{S}/D$, where $\Delta[n] \times (X, p) = (\Delta[n] \times X, pp_2)$, where p_2 is the projection $\Delta[n] \times X \to X$. The enriched category $\mathcal{S}/D$ admits tensor and cotensor products. The *tensor product* of an object $X = (X, p)$ in $\mathcal{S}/D$ with a simplicial set A is the objects

$$A \times X = (A \times X, pp_2).$$

The *cotensor product* of X by A is an object of $\mathcal{S}/D$ denoted $(X)^{[A]}$. If $q : (X)^{[A]} \to N(\Gamma^{op})$ is the structure map, then a simplex $x : \Delta[n] \to (X)^{[A]}$ over a simplex $y = qx : \Delta[n] \to N(D)$ is a map $A \times (\Delta[n], y) \to (X, p)$. The object $((X)^{[A]}, q)$ can be constructed by the following pullback square in $\mathcal{S}$:

(4)
$$
\begin{array}{ccc}
(X)^{[A]} & \longrightarrow & [A, X] \\
\downarrow{\scriptstyle q} & & \downarrow{\scriptstyle [A,p]} \\
N(D) & \longrightarrow & [A, D]
\end{array}
$$

where the bottom map is the diagonal. There are canonical isomorphisms:

(5) $$[A \times X, Y]_D \cong [A, [X, Y]_D] \cong [X, (Y)^{[A]}]_D$$

We now define a functor $\mathfrak{R}_D : \mathcal{S}/D \to [D, \mathcal{S}]$. For each $Y \in \mathcal{S}/N(D)$, the functor $\mathfrak{R}_D(Y)$ is defined as follows:

(6) $$\mathfrak{R}_D(Y)(d) := [N(d/D), Y]_D$$

The contravariant functor $N(-/D)$, see (**??**), ensures that this defines a functor $\mathfrak{R}_D(Y) : D \to \mathcal{S}$.

Notation 2.9. For a simplicial map $p : X \to B$, we denote the *fiber* of p over an n-simplex $\sigma \in B_n$ by $X(\sigma)$. In other words, the simplicial set $X(\sigma)$ is defined by the following pullback square:

$$
\begin{array}{ccc}
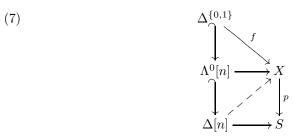X(\sigma) & \longrightarrow & X \\
\downarrow & & \downarrow{\scriptstyle p} \\
\Delta[n] & \xrightarrow{\ \sigma\ } & B
\end{array}
$$

3. Rectification of coCartesian fibrations

In this section we will prove a rectification theorem for *coCartesian* fibrations of simplicial sets over the nerve of a small category D. We will do so along the lines of a similar theory developed in appendix **??** for left fibrations. More precisely, we will show that a marked version of our Grothendieck construction functor is a left Quillen functor of a Quillen equivalence between the *coCartesian* model category $\mathcal{S}^+/N(D)$ and the projective model category $[D, (\mathcal{S}, \mathbf{Q})]$. We begin with a review of coCartesin fibrations over the simplicial set $N(D)$. We will also review a model category structure on the category $\mathcal{S}^+/N(D)$ in which the fibrant objects are (essentially) coCartesian fibrations.

Definition 3.1. Let $p : X \to S$ be an inner fibration of simplicial sets. Let $f : x \to y \in (X)_1$ be an edge in X. We say that f is p-coCartesian if, for all $n \geq 2$ and every (outer) commutative diagram, there exists a (dotted) lifting arrow which makes the entire diagram commutative:

(7)
$$
\begin{array}{ccc}
\Delta^{\{0,1\}} & & \\
\downarrow & \searrow^{f} & \\
\Lambda^0[n] & \longrightarrow & X \\
\downarrow & \nearrow \quad \big\downarrow^{p} & \\
\Delta[n] & \longrightarrow & S
\end{array}
$$

Remark 9. Let M be a (ordinary) category equipped with a functor $p : M \to I$, then an arrow f in M, which maps isomorphically to I, is coCartesian in the usual sense if and only if f is $N(p)$-coCartesian in the sense of the above definition, where $N(p) : N(M) \to \Delta[1]$ represents the nerve of p.

This definition leads us to the notion of a coCartesian fibration of simplicial sets:

Definition 3.2. A map of simplicial sets $p : X \to S$ is called a *coCartesian* fibration if it satisfies the following conditions:

(1) p is an inner fibration of simplicial sets.
(2) for each edge $p : x \to y$ of S and each vertex $\underline{x}$ of X with $p(\underline{x}) = x$, there exists a p-coCartesian edge $\underline{f} : \underline{x} \to \underline{y}$ with $p(\underline{f}) = f$.

A coCartesin fibration roughly means that it is upto weak-equivalence determined by a *functor* from S to a suitably defines ∞-category of ∞-categories. This idea is explored in detail in [Lur09, Ch. 3].

Notation 3.3. To each coCartesian fibration $p : X \to N(D)$ we can associate a marked simplicial set denoted $X^\natural$ which is composed of the pair $(X, \mathcal{E})$, where $\mathcal{E}$ is the set of p-coCartesian edges of X

Notation 3.4. Let $(X, p), (Y, q)$ be two objects in $\mathcal{S}^+/N(D)$. We denote by $[X, Y]_D^+$, the full (marked) simplicial subset of $[X, Y]^+$ spanned by maps in $\mathcal{S}^+/N(D)(X, Y)$, namely spanned by maps in $[X, Y]^+$ which are compatible with the projections p and q. We denote by $[X, Y]_D^\flat$, the full simplicial subset of $[X, Y]^\flat$ spanned by maps in $\mathcal{S}^+/N(D)(X, Y)$. We denote by $[X, Y]_D^\sharp \subseteq [X, Y]^\sharp$ the simplicial subsets spanned by maps in $\mathcal{S}^+/N(D)$.

Definition 3.5. A morphism $F : X \to Y$ in the category $\mathcal{S}^+/N(D)$ is called a *coCartesian*-equivalence if for each coCartesian fibration $p : Z \to N(D)$, the induced simplicial map

$$\left[F, Z^\natural\right]_D^\flat : \left[Y, Z^\natural\right]_D^\flat \to \left[X, Z^\natural\right]_D^\flat$$

is a categorical equivalence of simplicial-sets(quasi-categories).

Proposition 3.6. *Let $u : X \to Y$ be a map in $\mathcal{S}^+/N(D)$, then the following are equivalent*

(1) *u is a coCartesian equivalence.*
(2) *For each functor $Z : D \to \mathcal{S}^+$, such that $Z(d)$ is a quasi-category whose marked edges are equivalences, the following (simplicial) map is a categorical equivalence:*

$$\left[u, \int_+^{d \in D} Z\right]_D^\flat : \left[Y, \int_+^{d \in D} Z\right]_D^\flat \to \left[X, \int_+^{d \in D} Z\right]_D^\flat$$

(3) *For each functor $Z : D \to \mathcal{S}^+$, such that $Z(d)$ is a quasi-category whose marked edges are equivalences, the following map is a bijection:*

$$\pi_0\left[u, \int_+^{d \in D} Z\right]_D^\sharp : \pi_0\left[Y, \int_+^{d \in D} Z\right]_D^\sharp \to \pi_0\left[X, \int_+^{d \in D} Z\right]_D^\sharp$$

Proof. $(1 \Rightarrow 2)$ Follows from the definition of coCartesian equivalence because $\int_+^{d \in D} Z$ is a coCartesian fibration under the given hypothesis.

Let us assume that $\left[u, \int_+^{d \in D} Z\right]_D^\flat$ is a categorical equivalence of quasi-categories for each functor Z satisfying the given hypothesis. This imples that $\left[u, T^\natural\right]_D^\flat$ is a categorical equivalence if and only if $\left[u, \int_+^{d \in D} Z(T)\right]_D^\flat$ is one.

$(2 \Rightarrow 3)$ We recall from [Lur09, Prop. 3.1.3.3] and [Lur09, Prop. 3.1.4.1] that, for any coCartesian fibration $T^\natural \in \mathcal{S}^+/N(D)$, the simplicial map $\left[u, T^\natural\right]_D^\flat$ is a categorical equivalence if and only if the map $\left[u, T^\natural\right]_D^\sharp$ is a homotopy equivalence of Kan complexes. This implies that $\pi_0\left[u, \int_+^{d \in D} Z\right]_D^\sharp$ is a bijection.

$(3 \Rightarrow 1)$ We recall from [Lur09, Cor. 3.1.4.4] that the coCartesian model category is a simplicial model category with simplicial function object given by the bifunctor $[-,-]_D^\sharp$. This implies that u is a coCartesian equivalence if and only if $\pi_0\left[u, W^\natural\right]_D^\sharp$ is a bijection for each fibrant object W of the coCartesian model category. By [Lur09, Prop. 3.1.4.1] we may replace W by a coCartesian fibration $W \cong T^\natural$. Further, it follows from [Lur09, Prop. 3.2.5.18(2)] that for each cocartesian fibration $T^\natural$ there exists a functor $Z(T) : D \to \mathcal{S}^+$, which satisfies the assumptions of the functor in the statement of the proposition, such that there is map

$$F_T : T^\natural \to \int_+^{d \in D} Z(T)$$

which is a coCartesian equivalence. Now it follows that u is a coCartesian equivalence if and only if $\pi_0\left[u, \int_+^{d \in D} Z(T)\right]_D^\sharp$ is a bijection for each functor Z satisfying the conditions mentioned in the statement of the proposition.

$\square$

Next we will recall a model category structure on the overcategory $\mathcal{S}^+/N(D)$ from [Lur09, Prop. 3.1.3.7.] in which fibrant objects are (essentially) coCartesian fibrations.

Theorem 3.7. *There is a left-proper, combinatorial model category structure on the category $\mathcal{S}^+/N(D)$ in which a morphism is*

(1) *a cofibration if it is a monomorphism when regarded as a map of simplicial sets.*

(2) *a weak-equivalences if it is a coCartesian equivalence.*

(3) *a fibration if it has the right lifting property with respect to all maps which are simultaneously cofibrations and weak-equivalences.*

We have defined a function object for the category $\mathcal{S}^+/N(D)$ above. The simplicial set $[X,Y]_D^\flat$ has verices, all maps from X to Y in $\mathcal{S}^+/N(D)$. An n-simplex in $[X,Y]_D^\flat$ is a map $\Delta[n]^\flat \times X \to Y$ in $\mathcal{S}^+/N(D)$, where $\Delta[n]^\flat \times (X,p) = (\Delta[n]^\flat \times X, pp_2)$, where p_2 is the projection $\Delta[n]^\flat \times X \to X$. The enriched category $\mathcal{S}^+/N(D)$ admits tensor and cotensor products. The *tensor product* of an object $X = (X,p)$ in $\mathcal{S}^+/N(D)$ with a simplicial set A is the objects

$$A^\flat \times X = (A^\flat \times X, pp_2).$$

The *cotensor product* of X by A is an object of $\mathcal{S}^+/N(D)$ denoted $X^{[A]}$. If $q : X^{[A]} \to N(D)^\sharp$ is the structure map, then a simplex $x : \Delta[n]^\flat \to X^{[A]}$ over a simplex $y = qx : \Delta[n] \to N(D)^\sharp$ is a map $A^\flat \times (\Delta[n]^\flat, y) \to (X,p)$. The object $(X^{[A]}, q)$ can be constructed by the following pullback square in $\mathcal{S}^+$:

$$
\begin{array}{ccc}
X^{[A]} & \longrightarrow & [A^\flat, X]^+ \\
\downarrow{\scriptstyle q} & & \downarrow{\scriptstyle [A^\flat,p]^+} \\
N(D)^\sharp & \longrightarrow & [A^\flat, N(D)^\sharp]^+
\end{array}
$$

where the bottom map is the diagonal. There are canonical isomorphisms:

$$(8) \qquad \left[A^\flat \times X, Y\right]_D^\flat \cong \left[A, [X,Y]_D^\flat\right] \cong \left[X, Y^{[A]}\right]_D^\flat$$

Remark 10. The coCartesian model category structure on $\mathcal{S}^+/N(D)$ is a simplicial model category structure with the simplicial Hom functor:

$$[-,-]_D^\sharp : \mathcal{S}^+/N(D)^{op} \times \mathcal{S}^+/N(D) \to \mathcal{S}.$$

This is proved in [Lur09, Corollary 3.1.4.4.]. The coCartesian model category structure is a $(\mathcal{S}, \mathbf{Q})$-model category structure with the function object given by:

$$[-,-]_D^\flat : \mathcal{S}^+/N(D)^{op} \times \mathcal{S}^+/N(D) \to \mathcal{S}.$$

This is remark [Lur09, 3.1.4.5.].

Remark 11. The coCartesian model category is a $(\mathcal{S}^+, \mathbf{Q})$-model category with the Hom functor:

$$[-,-]_D^+ : \mathcal{S}^+/N(D)^{op} \times \mathcal{S}^+/N(D) \to \mathcal{S}^+.$$

This follows from [Lur09, Corollary 3.1.4.3] by taking $S = N(D)$ and $T = \Delta[0]$, where S and T are specified in the statement of the corallary.

Definition 3.8. Let $F : D \to \mathcal{S}^+$ be a functor. We can compose it with the forgetful functor U to obtain a composite functor $F : D \xrightarrow{F} \mathcal{S}^+ \xrightarrow{U} \mathcal{S}$. The *marked* Grothendieck construction of F, denoted $\int_+^{d \in D} F$, is the marked simplicial set $\left(\int_+^{d \in D} F, \mathcal{E} \right)$, where the set $\mathcal{E}$ consists of those edges $\underline{e} = (e, h)$ of $\int_+^{d \in D} F$, see remark 8, which determines a marked edge of the marked simplicial set $F(d)$, where $e : c \to d$ is an arrow in D.

The above construction of the marked Grothendieck construction determines a functor

$$(9) \qquad \int_+^{d \in D} - : [D, \mathcal{S}^+] \to \mathcal{S}^+/N(D).$$

Now we will define a marked version of the functor $\mathfrak{R}_D$, denoted $\mathfrak{R}_D^+$:

$$\mathfrak{R}_D^+(X)(d) := [N(d/D)^\sharp, X]_D^+$$

where X is an object of $\mathcal{S}^+/N(D)$. This functor has a left adjoint which we denote by $\mathfrak{L}_D^+$.

Definition 3.9. Let $X : D \to \mathcal{S}^+$ be a functor. For each $d \in D$ we define a map of marked simplicial sets

$$\eta_X^+(d) : X(d) \to [N(d/D), \int_+^{d \in D} X]_D^+.$$

Let $x \in X(d)_n$ be an n-simplex in $X(d)$. This n-simplex defines a canonical map $\eta_X^+(d)(x) : N(d/D) \times \Delta[n] \to \int_+^{d \in D} X$ in $\mathcal{S}^+/N(D)$ whose value on $(\mathbf{id}(d)_n, id_n) \in (N(d/D) \times \Delta[n])_n$ is the image of x in $\int_+^{d \in D} X$, namely the n-simplex $(\underline{x}, x)$, where $\underline{x} = (x_{n-1}, d_n(x))$. We recall that a k-simplex in $\Delta[n]$ is a map $\alpha : [k] \to [n]$ in the category Δ and therefore it can be written as $\Delta[n](\alpha)(id_n)$. For a k-simplex $((g, f_1, \ldots, f_{k+1}), \alpha)$ in $N(d/D) \times \Delta[n]$, we define

$$\eta_X^+(d)(x)((g, f_1, f_2, \ldots, f_{k+1}), \alpha) := X(f_{k+1} \circ f_k \circ \cdots \circ g)(X(d)(\alpha)(x)).$$

This defines the (simplicial) map $\eta_X^+(d)(x)$. These simplicial maps glue together into a natural transformation η_X^+.

Now we define a map ι_d^+ in $\mathcal{S}^+/N(D)$:

$$(10)$$

Lemma 3.10. *For each $d \in D$ the morphism ι_d^+ defined in (10) is a coCartesian equivalence.*

Proof. We will show that for each functor $Z : D \to \mathcal{S}^+$ such that, for each $d \in D$, $Z(d)$ is a quasi-category whose marked edges are equivalences, we have the following bijection:

$$\pi_0 \left[\iota_d^+, \int^{d \in D} Z \right]_D^\sharp : \pi_0 \left[N(d/D), \int^{d \in D} Z \right]_D^\sharp \to \pi_0 \left[\Delta[0], \int^{d \in D} Z \right]_D^\sharp \cong \pi_0(J(Z(d))),$$

where $J(Z(d))$ is the largest Kan complex contained in $Z(d)$. Let $z \in J(Z(d))_0$ be a vertex of $J(Z(d))^\sharp$. We will construct a morphism $F_z : N(d/D) \to \int^{d \in D} Z$ in the category $\mathcal{S}^+/N(D)$. The vertex z represents a natural transformation

$$T_z : D(d, -) \Rightarrow Z$$

such that $T_z(id_d) = z$. Since $N(d/D) \cong \int^{d \in D} D(d, -)$ therefore we have a map

$$F_z : N(d/D) \cong \int^{d \in D} D(d, -) \overset{\int^{d \in D} T_z}{\to} \int^{d \in D} Z$$

in $\mathcal{S}^+/N(D)$ such that $F_z(id_d) = z$. Thus we have shown that the map $\pi_0 \left[\iota_d^+, \int^{d \in D} Z \right]_D^\sharp$ is a surjection.

Let $f : y \to z$ be an edge of $J(Z(d))$, then by the (enriched) Yoneda's lemma followed by an application of the Grothendieck construction functor, this edge uniquely determines a map

$$T_f : N(d/D) \times \Delta[1] \to \int^{d \in D} Z$$

in $\mathcal{S}^+/N(D)$ such that $F_z((id_d, id_1)) = f$. Thus we have shown that the map $\pi_0 \left[\iota_d^+, \int^{d \in D} Z \right]_D^\sharp$ is also an injection.

$\square$

Lemma 3.11. *For any projectively fibrant functor $X : D \to \mathcal{S}^+$, the map η_X^+ defined in 3.9 is an objectwise categorical equivalence of marked simplicial sets.*

Proof. Under the hypothesis of the lemma, it follows from [Lur09, Prop. 3.2.5.18(2)] and lemma 2.8 that $\int_+^{d \in D} X$ is a fibrant object in the coCartesian model category. Now lemma 3.10 and remark 11 gives us, for each $d \in D$, the following homotopy equivalence in $(\mathcal{S}^+, \mathbf{Q})$:

$$[\iota_d^+, \int^{d \in D} X]_D^+ : [N(d/D)^\sharp, \int_+^{d \in D} X]_D^+ \to [\Delta[0]^\flat, \int_+^{d \in D} X]_D^+ \cong X(d)$$

such that $c \circ [\iota_d, \int^{d \in D} X]_D \circ \eta_X(d) = id_{X(d)}$, where c is the canonical isomorphism between the fiber of $p : \int^{d \in D} X \to N(D)$ over $d \in D$ and $X(d)$ *i.e.* the value of the functor X on d. Now the 2 out of 3 property of weak equivalences in a model category tells us that $\eta_X(d)$ is a homotopy equivalence for each $d \in D$ therefore η_X^+ is an objectwise homotopy equivalence in $[D, (\mathcal{S}^+, \mathbf{Q})]$.

$\square$

An immediate consequence of the definition of the right adjoint functor $\mathfrak{R}_D^+$ is the following lemma:

Lemma 3.12. *The adjunction $(\mathfrak{L}_D^+, \mathfrak{R}_D^+)$ is a Quillen adjunction between the projective model category structure on $[D, (\mathcal{S}^+, \mathbf{Q})]$ and coCartesian model category $\mathcal{S}^+/N(D)$.*

Proof. The coCartesian model category is a $(\mathcal{S}^+, \mathbf{Q})$-model category, see remark 11. This implies that $\mathfrak{R}_D^+$ maps (acyclic) fibrations in the coCartesian model category to (acyclic) projective fibrations in $[D, \mathcal{S}^+]$ which are objectwise (acyclic) fibrations of marked simplicial sets.

$\square$

Now we get to the main result of this note:

Theorem 3.13. *The Quillen pair $(\mathfrak{L}_D^+, \mathfrak{R}_D^+)$ is a Quillen equivalence.*

Proof. We will prove this proposition by showing that the (right) derived functor of $\mathfrak{R}_D^+$ induces an equivalence of categories between the two homotopy categories in context. It follows from [Lur09, Prop. 3.2.5.18(2)] that each fibrant object Z in the coCartesian model category is equipped with a coCartesian equivalence

$$\delta(Z) : Z \to \int_+^{d \in D} \mathfrak{F}_\bullet^+(D)(\mathfrak{R}_D^+(Z)).$$
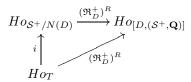
We observe that $\mathfrak{R}_D^+(Z)$ is a fibrant object of $[D, (\mathcal{S}^+, \mathbf{Q})]$.

Let T denote the full subcategory of $\mathcal{S}^+/N(D)$ in which every object W is fibrant in the coCartesian model category and lies in the image of $\int_+^{d \in D} -$, *i.e.* for any $W \in T$, there exists a fibrant V in $[D, (\mathcal{S}^+, \mathbf{Q})]$ such that $W = \int_+^{d \in D} V$. We denote the full subcategory of $Ho_{\mathcal{S}^+/N(D)}$ whose objects are those of T by Ho_T. We will refer to it as the homotopy category of T. We observe that the inclusion map $i : Ho_T \to Ho_{\mathcal{S}^+/N(D)}$ is an equivalence of categories. Now it is sufficient to show that the (right) derived functor of $\mathfrak{R}_D^+$ induces an equivalence of categories between Ho_T and $Ho_{[D,(\mathcal{S}^+,\mathbf{Q})]}$. It follows from [Lur09, Prop. 3.2.5.18(2)], lemma 3.11 and lemma 2.8 that for each fibrant X in the projective model category $[D, (\mathcal{S}^+, \mathbf{Q})]$ we have the following composite map which is a weak-eqivalence in $[D, (\mathcal{S}^+, \mathbf{Q})]$:

$$\mathfrak{F}_\bullet^+(D)\left(Q\left(\int_+^{d \in D} X\right)\right) \xrightarrow{q} \mathfrak{F}_\bullet^+(D)\left(\int_+^{d \in D} X\right) \xrightarrow{\epsilon} X \xrightarrow{\eta_X^+}$$

$$\mathfrak{R}_D^+\left(\int_+^{d \in D} X\right) \xrightarrow{r} \mathfrak{R}_D^+\left(R\left(\int_+^{d \in D} X\right)\right),$$

where ϵ is the counit map of the Quillen equivalence [Lur09, Prop. 3.2.5.18(2)], which is a weak equivalence, $r : id \Rightarrow R$ is a chosen fibrant replacement functor and $q : Q \Rightarrow id$ is a chosen cofibrant replacement functor. We claim that this defines a natural isomorphism between the (restriction to Ho_T of) left derived functor of $\mathfrak{F}_\bullet^+(D)$, which we also denote $(\mathfrak{F}_\bullet^+(D))^L : Ho_T \to Ho_{[D,(\mathcal{S}^+,\mathbf{Q})]}$ and the (restriction to Ho_T of) right derived functor of $\mathfrak{R}_D^+$, which we also denote $(\mathfrak{R}_D^+)^R : Ho_T \to Ho_{[D,(\mathcal{S}^+,\mathbf{Q})]}$. The Quillen equivalence [Lur09, Prop. 3.2.5.18(2)] implies that the right derived functor of $\int_+^{d \in D} -$ is fully-faithful. Since X and Y are fibrant, this means that for each equivalence class $[u]$ representing an arrow $[u] : R\left(\int_+^{d \in D} X\right) \to R\left(\int_+^{d \in D} Y\right)$ in Ho_T, there exists an arrow $v : X \to Y$ in $[D, (\mathcal{S}^+, \mathbf{Q})]$ such that the arrow $R\left(\int_+^{d \in D} v\right)$ is a representative of $[u]$. This implies that we have defined the desired natural isomorphism. Since $(\mathfrak{F}_\bullet^+(D))^L : Ho_T \to Ho_{[D,(\mathcal{S}^+,\mathbf{Q})]}$ is an equivalence of categories, the above natural isomorphism implies that the functor $(\mathfrak{R}_D^+)^R : Ho_T \to Ho_{[D,(\mathcal{S}^+,\mathbf{Q})]}$ is also one.

We have the following commutative triangle of functors between homotopy categories:

$$Ho_{\mathcal{S}^+/N(D)} \xrightarrow{(\mathfrak{R}_D^+)^R} Ho_{[D,(\mathcal{S}^+,\mathbf{Q})]}$$

with i arrow from Ho_T and $(\mathfrak{R}_D^+)^R$ arrow.

Now the 2 out of 3 property of weak equivalences in the model category **Cat** tells us that the right derived functor of $\mathfrak{R}_D^+$ induces an equivalence of categories between the homotopy categories. Thus we have proved that the Quillen pair $(\mathfrak{L}_D^+, \mathfrak{R}_D^+)$ is a Quillen equivalence.

$\square$

The natural isomorphism constructed in the proof of the above theorem implies the following proposition:

Proposition 3.14. *The (total) left derived functor of the left Quillen functor $\mathfrak{F}_\bullet^+(D)$ is naturally isomorphic to the (total) right derived functor of $\mathfrak{R}_D^+$.*

The proposition has the following corollary:

Corollary 3.15. *The (total) left derived functor of $\mathfrak{L}_D^+$ is naturally isomorphic to the (total) right derived functor of $\int_+^{d \in D} -$.*

Notation 3.16. The total (total) right derived functor of $\int_+^{d \in D} -$ refers to the total right derived functor of the relative nerve functor for marked simplicial sets, see [Lur09, Prop. 3.2.5.18(2)].

Proposition 3.17. *For all functor $X : D \to \mathcal{S}^+$ and $K \in \mathcal{S}$ we have the following isomorphism*
$$\mathfrak{L}_D^+(X \otimes K) \cong \mathfrak{L}_D^+(X) \otimes K.$$

APPENDIX A. A REVIEW OF MARKED SIMPLICIAL SETS

In this appendix we will review the theory of marked simplicial sets. Later in this paper we will develop a theory of coherently commutative monoidal objects in the category of marked simplicial sets.

Definition A.1. A *marked* simplicial set is a pair $(X, \mathcal{E})$, where X is a simplicial set and $\mathcal{E}$ is a set of edges of X which contains every degenerate edge of X. We will say that an edge of X is *marked* if it belongs to $\mathcal{E}$. A morphism $f : (X, \mathcal{E}) \to (X', \mathcal{E}')$ of marked simplicial sets is a simplicial map $f : X \to X'$ having the property that $f(\mathcal{E}) \subseteq \mathcal{E}'$. We denote the category of marked simplicial sets by $\mathcal{S}^+$.

Every simplicial set S may be regarded as a marked simplicial set in many ways. We mention two extreme cases: We let $S^\sharp = (S, S_1)$ denote the marked simplicial set in which every edge is marked. We denote by $S^\flat = (S, s_0(S_0))$ denote the marked simplicial set in which only the degerate edges of S have been marked.

The category $\mathcal{S}^+$ is *cartesian-closed*, i.e. for each pair of objects $X, Y \in Ob(\mathcal{S}^+)$, there is an internal mapping object $[X, Y]^+$ equipped with an *evaluation map* $[X, Y]^+ \times X \to Y$ which induces a bijection:

$$\mathcal{S}^+(Z, [X, Y]^+) \overset{\cong}{\to} \mathcal{S}^+(Z \times X, Y),$$

for every $Z \in \mathcal{S}^+$.

Notation A.2. We denote by $[X, Y]^\flat$ the underlying simplicial set of $[X, Y]^+$.

The mapping space $[X, Y]^\flat$ is charaterized by the following bijection:

$$\mathcal{S}(K, [X, Y]^\flat) \overset{\cong}{\to} \mathcal{S}^+(K^\flat \times X, Y),$$

for each simplicial set K.

Notation A.3. We denote by $[X, Y]^\sharp$ the simplicial subset of $[X, Y]^\flat$ consisting of all simplices $\sigma \in [X, Y]^\flat$ such that every edge of σ is a marked edge of $[X, Y]^+$.

The mapping space $[X, Y]^\sharp$ is charaterized by the following bijection:

$$\mathcal{S}(K, [X, Y]^\sharp) \overset{\cong}{\to} \mathcal{S}^+(K^\sharp \times X, Y),$$

for each simplicial set K.

The Joyal model category structure on $\mathcal{S}$ has the following analog for marked simplicial sets:

Theorem A.4. *There is a left-proper, combinatorial model category structure on the category of marked simplicial sets $\mathcal{S}^+$ in which a morphism $p : X \to Y$ is a*

  (1) *cofibration if the simplicial map between the underlying simplicial sets is a cofibration in $(\mathcal{S}, \mathbf{Q})$, namely a monomorphism.*
  (2) *a weak-equivalence if the induced simplicial map on the mapping spaces*

$$[p, K^{\natural}]^\flat : [X, K^{\natural}]^\flat \to [Y, K^{\natural}]^\flat$$

  is a weak-categorical equivalence, for each quasi-category K.
  (3) *fibration if it has the right lifting property with respect to all maps in $\mathcal{S}^+$ which are simultaneously cofibrations and weak equivalences.*

Further, the above model category structure is enriched over the Joyal model category, i.e. it is a $(\mathcal{S}, \mathbf{Q})$-model category.

The above theorem follows from [Lur09, Prop. 3.1.3.7].

Notation A.5. We will denote the model category structure in Theorem A.4 by $(\mathcal{S}^+, \mathbf{Q})$ and refer to it either as the *Joyal* model category of *marked* simplicial sets or as the model category of marked quasi-categories.

Theorem A.6. *The model category $(\mathcal{S}^+, \mathbf{Q})$ is a cartesian closed model category.*

Proof. The theorem follows from [Lur09, Corollary 3.1.4.3] by taking $S = T = \Delta[0]$. $\square$

There is an obvious forgetful functor $U : \mathcal{S}^+ \to \mathcal{S}$. This forgetful functor has a left adjoint $(-)^\flat : \mathcal{S} \to \mathcal{S}^+$.

Theorem A.7. *The adjoint pair of functors $((-)^\flat, U)$ determine a Quillen equivalence between the Joyal model category of marked simplicial sets and the Joyal model category of simplicial sets.*

The proof of the above theorem follows from [Lur09, Prop. 3.1.5.3].

Remark 12. A marked simplicial set X is fibrant in $(\mathcal{S}^+, \mathbf{Q})$ if and only if it is a quasi-category with the set of all its equivalences as the set of marked edges.

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