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# AXIOMATIZING SUBCATEGORIES OF ABELIAN CATEGORIES

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Abstract. We investigate how to characterize subcategories of abelian categories in terms of intrinsic axioms. In particular, we find intrinsic axioms which characterize generating cogenerating functorially finite subcategories, precluster tilting subcategories, and cluster tilting subcategories of abelian categories. As a consequence of this we prove that any $d$-abelian category is equivalent to a $d$-cluster tilting subcategory of an abelian category, without any assumption on the categories being projectively generated.

## Contents



## 1. Introduction

Higher Auslander-Reiten theory was introduced by Iyama in [16] and further developed in [15, 17]. It has several connections to other areas, for example non-commutative algebraic geometry [12, 13, 15], combinatorics [26], higher category theory [4], and symplectic geometry [5]. One of the main objects of study are subcategories of abelian categories called $d$-cluster tilting. The study of intrinsic properties of $d$-cluster tilting subcategories is an active area of research, see [7, 8, 14, 19, 20, 23, 27]. This approach, called higher homological algebra, was catalysed by the papers [11] and [21], where they introduced $d$-abelian, $d$-exact, and $(d+2)$-angulated categories and showed that $d$-cluster tilting subcategories of abelian, exact or triangulated categories are $d$-abelian, $d$-exact, or $(d+2)$-angulated,

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respectively. Furthermore, they also showed that any projectively generated $d$-abelian category is equivalent to a $d$-cluster tilting subcategory of an abelian category.

The idea of axiomatizing subcategories of abelian categories is also closely related to earlier work on characterizing $\Lambda$-modules $M$, where $\Lambda$ is an Artin algebra, in terms of properties of the endomorphism algebra $\Gamma := \mathrm{End}_\Lambda(M)$. In an unpublished manuscript [25] Morita and Tachikawa showed that $M \mapsto \Gamma$ gives a correspondence between generating and cogenerating modules $M$ and algebras $\Gamma$ with dominant dimension $\geq 2$. Auslander [2] showed that this specialize to the case where $M$ is an additive generator of a module category and $\Gamma$ is an algebra with dominant dimension $\geq 2$ and global dimension $\leq 2$. This is typically called the Auslander-correspondence. It was later extended by Iyama [15] to a bijection between $d$-cluster tilting modules $M$ and algebras $\Gamma$ with dominant dimension $\geq d+1$ and global dimension $\leq d+1$. Recently Iyama and Solberg [18] introduced $d$-precluster tilting modules $M$ and showed that the assignment $M \mapsto \Gamma$ gives a bijection to algebras of dominant dimension $\geq d+1$ and selfinjective dimension $\leq d+1$. They also showed that $d$-precluster tilting subcategories have a higher Auslander-Reiten theory, similar to $d$-cluster tilting subcategories.

In this paper we continue the idea of axiomatizing subcategories of abelian categories and study their properties. The following definition clarifies what we mean:

**Definition 1.1.** Let $\mathtt{P}$ be a set of axioms of additive categories, and let $\mathtt{S}$ be a class of subcategories of abelian categories. We say that $\mathtt{P}$ ***axiomatizes*** subcategories in $\mathtt{S}$ if the following hold:

(i) If $\mathcal{X}$ is in $\mathtt{S}$, then $\mathcal{X}$ satisfies $\mathtt{P}$ as an additive category;
(ii) If $\mathcal{X}$ satisfies $\mathtt{P}$, then there exists an abelian category $\mathcal{A}$ and a fully faithful functor $\mathcal{X} \to \mathcal{A}$ such that its essential image is in $\mathtt{S}$;
(iii) If $\mathcal{A}$ and $\mathcal{A}'$ are abelian categories and $\Phi\colon \mathcal{X} \to \mathcal{A}$ and $\Phi'\colon \mathcal{X} \to \mathcal{A}'$ are two fully faithful functors such that the essential images of $\Phi$ and $\Phi'$ are in $\mathtt{S}$, then there exists an equivalence $\Psi\colon \mathcal{A} \xrightarrow{\cong} \mathcal{A}'$ and a natural isomorphism $\Psi \circ \Phi \cong \Phi'$.

Part (i) and (ii) tells us that an additive category $\mathcal{X}$ satisfies $\mathtt{P}$ if and only if it is equivalent to a subcategory in $\mathtt{S}$. Part (iii) tells us that the ambient abelian category of a subcategory in $\mathtt{S}$ must be unique up to equivalence.

We prove the following theorem, which gives examples of axioms $\mathtt{P}$ and classes of subcategories $\mathtt{S}$. See the end of the introduction for the list of axioms.

**Theorem 1.2.** *Let* $\mathtt{P}$ *be a set of axioms for an additive category, and let* $\mathtt{S}$ *be a class of subcategories of abelian categories. We have that* $\mathtt{P}$ *axiomatizes subcategories in* $\mathtt{S}$ *in the following cases:*

*(i)* $\mathtt{P}$ *consists of the axioms (A1), (A1)*$^{\mathrm{op}}$*, (A2), (A2)*$^{\mathrm{op}}$*, (A3), and (A3)*$^{\mathrm{op}}$*, and* $\mathtt{S}$ *is the class of generating cogenerating functorially finite subcategories;*
*(ii)* $\mathtt{P}$ *consists of the axioms (A1), (A1)*$^{\mathrm{op}}$*, (A2), (A2)*$^{\mathrm{op}}$*, (A3), and (A3)*$^{\mathrm{op}}$*, and (d-Rigid) and* $\mathtt{S}$ *is the class of generating cogenerating functorially finite subcategories* $\mathcal{X}$ *of an abelian category* $\mathcal{A}$ *satisfying*

$$\mathrm{Ext}^i_{\mathcal{A}}(X, X') = 0 \text{ for all } X, X' \in \mathcal{X} \text{ and } 0 < i < d.$$

(iii) P *consists of the axioms (A0), (A1), (A1)*$^{\mathrm{op}}$*, (A2), (A2)*$^{\mathrm{op}}$*, (A3), and (A3)*$^{\mathrm{op}}$*, (d-Rigid), (A4), and (A4)*$^{\mathrm{op}}$*, and* S *is the class of* $d$*-precluster tilting subcategories.*

(iv) P *consists of the axioms (A0), (A1), (A1)*$^{\mathrm{op}}$*, (A2), (A2)*$^{\mathrm{op}}$*, (A3), and (A3)*$^{\mathrm{op}}$*, (d-Rigid), (d-Ker), and (d-Coker) and* S *is the class of* $d$*-cluster tilting subcategories.*

We even prove that there exists an axiomatiziation when only assuming (A1) and (A2), see Theorem 4.1. Note that the definition of $d$-precluster tilting subcategories in [18] can be reformulated in a way that makes sense for any abelian category, see Theorem 8.5, and this reformulated definition is what we use in Theorem 1.2 (iii).

As a corollary of Theorem 1.2 (iv) we show that any $d$-abelian category is equivalent to a $d$-cluster tilting subcategory, without the assumption of being projectively generated. Since there exists examples of cluster tilting subcategories without any non-zero projective or injective objects, see [22], the result is necessary to complete the axiomatization of $d$-cluster tilting subcategories in terms of $d$-abelian categories.

**Corollary 1.3.** *Let* $\mathcal{X}$ *be an additive category. The following holds:*

(i) $\mathcal{X}$ *is d-abelian if and only if it satisfies (A0), (A1), (A1)*$^{\mathrm{op}}$*, (A2), (A2)*$^{\mathrm{op}}$*, (A3), and (A3)*$^{\mathrm{op}}$*, (d-Rigid), (d-Ker), and (d-Coker).*

(ii) *If* $\mathcal{X}$ *is d-abelian, then there exists an abelian category* $\mathcal{A}$ *and a fully faithful functor* $\mathcal{X} \to \mathcal{A}$ *such that its essential image is d-cluster tilting.*[1]

We end the introduction by giving the list of axioms we use:

(A0) $\mathcal{X}$ is idempotent complete;

(A1) $\mathcal{X}$ has weak kernels;

(A1)$^{\mathrm{op}}$ $\mathcal{X}$ has weak cokernels;

(A2) Any epimorphism in $\mathcal{X}$ is a weak cokernel;

(A2)$^{\mathrm{op}}$ Any monomorphism in $\mathcal{X}$ is a weak kernel;

(A3) Consider the following diagram

$$\begin{tikzcd} X_2 \arrow[rr, "f"] \arrow[dr, "l"'] & & X_1 \arrow[r, "g"] & X_0 \\ & X_2' \arrow[ur, "h"'] & & \end{tikzcd}$$

where $f$ is an arbitrary morphism in $\mathcal{X}$, where $g$ is a weak cokernel of $f$, where $h$ is a weak kernel of $g$, and where $l$ is an induced map satisfying $h \circ l = f$. Then for any weak kernel $k\colon X_3' \to X_2'$ of $h$ the map $\begin{bmatrix} l & k \end{bmatrix} : X_2 \oplus X_3' \to X_2'$ is an epimorphism;

(A3)$^{\mathrm{op}}$ Consider the following diagram

$$\begin{tikzcd} X_0 \arrow[r, "g"] & X_1 \arrow[rr, "f"] \arrow[dr, "h"'] & & X_2 \\ & & X_2' \arrow[ur, "l"'] & \end{tikzcd}$$

[1] The author gave a talk about this result in the LMS Northern Regional Meeting and Workshop on Higher Homological Algebra in 2019

where $f$ is an arbitrary morphism in $\mathcal{X}$, where $g$ is a weak kernel of $f$, where $h$ is a weak cokernel of $g$, and where $l$ is a map satisfying $l \circ h = f$. Then for any weak cokernel $k\colon X_2' \to X_3'$ of $h$ the map $\begin{bmatrix} l \\ k \end{bmatrix} : X_2' \to X_2 \oplus X_3'$ is a monomorphism;

(A4) Let
$$X_{d+1} \xrightarrow{f_{d+1}} X_d \xrightarrow{f_d} \cdots \xrightarrow{f_3} X_2 \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0 \xrightarrow{f_0} X_{-1}$$
be a sequence with $f_{i+1}$ a weak kernel of $f_i$ for all $0 \leq i \leq d$. Then $f_{d+1}$ is a weak cokernel;

(A4)$^{\mathrm{op}}$ Let
$$X_{d+1} \xrightarrow{f_{d+1}} X_d \xrightarrow{f_d} \cdots \xrightarrow{f_3} X_2 \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0 \xrightarrow{f_0} X_{-1}$$
be a sequence with $f_i$ a weak cokernel of $f_{i+1}$ for all $0 \leq i \leq d$. Then $f_0$ is a weak kernel;

(d-Rigid) For all epimorphism $f_1\colon X_1 \to X_0$ in $\mathcal{X}$ there exists a sequence
$$X_{d+1} \xrightarrow{f_{d+1}} X_d \xrightarrow{f_d} \cdots \xrightarrow{f_3} X_2 \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0$$
with $f_{i+1}$ a weak kernel of $f_i$ and $f_i$ a weak cokernel of $f_{i+1}$ for all $1 \leq i \leq d$;

(d-Ker) Any map in $\mathcal{X}$ has a $d$-kernel;

(d-Coker) Any map in $\mathcal{X}$ has a $d$-cokernel.

1.1. **Conventions.** All categories are assumed to additive, i.e. enriched over abelian groups and admitting direct sums. For an additive category $\mathcal{X}$ we let $\mathcal{X}(X, X')$ denote the set of morphism between two objects $X, X' \in \mathcal{X}$ and $\mathrm{Hom}_{\mathcal{X}}(F_1, F_2)$ the set of natural transformations between two additive functors $F_1, F_2\colon \mathcal{X}^{\mathrm{op}} \to \mathrm{Ab}$. A subcategory $\mathcal{X}$ of an abelian category $\mathcal{A}$ is called generating (resp cogenerating) if for any object $A \in \mathcal{A}$ there exists an epimorphism $X \to A$ (resp a monomorphism $A \to X$ with $X \in \mathcal{X}$.

1.2. **Acknowledgement.** Corollary 1.3 (ii) is proved independently by Ramin Ebrahimi and Alireza Nasr-Isfahani in [6].

## 2. Serre subcategories

Let $\mathcal{A}$ be an abelian category. A subcategory $\mathcal{S}$ of $\mathcal{A}$ is called a *Serre subcategory* if for any exact sequence in $\mathcal{A}$
$$0 \to A_1 \to A_2 \to A_3 \to 0$$
we have that $A_1 \in \mathcal{S}$ and $A_3 \in \mathcal{S}$ if and only if $A_2 \in \mathcal{S}$. If $\mathcal{S}$ be a Serre subcategory of $\mathcal{A}$, we define the category $\mathcal{A}/\mathcal{S}$ to be the localization of $\mathcal{A}$ by the class of morphisms $f\colon X \to X'$ satisfying
$$\mathrm{Ker}\, f \in \mathcal{S} \quad \text{and} \quad \mathrm{Coker}\, f \in \mathcal{S}.$$
Note that the objects in $\mathcal{A}/\mathcal{S}$ are the same as the objects in $\mathcal{A}$. We need the following results for this localization:

**Theorem 2.1.** *Let $\mathcal{S}$ be a Serre subcategory of $\mathcal{A}$, and let $q\colon \mathcal{A} \to \mathcal{A}/\mathcal{S}$ denote the canonical functor to the localization. The following hold:*

*(i)* $\mathcal{A}/\mathcal{S}$ *is an abelian category;*
*(ii)* $q$ *is an exact functor;*
*(iii) For a morphism* $f\colon A_1 \to A_2$ *in* $\mathcal{A}$*, we have that* $q(f) = 0$ *if and only if* $\operatorname{im} f \in \mathcal{S}$.

*Proof.* This follows from Proposition 1 and Lemma 2 in Chapter 3 in [10]. □

**Remark 2.2.** Note that the class of morphisms $\Sigma$ forms a multiplicative system in $\mathcal{A}$, see [28, Exercise 10.3.2 (1)]. We will not use this fact in our proofs.

## 3. Weak kernels and weak cokernels

Let $\mathcal{X}$ be an additive category. The category of additive functors from $\mathcal{X}^{\mathrm{op}}$ to Ab is denoted by $\operatorname{Mod}\mathcal{X}$. The Yoneda embedding gives a fully faithful functor

$$\mathcal{X} \to \operatorname{Mod}\mathcal{X} \qquad X \mapsto \mathcal{X}(-,X)$$

A functor $F\colon \mathcal{X}^{\mathrm{op}} \to \mathrm{Ab}$ is called *finitely presented* if there exists an exact sequence

$$\mathcal{X}(-,X_1) \to \mathcal{X}(-,X_0) \to F \to 0$$

in $\operatorname{Mod}\mathcal{X}$. The subcategory of finitely presented functors is denoted by $\operatorname{mod}\mathcal{X}$.

Let $g\colon X'' \to X'$ and $f\colon X' \to X$ be two composable morphisms in $\mathcal{X}$. We say that $g$ is a *weak kernel* of $f$ if

$$\mathcal{X}(Y,X'') \xrightarrow{g\circ -} \mathcal{X}(Y,X') \xrightarrow{f\circ -} \mathcal{X}(Y,X)$$

is an exact sequence of abelian groups for all $Y \in \mathcal{X}$. Dually, we say that $f$ is a *weak cokernel* of $g$ if

$$\mathcal{X}(X,Y) \xrightarrow{-\circ f} \mathcal{X}(X',Y) \xrightarrow{-\circ g} \mathcal{X}(X'',Y)$$

is an exact sequence of abelian groups for all $Y \in \mathcal{X}$. The category $\mathcal{X}$ has *weak kernels* or *weak cokernels* if any morphism in $\mathcal{X}$ has a weak kernel or weak cokernel, respectively. In the following theorem we relate these notions to $\operatorname{mod}\mathcal{X}$.

**Theorem 3.1** ([9])**.** *Assume* $\mathcal{X}$ *has weak kernels. Then* $\operatorname{mod}\mathcal{X}$ *is an abelian category and the inclusion functor* $\operatorname{mod}\mathcal{X} \to \operatorname{Mod}\mathcal{X}$ *is an exact functor*

In the following we call a morphism $f$ for a weak kernel or weak cokernel if there exists a morphism for which $f$ is a weak kernel or cokernel, respectively.

**Lemma 3.2.** *Let* $f\colon X \to X'$ *be a morphism in* $\mathcal{X}$*. The following hold:*

*(i) If* $f$ *is a weak kernel and admits a weak cokernel, then it is a weak kernel of its weak cokernel;*
*(ii) If* $f$ *is a weak cokernel and admits a weak kernel, then it is a weak cokernel of its weak kernel*

*Proof.* We prove (i), (ii) is proved dually. Assume $f$ is a weak kernel of $g\colon X' \to X''$, and let $h\colon X' \to \tilde{X}$ be the weak cokernel of $f$. Since $g \circ f = 0$, there exists a morphism $k\colon \tilde{X} \to X''$ such that $k \circ h = g$. Hence, if a morphism $X' \to \tilde{X}'$ factors through $g$, then it factors through $h$. Since $h \circ f = 0$, this proves the claim. □

Now assume $\mathcal{X}$ is an additive subcategory of an abelian category $\mathcal{A}$. A morphism $X \xrightarrow{f} A$ with $X \in \mathcal{X}$ and $A \in \mathcal{A}$ is called a *right $\mathcal{X}$-approximation* of $A$ if any map $X' \to A$ with $X' \in \mathcal{X}$ factors through $f$. Dually, a morphism $g\colon A \to X$ with $A \in \mathcal{A}$ and $X \in \mathcal{X}$ is a *left $\mathcal{X}$-approximation* of $A$ if it is a right $\mathcal{X}^{\mathrm{op}}$-approximation of $A$ in $\mathcal{A}^{\mathrm{op}}$. We say that $\mathcal{X}$ is *contravariantly finite* (resp *covariantly finite*) if any object $A$ in $\mathcal{A}$ admits a right (resp left) $\mathcal{X}$-approximation. We say that $\mathcal{X}$ is *functorially finite* if it is both contravariantly finite and covariantly finite. Note that any contravariantly finite category has weak kernels; if $f\colon X' \to X$ is a morphism in $\mathcal{X}$, then the composite $X'' \to \operatorname{Ker} f \to X$ gives a weak kernel of $f$, where $X'' \to \operatorname{Ker} f$ is a right $\mathcal{X}$-approximation. Dually, any covariantly finite subcategory has weak cokernels.

## 4. Embeddings into abelian categories

In this section we compare intrinsic axioms of additive categories with properties of subcategories of abelian categories. For an additive category $\mathcal{X}$, the intrinsic axioms we consider are:

(A1) $\mathcal{X}$ has weak kernels;
(A2) Any epimorphism in $\mathcal{X}$ is a weak cokernel.

For an abelian category $\mathcal{A}$ and a full subcategory $\mathcal{X}$ of $\mathcal{A}$, the properties we consider are:

(B1) $\mathcal{X}$ is a generating subcategory of $\mathcal{A}$;
(B2) If $A \in \mathcal{A}$ satisfies $\mathcal{A}(A, X) = 0$ for all $X \in \mathcal{X}$, then $A = 0$;
(B3) Any $A \in \Omega^2_{\mathcal{X}}(\mathcal{A})$ admits a right $\mathcal{X}$-approximation.

Here $\Omega^i_{\mathcal{X}}(\mathcal{A})$ denotes the subcategory of $\mathcal{A}$ consisting of all objects $A$ for which there exists an exact sequence

$$0 \to A \to X_1 \to \cdots \to X_n$$

where $X_i \in \mathcal{X}$ for all $1 \leq i \leq n$. Our main goal is to prove the following theorem:

**Theorem 4.1.** *Let $\mathcal{X}$ be an additive category. The following hold:*

*(i) Assume $\mathcal{X} \subseteq \mathcal{A}$ is a full subcategory of an abelian category $\mathcal{A}$ satisfying (B1), (B2) and (B3). Then $\mathcal{X}$ satisfies (A1) and (A2) as an additive category;*
*(ii) Assume $\mathcal{X}$ satisfies (A1) and (A2). Then there exists an abelian category $\mathcal{A}$ and a fully faithful functor $\mathcal{X} \to \mathcal{A}$ such that its essential image satisfies (B1), (B2) and (B3);*
*(iii) Let $\mathcal{A}$ and $\mathcal{A}'$ be abelian categories and $\Phi\colon \mathcal{X} \to \mathcal{A}$ and $\Phi'\colon \mathcal{X} \to \mathcal{A}'$ be two fully faithful functors. Assume that the essential images of $\Phi$ and $\Phi'$ satisfy (B1), (B2) and (B3). Then there exists an equivalence $\Psi\colon \mathcal{A} \xrightarrow{\cong} \mathcal{A}'$ and a natural isomorphism $\Psi \circ \Phi \cong \Phi'$.*

*Proof of Theorem 4.1 (i).* Assume $\mathcal{X} \subseteq \mathcal{A}$ satisfy (B1), (B2) and (B3). Let $X_1 \xrightarrow{f} X_0$ be a morphism in $\mathcal{X}$. Then $\operatorname{Ker} f \in \Omega^2_{\mathcal{X}}(A)$, and hence there exists a right $\mathcal{X}$-approximation $X_2 \to \operatorname{Ker} f$. It follows that the composite $X_2 \to \operatorname{Ker} f \to X_1$ is a weak kernel of $f$. This proves (A1). To prove (A2), let $X_3 \xrightarrow{g} X_2$ be an epimorphism in $\mathcal{X}$. Let $\operatorname{Coker} g$ be the cokernel of $g$ in $\mathcal{A}$. Applying $\mathcal{A}(-, X)$ for $X \in \mathcal{X}$ to the right exact sequence

$$X_2 \xrightarrow{g} X_3 \to \operatorname{Coker} g \to 0$$

gives a left exact sequence

$$0 \to \mathcal{A}(\operatorname{Coker} g, X) \to \mathcal{A}(X_3, X) \xrightarrow{-\circ g} \mathcal{A}(X_2, X)$$

If $X \in \mathcal{X}$, then since $g$ is an epimorphism in $\mathcal{X}$ it follows that $- \circ g$ is a monomorphism, and hence $\mathcal{A}(\operatorname{Coker} g, X) = 0$. By (B2) it follows that $\operatorname{Coker} g = 0$, so $g$ is an epimorphism in $\mathcal{A}$. Next choose an epimorphism $X_4 \to \operatorname{Ker} g$ with $X_4 \in \mathcal{X}$, which exists by (B1). Then we have a right exact sequence

$$X_4 \to X_3 \xrightarrow{g} X_2 \to 0$$

in $\mathcal{A}$. Applying $\mathcal{A}(-, X)$ with $X \in \mathcal{X}$ gives a left exact sequence

$$0 \to \mathcal{A}(X_2, X) \xrightarrow{-\circ g} \mathcal{A}(X_3, X) \to \mathcal{A}(X_4, X)$$

Hence $g$ is a weak cokernel of $X_4 \to X_3$ which proves (A2). □

Now assume $\mathcal{X}$ is an additive category satisfying (A1) and (A2). Since $\mathcal{X}$ has weak kernels, the category of finitely presented functors $\operatorname{mod} \mathcal{X}$ is abelian. Let $\operatorname{eff} \mathcal{X}$ denote the subcategory of $\operatorname{mod} \mathcal{X}$ consisting of all functors $F$ for which there exists an exact sequence

$$\mathcal{X}(-, X_1) \xrightarrow{f\circ -} \mathcal{X}(-, X_0) \to F \to 0$$

where $f \colon X_1 \to X_0$ is an epimorphism in $\mathcal{X}$.

**Proposition 4.2.** *Let $\mathcal{X}$ be an additive category satisfying (A1) and (A2). The following hold:*

*(i) If*

$$\mathcal{X}(-, X_1') \xrightarrow{f'\circ -} \mathcal{X}(-, X_0') \to F \to 0$$

*is exact with $F \in \operatorname{eff} \mathcal{X}$, then $f' \colon X_1' \to X_0'$ is an epimorphism;*

*(ii)* $\operatorname{eff} \mathcal{X} = \{F \in \operatorname{mod} \mathcal{X} \mid \operatorname{Hom}_{\mathcal{X}}(F, \mathcal{X}(-, X)) = 0$ *for all* $X \in \mathcal{X}\}$*;*

*(iii) If $F \in \operatorname{eff} \mathcal{X}$ then $\operatorname{Ext}^1_{\mathcal{X}}(F, \mathcal{X}(-, X)) = 0$ for all $X \in \mathcal{X}$;*

*(iv)* $\operatorname{eff} \mathcal{X}$ *is a Serre subcategory of* $\operatorname{mod} \mathcal{X}$*.*

*Proof.* If $F \in \operatorname{eff} \mathcal{X}$, then there exists an exact sequence

$$\mathcal{X}(-, X_1) \xrightarrow{f\circ -} \mathcal{X}(-, X_0) \to F \to 0$$

where $f \colon X_1 \to X_0$ is an epimorphism in $\mathcal{X}$. Applying $\operatorname{Hom}_{\mathcal{X}}(-, \mathcal{X}(-, X))$ gives the exact sequence

$$0 \to \operatorname{Hom}_{\mathcal{X}}(F, \mathcal{X}(-, X)) \to \mathcal{X}(X_0, X) \xrightarrow{-\circ f} \mathcal{X}(X_1, X)$$

Since $f$ is an epimorphism in $\mathcal{X}$, the map $- \circ f$ is a monomorphism, and hence

$$\operatorname{Hom}_{\mathcal{X}}(F, \mathcal{X}(-, X)) = 0.$$

Conversely, assume $\mathrm{Hom}_{\mathcal{X}}(F, \mathcal{X}(-,X)) = 0$ for all $X \in \mathcal{X}$. Applying $\mathrm{Hom}_{\mathcal{X}}(F, \mathcal{X}(-,X))$ to an exact sequence of the form

$$\mathcal{X}(-,X_1') \xrightarrow{f'\circ -} \mathcal{X}(-,X_0') \to F \to 0$$

gives an exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{X}}(F, \mathcal{X}(-,X)) \to \mathcal{X}(X_0, X) \xrightarrow{-\circ f'} \mathcal{X}(X_1, X)$$

Since $\mathrm{Hom}_{\mathcal{X}}(F, \mathcal{X}(-,X)) = 0$, the map $-\circ f'$ is a monomorphism, and hence $f' \colon X_1' \to X_0$ is an epimorphism. This proves (i) and (ii). For (iii), assume again we have an exact sequence $\mathcal{X}(-,X_1) \xrightarrow{f\circ -} \mathcal{X}(-,X_0) \to F \to 0$ where $f \colon X_1 \to X_0$ is an epimorphism in $\mathcal{X}$. Let $g \colon X_2 \to X_1$ be a weak kernel of $f$. Then

$$\mathcal{X}(-,X_2) \xrightarrow{g\circ -} \mathcal{X}(-,X_1) \xrightarrow{f\circ -} \mathcal{X}(-,X_0) \to F \to 0$$

is exact. Applying $\mathrm{Hom}_{\mathcal{X}}(-, \mathcal{X}(-,X))$ to this where $X \in \mathcal{X}$ gives a complex

$$\mathrm{Hom}_{\mathcal{X}}(F, \mathcal{X}(-,X)) \to \mathcal{X}(X_0, X) \xrightarrow{-\circ f} \mathcal{X}(X_1, X) \xrightarrow{-\circ g} \mathcal{X}(X_2, X)$$

Since $f$ is an epimorphism, it is a weak cokernel by (A2). By Lemma 3.2 (ii) we know that $f$ must be a weak cokernel of $g$. Therefore the sequence

$$\mathcal{X}(X_0, X) \xrightarrow{-\circ f} \mathcal{X}(X_1, X) \xrightarrow{-\circ g} \mathcal{X}(X_2, X)$$

must be exact. This shows that $\mathrm{Ext}^1_{\mathrm{mod}\,\mathcal{X}}(F, \mathcal{X}(-,X)) = 0$.

Finally, we show that $\mathcal{S}$ is a Serre subcategory. Let

$$0 \to F_1 \to F_2 \to F_3 \to 0$$

be an exact sequence in $\mathrm{mod}\,\mathcal{X}$. Applying $\mathrm{Hom}_{\mathcal{X}}(-, \mathcal{X}(-,X))$, we get an exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}_{\mathcal{X}}(F_3, \mathcal{X}(-,X)) \to \mathrm{Hom}_{\mathcal{X}}(F_2, \mathcal{X}(-,X)) \to \mathrm{Hom}_{\mathcal{X}}(F_1, \mathcal{X}(-,X)) \\ \to \mathrm{Ext}^1_{\mathrm{mod}\,\mathcal{X}}(F_3, \mathcal{X}(-,X)) \end{aligned}$$

Now if $F_1 \in \mathrm{eff}\,\mathcal{X}$ and $F_3 \in \mathrm{eff}\,\mathcal{X}$, then $\mathrm{Hom}_{\mathcal{X}}(F_1, \mathcal{X}(-,X)) = 0$ and $\mathrm{Hom}_{\mathcal{X}}(F_3, \mathcal{X}(-,X)) = 0$ for all $X \in \mathcal{X}$ by part (ii) of this theorem. Therefore, $\mathrm{Hom}_{\mathcal{X}}(F_2, \mathcal{X}(-,X)) = 0$ for all $X \in \mathcal{X}$ by the exact sequence above. Again by (ii) we get that $F_2 \in \mathrm{eff}\,\mathcal{X}$. Conversely, assume $F_2 \in \mathrm{eff}\,\mathcal{X}$. Since $\mathrm{Hom}_{\mathcal{X}}(F_2, \mathcal{X}(-,X)) = 0$ for all $X \in \mathcal{X}$ by (ii) and $\mathrm{Ext}^1_{\mathcal{X}}(F_2, \mathcal{X}(-,X)$ by (iii), the exact sequence above correspondence to the exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{X}}(F_3, \mathcal{X}(-,X)) \to 0 \to \mathrm{Hom}_{\mathcal{X}}(F_1, \mathcal{X}(-,X)) \to 0$$

Hence $\mathrm{Hom}_{\mathcal{X}}(F_3, \mathcal{X}(-,X)) = 0$ and $\mathrm{Hom}_{\mathcal{X}}(F_1, \mathcal{X}(-,X)) = 0$ for all $X \in \mathcal{X}$. This shows that $F_1 \in \mathcal{X}$ and $F_3 \in \mathcal{X}$. □

Since $\mathrm{eff}\,\mathcal{X}$ is a Serre subcategory of $\mathrm{mod}\,\mathcal{X}$, the localization $\mathcal{A} := \mathrm{mod}\,\mathcal{X}/\,\mathrm{eff}\,\mathcal{X}$ is an abelian category. Furthermore, we have a functor $\Phi$ gives as the composite

$$\mathcal{X} \to \mathrm{mod}\,\mathcal{X} \xrightarrow{q} \mathcal{A}$$

where $\mathcal{X} \to \operatorname{mod} \mathcal{X}$ is the Yoneda functor and $q$ is the canonical functor to the localization. The functor $\Phi$ plays the role of the fully faithful functor in Theorem 4.1 (ii).

**Lemma 4.3.** *Let $\mathcal{X}$ be an additive category satisfying (A1) and (A2). Let $\phi\colon F_0 \to F_1$ be a morphism in $\operatorname{mod} \mathcal{X}$ with $\operatorname{Ker} \phi \in \mathcal{X}$ and $\operatorname{Coker} \phi \in \mathcal{X}$. Then*

$$\operatorname{Hom}_{\mathcal{X}}(F_1, \mathcal{X}(-, X)) \xrightarrow{-\circ\phi} \operatorname{Hom}_{\mathcal{X}}(F_0, \mathcal{X}(-, X))$$

*is an isomorphism for all $X \in \mathcal{X}$.*

*Proof.* Consider the exact sequence

$$0 \to \operatorname{im} \phi \to F_1 \to \operatorname{Coker} \phi \to 0$$

Applying $\operatorname{Hom}_{\mathcal{X}}(-, \mathcal{X}(-, X))$ gives an exact sequence

$$\begin{aligned} 0 \to \operatorname{Hom}_{\mathcal{X}}(\operatorname{Coker} \phi, \mathcal{X}(-, X)) &\to \operatorname{Hom}_{\mathcal{X}}(F_1, \mathcal{X}(-, X)) \\ &\to \operatorname{Hom}_{\mathcal{X}}(\operatorname{im} \phi, \mathcal{X}(-, X)) \to \operatorname{Ext}^1_{\operatorname{mod} \mathcal{X}}(\operatorname{Coker} \phi, \mathcal{X}(-, X)) \end{aligned}$$

Since $\operatorname{Coker} \phi \in \operatorname{eff} \mathcal{X}$, it follows that

$$\operatorname{Hom}_{\mathcal{X}}(\operatorname{Coker} \phi, \mathcal{X}(-, X)) = 0 \quad \text{and} \quad \operatorname{Ext}^1_{\operatorname{mod} \mathcal{X}}(\operatorname{Coker} \phi, \mathcal{X}(-, X)) = 0$$

by Theorem 4.2. Hence the map

$$\operatorname{Hom}_{\mathcal{X}}(F_1, \mathcal{X}(-, X)) \to \operatorname{Hom}_{\mathcal{X}}(\operatorname{im} \phi, \mathcal{X}(-, X))$$

is an isomorphism. Now consider the exact sequene

$$0 \to \operatorname{Ker} \phi \to F_0 \to \operatorname{im} \phi \to 0$$

Again applying $\operatorname{Hom}_{\mathcal{X}}(-, \mathcal{X}(-, X))$ and using that

$$\operatorname{Hom}_{\mathcal{X}}(\operatorname{Ker} \phi, \mathcal{X}(-, X)) = 0 = \operatorname{Ext}^1_{\operatorname{mod} \mathcal{X}}(\operatorname{Ker} \phi, \mathcal{X}(-, X)) = 0$$

by Theorem 4.2, it follows that the map

$$\operatorname{Hom}_{\mathcal{X}}(\operatorname{im} \phi, \mathcal{X}(-, X)) \to \operatorname{Hom}_{\mathcal{X}}(F_0, \mathcal{X}(-, X))$$

is an isomorphism. Since $\operatorname{Hom}_{\mathcal{X}}(F_1, \mathcal{X}(-, X)) \xrightarrow{-\circ\phi} \operatorname{Hom}_{\mathcal{X}}(F_0, \mathcal{X}(-, X))$ is the composite

$$\operatorname{Hom}_{\mathcal{X}}(F_1, \mathcal{X}(-, X)) \xrightarrow{\cong} \operatorname{Hom}_{\mathcal{X}}(\operatorname{im} \phi, \mathcal{X}(-, X)) \xrightarrow{\cong} \operatorname{Hom}_{\mathcal{X}}(F_0, \mathcal{X}(-, X))$$

of two isomorphisms, it must be an isomorphism itself. □

The localization functor $q\colon \operatorname{mod} \mathcal{X} \to \mathcal{A}$ induces a natural transformation

$$q\colon \operatorname{Hom}_{\mathcal{X}}(-, \mathcal{X}(-, X)) \to \mathcal{A}(-, \mathcal{X}(-, X)) \quad f \mapsto q(f)$$

for all $X \in \mathcal{X}$, which we also denote by $q$ by abuse of notation. We use the previous lemma to show that this is an isomorphism

**Lemma 4.4.** *Let $\mathcal{X}$ be an additive category satisfying (A1) and (A2). The natural transformation*

$$q\colon \operatorname{Hom}_{\mathcal{X}}(-, \mathcal{X}(-, X)) \to \mathcal{A}(-, \mathcal{X}(-, X)) \quad f \mapsto q(f)$$

*is an isomorphism.*

*Proof.* The functor

$$\mathrm{Hom}_{\mathcal{X}}(-,\mathcal{X}(-,X))\colon (\mathrm{mod}\,\mathcal{X})^{\mathrm{op}} \to \mathrm{Ab}$$

induces a well-defined functor

$$\mathrm{Hom}_{\mathcal{X}}(-,\mathcal{X}(-,X))\colon \mathcal{A}^{\mathrm{op}} \to \mathrm{Ab}$$

by Lemma 4.3 and the universal property of the localization. Furthermore, the natural transformation $q\colon \mathrm{Hom}_{\mathcal{X}}(-,\mathcal{X}(-,X)) \to \mathcal{A}(-,\mathcal{X}(-,X))$ also becomes a natural transformation of functors $\mathcal{A}^{\mathrm{op}} \to \mathrm{Ab}$. By Yoneda's Lemma, the element $1_{\mathcal{X}(-,X)} \in \mathrm{Hom}_{\mathcal{X}}(\mathcal{X}(-,X),\mathcal{X}(-,X))$ corresponds to a natural transformation

$$\mu\colon \mathcal{A}(-,\mathcal{X}(-,X)) \to \mathrm{Hom}_{\mathcal{X}}(-,\mathcal{X}(-,X))$$

of functors $\mathcal{A}^{\mathrm{op}} \to \mathrm{Ab}$. The composite

$$\mathcal{A}(-,\mathcal{X}(-,X)) \xrightarrow{\mu} \mathrm{Hom}_{\mathcal{X}}(-,\mathcal{X}(-,X)) \xrightarrow{q} \mathcal{A}(-,\mathcal{X}(-,X))$$

sends $1_{\mathcal{X}(-,X)} \in \mathcal{A}(\mathcal{X}(-,X),\mathcal{X}(-,X))$ to itself, and must therefore be the identity map. Furthermore, the composite

$$\mu\colon q\colon\ \mathrm{Hom}_{\mathcal{X}}(-,\mathcal{X}(-,X)) \to \mathrm{Hom}_{\mathcal{X}}(-,\mathcal{X}(-,X))$$

also becomes a natural transformation when we consider $\mathrm{Hom}_{\mathcal{X}}(-,\mathcal{X}(-,X))$ as a functor $(\mathrm{mod}\,\mathcal{X})^{\mathrm{op}} \to \mathrm{Ab}$. Since it also sends $1_{\mathcal{X}(-,X)} \in \mathrm{Hom}_{\mathcal{X}}(\mathcal{X}(-,X),\mathcal{X}(-,X))$ to itself, it must also be the identity. This shows that $q$ is an isomorphism. □

Now we are finally ready to prove Theorem 4.1 (ii). Recall that $\Phi\colon \mathcal{X} \to \mathcal{A}$ is the functor given by the composite of the Yoneda embedding $\mathcal{X} \to \mathrm{mod}\,\mathcal{X}$ and the localization functor $q\colon\ \mathrm{mod}\,\mathcal{X} \to \mathcal{A}$. For simplicity, in the proof we also denote the essential image of $\Phi$ by $\mathcal{X}$.

**Theorem 4.5.** *The functor $\Phi\colon \mathcal{X} \to \mathcal{A}$ is fully faithful, and its essential image satisfies (B1), (B2) and (B3).*

*Proof.* By Lemma 4.4 the canonical map

$$\mathrm{Hom}_{\mathcal{X}}(\mathcal{X}(-,X'),\mathcal{X}(-,X)) \to \mathcal{A}(\mathcal{X}(-,X'),\mathcal{X}(-,X))$$

is an isomorphism. Since the Yoneda embedding is fully faithful, the functor $\Phi$ must therefore be fully faithful. Next, recall that the objects in $\mathcal{A}$ can be identified with the objects in $\mathrm{mod}\,\mathcal{X}$. Let $F \in \mathrm{mod}\,\mathcal{X}$ be arbitrary. Choose an epimorphism $\mathcal{X}(-,X) \to F$ in $\mathrm{mod}\,\mathcal{X}$. Since $q$ is exact, the map also becomes an epimorphism in $\mathcal{A}$. This shows (B1), i.e. that $\mathcal{X}$ is generating. Next, if $\mathcal{A}(F,\mathcal{X}(-,X)) = 0$ for all $X \in \mathcal{X}$, then $\mathrm{Hom}_{\mathcal{X}}(F,\mathcal{X}(-,X)) = 0$ by Lemma 4.4. Therefore $F \in \mathrm{eff}\,\mathcal{X}$ by Lemma 4.2. It follows that $F \cong 0$ considered as an object in $\mathcal{A}$, which shows (B2). Finally, to prove (B3), assume $F \in \Omega^2_{\mathcal{X}}(\mathcal{A})$. Choose an exact sequence

$$0 \to F \xrightarrow{\phi} \mathcal{X}(-,X_1) \xrightarrow{\psi} \mathcal{X}(-,X_0)$$

in $\mathcal{A}$. By Lemma 4.4 it follows that $\phi$ and $\psi$ can be lifted to morphisms in $\mathrm{mod}\,\mathcal{X}$ (which we denote by the same name). In particular, we have a morphism $g\colon X_1 \to X_0$ in $\mathcal{X}$ such that $\psi = g \circ -\colon \mathcal{X}(-, X_1) \to \mathcal{X}(-, X_0)$. Let $f$ be the weak kernel of $g$ in $\mathcal{X}$. Then the sequence of functors

$$\mathcal{X}(-, X_2) \xrightarrow{f\circ -} \mathcal{X}(-, X_1) \xrightarrow{g\circ -} \mathcal{X}(-, X_0)$$

is exact in $\operatorname{mod}\mathcal{X}$. Since $q$ is an exact functor, the sequence is also exact in $\mathcal{A}$. Furthermore, since $F$ is the kernel of $g \circ -$ in $\mathcal{A}$, there exists an epimorphism $\xi\colon \mathcal{X}(-, X_2) \to F$ in $\mathcal{A}$ such that $f \circ - = \phi \circ \xi$. Now let $\kappa\colon \mathcal{X}(-, X) \to F$ be an arbitrary morphism in $\mathcal{A}$ with $X \in \mathcal{X}$. Then the composite $\phi \circ \kappa\colon \mathcal{X}(-, X) \to \mathcal{X}(-, X_1)$ can be written as $h \circ -\colon \mathcal{X}(-, X) \to \mathcal{X}(-, X_1)$ for some morphism $h\colon X \to X_1$ in $\mathcal{X}$, by Lemma 4.4. Since $g \circ h = 0$, and $f$ is a weak kernel of $g$, it follows that $h$ factors through $f$. Therefore the map $\phi \circ \kappa$ factors through $f \circ -\colon \mathcal{X}(-, X_2) \to \mathcal{X}(-, X_1)$. Since $\phi$ is a monomorphism, it follows that $\kappa$ factors through $\xi$. Hence $\xi$ is a right $\mathcal{X}$ approximation, which proves the claim. $\square$

Now we are ready to prove Theorem 4.1 (iii). We show that the category satisfies a universal property.

**Proposition 4.6.** *Let $\mathcal{A}'$ be an abelian category and $\mathcal{X}$ an additive subcategory which satisfies (B1), (B2) and (B3). The following holds:*

*(i) There exists an equivalence $\operatorname{mod}\mathcal{X}/\operatorname{eff}\mathcal{X} \xrightarrow{\cong} \mathcal{A}'$ unique up to natural isomorphism which makes the diagram*

$$\begin{tikzcd} \operatorname{mod}\mathcal{X}/\operatorname{eff}\mathcal{X} \arrow[rr, "\cong"] & & \mathcal{A}' \\ & \mathcal{X} \arrow[ul, "\Phi"] \arrow[ur] & \end{tikzcd}$$

*commute.*

*(ii) Let $\mathcal{B}$ be an abelian category and let $\Psi\colon \mathcal{X} \to \mathcal{B}$ be an additive functor which preserves epimorphisms and sends a sequence $X_2 \xrightarrow{f} X_1 \xrightarrow{g} X_0$ where $f$ is a weak kernel of $g$ to an exact sequence*

$$\Psi(X_2) \xrightarrow{\Psi(f)} \Psi(X_1) \xrightarrow{\Psi(g)} \Psi(X_0).$$

*in $\mathcal{B}$. Then there exists an exact functor $\mathcal{A}' \to \mathcal{B}$ extending $\Psi$, which is unique up to natural isomorphism.*

*Proof.* The uniqueness statements are obvious since $\mathcal{X}$ is generating. We first prove part (ii) for $\mathcal{A}' = \operatorname{mod}\mathcal{X}/\operatorname{eff}\mathcal{X}$ and where $\mathcal{X}$ is considered as a subcategory via the functor $\Phi$ in Theorem 4.5. Let $\Psi$ be a functor as in (ii). Then there exists a right exact functor $\tilde{\Psi}\colon \operatorname{mod}\mathcal{X} \to \mathcal{B}$ extending $\Psi$, see Property 2.1 in [24]. Since $\Psi$ sends weak kernels to exact sequences it follows that $\tilde{\Psi}$ is an exact functor by Lemma 2.5 in [24]. Now let $F \in \operatorname{mod}\mathcal{X}$ be arbitrary, and choose an exact sequence $\mathcal{X}(-, X_1) \xrightarrow{f\circ -} \mathcal{X}(-, X_0) \to F \to 0$. If $F \in \operatorname{eff}\mathcal{X}$, then it follow from Proposition 4.2 (i) that $f\colon X_1 \to X_0$ is an epimorphism in $\mathcal{X}$. Therefore $\Psi(f)$ is an epimorphism in $\mathcal{B}$ by assumption, which implies that $\tilde{\Psi}(F) = 0$. This shows that $\operatorname{eff}\mathcal{X} \subseteq \operatorname{Ker}\tilde{\Psi}$. Then, by [10, Chapitre III, Corollaire 2 and 3] there exists an exact functor

$$\overline{\Psi}\colon \operatorname{mod}\mathcal{X}/\operatorname{eff}\mathcal{X} \to \mathcal{B}$$

satisfying $q \circ \overline{\Psi} = \tilde{\Psi}$. This shows (ii) for $\mathcal{A}' = \operatorname{mod}\mathcal{X}/\operatorname{eff}\mathcal{X}$. It now suffices to prove part (i), as this will imply part (ii) for any $\mathcal{A}'$. To this end, assume $\mathcal{X}$ is a subcategory of an abelian category $\mathcal{A}'$ satisfying (B1), (B2) and (B3). As above, we get exact functors $\tilde{\Psi}\colon \operatorname{mod}\mathcal{X} \to \mathcal{A}'$ and $\overline{\Psi}\colon \operatorname{mod}\mathcal{X}/\operatorname{eff}\mathcal{X} \to \mathcal{A}'$ extending the inclusion $\mathcal{X} \subseteq \mathcal{A}'$. We show that $\overline{\Psi}$ is an equivalence. Let $F \in \operatorname{mod}\mathcal{X}$ be arbitrary and let $\mathcal{X}(-,X_1) \xrightarrow{f\circ-} \mathcal{X}(-,X_0) \to F \to 0$ be an exact sequence in $\operatorname{mod}\mathcal{X}$. Applying $\tilde{\Psi}$ to this, we get an exact sequence

$$X_1 \xrightarrow{f} X_0 \to \tilde{\Psi}(F) \to 0$$

in $\mathcal{A}'$. Hence, if $\tilde{\Psi}(F) \cong 0$, then $X_1 \xrightarrow{f} X_0$ is surjective in $\mathcal{A}$. Therefore, $f$ must be surjective in $\mathcal{X}$, so $F \in \operatorname{eff}\mathcal{X}$. This shows that $\operatorname{Ker}\tilde{\Psi} = \operatorname{eff}\mathcal{X}$. In other words, for the functor $\overline{\Psi}\colon \operatorname{mod}\mathcal{X}/\operatorname{eff}\mathcal{X} \to \mathcal{A}'$ we have that $\overline{\Psi}(F) \cong 0$ if and only if $F \cong 0$ in $\operatorname{mod}\mathcal{X}/\operatorname{eff}\mathcal{X}$. Since $\overline{\Psi}$ is also exact, it follows that $\overline{\Psi}$ must be faithful. Also, since $\mathcal{X} \subseteq \mathcal{A}'$ is generating, it follows immediately that $\overline{\Psi}$ is dense. It therefore only remains to show that $\overline{\Psi}$ is full. Let $F, F' \in \operatorname{mod}\mathcal{X}$, and let

$$\mathcal{X}(-,X_1) \xrightarrow{f\circ-} \mathcal{X}(-,X_0) \xrightarrow{\pi} F \to 0 \quad \text{and} \quad \mathcal{X}(-,X_1') \xrightarrow{f'\circ-} \mathcal{X}(-,X_0') \xrightarrow{\pi'} F' \to 0$$

be exact in $\operatorname{mod}\mathcal{X}$. Let $\phi\colon \overline{\Psi}(F) \to \overline{\Psi}(F')$ be a morphism in $\mathcal{A}'$. We get a commutative diagram

$$\begin{tikzcd}
X_1 \arrow[r, "f"] \arrow[d] & X_0 \arrow[r, "\overline{\Psi}(\pi)"] \arrow[d, "\begin{bmatrix}1\\0\end{bmatrix}"] & \overline{\Psi}(F) \arrow[r] \arrow[d, "\phi"] & 0 \\
K \arrow[r] & X_0 \oplus X_0' \arrow[r, "\begin{bmatrix}\phi\circ\overline{\Psi}(\pi) & \overline{\Psi}(\pi')\end{bmatrix}"] & \overline{\Psi}(F') \arrow[r] & 0 \\
X_1' \arrow[r] \arrow[u] & X_0' \arrow[r, "\overline{\Psi}(\pi')"] \arrow[u, "\begin{bmatrix}0\\1\end{bmatrix}"'] & \overline{\Psi}(F') \arrow[r] \arrow[u, "1"'] & 0
\end{tikzcd}$$

where the rows are exact and $K = \operatorname{Ker}\begin{bmatrix}\phi\circ\overline{\Psi}(\pi) & \overline{\Psi}(\pi')\end{bmatrix}$. Choose an epimorphism $X_1'' \to K$ such that $X_1 \oplus X_1' \to K$ factor through $X_1''$, and let $\begin{bmatrix}g\\h\end{bmatrix}\colon X_1'' \to X_0 \oplus X_0'$ denote the map obtained by composing with the inclusion $K \to X_0 \oplus X_0'$. Let

$$F'' = \operatorname{Coker}(\mathcal{X}(-,X_1'') \xrightarrow{\begin{bmatrix}g\\h\end{bmatrix}\circ-} \mathcal{X}(-,X_0 \oplus X_0'))$$

and let $\pi''\colon \mathcal{X}(-, X_0 \oplus X_0') \to F''$ denote the projection. Then we get morphisms $\phi'\colon F \to F''$ and $\phi''\colon F' \to F''$ and a commutative diagram

$$\begin{tikzcd}
\mathcal{X}(-,X_1) \arrow[r, "f\circ -"] \arrow[d] & \mathcal{X}(-,X_0) \arrow[r, "\pi"] \arrow[d, "\begin{bmatrix}1\\0\end{bmatrix}\circ -"] & F \arrow[r] \arrow[d, "\phi'"] & 0 \\
\mathcal{X}(-,X_1'') \arrow[r, "\begin{bmatrix}g\\h\end{bmatrix}\circ -"] & \mathcal{X}(-,X_0\oplus X_0') \arrow[r, "\pi''"] & F'' \arrow[r] & 0 \\
\mathcal{X}(-,X_1') \arrow[r, "f'\circ -"] \arrow[u] & \mathcal{X}(-,X_0') \arrow[r, "\pi'"] \arrow[u, "\begin{bmatrix}0\\1\end{bmatrix}\circ -"'] & F' \arrow[r] \arrow[u, "\phi''"'] & 0
\end{tikzcd}$$

Note furthermore that by definition $\tilde{\Psi}(\phi'')$ is an isomorphism, hence

$$\tilde{\Psi}(\operatorname{Ker}\phi'') \cong 0 \cong \tilde{\Psi}(\operatorname{Coker}\phi'')$$

since $\tilde{\Psi}$ is exact. This implies that $\operatorname{Ker}\phi'' \in \mathcal{X}$ and $\operatorname{Coker}\phi'' \in \mathcal{X}$. Therefore $\phi''$ is an isomorphism in $\operatorname{mod}\mathcal{X}/\operatorname{eff}\mathcal{X}$, and therefore admits an inverse $\phi''^{-1}$. It is now straightforward to see that

$$\phi = \overline{\Psi}(\phi''^{-1}) \circ \overline{\Psi}(\phi') = \overline{\Psi}(\phi''^{-1} \circ \phi')$$

which proves the proposition. □

## 5. Functorially finite generating cogenerating subcategories

Let $\mathcal{X}$ be an additive category. We would like to find intrinsic axioms on $\mathcal{X}$ which ensures that it can be realized as a functorially finite generating and cogenerating subcategory of an abelian category, similar to Theorem 4.1. To this end, by Theorem 4.1 (i) we know that $\mathcal{X}$ must satisfy (A1) and (A2) and their duals

(A1)$^{\mathrm{op}}$ $\mathcal{X}$ has weak cokernels;
(A2)$^{\mathrm{op}}$ Any monomorphism in $\mathcal{X}$ is a weak kernel.

For each $F \in \operatorname{mod}\mathcal{X}$ choose a projective presentation

$$\mathcal{X}(-,X_1) \xrightarrow{f\circ -} \mathcal{X}(-,X_0) \to F \to 0$$

in $\operatorname{mod}\mathcal{X}$. Applying the contravariant Yoneda functor to $f\colon X_1 \to X_0$ gives a map $-\circ f\colon \mathcal{X}(X_0,-) \to \mathcal{X}(X_1,-)$ in $\operatorname{mod}\mathcal{X}^{\mathrm{op}}$. We define $F^* = \operatorname{Ker}(-\circ f)$ and $\operatorname{Tr} F = \operatorname{Coker}(-\circ f)$, so that we have an exact sequence

$$0 \to F^* \to \mathcal{X}(X_0,-) \xrightarrow{-\circ f} \mathcal{X}(X_1,-) \to \operatorname{Tr} F \to 0.$$

in $\operatorname{mod}\mathcal{X}$. Dually, for $F' \in \operatorname{mod}\mathcal{X}^{\mathrm{op}}$ we choose a projective presentation

$$\mathcal{X}(X_0',-) \xrightarrow{-\circ f'} \mathcal{X}(X_1',-) \to F' \to 0$$

and define $\operatorname{Tr} F'$ and $F'^*$ by the exact sequence

$$0 \to F'^* \to \mathcal{X}(-,X_1') \xrightarrow{f'\circ -} \mathcal{X}(-,X_0') \to \operatorname{Tr} F' \to 0.$$

Note that we can identify

$$(-)^* = \operatorname{Hom}_{\mathcal{X}}(-, \mathcal{X})\colon\ \operatorname{mod}\mathcal{X} \to \operatorname{mod}\mathcal{X}^{\mathrm{op}} \qquad F \mapsto F^*.$$

and

$$(-)^* = \operatorname{Hom}_{\mathcal{X}^{\mathrm{op}}}(-, \mathcal{X}^{\mathrm{op}})\colon\ \operatorname{mod}\mathcal{X}^{\mathrm{op}} \to \operatorname{mod}\mathcal{X} \qquad F' \mapsto F'^*.$$

as contravariant functors, which implies that they form an adjoint pair. It is well know that the unit and counit are part of exact sequences (see Proposition 6.3 in [1] in the case $\mathcal{X}$ is a ring)

$$0 \to \operatorname{Ext}^2_{\operatorname{mod}\mathcal{X}^{\mathrm{op}}}(\operatorname{Tr} F, \mathcal{X}^{\mathrm{op}}) \to F \to F^{**} \to \operatorname{Ext}^1_{\operatorname{mod}\mathcal{X}^{\mathrm{op}}}(\operatorname{Tr} F, \mathcal{X}^{\mathrm{op}}) \to 0 \tag{5.1}$$

$$0 \to \operatorname{Ext}^2_{\operatorname{mod}\mathcal{X}}(\operatorname{Tr} F, \mathcal{X}) \to F \to F^{**} \to \operatorname{Ext}^1_{\operatorname{mod}\mathcal{X}}(\operatorname{Tr} F, \mathcal{X}) \to 0 \tag{5.2}$$

Now by Lemma 4.3 the functors $(-)^*$ induces contravariant functors

$$\operatorname{mod}\mathcal{X}/\operatorname{eff}\mathcal{X} \to \operatorname{mod}\mathcal{X}^{\mathrm{op}} \quad \text{and} \quad \operatorname{mod}\mathcal{X}^{\mathrm{op}}/\operatorname{eff}\mathcal{X}^{\mathrm{op}} \to \operatorname{mod}\mathcal{X}$$

such that the following diagrams commutes

$$\begin{tikzcd} \operatorname{mod}\mathcal{X} \arrow[r, "(-)^*"] \arrow[d, "q"] & \operatorname{mod}\mathcal{X}^{\mathrm{op}} \\ \operatorname{mod}\mathcal{X}/\operatorname{eff}\mathcal{X} \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} \operatorname{mod}\mathcal{X}^{\mathrm{op}} \arrow[r, "(-)^*"] \arrow[d, "q"] & \operatorname{mod}\mathcal{X} \\ \operatorname{mod}\mathcal{X}^{\mathrm{op}}/\operatorname{eff}\mathcal{X}^{\mathrm{op}} \arrow[ur] & \end{tikzcd}$$

Composing with the localization functors

$$\operatorname{mod}\mathcal{X} \to \operatorname{mod}\mathcal{X}/\operatorname{eff}\mathcal{X} \quad \text{and} \quad \operatorname{mod}\mathcal{X}^{\mathrm{op}} \to \operatorname{mod}\mathcal{X}^{\mathrm{op}}/\operatorname{eff}\mathcal{X}^{\mathrm{op}}$$

we get contravariant functors

$$(-)^*\colon\ \operatorname{mod}\mathcal{X}/\operatorname{eff}\mathcal{X} \to \operatorname{mod}\mathcal{X}^{\mathrm{op}}/\operatorname{eff}\mathcal{X}^{\mathrm{op}}$$
$$(-)^*\colon\ \operatorname{mod}\mathcal{X}^{\mathrm{op}}/\operatorname{eff}\mathcal{X}^{\mathrm{op}} \to \operatorname{mod}\mathcal{X}/\operatorname{eff}\mathcal{X}$$

denoted similarly. The natural transformations $\operatorname{Id} \to (-)^{**}$ satisfy the triangular identities in $\operatorname{mod}\mathcal{X}$ and $\operatorname{mod}\mathcal{X}^{\mathrm{op}}$, and hence also in $\operatorname{mod}\mathcal{X}/\operatorname{eff}\mathcal{X}$ and $\operatorname{mod}\mathcal{X}^{\mathrm{op}}/\operatorname{eff}\mathcal{X}^{\mathrm{op}}$. This implies that the functors $(-)^*$ still form an adjoint pair between $\operatorname{mod}\mathcal{X}/\operatorname{eff}\mathcal{X}$ and $\operatorname{mod}\mathcal{X}^{\mathrm{op}}/\operatorname{eff}\mathcal{X}^{\mathrm{op}}$.

We want to find conditions on $\mathcal{X}$ for which $(-)^*$ induces an equivalence

$$\operatorname{mod}\mathcal{X}/\operatorname{eff}\mathcal{X} \xrightarrow{\cong} \operatorname{mod}\mathcal{X}^{\mathrm{op}}/\operatorname{eff}\mathcal{X}^{\mathrm{op}}.$$

Note first that for $F \in \operatorname{mod}\mathcal{X}$ we have an isomorphism

$$\operatorname{Ext}^2_{\operatorname{mod}\mathcal{X}}(F, G) \cong \operatorname{Ext}^1_{\operatorname{mod}\mathcal{X}}(F', G)$$

where $F'$ is a syzygy of $F$, i.e. fits in an exact sequence

$$0 \to F' \to \mathcal{X}(-, X) \to F \to 0$$

Hence, $(-)^*$ induce equivalences on the localizations if

$$\operatorname{Ext}^1_{\operatorname{mod}\mathcal{X}^{\mathrm{op}}}(F,\mathcal{X}^{\mathrm{op}}) \in \operatorname{eff}\mathcal{X} \quad \text{for all } F \in \operatorname{mod}\mathcal{X}^{\mathrm{op}}$$
$$\operatorname{Ext}^1_{\operatorname{mod}\mathcal{X}}(F,\mathcal{X}) \in \operatorname{eff}\mathcal{X}^{\mathrm{op}} \quad \text{for all } F \in \operatorname{mod}\mathcal{X}$$

by the exact sequences (5.1) and (5.2). To capture these properties we introduce the following axiom:

(A3) Consider the following diagram

$$\begin{tikzcd} X_2 \arrow[rr, "f"] \arrow[dr, "l"'] & & X_1 \arrow[r, "g"] & X_0 \\ & X_2' \arrow[ur, "h"'] & & \end{tikzcd}$$

where $f$ is an arbitrary morphism in $\mathcal{X}$, where $g$ is a weak cokernel of $f$, where $h$ is a weak kernel of $g$, and where $l$ is an induced map satisfying $h \circ l = f$ (which exists since $g \circ f = 0$ and $h$ is a weak kernel of $g$). Then for any weak kernel $k\colon X_3' \to X_2'$ of $h$ the map $\begin{bmatrix} l & k \end{bmatrix} : X_2 \oplus X_3' \to X_2'$ is an epimorphism.

**Proposition 5.3.** *Assume $\mathcal{X}$ is an additive category satsifying (A1), (A1)$^{\mathrm{op}}$, (A2), (A2)$^{\mathrm{op}}$. Then the following statements hold:*

*(i) $\mathcal{X}$ satisfy (A3) if and only if $\operatorname{Ext}^1_{\operatorname{mod}\mathcal{X}^{\mathrm{op}}}(F,\mathcal{X}) \in \operatorname{eff}\mathcal{X}$ for all $F \in \operatorname{mod}\mathcal{X}^{\mathrm{op}}$;*
*(ii) $\mathcal{X}$ satisfy (A3)$^{\mathrm{op}}$ if and only if $\operatorname{Ext}^1_{\operatorname{mod}\mathcal{X}}(F,\mathcal{X}) \in \operatorname{eff}\mathcal{X}^{\mathrm{op}}$ for all $F \in \operatorname{mod}\mathcal{X}$;*

*Proof.* We prove (i), (ii) is proved dually. Let $f\colon X_2 \to X_1$ be arbitrary, and choose $g$, $h$, $k$, $l$ as in (A3). Let $F$ be the cokernel of $-\circ f\colon \mathcal{X}(X_1,-) \to \mathcal{X}(X_2,-)$. Applying $(-)^*$ to the exact sequence

$$\mathcal{X}(X_0,-) \xrightarrow{-\circ g} \mathcal{X}(X_1,-) \xrightarrow{-\circ f} \mathcal{X}(X_2,-) \to F \to 0$$

we get a complex

$$F^* \to \mathcal{X}(-,X_2) \xrightarrow{f\circ} \mathcal{X}(-,X_1) \xrightarrow{g\circ-} \mathcal{X}(-,X_0) \tag{5.4}$$

Let $K$ the kernel of $g \circ -$. Since $g \circ h = 0$, it follows that the map $h \circ -\colon \mathcal{X}(-,X_2') \to \mathcal{X}(-,X_1)$ factors through $K$ via a morphism $p\colon \mathcal{X}(-,X_2') \to K$. Since $h$ is a weak kernel of $g$, it follows that any map $\mathcal{X}(-,X) \to K$ must factor through $p$. Hence $p$ must be an epimorphism. Similarly, if we let $K' = \operatorname{Ker}\mathcal{X}(-,X_2') \xrightarrow{h\circ-} \mathcal{X}(-,X_1)$, then since $k\colon X_3' \to X_2'$ is a weak kernel of $h$, it follows that $\mathcal{X}(-,X_3') \xrightarrow{k\circ-} \mathcal{X}(-,X_2')$ factors through $K'$ via an epimorphism $\mathcal{X}(-,X_3) \xrightarrow{q} K'$. Hence, we get a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{X}(-,X_3') \arrow[r, "\begin{bmatrix} 0 \\ 1 \end{bmatrix}"] \arrow[d, "q"] & \mathcal{X}(-,X_2)\oplus\mathcal{X}(-,X_3') \arrow[r, "\begin{bmatrix} 1 & 0 \end{bmatrix}"] \arrow[d, "r"] & \mathcal{X}(-,X_2) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & K' \arrow[r] & \mathcal{X}(-,X_2') \arrow[r, "p"] & K \arrow[r] & 0 \end{tikzcd}$$

with exact rows, where $r = \begin{bmatrix} l\circ - & k\circ - \end{bmatrix}$. Note that the cokernel of $\mathcal{X}(-,X_2)\to K$ is $\mathrm{Ext}^1_{\mathrm{mod}\,\mathcal{X}^{\mathrm{op}}}(F,\mathcal{X})$. By the snake lemma, it follows that the cokernel of

$$r\colon \mathcal{X}(-,X_2)\oplus\mathcal{X}(-,X_3')\to\mathcal{X}(-,X_2')$$

is also $\mathrm{Ext}^1_{\mathrm{mod}\,\mathcal{X}^{\mathrm{op}}}(F,\mathcal{X})$. Hence, $\mathrm{Ext}^1_{\mathrm{mod}\,\mathcal{X}^{\mathrm{op}}}(F,\mathcal{X})\in\operatorname{eff}\mathcal{X}$ if and only if $r$ is an epimorphism in $\operatorname{mod}\mathcal{X}/\operatorname{eff}\mathcal{X}$. Using that $\mathcal{X}$ satisfy property (B2) as a subcategory of $\operatorname{mod}\mathcal{X}/\operatorname{eff}\mathcal{X}$ by Theorem 4.5, it follows that $r$ is an epimorphism in $\operatorname{mod}\mathcal{X}/\operatorname{eff}\mathcal{X}$ if and only if $\begin{bmatrix} l & k\end{bmatrix}\colon X_2\oplus X_3'\to X_2'$ is an epimorphism in $\mathcal{X}$. This proves the claim. □

We finish by giving a characterization of generating cogenerating functorially finite subcategories in terms of intrinsic axioms.

**Theorem 5.5.** *Let $\mathcal{X}$ be an additive category. The following hold:*

*(i) Assume $\mathcal{X}$ satisfies (A1), (A1)$^{\mathrm{op}}$, (A2), (A2)$^{\mathrm{op}}$, (A3) and (A3)$^{\mathrm{op}}$. Then there exists an abelian category $\mathcal{A}$ and a fully faithful functor $\mathcal{X}\to\mathcal{A}$ such that its essential image is a generating cogenerating functorially finite subcategory of $\mathcal{A}$;*

*(ii) Assume $\mathcal{X}\subseteq\mathcal{A}$ is a generating cogenerating functorially finite subcategory of an abelian category $\mathcal{A}$. Then $\mathcal{X}$ satisfies (A1) (A1)$^{\mathrm{op}}$, (A2), (A2)$^{\mathrm{op}}$,(A3) and (A3)$^{\mathrm{op}}$.*

*Proof.* If $\mathcal{X}$ satisfies (A1), (A1)$^{\mathrm{op}}$, (A2), (A2)$^{\mathrm{op}}$, (A3) and (A3)$^{\mathrm{op}}$, then by Proposition 5.3 and the exact sequences (5.1) and (5.2) it follows that $(-)^*$ induces an equivalence

$$\operatorname{mod}\mathcal{X}/\operatorname{eff}\mathcal{X}\cong(\operatorname{mod}\mathcal{X}^{\mathrm{op}}/\operatorname{eff}\mathcal{X}^{\mathrm{op}})^{\mathrm{op}}$$

which commutes with the natural inclusions

$$\mathcal{X}\to\operatorname{mod}\mathcal{X}/\operatorname{eff}\mathcal{X}\quad\text{and}\quad\mathcal{X}\to(\operatorname{mod}\mathcal{X}^{\mathrm{op}}/\operatorname{eff}\mathcal{X}^{\mathrm{op}})^{\mathrm{op}}.$$

By Theorem 4.5 and its dual it follows that $\mathcal{X}$ is a generating cogenerating subcategory of $\operatorname{mod}\mathcal{X}/\operatorname{eff}\mathcal{X}$. Therefore $\Omega^2_{\mathcal{X}}(\operatorname{mod}\mathcal{X}/\operatorname{eff}\mathcal{X})=\operatorname{mod}\mathcal{X}/\operatorname{eff}\mathcal{X}$, and since $\mathcal{X}$ satisfy (B3) it follows that $\mathcal{X}$ is contravariantly finite in $\operatorname{mod}\mathcal{X}/\operatorname{eff}\mathcal{X}$. Dually, one can show that $\mathcal{X}$ is covariantly finite in $\operatorname{mod}\mathcal{X}/\operatorname{eff}\mathcal{X}$. This shows (i).

It follows by Theorem 4.1 (i) that to prove part (ii) we only need to show (A3). Note first that a morphism $g\colon X_1\to X_0$ in $\mathcal{X}$ is a weak cokernel of $f\colon X_2\to X_1$ if and only if the induced map $\operatorname{Coker} f\to X_0$ is a right $\mathcal{X}$-approximation (and therefore also a monomorphism). The dual statement holds for weak kernels. Now assume we are given $f$,$g$,$h$,$k$ as in (A3). Since $\operatorname{Ker} g\cong\operatorname{Ker}(X_1\to\operatorname{Coker} f)$ by the discussion above, it follows that $f$ factors through $\operatorname{Ker} g$ via an epimorphism $p\colon X_2\to\operatorname{Ker} g$. Also, since $h$ is a weak kernel of $g$, it follows that $h$ factors through $\operatorname{Ker} g$ via an epimorphism $q\colon X_2'\to\operatorname{Ker} g$. Similarly, $k$ factors through $\operatorname{Ker} h$ via an epimorphism $p'\colon X_3'\to\operatorname{Ker} h$. Hence we get a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & X_3' \arrow[r, "\begin{bmatrix}0\\1\end{bmatrix}"] \arrow[d, "p'"] & X_2\oplus X_3' \arrow[r, "\begin{bmatrix}1 & 0\end{bmatrix}"] \arrow[d, "\begin{bmatrix}l & k\end{bmatrix}"] & X_2 \arrow[r] \arrow[d, "p"] & 0 \\
0 \arrow[r] & \operatorname{Ker} h \arrow[r] & X_2' \arrow[r, "q"] & \operatorname{Ker} g \arrow[r] & 0
\end{tikzcd}.$$

Since the leftmost and rightmost vertical maps are epimorphism, the middle map must be an epimorphisms. This shows that $\mathcal{X}$ satisfy (A3). □

## 6. Rigid subcategories

In this section we assume $\mathcal{X}$ is a generating cogenerating functorially finite subcategory of an abelian category $\mathcal{A}$. By Theorem 5.5 we know that this is equivalent to the intrinsic axioms (A1), (A1)$^{\mathrm{op}}$, (A2), (A2)$^{\mathrm{op}}$, (A3), (A3)$^{\mathrm{op}}$. Now we want to determine the intrinsic axiom which is equivalent to the property

$$\operatorname{Ext}^i_{\mathcal{A}}(X,X')=0 \text{ for } 0<i<d \text{ and } X,X'\in\mathcal{X}. \tag{6.1}$$

The axiom we consider is the following:

($d$-Rigid) For all epimorphism $f_1\colon X_1\to X_0$ in $\mathcal{X}$ there exists a sequence

$$X_{d+1}\xrightarrow{f_{d+1}} X_d\xrightarrow{f_d}\cdots\xrightarrow{f_3} X_2\xrightarrow{f_2} X_1\xrightarrow{f_1} X_0$$

with $f_{i+1}$ a weak kernel of $f_i$ and $f_i$ a weak cokernel of $f_{i+1}$ for all $1\le i\le d$.

The following theorem relates these two notions:

**Theorem 6.2.** *Let $\mathcal{X}$ be a generating cogenerating functorially finite subcategory of an abelian category $\mathcal{A}$. Then $\mathcal{X}$ satisfy (d-Rigid) if and only if*

$$\operatorname{Ext}^i_{\mathcal{A}}(X,X')=0$$

*for all $0<i<d$ and all $X,X'\in\mathcal{X}$.*

**Remark 6.3.** By Lemma 3.2 it follows that ($d$-Rigid) with $d=1$ is equivalent to axiom (A2) (under the assumptions of existence of weak kernels and cokernels). Hence, it holds automatically for a generating cogenerating functorially finite subcategory by Theorem 4.1 (i). This is reflected by the fact that condition (6.1) is empty for $d=1$.

*Proof of "if" part of Theorem 6.2.* Assume $f_1\colon X_1\to X_0$ is an epimorphism in $\mathcal{X}$. Since $\mathcal{X}$ is cogenerating, it follows that $f_1$ is an epimorphism in $\mathcal{A}$. Choose a right $\mathcal{X}$ approximation $X_2\to\operatorname{Ker} f$, which must be an epimorphism since $\mathcal{X}$ is generating. Let $f_2$ denote the composite $X_2\to\operatorname{Ker} f\to X_1$. We continue this construction iteratively for $1\le i\le d$, i.e. we choose a right $\mathcal{X}$ approximation $X_{i+1}\to\operatorname{Ker} f_i$ and we let $f_{i+1}$ denote the composite $X_{i+1}\to\operatorname{Ker} f_i\to X_i$. Then we get an exact sequence

$$X_{d+1}\xrightarrow{f_{d+1}} X_d\xrightarrow{f_d}\cdots\xrightarrow{f_3} X_2\xrightarrow{f_2} X_1\xrightarrow{f_1} X_0\to 0$$

in $\mathcal{A}$ where $f_{i+1}$ is a weak kernel of $f_i$ for all $1\le i\le d$. Furthermore, since $\operatorname{Ext}^i_{\mathcal{A}}(X,X')=0$ for all $0<i<d$ and all $X,X'\in\mathcal{X}$, it follows that that applying $\mathcal{A}(-,X')$ with $X'\in\mathcal{X}$ to the exact sequence gives an exact sequences

$$\mathcal{A}(X_0,X')\xrightarrow{-\circ f_1}\mathcal{A}(X_1,X')\xrightarrow{-\circ f_2}\cdots\xrightarrow{-\circ f_d}\mathcal{A}(X_d,X')\xrightarrow{-\circ f_{d+1}}\mathcal{A}(X_{d+1},X')$$

In particular, since the sequences $\mathcal{A}(X_{i-1},X')\xrightarrow{-\circ f_i}\mathcal{A}(X_i,X')\xrightarrow{-\circ f_{i+1}}\mathcal{A}(X_{i+1},X')$ are exact for all $1\le i\le d$ and all $X'\in\mathcal{X}$, it follows that $f_{i+1}$ is a weak cokernel of $f_i$ for all $1\le i\le d$. This proves the claim. □

The goal for the remaining part of the section is prove the converse.

**Lemma 6.4.** *Let $\mathcal{X}$ be a generating cogenerating functorially finite subcategory of an abelian category $\mathcal{A}$. Assume $\mathcal{X}$ satisfy (d-Rigid) for some $d > 0$. Then $\operatorname{Ext}^1_{\mathcal{A}}(X, X') = 0$ for all $X, X' \in \mathcal{X}$.*

*Proof.* Let $0 \to X' \xrightarrow{f} F \xrightarrow{f} X \to 0$ be an exact sequence in $\mathcal{A}$ with $X, X' \in \mathcal{X}$. Choose a right $\mathcal{X}$-approximation $p\colon X_1 \to F$. Then $g$ factors through $p$ via a monomorphism $i\colon X' \to X_1$. We therefore get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & X' \arrow[r, "i"] \arrow[d, "1"] & X_1 \arrow[r] \arrow[d, "p"] & \operatorname{Coker} i \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X' \arrow[r, "g"] & F \arrow[r, "f"] & X \arrow[r] & 0
\end{tikzcd}.$$

with exact rows. This gives an exact sequence

$$0 \to X' \oplus \operatorname{Ker} p \xrightarrow{\begin{bmatrix} i & j \end{bmatrix}} X_1 \xrightarrow{f \circ p} X \to 0$$

By ($d$-Rigid), there exists a sequence

$$X_3 \xrightarrow{l} X_2 \xrightarrow{k} X_1 \xrightarrow{f \circ p} X \to 0$$

where $l$ is a weak kernel of $k$ and $k$ is a weak kernel of $f \circ p$, and where $k$ is weak cokernel of $l$ and $f \circ p$ is a weak cokernel of $k$, and where $X_3, X_2 \in \mathcal{X}$. Then $\operatorname{im} l = \operatorname{Ker} k$ and $\operatorname{Coker} l = \operatorname{im} k = \operatorname{Ker}(f \circ p) = X' \oplus \operatorname{Ker} p$. Since $k$ is a weak cokernel it follows that the induced map

$$X' \oplus \operatorname{Ker} p \cong \operatorname{im} k \to X_1$$

is a left $\mathcal{X}$-approximation. Hence $X' \oplus \operatorname{Ker} p \xrightarrow{\begin{bmatrix} 1 & 0 \end{bmatrix}} X'$ factors through $X_1$. This means that there exists a map $s\colon X_1 \to X'$ such that $s \circ i = 1_{X'}$ and $s \circ j = 0$. Since $F \cong \operatorname{Coker} j$, we get an induced map $t\colon F \to X'$ satisfying $t \circ g = 1_{X'}$. This proves the claim. □

**Lemma 6.5.** *Let $\mathcal{X}$ be a generating cogenerating functorially finite subcategory of an abelian category $\mathcal{A}$. Assume $\mathcal{X}$ satisfy (d-Rigid) for some $d > 0$. Let*

$$X_2 \xrightarrow{g} X_1 \xrightarrow{f} X_0 \xrightarrow{p} A \to 0$$

*be an exact sequence with $f$ a weak cokernel of $g$ and $X_2, X_1, X_0 \in \mathcal{X}$. Then $\operatorname{Ext}^1_{\mathcal{A}}(A, X) = 0$ for all $X \in \mathcal{X}$.*

*Proof.* Since $f$ is a weak cokernel, it follows that $\operatorname{Ker} p \to X_0$ is a left $\mathcal{X}$-approximation. Applying $\mathcal{A}(-, X)$ for $X \in \mathcal{X}$ to the exact sequence $0 \to \operatorname{Ker} p \to X_0 \to A \to 0$ gives a long exact sequence

$$0 \to \mathcal{A}(A, X) \to \mathcal{A}(X_0, X) \to \mathcal{A}(\operatorname{Ker} p, X) \to \operatorname{Ext}^1_{\mathcal{A}}(A, X) \to \operatorname{Ext}^1_{\mathcal{A}}(X_0, X) \to \cdots$$

Now $\operatorname{Ext}^1_{\mathcal{A}}(X_0, X) = 0$ by Lemma 6.4 and $\mathcal{A}(X_0, X) \to \mathcal{A}(\operatorname{Ker} p, X)$ is surjective since $p$ is a left $\mathcal{X}$-approximation. Hence $\operatorname{Ext}^1_{\mathcal{A}}(A, X) = 0$, which proves the claim. □

*Proof of "only if" part of Theorem 6.2.* We prove that $\operatorname{Ext}^i_{\mathcal{A}}(X',X)=0$ for all $X,X'\in\mathcal{X}$ and $0<i<d$ by induction on $i$. For $i=1$ this follows from Lemma 6.4. Assume $\operatorname{Ext}^i_{\mathcal{A}}(X',X)=0$ for all $X,X'\in\mathcal{X}$ and all $0<i\leq j$ with $j<d-1$. We prove that $\operatorname{Ext}^{j+1}_{\mathcal{A}}(X',X)=0$ for all $X,X'\in\mathcal{X}$. Let

$$0\to X\xrightarrow{f_{j+2}}A_{j+1}\xrightarrow{f_{j+1}}\cdots\xrightarrow{f_2}A_1\xrightarrow{f_1}X'\to 0$$

be an exact sequence with $X,X'\in\mathcal{X}$. Choose an epimorphism $g\colon X_1\to A_1$ with $X_1\in\mathcal{X}$, and let $g_1=f_1\circ g$. Next take the pullback square

$$\begin{tikzcd} A_2' \arrow[r] \arrow[d] & \operatorname{Ker} g_1 \arrow[d] \\ A_2 \arrow[r] & \operatorname{Ker} f_1 \end{tikzcd}.$$

and let $g_2$ be the composite $A_2'\to\operatorname{Ker}g_1\to X_1$. Now constrict iteratively $A_k'$ and $g_k\colon A_k'\to A_{k-1}'$ for $k\leq j+1$ such that

$$\begin{tikzcd} A_k' \arrow[r] \arrow[d] & \operatorname{Ker} g_{k-1} \arrow[d] \\ A_k \arrow[r] & \operatorname{Ker} f_{k-1} \end{tikzcd}.$$

is a pullback square and $g_k$ is the composite $A_k'\to\operatorname{Ker}g_{k-1}\to A_{k-1}'$ (note that $\operatorname{Ker}g_k\cong\operatorname{Ker}f_k$ for $k\geq 2$ and $A_k'\cong A_k$ for $k\geq 3$). Then we get a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "g_{j+2}"] \arrow[d, "1"] & A_{j+1}' \arrow[r, "g_{j+1}"] \arrow[d] & \cdots \arrow[r, "g_3"] & A_2' \arrow[r, "g_2"] \arrow[d] & X_1 \arrow[r, "g_1"] \arrow[d] & X' \arrow[r] \arrow[d, "1"] & 0 \\ 0 \arrow[r] & X \arrow[r, "f_{j+2}"] & A_{j+1} \arrow[r, "f_{j+1}"] & \cdots \arrow[r, "f_3"] & A_2 \arrow[r, "f_2"] & A_1 \arrow[r, "f_1"] & X' \arrow[r] & 0 \end{tikzcd}.$$

Hence, both exact sequences represents the same element in the Yoneda Ext-group $\operatorname{Ext}^{j+1}_{\mathcal{A}}(X',X)$. Therefore, it is sufficient to show that the lower exact sequence is 0 as an element in $\operatorname{Ext}^{j+1}_{\mathcal{A}}(X',X)$. For this, we only need to show that $\operatorname{Ext}^{j}_{\mathcal{A}}(\operatorname{Ker}g_1,X)=0$. Now by axiom ($d$-Rigid) it follows that there exists an exact sequence

$$X_{j+3}\xrightarrow{h_{j+3}}\cdots\xrightarrow{h_4}X_3\xrightarrow{h_3}X_2\xrightarrow{h_2}X_1\xrightarrow{h_1}X'\to 0$$

where $g_1=h_1$, and where $h_{i+1}$ is a weak kernel of $h_i$ and $h_i$ is a weak cokernel of $h_{i+1}$ for $1\leq i\leq j+2$, and where $X_i\in\mathcal{X}$ for $1\leq i\leq j+3$. By Lemma 6.5 we know that $\operatorname{Ext}^1_{\mathcal{A}}(\operatorname{Coker}h_{j+2},X)=0$. Now consider the exact sequence

$$0\to\operatorname{Coker}h_{i+1}\to X_i\to\operatorname{Coker}h_i\to 0$$

for $3 \leq i \leq j+1$. Applying $\mathcal{A}(-, X)$, we get exact sequences

$$0 \to \mathcal{A}(\operatorname{Coker} h_i, X) \to \mathcal{A}(X_i, X) \to \mathcal{A}(\operatorname{Coker} h_{i+1}, X) \to \cdots \to \operatorname{Ext}^{2+j-i}_{\mathcal{A}}(X_i, X)$$
$$\to \operatorname{Ext}^{2+j-i}_{\mathcal{A}}(\operatorname{Coker} h_{i+1}, X) \to \operatorname{Ext}^{3+j-i}_{\mathcal{A}}(\operatorname{Coker} h_i, X) \to \operatorname{Ext}^{3+j-i}_{\mathcal{A}}(X_i, X) \to \cdots$$

Since $\operatorname{Ext}^{2+j-i}_{\mathcal{A}}(X_i, X) = 0 = \operatorname{Ext}^{3+j-i}_{\mathcal{A}}(X_i, X)$ for $3 \leq i \leq j+1$ by the induction hypothesis, we get that

$$\operatorname{Ext}^{3+j-i}_{\mathcal{A}}(\operatorname{Coker} h_i, X) \cong \operatorname{Ext}^{2+j-i}_{\mathcal{A}}(\operatorname{Coker} h_{i+1}, X)$$

Hence

$$\operatorname{Ext}^{j}_{\mathcal{A}}(\operatorname{Coker} h_3, X) \cong \operatorname{Ext}^{j-1}_{\mathcal{A}}(\operatorname{Coker} h_4, X) \cong \cdots \cong \operatorname{Ext}^{1}_{\mathcal{A}}(\operatorname{Coker} h_{j+2}, X) = 0$$

Since $\operatorname{Coker} h_3 \cong \operatorname{Ker} h_1 = \operatorname{Ker} g_1$, the claim follows. □

## 7. d-abelian categories are d-cluster tilting

In this section we show that any $d$-abelian category is equivalent to a $d$-cluster tilting subcategory of an abelian category. The idea is to show that any $d$-abelian category satisfy axioms (A1), (A1)$^{\mathrm{op}}$, (A2), (A2)$^{\mathrm{op}}$, (A3), (A3)$^{\mathrm{op}}$, ($d$-Rigid) and has $d$-kernels and $d$-cokernels, and to show that such subcategories are equivalent to $d$-cluster tilting subcategories.

We first recall the definition of a $d$-cluster tilting subcategory:

**Definition 7.1.** Let $\mathcal{X}$ be a full subcategory of an abelian category $\mathcal{A}$, and let $d > 0$ be a positive integer. We say that $\mathcal{X}$ is ***d-cluster tilting*** in $\mathcal{A}$ if the following hold:

(i) $\mathcal{X}$ is a generating cogenerating functorially finite subcategory of $\mathcal{A}$;

(ii) We have

$$\mathcal{X} = \{A \in \mathcal{A} \mid \operatorname{Ext}^i_{\mathcal{A}}(A, X) = 0 \text{ for } 1 \leq i \leq d-1 \text{ and } X \in \mathcal{X}\}$$
$$= \{A \in \mathcal{A} \mid \operatorname{Ext}^i_{\mathcal{A}}(X, A) = 0 \text{ for } 1 \leq i \leq d-1 \text{ and } X \in \mathcal{X}\};$$

We need the following result on $d$-cluster tilting subcategories

**Lemma 7.2.** *Let $\mathcal{X}$ be a generating cogenerating functorially finite subcategory of an abelian category $\mathcal{A}$. Assume $\operatorname{Ext}^i_{\mathcal{A}}(X, X') = 0$ for $1 \leq i \leq d-1$ and $X, X' \in \mathcal{X}$. The following are equivalent:*

*(i) $\mathcal{X}$ is $d$-cluster tilting in $\mathcal{A}$;*

*(ii) $\mathcal{X}$ is closed under direct summands, and for any $A \in \mathcal{A}$ there exists exact sequences*

$$0 \to A \to X_{-1} \to \cdots \to X_{-d} \to 0 \quad \textit{and} \quad 0 \to X'_d \to \cdots \to X'_1 \to A \to 0$$

*where $X_i, X'_i \in \mathcal{X}$ for $1 \leq i \leq d$.*

*Proof.* This follows from [16, Proposition 2.2.2]. □

Let $f\colon X_1 \to X_0$ be a morphism in $\mathcal{X}$. Following [21], we say that a sequence

$$X_{d+1} \to X_d \to \cdots \to X_1$$

in $\mathcal{X}$ is a $d$-***kernel*** of $f$ if the sequence of abelian groups

$$0 \to \mathcal{X}(X, X_{d+1}) \to \mathcal{X}(X, X_d) \to \cdots \mathcal{X}(X, X_1) \xrightarrow{f\circ -} \mathcal{X}(X, X_0)$$

is exact for all $X \in \mathcal{X}$. Dually, sequence

$$X_0 \to X_{-1} \to \cdots \to X_{-d}$$

is a $d$-***cokernel*** of $f$ if the sequence of abelian groups

$$0 \to \mathcal{X}(X_{-d}, X) \to \mathcal{X}(X_{-d+1}, X) \to \cdots \to \mathcal{X}(X_0, X) \xrightarrow{-\circ f} \mathcal{X}(X_1, X)$$

is exact for all $X \in \mathcal{X}$.

**Theorem 7.3.** *Let $\mathcal{X}$ be an idempotent complete additive category satisfying (A1), (A1)$^{\mathrm{op}}$, (A2), (A2)$^{\mathrm{op}}$, (A3), (A3)$^{\mathrm{op}}$, and (d-Rigid). Assume furthermore that every morphism in $\mathcal{X}$ has a $d$-kernel and a $d$-cokernel. Then $\mathcal{X}$ is equivalent to a $d$-cluster tilting subcategory of an abelian category.*

*Proof.* By Theorem 5.5 and Theorem 6.2 we can assume $\mathcal{X}$ is a generating cogenerating functorially finite subcategory of an abelian category $\mathcal{A}$ satisfying $\mathrm{Ext}^i_{\mathcal{A}}(X, X') = 0$ for all $0 < i < d$ and $X, X' \in \mathcal{X}$. In particular, since $\mathcal{X}$ is generating and cogenerating, for any object $A \in \mathcal{A}$ there exists morphisms $f\colon X_1 \to X_0$ and $g\colon X'_0 \to X'_{-1}$ in $\mathcal{X}$ with $\operatorname{Coker} f \cong A$ and $\operatorname{Ker} g \cong A$. Taking the $d$-cokernel of $f$ and the $d$-kernel of $g$, we see that condition (ii) in Lemma 7.2 holds for $\mathcal{X}$. Hence $\mathcal{X}$ must be $d$-cluster tilting in $\mathcal{A}$. ☐

Next we recall the definition of $d$-abelian categories. Following [21], we say that a complex

$$X_{d+1} \xrightarrow{f_{d+1}} \cdots \to X_1 \xrightarrow{f_1} X_0$$

is $d$-***exact*** if $X_{d+1} \xrightarrow{f_{d+1}} \cdots \xrightarrow{f_2} X_1$ is a $d$-kernel of $f_1$ and $X_d \xrightarrow{f_d} \cdots \to X_1 \xrightarrow{f_1} X_0$ is a $d$-cokernel of $f_{d+1}$.

**Definition 7.4** (Definition 3.1 in [21])**.** Let $d$ be a positive integer, and let $\mathcal{X}$ be an additive category. We say that $\mathcal{X}$ is a $d$-abelian category if it satisfies the following

(i) $\mathcal{X}$ is idempotent complete;
(ii) Every morphism in $\mathcal{X}$ has a $d$-kernel and a $d$-cokernel;
(iii) Any complex

$$X_{d+1} \xrightarrow{f_{d+1}} \cdots \to X_1 \xrightarrow{f_1} X_0$$

where $f_1$ is an epimorphism and $X_{d+1} \xrightarrow{f_{d+1}} \cdots \xrightarrow{f_2} X_1$ is a $d$-kernel of $f$ must be $d$-exact;
(iv) Any complex

$$X_{d+1} \xrightarrow{f_{d+1}} \cdots \to X_1 \xrightarrow{f_1} X_0$$

where $f_{d+1}$ is a monomorphism and $X_d \xrightarrow{f_d} \cdots \to X_1 \xrightarrow{f_1} X_0$ is a $d$-cokernel of $f_{d+1}$ must be $d$-exact.

**Proposition 7.5.** *Let $\mathcal{X}$ be an additive category. Then $\mathcal{X}$ is $d$-abelian if and only if it is idempotent complete, has $d$-kernels and cokernels, and satisifies (A1), (A1)$^{\mathrm{op}}$, (A2), (A2)$^{\mathrm{op}}$, (A3), (A3)$^{\mathrm{op}}$and ($d$-Rigid).*

*Proof.* The "if" direction follows from Theorem 7.3 and the fact that any $d$-cluster tilting subcategory is $d$-abelian by [21, Theorem 3.16]. Conversely, assume $\mathcal{X}$ is a $d$-abelian category. Since $\mathcal{X}$ has $d$-kernels and $d$-cokernels, axioms (A1), (A1)$^{\mathrm{op}}$ holds automatically. Also by Definition 7.4 (iii) and the fact that $\mathcal{X}$ has $d$-kernels we get that (A2) and ($d$-Rigid) holds, and dually by Definition 7.4 (iv) and the fact that $\mathcal{X}$ has $d$-cokernels we get that (A2)$^{\mathrm{op}}$ holds. Hence, it remains to show that (A3) holds. To this end, let $f^0\colon X^0 \to X^1$ be a morphism, and let $f^1\colon X^1 \to X^2$ be a weak cokernel of $f^0$. Then by [21, Proposition 3.13] there exists objects $Y^1$ and $Y^2$ in $\mathcal{X}$ and morphisms $g^1_1\colon Y^1_1 \to X^1$ and $g^2_1\colon Y^2_1 \to Y^1_1$ and $p^0_0\colon X^0 \to Y^1_1$ such that

(i) $g^1_1$ is a weak kernel of $f^1$, $g^2_1$ is a weak kernel of $g^1_1$;
(ii) $g^1_1 \circ p^0_0 = f^0$;
(iii) The map $\begin{bmatrix} p^0_0 & g^2_1 \end{bmatrix} : X^0 \oplus Y^2_1 \to Y^1_1$ is an epimorphism.

This is precisely axiom (A3). Axiom (A3)$^{\mathrm{op}}$ is proved dually. □

## 8. Precluster tilting subcategories

Let $\mathcal{X}$ be an additive category satisfying (A1), (A1)$^{\mathrm{op}}$, (A2), (A2)$^{\mathrm{op}}$, (A3), (A3)$^{\mathrm{op}}$ and ($d$-Rigid). The goal in this section is to find additional axioms on $\mathcal{X}$ so that it gives an axiomatization of $d$-precluster tilting subcategories as introduced in [18]. More precisely, we want to show that any $d$-precluster tilting subcategory satisfy the axioms, and that any category which satisfy the axioms is equivalent to a $d$-precluster tilting subcategory. In order to do this, we need to reformulate the definition of precluster tilting subcategories so that it makes sense for any abelian category.

In the following we fix an Artin algebra $\Lambda$, and we let $\operatorname{mod}\Lambda$ denote the category of finitely generated (right) $\Lambda$ modules. We consider the $d$-Auslander-Reiten translations defined by

$$\tau_d := \tau \circ \Omega^{d-1}\colon \underline{\operatorname{mod}}\Lambda \to \overline{\operatorname{mod}}\Lambda \quad \text{and} \quad \tau_d^- := \tau^- \circ \Omega^{-(d-1)}\colon \overline{\operatorname{mod}}\Lambda \to \underline{\operatorname{mod}}\Lambda$$

Here $\underline{\operatorname{mod}}\Lambda$ and $\overline{\operatorname{mod}}\Lambda$ denote the stable categories modulo projectives and injectives, respectively, and $\Omega^{d-1}$ and $\Omega^{-(d-1)}$ denote the syzygy and cosyzygy functor applied $d-1$ times, respectively, and $\tau\colon \underline{\operatorname{mod}}\Lambda \to \overline{\operatorname{mod}}\Lambda$ denotes the classical Auslander-Reiten translation with quasi-inverse $\tau^-\colon \overline{\operatorname{mod}}\Lambda \to \underline{\operatorname{mod}}\Lambda$.

**Definition 8.1** (Definition 3.2 in [18])**.** Let $\mathcal{X}$ be an additive subcategory of $\operatorname{mod}\Lambda$. Assume $\mathcal{X}$ is closed under direct summands. We say that $\mathcal{X}$ is a ***d-precluster tilting subcategory*** if it satisfies the following:

(i) $\mathcal{X}$ is a generating cogenerating subcategory of $\operatorname{mod}\Lambda$;
(ii) $\tau_d(X) \in \overline{\mathcal{X}}$ and $\tau_d^-(X) \in \underline{\mathcal{X}}$ for all $X \in \mathcal{X}$;
(iii) $\operatorname{Ext}^i_\Lambda(X, X') = 0$ for all $X, X' \in \mathcal{X}$ and $0 < i < d$;
(iv) $\mathcal{X}$ is a functorially finite subcategory of $\operatorname{mod}\Lambda$.

The appearance of $\tau_d$ and $\tau_d^-$ makes precluster tilting subcategories difficult to axiomatize in terms of intrinsic properties of $\mathcal{X}$. Luckily, criterium (ii) in Definition 8.1 can be reformulated in purely homological terms. Our first goal is to show this. For simplicity we set

$$^{\perp_d}\mathcal{X} := \{M \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^i_\Lambda(M, X) = 0 \text{ for all } 0 < i < d \text{ and } X \in \mathcal{X}\}$$

and

$$\mathcal{X}^{\perp_d} := \{M \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^i_\Lambda(X, M) = 0 \text{ for all } 0 < i < d \text{ and } X \in \mathcal{X}\}.$$

**Lemma 8.2.** *Let $d > 1$ be an integer. Assume $\mathcal{X}$ is an additive subcategory of $\operatorname{mod}\Lambda$ closed under direct summands and satisfying (i), (iii), (iv) in Definition 8.1. Then the following are equivalent:*

*(i) $\mathcal{X}$ is $d$-precluster tilting;*

*(ii) $^{\perp_d}\mathcal{X} = \mathcal{X}^{\perp_d}$.*

*Proof.* This follows from [18, Proposition 3.8 part b)] □

We also need the following lemma which give a simpler criteria for when $^{\perp_d}\mathcal{X} = \mathcal{X}^{\perp_d}$.

**Lemma 8.3.** *Let $d > 1$ be a positive integer, and assume $\mathcal{X}$ is an additive subcategory of $\operatorname{mod}\Lambda$ closed under direct summands and satisfying (i), (iii), (iv) in Definition 8.1. Assume furthermore that $\mathcal{X}^{\perp_d} \subseteq {}^{\perp_1}\mathcal{X}$ and $^{\perp_d}\mathcal{X} \subseteq \mathcal{X}^{\perp_1}$. Then $^{\perp_d}\mathcal{X} = \mathcal{X}^{\perp_d}$.*

*Proof.* We prove by induction on $1 \leq i \leq d$ that $\mathcal{X}^{\perp_d} \subseteq {}^{\perp_i}\mathcal{X}$. For $i = 1$ this follows by assumption. Assume the claim holds for $1 \leq i < d$, and we want to show that it holds for $i + 1$. Let $M \in \mathcal{X}^{\perp_d}$, choose a right $\mathcal{X}$-approximation $f : X \to M$, and let $M' = \operatorname{Ker} f$. Applying $\operatorname{Hom}_\Lambda(X', -)$ with $X' \in \mathcal{X}$ to the exact sequence

$$0 \to M' \to X \xrightarrow{f} M \to 0 \tag{8.4}$$

gives a long exact sequence

$$\begin{aligned} 0 \to \operatorname{Hom}_\Lambda(X', M') \to \operatorname{Hom}_\Lambda(X', X) &\xrightarrow{f\circ -} \operatorname{Hom}_\Lambda(X', M) \to \operatorname{Ext}^1_\Lambda(X', M') \\ \to \operatorname{Ext}^1_\Lambda(X', X) \to \cdots \to \operatorname{Ext}^{j-1}_\Lambda(X', M) &\to \operatorname{Ext}^j_\Lambda(X', M') \to \operatorname{Ext}^j_\Lambda(X', X) \to \cdots \end{aligned}$$

Since $f$ is a right $\mathcal{X}$-approximation, it follows that the map

$$\operatorname{Hom}_\Lambda(X', X) \xrightarrow{f\circ -} \operatorname{Hom}_\Lambda(X', M)$$

is an epimorphism. Also, since $\operatorname{Ext}^j_\Lambda(X', X) = 0$ for $0 < j < d$ by Definition 8.1 (iii) and $\operatorname{Ext}^j_\Lambda(X', M) = 0$ for $0 < j < d$ by assumption, it follows that $\operatorname{Ext}^j_\Lambda(X', M') = 0$ for $1 \leq j \leq d$. Hence $M' \in \mathcal{X}^{\perp_d}$, and therefore $M' \in {}^{\perp_i}\mathcal{X}$ by induction hypothesis. Now applying $\operatorname{Hom}_\Lambda(-, X')$ to (8.4) and considering the long exact sequence we get

$$\operatorname{Ext}^{i+1}_\Lambda(M, X') \cong \operatorname{Ext}^i_\Lambda(M', X') = 0.$$

This show that $M \in {}^{\perp_{i+1}}\mathcal{X}$. Therefore, by induction we get that $\mathcal{X}^{\perp_d} \subseteq {}^{\perp_d}\mathcal{X}$. The inclusion $^{\perp_d}\mathcal{X} \subseteq \mathcal{X}^{\perp_d}$ is proved dually. Combining the inclusions, we get that $^{\perp_d}\mathcal{X} = \mathcal{X}^{\perp_d}$, which proves the claim. □

Note that Lemma 8.2 only holds when $d > 1$, so we still still need a homological reformulation of Definition 8.1 (ii) when $d = 1$. This is done by the following result.

**Theorem 8.5.** *Let $d$ be a positive integer, and assume $\mathcal{X}$ is an additive subcategory of* $\operatorname{mod}\Lambda$ *closed under direct summands and satisfying (i), (iii) and (iv) in Definition 8.1. The following are equivalent:*

*(i) $\mathcal{X}$ is a $d$-precluster tilting subcategory;*

*(ii) For any exact sequence in* $\operatorname{mod}\Lambda$

$$0 \to M' \xrightarrow{f_{d+1}} X_d \xrightarrow{f_d} \cdots \xrightarrow{f_3} X_2 \xrightarrow{f_2} X_1 \xrightarrow{f_1} M \to 0$$

*with $X_i \in \mathcal{X}$ for $1 \leq i \leq d$, the following hold:*

*(a) If the induced map $X_i \to \operatorname{im} f_i$ is a right $\mathcal{X}$-approximation for all $1 \leq i \leq d$, then $f_{d+1}\colon M' \to X_d$ is a left $\mathcal{X}$-approximation;*

*(b) If the induced map $\operatorname{im} f_i \to X_{i-1}$ is a left $\mathcal{X}$-approximation for all $2 \leq i \leq d+1$, then $f_1\colon X_1 \to M$ is a right $\mathcal{X}$-approximation.*

*Proof.* We prove the cases $d > 1$ and $d = 1$ separately. First assume $d > 1$ and that $\mathcal{X}$ is a $d$-precluster tilting subcategory. Assume furthermore that

$$0 \to M' \xrightarrow{f_{d+1}} X_d \xrightarrow{f_d} \cdots \xrightarrow{f_3} X_2 \xrightarrow{f_2} X_1 \xrightarrow{f_1} M \to 0$$

is an exact sequences such that $X_i \in \mathcal{X}$ and $X_i \to \operatorname{im} f_i$ is a right $\mathcal{X}$-approximation for all $1 \leq i \leq d$. Applying $\operatorname{Hom}_\Lambda(X, -)$ with $X \in \mathcal{X}$ to the exact sequence

$$0 \to \operatorname{im} f_{i+1} \to X_i \to \operatorname{im} f_i \to 0$$

and using that $\operatorname{Ext}^j_\Lambda(X, X_i) = 0$ for $1 \leq j < d$, we get that

$$\operatorname{Ext}^1_\Lambda(X, \operatorname{im} f_{i+1}) = 0 \quad \text{and} \quad \operatorname{Ext}^j_\Lambda(X, \operatorname{im} f_i) \cong \operatorname{Ext}^{j+1}_\Lambda(X, \operatorname{im} f_{i+1})$$

for $1 \leq i \leq d$ and $1 \leq j < d-1$. Hence, we have that

$$\operatorname{Ext}^j_\Lambda(X, \operatorname{im} f_d) \cong \operatorname{Ext}^{j-1}_\Lambda(X, \operatorname{im} f_{d-1}) \cong \cdots \cong \operatorname{Ext}^1_\Lambda(X, \operatorname{im} f_{d-j+1}) = 0$$

for $0 < j < d$. This shows that $\operatorname{im} f_d \in \mathcal{X}^{\perp_d}$, so $\operatorname{im} f_d \in {}^{\perp_1}\mathcal{X}$ by Lemma 8.2, and therefore $f_{d+1}\colon M' \to X_d$ is a left $\mathcal{X}$ approximation. This together with its dual argument shows the implication (i) $\implies$ (ii) for $d > 1$.

Conversely, assume $d > 1$ and that $\mathcal{X}$ satisfy part (ii) of the theorem. Let $M' \in \mathcal{X}^{\perp_d}$, and choose a right $\mathcal{X}$-approximation $X_d \to M'$ and an exact sequence

$$0 \to M' \to X_{d-1} \xrightarrow{f_{d-1}} X_{d-2} \xrightarrow{f_{d-2}} \cdots \xrightarrow{f_2} X_1 \xrightarrow{f_1} M \to 0$$

Since $\operatorname{Ext}^i_\Lambda(X, M') = 0$ for $0 < i < d$, it follows that the sequence

$$\operatorname{Hom}_\Lambda(X, X_{d-1}) \xrightarrow{f_{d-1}\circ -} \cdots \xrightarrow{f_2\circ -} \operatorname{Hom}_\Lambda(X, X_1) \xrightarrow{f_1\circ -} \operatorname{Hom}_\Lambda(X, M) \to 0$$

is exact for $X \in \mathcal{X}$, and hence the canonical map $X_i \to \operatorname{im} f_i$ is a right $\mathcal{X}$-approximation for $1 \leq i \leq d-1$. Therefore, by assumption we get that the map $\operatorname{Ker} f_d \to X_d$ is a left $\mathcal{X}$-approximation, where $f_d$ is the composite $X_d \to M' \to X_{d-1}$. Hence, applying $\mathrm{Hom}_\Lambda(-, X)$ with $X \in \mathcal{X}$ to the exact sequence

$$0 \to \operatorname{Ker} f_d \to X_d \to M' \to 0$$

we get that $\mathrm{Ext}^1_\Lambda(M', X) = 0$ so $M' \in {}^{\perp_1}\mathcal{X}$. Since $M' \in \mathcal{X}^{\perp_d}$ was arbitrary, this shows that $\mathcal{X}^{\perp_d} \subseteq {}^{\perp_1}\mathcal{X}$. The inclusion ${}^{\perp_d}\mathcal{X} \subseteq \mathcal{X}^{\perp_1}$ is proved dually, and the fact that $\mathcal{X}$ is $d$-precluster tilting follows from Lemma 8.2 and Lemma 8.3.

Now we assume $d = 1$ and $\mathcal{X}$ is a 1-precluster tilting subcategory. Let

$$0 \to M \xrightarrow{f} X \xrightarrow{g} M' \to 0$$

be an exact sequence with $f$ a left $\mathcal{X}$-approximation. Then since $\tau(X') \in \overline{\mathcal{X}}$ for all $X' \in \underline{\mathcal{X}}$, we have that all morphisms $M \to \tau(X')$ will factor through $f$. Hence by [3, Corollary 4.4] all morphisms $X' \to M'$ with $X' \in \mathcal{X}$ will factor through $g$. Therefore $g$ is a right $\mathcal{X}$-approximation. Together with the dual argument this shows that $\mathcal{X}$-being 1-precluster tilting implies that $\mathcal{X}$ satisfies condition (ii) in the theorem for $d = 1$.

Finally, assume $d = 1$ and that condition (ii) holds for $\mathcal{X}$. Let $X \in \mathcal{X}$ be an indecomposable module, and assume that $Y = \tau(X) \notin \overline{\mathcal{X}}$. By abuse of notation we also let $\tau(X)$ denote the unique indecomposable object in $\operatorname{mod}\Lambda$ corresponding $Y$ in $\overline{\operatorname{mod}}\Lambda$. Consider the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \tau(X) \arrow[r] \arrow[d, "1"] & E \arrow[r] \arrow[d, dashed] & X \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & \tau(X) \arrow[r] & X' \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

where the top row is an almost split sequence, and where $\tau(X) \to X'$ is a left $\mathcal{X}$-approximation with cokernel $M$. Since $\tau(X) \notin \mathcal{X}$, the morphism $\tau(X) \to X$ is not a split monomorphism and therefore it factors through $\tau(X) \to E$. Hence we obtain vertical maps $E' \to X'$ and $X \to M$ making the diagram commute. By assumption we have that $X' \to M$ is a right $\mathcal{X}$-approximation, and hence $X \to M$ factors through $X' \to M$. Since the rightmost square is a pushout square, it follows that $X' \to M$ is a split epimorphism. Therefore the sequence $0 \to \tau(X) \to E \to X \to 0$ must be split, which is a contradiction. This shows that $\tau(X) \in \mathcal{X}$. The implication $X \in \overline{\mathcal{X}} \implies \tau^-(X) \in \underline{\mathcal{X}}$ is proved dually. $\square$

Motivated by this, we define $d$-precluster tilting subcategories for arbitrary abelian categories. By Theorem 8.5 it coincides with the normal definition when $\mathcal{A} = \operatorname{mod}\Lambda$.

**Definition 8.6.** Let $\mathcal{X}$ be an additive subcategory of an abelian category $\mathcal{A}$. Assume $\mathcal{X}$ is closed under direct summands. We say that $\mathcal{X}$ is a ***$d$-precluster tilting subcategory*** if it satisfies the following:

(i) $\mathcal{X}$ is a generating cogenerating subcategory of $\mathcal{A}$;
(ii) For any exact sequence in $\mathcal{A}$

$$0 \to A' \xrightarrow{f_{d+1}} X_d \xrightarrow{f_d} \cdots \xrightarrow{f_3} X_2 \xrightarrow{f_2} X_1 \xrightarrow{f_1} A \to 0$$

with $X_i \in \mathcal{X}$ for $1 \leq i \leq d$, the following hold:

(a) If the induced map $X_i \to \operatorname{im} f_i$ is a right $\mathcal{X}$-approximation for all $1 \leq i \leq d$, then $f_{d+1}\colon A' \to X_d$ is a left $\mathcal{X}$-approximation;

(b) If the induced map $\operatorname{im} f_i \to X_{i-1}$ is a left $\mathcal{X}$-approximation for all $2 \leq i \leq d+1$, then $f_1\colon X_1 \to A$ is a right $\mathcal{X}$-approximation.

(iii) $\operatorname{Ext}^i_{\mathcal{A}}(X, X') = 0$ for all $X, X' \in \mathcal{X}$ and $0 < i < d$;

(iv) $\mathcal{X}$ is a functorially finite subcategory of $\mathcal{A}$.

We now introduce the necessary axiom to capture Definition 8.6 (ii).

(A4) Consider a sequence

$$X_{d+1} \xrightarrow{f_{d+1}} X_d \xrightarrow{f_d} \cdots \xrightarrow{f_3} X_2 \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0 \xrightarrow{f_0} X_{-1}$$

with $f_{i+1}$ a weak kernel of $f_i$ for all $0 \leq i \leq d$. Then $f_{d+1}$ is a weak cokernel.

**Theorem 8.7.** *Let $\mathcal{X}$ be a generating cogenerating functorially finite subcategory of an abelian category $\mathcal{A}$ satisfying* $\operatorname{Ext}^i_{\mathcal{A}}(X, X') = 0$ *for all $X, X' \in \mathcal{X}$ and $0 < i < d$. Then $\mathcal{X}$ satisfy (A4) and (A4)*$^{\mathrm{op}}$ *if and only if it is d-precluster tilting*

*Proof.* This follows immediately from the fact that a weak kernel of a map $X \xrightarrow{f} X'$ is a weak kernel or weak cokernel if and only if the projection $X \to \operatorname{im} f$ is a right $\mathcal{X}$-approximation or the inclusion $\operatorname{im} f \to X'$ is a left $\mathcal{X}$-approximation, respectively. □

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