GENERAL COMODULE-CONTRAMODULE CORRESPONDENCE

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Abstract. This paper is a fundamental study of comodules and contramodules over a comonoid in a closed monoidal category. We study both algebraic and homotopical aspects of them. Algebraically, we enrich both comodule and contramodule categories over the original category, construct enriched functors between them and enriched adjunctions between the functors. Homotopically, for simplicial sets and topological spaces, we investigate the categories of comodules and contramodules and relations between them.

Comodules and contramodules appear in the work of Eilenberg and Moore [9] in the 1960's. Comodules have proved to be important in many areas of mathematics, for example Hopf algebras, representations of algebraic groups, combinatorics. However, contramodules had very little impact, and they remained a curiosity for 40 years. In the 2000's Positselski took up the theory of contramodules because they were a key technical tool in his work on the semi-infinite cohomology in the geometric Langlands program. Much of this work appears in [25], published in 2010. A key concept of the theory is the comodule-contramodule correspondence. In the current paper we study both the categorical and homotopical aspects of the comodule-contramodule correspondence in general categories.

Our work builds on that of Böhm, Brzeziński and Wisbauer [5], and also Hyland, López Franco, and Vasilakopoulou [17]. They set up a framework for studying comodules and contramodules in general categories in terms of an adjoint pair of endofunctors $(T \dashv F)$ on a category $\mathcal{C}$ such that T is a comonad and F is monad.

In Chapter 1, we describe the conceptual categorical framework for studying the comodule-contramodule correspondence. A key observation is that we need to assume that $\mathcal{C}$ is a biclosed monoidal category. This means that if X and Y are objects in $\mathcal{C}$, then in addition to the usual *external hom-set* $\mathcal{C}(X,Y)$ we have an *internal hom-object* $[X,Y]_{\mathcal{C}}$ which is an object in $\mathcal{C}$. Now we need to be clear that there are two notions of an adjoint pair of functors – the usual one, which we will call *externally adjoint*, involving external hom-sets, and the notion of *internally adjoint*, involving internal hom-objects. Being internally adjoint implies being externally adjoint but the converse is not true. Indeed, the notion of internally adjoint is surprisingly restrictive.

Now suppose we have a biclosed monoidal category $\mathcal{C}$ and a pair $(T \dashv F)$ of internally adjoint endofunctors on $\mathcal{C}$ such that T is a comonad and F is a monad. Then, as in [5], we can set up the category of T-comodules $\mathcal{C}_T$ and the category

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of F-modules $\mathcal{C}^F$. We will usually refer to $\mathcal{C}_T$ as the *category of comodules* and to $\mathcal{C}^F$ as the category of *contramodules*. Our first main result is combination of Propositions 1.20 and 1.22:

Hauptsatz 1. *Both $\mathcal{C}_T$ and $\mathcal{C}^F$ are categories enriched in $\mathcal{C}$.*

Our next piece of work is to construct a pair of enriched functors

$$\mathcal{L} : \mathcal{C}^F \rightleftarrows \mathcal{C}_T : \mathcal{R} .$$

Hauptsatz 2. *(Theorem 1.24) The pair $(\mathcal{L} \dashv \mathcal{R})$ is a $\mathcal{C}$-enriched adjoint pair.*

Here the term *$\mathcal{C}$-enriched adjoint* refers to the appropriate notion of adjointness for enriched categories. These two functors are what we mean by the comodule-contramodule correspondence in a biclosed monoidal category.

The proofs require some subtle work on extranaturality in enriched categories which appears in Section 1.6. It is of independent interest.

In Chapter 2 we discuss examples. In Sections 2.1 and 2.2 the base category is the category of vector spaces. Since it is the main example influencing Positselski, we go into further details about comodules and contramodules in this category. In Sections 2.3 and 2.4, the base category is the category of sets $\mathcal{S}ets$. There the main result can be summarised as follows:

Hauptsatz 3. *Suppose $(T \dashv F)$ originate from a comonoid ✠ in $\mathcal{S}ets$.*

1. *Every object of $\mathcal{S}ets_T$ is isomorphic to $\coprod_{x\in ✠} M_x$, where M_x is a collection of sets, parametrised by ✠.*
2. *Every object of $\mathcal{S}ets^F$ is isomorphic to $\prod_{x\in ✠} P_x$, where P_x is a collection of sets, parametrised by ✠.*
3. *The functors $\mathcal{L}$ and $\mathcal{R}$ can be described as*

$$\mathcal{L}(\prod_{x\in ✠} P_x) = \coprod_{x\in ✠} P_x\,, \quad \mathcal{R}(\coprod_{x\in ✠} M_x) = \prod_{x\in ✠} M_x\,.$$

The final section 2.5 contains a brief discussion of simplicial sets.

In Chapter 3 we start getting into homotopy theory. The motivation for this is that one of the main theorems due to Positselski is that in the algebraic context of comodules over a coalgebra the comodule-contramodule correspondence defines an equivalence between the coderived category of comodules and the contraderived category of contramodules. It is natural to think about such a theorem in terms of Quillen's model categories.

A model category is a category $\mathcal{M}$ together with three distinguished classes of morphisms: *cofibrations*, *fibrations* and *weak equivalences* satisfying appropriate axioms. If $\mathcal{M}$, $\mathcal{N}$ are model categories, a Quillen adjunction between them is a pair of adjoint functors $L : \mathcal{N} \rightleftarrows \mathcal{M} : R$, satisfying certain axioms. Further axioms turn a Quillen adjunction into a Quillen equivalence.

Let $\mathcal{C}$ be our base biclosed *monoidal model category* with an internally adjoint pair of functors $(T \dashv F)$ that define comodules and contramodules. A *monoidal model category* is the natural notion of a category with a compatible monoidal structure and model structure. We can form the categories $\mathcal{C}_T$ and $\mathcal{C}^F$. There are forgetful functors $\mathcal{C}_T \to \mathcal{C}$ and $\mathcal{C}^F \to \mathcal{C}$. Under mild restrictions, we can use the idea of transferring model structures, utilising these functors to define model structures on $\mathcal{C}_T$ and $\mathcal{C}^F$. Our main result of the section is Theorem 3.7 and its simplicial counterpart Theorem 3.8:

Hauptsatz 4. *Under certain restrictions, for instance, satisfied by the category of simplicial sets, there exist a left Bousfield localisation $L(\mathcal{C}^F)$ and a right Bousfield localisation $R(\mathcal{C}_T)$ such that the comodule-contramodule correspondence $(\mathcal{L} \dashv \mathcal{R})$ induces a Quillen equivalence between them.*

Clearly, a lot is going on here. Hence, the main aim of Chapter 3 is to organise this into a coherent picture.

In Chapter 4 the base category is the category $\mathcal{W}$ of compactly generated, weakly Hausdorff spaces, the most standard convenient category of topological spaces. A comonoid in $\mathcal{W}$ is a topological space ✠ with comultiplication given by its diagonal embedding. Most of the chapter is devoted to the general study of comodules and contramodules in $\mathcal{W}$. One non-obvious fact about this category is Theorem 4.8:

Hauptsatz 5. *The category of contramodules $\mathcal{W}^F$ is cocomplete.*

The conditions of Theorem 3.7 do not hold in $\mathcal{W}$ for set-theoretic reasons. Yet we can prove some interesting facts about the topological comodule-contramodule correspondence (Propositions 4.10, 4.12 and Theorem 4.14).

Hauptsatz 6.
(1) *The comodule-contramodule correspondence $(\mathcal{L} \dashv \mathcal{R})$ is a Quillen adjunction between $\mathcal{W}_T$ and $\mathcal{W}^F$.*
(2) *If all topological spaces are subsets of a Grothendieck universe, $(\mathcal{L} \dashv \mathcal{R})$ induces a Quillen equivalence between a left Bousfield localisation $L(\mathcal{W}_T)$ and a right Bousfield localisation $R(\mathcal{W}^F)$.*
(3) *If $X, Y \in \mathcal{W}_T$ are CW-complexes and $f \in \mathcal{W}_T(X,Y)$ is a weak equivalence, then $\mathcal{R}(f)$ is a weak equivalence.*
(4) *Suppose that ✠ is a CW-complex of finite type. If $X, Y \in \mathcal{W}_T$ are fibrant and $f \in \mathcal{W}_T(X,Y)$ is a weak equivalence such that $\pi_0(\mathcal{R}f)$ is an isomorphism, then $\mathcal{R}f$ is a weak equivalence.*

1. Monad-Comonads Adjoint Pairs over Closed Categories

1.1. **Closed categories.** Let us consider a closed monoidal category $\mathcal{C}$ with hom-sets $\mathcal{C}(X,Y)$, tensor product $\otimes$, unit object $\star$, and associators a. For any object $X \in \mathcal{C}$, we write λ_X and ϖ_X for the left and right unitors, i.e., the natural isomorphisms $\star \otimes X \xrightarrow{\cong} X$ and $X \otimes \star \xrightarrow{\cong} X$ respectively. Recall that a closed monoidal category means that for any object $X \in \mathcal{C}$ the endofunctor $- \otimes X$ admits a right adjoint endofunctor $[X,-]_{\mathcal{C}}$ called the internal hom [20]. When the category in question is clear, we use the shorthand notation $[X,Y]$ for $[X,Y]_{\mathcal{C}}$.

We say that $\mathcal{C}$ is *biclosed* if $\mathcal{C}$ is closed and additionally the functor $X \otimes -$ has a right adjoint endofunctor, denoted by $\widetilde{[X,-]}$. Note that every closed symmetric monoidal category automatically becomes biclosed.

1.2. **Adjoint functors.** Let us discuss adjointness in the context of closed categories. Consider an adjoint pair of endofunctors $(L \dashv R)$ on $\mathcal{C}$. There are two different notions of adjointness in play. Besides the usual notion, which we also call *external adjointness*, involving a natural equivalence of bifunctors

$$\mathcal{C}(L-,-), \mathcal{C}(-,R-) : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{S}ets,$$

we can talk about an *internally adjoint pair* of endofunctors $(L \dashv R)$. This involves a natural equivalence of bifunctors

$$[L-,-], [-,R-] : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}.$$

These notions are related.

Lemma 1.1. *An internally adjoint pair of endofunctors is (externally) adjoint.*

Proof. Recall the *functor of global sections*:

$$\Gamma : \mathcal{C} \to \mathcal{S}ets, \quad \Gamma(X) \coloneqq \mathcal{C}(\star, X).$$

The claim then follows from the standard property of the functor Γ: there are natural isomorphisms of bifunctors

$$\mathcal{C}(L-, -) \cong \Gamma([L-, -]) : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{S}ets,$$
$$\mathcal{C}(-, R-) \cong \Gamma([-, R-]) : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{S}ets\ .$$

It remains to apply this isomorphism to the internal adjunction to derive a usual (external) adjunction. $\square$

Definition 1.2. Let $(L \dashv R)$ be an internally adjoint pair of endofunctors on $\mathcal{C}$. We define *the chief* (or *the chief object*) of the pair $(L \dashv R)$ as ✠ $\coloneqq L\star$.

The following lemma, motivating our interest in the chief, is *surprising*:

Lemma 1.3. *Let $(L \dashv R)$ be an internally adjoint pair of endofunctors of $\mathcal{C}$, ✠ their chief. Then there are natural isomorphisms of functors*

$$R \cong [\text{✠}, -], \quad L \cong\ - \otimes \text{✠}.$$

Proof. Using the isomorphism $i_X : X \to \mathcal{C}(\star, X)$, we obtain the first natural isomorphism as the composite

$$[\text{✠}, X] \cong [L\star, X] \cong [\star, RX] \cong RX.$$

To derive the second one, start with the natural isomorphisms

$$\mathcal{C}(X \otimes \text{✠}, Y) \cong \mathcal{C}(X, [\text{✠}, Y]) \cong \mathcal{C}(X, RY) \cong \mathcal{C}(LX, Y).$$

This gives a natural isomorphism of representable functors

$$\alpha_X : \mathcal{C}(X \otimes \text{✠}, -) \xrightarrow{\cong} \mathcal{C}(LX, -)$$

and, therefore, by the Yoneda Lemma an isomorphism of representing objects

$$\beta_X : X \otimes \text{✠} \xrightarrow{\cong} LX.$$

Now β_X is also natural in X. So the Yoneda embedding ensures that β_X is a natural isomorphism of functors. $\square$

1.3. **Monads and comonads.** Let us now investigate monadic properties of an internally adjoint pair $(L \dashv R)$ of endofunctors. Recall the following notions for a monoidal category $\mathcal{C}$:

- A *monad* on $\mathcal{C}$ is a triple (R, μ, η), where R is an endofunctor on $\mathcal{C}$ and $\mu : RR \Longrightarrow R$ and $\eta : \mathrm{Id}_{\mathcal{C}} \Longrightarrow R$ are natural transformations, satisfying associativity and unitality conditions [5, 2.3], [8, §2].
- Dually, a *comonad* is a triple (L, Δ, ϵ), where L is an endofunctor on $\mathcal{C}$ and $\Delta : L \Longrightarrow LL$ and $\epsilon : L \Longrightarrow \mathrm{Id}_{\mathcal{C}}$ are natural transformations satisfying coassociativity and counitality conditions [5, 2.4], [8, §2].
- A *monoid* in $\mathcal{C}$ is an object $M \in \mathcal{C}$ with a multiplication $\mu_M : M \otimes M \to M$ and a unit $\eta_M : \star \to M$ satisfying associativity and unitality axioms.

- Dually, a *comonoid* in $\mathcal{C}$ is an object $C \in \mathcal{C}$ with a comultiplication $\Delta_C : C \to C \otimes C$ and a counit $\epsilon_C : C \to \star$ satisfying associativity and unitality axioms.

Lemma 1.4. *The following statements are equivalent:*
(1) *L is a monad.*
(2) *R is a comonad.*
(3) *$\maltese$ is a monoid in $\mathcal{C}$.*

The equivalence of statements (1) and (2) can be found in [5, 2.6], [8, Prop. 3.1]. The rest of the proof of Lemma 1.4 is similar to the proof of Lemma 1.5, so left to the reader. We are less interested in the monad-comonad adjoint pairs because the categories $\mathcal{C}^L$ of L-modules and $\mathcal{C}_R$ of R-comodules are equivalent [5, 2.6]. Often these are also called R-coalgebras and L-algebras [11]. These alternative names are justified in the context of the following example: $\mathcal{C}$ is the category of vector spaces, L is the free algebra functor, so that $\mathcal{C}^L$ is the category of algebras. We use the same terminology as Böhm, Brzeziński and Wisbauer, since it is more justified in the context of internally adjoint endofunctors [5].

Lemma 1.5. *The following statements are equivalent:*
(1) *L is a comonad.*
(2) *R is a monad.*
(3) *$\maltese$ is a comonoid in $\mathcal{C}$.*

Proof. For a comonad (L, Δ, ϵ), we can obtain a monadic structure on R in the following way: start with a natural transformation $LRR \Longrightarrow LLRR \Longrightarrow LR \Longrightarrow I_{\mathcal{C}}$. Using the adjunction we obtain a natural transformation $\mu : RR \Longrightarrow R$. The unit morphism η can be constructed easily using the adjunction. All axioms follow routinely. To go in the opposite way, if (R, μ, η) is a monad, we have a natural transformation $I_{\mathcal{C}} \Longrightarrow RL \Longrightarrow RRLL \Longrightarrow RLL$. Applying the adjunction gives us a natural transformation $\Delta : L \Longrightarrow LL$. The counit morphism can easily be constructed using the adjunction. Again the axioms follow routinely. This shows the equivalence of (1) and (2).

To show $\maltese = L\star$ is a comonoid, we need an associative comultiplication map and a counit map. Consider

$$\Delta_\star : L\star \to LL\star\,.$$

By Lemma 1.3 there is a natural isomorphism $L \cong -\otimes\maltese$. Since $L\star = \maltese$ and $\star\otimes\maltese = \maltese$, $\Delta_\star$ gives rise to a map $\Delta_\maltese : \maltese \to \maltese\otimes\maltese$ which is coassociative since L is a comonad. Similarly, one can define a map $\epsilon_\maltese : \maltese \to \mathrm{Id}_{\mathcal{C}}$, making $(\maltese, \Delta_\maltese, \epsilon_\maltese)$ a comonoid. The opposite direction is similar: if $(\maltese, \Delta_\maltese, \epsilon_\maltese)$ is a comonoid, we obtain a comonad structure on L by defining the natural transformations Δ, ϵ explicitly:

$$\Delta_X : LX = X \otimes \maltese \xrightarrow{\mathrm{Id}_X \otimes \Delta_\maltese} X \otimes \maltese \otimes \maltese = LLX,\ \ \epsilon_X : LX = X \otimes \maltese \xrightarrow{\mathrm{Id}_X \otimes \epsilon_\maltese} X,$$

for all $X \in \mathcal{C}$. Repeating the same construction on morphisms, we obtain the natural transformations making (L, Δ, ϵ) a comonad on $\mathcal{C}$. $\square$

With the conditions of Lemma 1.5 the categories $\mathcal{C}_L$ of L-comodules and $\mathcal{C}^R$ of R-modules are not necessarily equivalent. We aim to compare them in the context of the comodule-contramodule correspondence.

1.4. **Further categorical assumptions.** From now on we assume that the closed monoidal category $\mathcal{C}$ is complete and cocomplete. If $\mathcal{C}$ is required to be biclosed, we will explicitly state it.

Occasionally we will assume that $\mathcal{C}$ is *accessible* or *locally presentable*. We follow Adámek and Rosicky [1] with our terminology. For the convenience of the reader, we recall that, given a regular cardinal λ, an object X of some category $\mathcal{B}$ is λ-*presentable*, if $\mathcal{B}(X, -)$ preserves λ-directed colimits. The category $\mathcal{B}$ is *locally* λ*-presentable*, if it is cocomplete and admits a set $\mathfrak{A}$ of λ-presentable objects such that every object is a λ-directed colimit of objects from $\mathfrak{A}$. The category $\mathcal{B}$ is *locally presentable*, if it is *locally* λ*-presentable* for a regular cardinal λ.

We omit the slightly weaker notion of an *accessible category* because a (co)complete accessible category is locally presentable [1, 2.47]

1.5. **(Co)completeness of (co)modules.** The following lemma is an immediate consequence of the completeness and cocompleteness of $\mathcal{C}$.

Lemma 1.6. *Let $\mathcal{C}$ be as in Section 1.4. Let $(Q \dashv H)$ be an adjoint (internally or externally) comonad-monad pair on $\mathcal{C}$. Then $\mathcal{C}_Q$ is cocomplete and $\mathcal{C}^H$ is complete.*

Proof. Since H is a monad on $\mathcal{C}$, the forgetful functor $\mathcal{F}^H : \mathcal{C}^H \to \mathcal{C}$ creates limits [3]. Hence, as $\mathcal{C}$ is complete, so is $\mathcal{C}^H$. Similarly, since Q is a comonad the forgetful functor $\mathcal{F}_Q : \mathcal{C}^Q \to \mathcal{C}$ creates colimits, so $\mathcal{C}_Q$ is cocomplete [11]. □

The question of cocompleteness of $\mathcal{C}^F$ and completeness of $\mathcal{C}_T$ is subtle. Without any additional assumptions on $\mathcal{C}$, we can only write some obvious sufficient conditions.

Lemma 1.7. [3] *Let H be a monad on $\mathcal{C}$. The category $\mathcal{C}^H$ is cocomplete if one of the following conditions is satisfied:*

(1) *The monad H is cocontinuous.*
(2) *The category $\mathcal{C}^H$ has reflexive coequalisers.*

Lemma 1.8. [11] *Let Q be a comonad on $\mathcal{C}$. The category $\mathcal{C}_Q$ is complete if one of the following conditions is satisfied:*

(1) *The comonad Q is continuous.*
(2) *The category $\mathcal{C}_H$ has coreflexive equalisers.*

A more useful (for us) criterion for cocompleteness of $\mathcal{C}^H$ has been devised by Barr [2]. The following theorem has been influenced by it.

Theorem 1.9. *Suppose that $\mathcal{C}$ is locally presentable.*

(1) *If H is a continuous accessible monad on $\mathcal{C}$, then the category $\mathcal{C}^H$ is complete and locally presentable.*
(2) *If Q is a cocontinuous monad on $\mathcal{C}$, then the category $\mathcal{C}_Q$ is complete and locally presentable.*

Proof. **(1)** By [17, Rem. 2.5], H admits a left adjoint Q. The functor Q is a comonad by Lemma 1.5. By Lemma 1.6, $\mathcal{C}^H$ is complete.

The accessibility of H implies that $\mathcal{C}^H$ is accessible [1, Th. 2.78]. A complete accessible category is cocomplete and locally presentable [1, Cor. 2.47].

(2) By [17, Rem. 2.5], Q admits a right adjoint H. The functor H is a monad by Lemma 1.5. By Lemma 1.6, $\mathcal{C}_Q$ is cocomplete.

The comonad Q is accessible since it is cocontinuous. By [17, Cor. 2.8], $\mathcal{C}_Q$ is accessible. A cocomplete accessible category is complete and locally presentable [1, Cor. 2.47]. □

Let us state a stand-alone corollary for the case of an internally adjoint pair of functors. Notice that the functor $F = [\maltese, -]$ is accessible if and only if the chief $\maltese$ is presentable.

Corollary 1.10. (1) *If $\mathcal{C}$ is locally presentable, then the category $\mathcal{C}_T$ is complete and locally presentable.*
(2) *If, furthermore, $\maltese$ is presentable, then $\mathcal{C}^F$ is complete and locally presentable.*

1.6. **Extranaturality.** Let $\mathcal{A}_1, \dots, \mathcal{A}_n, \mathcal{B}$ be categories. By *a $\mathcal{B}$-formula* $\mathfrak{A}(X) = \mathfrak{A}(X_1, \dots, X_n)$ we understand a "natural" assignment of an object $\mathfrak{A}(A_1, \dots, A_n) \in \mathcal{B}$ to each n-tuple of objects $A_j \in \mathcal{A}_j$. Speaking precisely, a (k_j)-contravariant, (m_j)-covariant (or k_j-contravariant, m_j-covariant in variable X_j) $\mathcal{B}$-formula is a functor

$$\mathfrak{A}^\sharp : \prod_{j=1}^{n} \mathcal{A}_j^{op\ k_j} \times \mathcal{A}_j^{m_j} \to \mathcal{B}$$

that we use by plugging the same object $X_j \in \mathcal{A}_j$ into every appearance of $\mathcal{A}_j$.

For instance, the following

$$\mathfrak{A}(X_1, X_2, X_3) = ([X_1, X_2] \otimes [X_2, X_1]) \oplus ((X_3 \otimes \mathcal{F}(X_3)) \otimes Y \oplus (X_3 \otimes X_1)$$

is a $(1, 1, 0)$-contravariant, $(2, 1, 3)$-covariant $\mathcal{C}$-formula for a cocomplete closed monoidal category $\mathcal{C}$ where $\mathcal{A}_1 = \mathcal{A}_2 = \mathcal{A}_3 = \mathcal{B} = \mathcal{C}$ that uses an endofunctor $\mathcal{F}$ and an object $Y \in \mathcal{C}$. The formula comes from a functor

$$\mathfrak{A}^\sharp(Z_1, \dots, Z_8) = ([Z_1, Z_4] \otimes [Z_5, Z_2]) \oplus ((Z_6 \otimes \mathcal{F}(Z_7)) \otimes Y) \oplus (Z_8 \otimes Z_3).$$

The separate notation $\mathfrak{A}^\sharp$ helps us to recognise a situation where we regard $\mathfrak{A}$ as a functor by plugging distinct objects instead of repeated ones.

Given two $\mathcal{B}$-formulas $\mathfrak{A}(X)$ and $\mathfrak{B}(X)$, by a *transformation from $\mathfrak{A}$ to $\mathfrak{B}$* we understand an assignment of a morphism in $\mathcal{B}$ to each n-tuple of objects

$$\mathcal{N}_X = \{\mathcal{N}_{A_1, \dots A_n} \in \mathcal{B}(\mathfrak{A}(A_1, \dots, A_n), \mathfrak{B}(A_1, \dots, A_n))\}.$$

A transformation $\mathcal{N}_X$ is called *natural in X_j* if

- both $\mathfrak{A}(X)$ and $\mathfrak{B}(X)$ are 1-covariant, 0-contravariant or 0-covariant, 1-contravariant in X_j and
- for every choice of objects $A_1, \dots A_{j-1}, A_{j+1}, \dots A_n$ the transformation

  $$\mathcal{N}^{(j)} = \mathcal{N}_{A^{(j)}} : \mathfrak{A}(A^{(j)}) \Longrightarrow \mathfrak{B}(A^{(j)}),$$

  where $A^{(j)} := (A_1, \dots A_{j-1}, -, A_{j+1}, \dots A_n)$, is a natural transformation from the functor $\mathfrak{A}(A_1, \dots A_{j-1}, -, A_{j+1}, \dots A_n) : \mathcal{A}_j \to \mathcal{B}$ to the functor $\mathfrak{B}(A_1, \dots A_{j-1}, -, A_{j+1}, \dots A_n) : \mathcal{A}_j \to \mathcal{B}$.

The last condition means commutativity for every morphism $f \in \mathcal{A}_j(B, C)$ and every choice of $A_i \in \mathcal{A}_i$, $i \neq j$ of the following diagram on the left (right) in the

1-covariant (1-contravariant correspondingly) case

$$\begin{tikzcd}
\mathfrak{A}(B^{(j)}) \arrow[r, "\mathcal{N}_{B^{(j)}}"'] \arrow[d, "\mathfrak{A}(f^{(j)})"'] & \mathfrak{B}(B^{(j)}) \arrow[d, "\mathfrak{B}(f^{(j)})"] \\
\mathfrak{A}(C^{(j)}) \arrow[r, "\mathcal{N}_{C^{(j)}}"] & \mathfrak{B}(C^{(j)})
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathfrak{A}(B^{(j)}) \arrow[r, "\mathcal{N}_{B^{(j)}}"'] & \mathfrak{B}(B^{(j)}) \\
\mathfrak{A}(C^{(j)}) \arrow[r, "\mathcal{N}_{C^{(j)}}"] \arrow[u, "\mathfrak{A}(f^{(j)})"] & \mathfrak{B}(C^{(j)}) \arrow[u, "\mathfrak{B}(f^{(j)})"']
\end{tikzcd}$$

where $f^{(j)} = (f_i)$ with $f_j = f$ and $f_i = \mathrm{Id}_{\mathcal{A}_i}$, $i \neq j$.

Definition 1.11. A transformation $\mathcal{N}_X$ is called *extranatural in* X_j if

- $\mathfrak{A}(X)$ is 1-covariant, 1-contravariant in X_j, while $\mathfrak{B}(X)$ is 0-covariant, 0-contravariant in X_j and
- for every choice of objects $A_1, \ldots, A_{j-1}, A_{j+1}, \ldots A_n$ and every morphism $f \in \mathcal{A}_j(B, C)$ the diagram

$$\begin{tikzcd}
\mathfrak{A}^\sharp(CB^{(j)}) \arrow[r, "\mathfrak{A}^\sharp(\mathrm{Id}_{A_1},\ldots,\mathrm{Id}_C,f,\ldots)"'] \arrow[d, "\mathfrak{A}^\sharp(\mathrm{Id}_{A_1},\ldots,f,\mathrm{Id}_B,\ldots)"'] & \mathfrak{A}(C^{(j)}) \arrow[d, "\mathcal{N}_{C^{(j)}}"] \\
\mathfrak{A}(B^{(j)}) \arrow[r, "\mathcal{N}_{B^{(j)}}"] & \mathfrak{B}(B^{(j)}) = \mathfrak{B}(C^{(j)}).
\end{tikzcd}$$

  is commutative where $CB^{(j)} = (\underline{A_1}, \ldots, \underline{A_{j-1}}, C, B, \underline{A_{j+1}}, \ldots \underline{A_n})$, where $\underline{A_i} = \underbrace{\underline{A_i, \ldots, A_i}}_{k_i+m_i}$.

Notice that $\mathfrak{B}(X)$ being 0-covariant, 0-contravariant means that $\mathfrak{B}(X)$ is independent of X_j. Since $B^{(j)}$ and $C^{(j)}$ disagree only in position j, $\mathfrak{B}(B^{(j)}) = \mathfrak{B}(C^{(j)})$.

Similarly, one can define extranaturality in X_j for a transformation in the opposite direction $\mathfrak{B} \Longrightarrow \mathfrak{A}$. There $\mathfrak{B}$ is independent of X_j. We will not use it, so we do not go into details.

Example 1.12. Let $\mathcal{A}, \mathcal{B}, \mathcal{D}$ be categories, $\mathcal{F} : \mathcal{A}^{op} \times \mathcal{A} \to \mathcal{D}$ and $\mathcal{G} : \mathcal{B} \to \mathcal{D}$ functors. Consider the $\mathcal{D}$-formulas

$$\mathfrak{A}(X_1, X_2) = \mathcal{F}(X_1, X_1) \text{ and } \mathfrak{B}(X_1, X_2) = \mathcal{G}(X_2)\,.$$

A transformation from $\mathfrak{A}$ to $\mathfrak{B}$ is a family of morphisms $\mathcal{E}_X = \{\mathcal{E}_{(A,B)} \in \mathcal{D}(\mathcal{F}(A, A), \mathcal{G}(B)\}$, where $A \in \mathcal{A}, B \in \mathcal{B}$. Extranaturality in X_1 according to Definition 1.11 asserts that the following diagram commutes for every morphism $f \in \mathcal{A}(A, A')$

$$\begin{tikzcd}
\mathcal{F}(A', A) \arrow[r, "\mathcal{F}(\mathrm{Id},f)"] \arrow[d, "\mathcal{F}(f,\mathrm{Id})"] & \mathcal{F}(A', A') \arrow[d, "\mathcal{E}_{(A',B)}"] \\
\mathcal{F}(A, A) \arrow[r, "\mathcal{E}_{(A,B)}"] & \mathcal{G}(B).
\end{tikzcd}$$

This is precisely the usual notion of extranaturality.

We observe a useful coherence condition, which we formulate as the following *extranaturality property*. This can also be viewed as *extraassociativity*: given $g \in \mathcal{A}(B, C)$, it asserts that the two possible compositions $[C, D] \otimes [A, B] \to [A, D]$ are equal. With normal instead of enriched homs, the equality reads as $(fg)h = f(gh)$.

Let ${}^{\alpha}\bigotimes_{k=n}^{1} X_k = X_n \otimes X_{n-1} \otimes \cdots$ be an iterated tensor product where α stands for one of the $\frac{1}{n}\binom{2n-2}{n-1}$ possible choices of bracketing.

Proposition 1.13. *Let $A_1, A_2, \dots A_{2n}$ be objects of $\mathcal{C}$, $f_i \in \mathcal{C}(A_{2i-1}, A_{2i})$, $i = 1, \dots n$, α is a choice of bracketing. Then all morphisms in*

$$\mathcal{C}\Big({}^{\alpha}\bigotimes_{k=n-1}^{1}[A_{2k}, A_{2k+1}],\ [A_1, A_{2n}]\Big),$$

obtained by applying morphisms f_i and categorical compositions

$$c_{j,j+1,j+2} : [A_{j+1}, A_{j+2}] \otimes [A_j, A_{j+1}] \to [A_j, A_{j+2}]$$

in all possible ways, are equal. Moreover, given the two choices of bracketing α and β, the corresponding morphisms f_α and f_β are related via the associativity constraint:

$$f_\alpha : {}^{\alpha}\bigotimes_{k=n-1}^{1}[A_{2k}, A_{2k+1}] \xrightarrow{a_{\alpha,\beta}} {}^{\beta}\bigotimes_{k=n-1}^{1}[A_{2k}, A_{2k+1}] \xrightarrow{f_\beta} [A_1, A_{2n}].$$

We need the following lemma to prove Proposition 1.13.

Lemma 1.14. *Let A, B, C be objects of $\mathcal{C}$. The categorical composition*

$$c_{A,B,C} : [B, C] \otimes [A, B] \to [A, C],$$

is a transformation, natural in A and C and extranatural in B.

Proof. Let

$$\Phi : \mathrm{Hom}(A \otimes B, C) \xrightarrow{\cong} \mathrm{Hom}(A, [B, C])$$

denote the adjunction between the functors $- \otimes B$ and $[B, -]$ and let

$$\mathrm{ev}_{A,B} : [A, B] \otimes A \to B$$

be the evaluation map. Recall that Φ is a natural isomorphism in all three variables, and $\mathrm{ev}_{A,B}$ is natural in B and extranatural in A [18]. This means that for morphisms $f : A \to A'$ and $h : B \to B'$ we have commutative diagrams:

$$\begin{array}{ccc} [A', B] \otimes A & \xrightarrow{[f,\mathrm{Id}]\otimes \mathrm{Id}} & [A, B] \otimes A \\ \downarrow{\scriptstyle \mathrm{Id}\otimes f} & & \downarrow{\scriptstyle \mathrm{ev}_{A,B}} \\ [A', B] \otimes A' & \xrightarrow{\mathrm{ev}_{A',B}} & B \end{array} \quad \text{and} \quad \begin{array}{ccc} [A, B] \otimes A & \xrightarrow{\mathrm{ev}_{A,B}} & B \\ \downarrow{\scriptstyle [\mathrm{Id},h]\otimes \mathrm{Id}} & & \downarrow{\scriptstyle h} \\ [A, B'] \otimes A & \xrightarrow{\mathrm{ev}_{A,B'}} & B' \end{array}$$

Now let us investigate the naturality properties of the categorical composition $c_{A,B,C}$.

Naturality in A. To obtain naturality in A we want to show that the maps

$$f_1 : [B, C] \otimes [A', B] \xrightarrow{c_{A',B,C}} [A', C] \xrightarrow{[f,\mathrm{Id}]} [A, C]$$

and

$$f_2 : [B, C] \otimes [A', B] \xrightarrow{\mathrm{Id}\otimes[f,\mathrm{Id}]} [B, C] \otimes [A, B] \xrightarrow{c_{A,B,C}} [A, C]$$

are equal. Let us compute their adjuncts under Φ. These will be the maps $\Phi^{-1}(f_1)$ and $\Phi^{-1}(f_2)$, given by the compositions

$$\Phi^{-1}(f_1) : [B, C] \otimes [A', B] \otimes A \xrightarrow{c_{A',B,C}\otimes \mathrm{Id}} [A', C] \otimes A \xrightarrow{\mathrm{Id}\otimes f} [A', C] \otimes A' \xrightarrow{\mathrm{ev}_{A',C}} C$$

and

$$\Phi^{-1}(f_2) : [B, C] \otimes [A', B] \otimes A \xrightarrow{\mathrm{Id}\otimes f \otimes \mathrm{Id}} [B, C] \otimes [A, B] \otimes A \xrightarrow{\mathrm{ev}_{B,C}\circ(\mathrm{Id}\otimes \mathrm{ev}_{A,B})} C.$$

Look at $\Phi^{-1}(f_1)$. It is equal to

$$\mathrm{ev}_{A',C}(\mathrm{Id}\otimes f)(c_{A',B,C}\otimes \mathrm{Id}) = \mathrm{ev}_{A',C}(c_{A',B,C}\otimes f) = \mathrm{ev}_{A',C}(c_{A',B,C}\otimes \mathrm{Id})(\mathrm{Id}\otimes f).$$

Using the facts that the evaluation map is the unit of the adjunction Φ and that $c_{A',B,C} = \Phi^{-1}(\mathrm{ev}_{B,C}(\mathrm{Id}\otimes\mathrm{ev}_{A',B}))$, we can further rewrite

$$\mathrm{ev}_{A',C}(c_{A',B,C}\otimes\mathrm{Id})(\mathrm{Id}\otimes f) = \mathrm{ev}_{B,C}(\mathrm{Id}\otimes\mathrm{ev}_{A',B})(\mathrm{Id}\otimes f) = \Phi^{-1}(f_2),$$

completing the proof of the naturality in A.

Extranaturality in B. Similarly, for extranaturality of $c_{A,B,C}$ in B, we want to show that for a morphism $h : B \to B'$ in $\mathcal{C}$ the maps:

$$h_1 : [B',C]\otimes[A,B] \xrightarrow{[h,\mathrm{Id}]\otimes\mathrm{Id}} [B,C]\otimes[A,B] \xrightarrow{c_{A,B,C}} [A,C]$$

and

$$h_2 : [B',C]\otimes[A,B] \xrightarrow{\mathrm{Id}\otimes[\mathrm{Id},h]} [B',C]\otimes[A,B'] \xrightarrow{c_{A,B',C}} [A,C]$$

are equal.

The adjuncts of h_1 and h_2 under Φ are the composites:

$$\Phi^{-1}(h_1) : [B',C]\otimes[A,B]\otimes A \xrightarrow{[h,\mathrm{Id}]\otimes\mathrm{Id}\otimes\mathrm{Id}} [B,C]\otimes[A,B]\otimes A \xrightarrow{\mathrm{ev}_{B,C}\circ(\mathrm{Id}\otimes\mathrm{ev}_{A,B})} C$$

and

$$\Phi^{-1}(h_2) : [B',C]\otimes[A,B]\otimes A \xrightarrow{\mathrm{Id}\otimes[\mathrm{Id},h]\otimes\mathrm{Id}} [B',C]\otimes[A,B']\otimes A \xrightarrow{\mathrm{ev}_{B',C}\circ(\mathrm{Id}\otimes\mathrm{ev}_{A,B'})} C.$$

Consider the diagram

$$\begin{tikzcd}
{[B',C]\otimes[A,B]\otimes A} \arrow[r, "\mathrm{Id}\otimes\mathrm{ev}_{A,B}"] \arrow[d, "\mathrm{Id}\otimes[\mathrm{Id},h]\otimes\mathrm{Id}"] & {[B',C]\otimes B} \arrow[r, "{[h,\mathrm{Id}]\otimes\mathrm{Id}}"] \arrow[d, "\mathrm{Id}\otimes h"] & {[B,C]\otimes B} \arrow[d, "\mathrm{ev}_{B,C}"] \\
{[B',C]\otimes[A,B']\otimes A} \arrow[r, "\mathrm{Id}\otimes\mathrm{ev}_{A,B'}"] & {[B',C]\otimes B'} \arrow[r, "\mathrm{ev}_{B',C}"] & C.
\end{tikzcd}$$

The first square is commutative since it is obtained by applying $[B',C]\otimes -$ to the commutative square which represents the extranaturality of $\mathrm{ev}_{A,B}$ in B. The second square is exactly extranaturality of $\mathrm{ev}_{B,C}$ in B. Thus, the outer square commutes. Composing the maps from top left corner to bottom right corner along the top arrows gives $\Phi^{-1}(h_1)$ and composing the maps from top left corner to bottom right corner along the bottom arrows gives $\Phi^{-1}(h_2)$. Thus, $\Phi^{-1}(h_1) = \Phi^{-1}(h_2)$ and so are h_1 and h_2.

Naturality in C. It follows from naturality of $\mathrm{ev}_{B,C}$ in C and naturality of Φ. $\square$

Now we are ready to prove Proposition 1.13.

Proof. (*of Proposition 1.13*) The morphisms in $\mathcal{C}\Big({}^{\alpha}\bigotimes_{k=n-1}^{1}[A_{2k},A_{2k+1}],[A_1,A_{2n}]\Big)$ which are combinations of various f_i and categorical compositions are obtained by applying the functors $[f_i,-]$, $[-,f_i]$ and categorical compositions $c_{k,j,l}$ in all possible ways. For simplicity, we write $c_{k,j,l}$ for $c_{A_{2i+k},A_{2i+j},A_{2i+l}}$. To show that all desired morphisms coincide it is enough to show that the following diagrams, which we call *basic moves*, commute for every $i = 1,..,n$:

- **Basic move (1)**:

$$\begin{tikzcd}
{[A_{2i+1},A_{2i+2}]\otimes[A_{2i},A_{2i+1}]} \arrow[r, "c_{0,1,2}"] \arrow[d, "{\mathrm{Id}\otimes[f_i,\mathrm{Id}]}"'] & {[A_{2i},A_{2i+3}]} \arrow[d, "{[f_i,\mathrm{Id}]}"] \\
{[A_{2i+1},A_{2i+3}]\otimes[A_{2i-1},A_{2i+1}]} \arrow[r, "c_{-1,1,3}"] & {[A_{2i-1},A_{2i+3}]}.
\end{tikzcd}$$

- **Basic move (2)**:

$$\begin{tikzcd}
[A_{2i},A_{2i+1}]\otimes[A_{2i-2},A_{2i-1}] \arrow[r, "\mathrm{Id}\otimes[\mathrm{Id},f_i]"] \arrow[d, "{[f_i,\mathrm{Id}]\otimes\mathrm{Id}}"'] & {[A_{2i},A_{2i+1}]\otimes[A_{2i-2},A_{2i}]} \arrow[d, "c_{-2,0,1}"] \\
{[A_{2i-1},A_{2i+1}]\otimes[A_{2i-2},A_{2i-1}]} \arrow[r, "c_{-2,-1,1}"] & {[A_{2i-2},A_{2i+1}].}
\end{tikzcd}$$

- **Basic move (3)**:

$$\begin{tikzcd}
{[A_{2i-2},A_{2i}]\otimes[A_{2i-3},A_{2i-2}]} \arrow[r, "c_{-3,0,-2}"] \arrow[d, "{[\mathrm{Id},f_i]\otimes\mathrm{Id}}"'] & {[A_{2i-3},A_{2i}]} \arrow[d, "{[\mathrm{Id},f_i]}"] \\
{[A_{2i-2},A_{2i-1}]\otimes[A_{2i-3},A_{2i-2}]} \arrow[r, "c_{-3,-2,-1}"] & {[A_{2i-3},A_{2i-1}].}
\end{tikzcd}$$

Note that basic move (1) is equivalent to the categorical composition being natural in the first variable, basic move (2) is exactly the extranaturality of the composition in the second variable and basic move (3) is naturality in the third variable. Thus, Lemma 1.14 establishes the commutativity of all diagrams above. This argument works for any bracketing α since choosing a bracketing dictates the order in which we are allowed to compose morphisms. However, all possible cases are covered by our basic moves, so we are done. □

1.7. **Enriched categories.** Recall that for categories $\mathcal{A},\mathcal{B}$ enriched in a (closed) monoidal category $\mathcal{C}$ with hom objects denoted by $[-,-]_{\mathcal{A}}$ and $[-,-]_{\mathcal{B}}$ respectively, a *$\mathcal{C}$-enriched functor* $\mathcal{F}:\mathcal{A}\to\mathcal{B}$ consists of the following data:

- a map $\mathcal{F}:\mathcal{A}\to\mathcal{B}$ between the objects of $\mathcal{A}$ and $\mathcal{B}$,
- an $\mathcal{A}\times\mathcal{A}$-indexed family of morphisms in $\mathcal{C}$
$$\mathcal{F}_{X,Y}:[X,Y]_{\mathcal{A}}\to[\mathcal{F}X,\mathcal{F}Y]_{\mathcal{B}},$$
which respect the enriched composition and units in $\mathcal{A}$ and $\mathcal{B}$.

In Section 1.6 we discuss extranaturality properties of morphisms in $\mathcal{C}$. There is an analogue of those in the setting of enriched categories [19]. More precisely, if $\mathcal{A},\mathcal{B}$ are as above and $\mathcal{F},\mathcal{G}:\mathcal{A}\to\mathcal{B}$ are $\mathcal{C}$-enriched functors, a *$\mathcal{C}$-natural transformation* $\mathcal{F}\implies\mathcal{G}$ is an $\mathcal{A}$-indexed family of morphisms, such that for every $X,Y\in\mathcal{A}$ the following diagram commutes:

$$\begin{tikzcd}
{[X,Y]_{\mathcal{A}}} \arrow[r, "\mathcal{F}_{X,Y}"] \arrow[d, "\mathcal{G}_{X,Y}"] & {[\mathcal{F}X,\mathcal{F}Y]_{\mathcal{B}}} \arrow[d, "(\alpha_Y)_\star"] \\
{[\mathcal{G}X,\mathcal{G}Y]_{\mathcal{B}}} \arrow[r, "(\alpha_X)^\star"] & {[\mathcal{G}X,\mathcal{F}Y]_{\mathcal{B}}}
\end{tikzcd}$$

where

$$(\alpha_X)^\star:[\mathcal{G}X,\mathcal{G}Y]_{\mathcal{B}}\cong[\mathcal{G}X,\mathcal{G}Y]_{\mathcal{B}}\otimes\star\xrightarrow{\mathrm{Id}\otimes\alpha_X}[\mathcal{G}X,\mathcal{G}Y]_{\mathcal{B}}\otimes[\mathcal{F}X,\mathcal{G}X]_{\mathcal{B}}\xrightarrow{c^{\mathcal{B}}_{\mathcal{F}X,\mathcal{G}X,\mathcal{G}Y}}[\mathcal{F}X,\mathcal{G}Y]_{\mathcal{B}},$$

and

$$(\alpha_Y)_\star:[\mathcal{F}X,\mathcal{F}Y]_{\mathcal{B}}\cong\star\otimes[\mathcal{G}X,\mathcal{G}Y]_{\mathcal{B}}\xrightarrow{\alpha_Y\otimes\mathrm{Id}}$$
$$\xrightarrow{\alpha_Y\otimes\mathrm{Id}}[\mathcal{F}X,\mathcal{G}Y]_{\mathcal{B}}\otimes[\mathcal{F}X,\mathcal{F}Y]_{\mathcal{B}}\xrightarrow{c^{\mathcal{B}}_{\mathcal{F}X,\mathcal{F}Y,\mathcal{G}Y}}[\mathcal{F}X,\mathcal{G}Y]_{\mathcal{B}}.$$

Similarly to Definition 1.11 one can talk about $\mathcal{C}$-extranaturality (or extraordinarly $\mathcal{C}$-naturality) when dealing with enriched functors of the form $\mathcal{F} : \mathcal{A} \times \mathcal{A}^{op} \to \mathcal{B}$. We do not include the full definition, so for further details we refer the reader to [19, 1.7]. The following observations are useful.

Lemma 1.15. *(cf.* [19, 1.7, 1.8]*) Let $\mathcal{C}$ be a closed monoidal category and $\mathcal{A}$ a $\mathcal{C}$-enriched category. The following statements hold:*

(1) *The internal hom $[-,-]_{\mathcal{A}}$ is $\mathcal{C}$-natural in both variables.*
(2) *The enriched composition $c^{\mathcal{A}}_{-,-,-}$ is $\mathcal{C}$-natural in the first and third variable, and $\mathcal{C}$-extranatural in the second.*
(3) *If $\mathcal{B}$ is another $\mathcal{C}$-enriched category and $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ a $\mathcal{C}$-enriched functor, the maps $\mathcal{F}_{X,Y}$ are $\mathcal{C}$-natural in both X and Y.*

Lemma 1.16. *Let $\mathcal{C}$ be a closed monoidal category with internal hom $[-,-]$. If $f \in \mathcal{C}(X,Y)$ is a monomorphism, then the corresponding evaluation map $\widetilde{f}_Z : [Z,X] \to [Z,Y]$ is a monomorphism for all $Z \in \mathcal{C}$.*

Proof. Let $W \in \mathcal{C}$ and consider two morphisms $g, h \in \mathcal{C}(W,[Z,X])$, such that $\widetilde{f}_Z g = \widetilde{f}_Z h$. The adjunct of $\widetilde{f}_Z h$ decomposes with the adjuncts of g and h:

$$W \otimes Z \rightrightarrows X \xrightarrow{f} Y.$$

Since f is a monomorphism, the adjuncts of g and h are equal. Hence, $g = h$ and $\widetilde{f}_Z$ is a monomorphism. □

Lemma 1.17. *Let $\mathcal{C}$ be a closed monoidal category with internal hom $[-,-]$. If $f \in \mathcal{C}(X,Y)$ is an epimorphism, then the corresponding map $\overline{f}_Z : X \otimes Z \to Y \otimes Z$ is an epimorphism for all $Z \in \mathcal{C}$.*

Proof. Fix an object Z of $\mathcal{C}$. By definition, the functor $- \otimes Z$ is the left adjoint of $[Z,-]$. It is a standard fact that left adjoints preserve epimorphisms. □

As $\mathcal{C}$ is a closed monoidal category, it is in fact enriched in itself [27, Lemma 3.4.9]. Since $\mathcal{C}$ is complete and cocomplete, it has kernels and cokernels of pairs. The kernel of a pair $f, g : X \rightrightarrows Y$ represents a functor

$$\mathcal{C}^{op} \to \mathcal{S}ets, \quad Z \mapsto \ker(f\circ, g\circ : \mathcal{C}(Z,X) \rightrightarrows \mathcal{C}(Z,Y)).$$

Similarly, *an enriched kernel* is a map $h : K \to X$ such that the functor

$$E : \mathcal{C}^{op} \to \mathcal{C}, \quad Z \mapsto \ker(\widetilde{f}_Z, \widetilde{g}_Z : [Z,X] \rightrightarrows [Z,Y]).$$

is represented by K with the natural isomorphism $[-,K] \to E$ given by the evaluation $\widetilde{f}_-$. Similarly, *an enriched cokernel* of the pair $f, g : X \rightrightarrows Y$ is a map $d : Y \to C$ such the functor

$$F : \mathcal{C} \to \mathcal{C}, \quad Z \mapsto \operatorname{coker}({}_Z\widetilde{f}, {}_Z\widetilde{g} : [Y,Z] \rightrightarrows [X,Z]),$$

where ${}_Z\widetilde{f}$ and ${}_Z\widetilde{g}$ are evaluations on the other side, is represented by C with the natural isomorphism $[C,-] \to F$ is given by the evaluation ${}_-\widetilde{d}$.

Lemma 1.18. *In a closed complete cocomplete monoidal category $\mathcal{C}$ kernels coincide with enriched kernels. If, furthermore, $\mathcal{C}$ is biclosed, then cokernels coincide with enriched cokernels.*

Proof. Suppose $h : K \to X$ is a kernel of a pair $f, g : X \rightrightarrows Y$. The functor $[Z, -]$ preserves limits because it is a right adjoint. Thus, $[Z, K]$ is an enriched kernel of the pair $\widetilde{f}_Z, \widetilde{g}_Z : [Z, X] \rightrightarrows [Z, Y]$, which means that $f : K \to X$ is an enriched kernel.

The proof for cokernels is similar but requires biclosedness. Let $d : Y \to C$ be a cokernel of a pair $f, g : X \rightrightarrows Y$. The functor

$$[-, Z] : \mathcal{C}^{op} \to \mathcal{C}$$

preserves limits because it is a right adjoint. Indeed,

$$\mathcal{C}(X, [Y, Z]) \cong \mathcal{C}(X \otimes Y, Z) \cong \mathcal{C}(Y, \widetilde{[X, Z]}) = \mathcal{C}^{op}(\widetilde{[X, Z]}, Y)$$

so that its left adjoint is

$$\widetilde{[-, Z]} : \mathcal{C}^{op} \to \mathcal{C} .$$

Thus, $[C, Z]$ is an enriched kernel of the pair ${}_Z\widetilde{f}, {}_Z\widetilde{g} : [Y, Z] \rightrightarrows [X, Z]$, which means that $d : Y \to C$ is an enriched cokernel. □

1.8. **Internal homs for modules and comodules.** The objective of this section is to show that both comodules and contramodules form categories enriched in $\mathcal{C}$.

To achieve this objective, we look further at the internally adjoint pair $(T \dashv F)$ of endofunctors on $\mathcal{C}$ where T is a comonad and F is a monad. The corresponding external adjunction has unit and counit

$$\iota : \mathrm{Id} \to FT, \quad \epsilon : TF \to \mathrm{Id} .$$

Given T-comodules (X, ρ_X) and (Y, ρ_Y) (where $\rho_X : X \to TX$ is the structure map), let us consider the following two morphisms. The first morphism is the internal analogue of the composition with ρ_Y:

$$\alpha^T_{X,Y} : [X, Y] \xrightarrow{[\mathrm{Id}_X, \rho_Y]} [X, TY].$$

The second morphism utilises ρ_X and appears more involved (cf. Section 2.1 for an example of these maps for vector spaces):

$$\beta^T_{X,Y} : [X, Y] \xrightarrow{[\mathrm{Id}_X, \iota_Y]} [X, FTY] \cong [TX, TY] \xrightarrow{[\rho_X, \mathrm{Id}_{TY}]} [X, TY].$$

By definition, the C-comodule homomorphisms from X to Y is the equaliser of $\alpha^T_{X,Y}$ and $\beta^T_{X,Y}$. More generally, we have the following:

Definition 1.19. Let $\mathcal{C}$ and T be as in the beginning of this section. Denote by $\mathcal{C}_T$ the collection of objects in $\mathcal{C}$ which are T-comodules. The T-comodule maps object between objects $X, Y \in \mathcal{C}_T$, denoted $[X, Y]_{\mathcal{C}_T}$, or in shorthand $[X, Y]_T$, is the equaliser of the maps $\alpha^T_{X,Y}$ and $\beta^T_{X,Y}$ defined above.

Proposition 1.20. *With notation and conventions as above $\mathcal{C}_T$ becomes a category enriched in $\mathcal{C}$.*

Proof. Pick three objects $X, Y, Z \in \mathcal{C}_T$. We will observe that the morphism

$$\gamma : [Y, Z]_T \otimes [X, Y]_T \to [Y, Z] \otimes [X, Y] \to [X, Z]$$

has the equalising property

$$\alpha^T_{X,Z}\gamma = \beta^T_{X,Z}\gamma : [Y, Z]_T \otimes [X, Y]_T \to [X, TZ] .$$

Hence, γ uniquely factors through the equaliser producing the internal composition

$$c^T_{X,Y,Z} : [Y, Z]_T \otimes [X, Y]_T \to [X, Z]_T .$$

The key observation proceeds in five steps. First, Proposition 1.13 tells us that $\alpha^T_{X,Z}\gamma$ is equal to the composition

$$\gamma_1 : [Y,Z]_T \otimes [X,Y]_T \to [Y,Z] \otimes [X,Y] \xrightarrow{[Y,\rho_Z]\otimes \mathrm{Id}} [Y,TZ] \otimes [X,Y] \to [X,TZ].$$

Second, by the equalising property of $[Y,Z]_T$, γ_1 is equal to the composition

$$\gamma_2 : [Y,Z]_T \otimes [X,Y]_T \to [Y,Z] \otimes [X,Y] \to [TY,TZ] \otimes [X,Y] \xrightarrow{[\rho_Y,Z]\otimes \mathrm{Id}} [Y,TZ] \otimes [X,Y] \to [X,TZ].$$

Third, Proposition 1.13 strikes again: γ_2 is equal to

$$\gamma_3 : [Y,Z]_T \otimes [X,Y]_T \to [TY,TZ] \otimes [X,Y] \xrightarrow{\mathrm{Id}\otimes[X,\rho_Y]} [TY,TZ] \otimes [X,TY] \to [X,TZ].$$

Fourth, by the equalising property of $[X,Y]_T$, γ_3 is equal to

$$\gamma_4 : [Y,Z]_T \otimes [X,Y]_T \to [TY,TZ] \otimes [TX,TY] \xrightarrow{\mathrm{Id}\otimes[\rho_X,TY]} [TY,TZ] \otimes [X,TY] \to [X,TZ].$$

Finally, Proposition 1.13 finishes the job: γ_4 is equal to $\beta^T_{X,Z}\gamma$.

Let us now verify the properties of the internal composition $c^T_{X,Y,Z}$, in particular, associativity and unitality. For $X, Y \in \mathcal{C}_T$ denote by $\gamma^T_{X,Y} : [X,Y]_T \to [X,Y]$ the natural map (coming from the equaliser). Let us start with associativity. Consider the following diagram:

$$\begin{tikzcd}
([Y,Z]_T \otimes [X,Z]_T) \otimes [W,X]_T \arrow[rrrr, "a"] \arrow[dr, "g_1"] \arrow[ddd, "g_1'"'] & & & & {[Y,Z]_T \otimes ([X,Y]_T \otimes [W,X]_T)} \arrow[dl, "f_1"'] \arrow[ddd, "f_1'"] \\
& ([Y,Z] \otimes [X,Y]) \otimes [W,X] \arrow[rr, "a"] \arrow[d, "g_2"] & & {[Y,Z] \otimes ([X,Y] \otimes [W,X])} \arrow[d, "f_2"] & \\
& {[X,Z] \otimes [W,X]} \arrow[dr, "c_{W,Z,X}"] & & {[Y,Z] \otimes [W,Y]} \arrow[dl, "c_{W,Y,Z}"'] & \\
{[X,Z]_T \otimes [W,X]_T} \arrow[ur, "g_2'"] \arrow[drr, "c^T_{W,X,Z}"'] & & {[W,Z]} & & {[Y,Z]_T \otimes [W,Y]_T} \arrow[ul, "f_2'"'] \arrow[dll, "c^T_{W,Y,Z}"] \\
& & {[W,Z]_T} \arrow[u, "\gamma^T_{W,Z}"] & &
\end{tikzcd}$$

where

$$f_1 := \gamma^T_{X,Y} \otimes \gamma^T_{X,Y} \otimes \gamma^T_{W,X}, \;\; f_2 := \mathrm{Id} \otimes c_{W,Y,X},$$
$$f_1' := \mathrm{Id} \otimes c^T_{W,X,Y}, \;\; f_2' := \gamma^T_{Y,Z} \otimes \gamma^T_{W,Y}.$$

The maps g_1, g_2, g_1', g_2' are defined analogously.

To show that internal composition is associative, we need to show that the outer pentagon commutes. Now, the top, outer right, outer left, bottom right and bottom left squares commute by the definition of T-comodule maps. The inner pentagon commutes since a closed monoidal category is enriched in itself [27, Lemma 3.4.9]. Thus, the outer pentagon commutes: let $\psi := c_{W,Y,Z} f_2' f_1' a$. We have:

$$\psi = c_{W,Y,Z}((f_2 f_1)a) = c_{W,Y,Z}(f_2(a g_1)) =$$
$$= c_{W,Z,X}(g_2 g_1) = c_{W,Z,X}(g_2' g_1') = (c_{W,Z,X} g_2') g_1' = (\gamma^T_{W,Z} c^T_{W,X,Z}) g_1'.$$

However, following a different route we have:

$$\psi = (c_{W,Y,Z} f_2') f_1' a = (\gamma^T_{W,Z} c^T_{W,Y,Z}) f_1' a.$$

As $\gamma^T_{W,Z}$ is a monomorphism, $c^T_{W,X,Z} g'_1 = c^T_{W,Y,Z} f'_1 a$, finishing the proof of the associativity part. For unitality, let $j^{\mathcal{C}}_X : \star \to [X,X]$ be the adjunct of the left unitor $\lambda_X : \star \otimes X \to X$ in $\mathcal{C}$. Observe that $j^{\mathcal{C}}_X$ equalises the morphisms $\alpha^T_{X,X}$ and $\beta^T_{X,X}$: $\alpha^T_{X,X} \circ j^{\mathcal{C}}_X$ and $\beta^T_{X,X} \circ j^{\mathcal{C}}_X$ are adjuncts of

$$\star \otimes X \to [X,X] \otimes X \xrightarrow{[\mathrm{Id},\rho_X]\otimes \mathrm{Id}} [X,TX] \otimes X \xrightarrow{\mathrm{ev}_{X,TX}} TX$$

and

$$\star \otimes X \to [X,X] \otimes X \xrightarrow{\mathrm{ev}_{X,X}} X \xrightarrow{\rho_X} TX$$

respectively. These two morphisms are equal by the naturality property of the evaluation morphism. Thus, by the universal property of the equaliser, there exists a morphism

$$j^T_X : \star \to [X,X]_T.$$

We claim that this morphism is the identity in $\mathcal{C}_T$. Let us check that composition c^T is unital with respect to this identity. We show it is left unital. The proof for right unital is analogous. Consider the diagram:

$$\begin{tikzcd}
{[Y,Y]_T \otimes [X,Y]_T} \arrow[rrr, "\gamma^T_{Y,Y} \otimes \gamma^T_{X,X}"] \arrow[dr, "c^T_{X,Y,Y}"] & & & {[Y,Y] \otimes [X,Y]} \arrow[dl, "c_{X,Y,Y}"'] \\
& {[X,Y]_T} \arrow[r, "\gamma^T_{X,Y}"] & {[X,Y]} & \\
\star \otimes [X,Y]_T \arrow[uu, "j^T_Y \otimes \mathrm{Id}"] \arrow[ur, "\lambda_{[X,Y]_T}"'] \arrow[rrr, "\mathrm{Id} \otimes \gamma^T_{X,Y}"'] & & & \star \otimes [X,Y] \arrow[uu, "j^{\mathcal{C}}_Y \otimes \mathrm{Id}"'] \arrow[ul, "\lambda_{[X,Y]}"]
\end{tikzcd}$$

where $\lambda_{\_}$ is the left unitor in $\mathcal{C}$ and φ_i are the equaliser maps. The top and bottom trapezoid commute by the equaliser property. The right triangle commutes since $\mathcal{C}$ is enriched in itself, and the outer square commutes since the morphisms $j^{\mathcal{C}}$ factor through the equaliser. Thus, the left triangle commutes too, finishing the proof. $\square$

We can repeat all of this for F-modules (X, θ_X) and (Y, θ_Y) where $\theta_X : FX \to X$ is the structure map. The first key morphism, which utilises θ_Y, appears more involved this time:

$$\alpha^F_{X,Y} : [X,Y] \xrightarrow{[\epsilon_X, \mathrm{Id}_Y]} [TFX, Y] \cong [FX, FY] \xrightarrow{[\mathrm{Id}_{FX}, \theta_Y]} [FX, Y].$$

The second key morphism is just the internal analogue of the composition with θ_X:

$$\beta^F_{X,Y} : [X,Y] \xrightarrow{[\theta_X, \mathrm{Id}_Y]} [FX, Y].$$

Definition 1.21. With $\mathcal{C}$ and F as in the beginning of this section, denote by $\mathcal{C}^F$ the collection of objects in $\mathcal{C}$ which are F-modules. The F-comodule maps object between objects $X, Y \in \mathcal{C}^F$, denoted $[X,Y]_{\mathcal{C}^F}$ or, in shorthand $[X,Y]^F$, is the equaliser of the maps $\alpha^F_{X,Y}$ and $\beta^F_{X,Y}$ defined above.

The reader is advised to observe that Definition 1.21 fully agrees with the example in Section 2.1.

Proposition 1.22. *$\mathcal{C}^F$ is a category enriched in $\mathcal{C}$.*

Proof. The proof is similar to the proof of Proposition 1.20. In the same way we show that the morphism

$$\gamma : [Y, Z]^F \otimes [X, Y]^F \to [Y, Z] \otimes [X, Y] \to [X, Z]$$

equalises $\alpha^F_{X,Y}$ and $\beta^F_{X,Y}$, producing the internal to C^F composition

$$c^F_{X,Y,Z} : [Y, Z]^F \otimes [X, Y]^F \to [X, Z]^F.$$

Defining the unit of the enriched category and verifying the axioms of an enriched category is done in the same way as in Proposition 1.20. □

1.9. **Comodule-contramodule correspondence.** Now we tackle the main results of Chapter 1.

Theorem 1.23. *The assignment $X \mapsto [\maltese, X]_T$ determines a functor $\mathcal{R} : \mathcal{C}_T \to \mathcal{C}^F$ of $\mathcal{C}$-enriched categories.*

Proof. We start by showing that $\mathcal{R}X = [\maltese, X]_T$ is indeed an F-module. We then construct the morphisms $\mathcal{R}_{X,Y}$ which are part of the data of the enriched functor $\mathcal{R}$ and show they satisfy the desired properties.

1. $\mathcal{R}X$ **is an** F**-module.** To show that $\mathcal{R}X$ is indeed an F-module we need to construct a map $\theta_{\mathcal{R}X} : F\mathcal{R}X \to \mathcal{R}X$ which satisfies the associativity and unitality conditions. By Lemma 1.3 $F \cong [\maltese, \ \ ]$. Thus, by the universal property of the equaliser it is enough to show that there exists a map $g : F\mathcal{R}X \to FX$, such that $\alpha^T_{\maltese,X} \circ g = \beta^T_{\maltese,X} \circ g$, where $\alpha^T_{\maltese,X}$ and $\beta^T_{\maltese,X}$ are as in the beginning of the section. Let $\gamma^T_X \coloneqq \gamma^T_{\maltese,X} : \mathcal{R}X \to FX$ be the natural equalisation map. Then $\theta_{\mathcal{R}X}$ is the map satisfying $\gamma^T_X \circ \theta_{\mathcal{R}X} = g$. Let g be the composition:

$$g : F\mathcal{R}X \xrightarrow{F\gamma^T_X} FFX \xrightarrow{\cong} [\maltese \otimes \maltese, X] \xrightarrow{[\rho_{\maltese}, \mathrm{Id}]} FX.$$

Consider the sequence of maps:

$$\Theta_\alpha, \Theta_\beta : F\mathcal{R}X \xrightarrow{F\gamma^T_X} FFX \overset{F\alpha^T_{\maltese,X}}{\underset{F\beta^T_{\maltese,X}}{\rightrightarrows}} FFTX \xrightarrow{\cong} [\maltese \otimes \maltese, TX] \xrightarrow{[\rho_{\maltese}, \mathrm{Id}]} FTX, \tag{1}$$

where Θ_α is the composition along $F\alpha^T_{\maltese,X}$ and Θ_β is the composition along $F\beta^T_{\maltese,X}$.

Using the fact that in this case $\alpha^T_{\maltese,X} = F\rho_X$ it is easy to observe that Θ_α is equal to:

$$F\mathcal{R}X \xrightarrow{F\gamma^T_X} FFX \cong [\maltese \otimes \maltese, X] \xrightarrow{[\rho_{\maltese}, \mathrm{Id}]} FX \xrightarrow{\alpha^T_{\maltese,X}} FTX \tag{2}$$

In particular, $\alpha^T_{\maltese,X} \circ g = \Theta_\alpha$.

Moreover, observe that $\Theta_\alpha = \Theta_\beta$ equal since F is a right adjoint and thus preserves equalisers. Thus, $\alpha^T_{\maltese,X} \circ g = \beta^T_{\maltese,X} \circ g$ establishing the existence of the map $\theta_{\mathcal{R}X} : F\mathcal{R}X \to \mathcal{R}X$ as requested.

To tackle associativity of $\theta_{\mathcal{R}X}$ consider the diagram

$$\begin{tikzcd}
FF\mathcal{R}X \arrow[r, "\mu_{\mathcal{R}X}"] \arrow[d, "F\theta_{\mathcal{R}X}"] & F\mathcal{R}X \arrow[d, "\theta_{\mathcal{R}X}"] \\
F\mathcal{R}X \arrow[r, "\theta_{\mathcal{R}X}"] \arrow[d, "F\gamma^T_X"] & \mathcal{R}X \arrow[d, "\gamma^T_X"] \\
FFX \arrow[r, "\mu_X"] & FX,
\end{tikzcd}$$

where $\mu : FF \to F$ is the natural transformation coming from the monad structure on F.

To prove the claim we need to show that the top square commutes. The bottom square commutes by naturality of the equaliser maps. The outer rectangle also commutes: $\gamma^T_X \circ \theta_{\mathcal{R}X} \circ \mu_{\mathcal{R}X} = g \circ \mu_{\mathcal{R}X} = \mu_X \circ Fg = \mu_X \circ F\theta_{\mathcal{R}X} \circ \gamma^T_X$, which hold by definition of $\theta_{\mathcal{R}X}$ and μ being a natural transformation. This yields our claim.

To show that the unitality axiom holds, we need to verify the commutativity of the diagram on the left:

$$\begin{tikzcd}
\mathcal{R}X \arrow[r, "\eta_{\mathcal{R}X}"] \arrow[dr, "\mathrm{Id}"'] & F\mathcal{R}X \arrow[d, "\theta_{\mathcal{R}X}"] \\
& \mathcal{R}X
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
FX \arrow[r, "\eta_{FX}"] \arrow[dr, "\mathrm{Id}"'] & FFX \arrow[d, "\mu_{FX}"] \\
& FX
\end{tikzcd}$$

It follows from the commutativity of the diagram on the right and from the fact that the natural morphism γ^T_X is a monomorphism (indeed, a kernel of a pair is always a monomorphism). The commutativity of the right diagram is the unitality property of the monad F.

2. **The maps** $\mathcal{R}_{X,Y}$. To establish that $\mathcal{R}$ is indeed an enriched functor, for every $(X, \rho_X), (Y, \rho_Y) \in \mathcal{C}_T$, we need maps

$$\mathcal{R}_{X,Y} : [X, Y]_T \to [\mathcal{R}X, \mathcal{R}Y]^F,$$

which commute with enriched composition and units. Let us construct those.

The functor F is the "free F-module" functor: we have maps

$$F'_{X,Y} : [X, Y] \xrightarrow{F_{X,Y}} [FX, FY]^F \to [FX, FY]$$

for arbitrary $X, Y \in \mathcal{C}$. Given $(X, \rho_X) \in \mathcal{C}_T$, the natural map $\gamma^T_X : \mathcal{R}X \to FX$ yields a map

$$\mathcal{R}'_{X,Y} : [X, Y]_T \to [X, Y] \to [FX, FY]^F \xrightarrow{[\gamma^T_X, \mathrm{Id}]} [\mathcal{R}X, FY]^F.$$

Its composition with the natural map $\gamma^F_{\mathcal{R}X,FY} : [\mathcal{R}X, FY]^F \to [\mathcal{R}X, FY]$ admits the adjunct that is the enriched composition

$$\overline{c}^T_{\maltese,X,Y} : [X, Y]_T \otimes [\maltese, X]_T \to [\maltese, Y].$$

Since the enriched composition preserves T-comodule maps, $\overline{c}^T_{\maltese,X,Y}$ factors through

$$c^T_{\maltese,X,Y} : [X, Y]_T \otimes [\maltese, X]_T \to [\maltese, Y]_T.$$

The corresponding adjunct map $\mathcal{R}^{\dagger}_{X,Y}$ yields a factorisation

$$\Psi_{\gamma,\mathcal{R}'} \coloneqq \gamma^F_{\mathcal{R}X,FY} \circ \mathcal{R}'_{X,Y} : [X,Y]_T \xrightarrow{\mathcal{R}^{\dagger}_{X,Y}} [\mathcal{R}X, \mathcal{R}Y] \xrightarrow{[\mathrm{Id},\gamma^T_Y]} [\mathcal{R}X, FY]. \tag{3}$$

The key question is whether $\mathcal{R}^{\dagger}_{X,Y}$ "ends up" in the F-module maps, i.e., factors through as

$$\mathcal{R}^{\dagger}_{X,Y} : [X,Y]_T \xrightarrow{\mathcal{R}_{X,Y}} [\mathcal{R}X, \mathcal{R}Y]^F \xrightarrow{\gamma^F_{\mathcal{R}X,\mathcal{R}Y}} [\mathcal{R}X, \mathcal{R}Y].$$

For this to be true, we require that $\alpha^F_{\mathcal{R}X,\mathcal{R}Y} \circ \mathcal{R}^{\dagger}_{X,Y} = \beta^F_{\mathcal{R}X,\mathcal{R}Y} \circ \mathcal{R}^{\dagger}_{X,Y}$. Let $\overline{\gamma_Y} \coloneqq [\mathrm{Id}, \gamma^T_Y]$.

$$\begin{tikzcd}
{[X,Y]_T} \arrow[r, "\Psi_{\gamma,\mathcal{R}'}"] \arrow[d, "="] & {[\mathcal{R}X, FY]} \arrow[r, shift left, "\alpha^F_{\mathcal{R}X,FY}"] \arrow[r, shift right, "\beta^F_{\mathcal{R}X,FY}"'] & {[F\mathcal{R}X, FY]} \\
{[X,Y]_T} \arrow[r, "\mathcal{R}^{\dagger}_{X,Y}"] & {[\mathcal{R}X, \mathcal{R}Y]} \arrow[u, "\overline{\gamma_Y}"] \arrow[r, shift left, "\alpha^F_{\mathcal{R}X,\mathcal{R}Y}"] \arrow[r, shift right, "\beta^F_{\mathcal{R}X,\mathcal{R}Y}"'] & {[F\mathcal{R}X, \mathcal{R}Y]} \arrow[u, "\overline{\gamma_Y}"]
\end{tikzcd}$$

Note that the right square commutes by naturality of the internal hom and the left square commutes by definition of $\Psi_{\gamma,\mathcal{R}'}$. Thus, $\overline{\gamma_Y} \circ \alpha^F_{\mathcal{R}X,\mathcal{R}Y} \circ \mathcal{R}^{\dagger}_{X,Y} = \alpha^F_{\mathcal{R}X,FY} \circ \overline{\gamma_Y} \circ \mathcal{R}^{\dagger}_{X,Y} = \alpha^F_{\mathcal{R}X,FY} \circ \Psi_{\gamma,\mathcal{R}'}$.

However, as $\gamma^F_{\mathcal{R}X,FY} \circ \mathcal{R}'_{X,Y} = \Psi_{\gamma,\mathcal{R}'}$ equalises the pair $(\alpha^F_{\mathcal{R}X,FY}, \beta^F_{\mathcal{R}X,FY})$, we deduce

$$\overline{\gamma_Y} \circ \alpha^F_{\mathcal{R}X,\mathcal{R}Y} \circ \mathcal{R}^{\dagger}_{X,Y} = \overline{\gamma_Y} \circ \beta^F_{\mathcal{R}X,\mathcal{R}Y} \circ \mathcal{R}^{\dagger}_{X,Y}.$$

By Lemma 1.16 the natural map $\overline{\gamma_Y} : [F\mathcal{R}X, \mathcal{R}Y] \to [F\mathcal{R}X, FY]$ is a monomorphism, finishing the proof of the statement. Moreover, by construction the maps $\mathcal{R}_{X,Y}$ automatically respect enriched composition and units. ◻

In the case when $\mathcal{C}$ is the category of vector spaces the functor $\mathcal{R}$ admits a left adjoint functor $\mathcal{L}$, given by the contratensor product $\mathcal{L}(Y) = C \odot_C Y$ (cf. Example 2.1). This can be pushed through in higher generality as well. Back to the case when $\mathcal{C}$ is an arbitrary complete cocomplete closed monoidal category and $(T \dashv F)$ is a comonad-monad adjoint pair on $\mathcal{C}$, we consider the following morphisms in $\mathcal{C}$:

$$\alpha_Y : TFY \xrightarrow{T\theta_Y} TY, \quad \beta_Y : TFY \xrightarrow{\Delta FY} TTFY \xrightarrow{T\epsilon\,\mathrm{Id}_Y} TY, \tag{4}$$

where (Y, θ_Y) is a given F-module.

Theorem 1.24. *Suppose that $\mathcal{C}$ is a bicomplete monoidal category. Then the assignment of the coequaliser of α_Y and β_Y to any F-module Y determines a functor $\mathcal{L} : \mathcal{C}^F \to \mathcal{C}_T$ of $\mathcal{C}$-enriched categories. Moreover, $(\mathcal{L} \dashv \mathcal{R})$ is a $\mathcal{C}$-enriched adjoint pair.*

Proof. We start by showing that $\mathcal{L}$ is a functor of enriched categories. The proof of this part is similar to the proof of Theorem 1.23. Thus, we need to first show that $\mathcal{L}Y$ is indeed a T-comodule for every $(Y, \theta_Y) \in \mathcal{C}^F$, i.e., we need an associative unital map $\rho_{\mathcal{L}Y} : \mathcal{L}Y \to T\mathcal{L}Y$. Let $\gamma_Y : TY \to \mathcal{L}Y$ denote the natural coequaliser map. To obtain the map $\rho_{\mathcal{L}Y}$ it is enough to construct a map $f : TY \to T\mathcal{L}Y$, such that $f \circ \alpha_Y = f \circ \beta_Y$. Let f be the composition

$$TY \xrightarrow{\mu_{TY}} TTY \xrightarrow{T\gamma_Y} T\mathcal{L}Y.$$

The fact that $\mu : T \to TT$ is a natural transformation allows us to rewrite:

$$f \circ \beta_Y = T\gamma_Y \circ T\beta_Y \circ \mu_{TFY} = T\gamma_Y\beta_Y \circ \mu_{TFY} = \\ = T\gamma_Y\alpha_Y \circ \mu_{TFY} = f \circ \alpha_Y .$$

Thus, we have a map $\rho_{\mathcal{L}Y} : \mathcal{L}Y \to T\mathcal{L}Y$. It is associative since μ is associative. The argument for unitality is similar to the argument in the proof of Theorem 1.23: we use the unitality of the comonad T and the fact that $\gamma_Y : TY \to T\mathcal{L}Y$ is an epimorphism (by Lemma 1.17).

Next we need to show that for every pair of objects $(X\theta_X), (Y, \theta_Y) \in \mathcal{C}^F$, there exists a map $\mathcal{L}_{X,Y} : [X, Y]^F \to [\mathcal{L}X, \mathcal{L}Y]_T$ respecting the enriched composition and units in $\mathcal{C}^F$.

To obtain such $\mathcal{L}_{X,Y}$ we first construct a map $\mathcal{L}^\dagger_{X,Y} : [X, Y]^F \to [\mathcal{L}X, \mathcal{L}Y]$. By adjunction this is equivalent to having a map $\epsilon_{X,Y} : [X, Y]^F \otimes \mathcal{L}X \to \mathcal{L}Y$. Let $\mathrm{ev}^F_{X,Y}$ be the map

$$\mathrm{ev}^F_{X,Y} : [X, Y]^F \otimes X \otimes \maltese \xrightarrow{\gamma^F_{X,Y}\otimes \mathrm{Id}} [X, Y] \otimes X \otimes \maltese \xrightarrow{\mathrm{ev}_{X,Y}} Y \otimes \maltese = TY.$$

Recall that for every $(Y, \theta_Y) \in \mathcal{C}^F$ we write $\gamma_Y : TY \to \mathcal{L}Y$ for the natural coequaliser maps. Let

$$d \coloneqq \gamma_Y \circ \mathrm{ev}^F_{X,Y} : [X, Y]^F \otimes X \otimes \maltese \to \mathcal{L}Y.$$

The desired map $\mathcal{L}^\dagger_{X,Y}$ exists if d coequalises $\alpha_X, \beta_X : TFX \rightrightarrows TX$. Since $(X, \theta_X), (Y, \theta_Y) \in \mathcal{C}^F$ by definition $([X, Y]^F, \gamma^F_{X,Y})$ is the equaliser of the pair $(\alpha^F_{X,Y}, \beta^F_{X,Y})$ defined earlier in the section. Applying the adjunction, the equaliser property becomes the commutativity of the square

$$\begin{tikzcd}
{[X,Y]^F \otimes FX} \arrow[r, "\gamma^F_{X,Y}\otimes \mathrm{Id}"] \arrow[d, "\mathrm{Id}\otimes\theta_X"] & {[X,Y] \otimes FX} \arrow[r, "c_{\maltese,X,Y}"] & FY \arrow[d, "\theta_Y"] \\
{[X,Y]^F \otimes X} \arrow[r, "\gamma^F_{X,Y}\otimes \mathrm{Id}"] & {[X,Y] \otimes X} \arrow[r, "\mathrm{ev}_{X,Y}"] & Y
\end{tikzcd}$$

Applying the functor T to the diagram above, by associativity we obtain that

$$\alpha_Y \circ c^F_{\maltese,X,Y} = \mathrm{ev}^F_{X,Y} \circ \mathrm{Id}_{[X,Y]^F} \otimes \alpha_X .$$

We repeat the same procedure for β_X, β_Y in place of α_X, α_Y. In particular,

$$d \circ \alpha_X = \gamma_Y \circ \alpha_Y \circ c^F_{\maltese,X,Y} = \gamma_Y \circ \beta_Y \circ c^F_{\maltese,X,Y} = d \circ \beta_X .$$

Thus, we have well-defined maps $\mathcal{L}^\dagger_{X,Y} : [X, Y]^F \to [\mathcal{L}X, \mathcal{L}Y]$.

Having defined $\mathcal{L}^\dagger_{X,Y}$, in order to obtain the maps $\mathcal{L}_{X,Y}$ we repeat the proof of Theorem 1.23 line-by-line using the functor T instead of F and $\mathcal{L}$ instead of $\mathcal{R}$.

We proceed to the proof of the second statement. To show that $(\mathcal{L}, \mathcal{R})$ is a $\mathcal{C}$-enriched adjoint pair we need to show that there is a $\mathcal{C}$-natural isomorphism of bifunctors

$$[\mathcal{L}X, Y]_T \cong [X, \mathcal{R}Y]^F .$$

Note that by Lemma 1.15 the internal hom bifunctor $[-, -]$ is a $\mathcal{C}$-natural transformation. Thus, the adjunction $(T \dashv F)$ becomes an isomorphism of $\mathcal{C}$-enriched bifunctors

$$[TX, Y] \cong [X, FY] .$$

Moreover, we have $[TX, Y]_T \cong [X, \mathcal{R}Y]$ and $[\mathcal{L}X, Y] \cong [X, FY]^F$ as objects in $\mathcal{C}$. Note that the maps $\alpha^T_{TX,TY}, \beta^T_{TX,TY} : [TX, Y] \rightrightarrows [TX, TY]$ are adjuncts of the maps $[\mathrm{Id}, \alpha^T_{\maltese,X}], [\mathrm{Id}, \beta^T_{\maltese,X}] : [X, FY] \rightrightarrows [X, FTY]$. Observe that the functor $[X, -]$ is a right adjoint and thus preserves kernels. Combined with the fact that $[TX, Y] \cong [X, FY]$ is an isomorphism of bifunctors, we can deduce that every map which equalises the pair $(\alpha^T_{TX,TY}, \beta^T_{TX,TY})$ also equalises $([\mathrm{Id}, \alpha^T_{\maltese,X}], [\mathrm{Id}, \beta^T_{\maltese,X}])$. This implies the isomorphism $[TX, Y]_T \cong [X, \mathcal{R}Y]$.

Again, by Lemma 1.15 this isomorphism is, in fact, a $\mathcal{C}$-enriched isomorphism of enriched bifunctors. The argument for $[\mathcal{L}X, Y] \cong [X, FY]^F$ is analogous.

We can complete the proof by observing that the following squares are cartesian in $\mathcal{C}$:

$$\begin{tikzcd}
{[TX,Y]} & {[\mathcal{L}X,Y]} \arrow[l, "{[\gamma_X,\mathrm{Id}]}"'] & & {[X,FY]} & {[X,FY]^F} \arrow[l, "\gamma^F_{X,FY}"'] \\
{[TX,Y]_T} \arrow[u, "\gamma^T_{TX,Y}"] & {[\mathcal{L}X,Y]_T} \arrow[l, "\psi"] \arrow[u, "\gamma^T_{\mathcal{L}X,Y}"] & \cong & {[X,\mathcal{R}Y]} \arrow[u, "{[\mathrm{Id},\gamma^T_{\maltese,X}]}"'] & {[X,\mathcal{R}Y]^F} \arrow[l, "\gamma^F_{X,\mathcal{R}Y}"] \arrow[u, "\phi"']
\end{tikzcd}$$

Let $d_1 \coloneqq [\gamma_X, \mathrm{Id}] \circ \gamma^T_{\mathcal{L}X,Y}$ and $d_2 \coloneqq [\mathrm{Id}, \gamma^T_{\maltese,X}] \circ \gamma^F_{X,\mathcal{R}Y}$. The maps d_1 and d_2 clearly equalise the pairs $(\alpha^T_{TX,TY}, \beta^T_{TX,TY})$ and $([\mathrm{Id}, \alpha^T_{\maltese,X}], [\mathrm{Id}, \beta^T_{\maltese,X}])$ respectively. Thus, by definition $d_1 = \gamma^T_{TX,Y} \circ \psi$, i.e., the left square commutes. The universal property of the equaliser implies that $[\mathcal{L}X, Y]_T$ is a pullback. A similar argument shows that the square on the right is cartesian. The existence of the $\mathcal{C}$-enriched isomorphisms of bifunctors explained above completes the proof.

□

Note that the first part of Theorem 1.24 would have been an immediate consequence of Theorem 1.23 if we had a well-defined enriched duality between the categories $\mathcal{C}_T$ and $\mathcal{C}^F$ and their opposites $(\mathcal{C}_T)^{op}$ and $(\mathcal{C}^F)^{op}$. More precisely, observe that if we were in a setting of usual, rather than enriched categories, the functor $\mathcal{L}$ would be precisely the functor $\mathcal{R}^{op}$. However, as the opposite category of an enriched category is enriched in $\mathcal{C}^{rev}$, not in $\mathcal{C}^{op}$, where $\mathcal{C}^{rev}$ is the category with the same objects and morphisms as $\mathcal{C}$ but with the opposite tensor product. Moreover, $\mathcal{C}^{rev}$ is not a closed monoidal category and thus the desired duality argument fails. We believe it can be fixed if one was to look at a two level comonad-monad pairs. However, we leave this problem for subsequent research and do not address it here.

1.10. **Connection with Kleisli categories.** Let $\widetilde{\mathcal{C}_T}$ and $\widetilde{\mathcal{C}^F}$ be Kleisli categories. These are full subcategories of $\mathcal{C}_T$ and $\mathcal{C}^F$ spanned by cofree comodules TX and free modules FX. These categories are isomorphic [5, 2.6]. The isomorphisms are given by

$$\widetilde{\mathcal{C}_T} \longleftrightarrow \widetilde{\mathcal{C}^F}, \quad TX \longleftrightarrow FX\,.$$

Observe that

$$\mathcal{R}(TX) = [\maltese, TX]_T \cong [\maltese, X] = FX\,.$$

The isomorphism between the Kleisli categories extends to an equivalence between their Karoubian completions [5, 2.8], which agree with the full subcategories of $\mathcal{C}_T$ and $\mathcal{C}^F$ spanned by $\mathcal{F}_T$-injective T-comodules and $\mathcal{F}^F$-projective F-modules. In one direction this equivalence is given by

$$\mathcal{R} : \mathcal{C}_T^{inj} \longrightarrow \mathcal{C}^F_{proj} \,. \tag{5}$$

Our results imply that this is a $\mathcal{C}$-enriched functor. The following two questions are worth further attention.

Question 1.25. What is the relation between $\mathcal{L}$ and a quasiinverse of $\mathcal{R}$ in (5)?

Question 1.26. Are $\mathcal{C}_T^{inj}$ and $\mathcal{C}^F_{proj}$ equivalent as $\mathcal{C}$-enriched in categories?

1.11. **Change of comonoid.** We would like to collect standard technical facts on the behaviour of comodules and contramodules under a morphism $f : \maltese \to \widehat{\maltese}$ of comonoids in $\mathcal{C}$. We leave their proofs as an exercise to the reader.

We denote the two comonad-monad adjoint pairs by $(T \dashv F)$ and $(\widehat{T} \dashv \widehat{F})$. Clearly, we have *restriction functors*

$$\mathcal{R}es : \mathcal{C}_T \to \mathcal{C}_{\widehat{T}}, \quad \mathcal{R}es(M, \rho : M \to TM) = (M, Tf \circ \rho),$$

$$\mathcal{R}es : \mathcal{C}^F \to \mathcal{C}^{\widehat{F}}, \quad \mathcal{R}es(M, \theta : FM \to M) = (M, \theta \circ Ff).$$

Besides the comodules and the contramodules, we would like to consider the overcategory (or slice category) $(\mathcal{C} \downarrow \maltese)$, although the assumptions of $\maltese$, $\widehat{\maltese}$ being comonoids and f being a comonoid morphism are unnecessary for the overcategory. Again there is a restriction functor

$$\mathcal{R}es : (\mathcal{C} \downarrow \maltese) \to (\mathcal{C} \downarrow \widehat{\maltese}), \quad \mathcal{R}es(M, \phi : M \to \maltese) = (M, f \circ \phi).$$

All three functors deserve the same notation because they are essentially the "same" functor, at least they are the same on objects. The similarity breaks down when we consider the existence of induction functors, forcing us to use different notations.

We start with the overcategory because it is the easiest one to understand.

Proposition 1.27. *(cf.* [13, Lemma 7.6.6]*) Let $\maltese$, $\widehat{\maltese}$ be any objects of $\mathcal{C}$, $f \in \mathcal{C}(\maltese, \widehat{\maltese})$. Then*

$$\mathcal{I}nd\!\downarrow : (\mathcal{C} \downarrow \widehat{\maltese}) \to (\mathcal{C} \downarrow \maltese), \quad \mathcal{I}nd\!\downarrow (P, \phi : P \to \widehat{\maltese}) = (P \times_{\widehat{\maltese}} \maltese, \pi_2),$$

where π_2 is the projection onto the second component, is a $\mathcal{C}$-enriched functor, internally right adjoint to $\mathcal{R}es$.

Our comodules are right comodules since $T = - \otimes \maltese$. Similarly, there is a category of left comodules, ${}_T\mathcal{C}$, comodules over the comonad $T' = \maltese \otimes -$. The comonoid $\maltese$ is naturally an object of both ${}_T\mathcal{C}$ and $\mathcal{C}_T$. In fact, it is a bicomodule in a suitable sense. If $\mathcal{C}$ is biclosed, then we can use Proposition 1.20 to equip ${}_T\mathcal{C}$ with enrichment in $\mathcal{C}$.

Proposition 1.28. *(cf.* [6, 11.1.9]*) Suppose that $\mathcal{C}$ is biclosed.*

(1) *There exists a cotensor product, an enriched in $\mathcal{C}$ bifunctor*

$$-\square_{\maltese}- : \mathcal{C}_T \times {}_T\mathcal{C} \to \mathcal{C},$$

where $M \square_{\maltese} N$ is the equaliser of the pair of maps

$$\rho_M \otimes \mathrm{Id}_N,\; a^{-1}_{M,\maltese,N} \circ (\mathrm{Id}_M \otimes \rho_N) \;:\; M \otimes N \rightrightarrows (M \otimes \maltese) \otimes N.$$

(2) *If f is a morphism of comonoids and the monad T preserves equalisers of pairs of morphisms, then*

$$\mathcal{Ind}_T : \mathcal{C}_{\widehat{T}} \to \mathcal{C}_T, \quad \mathcal{Ind}_T(M, \rho : M \to \widehat{T}(M)) = (M\Box_{\widehat{\maltese}}\maltese, \widetilde{\rho})$$

where the structure morphism $\widetilde{\rho}$ appears in the diagram

$$\begin{tikzcd}
M\Box_{\widehat{\maltese}}\maltese \arrow[r] \arrow[d, dashed, "\widetilde{\rho}"] & M\otimes\maltese \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "a^{-1}_{\cdot,\cdot,\cdot}\circ(\mathrm{Id}\otimes\Delta_{\maltese})"'] & (M\otimes\widehat{\maltese})\otimes\maltese \arrow[d, "a^{-1}_{\cdot,\cdot,\cdot}\circ(\mathrm{Id}\otimes\Delta_{\maltese})"'] \\
(M\Box_{\widehat{\maltese}}\maltese)\otimes\maltese \arrow[r] & (M\otimes\maltese)\otimes\maltese \arrow[r, shift left] \arrow[r, shift right] & ((M\otimes\widehat{\maltese})\otimes\maltese)\otimes\maltese
\end{tikzcd}$$

with equalisers in both rows and commutative squares, as soon as only the top or only the bottom arrows are taken in the right square, defines a $\mathcal{C}$-enriched functor, internally right adjoint to $\mathcal{Res}$.

If T is continuous, then it preserves the equalisers. Similarly in Proposition 1.29 below, if F is cocontinuous, then it preserves the coequalisers. In the category of chain complexes over a commutative ring $\mathbb{K}$ (see Section 3.7), these are conditions for $\maltese$ to be flat and projective correspondingly. See also Section 2.4.

Opposite to comodules, biclosedness of $\mathcal{C}$ is necessary even to define the left contramodules: these are objects Y with structure map $\widetilde{[\maltese, Y]} \to Y$. On the other hand, the left contramodules are not necessary for the construction of the coinduction.

Proposition 1.29. *(cf.* [24, 2.2]*)*

(1) *There exists a cohom, a $\mathcal{C}$-enriched bifunctor*

$$\mathrm{Cohom}_{\maltese}(-,-) : \mathcal{C}_T \times \mathcal{C}^F \to \mathcal{C},$$

where $\mathrm{Cohom}_{\maltese}(M,P)$ is the coequaliser of the pair of maps

$$ad_{M,P}\circ[\rho_M, \mathrm{Id}_P],\ [\mathrm{Id}_M, \theta_P]\ :\ [M, F(P)] \rightrightarrows [M,P],$$

where $ad_{M,P}$ is the internal adjunction map.

(2) *If $\mathcal{C}$ is biclosed, f is a morphism of comonoids and the comonad F is cocontinuous, then*

$$\mathcal{Coind}^F : \mathcal{C}^{\widehat{F}} \to \mathcal{C}^F, \quad \mathcal{Coind}^F(P, \theta : \widehat{F}(P) \to P) = (\mathrm{Cohom}_{\widehat{\maltese}}(\maltese, P), \widetilde{\theta})$$

where the structure morphism $\widetilde{\theta}$ appears in the diagram

$$\begin{tikzcd}
{[\maltese, [\widehat{\maltese}, P]]} \arrow[r, shift left] \arrow[r, shift right] & {[\maltese, P]} \arrow[r] & \mathrm{Cohom}_{\widehat{\maltese}}(\maltese, P) \\
{[\maltese, [\maltese\otimes\widehat{\maltese}, P]]} \arrow[r, shift left] \arrow[r, shift right] \arrow[u, "{[\Delta_{\maltese}, \mathrm{Id}]\circ a^{-1}_{\cdot,\cdot,\cdot}\circ ad_{\cdot,\cdot}}"'] & {[\maltese, [\maltese, P]]} \arrow[r] \arrow[u, "{[\Delta_{\maltese}, \mathrm{Id}]\circ ad_{\cdot,\cdot}}"'] & {[\maltese, \mathrm{Cohom}_{\widehat{\maltese}}(\maltese, P)]} \arrow[u, dashed, "\widetilde{\theta}"]
\end{tikzcd}$$

with coequalisers in both rows and commutative squares, as soon as only the top or only the bottom arrows are taken in the left square, defines a $\mathcal{C}$-enriched functor, internally left adjoint to $\mathcal{Res}$.

We finish this section with a question, reminiscent of the standard cohom-defining property in linear categories (cf. [24, 2.2]):

Question 1.30. Assuming that $\mathcal{C}$ is biclosed, does there exist a $\mathcal{C}$-enriched natural equivalence of trifunctors $\mathcal{C}_T \times {}_T\mathcal{C} \times \mathcal{C} \to \mathcal{C}$

$$[M\Box_{\maltese}N, X] \cong \mathrm{Cohom}_{\maltese}(M, [N, X])\,?$$

2. Examples

While the general theory of comodules and contramodules in categories is exhilarating, it is instructive to examine concrete examples. Each section in this chapter is devoted to an example.

2.1. **Vector spaces.** Let $\mathcal{C}$ be the category of vector spaces over a field $\mathbb{K}$. The internal hom is more-or-less the same as the hom: $\mathcal{C}(X,Y) = [X,Y] = \widetilde{[X,Y]}$. The only difference is that $\mathcal{C}(X,Y)$ is a set, while $[X,Y]$ comes with a natural vector space structure. The unit object $\star$ is the one-dimensional vector space $\mathbb{K}$.

A chief $\maltese$ of the pair $(T \dashv F)$ is just a coalgebra so that

$$T(X) = X \otimes \maltese, \quad F(X) = [\maltese, X].$$

Writing the comodule structure map in Sweedler's Σ-notation $\rho_X(\mathbf{x}) = \sum_{(\mathbf{x})} \mathbf{x}_{(0)} \otimes \mathbf{x}_{(1)}$, so that the two maps in Definition 1.19 are

$$\alpha^T_{X,Y}(f)(\mathbf{x}) = \sum_{(f(\mathbf{x}))} f(\mathbf{x})_{(0)} \otimes f(\mathbf{x})_{(1)}, \quad \beta^T_{X,Y}(f)(\mathbf{x}) = \sum_{(\mathbf{x})} f(\mathbf{x}_{(0)}) \otimes \mathbf{x}_{(1)}.$$

It follows that the category $\mathcal{C}_T$ of T-comodules (as defined in Section 1.8) is isomorphic to the usual category of $\maltese$-comodules.

We can no longer write the contramodule structure maps in the Sweedler's Σ-notation. Instead it is instructive to inspect the square

$$\begin{tikzcd} {[\maltese, X]} \arrow[r, "\theta_X"] \arrow[d, "f\circ"] & X \arrow[d, "f"'] \\ {[\maltese, Y]} \arrow[r, "\theta_Y"] & Y \end{tikzcd} \tag{6}$$

that depends on a linear map $f : X \to Y$. The left-bottom path of the square is $\alpha^F_{X,Y}(f)$ and the top-right path of the square is $\beta^F_{X,Y}(f)$. By definition, f is a $\maltese$-contramodule homomorphism if and only if $\alpha^F_{X,Y}(f) = \beta^F_{X,Y}(f)$ if and only if the square is commutative. Thus, the space of $\maltese$-contramodule homomorphisms from X to Y is the equaliser of $\alpha^F_{X,Y}$ and $\beta^F_{X,Y}$, exactly as in Definition 1.21. It follows that the category $\mathcal{C}^F$ of F-modules is isomorphic to the less well known category of $\maltese$-contramodules.

The adjoint functors $\mathcal{L}$ ans $\mathcal{R}$ are described by Positselski in this case [25]. They define an equivalence between the coderived category of $\maltese$-comodules and the contraderived category of $\maltese$-contramodules. See Section 3.7 for further details on the slightly more general case of DG-coalgebras.

2.2. **Specific coalgebra.** Let us consider the polynomial coalgebra $\maltese = \mathbb{K}[z]$, $\Delta(z) = 1 \otimes z + z \otimes 1$. A $\maltese$-comodule is a vector space V with a countable family of operators ρ_n, $n \in \mathbb{N}$ such that

$$\rho : V \to V \otimes \maltese, \quad \rho(v) = \sum_n \rho_n(v) \otimes z^n.$$

It needs to satisfy the unitality condition

$$\rho_0(v) = v, \tag{7}$$

the associativity condition

$$\rho_m(\rho_n(v)) = \binom{m+n}{n}\rho_{m+n}(v) \tag{8}$$

and the finiteness condition

$$\forall v \ \exists N \ \forall n > N \ \rho_n(v) = 0. \tag{9}$$

Notice that in characteristic zero this is just a vector space with a locally nilpotent operator ρ_1 such that $\rho_n = \rho_1^{(n)} = \frac{1}{n!}\rho_1^n$.

A $\maltese$-contramodule is a vector space V with a countable family of operators $\theta_n,\ n \in \mathbb{N}$ such that

$$\theta : [\maltese, V] \to V, \quad \theta(f) = \sum_n \theta_n(f(z^n)).$$

It is easy to see that the unitality and the associativity conditions for θ are the same as for ρ. In particular, in characteristic zero $\theta_n = \theta_1^{(n)} = \frac{1}{n!}\theta_1^n$ for all n. The finiteness condition is different: since $f(z^n)$ can be any sequence of elements of V, the condition can be stated as

$$\forall \text{ sequence } (v_n),\ v_n \in V \text{ the sum } \sum_n \theta_n(v_n) \text{ is well-defined.} \tag{10}$$

Such well-definedness may or may not result from series convergence in some topology. Positselski [26, 0.2] emphasises the point that it is an algebraic infinite summation operation that, in this case, is a linear map $s : U \to V$ where U is a subspace of $V[[t]]$ such that

$$\sum_n \theta_n(f(z^n))t^n \in U \quad \text{and} \quad \theta(f) = s\big(\sum_n \theta_n(f(z^n))t^n\big)$$

for all $f \in [\maltese, V]$.

For instance, $\mathbb{K}[x]$ with $\rho_n = \partial_x^{(n)} = \frac{1}{n!}\frac{\partial^n}{\partial x^n}$ is a $\maltese$-comodule but not a $\maltese$-contramodule. On the other hand, $\mathbb{K}[[x^{-1}]]$ with the same operators $\theta_n = \partial_x^{(n)}$ is a $\maltese$-contramodule but not a $\maltese$-comodule. In this case

$$U = \{\sum_n h_n t^n \ \mid \ h_n \in (x^{-n}) \lhd \mathbb{K}[[x^{-1}]]\}\,, \ s(\sum_n h_n t^n) = \sum_n h_n$$

is well-defined because the calculation of the coefficient in front of each x^{-n} requires only a finite sum.

If $\mathbb{K}$ is a field of characteristic zero, this comodule and this contramodule correspond to each other under the comodule-contramodule correspondence:

$$\mathcal{R}(\mathbb{K}[x]) \cong \mathbb{K}[[x^{-1}]] \cong F\star\,, \quad \mathcal{L}(\mathbb{K}[[x^{-1}]]) \cong \mathbb{K}[x] \cong T\star\ .$$

2.3. **Sets.** The category of sets $\mathcal{S}ets$ has a closed symmetric monoidal structure given by the product of sets and a monoidal unit given by a one point set $* = \{p\}$. In this category the internal hom and the external hom are the same set, denoted here by $[X, Y]$. Let $\psi = (\alpha, \beta) : X \to X \times X$ be a coproduct. The counital axiom immediately implies that $\alpha(x) = x = \beta(x)$ and so ψ is equal to the diagonal map Δ. Thus, each set X has a unique comonoid structure. We fix a base set $\maltese$ and identify this with the comonoid

$$(\maltese, \Delta, \epsilon)$$

where Δ is the diagonal map $\maltese \to \maltese \times \maltese$ and $\epsilon : \maltese \to \{p\}$ is the unique map.

By a $\maltese$-set we mean a pair (X,ϕ) where X is a set and $\phi : X \to \maltese$ is a function. A morphism of $\maltese$-sets is a function f making the following square commutative:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\phi_X"] & Y \arrow[d, "\phi_Y"] \\ \maltese \arrow[r, "\mathrm{Id}"] & \maltese \end{tikzcd}$$

A $\maltese$-set (X,ϕ) admits a canonical right $\maltese$-comodule structure, given by

$$\rho_\phi : X \to X \times \maltese,\ \ \rho_\phi(y) = (y, \phi(y)). \tag{11}$$

The counitality of ρ easily implies that any $\maltese$-comodule is of this form. We state this as a proposition, which allows us to identify the category $\mathcal{C}_T$ with the category of $\maltese$-sets from now on.

Proposition 2.1. *For any set $\maltese$ formula* (11) *defines an isomorphism from the category $(\mathcal{S}ets\downarrow\maltese)$ of $\maltese$-sets to the category $\mathcal{S}ets_\maltese$ of $\maltese$-comodules.*

The right $\maltese$-contramodules are a bit more intriguing. These are sets Y with a contramodule operation

$$\theta : [\maltese, Y] \to Y$$

subject to the contraassociativity and contraunitality. A good thought experiment to visualise these axioms is to write the structure map in the "integral" notation:

$$\theta(f) = \theta_x(f(x)) = \text{“}\int_\maltese f(x)\mathbf{d}x\text{”} \tag{12}$$

Further it is useful to identify $[\maltese,[\maltese,X]]$ with $[\maltese\times\maltese,X]$. A map $f : \maltese\times\maltese \to X$ is a two-variable function $f(x,y)$. Now every $a \in X$ admits a constant function $\mathrm{Const}_a(x) = a$. So we can express the contraassociativity and contraunitality axioms as the following "integral" identities:

$$\int_\maltese f(x,x)\mathbf{d}x = \int_\maltese(\int_\maltese f(x,y)\mathbf{d}y)\mathbf{d}x\,, \quad \theta(\mathrm{Const}_a) = a\,. \tag{13}$$

We will not use this bulky notation but we will use the middle notation in (12). It is useful for multi-variable functions. For simplicity, whenever possible, we will try to use x, x_1, x_2 for variables, while y and z are reserved for fixed elements of the chief. For instance, contraassociativity in (13) becomes

$$\theta_x(f(x,x)) = \theta_{x_1}(\theta_{x_2}(f(x_1,x_2)))\,.$$

Example 2.2. The empty set $\varnothing$ is both a $\maltese$-set and $\maltese$-contramodule in a unique way. Since $\maltese\times\varnothing$ is empty, $\mathcal{L}(\varnothing)=\varnothing$. Since $[\maltese,\varnothing]$ is empty, $\mathcal{R}(\varnothing)=\varnothing$.

Example 2.3. Let $|\maltese| \geqslant 2$, $z \in \maltese$, X any set. By (X,θ_z) we denote its $\maltese$-contramodule structure, supported at z, i.e., $\theta_z(f(x)) = f(z)$. Let us examine the following $\maltese$-set:

$$(X(z),\phi_z) := (X\coprod(\maltese\backslash\{z\}),\phi_z),\ \ \phi_z(y) = \begin{cases} z & \text{if } y \in X, \\ y & \text{if } y \in \maltese. \end{cases}$$

An easy calculation shows that

$$(X,\theta_z) \cong \mathcal{R}(X(z),\phi_z) \ \text{ and } \ (X(z),\phi_z) \cong \mathcal{L}(X,\theta_z)\,.$$

Example 2.4. Let $\maltese = \{y, z\}$ be a 2-element set. A $\maltese$-set (X, ϕ) is a set, split as a disjoint union of two subsets:

$$(X, \phi) = X_y \coprod X_z, \quad X_y = \phi^{-1}(y), \quad X_z = \phi^{-1}(z).$$

On the other hand, a $\maltese$-contramodule (P, θ) is just a set with a binary operation:

$$\theta(f) \coloneqq f(y) \diamond f(z) \ \text{ or } \ a \diamond b \coloneqq \theta(f_{a,b}),$$

where the function $f_{a,b}(x)$ is defined by $f_{a,b}(y) = a$ and $f_{a,b}(z) = b$. The contraassociativity and the contraunitality are equivalent to the following axioms of this binary operation:

$$(a \diamond b) \diamond (c \diamond d) = a \diamond d \ \text{ and } \ a \diamond a = a.$$

Suppose P is non-empty. Choose $p \in P$ and define

$$X_y \coloneqq P \diamond p, \ X_z \coloneqq p \diamond P.$$

This $\maltese$-set $(X, \phi) = X_y \coprod X_z$ yields a $\maltese$-contramodule

$$\mathcal{R}(X, \phi) = X_y \times X_z, \quad (a, s)\clubsuit(b, t) = (a, t).$$

Since X_y and X_z are subsets of P, we have a function

$$\Upsilon : \mathcal{R}(X) \to P, \quad (a, s) \mapsto a \diamond s,$$

which is a contramodule homomorphism since

$$\Upsilon((a, s)\clubsuit(b, t)) = \Upsilon(a, t) = a \diamond t = (a \diamond s) \diamond (b \diamond t) = \Upsilon(a, t) \diamond \Upsilon(b, s).$$

The function Υ is surjective because

$$q = q \diamond q = \Upsilon(q, q)$$

for all $q \in P$. Finally, the function Υ is injective. Suppose $\Upsilon(a, s) = \Upsilon(b, t)$. By definition, $a = a' \diamond p$, $b = b' \diamond p$ for some $a', b' \in P$, which lets us conclude that

$$a \diamond p = (a' \diamond p) \diamond (p \diamond p) = a' \diamond p = a \ \text{ and } \ b \diamond p = b.$$

It follows that

$$a = (a \diamond s) \diamond (p \diamond p) = \Upsilon(a, s) \diamond (p \diamond p) = \Upsilon(b, t) \diamond (p \diamond p) = (b \diamond t) \diamond (p \diamond p) = b$$

with a similar proof showing that $s = t$.

Definition 2.5. Let (X, ϕ) be a $\maltese$-set. Let us call a function $h(x_1, x_2) \in [\maltese^2, X]$ *apt* if for all $z \in \maltese$ the map $h_z : \maltese \to X$, defined as $h_z(x) \coloneqq h(z, x)$, is a morphism of $\maltese$-sets.

When we identify $[\maltese, [\maltese, X]]$ with $[\maltese^2, X]$ via $h(x_1, x_2) \leftrightarrow [x_1 \mapsto h_{x_1}(x_2)]$, the subset $[\maltese, [\maltese, X]_{\maltese}] \subseteq [\maltese, [\maltese, X]]$ gets identified with the subset of apt functions $[\maltese^2, X]_{apt} \subseteq [\maltese^2, X]$.

Theorem 2.6. *For any $\maltese$-contramodule (P, θ) there exists a $\maltese$-set (X, ϕ) such that $P \cong \mathcal{R}(X)$ in $\mathcal{S}ets^F$.*

Proof. Since $\varnothing \cong \mathcal{R}(\varnothing)$ in $\mathcal{S}ets^F$, we can assume that P is non-empty.

Fix $p \in P$. For each $z \in \maltese$, $q \in P$ we define a function

$$f^{z,q}(x) \in [\maltese, P], \quad f^{z,q}(x) = \begin{cases} q & \text{if } x = z, \\ p & \text{if } x \neq z. \end{cases} \tag{14}$$

Now consider a set $P_z := \{\theta(f^{z,q}) \mid q \in P\} \subseteq P$ for each $z \in \maltese$. This collection yields a $\maltese$-set (X, ϕ) and a $\maltese$-contramodule $(\mathcal{R}(X), \theta')$ where

$$X = \coprod_{z \in \maltese} P_z,\ \phi(P_z) = z,\ \mathcal{R}(X) = [\maltese, X]_{\maltese} = \prod_{z \in \maltese} P_z,\ \theta'(h(x_1, x_2)) = h(x, x). \tag{15}$$

The natural function $X \to P$ yields a function $\mathcal{R}(X) = [\maltese, X]_{\maltese} \to [\maltese, P]$, $f(x) \mapsto \overline{f}(x)$, which, in its turn, gives a a homomorphism of $\maltese$-contramodules

$$\Upsilon : \mathcal{R}(X) \to [\maltese, P] \xrightarrow{\theta} P\,, \quad \Upsilon(f(x)) = \theta(\overline{f})\,.$$

For any apt function $h = h(x_1, x_2)$

$$\Upsilon(\theta'([x \mapsto h_x])) = \Upsilon(\theta'(h)) = \Upsilon(h(x, x)) = \theta(\overline{h}(x, x))$$

and then, using with contraassociativity (cf. (13)),

$$\theta(\overline{h}(x, x)) = \theta_{x_1}(\theta_{x_2}(\overline{h}(x_1, x_2))) = \theta([x \mapsto \theta(\overline{h_x})]) = \theta([\mathrm{Id}_{\maltese}, \Upsilon]([x \mapsto \theta(\overline{h_x})])).$$

The function Υ is surjective. Indeed, given $q \in P$, consider $h^q(x_1, x_2) \in [\maltese^2, P]_{apt}$ defined by $h^q(x_1, x_2) = f^{x_1,q}(x_2)$. By contraunitality, $q = \theta(\mathrm{Const}_q)$. By equation (14), $\mathrm{Const}_q(x) = f^{x,q}(x) = h^q(x, x)$. We continue with contraassociativity:

$$q = \theta(h^q(x, x)) = \theta_{x_1}(\theta_{x_2}(h^q(x_1, x_2))) = \theta([x \mapsto \theta(f^{x,q})])\,.$$

Since $\theta(f^{x,q}) \in P_x$, the assignment $x \mapsto \theta(f^{x,q})$ defines an element of $\mathcal{R}(X)$, which we denote k_q. It follows that

$$q = \theta([x \mapsto \theta(f^{x,q})]) = \theta(\overline{k_q}) = \Upsilon(k_q). \tag{16}$$

Finally, the function Υ is injective. Indeed, if $\Upsilon(q) = \Upsilon(r)$, then $\theta(f^{z,q}) = \theta(f^{z,r})$ for all $z \in \maltese$. By equation (16),

$$q = \theta([x \mapsto \theta(f^{x,q})]) = \theta([x \mapsto \theta(f^{x,s})]) = s.$$

□

We say that a $\maltese$-set (X, ϕ) is *non-degenerate* if ϕ is surjective.

Corollary 2.7. *Let $\maltese$ be a set, together with its comonoid structure in $\mathcal{S}ets$. The following statements about the functors $\mathcal{L}$ and $\mathcal{R}$ between $\mathcal{S}ets_T$ and $\mathcal{S}ets^F$ holds true.*

1. *If $X \in \mathcal{S}ets_T$ is degenerate, then $\mathcal{R}(X)$ is an empty set.*
2. *The functors $\mathcal{L}$ and $\mathcal{R}$ are quasiinverse equivalences between the category of non-degenerate $\maltese$-sets and the category of non-empty $\maltese$-contramodules.*

Proof. **(1)** The structure map $\phi : X \to \maltese$ is not surjective, so it has no sections.

(2) By Theorem 1.24, the pair of functors $(\mathcal{L} \dashv \mathcal{R})$ is adjoint. Since all $\maltese$-contramodules are of the form $\mathcal{R}(X)$ (Theorem 2.6), it remains to show that the adjunction counit $\varepsilon_X : \mathcal{L}(\mathcal{R}(X)) \to X$ is an isomorphism of $\maltese$-sets for any non-degenerate $X \in \mathcal{C}_T$.

Let (X, ϕ) be a non-degenerate $\maltese$-set. Then $\mathcal{R}(X) = [\maltese, X]_{\maltese}$ is the set of sections of $\phi : X \to \maltese$. Since ϕ is surjective, a section exists, in particular, $[\maltese, X]_{\maltese}$ is non-empty. The contrastructure map of $[\maltese, X]_{\maltese}$ is defined as follows.

The $\maltese$-set $(\mathcal{L}(\mathcal{R}(X)), \phi')$ can be presented as

$$\mathcal{L}(\mathcal{R}(X)) = ([\maltese, X]_{\maltese} \times \maltese)/_{\sim}\,, \quad \phi'[(f(x), y)] = y\,,$$

where $\sim$ is an equivalence relation which we now describe in detail. The maps

$$\alpha, \beta : [\maltese, [\maltese, X]_{\maltese}] \times \maltese \to [\maltese, X]_{\maltese} \times \maltese\,,$$

in terms of apt functions $h(x_1, x_2)$, are

$$\alpha(h(x_1, x_2), z) = (h_z(x), z), \qquad \beta(h(x_1, x_2), z) = (h \circ \Delta_{\maltese}, z) = (h(x, x), z).$$

The equivalence relation $\sim$ on $[\maltese, X]_{\maltese} \times \maltese$, coequalising α and β, is the equivalence relation generated by the binary relations $\approx$. The following equivalent statements comprise its definition:

1. $(f(x), y) \approx (g(x), z)$,
2. there exists $(h(x_1, x_2), w) \in [\maltese^2, X]_{apt} \times \maltese$ such that $\alpha(h(x_1, x_2), w) = (f(x), y)$ and $\beta(h(x_1, x_2), w) = (g(x), z)$,
3. $y = z$ and there exists $h(x_1, x_2) \in [\maltese \times \maltese, X]$ such that $h_y(x) = f(x)$, $h(x, x) = g(x)$ and $h_w(x)$ is a morphism of $\maltese$-sets for all $w \in \maltese$.

Notice that the last statement implies that $f(y) = h_y(y) = h(y, y) = g(y)$. This is the key to unlocking the relation $\sim$. Indeed, the following statements are equivalent:

1. $(f(x), y) \sim (g(x), z)$,
2. $y = z$ and $f(z) = g(z)$.

In fact, the passage above contains a proof that (1) implies (2).

To prove the opposite implication, pick $z \in \maltese$ and $f(x), g(x) \in [\maltese, X]_{\maltese}$ such that $f(z) = g(z)$. Consider a function

$$h(x_1, x_2) \in [\maltese \times \maltese, X], \quad h(x_1, x_2) = \begin{cases} g(x_1) & \text{if } x_1 = x_2, \\ f(x_2) & \text{if } x_1 \neq x_2. \end{cases} \tag{17}$$

Clearly, $h(x_1, x_2)$ is apt, while $\alpha(h, z) = (f, z)$ and $\beta(h, z) = (g, z)$. It follows that the adjunction counit ϵ_X is bijective:

$$\mathcal{L}(\mathcal{R}(X)) \xrightarrow{\epsilon_X} X, \quad \epsilon_X([(f(x), z)]) = f(z), \quad \epsilon_X^{-1}(a) = [\mathrm{Const}_a, \phi(a)].$$

□

Notice that Corollary 2.7 is reminiscent of the known equivalence (5) (cf. Section 1.10 and [5, 2.8]). It is easy to see that the non-degenerate $\maltese$-sets are precisely $\mathcal{F}_T$-injective $\maltese$-sets. Together with the fact that $\varnothing$ is obviously $\mathcal{F}^F$-projective, this yields the following corollary:

Corollary 2.8. *All $\maltese$-contramodules are $\mathcal{F}^F$-projective in the sense of* [5, 2.7].

Given a $\maltese$-contramodule (P, θ), we can describe the $\maltese$-set $(\mathcal{L}(P), \phi)$ directly from the definition:

$$\mathcal{L}(P) = \coprod_{z \in \maltese} P_z, \; P_z = P/_{\overset{z}{\sim}}, \; \phi(P_z) = z$$

where the equivalence $\overset{z}{\sim}$ is generated by a binary relation $\overset{z}{\approx}$, defined as $f(z) \overset{z}{\approx} \theta(f)$ for all functions $f : \maltese \to P$. Coupled with the $\maltese$-contramodule $\mathcal{R}(X)$ in (15), we arrive at an explicit description of this equivalence:

Corollary 2.9. *Suppose $c \in \maltese$, $p, q \in P$. Then $p \overset{z}{\sim} q$ if and only if $\theta(f^{z,p}) = \theta(f^{z,q})$ (see* (14) *for the definition of $f^{z,q}$).*

2.4. **Induction for contrasets.** Observe that in the category $\mathcal{S}ets$ the comonad T is continuous for any $\maltese$. Thus, for any function $f : \maltese \to \widehat{\maltese}$, we have the induction functor for comodules as in Proposition 1.28.

This agrees well with the isomorphism of categories in Proposition 2.1. Indeed, the induction functor for the overcategories $(\mathcal{S}ets \downarrow \maltese)$ does not require any additional assumptions (cf. Proposition 1.27).

On the other hand, F is not cocontinuous if $|\maltese| \geqslant 2$. Let $\maltese$ be a 2-element set. In this case, $F(X) = X^2$ for any set X. Look at the coequaliser of two maps from a point

$$\star \rightrightarrows X \overset{coeq.}{\dashrightarrow} X/\sim \ .$$

Here $X/\sim$ is obtained from X by identifying the images of these two points. Apply F:

$$F(\star) = \star \rightrightarrows F(X) \overset{coeq.}{\dashrightarrow} (X^2)/\sim \neq (X/\sim)^2 = F(X/\sim)\,.$$

Thus, Proposition 1.29 gives us no coinduction for contramodules in $\mathcal{S}ets$.

Let us discuss restriction. In light of equation (15), a contramodule $(P, \theta_P) \in \mathcal{S}ets^{\maltese}$ is represented as a product $(P, \theta_P) = \prod_{x\in\maltese} P_x$. Its restriction has similar representation:

$$(\widehat{P}, \widehat{\theta_P}) = \mathcal{R}es(P, \theta_P) = \prod_{z\in\widehat{\maltese}} \widehat{P}_z\,, \text{ where } \widehat{P}_z = \prod_{y\in f^{-1}(z)} P_y\,. \tag{18}$$

Notice that if z is not in the image of f, then $\widehat{P}_z$ is a 1-element set. Now it is time to address induction.

Proposition 2.10. *Let* $\maltese, \widehat{\maltese} \in \mathcal{S}ets$, $f \in \mathcal{S}ets(\maltese, \widehat{\maltese})$. *Then there exists a functor*

$$\mathcal{I}nd^F : \mathcal{S}ets^{\widehat{\maltese}} \to \mathcal{S}ets^{\maltese},$$

left adjoint to $\mathcal{R}es$.

Proof. A function f is a composition of a surjection f_1 and an injection f_2:

$$f : \maltese \xrightarrow{f_1} \widetilde{\maltese} = \mathrm{Im}(f) \xrightarrow{f_2} \widehat{\maltese}\,.$$

It suffices to define a left adjoint functor to $\mathcal{R}es$ for injections and surjections separately. Then $\mathcal{I}nd$ is a composition of these two functors.

If f is surjective, we can define the induction functor as a composition

$$\mathcal{I}nd^F : \mathcal{S}ets^{\widehat{\maltese}} \xrightarrow{\mathcal{L}} \mathcal{S}ets_{\widehat{\maltese}} \xrightarrow{\mathcal{I}nd_T} \mathcal{S}ets_{\maltese} \xrightarrow{\mathcal{R}} \mathcal{S}ets_{\maltese}\,. \tag{19}$$

In this case a non-degenerate comodule remains non-degenerate after induction. Thus, the non-empty contramodules turn into non-degenerate comodules and vice versa. The empty contramodule $\varnothing$ remains empty, going through these functors. It follows from Proposition 1.27 and Corollary 2.7 that this is a left adjoint.

Now let us assume that f is injective. We can define induction explicitly as

$$(\widetilde{Q}, \widetilde{\theta_Q}) = \mathcal{I}nd^F(Q, \theta_Q) = \prod_{y\in\maltese} Q_{f(y)}\,, \text{ whence } (Q, \theta_Q) = \prod_{y\in\widehat{\maltese}} Q_y\,. \tag{20}$$

To prove that this is a left adjoint, we just need to translate the representation in equation (15) to an explicit calculation of homs:

$$[\widetilde{Q}, P]^{\maltese} = \prod_{z\in\maltese} [Q_{f(z)}, P_z] \overset{(*)}{=} \prod_{y\in\widehat{\maltese}} [Q_y, \widehat{P}_y] = [Q, \widehat{P}]^{\widehat{\maltese}}$$

where the equality $(\circledast)$ holds true because $\widehat{P}_y = P_y$ if $y \in \mathrm{Im}(f)$ and $\widehat{P}_y$ is a 1-element set otherwise. $\square$

It follows from Proposition 2.10 that equation (20) essentially defines the induced contramodule for a general f as well. If $(Q,\theta) = \prod_{y\in\hat{\maltese}} Q_y \in \mathcal{S}ets^{\hat{\maltese}}$, then

$$(\widetilde{Q},\widetilde{\theta_Q}) = \mathcal{I}nd^F(Q,\theta_Q) = \prod_{z\in\maltese} \widetilde{Q}_z\,, \text{ where } \widetilde{Q}_z = Q_{f(z)}\,. \tag{21}$$

2.5. **Simplicial sets.** Let $\mathcal{S}$ be the category of simplicial sets. This is a cartesian closed category, meaning that the monoidal product is given by the levelwise product of sets and $\mathcal{S}$ becomes a closed monoidal category with respect to this structure. As in the start of Section 2.3, a comonoid in $\mathcal{C}$ is a simplicial set $\maltese = (\maltese_n)$ with the diagonal map $\maltese \to \maltese \times \maltese$.

Similarly to (24) and Proposition 2.1, $\mathcal{S}_T$ is isomorphic to the overcategory $(\mathcal{S}\downarrow\maltese)$ (c.f. [14]). Thus, a $\maltese$-comodule $M = (M_n)$ is a simplicial set with a $\maltese_n$-set structure $\phi_n : M_n \to \maltese_n$ at each level n. The compatibility condition is commutation of ϕ with the simplicial set structure maps:

$$\phi_n \circ M(f) = \maltese(f) \circ \phi_m$$

for all non-decreasing functions $f : [n] \to [m]$. Let us now analyse a $\maltese$-contramodule $(X = (X_n), \theta)$. Its structure map $\theta = (\theta_n) \in \mathcal{S}([\maltese, X], X)$ consists of functions

$$\theta_n : [\maltese, X]_n = \mathcal{S}(\maltese \times \Delta[n], X) \to X_n$$

at each level n, where $\Delta[n] \in \mathcal{S}$ is the standard n-simplex.

Let us now contemplate a simplicial set Y, which carries a $\maltese_n$-contramodule structure $(Y_n, \psi_n) \in \mathcal{S}ets^F$ at each level. The set $\Delta[n]_n$ contains the unique non-degenerate simplex in ι_n that yields the natural restriction

$$\tau_n : \mathcal{S}(\maltese \times \Delta[n], Y) \to \mathcal{S}ets(\maltese_n \times \Delta[n]_n, Y_n) \to \mathcal{S}ets(\maltese_n \times \{\iota_n\}, Y_n)\,.$$

If we identify $\mathcal{S}ets(\maltese_n \times \{\iota_n\}, Y_n)$ with $\mathcal{S}ets(\maltese_n, Y_n)$, we get maps

$$\theta_n : \mathcal{S}(\maltese \times \Delta[n], Y) \xrightarrow{\tau_n} \mathcal{S}ets(\maltese_n, Y_n) \xrightarrow{\psi_n} Y_n\,.$$

Thus, (Y_n, ψ_n) with the appropriate compatibility conditions determines a $\maltese$-contramodule. On the other hand, the map τ_n is neither injective, nor surjective in general. Thus, some $\maltese$-contramodules are not of this form.

3. Model Categories

3.1. **Model structures.** Let $\mathcal{B}$ be a model category, which we assume to be complete and cocomplete [13, 16]. The structure classes of morphisms are denoted $\mathbb{C}$ for cofibrations, $\mathbb{W}$ for weak equivalences and $\mathbb{F}$ for fibrations. Given a morphism f, we write its factorisations in the following way:

$$f : X \xrightarrow{f'\ \mathbb{C}} Y \xrightarrow{f''\ \mathbb{FW}} Z,\ \ f : X \xrightarrow{f'\ \mathbb{CW}} Y \xrightarrow{f''\ \mathbb{F}} Z.$$

Unlike [16, Def. 1.1.4], we do not automatically assume that the factorisations are endofunctors on the category of maps $\mathcal{M}\mathrm{ap}(\mathcal{B})$ (also called the category of squares or the category of arrows). Recall that $\mathcal{M}\mathrm{ap}(\mathcal{B})$ has the maps in $\mathcal{B}$ as objects and commutative squares in $\mathcal{B}$ as morphisms.

An object $X \in \mathcal{B}$ is cofibrant if the map from the initial object $\varnothing_X : \varnothing \to X$ is a cofibration. Similarly, an object $X \in \mathcal{B}$ is fibrant if the map to the terminal object $\mathbf{1}_X : X \to \mathbf{1}$ is a fibration. The full subcategory of cofibrant (or fibrant, or cofibrant and fibrant) objects is denoted $\mathcal{B}_{\mathbb{C}}$ (or $\mathcal{B}_{\mathbb{F}}$, or $\mathcal{B}_{\mathbb{CF}}$).

3.2. **Model structures on closed monoidal categories.** Suppose now that the closed monoidal category $\mathcal{C}$ is also a model category. The category $\mathcal{C}$ is called *a monoidal model category* [16, Def. 4.2.6] if the model and monoidal structures are compatible in the sense that the following three conditions hold.

(1) The monoidal structure $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a Quillen bifunctor [16, 4.2], i.e., given two cofibrations $f, g \in \mathbb{C}$, $f \in \mathcal{C}(U, V)$, $g \in \mathcal{C}(X, Y)$, their pushout
$$f \square g : (V \otimes X) \coprod_{U \otimes X} (U \otimes Y) \to V \otimes Y$$
is a cofibration.

(2) If one of the cofibrations f, g is a trivial cofibration, then $f \square g$ is a trivial cofibration.

(3) For all cofibrant X and cofibrant replacements of the monoidal unit
$$\varnothing_\star : \varnothing \xrightarrow{\mathbb{C}} \star_{\mathbb{C}} \xrightarrow{f\ \mathbb{FW}} \star$$
the maps
$$f \otimes \mathrm{Id}_X : \star_{\mathbb{C}} \otimes X \to \star \otimes X, \qquad \mathrm{Id}_X \otimes f : X \otimes \star_{\mathbb{C}} \to X \otimes \star$$
are weak equivalences.

Notice that condition (3) holds automatically if $\star$ is cofibrant.

The upshot of this definition is that the homotopy category $\mathrm{Ho}(\mathcal{C})$ becomes a biclosed monoidal category under the left derived tensor product $\otimes^L$ and the right derived internal homs $\mathrm{R}[-,-]$ and $\mathrm{R}\widetilde{[-,-]}$ with the monoidal unit $[\![\star]\!]$ [16, 4.3.2]. Given an object $X \in \mathcal{C}$, by $[\![X]\!]$ we denote the corresponding object in $\mathrm{Ho}(\mathcal{C})$.

3.3. **Induced model structures for modules and comodules.** We would like to equip the category $\mathcal{C}_T$ with a left induced model structure and the category $\mathcal{C}^F$ with a right induced model structure. The forgetful functors to $\mathcal{C}$ are denoted $\mathcal{F}_T$ and $\mathcal{F}^F$ respectively. The maximal right (left) complementary class of a class of morphisms $\mathbb{X}$ is denoted $\mathbb{X}^{\boxslash}$ (${}^{\boxslash}\mathbb{X}$ correspondingly). Let us define the classes of maps

$$\mathbb{C}_T := \mathcal{F}_T^{-1}(\mathbb{C}),\ \mathbb{W}_T := \mathcal{F}_T^{-1}(\mathbb{W}),\ \mathbb{F}_T := (\mathbb{C}_T \cap \mathbb{W}_T)^{\boxslash},$$
$$\mathbb{C}^F := {}^{\boxslash}(\mathbb{F}^F \cap \mathbb{W}^F),\ \mathbb{W}^F := \mathcal{F}^{F\,-1}(\mathbb{W}),\ \mathbb{F}^F := \mathcal{F}^{F\,-1}(\mathbb{F}). \tag{22}$$

Even if the categories $\mathcal{C}_T$ and $\mathcal{C}^F$ are complete and cocomplete (see Lemmas 1.6, 1.7 and 1.8), these classes do not necessarily define model structures. The following proposition gives some sufficient conditions. Further sufficient conditions are known (cf. [11, Th. 5.8], [28, Th. 4.1]).

Proposition 3.1. *Suppose that the model category $\mathcal{C}$ is accessible.*

(1) *If the category $\mathcal{C}_T$ is locally presentable, then*
   - *$\mathcal{C}_T$ is complete and*
   - *equation (22) defines an accessible model structure on $\mathcal{C}_T$, called (left)-induced.*

(2) *If the category $\mathcal{C}^F$ is cocomplete, then*
   - *$\mathcal{C}^F$ is locally presentable and*

- *equation* (22) *defines an accessible model structure on* $\mathcal{C}^F$*, called (right)-induced.*

Proof. A locally presentable category is complete [1, Cor. 1.28]. Then part (1) follows immediately from [12, Cor. 3.3.4].

The category $\mathcal{C}^F$ admits small limits and colimits by our assumptions and Lemma 1.6. Now, the functor $F : \mathcal{C} \to \mathcal{C}$ is a right adjoint, hence, accessible by [1, Prop. 2.23]. By [1, Th. 1.20], $\mathcal{C}^F$ is accessible. Since $\mathcal{C}^F$ is complete, it is locally presentable [1, Cor. 2.47].

The second statement in (2) follows from [12, Cor. 3.3.4]. □

We finish the section with the following fact:

Corollary 3.2. *Suppose that the category* $\mathcal{C}$ *is locally presentable. Then the following statements hold.*

1. *Equation* (22) *defines an accessible (left-induced) model structure on* $\mathcal{C}_T$.
2. *If the chief* $\maltese$ *is presentable, then equation* (22) *defines an accessible (right-induced) model structure* $\mathcal{C}^F$.
3. *Furthermore, if* $\mathcal{C}$ *is cofibrantly generated or right proper, with generating set of trivial cofibrations* $\mathbb{J}$*, and if the functor* $\mathcal{F}^F$ *takes relative* $F\mathbb{J}$*-complexes to weak equivalences, then* $\mathcal{C}^F$ *is also cofibrantly generated or right proper, respectively.*

Proof. The first two statements follow from Proposition 3.1 and Corollary 1.10.

By Proposition 3.1 $\mathcal{C}^F$ is locally presentable. Thus, combined with our assumption on $\mathcal{F}^F$, it follows that $\mathcal{C}^F$ is cofibrantly generated by [13, Th. 11.3.2]. Since limits in $\mathcal{C}^F$ are inherited from $\mathcal{C}$, the model structure on $\mathcal{C}^F$ is right proper. □

3.4. **Comodule-contramodule correspondence for model categories.** Let us consider the following diagram of categories and the three pairs of $\mathcal{C}$-enriched adjoint functors $(F \dashv \mathcal{F}^F)$, $(\mathcal{F}_T \dashv T)$ and $(\mathcal{L} \dashv \mathcal{R})$ (cf. Theorem 1.24).

$$\begin{tikzcd}
& \mathcal{C} \arrow[dl, shift right, "F"'] \arrow[dr, shift left, "T"] & \\
\mathcal{C}^F \arrow[ur, shift right, "\mathcal{F}^F"'] \arrow[rr, shift left, "\mathcal{L}"] & & \mathcal{C}_T \arrow[ul, shift left, "\mathcal{F}_T"'] \arrow[ll, shift left, "\mathcal{R}"]
\end{tikzcd} \tag{23}$$

All these adjunctions are $\mathcal{C}$-enriched. Assuming that equation (22) defines model structures, the adjunctions $(F \dashv \mathcal{F}^F)$ and $(\mathcal{F}_T \dashv T)$ are Quillen adjunctions. What about the third adjunction $(\mathcal{L} \dashv \mathcal{R})$?

Problem 3.3.

1. Find necessary and sufficient conditions for the adjunction $(\mathcal{L} \dashv \mathcal{R})$ to be a Quillen adjunction (and/or a Quillen equivalence) between the right-induced model category $\mathcal{C}^F$ and the left-induced model category $\mathcal{C}_T$.
2. Investigate existence of other model category structures on $\mathcal{C}^F$ and $\mathcal{C}_T$ (or their co(completions)) under which the adjunction $(\mathcal{L} \dashv \mathcal{R})$ is a Quillen adjunction or a Quillen equivalence.

3.5. **An answer for cartesian closed categories.** In this section we assume that $\mathcal{C}$ is a cartesian closed category. This means that the monoidal product $\otimes$ in $\mathcal{C}$ is the categorical product. It follows that $\mathcal{C}$ is symmetric and the unit object $\star$

is the terminal object. Similarly to the start of Section 2.3, all comonoids in such category are objects X with the diagonal map $\Delta : X \to X \times X$.

Let $\maltese$ be a comonoid in $\mathcal{C}$. Similarly to Proposition 2.1, $\mathcal{C}_T$ is isomorphic to the overcategory (or slice category) $(\mathcal{C} \downarrow \maltese)$ (c.f. [14]):

$$(M, \rho : M \to T(M)) \leftrightarrow (M, \phi : M \to \maltese) \quad \text{where} \quad \rho = (\phi, \mathrm{Id}_M)\,. \tag{24}$$

Proposition 3.4. *The category $\mathcal{C}_T$ is complete and cocomplete.*

Proof. The slice category of a complete category is complete [21, IV.7, Th. 1]. It is cocomplete by Lemma 1.6. □

The left-induced model structure (see (22)) on $\mathcal{C}_T$ is, in fact, *induced*:

Proposition 3.5. *(cf.* [14]*) If $\mathcal{C}$ is cofibrantly generated, then the following is a cofibrantly generated model structure on $\mathcal{C}_T$:*

$$\mathbb{C}_T = \mathcal{F}_T^{-1}(\mathbb{C}),\ \mathbb{W}_T = \mathcal{F}_T^{-1}(\mathbb{W}),\ \mathbb{F}_T = \mathcal{F}_T^{-1}(\mathbb{F}). \tag{25}$$

If $\mathcal{C}$ is left or right proper, then so is $\mathcal{C}_T$.

Proof. We identify $\mathcal{C}_T$ with $(\mathcal{C} \downarrow \maltese)$. Since $\mathcal{C}$ is a cofibrantly generated model category, so is $(\mathcal{C} \downarrow \maltese)$ under the model structure (25) [14, Th. 1.5]. This proves the first statement.

The second statement is [14, Th. 1.7]. □

We do not know any special description of $\mathcal{C}^F$ in the cartesian case but the behaviour of the comodule-contramodule correspondence is distinctive.

Proposition 3.6. *Suppose that $\mathcal{C}$ is cartesian closed, the left-induced model structure exists on $\mathcal{C}_T$ and the right-induced model structure exists on $\mathcal{C}^F$. Then the pair $(\mathcal{L} \dashv \mathcal{R})$ is a Quillen adjunction.*

Proof. We need to show that the functor $\mathcal{R} : \mathcal{C}_T \to \mathcal{C}^F$ preserves fibrations and trivial fibrations. Let $f : (X, \phi_X) \to (Y, \phi_Y)$ be a (trivial) fibration in $\mathcal{C}_T$. Since the model structure on $\mathcal{C}^F$ is right-induced, we need to verify that $\mathcal{R}f$ is a (trivial) fibration in $\mathcal{C}$. Let us consider a commutative diagram in $\mathcal{C}$

$$\begin{tikzcd}
U \arrow[r] \arrow[d, "\mathbb{C}\cap\mathbb{W}\ni(\text{ or } \mathbb{C}\ni)"'] & \mathcal{R}X \arrow[d, "\mathcal{R}f"] \\
V \arrow[r, "g"'] \arrow[ur, dashed, "h"] & \mathcal{R}Y
\end{tikzcd}$$

where the left down arrow is a trivial cofibration (correspondingly, a cofibration) in $\mathcal{C}$. The diagonal filling h has not been found yet. Since $\mathcal{R}X$ is a subobject of $FX = [\maltese, X]$, we have the adjunct commutative diagram

$$\begin{tikzcd}
TU = U \times \maltese \arrow[r] \arrow[d, "\mathbb{C}\cap\mathbb{W}\ni(\text{ or } \mathbb{C}\ni)"'] & X \arrow[d, "f"] \\
TV = V \times \maltese \arrow[r, "\hat{g}"'] \arrow[ur, dashed, "\hat{h}"] & Y
\end{tikzcd}$$

where the left down arrow is also a trivial cofibration (a cofibration) in $\mathcal{C}$. Since the model structure on $\mathcal{C}_T$ is induced, f is a (trivial) fibration in $\mathcal{C}$. Thus, there exists a diagonal filling $\hat{h}$, whose adjunct map $h : V \to [\maltese, X]$ would be a diagonal filling of the first diagram if it were to factor through $\mathcal{R}X \hookrightarrow FX$. This would imply that $\mathcal{R}f$ is a (trivial) fibration, finishing the proof.

To prove the outstanding claim we need to show that h equalises the pair of maps $\alpha^T_{\maltese,X}, \beta^T_{\maltese,X} : [\maltese, X] \rightrightarrows [\maltese, TX] = [\maltese, X \times \maltese] \cong [\maltese, X] \times [\maltese, \maltese]$ from Section 1.8. The first components of these maps are equal so that we need to prove that

$$(\alpha^T_{\maltese,X})_1 \circ h = (\beta^T_{\maltese,X})_2 \circ h : [\maltese, X] \rightrightarrows [\maltese, \maltese]\,.$$

This follows from the fact that $g : V \to \mathcal{R}Y$ equalises the similar maps for Y and the commutativity of the following diagram:

$$\begin{tikzcd}
U \arrow[r] \arrow[d, "\mathbb{C}\cap\mathbb{W}\ni(\text{ or }\mathbb{C}\ni)"'] & \mathcal{R}X \arrow[r] \arrow[d, "h"'] & {[\maltese, X]} \arrow[r] \arrow[d, "Ff"] & {[\maltese, \maltese]} \arrow[d, "\mathrm{Id}_{[\maltese,\maltese]}"] \\
V \arrow[r, "g"'] \arrow[urr, dashed] & \mathcal{R}Y \arrow[r] & {[\maltese, Y]} \arrow[r, shift left] \arrow[r, shift right] & {[\maltese, \maltese]}
\end{tikzcd} \tag{26}$$

$\square$

For the pair $(\mathcal{L} \dashv \mathcal{R})$ to be a Quillen equivalence, the maps

$$u_X : X \to \mathcal{R}(\mathcal{L}X) \to \mathcal{R}(\mathcal{L}X_{\mathbb{F}}), \quad \epsilon_M : \mathcal{L}(\mathcal{R}M_{\mathbb{C}}) \to \mathcal{L}(\mathcal{R}M) \to M \tag{27}$$

for all $X \in (\mathcal{C}^F)_{\mathbb{C}}$, $M \in (\mathcal{C}_T)_{\mathbb{F}}$, derived from the unit and the counit of adjunction, must be weak equivalences. For this to be true it suffices to localise at the classes of maps $\mathfrak{A}$ and $\mathfrak{B}$ as constructed below. First start with factorising the maps u_X and ϵ_M:

$$u_X : X \xrightarrow{g_X\ \mathbb{C}} X' \xrightarrow{\mathbb{FW}} \mathcal{R}(\mathcal{L}X_{\mathbb{F}})$$
$$\epsilon_M : \mathcal{L}(\mathcal{R}M_{\mathbb{C}}) \xrightarrow{k_M\ \mathbb{CW}} M' \xrightarrow{\mathbb{F}} M.$$

Taking fibrant and cofibrant replacements $X'_{\mathbb{F}}$ and $M'_{\mathbb{C}}$ of the objects X' and M' respectively, we obtain maps:

$$r_X : X \xrightarrow{g_X} X' \to X'_{\mathbb{F}} \text{ and } q_M : M'_{\mathbb{C}} \to M' \xrightarrow{k_M} M.$$

Factorising these gives us our desired classes:

$$\mathfrak{A} := \{f_X \mid X \xrightarrow{\mathbb{CW}} X'' \xrightarrow{f_X\ \mathbb{F}} X'_{\mathbb{F}}\}, \tag{28}$$
$$\mathfrak{B} := \{h_M \mid M'_{\mathbb{C}} \xrightarrow{\mathbb{C}} M'' \xrightarrow{\mathbb{FW}} M\}.$$

Theorem 3.7. *Let us make the following assumptions:*

(1) *$\mathcal{C}$ is a locally presentable category,*
(2) *$\mathcal{C}$ is a cartesian closed monoidal model category,*
(3) *$\mathcal{C}$ is a left and right proper model category,*
(4) *the chief $\maltese$ is presentable.*

Then there exist a right Bousfield localisation $R_{\mathfrak{A}}(\mathcal{C}^F)$ and a left Bousfield localisation $L_{\mathfrak{B}}(\mathcal{C}_T)$, so that the comodule-contramodule correspondence $(\mathcal{L} \dashv \mathcal{R})$ induces a Quillen equivalence between them.

Proof. We engineer the localisation classes so that $(\mathcal{L} \dashv \mathcal{R})$ would induce a Quillen equivalence. The only thing we need to check is that the localisations actually exist.

First, instead of the localisation classes we can use localisation sets because the categories $\mathcal{C}_T$ and $\mathcal{C}^F$ are locally presentable by Corollary 1.10. We define

$$\mathfrak{A}^\flat := \{f_{Y_{\mathbb{C}}} \in \mathfrak{A} \mid Y \text{ is in the generator}\}, \ \mathfrak{B}^\flat := \{h_{N_{\mathbb{F}}} \in \mathfrak{B} \mid N \text{ is in the generator}\}.$$

These are sets of maps. If these maps are turned into weak equivalences, the adjunction units and counits for Y and N become isomorphisms in the homotopy categories. Recall that the Quillen adjunction $(\mathcal{L} \dashv \mathcal{R})$ descends to a pair of adjoint functors between the homotopy categories $\mathrm{Ho}(\mathcal{C}^F)$ and $\mathrm{Ho}(\mathcal{C}_T)$.

Observe that Y belongs to the set of generating objects of $\mathcal{C}^T$. The corresponding objects $[\![Y_{\mathbb{C}}]\!]$ form a set of generating objects of $\mathrm{Ho}(\mathcal{C}^F)$. Thus, the adjunction unit is an isomorphism for all objects in $\mathrm{Ho}(\mathcal{C}^F)$. A similar argument shows that the adjunction counit is an isomorphism for all objects in $\mathrm{Ho}(\mathcal{C}_T)$.

It remains to show the existence of the localisations. Since $\mathcal{C}_T$ is a slice category of a locally presentable category, then it is locally presentable [7, Rmk. 3]. Thus, Proposition 3.5 yields that $\mathcal{C}_T$ is a left proper combinatorial model category and so $L_{\mathfrak{B}^\flat}(\mathcal{C}_T)$ exists. Similarly, Corollary 3.2 in combination with the fact that $\mathcal{C}^F$ is locally presentable, implies that all the conditions for existence of $R_{\mathfrak{A}^\flat}(\mathcal{C}^F)$, stated in [13, Rmk. 5.1.2], are met.

Finally, it is clear that $L_{\mathfrak{B}}(\mathcal{C}_T) = L_{\mathfrak{B}^\flat}(\mathcal{C}_T)$ and $R_{\mathfrak{A}}(\mathcal{C}^F) = R_{\mathfrak{A}^\flat}(\mathcal{C}^F)$. □

3.6. **Simplicial sets.** A good example of a category satisfying all conditions of Theorem 3.7 is the category $\mathcal{S}$ of simplicial sets, briefly discussed in Section 2.5, with respect to the classical (Quillen) model structure (for the definition of this model structure cf. [10, V.1.7]). The category $\mathcal{S}$ is locally presentable as it is a presheaf category [1, 1.46], proper ([13, Thm. 13.1.13]) and cartesian closed.

Let $✠ = (✠_n) \in \mathcal{S}$, considered as a comonoid under the diagonal map. Let us summarise its comodule-contramodule correspondence:

Theorem 3.8. (1) *The adjoint pair $(\mathcal{L} \dashv \mathcal{R})$ is a Quillen adjunction between $\mathcal{S}_T$ and $\mathcal{S}^F$.*
(2) *The adjoint pair $(\mathcal{L} \dashv \mathcal{R})$ is a Quillen equivalence between the right Bousfield localisation $R_{\mathfrak{A}}(\mathcal{S}^F)$ and the left Bousfield localisation $L_{\mathfrak{B}}(\mathcal{S}_T)$.*
(3) *All ✠-contramodules are cofibrant.*
(4) *A ✠-comodule (M, ϕ) is fibrant if and only if $\phi : M \to ✠$ is a Kan fibration.*

Proof. Statement (1) is Proposition 3.6. Statement (4) is the definition.

It is clear that ✠ is λ-presentable where λ is a regular cardinal greater than the cardinality of the union $\cup_n ✠_n$. Thus, statement (2) is Theorem 3.7

Finally, observe that $\mathcal{R}(\Delta[1])$ is a cylinder object in $\mathcal{C}^F$. This yields the cylinder decomposition of the empty map

$$\varnothing_X : \varnothing \xrightarrow{\mathbb{C}^F} \mathrm{Cyl}(\varnothing \to X) \xrightarrow{\mathbb{W}^F} X$$

for all $X \in \mathcal{C}^F$. Since $\varnothing \times X = \varnothing$, the second map $\mathrm{Cyl}(\varnothing \to X) \to X$ must be the identity. This proves statement (3). □

Notice that $(\mathcal{L} \dashv \mathcal{R})$ is not a Quillen equivalence between $\mathcal{S}_T$ and $\mathcal{S}^F$ even for "nice" simplicial sets ✠. There exist ✠-comodules (M, ϕ) such that the map of geometric realisations $|\phi| : |M| \to |✠|$ has no continuous sections. It follows that $\mathcal{R}M$ is empty. See Section 3.8 for further discussion.

3.7. **Positselski's answer.** Let $\mathcal{B}$ be the category of chain complexes over a commutative ring $\mathbb{K}$ with the standard closed monoidal structure and the Quillen model structure [4, Th. 1.4], [16, Th. 2.3.11].

A comonoid in $\mathcal{B}$ is a DG-coalgebra ✠. One can easily show that $\mathcal{B}$ is locally presentable and any DG-coalgebra is presentable. By Corollary 1.10 and Lemma 1.6, both $\mathcal{B}^F$ and $\mathcal{B}_T$ are complete, cocomplete and locally presentable categories.

The Quillen model structure on $\mathcal{B}$ is compactly generated [4, Th. 1.4], hence, accessible. Proposition 3.1 yields the left-induced model structure $(\mathbb{C}_T, \mathbb{W}_T, \mathbb{F}_T)$ on $\mathcal{B}_T$ and the right-induced model structure $(\mathbb{C}^F, \mathbb{W}^F, \mathbb{F}^F)$ on $\mathcal{B}^F$. Positselski calls them *projective* and *injective* correspondingly. Since the category of chain complexes is not cartesian closed, neither Proposition 3.6, nor Theorem 3.7 are applicable. This makes the following variation of Question 3.3 interesting.

Problem 3.9. Find necessary and sufficient conditions on the commutative ring $\mathbb{K}$ and the DG-coalgebra ✠ for the adjunction $(\mathcal{L} \dashv \mathcal{R})$ to be a Quillen adjunction (and/or a Quillen equivalence) between the injective model category $\mathcal{B}^F$ and the projective model category $\mathcal{B}_T$.

Instead of answering this question, Positselski gives an alternative answer to Question 3.3(2). He makes an additional assumption that

(29) ✠ is $\mathbb{K}$-projective and $\mathbb{K}$ is of finite global dimension.

This assumption ensures that the categories $\mathcal{B}_T$ and $\mathcal{B}^F$ are abelian. Positselski proves that under this assumption $\mathcal{B}_T$ admits a semiprojective model structure $(\mathbb{C}^p_T, \mathbb{W}^p_T, \mathbb{F}^p_T)$ [25, 9.1] (the letter p in the notation stands for Positselski), while $\mathcal{B}^F$ admits a semiinjective model structure $(\mathbb{C}^F_p, \mathbb{W}^F_p, \mathbb{F}^F_p)$ with the following properties [25, Rmk. 9.2.2]:

(1) $\mathbb{C}^p_T = \mathbb{C}_T$, $\mathbb{W}^p_T \subseteq \mathbb{W}_T$, $\mathbb{F}^p_T \supseteq \mathbb{F}_T$,
(2) $\mathbb{C}^F_p \supseteq \mathbb{C}^F$, $\mathbb{W}^F_p \subseteq \mathbb{W}^F$, $\mathbb{F}^F_p = \mathbb{F}^F$,
(3) The comodule-contramodule correspondence $(\mathcal{L} \dashv \mathcal{R})$ is a Quillen eqiuvalence between $(\mathcal{B}_T, \mathbb{C}^p_T, \mathbb{W}^p_T, \mathbb{F}^p_T)$ and $(\mathcal{B}^F, \mathbb{C}^F_p, \mathbb{W}^F_p, \mathbb{F}^F_p)$.

A proof of this fact is only indicated in [25]. In our view, the model structures on $\mathcal{B}_T$ and $\mathcal{B}^F$ deserve a thorough investigation in the spirit of [4]. For instance, there are indications that imposing the condition (29) above is too strong.

Problem 3.10. For an arbitrary commutative ring $\mathbb{K}$ and a DG-coalgebra ✠, do there exist a semiinjective model category $\mathcal{B}^F$ and a semiprojective model category $\mathcal{B}_T$ that satisfy the three properties just above?

4. Topological Spaces

4.1. **A convenient category of topological spaces $\mathcal{W}$.** The category of topological spaces $\mathcal{T}$ is not closed monoidal. To remedy this issue, Steenrod suggested the notion of a convenient category [30]. The most common convenient category is the category $\mathcal{W}$ of compactly generated weakly Hausdorff topological spaces, introduced by McCord [23]. We follow a modern exposition by Schwede [29, Appendix A]. Consider subcategories

$$\mathcal{W} \xhookrightarrow{\mathbf{i}} \mathcal{K} \xhookrightarrow{\mathbf{i}} \mathcal{T}$$

where $\mathcal{T}$ is the category of topological $\mathcal{K}$ is the category of compactly generated topological spaces. The embedding functors have adjoint functors the Kellification functor $\mathbf{k}$ and the weak Hausdorffication functor $\mathbf{w}$:

$$\mathcal{W} \xleftarrow{\mathbf{w}} \mathcal{K} \xleftarrow{\mathbf{k}} \mathcal{T}, \quad (\mathbf{i} \dashv \mathbf{k}), (\mathbf{w} \dashv \mathbf{i}).$$

We use a subscript to denote the category in which a construction is taking place:

$$X \times Y := X \times_{\mathcal{W}} Y = X \times_{\mathcal{K}} Y = \mathbf{k}(X \times_{\mathcal{T}} Y), \tag{30}$$

$$\prod X_n = \prod\nolimits_{\mathcal{K}} X_n = \mathbf{k}(\prod\nolimits_{\mathcal{T}} X_n).$$

No subscript means that the construction is taking place in the default category $\mathcal{W}$. Formula (30) tells us how the products in different categories relate. A similar relation holds for arbitrary limits:

$$\varprojlim \mathcal{F} = \varprojlim{}_{\mathcal{K}} \mathcal{F} = \mathbf{k}(\varprojlim{}_{\mathcal{T}} \mathcal{F}) .$$

On the other hand, the coproducts are the same in all three categories:

$$\coprod X_n = \coprod{}_{\mathcal{K}} X_n = \coprod{}_{\mathcal{T}} X_n .$$

Since quotients of weakly Hausdorff spaces are no longer weakly Hausdorff, the relation for colimits is this:

$$\varinjlim \mathcal{F} = \mathbf{w}(\varinjlim{}_{\mathcal{K}} \mathcal{F}) = \mathbf{w}(\varinjlim{}_{\mathcal{T}} \mathcal{F}) .$$

Both categories $\mathcal{W}$ and $\mathcal{K}$ are closed symmetric monoidal categories [29, A.22, A.23] with products $X \times Y$ and $X \times_{\mathcal{K}} Y$ and internal homs

$$[X, Y]_{\mathcal{W}} = \mathbf{k}(C(X, Y)) = \mathbf{k}(C'(X, Y)),\ [X, Y]_{\mathcal{K}} = \mathbf{k}(C'(X, Y)),$$

where $C(X, Y) = C'(X, Y) = \mathcal{T}(X, Y)$ is the set of continuous functions $X \to Y$. The difference is the topology. The space $C(X, Y)$ carries the compact open topology, while $C'(X, Y)$ is equipped with the *modified* compact open topology. The basis of the latter is given by sets of the form

$$N(h, U) := \{f : X \to Y \mid f \text{ is continuous}, f(h(K)) \subseteq U\},$$

where U is open in Y, K is compact and $h : K \to X$ is a continuous map. Notice that if X is weakly Hausdorff, then $h(K)$ is closed and thus compact. So the two topologies on $\mathcal{T}(X, Y)$ coincide in this case.

4.2. **Homotopy theory in $\mathcal{W}$.** The *Quillen model structure* on $\mathcal{W}$ is defined as follows.

- $\mathbb{W}$ – **weak equivalences:** These are the maps $f : X \to Y$ satisfying:
  - (i) f induces an isomorphism of sets $\pi_0(X) \xrightarrow{\cong} \pi_0(Y)$,
  - (ii) for any $x \in X$ and $n \geqslant 1$ the induced homomorphism $f_* : \pi_n(X, x) \to \pi_n(Y, f(x))$ is an isomorphism of groups.
- $\mathbb{F}$ – **fibrations:** The fibrations are the Serre fibrations, that is, those maps $p : E \to B$ which have the homotopy lifting property with respect to any CW-complex.
- $\mathbb{C}$ – **cofibrations:** The cofibrations are the maps $f : X \to Y$ which are retracts of a map $f' : X \to Y'$, where Y' is a space obtained from X by attaching cells.

Note that $\mathcal{W}$ with the Quillen model structure is a cofibrantly generated model category with a set of generating cofibrations

$$\mathbb{I} = \{\mathbb{S}^{n-1} \to \mathbb{D}^n \mid n \geqslant 0\} \tag{31}$$

and a set of generating trivial cofibrations

$$\mathbb{J} = \{\mathbb{D}^n \times \{0\} \to \mathbb{D}^n \times [0, 1] \mid n \geqslant 0\}. \tag{32}$$

4.3. **Cospaces.** Let $\maltese \in \mathcal{W}$. We identify $\mathcal{W}_{\maltese}$ with the category of spaces over $\maltese$, which we also call *cospaces*. An object of $\mathcal{W}_{\maltese}$ is a pair (X, ϕ_X), where X is an object of $\mathcal{W}$ and $\phi_X : X \to \maltese$ is a map in $\mathcal{W}$. A morphism $f : (X, \phi_X) \to (Y, \phi_Y)$ of cospaces is a map $f : X \to Y$ over $\maltese$, in the sense that $\phi_X = \phi_Y f$. Now let

$$[X, Y]_{\maltese} \subseteq [X, Y]_{\mathcal{W}}$$

be the subset of maps over $\maltese$. Note that by definition the category of cospaces $\mathcal{W}_{\maltese}$ is exactly the overcategory $(\mathcal{W} \downarrow \maltese)$ (c.f. [14]).

Proposition 4.1. *$[X, Y]_{\maltese}$ is a closed subset of $[X, Y]_{\mathcal{W}}$.*

Proof. Pick $f \in [X, Y]_{\mathcal{W}} \backslash [X, Y]_{\maltese}$. There exists $x \in X$ such that $\phi_Y(f(x)) \neq \phi_X(x)$. Since $\phi_Y^{-1}(\phi_X(x))$ is closed, we can choose an open set $U \subseteq Y$ such that $f(x) \in U$ and $U \cap \phi_Y^{-1}(\phi_X(x)) = \varnothing$. Then $f \in N(\{x\}, U) \subseteq [X, Y]_{\mathcal{W}} \backslash [X, Y]_{\maltese}$ so that $[X, Y]_{\mathcal{W}} \backslash [X, Y]_{\maltese}$ is open and $[X, Y]_{\maltese}$ is closed. $\square$

It follows that $[X, Y]_{\maltese} \in \mathcal{W}$. This makes the category $\mathcal{W}_{\maltese}$ enriched in $\mathcal{W}$. The isomorphism of categories (24) between $\mathcal{W}_{\maltese}$ and $\mathcal{W}_T$ for the comonad $TX = X \times \maltese$ is enriched in $\mathcal{W}$. By Proposition 3.4 $\mathcal{W}_{\maltese}$ is complete and cocomplete. By Proposition 3.5, there exists a Quillen induced model structure on $\mathcal{W}_{\maltese}$.

4.4. **Contraspaces.** The cospaces reduce to something quite conceptually simple. However, at the moment we do not know any conceptually simpler definition of a contraspace other than the general one – a $\maltese$-contraspace X is a space X equipped with a map $\theta_X : [\maltese, X]_{\mathcal{W}} \to X$ satisfying the usual properties. In other words, the category of contraspaces $\mathcal{W}^{\maltese}$ is the category of modules $\mathcal{W}^F$ for the monad $FX = [\maltese, X]_{\mathcal{W}}$, defined by the diagonal comonoid $(\maltese, \Delta_{\maltese})$. By Proposition 1.22, $\mathcal{W}^{\maltese}$ is a category enriched in $\mathcal{W}$. As before, its enriched hom is denoted by $[X, Y]^F$.

To understand the space $[X, Y]^F$, we consider the subset

$$[X, Y]^{\maltese} \subseteq [X, Y]_{\mathcal{W}}$$

that consists of contramodule maps over $\maltese$. We equip $[X, Y]^{\maltese}$ with the subspace topology.

Proposition 4.2. (1) *$[X, Y]^{\maltese}$ is a weakly Hausdorff space.*
(2) *If Y is Hausdorff, then $[X, Y]^{\maltese}$ is a closed subset of $[X, Y]_{\mathcal{W}}$. Consequently, $[X, Y]^{\maltese} \in \mathcal{W}$.*

Proof. Any subspace of $[X, Y]_{\mathcal{W}}$ is weakly Hausdorff [29, Prop. A4(i)]. This proves (1).

To show (2), start with picking $f \in [X, Y]_{\mathcal{W}} \backslash [X, Y]^{\maltese}$. There exists $g \in [\maltese, X]_{\mathcal{W}}$ such that $\theta_Y(fg) \neq f(\theta_X(g))$. Since Y is Hausdorff, we can find non-intersecting open sets $U, V \subseteq Y$ such that $\theta_Y(fg) \in U$ and $f(\theta_X(g)) \in V$. Then f belongs to the open set $L_g^{-1}(\theta_Y^{-1}(U)) \cap N(\{\theta_X(g)\}, V)$ where $L_g^{-1}(\theta_Y^{-1}(U))$ is the inverse image of the open set $\theta_Y^{-1}(U) \subseteq [\maltese, Y]_{\mathcal{W}}$ under the continuous map

$$L_g : [X, Y]_{\mathcal{W}} \to [\maltese, Y]_{\mathcal{W}}, \quad h \mapsto hg\,.$$

Notice that no $h \in L_g^{-1}(\theta_Y^{-1}(U)) \cap N(\{\theta_X(g)\}, V)$ can be a $\maltese$-contramodule map since $\theta_Y(hg) \in U$ and $h(\theta_X(g)) \in V$. Hence, $[X, Y]_{\mathcal{W}} \backslash [X, Y]^{\maltese}$ is open and $[X, Y]^{\maltese}$ is closed.

Finally, a closed subspace of a space in $\mathcal{W}$ is in $\mathcal{W}$ [29, Prop. A5(i)]. $\square$

Armed with this proposition, we can understand $[X,Y]^F$ now. A proof is left to the reader.

Corollary 4.3. *There exists a natural homeomorphism between* $[X,Y]^F$ *and* $\mathbf{k}([X,Y]^{\maltese})$.

By Lemma 1.6 $\mathcal{W}^{\maltese}$ is complete. Furthermore, $\mathcal{W}^{\maltese}$ inherits limits from $\mathcal{W}$.

Proposition 4.4. *If* $\maltese$ *is connected, then* $\mathcal{W}^{\maltese}$ *inherits coproducts from* $\mathcal{W}$.

Proof. Let $X = \coprod_n (X_n, \theta_n)$. Since $\maltese$ is connected, a continuous function $f : \maltese \to X$ takes values in one particular X_{n_0}. This enables us to define $\theta_X(f) \coloneqq \theta_{n_0}(f)$:

$$\theta_X : [\maltese, X]_{\mathcal{W}} \xrightarrow{\cong} \coprod [\maltese, X_n]_{\mathcal{W}} \xrightarrow{\coprod \theta_n} \coprod X_n = X\,.$$

□

A category with coproducts is cocomplete if and only it admits coequalisers. However, coequalisers are not inherited from $\mathcal{W}$, even for a connected $\maltese$. Hence, it is impossible to use Theorem 1.9 to establish cocompleteness of $\mathcal{W}^{\maltese}$.

Lemma 4.5. *A space* X *is presentable if and only if* X *is discrete.*

Proof. If X is discrete, then $[X,-]_{\mathcal{W}}$ commutes with $|X|$-directed colimits.

Suppose that X is not discrete. Let X_d denote the set X with the discrete topology. Given a limit ordinal α and $\beta \in \alpha$, let $X_\beta \coloneqq X^\alpha$ as a set and $X_\beta \coloneqq (\prod_{\gamma<\beta} X) \times (\prod_{\gamma\geqslant\beta} X_d)$ as a topological space. The colimit $\varinjlim(\ldots X_\beta \xrightarrow{\mathrm{Id}} X_{\beta+1} \ldots)$ is X^α as a topological space but the diagonal map $\Delta : X \to X^\alpha$ does not factor through any X_β. □

By a *subcontraspace* of (X, θ_X) we understand a subset Y of X such that $\theta_X(f) \in Y$ for any continuous function $f : \maltese \to Y$. We denote a subcontraspace by $Y \leqslant X$.

Consider the subspace topology on $Y \leqslant X$. Clearly, $Y \in \mathcal{K}$. Since $\mathcal{W}$ is closed under closed subsets [29, A5], if Y is closed, Y is a contraspace itself. In general, $\mathbf{k}(Y)$ is a contraspace because $\mathcal{K}(\maltese, Y) = \mathcal{W}(\maltese, \mathbf{k}(Y))$ due to the adjunction $(\mathbf{i} \dashv \mathbf{k})$. Thus, θ_Y is obtained by restricting θ_X to $[\maltese, \mathbf{k}(Y)]_{\mathcal{W}} \subseteq [\maltese, X]_{\mathcal{W}}$. The continuity of θ_Y is clear.

The following lemma is obvious:

Lemma 4.6. *An arbitrary intersection of subcontraspaces is a subcontraspace.*

In particular, the empty set is a subcontraspace with structure map $\mathrm{Id}_\varnothing : [\maltese, \varnothing]_{\mathcal{W}} = \varnothing \to \varnothing$. Lemma 4.6 allows us to define, given a subset $Z \subseteq X$ of a contraspace X, the subcontraspace generated by Z:

$$Z^{\maltese} \coloneqq \bigcap_{Z \subseteq Y \leqslant X} Y\,.$$

Let us describe $Z^{\maltese}$ constructively. For an ordinal β we define by transfinite recursion

$$Z_0 \coloneqq Z, \quad Z_\beta \coloneqq \begin{cases} \theta_X(Z_{\beta-1}) & \text{if } \beta \text{ is a successor ordinal,} \\ \bigcup_{\gamma\leqslant\beta} Z_\gamma & \text{if } \beta \text{ is a limit ordinal.} \end{cases}$$

Proposition 4.7. *If* β *is a* $|\maltese|$*-filtered ordinal, then* $Z^{\maltese} = Z_\beta$.

Proof. The inclusion $Z^{\maltese} \supseteq Z_\beta$ is obvious.

To prove the opposite inclusion, we need to show that Z_β is a subcontramodule. A continuous function $f : \maltese \to Z_\beta$ corestricts to a function $f|^{Z_\alpha} : \maltese \to Z_\alpha$ for some $\alpha < \beta$ because β is $|\maltese|$-filtered. Thus, $\theta_X(f) = \theta_X(f|^{Z_\alpha}) \in Z_{\alpha+1} \subseteq Z_\beta$. $\square$

While $\maltese$ is not presentable in general (Lemma 4.5), the proof of Proposition 4.7 uses the fact that $[\maltese, -]_{\mathcal{W}}$ commutes with special colimits (cf. [16, Lemma 2.4.1]). This can be sharpened to prove the following theorem.

Theorem 4.8. *The category $\mathcal{W}^{\maltese}$ is cocomplete.*

Proof. Let $F : \mathcal{D} \to \mathcal{W}^{\maltese}$ be a small diagram, V its colimit in $\mathcal{W}$. Hence, given a cocone $\Psi_X : FX \to Y$, $X \in D$ in $\mathcal{W}^{\maltese}$, we have a unique mediating morphism $\Psi^\sharp : V \to Y$ in $\mathcal{W}$.

Clearly, the cocone factors through the subcontramodule, generated by the image of $\Psi^\sharp$:

$$\Psi_X : FX \xrightarrow{\Phi_X} (\Psi^\sharp(V))^{\maltese} \hookrightarrow Y\,.$$

The explicit construction in Proposition 4.7 gives an upper bound α on the cardinality of $(\Psi^\sharp(V))^{\maltese}$. It depends on $|\maltese|$ and $|V|$ but does not depend on $|Y|$.

Let us consider a category $\mathcal{D}^*$, whose objects are cocones $\Psi_X : FX \to Y$ in $\mathcal{W}^{\maltese}$ with $|Y| < \alpha$. The morphisms from $\Psi_X : FX \to Y$ to $\Phi_X : FX \to Z$ are such morphisms $f \in \mathcal{W}^{\maltese}(Y, Z)$ that $f\Psi_X = \Phi_X$ for all $X \in D$. Since the cardinalities of the cocone targets in $\mathcal{D}^*$ are bounded above, the skeleton $\mathcal{D}^*_0$ of $\mathcal{D}^*$ is a small category. Then

$$F^* : \mathcal{D}^*_0 \to \mathcal{W}^{\maltese}, \quad (\Psi_X : FX \to Y) \mapsto Y$$

is a small diagram, whose limit $\varprojlim F^*$ is the colimit $\varinjlim F$. $\square$

We finish this section by right-inducing the Quillen model structure to $\mathcal{W}^{\maltese}$. It does not follow from Proposition 3.1 because $\mathcal{W}$ is not accessible.

Proposition 4.9. *There exists a Quillen right-induced model structure on $\mathcal{W}^{\maltese}$, defined by equations* (22). *This structure is right proper.*

Proof. Since the Quillen model structure on $\mathcal{W}$ is cofibrantly generated, a right induced model structure on $\mathcal{W}^{\maltese}$ exists if (cf. [13, Th. 11.3.2])

(1) $F(\mathbb{I})$ and $F(\mathbb{J})$ permit the small object argument
(2) and $\mathcal{F}^F$ takes relative $F(\mathbb{J})$-complexes to weak equivalences,

where $\mathbb{I}$ and $\mathbb{J}$ are the sets of generating cofibrations and generating trivial cofibrations as defined in (31) and (32) respectively. The second statement is obvious because the inclusions in

$$F(\mathbb{J}) = \{[\maltese, \mathbb{D}^n \times \{0\}] \to [\maltese, \mathbb{D}^n \times [0,1]\,] \mid n \geqslant 0\}$$

admit deformation retracts. Hence, relative $F(\mathbb{J})$-complexes are weak equivalences topologically.

The first statement holds because relative $F(\mathbb{I})$-complexes and relative $F(\mathbb{J})$-complexes are topological inclusions and every topological space is small relative to the inclusions [16, Lemma 2.4.1].

The model structure described above is cofibrantly generated [13, Th. 11.3.2]. Since the model structure on $\mathcal{W}$ is right proper, then so is the induced one on $\mathcal{W}^{\maltese}$. $\square$

4.5. **Topological comodule-contramodule correspondence.** Since $\mathcal{W}$ is cartesian closed, the pair $(\mathcal{L} \dashv \mathcal{R})$ is a Quillen adjunction by Proposition 3.6. An analogue of Theorem 3.7 encounters set-theoretic difficulties. We can sweep them under the carpet and have the following result with an identical proof:

Proposition 4.10. *Suppose that all topological spaces are subsets of a Grothendieck universe. Then there exist a right Bousfield localisation $R_{\mathfrak{A}}(\mathcal{W}^{\maltese})$ and a left Bousfield localisation $L_{\mathfrak{B}}(\mathcal{W}_{\maltese})$, where the sets $\mathfrak{A}$ and $\mathfrak{B}$ are defined similarly to classes in* (28)*, so that the comodule-contramodule correspondence $(\mathcal{L} \dashv \mathcal{R})$ induces a Quillen equivalence between the localisations.*

Let $\maltese = \mathbb{S}^2$ be the 2-sphere. As a $\maltese$-comodule, consider the Hopf fibration $\phi : \mathbb{S}^3 \to \mathbb{S}^2$. The comodule $\mathbb{S}^3$ is fibrant, yet $\mathcal{R}\mathbb{S}^3 = \varnothing$. This shows that $(\mathcal{L} \dashv \mathcal{R})$ in Proposition 4.10 is not a Quillen equivalence between $\mathcal{W}^{\maltese}$ and $\mathcal{W}_{\maltese}$. This example suggest some "local" version of the functor $\mathcal{R}$ (local sections) may still be an equivalence.

Another instructive example is the 1-sphere $\maltese = \mathbb{S}^1$ and the figure-8 $X = \mathbb{S}^1 \vee \mathbb{S}^1$. The comodule structure is $\phi_X = \mathrm{Const} \vee \mathrm{Id}_{\mathbb{S}^1}$. Clearly, $\mathcal{R}X = \{\mathrm{Id}\}$ is the one-element set and $\mathcal{L}\mathcal{R}X = \maltese$. Taking local sections does not help: local sections near the singular point are not going to see the collapsing loop in X. On the other hand, the collapsing loop will be "seen" by the local sections of the fibrant replacement $X_{\mathbb{F}}$. These phenomena deserve further investigation.

4.6. **Relation to simplicial sets.** Most of the current chapter equally applies to the category $\mathcal{K}$ of compactly generated spaces, not only $\mathcal{W}$. An advantage of $\mathcal{K}$ is its direct relation to the category of simplicial sets: there is a Quillen equivalence between simplicial sets and topological spaces [16, Th. 3.6.7]

$$(|-| \dashv \mathscr{S}), \ \mathscr{S} : \mathcal{S} \rightleftarrows \mathcal{K} : |-| \tag{33}$$

where $|Q_\bullet|$ is the geometric realisation of a simplicial set $Q_\bullet$ and $\mathscr{S}(Y)_n = \mathcal{K}(\Delta^n, Y)$ is the singular complex of a topological space Y. Let $\maltese_\bullet = (\maltese_n) \in \mathcal{S}$, $\maltese = |\maltese_\bullet| \in \mathcal{W}$, $\widehat{\maltese}_\bullet = \mathscr{S}(\maltese) \in \mathcal{S}$, considered as comonoids in their categories. We denote the corresponding comonad-monad adjoint pairs by $(T \dashv F)$, $(T \dashv F)$ and $(\widehat{T} \dashv \widehat{F})$.

In light of the isomorphism of categories (24), we consider the overcategories in place of the comodule categories. The functors (33) and the induction (Proposition 1.27) give rise to the following functors:

$$|-| : (\mathcal{S}\downarrow \maltese_\bullet) \to (\mathcal{K}\downarrow \maltese), \ \mathscr{S} : (\mathcal{K}\downarrow \maltese) \to (\mathcal{S}\downarrow \widehat{\maltese}_\bullet), \ \mathcal{I}nd\!\downarrow : (\mathcal{S}\downarrow \widehat{\maltese}_\bullet) \rightleftarrows (\mathcal{S}\downarrow \maltese_\bullet).$$

Similarly, we can use the functors (33). The induction functor from Proposition 2.10 can be applied levelwise to some but not all simplicial contrasets (see Section 2.5). We expect that the induction exists in general. These considerations yield the functors between the contramodule categories:

$$|-| : \mathcal{S}^F \to \mathcal{K}^F, \ \mathscr{S} : \mathcal{K}^F \to \mathcal{S}^{\widehat{F}}, \ \mathcal{I}nd_\bullet^F : \mathcal{S}^{\widehat{F}} \to \mathcal{S}^F.$$

We can package all these functors in the following conjectural worldview of the relation between the topological and the simplicial comodule-contramodule correspondences:

Conjecture 4.11. For any simplicial set $\maltese$ there exists a commutative (in an appropriate sense) square of categories and Quillen adjunctions

$$
\begin{tikzcd}
\mathcal{S}^F \arrow[r, dashed, shift left, "\mathcal{L}"] \arrow[d, shift right, "|-|"'] & \mathcal{S}_T \arrow[l, dashed, shift left, "\mathcal{R}"] \arrow[d, shift right, "|-|"'] \\
\mathcal{K}^F \arrow[r, dashed, shift left, "\mathcal{L}"] \arrow[u, shift right, "\mathcal{I}nd_{\bullet}^{F} \circ \mathscr{S}"'] & \mathcal{K}_T \arrow[l, dashed, shift left, "\mathcal{R}"] \arrow[u, shift right, "\mathcal{I}nd\!\downarrow \circ \mathscr{S}"']
\end{tikzcd}
\tag{34}
$$

where the left adjoint functors are either on top or on the left and the vertical solid arrows are Quillen equivalences.

4.7. **Topological fact.** We finish the paper with a useful fact about the topological comodule-contramodule correspondence that does not follow from the general framework of model categories.

Proposition 4.12. *Suppose $X, Y \in (\mathcal{W}_T)_{\mathbb{F}}$ are CW-complexes. If $f \in \mathbb{W}_T(X, Y)$, then $\mathcal{R}f \in \mathcal{W}^F(FX, FY)$ and $Ff : \mathcal{W}([\maltese, X], [\maltese, Y])$ are homotopy equivalences.*

Proof. By Whitehead Theorem, f is a homotopy equivalence. Moreover, f is a fibrewise homotopy equivalence [22, 7.5]. The rest of the argument is clear. □

In particular, $\mathcal{R}f \in \mathbb{W}^F(\mathcal{R}X, \mathcal{R}Y)$. It is a refinement of the following easy observation. We would like to refine Proposition 4.12, replacing the CW-complex condition on X and Y with a condition on $\maltese$.

We need a standard topological lemma, which we could not find in the literature. Let X, Y be connected topological spaces in $\mathcal{W}$ and $f : X \to Y$ be a map. If $A \in \mathcal{W}$ is another topological pace, we write $f_A : \mathcal{W}(A, X) \to \mathcal{W}(A, Y)$ for the map of function spaces defined by composition with f (cf. Section 4.1). Next fix a map $\alpha : A \to X$ that will be a base point for $\mathcal{W}(A, X)$. As a base point for $\mathcal{W}(A, Y)$ we use the map $\beta = f \circ \alpha$ so that $f_A : \mathcal{W}(A, X) \to \mathcal{W}(A, Y)$ is a map of pointed spaces.

Lemma 4.13. *Suppose that A is a CW-complex of finite type and f is a weak homotopy equivalence. Then $(f_A)_n : \pi_n(\mathcal{W}(A, X), \alpha) \to \pi_n(\mathcal{W}(A, Y), \beta)$ is an isomorphism for all $n \geqslant 1$.*

Proof. The first step in the proof is to show that the result is true for the sphere $A = \mathbb{S}^n$ where $n \geqslant 1$. In this case the space $\mathcal{W}(\mathbb{S}^n, X)$ is usually denoted by $\Lambda^n(X)$. Choose a base point for $\mathbb{S}^n$. Evaluating maps at the base point gives us a map $\Lambda^n(X) \to X$. This map is a fibration and the fibre over $x \in X$ is the space $\Omega^n_x(X)$, the n-fold iterated based loop space of X, with base point x. The map f now gives a map of fibrations:

$$
\begin{tikzcd}
\Lambda^n(X) \arrow[r] \arrow[d] & \Lambda^n(Y) \arrow[d] \\
X \arrow[r] & Y
\end{tikzcd}
$$

The homotopy groups of $\Omega^n_x(X)$ are given by $\pi_k(\Omega^n_x(X)) = \pi_{k+n}(X, x)$ for $k \geqslant 0$ and trivial for $k < 0$. Under this identification, the map of homotopy groups π_k induced by the map

$$\Omega^n_x(f) : \Omega^n_x(X) \to \Omega^n_{f(x)}(Y)$$

is just

$$f_{k+n} : \pi_{k+n}(X, x) \to \pi_{k+n}(Y, f(x)).$$

So since f_* is a weak homotopy equivalence, it follows that the map of fibrations $\Lambda^n(X) \to \Lambda^n(Y)$ defines isomorphisms on the homotopy groups of the fibres. Since f is a weak homotopy equivalence this map of fibrations defines an isomorphism on the homotopy groups of the base spaces. A standard five lemma argument shows that it, therefore, gives an isomorphism on the homotopy groups of the total spaces.

The second step is to extend the result to finite CW-complexes by induction on the number of cells. Assume that the map $(f_A)_* : \pi_n(\mathcal{W}(A, X), \alpha) \to \pi_n(\mathcal{W}(A, Y), \beta)$ is an isomorphism for $n \geqslant 1$. Now replace A by $B = A \cup_\psi D^{p+1}$ with $\psi \in \mathcal{W}(\mathbb{S}^p, A)$. This gives a cofibration sequence

$$A \to B \to \mathbb{S}^{p+1}.$$

Applying $\mathcal{W}(-, X)$ and $\mathcal{W}(-, Y)$ to this cofibration sequence and using the map $f : X \to Y$, leads to the following commutative diagram:

$$\begin{tikzcd}
\mathcal{W}(A, X) \arrow[d] & \mathcal{W}(B, X) \arrow[l] \arrow[d] \\
\mathcal{W}(A, Y) & \mathcal{W}(B, Y). \arrow[l]
\end{tikzcd}$$

The horizontal arrows are fibrations. The fibres of the top map are copies of $\mathcal{W}(\mathbb{S}^{p+1}, X)$. The fibres of the bottom one are copies of $\mathcal{W}(\mathbb{S}^{p+1}, Y)$. By assumption this map of fibrations induces an isomorphism on the homotopy groups of the base spaces, and by the first step it induces an isomorphism on the homotopy groups of the fibres. It follows from the five lemma that it induces isomorphisms on the homotopy groups of the total spaces.

The final step is to extend the result to a CW-complex of finite type. Let A^n be the n-skeleton of A, $i_n : A^n \to A^{n+1}$ the inclusion. Then A is the direct limit of the A^n and each of the inclusions i_n is a cofibration. It follows that $\mathcal{W}(A, X)$ is the inverse limit of the sequence of maps $\mathcal{W}(A^{n+1}, X) \to \mathcal{W}(A^n)$ induced by i_n. Since each of the maps i_n is a cofibration, the maps in the inverse system are fibrations. Now suppose $f : X \to Y$ is a weak equivalence. We have proved that for each n the map $f_{A^n} : \mathcal{W}(A^n, X) \to \mathcal{W}(A^n, Y)$ is a weak homotopy equivalence. The map $f_A : \mathcal{W}(A, X) \to \mathcal{W}(A, Y)$ is the map of inverse limits defined by the sequence f_{A^n}. Hence, f_A is also a weak homotopy equivalence [15, Th. 2.2]. $\square$

Given a topological space X and a point $s \in X$, by X_s we denote the connected component of X that contains s. A map $f \in \mathcal{W}(X, Y)$ yields a map $f_s \in \mathcal{W}(X_s, Y_{f(s)})$ between components.

Theorem 4.14. *Let $\maltese$ be a CW-complex of finite type. Suppose that $(X, \phi), (Y, \psi) \in (\mathcal{W}_T)_{\mathbb{F}}$ are fibrant $\maltese$-comodules and $s \in \mathcal{R}X$. If $f \in \mathbb{W}_T(X, Y)$ is a weak homotopy equivalence, then the map $\mathcal{R}f_s$ is also a weak homotopy equivalence.*

Proof. Consider a part of the commutative diagram (26):

$$\begin{tikzcd}
\mathcal{R}X_s \arrow[r, "i"] \arrow[d, "\mathcal{R}f_s"] & {[\maltese, X]_s = FX_s} \arrow[r, "\phi_*"] \arrow[d, "Ff_s"] & {[\maltese, \maltese]_{\mathrm{Id}}} \arrow[d, "\mathrm{Id}_{[\maltese,\maltese]}"] \\
\mathcal{R}Y_{fs} \arrow[r, "j"] & {[\maltese, Y]_{fs} = FY_{fs}} \arrow[r, "\psi_*"] & {[\maltese, \maltese]_{\mathrm{Id}}}
\end{tikzcd}$$

Since both ϕ and ψ are fibrations, both $\phi_* = [\mathrm{Id}_{\maltese}, \phi]$ and ψ_* are also fibrations. Moreover, $\mathcal{R}X_s$ is the fibre of ϕ_* over the identity and $\mathcal{R}Y_{fs}$ is the fibre of ψ_* over the identity. All the spaces in the diagram have chosen base points. This yields a map from the homotopy exact sequence of ϕ_* to the homotopy exact sequence of ψ_*.

The map of the base spaces is the identity: it induces the identity of homotopy groups. By Lemma 4.13, the map of total spaces induces an isomorphism of homotopy groups. The five lemma tells us that it induces an isomorphism on the homotopy groups of the fibres. $\square$

If one shows $\pi_0(\mathcal{R}f)$ is an isomorphism, then Theorem 4.14 ensures that $\mathcal{R}f$ is a weak homotopy equivalence. Such a proof would involve Topological Obstruction Theory and may require additional assumptions on $\maltese$.

Theorem 4.14 is an indication that the correspondence is full of topological mysteries, waiting to be uncovered.

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