RELATIVE SINGULARITY CATEGORIES III: CLUSTER RESOLUTIONS

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ABSTRACT. We build foundations of an approach to study canonical forms of 2-Calabi–Yau triangulated categories with cluster-tilting objects, using dg algebras and relative singularity categories. This is motivated by cluster theory, singularity categories, Wemyss's Homological Minimal Model Program and the relations between these topics.

1. INTRODUCTION

Triangulated categories provide a natural framework for homological algebra and their structural properties are widely studied in mathematics and theoretical physics. The most important symmetry of a triangulated category is the build-in *shift functor* Σ. Using classical works on dualities of Nakayama [50, 51] and Serre [59], many triangulated categories arising in representation theory and algebraic geometry admit another very important automorphism called *Serre functor*, see Happel [27] and Bondal & Kapranov [14], respectively. A triangulated category is called *d-Calabi–Yau* if there is a simple relation between these two functors: namely, if the d-th power Σ^d of the shift functor is a Serre functor.

Famous examples include derived categories of coherent sheaves on smooth *Calabi-Yau varieties* (i.e. varieties with trivial canonical bundle), the related *Kuznetsov component* for Fano varieties [45], Buchweitz-Orlov *singularity categories* [17, 52, 53] of varieties with Gorenstein (isolated) singularities [7, 32] and *cluster categories*, providing key insights into Fomin–Zelevinsky's cluster algebras [24] via categorification [9, 10, 19, 54, 56].

The study of canonical forms for derived and triangulated categories led to the development of *Tilting theory* by Baer [8], Bondal [13], Happel [27] and Rickard [58], generalising Beilinson's seminal work on derived categories of projective spaces $\mathbb{P}^n$, [11]. However, the definition of Calabi–Yau categories, does not allow any *tilting objects*. *Cluster-tilting theory* overcomes this problem in many cases (see e.g. Keller & Reiten's recognition theorem [43], which we partly recover in Theorem 5.7), and also plays a key role in the study of cluster algebras and Wemyss's homological minimal model program [65]. Indeed, *cluster-tilting objects* in cluster categories and in singularity categories correspond to, respectively, seeds of cluster algebras [9] and Van den Bergh's noncommutative crepant resolutions (NCCRs) [63, 30]. Given their structural similarities, it is natural to ask for equivalences between cluster categories and singularity categories – this is identified as a main problem in Iyama's 2018 ICM talk [31]. Motivated by this, we study the structure of 2-Calabi–Yau categories with cluster tilting objects, approaching the following Morita-type conjecture of Amiot [1].

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Conjecture 1.1. *Let k be an algebraically closed field of characteristic zero. Let $\mathcal{C}$ be a 2-Calabi–Yau algebraic triangulated category with a cluster-tilting object T. Then $\mathcal{C}$ is triangle equivalent to the cluster category of a quiver with potential.*

One approach to attack this conjecture is developed in [2] and further pursued in [6, 5, 4]. An alternative strategy, using relative singularity categories, was suggested in [20, 34]. We state the main result of this paper, which builds the foundation of our approach.

Theorem 4.1. *Let k be an algebraically closed field of characteristic zero. Let $\mathcal{C}$ be a d-Calabi–Yau algebraic triangulated k-category with a d-cluster-tilting object T. Then there is a dg k-algebra B such that*

(a) *$H^p(B) = 0$ for $p > 0$, $H^0(B) \cong \mathsf{End}_{\mathcal{C}}(T)$ and $H^p(B) \cong \mathsf{Hom}_{\mathcal{C}}(T, \Sigma^p T)$ for $p < 0$;*
(b) *$\mathsf{per}(B) \supseteq \mathcal{D}_{fd}(B)$ and there is a triangle equivalence $\mathsf{per}(B)/\mathcal{D}_{fd}(B) \to \mathcal{C}$ which takes B to T;*
(c) *there is a bifunctorial isomorphism for $X \in \mathsf{mod}\, H^0(B)$ and $Y \in \mathcal{D}_{fd}(B)$*

$$D\,\mathsf{Hom}_{\mathcal{D}_{fd}(B)}(X, Y) \cong \mathsf{Hom}_{\mathcal{D}_{fd}(B)}(Y, \Sigma^{d+1} X).$$

Known examples of categories $\mathcal{C}$ satisfying the assumptions of Theorem 4.1 are singularity categories $D_{sg}(R)$ of $(d+1)$-dimensional Gorenstein isolated singularities R admitting an NCCR. In order to explain the relation between our Theorem and Amiot's Conjecture 1.1, we first recall the definition of a *cluster category* $\mathcal{C}_{(Q,W)}$ of a quiver Q with potential W:

$$\mathcal{C}_{(Q,W)} = \mathsf{per}(B)/\mathcal{D}_{fd}(B), \tag{1.1}$$

where $B = \widehat{\Gamma}(Q, W)$ is the *complete Ginzburg dg algebra* associated to Q and W, [25].

In particular, taking $d = 2$ in our Theorem reduces Conjecture 1.1 to the problem of finding quasi-equivalences between the dg algebras B appearing in Theorem 4.1 and *complete Ginzburg dg algebras* $\widehat{\Gamma}(Q, W)$. To show the latter, it is sufficient ([64, Theorem B]) to show that the dg algebras B are quasi-equivalent to *exact* 3-bimodule dg algebras in the sense of [64, Section 1, Definition], leading to Question 5.5. The problem of comparing different Calabi–Yau properties of dg algebras is also studied in [12, 25, 64].

Remark 1.2. *The dg algebra B in Theorem 4.1 is a dg universal localisation of a certain algebra (possibly with several objects) A* [34, 35, 15]. *This implies that if A is quasi-isomorphic to a complete Ginzburg dg algebra, so is B. For example, let R be a complete local Gorenstein isolated singularity with residue field k of dimension* 3 *admitting an NCCR A. Then by Van den Bergh's* [64], *A is quasi-isomorphic to a complete Ginzburg dg algebra. It follows that Amiot's conjecture holds for $\mathcal{C} = \mathcal{D}_{sg}(R)$, see* [20]. *This is used* [29, 15] *to prove a derived version of the Donovan-Wemyss conjecture* [21]*, which is a central question in the Homological Minimal Model Program* [65].

Moreover, Theorem 4.1 (a) was also observed by Booth [15, Theorem 6.4.2.] *in case $\mathcal{C} = \mathcal{D}_{sg}(R)$ for $d+1$-dimensional Gorenstein singularities R. In particular, if $d = 2$ and R is a hypersurface (which is the setting of* [21]*), then the shift functor is* 2*-periodic by work of Eisenbud* [22] *and Theorem 4.1 (a) implies that the cohomology of B satisfies*

$$H^i(B) = \begin{cases} \mathsf{End}_{\mathcal{D}_{sg}(R)}(T) & \text{if } i \leq 0 \text{ is even,} \\ 0 & \text{else.} \end{cases} \tag{1.2}$$

We also use Theorem 4.1, to give an alternative proof of Keller & Reiten's recognition theorem [43, Theorems 2.1 and 4.2] in the special case that the quiver Q is a tree.

Theorem 5.7. *Let $\mathcal{C}$ be a d-Calabi–Yau algebraic triangulated category. Assume that T is a d-cluster-tilting object of $\mathcal{C}$ such that $\mathsf{Hom}_{\mathcal{C}}(T, \Sigma^p T) = 0$ for $-d+2 \leq p \leq -1$ and that $\mathsf{End}_{\mathcal{C}}(T) \cong kQ$ for some finite tree quiver Q.*

Then there is an equivalence of triangulated categories

$$\mathcal{C} \cong \mathcal{C}_Q^{(d)},$$

where $\mathcal{C}_Q^{(d)}$ is a triangulated orbit category in the sense of Keller [39]*:*

$$\mathcal{C}_Q^{(d)} := \mathcal{D}^b(\mathsf{mod}\, kQ)/\tau^{-1} \circ \Sigma^{d-1}.$$

Throughout the paper, let k be a commutative ring.

2. DG algebras and A_∞-algebras

A *dg k-algebra A* is a $\mathbb{Z}$-graded k-algebra $A = \bigoplus_{p \in \mathbb{Z}} A^p$ endowed with a differential d of degree 1 such that the graded Leibniz rule holds

$$d(ab) = d(a)b + (-1)^p a d(b)$$

for all $a \in A^p$ ($p \in \mathbb{Z}$) and $b \in A$.

2.1. Complete dg-quiver algebras.

Assume that k is a field.

Let Q be a graded quiver such that that Q_0 is finite. The *complete graded path algebra* $\widehat{kQ}$ of Q is the completion of the graded path algebra kQ at the graded ideal $\mathfrak{m}$ generated by the arrows of Q in the category of graded k-algebras. So the degree n component of $\widehat{kQ}$ consists of elements of the form $\Sigma_p \lambda_p p$, where $\lambda_p \in k$ and p runs over all paths of Q of degree n. We refer to [18, Section II.3] for the theory on completions of graded rings. Let $\widehat{\mathfrak{m}}$ be the completion of $\mathfrak{m}$ in $\widehat{kQ}$, *i.e.* $\widehat{\mathfrak{m}}$ consists of elements of the form $\Sigma_p \lambda_p p$, where $\lambda_p \in k$ and p runs over all non-trivial paths of Q. Then the $\widehat{\mathfrak{m}}$-adic topology on $\widehat{kQ}$ is pseudo-compact ([37, 64]). For an ideal I of $\widehat{kQ}$ we denote by $\overline{I}$ the closure of I with respect to this topology.

Let $r \in \mathbb{N}$ and K be the direct product of r copies of k (with standard basis $e_1, \ldots, e_r$). Let V be a graded K-K-bimodule. Thencomplete tensor algebra $\widehat{T}_K V := \prod_{p \geq 0} V^{\otimes_K p}$ is isomorphic to the complete graded path algebra $\widehat{kQ}$ of the graded quiver Q which has vertex set $\{1, \ldots, r\}$ and which has $\dim_k e_j V^m e_i$ arrows of degree m from i to j.

We call a dg k-algebra a *complete dg-quiver algebra* if it is of the form $A = (\widehat{kQ}, d)$, where Q is a graded quiver with finitely many vertices and $d \colon \widehat{kQ} \to \widehat{kQ}$ is a continuous k-linear differential of degree 1 such that d takes all trivial paths to 0. It is *minimal* if d takes an arrow of Q into $\widehat{\mathfrak{m}}^2$. By the graded Leibniz rule, the differential d is determined by its value on arrows. Since d takes all trivial paths to 0, it follows (again by the graded Leibniz rule) that d takes an arrow α to a (possibly infinite) linear combination of paths with source $s(\alpha)$ and target $t(\alpha)$.

Lemma 2.1. *Let $A = (\widehat{kQ}, d)$ be a complete dg-quiver algebra such that Q is concentrated in non-positive degrees and let m be a positive integer. If $H^p(C) = 0$ for all $-m \leq p \leq -1$ and if $H^0(A)$ is hereditary, then Q has no arrows in degree p for any $-m \leq p \leq -1$.*

Proof. Let Q_1^p denote the set of arrows of Q of degree p. Since $H^0(A)$ is hereditary, we have $d|_{Q_1^{-1}} = 0$. We prove the statement by induction on m. Any arrow of degree -1 is contained in the kernel but not in the image of d. Thus $H^{-1}(A) = 0$ implies that Q_1^{-1} is empty, so the statement holds true for $m = 1$. Now assume $m \geq 2$. By induction hypothesis Q_1^p is empty for $-m+1 \leq p \leq -1$. Thus any arrow of degree -1 is contained in the kernel but not in the image of d. Thus $H^{-m}(A) = 0$ implies that Q_1^{-m} is empty. $\square$

2.2. **A Duality.** This subsection generalises part of [38, Section 10]. Assume that k is a field.

Let A be a dg k-algebra and M be a dg A-module. We assume that M is $\mathcal{H}$-projective or $\mathcal{H}$-injective and put $B = \mathcal{E}nd_A(M)$. Then M becomes a dg B-A-bimodule and we have an adjoint pair of triangle functors

$$\mathcal{D}(B) \underset{\mathsf{RHom}_A(M,?)}{\overset{?\overset{L}{\otimes}_B M}{\rightleftarrows}} \mathcal{D}(A).$$

Let $\tilde{M}$ be an $\mathcal{H}$-projective resolution of M over $B^{op} \otimes A$. Then $\tilde{M}$ is $\mathcal{H}$-projective over both B^{op} and A. So we have isomorphisms of triangle functors

$$? \overset{L}{\otimes}_B M \cong ? \overset{L}{\otimes}_B \tilde{M} \cong ? \otimes_B \tilde{M}$$

and

$$\mathsf{RHom}_A(M, ?) \cong \mathsf{RHom}_A(\tilde{M}, ?) \cong \mathcal{H}om_A(\tilde{M}, ?).$$

Let $M^* = \mathcal{H}om_A(\tilde{M}, D(A)) = D(\tilde{M})$. Then M^* is a dg $\mathcal{E}nd_A(D(A))$-B-bimodule. Let N be an $\mathcal{H}$-projective or $\mathcal{H}$-injective resolution of M^* over B and put $C = \mathcal{E}nd_B(N)$. Let $\tilde{N}$ be an $\mathcal{H}$-projective resolution of N over $C^{op} \otimes B$. Consider the dg functor

$$F = \mathcal{H}om_A(\tilde{M}, ?) \circ (? \otimes_{\mathcal{E}nd_A(D(A))} D(A)) \colon \mathcal{C}_{dg}(\mathcal{E}nd_A(D(A))) \to \mathcal{C}_{dg}(B),$$

which takes $\mathcal{E}nd_A(D(A))$ to M^* and whose left derived functor is

$$\mathbb{L}F = \mathcal{H}om_A(\tilde{M}, ?) \circ (? \overset{L}{\otimes}_{\mathcal{E}nd_A(D(A))} D(A)) \colon \mathcal{D}(\mathcal{E}nd_A(D(A))) \to \mathcal{D}(B).$$

By [38, Lemma 7.3(a)], $Y = \mathcal{H}om_B(\tilde{N}, M^*) = \mathcal{H}om_B(\tilde{N}, F(\mathcal{E}nd_A(D(A))))$ is a quasi-functor from $\mathcal{E}nd_A(D(A))$ to C, and it is a quasi-equivalence if and only if $\mathbb{L}F$ restricts to a triangle equivalence $\mathsf{per}(\mathcal{E}nd_A(D(A))) \to \mathsf{thick}_{\mathcal{D}(B)}(M^*)$, equivalently, $\mathcal{H}om_A(\tilde{M}, ?)$ restricts to a triangle equivalence $\mathsf{thick}_{\mathcal{D}(A)}(D(A)) \to \mathsf{thick}_{\mathcal{D}(B)}(M^*)$ because $? \overset{L}{\otimes}_{\mathcal{E}nd_A(D(A))} D(A)\colon \mathsf{per}(\mathcal{E}nd_A(D(A))) \to \mathsf{thick}_{\mathcal{D}(A)}(D(A))$ is always a triangle equivalence.

Lemma 2.2. *If $M \in \mathsf{per}(A)$ and $D(A) \in \mathsf{Loc}_{\mathcal{D}(A)}(M)$, then the quasi-functor Y above is a quasi-equivalence.*

Proof. If $M \in \mathsf{per}(A)$, then $\mathcal{H}om_A(\tilde{M}, ?) \cong \mathsf{RHom}_A(M, ?)$ restricts to a triangle equivalence $\mathsf{Loc}_{\mathcal{D}(A)}(M) \to \mathcal{D}(B)$. If in addition $D(A) \in \mathsf{Loc}_{\mathcal{D}(A)}(M)$, it restricts further to a triangle equivalence $\mathsf{thick}_{\mathcal{D}(A)}(D(A)) \to \mathsf{thick}_{\mathcal{D}(B)}(M^*)$. $\square$

2.3. **Koszul duality.** Assume that k is field. Fix $r \in \mathbb{N}$. Let K be the direct product of r copies of k and consider it as a k-algebra via the diagonal embedding.

A dg algebra over K is a dg k-algebra A together with a dg k-algebra homomorphism $\eta\colon K \to A$, called the *unit*. It is *augmented* if in addition there is a dg k-algebra homomorphism $\varepsilon\colon A \to K$, called the *augmentation map*, such that $\varepsilon\eta = \mathrm{id}_K$.

Let A be an augmented dg algebra over K. Denote by $\bar{A} = \ker \varepsilon$. Note that $\bar{A}$ is a dg ideal of A. The *bar construction* of A, denoted by BA, is the graded K-bimodule

$$T_K(\bar{A}[1]) = K \oplus \bar{A}[1] \oplus \bar{A}[1] \otimes_K \bar{A}[1] \oplus \dots.$$

It is naturally a coalgebra with comultiplication $\Delta\colon BA \to BA \otimes_K BA$ defined by splitting the tensors:

$$\Delta(a_1 \otimes \dots \otimes a_n) = \sum_{p=0}^{n} (a_1 \otimes \dots \otimes a_p) \otimes (a_{p+1} \otimes \dots \otimes a_n).$$

Moreover, the differential and multiplication on $\bar{A}$ uniquely extend to a K-bilinear differential on BA, making it a dg coalgebra over K:

$$\begin{aligned} d_{BA}(a_1 \otimes \dots \otimes a_n) = & \sum_{p=1}^{n-1} (-1)^{|a_1|+\ldots+|a_p|+1} a_1 \otimes \dots \otimes a_p \otimes d_A(a_{p+1}) \otimes a_{p+2} \otimes \dots \otimes a_n \\ & + \sum_{p=1}^{n-2} (-1)^{|a_1|+\ldots+|a_p|+|a_{p+1}|} a_1 \otimes \dots \otimes a_p \otimes a_{p+1}a_{p+2} \otimes a_{p+3} \otimes \dots \otimes a_n, \end{aligned}$$

where $a_1, \dots, a_n$ are homogeneous elements of $\bar{A}[1]$ of degree $|a_1|, \dots, |a_n|$, respectively. The dual bar construction of A is the graded k-dual of BA:

$$E(A) = B^{\#}A := D(BA).$$

As a graded algebra $E(A) = \widehat{T}_K(D(\bar{A}[1]))$ is the complete tensor algebra of $D(\bar{A}[1]) = \mathsf{Hom}_k(\bar{A}[1], k)$ over K. It is naturally an augmented dg algebra over K with differential d being the unique continuous K-bilinear map satisfying the graded Leibniz rule and taking $f \in D(\bar{A}[1])$ to $d(f) \in D(\bar{A}[1]) \oplus D(\bar{A}[1] \otimes_K \bar{A}[1])$, defined by

$$\begin{aligned} d(f)(a_1) &= -f(d_A(a_1)), \\ d(f)(a_1 \otimes a_2) &= (-1)^{|a_1|} f(a_1 a_2), \end{aligned}$$

where a_1, a_2 are homogeneous elements of $\bar{A}[1]$ of degree $|a_1|, |a_2|$, respectively.

There is a certain $\mathcal{H}$-projective resolution M, which we call the *bar resolution*, of K (viewed as a dg A-module via the augmentation map), such that $\mathcal{E}nd_A(M) = B^{\#}A$. See [60] and [23, Section 19 exercise 4].

2.4. A_∞**-algebras.** Assume that k is a field and fix $r \in \mathbb{N}$. Let K be the direct product of r copies of k and consider it as a k-algebra via the diagonal embedding.

An A_∞-*algebra* A over K is a graded K-bimodule endowed with a family of homogenous K-bilinear maps $\{m_n\colon A^{\otimes_K n} \to A | n \geq 1\}$ of degree $2-n$ (called *multiplications*) satisfying certain conditions. We need the following facts, see [40, 47, 48, 49, 34].

(a) A dg algebra over K is a strictly unital A_∞-algebra over K with m_1 being the differential, m_2 being the multiplication and $m_n = 0$ for $n \geq 3$.

(b) ([47, Corollarie 3.2.4.1]) A strictly unital A_∞-algebra A over K admits a minimal model, that is, a strictly unital minimal A_∞-algebra which is A_∞-quasi-isomorphic to A.

(c) ([47, Lemme 2.3.4.3]) Let A be an augmented A_∞-algebra over K. Then there is an augmented dg algebra $U(A)$ over K, called the *enveloping dg algebra of* A, together with an A_∞-quasi-isomorphism $\psi\colon A \to U(A)$ of augmented A_∞-algebras over K. It satisfies the following universal property: if B is a dg algebra over K and $f\colon A \to B$ is a strictly unital A_∞-morphism of strictly unital A_∞-algebras over K, then there is a unique homomorphism $f'\colon U(A) \to B$ of dg algebras over K such that $f = f' \circ \psi$.

(d) ([34, Section 2.7]) For an augmented A_∞-algebra A over K, one can define the dual bar construction $E(A)$, which is a dg algebra over K. If A is an augmented dg algebra over K, then it is exactly the one defined in in Section 2.3. If A and B are A_∞-quasi-isomorphic augmented A_∞-algebras over K, then $E(A)$ and $E(B)$ are quasi-isomorphic as dg algebras over K.

(e) ([61, Section 4.2]) A strictly unital minimal A_∞-algebra A is said to be *positive* if
  - $A^p = 0$ for all $p < 0$,
  - $A^0 = K$ and the unit is the embedding $K = A^0 \hookrightarrow A$.

2.5. **Non-positive dg algebra: augmentation.** Assume that k is a field. Let A be a non-positive dg k-algebra satisfying the following three conditions:

(FD) $H^0(A)$ is finite-dimensional over k,

(BE) $H^0(A)$ is elementary, *i.e.* $H^0(A)$ is isomorphic to the quotient of the path algebra of a finite quiver by an admissible ideal,

(HS) $\mathsf{per}(A) \supseteq \mathcal{D}_{fd}(A)$.

In this subsection we will show that A is quasi-equivalent to the dual bar construction of the A_∞-Koszul dual of A.

Let $K = H^0(A)/\mathrm{rad}H^0(A)$ and $r = \dim_k K$. As an algebra, K is the direct product of r copies of k. View K as a dg A-module via the projection $A \to H^0(A) \to K$ and denote this dg module by S. We have a decomposition $S = S_1 \oplus \ldots \oplus S_r$ such that $S_1, \ldots, S_r$ form a complete set of pairwise non-isomorphic simple $H^0(A)$-modules.

Take (arbitrary) $\mathcal{H}$-projective resolutions $X_1, \ldots, X_r$ of $S_1, \ldots, S_r$ over A. Put $X = X_1 \oplus \ldots \oplus X_r$ and $B = \mathcal{E}nd_A(X)$. Then B is a dg algebra over K, with K being identified with $k\{\mathrm{id}_{X_1}\} \times \cdots \times k\{\mathrm{id}_{X_r}\}$. Denote by A^* the minimal model of B (Section 2.4(b)) and call it the A_∞-*Koszul dual* of A. By definition A^* is a strictly unital minimal A_∞-algebra over K and there is an A_∞-quasi-isomorphism $\varphi : A^* \to B$ of strictly unital A_∞-algebras over K. As a graded algebra over K, A^* is the graded endomorphism algebra $\bigoplus_{p\in\mathbb{Z}} \mathsf{Hom}_{\mathcal{D}(A)}(S, \Sigma^p S)$ of S. It follows from [16, Theorem A.1(c)] that A^* is positive. In particular, it is augmented with augmentation map the projection $A^* \to (A^*)^0 = K$.

Theorem 2.3. *A is quasi-equivalent to $E(A^*)$.*

Proof. Let U be the enveloping dg algebra of A^* and $\psi : A^* \to U$ be the associated quasi-isomorphism of augmented A_∞-algebras over K (Section 2.4(c)). Then there is a quasi-isomorphism $E(A^*) \to E(U)$ of dg algebras over K (Section 2.4(d)). Consider K as a dg U-module via the augmentation map and let Z be the bar resolution of K over U (Section 2.3). Then $E(U) = \mathcal{E}nd_U(Z)$.

By the universal property of U, there is a quasi-isomorphism $f : U \to B$ of dg algebras over K such that $f \circ \psi = \varphi$. It induces a quasi-isomorphism $E(U) = \mathcal{E}nd_U(Z) \to \mathcal{E}nd_B(Z \otimes_U B)$.

Let $\tilde{X}$ be an $\mathcal{H}$-projective resolution of X over $B^{op} \otimes A$. Put $S^* = \mathcal{H}om_A(\tilde{X}, D(A)) = D(\tilde{X})$, whose total cohomology is concentrated in degree 0 and is isomorphic to $H^0(B)$ as a graded module over $H^*(B)$. So as a dg module over U via the quasi-isomorphism $f : U \to B$, S^* is quasi-isomorphic to K. As a consequence, $Z \otimes_U B$ is an $\mathcal{H}$-projective resolution of S^* over B and there is a quasi-isomorphism $\mathcal{E}nd_U(Z) \to \mathcal{E}nd_B(Z \otimes_U B)$ of dg algebras. We claim that $D(A) \in \mathsf{Loc}_{\mathcal{D}(A)}(S)$. Since $S \in \mathcal{D}_{fd}(A) \subseteq \mathsf{per}(A)$, it follows from Lemma 2.2 that $\mathcal{E}nd_A(D(A))$ is quasi-equivalent to $\mathcal{E}nd_B(Z \otimes_U B)$, and hence to $E(A^*)$. Further, as A has finite-dimensional cohomology in each degree by [34, Proposition 2.5], A is quasi-isomorphic to $\mathcal{E}nd_A(D(A)) = DD(A)$. Therefore, A is quasi-equivalent to $E(A^*)$.

To prove the claim, consider the chain of dg submodules

$$\sigma^{\leq 0}D(A) \longrightarrow \sigma^{\leq 1}D(A) \longrightarrow \sigma^{\leq 2}D(A) \longrightarrow \ldots \longrightarrow D(A).$$

We have $D(A) = \bigcup_{p \geq 0} \sigma^{\leq p}D(A)$, so $D(A)$ is the homotopy colimit of $\sigma^{\leq p}D(A)$, *i.e.* there is a triangle

$$\bigoplus_{p\geq 0} \sigma^{\leq p}D(A) \xrightarrow{\text{id}-\text{shift}} \bigoplus_{p\geq 0} \sigma^{\leq p}D(A) \longrightarrow D(A) \longrightarrow \Sigma \bigoplus_{p\geq 0} \sigma^{\leq p}D(A).$$

Since $\sigma^{\leq p}D(A)$ belongs to $\mathcal{D}_{fd}(A) = \mathsf{thick}_{\mathcal{D}(A)}(S)$ (the equality is a consequence of [34, Proposition 2.1(b)]), it follows that $D(A) \in \mathsf{Loc}_{\mathcal{D}(A)}(S)$. $\square$

We remark that $E(A^*)$ has the form $(\widehat{kQ}, d)$, which is a complete dg-quiver algebra. The graded quiver Q is determined by the graded K-bimodule structure on $A^* = \bigoplus_{p\in\mathbb{Z}} \mathsf{Hom}_{\mathcal{D}(A)}(S, \Sigma^p S)$. More precisely, $Q_0 = \{1, \ldots, r\}$ and the number of arrows from i to j in degree p is the dimension of $\mathsf{Hom}_{\mathcal{D}(A)}(S_j, \Sigma^{1-p}S_i)$ over K. In particular, Q is finite and concentrated in non-positive degrees. The differential d is continuous and is determined by the A_∞-structure on A^*.

2.6. **Trivial extensions.** Assume that k is a field. Let Q be a finite tree quiver, and let R be the corresponding algebra with radical squared zero. For an R-bimodule M, define the trivial extension $A \ltimes M$ of A by M as follows. As a vector space it is $A \otimes M$. The multiplication is given by

$$(a, m)(a', m') = (aa', am' + ma').$$

Let $\lambda, \mu \colon Q_1 \to k^\times$ be two functions. For an R-bimodule M, define a new bimodule ${}^\lambda M^\mu$ by

$$\alpha \cdot m = \lambda(\alpha)\alpha m, \;\; m \cdot \alpha = \mu(\alpha)m\alpha, \;\text{ where } m \in M, \alpha \in Q_1.$$

Put $A(Q, \lambda, \mu) = A \ltimes {}^\lambda D(A)^\mu$.

Lemma 2.4. *$A(Q, \lambda, \mu)$ is isomorphic to $A(Q, \mathbf{1}, \mathbf{1})$, where $\mathbf{1} \colon Q_1 \to k^\times$ is the function with constant value* 1.

Proof. It is enough to show that as an R-bimodule ${}^\lambda D(A)^\mu$ is isomorphic to $D(A)$.

Fix a vertex $i \in Q_0$. Define $\lambda', \mu' \colon Q_1 \to k^\times$ by

$$\lambda'(\beta) = \begin{cases} \lambda(\beta) & \text{if } t(\beta) \neq i \\ 1 & \text{if } t(\beta) = i \end{cases}, \qquad \mu'(\beta) = \begin{cases} \mu(\beta) & \text{if } s(\beta) \neq i \\ 1 & \text{if } s(\beta) = i \end{cases}.$$

We claim that ${}^{\lambda}D(A)^{\mu} \cong {}^{\lambda'}D(A)^{\mu'}$ as R-bimodules. Then fixing an ordering $i_1 \cdots i_r$ of Q_0 and repeatedly applying the claim we obtain the desired result.

Now we prove the claim. Because Q is a true, there is at most one walk from any vertex to i. For $j \in Q_0$, put

$$f(j) = \begin{cases} \lambda(\alpha) & \text{if there is a walk from } j \text{ to } i \text{ ending with an arrow } \alpha, \\ \mu(\alpha) & \text{if there is a walk from } j \text{ to } i \text{ ending with the inverse of an arrow } \alpha, \\ 1 & \text{otherwise.} \end{cases}$$

For $\beta \in Q_1$, put

$$f(\beta) = \begin{cases} f(t(\beta)) & \text{if } t(\beta) \neq i, \\ f(s(\beta)) & \text{if } s(\beta) \neq i, \\ \mu(\beta) & \text{if } s(\beta) = i, \\ \lambda(\beta) & \text{if } t(\beta) = i. \end{cases}$$

Let $\{e_j^* | j \in Q_0\} \cup \{\beta^* | \beta \in Q_1\} \subseteq D(A)$ be the dual basis of $\{e_i | i \in Q_0\} \cup \{\beta | \beta \in Q_1\}$. Let $\varphi \colon {}^{\lambda}D(A)^{\mu} \cong {}^{\lambda'}D(A)^{\mu'}$ be the linear extension of $\beta^* \mapsto f(\beta)\beta^*$ and $e_j^* \mapsto f(j)e_j^*$. It is straightforward to check that φ is an A-bimodule isomorphism. $\square$

3. Silting reduction

Let $\mathcal{E}$ be a Frobenius k-category and $\mathcal{P}$ the full subcategory of projective-injective objects of $\mathcal{E}$. Put $\mathcal{D} = \mathcal{H}^b(\mathcal{E})/\mathcal{H}^b(\mathcal{P})$.

Lemma 3.1. *For $X, Y \in \mathcal{E}$, we have*

$$\begin{aligned} \mathsf{Hom}_{\mathcal{D}}(X, \Sigma^n Y) &= 0 \text{ for } n > 0, \\ \mathsf{Hom}_{\mathcal{D}}(X, Y) &= \mathsf{Hom}_{\underline{\mathcal{E}}}(X, Y), \\ \mathsf{Hom}_{\mathcal{D}}(X, \Sigma^n Y) &= \mathsf{Hom}_{\underline{\mathcal{E}}}(X, \Omega^{-n}(Y)) \text{ for } n < 0. \end{aligned}$$

Proof. The first formula will be proved in Step 2. The second and third formulas will be proved in Step 5.

Step 1: *In a left fraction $X \xleftarrow{s} Z \xrightarrow{f} \Sigma^n Y$, where $\mathsf{Cone}(s) \in \mathcal{H}^b(\mathcal{P})$, up to equivalence we can take Z to be a complex of the form*

$$X \xrightarrow{\alpha} P^0 \xrightarrow{d^0} P^1 \to \ldots \xrightarrow{d^{l-1}} P^l,$$

where $P^i \in \mathcal{P}$ and X is put in degree 0, and take s to be the natural projection from Z to X.

Form a triangle

$$Z \xrightarrow{s} X \xrightarrow{u} P \to \Sigma Z,$$

where $P \in \mathcal{H}^b(\mathcal{P})$ is of the form

$$P^m \xrightarrow{d^m} \ldots \to P^{-1} \xrightarrow{d^{-1}} P^0 \xrightarrow{d^0} P^1 \to \ldots \to P^l.$$

The morphism $u\colon X \to P$ is represented by a chain map $X \to P$, which is given by a morphism $\alpha\colon X \to P^0$ in $\mathcal{E}$ such that $d^0 \circ \alpha = 0$:

$$\begin{tikzcd} & & & X \arrow[d, "\alpha"] & & & \\ P^m \arrow[r, "d^m"] & \dots \arrow[r] & P^{-1} \arrow[r, "d^{-1}"] & P^0 \arrow[r, "d^0"] & P^1 \arrow[r] & \dots \arrow[r] & P^l \end{tikzcd}$$

So up to isomorphism $Z = \Sigma^{-1}\,\mathsf{Cone}(u)$ takes the form

$$P^m \xrightarrow{d^m} \dots \xrightarrow{\binom{0}{d^{-2}}} X \oplus P^{-1} \xrightarrow{(\alpha, d^{-1})} P^0 \xrightarrow{d^0} P^1 \longrightarrow \dots \longrightarrow P^l$$

and s is of the form

$$\begin{tikzcd} P^m \arrow[r, "d^m"] & \dots \arrow[r, "\binom{0}{d^{-2}}"] & X \oplus P^{-1} \arrow[r, "{(\alpha, d^{-1})}"] \arrow[d, "{(\mathrm{id}_X, 0)}"] & P^0 \arrow[r, "d^0"] & P^1 \arrow[r] & \dots \arrow[r] & P^l \\ & & X & & & & \end{tikzcd}$$

The complex Z'

$$X \xrightarrow{\alpha} P^0 \xrightarrow{d^0} P^1 \longrightarrow \dots \longrightarrow P^l$$

is a subcomplex of Z. The inclusion $Z' \hookrightarrow Z$ splits in each degree and the quotient complex lies in $\mathcal{H}^b(\mathcal{P})$. Let $t\colon Z' \to X$ be the natural projection and $g\colon Z' \hookrightarrow Z \xrightarrow{f} \Sigma^n Y$. Then $X \xleftarrow{s} Z \xrightarrow{f} \Sigma^n Y$ is equivalent to $X \xleftarrow{t} Z' \xrightarrow{g} \Sigma^n Y$.

Step 2: Let $X \xleftarrow{s} Z \xrightarrow{f} \Sigma^n Y$ be as in the statement of Step 1. If $n > 0$, then $\mathsf{Hom}_{\mathcal{H}^b(\mathcal{E})}(Z, \Sigma^n Y) = 0$, and hence $\mathsf{Hom}_{\mathcal{D}}(X, \Sigma^n Y) = 0$.

Step 3: *Let $n \leq 0$. In a left fraction $X \xleftarrow{s} Z \xrightarrow{f} \Sigma^n Y$, where $\mathsf{Cone}(s) \in \mathcal{H}^b(\mathcal{P})$, up to equivalence we can take Z to be a complex of the form*

$$X \xrightarrow{\alpha} P^0 \xrightarrow{d^0} P^1 \to \dots \xrightarrow{d^{-n-2}} P^{-n-1},$$

where $P^i \in \mathcal{P}$ and X is put in degree 0, and take s to be the natural projection from Z to X.

Let $X \xleftarrow{s} Z \xrightarrow{f} \Sigma^n Y$ be as in the statement of Step 1. Let Z'' be the complex

$$X \xrightarrow{\alpha} P^0 \xrightarrow{d^0} P^1 \to \dots \xrightarrow{d^{-n-2}} P^{-n-1}.$$

Then the quotient map $Z \to Z''$ splits in each degree and its kernel is in $\mathcal{H}^b(\mathcal{P})$. Moreover, any chain map $g\colon Z \to \Sigma^n Y$ is given by a morphism $\beta\colon P^{-n-1} \to Y$ in $\mathcal{E}$ such that $\beta \circ d^{-n-2} = 0$:

$$\begin{tikzcd} X \arrow[r, "\alpha"] & P^0 \arrow[r, "d^0"] & P^1 \arrow[r] & \dots \arrow[r, "d^{-n-2}"] & P^{-n-1} \arrow[r] \arrow[d, "\beta"] & \dots \arrow[r, "d^{l-1}"] & P^l \\ & & & & Y & & \end{tikzcd}$$

and factors through the quotient map $Z \to Z''$ to yield a chain map $g\colon Z'' \to Y$

$$\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[d, "\mathrm{id}"] & P^0 \arrow[r, "d^0"] \arrow[d, "\mathrm{id}"] & P^1 \arrow[r] \arrow[d, "\mathrm{id}"] & \dots \arrow[r, "d^{-n-2}"] & P^{-n-1} \arrow[r] \arrow[d, "\mathrm{id}"] & \dots \arrow[r, "d^{l-1}"] & P^l \\
X \arrow[r, "\alpha"] & P^0 \arrow[r, "d^0"] & P^1 \arrow[r] & \dots \arrow[r, "d^{-n-2}"] & P^{-n-1} \arrow[d, "\beta"] & & \\
& & & & Y & &
\end{tikzcd}$$

Let $t\colon Z'' \to X$ be the natural projection. Then $X \overset{s}{\leftarrow} Z \overset{f}{\to} \Sigma^n Y$ is equivalent to $X \overset{t}{\leftarrow} Z'' \overset{g}{\to} \Sigma^n Y$.

Step 4: Because $\mathcal{E}$ has enough projective-injective objects, there is a conflation $X \overset{\iota^0}{\to} I_X^0 \overset{\pi^0}{\to} K^1$ with $I_X^0 \in \mathcal{P}$. Put $K^0 = X$. By induction, there exist objects K^i ($i \in \mathbb{N}$) and conflations $K^i \overset{\iota^i}{\to} I_X^i \overset{\pi^i}{\to} K^{i+1}$ with $I_X^i \in \mathcal{P}$. Put $X(0) = X$ and let $X(l)$ ($l \geq 1$) be the complex

$$X \overset{\alpha_X}{\to} I_X^0 \overset{d_X^0}{\to} I_X^1 \to \dots \overset{d_X^{l-2}}{\to} I_X^{l-1},$$

where $\alpha_X = \iota^0$ and $d_X^i = \iota^{i+1} \circ \pi^i$.

Let $n \leq 0$. In a left fraction $X \overset{s}{\leftarrow} Z \overset{f}{\to} \Sigma^n Y$, where $\mathsf{Cone}(s) \in \mathcal{H}^b(\mathcal{P})$, up to equivalence we can take $Z = X(-n)$ and take s to be the natural projection from Z to X.

Let $X \overset{s}{\leftarrow} Z \overset{f}{\to} \Sigma^n Y$ be as in the statement of Step 3. Then there is a chain map

$$\begin{tikzcd}
X \arrow[r, "\iota^0"] \arrow[d, "\mathrm{id}"] & I_X^0 \arrow[r, "\pi^0"] \arrow[d] & K^1 \arrow[d] & & \\
X \arrow[r, "\alpha"] & P^0 \arrow[r, "d_0"] & P^1 \arrow[r] & \dots \arrow[r, "d^{-n-2}"] & P^{-n-1}
\end{tikzcd}$$

and by induction we obtain a chain map $u\colon X(-n) \to Z$

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[d, "\mathrm{id}"] & I_X^0 \arrow[r, "d_X^0"] \arrow[d] & I_X^1 \arrow[r] \arrow[d] & \dots \arrow[r, "d_X^{-n-2}"] & I_X^{-n-1} \arrow[d] \\
X \arrow[r, "\alpha"] & P^0 \arrow[r, "d_0"] & P^1 \arrow[r] & \dots \arrow[r, "d^{-n-2}"] & P^{-n-1}
\end{tikzcd}$$

Let $t\colon X(-n) \to X$ be the natural projection and let $g = f \circ u$. Then $X \overset{s}{\leftarrow} Z \overset{f}{\to} \Sigma^n Y$ is equivalent to $X \overset{t}{\leftarrow} X(-n) \overset{g}{\to} \Sigma^n Y$.

Step 5: Let $n \geq 0$. A chain map $X(-n) \to \Sigma^n Y$ is given by a morphism $\beta\colon I_X^{-n-1} \to Y$ in $\mathcal{E}$ such that $\beta \circ d_X^{-n-2} = 0$:

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] & I_X^0 \arrow[r, "d_X^0"] & I_X^1 \arrow[r] & \dots \arrow[r, "d_X^{-n-2}"] & I_X^{-n-1} \arrow[d, "\beta"] \\
& & & & Y
\end{tikzcd}$$

and thus it is in bijection with morphisms $\gamma\colon K^{-n} \to Y$. Therefore by Step 4, there is a surjective map

$$\mathsf{Hom}_{\mathcal{E}}(K^{-n}, Y) \longrightarrow \mathsf{Hom}_{\mathcal{D}}(X, \Sigma^n Y).$$

We claim that the kernel consists of the morphisms factoring though a projective-injective, which completes the proof.

Let $\gamma\colon K^{-n} \to Y$ be in the kernel of the above map, and $X \xleftarrow{s} X(-n) \xrightarrow{f} \Sigma^n Y$ be the corresponding left fraction. Then there exists a chain $t\colon Z \to X(-n)$ such that $f \circ t = 0$, *i.e.* $\gamma \circ \pi^{-n-1} \circ t^{-n} = 0$. Following the constructions in Steps 1, 3 and 4, we obtain a diagram of chain maps

$$\begin{tikzcd}
 & Z' \arrow[dl] \arrow[dr] & & X(-n) \arrow[dl, "u"'] \\
Z \arrow[dr, "t"] & & Z'' \arrow[dl, "t'"'] & \\
 & X(-n) & &
\end{tikzcd}$$

such that the square is commutative and $t'^0 \circ u^0 = \mathrm{id}_X$. So there is a chain map

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[d, "\mathrm{id}"] & I_X^0 \arrow[r, "d_X^0"] \arrow[d, "t^1 \circ u^1"] & I_X^1 \arrow[r] \arrow[d, "t^2 \circ u^2"] & \dots \arrow[r, "d_X^{-n-2}"] & I_X^{-n-1} \arrow[d, "t^{-n} \circ u^{-n}"] \\
X \arrow[r, "\alpha_X"] & I_X^0 \arrow[r, "d_X^0"] & I_X^1 \arrow[r] & \dots \arrow[r, "d_X^{-n-2}"] & I_X^{-n-1}
\end{tikzcd}$$

Therefore there exist morphisms $a\colon I_X^{-n-1} \to I_X^{-n-2}$ and $b\colon I_X^{-n} \to I_X^{-n-1}$ such that $\mathrm{id}_{I_X^{-n-1}} - t^{-n} \circ u^{-n} = d_X^{-n-2} \circ a + b \circ d_X^{-n-1}$. So

$$\begin{aligned}
\gamma \circ \pi^{-n-1} &= \gamma \circ \pi^{-n-1} \circ (\mathrm{id}_{I_X^{-n-1}} - t^{-n} \circ u^{-n}) \\
&= \gamma \circ \pi^{-n-1} \circ (d_X^{-n-2} \circ a + b \circ d_X^{-n-1}) \\
&= \gamma \circ \pi^{-n-1} \circ b \circ d_X^{-n-1} \\
&= \gamma \circ \pi^{-n-1} \circ b \circ \iota^{-n} \circ \pi^{-n-1}.
\end{aligned}$$

Since π^{-n-1} is an epimorphism, it follows that $\gamma = \gamma \circ \pi^{-n-1} \circ b \circ \iota^{-n}$, which factors through I_X^{-n}. $\square$

Corollary 3.2. *Let $\mathcal{M}$ be an additive subcategory of $\mathcal{E}$ containing $\mathcal{P}$. Then the essential image of the composite functor $\mathcal{M} \to \mathcal{H}^b(\mathcal{M}) \to \mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P})$ is a silting subcategory and is equivalent to the additive quotient $\frac{\mathcal{M}}{\mathcal{P}}$.*

4. Relative singularity category

Let k be an algebraically closed field and $d \geq 1$. A Hom-finite Krull–Schmidt triangulated k-category $\mathcal{C}$ is said to be *d-Calabi–Yau* if Σ^d is a Serre functor, that is, there is a bifunctorial isomorphism for $M, N \in \mathcal{C}$

$$D\,\mathsf{Hom}(M, N) \cong \mathsf{Hom}(N, \Sigma^d M).$$

Let $\mathcal{E}$ be a Frobenius k-category, $\mathcal{P}$ its full subcategory of projective-injective objects and $\mathcal{C} = \underline{\mathcal{E}}$ the stable category. Assume that $\mathcal{C}$ is a Hom-finite Krull–Schmidt d-Calabi–Yau algebraic triangulated category, and $T \in \mathcal{C}$ a basic d-cluster-tilting object. Then T is a classical generator of $\mathcal{C}$, by [42, Theorem 5.4 (a)].

Theorem 4.1. *There is a dg k-algebra B such that*

(a) $H^n(B) = 0$ *for* $n > 0$, $H^0(B) \cong \mathsf{End}_{\mathcal{C}}(T)$ *and* $H^p(B) \cong \mathsf{Hom}_{\mathcal{C}}(T, \Sigma^n T)$ *for* $n < 0$*;*

(b) $\mathsf{per}(B) \supseteq \mathcal{D}_{fd}(B)$*;*

(c) *there is a triangle equivalence* $\mathsf{per}(B)/\mathcal{D}_{fd}(B) \to \mathcal{C}$ *which takes* B *to* T*;*

(d) *there is a bifunctorial isomorphism for* $X \in \mathsf{mod}\, H^0(B)$ *and* $Y \in \mathcal{D}_{fd}(B)$

$$D\,\mathsf{Hom}_{\mathcal{D}_{fd}(B)}(X, Y) \cong \mathsf{Hom}_{\mathcal{D}_{fd}(B)}(Y, \Sigma^{d+1} X).$$

Remark 4.2. *If $\mathcal{P}$ is skeletally small, we could prove (a), (b) and (c) by establishing several-object versions of some results in* [34, 35]*. Here we use* [55]*. It is claimed in* [62] *that $\mathcal{D}_{fd}(B)$ (equivalent to $H^0(\mathcal{B})$ there) is $(d+1)$-Calabi–Yau. This turns out to be misunderstanding of* [42, Proposition 5.4].

Proof. Let $\mathcal{M}$ be the preimage of $\mathcal{T} = \mathsf{add}_{\mathcal{C}}(T)$ in $\mathcal{E}$. A complex X in $\mathcal{H}^b(\mathcal{M})$ is said to be $\mathcal{E}$-*acyclic* if there are conflations $Z^i \xrightarrow{\iota^i} X^i \xrightarrow{\pi^i} Z^{i+1}$ such that $d_X^i = \iota^{i+1} \circ \pi^i$ for $i \in \mathbb{Z}$. Let $\mathcal{H}^b_{\mathcal{E}-ac}(\mathcal{M})$ be the full subcategory of $\mathcal{H}^b(\mathcal{M})$ of $\mathcal{E}$-acyclic complexes. Then $\mathsf{Hom}_{\mathcal{H}^b(\mathcal{M})}(P, X) = 0$ for $P \in \mathcal{P}$ and $X \in \mathcal{H}^b_{\mathcal{E}-ac}(\mathcal{M})$. So $\mathcal{H}^b(\mathcal{P})$ is left orthogonal to $\mathcal{H}^b_{\mathcal{E}-ac}(\mathcal{M})$, and we can view $\mathcal{H}^b_{\mathcal{E}-ac}(\mathcal{M})$ as a full subcategory of the triangle quotient $\mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P})$.

By [55, Proposition 3], there is a short exact sequence of triangulated categories

$$0 \longrightarrow \mathcal{H}^b_{\mathcal{E}-ac}(\mathcal{M}) \longrightarrow \mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) \longrightarrow \mathcal{C} \longrightarrow 0,$$

which induces a triangle equivalence $(\mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}))/\mathcal{H}^b_{\mathcal{E}-ac}(\mathcal{M}) \to \mathcal{C}$. Let $\mathsf{per}(\mathcal{M})$ be the full subcategory of the derived category of modules over $\mathcal{M}$ classically generated by all representable functors and let $\mathsf{per}_{\underline{\mathcal{M}}}(\mathcal{M})$ be its full subcategory consisting of complexes whose cohomologies are in $\mathsf{mod}\,\underline{\mathcal{M}}$ and $\mathsf{per}(\mathcal{P})$ be its thick subcategory generated by $P^\wedge$, $P \in \mathcal{P}$. By [55, Lemma 7], the triangle equivalence $\mathcal{H}^b(\mathcal{M}) \to \mathsf{per}(\mathcal{M})$ induces a triangle equivalence $\mathcal{H}^b_{\mathcal{E}-ac}(\mathcal{M}) \to \mathsf{per}_{\underline{\mathcal{M}}}(\mathcal{M})$. So there is a triangle equivalence $(\mathsf{per}(\mathcal{M})/\mathsf{per}(\mathcal{P}))/\mathsf{per}_{\underline{\mathcal{M}}}(\mathcal{M}) \to \mathcal{C}$.

By Lemma 3.1, the representable functors form a silting subcategory of $\mathsf{per}(\mathcal{M})/\mathsf{per}(\mathcal{P})$ and is equivalent to $\underline{\mathcal{M}}$. Let $\tilde{T}$ be a basic additive generator of this silting subcategory corresponding to T. Since $\mathsf{per}(\mathcal{M})/\mathsf{per}(\mathcal{P})$ is an algebraic triangulated category, it follows that there is a dg k-algebra B together with a triangle equivalence $F\colon \mathsf{per}(\mathcal{M})/\mathsf{per}(\mathcal{P}) \to \mathsf{per}(B)$ taking $\tilde{T}$ to B. As a consequence $H^n(B) = 0$ for $n > 0$, $H^0(B) \cong \mathsf{End}_{\mathcal{C}}(T)$, and $H^p(B) \cong \mathsf{Hom}_{\mathcal{C}}(T, \Sigma^n T)$ for $n < 0$. This proves (a).

Next we show that the equivalence $F\colon \mathsf{per}(\mathcal{M})/\mathsf{per}(\mathcal{P}) \to \mathsf{per}(B)$ restricts to an equivalence $\mathsf{per}_{\underline{\mathcal{M}}}(\mathcal{M}) \to \mathcal{D}_{fd}(B)$. Then (b) and (c) follows.

For $M \in \mathcal{M}$ and $X \in \mathsf{per}_{\underline{\mathcal{M}}}(\mathcal{M})$, the space $\bigoplus_{n\in\mathbb{Z}} \mathsf{Hom}_{\mathsf{per}(\mathcal{M})/\mathsf{per}(\mathcal{P})}(M^\wedge, \Sigma^n X)$, being isomorphic to $\bigoplus_{n\in\mathbb{Z}} \mathsf{Hom}_{\mathsf{per}(\mathcal{M})}(M^\wedge, \Sigma^n X)$, is finite-dimensional. It follows that

$$\bigoplus_{n\in\mathbb{Z}} \mathsf{Hom}_{\mathsf{per}(B)}(B, \Sigma^n F(X))$$

is finite-dimensional, implying that $F(X) \in \mathcal{D}_{fd}(B)$. To show that the restriction is dense, take a simple module S over $\underline{\mathcal{M}}$. Then $\mathsf{Hom}_{\mathsf{per}(\mathcal{M})}(\tilde{T}, \Sigma^n S) \cong \mathsf{Hom}_{\mathsf{per}(\mathcal{M})/\mathsf{per}(\mathcal{P})}(\tilde{T}, \Sigma^n S)$ is 1-dimensional for $n = 0$ and is trivial for $n \neq 0$. Therefore $\mathsf{Hom}_{\mathsf{per}(B)}(B, \Sigma^n F(S))$ is 1-dimensional for $n = 0$ and is trivial for $n \neq 0$. So $F(S)$ is a simple module over $H^0(B)$.

But $\underline{\mathcal{M}}$ is equivalent to $\mathsf{proj}\, H^0(B)$, implying that F takes a complete set of pairwise non-isomorphic simple modules over $\underline{\mathcal{M}}$ to a complete set of pairwise non-isomorphic simple modules over $H^0(B)$. Now $\mathcal{D}_{fd}(B)$ is classically generated by simple $H^0(B)$-modules. It follows that F restricts to an equivalence $\mathsf{per}_{\underline{\mathcal{M}}}(\mathcal{M}) \to \mathcal{D}_{fd}(B)$.

Finally it follows by [42, Proposition 5.4] that there is a bifunctorial isomorphism for $X \in \mathsf{mod}\, \underline{\mathcal{M}}$ and $Y \in \mathsf{per}_{\underline{\mathcal{M}}}(\mathcal{M})$

$$D\,\mathsf{Hom}(X, Y) \cong \mathsf{Hom}(Y, \Sigma^{d+1} X).$$

We obtain (d) by applying the equivalence F. □

If $\mathcal{M}$ has an additive generator M and $\mathcal{P}$ has an additive generator P, then there is a canonical triangle equivalence $\mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) \to \mathcal{H}^b(\mathsf{proj}\, A)/\,\mathsf{thick}(eA)$, where $A = \mathsf{End}(M)$, and $e = \mathrm{id}_P$. In this case, B can be obtained using the constructions in [35, Section 7].

5. Calabi–Yau categories with cluster-tilting objects

Let k be a field. In this section we will apply our previous results to study the canonical form of Calabi–Yau triangulated categories with cluster-tilting objects.

5.1. **Ginzburg dg algebras.** Let A be a pseudo-compact dg k-algebra, see [44, 64]. Let $A^e = A^{op} \widehat{\otimes} A$ be the enveloping algebra, *i.e.* the completion of $A^{op} \otimes A$ with respect to the topology induced from A. The dg algebra A is *topologically homologically smooth* if $A \in \mathsf{per}(A^e)$, and is *bimodule d-Calabi–Yau* if in addition there is an isomorphism $\eta\colon \mathsf{RHom}_{A^e}(A, A^e) \xrightarrow{\cong} \Sigma^d A$ in $\mathcal{D}(A^e)$. In the original definition of Ginzburg, η was assumed to be self-dual, but this turns out to be automatic, see [64, Appendix 14].

Lemma 5.1. *Let A be a bimodule d-Calabi–Yau pseudo-compact dg algebra. Then $\mathcal{D}_{fd}(A)$ is d-Calabi–Yau as a triangulated category.*

Proof. This immediately follows from [44, Lemma A.16]. □

Nice examples of bimodule Calabi–Yau dg algebras include complete Ginzburg dg algebras of quivers with potential. Let Q be a finite quiver and W be a formal combination of cycles of Q. The pair (Q, W) is called a *quiver with potential.* From Q we define a graded quiver $\tilde{Q}$, which has the same vertices as Q and whose arrows are

- the arrows of Q (they all have degree 0),
- an arrow $a^* : j \to i$ of degree -1 for each arrow $a : i \to j$ of Q,
- a loop $t_i : i \to i$ of degree -2 for each vertex i of Q.

The *complete Ginzburg dg algebra* $\widehat{\Gamma}(Q, W)$, introduced by Ginzburg in [25], is the dg algebra whose underlying graded algebra is the complete path algebra $\widehat{k\tilde{Q}}$ and whose differential is the unique continuous linear differential which satisfies the graded Leibniz rule and which takes the following value on the arrows of $\tilde{Q}$:

- $d(a) = 0$ for each arrow a of Q,
- $d(a^*) = \partial_a W$ for each arrow a of Q,
- $d(t_i) = e_i(\sum_a (aa^* - a^*a))e_i$ for each vertex i of Q, where e_i is the trivial path at i and the sum runs over the set of arrows of Q.

Here for an arrow a of Q, the *cyclic derivative* ∂_a is the unique continuous linear map which takes a cycle c to the sum $\sum_{c=uav} vu$ taken over all decompositions of the cycle c (where u and v are possibly trivial paths).

The *complete Jacobian algebra* $\widehat{J}(Q,W)$ is the 0-th cohomology of $\widehat{\Gamma}(Q,W)$. More precisely,

$$\widehat{J}(Q,W) = \widehat{kQ}/\overline{(\partial_a W \mid a \in Q_1)}.$$

Theorem 5.2. ([44, Theorem A.17], [41, Theorem 6.3]) *The dg algebra $\widehat{\Gamma}(Q,W)$ is topologically homologically smooth and bimodule 3-Calabi–Yau.*

5.2. **Cluster categories.** Let A be a pseudo-compact dg k-algebra such that

- A is non-positive,
- A is topologically homologically smooth,
- A is bimodule 3-Calabi–Yau,
- $H^0(A)$ is finite-dimensional.

Then $\mathcal{D}_{fd}(A) \subseteq \mathsf{per}(A)$. Set

$$\mathcal{C}_A := \mathsf{per}(A)/\mathcal{D}_{fd}(A),$$

and call it the *cluster category* of A.

Theorem 5.3. ([2, Theorem 2.1], [44, Theorem A.21], [26, Theorem 2.2], [33, Theorem 5.8]) *The category $\mathcal{C}_A$ is 2-Calabi–Yau, and the image of A under the canonical projection $\mathsf{per}(A) \to \mathcal{C}_A$ is a cluster-tilting object whose endomorphism algebra is $H^0(A)$.*

For example, for a quiver with potential (Q,W) such that the complete Jacobian algebra $\widehat{J}(Q,W)$ is finite-dimensional, the cluster category

$$\mathcal{C}_{(Q,W)} := \mathcal{C}_{\widehat{\Gamma}(Q,W)}$$

is 2-Calabi–Yau, and the image of $\widehat{\Gamma}(Q,W)$ in $\mathcal{C}_{(Q,W)}$ is a (2-)cluster-tilting object whose endomorphism algebra is $\widehat{J}(Q,W)$.

We will call the image of A in $\mathcal{C}_A$ the *standard cluster-tilting object.*

5.3. **Calabi–Yau categories with cluster-tilting objects.** We propose the following conjecture (*cf.* [1, Summary of results, Part 2, Perspectives])..

Conjecture 5.4. *Let k be algebraically closed of characteristic zero. Let $\mathcal{C}$ be a 2-Calabi–Yau algebraic triangulated k-category with a cluster-tilting object T. Then $\mathcal{C}$ is triangle equivalent to the cluster category of some quiver with potential, with the equivalence sending T to the standard cluster-tilting object.*

Let B be as in Theorem 4.1. Then B is non-positive and topologically homologically smooth such that $\mathcal{C}$ is triangle equivalent to $\mathsf{per}(B)/\mathcal{D}_{fd}(B)$. If B is quasi-equivalent to an exact 3-bimodule dg algebra in the sense [64, Section 1, Definition], then by [64, Theorem B], B is quasi-equivalent to the complete Ginzburg dg algebra of some quiver with potential (Q,W), and therefore $\mathcal{C}$ is triangle equivalent to the cluster category $\mathcal{C}_{(Q,W)}$. Therefore we propose the following question. If it has a positive answer, then Conjecture 5.4 holds true. Note that (ii) implies (i) by Lemma 5.1.

Question 5.5. *Let A be a non-positive pseudo-compact topologically homologically smooth dg k-algebra and let K be the direct product of a finite copies of k. Assume that*

(a) *there is an injective homomorphism* $\eta : K \to A$ *and a surjective homomorphism* $\varepsilon : A \to K$ *of dg algebras such that* $\varepsilon \circ \eta = id_K$*,*
(b) $\mathcal{D}_{fd}(A) = \mathsf{thick}(K)$*, where* K *is viewed as a dg* A*-module via* ε*,*
(c) $H^0(A)$ *is finite-dimensional over* k*.*

Are the following conditions equivalent?

(i) [1] *there is a bifunctorial isomorphism for* $X \in \mathsf{mod}\, H^0(A)$ *and* $Y \in \mathcal{D}_{fd}(A)$

$$D\,\mathsf{Hom}_{\mathcal{D}_{fd}(A)}(X, Y) \cong \mathsf{Hom}_{\mathcal{D}_{fd}(A)}(Y, \Sigma^3 X),$$

(ii) A *is quasi-equivalent to an exact* 3*-Calabi–Yau dg algebra.*

In [3], Amiot asked the following question. If Conjecture 5.4 holds true, then this question has a positive answer.

Question 5.6. ([3, Question 2.20.1]) *Let* k *be algebraically closed of characteristic zero. Let* $\mathcal{C}$ *be a 2-Calabi–Yau algebraic triangulated category with a cluster-tilting object* T*. Is* $\mathsf{End}_{\mathcal{C}}(T)$ *isomorphic to the complete Jacobian algebra of some quiver with potential?*

Note that we have replaced 'Jacobian algebra' by 'complete Jacobian algebra'. The original question has a negative answer, as there are quivers with potentials whose complete Jacobian algebras are not non-complete Jacobian algebras of any quiver with potential, see [57, Example 4.3] for an example.

Moreover, we have put an extra assumption on the characteristic of the field, as when k is of positive characteristic, the answer to the question is negative. For example, if k is of characteristic $p > 0$, then $k[x]/(x^{p-1})$ is not a Jacobian algebra. However, take Γ as the dg algebra whose underlying graded algebra is $k\langle\langle x, x^*, t\rangle\rangle$ with $\deg(x) = 0$, $\deg(x^*) = -1$ and $\deg(t) = -2$, and whose differential d is defined by

$$d(x) = 0,\ d(x^*) = x^{p-1},\ d(t) = xx^* - x^*x.$$

It is straightforward to check that Γ satisfies the assumptions of Theorem 5.3, so $k[x]/(x^{p-1}) = H^0\Gamma$ is the endomorphism of a cluster-tilting object in a 2-Calabi–Yau algebraic triangulated category. More examples can be found in [46].

5.4. **Keller–Reiten's recognition theorem.** Let Q be an acyclic quiver and $d \geq 2$. Define the d-cluster category of Q as the orbit category

$$\mathcal{C}_Q^{(d)} := \mathcal{D}^b(\mathsf{mod}\, kQ)/\tau^{-1} \circ \Sigma^{d-1}.$$

It is a d-Calabi–Yau triangulated category ([39, Section 8] and [36]), and the image T of kQ is a d-cluster-tilting object with endomorphism algebra $\mathsf{End}_{\mathcal{C}_Q}(T) = kQ$ ([9, Proposition 1.7(d) and Theorem 3.3(b)], [42, Proposition 5.6]). Moreover, $\mathsf{Hom}_{\mathcal{C}_Q}(T) = 0$ for $-d+2 \leq p \leq -1$ ([43, Lemma 4.1]).

Theorem 5.7 ([43, Theorems 2.1 and 4.2])**.** *Let* $\mathcal{C}$ *be a* d*-Calabi–Yau algebraic triangulated category. Assume that* T *is a* d*-cluster-tilting object of* $\mathcal{C}$ *such that* $\mathsf{Hom}_{\mathcal{C}}(T, \Sigma^p T) = 0$ *for* $-d+2 \leq p \leq -1$ *and that* $\mathsf{End}_{\mathcal{C}}(T) \cong kQ$ *for some finite acyclic quiver* Q*. Then* $\mathcal{C}$ *is triangle equivalent to* $\mathcal{C}_Q^{(d)}$*.*

We give a proof of this theorem under the extra conditions that k is algebraically closed and that Q is a tree quiver, that is, there are no cycles in the underlying graph of Q.

[1] We point out that this condition is weaker than the one claimed in [34, Section 7.2].

Proof of Theorem 5.7. Assume that k is algebraically closed and that Q is a tree quiver with vertices $\{1,\ldots,r\}$.

Let B be the dg algebra obtained in Theorem 4.1. Then $\mathcal{C}$ is triangle equivalent to $\mathsf{per}(B)/\mathcal{D}_{fd}(B)$. We will show that up to quasi-equivalence B depends not on $\mathcal{C}$ but on Q only. As $\mathcal{C}_Q^{(d)}$ satisfies all the assumptions, it is also triangle equivalent to $\mathsf{per}(B)/\mathcal{D}_{fd}(B)$. Therefore $\mathcal{C}$ is triangle equivalent to $\mathcal{C}_Q^{(d)}$.

By Theorem 4.1, B is non-positive and satisfies the three conditions (FD), (BE) and (HS) in Section 2.5. Thus by Theorem 2.3, we may assume that B is the dual bar construction of its A_∞-Koszul dual B^*, and therefore $B=(kQ',d)$ is a complete dg-quiver algebra. By Theorem 4.1(a), $H^0(B)\cong kQ$ is hereditary, and $H^p(B)=0$ for all $-d+2\leq p\leq -1$. So applying Lemma 2.1 we know that Q' has no arrows in degrees $-d+2\leq p\leq -1$. This implies that $(B^*)^p=0$ for $2\leq p\leq d-1$.

As a graded vector space $B^*=\bigoplus_{p\in\mathbb{Z}}\bigoplus_{i,j=1}^r \mathsf{Hom}_{\mathcal{D}(B)}(S_i,\Sigma^pS_j)$. By Theorem 4.1(d) there is an isomorphism

$$D\,\mathsf{Hom}(S_i,\Sigma^pS_j)\cong \mathsf{Hom}(S_j,\Sigma^{d+1-p}S_i). \tag{5.1}$$

Since $(B^*)^p=0$ for $p<0$, this implies that $(B^*)^p=0$ for $p\neq 0,1,d,d+1$. Let $e_i=\mathrm{id}_{S_i}$. Then $K:=(B^*)^0$ has a basis $\{e_1,\ldots,e_r\}$. Let $\{e_1^*,\ldots,e_r^*\}$ be the dual basis in $(B^*)^{d+1}$. Let $Q^{gr,op}$ be the opposite quiver of Q with all arrows in degree 1. Since $H^0(B)\cong kQ$, it follows from the dual bar construction that $(B^*)^1$ has a basis $Q_1^{gr,op}$ and that $m_n|_{((B^*)^1)^{\otimes_K n}}\colon ((B^*)^1)^{\otimes_K}\to (B^*)^2$ is trivial for any $n\geq 2$. Let $\{\alpha^*\mid \alpha\in Q_1^{gr,op}\}$ be the dual basis of $Q_1^{gr,op}$ in $(B^*)^d$.

Recall that B^* is strictly unital over K and $m_n|_{((B^*)^1)^{\otimes_K n}}\colon ((B^*)^1)^{\otimes_K}\to (B^*)^2$ is trivial for any $n\geq 2$. Therefore for degree reasons and due to the form of $(B^*)^{d+1}$, the only possible non-trivial multiplication on $(B^*)^{\geq 1}$ is of the form

$$m_n(\alpha_1\otimes\cdots\otimes\alpha_{s-1}\otimes\alpha_s^*\otimes\alpha_{s+1}\otimes\cdots\otimes\alpha_n),$$

where $\alpha_{s+1}\cdots\alpha_n\alpha_1\cdots\alpha_{s-1}$ is a path parallel to α_s. Since Q is a tree quiver, this is not possible unless $n=2$ and $\alpha_1=\alpha_2$. Because the isomorphism (5.1) is compatible with compositions (see for example [28, Section 2]), we have $\alpha\alpha^*\neq 0$ and $\alpha^*\alpha\neq 0$ for any $\alpha\in Q_1^{gr,op}$. In other words, B^* is a graded algebra and forgetting the grading $B^*=A(Q^{gr,op},\lambda,\mu)$ for some functions $\lambda,\mu\colon Q_1^{gr,op}\to k$. Therefore $B^*\cong A(Q^{gr,op},\mathbf{1},\mathbf{1})$ depends on Q only. So does B. □

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