CLASSICAL SET THEORY: THEORY OF SETS AND CLASSES

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Introduction

Make things as simple as possible, but not simpler.

Albert Einstein

This is a short introductory course to Set Theory and Category Theory, based on axioms of von Neumann–Bernays–Gödel (briefly NBG). The text can be used as a base for a lecture course in Foundations of Mathematics, and contains a reasonable minimum which a good (post-graduate) student in Mathematics should know about foundations of this science.

My aim is to give strict definitions of all set-theoretic notions and concepts that are widely used in mathematics. In particular, we shall introduce the sets \mathbb{N} , \mathbb{Z} , \mathbb{Q} , \mathbb{R} of numbers (natural, integer, rational and real) and will prove their basic order and algebraic properties. Since the system of NBG axioms is finite and does not involve advanced logics, it is more friendly for beginners than other axiomatic set theories (like ZFC).

The legal use of classes in NBG will allow us to discuss freely Conway's surreal numbers that form an ordered field **No**, which is a proper class and hence is not "visible" in ZFC. Also the language of NBG allows to give natural definitions of some basic notions of Category Theory: category, functor, natural transformation, which is done in the last part of this book.

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- To be added.

Part 1. Naive Set Theory

A set is a Many that allows itself to be thought of as a One. Georg Cantor

1. Origins of Set Theory

The origins of (naive) Set Theory were created at the end of XIX century by Georg Cantor¹ (1845–1918) in his papers published in 1874–1897.

Cantors ideas made the notion of a set the principal (undefined) notion of Mathematics, which can be used to give precise definitions of all other mathematical concepts such as numbers or functions.

According to Cantor, a set is an arbitrary collection of objects, called *elements* of the set. In particular, sets can be elements of other sets. The fact that a set x is an element of a set y is denoted by the symbol $x \in y$. If x is not an element of y, then we write $x \notin y$.

A set consisting of finitely many elements x_1, \ldots, x_n is written as $\{x_1, \ldots, x_n\}$. Two sets are equal if they consist of the same elements. For example, the sets $\{x, y\}$ and $\{y, x\}$ both have elements x, y and hence are equal.

A set containing no elements at all is called *the empty set* and is denoted by \emptyset . Since sets with the same elements are equal, the empty set is unique.

The theory developed so far, allows us to give a precise meaning to natural numbers (which are abstractions created by humans to facilitate counting):

$$0 = \emptyset,$$

$$1 = \{0\},$$

$$2 = \{0, 1\},$$

$$3 = \{0, 1, 2\},$$

$$4 = \{0, 1, 2, 3\},$$

$$5 = \{0, 1, 2, 3, 4\},$$

The set $\{0,1,2,3,4,5,\dots\}$ of all natural numbers² is denoted by ω . The set $\{1,2,3,4,5,\dots\}$ of non-zero natural numbers is denoted by \mathbb{N} .

Very often we need to create a set of objects possessing some property (for example, the set of odd numbers). In this case we use the *constructor* $\{x : \varphi(x)\}$, which yields exactly what we need: the set $\{x : \varphi(x)\}$ of all objects x that have certain property $\varphi(x)$.

Using the constructor we can define some basic "algebraic" operations over sets X,Y:

• the intersection $X \cap Y = \{x : x \in X \land x \in Y\}$ whose elements are objects that belong to X and Y;

¹Exercise: Read about Georg Cantor in Wikipedia.

²Remark: There are two meanings (Eastern and Western) of what to understand by a natural number. The western approach includes zero to natural numbers whereas the eastern tradition does not. This difference can be noticed in numbering floors in buildings in western or eastern countries.

- the union $X \cup Y = \{x : x \in X \lor x \in Y\}$ consisting of the objects that belong to X or Y or to both or them;
- the difference $X \setminus Y = \{x : x \in X \land x \notin Y\}$ consisting of elements that belong to X but not to Y;
- the symmetric difference $X \triangle Y = (X \cup Y) \setminus (X \cap Y)$ consisting of elements that belong to the union $X \cup Y$ but not to the intersection $X \cap Y$.

In the formulas for the union and intersection we used the logical connectives \land and \lor denoting the logical operations and and or. Below we present the truth table for these logical operations and also for three other logical operations: the negation \neg , the implication \Rightarrow , the equivalence \Leftrightarrow .

\boldsymbol{x}	y	$x \wedge y$	$x \vee y$	$x \Rightarrow y$	$x \Leftrightarrow y$	$\neg x$
0	0	0	0	1	1	1
0	1	0	1	1	0	1
1	0	0	1	0	0	0
1	1	1	1	1	1	0

Therefore, we have four basic operations over sets X, Y:

$$\begin{split} X \cap Y &= \{x: x \in X \ \land \ x \in Y\}, \quad X \cup Y = \{x: x \in X \ \lor \ x \in Y\}, \\ X \setminus Y &= \{x: x \in X \ \land \ x \not\in Y\}, \quad X \triangle Y = (X \cup Y) \setminus (X \cap Y). \end{split}$$

Exercise 1.1. For the sets $X = \{0, 1, 2, 4, 5\}$ and $Y = \{1, 2, 3, 4\}$, find $X \cap Y$, $X \cup Y$, $X \setminus Y$, $X \triangle Y$.

2. Berry's Paradox

"The essence of mathematics is its freedom" $Georg\ Cantor$

In 1906 Bertrand Russell, a famous British philosopher, published a paradox, which he attributed to G. Berry (1867–1928), a junior librarian at Oxford's Bodleian library.

To formulate this paradox, observe that each natural number can be described by some property. For example, zero is the smallest natural number, one is the smallest nonzero natural number, two is the smallest prime number, three is the smallest odd prime number, four is the smallest square, five is the smallest odd prime number which is larger than the smallest square and so on.

Since there are only finitely many sentences of a given length, such sentences (of given length) can describe only finitely many numbers³. Consequently, infinitely many numbers cannot be described by short sentences, consisting of less than 100 symbols. Among such numbers take the smallest one and denote it by s. Now consider the characteristic property of this number: s is the smallest number that cannot be described by a sentence consisting less than 100 symbol. But the latter sentence consists of 96 symbols, which is less that 100, and uniquely defines the number s.

 $^{^3}$ The list of short descriptions of the first 10000 numbers can be found here: https://www2.stetson.edu/~efriedma/numbers.html

Now we have a paradox⁴ on one hand, the numbers s belongs to the set of numbers that cannot be described by short sentences, and on the other hand, it has a short description. Where is the problem?

The problem is that the description of s contains a quantifier that runs over all sentences including itself. Using such self-referencing properties can lead to paradoxes, in particular, to Berry's Paradox. This means that not all properties $\varphi(x)$ can be used for defining mathematical objects, in particular, for constructing sets of form $\{x : \varphi(x)\}$.

In order to avoid the Berry Paradox at constructing sets $\{x : \varphi(x)\}$, mathematicians decided to use only precisely defined properties $\varphi(x)$, which do not include the property appearing in Berry's paradox. Correct properties are described by formulas with one free variable in the language of Set Theory.

3. The Language and formulas of Set Theory

For describing the language of Set Theory we use our natural language, which will be called the *metalanguage* (with respect to the language of Set Theory).

We start describing the language of Set Theory with describing its *alphabet*, which is an infinite list of symbols that necessarily includes the following *special symbols*:

- the symbols of binary relations: the equality "=" and memberships " \in ";
- logical connectives: $\neg, \land, \lor, \Rightarrow, \Leftrightarrow$;
- quantifiers: \forall and \exists ;
- parentheses: "(" and ")".

All remaining (that is, non-special) symbols of the alphabet are called the *symbols of variables*. The list of those symbols is denoted by Var. For symbols of variables we shall use small and large symbols of Latin alphabet: a, A, b, B, c, C, u, U, v, V, w, W, x, X, y, Y, z, Z etc.

Sequences of symbols of the alphabet are called *words*. Well-defined words are called *formulas*.

Definition 3.1. Formulas are defined inductively by the following rules:

- (1) if x, y are symbols of variables, then the words (x = y) and $(x \in y)$ are formulas (called the *atomic formulas*);
- (2) if φ, ψ are formulas, then $(\neg \varphi)$, $(\varphi \land \psi)$, $(\varphi \lor \psi)$, $(\varphi \Rightarrow \psi)$, $(\varphi \Leftrightarrow \psi)$ are formulas;
- (3) if x is a symbol of a variable and φ is a formula, then $(\forall x \varphi)$ and $(\exists x \varphi)$ are formulas;
- (4) there are no other formulas than those constructed from atomic formulas using the rules (2) and (3).

Writing formulas we shall often omit parentheses when there will be no ambiguity. Putting back parentheses, we use the following preference order for logical operations:

$$\neg \land \lor \Rightarrow \Leftrightarrow .$$

Example 3.2. The formula $(\exists u \ ((\neg(u=v) \land (u \in x)) \land (v \in x))))$ can be written shortly as $\exists u \ \exists v \ (u \neq v \land u \in x \land v \in x)$. This formula describes the property of x to have at least two elements.

⁴A similar argument can be used to prove that every natural number have some interesting property. Assuming that the exist natural numbers without interesting properties, we can consider the smallest element of the set of "non-interesting" numbers and this number has an interesting property: it is the smallest non-interesting number.

For a formula φ and symbols of variables x, y the formulas

$$(\forall x \in y) \varphi \text{ and } (\exists x \in y) \varphi$$

are short versions of the formulas

$$\forall x \ (x \in y \Rightarrow \varphi) \text{ and } \exists x \ (x \in y \land \varphi),$$

respectively. In this case we say that the quantifiers $\forall x \in y$ and $\exists x \in y$ are y-bounded.

Properties $\varphi(x)$ of sets that can be used in the constructors $\{x:\varphi(x)\}$ corresponds to formulas with a unique free variable x.

Definition 3.3. For any formula φ its *set of free variables* $Free(\varphi)$ is defined by induction on the complexity of the formula according to the following rules:

- (1) If φ is the atomic formula x = y or $x \in y$, then $Free(\varphi) = \{x, y\}$;
- (2) for any formulas φ, ψ we have $\mathsf{Free}(\neg \varphi) = \mathsf{Free}(\varphi)$ and $\mathsf{Free}(\varphi \land \psi) = \mathsf{Free}(\varphi \lor \psi) = \mathsf{Free}(\varphi \Rightarrow \psi) = \mathsf{Free}(\varphi \Leftrightarrow \psi) = \mathsf{Free}(\varphi) \cup \mathsf{Free}(\psi)$.
- (3) for any formula φ we have $\mathsf{Free}(\forall x \ \varphi) = \mathsf{Free}(\exists x \ \varphi) = \mathsf{Free}(\varphi) \setminus \{x\}.$

Example 3.4. The formula $\exists u \ (u \in x)$ has x as a unique free variable and describes the property of a set x to be non-empty.

Example 3.5. The formula $\exists u \ (u \in x \land \forall v \ (v \in x \Rightarrow v = u))$ has x as a unique free variable and describes the property of a set x to be a singleton.

Exercise 3.6. Write down a formula representing the property of a set x to contain

- at least two elements;
- exactly two elements;
- exactly three elements.

Exercise 3.7. Write down formulas representing the property of a set x to be equal to 0, 1, 2, etc.

Exercise* 3.8. Suggest a formula $\varphi(x)$ such that each set x satisfying this formula is infinite.

Exercise 3.9. Explain why the property appearing in Berry's Paradox cannot be described by a formula of Set Theory.

4. Russell's Paradox

As we already know, Berry's Paradox can be avoided by formalizing the notion of a property. A much more serious problem for foundations of Set Theory was discovered in 1901 by Bertrand Russell (1872–1970) who suggested the following paradox.

Russell's Paradox. Consider the property φ of a set x to not contain itself as an element. This property is represented by the well-defined formula $x \notin x$. Many sets, for example, all natural numbers, have the property φ . Next, consider the set $A = \{x : x \notin x\}$ of all sets x that have the property φ . For this set A two cases are possible:

- (1) A has the property φ and hence belongs to the set $A = \{x : \varphi(x)\}$, which contradicts the property φ (saying that $A \notin A$);
- (2) A fails to have the property φ and then $A \notin \{x : \varphi(x)\} = A$, which means that $A \notin A$ and A has the property φ .

In both cases we have a contradiction.

There are at least three ways to avoid Russell's paradox. The most radical one is to exclude the Law of the Excluded Middle from laws of Logic. The mathematicians following this idea formed the schools of intuicionists and constructivists⁵.

Less radical ways of avoiding Russell's paradox were suggested by the school of formalists leaded by David Hilbert (1862–1943). One branch of this school (Zermelo, Fraenkel, etc) suggested to forbid the constructor $\{x : \varphi(x)\}$ replacing it by its restricted version

$$\{x \in y : \varphi(x)\}.$$

The class $\{x \in y : \varphi(x)\}$ consists of all elements of the class y that have property $\varphi(x)$. This approach resulted in appearance of the Axiomatic Set Theory of Zermelo–Fraenkel, used by many modern mathematicians.

The other branch (von Neuman, Robinson, Bernays, Gödel) suggested to resolve Russell's Paradox by introducing a notion of class for describing families of sets that are too big to be elements of other classes. An example of a class is the family $\{x:x\notin x\}$ appearing in the Russell's Paradox. Then sets are defined as "small" classes. They are elements of other classes. This approach resulted in appearance of the Axiomatic Set Theory of von Neumann–Bernays–Gödel, abbreviated by NBG by Mendelson [18]. Exactly this axiomatic system NBG will be taken as a base for presentation of Set Theory in this textbook. Since NBG allows us to speak about sets and classes and NBG was created by classics (von Neumann, Bernays, Gödel), we will call it the Classical Set Theory.

⁵Exercise: Read about intuicionists and constructivists in Wikipedia.

Part 2. Axiomatic Theories of Sets

Aus dem Paradies, das Cantor uns geschaffen, soll uns niemand vertreiben können

David Hilbert

In this part we present axioms of von Neumann–Bernays–Gödel and discuss the relation of these axioms to the Zermelo–Fraenkel axioms of Set Theory.

5. Axioms of von Neumann-Bernays-Gödel

In this section we shall list 15 axioms of the von Neumann–Bernays–Gödel and also introduce some new notions and notations on the base of these axioms.

In fact, the language of the Classical Set Theory has been described in Section 3. We recommend the reader (if he or she is not fluent in Logics) to return back to this section and read it once more.

The unique undefined notions of Set Theory NBG are the notions of a *class* and an *element*. So, classes can be elements of other class. The fact that a class X is an element of a class Y is written as $X \in Y$. The negation of $X \in Y$ is written as $X \notin Y$. So, $X \notin Y$ is a short version of $\neg(X \in Y)$. Also $X \neq Y$ is a short version of $\neg(X = Y)$.

Definition 5.1. A class X is defined to be a *set* if X is an element of some other class. More formally, a class X is a set if $\exists Y (X \in Y)$.

Definition 5.2. A class which is not a set is called a *proper class*.

To distinguish sets from proper classes, we shall use small characters (like x, y, z, u, v, w, a, b, c) for denoting sets, capital letters (like X, Y, Z, U, V, W, A, B, C) for denoting classes and bold-face characters (like $\mathbf{X}, \mathbf{Y}, \mathbf{U}, \mathbf{E}$) for denoting proper classes. In particular, the class of all sets \mathbf{U} is a proper class.

Now we start listing the axioms of NBG.

Axiom of Extensionality:
$$\forall X \ \forall Y \ (X = Y \Leftrightarrow \forall z \ (z \in X \Leftrightarrow z \in Y))$$

The axiom of extentionality says that two classes are equal if and only if they consist of the same elements.

Definition 5.3. Given two classes X, Y, we write $X \subseteq Y$ and say that the class X is a *subclass* of a class Y if $\forall z \ (z \in X \Rightarrow z \in Y)$, i.e., each element of the class X is an element of the class Y. If X is not a subclass of Y, then we write $X \not\subseteq Y$. If a subclass X of a class Y is a set, then X is called a *subset* of the class Y.

For two classes X, Y we write $X \subset Y$ iff $X \subseteq Y$ and $X \neq Y$.

Observe that the axiom of extensionality is equivalent to the formula

$$\forall X \ \forall Y \ (X = Y \Leftrightarrow (X \subseteq Y \land Y \subseteq X)).$$

Axiom of Universe:
$$\exists \mathbf{U} \ \forall x \ (x \in \mathbf{U} \Leftrightarrow \exists X \ (x \in X))$$

The Axiom of Universe says that the class **U** of all sets exists. By the axiom of extensionality, this class is unique. The definition of the universe **U** guarantees that $\forall X \ (X \subseteq \mathbf{U})$.

Axiom of Difference:
$$\forall X \ \forall Y \ \exists Z \ \forall u \ (u \in Z \iff (u \in X \ \land \ u \notin Y))$$

Axiom of Difference postulates that for any classes X, Y the class

• $X \setminus Y = \{x : x \in X \land x \notin Y\}$ exists.

Using the Axiom of difference, for any classes X, Y we can define their

- intersection $X \cap Y = X \setminus (X \setminus Y)$,
- union $X \cup Y = \mathbf{U} \setminus ((\mathbf{U} \setminus X) \cap (\mathbf{U} \setminus Y))$, and
- symmetric difference $X \triangle Y = (X \cup Y) \setminus (X \cap Y)$.

Applying the Axiom of Difference to the class U, we conclude that the *empty class*

$$\emptyset = \mathbf{U} \setminus \mathbf{U}$$

exists. The empty class contains no elements and is unique by the Axiom of Extensionality.

Axiom of Pair:
$$\forall x \in \mathbf{U} \ \forall y \in \mathbf{U} \ \exists z \in \mathbf{U} \ \forall u \ (u \in z \Leftrightarrow (u = x \lor u = y))$$

The Axiom of Pair says that for any sets x, y there exists a set z whose unique elements are x and y. By the Axiom of Extensionality, such set z is unique. It is called the *unordered* pair of the sets x, y and is denoted by $\{x, y\}$. The Axiom of Extensionality ensures that $\{x, y\} = \{y, x\}$ for any sets x, y.

For any set x, the unordered pair $\{x, x\}$ is denoted by $\{x\}$ and is called a *singeton*.

Definition 5.4 (Kuratowski, 1921). The ordered pair $\langle x, y \rangle$ of sets x, y is the set $\{\{x\}, \{x, y\}\}$.

Proposition 5.5. For sets x, y, u, v the ordered pairs $\langle x, y \rangle$ and $\langle u, v \rangle$ are equal if and only if x = u and y = v. More formally,

$$\forall x \ \forall y \ \forall u \ \forall v \ \big((\langle x, y \rangle = \langle u, v \rangle) \ \Leftrightarrow \ (x = u \ \land \ y = v) \big).$$

Proof. The "only if" part is trivial. To prove the "if" part, assume that

(5.1)
$$\{\{x\}, \{x,y\}\} = \langle x,y \rangle = \langle u,v \rangle = \{\{u\}, \{u,v\}\}\}$$

but $x \neq u$ or $y \neq v$.

First assume that $x \neq u$. By the Axiom of Extensionality, $\{x\} \neq \{u\}$. The equality (5.1) implies that $\{x\} = \{u,v\}$ and hence u = v = x, which contradicts our assumption. This contradiction shows that x = u. If x = y, then $\{\{u\}, \{u,v\}\} = \{\{x\}, \{x,y\}\} = \{\{x\}\}\}$ implies that u = v and hence y = x = u = v. If $x \neq y$, then $\{u\} \neq \{x,y\} \in \{\{u\}, \{u,v\}\}$ implies $\{x,y\} = \{u,v\}$ and hence y = v.

Using the notion of an ordered pair, we can introduce ordered triples, quadruples etc. Namely, for any sets x, y, z the ordered triple $\langle x, y, z \rangle$ is the set $\langle \langle x, y \rangle, z \rangle$.

The following 5 axioms are called the axioms of existence of classes.

Axiom of Product:
$$\forall X \ \forall Y \ \exists Z \ \forall z \ (z \in Z \ \Leftrightarrow \ \exists x \in X \ \exists y \in Y \ (z = \langle x, y \rangle))$$

The Axiom of Product guarantees that for any classes X, Y, their Cartesian product

$$X \times Y = \{\langle x, y \rangle : x \in X, y \in Y\}$$

exists.

The product $\mathbf{U} \times \mathbf{U}$ will be denoted by $\ddot{\mathbf{U}}$, and the product $\ddot{\mathbf{U}} \times \mathbf{U}$ by $\ddot{\mathbf{U}}$.

Definition 5.6. A class R is called a *relation* if its elements are ordered pairs. More formally,

$$R$$
 is a relation $\Leftrightarrow \forall z \in R \ \exists x \in \mathbf{U} \ \exists y \in \mathbf{U} \ (z = \langle x, y \rangle).$

Axiom of Inversion:
$$\forall X \ \exists Y \ \forall x \in \mathbf{U} \ \forall y \in \mathbf{U} \ (\langle x, y \rangle \in X \ \Leftrightarrow \ \langle y, x \rangle \in Y)$$

The Axiom of Inversion implies that for any relation R the relation

$$R^{-1} = \{ \langle y, x \rangle \in \ddot{\mathbf{U}} : \langle x, y \rangle \in R \}$$

exists. Observe that a class R is a relation if and only if $R = (R^{-1})^{-1}$.

Axiom of Domain:
$$\forall X \ \exists D \ \forall x \in \mathbf{U} \ (x \in D \ \Leftrightarrow \ \exists y \in \mathbf{U} \ (\langle x, y \rangle \in X))$$

By the Axioms of Domain and Inversion, for each class X its

- $domain dom[X] = \{x \in \mathbf{U} : \exists y \in \mathbf{U} \ \langle x, y \rangle \in X\}$ and
- $rangle \ rng[X] = \{ y \in \mathbf{U} : \exists x \in \mathbf{U} \ \langle x, y \rangle \in X \} = \mathsf{dom}[X^{-1}]$

exist.

Axiom of Membership:
$$\exists E \ \forall x \in \mathbf{U} \ \forall y \in \mathbf{U} \ (\langle x, y \rangle \in E \iff x \in y)$$

The Axioms of Membership, Difference and Product imply the existence of the class

$$\mathbf{E} = \{ \langle x, y \rangle \in \ddot{\mathbf{U}} : x \in y \}.$$

Exercise 5.7. Show that dom[E] = U.

Exercise 5.8. Find rng[E].

Now we can define the union and intersection of sets that belong to a given class of sets. Namely, for a class X consider its

- $union \bigcup X = \{z : \exists y \in X (z \in y)\},\$
- intersection $\bigcap X = \{z : \forall y \in X \ (z \in y)\}$, and
- power-class $\mathcal{P}(X) = \{y : y \subseteq X\}.$

Exercise 5.9. To show that the classes $\bigcup X, \bigcap X$ and $\mathcal{P}(X)$ exist, check that

$$\bigcup X = \mathsf{dom}[\mathbf{E} \cap (\mathbf{U} \times X)]$$

$$\bigcap X = \mathbf{U} \setminus \mathsf{dom}[(\ddot{\mathbf{U}} \setminus \mathbf{E}) \cap (\mathbf{U} \times X)], \text{ and }$$

$$\mathcal{P}(X) = \mathbf{U} \setminus \mathsf{dom}[\mathbf{E}^{-1} \cap [\mathbf{U} \times (\mathbf{U} \setminus X)].$$

Exercise 5.10. Prove that $\mathcal{P}(\mathbf{U}) = \mathbf{U}$.

For every relation R and class X, the Axioms of Product, Inversion and Domain allow us to define the class

$$R[X] = \{y: \exists x \in X \; (\langle x,y \rangle \in R)\} = \operatorname{rng}[R \cap (X \times \mathbf{U})]$$

called the *image* of the class X under the relation R. The class $R^{-1}[X]$ is called the *preimage* of X under the relation R.

Observe that a class R is a relation if and only if $R = (R^{-1})^{-1}$.

For a relation R let $R^{\pm} = R \cup R^{-1}$.

The class $dom[R^{\pm}] = rng[R^{\pm}]$ is called the *underlying class* of the relation R.

Definition 5.11. A relation F is called a function if for any ordered pairs $\langle x, y \rangle, \langle x', y' \rangle \in F$ the equality x = x' implies y = y'.

Therefore, for any function F and any $x \in \mathsf{dom}[F]$ there exists a unique set y such that $\langle x,y \rangle \in F$. This unique set y is called the *image* of x under the function F and is denoted by F(x). The round parentheses are used to distinguish the set F(x) from the image $F[x] = \{y : \exists z \in x \ \langle z,y \rangle \in F\}$ of the set x under the function F.

A function F is called *injective* if the relation F^{-1} also is a function.

If for some classes X, Y a function F has $\mathsf{dom}[F] = X$ and $\mathsf{rng}[F] \subseteq Y$, then we write $F: X \to Y$ and say that F is a function from X to Y. A function $F: X \to Y$ is called surjective if $\mathsf{rng}[F] = Y$, and $F: X \to Y$ is bijective if F is surjective and injective.

For a function F and a class X the function $F \cap (X \times \mathbf{U})$ is called the *restriction* of F to X and is denoted by $F \upharpoonright_X$.

The final axiom of existence of classes is

Axiom of Cycle:
$$\forall X \ \exists Y \ \forall u \in \mathbf{U} \ \forall v \in \mathbf{U} \ (\langle u, v, w \rangle \in X \iff \langle w, u, v \rangle \in Y)$$

The Axiom of Cycle implies that for every class X the classes

$$X^{\circlearrowleft} = \{\langle z, x, y \rangle : \langle x, y, z \rangle \in X\}$$
 and $X^{\circlearrowleft} = \{\langle y, z, x \rangle : \langle x, y, z \rangle \in X\}$

exist.

For two relations F, G their composition $G \circ F$ is the relation

$$G \circ F = \{ \langle x, z \rangle : \exists y \in \mathbf{U} \ (\langle x, y \rangle \in F \ \land \ \langle y, z \rangle \in G) \}.$$

The class $G \circ F$ exists since $G \circ F = \mathsf{dom}[T]$ where

$$T = \{ \langle x, z, y \rangle : \langle x, y \rangle \in F \ \land \ \langle y, z \rangle \in G \} = [F^{-1} \times \mathbf{U}]^{\circlearrowleft} \cap [G^{-1} \times \mathbf{U}]^{\circlearrowleft}.$$

Exercise 5.12. Prove that for any functions F, G the relation $G \circ F$ is a function.

The next three axioms are called the axioms of existence of sets.

Axiom of Replacement: For every function F and set x, the class F[x] is a set

Axiom of Union: For every set x, the class $\bigcup x = \{z : \exists y \in x \ (z \in y)\}$ is a set

Axiom of Power-set: For every set x, the class $\mathcal{P}(x) = \{y : y \subseteq x\}$ is a set

Exercise 5.13. Write the Axioms of Replacement, Union and Power-set as formulas.

Exercise 5.14. Show that for any sets x, y the class $x \cup y$ is a set.

Hint: Use the Axioms of Pair and Union.

For a set x the set $x \cup \{x\}$ is called the *successor* of x. The set $x \cup \{x\}$ is equal to $\cup \{x, \{x\}\}$ and hence exists by the Axiom of Union.

At the moment no axiom guarantees that at least one set exists. This is done by

Axiom of Infinity:
$$\exists x \in \mathbf{U} \ ((\emptyset \in x) \land \forall n \ (n \in x \Rightarrow n \cup \{n\} \in x))$$

A set x is called *inductive* if $(\emptyset \in x) \land \forall n \ (n \in x \Rightarrow n \cup \{n\} \in x)$. The Axiom of Infinity guarantees the existence of an inductive set. This axiom also implies that the empty class \emptyset is a set. So, it is legal to form sets corresponding to natural numbers:

$$0 = \emptyset$$
, $1 = \{0\}$, $2 = \{0, 1\}$, $3 = \{0, 1, 2\}$, $4 = \{0, 1, 2, 3\}$, $5 = \{0, 1, 2, 3, 4\}$, and so on.

Let **Ind** be the class of all inductive sets (the existence of the class **Ind** is established in Exercise 6.11). The intersection \bigcap **Ind** of all inductive sets is the smallest inductive set, which

is denoted by ω . Elements of the ω are called *natural numbers* or else *finite ordinals*. The set $\mathbb{N} = \omega \setminus \{\emptyset\}$ is the set of non-zero natural numbers.

The definition of the set ω as the smallest inductive set implies the well-known

Principle of Mathematical Induction: If a set X contains the empty set and for every $n \in X$ its successor $n \cup \{n\}$ belongs to X, then X contains all natural numbers.

A set x is called *finite* (resp. *countable*) if there exists a bijective function f such that dom[f] = x and $rng[f] \in \omega$ (resp. $rng[f] \in \omega \cup \{\omega\}$).

```
Axiom of Foundation: \forall x \in \mathbf{U} \ (x \neq \emptyset \Rightarrow \exists y \in x \ \forall z \in y \ (z \notin x))
```

The Axiom of Foundation says that each nonempty set x contains an element $y \in x$ such that $y \cap x = \emptyset$. This axiom forbids the existence of a set x such that $x \in x$. More generally, it forbids the existence of infinite sequences sets $x_1 \ni x_2 \ni x_3 \ni \ldots$

The final axiom is the

```
Axiom of Global Choice: \exists F ((F \text{ is a function}) \land \forall x \in \mathbf{U} (x \neq \emptyset \Rightarrow \exists y \in x (\langle x, y \rangle \in F)))
```

The Axiom of Global Choice postulates the existence of a function $F: \mathbf{U} \setminus \{\emptyset\} \to \mathbf{U}$ assigning to each nonempty set x some element F(x) of x.

The Axiom of Global Choice implies its weaker version, called the

```
Axiom of Choice: \forall x \in \mathbf{U} \exists f ((f \text{ is a function}) \land \forall y \in x (y \neq \emptyset \Rightarrow \exists z \in y (\langle y, z \rangle \in f)))
```

The Axiom of Choice says that for any set x there exists a function $f: x \setminus \{\emptyset\} \to \bigcup x$ assigning to every nonempty set $y \in x$ some element f(y) of y.

Therefore, the Classical Set Theory is based on 15 axioms.

Axioms of NBG

Exensionality: Two classes are equal if and only if they have the same elements.

Universe: The class $\mathbf{U} = \{x : \exists y \ (x \in y)\}\$ of all sets exists.

Difference: For any classes X, Y the class $X \setminus Y = \{x : x \in X \land x \notin Y\}$ exists.

Pair: For any sets x, y there exists the set $\{x, y\}$ exists.

Membership: The class $\mathbf{E} = \{\langle x, y \rangle : x \in y \in \mathbf{U}\}$ exists.

Product: For every classes X, Y the class $X \times Y = \{\langle x, y \rangle : x \in X, y \in Y\}$ exists.

Inversion: For every class X the class $X^{-1} = \{\langle y, x \rangle : \langle x, y \rangle \in X\}$ exists.

Domain: For every class X the class $\text{dom}[X] = \{x : \exists y \ (\langle x, y \rangle \in X\} \text{ exists.}$

Cycle: For every class X the class $X^{\circlearrowright} = \{\langle z, x, y \rangle : \langle x, y, z \rangle \in X\}$ exists.

Replacement: For every function F and set x the class $F[x] = \{F(y) : y \in x\}$ is a set.

Union: For every set x the class $\cup x = \{z : \exists y \in x \ (z \in y)\}$ is a set.

Power-set: For every set x the class $\mathcal{P}(x) = \{y : y \subseteq x\}$ is a set.

Infinity: There exists an inductive set.

Foundation: Every nonempty set x contains an element $y \in x$ such that $y \cap x = \emptyset$.

Global Choice: There is a function assigning to each nonempty set x some element of x.

6. Existence of Classes

In this section we prove the existence of some basic classes which will often appear in the remaining part of the textbook. Corresponding existence results are written as exercises with solutions (called hints). Nonetheless we strongly recommend the reader to try to do all exercises without looking at hints (and without use the Gödel's Theorem 7.2 on existence of classes).

Exercise 6.1. Prove that the class $\mathbf{S} = \{\langle x, y \rangle \in \ddot{\mathbf{U}} : x \subseteq y\}$ exists.

Hint: Observe that $\ddot{\mathbf{U}} \setminus \mathbf{S} = \mathsf{dom}[T]$ where

$$T = \{ \langle x, y, z \rangle \in \ddot{\mathbf{U}} : z \in x \land z \notin y \} = [\mathbf{E} \times \mathbf{U}]^{\circlearrowleft} \cap ((\ddot{\mathbf{U}} \setminus \mathbf{E}^{-1}) \times \mathbf{U})^{\circlearrowleft}.$$

Exercise 6.2. Prove that the identity function $\mathsf{Id} = \{\langle x, y \rangle \in \ddot{\mathbf{U}} : x = y\}$ exists.

Hint: Observe that $Id = S \cap S^{-1}$ where S is the class from Exercise 6.1.

Exercise 6.3. Prove that every subclass $Y \subseteq x$ of a set x is a set.

Hint: Observe that the identity function $F = \operatorname{Id} \cap (Y \times \mathbf{U})$ of Y exists and apply the Axiom of Reparement to conclude that the class $F[x] = Y \cap x = Y$ is a set.

Exercise 6.4. Prove that the universe U is a proper class.

Hint: Repeat the argument of Russell.

Exercise 6.5. Prove that the function dom: $\ddot{\mathbf{U}} \to \mathbf{U}$, dom: $\langle x, y \rangle \mapsto x$, exists.

Hint: Observe that dom = $\{\langle x, y, z \rangle \in \ddot{\mathbf{U}} : z = x\} = [\operatorname{Id} \times \mathbf{U}]^{\circlearrowleft}$

Exercise 6.6. Prove that the function rng: $\ddot{\mathbf{U}} \to \mathbf{U}$, rng: $\langle x, y \rangle \mapsto y$, exists.

Hint: Observe that $rng = \{\langle x, y, z \rangle \in \ddot{\mathbf{U}} : z = y\} = [Id \times \mathbf{U}]^{\circlearrowright}$.

Exercise 6.7. Prove that the function pair: $\ddot{\mathbf{U}} \to \mathbf{U}$, pair: $\langle x, y \rangle \mapsto \langle x, y \rangle$ exists.

Hint: Observe that $pair = \ddot{\mathbf{U}} \cap \mathsf{Id}$.

Exercise 6.8. Let R be a relation and F,G be functions. Prove that the class $F_RG = \{x \in \mathsf{dom}[F] \cap \mathsf{dom}[G] : \langle F(x), G(x) \rangle \in R\}$ exists.

Hint: Observe that $F_RG = \{x : \exists y \; \exists z \; (\langle x, y \rangle \in F \; \land \; \langle x, z \rangle \in G \; \land \; \langle y, z \rangle \in R)\} = \mathsf{dom}[\mathsf{dom}[T]],$ where

$$T = \{ \langle x, y, z \rangle \in \dddot{\mathbf{U}} : \langle x, y \rangle \in F \ \land \ \langle x, z \rangle \in G \ \land \ \langle y, z \rangle \in R \} = (F \times \mathbf{U}) \cap [G^{-1} \times \mathbf{U}]^{\circlearrowleft} \cap [R \times \mathbf{U}]^{\circlearrowright}.$$

Exercise 6.9. Prove that the function $Inv = \{\langle \langle x, y \rangle, \langle u, v \rangle \rangle \in \ddot{\mathbf{U}} \times \ddot{\mathbf{U}} : x = v \land y = u\}$ exists.

Hint: Observe that

 $\mathsf{Inv} = \{z \in \ddot{\mathbf{U}} \times \ddot{\mathbf{U}} : \mathsf{dom} \circ \mathsf{dom}(z) = \mathsf{rng} \circ \mathsf{rng}(z) \land \mathsf{rng} \circ \mathsf{dom}(z) = \mathsf{dom} \circ \mathsf{rng}(z)\} \text{ and apply Exercises 6.8 and 6.2.}$

Exercise 6.10. Prove that the function $Succ = \{\langle x, y \rangle \in \ddot{\mathbf{U}} : y = x \cup \{x\}\}$ exists.

Hint: Observe that $\ddot{\mathbf{U}} \setminus \mathsf{Succ} = \{ \langle x, y \rangle \in \ddot{\mathbf{U}} : \exists z \ \neg (z \in y \Leftrightarrow (z \in x \lor z = x)) \} = \mathsf{dom}[T],$ where $T = \{ \langle x, y, z \rangle \in \ddot{\mathbf{U}} : \neg (z \in y \Leftrightarrow (z \in x \lor z = x)) \} = T_1 \cup T_2 \cup T_3, \text{ and }$

$$T_1 = \{ \langle x, y, z \rangle \in \ddot{\mathbf{U}} : z \notin y \land z \in x \} = [(\ddot{\mathbf{U}} \setminus \mathbf{E}^{-1}) \times \mathbf{U}]^{\circlearrowleft} \cap (\mathbf{E} \times \mathbf{U});$$

$$T_2 = \{\langle x, y, z \rangle \in \ddot{\mathbf{U}} : z \notin y \ \land \ z = x\} = [(\ddot{\mathbf{U}} \setminus \mathbf{E}^{-1}) \times \mathbf{U}]^{\circlearrowright} \cap (\mathsf{Id} \times \ddot{\mathbf{U}});$$

$$T_3 = \{ \langle x, y, z \rangle \in \ddot{\mathbf{U}} : z \in y \ \land \ z \notin x \ \land \ z \neq x \} = [\mathbf{E}^{-1} \times \mathbf{U}]^{\circlearrowleft} \cap ((\ddot{\mathbf{U}} \setminus (\mathbf{E} \cup \mathsf{Id})) \times \mathbf{U}).$$

Exercise 6.11. The class Ind of all inductive sets exists.

Hint: Observe that $\mathbf{U} \setminus \mathbf{Ind} = \{x \in \mathbf{U} : \emptyset \notin x \lor (\exists y \in x \ (y \cup \{y\} \notin x))\} = \operatorname{rng}[(\{\emptyset\} \times \mathbf{U}) \setminus \mathbf{E}] \cup \operatorname{dom}[P], \text{ where } P = \{\langle x, y \rangle \in \ddot{\mathbf{U}} : (y \in x) \land (y \cup \{y\} \notin x)\} = \mathbf{E}^{-1} \cap P' \text{ and } P' = \{\langle x, y \rangle \in \ddot{\mathbf{U}} : y \cup \{y\} \notin x\} = \{\langle x, y \rangle \in \ddot{\mathbf{U}} : \exists z \ (z \notin x \land z = y \cup \{y\})\} = \operatorname{dom}[T \cap T'] \text{ where } T = \{\langle x, y, z \rangle : z \notin x\} = [(\ddot{\mathbf{U}} \setminus \mathbf{E}) \times \mathbf{U}]^{\circlearrowleft} \text{ and } T' = \{\langle x, y, z \rangle : z = y \cup \{y\}\} = [\operatorname{Succ} \times \mathbf{U}]^{\circlearrowleft}.$

7. GÖDEL'S THEOREM ON CLASS EXISTENCE

This section is devoted to a fundamental result of Gödel⁶ on the existence of the class $\{x:\varphi(x)\}$ for any **U**-bounded formula $\varphi(x)$ with one free variable x. It is formulated and proved in the metalanguage by induction on the complexity of a formula $\varphi(x)$. So it provides a scheme for proofs of concrete instances of the formula $\varphi(x)$, but some of them admit more simple and direct proofs, see (and solve) exercises throughout the book.

Definition 7.1. A formula φ of Set Theory is called **U**-bounded if each quantifier appearing in this formula is of the form $\forall x \in \mathbf{U}$ or $\exists x \in \mathbf{U}$, where x is a symbol of variable.

Restricting the domain of quantifiers to the class \mathbf{U} allows us to avoid Berry's paradox at forming classes by constructors.

For any natural number $n \geq 3$ define an ordered n-tuple $\langle x_1, \ldots, x_n \rangle$ of sets x_1, \ldots, x_n by the recursive formula: $\langle \langle x_1, \ldots, x_{n-1} \rangle, x_n \rangle$.

Theorem 7.2 (Gödel, 1940). Let $\varphi(x_1, \ldots, x_n, Y_1, \ldots, Y_m)$ be a **U**-bounded formula of Set Theory whose free variables belong to the list $x_1, \ldots, x_n, Y_1, \ldots, Y_m$. Then for any classes Y_1, \ldots, Y_m the class $\{\langle x_1, \ldots, x_n \rangle \in \mathbf{U}^n : \varphi(x_1, \ldots, x_n, Y_1, \ldots, Y_m)\}$ exists.

In this theorem the *n*-th power \mathbf{U}^n of the universe \mathbf{U} is defined inductively: $\mathbf{U}^1 = \mathbf{U}$ and $\mathbf{U}^{n+1} = \mathbf{U}^n \times \mathbf{U}$ for a natural number n. Here n is not an element of the set ω given by the Axiom of Infinity, but a natural number of our metalanguage used for description of formulas of the Classical Set Theory. So, the definition of \mathbf{U}^n is in fact, a metadefinition. Using the Axioms of Universe and Product, we can prove inductively that for every $n \in \mathbb{N}$ the class \mathbf{U}^n exists.

Theorem 7.2 is proved by induction on the complexity of the formula φ .

If the formula φ is atomic, then it is equal to one of the following atomic formulas:

$$x_i \in x_j, \ x_i = x_j, \ x_i \in Y_j, \ x_i = Y_j, \ Y_i \in x_j, \ Y_i = x_j, \ Y_i \in Y_j, \ Y_i = Y_j.$$

These cases are treated separately in the following lemmas.

Lemma 7.3. For every natural number n and positive numbers $i, j \leq n$, the classes

$$\{\langle x_1,\ldots,x_n\rangle\in\mathbf{U}^n:x_i\in x_j\}$$
 and $\{\langle x_1,\ldots,x_n\rangle\in\mathbf{U}^n:x_i=x_j\}$

exist.

Proof. Consider the functions

$$\mathsf{dom}: \mathbf{U}^2 \to \mathbf{U}, \; \mathsf{dom}: \langle x,y \rangle \mapsto x, \quad \text{and} \quad \mathsf{rng}: \mathbf{U}^2 \to \mathbf{U}, \; \mathsf{rng}: \langle x,y \rangle \mapsto y,$$

whose existence was established in Exercise 6.5 and 6.6.

Let $\mathsf{dom}^0 = \mathbf{Id}$ and $\mathsf{dom}^{n+1} = \mathsf{dom} \circ \mathsf{dom}^n$ for every natural number n (from the metalanguage). For every natural number n, the function dom^n assigns to any (n+1)-tuple $\langle x_1, \ldots, x_{n+1} \rangle$ its first element x_1 .

⁶Task: Read about Gödel in Wikipedia.

We can prove inductively that for every natural number n the function $dom^n : \mathbf{U}^{n+1} \to \mathbf{U}$ exists.

Now observe that for any numbers $i \leq n$ the function

$$\operatorname{Pr}_{i}^{n}: \mathbf{U}^{n} \to \mathbf{U}, \operatorname{Pr}_{i}^{n}: \langle x_{1}, \dots, x_{n} \rangle \to x_{i},$$

exists being equal to the composition $\operatorname{rng} \circ \operatorname{dom}^{n-i}|_{\mathbf{U}^n}$.

By the Axiom of Memberships and Exercise 6.2, the classes $\mathbf{E} = \{\langle x, y \rangle \in \ddot{\mathbf{U}} : x \in y\}$ and $\mathbf{Id} = \{\langle x, y \rangle \in \ddot{\mathbf{U}} : x = y\}$ exist.

Observing that for every non-zero natural numbers $i, j \leq n$

$$\{\langle x_1,\ldots,x_n\rangle:x_i\in x_j\}=\{z\in\mathbf{U}^n:\langle\mathsf{Pr}_i^n(z),\mathsf{Pr}_i^n(z)\rangle\in\mathbf{E}\}$$

and

$$\{\langle x_1,\ldots,x_n\rangle:x_i=x_j\}=\{z\in\mathbf{U}^n:\langle\mathsf{Pr}_i^n(z),\mathsf{Pr}_j^n(z)\rangle\in\mathbf{Id}\}$$

and applying Exercise 6.8, we conclude that these two classes exist.

Lemma 7.4. For every non-zero natural numbers $i \leq n$ and class Y the classes

$$\{\langle x_1, \dots, x_n \rangle : x_i \in Y\}$$
 and $\{\langle x_1, \dots, x_n \rangle : Y \in x_i\}$

exist.

Proof. Observing that

$$\{\langle x_1,\ldots,x_n\rangle:x_i\in Y\}=\{z\in\mathbf{U}^n:\exists y\in Y\;(\langle z,y\rangle\in\mathsf{Pr}_i^n)\}=\mathsf{dom}[(\mathbf{U}^n\times Y)\cap\mathsf{Pr}_i^n],$$

we see that the class $\{\langle x_1, \ldots, x_n \rangle : x_i \in Y\}$ exists by the Axioms of Universe, Product, Difference, and Domain.

If Y is a proper class, then the class $\{\langle x_1, \ldots, x_n \rangle : Y \in x_i\}$ is empty and hence exists by the Axioms of Universe and Difference.

If Y is a set, then we can consider the function $G = \mathbf{U} \times \{Y\}$ and conclude that the class $\{\langle x_1, \ldots, x_n \rangle : Y \in x_i\} = \{z \in \mathbf{U}^n : \langle G(z), \mathsf{Pr}_i^n(z) \rangle \in \mathbf{E}\}$ exists by Exercise 6.8.

By analogy we can prove the following lemma.

Lemma 7.5. For every non-zero natural numbers $i \leq n$ and class Y the classes

$$\{\langle x_1,\ldots,x_n\rangle:x_i=Y\}=\{\langle x_1,\ldots,x_n\rangle:Y=x_i\}$$

exist.

Lemma 7.6. For every classes Y, Z the classes

$$\{\langle x_1,\ldots,x_n\rangle:Y\in Z\},\ \{\langle x_1,\ldots,x_n\rangle:Y=Z\}$$

exist.

Proof. These classes are equal to \mathbf{U}^n or \emptyset and hence exist by the Axioms of Universe, Product and Difference.

By Lemmas 7.3–7.6, for any atomic formula φ with free variables in the list $x_1, \ldots, x_n, Y_1, \ldots, Y_m$ and any classes Y_1, \ldots, Y_m , the class $\{\langle x_1, \ldots, x_n \rangle : \varphi(x_1, \ldots, x_n)\}$ exists. Observe that each atomic formula has exactly 5 symbols (two variable, one relation and two parentheses).

Assume that for some natural number $k \geq 6$, Theorem 7.2 have been proved for all formulas φ of containing < k symbols. Let φ be a formula consisting of exactly k symbols. We also

assume that the free variables of the formula φ are contained in the list $x_1, \ldots, x_n, Y_1, \ldots, Y_m$. Since the formula φ is not atomic, there exist formulas ϕ, ψ such that φ is equal to one of the following formulas:

- 1) $(\neg \phi)$;
- $(\phi \wedge \psi);$
- 3) $(\phi \lor \psi)$;
- 4) $(\phi \Rightarrow \psi)$;
- 5) $(\phi \Leftrightarrow \psi)$;
- 6) $(\exists x \ \phi);$
- 7) $(\forall x \phi)$.

First assume the case 1. In this case the formula ϕ consists of k-3 < k symbols and has $\mathsf{Free}(\phi) = \mathsf{Free}(\varphi) \subseteq \{x_1, \ldots, x_n, Y_1, \ldots, Y_m\}$. Applying the inductive assumption, we conclude that for any classes Y_1, \ldots, Y_m the class

$$\Phi = \{ \langle x_1, \dots, x_n \rangle \in \mathbf{U}^n : \phi(x_1, \dots, x_n, Y_1, \dots, Y_m) \}$$

exists. Then the class

$$\{\langle x_1,\ldots,x_n\rangle\in\mathbf{U}^n:\varphi(x_1,\ldots,x_n,Y_1,\ldots,Y_m)\}=\mathbf{U}^n\setminus\Phi$$

exists by the Axiom of Difference.

Next, assume the case 2. In this case, φ is equal to the formula $\phi \wedge \psi$ and the formulas ϕ, ψ consist of $\langle k \rangle$ symbols and have $\mathsf{Free}(\phi) \cup \mathsf{Free}(\psi) = \mathsf{Free}(\varphi) \subseteq \{x_1, \ldots, x_n, Y_1, \ldots, Y_m\}$. Applying the inductive assumption, we conclude that for any classes Y_1, \ldots, Y_m the classes

$$\Phi = \{\langle x_1, \dots, x_n \rangle \in \mathbf{U}^n : \phi(x_1, \dots, x_n, Y_1, \dots, Y_m)\}$$

and

$$\Psi = \{\langle x_1, \dots, x_n \rangle \in \mathbf{U}^n : \psi(x_1, \dots, x_n, Y_1, \dots, Y_m)\}$$

exist. Then the class

$$\{\langle x_1,\ldots,x_n\rangle\in\mathbf{U}^n:\varphi(x_1,\ldots,x_n,Y_1,\ldots,Y_m)\}=\Phi\cap\Psi=\Phi\setminus(\Phi\setminus\Psi)$$

exists by the Axiom of Difference.

By analogy we can treat the cases (3)–(5)

Next, assume that φ is equal to the formula $\exists x \ \phi$ or $\forall x \ \phi$. If $x \in \{x_1, \dots, x_n, Y_1, \dots, Y_m\}$, then we can replace all free occurrences of the symbol x in the formula ϕ by some other symbol and assume that $x \notin \{x_1, \dots, x_n, Y_1, \dots, Y_m\}$. Then the formula ϕ has all its free variables in the list $x_1, \dots, x_n, x, Y_1, \dots, Y_n$. By the inductive assumption, the class

$$\Phi = \{\langle x_1, \dots, x_n, x \rangle \in \mathbf{U}^{n+1} \rangle : \phi(x_1, \dots, x_n, x, Y_1, \dots, Y_m) \}$$

exists. If φ is equal to the formula $\exists x \ \phi$, then the class

$$\{\langle x_1,\ldots,x_n\rangle\in\mathbf{U}^n:\varphi(x_1,\ldots,x_n,Y_1,\ldots,Y_m)\}$$

is equal to the class $dom[\Phi]$, which exists by the Axiom of Domain.

If φ is the formula $\forall x \ \phi$, then the class

$$\{\langle x_1,\ldots,x_n\rangle\in\mathbf{U}^n:\varphi(x_1,\ldots,x_n,Y_1,\ldots,Y_m)\}$$

is equal to the class $\mathbf{U}^n \setminus \mathsf{dom}[\mathbf{U}^{n+1} \setminus \Phi]$, which exists by the Axioms of Universe, Product, Difference and Domain. This completes the inductive step and also completes the proof of the theorem.

Exercise* 7.7. Find a formula $\varphi(x)$ for which the existence of the class $\{x:\varphi(x)\}$ cannot be proved.

8. Axiomatic Set Theory of Zermelo-Fraenkel

The Set Theory of Zermelo–Fraenkel (briefly ZF) is a part of the theory NBG, which speaks only about sets and identifies classes with formulas (which are used for defining those classes). The undefined notions of Zermelo–Fraenkel Set Theory are the notions of set and membership. The language of ZF theory the same as the language of NBG theory.

Since the classes formally do not exist in ZF, more axioms are necessary to ensure the existence of sufficiently many of sets. So, the list of ZF axioms is infinite. It includes two axiom schemas: of separation and replacement. The axiom schema of separation substitutes seven axioms of existence of classes and sets in NBG and the axiom schema of replacement is a substitute for the single axiom of replacement in the NBG axiom system.

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Axioms of Zermelo-Fraenkel:
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Axiom of Extensionality: $\forall x \ \forall y \ (x = y \Leftrightarrow \forall z \ (z \in x \Leftrightarrow z \in y))$

Axiom of Pair: $\forall x \ \forall y \ \exists z \ \forall u \ (u \in z \iff (u = x \lor u = y))$

Axiom of Union: $\forall x \; \exists y \; \forall z \; (z \in y \; \Leftrightarrow \; \exists u \; (z \in u \; \land \; u \in x))$

Axiom of Power-set: $\forall x \exists y \ \forall z \ (z \in y \Leftrightarrow \forall u \ (u \in z \Rightarrow u \in x))$

Axiom of Empty set: $\exists \emptyset \ \forall x \ (x \notin \emptyset)$

Axiom of Infinity: $\exists x \ (\emptyset \in x \land \forall n \ (n \in x \Rightarrow n \cup \{n\} \in x))$

Axiom of Foundation: $\forall x \ (\exists y \ (y \in x) \Rightarrow \exists z \ (z \in x \land \forall u \ (u \in x \Rightarrow u \notin z)))$

Axiom Schema of Separation: Let φ be a formula whose free variables are in the list

 x, z, c_1, \ldots, c_m and y is not free for φ . Then

 $\forall x \ \forall c_1 \dots \forall c_m \ \exists y \ \forall z \ (z \in y \ \Leftrightarrow \ (z \in x \ \land \ \varphi(z, c_1, \dots, c_m)))$

Axiom Schema of Replacement: Let φ be a formula whose free variables are in the list

 $x, u, v, c_1, \ldots, c_m$ and y is not free for φ . Then

 $\forall x \, \forall c_1 \cdots \forall c_m \, ((\forall u \in x \, \exists! v \, \varphi(u, v, c_1, \dots, c_m)) \Rightarrow \exists y \, \forall v \, (v \in y \, \Leftrightarrow \, \exists u \, (u \in x \, \land \, \varphi(u, v, c_1, \dots, c_m))))$

The axioms ZF with added Axiom of Choice form the axioms ZFC.

Replacing the quantifiers $\forall x$ and $\exists x$ in the axioms ZF by bounded quantifiers $\forall x \in \mathbf{U}$ and $\exists x \in \mathbf{U}$, we can see that obtained statements are theorems of NBG. This means that \mathbf{U} is a model of ZFC within NBG. So, consistency of NBG implies the consistency of ZFC. The converse is also true: the consistency of ZFC implies the consistency of NBG. So these two theories are equiconsistent. Moreover NBG is a conservative extension of ZFC, which means that a \mathbf{U} -bounded formula without free varable is a theorem of ZFC if and only if it is a theorem of NBG. This important fact was proved by Shoenfield, see [5, p.70]. Therefore, if we are interested only in sets, there is no difference (except aesthetic) which theory to use. On the other hand, NBG has essential advantages: it has finite list of axioms and allows to work freely with classes.

This is a reason why we have chosen NBG for presentation of Set Theory. Since NBG deals with sets and classes and it was created by the classics of Set Theory and Logic (von Neumann, Robinson, Bernays, Gödel) we refer to this theory as the Classical Set Theory (shortly, CST). From now on we accept the following list of axioms, called the *Axioms of Classical Set Theory*.

Axioms of Classical Set Theory

Exensionality: Two classes are equal if and only if they have the same elements.

Pair: For any sets x, y there exists the set $\{x, y\}$ exists.

Universe: The class $\mathbf{U} = \{x : \exists y \ (x \in y)\}\$ of all sets exists.

Membership: The class $\mathbf{E} = \{\langle x, y \rangle : x \in y \in \mathbf{U}\}$ exists.

Difference: For any classes X, Y the class $X \setminus Y = \{x : x \in X \land x \notin Y\}$ exists. **Product:** For every classes X, Y the class $X \times Y = \{\langle x, y \rangle : x \in X, y \in Y\}$ exists.

Domain: For every class X the class $\text{dom}[X] = \{x : \exists y \ (\langle x, y \rangle \in X\} \text{ exists.}$ **Inversion:** For every class X the class $X^{-1} = \{\langle y, x \rangle : \langle x, y \rangle \in X\}$ exists. **Cycle:** For every class X the class $X^{\circlearrowright} = \{\langle z, x, y \rangle : \langle x, y, z \rangle \in X\}$ exists.

Replacement: For every function F and set x the class $F[x] = \{F(y) : y \in x\}$ is a set.

Union: For every set x the class $\bigcup x = \{z : \exists y \in x \ (z \in y)\}$ is a set. **Power-set:** For every set x the class $\mathcal{P}(x) = \{y : y \subseteq x\}$ is a set.

Infinity: There exists an inductive set.

Whenever necessary, we will add to this list the **Axiom of Foundation** or the **Axiom of (Global) Choice**, which will be specially acknowledged.

Part 3. Fundamental Constructions

In this section we survey some fundamental constructions of the Classical Set Theory, which often appear in other areas of Mathematics: classes of relations, functions, indexed families of classes, Cartesian products, equivalence relations. Often we shall formulate the corresponding existence theorems as exercises with solutions or hints. We recall that $\ddot{\mathbf{U}} = \mathbf{U} \times \mathbf{U}$ and $\ddot{\mathbf{U}} = \ddot{\mathbf{U}} \times \mathbf{U}$.

9. Cartesian products of sets

By the Axiom of Product, for two classes X, Y the class

$$X \times Y = \{ \langle x, y \rangle : x \in X \ \land \ y \in Y \}$$

exists.

Exercise 9.1. Prove that for any sets X, Y, the class $X \times Y$ is a set.

Hint: Observe that $X \times Y \subseteq \mathcal{P}(\mathcal{P}(X \cup Y))$.

Exercise 9.2. Prove that for any sets X, Y, the class $X \times Y$ is a set, not using the Axiom of Power-Set.

Hint: Apply the Axioms of Replacement and Union.

10. Relations

We recall that a relation is a subclass of the class $\mathbf{U} \times \mathbf{U} = \ddot{\mathbf{U}}$.

Exercise 10.1. Prove that the class of relations $\mathbf{Rel} = \{r \in \mathbf{U} : r \text{ is a relation}\}\$ exists.

Hint: Observe that
$$\mathbf{U} \setminus \mathbf{Rel} = \{x \in \mathbf{U} : \exists y \in x \ (y \notin \ddot{\mathbf{U}})\} = \mathsf{dom}[P]$$
 where $P = \{\langle x, y \rangle \in \ddot{\mathbf{U}} : y \in x \ \land \ y \notin \ddot{\mathbf{U}}\} = \mathbf{E}^{-1} \cap (\mathbf{U} \times (\mathbf{U} \setminus \ddot{\mathbf{U}})).$

Exercise 10.2. Prove that for any class X the classes $\{r \in \mathbf{Rel} : \mathsf{dom}[r] \subseteq X\}$ and $\{r \in \mathbf{Rel} : \mathsf{rng}[r] \subseteq X\}$ exist.

Hint: Observe that $\mathbf{Rel} \setminus \{r \in \mathbf{Rel} : \mathsf{dom}[r] \not\subseteq X\} = \{r \in \mathbf{Rel} : \exists \langle x, y \rangle \in r \cap ((\mathbf{U} \setminus X) \times \mathbf{U})\} = \mathsf{rng}[((\mathbf{U} \setminus X) \times \mathbf{U}) \times \mathbf{Rel}) \cap \mathbf{E}].$

Exercise 10.3. Prove that for any class X the classes $\{r \in \mathbf{Rel} : X \subseteq \mathsf{dom}[r]\}$ and $\{r \in \mathbf{Rel} : X \subseteq \mathsf{rng}[r]\}$ exist.

Hint: These classes are empty if X is a proper class.

Exercise 10.4. Prove that for any class X the classes $\{r \in \mathbf{Rel} : \mathsf{dom}[r] = X\}$ and $\{r \in \mathbf{Rel} : \mathsf{rng}[r] = X\}$ exist.

For a relation R and a class X denote by $R \upharpoonright X$ the relation $R \cap (X \times X)$. The relation $R \upharpoonright X$ is called the *restriction* of the relation R to the class X. If R is a function with $R[X] \subseteq X$, then $R \upharpoonright X = R \upharpoonright_X$, where $R \upharpoonright_X = R \cap (X \times \mathbf{U})$.

11. Functions

We recall that a relation F is a function if

$$\forall x \in \mathbf{U} \ \forall y \in \mathbf{U} \ \forall z \in \mathbf{U} \ ((\langle x,y \rangle \in F \ \land \ \langle x,z \rangle \in F) \ \Rightarrow \ (y=z)).$$

Therefore, for any function F and any $x \in \mathsf{dom}[F]$ there exists a unique set y such that $\langle x, y \rangle \in F$. This unique set y is denoted by F(x) and called the value of the function F at x. The round parentheses are used to distinguish the element F(x) from the set

$$F[x] = \{y : \exists z \in x \ (\langle z, y \rangle \in F)\} = \{F(y) : y \in x\}.$$

Given a function F and two classes X, Y, we write $F : X \to Y$ and say that F is a function from X to Y if $\mathsf{dom}[F] = X$ and $\mathsf{rng}[F] \subseteq Y$. Often we shall use the notation

$$F: X \to Y, F: x \mapsto F(x),$$

indicating that F assigns to each element $x \in X$ some element F(x) of Y.

For any function $F: X \to Y$ and class A, the function $F \cap (A \times \mathbf{U})$ is denoted by $F \upharpoonright_A$ and is called the *restriction* of F to the class A. If $F[A] \subseteq A$, then $F \upharpoonright_A = F \upharpoonright A$ where $F \upharpoonright_A = F \cap (A \times A)$ is the restriction of the relation F.

Exercise 11.1. Prove that a function F is a set if and only if its domain dom[F] is a set if and only if dom[F] and rng[F] are sets.

A function $F: X \to Y$ is called

- surjective if rng[F] = Y;
- injective if F^{-1} is a function;
- \bullet bijective if F is surjective and injective.

Exercise 11.2. Prove that the class $\mathbf{Fun} = \{f \in \mathbf{Rel} : f \text{ is a function}\}\$ of all functions exists.

For two classes A, X denote by X^A the class of all functions f such that dom[f] = A and $rng[f] \subseteq X$. Therefore,

$$X^A = \{ f \in \mathbf{Fun} : \mathsf{dom}[f] = A, \ \mathsf{rng}[f] \subseteq X \}.$$

Exercise 11.3. Prove that for every class A the class U^A exists.

Hint: Observe that $\mathbf{U}^A = \mathbf{Fun} \cap \{f \in \mathbf{Rel} : \mathsf{dom}[f] = A\}$ and apply Exercise 10.4.

Exercise 11.4. Prove that for every classes A, X the class X^A exists.

Hint: Observe that $X^A = \mathbf{U}^A \cap \{f \in \mathbf{Rel} : \mathsf{rng}[f] \subseteq X\}$ and apply Exercise 10.2.

Exercise 11.5. Prove that for classes A, X the class X^A is empty if and only if one of the following holds:

- (1) A is a proper class;
- (2) $X = \emptyset$ and $A \neq \emptyset$.

Exercise 11.6. Prove that for any sets X, A the class X^A is a set.

Hint: Observe that $X^A \subseteq \mathcal{P}(A \times X)$ and apply Exercises 9.1, 6.3 and the Axiom of Power-Set.

We recall that $0 = \emptyset$, $1 = \{0\}$, $2 = \{0, 1\}$ and $3 = 2 \cup \{2\}$.

Exercise 11.7. Prove that the function $F_1: \mathbf{U} \to \mathbf{U}^1$, $F_1: x \mapsto \{\langle 0, x \rangle\}$ exists and is bijective.

Exercise 11.8. Prove that the function $F_2: \ddot{\mathbf{U}} \to \mathbf{U}^2$, $F_2: \langle x, y \rangle \mapsto \{\langle 0, x \rangle, \langle 1, y \rangle\}$, exists and is bijective.

Exercise 11.9. Prove that the function $F_3: \ddot{\mathbf{U}} \to \mathbf{U}^3$, $F_3: \langle x, y, z \rangle \mapsto \{\langle 0, x \rangle, \langle 1, y \rangle, \langle 2, z \rangle\}$ exists and is bijective.

12. Indexed families of classes

In spite of the fact that in the Classical Set Theory proper classes cannot be elements of other classes, we can legally speak about indexed families of classes. Namely, for any class A, any subclass $X \subseteq A \times \mathbf{U}$ can be identified with the indexed family $(X_{\alpha})_{\alpha \in A}$ of the classes $X_{\alpha} = X[\{\alpha\}]$, where $X[\{\alpha\}] = \operatorname{rng}[X \cap (\{\alpha\} \times \mathbf{U})]$ for any index $\alpha \in A$.

In this case we can define the union $\bigcup_{\alpha \in A} X_{\alpha}$ as the class $\operatorname{rng}[X]$ and the intersection $\bigcap_{\alpha \in A} X_{\alpha}$ as the class $\mathbf{U} \setminus \operatorname{rng}[\ddot{\mathbf{U}} \setminus X]$.

The indexed family $(X_{\alpha})_{\alpha \in A}$ can be also though as a multifunction $X : A \multimap \mathbf{U}$ assigning to each element $\alpha \in A$ the class $X[\{\alpha\}]$, and to each subclass $B \subseteq A$ the class $X[B] = \operatorname{rng}[X \cap (B \times \mathbf{U})]$.

If for every $\alpha \in A$ the class X_{α} is a set, then $(X_{\alpha})_{\alpha \in A}$ is an indexed family of sets and we can consider the function $X_A^{\bullet}: A \to \mathbf{U}$, assigning to each $\alpha \in A$ the set X_{α} .

The following theorem shows that the function X_A^{\bullet} exists.

Theorem 12.1. Let A, X be two classes such that for every $\alpha \in A$ the class $X_{\alpha} = X[\{\alpha\}]$ is a set. Then the function $X_{A}^{\bullet}: A \to \mathbf{U}, X_{A}^{\bullet}: \alpha \mapsto X_{\alpha}$, exists.

Proof. Observe that

$$X_A^{\bullet} = \{ \langle \alpha, y \rangle \in A \times \mathbf{U} : y = X_{\alpha} \} = \{ \langle \alpha, y \rangle \in A \times \mathbf{U} : \forall z \ (z \in y \iff z \in X_{\alpha}) \} = \{ \langle \alpha, y \rangle \in A \times \mathbf{U} : \forall z \ (z \in y \iff \langle \alpha, z \rangle \in X) \}$$

and

$$(A \times \mathbf{U}) \setminus X_A^{\bullet} = \{ \langle \alpha, y \rangle \in A \times \mathbf{U} : \exists z \ \neg (z \in y \iff \langle \alpha, z \rangle \in X) \} = \mathsf{dom}[T \cup T'],$$

where

$$T = \{ \langle \alpha, y, z \rangle \in (A \times \mathbf{U}) \times \mathbf{U} : z \notin y \ \land \ \langle \alpha, z \rangle \in X \} = [(\ddot{\mathbf{U}} \setminus \mathbf{E}^{-1}) \times A]^{\circlearrowleft} \cap [X^{-1} \times \mathbf{U}]^{\circlearrowleft}$$
 and

$$T' = \{ \langle \alpha, y, z \rangle \in (A \times \mathbf{U}) \times \mathbf{U} : z \in y \ \land \ \langle \alpha, z \rangle \not \in X \} = [\mathbf{E}^{-1} \times A]^{\circlearrowright} \cap [(\ddot{\mathbf{U}} \setminus X^{-1}) \times \mathbf{U}]^{\circlearrowleft}.$$

Now we see that the axioms of the Classical Set Theory guarantee the existence of the considered classes including the function X_A^{\bullet} .

If $X = (X_{\alpha})_{\alpha \in A}$ is an indexed family of sets, then the class $\{X_{\alpha} : \alpha \in A\}$ is equal to $X_A^{\bullet}[A]$ and hence exists. This justifies the use of the constructor $\{x_{\alpha} : \alpha \in A\}$ in our theory.

If A is a set, then the class $\{X_{\alpha} : \alpha \in A\} = X_A^{\bullet}[A]$ is a set by the Axiom of Replacement. In particular, each set x is equal to the set $\{y : y \in x\}$.

Exercise 12.2. Let A be a class and X be a subclass of $A \times \mathbf{U}$ thought as an indexed family $(X_{\alpha})_{\alpha \in A}$ of the classes $X_{\alpha} = X[\{\alpha\}]$. Observe that the class

$$B = \{ \alpha \in A : \alpha \notin X_{\alpha} \} = \{ \alpha \in A : \langle \alpha, \alpha \rangle \notin X \} = \mathsf{dom}[(\mathsf{Id} \cap (A \times A)) \setminus X]$$

exists. Repeating the argument of Russell's Paradox, prove that $B \neq X_{\alpha}$ for every $\alpha \in A$.

By a sequence of classes we understand a subclass $X \subseteq \omega \times \mathbf{U}$ identified with the indexed family of classes $(X_n)_{n \in \omega}$ where $X_n = \{x \in \mathbf{U} : \langle n, x \rangle \in X\}$. If each class X_n is a set, then the indexed family of sets $(X_n)_{n \in \omega}$ can be identified with the function $X_* : \omega \to \mathbf{U}$ assigning to each $n \in \omega$ the set X_n . By Theorem 12.1 such function exists.

For a natural number $n \in \mathbb{N}$ by an n-tuple of classes (X_0, \ldots, X_{n-1}) we understand the indexed family of classes $(X_i)_{i \in n}$. A pair of classes (X, Y) is identified with the 2-tuple $(X_i)_{i \in 2}$ such that $X_0 = X$ and $X_1 = Y$. By analogy we can introduce a triple of classes, a quarduple of classes, and so on.

Therefore, for any sets x, y we have three different notions related to pairs:

- (i) the unordered pair of sets $\{x, y\}$
- (ii) the ordered pair of sets $\langle x, y \rangle = \{\{x\}, \{x, y\}\},\$
- (iii) the pair of classes $(x,y) = (\{0\} \times x) \cup (\{1\} \times y) = \{\langle 0,u \rangle : u \in x\} \cup \{\langle 1,v \rangle : v \in y\}.$

The definition of a pair of classes uses ordered pairs of sets and the definition of an ordered pair of sets is based on the notion of an unordered pair of sets (which exists by the Axiom of Pair).

The following exercise shows that the notion of a pairs of classes has the characteristic property of an ordered pair.

Exercise 12.3. Prove that for any classes A, B, X, Y we have the equivalence

$$(A, B) = (X, Y) \Leftrightarrow (A = X \land B = Y).$$

13. Cartesian products of classes

In this section we define the Cartesian product $\prod_{\alpha \in A} X_{\alpha}$ of an indexed family of classes $X = (X_{\alpha})_{\alpha \in A}$. By definition, the class $\prod_{\alpha \in A} X_{\alpha}$ consists of all functions f such that dom[f] = A and $f(\alpha) \in X_{\alpha}$ for every $\alpha \in A$. Equivalently, the Cartesian product can be defined as the class

$$\prod_{\alpha \in A} X_{\alpha} = \{ f \in \mathbf{Fun} : (f \subseteq X) \ \land \ (\mathsf{dom}[f] = A) \}.$$

Proposition 13.1. For any class A and a subclass $X \subseteq A \times \mathbf{U}$, the Cartesian product $\prod_{\alpha \in A} X_{\alpha}$ of the indexed family $X = (X_{\alpha})_{\alpha \in A}$ exists. If A is a proper class, then $\prod_{\alpha \in A} X_{\alpha}$ is the empty class.

Proof. The class $\operatorname{rng}[X] = \bigcup_{\alpha \in A} X_{\alpha}$ exists by the Axioms of Domain and Inversion. By Exercise 11.4, the class $(\operatorname{rng}[X])^A$ of functions from A to $\operatorname{rng}[X]$ exists. Observe that

$$\prod_{\alpha \in A} X_\alpha = \{f \in (\operatorname{rng}[X])^A : f \subseteq X\}$$

and hence

$$(\operatorname{rng}[X])^A \setminus \prod_{\alpha \in A} X_\alpha = \{ f \in (\operatorname{rng}[X])^A : \exists z \ (z \in f \land z \notin X \} = \operatorname{dom}[((\operatorname{rng}[X])^A \times (\mathbf{U} \setminus X)) \cap \mathbf{E}^{-1}].$$

Now the axioms of the Classical Set Theory ensure that the class $\prod_{\alpha \in A} X_{\alpha}$ exists. If this class is not empty, then it contains some function f with dom[f] = A. Applying the Axiom of Replacement to the function dom, we conclude that the class A = dom[f] is a set.

Exercise 13.2. Show that for a set A and an indexed family of sets $X = (X_{\alpha})_{{\alpha} \in A}$ the Cartesian product $\prod_{{\alpha} \in A} X_{\alpha}$ is a set.

Hint: Since $\{X_{\alpha}\}_{{\alpha}\in A}$ is a set, its union $\bigcup_{{\alpha}\in A}X_{\alpha}=\operatorname{rng}[X]$ is a set by the Axiom of Union and then $\prod_{{\alpha}\in A}X_{\alpha}$ is a set, being a subclass of the set $(\operatorname{rng}[X])^A$, see Exercise 6.3.

Exercise 13.3. Let A be a class and $X \subseteq A \times \mathbf{U}$ be a class such that $\prod_{\alpha \in A} X_{\alpha}$ is not empty. Prove that the class A is a set and for every $\alpha \in A$ the class $X_{\alpha} = \{x : \langle \alpha, x \rangle \in X\}$ is not empty.

Exercise 13.3 motivates the following definition. For a class X by $\prod X$ we denote the Cartesian product $\prod_{\alpha \in \mathsf{dom}[X]} X[\{\alpha\}]$ of the indexed family of nonempty classes $(X[\{\alpha\}])_{\alpha \in \mathsf{dom}[X]}$.

Exercise 13.4. Prove that $\prod X = \{ f \in \mathbf{Fun} : (\mathsf{dom}[f] = \mathsf{dom}[X]) \land (f \subseteq X) \}.$

Exercise 13.5. Show that $X^A = \prod (A \times X)$ for any classes A, X.

Exercise 13.6. Show that $\prod \emptyset = \{\emptyset\}$ and hence $\prod \emptyset$ is not empty.

Exercise 13.7. Observe that the Axiom of Choice holds if and only if for any set X its Cartesian product $\prod X$ is not empty.

14. Reflexive and irreflexive relations

We recall that a relation is a class whose elements are ordered pairs of sets. For a relation R the class $dom[R^{\pm}]$ is called the *underlying class* of the relation. Here $R^{\pm} = R \cup R^{-1}$.

Definition 14.1. A relation R is called

- reflexive if $\mathbf{Id} \upharpoonright \mathsf{dom}[R^{\pm}] \subseteq R$;
- $irreflexive if R \cap \mathbf{Id} = \emptyset$.

Example 14.2. (1) The relation **Id** is reflexive.

(2) The relation $\mathbf{U} \setminus \mathbf{Id}$ is irreflexive.

Example 14.3. For any relation R the relation $R \setminus \mathbf{Id}$ is irreflexive and $R \cup \mathbf{Id} \upharpoonright \mathsf{dom}[R^{\pm}]$ is reflexive.

Exercise 14.4. Using the Axiom of Foundation, prove that the Membership relation **E** is irreflexive.

Exercise 14.5. Prove that the class $\{r \in \mathbf{Rel} : r \text{ is a reflexive relation}\}$ exists.

15. Equivalence relations

Definition 15.1. A relation R is called

- symmetric if $R = R^{-1}$;
- transitive if $\{\langle x, z \rangle \in \mathbf{U} : \exists y \in \mathbf{U} \ (\langle x, y \rangle \in R \ \land \ \langle y, z \rangle \in R)\} \subseteq R$;
- an equivalence relation if R is symmetric and transitive.

Usually equivalence relations are denoted by symbols =, \equiv , \approx , etc.

Example 15.2. The identity function $\mathbf{Id} = \{\langle x, x \rangle : x \in \mathbf{U}\}$ is an equivalence relation.

Exercise 15.3. Prove the existence of the classes of sets which are symmetric relations, transitive relations, equivalence relations.

Let R be an equivalence relation. The symmetry of R guarantees that dom[R] = rng[R].

Exercise 15.4. Prove that any equivalence relation R is reflexive.

Hint: Given any $x \in \text{dom}[R]$, find $y \in \mathbf{U}$ with $\langle x, y \rangle \in R$. By the symmetry of R, $\langle y, x \rangle \in R$ and by the transitivity, $\langle x, x \rangle \in R$.

Let R be an equivalence relation. For any set x, the class $R[\{x\}] = \{y : \langle x, y \rangle \in R\}$ is called the R-equivalence class of x. If $R[\{x\}]$ is not empty, then $x \in R[\{x\}]$ by Exercise 15.4.

Exercise 15.5. Prove that for any equivalence relation R and sets x, y, the R-equivalence classes $R[\{x\}]$ and $R[\{y\}]$ are either disjoint or coincide.

Let R be an equivalence relation. If for any set x its R-equivalence class $R[\{x\}]$ is a set, then by Theorem 12.1, the class $R^{\bullet} = \{\langle x, R[\{x\}] \rangle : x \in \mathsf{dom}[R] \}$ is a well-defined function assigning to each set $x \in \mathsf{dom}[R]$ its equivalence class $R^{\bullet}(x) = R[\{x\}]$. The range $\{R^{\bullet}(x) : x \in \mathsf{dom}[R]\}$ of this function is called the *quotient class* of the relation R. The quotient class is usually denoted by $\mathsf{dom}[R]/R$. The function $R^{\bullet} : \mathsf{dom}[R] \to \mathsf{dom}[R]/R$ is called the *quotient function*.

If the relation R is a set, then by the Axiom of Replacement, the quotient class $dom[R]/R = R^{\bullet}[dom[R]]$ is a set, called the *quotient set* of the relation R.

By an equivalence relation on a set X we understand any equivalence relation R with $dom[R^{\pm}] = X$. In this case $R \subseteq X \times X$ is a set and so are all R-equivalence classes $R^{\bullet}(x)$. Consequently, the quotient class X/R is a set, called the quotient set of X by the relation R.

Example 15.6. Consider the equivalence relation

$$|\cdot\cdot| = \{\langle x,y\rangle \in \mathbf{U} \times \mathbf{U} : \exists f \in \mathbf{Fun} \ (f^{-1} \in \mathbf{Fun} \ \land \ \mathsf{dom}[f] = x \ \land \ \mathsf{rng}[f] = y)\}.$$

The equivalence class of a set x by this equivalence relation is called the *cardinality* of the set x and is denoted by |x|.

16. Well-Founded relations

In this section we introduce and discuss well-founded relations, which play an extremely important role in Classical Set Theory.

Definition 16.1. A relation R is defined to be

- set-like if for every $x \in \mathbf{U}$ the class $\bar{R}(x) = R^{-1}[\{x\}] \setminus \{x\}$ is a set;
- well-founded if every nonempty class X contains an element $x \in X$ such that $R(x) \cap X = \emptyset$.

The set $\bar{R}(x)$ appearing in this definition is called the *initial R-interval* of x. It is equal to $\bar{R}(x) = \{z : \langle z, x \rangle \in R\} \setminus \{x\}$. The set $\bar{R}(x)$ is empty if $x \notin \text{rng}[R]$.

Remark 16.2. For the membership relation the initial **E**-interval $\mathbf{E}(x)$ of a set x coincides with the set $x \setminus \{x\}$. If the Axiom of Foundation holds, then $\mathbf{E}(x) = x$. The relation **E** is set-like.

Proposition 16.3. A transitive set-like relation R is well-founded if and only if every nonempty set $a \subseteq \text{rng}[R]$ contains an element $y \in a$ such that $\overline{R}(y) \cap y = \emptyset$.

Proof. The "only if" part is trivial. To prove the only "if" part, fix a nonempty class X. Take any element $x \in X$ and consider the class $a = (\bar{R}(x) \cup \{x\}) \cap X$, which is a set, being a subclass of the set $\bar{R}(x) \cup \{x\}$. If $a \not\subseteq \operatorname{rng}[R]$, then take any element $z \in a \setminus \operatorname{rng}[R]$ and observe that the class $\bar{R}(z) \subseteq R^{-1}[\{z\}] = \emptyset$ is empty and hence $\bar{R}(z)$ is a set with $\bar{R}(z) \cap X = \emptyset$.

So, we assume that $a \subseteq \operatorname{rng}[R]$. Since $x \in a$, the set a is not empty and by the assumption, there exists an element $y \in a \subseteq X$ such that $\overline{R}(y) \cap a = \emptyset$. If y = x, then

$$X \cap \overline{R}(x) \subseteq X \cap (\overline{R}(x) \cup \{x\}) \cap \overline{R}(x) = a \cap \overline{R}(y) = \emptyset$$

and the point $x = y \in X$ has the required property: $\bar{R}(x) \cap X = \emptyset$.

Now assume that $y \neq x$. In this case

$$y \in a \setminus \{x\} \subset (\overline{R}(x) \cup \{x\}) \setminus \{x\} \subseteq \overline{R}(x) \subseteq R^{-1}[\{x\}]$$

and then the transitivity of the relation R ensures that

$$\bar{R}(y) \subset R^{-1}[\{y\}] \subseteq R^{-1}[R^{-1}[\{x\}]] \subseteq R^{-1}[\{x\}] \subset R^{-1}[\{x\}] \cup \{x\}.$$

Then

$$X \cap \overline{R}(y) = X \cap (R^{-1}[\{x\}] \cup \{x\}) \cap \overline{R}(y) = a \cap \overline{R}(y) = \emptyset$$

and $y \in a \subset X$ is a required element such that $\overline{R}(y) \cap X = \emptyset$.

Well-founded relations allow us to generalize the Principle of Mathematical Induction to the Principle of Transfinite Induction. In fact, the Principle of Mathematical Induction has two forms.

Principle of Mathematical Induction:

Let X be a set of natural numbers. If $\emptyset \in X$ and for every $n \in X$ the number n+1 belongs to X, then $X = \omega$.

Proof. It follows that X is an inductive set and hence $\omega \subseteq X$ by the definition of ω . Since $X \subseteq \omega$, the Axiom of Extensionality ensures that $X = \omega$.

Principle of Mathematical Induction (metaversion): Let $\varphi(x)$ be formula with a free variables x, Y_1, \ldots, Y_m . Let Y_1, \ldots, Y_m be any classes. Assume that $\varphi(\emptyset, Y_1, \ldots, Y_m)$ holds and for every $n \in \omega$ if $\varphi(n, Y_1, \ldots, Y_m)$ holds, then $\varphi(n + 1, Y_1, \ldots, Y_m)$ holds. Then $\varphi(n, Y_1, \ldots, Y_m)$ holds for all $n \in \omega$.

Proof. Consider the class $X = \{n \in \omega : \varphi(n, Y_1, \dots, Y_m)\}$ which exists by Gödel's class existence Theorem 7.2. Since $X \subseteq \omega$, the class X is a set, see Exercise 6.3. Our assumptions on φ ensure that the set X is inductive. Then $X = \omega$ by the minimality of the inductive set ω .

The Principle of Transfinite induction also has two versions.

Principle of Transfinite Induction: Let Y be a subclass of a class X and R be a well-founded relation such that $dom[R] \subseteq X$. If each element $x \in X$ with $\overline{R}(x) \subseteq Y$ belongs to Y, then Y = X.

Proof. Assuming that $Y \neq X$, consider the nonempty class $X \setminus Y$ and by the well-foundedness of the relation R, find an element $x \in X \setminus Y$ such that $\overline{R}(x) \cap (X \setminus Y) = \emptyset$ and hence

Now the assumption ensures that $x \in Y$, which contradicts the choice of x.

Principle of Transfinite Induction (metaversion): Let $\varphi(x, Y_1, \ldots, Y_m)$ be formula with a free variables x, Y_1, \ldots, Y_m . Let Y_1, \ldots, Y_m be any classes. Let R be a well-founded relation and X be a class such that $\text{dom}[R] \subseteq X$. Assume that for any $x \in X$ the following implication holds:

$$(\forall z \in \overline{R}(x) \ \varphi(z, Y_1, \dots, Y_m)) \Rightarrow \varphi(x, Y_1, \dots, Y_m).$$

Then $\varphi(x, Y_1, \dots, Y_m)$ holds for every $x \in X$.

Proof. Consider the class $Y = \{x \in X : \varphi(x, Y_1, \dots, Y_m)\}$ which exists by Gödel's class existence Theorem 7.2. Applying the Principle of Transfinite Induction, we conclude that Y = X.

Part 4. Order

In this section we consider some notions related to order and introduce ordinals.

17. Order relations

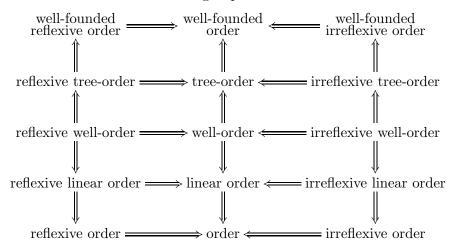
There exists a wide class of relations describing various types of order on classes and sets.

Definition 17.1. A relation R is called

- antisymmetric if $R \cap R^{-1} \subseteq \mathbf{Id}$;
- an *order* if the relation R is transitive and antisymmetric;
- a linear order if R is an order such that $dom[R^{\pm}] \times dom[R^{\pm}] \subseteq R \cup Id \cup R^{-1}$;
- a well-order if R is a well-founded linear order;
- a tree-order if R is an order such that for every $x \in R$ the order $R \upharpoonright \overline{R}(x)$ is a well-order.

Exercise 17.2. Prove that any well-founded transitive relation is antisymmetric and hence is an order relation.

For order relations we have the following implication.



Remark 17.3. Reflexive orders are called *partial orders*, and irreflexive orders are called *strict orders*.

Reflexive orders are usually denoted by \leq , \leq , \sqsubseteq and irreflexive orders by <, \prec , \sqsubseteq etc.

Exercise 17.4. Show that for any order R, the relation $R \setminus \mathbf{Id}$ is an irreflexive order and the relation $R \cup \mathbf{Id} \upharpoonright \mathsf{dom}[R^{\pm}]$ is a reflexive order.

Definition 17.5. For a reflexive order R (which is a set), the pair (dom[R], R) is called a partially ordered class (resp. a partially ordered set or briefly, a poset).

Exercise 17.6. Prove that the relation $\mathbf{S} = \{\langle x, y \rangle \in \ddot{\mathbf{U}} : x \subseteq y\}$ is a partial order.

Exercise 17.7. Prove that an order R is a well-order if and only if every non-empty class $X \subseteq \mathsf{dom}[R^{\pm}]$ contains an element $x \in X$ such that $\langle x, y \rangle \in R$ for every $y \in X \setminus \{x\}$.

Exercise 17.8. Prove that the class Lin of strict linear orders exists.

Exercise 17.9. Prove that the class WF of well-founded relations exists.

Exercise 17.10. Prove that the classes of sets which are orders (linear orders, well-orders) exist.

Let R be a relation and X be a class. An element $x \in X \cap \text{dom}[R^{\pm}]$ is called

- an R-minimal element of X if $\forall y \in X \ (\langle y, x \rangle \in R \Rightarrow y = x);$
- an R-maximal element of X if $\forall y \in X \ (\langle x, y \rangle \in R \Rightarrow y = x);$
- an R-least element of X if $\forall y \in X \ (\langle x, y \rangle \in R \cup \mathbf{Id});$
- an R-greatest element of X if $\forall y \in X \ (\langle y, x \rangle \in R \cup \mathbf{Id})$.

Exercise 17.11. Let X be a class and R be an antisymmetric relation. Show that every R-least element of X is R-minimal, and every R-greatest element of X is R-maximal in X.

Exercise 17.12. Let R be a linear order and $X \subseteq \mathsf{dom}[R^{\pm}]$. Show that an element $x \in X$ is

- R-minimal in X if and only if x is the R-least element of X;
- R-maximal in X if and only if x is the R-greatest element of X.

Example 17.13. Consider the partial order $S = \{\langle x, y \rangle : x \subseteq y\}$. Observe that every element of the set $X = \{\{0\}, \{1\}\}$ is S-minimal and S-maximal, but X contains no S-least and no S-greatest elements.

Exercise 17.14. Prove that for every natural number $n \in \mathbb{N}$ and any linear order R with $dom[R^{\pm}] = n$ there exist an R-minimal element $x \in n$ and an R-maximal element $y \in n$.

Hint: Apply the Principle of Mathematical Induction.

We recall that a set x is called finite if there exists an injective function f such that dom[f] = x and $rng[f] \in \omega$.

Exercise 17.15. Prove that for any linear order R on a finite set $X = \text{dom}[R^{\pm}]$ there exist an R-minimal element $x \in X$ and an R-maximal element $y \in X$.

Definition 17.16. Let L be an order on the class $X = \text{dom}[L^{\pm}]$. A subclass $A \subseteq X$ is called upper L-bounded (resp. lower L-bounded) if the set $\overline{A} = \{b \in X : A \times \{b\} \subseteq L \cup \mathbf{Id}\}$ (resp. the set $\underline{A} = \{b \in X : \{b\} \times A \subseteq L \cup \mathbf{Id}\}$) is not empty.

If the set \overline{A} (resp. \underline{A}) contains the L-least (resp. L-greatest) element, then this unique element is denoted by $\sup_L(A)$ (resp. $\inf_L(A)$) and called the least upper L-bound (resp. the greatest lower L-bound) of A.

For a well-founded order R and an element $x \in dom[R^{\pm}]$, the class

$$Succ_R(x) = min_R(R[\{x\}] \setminus \{x\})$$

of R-minimal elements of the class $R[\{x\}] = \{y : \langle x, y \rangle \in R\}$ is called the class of immediate R-successors of x in the well-founded order R.

Definition 17.17. A well-founded order R is called

- locally finite if for every $x \in dom[R^{\pm}]$ the class $Succ_R(x)$ is a finite set;
- locally countable if for every $x \in dom[R^{\pm}]$ the class $Succ_R(x)$ a countable set;
- locally set if for every $x \in \text{dom}[R^{\pm}]$ the class $\text{Succ}_R(x)$ is a set.

18. Transitivity

Definition 18.1. A class X is called *transitive* if $\forall y \in X \ \forall z \in y \ (z \in X)$.

Exercise 18.2. Prove that a class X is transitive if and only if $\forall x \in X \ (x \subseteq X)$ if and only if $\bigcup X \subseteq X$.

Exercise 18.3. Prove that the class Tr of transitive sets exists.

Hint: Observe that $\mathbf{U} \setminus \mathbf{Tr} = \{x : \exists y \; \exists z \; (y \in x \; \land \; z \in y \; \land \; z \notin x)\} = \operatorname{rng}[T_1 \cap T_2 \cap T_3]$ where $T_1 = \{\langle z, y, x \rangle : y \in x\} = [\mathbf{E} \times \mathbf{U}]^{\circlearrowright}, \; T_2 = \{\langle z, y, x \rangle : y \in z\} = \mathbf{E} \times \mathbf{U}, \text{ and } T_3 = \{\langle z, y, x \rangle : z \notin x\} = [(\ddot{\mathbf{U}} \setminus E^{-1}) \times \mathbf{U}]^{\circlearrowleft}.$

Remark 18.4. The membership relation E \textstyle Tr on the class Tr is transitive.

Exercise 18.5. Let $(X_{\alpha})_{\alpha \in A}$ be an indexed family of transitive classes. Prove that the union $\bigcup_{\alpha \in A} X_{\alpha}$ and intersection $\bigcap_{\alpha \in A} X_{\alpha}$ are transitive classes.

Let us recall that a class X is called *inductive* if $\emptyset \in X$ and for every $x \in X$ the set $x \cup \{x\}$ belongs to X. By the Axiom of Infinity there exists an inductive set. The intersection of all inductive sets is denoted by ω .

Proposition 18.6. The class **Tr** of transitive sets is inductive.

Proof. It is clear that the empty set is transitive. Assume that a set x is transitive and take any element $y \in x \cup \{x\}$. If $y \in x$, then for every $z \in y$ the element z belongs to $x \cup \{x\}$ by the transitivity of x. If y = x, then every element $z \in y = x$ belongs to $x \subset x \cup \{x\}$ as y = x.

Theorem 18.7. $\omega \subset \operatorname{Tr} \ and \ \omega \in \operatorname{Tr}$.

Proof. Since the classes Tr and ω are inductive, so is their intersection $\operatorname{Tr}\cap\omega$. By Exercise 6.3, the class $\operatorname{Tr}\cap\omega$ is a set and then $\omega\subseteq\omega\cap\operatorname{Tr}\subseteq\operatorname{Tr}$ by the definition of ω .

To prove that $\omega \in \mathbf{Tr}$, we need to show that each element of ω is a subset of ω . For this consider the class $T = \{n \in \omega : n \subseteq \omega\}$, which is equal to $(\omega \times \mathbf{U}) \cap \mathsf{dom}[\mathbf{S} \cap (\mathbf{U} \times \{\omega\})]$ and hence exists. The class T is a set, being a subclass of the set ω , see Exercise 6.3. Let us show that the set T is inductive. It is clear that $\emptyset \in T$. Assuming that $n \in T$, we conclude that $n \subseteq \omega$ and also $n \in T \subseteq \omega$. Then $n \cup \{n\} \subseteq \omega$, which means that $n \cup \{n\} \in T$ and the set T inductive. Since ω is the smallest inductive set, $\omega = T$. Consequently, ω is transitive set and $\omega \in \mathbf{Tr}$.

Given any set x, consider the class $\mathbf{Tr}(x) = \{y \in \mathbf{Tr} : x \subseteq y\}$ of transitive sets that contain x. This class is equal to $\mathbf{Tr} \cap \mathsf{rng}[\mathbf{S}]$ and hence exists by Exercises 18.3 and 6.1. The intersection

$$TC(x) = \bigcap Tr(x)$$

of the class $\mathbf{Tr}(x)$ is the smallest transitive class that contains x. It is called the *transitive closure* of X.

In Theorem 21.5 we shall prove that

$$\mathsf{TC}(x) = \bigcup_{n \in \omega} \bigcup^{\circ n} x$$

where $\bigcup^{0} x = x$ and $\bigcup^{0} (n+1) x = \bigcup (\bigcup^{n} x)$ for $n \in \omega$. Now we establish some properties of the transitive closure.

Proposition 18.8. $\mathsf{TC}(x) = x \cup \mathsf{TC}(\bigcup x) = x \cup \bigcup_{y \in x} \mathsf{TC}(y)$ for every set x.

Proof. The equality $\mathsf{TC}(x) = x \cup \mathsf{TC}(\bigcup x)$ will be established as soon as we check that the class $x \cup \mathsf{TC}(\bigcup x)$ is transitive and is a subclass of every transitive set t that contains x.

Given any sets $u \in x \cup \mathsf{TC}(x)$ and $v \in u$, we shall prove that $v \in x \cup \mathsf{TC}(\bigcup x)$. If $u \in x$, then $v \in \bigcup x \subseteq \mathsf{TC}(\bigcup x) \subseteq x \cup \mathsf{TC}(\bigcup x)$. If $u \in \mathsf{TC}(\bigcup x)$, then $v \in \mathsf{TC}(\bigcup x)$ by the transitivity of the class $\mathsf{TC}(\bigcup x)$. Therefore the class $x \cup \mathsf{TC}(\bigcup x)$ is transitive.

Now let t be any transitive set that contains x. Then $\bigcup x \subseteq t$ by transitivity of t and then $\mathsf{TC}(\bigcup x) \subseteq t$ since $\mathsf{TC}(\bigcup x)$ is the smallest transitive set that contains $\bigcup x$. Then $x \cup \mathsf{TC}(\bigcup x) \subseteq t$.

Exercise 18.9. Given any sets x, y, z, prove the following equalities:

- (1) $\mathsf{TC}(x \cup y) = \mathsf{TC}(x) \cup \mathsf{TC}(y)$
- (2) $\mathsf{TC}(\{x,y\}) = \{x,y\} \cup \mathsf{TC}(x \cup y).$
- (3) $\mathsf{TC}(\langle x, y \rangle) = \{ \{x\}, \{x, y\} \} \cup \{x, y\} \cup \mathsf{TC}(x \cup y).$
- $(4) \ \mathsf{TC}(x \times y) = (x \times y) \cup \{\{u\} : u \in x\} \cup \{\{u,v\} : u \in x \ \land \ v \in y\} \cup \mathsf{TC}(x \cup y).$
- (5) $\mathsf{TC}(\mathsf{dom}[x]) \subseteq \mathsf{TC}(x)$.
- (6) Calculate $\mathsf{TC}(\langle x, y, z \rangle)$.

19. Ordinals

Definition 19.1 (von Neumann). A set x is called an *ordinal* if x is transitive and the relation $\mathbf{E} \upharpoonright x = \mathbf{E} \cap (x \times x)$ is an irreflexive well-order on x.

Exercise 19.2. Show that under the Axiom of Foundation, a transitive set x is an ordinal if and only if the relation $\mathbf{E} \upharpoonright x$ is a linear order.

We recall that for two classes X, Y the notation $X \subset Y$ means that $X \subseteq Y$ and $X \neq Y$.

Theorem 19.3. 1) Each element of an ordinal is a transitive set.

- 2) Each element of an ordinal is an ordinal.
- 3) The intersection of two ordinals is an ordinal.
- 4) For any two ordinals α, β , we have the equivalence: $\alpha \in \beta \iff \alpha \subset \beta$.
- 5) For any ordinals α, β we have the dychotomy: $\alpha \subseteq \beta \vee \beta \subseteq \alpha$.
- 6) For any ordinals α, β we have the trichotomy: $\alpha \in \beta \vee \alpha = \beta \vee \beta \in \alpha$.

Proof. 1. Let β be an ordinal and $\alpha \in \beta$. To show that the set α is transitive, take any sets $y \in \alpha$ and $z \in y$. The transitivity of β guarantees that $y \in \beta$ and $z \in \beta$. Since $z \in y \in \alpha$, the transitivity of the relation $\mathbf{E} \upharpoonright \beta$ implies $z \in \alpha$, which means that the set α is transitive.

- 2. Let β be an ordinal and $\alpha \in \beta$. By the preceding statement, the set α is transitive. The transitivity of the set β implies that $\alpha \subseteq \beta$. Since the relation $\mathbf{E} \upharpoonright \beta$ is an irreflexive well-order, its restriction $\mathbf{E} \upharpoonright \alpha$ to the subset $\alpha \subseteq \beta$ is an irreflexive well-order, too. This means that the transitive set α is an ordinal.
 - 3. The definition of an ordinal implies that the intersection of two ordinals is an ordinal.
- 4. Let α, β be two ordinals. If $\alpha \in \beta$, then $\alpha \subseteq \beta$ by the transitivity of β , and $\alpha \neq \beta$ by the irreflexivity of the relation $\mathbf{E} \upharpoonright \beta$. Therefore, $\alpha \in \beta$ implies $\alpha \subset \beta$.

Now assume that $\alpha \subset \beta$. Since the set $\beta \setminus \alpha$ is not empty and $\mathbf{E} \upharpoonright \beta$ is a strict well-order, there exists an element $\gamma \in \beta \setminus \alpha$ such that $\gamma \cap (\beta \setminus \alpha) = \emptyset$. By the transitivity of the set β , the element γ is a subset of β . Taking into account that $\gamma \cap (\beta \setminus \alpha) = \emptyset$, we conclude

that $\gamma \subseteq \alpha$. We claim that $\gamma = \alpha$. In the opposite case, there exists an element $\delta \in \alpha \setminus \gamma$. Since $\delta \in \alpha \subset \beta$ and $\gamma \in \beta$, we can apply the linearity of the order $\mathbf{E} \upharpoonright \beta$ to conclude that $\delta \in \gamma \lor \delta = \gamma \lor \gamma \in \delta$. The assumption $\delta \in \gamma$, contradicts the choice of $\delta \in \alpha \setminus \gamma$. The equality $\delta = \gamma$ implies that $\gamma = \delta \in \alpha$, which contradicts the choice of $\gamma \in \beta \setminus \alpha$. The assumption $\gamma \in \delta$ implies $\gamma \in \delta \subseteq \alpha$ (by the transitivity of α) and this contradicts the choice of $\gamma \in \beta \setminus \alpha$. These contradictions imply that $\alpha = \gamma \in \beta$.

- 5. Let α, β be two ordinals. Assuming that neither $\alpha \subseteq \beta$ not $\beta \subseteq \alpha$, we conclude that the ordinal $\gamma = \alpha \cap \beta$ is a proper subset in α and β . Applying the preceding statement, we conclude that $\gamma \in \alpha$ and $\gamma \in \beta$. This implies that $\gamma \in \alpha \cap \beta = \gamma$. But this contradicts the irreflexivity of the well-order $\mathbf{E} \upharpoonright \alpha$ on the ordinal α .
- 6. Let α, β be two ordinals. By the preceding statement, $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$. This implies the trichotomy $\alpha \subset \beta \lor \alpha = \beta \lor \beta \subset \alpha$, which is equivalent to the trichotomy $\alpha \in \beta \lor \alpha = \beta \lor \beta \in \alpha$ according to the statement (4).

Exercise 19.4. Using the Axiom of Foundation prove that a set x is an ordinal if and only if x is transitive and each element y of x is a transitive set.

Now we establish some properties of the class of ordinals **On**.

Exercise 19.5. Prove that the class On exists.

Hint: Apply Exercises 18.3, 17.8, 17.9.

Theorem 19.6. (1) The class **On** is transitive.

- (2) The relation **E**[On is an irreflexive well-order on the class On.
- (3) $\emptyset \in \mathbf{On}$.
- (4) $\forall \alpha \ (\alpha \in \mathbf{On} \Rightarrow \alpha \cup \{\alpha\} \in \mathbf{On}).$
- (5) $\forall x \in \mathbf{U} \ (x \subseteq \mathbf{On} \Rightarrow \bigcup x \in \mathbf{On}).$
- (6) On is a proper class.
- (7) $\omega \subset \mathbf{On}$.
- (8) $\omega \in \mathbf{On}$.
- *Proof.* 1. The transitivity of the class **On** follows from Theorem 19.3(2) saying that any element of an ordinal is an ordinal.
- 2. The transitivity of ordinals implies the transitivity of the relation $\mathbf{E} \upharpoonright \mathbf{On}$. The irreflexivity of the relation $\mathbf{E} \upharpoonright \mathbf{On}$ follows from the irreflexivity of the relation $\mathbf{E} \upharpoonright \mathbf{On}$ for each ordinal. Aplying Theorem 19.3(6), we see that the relation $\mathbf{E} \upharpoonright \mathbf{On}$ is a linear order on the class \mathbf{On} . To see that $\mathbf{E} \upharpoonright \mathbf{On}$ is well-founded, take any nonempty subclass $A \subseteq \mathbf{On}$. We should find an ordinal $\alpha \in A$ such that $\alpha \cap A = \emptyset$. Take any ordinal $\beta \in A$. If $\beta \cap A = \emptyset$, then we are done. In the opposite case, $\beta \cap A$ is a nonempty subset of the ordinal β . Since $\mathbf{E} \upharpoonright \beta$ is an irreflexive well-order, there exists an ordinal $\alpha \in \beta \cap A$ such that $\alpha \cap (\beta \cap A) = \emptyset$. The transitivity of the set β guarantees that $\alpha \subseteq \beta$ and then $\alpha \cap A = (\alpha \cap \beta) \cap A = \alpha \cap (\beta \cap A) = \emptyset$.
 - 3. The inclusion $\emptyset \in \mathbf{On}$ is trivial.
- 4. Assume that α is an ordinal. Then α is a transitive set and by Proposition 18.6, its successor $\beta = \alpha \cup \{\alpha\}$ is transitive, too. It remains to prove that the relation $\mathbf{E} \upharpoonright \beta$ is an irreflexive well-order. The transitivity of the relation $\mathbf{E} \upharpoonright \beta$ follows from the transitivity of the set α and the transitivity of the relation $\mathbf{E} \upharpoonright \alpha$. The irreflexivity of this relation follows from the irreflexivity of the relation $\mathbf{E} \upharpoonright \alpha$, which implies also that $\alpha \notin \alpha$.

To see that $\mathbf{E} \upharpoonright \beta$ is a well-order, it suffices to show that any nonempty subset $X \subseteq \beta$ contains an element $x \in X$ such that $x \in y$ for every $y \in X \setminus \{x\}$. If $X = \{\alpha\}$, then $x = \alpha$

has the required property. If $X \neq \{\alpha\}$, then $X \cap \alpha$ is a non-empty set in α . Since α is an ordinal, there exists $x \in X \cap \alpha$ such that $x \in y$ for every $y \in X \cap \alpha \setminus \{x\}$. For $y = \alpha$ we have $x \in X \cap \alpha \subseteq \alpha = y$, too.

- 5. Let x be any subset of \mathbf{On} . By the Axiom of Union, the class $\cup x = \{z : \exists y \in x \ (z \in y)\}$ is a set. The transitivity of the class \mathbf{On} guarantees that $\cup x \subseteq \mathbf{On}$. Since $\mathbf{E} \upharpoonright \mathbf{On}$ is an irreflexive well-order, its restriction $\mathbf{E} \upharpoonright \cup x$ is an irreflexive well-order, too. Since the elements of x are transitive sets, the union $\cup x$ is a transitive set. Therefore, $\cup x$ is an ordinal and hence $\cup x \in \mathbf{On}$.
- 6. Assuming that the class \mathbf{On} is a set, we can apply the preceding statement and conclude that $\bigcup \mathbf{On} = \mathbf{On}$ is an ordinal and hence $\mathbf{On} \in \mathbf{On}$ and $\mathbf{On} \in \mathbf{On}$, which contradicts the irreflexivity of the relation $\mathbf{E} \upharpoonright \mathbf{On}$. This contradiction shows that \mathbf{On} is a proper class.
- 7. The statements (3) and (4) imply that the class **On** is inductive. Then the intersection $\omega \cap \mathbf{On}$ is an inductive class. By Exercise 6.3, the class $\omega \cap \mathbf{On}$ is a set and hence $\omega \subseteq \omega \cap \mathbf{On} \subseteq \mathbf{On}$. Since **On** is a proper class, $\omega \neq \mathbf{On}$ and hence $\omega \subset \mathbf{On}$.
- 8. By the statements (7) and (5), $\omega \subset \mathbf{On}$ and $\cup \omega \in \mathbf{On}$. By Theorem 18.7, the set ω is transitive and hence $\cup \omega \subseteq \omega$. On the other hand, for any $x \in \omega$ we have $x \in x \cup \{x\} \in \omega$ and hence $x \in \bigcup \omega$. Therefore, $\omega = \bigcup \omega \in \mathbf{On}$.

Exercise 19.7. Prove that under the Axiom of Foundation, **On** coincides with the smallest class X such that

- (1) $\emptyset \in X$;
- $(2) \ \forall x \in X \ (x \cup \{x\} \in X);$
- (3) $\forall x \in \mathbf{U} \ (x \subseteq X \Rightarrow \bigcup x \in X).$

Definition 19.8. An ordinal α is called

- a successor ordinal if $\alpha = \beta \cup \{\beta\}$ for some ordinal β ;
- a *limit ordinal* if α is not empty and is not a successor ordinal;
- a finite ordinal if every ordinal $\beta \in (\alpha \cup \{\alpha\}) \setminus \{\emptyset\}$ is a successor ordinal.

Exercise 19.9. Show that the classes of successor ordinals, limit ordinals, finite ordinals exist.

Theorem 19.10. 1) The ordinal ω is the smallest limit ordinal.

2) The set of natural numbers ω coincides with the set of finite ordinals.

Proof. Assuming that ω is a successor ordinal, we can find an ordinal α such that $\omega = \alpha \cup \{\alpha\}$. Then $\alpha \in \omega$ and by the inductivity of ω , we obtain $\omega = \alpha \cup \{\alpha\} \in \omega$, which is not possible as ω is an ordinal (by Theorem 19.3(8). This contradiction shows that the ordinal ω is limit. Assuming that ω is not the smallest nonempty limit ordinal, we can find a nonempty limit ordinal $\alpha \in \omega$. By Theorem 19.3(4), $\emptyset \in \alpha$ and $\alpha \subset \omega$. Since α is a limit ordinal, for every $x \in \alpha$, $x \cup \{x\} \neq \alpha$. By the transitivity of α , we obtain $x \subseteq \alpha$ and hence $x \cup \{x\} \subset \alpha$. Applying Theorem 19.3(4), we conclude that $x \cup \{x\} \in \alpha$, which means that the set α is inductive and hence $\omega \subseteq \alpha$ as ω is the smallest inductive set. Then $\omega \subseteq \alpha \subset \omega$ and Theorem 19.3(4), imply $\omega \in \omega$, which is not possible as $\omega \in \mathbf{On}$.

2. Denote by FO the class of finite ordinals (it exists by Exercise 19.9). It is easy to see that the class FO is inductive and hence $\omega \subseteq \mathsf{FO}$. On the other hand, the limit property of ω and Theorem 19.3(6) ensure that $\mathsf{FO} \subseteq \omega$. Therefore, $\omega = \mathsf{FO}$ is the set of all finite ordinals. \square

Some terminology and notation. Since the Memberships relation $\mathbf{E} \upharpoonright \mathbf{On}$ on the class \mathbf{On} is a linear order, it is often denoted by the symbol <. Therefore, given two ordinals α, β we write $\alpha < \beta$ if $\alpha \in \beta$ (which is equivalent to $\alpha \subset \beta$). In this case we say that α is *smaller* than β . Also we write $\alpha \leq \beta$ if $\alpha \in \beta$ or $\alpha = \beta$. The successor $\alpha \cup \{\alpha\}$ is often denoted by $\alpha + 1$. For a set A of ordinals let

$$\sup A = \min \{ \beta \in \mathbf{On} : \forall \alpha \in A \ (\alpha \subseteq \beta) \}.$$

Lemma 19.11. For any set $A \subseteq \mathbf{On}$,

$$\sup A = \bigcup A$$
.

Proof. By Theorem 19.6(5), the set $\bigcup A$ is an ordinal. For every $\alpha \in A$ the definition of the union $\bigcup A$ ensures that $\alpha \subseteq \bigcup A$, which implies $\sup A \subseteq \bigcup A$. Assuming that $\sup A \subset \bigcup A$, we would conclude that $\sup A \in \bigcup A$ and hence $\sup A \in \alpha$ for some $\alpha \in A$. By Theorem 19.3(4), $\sup A \in \alpha$ implies $\sup A \subset \alpha \subseteq \sup A$ and hence $\sup A \neq \sup A$, which is a contradiction showing that $\sup A = \bigcup A$.

In this section we introduce some notions related to trees and tree-orders. We recall that a tree-order is an order R such that for every $x \in \text{dom}[R]$ the restriction $R \upharpoonright \overline{R}(x)$ is a well-order.

Example 20.1. Let $\mathbf{U}^{<\mathbf{On}}$ be the class of functions f with $\mathsf{dom}[f] \in \mathbf{On}$. Then for the reflexive order relation $\mathbf{S} = \{\langle x, y \rangle \in \ddot{\mathbf{U}} : x \subseteq y\}$ the restriction $\mathbf{S} \upharpoonright \mathbf{U}^{<\mathbf{On}}$ is a tree-order.

Definition 20.2. A class T is called an *ordinary tree* if $T \subseteq \mathbf{U}^{<\mathbf{On}}$ and for every $t \in T$ and any ordinal α the function $t \upharpoonright_{\alpha}$ belongs to T.

The class $dom[T] = \{dom[t] : t \in T\}$ is called the *height* of an ordinary tree T.

Example 20.3. For every class A and every ordinal κ the class

$$A^{<\kappa} = \{t \in \mathbf{U^{$$

is an ordinary tree.

For every ordinal κ the ordinary tree $2^{<\kappa}$ is called the *full binary* κ -tree. Any ordinary tree $T \subseteq 2^{<\omega}$ is called a *binary tree*.

The ordinary tree $\mathbf{On}^{\leq \mathbf{On}}$ of all functions f with $\mathsf{dom}[f] \in \mathbf{On}$ and $\mathsf{rng}[f] \subseteq \mathbf{On}$ carries an irreflexive linear order W_{\leq} defined by the formula

$$\begin{aligned} \mathsf{W}_{<} &= \left\{ \langle f, g \rangle \in \mathbf{On^{< On}} \times \mathbf{On^{< On}} : \mathsf{dom}[f] \cup \bigcup \mathsf{rng}[f] \subset \mathsf{dom}[g] \cup \bigcup \mathsf{rng}[g] \; \vee \\ & (\mathsf{dom}[f] \cup \bigcup \mathsf{rng}[f] = \mathsf{dom}[g] \cup \bigcup \mathsf{rng}[g] \; \wedge \; \mathsf{dom}[f] \subset \mathsf{dom}[g]) \; \vee \\ & \left(\mathsf{dom}[f] \cup \bigcup \mathsf{rng}[f] = \mathsf{dom}[g] \cup \bigcup \mathsf{rng}[g] \; \wedge \; \mathsf{dom}[f] = \mathsf{dom}[g] \; \wedge \\ & \exists \alpha \in \mathsf{dom}[f] = \mathsf{dom}[g] \; \left(f \! \upharpoonright_{\alpha} = g \! \upharpoonright_{\alpha} \; \wedge \; f(\alpha) \in g(\alpha) \right) \right) \right\} \end{aligned}$$

and called the *canonical linear order* on $\mathbf{On}^{<\mathbf{On}}$.

Exercise 20.4. Prove that the restriction $W_{\leq} \upharpoonright \mathbf{On}^{\leq \omega}$ is a set-like well-order.

Exercise 20.5. Prove that the order $W_{\leq} \upharpoonright 2^{\omega}$ is not well-founded.

For an ordinary tree T and an element $x \in T$ the class

$$\mathsf{Succ}_T(x) = \{t \in T : \mathsf{dom}[t] = \mathsf{dom}[x] + 1$$

is called the class of immediate successors of x in the tree T.

Definition 20.6. An ordinary tree T is called

- locally finite if for every $x \in dom[R^{\pm}]$ the class $Succ_R(x)$ is a finite set;
- locally countable if for every $x \in dom[R^{\pm}]$ the class $Succ_R(x)$ a countable set;
- locally set if for every $x \in \text{dom}[R^{\pm}]$ the class $\text{Succ}_R(x)$ is a set.

Exercise* 20.7. Show that an ordinary tree T is a set if and only if it is set-like, locally set and its height dom[T] is a set.

Hint: See Exercise 22.4.

21. Recursion Theorem

In this section we prove an important theorem guaranteeing the existence of functions defined by recursive procedures, which are often used in Mathematics and Computer Science.

Theorem 21.1. Let X be a class, $F: X \times \mathbf{U} \to \mathbf{U}$ be a function and R be a set-like well-founded order such that $dom[R] \subseteq X$. Then there exists a unique function $G: X \to \mathbf{U}$ such that for every $x \in X$

$$G(x) = F(x, G[\overline{R}(x)]) \quad \text{where} \quad \overline{R}(x) = R^{-1}[\{x\}] \setminus \{x\}.$$

Proof. Since the relation R is set-like, for every $x \in X$ the class

$$\overline{R}(x) = R^{-1}[\{x\}] \setminus \{x\} = \{z : z \neq x \land \langle z, x \rangle \in R\}$$

is a set.

Let \mathbb{G} be the class consisting of all functions $f \in \mathbf{U}$ that satisfy the following conditions:

- (i) $\forall x \in \mathsf{dom}[f] \ (\overleftarrow{R}(x) \subseteq \mathsf{dom}[f] \subseteq X);$
- (ii) $f(x) = F(x, f[\overline{R}(x)])$ for every $x \in \text{dom}[f]$.

We claim that any two functions $f,g \in \mathbb{G}$ agree on the intersection of their domains. To derive a contradiction, assume that the set $A = \{x \in \mathsf{dom}[f] \cap \mathsf{dom}[g] : f(x) \neq g(x)\}$ is not empty. Since the relation R is well-founded, the set A contains an element a such that $\overline{R}(a) \cap A = \emptyset$. The condition (i) ensures that

$$\bar{R}(a) \subseteq (\mathsf{dom}[f] \cap \mathsf{dom}[g]) \setminus A = \{x \in \mathsf{dom}[f] \cap \mathsf{dom}[g] : f(x) = g(x)\}$$

and then

$$f(a) = F(a, f[\overleftarrow{R}(a)]) = F(a, g[\overleftarrow{R}(a)]) = g(a),$$

which contradicts $a \in A$. This contradiction shows that the functions f and g coincide on the intersection of their domains.

This property of the class G implies that the class

$$G = \bigcup \mathbb{G} = \{ \langle x, y \rangle : \exists f \in \mathbb{G} \ \langle x, y \rangle \in f \}$$

is a function.

By definition of $G = \bigcup \mathbb{G}$, for every $x \in \mathsf{dom}[G]$ there exists $y \in \mathbf{U}$ such that $\langle x, y \rangle \in G = \bigcup \mathbb{G}$ and hence $\langle x, y \rangle \in f$ for some $f \in G$. Taking into account that $G \cap (\mathsf{dom}[f] \times \mathbf{U}) = f$, we conclude that

$$G(x) = y = f(x) = F(x, f[\bar{R}(x)]) = F(x, G[\bar{R}(x)]).$$

Next, we prove that $\mathsf{dom}[G] = X$. The condition (i) guarantees that $\mathsf{dom}[G] \subseteq X$. Assuming that $\mathsf{dom}[G] \neq X$ and using the well-foundedness of the relation R, we can find an element $x \in X \setminus \mathsf{dom}[G]$ such that $\overline{R}(x) \cap (X \setminus \mathsf{dom}[G]) = \emptyset$ and hence $\overline{R}(x) \subseteq \mathsf{dom}[G]$. As R

is set-like, the class $\bar{R}(x)$ is a set. By the Axiom of Replacement, the class $G[\bar{R}(x)]$ is a set, too. Now consider the function

$$f = G \upharpoonright_{\overline{R}(x)} \cup \{\langle x, F(x, G[\overline{R}(x)]) \rangle\}.$$

with domain $\mathsf{dom}[f] = \bar{R}(x) \cup \{x\} = R^{-1}[\{x\}] \cup \{x\}$. The transitivity of the relation R ensures that the function f has properties (i),(ii) and hence $f \in \mathbb{G}$ and $x \in \mathsf{dom}[f] \subseteq \mathsf{dom}[G]$, which contradicts the choice of x.

Finally, we show that the function G is unique. Indeed, take any function $\Phi: X \to \mathbf{U}$ such that $\Phi(x) = F(x, \Phi[\bar{R}(x)])$ for all $x \in X$. Assuming that $\Phi \neq G$, we conclude that the set $D = \{x \in X : \Phi(x) \neq G(x)\}$ is not empty and by the well-foundedness of the relation R contains an element $a \in D$ such that $\bar{R}(a) \cap D = \emptyset$ and hence $\bar{R}(a) = \{x \in X : \Phi(x) = G(x)\}$. Then

$$\Phi(a) = F(a, \Phi[\vec{R}(a)]) = F(a, G[\vec{R}(a)]) = G(a),$$

which contradicts the chocie of a.

Now we apply the Recursion Theorem to legalize recursive definitions of (function) sequences.

Theorem 21.2. For every class X and functions $G_0: X \to \mathbf{U}$ and $F: (\omega \times X) \times \mathbf{U} \to \mathbf{U}$ there exists a unique function $G: \omega \times X \to \mathbf{U}$ such that

$$G(0,x) = G_0(x)$$
 and $G(n+1,x) = F(n+1,x,G(n,x))$

for every $\langle n, x \rangle \in X \times \omega$.

Proof. Consider the function $\Phi: (\omega \times X) \times \mathbf{U} \to (\omega \times \mathbf{U})$ such that for every $\langle n, x, y \rangle \in (\omega \times X) \times \mathbf{U}$ the following two conditions hold:

- $\Phi(0, x, y) = \langle 0, G_0(x) \rangle$ and
- $\Phi(n+1,x,y) = \begin{cases} \langle n+1,F(n+1,x,z) \rangle & \text{if z is a unique set such that } \langle n,z \rangle \in y; \\ \langle n+1,G_0(x) \rangle & \text{otherwise.} \end{cases}$

The function Φ exists by the Gödel's class existence theorem. Consider the set-like well-founded order

$$R = \{ \langle \langle k, x \rangle, \langle n, x \rangle \rangle : k \in n \in \omega \}$$

on the class $\omega \times \mathbf{U} = \mathsf{dom}[R^{\pm}].$

By the Recursion Theorem 21.1, there exists a unique function $\Psi: \omega \times X \to \omega \times \mathbf{U}$ such that

(21.1)
$$\Psi(n,x) = \Phi(n,x, \{\Psi(k,x) : k \in n\})$$

for all $\langle n, x \rangle \in \omega \times X$.

Consider the functions

$$\operatorname{rng}: \mathbf{U} \times \mathbf{U} \to \mathbf{U}, \quad \operatorname{rng}: \langle u, v \rangle \mapsto v$$

and

$$G = \operatorname{rng} \circ \Psi : \omega \times \mathbf{U} \to \mathbf{U}.$$

The equality (21.1) and the definition of the function Φ imply that for every $\langle n, x \rangle \in \omega \times X$ we have

$$\Psi(n,x) = \langle n, G(n,x) \rangle$$

and

$$G(n+1,x) = \operatorname{rng} \circ \Phi(n+1,x,\{\langle k,G(k,x)\rangle : k \in n+1\}) = F(n+1,x,G(n,x)).$$
 Also $G(0,x) = \operatorname{rng} \circ \Phi(0,x,\emptyset) = G_0(x).$

As a special case of Theorem 21.2 for $X = \{0\}$, we obtain the following corollary justifying the definition of sequences by recursive formulas.

Corollary 21.3. For every function $F: \omega \times \mathbf{U} \to \mathbf{U}$ and set z there exists a unique sequence $(x_n)_{n \in \omega}$ such that

$$x_0 = a$$
 and $x_{n+1} = F(n+1, x_n)$

for every $\langle n, x \rangle \in X \times \omega$.

Now we apply Theorem 21.2 to legalize the widely used procedure of iterations of functions. Let X be a class and $\Phi: X \to X$ be a function. Consider the sequence of functions $(\Phi^{\circ n})_{n \in \omega}$ defined by the recursive formula:

$$\Phi^{\circ 0} = \mathbf{Id} \upharpoonright X$$
 and $\Phi^{\circ (n+1)} = \Phi \circ \Phi^{\circ n}$ for every $n \in \omega$.

Let us recall that for two functions G, H their compositions $G \circ H$ is defined as the function

$$G \circ H = \{ \langle x, z \rangle : \exists y \ \langle z, y \rangle \in H \ \land \ \langle y, z \rangle \in H \}.$$

Theorem 21.4. For every class X and function $\Phi: X \to X$ the function sequence $(\Phi^{\circ n})_{n \in \omega}$ is well-defined.

Proof. Consider the function $F:(\omega \times X) \times \mathbf{U} \to \mathbf{U}$ defined by

$$F(n, x, y) = \begin{cases} \Phi(y) & \text{if } y \in X; \\ \emptyset & \text{otherwise,} \end{cases}$$

By Theorem 21.2, there exists a function $G: \omega \times X \to \mathbf{U}$ such that G(0,x) = x and $G(n+1,x) = F(n+1,x,\Psi(n,x))$ for all $\langle n,x \rangle \in \omega \times X$.

By induction we shall prove that for every $x \in X$ and $n \in \omega$ we have

(21.2)
$$G(n,x) \in X$$
 and $G(n,x) = \Phi^{\circ n}(x)$.

Observe that $G(0,x)=x\in X$ and $G(0,x)=\Phi^{\circ n}(x)$. Assume that for some $n\in\omega$ the equality (21.2) holds. Then

$$G(n+1,x) = F(n+1,x,G(n,x)) = \Phi(G(n,x)) = \Phi(\Phi^{\circ n}(x)) = \Phi^{\circ (n+1)}(x) \in X.$$

By the Principle of Matematical Induction, the equality (21.2) holds for all $n \in \omega$.

As an application of iterations let us prove the existence of transitive closures. Consider the function of taking union

$$\bigcup : \mathbf{U} \to \mathbf{U}, \quad \bigcup : x \mapsto \bigcup x,$$

and its iterations $\bigcup^{\circ n}$ for $n \in \omega$. Taking the union of those iterations, we obtain the function

$$\bigcup^{\circ\omega}:\mathbf{U}\to\mathbf{U},\quad\bigcup^{\circ\omega}:x\mapsto\bigcup\{\bigcup^{\circ n}x:n\in\omega\}.$$

Theorem 21.5. The function $\bigcup_{i=1}^{\infty}$ coincides with the function TC of transitive closure.

Proof. We need to show that for every set x the set $\bigcup^{\circ\omega} x$ is the smallest transitive set containing x as a subset.

To see that the set $\bigcup^{\circ \omega} x$ is transitive, take any element $y \in \bigcup^{\circ \omega} x = \bigcup_{n \in \omega} \bigcup^{\circ n} x$ and find $n \in \omega$ such that $y \in \bigcup^{\circ n} x$. Then $y \subseteq \bigcup(\bigcup^{\circ n} x) = \bigcup^{\circ (n+1)} x \subseteq \bigcup^{\circ \omega} x$. So, $\bigcup^{\circ \omega} x$ is transitive.

Next, we prove that $\bigcup^{\circ\omega} x \subseteq Y$ for any transitive class Y with $x \subseteq Y$. Since $\bigcup^{\circ\omega} x = \bigcup_{n \in \omega} \bigcup^{\circ n} x$, it suffices to show that $\forall n \in \omega \quad \bigcup^{\circ n} x \subseteq Y$. For n = 0 this follows from the equality $\bigcup^{\circ 0} x = x \subseteq Y$. Assume that for some $n \in \omega$ we proved that $\bigcup^{\circ n} x \subseteq Y$. Then $\bigcup^{\circ (n+1)} x = \bigcup(\bigcup^{\circ n} x) \subseteq \bigcup Y \subseteq Y$ by the transitivity of Y. Applying the Principle of Mathematical Induction, we conclude that $\bigcup^{\circ n} x \subseteq Y$ for all $\langle n, x \rangle \in \omega \times X$.

Finally, we apply Theorem 21.4 to prove a general form of the Recursion Theorem 21.1.

Theorem 21.6. Let X be a class, $F: X \times \mathbf{U} \to \mathbf{U}$ be a function and R be a set-like well-founded relation such that $\mathsf{dom}[R] \subseteq X$. Then there exists a unique function $G: X \to \mathbf{U}$ such that for every $x \in X$

$$G(x) = F(x, G[\bar{R}(x)])$$
 where $\bar{R}(x) = R^{-1}[\{x\}] \setminus \{x\}.$

Proof. Let \mathbb{G} be the family of functions considered in the proof of Theorem 21.1, and $G = \bigcup \mathbb{G}$ be its union. We claim that $\mathsf{dom}[G] = X$. Assuming that $X \neq \mathsf{dom}[G]$ and using the well-foundedness of the relation R, we can find an element $a \in X \setminus \mathsf{dom}[G]$ such that $\overline{R}(a) \cap (X \setminus \mathsf{dom}[G]) = \emptyset$ and hence $\overline{R}(a) \subseteq \mathsf{dom}[G]$.

Since the relation R is set-like, for every $x \in X$ the class

$$\overline{R}(x) = R^{-1}[\{x\}] \setminus \{x\} = \{z : z \neq x \land \langle z, x \rangle \in R\}$$

is a set. Using Theorem 7.2, it can be shown that

$$\ddot{R} = \{\langle z, y \rangle \in \ddot{\mathbf{U}} : y = \ddot{R}(x)\}$$

is a well-defined function. By the Axiom of Replacement, for every set x the class $\bar{R}[x] = \{\bar{R}(y) : y \in x\}$ is a set and by the Axiom of Union, the class $\bigcup \bar{R}[x]$ is a set, too. Using Theorem 7.2, it can be shown that $\Phi : \mathbf{U} \to \mathbf{U}$, $\Phi : x \mapsto x \cup \bigcup \bar{R}[x]$, is a well-defined function. By Corollary 21.3, there exists a sequence of sets $(A_n)_{n \in \omega}$ such that $A_0 = \bar{R}(a)$ and $A_{n+1} = A_n \cup \bigcup \bar{R}[A_n]$ for every $n \in \omega$.

By mathematical induction, we shall prove that $\forall n \in \omega \ (A_n \subseteq \mathsf{dom}[G])$. For n=0 we have $A_0 = \bar{R}(a) \subseteq \mathsf{dom}[G]$ by the choice of a. Assume that for some $n \in \omega$ we have $A_n \subseteq \mathsf{dom}[G]$. To prove that $A_{n+1} \subseteq \mathsf{dom}[G]$, take any element $x \in A_{n+1} = A_n \cup \bigcup \bar{R}[A_n]$. If $x \in A_n$, then $x \in A_n \subseteq \mathsf{dom}[G]$ and we are done. If $x \in \bigcup \bar{R}[A_n]$, then $x \in \bar{R}(y)$ for some $y \in A_n$. By the induction hypothesis, $y \in A_n \subseteq \mathsf{dom}[G]$. Consequently, there exists a function $f \in \mathbb{G}$ such that $y \in \mathsf{dom}[f]$. By the condition (i) from the definition of the family \mathbb{G} (see the proof of Theorem 21.1), $x \in \bar{R}(y) \subseteq \mathsf{dom}[f] \subseteq \mathsf{dom}[G]$ and we are done. By the Principle of Mathematical Induction, $\forall n \in \omega \ (A_n \subseteq \mathsf{dom}[G])$ and hence $A = \bigcup_{n \in \omega} A_n \subseteq \mathsf{dom}[G]$. Now consider the function

$$g = G {\restriction} A \cup \{\langle a, F(a, G[\tilde{R}(a)]) \rangle\}$$

and observe that $g \in \mathbb{G}$ and hence $a \in \mathsf{dom}[g] \subseteq \mathsf{dom}[G]$, which contradicts the choice of a. This contradiction shows that $\mathsf{dom}[G] = X$.

The uniqueness of the function G can be proved repeating the reasoning from Theorem 21.1.

22. Ranks

In this section we discuss the rank functions induced by set-like well-founded relations. The intuition behind this notions is the following.

Given any well-founded relation R on a set $X = \text{dom}[R^{\pm}]$, we can consider the set X_0 of elements $x \in X$ of whose initial interval $R(x) = \{z : \langle z, x \rangle \in R\} \setminus \{x\}$ is empty. The elements of the set X_0 are called R-minimal elements of X. The rank function rank_R assigns to elements of the set X_0 the ordinal $0 = \emptyset$. Then we consider the set X_1 of R-minimal elements of the set $X \setminus X_0$ and assign to them the ordinal $1 = \{0\}$. Next, consider the set X_2 of R-minimal elements of the set $X \setminus (X_0 \cup X_1)$. Continuing by induction, we represent X as the union $X = \bigcup_{\alpha \in \text{rank}(R)} X_\alpha$ of sets X_α indexed by ordinals α that belong to some ordinal rank(R), called the rank of the well-founded order R. The function $\text{rank}_R : X \to \text{rank}(R)$ assigns to each $x \in X$ a unique ordinal $\alpha \in \text{rank}(R)$ such that $x \in X_\alpha$. So, this is a rough idea.

Now let us give the precise definition of the rank function rank_R. In the definition for an ordinal α by $\alpha + 1$ we denote the successor $\alpha \cup \{\alpha\}$ of α , and for a set A of ordinals, $\sup A = \min\{\beta \in \mathbf{On} : \forall \alpha \in A \ (\alpha \subseteq \beta)\} = \bigcup A$, see Lemma 19.11.

Definition 22.1. For a set-like well-founded relation R, the R-rank is the function

$$\operatorname{rank}_R : \operatorname{dom}[R^{\pm}] \to \mathbf{On}, \quad \operatorname{rank}_R(x) = \sup \{\operatorname{rank}_R(y) + 1 : y \in \overline{R}(x)\}.$$

The existence of the function rank_R follows from the Recursion Theorem 21.6 applied to the function $F:\operatorname{dom}[R^\pm]\times \mathbf{U}\to\operatorname{On}, \quad F:\langle x,y\rangle\mapsto\bigcup\{z\cup\{z\}:z\in y\}.$ To see that the function F exists, observe $F=\Phi\circ\operatorname{rng}|_{\operatorname{dom}[R^\pm]}$ where $\Phi:\mathbf{U}\to\mathbf{U}, \ \Phi:y\mapsto\bigcup\{z\cup\{z\}:z\in y\}.$ The function Φ exists since $\Phi=\{\langle y,u\rangle\in\ddot{\mathbf{U}}:\exists z\in y\ (u\in z\cup\{z\})\}=\operatorname{dom}[T]$ where $T=\{\langle y,u,z\rangle:u\in z\ \land\ u=z\}=[(\mathbf{E}\cup\operatorname{Id})\times\mathbf{U}]^\circlearrowright.$

The rank function rank_R can be characterized as the smallest R-increasing function $\operatorname{dom}[R^{\pm}] \to \operatorname{On}$.

Definition 22.2. Let R, P be two relations and X, Y be two classes. A function $F: X \to Y$ is called R-to-P-increasing if for any distinct elements $x, x' \in X$ with $\langle x, x' \rangle \in R$ we have $\langle f(x), f(x') \rangle \in P$.

Theorem 22.3. Let R be a set-like well-founded relation and $X = \text{dom}[R^{\pm}]$.

- 1) The rank function $\operatorname{rank}_R: X \to \operatorname{On}$ is R-to-E-increasing.
- 2) For every R-to-E-increasing function $F: X \to \mathbf{On}$ we have $\operatorname{rank}_R(x) \leq F(x)$ for all $x \in \operatorname{dom}[R^{\pm}]$.
- 3) If $\operatorname{rank}_R[X] \neq \operatorname{On}$, then $\operatorname{rank}_R[X]$ is an ordinal.

Proof. 1. To see that the rank function rank_R is R-to-**E**-increasing, take any $x \in \mathbf{U}$ and $y \in \overline{R}(x)$. Then

$$\operatorname{rank}_R(y) < \operatorname{rank}_R(y) + 1 \le \sup \{\operatorname{rank}_R(z) + 1 : z \in \overline{R}(x)\} = \operatorname{rank}_R(x),$$

which means that $rank_R$ is R-to- \mathbf{E} -increasing.

2. Let $F: X \to \mathbf{On}$ be any R-to- \mathbf{E} -increasing function. To show that $\mathrm{rank}_R \leq F$, it suffices to show that the class $Z = \{x \in X : \mathrm{rank}_R(x) \not \leq F(x)\}$ is empty. To derive a contradiction, assume that the class Z is not empty and hence contains some element x. Then $0 \leq F(x) < \mathrm{rank}_R(x) = \sup\{\mathrm{rank}_R(y) + 1 : y \in \overline{R}(x)\}$, which implies $\overline{R}(x) \neq \emptyset$ and $x \in \mathrm{rng}[R] \subseteq \mathrm{dom}[R^{\pm}]$. Consequently, the class $A = \mathrm{dom}[R^{\pm}] \cap Z$ is not empty and

by the well-foundedness of the relation R, we can find an element $a \in A$ such that $\emptyset = \tilde{R}(a) \cap A = \tilde{R}(a) \cap \text{dom}[R^{\pm}] \cap Z = \tilde{R}(a) \cap Z$. Then $\text{rank}_R(y) \leq F(y)$ for all $y \in \tilde{R}(a)$. Since the function F is R-to- \mathbf{E} -increasing, for every $y \in \tilde{R}(a)$, we have F(y) < F(a) and hence $F(z) + 1 \leq F(a)$. Observe that for every $y \in \tilde{R}(a)$ the inequality $\text{rank}_R(y) \leq F(y)$ implies $\text{rank}_R(y) + 1 \leq F(y) + 1 \leq F(a)$ and hence $\text{rank}_R(a) = \sup\{\text{rank}_R(y) + 1 : y \in \tilde{R}(a)\} \leq F(a)$, which contradicts the choice of $a \in A \subseteq Z$.

3. Assuming that $\operatorname{rank}_R[X] \neq \operatorname{On}$, consider the smallest ordinal α in the class $\operatorname{On} \setminus \operatorname{rank}_R[X]$. We claim that $\operatorname{rank}_R[X] = \alpha$. First we show that $\operatorname{rank}_R[X] \subseteq \alpha$. Assuming that $\operatorname{rank}_R[X] \not\subset \alpha$, consider the smallest ordinal β in the set $\operatorname{rank}_R[X] \setminus \alpha$. Consider the function L: $\operatorname{On} \to \operatorname{On}$ such that $L(\gamma) = \gamma$ for any $\gamma \in \operatorname{On} \setminus [\alpha, \beta]$ and $L(\gamma) = \alpha$ for every $\gamma \in [\alpha, \beta]$, where $[\alpha, \beta] = \{x \in \operatorname{On} : \alpha \leq x \leq \beta]$. Taking into account that $[\alpha, \beta] \cap \operatorname{rank}_R[X] = \{\beta\}$, we can show that the function $L \circ \operatorname{rank}_R : \operatorname{U} \to \operatorname{On}$ is R-to- E -increasing and $L \circ \operatorname{rank}_R(x) < \operatorname{rank}_R(x)$ for any $x \in \operatorname{rank}_R^{-1}[\{\beta\}]$. But this contradicts the preceding statement. This contradiction shows that $\operatorname{rank}_R[X] \subseteq \alpha$.

Next we show that $\alpha \subseteq \operatorname{rank}_R[X]$. Assuming that this is not true, find an ordinal $\alpha' \in \alpha \backslash \operatorname{rank}_R[X]$. Since α is the smallest ordinal in the class $\operatorname{On}\backslash \operatorname{rank}_R[X]$, the set $[\alpha',\alpha)\cap \operatorname{rank}_R[X]$ is not empty and hence contains the smallest element β' . Then $\alpha' < \beta' < \alpha$. Now consider the function $L': \operatorname{On} \to \operatorname{On}$ such that $L'(\gamma) = \gamma$ for any $\gamma \in \operatorname{On} \setminus [\alpha',\beta']$ and $L(\gamma) = \alpha'$ for every $\gamma \in [\alpha',\beta']$. Taking into account that $[\alpha',\beta']\cap \operatorname{rank}_R[X] = \{\beta'\}$, we can show that the function $L' \circ \operatorname{rank}_R : X \to \operatorname{On}$ is R-to-E-increasing and $L' \circ \operatorname{rank}_R(x') < \operatorname{rank}_R(x')$ for any $x' \in \operatorname{rank}_R^{-1}[\{\beta'\}]$. But this contradicts the statement (2). This contradiction shows that $\alpha \subseteq \operatorname{rank}_R[X]$ and hence $\operatorname{rank}_R[X] = \alpha$ is an ordinal.

For every set-like well-founded relation R, let $\operatorname{rank}(R) = \operatorname{rng}[\operatorname{rank}_R] \subseteq \operatorname{On}$. By Theorem 22.3(3), the class $\operatorname{rank}(R)$ either coincides with the class On or is an ordinal. In the latter case this ordinal is called the rank of the well-founded relation R.

Exercise 22.4. Show that a well-founded relation R is a set if and only if it is locally set, set-like and its rank rank(R) is a set.

23. Well-orders

In this section we apply ranks to constructing isomorphisms between set-like well-orders.

Definition 23.1. Let R, P be two orders. A bijective function $F : dom[R^{\pm}] \to dom[P^{\pm}]$ is called an *order isomorphism* if the function F is R-to-P-increasing and F^{-1} is P-to-R-increasing. In this case the function F^{-1} is also an order isomorphism.

Two orders R, P are called *isomorphic* if there exists an order isomorphism $F: dom[R^{\pm}] \to dom[P^{\pm}]$.

Proposition 23.2. Let R, P be two linear orders. For an bijective function $F : dom[R^{\pm}] \to dom[P^{\pm}]$ the following conditions are equivalent:

- 1) F is an order isomorphism;
- 2) F is an R-to-P increasing;
- 3) F^{-1} is P-to-R-increasing.

Proof. The implication $(1) \Rightarrow (2)$ is trivial.

 $(2) \Rightarrow (3)$. Assume that the condition (2) holds but (3) does not. Then there exist distinct points $x, x' \in X$ such that $\langle F(x), F(x') \rangle \in P$ but $\langle x, x' \rangle \notin R$. Then $\langle x', x \rangle \in R$ by the

linearity of the order R. Applying the condition (2), we obtain $\langle F(x'), F(x) \rangle \in P$. Now the antisymmetry of the relation P ensures that F(x) = F(x') which contradicts our assumption.

 $(3) \Rightarrow (1)$ Assume that the condition (3) holds but (1) does not. Then there exist distinct points $x, x' \in \text{dom}[R^{\pm}]$ such that $\langle x, x' \rangle \in R$ but $\langle F(x), F(x') \rangle \notin P$. Then $F(x) \neq F(x')$ by the injectivity of the function F and $\langle F(x'), F(x) \rangle \in P$ by the linearity of the order P. Applying the condition (3), we obtain $\langle x', x \rangle \in R$. Now the antisymmetry of the relation R ensures that x = x', which contradicts the choice of x, x'.

Proposition 23.3. Let R be a well-order. Every order isomorphism $F : dom[R^{\pm}] \to dom[R^{\pm}]$ is equal to the identity function of $Id[dom[R^{\pm}]]$.

Proof. To derive a contradiction, assume that $F(x) \neq x$ for some $x \in \text{dom}[R^{\pm}]$. Then the class $A = \{z \in R^{-1}[\{x\}] : F(z) \neq z\}$ is not empty. Since R is a well-order, the class A contains an element $a \in A$ such that $\overline{R}(a) \cap A = \emptyset$. The transitivity of the relation R ensures that $\overline{R}(a) \subset R^{-1}[\{x\}]$. It follows from $\overline{R}(a) \cap A = \emptyset$ that $F(z) = z = F^{-1}(z)$ for all $z \in \overline{R}(a)$. It follows from $a \in A$ that $a \in A$

It follows from $a \in A$ that $F(a) \neq a$. Since the order R is linear, either $F(a) \in R(a)$ or $a \in \overline{R}(F(a))$. In the first case we get the equality F(F(a)) = F(a), which contradicts the injectivity of F. In the second case, the inclusion $a \in \overline{R}(F(a))$ implies $F^{-1}(a) \in \overline{R}(a)$ and then $F^{-1}(F^{-1}(a)) = F^{-1}(a)$, which contradicts the injectivity of the function F^{-1} .

Corollary 23.4. For any well-orders P, R there exists at most one order-isomorphism from P to R.

Proof. Let $\Phi, \Psi : \mathsf{dom}[P^{\pm}] \to \mathsf{dom}[R^{\pm}]$ be two order-isomorphisms. Then $\Phi^{-1} \circ \Psi : \mathsf{dom}[P^{\pm}] \to \mathsf{dom}[P^{\pm}]$ is an order isomorphism of the well-order P. By Proposition 23.3, $\Phi^{-1} \circ \Psi = \mathsf{Id} \upharpoonright \mathsf{dom}[P^{\pm}]$. Applying to this equality the bijective function Φ , we obtain the desired equality $\Psi = \Phi \circ \Phi^{-1} \circ \Psi = \Phi \circ \mathsf{Id} \upharpoonright \mathsf{dom}[P^{\pm}] = \Phi$.

Let R be an order. For an element $x \in \text{dom}[R^{\pm}]$, the set $\overline{R}(x) = R^{-1}[\{x\}] \setminus \{x\}$ is called the *initial interval* of $\text{dom}[R^{\pm}]$ and the partial order $R \upharpoonright \overline{R}(x)$ is called an *initial interval* of the partial order R.

Proposition 23.5. A well-order R cannot be isomorphic to its own initial interval.

Proof. Assume that for some $x \in \text{dom}[R^{\pm}]$ there exists an order isomorphism $F : \text{dom}[R^{\pm}] \to \overline{R}(x)$. Then $F(x) \in \overline{R}(x)$ and hence $F(x) \neq x$. Consider the class $A = \{z \in R^{-1}[\{x\}] : F(z) \neq z\}$ which contains x and hence is not empty. Since the order R is well-founded, the class A contains an element a such that $\overline{R}(a) \cap A = \emptyset$. The transitivity of R ensures that $\overline{R}(a) \subset R^{-1}[\{x\}] \setminus A$ and hence $F(z) = z = F^{-1}(z)$ for all $z \in \overline{R}(a)$. Since the order R is linear and $F(a) \neq a$, either $F(a) \in \overline{R}(a)$ or $a \in \overline{R}(F(a))$. In the first case we obtain that F(a) = F(F(a)), which contradicts the injectivity of F. If $a \in \overline{R}(F(a))$, then $a \in \overline{R}(F(a)) \subset \overline{R}(x) = F[\text{dom}[R]]$ and hence $F^{-1}(a) \in \text{dom}[R]$ exists. Since F is an order-isomorphism, $a \in \overline{R}(F(a))$ implies $F^{-1}(a) \in \overline{R}(a)$ and then $F^{-1}(F^{-1}(a)) = F^{-1}(a)$, which contradicts the injectivity of F^{-1} .

Theorem 23.6. For any set-like well-order R, the function $\operatorname{rank}_R : \operatorname{dom}[R^{\pm}] \to \operatorname{rank}(R)$ is an order isomorphism.

Proof. By Theorem 22.3, the function rank_R is R-to- \mathbf{E} -increasing. Since the order R is linear, for any distinct elements $x, x' \in \operatorname{dom}[R^{\pm}]$ we have $\langle x, x' \rangle \in R$ or $\langle x', x \rangle \in R$. Taking into account that rank_R is P-to- \mathbf{E} increasing, we conclude that $\operatorname{rank}_R(x) < \operatorname{rank}_R(x')$ or $\operatorname{rank}_R(x') < \operatorname{rank}_R(x)$. In both cases we have $\operatorname{rank}_R(x) \neq \operatorname{rank}_R(x')$, which means that the function rank_R is injective. By the choice of $\operatorname{rank}(R) = \operatorname{rng}[\operatorname{rank}_R]$, the function rank_R is surjective and hence bijective. The P-to- \mathbf{E} -increasing property and Proposition 23.2 imply that $\operatorname{rank}_R : \operatorname{dom}[R^{\pm}] \to \operatorname{rank}(R)$ is an order isomorphism.

Corollary 23.7 (Cantor). For set-like well-orders R, P one of the following conditions holds:

- 1) R and P are isomorphic;
- 2) R is isomorphic to a unique initial interval of P;
- 3) P is isomorphic to a unique initial interval of R.

Proof. The uniqueness of the initial intervals in the statements (2),(3) follows from Proposition 23.5. It remains to prove the existence of order-isomorphisms in one of the statements (1)–(3).

By Theorem 23.6, the functions $\operatorname{rank}_R : \operatorname{dom}[R^{\pm}] \to \operatorname{rank}(R)$ and $\operatorname{rank}_P : \operatorname{dom}[R^{\pm}] \to \operatorname{rank}(P)$ are order isomorphisms. Each of the ranks $\operatorname{rank}(R)$, $\operatorname{rank}(P)$ is either **On** or some ordinal. Consequently, three cases are possible.

- 1) rank(R) = rank(P). In this case the well-orders R, P are isomorphic.
- 2) $rank(R) \in rank(P)$. In this case the ordinal rank(R) is an initial interval of rank(P) and the well-order R is isomorphic to an initial interval of the well-order P.
- 3) $\operatorname{rank}(P) \in \operatorname{rank}(R)$. In this case the ordinal $\operatorname{rank}(P)$ is an initial interval of $\operatorname{rank}(R)$ and the well-order P is isomorphic to an initial interval of the well-order R.

Let **WO** be the class of well-orders which are sets. The function $\operatorname{rank}: \operatorname{WO} \to \operatorname{On}$ assigns to each well-order $R \in \operatorname{WO}$ the ordinal $\operatorname{rank}(R)$, called the order type of R. For any ordinal α the preimage $\operatorname{rank}^{-1}[\{\alpha\}]$ is the equivalence class of all well-orders that are isomorphic to α . Initially ordinals were thought as such equivalence classes (till John von Neumann discovered the notion of an ordinal we use nowadays).

Exercise 23.8. Prove that the class WO and the function rank: WO \rightarrow On exist.

The following theorem was proved by Friedrich Hartogs in 1915.

Theorem 23.9 (Hartogs). For any set x there exists an ordinal α admitting no injective function $f: \alpha \to x$.

Proof. Let WO(x) be the set whose elements are well-orders w with $\mathsf{dom}[w^{\pm}] \subseteq x$. Since $WO(x) \subseteq \mathcal{P}(x \times x)$, the class WO(x) is a set by the Axiom of Power-set and Exercises 9.1, 6.3. Let $\mathsf{rank} : WO(x) \to \mathbf{On}$ be the function assigning to each well-order $w \in WO(x)$ its rank $\mathsf{rank}(w) = \mathsf{rank}_W[\mathsf{dom}[w^{\pm}]]$. By the Axiom of Replacement, the image $\mathsf{rank}[WO(x)] \subseteq \mathbf{On}$ is a set and so is its union $\alpha = \bigcup \mathsf{rank}[WO(x)]$, which is an ordinal by Theorem 19.6(5).

We claim that the ordinal α admits no injective function $f: \alpha \to x$. In the opposite case, $w = \{\langle f(\gamma), f(\beta) \rangle : \gamma \in \beta \in \alpha\}$ would be a well-order in the set WO(x) such that $\mathsf{rank}(w) = \alpha \in \alpha$, which is forbidden by the definition of an ordinal.

Part 5. Foundation and Constructibility

In this part we constuct two special proper subclasses V and L, related to the Axiom of Foundation and the Axiom of Global Choice. Using inner models with shall prove the consistency of the equality U = L.

24. Foundation

In this section we construct a proper class V called the *von Neumann universe*. This class is the smallest class that contains all ordinals and is closed under the operations of taking power-set and union. The restriction E|V of the membership relation to this class is well-founded and the Axiom of Foundation is equivalent to the equality U = V. The class V is defined as the union of the von Neumann cumulative hierarchy.

Definition 24.1 (Cumulative hierarchy of von Neumann). The *cumulative hierarchy of von Neumann* is the transfinite sequence of sets $(V_{\alpha})_{\alpha \in \mathbf{On}}$, defined by the recursive formula

$$V_{\alpha} = \bigcup \{ \mathcal{P}(V_{\gamma}) : \gamma \in \alpha \}, \quad \alpha \in \mathbf{On}.$$

The class $\mathbf{V} = \bigcup_{\alpha \in \mathbf{On}} V_{\alpha}$ is called the von Neumann universe.

Theorem 24.2. The von Neumann cumulative hierarchy $(V_{\alpha})_{\alpha \in \mathbf{On}}$ is well-defined and has the following properties:

- 1) $\{\alpha\} \cup V_{\alpha} \subseteq V_{\alpha+1} = \mathcal{P}(V_{\alpha}) \text{ for every ordinal } \alpha.$
- 2) $V_{\alpha} = \bigcup \{V_{\gamma} : \gamma \in \alpha\}$ for any limit ordinal α .
- 3) For every ordinal α the set V_{α} is transitive.
- 4) The class $\mathbf{V} = \bigcup \{V_{\alpha} : \alpha \in \mathbf{On}\}$ is transitive, contains all ordinals and hence is proper.
- 5) The relation $\mathbf{E}[\mathbf{V} \text{ is set-like, well-founded and } \operatorname{rank}_{\mathbf{E}[\mathbf{V}]}[V_{\alpha}] \subseteq \alpha$.
- 6) Each subset of V is an element of V, which can be written as $\mathcal{P}(V) \subseteq V$.
- 7) V is a subclass of any class X such that $\mathcal{P}(X) \subseteq X$.

Proof. 0. The existence of the function

$$V_*: \mathbf{On} \to \mathbf{U}, \quad V_*: \alpha \mapsto V_{\alpha},$$

determining the von Neumann cumulative hierarchy follows from the Recursion Theorem 21.1 applied to the set-like well-order $\mathbf{E} \upharpoonright \mathbf{On}$ and the function

$$F: \mathbf{On} \times \mathbf{U} \to \mathbf{U}, \ F: \langle \alpha, y \rangle \mapsto \bigcup \{ \mathcal{P}(z) : z \in y \}.$$

The existence of the function V_* also implies the existence of the "inverse function"

$$\Lambda: \mathbf{V} \to \mathbf{On}, \ \Lambda: x \mapsto \min\{\alpha \in \mathbf{On}: x \in V_{\alpha}\},\$$

where $\mathbf{V} = \bigcup_{\alpha \in \mathbf{On}} V_{\alpha} = \bigcup V_*[\mathbf{On}]$. The function Λ exists since

$$\Lambda = \{ \langle x, \alpha \rangle \in \mathbf{V} \times \mathbf{On} : x \in V_{\alpha} \land \forall \gamma \in \alpha \ (x \notin V_{\gamma}) \}.$$

1. For any ordinal α , the definition of $V_{\alpha} = \bigcup \{ \mathcal{P}(V_{\gamma}) : \gamma \in \alpha \}$ implies that $V_{\alpha} \subseteq V_{\beta}$ and hence $\mathcal{P}(V_{\alpha}) \subseteq \mathcal{P}(V_{\beta})$ for any ordinals $\alpha \leq \beta$. Then

$$V_{\alpha+1} = \bigcup \{ \mathcal{P}(V_{\gamma}) : \gamma \leq \alpha \} = \mathcal{P}(V_{\alpha}).$$

Next we show that $\forall \alpha \in \mathbf{On} \ (\alpha \in V_{\alpha+1})$. Assuming that this is not true, we can use the well-foundedness of the order $\mathbf{E} \upharpoonright \mathbf{On}$ and find an ordinal α such that $\alpha \notin V_{\alpha+1}$ but

 $\forall \gamma \in \alpha \ (\gamma \in V_{\gamma+1} \subseteq V_{\alpha})$. Then $\alpha = \{\gamma : \gamma \in \alpha\} \subseteq V_{\alpha} \text{ and hence } \alpha \in \mathcal{P}(V_{\alpha}) = V_{\alpha+1}, \text{ which contradicts the choice of } \alpha.$

2. If α is a limit ordinal, then

$$V_{\alpha} = \bigcup \{ \mathcal{P}(V_{\gamma}) : \gamma \in \alpha \} = \bigcup \{ V_{\gamma+1} : \gamma \in \alpha \} = \bigcup \{ V_{\gamma} : \gamma \in \alpha \}.$$

- 3. For every ordinal α and every set $x \in V_{\alpha}$, we can find an ordinal $\beta \in \alpha$ such that $\alpha \in V_{\beta+1} = \mathcal{P}(V_{\beta})$. Then $x \subseteq V_{\beta} \subseteq V_{\alpha}$.
- 4. By the statement (1) that the class **V** contains all ordinals and hence is a proper class according to Theorem 19.6(6) and Exercise 6.3. The transitivity of the sets V_{α} , $\alpha \in \mathbf{On}$, implies the transitivity of the union $\mathbf{V} = \bigcup_{\alpha \in \mathbf{On}}$.
- 5. It is clear that the relation $\mathbf{E} \upharpoonright \mathbf{V}$ is set-like. To see that it is well-founded, take any nonempty subclass $X \subseteq \mathbf{V}$. We should find an element $x \in X$ such that $x \cap X = \emptyset$. Since the relation $\mathbf{E} \upharpoonright \mathbf{On}$ is well-founded, the nonempty subclass $\Lambda[X]$ of \mathbf{On} contains the smallest ordinal, denoted by α . For this ordinal α we have $X \cap V_{\alpha} \neq \emptyset$ but $X \cap V_{\gamma} = \emptyset$ for all $\gamma \in \alpha$. Take any set $x \in X \cap V_{\alpha}$. Since $V_0 = \bigcup \{\mathcal{P}(V_{\gamma}) : \gamma \in \emptyset\} = \bigcup \emptyset = \emptyset$, the ordinal α is not empty.

If α is a limit ordinal, then the statement (2) implies that $x \in V_{\gamma}$ for some $\gamma \in \alpha$, which contradicts the minimality of α . Therefore, $\alpha = \beta + 1$ and for some ordinal β . Then $x \in V_{\alpha} = V_{\beta+1} = \mathcal{P}(V_{\beta})$ and hence $x \subseteq V_{\beta}$. The choice of α guarantees that $x \cap X \subseteq V_{\beta} \cap X = \emptyset$. Since the relation $\mathbf{E} \upharpoonright \mathbf{V}$ is set-like and well-founded it has a well-defined function

$$\operatorname{rank}_{\mathbf{E}|\mathbf{V}}: \mathbf{V} \to \mathbf{On}, \quad \operatorname{rank}_{\mathbf{E}|\mathbf{V}}: x \mapsto \sup \{\operatorname{rank}_{\mathbf{E}|\mathbf{V}}(y) + 1: y \in x\}.$$

The embedding $\operatorname{\mathsf{rank}}_{\mathbf{E} \mid \mathbf{V}}[V_{\alpha}] \subseteq \alpha$ will be proved by transfinite induction. For $\alpha = 0$ we have $\operatorname{\mathsf{rank}}_{\mathbf{E} \mid \mathbf{V}}[V_0] = \operatorname{\mathsf{rank}}_{\mathbf{E} \mid \mathbf{V}}[\emptyset] = \emptyset = 0$. Assume that for some ordinal α and all its elements $\beta \in \alpha$ the embedding $\operatorname{\mathsf{rank}}_{\mathbf{E} \mid \mathbf{V}}[V_{\beta}] \subseteq \beta$ has been proved. If α is a limit ordinal, then

$$\mathsf{rank}_{\mathbf{E} \upharpoonright \mathbf{V}}[V_{\alpha}] = \mathsf{rank}_{\mathbf{E} \upharpoonright \mathbf{V}}[\bigcup_{\beta \in \alpha} V_{\beta}] = \bigcup_{\beta \in \alpha} \mathsf{rank}_{\mathbf{E} \upharpoonright \mathbf{V}}[V_{\beta}] \subseteq \bigcup_{\beta \in \alpha} \beta = \alpha.$$

If α is a successor ordinal, then $\alpha = \beta + 1$ for some ordinal $\beta \in \alpha$ and then for every $x \in V_{\alpha} = \mathcal{P}(V_{\beta})$ and $y \in x$ we have $y \in x \subseteq V_{\beta}$ and hence $\mathsf{rank}_{\mathbf{E} | \mathbf{V}}(y) \in \mathsf{rank}_{\mathbf{E} | \mathbf{V}}[V_{\beta}] \subseteq \beta$. So, $\mathsf{rank}_{\mathbf{E} | \mathbf{V}}(y) \in \beta$ and $\mathsf{rank}_{\mathbf{E} | \mathbf{V}}(y) + 1 \leq \beta$. Then

$$\operatorname{rank}_{\mathbf{E} \upharpoonright \mathbf{V}}(x) = \sup \{ \operatorname{rank}_{\mathbf{E} \upharpoonright \mathbf{V}}(y) + 1 : y \in x \} \le \beta \in \alpha$$

and hence $\operatorname{\mathsf{rank}}_{\mathbf{E} \upharpoonright \mathbf{V}}(x) \in \alpha$ and $\operatorname{\mathsf{rank}}_{\mathbf{E} \upharpoonright \mathbf{V}}(V_{\alpha}) \subseteq \alpha$.

- 6. Assume that x is a subset of \mathbf{V} . By the Axiom of Replacement, the image $\Lambda[x] \subset \mathbf{On}$ is a set and hence $\alpha = \bigcup \Lambda[x] = \bigcup \{\Lambda(y) : y \in x\}$ is an ordinal according to Theorem 19.3(5). Then for every $y \in x$ we have $\Lambda(y) \subseteq \bigcup \Lambda[x] = \alpha$ and thus $x \in V_{\Lambda(y)} \subseteq V_{\alpha}$ and $x \in V_{\alpha+1} \subseteq \mathbf{V}$.
- 7. Assume that \mathbf{X} is a class such that $\mathbf{On} \subseteq \mathbf{X}$ and $\mathcal{P}(\mathbf{X}) \subseteq \widetilde{\mathbf{X}}$. We claim that for every $\alpha \in \mathbf{On}$ the set V_{α} is a subset of \mathbf{X} . It is easy to show that the class $A = \{\alpha \in \mathbf{On} : V_{\alpha} \subseteq \mathbf{X}\}$ exists. If $A = \mathbf{On}$, then $\mathbf{V} = \bigcup_{\alpha \in \mathbf{On}} V_{\alpha} \subseteq \mathbf{X}$ and we are done. So assume that $A \neq \mathbf{On}$ and take the smallest ordinal α in the subclass $\mathbf{On} \setminus A$. Such an ordinal exists since the relation $\mathbf{E} \upharpoonright \mathbf{On}$ is well-founded. Then $\alpha \subseteq A$ and hence $V_{\gamma} \subseteq \mathbf{X}$ for all $\gamma \in \alpha$. If α is a limit ordinal, then $V_{\alpha} = \bigcup_{\gamma \in \alpha} V_{\gamma} \subseteq \mathbf{X}$ and hence $\alpha \in A$, which contradicts the choice of A. This contradiction shows that α is not limit and hence $\alpha = \gamma + 1$ for some ordinal γ . The choice of α guarantees that $V_{\gamma} \subseteq \mathbf{X}$. Then $V_{\alpha} = \mathcal{P}(V_{\gamma}) \subseteq \mathcal{P}(\mathbf{X}) \subseteq X$. But this contradicts the choice of α . This contradiction shows that $A = \mathbf{On}$ and $\mathbf{V} = \bigcup_{\alpha \in \mathbf{On}} V_{\alpha} \subseteq \mathbf{X}$.

Remark 24.3. Theorem 24.2(6,7) implies that the von Neumann class **V** is the smallest class **X** such that $\mathcal{P}(\mathbf{X}) \subseteq \mathbf{X}$.

Exercise 24.4. Prove that the set V_{ω} coincides with the set of all hereditarily finite sets.

Theorem 24.5. The Axiom of Foundation is equivalent to the equality U = V.

Proof. If U = V then the relation $E = E \upharpoonright V$ is well-founded by Theorem 24.2(5) and hence the Axiom of Foundation holds.

Now assumming the Axiom of Foundation, we shall prove that $\mathbf{U} = \mathbf{V}$. To derive a contradiction, assume that $\mathbf{U} \setminus \mathbf{V}$ is not empty and fix any set $a \in \mathbf{U} \setminus \mathbf{V}$. By Theorem 24.2(6), the set $a \setminus \mathbf{V}$ is not empty. By Theorem 21.5, the set $a \setminus \mathbf{V}$ is contained in some transitive set t. By the Axiom of Foundation, the set $t \setminus \mathbf{V}$ contains an element $u \in t \setminus \mathbf{V}$ such that $u \cap (t \setminus \mathbf{V}) = \emptyset$. By the transitivity of the set t, we have

$$u \subseteq t \setminus (t \setminus \mathbf{V}) = t \cap \mathbf{V} \subset \mathbf{V}.$$

Applying Theorem 24.2(6), we conclude that $u \in \mathbf{V}$, which contradicts the choice of u. \square

Exercise 24.6. Show that for every ordinal α and sets $x, y \in V_{\alpha}$ we have

- (1) $x \setminus y \in V_{\alpha}$;
- (2) $\bigcup x \in V_{\alpha}$;
- (3) $\operatorname{\mathsf{dom}}[x] \in V_{\alpha}$;
- (4) $x^{-1} \in V_{\alpha}$;
- (5) $x^{\circlearrowleft} \in V_{\alpha}$;
- (6) $\{x,y\} \in V_{\alpha+1}$;
- (7) $x \times y \in V_{\alpha+2}$.
- (8) $\mathbf{E} \cap (x \times y) \in V_{\alpha+2}$.

25. Constructibility

In this section we introduce the Gödel's constructible universe \mathbf{L} that consists of the sets that can be constructed from ordinals by applications of finitely many Gödel's operations $\mathsf{G}_1\mathsf{-}\mathsf{G}_8$.

Gödel's operations

Definition 25.1. By the *Gödel's operations* we understand the following nine functions:

- (0) $G_0: \mathbf{U} \times \mathbf{U} \to \mathbf{U}, G_0: \langle x, y \rangle \mapsto x$
- (1) $G_1: \mathbf{U} \times \mathbf{U} \to \mathbf{U}, G_1: \langle x, y \rangle \mapsto x \setminus y = \{u \in x : u \notin y\}$
- (2) $\mathsf{G}_2: \mathbf{U} \times \mathbf{U} \to \mathbf{U}, \, \mathsf{G}_2: \langle x, y \rangle \mapsto \{x, y\}$
- (3) $G_3: \mathbf{U} \times \mathbf{U} \to \mathbf{U}, G_3: \langle x, y \rangle \mapsto \mathbf{E} \cap (x \times y)$
- (4) $\mathsf{G}_4: \mathbf{U} \times \mathbf{U} \to \mathbf{U}, \; \mathsf{G}_4: \langle x, y \rangle \mapsto x^{-1} = \{\langle v, u \rangle : \langle u, v \rangle \in x\}$
- (5) $G_5: \mathbf{U} \times \mathbf{U} \to \mathbf{U}, G_5: \langle x, y \rangle \mapsto \mathsf{dom}[x] = \{u \in \mathbf{U}: \exists v \in \mathbf{U} \ (\langle u, v \rangle \in x)\}$
- (6) $G_6: \mathbf{U} \times \mathbf{U} \to \mathbf{U}, G_6: \langle x, y \rangle \mapsto x \times y = \{\langle u, v \rangle : u \in x \land v \in y\}$
- (7) $G_7: \mathbf{U} \times \mathbf{U} \to \mathbf{U}, G_7: \langle x, y \rangle \mapsto \bigcup x = \{z: \exists y \in x \ (z \in y)\}$
- (8) $\mathsf{G}_8: \mathbf{U} \times \mathbf{U} \to \mathbf{U}, \; \mathsf{G}_8: \langle x, y \rangle \mapsto x^{\circlearrowright} = \{\langle w, u, v \rangle : \langle u, v, w \rangle \in x\}.$

Observe that the functions G_4 – G_8 do not depend on the second variable. Nonetheless we have written them as functions of two variable for uniform treatment of all Gödel's operations.

Exercise 25.2. Prove that the functions G_0 – G_8 exist.

Remark 25.3. The enumeration of Gödel's operations agrees with the order of corresponding axioms of the Classical Set Theory.

Definition 25.4. The Gödel's extension is the function $G: U \to U$ assigning to every set x the set

$$\mathsf{G}(x) = \bigcup_{i=0}^8 \mathsf{G}_i[x \times x] = \{u, u \setminus v, \{u, v\}, \mathbf{E} \cap (u \times v), u^{-1}, \mathsf{dom}[u], u \times v, \bigcup u, u^{\circlearrowright} : u, v \in x\}.$$

So, the set G(x) consists of the results of application of Gödel's operations to elements of

Iterating the function G, we obtain the function sequence $(G^{\circ n})_{n \in \omega}$ such that $G^{\circ 0} = \mathbf{Id}$ and $\mathsf{G}^{\circ(n+1)} = \mathsf{G} \circ \mathsf{G}^{\circ n}$ for every $n \in \omega$. The function sequence $(\mathsf{G}^{\circ n})_{n \in \omega}$ exists by Theorem 21.4. Finally, consider the function

$$\mathsf{G}^{\circ\omega}:\mathbf{U}\to\mathbf{U},\quad \mathsf{G}^{\circ\omega}:x\mapsto \bigcup_{n\in\omega}\mathsf{G}^{\circ n}(x),$$

assigning to each set x the set $G^{\circ\omega}(x)$ called the Gödel's hull of x.

The principal result of this section is Theorem 25.10 saying that the Gödel's hull of any transitive set is a transitive set. We precede this theorem by several exercises and lemmas.

Exercise 25.5. Prove that the Gödel's extension is monotone in the sense that

$$G(x) \subseteq G(y)$$

for any sets $x \subseteq y$.

Exercise 25.6. Prove that

$$x \subseteq \mathsf{G}^{\circ n}(x) \subseteq \mathsf{G}^{\circ (n+1)}(x) \subseteq \mathsf{G}^{\circ \omega}(x)$$

for every set x and natural number n.

Lemma 25.7. For any sets x, y, z we have

- 1) $TC(\{x,y\}) = \{x,y\} \cup TC(x \cup y)$.
- 2) $\mathsf{TC}(\langle x, y \rangle) \subseteq \mathsf{G}(\{x, y\}) \cup \mathsf{TC}(x \cup y)$.
- 3) $\mathsf{TC}(\langle x,y,z\rangle) \subseteq \mathsf{G}^{\circ 3}(\{x,y,z\}) \cup \mathsf{TC}(x \cup y \cup z).$ 4) $\mathsf{TC}(x \times y) \subseteq \mathsf{G}^{\circ 2}(\{x,y\}) \cup \mathsf{G}(\mathsf{TC}(x \cup y)).$

Proof. 1. The equality $TC(\{x,y\}) = \{x,y\} \cup TC(x) \cup TC(y)$ follows from Proposition 18.8.

2. By the Kuratowski definition of the ordered pair $\langle x,y\rangle = \{\{x\},\{x,y\}\}$ and Proposition 18.8,

$$\mathsf{TC}(\langle x,y\rangle) = \{\{x\},\{x,y\}\} \cup \mathsf{TC}(\{x\}) \cup \mathsf{TC}(\{x,y\}) = \{\{x\},\{x,y\}\} \cup \mathsf{TC}(\{x,y\}) = \{\mathsf{G}_2(x,x),\mathsf{G}_2(x,y)\} \cup \{x,y\} \cup \mathsf{TC}(x) \cup \mathsf{TC}(y) \subseteq \mathsf{G}(\{x,y\}) \cup \mathsf{TC}(x \cup y).$$

3. Since $\langle x, y, z \rangle = \langle \langle x, y \rangle, z \rangle$ we can apply Proposition 18.8 and the preceding statement to conclude that

$$\begin{split} \mathsf{TC}(\langle x,y,z\rangle) &= \{ \{\langle x,y\rangle\}, \{\langle x,y\rangle,z\} \} \cup \mathsf{TC}(\langle x,y\rangle) \cup \mathsf{TC}(z) \subseteq \\ & \{ \{ \{x\}, \{x,y\}\}\}, \{ \{\{x\}, \{x,y\}\},z\} \} \cup \mathsf{G}(\{x,y\}) \cup \mathsf{TC}(x \cup y) \cup \mathsf{TC}(z) \subseteq \\ & \mathsf{G}^{\circ 3}(\{x,y,z\}) \cup \mathsf{TC}(x \cup y \cup z). \end{split}$$

4. By Proposition 18.8 and the second statement,

$$\begin{split} \mathsf{TC}(x\times y) &= (x\times y) \cup \bigcup_{\langle u,v\rangle \in x\times y} \mathsf{TC}(\langle u,v\rangle) \subseteq \\ &(\bigcup \{x\times y\}) \cup \bigcup_{\langle u,v\rangle \in x\times y} \left(\mathsf{G}(\{u,v\}) \cup \mathsf{TC}(u\cup v)\right) \subseteq \\ &\mathsf{G}^{\circ 2}(\{x,y\}) \cup \mathsf{G}(x\cup y) \cup \mathsf{TC}(x\cup y) \subseteq \mathsf{G}^{\circ 2}(\{x,y\}) \cup \mathsf{G}(\mathsf{TC}(x\cup y)). \end{split}$$

Lemma 25.8. For every set x we have $TC(G(x)) \subseteq G^{\circ 3}(TC(x))$.

Proof. By the monotonicity of Gödel's extension, $G(x) \subseteq G(TC(x)) \subseteq G^{\circ 3}(TC(x))$. By Proposition 18.8, $TC(G(x)) = G(x) \cup \bigcup \{TC(g) : g \in G(x)\}$. Therefore, it suffices to prove that $TC(g) \subseteq G^{\circ 3}(TC(x))$ for any

$$g \in \mathsf{G}(x) = \{u, u \setminus v, \{u, v\}, \mathbf{E} \cap (u \times v), u^{-1}, \mathsf{dom}[u], u \times v, \bigcup u, u^{\circlearrowright} : u, v \in x\}.$$

Nine cases are possible.

- 0. If g=u for some $u\in x$, then $\mathsf{TC}(g)\subset \mathsf{TC}(x)\subseteq \mathsf{G}^{\circ 3}(\mathsf{TC}(x))$ by the monotonicity of Gödel's extension G .
 - 1. If $g = u \setminus v$ for some $u, v \in x$, then $\mathsf{TC}(g) \subseteq \mathsf{TC}(u) \subseteq \mathsf{TC}(x)$.
- 2. If $g = \{u, v\}$ for some $u, v \in x$, then $\mathsf{TC}(g) = \mathsf{TC}(\{u, v\}) \cup \mathsf{TC}(u) \cup \mathsf{TC}(v) \subseteq \mathsf{TC}(x)$ according to Lemma 25.7(1).
 - 3. If $g = \mathbf{E} \cap (u \times v)$ for some $u, v \in x$, then $g \subseteq u \times v$ and

$$\mathsf{TC}(g) \subseteq \mathsf{TC}(u \times v) \subseteq \mathsf{G}^{\circ 2}(\{u,v\}) \cup \mathsf{G}(\mathsf{TC}(u \cup v)) \subseteq \mathsf{G}^{\circ 2}(\mathsf{TC}(x))$$

by Lemma 25.7(4).

4. If $g = u^{-1}$ for some $u \in x$, then

$$\begin{split} \mathsf{TC}(g) &= u^{-1} \cup \bigcup \{ \mathsf{TC}(\langle b, a \rangle) : \langle a, b \rangle \in u \} \subseteq \\ & (\bigcup \{u^{-1}\}) \cup \bigcup \{ \mathsf{G}(\{a, b\}) \cup \mathsf{TC}(a \cup b) : \langle a, b \rangle \in u \} \subseteq \\ & (\bigcup \mathsf{G}(x)) \cup \mathsf{G}(\bigcup \bigcup u) \cup \mathsf{TC}(\bigcup \bigcup u) \subseteq \\ & \mathsf{G}^{\circ 2}(x) \cup \mathsf{G}(\mathsf{TC}(x)) \cup \mathsf{TC}(x) \subseteq \mathsf{G}^{\circ 2}(\mathsf{TC}(x)) \end{split}$$

according to proposition 21.5 and Lemma 25.7(2).

5. If $g = \mathsf{dom}[u]$ for some $u \in x$, then $g = \mathsf{dom}[u] = \{a : \langle a, b \rangle \in u\}$. Applying Proposition 21.5 and the equality $\bigcup \bigcup \langle a, b \rangle = a \cup b$, we can see that

$$\begin{split} \mathsf{TC}(g) &= g \cup \bigcup \{ \mathsf{TC}(a) : \langle a,b \rangle \in u \} \subseteq \mathsf{dom}[u] \cup \bigcup \{ \mathsf{TC}(a \cup b) : \langle a,b \rangle \in u \} = \\ \mathsf{dom}[u] \cup \bigcup \{ \mathsf{TC}(\bigcup \bigcup \langle a,b \rangle) : \langle a,b \rangle \in u \} \subseteq \mathsf{dom}[u] \cup \bigcup \{ \mathsf{TC}(\bigcup \bigcup v) : v \in u \} \subseteq \\ \mathsf{dom}[u] \cup \mathsf{TC}(\bigcup \bigcup \bigcup u) \subseteq \mathsf{G}[x] \cup \mathsf{TC}(x) \subseteq \mathsf{G}(\mathsf{TC}(x)). \end{split}$$

6. If $g = u \times v$ for some $u, v \in x$, then by Lemma 25.7(4),

$$\mathsf{TC}(g) = \mathsf{TC}(u \times v) \subseteq \mathsf{G}^{\circ 2}(\{u,v\}) \cup \mathsf{G}(\mathsf{TC}(x) \cup \mathsf{TC}(y)) \subseteq \mathsf{G}^{\circ 2}(x) \cup \mathsf{G}(\mathsf{TC}(x)) \subseteq \mathsf{G}^{\circ 2}(\mathsf{TC}(x)).$$

- 7. If $g = \bigcup u$ for some $u \in x$, then $\mathsf{TC}(g) \subseteq \mathsf{TC}(u) \subseteq \mathsf{TC}(x)$.
- 3. Assume that $g = u^{\circlearrowleft}$ for some $u \in x$. Theorem 21.5 implies that for any set a its transitive closure TC(x) is equal to the union $\bigcup_{n\in\omega}\bigcup^n a$ where $\bigcup^0 a=a$ and $\bigcup^{n+1} a=\bigcup(\bigcup^n a)$.

Given any triple $\langle a,b,c\rangle \in u$ we can recover the union $a \cup b \cup c$ considering consecutive unions:

- $\bigcup \langle a, b, c \rangle = \bigcup \{ \{\langle a, b \rangle, \{\langle a, b \rangle, c\} \} = \{\langle a, b \rangle, c\};$
- $\bigcup^{2} \langle a, b, c \rangle = \bigcup \{ \langle a, b \rangle, c \} = \langle a, b \rangle \cup c;$
- $\bigcup^{\circ 3} \langle a, b, c \rangle = \{a, b\} \cup \bigcup c;$ $\bigcup^{\circ 4} \langle a, b, c \rangle = \bigcup^{\circ 2} (\langle a, b \rangle \cup c) = (a \cup b) \cup \bigcup c.$

Then

- $\bullet \ \{a,b,c\} \subseteq (\bigcup^{\circ 3} \langle a,b,c \rangle) \cup (\bigcup \langle a,b,c \rangle) \subseteq (\bigcup^{\circ 4} u) \cup (\bigcup^{\circ 2} u) \subseteq (\bigcup^{\circ 5} x) \cup (\bigcup^{\circ 3} x) \subseteq \mathsf{TC}(x)$
- $a \cup b \cup c = \bigcup \{a, b, c\} \subseteq (\bigcup^{\circ 6} x) \cup (\bigcup^{\circ 4} x) \subseteq \mathsf{TC}(x).$

Applying Proposition 18.8 and Lemma 25.7(3), we see that

$$\begin{split} \mathsf{TC}(g) &= u^{\circlearrowright} \cup \bigcup \{ \mathsf{TC}(\langle c, a, b \rangle) : \langle a, b, c \rangle \in u \} \subseteq \\ & (\bigcup \{u^{\circlearrowright}\}) \cup \bigcup \{ \mathsf{G}^{\circ 3}(\{c, a, b\}) \cup \mathsf{TC}(c \cup a \cup b) : \langle a, b, c \rangle \in u \} \subseteq \\ & (\bigcup \mathsf{G}(x)) \cup \mathsf{G}^{\circ 3}(\mathsf{TC}(x)) \cup \mathsf{TC}(\mathsf{TC}(x)) \subseteq \\ & \mathsf{G}^{\circ 2}(x) \cup \mathsf{G}^{\circ 3}(\mathsf{TC}(x)) \cup \mathsf{TC}(x) \subseteq \mathsf{G}^{\circ 3}(\mathsf{TC}(x)). \end{split}$$

Lemma 25.9. For every set x and natural number $n \in \omega$ we have

$$\mathsf{TC}(\mathsf{G}^{\circ n}(x)) \subseteq \mathsf{G}^{\circ (3n)}(\mathsf{TC}(x)).$$

Proof. For n=0 we have $\mathsf{TC}(\mathsf{G}^{\circ 0}(x))=\mathsf{TC}(x)=\mathsf{G}^{\circ 0}(x)$. Assume that for some $n\in\omega$ the embedding $TC(G^{\circ n}(x)) \subseteq G^{\circ(3n)}(TC(x))$ has been proved. By Lemma 25.8,

$$\begin{split} \mathsf{TC}(\mathsf{G}^{\circ(n+1)}(x)) &= \mathsf{TC}(\mathsf{G}(\mathsf{G}^{\circ n}(x)) \subseteq \mathsf{G}^{\circ 3})(\mathsf{TC}(\mathsf{G}^{\circ n}(x))) \subseteq \\ & \qquad \qquad \mathsf{G}^{\circ 3}(\mathsf{G}^{\circ(3n)}(\mathsf{TC}(x))) = \mathsf{G}^{\circ(3n+3)}(\mathsf{TC}(x)) = \mathsf{G}^{\circ(3(n+1))}(\mathsf{TC}(x)). \end{split}$$

The following theorem is the main result of this subsection.

Theorem 25.10. For every transitive set x its Gödel's hull $G^{\circ\omega}(x)$ is a transitive set.

Proof. Given any set $y \in \mathsf{G}^{\circ\omega}(x)$, find $n \in \omega$ such that $y \in \mathsf{G}^{\circ n}(x)$. Applying Lemma 25.9, we conclude that

$$\mathsf{TC}(y)\subseteq \mathsf{TC}(\mathsf{G}^{\circ n}(x))\subseteq \mathsf{G}^{\circ (3n)}(\mathsf{TC}(x))\subseteq \mathsf{G}^{\circ \omega}(x),$$

where the last embedding follows from the transitivity of the set x.

Gödel's Constructible Universe L

The universe is almost like a huge magic trick and scientists are trying to figure out how it does what it does.

Martin Gardner

Definition 25.11. The class

$$\mathbf{L} = \bigcup_{\alpha \in \mathbf{On}} \mathsf{G}^{\circ \omega}(\alpha) = \mathsf{G}^{\circ \omega}[\mathbf{On}]$$

is called the Gödel's constructible universe. Its elements are called constructible sets.

Therefore, constructible sets can be constructed from ordinals applying finitely many Gödel's operations.

The transitivity of ordinals and Theorem 25.10 imply the following important fact.

Theorem 25.12. The constructible universe L is a transitive proper class.

We say that a class X is closed under Gödel's operations if $G(x) \subseteq X$ for any subset $x \subseteq X$.

Theorem 25.13. The class of constructible sets \mathbf{L} is the smallest class that contains all ordinals and is closed under the Gödel's operations.

Proof. Given any ordinal α , observe that $\alpha \in \alpha + 1 \subset \mathsf{G}^{\circ\omega}(\alpha) \subseteq \mathbf{L}$.

To show that \mathbf{L} is closed under the Gödel's operation, take any subset $x \subset \mathbf{L}$. Consider the function $\lambda : \mathbf{L} \to \mathbf{On}$ assigning to every constructible set $c \in \mathbf{L}$ the smallest ordinal α such that $c \in \mathsf{G}^{\circ\omega}(\alpha) \neq \emptyset$. Using the Gödel's theorem on existence of classes, it can be shown that the function λ is well-defined. By the Axiom of Replacement, the class $\lambda[x]$ is a set and hence $\lambda[x] \subset \alpha$ for some ordinal α . Then $x \subseteq \mathsf{G}^{\circ\omega}(\alpha)$ by the definition of the function λ . Now we see that $\mathsf{G}(x) \subseteq \mathsf{G}(\mathsf{G}^{\circ\omega}(\alpha)) = \mathsf{G}^{\circ\omega}(\alpha) \subseteq \mathsf{L}$, which means that the class L is closed under the Gödel's operations.

Now take any class **X** that contains all ordinals and is closed under the Gödel's operations. Then for every ordinal α we have $\alpha \subseteq \mathbf{X}$. Using the Principle of Mathematical Induction, we will show that $\mathsf{G}^{\circ n}(\alpha) \subseteq \mathbf{X}$ for every $n \in \omega$. For n = 0 this follows from $\alpha \subseteq \mathbf{X}$. Assume that for some $n \in \omega$ we have the embedding $\mathsf{G}^{\circ n}(\alpha) \subseteq \mathbf{X}$. Since **X** is closed under the Gödel's operations,

$$\mathsf{G}^{\circ(n+1)}(\alpha) = \mathsf{G}(\mathsf{G}^{\circ n}(\alpha)) \subseteq \mathbf{X}.$$

By Exercise 24.6, the von Neumann universe V is closed under Gödel's operations. Since $On \subseteq V$, we can apply the preceding statement and obtain the following corollary.

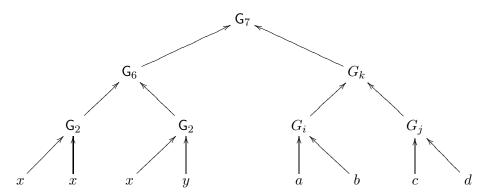
Corollary 25.14. $L \subseteq V$.

An important feature of the class \mathbf{L} is its well-orderability. In Theorem 25.21 below we shall prove that there exists a canonical set-like well-order $\mathbf{W}_{<}$ with $\mathsf{dom}[\mathbf{W}_{<}] = \mathbf{L}$.

To construct such a well-order, we first construct a canonical enumeration of all possible compositions of Gödels operations. There are only countably many such compositions. We shall enumerate them by the countable set $\bigcup_{n\in\omega} 9^{2^{< n}}$. Since we included the identity operation G_0 in the list of Gödel's operations, any composition of Gödel's operations can be encoded by a 9-labeled full binary tree $2^{< n}$ of some finite height n.

For example, operation of union $x \cup y = \bigcup \{x,y\}$ can be written as the composition $G_7(G_2(G_2(x,x),G_2(x,y)),G_k(G_i(a,b),G_j(c,d)))$ and represented by the full binary tree of

height 3:



Since the operation G_7 of union does not depend on the second coordinate, in the right-hand part of the tree we can write any operations and variables.

Exercise 25.15. Draw the corresponding tree for representing the operation of forming the ordered triple $\langle x, y, z \rangle = \langle \langle x, y \rangle, z \rangle$.

Such a representation suggests the idea of encoding of all possible compositions of Gödel's operations by binary trees whose vertices are labeled by numbers that belong to the set $9 = \{0, 1, \dots, 8\}$.

By a binary tree we understand any ordinary tree T which is a subset of the full binary ω -tree $2^{<\omega}$. We recall that $2^{<\omega}$ consists of all functions f with $\mathsf{dom}[f] \in \omega$ and $\mathsf{rng}[f] \subseteq 2 = \{0,1\}$. A subset $T \subseteq 2^{<\omega}$ is called an ordinary tree if for any $t \in T$ and $n \in \omega$ the function $t \upharpoonright_n = t \cap (n \times \mathbf{U})$ belongs to T.

For every $k \in \{0, 1\}$, the injective function

$$\vec{k}: 2^{<\omega} \to 2^{<\omega}, \quad \vec{k}: t \mapsto \{\langle 0, k \rangle\} \cup \{\langle \alpha + 1, y \rangle : \langle \alpha, y \rangle \in t\}$$

is called the k-transplantation of the full binary ω -tree.

Exercise 25.16. Show that $\operatorname{rng}[\vec{k}] = \{t \in 2^{<\omega} : t(0) = k\}$ and for every $n \in \omega$ we have $\vec{k}[2^n] = \{t \in 2^{n+1} : t(0) = k\}$. Deduce from this that for every function $x : 2^{n+1} \to \mathbf{U}$ the composition $x \circ \vec{k}$ is a function with $\operatorname{dom}[x \circ \vec{k}] = 2^n$. Consequently, the function $\mathbf{U}^{2^{n+1}} \to \mathbf{U}^{2^n}$, $x \mapsto x \circ \vec{k}$, is well-defined.

Exercise 25.17. For a function $x = \{\langle \langle 0, 0 \rangle, x_{00} \rangle, \langle \langle 0, 1 \rangle, x_{01} \rangle, \langle \langle 1, 0 \rangle, x_{10} \rangle, \langle 1, 1 \langle x_{11}, \rangle \} \in \mathbf{U}^{2^2},$ find the functions $x \circ \vec{0}$ and $x \circ \vec{1}$.

$$\mathit{Hint:}\ x \circ \vec{0} = \{\langle 0, x_{00} \rangle, \langle 1, x_{01} \rangle\} \ \text{and}\ x \circ \vec{1} = \{\langle 0, x_{10} \rangle, \langle 1, x_{11} \rangle\}.$$

By a 9-labeling of a binary tree T we understand any function $\lambda: T \to 9 = \{0, 1, \dots, 8\}$. Therefore a 9-labeling λ assigns to each vertex $t \in T$ of the tree some number $\lambda(t) \in 9$. Observe that for any $n \in \omega$ and a 9-labeling $\lambda: 2^{<(n+1)} \to 9$ of the full binary (n+1)-tree $2^{<(n+1)}$, the composition $\lambda \circ \vec{k}$ has $\mathsf{dom}[\lambda \circ \vec{k}] = 2^{< n}$, so $\lambda \circ \vec{k}$ is a labeling of the tree $2^{< n}$.

Now for every $n \in \omega$ and a 9-labeling $\lambda : 2^{< n} \to 9$ of the tree $2^{< n}$ we define the function $\ddot{\mathsf{G}}_{\lambda} : \mathbf{U}^{2^n} \to \mathbf{U}$ by the recursive formula:

- (1) If n = 0, then $\ddot{\mathsf{G}}_{\lambda}(\langle 0, x \rangle) = x$ for any $\{\langle 0, x \rangle\} \in \mathbf{U}^{2^0} = \mathbf{U}^1$;
- (2) If n = 1, then $\ddot{\mathsf{G}}_{\lambda}(x) = \mathsf{G}_{\lambda(0)}(x(0), x(1))$ for any $x \in \mathbf{U}^{2^1}$;
- (3) If $n \geq 2$, then $\ddot{\mathsf{G}}_{\lambda}(x) = \mathsf{G}_{\lambda(0)}(\mathsf{G}_{\lambda \circ \vec{0}}(x \circ \vec{0}), \mathsf{G}_{\lambda \circ \vec{1}}(x \circ \vec{1}))$ for every $x \in \mathbf{U}^{2^n}$.

Lemma 25.18. For every $n \in \omega \setminus \{0\}$ and set x we have

$$\mathsf{G}^{\circ n}(x) = \bigcup \{ \ddot{\mathsf{G}}_{\lambda}[x^{2^n}] : \lambda \in 9^{2^{< n}} \}.$$

Proof. For n = 1 we have $2^{<1} = \{f \in \mathbf{Fun} : \mathsf{dom}[f] \in 1\} = \{f \in \mathbf{Fun} : \mathsf{dom}[f] = \emptyset\} = \{\emptyset\} = 1$ and then

$$\mathsf{G}^{\circ 1}(x) = \mathsf{G}(x) = \bigcup_{i=0}^{8} [x \times x] = \bigcup \{ \ddot{\mathsf{G}}_{\lambda}[x^{2^{1}}] : \lambda \in 9^{1} \}.$$

Assume that for some $n \in \omega \setminus \{0\}$ the equality

$$\mathsf{G}^{\circ n}(x) = \bigcup \{ \ddot{\mathsf{G}}_{\lambda}[x^{2^n}] : \lambda \in 9^{2^{< n}} \}$$

has been proved. Then

$$\mathsf{G}^{\circ(n+1)}(x) = \mathsf{G}(\mathsf{G}^{\circ n}(x)) = \{G_i(u,v) : i \in 9, \ u,v \in \mathsf{G}^{\circ n}(x)\} = \{\mathsf{G}_i(u,v) : i \in 9, \ u,v \in \bigcup \{\ddot{\mathsf{G}}_{\lambda}[x^{2^n}] : \lambda \in 9^{2^{< n}}\}\} = \{\mathsf{G}_i(\ddot{\mathsf{G}}_{\mu}(f),\ddot{\mathsf{G}}_{\nu}(g)) : i \in 9, \ \mu,\nu \in 9^{2^{< n}}, \ f,g \in x^{2^n}\} = \{\ddot{\mathsf{G}}_{\lambda}(\varphi) : \lambda \in 9^{2^{< (n+1)}}, \ \varphi \in x^{2^{(n+1)}}\}.$$

Lemma 25.18 implies the following two corollaries.

Corollary 25.19. For every set x its Gödel's hull $G^{\circ\omega}(x)$ is equal to

$$\bigcup_{n \in \omega} \bigcup_{\lambda \in \Omega^{2^{< n}}} \ddot{\mathsf{G}}_{\lambda}[x^{2^{n}}].$$

$$\textbf{Corollary 25.20. } \mathbf{L} = \bigcup_{n \in \omega} \bigcup_{\lambda \in 9^{2^{< n}}} \big\{ \ddot{\mathsf{G}}_{\lambda}(f) : f \in \mathbf{On}^{2^n} \big\}.$$

Now we are able to prove the promised

Theorem 25.21 (Gödel). There exists a set-like well-order $\mathbf{W}_{<}$ such that $dom[\mathbf{W}_{<}] = \mathbf{L}$.

Proof. Observe that for every $n \in \omega$ the sets $2^{< n}$ and 2^n are subsets of the class $\mathbf{On}^{<\omega}$ that carries a canonical set-like well-order $\mathbf{W}_{<} \upharpoonright \mathbf{On}^{<\omega}$, see Section 20. So, $2^{< n}$ and 2^n can be identified with the corresponding natural numbers and then the classes $9^{2^{< n}}$ and \mathbf{On}^{2^n} can be identified with subclasses of the class $\mathbf{On}^{<\omega}$, which is well-ordered by the relation $\mathbf{W}_{<}$.

Consider the class

$$\mathbb{L} = \{ \langle n, \lambda, f \rangle : n \in \omega \ \land \ \lambda \in 9^{2^{< n}} \ \land \ f \in \mathbf{On}^{2^n} \}.$$

It is easy to see that the class \mathbb{L} exists.

Next, define the set-like well-order on L:

$$\mathbb{W}_{<} = \{ \langle \langle n, \lambda, f \rangle, \langle m, \mu, g \rangle \rangle \in \mathbb{L} \times \mathbb{L} : \\ (\langle f, g \rangle \in \mathbb{W}_{<}) \ \lor \ (f = g \ \land \ \langle \lambda, \mu \rangle \in \mathbb{W}_{<}) \ \lor \ (f = g \ \land \ n \in m) \}.$$

Consider the function

$$\mathsf{Code} = \{ \langle \langle n, \lambda, f \rangle, z \rangle \in \mathbb{L} \times \mathbb{L} : z = \ddot{\mathsf{G}}_{\lambda}(f) \}.$$

Corollary 25.20 implies that the function Code: $\mathbb{L} \to \mathbb{L}$ is surjective. Then

$$\mathbf{W}_{<} = \left\{ \langle x, y \rangle \in \mathbf{L} \times \mathbf{L} : \exists t \in \mathbb{L} \left(\mathsf{Code}(t) = x \ \land \ \forall u \in \mathbb{L} \left(\langle t, u \rangle \notin \mathbb{W}_{<} \ \Rightarrow \ \left(\mathsf{Code}(u) \neq y \right) \right) \right\}$$
 is a desired set-like well-order with $\mathsf{dom}[\mathbf{W}_{<}^{\pm}] = \mathbf{L}$.

Exercise 25.22. Check that $\mathbb{W}_{<}$ is indeed a well-order on the class $\mathbb{L} = \mathsf{dom}[\mathbb{W}_{<}^{\pm}]$. Show that the order

$$\mathbb{W}'_{<} = \{ \langle \langle n, \lambda, f \rangle, \langle m, \mu, g \rangle \rangle \in \mathbb{L} \times \mathbb{L} :$$

$$(n \in m) \lor (n = m \land \langle \lambda, \mu \rangle \in \mathbb{W}_{<}) \lor (n = m \land \lambda = \mu \land \langle f, g \rangle \in \mathbb{W}_{<}) \}.$$

is not set-like.

By Theorem 25.13 the constructible universe \mathbf{L} is the smallest class that contains all ordinals and is closed under Gödel's operations. In Corollary 26.5 we shall prove that the following statement does not contradict the axioms of the Classical Set Theory.

Axiom of Constructibility: U = L

The Axiom of Constructibility postulates that every set is constructible (from ordinals by applying finitely many Gödel's operations).

Theorem 25.23. The Axioms Classical Set Theory with added Axiom of Constructibility imply the Axiom of Foundation and the Axiom of Global Choice.

Proof. Since $\mathbf{L} \subseteq \mathbf{V} \subseteq \mathbf{U}$, the equality $\mathbf{L} = \mathbf{U}$ implies the equality $\mathbf{V} = \mathbf{U}$, which is equivalent to the Axiom of Foundation by Theorem 24.5.

To prove that $\mathbf{L} = \mathbf{U}$ implies the Axiom of Global Choice, define the choice function $C: 2^{\mathbf{U}} \setminus \{\emptyset\} \to \mathbf{U}$ assigning to every nonempty subset $x \subseteq \mathbf{U} = \mathbf{L}$ the unique $\mathbf{W}_{<}$ -least element of x.

26. Inner Models

But above all I wish to designate the following as the most important among the numerous questions which can be asked with regard to the axioms:

To prove that they are not contradictory, that is, that a definite number of logical steps based upon them can never lead to contradictory results.

David Hilbert

In this section we consider classes, called inner models, and will use them to prove that the consistency of CST implies the consistency of CST with added Gödel's Axiom of Constructibility $\mathbf{U} = \mathbf{L}$. By CST we denote the list of axioms of the Classical Set Theory (= NBG with removed Axioms of Foundation and Global Choice). We say that a system of axioms is *consistent* if it does not lead to a contradiction. It is known (by Gödel's incompleteness theorem) that any consistent list of axioms of Set Theory does not imply its own consistency. By the Gödel's completeness theorem, a system of axioms is consistent if and only if it has a model.

Definition 26.1. A transitive class **M** is called an *inner model* if it contains all ordinals and is closed under Gödel's operations.

Example 26.2. The von Neumann universe V is an inner model, see Exercise 24.6. By Theorem 25.13, the class **L** is the smallest inner model.

The aim of this section is to prove that for any inner model M the consistency of CST implies the consistency of CST + (U = M). So, the equality U = M can be added as a new axiom and this will not lead to a contradiction if the system CST does not lead to a contradiction.

Let us observe that all Gödel's operations except for the operation G_2 of producing an unordered pair $\{x,y\}$ are well-defined on classes. Namely, by the Axioms of the Classical Set Theory, for any classes X, Y the following classes are well-defined:

```
(0) G_0(X,Y) = X;
```

- (1) $G_1(X,Y) = X \setminus Y;$
- (3) $G_3(X,Y) = \mathbf{E} \cap (X \times Y);$
- (4) $G_4(X,Y) = X^{-1}$;
- (5) $G_5(X) = dom[X];$
- (6) $G_6(X,Y) = X \times Y$;
- (7) $\mathsf{G}_7(X,Y) = \bigcup X;$
- (8) $G_8(X,Y) = X^{\circ}$.

Then the compositions of these operations also act on indexed tuples of classes. In fact, with some care, we can also use the operation G_2 , under the condition that it applies only to sets.

For a set A let $\bar{\mathbf{M}}^A$ be the class of functions f such that $\mathsf{dom}[f] \subseteq A$ and $\mathsf{rng}[f] \subseteq \mathbf{M}$.

Now we every $n \in \omega$ by induction we define a class $\mathsf{Comp}_n \subseteq 9^{2^{< n}} \times \bar{\mathbf{M}}^{2^n}$ and then for every ordered pair $\langle \lambda, x \rangle \in \mathsf{Comp}_n$ we define a class $\ddot{\mathsf{G}}_{\lambda,x}$ such that the following conditions are satisfied:

- If n = 0, then $\mathsf{Comp}_0 = \emptyset$;
- If n=1, then $\mathsf{Comp}_1 = \{\langle \lambda, x \rangle \in 9^{2^{<1}} \times \bar{\mathbf{M}}^{2^1} : \lambda(\emptyset) = 2 \Rightarrow \mathsf{dom}[x] = 2^1\};$
- If n = 1 and $\langle \lambda, x \rangle \in \mathsf{Comp}_1$ and $\mathsf{dom}[x] = 2^1$, then $\ddot{\mathsf{G}}_{\lambda,x} = \mathsf{G}_{\lambda(\emptyset)}(x(\langle 0, 0 \rangle), x(\langle 0, 1 \rangle))$.
- If n = 1 and $\langle \lambda, x \rangle \in \mathsf{Comp}_1$ and $\mathsf{dom}[x] = \{\langle 0, 0 \rangle\}$, then $\ddot{\mathsf{G}}_{\lambda, x} = \mathsf{G}_{\lambda(\emptyset)}(x(\langle 0, 0 \rangle), \mathbf{M})$.
- If n=1 and $\langle \lambda, x \rangle \in \mathsf{Comp}_1$ and $\mathsf{dom}[x] = \{\langle 0, 1 \rangle\}$, then $\ddot{\mathsf{G}}_{\lambda,x} = \mathsf{G}_{\lambda(\emptyset)}(\mathbf{M}, x(\langle 0, 1 \rangle))$.
- If n = 1 and $\langle \lambda, x \rangle \in \mathsf{Comp}_1$ and $\mathsf{dom}[x] = \emptyset$, then $\ddot{\mathsf{G}}_{\lambda,x} = \mathsf{G}_{\lambda(\emptyset)}(\mathbf{M}, \mathbf{M})$. If n > 1, then $\mathsf{Comp}_n = \{\langle \lambda, x \rangle \in 9^{2^n} \times \mathbf{\bar{M}}^{2^n} :$

$$\lambda(\emptyset) = 2 \ \Rightarrow \ \forall k \in 2 \ (\lambda \circ \vec{k} \in \mathsf{Comp}_{n-1} \ \land \ \ddot{\mathsf{G}}_{\lambda \circ \vec{k}.x \circ \vec{k}} \in \mathbf{U}) \}.$$

 $\bullet \ \ \text{If} \ n>1 \ \ \text{and} \ \ \langle \lambda,x\rangle \in \mathsf{Comp}_n, \ \text{then} \ \ \ddot{\mathsf{G}}_{\lambda,x}=\mathsf{G}_{\lambda(\emptyset)}(\ddot{\mathsf{G}}_{\lambda\circ\vec{0},x\circ\vec{0}}, \ddot{\mathsf{G}}_{\lambda\circ\vec{1},x\circ\vec{1}}).$

This long recursive definition formalizes the idea that a 9-labeled tree encodes a composition of Gödel's operations that can be computed on the function \bar{x} which extends x and has value **M** at point where x is not defined. So, the ordered pair $\langle \lambda, x \rangle$ belongs to Comp, if in the process of computation there is no necessity to form an unordered pair of proper classes.

Definition 26.3. A class X is called M-definable if there exist $n \in \omega$ and an ordered pair $\langle \lambda, x \rangle \in \mathsf{Comp}_n \text{ such that } X = \ddot{\mathsf{G}}_{\lambda, x}.$

Now we construct a model M of Axioms of the Classical Set Theory restricting our classes only to M-definable subclasses of the class M. We claim that in this smaller model M all Axioms of the Classical Set Theory hold and **M** is the universe of sets in this model.

Observe that elements of the class M are sets in the model M because the singleton $\{x\}$ is M-definable. On the other hand, if an M-definable class X is a set, then we can find $n \in \omega$ and an ordered pair $\langle \lambda, x \rangle \in \mathsf{Comp}_n$ such that $X = \mathsf{G}_{\lambda,x}$. Since X is a set, it is legal to form a singleton $\{X\} = \mathsf{G}_2(X,X) = X_{\mu,y}$ for a suitable ordered pair $\langle \mu, x \rangle \in \mathsf{Comp}_{n+1}$. The singleton $\{X\}$ is an M-definable class witnessing that X is a set in the model M.

Now we check the Axioms of the Classical Set Theory.

The Axiom of Extensionality holds since the classes of the models are subclasses of M.

Since all classes of the model are subclasses of \mathbf{M} , the class \mathbf{M} plays the role of the universe in the model M. For the role of the class \mathbf{E} in the model M we take the class $\mathbf{E} \cap (\mathbf{M} \times \mathbf{M})$. It is definable being produced by the Gödel's operation $\mathsf{G}_3(\mathbf{M},\mathbf{M})$.

The Axiom of Difference holds because of the Gödels operation G_1 . The same concerns the Axioms of Pair, Domain, Product, and Cycle. Here we should remark that by the transitivity of \mathbf{M} , each ordered pair $\langle x, y \rangle$ can be decomposed into pieces and by the Gödel's operations G_2 we can construct an ordered pair $\langle y, x \rangle$ and this pair will belong to the class \mathbf{M} as \mathbf{M} is closed under Gödel's operations.

The Axiom of Union holds because we have the Gödel's operations G_7 and G_2 .

To see that the axiom of Replacment holds, take any set $x \in \mathbf{M}$ and an \mathbf{M} -definable function $F \subseteq \mathbf{M}$. It is easy to see that F remains a function in the original model. By the Axiom of Replacement, the image F[x] is a set. It remains to prove that this set is \mathbf{M} -definable. For this observe that $F[x] = \{v : \exists u \in x \ \langle x, v \rangle \in F\} = \mathsf{dom}[(F \cap (x \times \mathbf{M}))^{-1}]$ is an \mathbf{M} -definable class and being a set in \mathbf{U} remains a set in \mathbf{M} .

The same trick works for the Power-set $2_{\mathbf{M}}^x = \{y : y \subseteq x \text{ is an } \mathbf{M}\text{-definable set}\}$. For any $\mathbf{M}\text{-definable set } x$ we have $\mathbf{M} \setminus 2_{\mathbf{M}}^x = \{y : \exists z \in \mathbf{M} \ (z \in y \land z \notin x)\} = \mathsf{dom}[P]$ where $P = (\mathbf{E} \upharpoonright \mathbf{M})^{-1} \setminus (\mathbf{E} \upharpoonright \mathbf{M} \cap (\mathbf{M} \times \{x\})^{-1})$. It is clear that the set P can be written via Gödel's operations over classes.

The axiom of infinity holds because **M** contains all ordinals in particular, ω .

Therefore, we have finished the proof of the following important theorem of Gödel.

Theorem 26.4. For every inner model M, the consistency of CST implies the consistency of CST with added Axiom U = M.

Corollary 26.5. The consistency of CST implies the consistency of CST with added Gödel's Constructibility Axiom U = L.

Since the Axioms of Foundation and Global Choice hold in the Constructible Universe, we obtain the following (relative) consistency result.

Corollary 26.6. The consistency of CST implies the consistency of NBG.

Part 6. Choice and Global Choice

The Axiom of Choice is the most controversial axiom in mathematics. It has many valuable implications (Tychonoff's compactness theorem in Topology, Hahn-Banach Theorem in Functional Analysis) but it also implies some highly counter-intuitive statements like the Banach-Tarski Paradox⁷. In part we survey some implications or equivalents of Axiom of Choice and its stronger version, the Axiom of Global Choice.

27. Choice

The axiom of choice is obviously true, the well-ordering principle obviously false, and who can tell about Zorn's lemma? Jerry Lloyd Bona

In this section we discuss some statements related to the Axiom of Choice. In fact, this subject is immense and there are good and complete books covering this topic in many details, see for example, [4], [7], [8], [19]. So, we shall recall only the most important choice principles that have applications in mathematics.

Definition 27.1. We say that a class X

- can be well-ordered if there exists a well-order W such that $dom[W^{\pm}] = X$;
- is well-orderable if there exists a set-like well-order W such that $dom[W^{\pm}] = X$;
- has a *choice function* if there exists a function $F: X \setminus \{\emptyset\} \to \bigcup X$ such that $F(x) \in x$ for every nonempty set $x \in X$; the function F is called a *choice function* for X.

Observe that a set can be well-ordered if and only if it is well-orderable.

We recall that the **Axiom of Choice** postulates that each set has a choice function.

The following fundamental result is known as the well-ordering theorem of Zermelo.

Theorem 27.2 (Zermelo). For any set x the following statements are equivalent.

WO(x): The set x can be well-ordered.

 $AC(2^x)$: The power-set $\mathcal{P}(x)$ of x has a choice function.

Proof. $AC(2^x) \Rightarrow WO(x)$: Assume that there exists a choice function $c : \mathcal{P}(x) \setminus \{\emptyset\} \to \bigcup x$ for the power-set $\mathcal{P}(x)$ of x. By the Axiom of Union, the class $\bigcup x$ is a set. Since \mathbf{U} is a proper class, there exists an element $z \in \mathbf{U} \setminus \bigcup x$. Consider the function $\bar{c} = \{\langle \emptyset, z \rangle\} \cup c$.

Applying the Recursion Theorem 21.1 to the function

$$F: \mathbf{On} \times \mathbf{U} \to \{z\} \cup \bigcup x, \quad F: \langle \alpha, y \rangle \mapsto \bar{c}(x \setminus y),$$

and the set-like well-order $\mathbf{E} \upharpoonright \mathbf{On}$, we obtain a (unique) function $G: \mathbf{On} \to \{z\} \cup \bigcup x$ such that

$$G(\alpha) = F(\alpha, \{G(\beta) : \beta \in \alpha\}) = F(\alpha, G[\alpha])$$

for every $\alpha \in \mathbf{On}$.

We claim that $z \in G[\mathbf{On}]$. To derive a contradiction assume that $z \notin G[\mathbf{On}]$. In this case for every ordinals $\beta \in \alpha$ we have

$$z \neq G(\alpha) = F(\alpha, G[\alpha]) = c(x \setminus G[\alpha]) \in x \setminus G[\alpha] \subseteq \bigcup x \setminus \{G(\beta)\}\$$

⁷Exercise: Read about Banach-Tarski Paradox in Wikipedia.

and hence $G(\beta) \neq G(\alpha)$. The injectivity of the function G guarantees that G^{-1} is a function, too. Then $\mathbf{On} = G^{-1}[X]$ is a set by the Axiom Replacement. But this contradicts Theorem 19.6(6). This contradiction shows that $z \in G[\mathbf{On}]$. Since the order $E \upharpoonright \mathbf{On}$ is well-founded, for the nonempty set $G^{-1}[\{z\}] \subseteq \mathbf{On}$ there exists an ordinal $\alpha \in G^{-1}[\{z\}]$ such that $\alpha \cap G^{-1}[\{z\}] = \emptyset$. Then $z \notin G[\alpha]$ and hence $G[\alpha] \subseteq X$. Repeating the above argument, we can prove that the function $G \upharpoonright_{\alpha}$ is injective.

We claim that $G[\alpha] = x$. Assuming that $G[\alpha] \neq x$, we see that the set $x \setminus G[\alpha]$ is not empty and then the definition of the function F ensures that $G(\alpha) = F(\alpha, G[\alpha]) = c(x \setminus G[\alpha]) \in x \setminus G[\alpha] \subseteq \bigcup x$ and hence $G(\alpha) \neq z$, which contradicts the choice of α .

Therefore, the function $G \upharpoonright_{\alpha} : \alpha \to x$ is bijective and we can define an irreflexive well-order on x by the formula

$$w = \{ \langle G(\beta), G(\alpha) \rangle : \beta \in \alpha \}.$$

 $WO(x) \Rightarrow AC(2^x)$: If there exists a well-order w with $dom[w^{\pm}] = x$, then the formula

$$c: \mathcal{P}(x) \setminus \{\emptyset\} \to \bigcup x, \quad c: a \mapsto \min_{w}(a)$$

determines a choice function for $\mathcal{P}(x)$. In this formula by $\min_w(a)$ we denote the unique w-minimal element of a nonempty set $a \subseteq x$.

An important statement which is equivalent to the Axiom of Choice was found by Kuratowski in 1922 and (independently) Zorn in 1935. It concerns the existence of maximal elements in orders.

Let us recall that for an order R, an element $x \in dom[R^{\pm}]$ is called R-maximal if

$$\forall y \in \mathsf{dom}[R^{\pm}] \; (\langle x, y \rangle \in R \; \Rightarrow \; y = x).$$

A subclass $L \subseteq \mathsf{dom}[R^{\pm}]$ is called

- an R-chain if $L \times L \subseteq R^{\pm} \cup \mathbf{Id}$;
- an R-antichain if $(L \times L) \cap R \subseteq \mathbf{Id}$;
- a maximal R-chain if L is an R-chain and L is equal to any R-chain $L' \subseteq \mathsf{dom}[R^{\pm}]$ with $L \subseteq L'$;
- a maximal R-antichain if L is an R-antichain and L is equal to any R-antichain $L' \subseteq \text{dom}[R^{\pm}]$ with $L \subseteq L'$.

An element $b \in \mathsf{dom}[R^{\pm}]$ is called an *upper bound* of a set $L \subseteq \mathsf{dom}[R^{\pm}]$ if $L \times \{b\} \subseteq R \cup \mathbf{Id}$. We say that an order R is *chain-bounded* if each R-chain $L \subseteq \mathsf{dom}[R^{\pm}]$ has an upper bound in $\mathsf{dom}[R^{\pm}]$.

Lemma 27.3 (Kuratowski–Zorn). Let r be a chain-bounded order on a set $x = \text{dom}[r^{\pm}]$. If the power-set $\mathcal{P}(x)$ of x has a choice function, then there exists an R-maximal element $z \in \text{dom}[r^{\pm}]$.

Proof. To derive a contradiction, assume that $dom[r^{\pm}]$ contains no r-maximal elements.

Let c be the set of all r-chains in the set $x = \mathsf{dom}[r^{\pm}]$. Since the order r is chain-bounded, for every chain $\ell \in c$ the set $u(\ell) = \{b \in \mathsf{dom}[r^{\pm}] : \ell \times \{b\} \subseteq r \cup \mathbf{Id}\}$ of its upper bounds is not empty. We claim that the subset $v(\ell) = \{b \in \mathsf{dom}[r^{\pm}] : \ell \times \{b\} \subseteq r \setminus \mathbf{Id}\}$ of $u(\ell)$ is not empty, too. For this take any element $b \in u(\ell)$. By our assumption, the element b is not r-maximal. Consequently, there exists an element $d \in \mathsf{dom}[r^{\pm}]$ such that $\langle b, d \rangle \in r \setminus \mathbf{Id}$. The transitivity of the relation r and the inequality $b \neq d$ guarantees that $d \notin \ell$ and hence $d \in v(\ell)$.

By our assumption, the power-set $\mathcal{P}(x)$ has a choice function $f: \mathcal{P}(X) \setminus \{\emptyset\} \to \bigcup x$. Let $z \in \mathbf{U} \setminus \mathsf{dom}[r^{\pm}]$ be any set (which exists as $\mathsf{dom}[r^{\pm}]$ is a set and \mathbf{U} is a proper class). Consider the function $F: \mathbf{On} \times \mathbf{U} \to \mathbf{U}$ defined by the formula

$$F(\alpha, y) = \begin{cases} f(v(y)) & \text{if } y \in c; \\ z & \text{otherwise.} \end{cases}$$

By the Recursion Theorem 21.1, there exists a (unique) function $G: \mathbf{On} \to \mathbf{U}$ such that $G(\alpha) = F(\alpha, G[\alpha])$ for every ordinal α .

We claim that for every ordinal α the image $G[\alpha]$ is an r-chain. Assuming that this is not true, we can find the smallest ordinal α such that $G[\alpha]$ is not an r-chain but for every $\beta \in \alpha$ the set $G[\beta]$ is an r-chain. Since $G[\alpha]$ is not an r-chain, there two elements $y, z \in G[\alpha]$ such that $\langle y, z \rangle \notin R^{\pm} \cup \mathbf{Id}$. Find two ordinals $\beta, \gamma \in \alpha$ such that $y = G(\beta)$ and $z = G(\gamma)$. Since the relation $r^{\pm} \cup \mathbf{Id}$ is symmetric, we lose no generality assuming that $\beta \leq \gamma$. We claim that $\gamma + 1 = \alpha$. In the opposite case, the minimality of α guarantees that $G[\gamma + 1]$ is an r-chain and then $\langle x, y \rangle \in r^{\pm} \cup \mathbf{Id}$, which contradicts the choice of the elements y, z. Therefore, $\alpha = \gamma + 1$. Since $G[\gamma]$ is an r-chain, the definition of the function F guarantees that $G(\gamma) = F(\gamma, G[\gamma]) = f(v(G[\gamma]))$ and then $G[\alpha] = G[\gamma] \cup G(\gamma) = G[\gamma] \cup f(v(G[\gamma]))$ is an r-chain. But this contradicts the choice of α . This contradiction shows that for all ordinals α the set $G[\alpha]$ is an r-chain. In this case for every ordinal α we have $G(\alpha) = F(\alpha, G[\alpha]) = f(v(G[\alpha])) \in v(G[\alpha]) \subseteq \bigcup x \setminus G[\alpha]$, which implies that the function $G: \mathbf{On} \to \mathsf{dom}[r^{\pm}]$ is injective. Then G^{-1} is a function and $\mathbf{On} = G^{-1}[\mathsf{dom}[r^{\pm}]]$ is a set by the Axiom of Replacement. But this contradicts the properness of the class \mathbf{On} , see Theorem 19.6(6). \square

Another statement, which is equivalent to the Axiom of Choice is

Hausdorff's Maximality Principle

(MP): For every every order
$$r \in \mathbf{U}$$
, the set $\mathsf{dom}[r^{\pm}]$ contains a maximal r-chain.

Hausdorff's Maximality Principle restricted to ordinary trees is called the *Principle of Tree Choice* and is denoted by (TC).

Let us recall that an order R is called a tree-order if for every $x \in \mathsf{dom}[R^{\pm}]$ the initial interval R(t) is well-ordered by the relation $R \upharpoonright R$ (t). A standard example of a tree-order is the order $\mathbf{S} \upharpoonright \mathbf{U}^{<\mathbf{On}}$ where $\mathbf{S} = \{\langle x, y \rangle : x \subseteq y\}$ and $\mathbf{U}^{<\mathbf{On}}$ is the class of all function f with $\mathsf{dom}[f] \in \mathbf{On}$. An ordinary tree is a subclass $T \subseteq \mathbf{U}^{<\mathbf{On}}$ such that for every function $t \in T$ and ordinal α the function $t \upharpoonright_{\alpha} = t \cap (\alpha \times \mathbf{U})$ belongs to T. For an ordinary tree T, a subclass $C \subseteq T$ is called a (maximal) chain if it is a (maximal) $\mathbf{S} \upharpoonright T$ -chain.

Now we list some statements that are equivalent to the Axiom of Choice.

Theorem 27.4. The following statements are equivalent:

- (AC) Every set has a choice function.
- (KZ) Every chain-bounded order $r \in U$ has an r-maximal element.
- (MP) For every order $r \in \mathbf{U}$, the set $\mathsf{dom}[r^{\pm}]$ contains an r-maximal chain.
- (TC) Every ordinary tree $t \in \mathbf{U}$ contains a maximal chain.
- (WO) Every set x can be well-ordered.
 - (II) For any set a and indexed family of nonempty sets $(x_{\alpha})_{\alpha \in a}$, the Cartesian product $\prod_{\alpha \in a} x_{\alpha}$ is not empty.

Proof. The implication (AC \Rightarrow KZ) has been proved in the Kuratowski–Zorn Lemma 27.3.

 $(KZ) \Rightarrow (MP)$: Fix an order $r \in U$ and consider the set

$$c = \{ \ell \subseteq \mathsf{dom}[r^{\pm}] : \ell \text{ is an } r\text{-chain} \},$$

endowed with the partial order $\mathbf{S} \upharpoonright c$. It is easy to see that for any $\mathbf{S} \upharpoonright c$ -chain $c' \subseteq c$, the union $\bigcup c'$ is an r-chain, which is an upper bound of the chain c' in the set c endowed with the partial order $\mathbf{S} \upharpoonright c$. This means that the partial order $\mathbf{S} \upharpoonright c$ is chain-bounded. By (KZ), there exists an $\mathbf{S} \upharpoonright c$ -maximal element $c' \in c$, which is a required maximal r-chain in $\mathsf{dom}[r^{\pm}]$.

 $(\mathsf{MP}) \Rightarrow (\mathsf{TC})$: Given an ordinary tree t, consider the partial order $\mathbf{S} \upharpoonright t$ and applying (MP) , find a maximal $\mathsf{S} \upharpoonright t$ -chain $c \subseteq t$, which is a maximal chain in the ordinary tree t.

 $(TC) \Rightarrow (WO)$: Given any set x, consider the ordinary tree

$$T = \{ f \in \mathbf{U}^{<\mathbf{On}} : f^{-1} \in \mathbf{Fun} \ \land \ \mathsf{rng}[f] \subseteq x \}$$

consisting of all injective functions f with $\mathsf{dom}[f] \in \mathbf{On}$ and $\mathsf{rng}[f] \subseteq x$. By the Hartogs Theorem 23.9, there exists an ordinal α (equal to $\mathsf{rank}[WO(x)]$) that admits no injective function $f:\alpha \to x$. Since $T \subseteq \mathcal{P}(\alpha \times x)$, the tree T is a set. By (TC), the ordinary tree T contains a maximal chain $c \subseteq T$. It is easy to see that the union of this chain $f = \bigcup c$ is an injective function with $\mathsf{dom}[f] \in \mathbf{On}$ and $\mathsf{rng}[f] \subseteq x$. So, $f \in T$. We claim that $\mathsf{rng}[f] = x$. Assuming that $x \neq \mathsf{rng}[f]$, take any element $y \in x \setminus \mathsf{rng}[f]$ and consider the function $\overline{f} = f \cup \{\langle \mathsf{dom}[f], y \rangle\} \in T$. The maximality of the chain c guarantees that $c = c \cup \{\overline{f}\}$ and hence $\overline{f} \in c$. Then $\langle \mathsf{dom}[f], y \rangle \in \overline{f} \subseteq \bigcup c = f$ and $\mathsf{dom}[f] \in \mathbf{On}$, which contradicts the definition of an ordinal. Therefore, $f : \mathsf{dom}[f] \to x$ is a bijective function and

$$w = \{ \langle f(\gamma), f(\beta) \rangle : \gamma \in \beta \in \mathsf{dom}[f] \}$$

is a well-order w with $dom[w^{\pm}] = x$, witnessing that the set x is well-ordered.

(WO \Rightarrow II): Let a be a set and $(x_{\alpha})_{\alpha \in a}$ be an indexed family of sets. By the Axiom of Union, the class $x = \bigcup_{\alpha \in A} x_{\alpha}$ is a set. By (WO), the set x admits a well-order w with $\mathsf{dom}[w^{\pm}] = x$. Now consider the function $f: a \to x$ assigning to every $\alpha \in a$ the unique w-minimal element $\min_w(x_{\alpha})$ of the set x_{α} . Then $f \in \prod_{\alpha \in a} x_a$, witnessing that the Cartesian product $\prod_{\alpha \in a} x_a$ is not empty.

 $(\Pi) \Rightarrow (\mathsf{AC})$: By (Π) , for any set a, the Cartesian product $\prod_{x \in a \setminus \{\emptyset\}} x$ contains some function $f : a \setminus \{\emptyset\} \to \bigcup a$, which is a choice function for the set a.

Exercise 27.5. Prove that the Axiom of Choice is equivalent to the existence of a maximal r-antichain $a \subseteq \text{dom}[r^{\pm}]$ for each order $r \in \mathbf{U}$.

Now we present an application of the Kuratowski-Zorn Lemma to ultrafilters.

Definition 27.6 (Cartan). A class F is called a filter if the following conditions are satisfied:

- (1) $\emptyset \notin F$;
- (2) $\forall x \in F \ \forall y \in F \ (x \cap y \in F)$;
- (3) $\forall x \, \forall y \, ((x \in F \land x \subseteq y \subseteq \bigcup F) \Rightarrow y \in F).$

A filter F is called a filter on a class X if $X = \bigcup F$.

A filter F is called an *ultrafilter* if F is equal to any filter F' such that $F \subseteq F'$ and $\bigcup F = \bigcup F'$.

Example 27.7. The family $F = \{x \in \mathcal{P}(\omega) : \exists n \in \omega \ (\omega \setminus x \subseteq n)\}$ is a filter, called the Fréchet filter on ω .

The Kuratowski-Zorn Lemma has the following implication (the stetement UL below is called the *Ultrafilter Lemma*).

Lemma 27.8. The Axiom of Choice implies the following statement:

(UL): Each filter φ on a set x is a subset of some ultrafilter u on x.

Proof. Given a filter φ on a set x, consider the set $\hat{\varphi}$ of all filters on x that contain the filter φ as a subset. The set $\hat{\varphi}$ is a subset of the double exponent $\mathcal{P}(\mathcal{P}(x))$, so it exists by the Axiom of Power-Set. By the Kuratowski-Zorn Lemma, the set $\hat{\varphi}$ endowed with the partial order $\mathbf{S} \upharpoonright \hat{\varphi}$ contains an $\mathbf{S} \upharpoonright \hat{\varphi}$ -maximal element u, which is the required ultrafilter on x that contains φ .

It is known [4] that the statement (UF) appearing in Corollary 27.8 is not equivalent to the Axiom of Choice. Nonetheless, it implies that each set admits a linear order.

Proposition 27.9. The statement (UF) implies the following weak version of (WO):

(LO): for every set x there exists a linear order ℓ such that $x = \text{dom}[\ell^{\pm}]$.

Proof. Given a set x, consider the set λ of all linear orders ℓ , which are finite subsets of $x \times x$. Let $\mathcal{P}_{<\omega}(x)$ be the set of finite subsets of x. We recall that a set s is called *finite* if there exists an injective function f such that $\mathsf{dom}[f] = s$ and $\mathsf{rng}[f] \in \omega$.

For any finite subset $a \subseteq x$ let $\lambda_a = \{\ell \in \lambda : a \subseteq \mathsf{dom}[\ell^{\pm}]\}$. Consider the set

$$\varphi = \{ l \subseteq \lambda : \exists a \in \mathcal{P}_{\leq \omega}(x) \ (\lambda_a \subseteq l) \}.$$

It is easy to see that φ is a filter with $\bigcup \varphi = \lambda$. By (UF), the filter φ is contained in some ultrafilter u with $\bigcup u = \lambda$. It can be shown that $\ell = \bigcap_{l \in u} \bigcup l$ is a linear order with $\mathsf{dom}[\ell^{\pm}] = x$.

Exercise 27.10. Fill all the details in the proof of Proposition 27.9.

Remark 27.11. The statement (UL) is equivalent to many important statements in Mathematics. For example, it is equivalent to the compactness of the Tychonoff product of any family of compact Hausdorff spaces, see [4, 2.6.15]. For a long list of statements which are equivalent to (UL), see [8, Form 14].

Proposition 27.12. The statement (LO) implies the following statement

(AC^{$<\omega$}): For every set A and every indexed family of finite nonempty sets $(X_{\alpha})_{\alpha \in A}$, there exists a function $f: A \to \bigcup_{\alpha \in A} X_{\alpha}$ such that $f(\alpha) \in X_{\alpha}$ for every $\alpha \in A$.

Proof. By the statement LO, for the set $X = \bigcup_{\alpha \in A} X_{\alpha}$ there exists a linear order ℓ such that $\mathsf{dom}[\ell^{\pm}] = X$. Let $f : A \to X$ be the function assigning to every $\alpha \in A$ the unique ℓ -minimal element of the finite set X_{α} . It is clear that $f \in \prod_{n \in \omega} X_{\alpha}$.

Exercise 27.13. Prove that for every finite linear order ℓ , the finite set $dom[\ell^{\pm}]$ contains a unique ℓ -minimal element.

Hint: Apply the Principle of Mathematical Induction.

Observe that the statement (TC) from Theorem 27.4 implies the following weaker statement

$$(\mathsf{TC}_{\omega})$$
: Every ordinary tree $t \subseteq \mathbf{U}^{<\omega}$ with $t \in \mathbf{U}$ contains a maximal chain.

The statement (TC_{ω}) is equivalent to the Axiom of Dependent Choice (DC), introduced by Paul Bernays in 1942 whose aim was to suggest an axiom which is weaker than AC and

does not have strange consequences like the Banach-Tarski Paradox⁸ but still is sufficient for normal development of Matematical Analysis.

Proposition 27.14. The principle (TC_ω) is equivalent to the following statement

(DC): For any relation $r \in \mathbf{U}$ with $\mathsf{dom}[r] = \mathsf{dom}[r^{\pm}]$ there exists a function f such that $\mathsf{dom}[f] = \omega$ and $\langle f(n), f(n+1) \rangle \in r$ for all $n \in \omega$.

Proof. $(\mathsf{TC}_{\omega}) \Rightarrow (\mathsf{DC})$: Given any relation $r \in \mathbf{U}$ with $\mathsf{dom}[r] = \mathsf{dom}[r^{\pm}]$, consider the set t of all functions f such that $\mathsf{dom}[f] \in \omega$, $\mathsf{rng}[f] \subseteq \mathsf{dom}[r^{\pm}]$ and $\langle f(k), f(k+1) \rangle \in r$ for any $k \in \mathsf{dom}[f]$ with $k+1 \in \mathsf{dom}[f]$. It is clear that t is an ordinary tree with $t \subseteq \mathbf{U}^{<\omega}$. By (TC_{ω}) , t contains a maximal chain c. Then the union $f = \bigcup c$ is a required function with $\mathsf{dom}[f] = \omega$ and $\langle f(n), f(n+1) \rangle \in r$ for all $n \in \omega$.

 $(DC) \Rightarrow (TC_{\omega})$: Let $t \subseteq U^{<\omega}$ be an ordinary tree, which is a set. Then the relation

$$r = \left\{ \langle \langle f, k, f(k) \rangle, \langle f, k+1, f(k+1) \rangle \rangle : (f \in t) \ \land \ (k \in \mathsf{dom}[f]) \ \land \ (k+1 \in \mathsf{dom}[f]) \right\}$$

is a set, too. If t has a maximal chain, then we are done. If t has no maximal chains, then t contains no $\mathbf{S} \upharpoonright T$ -maximal elements and hence $\mathsf{dom}[r] = \mathsf{dom}[r^{\pm}]$. By (DC), there exists a function g such that $\mathsf{dom}[g] = \omega$ and $\langle g(n), g(n+1) \rangle \in r$ for all $n \in \omega$. Then $g(0) = \langle f, k, f(k) \rangle$ for some $f \in t$ and $k \in \omega$. We claim that $g(n) = \langle f, k+n, f(k+n) \rangle$ for all $n \in \omega$. For n = 0 this follows from the choice of k. Assume that for some $n \in \omega$ we proved that $g(n) = \langle f, k+n, f(k+n) \rangle$. Since $\langle g(n), g(n+1) \rangle \in r$, the definition of the relation r ensures that $g(n+1) = \langle f, k+n+1, f(k+n+1) \rangle$. By the Principle of Mathematical Induction, $g(n) = \langle f, k+n, f(k+n) \rangle$ for all $n \in \omega$. Then $\mathsf{dom}[f] = \omega$ and $\{f \upharpoonright_n : n \in \omega\}$ is a maximal chain in the tree t.

In its turns, the principle (TC_{ω}) implies the Axiom of Countable Choice (AC_{ω}) , introduced in the following proposition.

Proposition 27.15. The Principle (TC_ω) implies the following statement:

(AC_{\omega}): For any indexed sequence of nonempty sets $(X_n)_{n\in\omega}$ there exists a function $f:\omega\to\bigcup_{n\in\omega}X_n$ such that $f(n)\in X_n$ for every $n\in\omega$.

Proof. Consider an ordinary tree t consisting of functions f such that $\mathsf{dom}[f] \in \omega$ and $f(k) \in X_k$ for all $k \in \mathsf{dom}[f]$. By (TC_ω) , the tree t contains a maximal chain c. By the maximality of c, the union $f = \bigcup c$ is a function such that $\mathsf{dom}[f] = \omega$ and $f \upharpoonright_n \in t$ for all $n \in \omega$. The definition of the tree t ensures that $f(k) \in X_k$ for all $k \in \omega$.

Exercise 27.16. Prove that (AC_{ω}) is equivalent to the existence of a choice function $c: X \setminus \{\emptyset\} \to \bigcup X$ for any countable set X.

In its turn, the Axiom of Countable Choice implies the following statement (UT_{ω}) called the Countable Union Theorem.

Proposition 27.17. The Axiom of Countable Choice (AC_{ω}) implies the following statement. (UT_{ω}) : For any indexed family of countable sets $(X_n)_{n\in\omega}$ the set $\bigcup_{n\in\omega} X_n$ is countable.

Proof. By the Axiom of Union, the union $X = \bigcup_{n \in \omega} X_n$ is a set. Consider the function $\nu : X \to \omega$ assigning to every element $x \in X$ the smallest ordinal $\nu(x) \in \omega$ such that $x \in X_{\nu(x)}$.

⁸Task: Read about the Banach-Tarski Paradox in Wikipedia.

For every $n \in \omega$ consider the set F_n of all injective functions f such that $dom[f] = X_n$ and $\operatorname{rng}[f] \in \omega \cup \{\omega\}$. Since the set X_n are countable, the sets F_n are not empty. By the Axiom of Countable Choice, there exists a function $\varphi \in \prod_{n \in \omega} F_n$. For every $n \in \omega$ denote the function $\varphi(n) \in F_n$ by φ_n . Consider the function

$$\mu: X \to \omega, \quad \mu: x \mapsto \max\{\nu(x), \varphi_{\nu(x)}(x)\} = \mu(x) \cup \varphi_{\nu(x)}(x).$$

On the set X consider the irreflexive set-like well-order

$$W = \{ \langle x, y \rangle \in X \times X : \mu(x) < \mu(y) \lor (\mu(x) = \mu(y) \land \nu(x) < \nu(y)) \lor (\mu(x) = \mu(y) \land \nu(x) = \nu(y) \land \varphi_{\nu(x)}(x) < \varphi_{\nu(y)}(y) \} \}.$$

By Theorem 23.6, the function $\operatorname{rank}_W: X \to \operatorname{rank}(W) \in \mathbf{On}$ is an order isomorphism. The definition of the well-order W implies that each initial interval of W is finite 9 and hence $\operatorname{\mathsf{rank}}(W) \leq \omega$. Then $\operatorname{\mathsf{rank}}_W$ is an injective function with $\operatorname{\mathsf{dom}}[\operatorname{\mathsf{rank}}_W] = X$ and $\operatorname{\mathsf{rng}}[\operatorname{\mathsf{rank}}_W] = X$ $rank(W) \in \omega \cup \{\omega\}$, witnessing that the set X is countable.

Proposition 27.18. The Countable Union Theorem implies the following statement:

 (AC_{ω}^{ω}) For any indexed sequence of countable sets $(X_n)_{n\in\omega}$, there exists a function $f: \omega \to \bigcup_{n \in \omega} X_n$ such that $f(n) \in X_n$ for every $n \in \omega$.

Proof. Let $(X_n)_{n\in\omega}$ be an indexed sequence of nonempty countable sets. By Countable Union Theorem, the set $X = \bigcup_{n \in \omega} X_n$ is countable and hence $X = \text{dom}[W^{\pm}]$ for some well-order W. Consider the function $f:\omega\to X$ assigning to each $n\in\omega$ the unique W-minimal element of the nonempty set X_n .

Each of the statements (AC_{ω}^{ω}) or $(AC^{<\omega})$ imply the equivalent statements in the following theorem.

Theorem 27.19. The following statements are equivalent:

 $(AC_{\omega}^{<\omega})$: For any indexed sequence of nonempty finite sets $(X_n)_{n\in\omega}$, there exists a function $f: \omega \to \bigcup_{n \in \omega} X_n$ such that $f(n) \in X_n$ for every $n \in \omega$.

 $(\mathsf{UT}_{\omega}^{<\omega})$: For any indexed sequence of finite sets $(X_n)_{n\in\omega}$ the union $\bigcup_{n\in\omega} X_n$ is countable. $(\mathsf{TC}_{\omega}^{<\omega})$: Any locally finite ordinary tree $T\subseteq \mathbf{U}^{<\omega}$ contains a maximal chain.

(DC $^{<\omega}$): For any relation $r \in \mathbf{U}$ such that for every $x \in \mathsf{dom}[r]$ the set $\{y : \langle x,y \rangle \in r\}$ is finite and non-empty, there exists a function f such that $dom[f] = \omega$ and $\forall n \in \omega \ \langle f(n), f(n+1) \rangle \in r.$

Proof. $(\mathsf{AC}^{<\omega}_\omega) \ \Rightarrow \ (\mathsf{UT}^{<\omega}_\omega)$: Given any indexed sequence of finite sets $(X_n)_{n\in\omega}$, for every $n \in \omega$ consider the set

$$F_n = \{ f \in \mathbf{Fun} : f^{-1} \in \mathbf{Fun} \ \land \ \mathsf{dom}[f] = X_n \ \land \ \mathsf{rng}[f] \in \omega \}.$$

Since the sets X_n are finite, the sets F_n are finite and nonempty, see ??. By $(\mathsf{AC}_\omega^{<\omega})$, there exists a function $\varphi \in \prod_{n \in \omega} F_n$. Repeating the argument of the proof of Proposition 27.17, we can prove that the union $\bigcup_{n\in\omega} X_n$ is countable.

 $(\mathsf{UT}_{\omega}^{<\omega}) \Rightarrow (\mathsf{TC}_{\omega}^{<\omega})$: Let $T \subseteq \mathbf{U}^{<\omega}$ be a locally finite ordinary tree. For every $n \in \omega$ consider the class $T_n = \{t \in T : dom[t] = n\}$. Using the Principle of Mathematical Induction and the local finiteness of the tree T, one can prove that for every $n \in \omega$ the class T_n is a

⁹Exercise: Prove (by Mathematical Induction) that all initial intervals of the well-order W are finite.

finite set. By $(\mathsf{UT}_{\omega}^{<\omega})$, the union $T=\bigcup_{n\in\omega}T_n$ is a countable set. Consequently, the set T admits a well-order W such that $\mathsf{dom}[W^\pm]=T$.

Let L be the class of chains in the tree T. For every chain $\ell \in L$, the union $\bigcup \ell$ is a function with $\mathsf{dom}[\bigcup \ell] \subseteq \omega$. Let $\mathsf{Succ}_T(\bigcup \ell) = \{t \in T : \bigcup \ell \subset t \land \mathsf{dom}[t] = \mathsf{dom}[\bigcup \ell] + 1\}$ be the (finite) set of immediate successors of the function $\bigcup \ell$ in the tree T. Consider the function $F : \omega \times \mathbf{U} \to \mathbf{U}$ assigning to every ordered pair $\langle n, \ell \rangle \in \omega \times \mathbf{U}$ the set

$$F(n,\ell) = \begin{cases} \min_{W} (\mathsf{Succ}_{T}(\bigcup \ell)) & \text{if } \ell \in L \text{ and } \mathsf{Succ}_{T}(\bigcup \ell) \neq \emptyset; \\ \emptyset & \text{otherwise.} \end{cases}$$

By the Recursion Theorem 21.1, there exists a function $G: \omega \to \mathbf{U}$ such that G(n) = F(n, G[n]) for every $n \in \omega$. It can be shown that G[w] is a maximal chain in the tree T.

The implication $(\mathsf{TC}_\omega^{<\omega}) \Rightarrow (\mathsf{AC}_\omega^{<\omega})$ can be proved by analogy with Proposition 27.15, and the equivalence $(\mathsf{TC}_\omega^{<\omega}) \Leftrightarrow (\mathsf{DC}_\omega^{<\omega})$ can be proved by analogy with Proposition 27.14. \square

Theorem 27.20 (Kőnig, 1927). The statement $(AC_{\omega}^{<\omega})$ is equivalent to the statement $(TC_{\omega}^{<\omega})$: Every locally finite ordinary tree of countable height has a maximal chain.

Therefore, we have the following diagram of statements related to the Axiom of Choice.

$$\begin{array}{c} \mathsf{LO} \longleftarrow \mathsf{UL} \longleftarrow \mathsf{WO} \longleftarrow \mathsf{KZ} \longleftrightarrow \mathsf{AC} \longleftrightarrow \mathsf{MP} \longleftrightarrow \mathsf{TC} \longrightarrow \mathsf{DC} \\ \downarrow & & \downarrow \\ \mathsf{AC}_{\omega}^{<\omega} \longleftrightarrow \mathsf{TC}_{\omega}^{<\omega} \longleftrightarrow \mathsf{DC}_{\omega}^{<\omega} \longleftrightarrow \mathsf{UT}_{\omega}^{<\omega} \longleftarrow \mathsf{AC}_{\omega} \longleftarrow \mathsf{UT}_{\omega} \longleftarrow \mathsf{AC}_{\omega} \longleftarrow \mathsf{TC}_{\omega} \end{array}$$

Remark 27.21. All implications in this diagram are strict (i.e., cannot be reversed). The proof of this fact requires more advanced technique, see [4].

28. Global Choice

The strongest principle of growth lies in human choice George Eliot

In this section we study the interplay between global versions of Choice Principles that were analyzed in the preceding section.

We start with the statements (GWO) and (AGC) called the *Global Well-Orderability Principle* and the *Axiom of Global Choice*, respectively. These two statements are defined as follows:

(GWO): There exists a set-like well-order W such that $\mathsf{dom}[W^{\pm}] = \mathbf{U}$.

(AGC): There exists a function $F \colon \mathbf{U} \setminus \{\emptyset\} \to \mathbf{U}$ such that $F(x) \in x$ for any nonempty set x.

Let us recall that (AGC) (i.e., the Axiom of Global Choice) is the last axiom of the list NBG. It turns out that (GWO) is equivalent to the conjunction of (AGC) and the

Axiom of Cumulativity

(AV): $\mathbf{U} = \bigcup_{\alpha \in \mathbf{On}} U_{\alpha}$ for some indexed family of sets $(U_{\alpha})_{\alpha \in \mathbf{On}}$.

The Axiom of Cumulativity holds if and only if there exists a function $F: \mathbf{U} \to \mathbf{On}$ such that for every ordinal α the preimage $F^{-1}[\alpha]$ is a set. By the definition of von Neumann cumulative hierarchy $(V_{\alpha})_{\alpha \in \mathbf{On}}$, the Axiom of Cumulativity (AV) follows from the Axiom of Foundation $(\mathbf{U} = \mathbf{V})$.

Theorem 28.1. The Global Well-Orderability Principle holds if and only if the Axiom of Global Choice and Axiom Cumulativity hold simultaneously. This can be written as

$$(\mathsf{GWO}) \Leftrightarrow (\mathsf{AGC} + \mathsf{AV}).$$

Proof. (GWO) \Rightarrow (AGC + AV). Let W be a set-like well-order such that $\mathsf{dom}[W\pm] = \mathbf{U}$. Replacing W by $W \setminus \mathbf{Id}$, we can assume that the well-order W is irreflexive. By Theorem 23.6(1), the rank function $\mathsf{rank}_W : \mathbf{U} \to \mathbf{On}$ is well-defined and injective. Assuming that rank_W is not surjective, we can apply Theorems 23.6(3) and conclude that $\mathsf{rank}(W) = \mathsf{rank}_W[\mathbf{U}]$ is a set. The injectivity of the function rank_W and the Axiom of Replacement imply that the universe $\mathbf{U} = \mathsf{rank}_W^{-1}[\mathsf{rank}(\mathbf{U})]$ is a set, which a contradiction showing that the function $\mathsf{rank}_W : \mathbf{U} \to \mathbf{On}$ is bijective. By the Axiom of Replacement, for every ordinal α the preimage $\mathsf{rank}_W^{-1}[\alpha]$ is a set. Then $\mathbf{U} = \bigcup_{\alpha \in \mathbf{On}} \mathsf{rank}_W^{-1}[\alpha]$ and hence the Axiom of Cumulativity holds.

To see that the Axiom of Global Choice holds, consider the function $\min_W : \mathbf{U} \setminus \{\emptyset\} \to \mathbf{U}$ assigning to each nonempty set x its (unique) W-minimal element $\min_W(x)$.

 $(\mathsf{AGC} + \mathsf{AV}) \Rightarrow (\mathsf{GWO})$: By (AGC) , there exists a function $C : \mathbf{U} \to \mathbf{U}$ such that $C(x) \in x$ for every nonempty set x. By (AV) , $\mathbf{U} = \bigcup_{\alpha \in \mathbf{On}} U_{\alpha}$ for some indexed family of sets $(U_{\alpha})_{\alpha \in \mathbf{On}}$.

Consider the function $\mu: \mathbf{U} \to \mathbf{On}$ assigning to each set $y \in \mathbf{U}$ the smallest ordinal α such that $U_{\alpha} \setminus y \neq \emptyset$. Then $U_{\mu(y)} \setminus y$ is not empty and we can consider the element $C(U_{\mu(y)} \setminus y)$ given by the choice function C.

Then the function

$$F: \mathbf{On} \times \mathbf{U} \to \mathbf{U}, \quad F: \langle \alpha, y \rangle \mapsto C(U_{\mu(y)} \setminus y)$$

is well-defined. By the Recursion Theorem 21.1, there exists a function $G: \mathbf{On} \to \mathbf{U}$ such that

$$G(\alpha) = F(\alpha, G[\alpha]) = C(U_{\mu(G[\alpha])} \setminus G[\alpha]) \in \mathbf{U} \setminus G[\alpha]$$

for every $\alpha \in \mathbf{On}$. This property of G implies that G is injective. Next, we show that $G[\mathbf{U}] = \mathbf{On}$. Assuming that $G[\mathbf{U}] \neq \mathbf{U}$, we can find a set $z \notin G[\mathbf{U}]$ and an ordinal α such that $z \in U_{\alpha}$. It follows that for every ordinal β , the set $U_{\alpha} \setminus G[\beta] \ni z$ is not empty. The definition of the functions μ and F guarantee that $\mu(G[\beta]) \leq \alpha$ and $G(\beta) = F(\beta, G[\beta]) \in \bigcup_{\gamma \leq \alpha} U_{\gamma}$. Now we see that G is an injective function from \mathbf{On} to the set $\bigcup_{\gamma \leq \alpha} U_{\gamma}$ which contradicts the Axiom of Replacement. This contradiction shows that $G[\mathbf{On}] = \mathbf{U}$. Then we can define a set-like well-order W on \mathbf{U} by the formula

$$W = \{ \langle G(\alpha), G(\beta) \rangle : \alpha \in \beta \in \mathbf{On} \}.$$

In the following theorem we prove that under (AV), the Axiom of Global Choise is equivalent to many global versions of the statements, equivalent to the Axiom of Choice.

Theorem 28.2. If the Axiom of Cumulativity (AV) holds (which follows from U = V), then the following statements are equivalent:

(GWO): There exists a set-like well-order W such that $dom[R^{\pm}] = U$.

(GwO): There exists a well-order W such that $dom[R^{\pm}] = U$.

- (Gwo): There exists a linear order W such that $dom[R^{\pm}] = U$ and each nonempty set contains a W-minimal element.
- (GMP): For every order R there exists a maximal R-chain $C \subseteq \text{dom}[R^{\pm}]$.
- (GKZ): For every chain-bounded order R there exists an R-maximal element $x \in \text{dom}[R^{\pm}]$.
- (GTC): Every ordinary tree has a maximal chain.
- (TC^s): Every locally set ordinary tree has a maximal chain.
- (EC): For every equivalence relation R there exists a class C such that for every $x \in dom[R]$ the intersection $R[\{x\}] \cap C$ is a singleton.
- (AC_c): For every indexed family of non-empty classes $(X_{\alpha})_{\alpha \in A}$ there exists a function $f \colon A \to \bigcup_{\alpha \in A} X_{\alpha}$ such that $f(\alpha) \in X_{\alpha}$ for all $\alpha \in A$.
- (AC_c): For every indexed family of non-empty sets $(X_{\alpha})_{\alpha \in A}$ there exists a function $f: A \to \bigcup_{\alpha \in A} X_{\alpha}$ such that $f(\alpha) \in X_{\alpha}$ for all $\alpha \in A$.
- (AGC): There exists a function $F \colon \mathbf{U} \setminus \{\emptyset\} \to \mathbf{U}$ such that $F(x) \in x$ for every nonempty set x.

To prove Theorem 28.2, in Lemmas 28.3–28.10 we shall prove the following implications.

$$(\mathsf{AV} + \mathsf{TC^s}) \xrightarrow{} (\mathsf{GWO}) \xrightarrow{} (\mathsf{GwO}) \xrightarrow{} (\mathsf{Gwo}) \xrightarrow{} \mathsf{AGC}$$

$$28.5 \downarrow \qquad 28.7 \downarrow \qquad \qquad \downarrow \qquad$$

We recall that an ordinary tree $T \in \mathbf{U}^{<\mathbf{On}}$ is called *locally set* if for each $t \in T$ the class $\mathsf{Succ}_T(t)$ of its immediate successors in T is a set.

Lemma 28.3.
$$(AV + TC^s) \Rightarrow (GWO)$$
.

Proof. By (AV), there exists an indexed family of sets $(U_{\alpha})_{\alpha \in \mathbf{On}}$ such that $\mathbf{U} = \bigcup_{\alpha \in \mathbf{On}} U_{\alpha}$. This family induces the function $\mu : \mathbf{U} \to \mathbf{On}$ assigning to every set y the smallest ordinal α such that $U_{\alpha} \setminus y \neq \emptyset$. Let T be the class of functions f such that $\mathsf{dom}[f] \in \mathbf{On}$ and for every $\alpha \in \mathsf{dom}[f]$ with $\alpha + 1 \in \mathsf{dom}[f]$ we have $f(\alpha) \in U_{\mu(f[\alpha])} \setminus f[\alpha]$. It is easy to see that T is a locally set ordinary tree. By $(\mathsf{TC}^{\mathsf{s}})$, this tree contains a maximal chain C. Its union $F = \bigcup C$ is a function such that $\mathsf{dom}[f] \subseteq \mathbf{On}$ and $f(\alpha) \in U_{\mu(f[\alpha])} \setminus f[\alpha]$ for every $\alpha \in \mathsf{dom}[f]$. The latter condition ensures that f is injective.

We claim that $\operatorname{rng}[f] = \mathbf{U}$. Assuming that $\operatorname{rng}[f] \neq \mathbf{U}$, we can find the smallest ordinal α such that $U_{\alpha} \cap (\mathbf{U} \setminus \operatorname{rng}[f]) \neq \emptyset$. Then for every $\beta \in \mathbf{On}$ the set $U_{\alpha} \setminus f[\beta] \supseteq U_{\alpha} \setminus \operatorname{rng}[f]$ is not empty, which implies that $\mu(f[\beta]) \leq \alpha$ and $f(\beta) \in \bigcup_{\gamma \leq \alpha} U_{\gamma}$. Consequently, $\operatorname{rng}[f] \subseteq \bigcup_{\gamma \leq \alpha} U_{\gamma}$. By the injectivity of f and the Axiom of Replacement, $\operatorname{dom}[f] \subseteq \mathbf{On}$ is a set and hence $\operatorname{dom}[f] \in \mathbf{On}$. Now take any element $z \in U_{\alpha} \setminus \operatorname{rng}[f]$ and consider the chain $\overline{C} \cup \{\langle \operatorname{dom}[f], z \rangle\} \subseteq T$, witnessing that the chain C is not maximal. But this contradicts the choice of C. This contradiction shows that $\operatorname{rng}[f] = \mathbf{U}$. Now we see that

$$W = \{ \langle f(\beta), f(\gamma) \rangle : \beta \in \gamma \in \mathsf{dom}[f] \subseteq \mathbf{On} \}$$

is a set-like well-order with $\mathsf{dom}[W^{\pm}] = \mathbf{U}$.

Lemma 28.4. (GWO) \Rightarrow (GKZ).

Proof. The proof is a suitable modification of the Kuratowski-Zorn Lemma 27.3. By (GWO), there exists a set-like well-order W with $dom[W] = \mathbf{U}$. Since W is set-like, for every $x \in \mathbf{U}$ the initial interval $\widetilde{W}(x)$ is a set.

To prove (GKZ), fix any chain-bounded order R. If R is a set, then we can apply the Kuratowski–Zorn Lemma 27.3 and conclude that the set $\mathsf{dom}[R^{\pm}]$ contains an R-maximal element. So, we assume that R is a proper class. To derive a contradiction, assume that no element $x \in \mathsf{dom}[R^{\pm}]$ is R-maximal.

Let C be the class of all R-chains which are subsets of the class $X = \mathsf{dom}[R^{\pm}]$. Repeating the argument of the proof of Lemma 27.3, we can show that for every R-chain $\ell \in C$, the class $V(\ell) = \{b \in \mathsf{dom}[R^{\pm}] : \ell \times \{b\} \subseteq R \setminus \mathbf{Id}\}$ is not empty. The well-foundedness of W guarantees that the class $V(\ell)$ contains a unique W-minimal element $\min_W(V(\ell))$.

Consider the function $F: \mathbf{On} \times \mathbf{U} \to \mathbf{U}$ defined by the formula

$$F(\alpha, y) = \begin{cases} \min_{W}(V(y)) & \text{if } y \in C; \\ \emptyset & \text{otherwise.} \end{cases}$$

By the Recursion Theorem 21.1, there exists a (unique) function $G: \mathbf{On} \to \mathbf{U}$ such that $G(\alpha) = F(\alpha, G[\alpha])$ for every ordinal α .

Repeating the argument of the proof of Lemma 27.3, we can show that for every ordinal α the image $G[\alpha]$ is an R-chain and hence

$$G(\alpha) = F(\alpha, G[\alpha]) = \min_{W}(V(G[\alpha])) \in V(G[\alpha]) \subseteq \mathbf{U} \setminus G[\alpha],$$

which implies that the function $G: \mathbf{On} \to \mathsf{dom}[R^{\pm}]$ is injective. Since for every $\alpha \in \mathbf{On}$ the element $G(\alpha) = V(G[\alpha])$ is an upper bound of the R-chain $G[\alpha]$, the image $G[\mathbf{On}]$ is an R-chain in $\mathsf{dom}[R^{\pm}]$. Since R is chain-bounded, the R-chain $G[\mathbf{On}]$ has an upper bound b. It follows that for every ordinal α the element b belongs to the set $V(G[\alpha])$ and hence $G(\alpha) = \min_W (V(G[\alpha])) \in \overline{W}(b)$. Therefore $G[\mathbf{On}] \subseteq \overline{W}(b)$ and hence $\mathbf{On} = G^{-1}[\overline{W}(b)]$ is a set by the Axiom of Replacement. But this contradicts Theorem 19.6(6).

Lemma 28.5.
$$(GWO) \Rightarrow (GMP)$$
.

Proof. Assume that (GWO) holds and fix a set-like well-order W such that $dom[W^{\pm}] = \mathbf{U}$. By Theorem 28.1, the Axiom of Cumulativity holds, so we can find an indexed family of sets $(U_{\alpha})_{\alpha \in \mathbf{On}}$ such that $\mathbf{U} = \bigcup_{\alpha \in \mathbf{On}} U_{\alpha}$. Replacing each set U_{α} by the union $\bigcup_{\beta \leq \alpha} U_{\beta}$, we can assume that $U_{\beta} \subseteq U_{\alpha}$ for any ordinals $\beta \in \alpha$.

To prove the (GMP), take any order R. For every ordinal α let Λ_{α} be the set of all R-chains $\ell \subseteq U_{\alpha} \cap \mathsf{dom}[R^{\pm}]$. The set Λ_{α} is endowed with the partial order $\mathbf{S} \upharpoonright \Lambda_{\alpha}$. Let M_{α} be the subset of Λ_{α} consisting of $\mathbf{S} \upharpoonright \Lambda_{\alpha}$ -maximal chains. By the (GKZ) (which follows from (GWO) by Lemma 28.4), for every chain $\ell \in \Lambda_{\alpha}$ the set $M_{\alpha}(\ell) = \{\lambda \in M_{\alpha} : \ell \subseteq \lambda\}$ is not empty and hence contains a unique \min_{W} -minimal element $\min_{W}(M_{\alpha}(\ell))$.

So, we can define the function $F: \mathbf{On} \times \mathbf{U} \to \mathbf{U}$ by the formula

$$F: \langle \alpha, y \rangle = \begin{cases} \min_{W} (M_{\alpha}(\bigcup y)) & \text{if } \bigcup y \in \Lambda_{\alpha}; \\ \emptyset & \text{otherwise.} \end{cases}$$

By the Recursion Theorem 21.1, there exists a function $G: \mathbf{On} \to \mathbf{U}$ such that $G(\alpha) = F(\alpha, G[\alpha])$ for every $\alpha \in \mathbf{On}$.

We claim that for every ordinal α the set $G(\alpha)$ is an element of the set M_{α} and $G(\beta) \subseteq G(\alpha)$ for all $\beta \in \alpha$. For $\alpha = 0$, $G(0) = F(0, \emptyset) = \min_{W}(M_0(\emptyset)) \in M_0$. Assume that for some ordinal

 α we have proved that for every ordinal $\beta \in \alpha$ the set $G(\beta)$ is an element of M_{β} and for every ordinal $\gamma \in \beta$ we have $G(\gamma) \subseteq G(\beta)$. Then the union $\ell = \bigcup_{\beta \in \alpha} G(\beta)$ is a chain in the set $\operatorname{dom}[R^{\pm}] \cap \bigcup_{\beta \in \alpha} U_{\beta} \subseteq \operatorname{dom}[R^{\pm}] \cap U_{\alpha}$ and hence $\ell \in \Lambda_{\alpha}$. Now the definition of the function F guarantees that

$$G(\alpha) = F(\alpha, G[\alpha]) = \min_{W}(M_{\alpha}(\ell)) \in M_{\alpha}$$

is a maximal $\mathbf{S} \upharpoonright \Lambda_{\alpha}$ -chain containing the chain ℓ as a subset. Then for every $\beta \in \alpha$ we have $G_{\beta} \subseteq \ell \subseteq G_{\alpha}$.

Since the transifinite sequence of R-chains $(G(\alpha))_{\alpha \in \mathbf{On}}$ is increasing, its union

$$L = \bigcup_{\alpha \in \mathbf{On}} G(\alpha)$$

is an R-chain in $\mathsf{dom}[R^{\pm}]$. We claim that L is a maximal R-chain. In the opposite case, we could find an element $b \in \mathsf{dom}[R^{\pm}] \setminus L$ such that $L \cup \{b\}$ is an R-chain. Find an ordinal α such that $b \in U_{\alpha}$. Consider the R-chain $G(\alpha) \subseteq L$ and observe that $G(\alpha) \cup \{b\}$ is a chain in $\mathsf{dom}[R^{\pm}] \cap U_{\alpha}$, which implies that $G(\alpha) \notin M_{\alpha}$. But this contradicts the choice of $G(\alpha)$. This contradiction completes the proof of the maximality of the R-chain L.

Lemma 28.6. $(GMP) \Rightarrow (GKZ)$.

Proof. Assume that (GMP) holds. To prove (GKZ), take any chain-bounded order R. We need to find an R-maximal element $b \in \text{dom}[R^{\pm}]$. By (GMP), the class $\text{dom}[R^{\pm}]$ contains a maximal R-chain L. By the chain-boundedness of the order R, the chain L has an upper bound $b \in \text{dom}[R^{\pm}]$. We claim that the element b is R-maximal. In the opposite case we can find an element $b' \in \text{dom}[R^{\pm}]$ such that $\langle b, b' \rangle \in R \setminus \text{Id}$. Then the R-chain is contained in the strictly larger R-chain $C \cup \{b'\}$, which contradicts the maximality of L.

Lemma 28.7. $(GwO) \Rightarrow (EC)$.

Proof. Assume that (GwO) holds, which means that there exists a well-order W such that $\mathsf{dom}[W^{\pm}] = \mathbf{U}$. To prove (EC), take any equivalence relation R. Consider the function $F : \mathsf{dom}[R] \to \mathsf{dom}[R]$ assigning to every $x \in \mathsf{dom}[R]$ the unique W-minimal element of the class $R[\{x\}]$. It is easy to see that the class $C = \mathsf{rng}[F]$ has the required property: for every $x \in \mathsf{dom}[R]$ the intersection $R[\{x\}] \cap C$ is a singleton.

Lemma 28.8. (EC) \Rightarrow (AC_c).

Proof. Assume that (EC) holds. To prove (AC_c), take any indexed family of non-empty classes $X = (X_{\alpha})_{\alpha \in A}$. Consider the equivalence relation

$$R = \bigcup_{\alpha \in A} ((\{\alpha\} \times X_\alpha) \times (\{\alpha\} \times X_\alpha)).$$

By (EC), there exists a class C such that for every $\alpha \in A$ and $x \in X_{\alpha}$ the intersection $C \cap (\{\alpha\} \times X_{\alpha})$ is a singleton. Replacing C by $C \cap X$, we can assume that $C \subseteq X = \bigcup_{\alpha \in A} (\{\alpha\} \times X_{\alpha})$. In this case, C is a function such that $\mathsf{dom}[C] = A$ and $\forall \alpha \in A$ $C(\alpha) \in X_{\alpha}$.

Lemma 28.9. $(GKZ) \Rightarrow (GTC)$.

Proof. Assume that (GKZ) hold. To check (GTC), we should prove that any ordinary tree T has a maximal $\mathbf{S} \upharpoonright T$ -chain. To derive a contradiction, assume that T does not contain maximal $\mathbf{S} \upharpoonright T$ -chains. In this case we shall show that the partial order $\mathbf{S} \upharpoonright T$ is chain-bounded. Take any chain $L \subseteq T$ and consider its union $f = \bigcup L$, which is function with $\mathsf{dom}[f] \subseteq \mathbf{On}$.

Then $C = \{f \upharpoonright_{\alpha} : \alpha \in \mathsf{dom}[t]\}$ is an $\mathbf{S} \upharpoonright T$ -chain in T. By our assumption this chain is not maximal and hence there exists an $\mathbf{S} \upharpoonright T$ -chain $C' \subseteq T$ such that $C \subsetneq C'$. Take any element $c' \in C' \setminus C$ and observe that $f \subseteq c'$ for any function $f \in L$. This means that c' is an upper bound of the chain L, and hence the partial order $\mathbf{S} \upharpoonright T$ is chain-bounded. By (GKZ) , this order has a maximal element $t \in T$. This maximal element t generates the maximal chain $M = \{t\} \cup \{t \upharpoonright_{\alpha} : \alpha \in \mathsf{dom}[t]\}$ in T. But this contradicts our assumption.

Lemma 28.10.
$$(AC_c^c) \Rightarrow (GTC)$$
 and $(AC_c^s) \Rightarrow (TC^s)$.

Proof. Given a (locally set) ordinary tree $T \subseteq \mathbf{U}^{<\mathbf{On}}$, we should find a maximal chain in T. We endow T with the partial order $\mathbf{S} \upharpoonright T$. To derive a contradiction, assume that T contains no maximal chains. Observe that for every chain $C \subseteq T$ its union $f = \bigcup C \subseteq T$ is a function with $\mathsf{dom}[f] \subseteq \mathbf{On}$. If $\mathsf{dom}[f] = \mathbf{On}$, then $\{f \upharpoonright_{\alpha} : \alpha \in \mathbf{On}\}$ is a maximal chain in T, which contradicts our assumption. Then $\mathsf{dom}[f]$ is some ordinal, which implies that the chain C is a set. Therefore, the class L of all chains in T is well-defined. We say that a chain $\ell \in L$ is $\ell \in \mathcal{L}$ if $\ell \in \mathcal{L}$ is a limit ordinal. Let $\ell \in \mathcal{L}$ be the subclass of $\ell \in \mathcal{L}$ consisting of limit chains. By our assumption, every chain in $\ell \in \mathcal{L}$ is not empty. If the tree $\ell \in \mathcal{L}$ is locally set, then $\ell \in \mathcal{L}$ is a set.

Using (AC_c^c) (or (AC_c^s)), we can construct a function $\Psi: \mathbf{U} \to \mathbf{U}$ satisfying the following conditions:

- (1) $\Psi(\ell) = \emptyset$ if $\ell \in \mathbf{U} \setminus L$;
- (2) $\Psi(\ell) = \bigcup \ell \text{ if } \ell \in L';$
- (3) $\Psi(\ell) \in T_{\ell}$ if $\ell \in L \setminus L'$.

In fact, the (AC_c^c) (or (AC_c^s)) is necessary only for satisfying the condition (3). Consider the function

$$F: \mathbf{On} \times \mathbf{U} \to \mathbf{U}, \quad F: \langle \alpha, \ell \rangle \mapsto \Psi(\lfloor \rfloor \ell).$$

Applying the Recursion Theorem 21.1 to the function F and the well-order $\mathbf{E} \upharpoonright \mathbf{On}$, we can find a unique function $G : \mathbf{On} \to \mathbf{U}$ such that $G(\alpha) = F(\alpha, G[\alpha])$ for every $\alpha \in \mathbf{On}$. By transfinite induction it can be shown that for every α the set $G[\alpha]$ is a chain in T and $\mathsf{dom}[G(\alpha)] = \alpha$. Then $G[\mathbf{On}]$ is a maximal chain in T, which contradicts our assumption.

Exercise 28.11. (i) Prove that (GTC) implies the principle

 (AC_s^c) : For every set A and indexed family of nonempty classes $(X_\alpha)_{\alpha\in\omega}$ there exists a function $f:A\to\bigcup_{\alpha\in A}X_n$ such that $f(\alpha)\in X_\alpha$ for all $\alpha\in A$.

(ii) Prove that (AV + AC) implies AC_s^c .

Exercise 28.12. Prove that $(AV + AC_{\omega})$ implies the principle

 $(\mathsf{AC}^\mathsf{c}_\omega)$: For every indexed sequence of classes $(X_n)_{n\in\omega}$ there exists a function $f:\omega\to\bigcup_{n\in\omega}X_n$ such that $f(n)\in X_n$ for every $n\in\omega$.

Exercise 28.13. Prove that $(AV + TC_{\omega})$ implies the principle

 $(\mathsf{TC}^{\mathsf{c}}_{\omega})$: Every ordinary tree $T \subseteq \mathbf{U}^{<\omega}$ has a maximal chain.

Exercise 28.14. Prove that $(\mathsf{TC}^{\mathsf{c}}_{\omega}) \Rightarrow (\mathsf{AC}^{\mathsf{c}}_{\omega})$.

Exercise 28.15. Prove that $(GwO) \Leftrightarrow (Gwo + TC^{c}_{\omega})$.

Part 7. Ordinal Arithmetics

In this section we define algebraic operations on ordinals: addition, multiplication, exponentiation. These algebraic operations on ordinals are introduced with the help of transfinite iterations, considered in the next section.

29. Transfinite Dynamics

A set c is called an **S**-chain if **S**\[cap c is a linear order, i.e., $\forall y, z \in c \ (y \subseteq z \lor z \subseteq y).$

A class X is called *chain-inclusive* if for any S-chain $c \subseteq \mathcal{P}(X)$ its union $\bigcup c$ is an element of X. For example, the class of ordinal **On** is chain-inclusive (by Theorem 19.6(5)), and so is the universal class **U** (trivially).

A function Φ is called *expansive* if $x \subseteq \Phi(x)$ for any $x \in \mathsf{dom}[\Phi]$.

Given a chain-inclusive class X and an expansive function $\Phi: X \to X$, consider the transfinite sequence of functions $(\Phi^{\circ \alpha}: X \to X)_{\alpha \in \mathbf{On}}$ defined by the recursive formula

(29.1)
$$\Phi^{\circ \alpha}(x) = \begin{cases} x & \text{if } \alpha = 0; \\ \Phi(\Phi^{\circ \beta}(x)) & \text{if } \alpha = \beta + 1 \text{ is a successor ordinal;} \\ \bigcup_{\beta \in \alpha} \Phi^{\circ \beta}(x) & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

Theorem 29.1. For any chain-inclusive class X and an expansive function $\Phi: X \to X$ the transfinite sequence of functions $(\Phi^{\circ \alpha})_{\alpha \in \mathbf{On}}$ is well-defined and consists of functions $\Phi^{\circ \alpha}: X \to X$ such that

$$\Phi^{\circ \alpha}(x) = x \cup \bigcup \{\Phi(\Phi^{\circ \gamma}(x)) : \gamma \in \alpha\} \subseteq \Phi^{\circ \beta}(x)$$

for every set $x \in X$ and ordinals $\alpha \leq \beta$.

Proof. Consider the function $F:(X \times \mathbf{On}) \times \mathbf{U} \to \mathbf{U}$ assigning to each triple $\langle x, \alpha, y \rangle \in (X \times \mathbf{On}) \times \mathbf{U}$ the set

$$F(x,\alpha,y) = \begin{cases} x \cup \bigcup \{\Phi(z) : z \in y\} & \text{if } y \subseteq X \\ \emptyset & \text{otherwise.} \end{cases}$$

On the class $X \times \mathbf{On}$ consider the set-like well-founded order

$$R = \{ \langle \langle x, \alpha \rangle, \langle x, \beta \rangle \rangle : x \in X \land \alpha \in \beta \in \mathbf{On} \}$$

and observe that $dom[R^{\pm}] = X \times \mathbf{On}$ and $\overline{R}(x, \alpha) = \{x\} \times \alpha$ for any ordered pair $\langle x, \alpha \rangle \in X \times \mathbf{On}$.

By Recursion Theorem 21.1, there exists a unique function $\Psi: X \times \mathbf{On} \to \mathbf{U}$ such that $\Psi(x,\alpha) = F(x,\alpha,\{\Psi(x,\gamma): \gamma \in \alpha\})$ for every $\langle x,\alpha \rangle \in X \times \mathbf{On}$.

By transfinite induction we will prove that for every $\langle x, \alpha \rangle \in X \times \mathbf{On}$, the following conditions are satisfied:

- $(1_{\alpha}) \ \Psi(x,\alpha) \in X;$
- $(2_{\alpha}) \ \forall \beta \in \alpha \ (\Psi(x,\beta) \subseteq \Psi(x,\alpha));$
- $(3_{\alpha}) \ \Psi(x,\alpha) \subseteq \Phi(\Psi(x,\alpha)) = \Psi(x,\alpha+1).$

For $\alpha = 0$ the condition (1_0) is satisfied:

$$\Psi(x,0) = F(x,0,\{\Psi(x,\gamma) : \gamma \in \emptyset\}) = F(x,0,\emptyset) = x \cup \bigcup \emptyset = x \in X.$$

The condition (2_0) is trivially true, and (3_0) holds as

$$\Psi(x,1) = F(x,1,\{\Psi(x,0)\}) = F(x,1,\{x\}) = x \cup \bigcup \{\Phi(x)\} = x \cup \Phi(x) = \Phi(x) = \Phi(\Psi(x,0)) \supseteq \Psi(x,0)$$

by the expansive property of Φ .

Assume that for some ordinal α and all $\gamma \in \alpha$, the conditions $(1_{\gamma}) - (3_{\gamma})$ hold. By the Axiom of Replacement and conditions (1_{γ}) , $\gamma \in \alpha$, the class $\Psi[\{x\} \times \alpha] = \{\Psi(x,\gamma) : \gamma \in \alpha\}$ is a subset of X. Applying the function Φ to the elements of this subset, and using the inductive conditions (1_{γ}) , $\gamma \in \alpha$, we obtain the subset $\{\Phi(\Psi(x,\gamma))\}_{\gamma \in \alpha}$ of X. By the inductive conditions (2_{γ}) , (3_{γ}) , $\gamma \in \alpha$, the set $\{x\} \cup \{\Phi(\Psi(x,\gamma))\}_{\gamma \in \alpha} = \{\Psi(x,0)\} \cup \{\Psi(x,\gamma+1)\}_{\gamma \in \alpha}$ is an S-chain. Since X is chain-inclusive, the set $\Psi(x,\alpha) = x \cup \bigcup \{\Phi(\Psi(x,\gamma)) : \gamma \in \alpha\}$ is an element of X. Therefore, the condition (1_{α}) is satisfied. To see that (2_{α}) is satisfied, observe that for every $\beta \in \alpha$ we have $\beta \subseteq \alpha$ and hence

$$\Psi(x,\beta) = \{x\} \cup \bigcup \{\Phi(\Psi(x,\gamma)) : \gamma \in \beta\} \subseteq \{x\} \cup \bigcup \{\Phi(\Psi(x,\gamma)) : \gamma \in \alpha\} = \Psi(x,\alpha).$$

To see that (3_{α}) holds, observe that expansive property of Φ and the inductive conditions $(2_{\gamma}), (3_{\gamma})$ for $\gamma \in \alpha$ imply

$$\begin{split} \Psi(x,\alpha+1) &= \{x\} \cup \bigcup \{\Phi(\Psi(x,\gamma)) : \gamma \in \alpha+1\} = \\ \{x\} \cup \{\Psi(x,\gamma+1) : \gamma \in \alpha\} \cup \{\Phi(\Psi(x,\alpha))\} = \Psi(x,\alpha) \cup \Phi(\Psi(x,\alpha)) = \Phi(\Psi(x,\alpha)). \end{split}$$

By the Principle of Transfinite Induction, the conditions (1_{α}) – (3_{α}) hold for every ordinal α .

Now we prove that $\Psi(x,\alpha) = \Phi^{\circ\alpha}(x)$ for any $x \in X$ and $\alpha \in \mathbf{On}$. For $\alpha = 0$ this is true: $\Psi(x,0) = x = \Phi^{\circ 0}(x)$. Assume that for some ordinal α and all its elements $\gamma \in \alpha$ we have proved that $\Psi(x,\gamma) = \Phi^{\circ\gamma}(x)$. If α is a successor ordinal, then $\alpha = \beta + 1$ for some ordinal β and by the inductive condition (3_{α}) ,

$$\Psi(x,\alpha) = \Psi(x,\beta+1) = \Phi(\Psi(x,\beta)) = \Phi(\Phi^{\circ\beta}(x)) = \Phi^{\circ(\beta+1)}(x) = \Phi^{\circ\alpha}(x).$$

If α is a limit ordinal, then using the inductive assumption and the conditions $(3_{\gamma}), \gamma \in \alpha$, we obtain

$$\begin{split} \Psi(x,\alpha) &= \{x\} \cup \bigcup \{\Phi(\Psi(x,\gamma)) : \gamma \in \alpha\} = \{\Psi(x,0)\} \cup \{\Psi(x,\gamma+1) : \gamma \in \alpha\} = \\ &\qquad \qquad \bigcup \{\Psi(x,\gamma) : \gamma \in \alpha\} = \bigcup \{\Phi^{\circ\gamma} : \gamma \in \alpha\}. \end{split}$$

For any finitary expansive function Φ , the transfinite sequence $(\Phi^{\circ \alpha})_{\alpha \in \mathbf{On}}$ stabilizes at the step ω . An expansive function Φ is called *finitary* if for any set $x \in \mathsf{dom}[\Phi]$ and element $a \in \Phi(x)$ there exists a finite set $y \subseteq x$ such that $a \in \Phi(y) \subseteq \Phi(x)$. We recall that a set y is called finite if there exists a function f such that $\mathsf{dom}[f] = x$ and $\mathsf{rng}[f] \in \omega$.

Proposition 29.2. If X is a chain-inclusive class and $\Phi: X \to X$ is a finitary expansive function, then $\Phi^{\circ \omega} = \Phi^{\circ \alpha}$ for any ordinal $\alpha \geq \omega$.

Proof. It suffices to prove that $\Phi^{\circ(\omega+1)}(x) = \Phi^{\circ\omega}(x)$ for any $x \in \mathsf{dom}[\Phi]$. The inclusion $\Phi^{\circ\omega}(x) \subseteq \Phi(\Phi^{\circ\omega}(x)) = \Phi^{\circ(\omega+1)}(x)$ holds by the expansive property of Φ . On the other hand, by the finitarity of Φ , for any $a \in \Phi(\Phi^{\circ\omega}(x))$, there exists a finite subset $y \subseteq \Phi^{\circ\omega}(x) = \bigcup_{n \in \omega} \Phi^{\circ n}(x)$ such that $a \in \Phi(y)$. Since y is finite, there exists $n \in \omega$ such that $y \subseteq \Phi^{\circ n}(x)$. Then

$$a \in \Phi(y) \subseteq \Phi(\Phi^{\circ n}(x)) = \Phi^{\circ (n+1)}(x) \subseteq \Phi^{\circ \omega}(x)$$

and hence $\Phi(\Phi^{\circ\omega}(x)) = \Phi^{\circ\omega}(x)$.

Exercise 29.3. Let $(X_n)_{n\in\omega}$ be a sequence of sets such that $X_n\subseteq X_m$ for all $n\in m\in\omega$. Prove (by induction) that for any finite set $y\subseteq\bigcup_{n\in\omega}X_n$ there exists $n\in\omega$ such that $y\subseteq X_n$.

A function Φ is called *strictly expansive* if $x \subset \Phi(x)$ for any $x \in \text{dom}[\Phi]$.

Proposition 29.4. If X is a chain-inclusive class and $\Phi: X \to X$ is a strictly expansive function, then for any $x \in X$ and ordinals $\alpha < \beta$ we have $\Phi^{\circ \alpha}(x) \subset \Phi^{\circ \beta}(x)$.

Proof. By the strict expansivity of Φ and Theorem 29.1,

$$\Phi(x,\alpha) \subset \Phi(\Phi(x,\alpha)) = \Phi(x,\alpha+1) \subseteq \Phi(x,\beta).$$

Remark 29.5. By Propositions 29.2 and 29.4, strictly expansive functions cannot be finitary.

Example 29.6. For the expansive function $P: \mathbf{U} \to \mathbf{U}, P: x \mapsto x \cup \mathcal{P}(x)$, the transfinite iterations $P^{\circ \alpha}(\emptyset)$ are equal to the sets V_{α} of the von Neumann cumulative hierarchy $(V_{\alpha})_{\alpha \in \mathbf{On}}$.

In this section we analyze the operation

Succ :
$$\mathbf{U} \to \mathbf{U}$$
, Succ : $x \mapsto x \cup \{x\}$,

of taking the successor set. For a set x it successor $x \cup \{x\}$ will be denoted by x+1. It is clear that the function Succ is expansive. If the Axiom of Foundation holds, then this function is strictly expansive.

Let us observe some immediate properties of the function Succ.

Proposition 30.1. Let x, y be two sets.

- 1) $x + 1 \subseteq y$ if and only if $x \subseteq y$ and $x \in y$.
- 2) If x = x + 1, then $x \in x$;
- 3) If x + 1 = y + 1 and $x \neq y$, then $x \in y$ and $y \in x$.

Corollary 30.2. If the Axiom of Foundation holds, then for any sets x, y

- 1) $x \subset x + 1$;
- 2) x = y if and only if x + 1 = y + 1.

Since the membership relation $\mathbf{E} \upharpoonright \mathbf{On}$ is an an irreflexive well-order on \mathbf{On} , Proposition 30.1 has the following

Corollary 30.3. Let α, β be two ordinals.

- 1) $\alpha + 1 \leq \beta$ iff $\alpha < \beta$.
- 2) $\alpha < \alpha + 1$.
- 3) $\alpha + 1 = \beta + 1$ iff $\alpha = \beta$.
- 4) $\alpha + 1 < \beta + 1$ iff $\alpha < \beta$.

Applying Theorem 29.1 to the (expansive) function $Succ : U \to U$, we obtain the following

Corollary 30.4. There exists a transfinite sequence of functions $(Succ^{\circ \alpha})_{\alpha \in \mathbf{On}}$ such that for every set x and ordinal α the following conditions are satisfied:

- 1) $Succ^{\circ \alpha}(x) = x \cup \{ J\{Succ(Succ^{\circ \gamma}(x)) : \gamma \in \alpha \}; \}$
- 2) $Succ^{0}(x) = x$;

- 3) $Succ^{\circ(\alpha+1)}(x) = Succ(Succ^{\circ\alpha}(x));$
- 4) $Succ^{\circ\alpha}(x) = \sup\{Succ^{\circ\gamma}(x) : \gamma \in \alpha\}$ if the ordinal α is limit and nonzero.

31. Addition

The following theorem introduces the addition of ordinals.

Theorem 31.1. There exists a unique function

$$+: \mathbf{On} \times \mathbf{On} \to \mathbf{On}, +: \langle \alpha, \beta \rangle \mapsto \alpha + \beta$$

such that for every ordinals α, β the following conditions are satisfied:

- 0) $\alpha + 0 = \alpha$ for any set x;
- 1) $\alpha + (\beta + 1) = (\alpha + \beta) + 1$ for any set x and any ordinal α ;
- 2) $\alpha + \beta = \bigcup \{\alpha + \gamma : \gamma \in \beta\}$ if the ordinal β is limit and nonzero.

Proof. Define the addition letting $\alpha + \beta = \mathsf{Succ}^{\circ\beta}(\alpha)$ and apply Corollary 30.4. The conditions (1)–(3) and Theorem 19.6(3,4) imply that for any ordinals α, β their sum $\alpha + \beta$ is an ordinal.

Theorem 31.2. Let X be a chain-inclusive class and $\Phi: X \to X$ be an expansive function. Then for any set $x \in X$ and ordinals α, β we have

$$\Phi^{\circ\beta}(\Phi^{\circ\alpha}(x)) = \Phi^{\circ(\alpha+\beta)}(x).$$

Proof. This equality will be proved by transfinite induction on β . Fix $x \in X$ and an ordinal α . Observe that

$$\Phi^{\circ 0}(\Phi^{\circ \alpha}(x)) = \Phi^{\circ \alpha}(x) = \Phi^{\circ (\alpha+0)}(x).$$

Assume that for some ordinal β and all its elements $\gamma \in \beta$ we have proved that $\Phi^{\circ \gamma}(\Phi^{\circ \alpha}(x)) = \Phi^{\circ (\alpha + \gamma)}(x)$.

If β is a successor ordinal, then $\beta = \gamma + 1$ for some ordinal $\gamma \in \beta$ and then

$$\begin{split} \Phi^{\circ\beta}(\Phi^{\circ\alpha}(x)) &= \Phi^{\circ(\gamma+1)}(\Phi^{\circ\alpha}(x)) = \Phi(\Phi^{\circ\gamma}(\Phi^{\circ\alpha}(x))) = \\ \Phi(\Phi^{\circ(\alpha+\gamma)}(x)) &= \Phi^{\circ((\alpha+\gamma)+1)}(x) = \Phi^{\circ(\alpha+(\gamma+1))}(x) = \Phi^{\circ(\alpha+\beta)}(x). \end{split}$$

Next, assume that β is a nonzero limit ordinal. In this case the ordinal $\alpha + \beta = \bigcup \{\alpha + \gamma : \gamma \in \beta\}$ is also limit (since for any ordinal $\alpha + \gamma \in \beta$ the ordinal $(\alpha + \gamma) + 1 = \alpha + (\gamma + 1)$ also belongs to $\alpha + \beta$). Moreover, $\{\alpha + \gamma : \gamma \in \beta\} \subseteq \alpha + \beta$ and for every $\delta \in \alpha + \beta$ there exists $\gamma \in \beta$ such that $\delta \leq \alpha + \gamma$. Then

$$\Phi^{\circ\beta}(\Phi^{\circ\alpha}(x)) = \bigcup \{\Phi^{\circ\gamma}(\Phi^{\circ\alpha}(x)) : \gamma \in \beta\} = \bigcup \{\Phi^{\circ(\alpha+\gamma)}(x) : \gamma \in \beta\} = \bigcup \{\Phi^{\circ\delta} : \delta \in \alpha + \beta\} = \Phi^{\circ(\alpha+\beta)}(x).$$

Next we establish some properties of addition of ordinals.

Theorem 31.3. Let α, β, γ be ordinals.

- 1) $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$.
- 2) $0 + \alpha = \alpha = \operatorname{Succ}^{\circ \alpha}(0)$ for any ordinal.
- 3) If $\alpha \leq \beta$, then $\alpha + \gamma \leq \beta + \gamma$.
- 4) $\beta < \gamma$ if and only if $\alpha + \beta < \alpha + \gamma$.
- 5) For any ordinals $\alpha \leq \beta$ there exists a unique ordinal γ such that $\alpha + \gamma = \beta$.

Proof. 1. The equality $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$ is nothing else but the equality

$$\mathsf{Succ}^{\circ\gamma}(\mathsf{Succ}^{\circ\beta}(\alpha)) = \mathsf{Succ}^{\circ(\beta+\gamma)}(\alpha)$$

established in Theorem 31.2.

2. The equality $0 + \alpha = \mathsf{Succ}^{\circ \alpha}(0)$ follows from the definition of the addition. The equality $0 + \alpha = \alpha$ will be proved by transfinite induction. For $\alpha = 0$ the equality 0 + 0 = 0 holds. Assume that for some ordinal $\alpha > 0$ we have proved that $0 + \beta = \beta$ for all $\beta \in \alpha$. If $\alpha = \beta + 1$ is a successor ordinal, then

$$0 + \alpha = 0 + (\beta + 1) = (0 + \beta) + 1 = \beta + 1 = \alpha$$

by the induction hypothesis.

If α is a limit ordinal, then

$$0 + \alpha = \sup\{0 + \beta : \beta \in \alpha\} = \sup\{\beta : \beta \in \alpha\} = \alpha.$$

3. Assume that $\alpha \leq \beta$. The inequality $\alpha + \gamma \leq \beta + \gamma$ will be proved by Transfinite Induction. For $\gamma = 0$ the inequality $\alpha + 0 = \alpha \leq \beta + 0$ trivially holds. Assume that for some nonzero ordinal γ and all its elements $\delta \in \gamma$ we have proved that $\alpha + \delta \leq \beta + \delta$, which implies $(\alpha + \delta) + 1 \subseteq (\beta + \delta) + 1$. If $\gamma = \delta + 1$ for some ordinal δ , then

$$\alpha + \gamma = \alpha + (\delta + 1) = (\alpha + \delta) + 1 \subseteq (\beta + \delta) + 1 = \beta + (\delta + 1) = \beta + \gamma.$$

If γ is a limit ordinal, then $\alpha + \gamma = \sup\{\alpha + \delta : \delta \in \gamma\} \le \sup\{\beta + \delta : \delta \in \gamma\} = \beta + \gamma$.

- 4. If $\beta < \gamma$, then $\alpha + \beta \subset \alpha + \gamma$ by Proposition 29.4 and the irreflexivity of the relation $\mathbf{E} \upharpoonright \mathbf{On}$ (see Definition 19.1). Now we prove that $\alpha + \beta < \alpha + \gamma$ implies $\beta < \gamma$. In the oppositve case, we get $\gamma \leq \beta$ (by Theorem 19.3(6)) and hence $\alpha + \gamma \leq \alpha + \beta$, which contradicts our assumption.
- 5. Given two ordinals $\alpha \leq \beta$, consider the class $\Gamma = \{ \gamma \in \mathbf{On} : \alpha + \gamma \subseteq \beta \}$. By Theorem 31.3(3), for every $\gamma \in \Gamma$ we have $\gamma = 0 + \gamma \leq \alpha + \beta \leq \beta$ and hence $\Gamma \subseteq \mathcal{P}(\beta)$ is a set. By Theorem 19.6(5) and Lemma 19.11, the set $\gamma = \sup \Gamma = \bigcup \{ \delta + 1 : \delta \in \Gamma \}$ is an ordinal. First we prove that $\alpha + \gamma \leq \beta$.

If $\gamma = \sup \Gamma$ is a successor ordinal, then $\gamma \in \Gamma$ and hence $\alpha + \gamma \subseteq \beta$ by the definition of the set Γ . If γ is a limit ordinal, then

$$\alpha + \gamma = \alpha + \sup \Gamma = \sup \{\alpha + \delta : \delta \in \Gamma\} \le \beta$$

as $\alpha + \delta \subseteq \beta$ for all $\delta \in \Gamma$. Therefore, $\alpha + \gamma \leq \beta$. Assuming that $\alpha + \gamma \neq \beta$, we conclude that $\alpha + \gamma < \beta$ and hence $\alpha + (\gamma + 1) = (\alpha + \gamma) + 1 \leq \beta$ by Proposition 30.1(1). Then $\gamma + 1 \in \Gamma$ and $\gamma \in \gamma + 1 \subseteq \sup \Gamma = \gamma$, which contradicts the irreflexivity of the relation $\mathbf{E} \upharpoonright \mathbf{On}$. This contradiction shows that $\alpha + \gamma = \beta$. The uniqueness of the ordinal γ follows from Theorems 31.3(4) and 19.3(6).

Finally we establish the commutativity of addition for natural numbers.

Theorem 31.4. Let $k, n \in \omega$ be two natural numbers.

- 1) $k + n \in \omega$;
- 2) n+1=1+n;
- 3) n + k = k + n.

Proof. Fix any natural number $k \in \omega$.

- 1. The inclusion $k+n \in \omega$ will be proved by Mathematical Induction. For n=0, we have $k+0=k\in\omega$ by Theorem 31.3(2). Assume that for some $n\in\omega$ we have proved that $k+n\in\omega$. Taking into account that ω is a limit ordinal, we conclude that $k+(n+1)=(k+n)+1\in\omega$. By the Principle of Mathematical Induction, $\forall n\in\omega$ $(k+n\in\omega)$.
- 2. By Theorems 31.3(2), 1+0=1=0+1. Assume that for some $n \in \omega$ we have proved that 1+n=n+1. Then by Theorem 31.3(1), 1+(n+1)=(1+n)+1=(n+1)+1. By the Principle of Mathematical Induction, the equality 1+n=n+1 holds for all $n \in \omega$.
- 3. The equality k+n=n+k will be proved by induction on $n \in \omega$. For n=0 the equality 0+n=n=n+0 holds by Theorems 31.3(2). Assume that for some $n \in \omega$ the equality k+n=n+k has been proved. By the inductive assumption and Theorems 31.3(1), 31.4(2), we obtain

$$k + (n+1) = (k+n) + 1 = 1 + (k+n) = 1 + (n+k) = (1+n) + k = (n+1) + k.$$

Exercise 31.5. Find two ordinals α, β such that $\alpha + \beta \neq \beta + \alpha$.

Hint: Show that $1 + \omega = \omega \neq \omega + 1$.

Exercise 31.6. Prove that every ordinal α can be uniquely written as the sum $\alpha = \beta + n$ of a limit ordinal β and a natural number n.

In fact, the operation of addition can be defined "geometrically" for any relations. Namely, for two relations R, P their sum $R \uplus P$ is defined as the relation

$$\begin{split} R \uplus P &= \{ \langle \langle 0, x \rangle, \langle 0, y \rangle \rangle : \langle x, y \rangle \in R \} \cup \{ \langle \langle 1, x \rangle, \langle 1, y \rangle \rangle : \langle x, y \rangle \in P \} \cup \\ & \{ \langle \langle 0, x \rangle, \langle 1, y \rangle \rangle : \langle x, y \rangle \in \text{dom}[R^{\pm}] \times \text{dom}[P^{\pm}] \} \end{split}$$

with the underlying class $\operatorname{\mathsf{dom}}[(R+P)^\pm] = (\{0\} \times \operatorname{\mathsf{dom}}[R^\pm]) \cup (\{1\} \times \operatorname{\mathsf{dom}}[P^\pm]).$

Exercise 31.7. Given ordinals α, β , prove that

- (1) the order $\mathbf{E} \upharpoonright (\alpha + \beta)$ is isomorphic to $\mathbf{E} \upharpoonright \alpha \uplus \mathbf{E} \upharpoonright \beta$;
- (2) $\alpha + \beta = \operatorname{rank}(\mathbf{E} \upharpoonright \alpha \uplus \mathbf{E} \upharpoonright \beta)$.

32. Multiplication

The following theorem introduces the operation of multiplication of ordinals.

Theorem 32.1. There exists a unique function

$$\cdot : \mathbf{On} \times \mathbf{On} \to \mathbf{On}, \quad \cdot : \langle \alpha, \beta \rangle \mapsto \alpha \cdot \beta,$$

such that for any ordinals α, β the following properties are satisfied:

- 0) $\alpha \cdot 0 = 0$;
- 1) $\alpha \cdot (\beta + 1) = \alpha \cdot \beta + \alpha$;
- 2) $\alpha \cdot \beta = \sup{\{\alpha \cdot \gamma : \gamma \in \beta\}}$ if the ordinal β is limit.

Proof. The uniqueness of \cdot can be proved by transfinite induction on β . The existence of \cdot follows from Theorem 29.1 applied to the function

$$\Phi_{\alpha}: \mathbf{On} \to \mathbf{On}, \quad \Phi_{\alpha}: x \mapsto x + \alpha = \mathsf{Succ}^{\circ \alpha}(0).$$

Then
$$\alpha \cdot \beta = \Phi_{\alpha}^{\circ \beta}(0)$$
.

Lemma 32.2. $0 \cdot \alpha = 0$ for any ordinal α .

Proof. This equality will be proved by transfinite induction. For $\alpha = 0$ it follows from the definition of multiplication by zero. Assume that for for nonzero ordinal β we proved that $0 \cdot \gamma = \gamma$ for all $\gamma \in \beta$. If β is a successor ordinal, then $\beta = \gamma + 1$ for some ordinal $\gamma \in \beta$ and hence

$$0 \cdot \beta = 0 \cdot (\gamma + 1) = 0 \cdot \gamma + 0 = 0 + 0 = 0.$$

If β is a limit ordinal, then

$$0 \cdot \beta = \sup\{0 \cdot \gamma : \gamma \in \beta\} = \sup\{0\} = 0.$$

Theorem 32.3. Let X be a chain-inclusive class and $\Phi: X \to X$ be an expansive function. Then for any set $x \in X$ and ordinals α, β, γ we have

- 1) $(\Phi^{\circ \alpha})^{\circ \beta}(x) = \Phi^{\circ (\alpha \cdot \beta)}(x)$;
- 2) $\Phi^{\circ(\alpha\cdot(\beta+\gamma))}(x) = \Phi^{\circ(\alpha\cdot\beta+\alpha\cdot\gamma)}(x)$

Proof. 1. If $\alpha = 0$, then by Lemma 32.2,

$$(\Phi^{\circ 0})^{\circ \beta}(x) = x = \Phi^{\circ 0}(x) = \Phi^{\circ (\alpha \cdot 0)}(x).$$

So, we assume that $\alpha \neq 0$.

The equality $(\Phi^{\circ \alpha})^{\circ \beta}(x) = \Phi^{\circ (\alpha \cdot \beta)}(x)$ will be proved by transfinite induction on β . Fix $x \in X$ and an ordinal α . Observe that $(\Phi^{\circ \alpha})^{\circ 0}(x) = x = \Phi^{\circ 0}(x) = \Phi^{\circ (\alpha \cdot 0)}(x)$. Assume that for some ordinal β and all its elements $\gamma \in \beta$ we have proved that $(\Phi^{\circ \alpha})^{\circ \gamma} = \Phi^{\circ (\alpha \cdot \gamma)}(x)$.

If β is a successor ordinal, then $\beta = \gamma + 1$ for some ordinal γ and by the inductive assumption, Theorem 31.2 and the definition of ordinal multiplication,

$$\begin{split} (\Phi^{\circ\alpha})^{\circ\beta}(x) &= (\Phi^{\circ\alpha})^{\circ(\gamma+1)}(x) = \Phi^{\circ\alpha}((\Phi^{\circ\alpha})^{\circ\gamma}(x)) = \Phi^{\circ\alpha}(\Phi^{\circ(\alpha\cdot\gamma)}(x)) = \\ \Phi^{\circ((\alpha\cdot\gamma)+\alpha)}(x) &= \Phi^{\circ(\alpha\cdot(\gamma+1))}(x) = \Phi^{\circ(\alpha\cdot\beta)}(x). \end{split}$$

Next, assume that β is a nonzero limit ordinal. In this case the ordinal $\alpha \cdot \beta = \bigcup \{\alpha \cdot \gamma : \gamma \in A\}$ β } is also limit (since for any ordinal $\alpha \cdot \gamma \in \beta$ the ordinal $(\alpha \cdot \gamma) + 1 \le \alpha \cdot \gamma + \alpha = \alpha \cdot (\gamma + 1)$ also belongs to $\alpha + \beta$). Moreover, $\{\alpha \cdot \gamma : \gamma \in \beta\} \subseteq \alpha \cdot \beta$ and for every $\delta \in \alpha \cdot \beta$ there exists $\gamma \in \beta$ such that $\delta \leq \alpha \cdot \gamma$. Then

$$(\Phi^{\circ\alpha})^{\circ\beta}(x) = \bigcup\{(\Phi^{\circ\alpha})^{\circ\gamma}(x): \gamma \in \beta\} = \bigcup\{\Phi^{\circ(\alpha \cdot \gamma)}(x): \gamma \in \beta\} = \bigcup\{\Phi^{\circ\delta}: \delta \in \alpha \cdot \beta\} = \Phi^{\circ(\alpha \cdot \beta)}(x).$$

2. Applying Theorems 31.2 and 32.3(1), we see that

$$\Phi^{\circ(\alpha\cdot(\beta+\gamma))}(x) = (\Phi^{\circ\alpha})^{\circ(\beta+\gamma)}(x) = (\Phi^{\circ\alpha})^{\circ\gamma}((\Phi^{\circ\alpha})^{\circ\beta}(x)) = \Phi^{\circ(\alpha\cdot\gamma)}(\Phi^{\circ(\alpha\cdot\beta)}(x)) = \Phi^{\circ(\alpha\cdot\beta+\alpha\cdot\gamma)}(x).$$

Next we establish some properties of multiplication of ordinals.

Theorem 32.4. Let α, β, γ be ordinals.

- 1) $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$.
- 2) $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$.
- 3) If $\alpha \leq \beta$, then $\alpha \cdot \gamma \leq \beta \cdot \gamma$.
- 4) If $\alpha > 0$ and $\beta > 0$, then $\alpha \cdot \beta > 0$.
- 5) $\alpha > 0$ and $\beta < \gamma$, then $\alpha \cdot \beta < \alpha \cdot \gamma$.

6) For any ordinals $\alpha \neq 0$ and β there exist unique ordinals γ and δ such that $\beta = \alpha \cdot \gamma + \delta$ and $\delta < \alpha$.

Proof. 1. Applying Theorems 31.3(2) and 32.3(1), we obtain

$$\begin{split} \alpha \cdot (\beta \cdot \gamma) &= \mathsf{Succ}^{\circ (\alpha \cdot (\beta \cdot \gamma))}(0) = (\mathsf{Succ}^{\circ \alpha})^{\circ (\beta \cdot \gamma)}(0) = ((\mathsf{Succ}^{\circ \alpha})^{\circ \beta})^{\circ \gamma}(0) = \\ &(\mathsf{Succ}^{\circ (\alpha \cdot \beta)})^{\circ \gamma}(0) = \mathsf{Succ}^{\circ ((\alpha \cdot \beta) \cdot \gamma)}(0) = (\alpha \cdot \beta) \cdot \gamma. \end{split}$$

2. Applying Theorems 31.3(2) and 32.3(2), we obtain

$$\alpha \cdot (\beta + \gamma) = \mathsf{Succ}^{\circ(\alpha \cdot (\beta + \gamma))}(0) = \mathsf{Succ}^{\circ(\alpha \cdot \beta + \alpha \cdot \gamma)}(0) = \alpha \cdot (\beta + \gamma).$$

The equality $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$ is nothing else but the equality

$$\mathsf{Succ}^{\circ\gamma}(\mathsf{Succ}^{\circ\beta}(\alpha)) = \mathsf{Succ}^{\circ(\beta+\gamma)}(\alpha)$$

established in Theorem 31.2.

3. Assume that $\alpha \leq \beta$. The inequality $\alpha \cdot \gamma \leq \beta \cdot \gamma$ will be proved by Transfinite Induction. For $\gamma = 0$ the inequality $\alpha \cdot 0 = 0 = \beta \cdot 0$ trivially holds. Assume that for some nonzero ordinal γ and all its elements $\delta \in \gamma$ we have proved that $\alpha \cdot \delta \leq \beta \cdot \delta$, which implies $\alpha \cdot \delta + \alpha \leq \beta \cdot \delta + \alpha \leq \beta \cdot \delta + \beta$, see Theorem 31.3(3,4). If $\gamma = \delta + 1$ for some ordinal δ , then

$$\alpha \cdot \gamma = \alpha \cdot (\delta + 1) = \alpha \cdot \delta + \alpha \le \beta \cdot \delta + \beta = \beta \cdot (\delta + 1) = \beta \cdot \gamma.$$

If γ is a limit ordinal, then $\alpha \cdot \gamma = \sup\{\alpha \cdot \delta : \delta \in \gamma\} \leq \sup\{\beta \cdot \delta : \delta \in \gamma\} = \beta \cdot \gamma$.

4. Assume that α and β are nonzero ordinals. The inequality $\alpha \cdot \beta > 0$ will be proved by induction on β . For $\beta = 1$, we have $\alpha \cdot 1 = \alpha \cdot 0 + \alpha = 0 + \alpha = \alpha > 0$. Assume that for some ordinal $\beta > 1$ and all its nonzero elements $\gamma \in \beta$ we proved that $\alpha \cdot \beta > 0$. If β is a successor ordinal, then $\beta = \gamma + 1$ and by Theorem 31.3(3),

$$\alpha \cdot (\gamma + 1) = \alpha \cdot \gamma + \alpha \ge 0 + \alpha = 0 + \alpha = \alpha > 0.$$

If β is a limit ordinal, then

$$\alpha \cdot \beta = \sup \{\alpha \cdot \gamma : \gamma \in \beta\} \ge \alpha \cdot 1 = \alpha > 0.$$

5. If $\alpha > 0$ and $\beta < \gamma$, then by Theorem 31.3(5), we can find a unique ordinal δ such that $\beta + \delta = \gamma$. Since $\beta \neq \gamma$, the ordinal δ is nonzero. By Theorem 32.4(4), $0 < \alpha \cdot \delta$. By Theorems 31.3(4) and 32.4(2), we have

$$\alpha \cdot \beta = \alpha \cdot \beta + 0 < \alpha \cdot \beta + \alpha \cdot \delta = \alpha \cdot (\beta + \delta) = \alpha \cdot \gamma.$$

6. Given two ordinals $\alpha > 0$ and β , consider the class $\Gamma = \{ \gamma \in \mathbf{On} : \alpha \cdot \gamma \leq \beta \}$. By Theorem 32.4(3), for every $\gamma \in \Gamma$ we have $\gamma = 1 \cdot \gamma \leq \alpha \cdot \gamma \leq \beta$ and hence $\Gamma \subseteq \mathcal{P}(\beta)$ is a set. By Theorem 19.6(5) and Lemma 19.11, the set $\gamma = \sup \Gamma = \bigcup \{ \delta + 1 : \delta \in \Gamma \}$ is an ordinal. First we prove that $\alpha \cdot \gamma \leq \beta$.

If $\gamma = \sup \Gamma$ is a successor ordinal, then $\gamma \in \Gamma$ and hence $\alpha \cdot \gamma \subseteq \beta$ by the definition of the set Γ . If γ is a limit ordinal, then

$$\alpha \cdot \gamma = \alpha \cdot \sup \Gamma = \sup \{\alpha \cdot \delta : \delta \in \Gamma\} \le \beta$$

as $\alpha \cdot \delta \subseteq \beta$ for all $\delta \in \Gamma$. Therefore, $\alpha \cdot \gamma \leq \beta$. By Theorem 31.3(5), there exists a unique ordinal δ such that $\alpha \cdot \gamma + \delta = \beta$. We claim that $\delta < \alpha$. Assuming that $\delta \not< \alpha$ and applying

Theorem 19.3(6), we conclude that $\alpha \leq \delta$. By Theorem 31.3(5), there exists an ordinal δ' such that $\alpha + \delta' = \delta$. Then

$$\beta = \alpha \cdot \gamma + \delta = \alpha \cdot \gamma + (\alpha + \delta') = (\alpha \cdot \gamma + \alpha) + \delta' = \alpha \cdot (\gamma + 1) + \delta' \ge \alpha \cdot (\gamma + 1)$$

and hence $\gamma + 1 \in \Gamma$ and $\gamma + 1 \leq \sup \Gamma = \gamma$, which contradicts the irreflexivity of the relation $\mathbf{E} \upharpoonright \mathbf{On}$. This contradiction shows that $\delta < \alpha$.

It remains to show that the ordinals γ and δ are unique. Assume that γ', δ' are ordinals such that $\delta' < \alpha$ and $\alpha \cdot \gamma' + \delta' = \beta$. Since

$$\alpha \cdot \gamma' = \alpha \cdot \gamma' + 0 \le \alpha \cdot \gamma' + \delta' = \beta,$$

the ordinal γ' belongs to Γ and hence $\gamma' \leq \gamma$. Assuming that $\gamma' \neq \gamma$, we conclude that $\gamma' < \gamma$. By Theorem 31.3(5), there exists a unique ordinal $\delta'' > 0$ such that $\gamma' + \delta'' = \gamma$. Then

$$\alpha \cdot \gamma' + \delta' = \beta = \alpha \cdot \gamma + \delta = \alpha \cdot (\gamma' + \delta'') + \delta = (\alpha \cdot \gamma' + \alpha \cdot \delta'') + \delta = \alpha \cdot \gamma' + (\alpha \cdot \delta'' + \delta).$$

By Theorem 31.3(5,4), 32.4(5)

$$\delta' = \alpha \cdot \delta'' + \delta > \alpha \cdot \delta'' + 0 = \alpha \cdot \delta'' > \alpha \cdot 1 = \alpha,$$

which contradicts the chocie of $\delta' < \alpha$. This contradiction completes the proof of the equality $\gamma = \gamma'$. Now the equality

$$\alpha \cdot \gamma + \delta' = \alpha \cdot \gamma' + \delta' = \beta = \alpha \cdot \gamma + \delta$$

and Theorem 31.3(5) imply $\delta = \delta'$.

Exercise 32.5. Find two ordinals α, β such that $\alpha \cdot \beta \neq \beta \cdot \alpha$.

Hint: Show that $\omega \cdot 2 = \omega + \omega \neq \omega = 2 \cdot \omega$.

Theorem 32.6. For any natural numbers $n, k \in \omega$ the following conditions hold:

- 1) $n \cdot k \in \omega$;
- 2) $(n+1) \cdot k = n \cdot k + k$;
- 3) $n \cdot k = k \cdot n$.

Proof. 1. Fix any natural number n. The inclusion $n \cdot k \in \omega$ will be proved by induction on k. For k = 0 we have $n \cdot 0 = 0 \in \omega$. Assume that for some $k \in \omega$ we have proved that $n \cdot k \in \omega$. Then $n \cdot (k+1) = n \cdot k + n \in \omega$ by Theorem 31.4(1).

2. The equality $(n+1) \cdot k = n \cdot k + k$ will be proved by induction on k. For k = 0 we have $(n+1) \cdot 0 = 0 = n \cdot 0 + 0$. Assume that for some $k \in \omega$ we have proved that $(n+1) \cdot k = n \cdot k + k$. Taking into account that the addition of natural numbers is associative and commutative, we conclude that

$$(n+1)\cdot(k+1) = (n+1)\cdot k + (n+1) = n\cdot k + k + n + 1 = n\cdot k + n + k + 1 = n\cdot(k+1) + (k+1).$$

By the Principle of Mathematical Induction, the equality $(n+1) \cdot k = n \cdot k + k$ holds for all natural numbers.

3. The equality $n \cdot k = k \cdot n$ will be proved by induction on $n \in \omega$. For k = 0 we have $n \cdot 0 = 0 = 0 \cdot n$, by Lemma 32.2 and the definition of the multiplication. Assume that for some n we have proved that $k \cdot n = n \cdot k$. By the preceding statement,

$$(n+1) \cdot k = n \cdot k + k = k \cdot n + k = k \cdot (n+1).$$

In fact, the operation of multiplication can be defined "geometrically" for any relations. Namely, for two relations R, P we can defined their left and right lexicographic products $R \ltimes P$ and $R \rtimes P$ by the formulas:

$$R \ltimes P = \{ \langle \langle x, y \rangle, \langle x', y' \rangle \rangle : \\ (\langle x, x' \rangle \in R \ \land \ \{y, y'\} \subseteq \mathsf{dom}[P^{\pm}]) \ \lor \ (x = x' \in \mathsf{dom}[R^{\pm}] \ \land \ \langle y, y' \rangle \in P) \}$$

and

$$R \rtimes P = \{ \langle \langle x, y \rangle, \langle x', y' \rangle \rangle : \\ (\langle y, y' \rangle \in P \ \land \ \{x, x'\} \subseteq \mathsf{dom}[R^{\pm}]) \ \lor \ (y = y' \in \mathsf{dom}[P^{\pm}] \ \land \ \langle x, x' \rangle \in R) \}.$$

Exercise 32.7. Given any ordinals α, β , prove that

- (1) the well-order $\mathbf{E} \upharpoonright \alpha \cdot \beta$ is isomorphic to the orders $\mathbf{E} \upharpoonright \alpha \times \mathbf{E} \upharpoonright \beta$ and $\mathbf{E} \upharpoonright \beta \times \mathbf{E} \upharpoonright \alpha$;
- (2) $\alpha \cdot \beta = \operatorname{rank}(\mathbf{E} \upharpoonright \alpha \rtimes \mathbf{E} \upharpoonright \beta) = \operatorname{rank}(\mathbf{E} \upharpoonright \beta \ltimes \mathbf{E} \upharpoonright \alpha).$

33. Exponentiation

The following theorem introduces the operation of exponentiation of ordinals.

Theorem 33.1. There exists a unique function

$$\exp: \mathbf{On} \times \mathbf{On} \to \mathbf{On}, \quad \exp: \langle \alpha, \beta \rangle \mapsto \alpha^{\beta},$$

such that for any ordinals α, β the following properties are satisfied:

- 0) $\alpha^{0} = 1$;
- 1) $\alpha^{\cdot(\beta+1)} = \alpha^{\cdot\beta} \cdot \alpha$:
- 2) $\alpha^{\beta} = \sup\{\alpha^{\gamma} : \gamma \in \beta\}$ if the ordinal β is limit.

Proof. Given an ordinal α , consider the expanding function

$$\Phi_{\alpha}: \mathbf{On} \to \mathbf{On}, \quad \Phi_{\alpha}: x \mapsto x \cdot \alpha.$$

By Theorem 29.1 there exists a transfinite function sequence $(\Phi_{\alpha}^{\circ\beta})_{\beta\in\mathbf{On}}$ such that for any ordinals x and β the following conditions hold:

- $\bullet \ \Phi_{\alpha}^{\circ 0}(x) = x;$
- $\Phi_{\alpha}^{\circ(\beta+1)}(x) = \Phi_{\alpha}(\Phi_{\alpha}^{\circ\beta}(x)) = \Phi^{\circ\beta}(x) \cdot \alpha;$
- $\Phi_{\alpha}^{\circ\beta}(x) = \sup\{\Phi_{\alpha}^{\circ\gamma}(x) : \gamma \in \beta\}$ if the ordinal β is limit.

Comparing these there conditions with the definition of exponentiation, we can see that $\alpha^{\beta} = \Phi_{\alpha}^{\circ\beta}(1)$ for all α, β .

Remark 33.2. In most textbooks in Set Theory, the ordinal exponentiation is denoted by α^{β} , which unfortunately coincides with the notation α^{β} for the set of all functions from β to α . For distinguishing these two notions we denote the ordinal exponentiation $\alpha^{\cdot\beta}$ using the dot before β , which indicates that $\alpha^{\cdot\beta}$ is the result of repeated multiplication of 1 by α , β times.

Now we establish some properties of the exponentiation of ordinals.

Theorem 33.3. Let α, β, γ be any non-zero ordinals.

- 1) If $\alpha \leq \beta$, then $\alpha^{\gamma} \leq \beta^{\gamma}$.
- 2) $\alpha^{\cdot(\beta+\gamma)} = \alpha^{\cdot\beta} \cdot \alpha^{\cdot\gamma}$.

- 3) $\alpha^{\cdot(\beta\cdot\gamma)} = (\alpha^{\cdot\beta})^{\cdot\gamma}$.
- 4) If $\beta \leq \gamma$, then $\alpha^{\beta} \leq \alpha^{\gamma}$.
- 5) If $\alpha > 1$ and $\beta < \gamma$, then $\alpha^{\beta} < \alpha^{\gamma}$.
- 6) For any ordinals $\alpha > 1$ and $\beta \geq \alpha$ there exists unique ordinals x, y, z such that $\beta = \alpha^{\cdot x} \cdot y + z$, $0 < y < \alpha$, and $z < \alpha^{\cdot x}$.
- 7) For any ordinal $\beta \geq \omega$ there are unique ordinals x, y, z such that $0 < y < \omega, z < \omega^{\cdot x}$ and $\beta = \omega^{\cdot x} \cdot y + z$.

Proof. 1. Take any ordinals $\alpha \leq \beta$. The inequality $\alpha^{\cdot \gamma} \leq \beta^{\cdot \gamma}$ will be proved by induction on the ordinal γ . For $\gamma = 0$ the inequality $\alpha^{\cdot 0} = 1 = \beta^{\cdot 0}$ is true. Assume that for some ordinal γ and all ordinals $\delta \in \gamma$ we have proved that $\alpha^{\cdot \delta} \leq \beta^{\cdot \delta}$.

If γ is a successor ordinal, then $\gamma = \delta + 1$ for some $\delta \in \gamma$. By Theorem 32.4(3,5),

$$\alpha^{\cdot \gamma} = \alpha^{\cdot (\delta+1)} = \alpha^{\cdot \delta} \cdot \alpha \le \beta^{\cdot \delta} \cdot \beta = \beta^{\cdot (\delta+1)} = \beta^{\cdot \gamma}.$$

If the ordinal γ is limit, then

$$\alpha^{\cdot\gamma} = \sup\{\alpha^{\cdot\delta}: \delta \in \gamma\} \leq \sup\{\beta^{\cdot\delta}: \delta \in \gamma\} = \beta^{\cdot\gamma}.$$

2. Fix ordinals α, β . The equality $\alpha^{\cdot(\beta+\gamma)} = \alpha^{\cdot\beta} \cdot \alpha^{\cdot\gamma}$ will be proved by transfinite induction on γ . For $\gamma = 0$ we have

$$\alpha^{\cdot(\beta+0)} = \alpha^{\cdot\beta} = 0 + \alpha^{\cdot\beta} = \alpha^{\beta} \cdot (0+1) = \alpha^{\cdot\beta} \cdot 1 = \alpha^{\cdot\beta} \cdot \alpha^{\cdot0}.$$

Assume that for some ordinal γ and all its elements $\delta \in \gamma$ we have proved that $\alpha^{\cdot(\beta+\delta)} = \alpha^{\cdot\beta} \cdot \alpha^{\cdot\delta}$. If γ is a successor ordinal, then $\gamma = \delta + 1$ for some $\delta \in \gamma$ and then

$$\alpha^{\cdot(\beta+\gamma)} = \alpha^{\cdot(\beta+\delta+1)} = \alpha^{\cdot(\beta+\delta)} \cdot \alpha = (\alpha^{\cdot\beta} \cdot \alpha^{\cdot\delta}) \cdot \alpha = \alpha^{\cdot\beta} \cdot (\alpha^{\cdot\delta} \cdot \alpha) = \alpha^{\cdot\beta} \cdot \alpha^{\cdot(\delta+1)} = \alpha^{\cdot\beta} \cdot \alpha^{\cdot\gamma}.$$

If γ is a limit ordinal, then $\beta + \gamma = \sup\{\beta + \delta : \delta \in \gamma\}$ is limit, too and hence

$$\alpha^{\cdot(\beta+\gamma)} = \sup\{\alpha^{\cdot(\beta+\delta)} : \delta \in \gamma\} = \sup\{\alpha^{\cdot\beta} \cdot \alpha^{\cdot\delta} : \delta \in \gamma\} = \alpha^{\cdot\beta} \cdot \sup\{\alpha^{\cdot\delta} : \delta \in \gamma\} = \alpha^{\cdot\beta} \cdot \alpha^{\cdot\gamma}.$$

3. Fix ordinals α, β . If $\beta = 0$, then for any ordinal γ we have

$$\alpha^{\cdot(\beta\cdot\gamma)}=\alpha^{\cdot(0\cdot\gamma)}\alpha^{\cdot0}=1=1^{\cdot\gamma}=(\alpha^{\cdot0})^{\cdot\gamma}=(\alpha^{\cdot\beta})^{\cdot\gamma}.$$

So, we assume that $\beta > 0$.

The equality $\alpha^{\cdot(\beta\cdot\gamma)}=(\alpha^{\cdot\beta})^{\cdot\gamma}$ will be proved by transfinite induction on γ . For $\gamma=0$ we have

$$\alpha^{\cdot(\beta\cdot 0)} = \alpha^{\cdot 0} = 1 = (\alpha^{\cdot\beta})^{\cdot 0}.$$

Assume that for some ordinal γ and all its elements $\delta \in \gamma$ we have proved that $\alpha^{\cdot(\beta \cdot \delta)} = (\alpha^{\cdot \beta})^{\cdot \delta}$. If γ is a successor ordinal, then $\gamma = \delta + 1$ for some $\delta \in \gamma$ and then

$$\alpha^{\cdot(\beta\cdot\gamma)}=\alpha^{\cdot(\beta\cdot(\delta+1))}=\alpha^{\cdot(\beta\cdot\delta+\beta)}=\alpha^{\cdot(\beta\cdot\delta)}\cdot\alpha^{\cdot\beta}=(\alpha^{\cdot\beta})^{\cdot\delta}\cdot\alpha^{\cdot\beta}=(\alpha^{\cdot\beta})^{\cdot(\delta+1)}=(\alpha^{\cdot\beta})^{\cdot\gamma}.$$

If γ is a limit ordinal, then $\beta \cdot \gamma = \sup\{\beta \cdot \delta : \delta \in \gamma\}$ is limit, too and hence

$$\alpha^{\cdot(\beta\cdot\gamma)} = \sup\{\alpha^{\cdot(\beta\cdot\delta)} : \delta\in\gamma\} = \sup\{(\alpha^{\cdot\beta})^{\cdot\delta} : \delta\in\gamma\} = (\alpha^{\cdot\beta})^{\cdot\gamma}.$$

4. Let $\alpha > 1$ and $\beta < \gamma$ be any ordinals. By Theorem 31.3(5), there exists a unique ordinal $\delta \ge 1$ such that $\beta + \delta = \gamma$. By Theorem 32.4(5),

$$\alpha^{\cdot\beta} = \alpha^{\cdot\beta} \cdot 1 < \alpha^{\cdot\beta} \cdot \alpha \leq \alpha^{\cdot\beta} \cdot \alpha^{\cdot\delta} = \alpha^{\cdot(\beta+\delta)} = \alpha^{\cdot\gamma}.$$

5. Fix any ordinals $\alpha > 1$ and $\beta \ge \alpha$. Let $X = \{x \in \mathbf{On} : \alpha^{\cdot x} \le \beta\}$ and $x = \sup X$. It can be shown that $\alpha^{\cdot x} \le \beta$ and $\alpha^{\cdot (x+1)} > \beta$. Let $Y = \{y \in \mathbf{On} : \alpha^{\cdot x} \cdot y \le \beta\}$. It can be shown that for the ordinal $y = \sup Y$ we have $\alpha^{\cdot x} \cdot y \le \beta$ but $\alpha^{\cdot x} \cdot (y+1) > \beta$. The choice of x implies that

 $0 < y < \alpha$. Let $Z = \{z \in \mathbf{On} : \alpha^{x} \cdot y + z \leq \beta\}$. It can be shown that the ordinal $z = \sup Z$ has the required property: $\alpha^{\cdot x} \cdot y + z = \beta$. It follows from $\alpha^{\cdot x} \cdot (y+1) = \alpha^{\cdot x} \cdot y + \alpha^{\cdot x} > \beta = \alpha^{\cdot x} \cdot y + z$ that $z < \alpha^{x}$. The proof of the uniqueness of the ordinals x, y, z is left to the reader.

6. The sixth statement is a partial case of the fifth statement for $\alpha = \omega$.

Exercise 33.4. Prove that $\forall n \in \omega \ \forall k \in \omega \ (n^{\cdot k} \in \omega)$.

Exercise 33.5. Simplify:

- $(\omega + 2) \cdot \omega$;
- $(\omega + \omega^{\cdot 2}) \cdot (\omega^{\cdot 3} + \omega^{\cdot 4});$ $\omega + \omega^{\cdot 2} + \omega^{\cdot 3}.$

Exercise 33.6. Find $\alpha < \beta$ and γ such that

- $\alpha + \gamma = \beta + \gamma$;
- $\begin{array}{ll} \bullet & \alpha \cdot \gamma = \beta \cdot \gamma; \\ \bullet & \alpha^{\cdot \gamma} = \beta^{\cdot \gamma}. \end{array}$

Exercise 33.7. Consider the sequence of ordinals $(\alpha_n)_{n\in\omega}$ defined by the recursive formula $\alpha_0 = 1$ and $\alpha_{n+1} = \omega^{\alpha_n}$. Prove that $\varepsilon_0 = \sup_{n \in \omega} \alpha_n$ the smallest ordinal ε such that $\varepsilon = \omega^{\varepsilon}$.

An ordinal $\gamma > 1$ is called *indecomposable* if $\gamma \neq \alpha + \beta$ for any ordinals $\alpha, \beta < \gamma$.

Exercise 33.8. Prove that an ordinal $\alpha > 1$ is indecomposable if and only if $\alpha = \omega^{\beta}$ for some ordinal β .

The exponentiation of ordinals has a nice geometric model. Let L be a linear order on a set $X = \mathsf{dom}[L^{\pm}]$ and R be an order on a set $Y = \mathsf{dom}[R^{\pm}]$. Let $Y^{< X}$ be the set of all functions f such that f is a finite set with $dom[f] \subseteq X$ and $rng[X] \subseteq Y$. We endow the set $Y^{< X}$ with the irreflexive order W consisting of all ordered pairs $\langle f, g \rangle$ such that there exists $x \in \mathsf{dom}[g]$ such that $f \cap ((L[\{x\}] \setminus \{x\}) \times Y) = g \cap ((L[\{x\}] \setminus \{x\}) \times Y)$ and either $x \notin \mathsf{dom}[f]$ or $x \in \mathsf{dom}[f]$ and $\langle f(x), g(x) \rangle \in R \setminus \mathbf{Id}$.

Exercise 33.9. Prove that for any ordinals α, β the order $\mathbf{E} \upharpoonright \alpha^{\beta}$ is isomorphic to the order W on the set $\alpha^{<\beta}$.

34. Cantor's normal form

The Cantor's normal form of an ordinal is its power expansion with base ω .

Theorem 34.1. Each nonzero ordinal α can be uniquely written as

$$\alpha = \omega^{\beta_1} \cdot k_1 + \dots + \omega^{\beta_n} \cdot k_n$$

for some ordinals $\beta_1 > \cdots > \beta_n$ and nonzero natural numbers k_1, \ldots, k_n .

Proof. The proof is by induction on $\alpha > 0$. For $\alpha = 1$, we have the representation $\alpha = \omega^{0} \cdot 1$. Assume that the theorem has been proved for all ordinal smaller than some ordinal α . By Theorem 33.3(6), there are unique ordinals β_1, k_1, α_1 such that $\alpha = \omega^{\beta_1} \cdot k_1 + \alpha_1$ and $0 < \infty$ $k_1 < \omega$, $\alpha_1 < \omega^{\beta_1}$. Since $\alpha_1 < \alpha$ we can apply the inductive assumption and find ordinals $\beta_2 > \cdots > \beta_n$ and nonzero natural numbers k_2, \ldots, k_n such that $\alpha_1 = \omega^{\beta_2} \cdot k_2 + \cdots + \omega^{\beta_n} \cdot k_n$.

$$\alpha = \omega^{\beta_1} \cdot k_1 + \dots + \omega^{\beta_n} \cdot k_n$$

the required decomposition of α .

Theorem 34.1 allows to identify ordinals with functions $f:\omega^{\mathbf{On}}\to\omega$ defined on the class $\omega^{\mathbf{On}}=\{\omega^{\cdot\beta}:\beta\in\mathbf{On}\}$ such that $\mathrm{supp}(f)=\{\beta\in\omega^{\mathbf{On}}:f(\omega^{\cdot\beta})>0\}$ is finite. The coordinatewise addition of such functions induces the so-called *normal addition* $\alpha\oplus\beta$ of ordinals. The normal addition of ordinals is both associative and commutative.

Example 34.2. $(\omega^{\cdot 2} \cdot 3 + \omega \cdot 5 + 1) \oplus (\omega^{\cdot 3} \cdot 4 + \omega^{\cdot 2} \cdot 2 + 3) = \omega^{\cdot 3} \cdot 4 + \omega^{\cdot 2} \cdot 5 + \omega \cdot 5 + 4.$

Part 8. Cardinals

This part is devoted to cardinals and cardinalities. Since we will mostly speak about sets (and rarely about classes), we will deviate from our convention of denoting sets by small characters and shall use both small and capital letters for denoting sets.

35. Cardinalities and cardinals

Definition 35.1. Two classes X, Y are defined to be *equipotent* if there exists a bijective function $F: X \to Y$. In this case we write |X| = |Y| and say that x and y have the same cardinality.

Exercise 35.2. Prove that for any classes X, Y, Z

- 1) |X| = |X|;
- 2) $|X| = |Y| \Rightarrow |Y| = |X|$;
- 3) $(|X| = |Y| \land |Y| = |Z|) \Rightarrow |X| = |Z|$.

Exercise 35.2 witnesses that

$$\{\langle x, y \rangle \in \ddot{\mathbf{U}} : |x| = |y|\}$$

is an equivalence relation on the universe **U**. For every set x its cardinality |x| is the unique equivalence class $\{y \in \mathbf{U} : |y| = |x|\}$ of this relation, containing the set x. Let us write down this as a formal definition.

Definition 35.3. For a set x its *cardinality* |x| is the class

$$\{y \in \mathbf{U}: \exists f \in \mathbf{Fun} \ \land \ (f^{-1} \in \mathbf{Fun} \ \land \ \mathsf{dom}[f] = x \ \land \ \mathsf{rng}[f] = y)\}$$

consisting all sets y that are equiponent with x.

Example 35.4. 0) The cardinality |0| of the emptyset $0 = \emptyset$ is the singleton $\{\emptyset\} = 1$.

- 1) The cardinality |1| of the natural number $1 = \{\emptyset\}$ is the proper class of all singletons $\{\{x\} : x \in \mathbf{U}\}.$
- 2) The cardinality |2| of the natural number $2 = \{0, 1\}$ is the proper class of all doubletons $\{\{x, y\} : x \in \mathbf{U} \land y \in \mathbf{U} \land x \neq y\}.$

The Hartogs' Theorem 23.9 implies that for every set x the class $|x| \cap \mathbf{On}$ is a set. This intersection is not empty if and only if the set x can be well-ordered.

Definition 35.5. An ordinal α is called a *cardinal* if α is the **E**-least element in the set $|\alpha| \cap \mathbf{On}$. The class of cardinals will be denoted by **Card**.

If the Axiom of Choice holds, then by Zermelo Theorem 27.2, each set x can be well-ordered and then Theorem 23.6 ensures that the cardinality |x| contains an ordinal and hence contains a unique cardinal. This unique cardinal is called the *cardinal* of the set x. For example, the cardinal of the ordinal $\omega + 1$ is ω .

The cardinal of a set x is well-defined if and only of the cardinality |x| contains some ordinal.

Let us recall that for two classes X, Y we write |X| = |Y| iff there exists a bijective function $F: X \to Y$.

Given two classes X, Y we write $|X| \leq |Y|$ if there exists an injective function $F: X \to Y$. Also we write |X| < |Y| if $|X| \leq |Y|$ but $|X| \neq |Y|$.

Theorem 35.6 (Cantor-Bernstein-Schröder). For two classes X, Y,

$$|X| = |Y| \iff (|X| \le |Y| \land |Y| \le |X|).$$

Proof. The implication $|X| = |Y| \Rightarrow (|X| \leq |Y| \land |Y| \leq |X|)$ is trivial. To prove the reverse implication, assume that $|X| \leq |Y|$, $|Y| \leq |X|$ and fix injective functions $F: X \to Y$ and $G: Y \to X$. For every $n \in \omega$ let $(G \circ F)^{\cdot n}: X \to X$ and $(F \circ G)^{\circ n}: Y \to Y$ be the *n*-th iterations of the functions $G \circ F$ and $F \circ G$, respectively. These iterations exist by Theorem 21.4.

For every $n \in \omega$ let $X_{2n} = (G \circ F)^{\circ n}[X]$, $Y_{2n} = (F \circ G)^{\circ n}[Y]$, $X_{2n+1} = (F \circ G)^{\circ n}[G[Y]]$ and $Y_{2n+1} = (F \circ G)^{\circ n}[F[X]]$.

By induction we can prove that $X_{n+1} \subseteq X_n$ and $Y_{n+1} \subseteq Y_n$ for every $n \in \omega$. Let $X_{\omega} = \bigcap_{n \in \omega} X_n$ and $Y_{\omega} = \bigcap_{n \in \omega} Y_n$, and observe that

$$F[x_{\omega}] = \bigcap_{n \in \omega} F[x_n] = \bigcap_{n \in \omega} Y_{n+1} = Y_{\omega}.$$

It is easy to check that the function $H: X \to Y$ defined by

$$H(x) = \begin{cases} F(x) & \text{if } x \in \bigcup_{n \in \omega} (X_{2n} \setminus X_{2n+1}); \\ G^{-1}(x) & \text{if } x \in \bigcup_{n \in \omega} (X_{2n+1} \setminus X_{2n+2}); \\ F(x) & \text{if } x \in X_{\omega} \end{cases}$$

is bijective and witnesses that |X| = |Y|.

Since the cardinalities of sets |x| are proper classes, we cannot speak about the class of cardinalities. Nonetheless, the indexed family of cardinalities $(|x|)_{x\in \mathbb{U}}$ is absolutely legal. And Theorem 35.6 implies that \leq is a partial order on this indexed family.

Given two classes X, Y we write $|X| \leq^* |Y|$ if there exists a surjective function $f: Y \to X$ or X is empty.

The following proposition shows that for any sets x, y we have the implications

$$(|x| \le |y|) \Rightarrow (|x| \le^* |y|) \Rightarrow (|\mathcal{P}(x)| \le |\mathcal{P}(y)|).$$

Proposition 35.7. Let X, Y be nonempty classes.

- 1) If $|X| \leq |Y|$, then $|X| \leq^* |Y|$.
- 2) If Y is well-orderable, then $|X| \leq |Y|$ is equivalent to $|X| \leq^* |Y|$.
- 3) If X, Y are sets and $|X| \leq^* |Y|$, then $|\mathcal{P}(X)| \leq |\mathcal{P}(Y)|$.

Proof. 1. If $|X| \leq |Y|$, then there exists an injective function $F: X \to Y$. If $X = \emptyset$, then $|X| \leq^* |Y|$ by definition. If $X \neq \emptyset$, then we can choose an element $b \in X$ and consider the function $G: Y \to X$ defined by

$$G(y) = \begin{cases} F^{-1}(y) & \text{if } y \in F[X]; \\ b & \text{if } y \in Y \setminus F[X]. \end{cases}$$

Using Theorem 7.2, one can check that the function G is well-defined and surjective.

2. Assume that Y is well-orderable and fix a well-order W on $Y = \text{dom}[W^{\pm}]$. If $|X| \leq |Y|$, then $|X| \leq^* |Y|$ by the preceding statement. Now assume that $|X| \leq^* |Y|$. If X is empty, then the empty injective function $\emptyset : \emptyset \to Y$ witnesses that $|X| \leq |Y|$. If X is not empty, then there exists a surjective function $F: Y \to X$. For every $x \in X$ let G(x) be the unique W-minimal element of the nonempty set $\{y \in Y : F(y) = x\}$. Then $G: X \to Y$, $G: x \mapsto G(x)$, a well-define injective function witnessing that $|X| \leq |Y|$.

3. If $|X| \leq^* |Y|$ and X, Y are sets, then either $X = \emptyset$ or there exists a surjective function $G: Y \to X$. In the first case we have $|\mathcal{P}(X)| = |\{\emptyset\}| \leq |\mathcal{P}(Y)|$. In the second case we can consider the injective function $G^{-1}: \mathcal{P}(X) \to \mathcal{P}(Y), G^{-1}: a \mapsto g^{-1}[a]$, witnessing that $|\mathcal{P}(X)| \leq |\mathcal{P}(Y)|$.

Exercise 35.8. Show the equivalence of two statements:

- (i) For any sets x, y and a surjective function $f: x \to y$ there exists an injective function $g: y \to x$ such that $f \circ g = \mathbf{Id} \upharpoonright y$.
- (ii) The Axiom of Choice holds.

Prove that the equivalent conditions (i),(ii) imply the condition

(PP) For any sets x, y ($|x| \le |y| \Leftrightarrow |x| \le^* |y|$).

Remark 35.9. The statement (PP) is known is Set Theory as the Partition Principle. It is an open problem whether (PP) implies (AC), see https://karagila.org/2014/on-the-partition-principle/

Theorem 35.10 (Cantor). For any set x there is no surjective function $f: x \to \mathcal{P}(x)$.

Proof. Given any function $f: x \to \mathcal{P}(x)$, consider the set $a = \{z \in x : z \notin f(z)\} \in \mathcal{P}(x)$. Assuming that $a \in \mathsf{rng}[f]$, we could find an element $z \in x$ such that a = f(z). If $z \notin f(z)$, then $z \in a = f(z)$ which is a contradiction. If $z \in f(z)$, then $z \in a$ and hence $z \notin f(z)$. In both cases we obtain a contradiction, which shows that $a \notin \mathsf{rng}[f]$ and hence f is not surjective.

Corollary 35.11 (Cantor). For any set x we have $|x| < |\mathcal{P}(x)|$.

Proof. The inequality $|x| \leq |\mathcal{P}(x)|$ follows from the injectivity of the function $x \to \mathcal{P}(x)$, $z \mapsto \{z\}$. Assuming that $|x| = |\mathcal{P}(x)|$, we would get a bijective (and hence surjective) function $x \to \mathcal{P}(x)$, which is forbidden by Cantor's Theorem 35.10.

36. Finite and Countable sets

In this section we shall establish some elementary facts about finite and countable sets. Let us recall that a set x is *countable* (resp. *finite*) if $|x| = |\alpha|$ for some ordinal $\alpha \in \omega \cup \{\omega\}$ (resp. $\alpha \in \omega$). Elements of the set ω are called *natural numbers*.

Lemma 36.1. Let n be a natural number. Every injective function $f: n \to n$ is bijective.

Proof. This lemma will be proved by induction. For n=0 the unique function $\emptyset:\emptyset\to\emptyset$ is bijective. Assume that for some $n\in\omega$ we have proved that any injective function $f:n\to n$ is bijective.

Take any injective function $f: n+1 \to n+1$. If f(n)=n, then the injectivity of f guarantees that $n \notin f[n]$ and hence $f[n] \subseteq (n+1) \setminus \{n\} = n$. By the inductive assumption, the injective function $f \upharpoonright_n : n \to n$ is bijective and hence

$$f[n+1] = f[n \cup \{n\}] = f[n] \cup \{f(n)\} = n \cup \{n\} = n+1,$$

which means that f is surjective and hence bijective. If $f(n) \neq n$, then consider the bijective function $g: n+1 \rightarrow n+1$ defined by

$$g(x) = \begin{cases} f(n) & \text{if } x = n; \\ n & \text{if } x = f(n); \\ x & \text{otherwise.} \end{cases}$$

Then $g \circ f : n+1 \to n+1$ is an injective function with $g \circ f(n) = n$. As we already proved, such function is bijective. Now the bijectivity of g implies that $f = g^{-1} \circ g \circ f$ is bijective, too.

Lemma 36.2. Let $f: x \to n$ be a surjective function from a set x onto a natural number $n \in \omega$. Then there exists a function $g: n \to x$ such that $f \circ g = \mathbf{Id} \upharpoonright n$.

Proof. This lemma will be proved by induction on n. For $n = \emptyset$ the statement of the lemma is trivally true. Assume that the lemma has been proved for some natural number $n \in \omega$. Take any surjective function $f: x \to n+1$. Consider the subset $x' = f^{-1}[n] \subseteq x$ and observe that the function $f|_{x'}: x' \to n$ is surjective. By the inductive assumption, there exists a function $g: n \to x'$ such that $f|_{x'} \circ g = \operatorname{Id}|_{n}$. By the surjectivity of f, there exists an element $z \in x$ such that f(z) = n. Define the function $\bar{g}: n+1 \to x$ by the formula

$$\bar{g}(y) = \begin{cases} g(y) & \text{if } y \in n; \\ z & \text{if } y = n. \end{cases}$$

It is clear that $f \circ \bar{g} = \mathbf{Id} \upharpoonright n$.

Theorem 36.3. For any natural number n and function $f: n \to n$ the following conditions are equivalent:

- 1) f is injective;
- 2) f is surjective;
- 3) f is bijective.

Proof. The implication $(1) \Rightarrow (3)$ was proved in Lemma 36.1 and $(3) \Rightarrow (2)$ is trivial. To prove $(2) \Rightarrow (1)$, assume that the function $f: n \to n$ is surjective. By Lemma 36.2, there exists a function $g: n \to n$ such that $f \circ g = \mathbf{Id} \upharpoonright n$. The function g is injective since for any distinct elements $x, y \in n$ we have $f(g(x)) = x \neq y = f(g(y))$ and hence $g(x) \neq g(y)$. By Lemma 36.1, the injective function $g: n \to n$ is bijective. Then for any distinct elements $x, y \in n$ the injectivity of g implies that $g^{-1}(x) \neq g^{-1}(y)$ and finally

$$f(x) = f \circ g(g^{-1}(x)) = \mathbf{Id}(g^{-1}(x)) = g^{-1}(x) \neq g^{-1}(y) = f \circ g(g^{-1}(y)) = f(y),$$

which means that f is injective.

Corollary 36.4. For any finite set x and function $f: x \to x$ the following conditions are equivalent:

- 1) f is injective;
- 2) f is surjective;
- 3) f is bijective.

Theorem 36.5. Every natural number is a cardinal.

Proof. We need to prove that every natural number n is the smallest ordinal in the set $|n| \cap \mathbf{On}$. Assuming the opposite, we can find an ordinal k < n and a bijective function $f : n \to k$. By Theorem 19.3(4), $k \subset n$ and hence the function $f : n \to k \subset n$ is injective but not surjective. But this contradicts Theorem 36.3.

Theorem 36.6. The ordinal ω is a cardinal.

Proof. Assuming that ω is not cardinal, we could find a bijective function $f:\omega\to n$ to some natural number $n\in\omega$. Then for the natural number n+1 the function $f|_{n+1}:n+1\to n\subset n+1$ is injective but not surjective, which contradicts Theorem 36.3.

Theorem 36.7. For a set x and a natural number n we have $|n| \le |x|$ or $|x| \le |n|$.

Proof. This theorem will be proved by induction on n. For n=0 the inequality $|\emptyset| \leq |x|$ holds for every set x as the empty function $\emptyset : \emptyset \to x$ is injective.

Assume that for some natural number n and all sets x we have proved that $|x| \leq |n|$ or $|n| \leq |x|$. Now consider the natural number n+1 and take any set x. If $|x| \leq n+1$, then we are done. So, assume that $|x| \not\leq n+1$. By the inductive assumption, $|x| \leq |n| \vee |n| \leq |x|$. The first case is impossible since it lead to the contradiction: $|x| \leq |n| \leq |n+1|$. Therefore, $|n| \leq |x|$ and hence there exists an injective function $f: n \to x$. If f is surjective, then f is bijective and hence $|x| = |n| \leq |n+1|$, which contradicts our assumption. Therefore, f is not surjective and we can choose an element $z \in x \setminus f[n]$ and extend the function f to the injective function $\bar{f} = f \cup \{\langle n, z \rangle \rangle$ from n+1 to x, witnessing that $|n+1| \leq |x|$.

Proposition 36.8. If for some set x we have $|x| < |\omega|$, then x is finite.

Proof. The inequality $|x| < |\omega|$ implies that x admits an injective function $x \to \omega$ and hence is well-orderable. By Theorem 23.6, there exists a bijective function $f: x \to \alpha$ to some ordinal α . We can assume that this ordinal is the smallest possible. We claim that $\alpha < \omega$. Assuming that $\alpha \not< \omega$ and applying Theorem 19.3(6), we conclude that $\omega \le \alpha$ and hence $|\omega| \le |\alpha| = |x|$. Since $|x| \le |\omega|$, we can apply Theorem 35.6 and conclude that $|x| = |\omega|$, which contradicts our assumption. This contradiction shows that $\alpha < \omega$ and then $|x| = |\alpha| \in \omega$, which means that the set x is finite.

Corollary 36.9. A set x is countable if and only if $|x| \leq |\omega|$.

Proposition 36.10. For a set x the following conditions are equivalent:

- 1) $|\omega| \leq |x|$;
- 2) there exists an injective function $f: x \to x$ which is not surjective.

Proof. (1) \Rightarrow (2) If $|\omega| \leq |x|$, then there exists an injective function $g: \omega \to x$. Define a function $f: x \to x$ by the formula

$$f(z) = \begin{cases} g(g^{-1}(z) + 1) & \text{if } z \in g[\omega]; \\ z & \text{if } z \in x \setminus g[\omega]; \end{cases}$$

and observe that f is injective but $g(0) \notin f[x]$, so f is not surjective.

(2) \Rightarrow (1) Assume that there exists an injective function $f: x \to x$ which is not surjective. Choose any element $x_0 \in x \setminus f[x]$ and consider the sequence $(x_n)_{n \in \omega}$ defined by the recursive formula $x_{n+1} = f(x_n)$ for $n \in \omega$. We claim that the function $g: \omega \to x$, $g: n \mapsto x_n$ is injective. This will follow from Theorem 19.3(6) as soon as we check that $g(k) \neq g(n)$ for any natural numbers k < n. This will be proved by induction on $k \in \omega$. For k = 0 this follows from the choice of $x_0 \notin f[x]$. Assume that for some $k \in \omega$ we have proved that $g(k) \neq g(n)$ for all n > k. Take any natural number n > k + 1. By Theorem 31.3(5), there exists an ordinal $\alpha > 0$ such that $(k+1) + \alpha = n$. Theorem 31.3(3) implies that $\alpha = 0 + \alpha \leq (k+1) + \alpha = n$ and hence $\alpha \in \omega$. By Theorem 31.4, $n = (k+1) + \alpha = (k+\alpha) + 1$. By Theorem 31.3(4), $k = k + 0 < k + \alpha$ and then $g(k) \neq g(k + \alpha)$ by the inductive assumption. The injectivity of f guarantees that $g(k+1) = x_{k+1} = f(x_k) = f(g(k)) \neq f(g(k+\alpha)) = g(k+\alpha+1) = g(n)$. This completes the proof of the inductive step. Now the Principle of Mathematical Induction implies that $g(k) \neq g(n)$ for all natural numbers k < n. Finally, Theorem 19.3(6) implies that the function $g: \omega \to x$ is injective and hence $|\omega| \leq |x|$.

Definition 36.11. A set X is called *Dedekind-finite* if every injective function $f: x \to x$ is surjective.

By Proposition 36.10, a set x is Dedekind-finite if and only if $|\omega| \leq |x|$. By Theorem 36.3, every finite set is Dedekind-finite.

Proposition 36.12. Assume that the Axiom of Dependent Choice (DC) holds. A set x is Dedekind-finite if and only if it is finite.

Proof. The "only if" part follows from Theorem 36.3. To prove the "if" part, assume that a set x is not finite. Consider the ordinary tree $T \subseteq x^{<\omega}$ whose elements are injective functions f with $\mathsf{dom}[f] \in \omega$ and $\mathsf{rng}[f] \subseteq x$. By Proposition 27.14, the (DC) is equivalent to (TC_ω) and hence the tree T contains a maximal chain $C \subseteq T$. It follows that $f = \bigcup C$ is an injective function such that $\mathsf{dom}[f] \in \omega \cup \{\omega\}$ and $\mathsf{rng}[f] \subseteq x$. We claim that $\mathsf{dom}[f] = \omega$. To derive a contradiction, assume that $\mathsf{dom}[f] = n \in \omega$. Since x is not finite, the injective function $f: n \to X$ is not surjective. Consequently, we can find an element $z \in x \setminus f[n]$ and consider the function $\bar{f} = f \cup \{\langle n, z \rangle\} \in T$ and the chain $\bar{C} = C \cup \{\bar{f}\}$, which is strictly larger than C. But this contradicts the maximality of C. This contradiction shows that $\mathsf{dom}[f] = \omega$ and hence $f: \omega \to x$ is an injective function witnessing that $|\omega| \leq |x|$ and hence x is not Dedekind-finite.

Theorem 36.7 and Propositions 36.8, 36.10 imply

Corollary 36.13. If a set x is Dedekind-finite but not finite, then

- 1) $|n| \le |x|$ for all $n \in \omega$;
- 2) $|\omega| \not \leq |x|$;
- 3) $|x| \not\leq |\omega|$.

Exercise 36.14. Show that $|\omega|$ and the cardinalities of Dedekind-finite sets are unique \leq -minimal cardinalities in the family of cardinalities $(|x|)_{x\in \mathbf{U}_{\infty}}$, indexed by the class \mathbf{U}_{∞} of infinite sets. Under (DC) the cardinality $|\omega|$ is a unique \leq -minimal cardinality in the family $(|x|)_{x\in \mathbf{U}_{\infty}}$.

Remark 36.15. The existence of infinite Dedekind-finite sets does not contradicts the Axioms of CST, see $[4, \S 4.6]$ for the proof of this fact. On the other hand, such sets do not exist under the axioms (CST + DC), see 36.12.

Next, we show that the countability is preserved by some operations over sets.

Proposition 36.16. For any function F and a countable set x the image F[x] is a countable set.

Proof. Let y = F[x]. Since x is countable, there exists an injective function g such that $\mathsf{dom}[g] \in \omega \cup \{\omega\}$ and $\mathsf{rng}[g] = x$. Then $f = F \circ g$ is a function such that $y = \mathsf{rng}[f]$ and $\mathsf{dom}[f] \subseteq \omega$. Consider the function $h: y \to \omega$ assigning to each element $v \in y$ the unique **E**-minimal element of the set $f^{-1}[\{v\}] \subseteq \omega$. It is easy to see that the function $h: y \to \omega$ is injective and hence $|y| \le |\omega|$. By Proposition 36.8, the set y is countable.

Corollary 36.17. If x is a countable set, then each subset of x is countable.

Proof. Observe that each subset $y \subseteq x$ has $|y| \le |x| \le |\omega|$. Applying Proposition 36.8, we obtain that |y| = |n| for some $n \in \omega \cup \{\omega\}$.

Proposition 36.18. The set $\omega^{<\omega} = \bigcup_{n \in \omega} \omega^n$ is countable.

Proof. Observe that the set $\omega^{<\omega}$ consists of all functions f such that $\mathsf{dom}[f] \in \omega$ and $\mathsf{rng}[f] \subseteq \omega$. It follows that the set $\mathsf{rng}[f]$ is finite and hence has an **E**-maximal element $\max \mathsf{rng}[f] \in \omega$, see Exercise 17.14. Then the function $\mu : \omega^{<\omega} \to \omega, \ \mu : f \mapsto \max\{\mathsf{dom}[f], \max \mathsf{rng}[f]\}$, is well-defined.

On the set $\omega^{<\omega}$ consider the well-order

$$W = \{ \langle f, g \rangle \in \omega^{<\omega} \times \omega^{<\omega} : \mu(f) < \mu(g) \ \lor \ (\mu(f) = \mu(g) \land \ \operatorname{dom}[f] < \operatorname{dom}[g]) \lor (\mu(f) = \mu(g) \land \operatorname{dom}[f] = \operatorname{dom}[g] \land \exists n \in \operatorname{dom}[f] = \operatorname{dom}[g] \ (f(n) < g(n) \land \forall i \in n \ f(i) = g(i)) \}.$$

By Theorem 23.6, the function $\mathsf{rank}_W : \omega^{<\omega} \to \mathsf{rank}(W)$ is bijective. Observe that for any $f \in \omega^{<\omega}$ the initial interval $\bar{W}(f)$ is contained in the finite set k^k for some natural number $k \in \omega$. This implies that $\mathsf{rank}(W) \leq \omega$. Then $|\omega^{<\omega}| = |\mathsf{rank}(W)| \leq |\omega|$ and the set $\omega^{<\omega}$ is countable.

Corollary 36.19. For any countable set x the set $x^{<\omega}$ is countable.

Exercise 36.20. Prove that for any countable ordinals α, β the ordinals $\alpha + \beta$, $\alpha \cdot \beta$ and α^{β} are countable.

Proposition 36.21. For any natural number n and an indexed family of countable sets $(x_i)_{i\in n}$ the union $\bigcup_{i\in n} x_i$ is countable.

Proof. The proof is inductive. For n=0 the union $\bigcup_{i\in k} x_i$ is empty and hence countable. Assume that for some $n\in\omega$ we proved that the union $\bigcup_{i\in n} x_i$ of any family $(x_i)_{i\in n}$ of countable sets is countable.

Take any family of countable sets $(x_i)_{i \in n+1}$ and consider its union $x = \bigcup_{i \in n+1} x_i$. Using Theorems 32.4(6) it can be shown that the functions $e : \omega \to \omega$, $e : n \mapsto 2n$, and $o : \omega \to \omega$, $o : n \mapsto 2n + 1$, are injective and have disjoint images.

By the inductive assumption, the set $\bigcup_{i\in n} x_i$ is countable and hence admits an injective function $f:\bigcup_{i\in n} x_i\to\omega$. Since the set x_n is countable there exists an injective function $g:x_n\to\omega$. Consider the injective function $h:x\to\omega$ defined by the formula

$$h(u) = \begin{cases} 2 \cdot f(z) & \text{if } u \in \bigcup_{i \in n} x_i; \\ 2 \cdot g(z) + 1 & \text{if } u \in x_n \setminus \bigcup_{i \in n} x_i. \end{cases}$$

The injective function h witnesses that $|x| \leq |\omega|$. By Corollary 36.9, the set x is countable. \square

Remark 36.22. The axioms of CST do not imply that the union $\bigcup_{n\in\omega} x_n$ of an indexed family $(x_n)_{n\in\omega}$ of countable sets is countable. This holds only under the axiom (UT_ω) , which is weak version of the Axiom of Choice, see Chapter 27.

Proposition 36.23. For two countable sets x, y their Cartesian product $x \times y$ is countable.

Proof. By Proposition 36.21 and Corollary 36.19, the sets $x \cup y$ and $(x \cup y)^{<\omega}$ are countable. Since $|x \times y| \le |(x \cup y)^2| \le |(x \cup y)^{<\omega}| \le |\omega|$, the set $x \times y$ is countable Corollary 36.9. \square

Definition 36.24. A set x is called *hereditarily countable* (resp. *hereditarily finite*) if each set $y \in \mathsf{TC}(x)$ is countable (resp. finite).

Exercise 36.25. Prove that a set $x \in \mathbf{V}$ is hereditarily finite if and only if its transitive closure $\mathsf{TC}(x)$ is finite.

Exercise 36.26. Assuming the principle (UT_{ω}) , prove that a set x is hereditarily countable if and only if its transitive closure TC(x) is countable.

37. Successor Cardinals and Alephs

In the following theorem for a set x by WO(x) we denote the set of all well-orders $w \subseteq x \times x$.

Theorem 37.1 (Hartogs–Sierpiński). There exists a function $(\cdot)^+: \mathbf{U} \to \mathbf{Card}$ assigning to every set x its successor cardinal $x^+ = \{\alpha \in \mathbf{On} : |\alpha| \le |x|\}$. For this cardinal we have the upper bounds

$$|x^{+}| \le |WO(x)| \le |\mathcal{P}(x \times x)|$$
 and $|x^{+}| \le \min\{|\mathcal{P}^{\circ 2}(x \times x)|, |\mathcal{P}^{\circ 3}(x)|\}.$

Proof. By Theorem 23.9 for any set x there exists an ordinal α admitting no injective map $\alpha \to x$. So, we can define x^+ as the smallest ordinal with this property. The minimality of the ordinal x^+ ensures that x^+ is a cardinal and for any $\alpha \in x^+$ there exists an injective function $\alpha \to x$ and hence $|\alpha| \le |x|$. The function $(\cdot)^+ : \mathbf{U} \to \mathbf{Card}$, $(\cdot)^+ : x \mapsto x^+$, exists by Theorem 7.2 on the existence of classes.

Next we prove the upper bounds for the successor cardinal x^+ of a set x. Let WO(x) be the set of all well-orders $w \subseteq x \times x$. It is easy to see that $WO(x) \subseteq \mathcal{P}(x \times x)$ and $x^+ = \mathsf{rank}[WO(x)]$, which implies

$$|x^+| \le |WO(x)| \le \mathcal{P}(x \times x)$$
 and hence $|x^+| \le |\mathcal{P}(\mathcal{P}(x \times x))| = |\mathcal{P}^{\circ 2}(x \times x)|$

by Proposition 35.7(3).

On the other hand, any injective function f with $\mathsf{dom}[f] \in x^+$ and $\mathsf{rng}[f] \subseteq x$ is uniquely determined by the chain of subsets $C_f = \{f[\beta] : \beta \leq \mathsf{dom}[f]\} \subseteq \mathcal{P}(x)$ and each ordinal $\alpha \in x^+$ is uniquely determined by the subset

$$F_{\alpha} = \{C_f : f \in \mathbf{Fun} \land f^{-1} \in \mathbf{Fun} \land \mathsf{dom}[f] = \alpha \land \mathsf{rng}[f] \subseteq x\} \subseteq \mathcal{P}(\mathcal{P}(x)),$$

which implies that

$$|x^+| \le |\{F_\alpha : \alpha \in x^+\}| \le |\mathcal{P}(\mathcal{P}(\mathcal{P}(x)))| = |\mathcal{P}^{\circ 3}(x)|.$$

Theorem 36.3 implies

Corollary 37.2. For every natural number n its successor cardinal n^+ is equal to n+1.

The Hartogs–Sierpiński Theorem 37.1 has an interesting implication. Given a cardinality κ , we say that a cardinality κ^+ is a successor cardinality of κ if $\kappa < \kappa^+$ and $\kappa^+ \le \lambda$ for any cardinality λ with $\kappa < \lambda$. By Theorem 35.6 a successor cardinality if exists, then it is unique.

Theorem 37.3 (Tarski). The following statements are equivalent:

- 1) The Axiom of Choice holds.
- 2) For any sets x, y we have $|x| \leq |y|$ or $|y| \leq |x|$.
- 3) For any set x we have $|x| < |x^+|$.
- 4) For any set x the cardinality $|x^+|$ of the successor cardinal x^+ is the successor cardinality of |x|.

Proof. (1) \Rightarrow (2) If the Axiom of Choice holds, then by Zermelo Theorem 27.2, every set is equipotent to some ordinal. Now the comparability of cardinalities follows from the comparability of ordinals, see Theorem 19.3(6).

 $(2) \Rightarrow (3)$ Assume that any two cardinalities are comparable. For any set x, the definition of the successor cardinal x^+ implies that $|x^+| \nleq |x|$ and hence $|x| < |x^+|$.

- $(3) \Rightarrow (1)$ If for every x we have $|x| < |x^+|$, then there exists an injective function $f: X \to (3)$ x^{+} and hence x is well-orderable. By Theorem 27.4, the Axiom of Choice holds.
- $(2) \Rightarrow (4)$ Take any set x and consider its successor cardinal x^+ . The definition of x^+ guarantees that $|x^+| \leq |x|$. Now the condition (2) implies that $|x| < |x^+|$. Assuming that $|x^+|$ is not a successor cardinality of |x|, we can find a set y such that |x| < |y| but $|x^+| \le |y|$. The comparability of the cardinals $|x^+|$ and |y| implies that $|y| < |x^+|$. Then the set y admits an injective function to x^+ and hence is well-orderable. By Theorem 23.6, y admits a bijective function on some ordinal α . We claim that $\alpha < x^+$. In the opposite case, we can apply Theorem 19.3(6) and conclude that $x^+ \le \alpha$ and then $|x^+| \le |\alpha| = |y|$. Since $|y| < |x^+|$ we can apply Theorem 35.6 and conclude that $|y| = |x^+|$ which contradicts our assumption. This contradiction shows that $\alpha < x^+$. Since $|x| < |y| = |\alpha|$, the cardinal α admits no injective function into x and hence $x^+ \leq \alpha$ as x^+ is the smallest ordinal with this property. But this contradicts the strict inequality $\alpha < x^+$ established earlier. This contradiction shows that $|x^+|$ is the successor cardinality of |x|.
- $(4) \Rightarrow (1)$ If for every set x, the cardinality $|x^+|$ is a successor cardinality of |x|, then $|x| < |x^+|$ and hence x admits an injective function $f: x \to x^+$ to the cardinal x^+ , which imlies that x can be well-ordered. By Theorem 27.4, the Axiom of Choice holds.

Transfinite iterations of the operation of taking the successor cardinals yield a nice parametrization of the class **Card** of cardinals by ordinals.

Consider the transfinite sequence of cardinals $(\omega_{\alpha})_{\alpha \in \mathbf{On}}$ defined by the recursive formula:

- $\omega_{\alpha+1} = \omega_{\alpha}^+$ for any ordinal α ; $\omega_{\alpha} = \sup\{\omega_{\beta} : \beta \in \alpha\}$ for any limit ordinal $\alpha > 0$.

Therefore, $\omega_{\alpha} = \omega^{+\circ \alpha}$ for every ordinal α .

Proposition 37.4. The function $\omega_* : \mathbf{On} \to \mathbf{Card}$, $\omega_* : \alpha \to \omega_{\alpha}$, is well-defined. For every ordinal α we have $\alpha \leq \omega_{\alpha} < \omega_{\alpha+1}$.

Proof. The existence of the function ω_* follows from Theorem 29.1 applied to the Hartogs' function $\mathbf{On} \to \mathbf{On}$, $\alpha \mapsto \alpha^+$, of taking the successor cardinal.

The inequality $\alpha \leq \omega_{\alpha}$ will be proved by transfinite induction on α . For $\alpha = 0$ we have $0 < \omega$. Assume that for some ordinal α and all its elements $\beta \in \alpha$ we proved that $\beta \leq \omega_{\beta}$. If α is a successor ordinal, then $\alpha = \beta + 1$ for some $\beta \in \alpha$ and hence $\omega_{\alpha} = \omega_{\beta+1} = \omega_{\beta}^+ > \omega_{\beta} \geq \beta$, which implies $\alpha = \beta + 1 \leq \omega_{\alpha}$.

If α is a limit ordinal, then

$$\omega_{\alpha} = \sup\{\omega_{\beta} : \beta \in \alpha\} > \sup\{\beta : \beta \in \alpha\} = \alpha$$

by the inductive assumption. By the Principle of Transfinite Induction, the inequality $\alpha \leq \omega_{\alpha}$ is true for all ordinals α .

The strict inequality $\omega_{\alpha} < \omega_{\alpha+1} = \omega_{\alpha}^+$ follows from the definition of the successor cardinal $\omega_{\alpha}^{+} > \omega_{\alpha}$.

Theorem 37.5. For every infinite cardinal κ there exists an ordinal α such that $\kappa = \omega_{\alpha}$.

Proof. Given a cardinal κ , consider the class $A = \{\alpha \in \mathbf{On} : \omega_{\alpha} \leq \kappa\}$.

Since the function $\omega_*: \mathbf{On} \to \mathbf{Card}, \ \omega_*: \alpha \mapsto \omega_{\alpha}$, is injective, the class A is a set by the Axiom of Replacement. So, we can consider the ordinal $\alpha = \sup A$. If α is a limit ordinal, then $\omega_{\alpha} = \sup_{\beta \in \alpha} \omega_{\beta} = \sup_{\beta \in A} \omega_{\beta} \leq \sup_{\beta \in A} \kappa = \kappa$. If $\alpha = \sup A$ is a successor ordinal, then $\alpha \in A$ and again $\omega_{\alpha} \leq \kappa$. In both cases we obtain $\omega_{\alpha} \leq \kappa$. Assuming that $\omega_{\alpha} \neq \kappa$, we conclude that $\omega_{\alpha} < \kappa$. Since κ is a cardinal, there exists no bijective function $\kappa \to \omega_{\alpha}$. By Theorem 35.6, there is no injective functions from $\kappa \to \omega_{\alpha}$. Then $\omega_{\alpha+1} = \omega_{\alpha}^+ \leq \kappa$ by the definition of the successor cardinal ω_{α}^+ . Then $\alpha + 1 \in A$ and hence $\alpha \in \alpha + 1 \leq \sup A = \alpha$, which contradicts the irreflexivity of the relation $\mathbf{E} \upharpoonright \mathbf{On}$. This contradiction shows that $\kappa = \omega_{\alpha}$.

For every ordinal α , denote by \aleph_{α} the cardinality $|\omega_{\alpha}|$ of the cardinal ω_{α} . Theorems 27.2, 23.6 and 37.5 imply the following characterization.

Corollary 37.6. The following statements are equivalent:

- 1) For every infinite set x there exists an ordinal α such that $|x| = \aleph_{\alpha}$.
- 2) The Axiom of Choice holds.

38. Arithmetics of Cardinals

In this section we study the operations of sum, product and exponent of cardinalities. Namely, for any cardinalities |x|, |y| we put

- $|x| + |y| = |(\{0\} \times x) \cup (\{1\} \times y)|;$
- $\bullet |x| \cdot |y| = |x \times y|;$
- $|x|^{|y|} = |x^y|$.

Exercise 38.1. Show that the sum, product and exponent of cardinalities are well-defined, i.e., do not depend on the choice of sets in the corresponding equivalence classes.

Exercise 38.2. Prove that for any ordinals α, β we have $|\alpha| + |\beta| = |\alpha + \beta|$ and $|\alpha| \cdot |\beta| = |\alpha \cdot \beta|$. If the ordinal β is finite, then $|\alpha|^{|\beta|} = |\alpha^{\cdot \beta}|$.

Exercise 38.3. Find two ordinals α, β such that $|\alpha|^{|\beta|} \neq |\alpha^{\beta}|$.

Hint: Observe that $|2^{\cdot\omega}| = |\omega| < |2^{\omega}|$.

Exercise 38.4. Given cardinalities κ, λ, μ , prove that

- (1) $\kappa + \lambda = \lambda + \kappa$;
- (2) $(\kappa + \lambda) + \mu = \kappa + (\lambda + \mu)$;
- (3) $\kappa \cdot \lambda = \lambda \cdot \kappa$;
- (4) $(\kappa \cdot \lambda) \cdot \mu = \kappa \cdot (\lambda \cdot \mu)$;
- (5) $\kappa \cdot (\lambda + \mu) = (\kappa \cdot \lambda) + (\kappa \cdot \mu);$
- (6) If $\kappa \leq \lambda$, then $\kappa + \mu \leq \lambda + \mu$ and $\kappa \cdot \mu \leq \lambda \cdot \mu$;
- (7) $\kappa + |\emptyset| = \kappa = |\kappa| \cdot |1|;$
- (8) if $|2| \le \kappa$ and $|2| \le \lambda$, then $\kappa + \lambda \le \kappa \cdot \lambda$.

Hint to (8): Fix two sets x, y with $|x| = \kappa \ge |2|$ and $|y| = \lambda \ge |2|$. Fix points $a, b \in x$ and $c, d \in y$ with $a \ne b$ and $c \ne d$. Consider the injective function $f: (\{0\} \times x) \cup (\{1\} \times y) \to x \times y$ assigning to each point $\langle 0, z \rangle \in \{0\} \times x$) the ordered pair $\langle z, c \rangle$, to each point $\langle 1, z \rangle \in \{1\} \times (y \setminus \{c\})$ the ordered pair $\langle a, z \rangle$, and to the ordered pair $\langle 1, c \rangle$ the ordered pair $\langle b, d \rangle$.

Theorem 38.5. For any ordinal α we have $\aleph_{\alpha} \cdot \aleph_{\alpha} = \aleph_{\alpha}$.

Proof. This theorem will be proved by transfinite induction on α . Assume that for some ordinal α and all its elements $\beta \in \alpha$ we have proved that $\aleph_{\beta} \cdot \aleph_{\beta} = \aleph_{\beta}$.

Consider the cardinal ω_{α} and its square $\omega_{\alpha} \times \omega_{\alpha}$ endowed with the canonical well-order

$$W = \{ \langle \langle x, y \rangle, \langle x', y' \rangle \rangle \in (\omega_{\alpha} \times \omega_{\alpha}) \times (\omega_{\alpha} \times \omega_{\alpha}) : (x \cup y \subset x' \cup y') \lor (x \cup y = x' \cup y' \land x \in x') \lor (x \cup y = x' \cup y' \land x = x' \land y \in y') \}.$$

By Theorem 23.6, there exists an order isomorphism $\operatorname{rank}_W:\operatorname{dom}[W^\pm]\to\operatorname{rank}(W)$. The definition of the well-order W guarantees that for any $z\in\omega_\alpha\times\omega_\alpha$ the initial interval $\bar{W}(z)$ is contained in the square $\beta\times\beta$ of some ordinal $\beta\in\omega_\alpha$. Since ω_α is a cardinal, $|\beta|<|\omega_\alpha|=\aleph_\alpha$. By Theorem 37.5, there exists an ordinal γ such that $|\beta|=\aleph_\gamma$. Taking into account that $\aleph_\gamma=|\beta|<|\omega_\alpha|=\aleph_\alpha$ and applying Proposition 37.4 and Theorem 19.3(6), we conclude that $\gamma<\alpha$. Then by the inductive assumption, $|\beta\times\beta|=\aleph_\gamma\cdot\aleph_\gamma=\aleph_\gamma<\aleph_\alpha=|\omega_\alpha|$ and consequently, $|\bar{W}(z)|\leq |\beta\times\beta|<|\omega_\alpha|$. Since $\operatorname{rank}_W:\omega_\alpha\times\omega_\alpha\to\operatorname{rank}(W)\subset\operatorname{On}$ is an order isomorphism, for every $z\in\operatorname{rank}(W)$ the initial interval $\bar{E}(z)=z\in\operatorname{On}$ has cardinality $|z|<|\omega_\alpha|$ and hence $z\subset\omega_\alpha$. Then $\operatorname{rank}(W)=\bigcup\{z:z\in\operatorname{rank}(W)\}\subseteq\omega_\alpha$ and hence $\aleph_\alpha\cdot\aleph_\alpha\leq|\omega_\alpha\times\omega_\alpha|=|\operatorname{rank}(W)|\leq|\omega_\alpha|=\aleph_\alpha$. The inequality $\aleph_\alpha\leq\aleph_\alpha\cdot\aleph_\alpha$ is trivial. By Theorem 35.6, $\aleph_\alpha\cdot\aleph_\alpha=\aleph_\alpha$.

Theorems 38.5 and 35.6 imply:

Corollary 38.6. For any ordinals $\alpha \leq \beta$ we have

$$\aleph_{\alpha} + \aleph_{\beta} = \aleph_{\alpha} \cdot \aleph_{\beta} = \aleph_{\beta}.$$

In its turn, Corollaries 38.6 and 37.6 imply

Corollary 38.7. Assume that Axiom of Choice. Then for any infinite cardinalities κ, λ we have

$$\kappa + \lambda = \kappa \cdot \lambda = \max{\{\kappa, \lambda\}}.$$

We are going to show that the Axiom of Choice cannot be removed from Corollary 38.7.

Lemma 38.8. If for some sets x, y, z, α we have $|x| + |\alpha| = |y \times z|$, then $|z| \leq |x|$ or $|y| \leq^* |\alpha|$.

Proof. The equality $|x| + |\alpha| = |y \times z|$ implies that $y \times z = f[x] \cup g[\alpha]$ for some injective functions $f: x \mapsto y \times z$ and $g: \alpha \to y \times z$ with $f[x] \cap g[\alpha] = \emptyset$. If for some $v \in y$ the set $\{v\} \times z$ is a subset of f[x], then the injective function $f^{-1}|_{\{v\} \times z}$ witnesses that $|z| \leq |x|$. So, assume that for every $v \in y$ the set $\{v\} \times z$ is not contained in f[x]. Then it intersects the set $g[\alpha] = (y \times z) \setminus f[x]$ and the function $\operatorname{dom} \circ g: \alpha \to y$ is surjective, witnessing that $|y| \leq^* |\alpha|$.

Combining Lemma 38.8 with Proposition 35.7(2), we obtain the following lemma.

Lemma 38.9. If for some sets x, y, z and ordinal α we have $|x| + |\alpha| = |y \times z|$, then $|z| \le |x|$ or $|y| \le |\alpha|$.

Theorem 38.10 (Tarski). The following conditions are equivalent.

- 1) For any infinite sets x, y we have $|x| + |y| = |x| \cdot |y|$;
- 2) For any infinite set x we have $|x| + |x^+| = |x| \cdot |x^+|$;
- 3) For any infinite set x we have $|x \times x| = |x|$;
- 4) The Axiom of Choice holds.

Proof. The implication $(1) \Rightarrow (2)$ is trivial.

 $(2) \Rightarrow (4)$: Assuming some set x has $|x| + |x^+| = |x| \cdot |x^+|$, we can apply Lemma 38.9 and conclude that $|x^+| \le |x|$ or $|x| \le |x^+|$. The inequality $|x^+| \le |x|$ contradicts the definition of the ordinal x^+ . Therefore, $|x| \leq |x^+|$ which implies that the set x admits an injective function into the ordinal x^+ and hence x can be well-ordered. By Theorem 27.4, the Axiom of Choice holds.

The implication $(4) \Rightarrow (1)$ has been proved in Corollary 38.7.

The implication $(4) \Rightarrow (3)$ follows from Corollary 38.7.

 $(3) \Rightarrow (2)$: Given any infinite set x, consider the set $y = (\{0\} \times x) \cup (\{1\} \times x^+)$. By (3) and Exercise 38.4(8), we have

$$|y \times y| = |y| = |x| + |x^{+}| \le |x \times x^{+}| \le |y \times y|$$

and hence $|x| + |x^{+}| = |x \times x^{+}|$.

Theorem 38.11. Let I be a set and $(x_i)_{i\in I}$ be an indexed family of sets and κ be an infinite cardinal such that $|I| \leq |\kappa|$ and $|x_i| \leq |\kappa|$ for all $i \in I$. If the Axiom of Choice holds, then $\left|\bigcup_{i\in I} x_i\right| \leq \kappa.$

Proof. For every $i \in I$ consider the set F_i of all injective functions from the set x_i to the cardinal κ . By our assumption, $|x_i| \leq |\kappa|$ and hence the set F_i is not empty. By the Axiom of Choice, the Cartesion product $\prod_{i \in I} F_i$ is not empty and hence contains some indexed family of injective functions $(f_i)_{i\in I}$. Since $|I|\leq \kappa$, there exists an injective function $g:I\to\kappa$. The function g induced the well-order $W = \{\langle i,j \rangle \in I \times I : g(i) \in g(j)\}$ on the index set I. For every $u \in \bigcup_{i \in I} X_i$ let $\mu(u)$ be the unique W-minimal element of the non-empty set $\{i \in I : u \in x_i\}$. Then the injective function

$$h: \bigcup_{i \in I} x_i \to \kappa \times \kappa, \quad h: u \mapsto \langle g(u), f_{\mu(u)}(u) \rangle$$

witnesses that $|\bigcup_{i\in I} x_i| \leq |\kappa \times \kappa| = |\kappa|$, where the last equality follows from Theorems 38.5 and 37.5.

Next, we consider the exponentiation of cardinalities.

Exercise 38.12. For any nonzero cardinalities κ, λ, μ , the exponentiation has the following properties:

- $\kappa^{\lambda+\mu} = \kappa^{\lambda} \cdot \kappa^{\mu}$:
- $\kappa^{\lambda \cdot \mu} = (\kappa^{\lambda})^{\mu}$;
- If κ ≤ λ, then κ^μ ≤ κ^μ;
 If λ ≤* μ, then κ^λ ≤ κ^μ.

Exercise 38.13. Prove that $|2^x| = |\mathcal{P}(x)|$ for every set x.

Hint: Observe that the function $\xi: 2^x \to \mathcal{P}(x), \, \xi: f \mapsto f^{-1}[\{1\}]$, is bijective.

Exercise 38.14. Prove that $|x| < |2^x|$ for any set x.

Many results on exponents of cardinalities can be derived from Kőnig's Theorem 38.17, which compares the cardinalities of sum and products of cardinals.

For an indexed family of classes $(X_i)_{i \in I}$, consider the class

$$\sum_{i \in I} X_i = \bigcup_{i \in I} \{i\} \times X_i.$$

Lemma 38.15. Let I be a set and $(\kappa_i)_{i\in I}$ and $(\lambda_i)_{i\in I}$ be two indexed families of cardinals. If $\forall i \in I \ (\max\{2, \kappa_i\} \leq \lambda_i)$, then $|\sum_{i \in I} \kappa_i| \leq |\prod_{i \in I} \lambda_i|$.

Proof. If $|I| \leq |2|$ then the inequality $|\sum_{i \in I} \kappa_i| \leq |\prod_{i \in I} \lambda_i|$ follows from Exercise 38.4(8). So, we assume that $|I| \geq |3|$.

In this case the inequality $|\sum_{i\in I} \kappa_i| \leq |\prod_{i\in I} \lambda_i|$ is witnessed by the injective function $f: \sum_{i\in I} \kappa_i \to \prod_{i\in I} \kappa_i$ assigning to every ordered pair $\langle i, x \rangle \in \bigcup_{j\in I} (\{j\} \times \kappa_j)$ the function $f_{\langle i, x \rangle}: I \to \bigcup_{i\in I} \lambda_i$ such that

$$f_{\langle i,x\rangle}(j) = \begin{cases} 0 & \text{if } x > 0 \text{ and } j \in I \setminus \{i\}; \\ x & \text{if } x > 0 \text{ and } j = i; \\ 1 & x = 0 \text{ and } j \in I \setminus \{i\}; \\ 0 & x = 0 \text{ and } j = i. \end{cases}$$

Lemma 38.16. Let I be a se,t $(x_i)_{i\in I}$ be an indexed family of sets and $(\lambda_i)_{i\in I}$ an indexed family of cardinals. If $\forall i \in I \ |\lambda_i| \leq^* |x_i|$, then $|\prod_{i\in I} \lambda_i| \leq^* |\sum_{i\in I} x_i|$.

Proof. To derive a contradiction, assume that $|\prod_{i\in I}\lambda_i| \leq^* |\sum_{i\in I}x_i|$ and find a surjective function $f:\prod_{i\in I}x_i\to\sum_{i\in I}\lambda_i$. For every $i\in I$ denote by

$$\operatorname{pr}_i: \prod_{j\in I} \lambda_j \to \lambda_i, \quad \operatorname{pr}_i: g \mapsto g(i),$$

the projection onto the *i*-th coordinate. It follows from $|\lambda_i| \not\leq^* |x_i|$ that $\operatorname{pr}_i \circ f[\{i\} \times x_i] \neq \lambda_i$. So, we can define a function $g \in \prod_{j \in I} \lambda_j$ assigning to every $i \in I$ the unique **E**-minimal element of the nonempty subset $\lambda_i \setminus \operatorname{pr}_i \circ f[\{i\} \times x_i]$ of the cardinal λ_i .

By the surjectivity of f, there exists $i \in I$ and $z \in x_i$ such that g = f(i, z). Then $g(i) = \operatorname{pr}_i \circ f(i, z) \in \operatorname{pr}_i \circ f[\{i\} \times x_i]$, which contradicts the definition of g.

Lemmas 38.15 and 38.16 imply the following

Theorem 38.17 (Kőnig). Let I be a set, and $(\kappa_i)_{i \in I}$ and $(\lambda_i)_{i \in I}$ be two indexed families of cardinals. If $\forall i \in I \ (0 < \kappa_i < \lambda_i)$, then $|\sum_{i \in I} \kappa_i| < |\prod_{i \in I} \lambda_i|$.

Remark 38.18. For $\kappa_i = 1$ and $\lambda_i = 2$, Kőnig's Theorem 38.17 implies the strict inequality $|I| = |\sum_{i \in I} \kappa_i| < |\prod_{i \in I} \lambda_i| = |2^I|$, which has been established in Corollary 35.11.

39. Cofinality of Cardinals

Definition 39.1. The cofinality $cf(\alpha)$ of an ordinal α is the smallest cardinal λ for which there exists a function $f: \lambda \to \alpha$ which is unbounded in the sense that for every $\beta < \kappa$ there exists $\gamma \in \lambda$ such that $\beta \leq f(\gamma)$.

Exercise 39.2. Check that:

- cf(0) = 0:
- cf(n) = 1 for any natural number n > 0;
- $cf(\omega) = \omega$;
- $cf(\alpha + 1) = 1$ for any ordinal α ;
- $\operatorname{cf}(\omega_{\alpha}) \leq \operatorname{cf}(\alpha)$ for any limit ordinal α .
- $cf(\alpha) \leq \alpha$ for any ordinal α .

Definition 39.3. An infinite cardinal κ is called

- regular if $cf(\kappa) = \kappa$;
- singular if $cf(\kappa) < \kappa$.

Example 39.4. The cardinal ω_{ω} is singular because $\omega_{\omega} > \mathrm{cf}(\omega_{\omega}) = \omega$,

Proposition 39.5. For any limit ordinal α its cofinality $cf(\alpha)$ is a regular cardinal.

Proof. By the definition of the cofinality $cf(\alpha)$, there exists an unbounded function $f : cf(\alpha) \to \alpha$.

Since the ordinal α is limit, the cardinal $\operatorname{cf}(\alpha)$ is infinite and hence is a limit ordinal. Assuming that $\operatorname{cf}(\alpha)$ is a singular, we can find a cardinal $\kappa < \operatorname{cf}(\alpha)$ and an unbounded function $g: \kappa \to \operatorname{cf}(\kappa)$.

By the minimality of $\operatorname{cf}(\alpha)$, for every $\beta \in \operatorname{cf}(\alpha)$ the function $f \upharpoonright_{\beta}$ is bounded in α . Consequently, the ordinal $h(\beta) = \sup f[\beta]$ is an element of α . We claim that the function $h \circ g : \kappa \to \alpha$ is unbounded. Indeed, for any ordinal $\beta \in \alpha$ by the unboundedness of f, there exists an ordinal $\gamma \in \operatorname{cf}(\alpha)$ such that $\beta \leq f(\gamma)$. Since the function $g : \kappa \to \operatorname{cf}(\alpha)$ is unbounded, there exists an ordinal $\delta \in \kappa$ scuh that $\gamma < g(\delta)$. Then $\beta \leq f(\gamma) \leq \sup f[\gamma+1] = h(\gamma+1) \leq h(g(\delta))$.

Since the function $h \circ g : \kappa \to \alpha$ is unbounded, the minimality of the cardinal $cf(\alpha)$ implies that $cf(\alpha) \leq \kappa$, which contradicts the choice of κ . This contradiction shows that the cardinal $cf(\alpha)$ is regular.

Theorem 39.6. Under the Axiom of Choice, for every ordinal α the cardinal $\omega_{\alpha+1}$ is regular.

Proof. To derive a contradiction, assume that the cardinal $\omega_{\alpha+1}$ is singular and hence the cardinal $\kappa = \operatorname{cf}(\omega_{\alpha+1})$ is strictly smaller than $\omega_{\alpha+1}$. Then $|\kappa| \leq \omega_{\alpha}$ by the definition of $\omega_{\alpha+1} = \omega_{\alpha}^+$.

By the definition of the cofinality $\operatorname{cf}(\omega_{\alpha+1}) = \kappa$, there exists an unbounded function $f: \kappa \to \omega_{\alpha+1}$. Since $\omega_{\alpha+1}$ is a cardinal, for every $\gamma \in \kappa$, the ordinal $f(\gamma) \subset \omega_{\alpha+1}$ has cardinality $|f(\gamma)| < |\omega_{\alpha+1}|$. By the definition of the successor cardinal $\omega_{\alpha}^+ = \omega_{\alpha+1}$, the ordinal $f(\gamma)$ admits an injective function to ω_{α} and hence $|f(\gamma)| \le \omega_{\alpha}$.

Applying Theorem 38.11, we conclude that

$$|\omega_{\alpha}^{+}| = |\omega_{\alpha+1}| = |\bigcup_{\alpha \in \kappa} f(\alpha)| \le |\omega_{\alpha}|,$$

which contradicts the definition of the successor cardinal ω_{α}^{+} . This contradiction shows that the cardinal $\omega_{\alpha+1}$ is regular.

Remark 39.7. Without the Axiom of Choice it is not possible to prove that the cardinals $\omega_{\alpha+1}$ are regular: by [5, Theorem 10.6] it is consistent with the axioms of ZF that the cardinal ω_1 is a countable union of countable sets and hence $cf(\omega_1) = \omega$.

Now derive some corollaries of Kőnig's Theorem 38.17 that involve the cofinality.

Corollary 39.8. Every infinite cardinal κ has cardinality $|\kappa| < |\kappa^{\mathrm{cf}(\kappa)}|$.

Proof. By the definition of the cardinal $\operatorname{cf}(\kappa)$, there exists an unbounded function $f:\operatorname{cf}(\kappa)\to \kappa$. The unboundedness of f guarantees that $\kappa=\bigcup_{\alpha\in\operatorname{cf}(\kappa)}f(\alpha)$. For every $x\in\kappa$ let $\alpha(x)$ be the smallest ordinal such that $x\in f(\alpha(x))$. The injective map

$$g: \kappa \to \sum_{\alpha \in \mathrm{cf}(\kappa)} f(\alpha), \quad g: x \mapsto \langle \alpha(x), x \rangle$$

witnesses that $|\kappa| \leq |\sum_{\alpha \in cf(\kappa)} f(\alpha)|$. Since κ is a cardinal, for every $\alpha \in cf(\kappa)$, the ordinal $f(\alpha) \in \kappa$ has cardinality $|f(\alpha)| < |\kappa|$. Applying Theorem 38.17, we obtain that

$$|\kappa| \le |\sum_{\alpha \in \mathrm{cf}(\kappa)} f(\alpha)| < |\prod_{\alpha \in \mathrm{cf}(\kappa)} \kappa| = |\kappa^{\mathrm{cf}(\kappa)}|.$$

Corollary 39.9. Let κ, λ be infinite cardinals. If $|\lambda| = |a^{\kappa}|$ for some set a, then $\kappa < \operatorname{cf}(\lambda)$.

Proof. Assuming that $\operatorname{cf}(\lambda) \leq \kappa$, find an unbounded function $f : \kappa \to \lambda$. For every $i \in \kappa$ consider the ordinal $\kappa_i = f(i) \subset \lambda$. Since λ is a cardinal, $|\kappa_i| < |\lambda|$. Since f is unbounded, $\lambda = \bigcup_{i \in \kappa} f(i)$. By Theorems 38.17 and 38.5,

$$|\lambda| = |\bigcup_{i \in \kappa} f(i)| \le |\sum_{i \in \kappa} \kappa_i| < |\prod_{i \in \kappa} \lambda| = |\lambda^{\kappa}| = |(a^{\kappa})^{\kappa}| = |a^{\kappa \cdot \kappa}| = |a^{\kappa}| = |\lambda|,$$

which is a desired contradiction.

40. (Generalized) Continuum Hypothesis

Wir müssen wissen, wir werden wissen David Hilbert

By Cantor's Theorem 35.10, for every set x the cardinality of its power-set is strictly larger than the cardinality of x. Observe that for a finite set x the set $\{y \in \mathbf{Card} : |x| \le |y| \le |\mathcal{P}(x)|\}$ has cardinality $|2^x| - |x|$ and hence contains many cardinals.

In 1878 Cantor made a conjecture that for infinite sets the situation is different: there is no cardinality |x| such that $|\omega| < |x| < |\mathcal{P}(\omega)|$. This conjecture is known as the Continuum Hypothesis (briefly (CH)). At the presence of the Axiom of Choice the Continuum Hypothesis is equivalent to the equality $\aleph_1 = \mathfrak{c}$, where \mathfrak{c} denotes the cardinality $|\mathcal{P}(\omega)|$ of the power-set $\mathcal{P}(\omega)$ and is called the *cardinality of continuum*.

Cantor himself tried to prove the Continuum Hypothesis but without success. David Hilbert included the Continuum Hypothesis as problem number one in his famous list of open problems announced in the II World Congress of mathematicians in 1900. For the complete solution, this problem waited more than 60 years. The final solution appeared to be a bit unexpected. First, in 1939 Kurt Gödel proved that the Continuum Hypothesis does not contradict the axioms NBG or ZFC. Twenty four years later, in 1963 Paul Cohen proved that the negation CH does not contradict the axioms ZFC. So, CH turned out to be independent of ZFC. It can be neither proved nor disproved. To prove that CH does not contradicts the axioms NBG, Gödel established that it holds in the constructible universe. Moreover, Gödel showed that his Axiom of Constructibility $\mathbf{U} = \mathbf{L}$ implies the following more general statement, called the *Generalized Continuum Hypothesis*

(GCH): For any infinite set
$$x$$
 there is no cardinality κ such that $|x| < \kappa < |2^x|$.

Under the Axiom of Choice the Generalized Continuum Hypothesis is equivalent to the statement that $|\kappa^+| = |2^{\kappa}|$ every infinite cardinal κ .

Theorem 40.1 (Gödel). $(U = L) \Rightarrow (GCH)$.

Proof. Assume that $\mathbf{U} = \mathbf{L}$. By Theorem 25.21, $(\mathbf{U} = \mathbf{L})$ implies the Axiom of (Global) Choice. So, (GCH) will be established as soon as we prove the equality $|\mathcal{P}(\kappa)| = |\kappa^+|$ for every infinite cardinal κ . Given any constructible subset $y \subseteq \kappa$, we can find $n \in \omega$, $\lambda \in 9^{2^{< n}}$ and $f \in \mathbf{On}^{2^n}$ such that $y = \ddot{\mathsf{G}}_{\lambda}(f)$, see Corollary 25.20. It can be shown (but it is difficult and requires more advanced tools, see e.g. [5, 13.20] or [9]) that there exists function $g: 2^n \to \kappa^+$ such that $\ddot{\mathsf{G}}_{\lambda}(g) = \ddot{\mathsf{G}}_{\lambda}(f) = y$. This implies that

$$|\mathcal{P}(\kappa)| \le |\{\ddot{\mathsf{G}}_{\lambda}(g) : n \in \omega, \ \lambda \in 9^{2^{< n}}, \ g \in (\kappa^+)^{2^n}\}| = |\kappa^+|.$$

Exercise* **40.2.** ¹⁰ Prove that for any $\kappa \in \mathbf{Card}$, $n \in \omega$, $\lambda \in 9^n$ and $f \in \mathbf{U}^{2^n}$ with $\ddot{\mathsf{G}}_{\lambda}(f) \subseteq \kappa$ there exists a function $g \in (\kappa)^+$ such that $\ddot{\mathsf{G}}_{\lambda}(g) = \ddot{\mathsf{G}}_{\lambda}(f)$.

It turns out that (GCH) implies (AC). The following theorem was announced by Lindenbaum and Tarski in 1926 but the first written proof was published only in 1947 by Sierpiński [21].

Theorem 40.3 (Sierpiński). The Generalized Continuum Hypothesis implies the Axiom of Choice, i.e., (GCH) \Rightarrow (AC).

Theorem 40.3 will be derived from its local version.

Lemma 40.4. A set x can be well-ordered if every $n \in \{4, 5, 6\}$ the class

$$\{y \in \mathbf{U} : |\mathcal{P}^{\circ n}(x)| < |y| < |\mathcal{P}^{\circ (n+1)}(x)|\}$$

is empty.

Proof. If the set x is finite, then it can be well-ordered without any additional assumptions. So, assume that x is infinite and for every $i \in \{4,5,6\}$ every cardinality κ with $|\mathcal{P}^{\circ i}(x)| \leq \kappa \leq |\mathcal{P}^{\circ (i+1)}(x)|$ is equal either to $|\mathcal{P}^{\circ i}(x)|$ or to $|\mathcal{P}^{\circ (i+1)}(x)|$.

By Theorem 36.7, the successor cardinal x^+ of x is infinite and hence $\omega \leq x^+$. By the Hartogs–Sierpiński Theorem 37.1, $|x^+| \leq |\mathfrak{P}^{\circ 3}(x)|$.

For every $n \in \omega$ consider the iterated power-set $p_n = \mathcal{P}^{\circ n}(x)$ of x. The inequality $|\omega| \le |x^+| \le |\mathcal{P}^{\circ 3}(x)| = |p_3|$ implies that $1 + |p_3| = |p_3|$.

Claim 40.5. For every natural number $n \ge 4$ we have $2 \cdot |p_n| = |p_n|$.

Proof. For n = 4 we have

$$2 \cdot |p_4| = 2 \cdot |2^{p_3}| = 2^{1+|p_3|} = 2^{|p_3|} = |p_4|.$$

Assume that for some $n \ge 4$ we proved that $2 \cdot |p_n| = |p_n|$. Then $|p_n| \le 1 + |p_n| \le |p_n| + |p_n| = 2 \cdot |p_n| = |p_n|$ and hence $1 + |p_n| = |p_n|$ by Theorem 35.6. Finally,

$$2 \cdot |p_{n+1}| = 2 \cdot 2^{|p_n|} = 2^{1+|p_n|} = 2^{|p_n|} = |p_{n+1}|.$$

Claim 40.6. For every natural number $n \ge 4$ and every ordinal α the inequality $|\alpha| + |p_n| = |p_{n+1}|$ implies $|p_{n+1}| \le |\alpha|$.

 $^{^{10}}$ In case you find an elementary proof of this fact, write for a prize to t.o.banakh@gmail.com.

Proof. By Claim 40.5,

$$|p_{n+1}| = 2^{|p_n|} = 2^{|p_n| + |p_n|} = 2^{|p_n|} \cdot 2^{|p_n|} = |p_{n+1}| \cdot |p_{n+1}|$$

and hence $|\alpha| + |p_n| = |p_{n+1}| \cdot |p_{n+1}|$. By Lemma 38.9, either $|p_{n+1}| \leq |p_n|$ or $|p_{n+1}| \leq |\alpha|$. The first case is excluded by Cantor's Theorem 35.10. Therefore $|p_{n+1}| \leq |\alpha|$.

By Hartogs-Sierpiński Theorem 37.1, the successor cardinal $\alpha = p_4^+$ of the set p_4 has cardinality $|\alpha| \leq |\mathcal{P}^{\circ 3}(p_4)| = |p_7|$.

Then $|p_6| \leq |\alpha| + |p_6| \leq 2 \cdot |p_7| = |p_7|$. By our assumption, either $|\alpha| + |p_6| = |p_7|$ or $|\alpha| + |p_6| = |p_6|$. In the first case we can apply Claim 40.6 and conclude that $|x| \leq |p_7| \leq |\alpha|$, which implies that x admits an injective function into the cardinal $\alpha = p_4^+$ and hence x can be well-ordered.

So, consider the second case $|\alpha| + |p_6| = |p_6|$. In this case $|p_5| \le |\alpha| + |p_5| \le |p_6| + |p_6| = |p_6|$ and by our assumption, the cardinality $|\alpha| + |p_5|$ is equal either to $|p_6|$ or to $|p_5|$. If $|\alpha| + |p_5| = |p_6|$, then by Claim 40.6, $|x| \le |p_6| \le |\alpha|$ and hence x can be well-ordered.

It remains to consider the case $|\alpha| + |p_5| = |p_5|$. Then $|p_4| \le |\alpha| + |p_4| \le |p_5| + |p_5| = |p_5|$ and by our assumption, either $|\alpha| + |p_4| = |p_5|$ or $|\alpha| + |p_4| = |p_4|$. In fact, the latter case is not possible as $|\alpha| = |p_4^+| \le |p_4|$. So, $|\alpha| + |p_4| = |p_5|$ and by Claim 40.6, $|x| \le |p_5| \le |\alpha|$ and x can be well-ordered.

Under (GCH) the exponentiation of cardinals can be desribed by a simple formula, presented in the following theorem.

Theorem 40.7. Assume (GCH). Let κ, λ be infinite cardinals.

- 1) If $\kappa \leq \lambda$, then $|\kappa^{\lambda}| = |\lambda^{+}|$.
- 2) If $cf(\kappa) \leq \lambda < \kappa$, then $|\kappa^{\lambda}| = |\kappa^{+}|$.
- 3) If $\lambda < \operatorname{cf}(\kappa)$, then $|\kappa^{\lambda}| = |\kappa|$.

Proof. 1. If $\kappa \leq \lambda$, then by (GCH),

$$|\lambda^{+}| = |2^{\lambda}| \le |\kappa^{\lambda}| \le |\lambda^{\lambda}| \le |(2^{\lambda})^{\lambda}| = |2^{\lambda \times \lambda}| = |2^{\lambda}| = |\lambda^{+}|.$$

2. If $cf(\kappa) \leq \lambda < \kappa$, then by Corollary 39.8,

$$|\kappa| < |\kappa^{\mathrm{cf}(\kappa)}| \le |\kappa^{\lambda}| \le |(2^{\kappa})^{\lambda}| = 2^{|\kappa \times \lambda|} = 2^{|\kappa|} = |\kappa^{+}|$$

and hence $|\kappa^{\lambda}| = |\kappa^{+}|$.

3. Finally, assume that $\lambda < \operatorname{cf}(\kappa)$. Observe that for any cardinal $\mu < \kappa$ and the cardinal $\nu = \max\{\mu, \lambda\} < \kappa$ we have $|\mu^{\lambda}| \leq |\nu^{\nu}| \leq |(2^{\nu})^{\nu}| = 2^{|\nu \times \nu|} = |2^{\nu}| = |\nu^{+}| \leq \kappa$. The strict inequality $\lambda < \operatorname{cf}(\kappa)$ implies that $\kappa^{\lambda} = \bigcup_{\alpha \in \kappa} \alpha^{\lambda}$ and hence

$$|\kappa| \le |\kappa^{\lambda}| \le |\kappa| \cdot \sup_{\alpha \in \kappa} |\alpha^{\lambda}| \le |\kappa| \cdot |\kappa| = |\kappa|.$$

41. INACCESSIBLE AND MEASURABLE CARDINALS

Definition 41.1. An uncountable cardinal κ is called

- weakly inaccessible if κ is a regular and $|\lambda^+| < |\kappa|$ for every cardinal $\lambda < \kappa$;
- strongly inaccessible if κ is regular and $|2^{<\kappa}| = |\kappa|$.

Remark 41.2. Under (GCH) a cardinal is weakly inaccessible if and only if it is strongly inaccessible. The existence of weakly inaccessible or strongly inaccessible cardinals can not be proved within the axioms ZFC since for the smallest strongly inaccessible cardinal κ the set V_{κ} and its elements is a model of ZFC (in which strongly inaccessible cardinals do not exist).

Inaccessible cardinals are examples of large cardinals, i.e., cardinals that are so large that their existence cannot be derived from the axioms of NBG or ZFC. Important examples of large cardinals are measurable cardinals, defined with the help of 2-valued measures.

Definition 41.3. A function $\mu: \mathcal{P}(x) \to \{0,1\}$ is called a 2-valued measure on a set x if

- (1) $\mu(x) = 1$;
- (2) for any disjoint subsets $a, b \subseteq x$ we have $\mu(a \cup b) = \mu(a) + \mu(b)$;
- (3) any finite subset $a \subseteq x$ has measure $\mu(a) = 0$.

Exercise 41.4. Show that for any 2-valued measure $\mu : \mathcal{P}(x) \to 2$ the family $U = \{a \in \mathcal{P}(x) : \mu(a) = 1\}$ is an ultrafilter with $\bigcup U = x$ and $\bigcap U = \emptyset$.

Exercise 41.5. Show that for any ultrafilter U with $\bigcap U = \emptyset$, the function $\mu : \mathcal{P}(\bigcup U) \to 2$ such that $\mu^{-1}[\{1\}] = U$ is a 2-valued measure on the set $\bigcup U$.

Exercise 41.6. Show that under the Axiom of Choice for every infinite set x there exists a 2-valued measure $\mu: \mathcal{P}(x) \to \{0,1\}$.

Definition 41.7. A 2-valued measure $\mu: \mathcal{P}(x) \to 2$ is called

- κ -additive if for any subset $y \subseteq \{a \in \mathcal{P}(x) : \mu(a) = 0\}$ of cardinality $|y| \le |\kappa|$ the union $\bigcup y$ has measure $\mu(\bigcup y) = 0$;
- $\kappa^{<}$ -additive if μ is λ -additive for every cardinal $\lambda < \kappa$.

The existence of a κ -additive 2-valued measure on a set x imposes the following restriction on the cardinality of x.

Lemma 41.8. Let κ be a cardinal. If a 2-valued measure $\mu : \mathcal{P}(x) \to 2$ on some set x is κ -additive, then for $|x| \not\leq |2^{\kappa}|$.

Proof. To derive a contradiction, assume that $|x| \leq |2^{\kappa}|$. Then there exists an injective function $f: x \to 2^{\kappa}$. For every $i \in \kappa$ consider the projection $\operatorname{pr}_i: 2^{\kappa} \to 2$, $\operatorname{pr}_i: g \mapsto g(i)$, onto the i-th coordinate. Next, for every $i \in \kappa$ and $k \in 2$, consider the set $a_{i,k} = \{y \in x : \operatorname{pr}_i \circ f(y) = k\}$. Since $x = a_{i,0} \cup a_{i,1}$ and $1 = \mu(x) = \mu(a_{i,0}) + \mu(a_{i,1})$ there exists a number $k_i \in 2$ such that $\mu(a_{i,k_i}) = 1$. By the κ -additivity of the measure μ , the intersection $a = \bigcap_{i \in \kappa} a_{i,k_i}$ has measure $\mu(a) = 1$. On the other hand, the injectivity of f implies that $|a| \leq 1$ and hence $\mu(a) = 0$.

Definition 41.9. An uncountable cardinal κ is defined to be *measurable* if it carries a $\kappa^{<}$ -additive 2-valued measure $\mu : \mathcal{P}(\kappa) \to 2$.

Theorem 41.10 (Tarski–Ulam). Under Axiom of Choice, each measurable cardinal is strongly inaccessible.

Proof. Let κ be a smallest measurable cardinal and $\mu: \mathcal{P}(\kappa) \to 2$ be a 2-valued measure which is λ -additive for every cardinal $\lambda < \kappa$. By Lemma 41.8 and Theorem 37.3, for every cardinal $\lambda < \kappa$ we have $|2^{\lambda}| < |\kappa|$. To show that κ is strongly inaccessible, it remains to prove that κ is regular. To derive a contradiction, assume that $\mathrm{cf}(\kappa) < \kappa$ and choose an unbounded

function $f: \operatorname{cf}(\kappa) \to \kappa$. Then $\kappa = \bigcup_{\alpha \in \operatorname{cf}(\kappa)} f(\alpha)$. Since κ is a cardinal, for every $\alpha \in \operatorname{cf}(\kappa)$ the ordinal $f(\alpha) \in \kappa$ has cardinality $|f(\alpha)| < |\kappa|$. The $|f(\alpha)|$ -additivity of the measure μ ensures that $\mu(f(\alpha)) = \bigcup_{\beta \in f(\alpha)} \mu(\{\beta\}) = 0$. Applying the $\operatorname{cf}(\kappa)$ -additivity of μ , we conclude that $1 = \mu(\kappa) = \mu(\bigcup_{\alpha \in \operatorname{cf}(\mu)} f(\alpha)) = 0$, which is a desired contradiction showing that the cardinal κ is regular and hence strongly inaccessible.

The cardinality of a set carrying an ω -additive 2-valued measure still is very large.

Proposition 41.11. The smallest cardinal κ carrying an ω -additive 2-valued measure is measurable and hence κ strongly inaccessible under the Axiom of Choice.

Proof. By our assumption, there exists a ω -additive 2-valued measure $\mu: \mathcal{P}(\kappa) \to 2$. To show that κ is measurable, it suffices to prove that the measure μ is $\kappa^{<}$ -additive. To derive a contradiction, assume that μ is not λ -additive for some cardinal $\lambda < \kappa$. Then there exists a family $(x_i)_{i \in \lambda}$ of sets of measure $\mu(x_i) = 0$ such that $\mu(\bigcup_{i \in \lambda} x_i) = 1$. Replacing each set x_i by $x_i \setminus \bigcup_{j \in i} x_j$, we can assume that the family $(x_i)_{i \in \lambda}$ consists of pairwise disjoint sets. Observe that the function $\nu: \mathcal{P}(\lambda) \to 2$ defined by $\nu(a) = \mu(\bigcup_{i \in a} x_i)$ is an ω -additive 2-valued measure on the cardinal $\lambda < \kappa$. But this contradicts the minimality of κ .

Exercise* **41.12.** Prove that for every measurable cardinal κ and function $h: \mathcal{P}_2(\kappa) \to 2$ on the set $\mathcal{P}_2(\kappa) = \{x \in \mathcal{P}(\kappa) : |x| = |2|\}$ there exists a subset $a \subseteq \kappa$ of cardinality $|a| = |\kappa|$ such that $|h[\mathcal{P}_2(a)]| = 1$.

Remark 41.13. It is consistent with ZF that the cardinal ω_1 is measurable, see [4, 12.2].

The existence of a measurable cardinal contradicts the Axiom of Constructibility. This non-trivial fact was discovered by D.S. Scott [20] in 1961.

Theorem 41.14 (Scott). If a measurable cardinal exists, then $U \neq L$.

The proof of Theorem 41.14 is not elementary and can be found in [5, 17.1].

Part 9. Linear orders

In this part we study linear orders. Linear orders often arise in mathematical practice. For example, the natural order on numbers (integer, rational or real) is a linear order, which fails to be a well-order.

42. Completeness

In this section we study complete and boundedly complete linear orders.

Definition 42.1. An order R is called *complete* if every subclass $A \subseteq dom[R^{\pm}]$ has $sup_R(A)$ and $inf_R(A)$.

We recall that $\sup_R(A)$ is the unique R-least element of the class $\{b \in \mathsf{dom}[R^{\pm}] : A \times \{b\} \subseteq R \cup \mathbf{Id}\}$ and $\inf_R(A)$ the unique R-greatest element of the class $\{b \in \mathsf{dom}[R^{\pm}] : \{b\} \times A \subseteq R\}$.

Proposition 42.2. For an order R the following conditions are equivalent:

- 1) R is complete;
- 2) each subclass $A \subseteq dom[R^{\pm}]$ has $sup_R(A)$;
- 3) each subclass $B \subseteq \text{dom}[R^{\pm}]$ has $\inf_{R}(B)$.

Proof. The implications $(1) \Rightarrow (2,3)$ are trivial.

 $(2) \Rightarrow (3)$ Assume that every subclass $A \subseteq \mathsf{dom}[R^{\pm}]$ has $\sup_R(A)$. Given any subclass $B \subseteq \mathsf{dom}[R^{\pm}]$, consider the subclass $A = \{a \in \mathsf{dom}[L^{\pm}] : \{a\} \times B \subseteq R \cup \mathbf{Id}\}$ of lower R-bounded of B in $\mathsf{dom}[R^{\pm}]$. By our assumption, the class A has $\sup_R(A)$, which is the R-least element of the class $\overline{A} = \{b \in \mathsf{dom}[L^{\pm}] : A \times \{b\} \subseteq R \cup \mathbf{Id}\}$ of upper R-bounds of A in $\mathsf{dom}[R^{\pm}]$. Since $B \subseteq \overline{A}$, the element $\sup_R(A)$ is a lower R-bound for B. On the other hand, each lower R-bound $b \in \mathsf{dom}[R^{\pm}]$ for B belongs to the set \overline{A} and hence $\langle b, \sup_R(A) \rangle \in R \cup \mathbf{Id}$ by the definition of $\sup_R(A)$. This means that $\sup_R(A) = \inf_R(B)$, so B has $\inf_R(B)$.

By analogy we can prove that $(3) \Rightarrow (2)$.

Definition 42.3. An order R is called boundedly complete if

- each upper R-bounded nonempty subclass $A \subseteq \mathsf{dom}[R^{\pm}]$ has $\sup_{R}(A)$, and
- each lower L-bounded nonempty subclass $A \subseteq \mathsf{dom}[R^{\pm}]$ has $\inf_{R}(A)$.

Proposition 42.4. For an order R the following conditions are equivalent:

- 1) R is boundedly complete;
- 2) each upper R-bounded nonempty subclass $A \subseteq \text{dom}[R^{\pm}]$ has $\sup_{R}(A)$;
- 3) each lower R-bounded nonempty subclass $B \subseteq \text{dom}[R^{\pm}]$ has $\inf_{R}(B)$.

Proof. The implications $(1) \Rightarrow (2,3)$ are trivial.

 $(2)\Rightarrow (3)$ Assume that every upper R-bounded nonempty subclass $A\subseteq \mathsf{dom}[R^\pm]$ has $\sup_R(A)$. Let B be a non-empty lower R-bounded subclass of $\mathsf{dom}[R^\pm]$. Then the subclass $A=\{a\in \mathsf{dom}[L^\pm]: \{a\}\times B\subseteq R\cup \mathbf{Id}\}$ of $\mathsf{dom}[R^\pm]$ is not empty and upper R-bounded. By our assumption, the class A has $\sup_R(A)$, which is the R-least element of the class $\overline{A}=\{b\in \mathsf{dom}[L^\pm]: A\times \{b\}\subseteq R\cup \mathbf{Id}\}$ of upper R-bounds of A in $\mathsf{dom}[R^\pm]$. Since $B\subseteq \overline{A}$, the element $\sup_R(A)$ is a lower R-bound for B. On the other hand, each lower R-bound $b\in \mathsf{dom}[R^\pm]$ for B belongs to the class \overline{A} and hence $\langle b, \sup_R(A) \rangle \in R \cup \mathbf{Id}$ by the definition of $\sup_R(A)$. This means that $\sup_R(A) = \inf_R(B)$, so B has $\inf_R(B)$.

By analogy we can prove that $(3) \Rightarrow (2)$.

By Exercise 17.15, any finite linear order is complete. The following theorem implies that completeness is preserved by lexicographic powers of linear orders.

Theorem 42.5. For any complete linear order L on a set $X = dom[L^{\pm}]$ and ordinal α the lexicographic order

$$L_{\alpha} = \{ \langle f, g \rangle \in X^{\alpha} \times X^{\alpha} : \exists \beta \in \alpha \ (f \upharpoonright_{\beta} = g \upharpoonright_{\beta} \ \land \ \langle f(\beta), g(\beta) \rangle \in L) \}$$

on the set X^{α} is complete.

Proof. This theorem will be proved by transfinite induction. Observe that the set X^0 is a singleton and the order $L_0 \subseteq X^0 \times X^0$ complete. Assume that for some ordinal α and all its elements $\beta \in \kappa$ we have proved that the order L_{β} is complete. To show that the order L_{α} is complete, take any subset $A \subseteq X^{\alpha}$. For every $\beta \in \alpha$, consider the projection $\operatorname{pr}_{\beta}: X^{\alpha} \to X^{\beta}$, $\operatorname{pr}_{\beta}: f \mapsto f \upharpoonright_{\beta}$.

By the induction hypothesis, for every $\beta \in \alpha$ the linear order L_{β} is complete and hence the set $\operatorname{pr}_{\beta}[A] \subseteq X^{\beta}$ has the smallest upper L_{β} -bound $s_{\beta} = \sup_{L_{\beta}}(\operatorname{pr}_{\beta}[A]) \in X^{\beta}$.

If α is a successor ordinal, then $\alpha = \beta + 1$ for some ordinal $\beta \in \alpha$. Consider the function $s_{\beta} = \sup_{L_{\beta}}(\operatorname{pr}_{\beta}[A])$ and the subset $A' = \{x \in X : s_{\beta} \cup \{\langle \beta, x \rangle\} \in A\}$. Since the order L is complete, the set A' has $\sup_{L}(A') \in X$. It can be shown that the function

$$s_{\alpha} = s_{\beta} \cup \{ \langle \beta, \sup_{L} (A') \rangle \}$$

is the required least upper bound $\sup_{L_{\alpha}}(A)$ on the set A in X^{α} .

If α is a limit ordinal, then we can show that the union $s_{\alpha} = \bigcup_{\beta \in \alpha} s_{\beta}$ is a function, which is the required least upper bound $\sup_{L_{\alpha}}(A)$ of the set A.

By analogy we can prove that A has the greatest lower L_{α} -bound $\inf_{L_{\alpha}}(A)$.

Corollary 42.6. For every ordinal α the lexicographic order

$$L = \{ \langle f, g \rangle \in 2^{\alpha} \times 2^{\alpha} : \exists \beta \in \alpha \ (f \upharpoonright_{\beta} = g \upharpoonright_{\beta} \ \land \ \langle f(\beta), g(\beta) \rangle \in C) \}$$

on 2^{α} is complete.

Exercise 42.7. Complete all omitted details in the proof of Theorem 42.5.

43. Universality

Definition 43.1. A linear order L is called *universal* if for any subsets a, b of $\mathsf{dom}[L^{\pm}]$ with $|a \cup b| < |\mathsf{dom}[L^{\pm}]|$ and $a \times b \subseteq L \setminus \mathbf{Id}$ there exists an elements $x \in \mathsf{dom}[L^{\pm}]$ such that $(a \times \{x\}) \cup (\{x\} \times b) \subseteq L \setminus \mathbf{Id}$.

Examples of universal orders can be constructed as follows.

Given a subclass $\kappa \subseteq \mathbf{On}$, consider the class $2^{<\kappa}$ of all functions f with $\mathsf{dom}[f] \in \kappa$ and $\mathsf{rng}[f] \subseteq 2 = \{0,1\}$, endowed with the linear order

$$\begin{split} \mathsf{U}_{2^{<\kappa}} = \{\langle f,g \rangle \in 2^{<\kappa} \times 2^{<\kappa} : \langle \mathsf{dom}[g],0 \rangle \in f \ \lor \ \langle \mathsf{dom}[f],1 \rangle \in g \ \lor \\ \exists \alpha \in \mathsf{dom}[f] \cap \mathsf{dom}[g] \ (f \upharpoonright_{\alpha} = g \upharpoonright_{\alpha} \ \land \ f(\alpha) = 0 \ \land \ g(\alpha) = 1) \}. \end{split}$$

Theorem 43.2. The linear order $\bigcup_{2 \le \kappa}$ is universal if $\kappa = \mathbf{On}$ or κ is a regular cardinal with $|\kappa| = |2^{\le \kappa}|$.

Proof. Assume that $\kappa = \mathbf{On}$ or κ is a regular cardinal with $|\kappa| = |2^{<\kappa}|$. Given any subsets $a, b \subseteq 2^{<\kappa}$ with $|a \cup b| < |\mathsf{dom}[L^{\pm}]|$ and $a \times b \subseteq \mathsf{U}_{2^{<\kappa}}$, we need to find an element $z \in 2^{<\kappa}$ such that $(a \times \{z\}) \cup (\{z\} \times b) \subseteq \mathsf{U}_{2^{<\kappa}}$. By the Axiom of Replacement, the set $\gamma = \bigcup \{\mathsf{dom}[f] : f \in a \cup b\} \subseteq \mathbf{On}$ is an ordinal.

We claim that $\gamma \subset \kappa$. This is clear if $\kappa = \mathbf{On}$. If κ is a regular cardinal with $|\kappa| = |2^{<\kappa}|$, then $|a \cup b| < |2^{<\kappa}| = |\kappa|$. In this case the set $2^{<\kappa}$ is well-orderable and so is the set $a \cup b$. It follows that the set $\Gamma = \{\mathsf{dom}[f] : f \in a \cup b\} \subseteq \kappa$ has cardinality $|\Gamma| \leq |a \cup b| < |\kappa|$, see Proposition 35.7(2). By the regularity of the cardinal κ , the union $\gamma = \bigcup \Gamma = \bigcup \{\mathsf{dom}[f] : f \in a \cup b\}$ is a proper subset of κ . Since κ is a limit ordinal, $\gamma \subset \kappa$ implies $\gamma + 1 \in \kappa$.

If $a = \emptyset$ (resp. $b = \emptyset$), then the constant function $z = (\gamma + 1) \times \{0\}$ (resp. $z = (\gamma + 1) \times \{1\}$) is an element of $2^{<\kappa}$ that has the required property: $(a \times \{z\}) \cup (\{z\} \times b) \subseteq U_{2^{<\kappa}}$.

So, we assume that the sets a,b are not empty. In this case, consider the set $u=\bigcup\{f\cap g:f\in a,\ g\in b\}$ and observe that it is a subset of $\gamma\times 2$. Assuming that the set u is not a function, we can find the smallest ordinal α such that $u\!\upharpoonright_{\alpha}$ is not a function. It is easy to see that α is a successor ordinal and hence $\alpha=\beta+1$ for some ordinal $\beta\in\alpha$. It follows that $u\!\upharpoonright\!\beta$ is a function but $u\!\upharpoonright\!\alpha$ is not a function. Then both pairs $\langle\beta,0\rangle$ and $\langle\beta,1\rangle$ belong to the set u and hence $\langle\beta,0\rangle\in f\cap g$ and $\langle\beta,1\rangle\in f'\cap g'$ for some functions $f,f'\in a$ and $g,g'\in b$. Then $\langle g,f'\rangle\in \mathsf{U}_{2^{<\kappa}}$ which contradicts $\langle f',g\rangle\in a\times b\subseteq \mathsf{U}_{2^{<\kappa}}$. This contradiction shows that u is a function and hence $u\in 2^{<\kappa}$.

Now three cases are possible.

1. There exist functions $f \in a$, $g \in b$ such that $\mathsf{dom}[u] \subset \mathsf{dom}[f] \cap \mathsf{dom}[g]$ and $u = f \upharpoonright \mathsf{dom}[u] = g \upharpoonright \mathsf{dom}[u]$. Taking into account that $f \cap g \subseteq u$ and $\langle f, g \rangle \in a \times b \subseteq \mathsf{U}_{2^{<\kappa}}$, we conclude that $f(\mathsf{dom}[u]) = 0$ and $g(\mathsf{dom}[u]) = 1$.

Consider the function $z \in 2^{\gamma+1} \subset 2^{<\kappa}$ such that

$$z(\alpha) = \begin{cases} f(\alpha) & \text{if } \alpha \leq \text{dom}[u]; \\ 1 & \text{if } \text{dom}[u] < \alpha \leq \gamma. \end{cases}$$

It can be shown that the function z has the required property: $(a \times \{z\}) \cup (\{z\} \times b) \subseteq U_{2 \le \kappa}$.

2. There exist no functions $f \in a$ such that $dom[u] \subset dom[f]$ and $u = f \upharpoonright_{dom[u]}$. In this case define the function $z \in 2^{\gamma+1} \subset 2^{<\kappa}$ by the formula

$$z(\alpha) = \begin{cases} u(\alpha) & \text{if } \alpha < \text{dom}[u]; \\ 0 & \text{if } \text{dom}[u] \le \alpha \le \gamma; \end{cases}$$

and prove that z has the required property: $(a \times \{z\}) \cup (\{z\} \times b) \subseteq \mathsf{U}_{2^{<\kappa}}$.

3. There exist no functions $g \in b$ such that $dom[u] \subseteq dom[g]$ and $u = g \upharpoonright_{dom[u]}$. In this case define the function $z \in 2^{\gamma+1} \subset 2^{<\kappa}$ by the formula

$$z(\alpha) = \begin{cases} u(\alpha) & \text{if } \alpha < \text{dom}[u]; \\ 1 & \text{if } \text{dom}[u] \leq \alpha \leq \gamma; \end{cases}$$

and prove that z has the required property: $(a \times \{z\}) \cup (\{z\} \times b) \subseteq \mathsf{U}_{2^{<\kappa}}$.

Remark 43.3. Under (GCH), every infinite cardinal κ satisfies the equality $|\kappa| = |2^{<\kappa}|$.

The following theorem explains why universal orders are called universal.

Theorem 43.4. Let U be a universal linear order whose underlying class $dom[U^{\pm}]$ is well-orderable. For a linear order L the following conditions are equivalent.

- 1) There exists an L-to-U-increasing function $dom[L^{\pm}] \rightarrow dom[U^{\pm}]$.
- 2) There exists an injective function $dom[L^{\pm}] \rightarrow dom[U^{\pm}]$.

Proof. The implication $(1) \Rightarrow (2)$ is trivial. To prove that $(2) \Rightarrow (1)$, assume that there exists an injective function $J: \mathsf{dom}[L^{\pm}] \to \mathsf{dom}[U^{\pm}]$. The well-orderability of the class $\mathsf{dom}[U^{\pm}]$ and the injectivity of the function J imply that the class $dom[L^{\pm}]$ is well-orderable. If $dom[L^{\pm}]$ is a proper class, then put $\kappa_1 = \mathbf{On}$. If $\mathsf{dom}[L^{\pm}]$ is a set, then let κ_1 be the unique cardinal such that $|\kappa_1| = |\mathsf{dom}[L^{\pm}]|$ (the cardinal κ_1 exists since the set $\mathsf{dom}[L^{\pm}]$ is well-orderable). Using Theorem 23.6, find a bijective function $N_1 : \kappa_1 \to \mathsf{dom}[L^{\pm}]$.

Now do the same with the order U. If $dom[U^{\pm}]$ is a proper class, then put $\kappa_2 = \mathbf{On}$ and if $dom[U^{\pm}]$ is a set, then let κ_2 be the unique cardinal such that $|\kappa_2| = |dom[U^{\pm}]$. Using Theorem 23.6, find a bijective function $N_2 : \kappa_2 \to \mathsf{dom}[U^{\pm}]$.

Let I be the class whose elements are injective functions $\varphi \subseteq \mathsf{dom}[L^{\pm}] \times \mathsf{dom}[U^{\pm}]$ such that $|\varphi| < |\kappa_1|$ and φ is an isomorphism of the linear orders $L \upharpoonright \mathsf{dom}[\varphi]$ and $U \upharpoonright \mathsf{rng}[\varphi]$.

Let $\Phi: \kappa_1 \times I \to I$ be the function assigning to each ordered pair $\langle \alpha, \varphi \rangle \in \kappa_1 \times I$ the function $\Phi(\alpha,\varphi) \in I$ defined as follows. If $N_1(\alpha) \in \mathsf{dom}[\varphi]$, then $\Phi(\alpha,\varphi) = \varphi$. If $N_1(\alpha) \notin \mathsf{dom}[\varphi]$ then let $\beta(\alpha, \varphi)$ be the smallest ordinal $\beta \in \kappa_2$ such that the function $\varphi \cup \{\langle N_1(\alpha), N_2(\beta) \rangle\}$ is an element of the class I. Let us show that the ordinal $\beta(\alpha, \varphi)$ exists. Consider the sets $a = \{x \in \mathsf{dom}[\varphi] : \langle x, N_1(\alpha) \rangle \in L\}$ and $b = \{y \in \mathsf{dom}[\varphi] : \langle N_1(\alpha), y \rangle \in L\}$ and observe that $a \times b \subseteq L$. Taking into account that the function φ is an isomorphism of the orders $L \upharpoonright \mathsf{dom}[\varphi]$ and $U \upharpoonright \mathsf{rng}[\varphi]$, we conclude that $\varphi[a] \times \varphi[b] \subseteq U \setminus \mathsf{Id}$. It follows that $|\varphi[a] \cup \varphi[b]| = |a \cup b| < |\kappa_1| \le |\kappa_2| \le |U|$. By the universality of the linear order U, there exists an element $z \in \mathsf{dom}[U^{\pm}]$ such that $(\varphi[a] \times \{z\}) \cup (\{z\} \times \varphi[b]) \subseteq U \setminus \mathbf{Id}$. Then the function $\varphi \cup \{\langle N_1(\alpha+1), z \rangle\}$ is an element of the class I. Since the function $N_2 : \kappa_2 \to \mathsf{dom}[U^{\pm}]$ is surjective, $z = N_2(\beta)$ for some ordinal $\beta \in \kappa_2$. Then $\varphi \cup \{\langle N_1(\alpha), N_2(\beta) \rangle\} \in I$, which completes the proof of the existence of the ordinal $\beta(\alpha, \varphi)$.

Then put $\Phi(\alpha, \varphi) = \varphi \cup \{\langle N_1(\alpha), N_2(\beta(\alpha, \varphi)) \rangle\}$. Observe that the function $\Phi(\alpha, \varphi)$ has the properties: $\varphi \subseteq \Phi(\alpha, \varphi) \in I$ and $N_1(\alpha) \in \text{dom}[\Phi(\alpha, \varphi)]$.

Finally, consider the function $F: \kappa_1 \times \mathbf{U} \to \mathbf{U}$ assigning to every ordered pair $\langle \alpha, x \rangle \in \kappa_1 \times \mathbf{U}$ the set

$$F(\alpha, x) = \begin{cases} \Phi(\alpha, \bigcup x) & \text{if } \bigcup x \in I; \\ \emptyset & \text{otherwise.} \end{cases}$$

By Recursion Theorem 21.1, there exists a transfinite sequence $(\varphi_{\alpha})_{\alpha \in \kappa_1}$ such that $\varphi_0 =$ \emptyset and $\varphi_{\alpha} = F(\alpha, \{\varphi_{\beta}\}_{{\beta} \in \alpha})$ for every ordinal $\alpha \in \kappa_1$. Using the Principle of Transfinite Induction, it can be proved that for every ordinal $\alpha \in \kappa_1$ the following conditions are satisfied:

- $\varphi_{\alpha} \in I$;
- $\forall \beta \in \alpha \ (\varphi_{\beta} \subseteq \varphi_{\alpha});$ $\varphi_{\alpha} = F(\alpha, \{\varphi_{\beta}\}_{\beta \in \alpha+1}) = \Phi(\alpha, \bigcup_{\beta \in \alpha} \varphi_{\beta});$
- $N_1(\alpha) \in \text{dom}[\varphi_{\alpha}].$

Then $\varphi = \bigcup_{\alpha \in \kappa_1} \varphi_{\alpha}$ is a required L-to-U-increasing function from $dom[L^{\pm}]$ to $dom[U^{\pm}]$.

Theorem 43.5. Let U be a universal linear order whose underlying class $dom[U^{\pm}]$ is wellorderable. For a linear order L the following conditions are equivalent:

- 1) there exists an isomorphism $dom[L^{\pm}] \rightarrow dom[U^{\pm}]$ of the orders L, U;
- 2) there exists a bijective function $F: dom[L^{\pm}] \to dom[U^{\pm}]$ and the order L is universal.

Proof. The implication $(1) \Rightarrow (2)$ is trivial. To prove that $(2) \Rightarrow (1)$, assume that there exists a bijective function $J: \mathsf{dom}[L^{\pm}] \to \mathsf{dom}[U^{\pm}]$ and the order L is universal. If $\mathsf{dom}[U^{\pm}]$ is a set, then let κ be a unique cardinal such that $|\kappa| = |\mathsf{dom}[U^{\pm}]| = |\mathsf{dom}[L^{\pm}]|$. If $\mathsf{dom}[U^{\pm}]$ is a proper class, then put $\kappa = \mathbf{On}$. Using Theorem 23.6, find bijective functions $N_1 : \kappa \to \mathsf{dom}[L^{\pm}]$ and $N_2: \kappa \to \mathsf{dom}[U^{\pm}].$

Let I be the class whose elements are injective functions $\varphi \subseteq \mathsf{dom}[L^{\pm}] \times \mathsf{dom}[U^{\pm}]$ such that $|\varphi| < |\kappa|$ and φ is an isomorphism of the linear orders $L \upharpoonright \mathsf{dom}[\varphi]$ and $U \upharpoonright \mathsf{rng}[\varphi]$.

Let $\Phi: \kappa \times I \to I$ be the function assigning to each ordered pair $\langle \alpha, \varphi \rangle \in \kappa_1 \times I$ the function $\Phi(\alpha,\varphi) \in I$ defined as follows. If $N_1(\alpha) \in \mathsf{dom}[\varphi]$, then $\Phi(\alpha,\varphi) = \varphi$. If $N_1(\alpha) \notin \mathsf{dom}[\varphi]$ then let $\beta(\alpha, \varphi)$ be the smallest ordinal $\beta \in \kappa$ such that the function $\varphi \cup \{\langle N_1(\alpha), N_2(\beta) \rangle\}$ is an element of the class I. Repeating the argument from the proof of Theorem 43.4, we can show that the ordinal $\beta(\alpha, \varphi)$ is well-defined.

By analogy define a function $\Psi: \kappa \times I \to I$ such that $\varphi \subseteq \Psi(\alpha, \varphi) \in I$ and $N_2(\alpha) \in I$ $\operatorname{rng}[\Psi(\alpha,\varphi)]$ for any $\langle \alpha,\varphi\rangle \in \kappa \times I$.

Finally, consider the function $F: \mathbf{On} \times \mathbf{U} \to \mathbf{U}$ assigning to every ordered pair $\langle \alpha, x \rangle \in$ $\mathbf{On} \times \mathbf{U}$ the set

$$F(\alpha, x) = \begin{cases} \Psi(\alpha, \Phi(\alpha, \bigcup x)) & \text{if } \bigcup x \in I; \\ \emptyset & \text{otherwise.} \end{cases}$$

By Recursion Theorem 21.1, there exists a transfinite sequence $(\varphi_{\alpha})_{\alpha \in \kappa}$ such that $\varphi_0 = \emptyset$ and $\varphi_{\alpha} = F(\alpha, \{\varphi_{\beta}\}_{{\beta} \in \alpha})$ for every ordinal $\alpha \in \kappa$. By the Transfinite Induction it can be proved that for every ordinal $\alpha \in \kappa$ the following conditions are satisfied:

- $\varphi_{\alpha} \in I$;
- $\varphi_{\alpha} \in I$, $\forall \beta \in \alpha \ (\varphi_{\beta} \subseteq \varphi_{\alpha})$; $\varphi_{\alpha} = \Psi(\alpha, \Phi(\alpha, \bigcup_{\beta \in \alpha} \varphi_{\beta}))$;
- $N_1(\alpha) \in \text{dom}[\varphi_{\alpha}] \text{ and } N_2(\alpha) \in \text{rng}[\varphi_{\alpha}].$

Then $\varphi = \bigcup_{\alpha \in \kappa} \varphi_{\alpha}$ is a required isomorphism of the linear orders L and U.

Corollary 43.6. Under (GWO) every linear order L admits an L-to- $U_{2\leq On}$ -increasing func $tion f : dom[L] \rightarrow 2^{<On}$.

Exercise 43.7. Prove that a nonempty countable order L is universal if and only if it has

- 1) for any elements $x <_L y$ of $dom[L^{\pm}]$ there exists $z \in dom[L^{\pm}]$ such that $x <_L z <_L y$;
- 2) for any element $z \in \text{dom}[L^{\pm}]$ there are $x, y \in \text{dom}[L^{\pm}]$ such that $x <_L z <_L y$.

In this exercise we write $x <_L y$ instead of $\langle x, y \rangle \in L \setminus \mathbf{Id}$.

For countable orders, Theorems 43.4 and 43.5 have the following corollaries, proved by Georg Cantor.

Corollary 43.8 (Cantor). Any countable linear order L admits L-to- $\bigcup_{2 < \omega}$ -increasing function $f: \mathsf{dom}[L] \to 2^{<\omega}$.

Corollary 43.9 (Cantor). An order L is isomorphic to the universal linear order $U_{2<\omega}$ if and only if L is nonempty, countable, and has two properties:

- 1) for any elements $x <_L y$ of $dom[L^{\pm}]$ there exists $z \in dom[L^{\pm}]$ such that $x <_L z <_L y$; 2) for any element $z \in dom[L^{\pm}]$ there are $x, y \in dom[L^{\pm}]$ such that $x <_L z <_L y$.

44. Cuts

In this section we study cuts of linear orders.

Let L be a linear order. An ordered pair of sets $\langle a, b \rangle$ is called an L-cut if

$$a \cup b = \mathsf{dom}[L^{\pm}], \quad a \cap b = \emptyset \quad \text{and} \quad a \times b \subseteq L.$$

Let $\operatorname{Cut}(L)$ be the class of all L-cuts. For every L-cut $x = \langle a, b \rangle$ the sets a and b will be denoted by \overline{x} and \overline{x} , respectively.

In the following lemma, $(V_{\alpha})_{\alpha \in \mathbf{On}}$ is von Neumann's cumulative hierarchy, studied in Section 24.

Lemma 44.1. Let L be a linear order.

- 1) The class Cut(L) exists and is a set.
- 2) $Cut(L) \neq \emptyset$ if and only if L is a set.
- 3) If $L \subseteq V_{\alpha}$ for some ordinal α , then $Cut(L) \subseteq V_{\alpha+3}$ and $L \cap Cut(L) = \emptyset$.

Proof. The class $\operatorname{Cut}(L)$ exists by Theorem 7.2. If L is a proper class, then $\operatorname{dom}[L^{\pm}]$ is a proper class and then $\operatorname{Cut}(L)$ is the empty set (since the proper class $\operatorname{dom}[L^{\pm}]$ cannot be represented as the union of two sets). If L is a set, then the class $\operatorname{Cut}(L)$ is a set, being a subclass of the set $\mathcal{P}(\operatorname{dom}[L^{\pm}]) \times \mathcal{P}(\operatorname{dom}[L^{\pm}])$. The set $\operatorname{Cut}(L)$ contains the L-cuts $\langle L, \emptyset \rangle$ and $\langle \emptyset, L \rangle$, witnessing that $\operatorname{Cut}(L) \neq \emptyset$.

If $\mathsf{Cut}(L)$ is a nonempty set, then for any ordered pair $\langle a,b\rangle\in\mathsf{Cut}(L)$, the union $a\cup b=\mathsf{dom}[L^\pm]$ is a set and so is the linear order $L\subseteq\mathsf{dom}[L^\pm]\times\mathsf{dom}[L^\pm]$.

If $L \subseteq V_{\alpha}$ for some ordinal, then $\mathsf{dom}[L^{\pm}] = \mathsf{dom}[L] \cup \mathsf{rng}[L] \subseteq \bigcup \bigcup L \subseteq V_{\alpha}$ by the transitivity of the set V_{α} , see Theorem 24.2(3). For every ordered pair $\langle a,b \rangle \in \mathsf{Cut}(L)$ we have $a,b \subseteq \mathsf{dom}[L^{\pm}] \subseteq V_{\alpha}$ and hence $a,b \in \mathcal{P}(V_{\alpha}) = V_{\alpha+1}$ and $\langle a,b \rangle \in V_{\alpha+3}$. Therefore, $\mathsf{Cut}(L) \subseteq V_{\alpha+3}$ and $\mathsf{Cut}(L) \in V_{\alpha+4}$.

Assuming that $L \cap \mathsf{Cut}(L) \neq \emptyset$, we would find an ordered pair $\langle a, b \rangle \in \mathsf{Cut}(L) \cap L$ and conclude that $a, b \in \mathsf{dom}[L^{\pm}] = a \cup b$. Taking into account that $a, b \in \mathbf{V}$ and the relation $\mathbf{E} \upharpoonright \mathbf{V}$ is well-founded (see Theorem 24.2(5)), we conclude that $a \notin a$ and $b \notin b$. Then $a, b \in a \cup b$ implies $a \in b \in a$, which contradicts the well-foundedness of the relation $\mathbf{E} \upharpoonright \mathbf{V}$. This contradiction shows that $L \cap \mathsf{Cut}(L) = \emptyset$.

The cut extension of a linear order L is the relation

$$\Xi(L) = L \cup \{ \langle \langle a, b \rangle, \langle a', b' \rangle \rangle \in \operatorname{Cut}(L) \times \operatorname{Cut}(L) : a \subseteq a' \} \cup \{ \langle x, \langle a, b \rangle \rangle \in \operatorname{dom}[L^{\pm}] \times \operatorname{Cut}(L) : x \in a \} \cup \{ \langle \langle a, b \rangle, y \rangle \in \operatorname{Cut}(L) \times \operatorname{dom}[L^{\pm}] : y \in b \}$$

Theorem 44.2. For any linear order $L \subseteq \mathbf{V}$ its cut extension $\Xi[L]$ has the following properties:

- 1) $L \subset \Xi(L) \subset \mathbf{V}$.
- 2) $\operatorname{\mathsf{dom}}[\Xi(L)^{\pm}] = \operatorname{\mathsf{dom}}[L^{\pm}] \cup \operatorname{\mathsf{Cut}}(L).$
- 3) If L is a set, then $L \neq \Xi(L)$.
- 4) The relation $\Xi(L)$ is transitive.
- 5) The relation $\Xi(L)$ is antisymmetric.
- 6) The relation $\Xi(L)$ is a linear order.
- 7) If L is reflexive, then $\Xi(L)$ is a reflexive linear order.

Proof. The first three statements follow from the definition of the relation $\Xi(L)$ and Lemma 44.1.

- 4. To prove that the relation $\Xi(L)$ is transitive, fix any elements $x, y, z \in \mathsf{dom}[\Xi(L)] = \mathsf{dom}[L^{\pm}] \cup \mathsf{Cut}(L)$ with $\langle x, y \rangle, \langle y, z \rangle \in \Xi(L)$. We need to check that $\langle x, z \rangle \in \Xi(L)$. Since $y \in \mathsf{dom}[L^{\pm}] \cup \mathsf{Cut}(L)$, two cases are possible.
 - 4a. First we assume that $y \in dom[L^{\pm}]$. This case has four subcases.
- 4a1. If $x \in dom[L^{\pm}]$ and $z \in dom[L^{\pm}]$, then $\langle x, z \rangle \in L \subseteq \Xi(L)$ by the transitivity of the linear order L.
- 4a2. If $x \in \mathsf{dom}[L^{\pm}]$ and $z \in \mathsf{Cut}(L)$, then $z = \langle a, b \rangle$ for some L-cut $\langle a, b \rangle$ such that $y \in a$ (the latter inclusion follows from $\langle y, z \rangle \in \Xi(L)$). We claim that $x \in a$, too. In the opposite case $x \in b$ and then $\langle y, x \rangle \in a \times b \subseteq L$ and $x = y \in a$ by the antisymmetricity of L. But the inclusion $x \in a$ contradicts our assumption. This contradiction shows that $x \in a$ and hence $\langle x, z \rangle = \langle x, \langle a, b \rangle \rangle \in \Xi(L)$.
- 4a3. If $x \in \operatorname{Cut}(L)$ and $z \in \operatorname{dom}[L^{\pm}]$, then $x = \langle a, b \rangle$ for some L-cut $\langle a, b \rangle$ with $y \in b$ (the latter inclusion follows from $\langle x, y \rangle \in \Xi(L)$). We claim that $z \in b$. In the opposite case $z \in a$ and then $\langle z, y \rangle \in a \times b \subseteq L$. On the other hand, $\langle y, z \rangle \in \Xi(L) \cap (\operatorname{dom}[L^{\pm}] \times \operatorname{dom}[L^{\pm}]) = L$ and the antisymetricity of L imply that $z = y \in b$, which contradict our assumption. This contradiction shows that $z \in b$ and hence $\langle x, z \rangle = \langle \langle a, b \rangle, z \rangle \in \Xi(L)$.
- 4a4. If $x \in \text{Cut}(L)$ and $z \in \text{Cut}(L)$, then $x = \langle a, b \rangle$ and $z = \langle a', b' \rangle$ for some L-cuts $\langle a, b \rangle$ and $\langle a', b' \rangle$. It follows from $\langle x, y \rangle \in \Xi(L)$ and $\langle y, z \rangle \in \Xi(L)$ that $y \in b$ and $y \in a'$. To show that $\langle x, z \rangle \in \Xi(L)$ we need to check that $a \subseteq a'$. Given any element $\alpha \in a$, observe that $\langle \alpha, y \rangle \in a \times b \subseteq L$. Assuming that $\alpha \notin a'$, we conclude that $\alpha \in b'$ and hence $\langle y, \alpha \rangle \in a' \times b' \subseteq L$. The antisymmetry of L implies $\alpha = y \in a'$, which contradicts our assumption. This contradiction shows that $\alpha \in a'$ and hence $a \subseteq a'$ and finally $\langle x, z \rangle = \langle \langle a, b \rangle, \langle a', b' \rangle \rangle \in \Xi(L)$.

Now consider the second case.

- 4b. $y \in Cut(L)$. In this case $y = \langle a, b \rangle$ for some L-cut. This case also has four subcases.
- 4b1. $x, z \in \mathsf{dom}[L^{\pm}]$. In this subcase, $\langle x, y \rangle \in \Xi(L)$ and $\langle y, z \rangle \in \Xi(L)$ imply that $x \in a$ and $z \in b$. Then $\langle x, z \rangle \in a \times b \subseteq L$.
- 4b2. $x \in \mathsf{dom}[L^{\pm}]$ and $z \in \mathsf{Cut}(L)$. In this subcase $x \in a$ and $z = \langle a', b' \rangle$ for some L-cut $\langle a', b' \rangle$ such that $a \subseteq a'$ (the latter embedding follows from $\langle y, z \rangle \in \Xi(L)$). Then $x \in a \subseteq a'$ implies $\langle x, z \rangle = \langle x, \langle a', b' \rangle \rangle \in \Xi(L)$.
- 4b3. $x \in \operatorname{Cut}(L)$ and $z \in \operatorname{dom}[L^{\pm}]$. In this subcase $x = \langle a', b' \rangle$ for some L-cut $\langle a', b' \rangle$. It follows from $\langle x, y \rangle \in \Xi(L)$ and $\langle y, z \rangle \in \Xi(L)$ that $a' \subseteq a$ and $z \in b$. Taking into account that $a' \subseteq a$ and $a \cup b = a' \cup b' = \operatorname{dom}[L^{\pm}]$, we conclude that $z \in b \subseteq b'$ and hence $\langle x, z \rangle = \langle \langle a', b' \rangle, z \rangle \in \Xi(L)$.
- 4b4. $x, z \in Cut(L)$. In this subcase $x = \langle a', b' \rangle$ and $y = \langle a'', b'' \rangle$ for some L-cuts $\langle a', b' \rangle$ and $\langle a'', b'' \rangle$. It follows from $\langle x, y \rangle \in \Xi(L)$ and $\langle y, z \rangle \in \Xi(L)$ that $a' \subseteq a \subseteq a''$ and hence $a' \subseteq a''$, which means that $\langle x, z \rangle = \langle \langle a', b' \rangle, \langle a'', b'' \rangle \rangle \in \Xi(L)$.
- Therefore, we have considered all 8 cases and thus proved the transitivity of the relation $\Xi(L)$.
- 5. To show that the relation $\Xi(L)$ is antisymmetric, take any elements $x, y \in \mathsf{dom}[\Xi(L)^{\pm}] = \mathsf{dom}[L^{\pm}] \cup \mathsf{Cut}(L)$ and assume that $\langle x, y \rangle \in \Xi(L)$ and $\langle y, x \rangle \in \Xi(L)$. By Lemma 44.1(3), the sets $\mathsf{dom}[L^{\pm}]$ and $\mathsf{Cut}(L)$ are disjoint. Now we consider four possible cases.
- 5a. If $x, y \in \text{dom}[L^{\pm}]$ then $\langle x, y \rangle, \langle y, z \rangle \in \Xi(L) \cap (\text{dom}[L^{\pm}] \times \text{dom}[L^{\pm}]) = L$ and x = y by the antisymmetricity of L.
- 5b. If $x \in \mathsf{dom}[L^{\pm}]$ and $y \in \mathsf{Cut}(L)$, then $y = \langle a, b \rangle$ for some L-cut $\langle a, b \rangle$ and the inclusions $\langle x, y \rangle, \langle y, x \rangle \in \Xi(L)$ imply $x \in a$ and $x \in b$ which is not possible as $a \cap b = \emptyset$.

- 5c. By analogy we can show that the case $x \in Cut(L)$, $y \in dom[L^{\pm}]$ is incompatible with $\langle x, y \rangle, \langle y, x \rangle \in \Xi(L)$.
- 5d. If $x, y \in \mathsf{Cut}(L)$, then $x = \langle a, b \rangle$ and $y = \langle a', b' \rangle$ for some L-cuts $\langle a, b \rangle$ and $\langle a', b' \rangle$. The inclusions $\langle x, y \rangle, \langle y, x \rangle \in \Xi(L)$ imply $a \subseteq a' \subseteq a$ and hence a = a' and $b = \mathsf{dom}[L^{\pm}] \setminus a = \mathsf{dom}[L^{\pm}] \setminus a' = b'$, which implies $x = \langle a, b \rangle = \langle a', b' \rangle = y$.
- 6. The statements 4,5 imply that the relation $\Xi(L)$ is a partial order. To show that it is a linear order, we should prove that for any $x, y \in \mathsf{dom}[\Xi(L)^{\pm}] = \mathsf{dom}[L^{\pm}] \cup \mathsf{Cut}(L)$ we have $\langle x, y \rangle \in \Xi(L)^{\pm} \cup \mathbf{Id}$. Four cases are possible.
 - 6a. If $x, y \in \text{dom}[\Xi(L)^{\pm}]$, then $\langle x, y \rangle \in L^{\pm} \cup \text{Id}$ since L is a linear order.
- 6b. If $x \in \mathsf{dom}[\Xi(L)^{\pm}]$ and $y \in \mathsf{Cut}(L)$, then $y = \langle a, b \rangle$ for some L-cut $\langle a, b \rangle$. Since $x \in \mathsf{dom}[L^{\pm}] = a \cup b$, either $x \in a$ and then $\langle x, y \rangle \in \Xi(L)$ or $x \in b$ and then $\langle y, x \rangle \in \Xi(L)$.
 - 6c. By analogy with (6b) we can treat the case $x \in Cut(L)$ and $y \in dom[L^{\pm}]$.
- 6d. If $x,y \in \operatorname{Cut}(L)$, then $x = \langle a,b \rangle$ and $y = \langle a',b' \rangle$ for some L-cuts $\langle a,b \rangle$ and $\langle a',b' \rangle$. We claim that either $a \subseteq a'$ or $a' \subseteq a$. In the opposite case we can find elements $x \in a \setminus a'$ and $x' \in a' \setminus a$. It follows from $a \cup b = \operatorname{dom}[L^{\pm}] = a' \cup b'$ that $x \in b'$ and $x' \in b$. Then $\langle x,x' \rangle \in a \times b \subseteq L$ and $\langle x',x \rangle \in a' \times b' \subseteq L$. The antisymmetricity of L ensures that $x = x' \in a \cap b = \emptyset$ which is a desired contradiction showing that $a \subseteq a'$ or $a' \subseteq a$ and hence $\langle x,y \rangle \in \Xi(L)$ or $\langle y,x \rangle \in \Xi(L)$.
- 7. If the linear order L is reflexive, then the linear order $\Xi(L)$ is reflexive by the definition of $\Xi(L)$.

45. Gaps

Let L be a linear order. An L-cut $\langle a, b \rangle$ is called an L-gap if the sets a, b are not empty and for any $x \in a$ and $y \in b$ there are elements $x' \in a$ and $y' \in b$ such that $a <_L a'$ and $b' <_L b$. By $\mathsf{Gap}(L)$ we denote the set of L-gaps. It is a subset of the set $\mathsf{Cut}(L)$ of L-cuts.

The linear order

$$\Theta(L) = \{ \langle x, y \rangle \in \Xi(L) : x, y \in \mathsf{dom}[L^{\pm}] \cup \mathsf{Gap}(L) \}$$

is called the *gap extension* of the linear order.

Theorem 45.1. For any linear order $L \in \mathbf{V}$ its gap extension has the following properties:

- 1) $L \subseteq \Theta(L) \subseteq \Xi(L) \in \mathbf{V}$.
- 2) $\operatorname{\mathsf{dom}}[\Theta(L)^{\pm}] = \operatorname{\mathsf{dom}}[L^{\pm}] \cup \operatorname{\mathsf{Gap}}(L).$
- 3) The relation $\Theta(L)$ is a linear order.
- 4) If L is reflexive, then $\Theta(L)$ is a reflexive linear order.
- 5) $\mathsf{Gap}(\Theta(L)) = \emptyset$ and hence $\Theta(\Theta(L)) = \Theta(L)$.

Proof. The first four statements follow from Lemma 44.1 and Theorem 44.2. It remains to prove that the gap extension $\Theta(L)$ of any linear order $L \in \mathbf{V}$ has no $\Theta(L)$ -gaps. Let $X = \mathsf{dom}[L^{\pm}]$ be the underlying set of the linear order L.

To derive a contradiction, assume that the linear order $\Theta(L)$ has a $\Theta(L)$ -gap $\langle A, B \rangle$. Consider the sets $a = A \cap X$ and $b = B \cap X$ and observe that $a \cap b \subseteq A \cap B = \emptyset$, $a \cup b = X \cap (A \cup B) = X$ and $a \times b = (A \times B) \cap (X \times X) \subseteq \Theta(L) \cap (X \times X) = L$ by Lemma 44.1. Therefore, $\langle a, b \rangle$ is an L-cut. We claim that $\langle a, b \rangle$ is an L-gap.

First we prove that the sets a, b are not empty. To derive a contradiction, assume that the set $a = A \cap X$ is empty. Since the left set $A \subseteq X \cup \mathsf{Gap}(L)$ of the $\Theta(L)$ -gap $\langle A, B \rangle$ is not empty, it contains some L-gap $\langle u, v \rangle$, whose left side u is a non-empty subset of X. Then for

every $x \in u$ we have $\langle x, \langle u, v \rangle \rangle \in \Theta(L)$. On the other hand, $x \in u \subseteq a \cup b = \emptyset \cup b = b \subseteq B$ and hence $\langle \langle u, v \rangle, x \rangle \in A \times B \subseteq \Theta(L)$, which implies $x = \langle u, v \rangle$ by the antisymmetricity of the linear order $\Theta(L)$. Then $x = \langle u, v \rangle \in X \cap \mathsf{Gap}(L) = \emptyset$, which is a contradiction showing that $a \neq \emptyset$. By analogy we can prove that $b \neq \emptyset$.

To show that $\langle a,b \rangle$ is an L-gap, fix any elements $x \in a$ and $y \in b$. We need to find elements $x' \in a$ and $y' \in b$ such that $x <_L x'$ and $y' <_L y$. Since $\langle A,B \rangle$ is a $\Theta(L)$ -gap, for the elements $x \in a \subseteq A$ and $y \in b \subseteq B$ there are elements $x'' \in A$ and $y'' \in B$ such that $\langle x,x'' \rangle \in \Theta(L) \setminus \mathbf{Id}$ and $\langle y'',y \rangle \in \Theta(L) \setminus \mathbf{Id}$. If $x'' \in X$, then $x' = x'' \in A \cap X = a$ is a required element of a with $\langle x,x' \rangle \in (X \times X) \cap \Theta(L) \setminus \mathbf{Id} = L \setminus \mathbf{Id}$. If $x'' \notin X$, then $x'' = \langle u,v \rangle$ is an L-gap. It follows from $\langle x,\langle u,v \rangle \rangle = \langle x,x'' \rangle \in \Theta(L)$ that $x \in u$. Since $\langle u,v \rangle$ is an L-gap, there exists an element $x' \in u$ such that $x <_L x'$. Then $\langle x',x'' \rangle = \langle x',\langle u,v \rangle \rangle \in \Theta(L)$ and $x'' \in A$ imply $x' \in A \cap X = a$. By analogy we can prove that the set b contains an element y' such that $y' <_L y$.

Therefore, $\langle a,b \rangle$ is an L-gap and $\langle a,b \rangle \in \mathsf{Gap}(L) \in A \cup B$. If $\langle a,b \rangle \in A$, then by the definition of a $\Theta(L)$ -gap, we can find an element $g \in A$ such that $\langle \langle a,b \rangle,g \rangle \in \Theta(L) \setminus \mathbf{Id}$. If $g \in X$, then $g \in A \cap X = a$ and hence $\langle g,\langle a,b \rangle \rangle \in \Theta(L)$, which contradicts the antisymmetricity of $\Theta(L)$. So, $g \notin X$ and hence $g = \langle u,v \rangle$ is an L-gap. Then $\langle \langle a,b \rangle,g \rangle \in \Theta(L) \setminus \mathbf{Id}$ and the definition of the linear order $\Theta(L)$ implies that $a \subset u$. Choose any $g \in U \setminus a \subseteq X \setminus a = b \subseteq B$ and conclude that $\langle g,g \rangle \in A \times B \subseteq \Theta(L)$. On the other hand, $g \in U$ implies $\langle g,g \rangle = \langle g,\langle u,v \rangle \in \Theta(L)$ and hence $g = g \in X$ by the antisymmetricity of $\Theta(L)$, which contradicts our assumption. By analogy we can prove that the inclusion $\langle a,b \rangle \in B$ leads to a contradiction.

Definition 45.2. A linear order L is called gapless if $Gap(L) = \emptyset$.

By Theorem 45.1, a linear order $L \subseteq \mathbf{V}$ is gapless if and only if $L = \Theta(L)$.

Proposition 45.3. A linear order L (on a set $X = dom[L^{\pm}]$) is gapless if (and only if) it is boundedly complete.

Proof. Let L be a boundedly complete linear order. To derive a contradiction, assume that L has an L-gap $\langle a,b\rangle$. Then the sets a, b are not empty and hence the set a is upper L-bounded. By the bounded completeness of L, the set a has $\sup_L(a)$. If $\sup_L(a) \in a$, then $\sup_L(a)$ is the L-greatest element of a. If $\sup_L(a) \in b$, then $\sup_L(a)$ is the L-least element of b. In both cases, $\langle a,b\rangle$ is not an L-gap.

Now assume that the linear order L is gapless and $X = \mathsf{dom}[L^{\pm}]$ is a set. Assuming that L is not boundedly complete and applying Proposition 42.4, we conclude that $\mathsf{dom}[L^{\pm}]$ contains an upper L-bounded subset $A \subseteq X$ that has no $\sup_L(A)$. Then A does not have an L-maximal element. Since A is upper L-bounded, then set $b = \{x \in X : a \times \{x\} \subseteq L \cup \mathbf{Id}\}$ is not empty. Consider the set $a = L^{-1}[A]$. Since L is a linear order, $a \cup b = \mathsf{dom}[L^{\pm}]$. Since L is gapless, the ordered pair $\langle a, b \rangle$ is not an L-gap and hence the set b has an L-minimal element which is equal to $\sup_A(L)$.

Definition 45.4. Let κ be a cardinal. A linear order L is called κ -universal if there exists a subset $U \subseteq \mathsf{dom}[L^{\pm}]$ of cardinality $|U| = \kappa$ such that for any subsets $a, b \subseteq \mathsf{dom}[L^{\pm}]$ with $|a \cup b| < |\kappa|$ and $a \times b \subseteq L \setminus \mathbf{Id}$ there exits an element $u \in U$ such that $(a \times \{u\}) \cup (\{u\} \times b) \subseteq L \setminus \mathbf{Id}$.

Remark 45.5. Each universal linear order L with $|\mathsf{dom}[L^{\pm}]| = \kappa$ is κ -universal.

It can be shown that for any infinite cardinal κ and universal linear order $L \in \mathbf{V}$ with $|\mathsf{dom}[L^{\pm}]| = \kappa$, its gap extension $\Theta(L)$ is κ -universal. The following theorem shows that all such orders are pairwise isomorphic.

Theorem 45.6. Let κ be an infinite cardinal. Any κ -universal gapless linear orders are isomorphic.

Proof. Fix two κ -universal gapless linear orders L_1, L_2 . For every $i \in \{1, 2\}$, the underlying set $X_i = \mathsf{dom}[L_i^{\pm}]$ of the order L_i contains a subset $D_i \subseteq X_i$ of cardinality $|D_i| = |\kappa|$ such that for any sets $a, b \subseteq \mathsf{dom}[L_i]$ of cardinality $|a \cup b| < |\kappa|$ with $a \times b \subseteq L_i \setminus \mathbf{Id}$ there exists an element $z \in D_i$ such that $(a \times \{z\}) \cup (\{z\} \times b) \subseteq L_i \setminus \mathbf{Id}$. This implies that the linear order $U_i = L_i \upharpoonright D_i$ is universal.

By Theorem 43.5, there exists an isomorphism $f: U_1 \to U_2$ of the universal linear orders U_1 and U_2 . Now we show that f admits a unique extension F to an isomorphism of the linear orders L_1 and L_2 .

For every $x \in X_1 \setminus D_1$, consider the U_1 -gap $\langle \bar{x}, \bar{x} \rangle$ consisting of the sets $\bar{x} = \{y \in D_1 : \langle y, x \rangle \in L_1\}$ and $\bar{x} = \{y \in D_1 : \langle x, y \rangle \in L_1\}$. Since f is an order isomorphism, the pair $\langle f[\bar{x}], f[\bar{y}] \rangle$ is a U_2 -gap. Consider the sets $\downarrow f[\bar{x}] = L_2^{-1}[f[\bar{x}]]$ and $\uparrow f[\bar{x}] = L_2[f[\bar{x}]]$. Since the order L_2 is gapless, the ordered pair $\langle \downarrow f[\bar{x}], \uparrow f[\bar{x}] \rangle$ is not an L_2 -gap, which implies that the complement $X_2 \setminus (\downarrow f[\bar{x}] \cup \uparrow f[\bar{x}]) \subseteq X_2 \setminus D_2$ is not empty. The choice of the set D_2 ensures that this complement contains a unique point. We denote this unique point by F(x). Therefore, we have constructed an extension $F = f \cup \{\langle x, F(x) \rangle : x \in X_1 \setminus D_1\}$ of the function f to a function $F: X_1 \to X_2$. It can be shown that the function F is injective and L_1 -to- L_2 -increasing. By analogy we can extend the function $f^{-1}: D_2 \to D_1$ to an injective L_2 -to- L_1 -increasing function $G: X_2 \to X_1$. Then the composition $G \circ F: X_1 \to X_1$ is an L_1 -to- L_1 -increasing function such that $G \circ F(x) = x$ for every $x \in D_1$. The L_1 -density of the set D_1 implies that $G \circ F(x) = x$ for all $x \in X_1$. By analogy we can prove that $F \circ G$ is the identity function of the set X_2 . Therefore, $F: X_1 \to X_2$ is an isomorphism of the linear orders L_1 and L_2 .

Corollary 45.7 (Cantor). Any ω -universal gapless linear orders are isomorphic.

Theorem 45.8. Let κ be a regular cardinal with $|\kappa| = |2^{<\kappa}|$. Any κ -universal gapless linear order L has cardinality

$$|L| = |\mathsf{dom}[L^{\pm}]| = |2^{\kappa}|.$$

Proof. By Theorem 43.2, the linear order $\mathsf{U}_{2^{<\kappa}}$ is universal. By Theorem 45.7, the gap extension $\Theta(\mathsf{U}_{2^{<\kappa}})$ of $\mathsf{U}_{2^{<\kappa}}$ is isomorphic to the order L. Therefore,

$$|\mathrm{dom}[L^\pm]| = |\mathrm{dom}[\Theta(\mathsf{U}_{2^{<\kappa}})^\pm]| = |\mathrm{dom}[\mathsf{U}_{2^{<\kappa}}]| + |\mathrm{Gap}(\mathsf{U}_{2^{<\kappa}})| \leq |2^{<\kappa}| + |\mathcal{P}(2^{<\kappa}) \times \mathcal{P}(2^{<\kappa})| = |\kappa| + |\mathcal{P}(\kappa) \times \mathcal{P}(\kappa)| \leq |2^\kappa| + 2^{|\kappa| + |\kappa|} = |2^\kappa|$$

and $|L| \leq |\mathsf{dom}[L^{\pm}] \times \mathsf{dom}[L^{\pm}]| \leq |2^{\kappa} \times 2^{\kappa}| = |2^{\kappa}|.$

To prove that $|\mathsf{dom}[L^{\kappa}]| \ge |2^{\kappa}|$ and $|L| \ge |2^{\kappa}|$, consider the set $F = (2^{<\kappa})^{\kappa}$, endowed with the lexicographic linear order

$$R = \{ \langle f,g \rangle F : \exists \alpha \in \kappa \; (f \! \upharpoonright_{\alpha} = g \! \upharpoonright_{\alpha} \; \wedge \; \langle f(\alpha),g(\alpha) \rangle \in \mathsf{U}_{2^{<\kappa}} \}.$$

The set

$$D = \{ f \in F : |\{ \alpha \in \kappa : f(\alpha) \neq \emptyset \}| < \kappa \}$$

witnesses that the order R is κ -universal. Then its gap extension $\Theta(R)$ is gapless and κ -universal. By Theorem 45.7, the orders $\Theta(R)$ and L are isomorphic. Then $|\mathsf{dom}[L^{\pm}]| = |\mathsf{dom}[\Theta(R)]| \geq |\mathsf{dom}[R]| = |F| \geq |2^{\kappa}|$ and $|L| = |\Theta(R)| \geq |R| \geq |2^{\kappa}|$. Therefore, we have the inequalities

$$|2^{\kappa}| \leq |\mathsf{dom}[L^{\pm}]| \leq |2^{\kappa}| \quad \text{and} \quad |2^{\kappa}| \leq |L| \leq |2^{\kappa}|.$$
 By Theorem 35.6, $|L| = |\mathsf{dom}[L^{\pm}]| = |2^{\kappa}|.$

Part 10. Numbers

The aim of this part is to introduce the sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ which are of crucial importance for whole mathematics. The elements of those sets are numbers: natural, integer, rational, real, respectively. In fact, the set \mathbb{N} of nonzero natural numbers has been introduced in Section 5 as the set $\omega \setminus \{0\}$.

46. Integer numbers

Theorem 31.3(5) implies that for any natural numbers $n \leq m$ there exists a unique natural number k such that m = n + k. This natural number k is denoted by m - n and called the result of subtraction of n from m. If m < n, then m - n is not a natural number but is a negative integer. But what is a negative integer? For example, what is -1? It should be equal to 1 - 2, but also to 2 - 3 and 3 - 4 and so on. So, it is natural to define the negative integer -1 as the set of ordered pairs $\{\langle 0,1\rangle,\langle 1,2\rangle,\langle 2,3\rangle,\dots\}$. The simplest (in the sense of von Neumann hierarchy) element of this set is the ordered pair $\langle 0,1\rangle$. We take this simplest pair $\langle 0,1\rangle$ to represent the negative integer -1. As a result, the negative number -1 becames a relative simple set $\langle 0,1\rangle = \{\{0\},\{0,1\}\} = \{\{\emptyset\},\{\emptyset,\{\emptyset\}\}\}\}$, which is an element of the set V_4 of the von Neumann hierarchy. On the other hand, the set of ordered pairs $\{\langle n,m\rangle \in \omega \times \omega: n+1=m\}$ which can be taken as an alternative definition of -1 appears only at the stage $V_{\omega+1}$ of von Neumann hierarchy.

Realizing this idea, for every nonzero natural number n define the negative integer $\neg n$ as the ordered pair $\langle 0, n \rangle$. Then $\neg \mathbb{N} = \{\langle 0, n \rangle : n \in \mathbb{N}\} = \{0\} \times \mathbb{N}$ is the set of negative integers and the union

$$\mathbb{Z} = -\mathbb{N} \cup \{0\} \cup \mathbb{N}$$

is the set of integer numbers.

Exercise 46.1. Show that the sets $-\mathbb{N}$ and ω are disjoint.

Hint: A unique two-element set in ω is $2 = \{\emptyset, \{\emptyset\}\}\$, which does not belong to the class $\ddot{\mathbf{U}} \supset -\mathbb{N}$.

The reflexive linear order $\mathbf{S} \upharpoonright \omega$ on the set ω can be extended to the reflexive linear order

$$\leq_{\mathbb{Z}} = (-\mathbb{N} \times \omega) \cup \{\langle n, m \rangle : n \subseteq m\} \cup \{\langle -n, -m \rangle : m \subseteq n\}$$

on the set $\mathbb{Z} = \mathsf{dom}[\leq_{\mathbb{Z}}] = \mathsf{rng}[\leq_{\mathbb{Z}}] = \mathsf{dom}[\leq_{\mathbb{Z}}^{\pm}]$. The irreflexive order

$$<_{\mathbb{Z}} = <_{\mathbb{Z}} \setminus \mathbf{Id}$$

is called the *strict linear order* on \mathbb{Z} .

Exercise 46.2. Show that $-\mathbb{N} = \stackrel{\checkmark}{\leq}_{\mathbb{Z}}(0) = \stackrel{\checkmark}{<}_{\mathbb{Z}}(0)$.

Now we introduce some arithmetic operations on the set \mathbb{Z} . The first one is *additive* inversion $\neg: \mathbb{Z} \to \mathbb{Z}$, $\neg: x \mapsto \neg z$, defined by the formula

$$-z = \begin{cases} \langle 0, z \rangle & \text{if } z \in \mathbb{N}; \\ 0 & \text{if } z = 0; \\ n & \text{if } z = \langle 0, n \rangle \text{ for some } n \in \mathbb{N}. \end{cases}$$

The definition of the function – implies that -(-z) = z for any integer number z.

Exercise 46.3. Show that $\forall x, y \in \mathbb{Z} \ (x \leq_{\mathbb{Z}} y \Leftrightarrow \neg y \leq_{\mathbb{Z}} \neg x) \ \land \ (x <_{\mathbb{Z}} y \Leftrightarrow \neg y <_{\mathbb{Z}} \neg x).$

Now we shall extend the operation of addition to integer numbers. Since the addition of natural numbers is already defined, we need to define the sum x + y only for pairs

$$\langle x,y\rangle = (\mathbb{Z}\times\mathbb{Z})\setminus (\omega\times\omega) = ((-\mathbb{N})\times(-\mathbb{N}))\cup (\omega\times(-\mathbb{N}))\cup ((-\mathbb{N})\times\omega).$$

This is done by the formulas

$$(-m) + (-n) = -(m+n)$$
 and $m + (-n) = (-n) + m = \begin{cases} m-n & \text{if } n \le m, \\ -(n-m) & \text{if } m \le n, \end{cases}$

for any $n, m \in \omega$.

Exercise 46.4. Check that the addition $+: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ has the following properties for every integer numbers x, y, z:

- (1) (x + y) + z = x + (y + z);
- (2) x+y=y+x;
- (3) x + 0 = x;
- (4) x + (-x) = 0;
- (5) $x <_{\mathbb{Z}} y \Leftrightarrow x + z <_{\mathbb{Z}} y + z$.

Hint: Prove these properties for the isomorphic copy of \mathbb{Z} , which is the quotient set $\tilde{\mathbb{Z}} = (\omega \times \omega)/Z$ of $\omega \times \omega$ by the equivalence relation

$$Z = \{ \langle \langle k, l \rangle, \langle m, n \rangle \rangle \in (\omega \times \omega) \times (\omega \times \omega) : k + n = m + l \}.$$

For every ordered pair $\langle m, n \rangle \in \omega \times \omega$ its equivalence class $Z^{\bullet}(m, n)$ represents the integer number m - n. For two equivalence classes $Z^{\bullet}(m, n), Z^{\bullet}(k, l)$ their sum $Z^{\bullet}(m, n) + Z^{\bullet}(k, l)$ is defined as the equivalence class $Z^{\bullet}(m + k, n + l)$.

Next, we extend the multiplication to integer numbers. Since the multiplication of natural numbers is already defined, we need to define the product $x \cdot y$ only for pairs

$$\langle x, y \rangle = (\mathbb{Z} \times \mathbb{Z}) \setminus (\omega \times \omega) = ((-\mathbb{N}) \times (-\mathbb{N})) \cup (\omega \times (-\mathbb{N})) \cup ((-\mathbb{N}) \times \omega).$$

This is done by the formulas

$$(-m) \cdot (-n) = m \cdot n$$
 and $m \cdot (-n) = (-n) \cdot m = -(n \cdot m)$

for any $n, m \in \omega$.

Exercise 46.5. Check that the multiplication $\cdot : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ has the following properties for every $x, y, z \in \mathbb{Z}$:

- $(1) (x \cdot y) \cdot z = x \cdot (y \cdot z);$
- (2) $x \cdot y = y \cdot x$;
- (3) $x \cdot 0 = x$;
- (4) $x \cdot 1 = x$;
- (5) $x \cdot (-1) = -x$;
- (6) If $0 <_{\mathbb{Z}} x$ and $0 <_{\mathbb{Z}} y$, then $0 <_{\mathbb{Z}} x \cdot y$;
- $(7) x \cdot (y+z) = x \cdot y + x \cdot z;$
- (8) If $0 <_{\mathbb{Z}} x$, then $y <_{\mathbb{Z}} z \Leftrightarrow x \cdot y <_{\mathbb{Z}} x \cdot z$;
- $(9) x \cdot y = 0 \Leftrightarrow (x = 0 \land y = 0).$

Hint: Prove these properties for the isomorphic copy $\tilde{\mathbb{Z}}$ of \mathbb{Z} , considered in the hint to Exercise 46.4. In the proof apply Theorems 32.4 and 32.6.

To introduce rational numbers, we shall need some standard facts about the divisibility of integer numbers. We say that a natural number d divides an integer number z if $z = d \cdot k$ for some integer number k, which is denoted by $\frac{z}{d}$.

For an integer number z by $\mathsf{Div}(z)$ we denote the set of natural numbers dividing z. The set $\mathsf{Div}(z)$ contains 1 and hence is not empty. Two integer numbers a, b are called *coprime* if $\mathsf{Div}(a) \cap \mathsf{Div}(b) = \{1\}$. For two numbers a, b the largest element of the set $\mathsf{Div}(a) \cap \mathsf{Div}(b)$ is called the *greatest common divisor* of a and b and is denoted by $\gcd(a, b)$. If d is the largest common divisor of integer numbers a, b, then the integer numbers $\frac{a}{d}$ and $\frac{b}{d}$ are coprime.

47. RATIONAL NUMBERS

Rational numbers are introduced to "materialize" the result of division of integer numbers. For example, $\frac{1}{2}$ represents the result of division of 1 by 2 but also 2 by 4 and 3 by 6, etc. So, $\frac{1}{2}$ can be defined as the set of pairs $\{\langle 1,2\rangle,\langle 2,4\rangle,\langle 3,6\rangle,\dots\}$. Among such pairs the simplest (in the sense of von Neumann hierarchy) is the pair $\langle 1,2\rangle$, which can be taken as the representative of $\frac{1}{2}$.

More generally, for any pair $\langle a,b\rangle \in \mathbb{Z} \times \mathbb{N}$ the fraction $\frac{a}{b}$ can be defined as the set of ordered pairs $\{\langle m,n\rangle \in \mathbb{Z} \times \mathbb{N} : a \cdot n = b \cdot m\}$. The simplest element of this set is the ordered pair $\langle \frac{a}{\gcd(a,b)}, \frac{b}{\gcd(a,b)} \rangle$ consisting of relatively prime integers $\frac{a}{\gcd(a,b)}$ and $\frac{b}{\gcd(a,b)}$. The pairs $\langle m,n\rangle \in \mathbb{Z} \times \mathbb{N}$ with relatively prime numbers m,n thus encode all rational numbers $\frac{m}{n}$.

This suggests to define the set of rational numbers as the set

$$\mathbb{Q} = \mathbb{Z} \cup \{\langle m, n \rangle \in \mathbb{Z} \times \mathbb{N} : m \text{ and } n \text{ are coprime and } n \geq 2\}.$$

Pairs $\langle m, n \rangle \in \mathbb{Q} \setminus \mathbb{Z}$ will be denoted as fractions $\frac{m}{n}$, and integers $z \in \mathbb{Z}$ as fractions $\frac{z}{1}$. Therefore, the set \mathbb{Q} can be written more uniformly as

$$\left\{\frac{m}{n}: m \in \mathbb{Z} \text{ and } n \in \mathbb{N} \text{ are coprime}\right\}.$$

The linear order $\leq_{\mathbb{Z}}$ can be extended to the linear order

$$\leq_{\mathbb{Q}} = \{\langle \frac{m}{n}, \frac{k}{l} \rangle \in \mathbb{Q} \times \mathbb{Q} : m \cdot l \leq_{\mathbb{Z}} k \cdot n \}.$$

on the set \mathbb{Q} .

Exercise 47.1. Show that $\leq_{\mathbb{Q}}$ is indeed a reflexive linear order on \mathbb{Q} .

The irreflexive linear order

$$<_{\mathbb{O}} = \leq_{\mathbb{O}} \backslash \mathbf{Id}$$

is called the *strict linear order* on \mathbb{Q} .

Next we extend the arithmetic operations from \mathbb{Z} to \mathbb{Q} . The additive inversion is extended to \mathbb{Q} letting $\neg \langle m, n \rangle = \langle \neg m, n \rangle$ for any $\langle m, n \rangle \in \mathbb{Q} \setminus \mathbb{Z}$.

Let $\div : \mathbb{Z} \times \mathbb{N} \to \mathbb{Q}$ be the function assigning to any ordered pair $\langle m, n \rangle \in \mathbb{Z} \times \mathbb{N}$ the rational number

$$\frac{m}{n} = \frac{\frac{m}{\gcd(m,n)}}{\frac{n}{\gcd(m,n)}} = \begin{cases} \langle \frac{m}{\gcd(m,n)}, \frac{n}{\gcd(m,n)} \rangle & \text{if } \gcd(m,n) < n; \\ \frac{m}{\gcd(m,n)} & \text{if } \gcd(m,n) = n. \end{cases}$$

We recall that for a nonzero integer d dividing an integer number z the fraction $\frac{z}{d}$ denotes the unique integer number k such that $d \cdot k = z$.

Given any rational numbers $\frac{m}{n}$, $\frac{k}{l} \in \mathbb{Q}$ define their sum $\frac{m}{n} + \frac{k}{l}$ and product $\frac{m}{n} \cdot \frac{k}{l}$ as

$$\frac{m}{n} + \frac{k}{l} = \frac{m \cdot l + k \cdot n}{n \cdot l}$$
 and $\frac{m}{n} \cdot \frac{k}{l} = \frac{m \cdot k}{n \cdot l}$.

Exercise 47.2. Given any $x, y, z \in \mathbb{Q}$, check the following properties of the addition and multiplication of rational numbers:

- (1) x + (y + z) = (x + y) + z;
- (2) x + y = y + x;
- (3) x + 0 = x;
- (4) x + (-x) = 0;
- (5) $x \cdot (y \cdot z) = (x \cdot y) \cdot z;$
- (6) $x \cdot y = y \cdot x$;
- (7) $x \cdot 1 = x$;
- (8) $(x \neq 0) \Rightarrow (x \cdot \frac{1}{x} = 1);$ (9) $x \cdot (y + z) = (x \cdot y) + (x \cdot z);$
- $(10) \ x <_{\mathbb{Q}} y \le \Rightarrow \ x + z <_{\mathbb{Q}} y + z;$
- $(11) (0 <_{\mathbb{O}} x \land 0 <_{\mathbb{O}} y) \Rightarrow 0 <_{\mathbb{O}} x \cdot y.$

Since the linear orders $\leq_{\mathbb{Q}}$ and $<_{\mathbb{Q}}$ are countable universal, Cantor's Theorem ?? implies the following characterization.

Theorem 47.3 (Cantor). An (ir)reflexive order L is isomorphic to the (strict) linear order on \mathbb{Q} if and only if L is nonempty, countable and has two properties:

- 1) for any elements $x <_L y$ of $dom[L^{\pm}]$ there exists $z \in dom[L^{\pm}]$ such that $x <_L z <_L y$;
- 2) for any element $z \in \text{dom}[L^{\pm}]$ there are $x, y \in \text{dom}[L^{\pm}]$ such that $x <_L z <_L y$.

48. Real numbers

The set of real numbers is defined as the set

$$\mathbb{R} = \mathbb{Q} \cup \mathsf{Gap}(\leq_{\mathbb{Q}})$$

where $\mathsf{Gap}(\leq_{\mathbb{Q}})$ is the set of $\leq_{\mathbb{Q}}$ -gaps. We recall that a $\leq_{\mathbb{Q}}$ -gap is an ordered pair $\langle a,b\rangle$ of nonempty disjoint sets a, b such that $a \cup b = \mathbb{Q}$ and for any elements $x \in a$ and $y \in b$ there are elements $x' \in a$ and $y' \in b$ such that $x <_{\mathbb{Q}} x' <_{\mathbb{Q}} y' <_{\mathbb{Q}} y$.

The set \mathbb{R} is called the *real line* and its elements are called *real numbers*. The set \mathbb{R} carries the reflexive linear order $\leq_{\mathbb{R}}$ equal to the gap extension $\Theta(\leq_{\mathbb{Q}})$ of the linear order $\leq_{\mathbb{Q}}$. Also \mathbb{R} carries the strict linear order $\leq_{\mathbb{R}} \leq_{\mathbb{R}} \backslash \mathbf{Id}$. By Theorem 45.1(5), the linear orders $\leq_{\mathbb{R}}$ and $\leq_{\mathbb{R}}$ are gapless.

Now we extend the arithmetic operations from the set of rationals \mathbb{Q} to the set of reals \mathbb{R} . The operation of additive inverse $-: \mathbb{Q} \to \mathbb{Q}$ is extended to the operation $-: \mathbb{R} \to \mathbb{R}$ assigning to each $\leq_{\mathbb{Q}}$ -gap $\langle a, b \rangle$ the $\leq_{\mathbb{Q}}$ -gap $\langle -b, -a \rangle$, where $-a = \{-x : x \in a\}$ and $-b = \{-y : y \in b\}$.

To define the addition and multiplication of real numbers, we need some preparation.

For every real number $x \in \mathbb{R}$, consider its left and right sets

$$\bar{x} = \begin{cases} a & \text{if } x = \langle a, b \rangle \in \mathsf{Gap}(\leq_{\mathbb{Q}}); \\ \{ y \in \mathbb{Q} : y <_{\mathbb{Q}} x \} & \text{if } x \in \mathbb{Q}; \end{cases}$$

and

$$\vec{x} = \begin{cases} b & \text{if } x = \langle a, b \rangle \in \mathsf{Gap}(\leq_{\mathbb{Q}}); \\ \{ y \in \mathbb{Q} : x <_{\mathbb{Q}} y \} & \text{if } x \in \mathbb{Q}. \end{cases}$$

Let $\check{\mathbb{R}} = \{\langle \bar{x}, \vec{x} \rangle : x \in \mathbb{R}\}$ and $f : \mathbb{R} \to \check{\mathbb{R}}$ be the function assigning to each real number x the ordered pair $\langle \bar{x}, \vec{x} \rangle$. It is clear that this function is bijective and $f \upharpoonright_{\mathsf{Gap}(\leq_{\mathbb{Q}})} = \mathbf{Id} \upharpoonright_{\mathsf{Gap}(\leq_{\mathbb{Q}})}$. The inverse function $f^{-1} : \check{\mathbb{R}} \to \mathbb{R}$ assigns to each ordered pair $\langle a, b \rangle$ the unique real number y such that $\langle a, b \rangle = \langle \bar{y}, \vec{y} \rangle$. This unique real number y will be denoted by $a \curlyvee b$. Therefore, $x = \bar{x} \curlyvee \vec{x}$ for every real number x.

For subsets $a, b \subseteq \mathbb{Q}$ let $a + b = \{x + y : x \in a, y \in b\}$. In these notations the addition of real numbers $x, y \in \mathbb{R}$ is defined by the simple formula:

$$x + y = (\overline{x} + \overline{y}) \lor (\overline{x} + \overline{y}).$$

Exercise 48.1. Show that the addition of real numbers is well-defined and prove the following its properties for any real numbers $x, y, z \in \mathbb{R}$:

- (1) (x + y) + z = x + (y + z);
- (2) x + y = y + x;
- (3) x + 0 = x;
- (4) x + (-x) = 0;
- (5) If $x <_{\mathbb{R}} y$, then $x + z <_{\mathbb{R}} y + z$.

Now we define the multiplication of real numbers x, y. If one of these numbers is equal to zero, then we put $x \cdot y = 0$. If $0 <_{\mathbb{R}} x$ and $0 <_{\mathbb{R}} y$, then $x \cdot y = a \land b$ where

$$a = \{ z \in \mathbb{Q} : \exists u, v \in \mathbb{Q} \ ((0 <_{\mathbb{R}} u <_{\mathbb{R}} x) \ \land \ (0 <_{\mathbb{R}} v <_{\mathbb{R}} y) \ \land \ (z < u \cdot v)) \}$$

and

$$b = \{ z \in \mathbb{Q} : \exists u, v \in \mathbb{Q} \ ((x <_{\mathbb{R}} u) \ \land \ (y <_{\mathbb{R}} v) \ \land \ (u \cdot v < z) \}.$$

Having defined the product of $x \cdot y$ of strictly positive real numbers x, y, we also put

$$(-x) \cdot y = x \cdot (-y) = -(x \cdot y)$$
 and $(-x) \cdot (-y) = x \cdot y$.

Those formulas define the product of arbitrary real numbers.

Exercise 48.2. Show that the multiplication of real numbers is well-defined and prove the following its properties for any real numbers $x, y, z \in \mathbb{R}$:

- (1) $x \cdot (y \cdot z) = (x \cdot y) \cdot z$;
- (2) $x \cdot y = y \cdot x$;
- (3) $x \cdot 1 = x$;
- $(4) x \cdot (y+z) = (x \cdot y) + (x \cdot z);$
- $(5) (0 <_{\mathbb{R}} x \land 0 <_{\mathbb{R}} y) \Rightarrow 0 <_{\mathbb{R}} x \cdot y.$

Exercise* **48.3.** Show that the real line \mathbb{R} is *real closed* in the sense that for every odd number $n \in \omega$ and any real numbers a_0, \ldots, a_n with $a_n \neq 0$ there exists a real number x such that

$$a_0 + a_1 x + \dots + a_n x^n = 0.$$

In particular, for any nonzero real number x there exists a unique real number y such that $x \cdot y = 1$.

A subset $a \subseteq \mathbb{R}$ is called *upper-bounded* if $\exists y \in \mathbb{R} \ \forall x \in a \ (x \leq_{\mathbb{R}} y)$.

Exercise 48.4. Prove that for every nonempty upper bounded set $A \subseteq \mathbb{R}$ the set $B = \{b \in \mathbb{R} : \forall x \in A \ (x \leq_{\mathbb{R}} b)\}$ has the smallest element. This smallest element is denoted by $\sup A$.

Exercise 48.5 (Axiom of Archimedes). Prove that for every positive real numbers a and ε there exists a natural number n such that $a \leq n \cdot \varepsilon$.

Since the linear orders $\leq_{\mathbb{R}}$ and $<_{\mathbb{R}}$ are countably dense, endless and gapless, Proposition 45.3 and Cantor's Theorem 45.7 imply the following characterization.

Theorem 48.6. A (ir)reflexive order L is isomorphic to the (strict) linear order of the real line \mathbb{R} if and only if L is boundedly complete and there exists a nonempty countable subset $D \subseteq \mathsf{dom}[L^{\pm}]$ that has two properties:

- 1) for any elements $x <_L y$ of $dom[L^{\pm}]$ there exists $z \in D$ such that $x <_L z <_L y$;
- 2) for any element $z \in D$ there are $x, y \in \text{dom}[L^{\pm}]$ such that $x <_L z <_L y$.

Theorem 45.8 implies

Corollary 48.7. The real line has cardinality $|\mathbb{R}| = |2^{\omega}|$.

Exercise 48.8. Show that $\mathbb{R} \subset V_{\omega+3}$ and $\mathbb{R} \in V_{\omega+3}$.

49. Surreal numbers

An empty hat rests on a table made of a few axioms of standard set theory. Conway waves two simple rules in the air, then reaches into almost nothing and pulls out an infinitely rich tapestry of numbers.

Martin Gardner

I walked around for about six weeks after discovering the surreal numbers in a sort of permanent daydream, in danger of being run over

John Horton Conway

The usual numbers are very familiar, but at root they have a very complicated structure. Surreals are in every logical, mathematical and aesthetic sense better. Martin Kruskal

In this section we make a short introduction to Conway's surreal numbers. Surreal numbers are elements of a proper class \mathbf{No} called the surreal line. The surreal line carries a natural structure of an ordered field, which contains an isomorphic copy of the field $\mathbb R$ but also contains surreal numbers that can be identified with arbitrary ordinals.

Surreal numbers were introduced by John Horton Conway [3] who called them numbers (the adjective "surreal" was suggested by Donald Knuth [10]).

The surreal line is obtained by transfinite iterations of cut extensions of linear orders, starting from the empty order. We recall (see Section 44) that for a linear order L an L-cut is an ordered pair of sets $\langle a,b\rangle$ such that $a\cap b=\emptyset$, $a\cup b=\mathsf{dom}[L^\pm]$ and $a\times b\subseteq L$. For an L-cut $x=\langle a,b\rangle$ the sets a and b will be denoted by \tilde{x} and \tilde{x} and called the *left* and *right parts* of the cut x.

If the set L is well-founded (i.e., $L \in \mathbf{V}$), then the set $\mathsf{dom}[L^{\pm}]$ is disjoint with the set $\mathsf{Cut}(L)$ of L-cuts and the set $\mathsf{dom}[L^{\pm}] \cup \mathsf{Cut}(L)$ carries the linear order

$$\Xi(L) = L \cup \{\langle\langle a,b\rangle,\langle a',b'\rangle\rangle \times \operatorname{Cut}(L) \times \operatorname{Cut}(L) : a \subseteq a'\} \cup \\ \{\langle x,\langle a,b\rangle\rangle \in \operatorname{dom}[L^{\pm}] \times \operatorname{Cut}(L) : x \in a\} \cup \{\langle\langle a,b\rangle,y\rangle \in \operatorname{Cut}(L) \times \operatorname{dom}[L^{\pm}] : y \in b\}$$

called the *cut extension* of L, see Section 44.

Let $(L_{\alpha})_{\alpha \in \mathbf{On}}$ be the transfinite sequence of reflexive linear orders defined by the recursive formula

$$L_{\alpha} = \bigcup_{\beta \in \alpha} \Xi(L_{\beta}),$$

where $\Xi(L_{\beta})$ is the cut extension of the linear order L_{β} . So,

- $L_0 = \emptyset$,
- $L_{\alpha+1} = \Xi(L_{\alpha})$ for any ordinal α ;
- $L_{\alpha} = \bigcup_{\beta \in \alpha} L_{\beta}$ for any limit ordinal α .

The existence of the transfinite sequence $(L_{\alpha})_{\alpha \in \mathbf{On}}$ follows from Theorem 29.1 applied to the expansive function

$$\Phi: \mathbf{V} \to \mathbf{V}, \quad \Phi(y) = \begin{cases} \Xi(y) & \text{if } y \text{ is a linear order;} \\ y & \text{otherwise.} \end{cases}$$

Using Lemma 44.1 it can be shown that for every ordinal α the underlying set $\mathsf{No}_{\alpha} = \mathsf{dom}[L_{\alpha}^{\pm}]$ of the linear order L_{α} can be written as the union $\bigcup_{\beta \in \alpha} \mathsf{Cut}(L_{\beta})$ of the indexed family of pairwise disjoint sets $(\mathsf{Cut}(L_{\beta}))_{\beta \in \alpha}$.

The union

$$\mathbf{No} = \bigcup_{\alpha \in \mathbf{On}} \mathsf{No}_{\alpha} = \bigcup_{\alpha \in \mathbf{On}} \mathsf{Cut}(L_{\alpha})$$

is called the $surreal\ line$ and its elements are called $surreal\ numbers$. The surrreal line ${\bf No}$ carries the reflexive linear order

$$\leq_{\mathbf{No}} = \bigcup_{\alpha \in \mathbf{On}} L_{\alpha}$$

and the strict linear order

$$<_{\mathbf{No}} = \leqslant_{\mathbf{No}} \backslash \mathbf{Id}.$$

Let us look at the structure of the sets No_{α} for small ordinals α . The set No_0 is empty and carries the empty linear order L_0 whose set of cuts $Cut(L_0) = \{\langle \emptyset, \emptyset \rangle\}$ contains the unique ordered pair $\langle \emptyset, \emptyset \rangle$ denoted by 0.

Therefore, $No_1 = Cut(L_0) = \{0\}$. This set carries the linear order $L_1 = \{\langle 0, 0 \rangle\}$ that has two cuts denoted by

$$-1 := \langle \emptyset, \{0\} \rangle$$
 and $1 := \langle \{0\}, \emptyset \rangle$.

The set $No_2 = No_1 \cup Cut(L_1) = \{-1, 0, 1\}$ carries the linear order

$$L_2 = \{\langle -1, -1 \rangle, \langle -1, 0 \rangle, \langle -1, 1 \rangle, \langle 0, 0 \rangle, \langle 0, 1 \rangle, \langle 1, 1 \rangle\}$$

that has four cuts denoted by

$$-2 = \langle \emptyset, \{-1,0,1\} \rangle, \quad -\frac{1}{2} := \langle \{-1\}, \{0,1\} \rangle, \quad \frac{1}{2} := \langle \{-1,0\}, \{1\} \rangle, \quad 2 = \langle \{-1,0,1\}, \emptyset \rangle.$$

Therefore, $No_3 = \{-2, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, 2\}.$

The set No₄ = $\{-3, -2, -\frac{3}{2}, -1, -\frac{3}{4}, -\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \frac{3}{2}, 2, 3\}$ has 15 elements, in particlular: $-3 = \langle \emptyset, \{-2, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, 2\} \rangle, \quad -\frac{3}{2} = \langle \{-2\}, \{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1, 2\} \rangle,$ $-\frac{3}{4} = \langle \{-2, -1\}, \{-\frac{1}{2}, 0, \frac{1}{2}, 1, 2\} \rangle, \quad -\frac{1}{4} = \langle \{-2, -1, -\frac{1}{2}\}, \{0, \frac{1}{2}, 1, 2\} \rangle,$ $\frac{1}{4} = \langle \{-2, -1, -\frac{1}{2}, 0\}, \{\frac{1}{2}, 1, 2\} \rangle, \quad \frac{3}{4} = \langle \{-2, -1, -\frac{1}{2}, 0, \frac{1}{2}\}, \{1, 2\} \rangle,$ $\frac{3}{2} = \langle \{-2, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1\}, \{2\} \rangle, \quad 3 = \langle \{-2, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, 2\}, \emptyset \rangle.$

The set No_5 consists of 31 elements:

$$-4, -3, -\frac{5}{2}, -2, -\frac{7}{4}, -\frac{3}{2}, -\frac{5}{4}, -1, -\frac{7}{8}, -\frac{3}{4}, -\frac{5}{8}, -\frac{1}{2}, -\frac{3}{8}, -\frac{1}{4}, -\frac{1}{8}, 0, \frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{1}{2}, \frac{5}{8}, \frac{3}{4}, \frac{7}{8}, 1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 2, \frac{5}{2}, 3, \frac{1}{8}, \frac{1}{8}$$

Now we see that the set No_{ω} is countable and its elements can be labeled by dyadic rational numbers (with preservation of order).

The set $\mathsf{No}_{\omega+1} = \mathsf{No}_{\omega} \cup \mathsf{Cut}(L_{\omega})$ contains all L_{ω} -cuts which include all L_{ω} -gaps that can be interpreted as real numbers. Besides the real numbers the set $\mathsf{No}_{\omega+1}$ contains the infinitely large number $\langle \mathsf{No}_{\omega}, \emptyset \rangle$ that can be identified with the ordinal ω and also its additive inverse $-\omega = \langle \emptyset, \mathsf{No}_{\omega} \rangle$. Also for any dyadic rational x the set $\mathsf{No}_{\omega+1}$ contains two L_{ω} -cuts

$$x_{-} = \langle \{ y \in \mathsf{No}_{\omega} : y <_{\mathsf{No}} x \}, \{ z \in \mathsf{No}_{\omega} : x \leq_{\mathsf{No}} z \} \rangle$$

and

$$x_{+} = \langle \{ y \in \mathsf{No}_{\omega} : y \leq_{\mathbf{No}} x \}, \{ z \in \mathsf{No}_{\omega} : x <_{\mathbf{No}} z \} \rangle$$

that can be interpreted as numbers that are infinitely close to x.

The exists a natural injective function $\mathbf{On} \to \mathbf{No}$ which assigns to every ordinal α the L_{α} -cut $\langle \mathsf{No}_{\alpha}, \emptyset \rangle$. Therefore, the surreal line contains a copy of the ordinal line, which implies that \mathbf{No} is a proper class.

Let birth: $\mathbf{No} \to \mathbf{On}$ be the function assigning to each element $x \in \mathbf{No}$ the unique ordinal α such that $x \in \mathsf{Cut}(L_{\alpha})$. The function birth is called the *birthday function*.

Consider the class of ordered pairs

$$\mathcal{P}_{<}(\mathbf{No}) = \{ \langle a, b \rangle : a, b \in \mathcal{P}(\mathbf{No}), \ a \times b \subset <_{\mathbf{No}} \}$$

and observe that

$$\mathbf{No} = \bigcup_{\alpha \in \mathbf{On}} \mathsf{Cut}(L_\alpha) \subseteq \mathcal{P}_<(\mathbf{No}).$$

The following property of the birthday function is crucial for introducing various algebraic structures on the surreal line.

Theorem 49.1. For any ordered pair $\langle a, b \rangle \in \mathcal{P}_{<}(\mathbf{No})$ there exists a unique number $x \in \mathbf{No}$ such that

- 1) $(a \times \{x\}) \cup (\{x\} \times b) \subseteq \langle_{\mathbf{No}} \ and$
- 2) $(a \times \{y\}) \cup (\{y\} \times b) \not\subseteq <_{\mathbf{No}} \text{ for any element } y \in \mathbf{No} \text{ with } \mathsf{birth}(y) < \mathsf{birth}(x).$

This unique number x will be denoted by a
ightharpoonup b.

Proof. First we show that the class

$$\mathbf{No}(a,b) = \{z \in \mathbf{No} : (a \times \{z\}) \cup (\{z\} \times b) \subseteq \langle \mathbf{No} \rangle \}$$

is not empty.

By the Axiom of Replacement, the image $\operatorname{birth}[a \cup b] \subseteq \mathbf{On}$ of the set $a \cup b$ under the birthday function is a set, which implies that $a \cup b \subseteq \operatorname{No}_{\alpha}$ for some ordinal α . Consider the sets $\downarrow a = \{x \in \operatorname{No}_{\alpha} : \langle x, a \rangle \in L_{\alpha}\}$ and $\uparrow b = \{y \in \operatorname{No}_{\alpha} : \langle b, y \rangle \in L_{\alpha}\}$. Since $a \times b \subseteq L_{\alpha} \setminus \operatorname{Id}$, the transitivity of the order L_{α} implies that $\downarrow a \cap \uparrow b = \emptyset$. If $\downarrow a \cup \uparrow b \neq L_{\alpha}$, then $\emptyset \neq L_{\alpha} \setminus (\downarrow a \cup \uparrow b) \subseteq \operatorname{No}(a,b)$, witnessing that the class $\operatorname{No}(a,b)$ is nonempty. If $\downarrow a \cup \uparrow b = L_{\alpha}$, then $z = \langle \downarrow a, \uparrow b \rangle \in \operatorname{Cut}(L_{\alpha})$ and $(a \times \{z\}) \cup (\{z\} \times b) \subseteq L_{\alpha+1} \setminus \operatorname{Id} \subset \langle \mathbf{No} \text{ by the definition of the cut extension } L_{\alpha+1} = \Xi(L_{\alpha})$. Therefore, $z \in \operatorname{No}(a,b)$ and again the class $\operatorname{No}(a,b)$ is not empty. Since the relation $\operatorname{E} \upharpoonright \operatorname{On}$ is well-founded, the nonempty class $\operatorname{birth}[\operatorname{No}(a,b)]$ contains the smallest ordinal α . Since $\operatorname{No}_{\gamma} = \bigcup_{\beta \in \gamma} \operatorname{No}_{\beta}$ for any limit ordinal γ , the ordinal α is not limit and hence hence $\alpha = \beta + 1$ for some ordinal $\beta \in \alpha$.

To finish the proof, it suffices to check that the class $\mathbf{No}(a,b) \cap \mathsf{No}_{\alpha}$ is a singleton. To derive a contradiction, assume that $\mathbf{No}(a,b) \cap \mathsf{No}_{\alpha}$ contains two distinct elements x,y. The minimality of α ensures that $x,y \notin \mathsf{No}_{\beta}$. Then $x,y \in \mathsf{No}_{\beta+1} \setminus \mathsf{No}_{\beta} = \mathsf{Cut}(L_{\beta})$ are two distinct L_{β} -cuts. So, $x = \langle a',b' \rangle$ and $y = \langle a'',b'' \rangle$ for some L_{β} -cuts $\langle a',b' \rangle$ and $\langle a'',b'' \rangle$. Since L_{α} is a linear order, either $\langle x,y \rangle \in L_{\alpha}$ or $\langle y,x \rangle \in L_{\alpha}$. We lose no generality assuming that $\langle x,y \rangle \in L_{\alpha}$ and hence $a' \subset a''$ by the definition of the linear order $L_{\alpha} = L_{\beta+1}$. Choose any point $z \in a'' \setminus a' \subseteq \mathsf{No}_{\beta}$ and observe that $z \in b'$ and hence $x = \langle a',b' \rangle < z < \langle a'',b'' \rangle = y$. For every $u \in a$ and $v \in b$, we have $u <_{\mathsf{No}} x <_{\mathsf{No}} z <_{\mathsf{No}} y <_{\mathsf{No}} v$, which implies that $z \in \mathsf{No}(a,b) \cap \mathsf{No}_{\beta}$. But this contradicts the minimality of the ordinal α .

Exercise 49.2. Given any ordinal α and an L_{α} -cut $x = \langle a, b \rangle \in \text{Cut}(L_{\alpha}) \subset \text{No}$, show that $x = a \curlyvee b$.

Theorem 49.1 implies that the (strict) linear order of the surreal line is universal. Under the Global Well-Orderability Principle (GWO) the surreal line in well-orderable. Applying Theorem 43.5, we obtain the following characaterization of the (strict) linear order of the surreal line.

Theorem 49.3. Assume (GWO). An (ir)reflexive linear order L is isomorphic to the (strict) linear order of the surreal line if and only if L is universal and dom[L^{\pm}] is a proper class.

Exercise* **49.4.** Show that the linear order of the sureal line is isomorphic to the universal linear order $U_{2\leq \mathbf{On}}$ on $2^{\leq \mathbf{On}}$ (this gived so-called sign representation of surreal numbers).

It turns out that the surreal line carries a natural structure of an ordered field. The operation of addition on No is defined by the recursive formula:

$$x + y = ((\overleftarrow{x} + y) \cup (x + \overleftarrow{y})) \land ((\overrightarrow{x} + y) \cup (x + \overrightarrow{y})).$$

In this formula $x = \langle \overline{x}, \overrightarrow{x} \rangle$, $y = \langle \overline{y}, \overrightarrow{y} \rangle$, $\overline{x} + y = \{z + y : z \in \overline{x}\}$ etc.

Exercise 49.5. Check that 0 + 0 = 0.

$$\begin{array}{l} \textit{Solution:} \ 0+0=\langle\emptyset,\emptyset\rangle+\langle\emptyset,\emptyset\rangle = \\ ((\overline{0}+0)\cup(0+\overline{0})) \ \\ \top \ ((\overline{0}+0)\cup(0+\overline{0})) = ((\emptyset+0)\cup(0+\emptyset)) \ \\ \top \ ((\emptyset+0)\cup(0+\emptyset)) = \emptyset \ \\ \top \ \emptyset = 0. \end{array}$$

Exercise 49.6. Check that 0 + 1 = 1.

Exercise 49.7. Check that 1 + 1 = 2.

Solution:
$$1 + 1 = \langle \{0\}, \emptyset \rangle + \langle \{0\}, \emptyset \rangle = ((\overline{1} + 1) \cup (1 + \overline{1})) \land ((\overline{1} + 1) \cup (1 + \overline{1})) = ((\{0\} + 1) \cup (1 + \{0\})) \land ((\emptyset + 1) \cup (1 + \emptyset)) = (\{0 + 1, 1 + 0\} \land \emptyset = \{1\} \land \emptyset = 2.$$

By transfinite induction the following properties of the addition can be established.

Proposition 49.8. For every numbers $x, y, z \in \mathbf{No}$ we have

- 1) x + 0 = x;
- 2) x + y = y + x;
- 3) x + (y + z) = (x + y) + z;
- 4) $x <_{\mathbf{No}} y \Rightarrow x + z <_{\mathbf{No}} y + z$.

To introduce the subtraction of Conway's numbers, for every surreal number $x \in \mathbf{No}$ consider its inverse -x defined by the recursive formula $-x = -\langle \bar{x}, \vec{x} \rangle := \{-z : z \in \vec{x}\} \land \{-z : z \in \bar{x}\} \rangle$.

Exercise 49.9. Show that -0 = 0.

Solution:
$$-0 = -\langle \emptyset, \emptyset \rangle = \{-z : z \in \emptyset\} \land \{-z : z \in \emptyset\} = \emptyset \land \emptyset = 0.$$

Example 49.10. Show that -1 = -1.

The following proposition can be proved by transfinite induction.

Proposition 49.11. For every number $x \in \mathbf{No}$ we have x + (-x) = 0.

Proposition 49.8 and 49.11 imply that the surreal line **No** endowed with the operation of addition has the structure of an ordered commutative group.

The multiplication of surreal numbers is defined by the recursive formula

$$xy = L \land R$$

where

$$L = \{\dot{x}y + x\dot{y} - \dot{x}\dot{y} : \dot{x} \in \overline{x}, \ \dot{y} \in \overline{y}\} \cup \{\ddot{x}y + x\ddot{y} - \ddot{x}\ddot{y} : \ddot{x} \in \vec{x}, \ \ddot{y} \in \overline{y}\},$$

$$R = \{\dot{x}y + x\ddot{y} - \dot{x}\ddot{y} : \dot{x} \in \overline{x}, \ \ddot{y} \in \overline{y}\} \cup \{\dot{x}\dot{y} + \ddot{x}\dot{y} - \ddot{x}\dot{y} : \ddot{x} \in \vec{x}, \ \dot{y} \in \overline{y}\}.$$

Exercise 49.12. Prove that $0 \cdot 0 = 0$, $1 \cdot 1 = 1$, $1 \cdot 2 = 1$.

Exercise 49.13. Prove that $2 \cdot 2 = 4$.

Solution:
$$2 \cdot 2 = \{1 \cdot 2 + 2 \cdot 1 - 1 \cdot 1\} \land \emptyset = \{3\} \land \emptyset = 4.$$

Exercise* **49.14.** Prove that the multiplication of surreal numbers is well-defined and has the following properties for any $x, y, z \in \mathbf{No}$:

- (1) $x \cdot (y \cdot z) = (x \cdot y) \cdot z$;
- (2) $x \cdot y = y \cdot x$;
- (3) $x \cdot 1 = x$;
- (4) $x \cdot (y+z) = (x \cdot y) + (x \cdot z);$
- $(5) (0 <_{\mathbb{R}} x \land 0 <_{\mathbb{R}} y) \Rightarrow 0 <_{\mathbb{R}} x \cdot y.$

Hint: See [3, pp.19–20].

Exercise* **49.15.** Prove that the surreal line is real closed in the sense that for every odd number $n \in \omega$ and any surreal numbers a_0, \ldots, a_n with $a_n \neq 0$ there exists a surreal number x such that

$$a_0 + a_1 x + \dots + a_n x^n = 0.$$

In particular, for any nonzero number $x \in \mathbf{No}$ there exists a unique surreal number $y \in \mathbf{No}$ such that $x \cdot y = 1$.

Hint: See [3, Theorem 25].

Part 11. Mathematical Structures

Mathematics has less than ever been reduced to a purely mechanical game of isolated formulas; more than ever does intuition dominate in the genesis of discoveries.

But henceforth, it possesses the powerful tools furnished by the theory of the great types of structures; in a single view, it sweeps over immense domains, now unified by the axiomatic method, but which were formerly in a completely chaotic state.

"The Architecture of Mathematics" Bourbaki, 1950

According to Nicolas Bourbaki¹¹, the mathematics is a science about mathematical structures. Many mathematical structures are studied by various areas of mathematics: graphs, partially ordered sets, semigroups, monoids, groups, rings, ordered fields, topological spaces, topological groups, topological vector spaces, Banach spaces, Banach algebras, Boolean algebras, etc etc.

A particular type of a mathematical structure is determined by a list of axiom \mathcal{A} it should satisfy. By an axiom in the list \mathcal{A} we understand any formula $\varphi(x, s, C_1, \ldots, C_n)$ in the language of CST with free variables x, s, and parameters C_1, \ldots, C_n which are some fixed classess.

Definition. A mathematical structure satisfying a list of axioms \mathcal{A} is any pair of classes (X, S) such that for every axiom $\varphi(x, s, C_1, \ldots, C_n)$ in the list \mathcal{A} , the formula $\varphi(X, S, C_1, \ldots, C_n)$ is true. The classes X and S are called respectively the underlying class and the structure of the mathematical structure (X, S).

Mathematical structures satisfying certain specific lists of axioms have special names and are studied by the corresponding fields of mathematics.

50. Examples of Mathematical Structures

In this section we present some important examples of mathematical structures.

Example 50.1. The structure of a set can be considered as a mathematical structure (x, S) with empty structure $S = \emptyset$, i.e., sets are mathematical structures without structure.

Next, we consider some relation structures.

Example 50.2. A graph is a mathematical structure (X, S) satisfying the axiom

$$\bullet \ S \subseteq \{\{u,v\} : u,v \in X\}.$$

The list \mathcal{A} of axioms for the structure of a graph consists of a unique formula $\varphi(x,s)$:

$$\forall z \ (z \in s \Rightarrow \exists u \ \exists v \ (u \in x \land v \in x \land \forall w \ (w \in z \Leftrightarrow (w = u \lor w = v))))$$

expressing the fact that elements of s are unordered pairs of elements of x. A less formal way of writing this formula is $\forall z \in s \ \exists u \in x \ \exists v \in x \ (s = \{u, v\}).$

¹¹Task: Read about Bourbaki in Wikipedia

For our next examples of mathematical structures we shall use such shorthand versions of formulas in the axiom lists.

Example 50.3. A directed graph (or else a digraph is a mathematical structure (X, S) satisfying the axiom

• $S \subseteq X \times X$.

A directed graph (X, S) is *simple* if $\forall x \in X \ (\langle x, x \rangle \notin S)$.

Example 50.4. An ordered class is a mathematical structure (X, S) consisting of a class X and an order $S \subseteq X \times X$. The list of axioms determining this mathematical structure consists of three axioms:

- $S \subseteq X \times X$;
- $S \cap S^{-1} \subseteq \mathbf{Id}$;
- $S \circ S \subset S$.

An ordered set is an ordered class (X, S) such that $X \in \mathbf{U}$.

Example 50.5. A partially ordered class is a mathematical structure (X, S) consisting of a class and an order S such that $\mathbf{Id} \upharpoonright X \subseteq S \subseteq X \times X$. The list of axioms determining this mathematical structure consists of three axioms:

- $S \subseteq X \times X$;
- $S \cap S^{-1} \subseteq \operatorname{Id} \upharpoonright X$;
- $S \circ S \subset S$.

A partially ordered set is a partially ordered class (X, S) such that $X \in \mathbf{U}$.

Example 50.6. A linearly ordered class is a mathematical structure (X, S) satisfying the axioms

- $S \subseteq X \times X \subseteq S \cup S^{-1} \cup \mathbf{Id}$;
- $S \cap S^{-1} \subseteq \mathbf{Id}$;
- $S \circ S \subseteq S$.

A linearly ordered set is a linear ordered class (X, S) such that $X \in \mathbf{U}$.

Example 50.7. A well-ordered class is a mathematical structure (X, S) satisfying the axioms:

- $S \subseteq X \times X \subseteq S \cup S^{-1} \cup \mathbf{Id}$;
- $S \cap S^{-1} \subseteq \mathbf{Id}$;
- $S \circ S \subseteq S$;
- $\forall Y \ (\emptyset \neq Y \subseteq X \Rightarrow \exists y \in Y \ \forall x \in X \ (\langle x, y \rangle \in S \Rightarrow x = y)).$

A well-ordered set is a well-ordered class (X, S) such that $X \in \mathbf{U}$.

These mathematical structures relate as follows:

well-ordered class \Rightarrow linearly ordered class \Rightarrow ordered class \Rightarrow directed graph.

Exercise 50.8. Find examples of:

- (1) an ordered set which is not partially ordered;
- (2) an ordered class which is not an ordered set;
- (3) a partially ordered set which is not linearly ordered;
- (4) a linearly ordered set which is not well-ordered.
- (5) a well-ordered class which is not a well-ordered set.

For two directed graphs (X, S_X) , (Y, S_Y) a function $f : X \to Y$ is called *increasing* if $\forall x \in X, \ \forall y \in Y \ (\langle x, y \rangle \in S_X \setminus \mathbf{Id}) \Rightarrow \ \langle f(x), f(y) \rangle \in S_Y \setminus \mathbf{Id}).$

Next, we present examples of some elementary mathematical structures arising in Algebra.

Example 50.9. A magma is a mathematical structure (X, S) such that S is a function with $dom[S] = X \times X$ and $rng[S] \subseteq X$. This mathematical structure is determined by two axioms:

- $\forall t \ (t \in S \Rightarrow \exists x \exists y \exists z \ (x \in X \land y \in X \land z \in X \land (\langle \langle x, y \rangle, z \rangle = t));$
- $\forall x \ \forall y \ \forall u \ \forall v \ (\langle \langle x, y \rangle, u \rangle \in S \ \land \ \langle \langle x, y \rangle, v \rangle \in S \ \Rightarrow \ u = v).$

The structure S of a magma (X, S) is called a binary operation on X. Binary operations are usually denoted by symbols: $+, \cdot, *, \star$, etc.

Definition 50.10. For two magmas (X, M_X) and (Y, M_Y) a function $f: X \to Y$ is called a magma homomorphism if $\forall x \in X \ \forall y \in X \ M_Y(f(x), f(y)) = f(M_X(x, y))$.

Example 50.11. A semigroup is a magma (X, S) whose binary operation S is associative in the sense that S(S(x, y), z) = S(x, S(y, z)) for all $x, y, z \in X$.

Example 50.12. A regular semigroup is a semigroup (X, S) such that

•
$$\forall x \in X \ \exists y \in X \ (S(S(x,y),x) = x; \land S(S(y,x),y) = y).$$

Example 50.13. An inverse semigroup is a semigroup (X, S) such that for every $x \in X$ there exists a unique element $y \in X$ such that S(S(x, y), x) = x and S(S(y, x), y) = y. This unique element y is denoted by x^{-1} .

Example 50.14. A Clifford semigroup is a regular semigroup (X, S) such that

•
$$\forall x \in X \ (S(x,x) = x \Rightarrow \forall y \in X \ (S(x,y) = S(y,x)).$$

Exercise* 50.15. Prove that every Clifford semigroup is inverse.

Example 50.16. A monoid is a semigroup (X, S) satisfying the axiom

$$\bullet \exists e \in X \ \forall x \in X \ S(x,e) = x = S(e,x).$$

The element e is unique and is called the *identity* of the monoid (X, S).

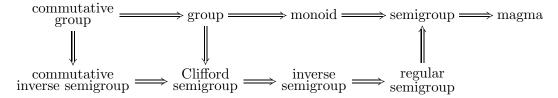
Example 50.17. A group is a semigroup (X, S) satisfying the axiom

•
$$\exists e \in X \ \forall x \in X \ \exists y \in X \ (S(x,e) = x = S(e,x) \ \land \ S(x,y) = e = S(y,x)).$$

Example 50.18. A commutative magma is a magma (X, S) such that

•
$$\forall x \ \forall y \ S(x,y) = S(y,x)$$
.

For these algebraic structures we have the implications:



Exercise 50.19. Find examples of:

- (1) a magma which is not a semigroup;
- (2) a semigroup which is not a monoid;

- (3) a monoid which is not a group;
- (4) a group which is not a commutative group;
- (5) a semigroup which is not regular;
- (6) a regular semigroup which is not inverse;
- (7) an inverse semigroup which is not Clifford;
- (8) a Clifford semigroup which is not a group.

Next we consider some mathematical structures arising in Geometry and Topology.

Example 50.20. A metric space is a mathematical structure (X, S) satisfying the axioms

- S is a function with $dom[S] = X \times X$ and $rng[S] \subseteq \mathbb{R}$;
- $\forall x \in X \ \forall y \in X \ (S(x,y) = 0 \ \Rightarrow \ x = y);$
- $\forall x \in X \ \forall y \in X \ (S(x,y) = S(y,x);$
- $\forall x \in X \ \forall y \in X \ \forall z \in X; (S(x,z) \le S(x,y) + S(y,z)).$

Observe that the axioms of a metric space contain as parameters the following sets: the set of real numbers \mathbb{R} , the natural number $0 = \emptyset$, the linear order \leq of the real line, and the addition operation $+: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ on the real line.

Another structure whose definition involves the real line as a parameter is the structure of a measure space.

Example 50.21. A measure space is a mathematical structure (X, S) satisfying the following axioms

- S is a function with $dom[S] \subseteq \mathcal{P}(X)$ and $rng[S] \subset \mathbb{R}$;
- $\forall u \ \forall v \ (u \in \mathsf{dom}[S] \ \land \ v \in \mathsf{dom}[S] \ \Rightarrow \ u \setminus v \in \mathsf{dom}[S]);$
- $\forall u \ (u \in \mathsf{dom}[S] \Rightarrow 0 \leq S(u));$
- $\forall u \ \forall v \ ((u \in \mathsf{dom}[S] \ \land \ v \in \mathsf{dom}[S] \ u \cap v = \emptyset) \ \Rightarrow \ S(u \cup v) = S(u) + S(v)).$

Example 50.22. A topological space is a mathematical structure (X, S) satisfying the following axioms:

- $\{\emptyset, X\} \subseteq S \subseteq \mathcal{P}(X)$;
- $\forall x \ \forall y \ (x \in S \ \land \ y \in S \ \Rightarrow \ x \cap y \in S);$
- $\forall u \ (u \subseteq S \Rightarrow \bigcup u \in S).$

For two topological spaces (X, S_X) , (Y, S_Y) a function $f: X \to Y$ is called *continuous* if $\forall u \ (u \in S_Y \Rightarrow f^{-1}[u] \in S_X)$.

Example 50.23. A bornological space is a mathematical structure (X, S) satisfying the following axioms:

- $\bullet \bigcup S = X;$
- $\forall x \ \forall y \ (x \in S \ \land \ y \in S \ \Rightarrow \ x \cup y \in S);$
- $\forall x \ \forall y \ (x \subseteq y \in S \subseteq S \Rightarrow \bigcup x \in S).$

Example 50.24. A uniform space is a mathematical structure (X, S) satisfying the following axioms:

- $\forall u \in S \ (\mathbf{Id} \upharpoonright X \subseteq u \subseteq X \times X);$
- $\forall u \in S \ \forall v \in S \ \exists w \in S \ (w \circ w \subseteq u \cap v^{-1});$
- $\forall u \in S \ \forall v \ (u \subseteq v \subseteq X \times X \Rightarrow v \in S).$

Example 50.25. A coarse space is a mathematical structure (X, S) satisfying the following axioms:

- $\forall u \in S \ (\mathbf{Id} \upharpoonright X \subseteq u \subseteq X \times X);$
- $\forall u \in S \ \forall v \in S \ \exists w \in S \ (u \circ v^{-1} \subseteq w);$
- $\forall u \in S \ \forall v \ (\mathbf{Id} \upharpoonright X \subseteq v \subseteq u \Rightarrow v \in S).$

Example 50.26. A *duoform space* is a mathematical structure (X, S) satisfying the following axioms:

- $\forall u \in S \ (\mathbf{Id} \upharpoonright X \subseteq u \subseteq X \times X);$
- $\forall u \in S \ \forall v \in S \ \exists w \in S \ (w \circ w \subseteq u \cap v^{-1});$
- $\forall u \in S \ \forall v \in S \ \exists w \in S \ (u \circ v^{-1} \subseteq w);$
- $\forall u \in S \ \forall v \in S \ \forall w \ u \subseteq w \subseteq v \Rightarrow w \in S$).

Exercise 50.27. Prove that a duoform space (X, S) is

- (1) a uniform space if and only if $X \times X \in S$;
- (2) a coarse space if and only if $\mathbf{Id} \upharpoonright X \in S$.

Complex mathematical structures consists of several substructures which can be related each with the other. A typical example is the structures of an ordered group.

Example 50.28. An *ordered group* is a mathematical structure (X, S) whose structure S is a pair of classes (+, <) such that (X, +) is a commutative group, and (X, <) is a linearly ordered class such that

• $\forall x \in X \ \forall y \in X \ \forall z \in X \ (x < y \implies x + z < y + z).$

Example 50.29. A partially ordered space is a mathematical structure (X, S) whose structure S is a pair of sets (\leq, τ) such that (X, \leq) is a partially ordered set, (X, τ) is a topological space and the following conditions are satisfied:

•
$$\forall x \in X \ \forall y \in X \ (x \nleq y \Rightarrow \exists U \in \tau \ \exists V \in \tau \ (\langle x, y \rangle \in U \times V \subseteq (X \setminus X) \setminus \leq).$$

Example 50.30. A ring is a mathematical structure (X, S) whose structure S is a pair of classes $(+, \cdot)$ such that (X, +) is a commutative group, (X, \cdot) is a monoid and the following axiom (called the *distributivity*) is satisfied:

(1)
$$\forall x \in X \ \forall y \in X \ \forall z \in X \ (x \cdot (y+z) = (x \cdot y) + (x \cdot z) \ \land \ (x+y) \cdot z = (z \cdot z) + (y \cdot z)).$$

Example 50.31. A commutative ring is a ring $(X, (+, \cdot))$ such that the monoid (X, \cdot) is commutative.

Example 50.32. A *field* is a commutative ring $(X, (+, \cdot))$ such that

(1)
$$\forall x \in X \ (x + x \neq x \Rightarrow \exists y \in X \ \forall z \in X \ z \cdot (x \cdot y) = z)$$
).

Example 50.33. An ordered field is a mathematical structure (X, S) whose structure S is a triple of classes $(+, \cdot, <)$ such that $(X, (+, \cdot))$ is a field, (X, (+, <)) is an ordered group and $\forall x \in X \ \forall y \in X \ \forall z \in X \ ((z + z = z \ \land \ z < x \ \land \ z < y) \Rightarrow \ \langle z < x \cdot y).$

Exercise 50.34. Show that $(\mathbb{R}, (+, \cdot, <_{\mathbb{R}}))$ is an ordered field.

Exercise 50.35. Show that $(No, (+, \cdot, <_{No}))$ is an ordered field.

The most general algebraic structure is that of universal algebra of a given signature.

Definition 50.36. Let σ be a function with $\operatorname{rng}[\sigma] \subseteq \omega$. A universal algebra of signature σ is a mathematical structure (X, S) whose structure is an indexed family of classes $(S_i)_{i \in \operatorname{\mathsf{dom}}[\sigma]}$ such that for every $i \in \operatorname{\mathsf{dom}}[\sigma]$, S_i is a function with $\operatorname{\mathsf{dom}}[S_i] = X^{\sigma(i)}$ and $\operatorname{\mathsf{rng}}[S_i] \subseteq X$.

Even more general is a universal relation structure of a given signature.

Definition 50.37. Let σ be a function with $\operatorname{rng}[\sigma] \subseteq \omega$. A universal relation structure of signature σ is a mathematical structure (X, S) whose structure is an indexed family $(S_i)_{i \in \operatorname{dom}[\sigma]}$ such that for every $i \in \operatorname{dom}[\sigma]$, $S_i \subseteq X^{\sigma(i)}$.

Finally we consider the structure of an objectless category.

Definition 50.38. An *objectless category* is a mathematical structure (X, S) whose underlying class X is called the *class of arrows* of the objectless category and the structure S is a triple $(\star, +, \circ)$ consisting of functions $\star : X \to X$, $+ : X \to X$ assigning to each arrow $x \in X$ its *source* $\star(x)$ and *target* +(x), and a function \circ called the function of *composition* of arrows satisfying the following axioms:

```
• \forall x \in X \ (\star(\star(x)) = \dagger(\star(x)) = \star(x) \land \star(\dagger(x)) = \dagger(\dagger(x)) = \dagger(x));
```

- dom $[\circ] = \{\langle x, y \rangle : t(x) = \star(y)\}$ and rng $[\circ] \subseteq X$;
- $\forall x \in X \ (\circ(\star(x), x) = x = \circ(x, \dagger(x));$
- $\forall x \in X \ \forall y \in X \ (\dagger(x) = \star(y) \ \Rightarrow \ \star(\circ(x,y)) = \star(x) \ \land \ \dagger(\circ(x,y)) = \dagger(y));$
- $\forall x \in X \ \forall y \in X \ \forall z \in X \ ((\dagger(x) = \star(y) \ \land \ \dagger(y) = \star(z)) \ \Rightarrow \ \circ(\circ(x,y),z) = \circ(x,\circ(y,z)).$

51. Morphisms of Mathematical Structures

For two mathematical structures (X, S), (X', S') a function $f: X \to X'$ is called a *morphism* of the mathematical structures if f respects the structures in a certain sense (depending on the type of the structures). Definitions of morphisms should be chosen so that the composition of two morphisms between mathematical structures of the same type remain a morphism of mathematical structures of that type.

For the structure of magma and its specifications (semigroups, monoids, groups) morphisms are called homomorphims and are defined as follows.

Definition 51.1. For two magmas (X, S_X) and (Y, S_Y) a function $f: X \to Y$ is called a homomorphism if $\forall x \in X \ \forall y \in X \ (f(S_X(x,y)) = S_Y(f(x), f(y))).$

For the structure of a ring (and field) homomorphisms should preserve both operations (of addition and multiplication).

Definition 51.2. For two rings $(X, (+, \cdot))$ and $(Y, (\oplus, \odot))$ a function $f: X \to Y$ is called a homomorphism if $\forall x \in X \ \forall y \in X \ \big(f(x+y) = f(x) \oplus f(y) \ \land \ f(x \cdot y) = f(x) \odot f(y) \big)$.

A far generalization of a magma homomorphisms are homomorphisms of universal algebras.

Definition 51.3. Let σ be a function with $\operatorname{rng}[\sigma] \subseteq \omega$ and (X, S), (Y, S') be two universal algebras of signature σ . A function $f: X \to Y$ is called a homomorphism of universal algebras if

• $\forall i \in \text{dom}[\sigma] \ \forall x \in X^{\sigma(i)} \ (f(S_i(x)) = S_i'(f \circ x)).$

Definition 51.4. Let σ be a function with $\operatorname{rng}[\sigma] \subseteq \omega$ and (X, S), (Y, S') be two universal relation structures of signature σ . A function $f: X \to Y$ is called a homomorphism of relation structures if

• $\forall i \in \mathsf{dom}[\sigma] \ \forall x \ (x \in S_i \Rightarrow f \circ x \in S'_i)$.

For the structure of a topological space, morphisms are defined as continuous functions.

Definition 51.5. For two topological spaces (X, S_X) and (Y, S_Y) a function $f: X \to Y$ is continuous if $\forall u \ (u \in S_Y \Rightarrow f^{-1}[u] \in S_X)$.

For the structure of a duoform space, morphisms are defined as duomorph functions.

Definition 51.6. For two duoform spaces (X, S_X) and (Y, S_Y) a function $f: X \to Y$ is called duoform if

- $\forall u \in S_Y \ \exists v \in S_X \ \forall x \ \forall y \ (\langle x, y \rangle \in v \ \Rightarrow \ \langle f(x), f(y) \rangle \in u);$ $\forall v \in S_X \ \exists u \in S_Y \ \forall x \ \forall y \ (\langle x, y \rangle \in v \ \Rightarrow \ \langle f(x), f(y) \rangle \in u).$

Definition 51.7. For two objectness categories $(X, (\star, +, \circ))$ and $(X', (\star', +', \circ'))$ a function $F: X \to X'$ is called a functor between objectless categories if

- $\forall x \in X \ (\star'(F(x)) = F(\star(x)) \land +'(F(x)) = F(+(x)));$
- $\forall x \in X \ \forall y \in X \ (\dagger(x) = \star(y) \Rightarrow F(\circ(x,y)) = \circ'(F(x),F(y)).$

Exercise 51.8. Prove that the compositions of morphisms considered in Definitions 51.1–51.6 remain morphisms in the sense of those definitions.

Definition 51.9. Let (X, Γ_X) and (Y, Γ_Y) be two digraphs. A function $f: X \to Y$ is called

- a digraph homomorphism if $\forall x \ \forall x' \in X \ (\langle x, x' \rangle \in \Gamma_X \ \Rightarrow \ \langle f(x), f(x') \rangle \in \Gamma_Y);$
- a digraph isomomorphism if f is bijective and the functions f and f^{-1} are digraph homomorphisms.

A digraph (X, Γ) is called *extensional* if

$$\forall x \in X \ \forall y \in X \ (x = y \ \Leftrightarrow \ \overleftarrow{\Gamma}(x) = \overleftarrow{\Gamma}(y), \quad \text{where} \quad \overleftarrow{\Gamma} = \{x' \in X : \langle x', x \rangle \in \Gamma\} \setminus \{x\}.$$

Theorem 51.10 (Mostowski-Shepherdson collapse). Let (X,Γ) be a simple directed graph such that the relation Γ is set-like and well-founded. Then there exists a unique function $f: X \to \mathbf{U}$ such that $f(x) = f[\bar{\Gamma}(x)]$ for all $x \in X$ and $Y = f[X] \subset \mathbf{V}$ is a transitive class. The function f is a homomorphism between the digraphs (X,Γ) and $(Y,\mathbf{E}|Y)$. The relation Γ is extensional if and only if f is a isomorphism of the digraphs (X,Γ) and $(Y,\mathbf{E}\upharpoonright Y)$.

Proof. By Recursion Theorem 21.6, for the function

$$F: \Gamma \times \mathbf{U} \to \mathbf{U}, \quad F: \langle \gamma, u \rangle \mapsto u,$$

there exists a unique function $f: X \to \mathbf{U}$ such that

$$f(x) = F(x, \{f(x') : x' \in \overline{\Gamma}(x)\}) = \{f(x') : x' \in \overline{\Gamma}(x)\} = f[\overline{\Gamma}(x)]$$

for all $x \in X$.

Consider the class Y = f[X]. Given any set $y \in Y$, find $x \in X$ such that y = f(x) and observe that $y = f(x) = f[\overline{\Gamma}(x)] \subseteq f[X] = Y$, which means that the class Y is transitive.

Assuming that $Y \not\subseteq \mathbf{V}$, we conclude that the set $A = \{x \in X : f(x) \notin \mathbf{V}\}$ is not empty. By the well-foundedness of the relation Γ , there exists an element $a \in A$ such that $\overline{\Gamma}(a) \cap A = \emptyset$ and hence $f[\bar{\Gamma}(a)] \subseteq \mathbf{V}$ and $f(a) = f[\bar{\Gamma}(a)] \in \mathbf{V}$, see Theorem 24.2(6). But the inclusion $f(a) \in \mathbf{V}$ contradicts the choice of a.

To see that the function $f: X \to Y$ is a homomorphism of the digraphs (X, Γ) and $(Y, \mathbf{E} \upharpoonright Y)$, take any pair $\langle x', x \rangle \in \Gamma$. Then $x' \in \overline{\Gamma}(x)$ and hence $f(x) \in f(x')$, which is equivalent to $\langle f(x), f(x') \rangle \in \mathbf{E}.$

If the function $f: X \to Y$ is an isomorphism of the digraphs (X, Γ) and $(Y, \mathbf{E} \upharpoonright Y)$, then the relation Γ is extensional by the extensionality of the membership relation, which is postulated by the Axiom of Extensionality. Now assume that the relation Γ is extensional.

We claim that the function f is injective. In the opposite case we can find a set $z \in Y \subseteq \mathbf{V}$ such that $z = f(x) \neq f(x')$. Find an ordinal α such that $z \in V_{\alpha}$. We can assume that α is the smallest possible, i.e., for any $y \in Y \cap \bigcup_{\beta \in \alpha} V_{\beta}$ with y = f(x) = f(x') we have x = x'. The minimality of α ensures that $\alpha = \beta + 1$ for some $\beta \in \alpha$. By the extensionality of the relation Γ , the sets $\bar{\Gamma}(x)$ and $\bar{\Gamma}(x')$ are distinct. Consequently, there exists $x'' \in X$ such that $x'' \in \bar{\Gamma}(x) \setminus \bar{\Gamma}(x')$ or $x'' \in \bar{\Gamma}(x') \setminus \bar{\Gamma}(x)$. In the first case we have $f(x'') \in f(x) \in V_{\beta+1} = \mathcal{P}(V_{\beta})$ and hence $f(x'') \in V_{\beta}$. The minimality of α , and $x' \notin \bar{\Gamma}(x')$ imply $f(x'') \notin f[\bar{\Gamma}(x')] = f(x') = f(x)$ which is a contradiction. By analogy we can derive a contradiction from the assumption $x'' \in \bar{\Gamma}(x') \setminus \bar{\Gamma}(x)$. This contradiction completes the proof of the injectivity of f. Since Y = f[X], the function $f: X \to Y$ is bijective.

It remains to prove that $f^{-1}: Y \to X$ is a digraph homomorphism. Indeed, for any $x', x \in X$ with $\langle f(x'), f(x) \rangle \in \mathbf{E} \upharpoonright Y$ we have $f(x') \in f(x) = f[\tilde{\Gamma}(x)]$ and by the injectivity of $f, x' \in \tilde{\Gamma}(x)$ and finally $\langle x', x \rangle \in \Gamma$.

Remark 51.11. The Mostowski–Shepherdson collapse is often applied to extensional digraphs (X, Γ) whose relation Γ satisfies the axioms of ZFC or NBG. Such digraphs are called *models* of ZFC or NBG, respectively. In this case Mostowski–Shepherdson collapse says that each model of ZFC or NBG is isomorphic to a model $(X, \mathbf{E} | X)$ for a suitable class X.

52. A CHARACTERIZATION OF THE REAL LINE

The real line is the most fundamental object in mathematics. It carries a bunch of mathematical structures: additive group, multiplicative semigroup, ring, field, linearly ordered set, metric space, topological space, bornological space, topological field etc etc. Some combinations of these structures determine the real line uniquely up to an isomorphisms. For Analysis the most important structure determining the real line uniquely is the structure of an ordered field.

We recall that an ordered field is a mathematical structure (X, S) whose structure is a triple $S = (+, \cdot, \leq)$ consisting of two binary operations and a linear order on X satisfying the axioms described in Example 50.33.

We say that two ordered fields $(X, (+, \cdot, <))$ and $(Y, (\oplus, \odot, \preceq))$ are isomorphic if there exists a bijective function $f: X \to X'$ that preserves the structure in the sense that

$$f(x+y) = f(x) \oplus f(y), \quad f(x \cdot y) = f(x) \odot f(y) \quad \text{and} \quad x \le y \iff f(x) \le f(y)$$

for any elements $x, y \in X$. If an isomorphism f between ordered fields exists and is unique, then we say that these fields are uniquely isomorphic.

Theorem 52.1. An ordered field $(X, (\oplus, \otimes, \prec))$ is (uniquely) isomorphic to the real line $(\mathbb{R}, (+, \cdot, \leq))$ if and only if its order \prec is boundedly complete.

Proof. The "only if" part follows from the bounded completeness of the linear order on the real line, which was established in Theorem 48.6.

To prove the "if" part, assume that $(X, (\oplus, \otimes, \prec))$ is an ordered field with boundedly complete linear order \prec . Let 0 and 1 be the identity element of the group (X, \oplus) and 1 be the multiplicative element of the semigroup (X, \odot) . Let $f_{\omega} : \omega \to X$ be the function defined by the recursive formula: $f_{\omega}(0) = 0$ and $f_{\omega}(n+1) = f_{\omega}(n) \oplus 1$ for every $n \in \omega$. Using the

connection between the addition and order in the ordered group $(X, (\oplus, \prec))$ we can prove (by Mathematical Induction) that $f_{\omega}(n) \prec f_{\omega}(n+1)$ for all $n \in \omega$. This implies that the function f_{ω} is injective. By Mathematical Induction, it can be shown that $f_{\omega}(n+m) = f_{\omega}(n) \oplus f_{\omega}(m)$ and $f_{\omega}(n \cdot m) = f(n) \odot f(m)$ for any $n, m \in \omega$.

Extend f_{ω} to a function $f_{\mathbb{Z}}: \mathbb{Z} \to X$ such that $f_{\mathbb{Z}}(\neg n) = -f_{\omega}(n)$ for every $n \in \mathbb{N}$. Here $-f_{\omega}(n)$ is the additive inverse of $f_{\omega}(n)$ in the group (X, \oplus) . Using suitable properties of the ordered group $(X, (\oplus, \prec))$ we can show that $f_{\mathbb{Z}}(\neg n) \prec 0$ for every $n \in \mathbb{N}$. Using the properties of the addition and multiplication in the field $(X, (\oplus, \odot))$ it can be shown that $f_{\mathbb{Z}}(n+m) = f_{\mathbb{Z}}(n) \oplus f_{\mathbb{Z}}(m)$ and $f_{\mathbb{Z}}(n \cdot m) = f_{\mathbb{Z}}(n) \odot f_{\mathbb{Z}}(m)$ for all $n, m \in \mathbb{Z}$.

Next, extend the function $f_{\mathbb{Z}}$ to a function $f_{\mathbb{Q}}: \mathbb{Q} \to X$ letting $f_{\mathbb{Q}}(\frac{m}{n}) = f_{\mathbb{Z}}(m) \odot (f_{\omega}(n))^{-1}$ for any rational number $\frac{m}{n} \in \mathbb{Q} \setminus \mathbb{Z}$. In this formula $(f_{\omega}(n))^{-1}$ is the multiplicative inverse of $f_{\omega}(n)$ in the multiplicative group $(X \setminus \{0\}, \odot)$. Using algebraic properties of the filed $(X, (\oplus, \odot))$, it can be shown that the function $f_{\mathbb{Q}}: \mathbb{Q} \to X$ is injective and preserves the field operations and the order.

The bounded completeness of the orderd field $(X, (\oplus, \odot, \prec))$ implies that this field is Archimedean in the sense that for any $x \in X$ there exists $n \in \omega$ such that $x \prec f(n)$. Assuming that such a number n does not exist, we conclude that the set $f_{\omega}[\omega] \subseteq X$ is upper bounded by x in the linear order (X, \prec) and by the bounded completeness, $f_{\omega}[\omega]$ has the least upper bound $\sup f_{\omega}[\omega]$. On the other hand, $\sup f_{\omega}[\omega] \oplus (-1) \prec \sup f_{\omega}[\omega]$ also is an upper bound for $f_{\omega}[\omega]$, which a contradiction showing that the field $(X, (\oplus, \odot, \prec))$ is Archimedean.

Now extend $f_{\mathbb{Q}}$ to a function $f_{\mathbb{R}}: \mathbb{R} \to X$ assigning to each real number $r \in \mathbb{R} \setminus \mathbb{Q}$ the element $\sup f_{\mathbb{Q}}(\{q \in \mathbb{Q}: q < r\})$ of the set X. This element exists by the bounded completenes of X. Using the Archimedean property of the field $(X, (\oplus, \odot, \prec))$, it can be shown that $f_{\mathbb{R}}$ is a required isomorphism of the fields $(\mathbb{R}, (+, \cdot, <))$ and $(X, (\oplus, \odot, \prec))$. The uniqueness of $f_{\mathbb{R}}$ follows from the construction: at each step there is a unique way to extend the function so that the field operations and the order are preserved.

Exercise 52.2 (Tarski). Prove that a bijective function $f : \mathbb{R} \to \mathbb{R}$ is the identity function of \mathbb{R} if and only if it has the following three properties:

- (1) f(1) = 1;
- (2) $\forall x \in \mathbb{R} \ \forall y \in \mathbb{R} \ (f(x+y) = x+y);$
- (3) $\forall x \in \mathbb{R} \ \forall y \in \mathbb{R} (x < y \iff f(x) < f(y)).$

Part 12. Elements of Category Theory

A mathematician is a person who can find analogies between theorems; a better mathematician is one who can see analogies between proofs and the best mathematician can notice analogies between theories. One can imagine that the ultimate mathematician is one who can see analogies between analogies.

Stefan Banach

The language of categories is affectionately known as "abstract nonsense", so named by Norman Steenrod. This term is essentially accurate and not necessarily derogatory: categories refer to "nonsense" in the sense that they are all about the "structure", and not about the "meaning", of what they represent.

Paolo Aluffi, 2009

Theory of Categories was founded in 1942–45 by Samuel Eilenberg and Saunders Mac Lane who worked in Algebraic Topology and needed tools for describing common patterns appearing in topology and algebra. Category Theory was created as a science about structures and patterns appearing in mathematics. Rephrasing Stefan Banach, we could say that Category Theory is a science about analogies between analogies.

By some mathematicians, Category Theory is considered as an alternative (to Set Theory) foundation for mathematics. In this respect, Saunders MacLane, one of founders of Category Theory writes the following [16].

... the membership relation for sets can often be replaced by the composition operation for functions. This leads to an alternative foundation for Mathematics upon categories specifically, on the category of all functions. Now much of Mathematics is dynamic, in that it deals with morphisms of an object into another object of the same kind. Such morphisms (like functions) form categories, and so the approach via categories fits well with the objective of organizing and understanding Mathematics. That, in truth, should be the goal of a proper philosophy of Mathematics.

The standard "foundation" for mathematics starts with sets and their elements. It is possible to start differently, by axiomatising not elements of sets but functions between sets. This can be done by using the language of categories and universal constructions.

In this chapter we discuss some basic concepts of Category Theorem, but define them using the language of the Classical Set Theory. The culmination result of this part are Theorem 58.13 and 58.14 characterizing categories that are isomorphic to the category of sets.

53. Categories

Definition 53.1. A category is a 6-tuple $\mathcal{C} = (\mathsf{Ob}, \mathsf{Mor}, \star, \dagger, 1, \circ)$ consisting of

- a class Ob whose elements are called *objects* of the category \mathcal{C} (briefly, \mathcal{C} -objects);
- a class Mor whose elements are called *morphisms* of the category C (briefly, C-morphisms);
- two functions \star : Mor \to Ob and \dagger : Mor \to Ob assigning to each morphism $f \in$ Mor its source $\star(f) \in$ Ob and target $\dagger(f) \in$ Ob;
- a function 1 : Ob \rightarrow Mor assigning to each object $X \in$ Ob a morphism $1_X \in$ Mor, called the *identity morphism* of X, and satisfying the equality $\star(1_X) = X = +(1_X)$;

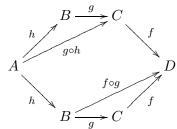
- a function \circ with domain $\mathsf{dom}[\circ] = \{\langle f, g \rangle \in \mathsf{Mor} \times \mathsf{Mor} : \mathsf{t}(g) = \mathsf{t}(f)\}$ and range $\mathsf{rng}[\circ] \subseteq \mathsf{Mor}$ assigning to any $\langle f, g \rangle \in \mathsf{dom}[\circ]$ a morphism $f \circ g \in \mathsf{Mor}$ such that $\mathsf{t}(f \circ g) = \mathsf{t}(g)$ and $\mathsf{t}(f \circ g) = \mathsf{t}(f)$ and the following axioms are satisfied:
 - (A) for any morphisms $f, g, h \in Mor$ with $\star(f) = \dagger(g)$ and $\star(g) = \dagger(h)$ we have $(f \circ g) \circ h = f \circ (g \circ h)$;
 - (U) for any morphism $f \in \mathsf{Mor}$ we have $1_{\dagger(f)} \circ f = f = f \circ 1_{\star(f)}$.

The function \circ is called the *operation of composition* of morphisms and the axiom (A) is called the *associativity* of the composition.

Discussing several categories simultaneously, it will be convenient to label the classes of objects and and morphisms of a category \mathcal{C} with subscripts writing $\mathsf{Ob}_{\mathcal{C}}$ and $\mathsf{Mor}_{\mathcal{C}}$.

For explaining definitions and results of Category Theory it is convenient to use arrow notations. Morphisms between objects are denoted by arrows with subscripts or supersripts, and equalities of compositions of morphisms are expresses by commutative diagrams.

For example, the associativity of the composition can be expresses as the commutativity of the diagram



Definition 53.2. A category $\mathcal{C} = (\mathsf{Mor}, \mathsf{Ob}, \star, +, \circ)$ is called

- *small* if its class of morphisms Mor is a set;
- locally small if for any objects $X, Y \in \mathsf{Ob}$ the class of morphisms $\mathsf{Mor}(X,Y) = \{ f \in \mathsf{Mor} : \star(f) = X, \ \dagger(f) = Y \}$ is a set;
- discrete if for any objects $X, Y \in \mathsf{Ob}$

$$\mathsf{Mor}(X,Y) = \begin{cases} \{1_X\} & \text{if } X = Y; \\ \emptyset & \text{if } X \neq Y. \end{cases}$$

Mathematics is literally saturated with categories. We start with the category of sets, one of the most important categories in Mathematics.

Example 53.3. The category of sets **Set** is the 6-tuple (Ob, Mor, \star , \dagger , 1, \circ) consisting of

- the class Ob = U:
- the class $\mathsf{Mor} = \{ \langle X, f, Y \rangle \in \mathbf{U} \times \mathbf{Fun} \times \mathbf{U} : X = \mathsf{dom}[f], \, \mathsf{rng}[f] \subseteq Y \};$
- the function $\star : \mathsf{Mor} \to \mathsf{Ob}, \, \star : \langle X, f, Y \rangle \mapsto X;$
- the function $+: \mathsf{Mor} \to \mathsf{Ob}, +: \langle X, f, Y \rangle \mapsto Y;$
- the function $1: \mathsf{Ob} \to \mathsf{Mor}, \, 1: X \mapsto \langle X, \mathbf{Id} \upharpoonright_X, X \rangle;$
- the function $\circ = \{ \langle \langle \langle A, f, B \rangle, \langle C, g, D \rangle \rangle, \langle A, gf, D \rangle \rangle \in (\mathsf{Mor} \times \mathsf{Mor}) \times \mathsf{Mor} : B = C \}$ where $gf = \{ \langle x, z \rangle : \exists y \ (\langle x, y \rangle \in f \ \land \ \langle y, z \rangle \in g) \}.$

Taking for morphisms the class of functions **Fun**, we obtain the category of sets and their surjective maps.

Example 53.4. The category of sets and their surjective functions is the 6-tuple (Ob, Mor, \star , \dagger , 1, \circ) consisting of

- the class Ob = U;
- the class Mor = Fun;
- the function $\star = \mathsf{dom} \upharpoonright_{\mathbf{Fun}}$;
- the function $\dagger = \mathsf{rng} \upharpoonright_{\mathbf{Fun}}$;
- the function $1:\mathsf{Ob}\to\mathsf{Mor},\, 1:X\mapsto\mathbf{Id}\!\upharpoonright_X;$
- the function $\circ = \{ \langle \langle f, g \rangle, gf \rangle \in (\mathsf{Mor} \times \mathsf{Mor}) \times \mathsf{Mor} : \mathsf{rng}[f] = \mathsf{dom}[g] \}$ where $gf = \{ \langle x, z \rangle : \exists y \ (\langle x, y \rangle \in f \ \land \ \langle y, z \rangle \in g) \}.$

Definition 53.5. A category $C = (\mathsf{Ob}, \mathsf{Mor}, \star, \dagger, 1, \circ)$ is a *subcategory* of a category $C' = (\mathsf{Ob'}, \mathsf{Mor'}, \star', \dagger', 1', \circ')$ if

$$\mathsf{Ob} \subseteq \mathsf{Ob}', \; \mathsf{Mor} \subseteq \mathsf{Mor}', \; \star = \star' \upharpoonright_{\mathsf{Mor}}, \; \dagger = \dagger' \upharpoonright_{\mathsf{Mor}}, \; 1 = 1' \upharpoonright_{\mathsf{Ob}}, \; \mathrm{and} \; \circ = \circ' \upharpoonright_{\mathsf{Mor} \times \mathsf{Mor}}.$$

A subcategory \mathcal{C} of \mathcal{C}' is called *full* if $\forall X \in \mathsf{Ob} \ \forall Y \in \mathsf{Ob} \ \mathsf{Mor}(X,Y) = \mathsf{Mor}'(X,Y)$.

A full subcategory is fully determined by its class of objects.

Example 53.6. Let **FinSet** be the full subcategory of the category **Set**, whose class of objects coincides with the class of finite sets.

Example 53.7. Let **Card** be the full subcategory of the category **Set**, whose class of objects coincides with the class of cardinals.

An important example of a category is the category of mathematical structures. We recall that a mathematical structure is a pair of classes (X,S) satisfying certain list of axioms. If X and S are sets, then the pair (X,S) can be identified with the ordered pair $\langle X,S\rangle$, which is an element of the class $\ddot{\mathbf{U}} = \mathbf{U} \times \mathbf{U}$. The underlying set X and the structure S can be recovered from the ordered pair $\langle X,S\rangle$ using the functions dom and rng as $X = \text{dom}(\langle X,S\rangle)$ and $S = \text{rng}(\langle X,S\rangle)$.

Example 53.8. The category of mathematical structures **MS** is the 6-tuple (Ob, Mor, \star , \dagger , 1, \circ) consisting of

- the class $Ob = U \times U$;
- the class $\mathsf{Mor} = \{ \langle X, f, Y \rangle \in \mathsf{Ob} \times \mathsf{Fun} \times \mathsf{Ob} : \mathsf{dom}[f] = \mathsf{dom}(X) \wedge \mathsf{rng}[f] \subseteq \mathsf{rng}(Y) \};$
- the function $\star : \mathsf{Mor} \to \mathsf{Ob}, \, \star : \langle X, f, Y \rangle \mapsto X;$
- the function $+: \mathsf{Mor} \to \mathsf{Ob}, \langle X, f, Y \rangle \mapsto Y;$
- the function $1: \mathsf{Ob} \to \mathsf{Mor}, \ 1: X \mapsto \langle X, \mathbf{Id} \upharpoonright_{\mathsf{dom}(X)}, X \rangle;$
- the function $\circ = \{ \langle \langle A, f, B \rangle, \langle C, g, D \rangle \rangle, \langle A, gf, D \rangle \rangle \in (\mathsf{Mor} \times \mathsf{Mor}) \times \mathsf{Mor} : B = C \}$ where $gf = \{ \langle x, z \rangle : \exists y \ (\langle x, y \rangle \in f \ \land \ \langle y, z \rangle \in g) \}.$

Many important category arise as subcategories of the category MS.

Example 53.9. The category of magmas Mag is a subcategory of the category MS. Its objects are magmas and morphisms are triples $\langle X, f, Y \rangle$ where f is a homomorphism of magmas X, Y.

Example 53.10. The categories of semigroups, inverse semigroups, Clifford semigroups, monoids, groups, commutative groups are full subcategories of the category of magmas. The objects of these categories are semigroups, inverse semigroups, Clifford semigroups, monoids, groups, commutative groups, respectively.

Example 53.11. The category of topological spaces **Top** is the subcategory of the category **MS**. The object of the category **Top** are topological spaces and morphisms are triples $\langle X, f, Y \rangle$ where f is a continuous function between topological spaces X and Y.

Example 53.12. The category of directed graphs is the subcategory of the category **MS**. The object of this category are directed graphs and morphisms are triples $\langle X, f, Y \rangle$ where f is an increasing function between directed graphs X and Y. The category of directed graphs contains full subcategories of ordered sets, partially ordered sets, linearly ordered sets, well-ordered sets.

Example 53.13. Each monoid (X, S) can be identified with the category $(\mathsf{Ob}, \mathsf{Mor}, \star, \dagger, 1, \circ)$ such that

```
• Ob = \{X\};
```

- Mor = X;
- $\star = \dagger = \mathsf{Mor} \times \mathsf{Ob};$
- $1 = \mathsf{Ob} \times \{e\}$ where $e \in X$ is the unit of the monoid (X, S);
- $\bullet \circ = S$

On the other hand, for any category $\mathcal{C} = (\mathsf{Ob}, \mathsf{Mor}, \star, \dagger, 1, \circ)$ with a single object, the pair (Mor, \circ) is a monoid.

Example 53.14. Each partially ordered class (X, S) can be identified with the category $(\mathsf{Ob}, \mathsf{Mor}, \star, \dagger, 1, \circ)$ such that

```
• \mathsf{Ob} = X;
```

- Mor = S;
- $\star = \mathsf{dom} \upharpoonright_X$;
- $\dagger = \operatorname{rng} \upharpoonright_X;$
- $1 = \{\langle x, \langle x, x \rangle \rangle : x \in X\};$
- \circ is the function assigning to any pair of pairs $\langle \langle x, y \rangle, \langle u, v \rangle \rangle \in S \times S$ with y = u the pair $\langle x, v \rangle$ which belongs to the order S by the transitivity of S.

Definition 53.15. For any category $C = (\mathsf{Ob}, \mathsf{Mor}, \star, \dagger, 1, \circ)$ the *dual* (or else *opposite*) category to C is the category $C^{\mathsf{op}} = (\mathsf{Ob}^{\mathsf{op}}, \mathsf{Mor}^{\mathsf{op}}, \star^{\mathsf{op}}, \dagger^{\mathsf{op}}, 1^{\mathsf{op}}, \circ^{\mathsf{op}})$ such that

```
• Ob^{op} = Ob, Mor^{op} = Mor, and 1^{op} = 1;
```

- $\star^{\text{op}} = \dagger$, $\dagger^{\text{op}} = \star$;
- $\circ^{\text{op}} = \{ \langle \langle g, f \rangle, h \rangle : \langle \langle f, g \rangle, h \rangle \in \circ \}.$

In the dual category all arrows are reverted and the composition of arrows is taken in the reverse order.

The philosophy of Category Theory is to derive some properties of objects from the information about morphisms related to these objects. A category is a kind of algebraic structure that operates with morphisms, not objects. Without any loss of information, objects can be identified with their unit morphisms. After such reduction a category becames a typical algebraic structure on the class Mor of morphisms.

Definition 53.16. Let \mathcal{C} be a category. A morphism $f \in \mathsf{Mor}_{\mathcal{C}}(X,Y)$ between \mathcal{C} -objects X,Y is called an *isomorphism* (more precisely, a \mathcal{C} -isomorphism) if there exists a morphism $g \in \mathsf{Mor}_{\mathcal{C}}(Y,X)$ such that $g \circ f = 1_X$ and $f \circ g = 1_Y$. The morphism g is unique and is denoted by f^{-1} . For \mathcal{C} -objects X,Y by $\mathsf{Iso}_{\mathcal{C}}(X,Y)$ we shall denote the subclass of $\mathsf{Mor}_{\mathcal{C}}(X,Y)$ consisting of isomorphisms.

Exercise 53.17. Prove that for any isomorphism $f \in \mathsf{Mor}_{\mathcal{C}}(X,Y)$ of a category \mathcal{C} , the morphism f^{-1} is unique.

```
Hint: If g \in \mathsf{Mor}_{\mathcal{C}}(Y,X) is a morphism such that g \circ f = 1_X and f \circ g = 1_Y, then g = g \circ 1_Y = g \circ (f \circ f^{-1}) = (g \circ f) \circ f^{-1} = 1_X \circ f^{-1} = f^{-1}.
```

Definition 53.18. Two objects $X, Y \in \mathsf{Ob}_{\mathcal{C}}$ of a category \mathcal{C} are called *isomorphic* (more precisely, \mathcal{C} -isomorphic) if there exists an isomorphism $f \in \mathsf{Mor}(X,Y)$. The isomorphness of X, Y will be denoted as $X \cong Y$ or $X \cong_{\mathcal{C}} Y$.

From the point of view of Category Theory, isomorphic objects have the same properties (which can be expresses in the language of morphisms).

Exercise 53.19. Prove that (i) in the category of sets, isomorphisms are bijective maps; (ii) in the category of magmas, isomorphisms are bijective homomorphisms of magmas.

Example 53.20. Any inverse semigroup (X, S) can be identified with the category $(\mathsf{Ob}, \mathsf{Mor}, \star, +, 1, \circ)$ such that

```
 \begin{split} \bullet & \text{ Ob} = \{S(x,x^{-1}): x \in X\}; \\ \bullet & \text{ Mor} = X; \\ \bullet & \star = \{\langle x,S(x^{-1},x)\rangle: x \in X\}; \\ \bullet & + = \{\langle x,S(x,x^{-1}x)\rangle: x \in X\}; \\ \bullet & 1 = \mathbf{Id} \upharpoonright_{\mathbf{Ob}}; \\ \bullet & \circ = S. \end{split}
```

Each morphism of this category is an isomorphism.

Now we define category analogs of injective and surjective functions.

Definition 53.21. Let $\mathcal{C} = (\mathsf{Ob}, \mathsf{Mor}, \star, \dagger, 1, \circ)$ be a category. A morphism $f \in \mathsf{Mor}(X, Y)$ between two \mathcal{C} -objects X, Y is called

- a monomorphism if $\forall Z \in \mathsf{Ob} \ \forall g, h \in \mathsf{Mor}(Z, X) \ (f \circ g = f \circ h \ \Rightarrow \ g = h);$
- a epimorphism if $\forall Z \in \mathsf{Ob} \ \forall g, h \in \mathsf{Mor}(Y, Z) \ (g \circ f = h \circ f \ \Rightarrow \ g = h);$
- \bullet a bimorphism if f is both monomorphism and epimorphism.

For two \mathcal{C} -objects X, Y by $\mathsf{Mono}_{\mathcal{C}}(X, Y)$ and $\mathsf{Epi}_{\mathcal{C}}(X, Y)$ we denote the subclasses of $\mathsf{Mor}_{\mathcal{C}}(X, Y)$ constisting of monomorphisms and epimorphisms from a to b, respectively.

Definition 53.22. A category is *balanced* if each bimorphism of this category is an isomorphism.

Exercise 53.23. Prove that a morphism f of a category \mathcal{C} is a monomorphism if and only if f is an epimorphism of the dual category \mathcal{C}^{op} .

Exercise 53.24. Prove that that in the category of sets (and in the category of topological spaces) monomorphisms are injective functions and epimorphisms are surjective functions.

Exercise 53.25. Prove that the category of sets is balances but the category of topological spaces is not balanced.

Exercise 53.26. Find a homomorphism $h: X \to Y$ of two monoids, which is not a surjective function but is an epimorphism in the category of monoids.

Hint: Consider the identity function $\mathbf{Id} \upharpoonright_{\mathbb{N}} : \mathbb{N} \to \mathbb{Z}$ of the monoids $(\mathbb{N}, +)$ and $(\mathbb{Z}, +)$.

Exercise 53.27. Given a monoid (X, M) characterize monomorphisms and epimorphisms of the category described in Exercise 53.13.

Now using the properties of morphisms we distinguish two special types of objects.

Definition 53.28. Let \mathcal{C} be a category. A \mathcal{C} -object X is called

- initial (more precisely, C-initial) if for any C-object Y there exists a unique C-morphism $X \to Y$:
- terminal (more precisely, C-terminal) if for any C-object Z there exists a unique Cmorphism $Z \to X$.

Exercise 53.29. Prove that any initial (resp. terminal) objects of a category are isomorphic.

Exercise 53.30. Prove that an object of a category \mathcal{C} is terminal if and only if it is an initial object of the dual category \mathcal{C}^{op} .

Exercise 53.31. Prove that a set X is an initial (resp. terminal) object of the category of sets if and only if X is empty (resp. a singleton).

Exercise 53.32. Prove that a group G = (X, S) is an initial object of the category of groups **Grp** if and only if G is a terminal object of the category **Grp** if and only if X is a trivial group.

Exercise 53.33. Describe initial and terminal objects in the category of magmas, semigroups, inverse semigroups, Clifford semigroups, monoids.

Definition 53.34. A global element of an object X of a category $\mathcal C$ is any $\mathcal C$ -morphism $f: \mathbf{1} \to X$ form a terminal object 1 of \mathcal{C} to X.

Exercise 53.35. Describe global elements in the categories of sets, topological spaces, magmas, semigroups, inverse semigroups, Clifford semigroups, monoids, groups.

Finally, we define two operations on categories: product of categories and taking the category of morphisms.

Definition 53.36. For two categories $\mathcal{C} = (\mathsf{Ob}, \mathsf{Mor}, \star, \dagger, 1, \circ)$ and $\mathcal{C}' = (\mathsf{Ob'}, \mathsf{Mor'}, \star', \dagger', 1', \circ')$ their product $\mathcal{C} \times \mathcal{C}'$ is the category $(\mathsf{Ob''}, \mathsf{Mor''}, \star'', +'', 1'', \circ'')$ with

- $Ob'' = Ob \times Ob'$;
- $Mor'' = Mor \times Mor'$:
- $\star'' = \{ \langle \langle f, f' \rangle, \langle X, X' \rangle \rangle : \langle f, X \rangle \in \star \land \langle f', X' \rangle \in \star' \};$
- $+'' = \{ \langle \langle f, f' \rangle, \langle Y, Y' \rangle \rangle : \langle f, Y \rangle \in + \land \langle f', Y' \rangle \in +' \};$ $1'' = \{ \langle \langle X, X' \rangle, \langle f, f' \rangle \rangle : \langle X, f \rangle \in 1 \land \langle X', f' \rangle \in 1' \};$
- $\circ'' = \{ \langle \langle f, f' \rangle, \langle g, g' \rangle, \langle h, h' \rangle \rangle : (\langle f, g, h \rangle \in \circ) \land (\langle f', g', h' \rangle \in \circ') \}.$

Definition 53.37. For a category $\mathcal{C} = (\mathsf{Ob}, \mathsf{Mor}, \star, \dagger, 1, \circ)$ the category of \mathcal{C} -morphisms is the category $C^{\rightarrow} = (\mathsf{Ob}', \mathsf{Mor}', \star', +', 1', \circ')$ where

- Ob' = Mor:
- $\mathsf{Mor}' = \{ \langle f, \langle \alpha, \beta \rangle, g \rangle : f, g, \alpha, \beta \in \mathsf{Mor} \ \land \star(g) = \dagger(\alpha) \ \land \star(\beta) = \dagger(f) \ \land \ g \circ \alpha = \beta \circ f \};$
- $\star' = \{ \langle \langle f, \langle \alpha, \beta \rangle, g \rangle, h \rangle \in \mathsf{Mor}' \times \mathsf{Mor} : h = \widehat{f} \};$
- $\dagger' = \{ \langle \langle f, \langle \alpha, \beta \rangle, g \rangle, h \rangle \in \mathsf{Mor}' \times \mathsf{Mor} : h = g \};$
- $1' = \{\langle f, \langle f, \langle 1_{\star(f)}, 1_{\dagger(f)} \rangle, f \rangle : f \in \mathsf{Ob}'\};$
- $\circ' = \{ \langle \langle f, \langle \alpha, \beta \rangle, g \rangle, \langle g, \langle \alpha', \beta' \rangle, h \rangle, \langle f, \langle \alpha' \circ \alpha, \beta' \circ \beta \rangle, h \rangle \} :$

$$\langle f, \langle \alpha, \beta \rangle, g \rangle \in \mathsf{Mor}' \ \land \ \langle g, \langle \alpha', \beta' \rangle, h \rangle \in \mathsf{Mor}' \}.$$

Exercise 53.38. Illustrate the composition of morphisms of the category $\mathcal{C}^{\rightarrow}$ by commutative diagrams.

Each category can be identified with an objectless category, see Definition 50.38.

Remark 53.39. For each category (Ob, Mor, \star , \dagger , 1, \circ), the mathematical structure (Mor, (\star' , \dagger' , \circ')) where

- $\bullet \ \star' = \{\langle x, 1_{\star(x)} \rangle : x \in \mathsf{Mor}\},\$
- $t' = \{\langle x, 1_{t(x)} \rangle : x \in \mathsf{Mor}\},\$
- $\circ' = \{\langle x, y, z \rangle : \langle y, x, z \rangle \in \circ \}$

is an objectless category.

Conversely, for each objectless category $(X, (\star', +', \circ'))$ the 6-tuple $(\mathsf{Ob}, \mathsf{Mor}, \star, +, 1, \circ)$ consisting of

- the class $\mathsf{Ob} = \{x \in X : \star'(x) = x = +'(x)\};$
- the class Mor = X;
- the functions $\star = \star'$ and $\dagger = \dagger'$;
- the function $1 = \mathbf{Id} \upharpoonright_{\mathsf{Ob}}$;
- the function $\circ = \{\langle x, y, z \rangle : \langle y, x, z \rangle \in \circ' \}$

is a category.

In fact, without loss of information, the theory of categories can be well developed in its objectless form, but human intuition is better fit to object version of category theory.

54. Functors

Functors are functions between categories. The formal definition follows.

Definition 54.1. A functor $F: \mathcal{C} \to \mathcal{C}'$ between two categories $\mathcal{C} = (\mathsf{Ob}, \mathsf{Mor}, \star, +, 1, \circ)$ and $\mathcal{C}' = (\mathsf{Ob'}, \mathsf{Mor'}, \star', +', 1', \circ')$ is a pair $F = (\dot{F}, \ddot{F})$ of two functions $\dot{F}: \mathsf{Ob} \to \mathsf{Ob'}$ and $\ddot{F}: \mathsf{Mor} \to \mathsf{Mor'}$ such that

- $\bullet \ \forall X \in \mathsf{Ob} \ \big(\mathbf{1}'_{\dot{F}(X)} = \ddot{F}(\mathbf{1}_X) \big);$
- $\bullet \ \forall X,Y \in \mathsf{Ob} \ \big(\ddot{F}(\mathsf{Mor}(X,Y)) \subseteq \mathsf{Mor}'(\dot{F}(X),\dot{F}(Y))\big);$
- $\forall f, g \in \mathsf{Mor} \ (\star(g) = +(f) \Rightarrow \ddot{F}(g \circ f) = \ddot{F}(g) \circ '\ddot{F}(f)).$

In the sequel, we shall write FX and Ff instead of $\dot{F}(X)$ and $\ddot{F}(f)$, respectively.

Definition 54.2. A functor $F: \mathcal{C} \to \mathcal{C}'$ is called *faithful* (resp. *full*) if for any \mathcal{C} -objects X, Y, the function $F \upharpoonright_{\mathsf{Mor}_{\mathcal{C}}(X,Y)} : \mathsf{Mor}_{\mathcal{C}}(X,Y) \to \mathsf{Mor}_{\mathcal{C}'}(FX,FY)$ is injective (resp. surjective).

Example 54.3 (The embedding functor). For any (full) subcategory \mathcal{C} of a category \mathcal{C}' the identity embedding functor $1_{\mathcal{C},\mathcal{C}'}:\mathcal{C}\to\mathcal{C}'$ is the pair of functions $(\mathbf{Id}\!\upharpoonright_{\mathsf{Ob}_{\mathcal{C}}},\mathbf{Id}\!\upharpoonright_{\mathsf{Mor}_{\mathcal{C}}})$. This functor is faithful (and full). If $\mathcal{C}=\mathcal{C}'$, then the functor $1_{\mathcal{C},\mathcal{C}'}$ is denoted by $1_{\mathcal{C}}$.

Example 54.4 (Forgetful functor). Consider the functor $U: \mathbf{MS} \to \mathbf{Set}$ assigning to each mathematical structure $\langle X, S \rangle \in \mathbf{U} \times \mathbf{U}$ its underlying set X and to each morphism $\langle X, f, Y \rangle$ of the category \mathbf{MS} the morphism $\langle UX, f, UY \rangle$ of the category \mathbf{Set} . The functor U is called the *forgetful functor*. It is easy to see that this functor is faithfull. Then the restriction of the functor U to any subcategory of \mathbf{MS} also is a faithful functor.

This example motivates the following definition.

Definition 54.5. A category C is called *concrete* if it admits a faithful functor $F: C \to \mathbf{Set}$ to the category of sets \mathbf{Set} .

So, categories of mathematical structures are concrete.

Exercise* 54.6. Give an example of a category which is not concrete.

Hint: Such categories exist in Algebraic Topology (for example, the category of topological spaces and classes of homotopic maps).

Next, we consider two important functors reflecting the structure of any category in the categories **Set** and **Set**^{op}.

Example 54.7. For any locally small category $C = (Mor, Ob, \star, \dagger, 1, \circ)$ and any C-object C consider

- 1) the functor $\mathsf{Mor}(C,-): \mathcal{C} \to \mathbf{Set}$ assigning to any object $X \in \mathsf{Ob}$ the set $\mathsf{Mor}(C,X)$ and to any morphism $f \in \mathsf{Mor}$ the function $\mathsf{Mor}(C,f): \mathsf{Mor}(C,\star(f)) \to \mathsf{Mor}(C,+(f))$, $\mathsf{Mor}(C,f): g \mapsto f \circ g \in \mathsf{Mor}(C,+(f))$.
- 2) the functor $\mathsf{Mor}(-,C): \mathcal{C} \to \mathbf{Set}^\mathrm{op}$ assigning to any object $X \in \mathsf{Ob}$ the set $\mathsf{Mor}(X,C)$ and to any morphism $f \in \mathsf{Mor}$ the function $\mathsf{Mor}(f,C): \mathsf{Mor}(+(f),C) \to \mathsf{Mor}(\star(f),C)$, $\mathsf{Mor}(f,C): g \mapsto g \circ f$.

Exercise 54.8. Study the functors Mor(c, -) and Mor(-, c) for categories defined in Example 53.10.

Let $F: \mathcal{C} \to \mathcal{C}'$ and $G: \mathcal{C}' \to \mathcal{C}''$ be functors between categories $\mathcal{C}, \mathcal{C}', \mathcal{C}''$. We recall that these functors are pairs of functions (\dot{F}, \ddot{F}) and (\dot{G}, \ddot{G}) . Taking compositions of the corresponding components, we obtain a functor $(\dot{G} \circ \dot{F}, \ddot{G} \circ \ddot{F}): \mathcal{C} \to \mathcal{C}''$ denoted by GF and called the *composition* of the functors F, G.

Exercise 54.9. Show the composition of faithful (resp. full) functors is a faithful (resp. full) functor.

Any functor between small categories is a set. So, it is legal to consider the category **Cat** whose objects are small categories and morphisms are functors between small categories. Applying to this category the general notion of an isomorphism, we obtain the notion of isomorphic categories, which can be defined for any (not necessarily small) categories.

Definition 54.10. Two categories C and C' are called *isomorphic* if there exist functors $F: C \to C'$ and $G: C' \to C$ such that $FG = 1_{C'}$ and $GF = 1_{C}$. In this case we write $C \cong C'$.

A weaker notion is that of equivalent categories. To introduce this notion we need the notion of a natural transformation of functors.

55. Natural transformations

Definition 55.1. Let $\mathcal{C}, \mathcal{C}'$ be two categories and $F, G : \mathcal{C} \to \mathcal{C}'$ be two functors. A natural transformation $\eta : F \to G$ of the functors F, G is a function $\eta : \mathsf{Ob}_{\mathcal{C}} \to \mathsf{Mor}_{\mathcal{C}'}$ assigning to each \mathcal{C} -object X a \mathcal{C}' -morphism $\eta_X \in \mathsf{Mor}_{\mathcal{C}'}(FX, GX)$ so that for any \mathcal{C} -objects X, X and \mathcal{C} -morphism $f \in \mathsf{Mor}_{\mathcal{C}}(X, Y)$ the following diagram commutes.

$$FX \xrightarrow{\eta_X} GX$$

$$Ff \downarrow \qquad \qquad \downarrow Gf$$

$$FY \xrightarrow{\eta_X} GY$$

A natural transformation $\eta: F \to G$ is called an *isomorphism* of the funtors F, G if for every C-object X the morphism $\eta_X: FX \to GX$ is an isomorphism of the category C'.

Two functors $F, G : \mathcal{C} \to \mathcal{C}'$ are called *isomorphic* if there exists an isomorphism $\eta : F \to G$. In this case we write $F \cong G$.

Exercise 55.2. Prove that for any functors $F, G, H : \mathcal{C} \to \mathcal{C}'$ between categories $\mathcal{C}, \mathcal{C}'$ the following properties hold:

- \bullet $F \cong F$:
- $F \cong G \Rightarrow G \cong F$;
- $(F \cong G \land G \cong H) \Rightarrow F \cong H$.

Now we can introduce the notion of equivalence for categories.

Definition 55.3. Two categories \mathcal{C} and \mathcal{C}' are defined to be *equivalent* (denoted by $\mathcal{C} \simeq \mathcal{C}'$) if there are two functors $F: \mathcal{C} \to \mathcal{C}'$ and $G: \mathcal{C}' \to \mathcal{C}$ such that $FG \cong 1_{\mathcal{C}'}$ and $GF \cong 1_{\mathcal{C}}$.

Exercise 55.4. Prove that for any categories $\mathcal{C}, \mathcal{C}', \mathcal{C}''$ the following properties hold:

- $\mathcal{C} \simeq \mathcal{C}$;
- $\mathcal{C} \cong \mathcal{C}' \Rightarrow \mathcal{C}' \cong \mathcal{C};$
- $\bullet \ (\mathcal{C} \cong \mathcal{C}' \ \land \ \mathcal{C}' \cong \mathcal{C}'') \ \Rightarrow \ \mathcal{C} \cong \mathcal{C}''.$

Here by \simeq we denote the equivalence of categories.

Let \mathcal{X}, \mathcal{Y} be two categories. If the category \mathcal{X} is small, then any functor $F: \mathcal{X} \to \mathcal{Y}$ is a set, so it is legal to consider the category $\mathcal{Y}^{\mathcal{X}}$ whose objects are functors from \mathcal{X} to \mathcal{Y} and whose morphisms are natural transformations between functors. If the categories \mathcal{X} and \mathcal{Y} are small, then the set $\mathcal{Y}^{\mathcal{X}}$ coincides with the set of morphisms $\mathsf{Mor}(\mathcal{X}, \mathcal{Y})$ of the category Cat of small categories.

Exercise 55.5. Let $F: \mathcal{C} \to \mathcal{C}'$ be a functor. Prove that for any \mathcal{C} -isomorphism f the morphism Ff is a \mathcal{C}' -isomorphism.

Exercise 55.6. Let $F, G: \mathcal{C} \to \mathcal{C}'$ be isomorphic functors. Prove that for any functor

- 1) $H: \mathcal{C}' \to \mathcal{C}''$ the functors HF and HG are isomorphic;
- 2) $H: \mathcal{C}'' \to \mathcal{C}$ the functors FH and GH are isomorphic.

56. Skeleta and equivalence of categories

A category \mathcal{C} is called *skeletal* if any isomorphic objects in \mathcal{C} coincide.

A category S is called a *skeleton* of a category C if S is a full subcategory of C such that for any C-object X there exists a unique S-object Y, which is C-isomorphic to X. This definition implies that each skeleton of a category is a skeletal category.

The existence of skeleta in various categories implies from suitable forms of the Axiom of Choice.

Exercise 56.1. Using the Principle of Mathematical Induction, show that every finite category has a skeleton.

To prove the existence of skeleta in arbitrary categories we shall apply the choice principle (EC). This principle asserts that for every equivalence relation R there exists a class C such that for every $x \in \text{dom}[R]$ the intersection $R[\{x\}] \cap C$ is a singleton. The principle (GMP) is weaker than the Global Well-Orderability Principle (GWO) but stronger than the Axiom of Global Choice (AGC). On the other hand, (GWO) \Leftrightarrow (EC) \Leftrightarrow (AGC) under the Axiom of Cumulativity (AV) that follows from the Axiom of Foundation, see Section 28.

Theorem 56.2. Under (EC), each category C has a skeleton S. Moreover, there exists a full faithful functor $F: C \to S$ such that for the identity embedding functor $J: S \to C$ we have $FJ = 1_S$ and $JF \cong 1_C$.

Proof. Consider the equivalence relation $R = \{\langle x, y \rangle \in \mathsf{Ob}_{\mathcal{C}} \times \mathsf{Ob}_{\mathcal{C}} : x \cong_{\mathcal{C}} y \}$ on the class $\mathsf{Ob}_{\mathcal{C}} = \mathsf{dom}[R^{\pm}]$. By (EC), there exists a subclass $S \subseteq \mathsf{Ob}_{\mathcal{C}}$ such that for every object $x \in \mathsf{Ob}_{\mathcal{C}}$ the intersection $R[\{x\}] \cap S$ is a singleton. Let \mathcal{S} be the full subcategory of the category \mathcal{C} whose class of objects coincides with S. It follows that any \mathcal{S} -isomorphic objects in the class $S = \mathsf{Ob}_{\mathcal{S}}$ are equal, which means that \mathcal{S} is a skeleton of the category \mathcal{C} .

Consider the function $\dot{F}: \mathsf{Ob}_{\mathcal{C}} \to \mathsf{Ob}_{\mathcal{S}}$ assigning to each $\mathcal{C}\text{-object } x$ the unique element of the intersection $R[\{x\}] \cap S$. For every object $x \in \mathsf{Ob}_{\mathcal{C}}$, consider the class $\mathsf{Iso}(x, \dot{F}(x))$ of $\mathcal{C}\text{-isomorphisms } f: x \to \dot{F}(x)$. By Lemma 28.8, (EC) \Rightarrow (AC_c) and (AC_c) implies the existence of a function $i_*: \mathsf{Ob}_{\mathcal{C}} \to \mathsf{Mor}_{\mathcal{C}}$ assigning to every $\mathcal{C}\text{-object } x$ some isomorphism $i_x \in \mathsf{Iso}_{\mathcal{C}}(x, \dot{F}(x))$. Define a function $\ddot{F}: \mathsf{Mor}_{\mathcal{C}} \to \mathsf{Mor}_{\mathcal{S}}$ assigning to any $\mathcal{C}\text{-objects } a, b$ and $\mathcal{C}\text{-morphism } f \in \mathsf{Mor}_{\mathcal{C}}(a, b)$ the morphism

$$i_b \circ f \circ i_a^{-1} \in \mathsf{Mor}_{\mathcal{S}}(\dot{F}(a), \dot{F}(b)) = \mathsf{Mor}_{\mathcal{C}}(\dot{F}(a), \dot{F}(b)).$$

It is easy to check that $F = (\dot{F}, \ddot{F}) : \mathcal{C} \to \mathcal{S}$ is a full faithful functor such that for the identity embedding functor $J = 1_{\mathcal{S},\mathcal{C}} : \mathcal{S} \to \mathcal{C}$ we have $FJ = 1_{\mathcal{S}}$ and $JF \cong 1_{\mathcal{C}}$. The natural transformation $i = (i_x)_{x \in \mathsf{Ob}_{\mathcal{C}}} : 1_{\mathcal{C}} \to JF$ witnesses that $1_{\mathcal{C}} \cong JF$, which means that the categories \mathcal{C} and \mathcal{C}' are equivalent.

For small categories, we can replace the principle (EC) in Theorem 56.2 by the Axiom of Choice and obtain the following "small" version of Theorem 56.2.

Theorem 56.3. Under (AC) each small category has a skeleton.

For locally small categories the second part of Theorem 56.2 can be proved using the Axiom of Global Choice instead of (EC).

Theorem 56.4. Let S be a skeleton of a locally small category C. Under (AGC), the categories C and S are equivalent.

Let **Card** be the full subcategory of the category **Set**, whose class of objects coincides with the class of cardinals. It is clear that **Card** is a skeletal category and under (AC), **Card** is a skeleton of the category **Set**. Applying Theorem 56.4 to this skeleton, we obtain the following theorem.

Theorem 56.5. Under (AGC), the category **Set** is equivalent to its skeleton **Card**.

Next, we prove some criteria of equivalence and isomorphness of categories.

We recall that two categories \mathcal{C} and \mathcal{C}' are isomorphic if there exist functors $F: \mathcal{C} \to \mathcal{C}'$ and $G: \mathcal{C}' \to \mathcal{C}$ such that $GF = 1_{\mathcal{C}}$ and $FG = 1_{\mathcal{C}'}$.

Theorem 56.6. Two categories $C = (\mathsf{Ob}, \mathsf{Mor}, \star, +, 1, \circ)$ and $C' = (\mathsf{Ob'}, \mathsf{Mor'}, \star', +', 1', \circ')$ are isomorphic if and only if there exists a full faithful functor $F : C \to C'$ whose object part $\dot{F} : \mathsf{Ob} \to \mathsf{Ob'}$ is bijective.

Proof. The "only if" part is trivial. To prove the "if" part, assume that $F: \mathcal{C} \to \mathcal{C}'$ is a full faithful functor whose object part $\dot{F}: \mathsf{Ob} \to \mathsf{Ob}'$ is bijective. Consider the function $\dot{G} = (\dot{F})^{-1}: \mathsf{Ob}' \to \mathsf{Ob}$. We claim that the morphism part $\ddot{F}: \mathsf{Mor} \to \mathsf{Mor}'$ of the functor F is bijective, too. Given two distinct morphisms $f, g \in \mathsf{Mor}$ consider the following cases.

- 1. If $\star(f) \neq \star(g)$, then $\star'(\ddot{F}f) = \dot{F}(\star(f)) \neq \dot{F}(\star(g)) = \star'(\ddot{F}(g))$ and hence $\ddot{F}f \neq \ddot{F}g$.
- 2. If $t(f) \neq t(g)$, then $t'(\ddot{F}f) = \dot{F}(t(f)) \neq \dot{F}(t(g)) = t'(\ddot{F}(g))$ and hence $\ddot{F}f \neq \ddot{F}g$.
- 3. If $\star(f) = \star(g)$ and $\dagger(f) = \dagger(g)$, then $\ddot{F}(f) \neq \ddot{F}(g)$ since the functor F is faithful.

Therefore, the function $\ddot{F}: \mathsf{Mor} \to \mathsf{Mor}'$ is injective. To see that it is surjective, take any morphism $f' \in \mathsf{Mor}'$. Since the function $\dot{F}: \mathsf{Ob} \to \mathsf{Ob}'$ is bijective, there are \mathcal{C} -objects X,Y such that $\dot{F}(X) = \star'(f')$ and $\dot{F}(Y) = +'(f')$. Since the functor F is full, the function $\ddot{F} \upharpoonright_{\mathsf{Mor}(X,Y)} : \mathsf{Mor}(X,Y) \to \mathsf{Mor}'(\star'(f'),+'(f'))$ is surjective, so there exists a morphism $f \in \mathsf{Mor}(X,Y)$ such that $\ddot{F}(f) = f'$. Therefore, the function $\ddot{F}: \mathsf{Mor} \to \mathsf{Mor}'$ is bijective and we can consider the function $\ddot{G} = (\ddot{F})^{-1}: \mathsf{Mor}' \to \mathsf{Mor}'$.

It is easy to check that $G = (\dot{G}, \ddot{G}) : \mathcal{C}' \to \mathcal{C}$ is a functor such that $GF = 1_{\mathcal{C}}$ and $FG = 1_{\mathcal{C}'}$.

We recall that two categories $\mathcal{C}, \mathcal{C}'$ are called *equivalent* if there exist functors $F: \mathcal{C} \to \mathcal{C}'$ and $G: \mathcal{C}' \to \mathcal{C}$ such that $GF \cong 1_{\mathcal{C}}$ and $FG \cong 1_{\mathcal{C}'}$ where \cong stands for the isomorphism of functors.

Proposition 56.7. Two skeletal categories C, C' are isomorphic if and only if they are equivalent.

Proof. If categories $\mathcal{C}, \mathcal{C}'$ are equivalent, then there exist functors $F: \mathcal{C} \to \mathcal{C}'$ and $G: \mathcal{C}' \to \mathcal{C}$ such that $GF \cong 1_{\mathcal{C}}$ and $FG \cong 1_{\mathcal{C}'}$. The latter isomorphisms imply that the functors GF and FG are full and faithful and so are the functors F and G. Since the categories $\mathcal{C}, \mathcal{C}'$ are skeletal, for any \mathcal{C} -object X and \mathcal{C}' -object X', the isomorphisms $GFX \cong X$ and $FGX' \cong X'$ imply GFX = X and FGX' = X'. This means that the functors F and G are bijective on objects. By Theorem 56.6, the categories $\mathcal{C}, \mathcal{C}'$ are isomorphic.

A functor $F: \mathcal{C} \to \mathcal{C}'$ is defined to be (essentially) surjective on objects if for any \mathcal{C}' -object Y there exists a \mathcal{C} -object X such that FX = Y (resp. $FX \cong Y$).

Theorem 56.8. Under (EC), two categories C, C' are equivalent if and only if there exists a full faithful functor $F: C \to C'$ which is essentially surjective on objects.

Proof. The "only if" part is trivial. To prove the "if" part, assume that here exists a full faithful functor $F: \mathcal{C} \to \mathcal{C}'$ which is essentially surjective on objects. Write the categories \mathcal{C} and \mathcal{C}' in expanded form as 6-tuples: $\mathcal{C} = (\mathsf{Ob}, \mathsf{Mor}, \star, \dagger, 1, \circ)$ and $\mathcal{C}' = (\mathsf{Ob}', \mathsf{Mor}', \star', \dagger', 1, \circ')$.

By Theorem 56.2, under (EC), the categories $\mathcal{C}, \mathcal{C}'$ have skeleta $\mathcal{S} \subseteq \mathcal{C}$, $\mathcal{S}' \subseteq \mathcal{C}$, and there are full faithful functors $R: \mathcal{C} \to \mathcal{S}$ and $R': \mathcal{C}' \to \mathcal{S}'$ such that for the indentity embeddings $J: \mathcal{S} \to \mathcal{C}$ and $J': \mathcal{S}' \to \mathcal{C}'$ we have $RJ = 1_{\mathcal{S}}$, $JR \cong 1_{\mathcal{C}}$, $R'J' = 1_{\mathcal{S}'}$, $J'R' \cong 1_{\mathcal{C}'}$.

Since the functors R', F, J are full and faithful, so is their composition $\Phi = R'FJ : \mathcal{S} \to \mathcal{S}'$. Since \mathcal{S}' is a skeleton of \mathcal{C}' and the functor F is essentially surjective on objects, the functor Φ is surjective on objects. Next, we show that Φ is injective on objects. Assuming that $\Phi X = \Phi Y$ for some \mathcal{S} -objects X, Y and using the faithful property of Φ , we conclude that the functions $\Phi \upharpoonright_{\mathsf{Mor}(X,Y)} : \mathsf{Mor}(X,Y) \to \mathsf{Mor}'(\Phi X,\Phi Y)$ and $\Phi \upharpoonright_{\mathsf{Mor}(Y,X)} : \mathsf{Mor}(Y,X) \to \mathsf{Mor}'(\Phi Y,\Phi X)$ are bijective and hence there exist \mathcal{S} -morphisms $f \in \mathsf{Mor}(X,Y)$ and $g \in \mathsf{Mor}(Y,X)$ such that $\Phi f = 1_{\Phi X}$ and $\Phi g = 1_{\Phi Y}$. Then $\Phi (f \circ g) = \Phi f \circ' \Phi g = 1_{\Phi X} \circ 1_{\Phi Y} = \Phi (1_Y)$ and hence $f \circ g = 1_Y$ by the injectivity of the restriction $\Phi \upharpoonright_{\mathsf{Mor}(Y,Y)}$. By analogy we can prove that $g \circ f = 1_X$. This means that $f : X \to Y$ is a \mathcal{C} -isomorphism. Since the category \mathcal{S} is skeletal, X = Y. Therefore, the function $\Phi \hookrightarrow \mathsf{Mor}(Y,Y)$ is by (the proof of) Theorem 56.6, there exists a functor $\Psi : \mathcal{S}' \to \mathcal{S}$ such that $\Psi \Phi = 1_{\mathcal{S}}$ and $\Phi \Psi = 1_{\mathcal{S}'}$. Then for the functor

 $G = J\Psi R' \colon \mathcal{C}' \to \mathcal{C}$ we have $FG = FJ\Psi R' \cong J'R'FJ\Psi R' = J'\Phi\Psi R' = J'R' \cong 1_{\mathcal{C}'}$ and $GF = J\Psi R'F \cong J\Psi R'FJR = J\Psi\Phi R = JR \cong 1_{\mathcal{C}}$, witnessing that the categories $\mathcal{C}, \mathcal{C}'$ are equivalent.

In the following theorem we use (GWO), the principle of Global Well-Orderability, which is the strongest among Choice Principles, considered in Section 28.

Theorem 56.9. Under (GWO), two categories C and C' are isomorphic if and only if there exists a full faithful functor $F: C \to C'$ such that F is essentially surjective on objects and for any object $A \in \mathsf{Ob}_{C}$ we have $|\{X \in \mathsf{Ob}_{C}: X \cong_{C} A\}| = |\{Y \in \mathsf{Ob}_{C'}: Y \cong_{C'} FA\}|$.

Proof. The "only if" part is trivial. To prove the "if" part, assume that there exists a full faithful functor $F: \mathcal{C} \to \mathcal{C}'$ such that F is essentially surjective on objects, and for any \mathcal{C} -object a we have $|\{x \in \mathsf{Ob}_{\mathcal{C}} : x \cong_{\mathcal{C}} a\}| = |\{y \in \mathsf{Ob}_{\mathcal{C}'} : y \cong_{\mathcal{C}'} Fa\}|$.

Since (GWO) \Rightarrow (EC), we can apply Theorem 56.2 and conclude that the categories $\mathcal{C}, \mathcal{C}'$ have skeleta $\mathcal{S} \subseteq \mathcal{C}$ and $\mathcal{S}' \subseteq \mathcal{C}'$, and there exist full faithful functors $R: \mathcal{C} \to \mathcal{S}$ and $R': \mathcal{C}' \to \mathcal{S}'$ such that for the identity embeddings $J: \mathcal{S} \to \mathcal{C}$ and $J': \mathcal{S}' \to \mathcal{C}'$ we have $RJ = 1_{\mathcal{S}}, JR \cong 1_{\mathcal{C}}, R'J' = 1_{\mathcal{S}'}, J'R' \cong 1_{\mathcal{C}'}$.

Consider the functor $\Phi = R'FJ : \mathcal{S} \to \mathcal{S}'$. By (the proof of) Theorem 56.8, there exists a functor $\Psi : \mathcal{S}' \to \mathcal{S}$ such that $\Psi \Phi = 1_{\mathcal{S}}$ and $\Phi \Psi = 1_{\mathcal{S}'}$. Observe that for every $a \in \mathsf{Ob}_{\mathcal{S}}$ we have $\Phi a \cong Fa$ and hence

$$|\{x \in \mathsf{Ob}_{\mathcal{C}} : x \cong_{\mathcal{C}} a\}| = |\{y \in \mathsf{Ob}_{\mathcal{C}'} : y \cong_{\mathcal{C}'} Fa\}| = |\{y \in \mathsf{Ob}_{\mathcal{C}'} : y \cong_{\mathcal{C}'} \Phi a\}|.$$

Consider the classes $\mathsf{Ob}_{\mathcal{C}}^{\mathsf{s}} = \{x \in \mathsf{Ob}_{\mathcal{C}} : \{y \in \mathsf{Ob}_{\mathcal{C}} : y \cong_{\mathcal{C}} x\} \in \mathbf{U}\}$ and $\mathsf{Ob}_{\mathcal{S}}^{\mathsf{s}} = \mathsf{Ob}_{\mathcal{C}}^{\mathsf{s}} \cap \mathsf{Ob}_{\mathcal{S}}$. The existence of these classes follows from Theorem 7.2. Using the Axiom of Global Choice (which follows from (GWO)) and the equality (56.1), we can find a function $\Theta_* : \mathsf{Ob}_{\mathcal{S}}^{\mathsf{s}} \to \mathbf{U}$ assigning to each object $a \in \mathsf{Ob}_{\mathcal{S}}^{\mathsf{s}}$ a bijective function Θ_a such that $\mathsf{dom}[\Theta_a] = \{x \in \mathsf{Ob}_{\mathcal{C}} : x \cong_{\mathcal{C}} a\}$, $\mathsf{rng}[\Theta_a] = \{y \in \mathsf{Ob}_{\mathcal{C}'} : y \cong_{\mathcal{C}'} \Phi a\}$ and $\Theta_a(a) = \Phi a$.

By (GWO) there exists a set-like well-order \mathbf{W} with $\mathsf{dom}[\mathbf{W}^{\pm}] = \mathbf{U}$. This well-order induces the set-like well-founded orders

$$W = \{\langle x, y \rangle \in \mathbf{W} : x, y \in \mathsf{Ob}_{\mathcal{C}} \setminus \mathsf{Ob}_{\mathcal{S}}, \ x \cong_{\mathcal{C}} y\} \cup \{\langle x, y \rangle \in \mathsf{Ob}_{\mathcal{S}} \times \mathsf{Ob}_{\mathcal{C}} : x \cong_{\mathcal{C}} y\}$$

and

$$W' = \{ \langle x, y \rangle \in \mathbf{W} : x, y \in \mathsf{Ob}_{\mathcal{C}'} \setminus \mathsf{Ob}_{\mathcal{S}'}, \ x \cong_{\mathcal{C}'} y \} \cup \{ \langle x, y \rangle \in \mathsf{Ob}_{\mathcal{S}'} \times \mathsf{Ob}_{\mathcal{C}'} : x \cong_{\mathcal{C}'} y \}.$$

For every \mathcal{C} -object a the well-order W determines a set-like well-order on the equivalence class $[a]_{\cong} = \{x \in \mathsf{Ob}_{\mathcal{C}} : a \cong_{\mathcal{C}} x\}$ such that the unique element of the intersection $\mathsf{Ob}_{\mathcal{S}} \cap [a]_{\cong}$ is the unique W-minimal element of $[a]_{\cong}$. If $a \notin \mathsf{Ob}_{\mathcal{C}}^{f}$, then $[a]_{\cong}$ is not a set and hence the well-ordered class $([a]_{\cong}, W \upharpoonright [a]_{\cong})$ is order-isomorphic to \mathbf{On} by Theorem 23.6. By (56.1), the same is true for the object $b = \Phi a$ and its equivalnce class $[b]_{\cong} = \{y \in \mathsf{Ob}_{\mathcal{C}'} : y \cong_{\mathcal{C}'} b\}$. Since $|[b]_{\cong}| = |[a]_{\cong}|$, $[b]_{\cong}$ is not a set and then the well-ordered class $([b]_{\cong}, W' \upharpoonright [b]_{\cong})$ is order isomorphic to \mathbf{On} .

Let $\operatorname{\mathsf{rank}}_W: \mathbf{U} \to \mathbf{On}$ and $\operatorname{\mathsf{rank}}_{W'}: \mathbf{U} \to \mathbf{On}$ be the rank functions of the well-founded setlike orders W and W'. The definition of the orders W, W' implies that for every $a \in \mathsf{Ob}_{\mathcal{C}} \setminus \mathsf{Ob}_{\mathcal{C}}^{\mathsf{s}}$ and $b = \Phi a$ the restrictions $\operatorname{\mathsf{rank}}_{W} \upharpoonright_{[a]_{\cong}} : [a]_{\cong} \to \mathbf{On}$ and $\operatorname{\mathsf{rank}}_{W'} \upharpoonright_{[b]_{\cong}} : [b]_{\cong} \to \mathbf{On}$ are bijections.

Now consider the function $\dot{G}: \mathsf{Ob}_{\mathcal{C}} \to \mathsf{Ob}_{\mathcal{C}'}$ assigning to each object $a \in \mathsf{Ob}_{\mathcal{C}}^{\mathsf{s}}$ the object $\Theta_{Ra}(a)$ and to each object $a \in \mathsf{Ob}_{\mathcal{C}} \setminus \mathsf{Ob}_{\mathcal{C}}^{\mathsf{s}}$ the unique object $b \in \mathsf{Ob}_{\mathcal{C}'} \setminus \mathsf{Ob}_{\mathcal{C}'}^{\mathsf{s}}$ such that $b \cong \Phi Ra$ and $\mathsf{rank}_{W'}(b) = \mathsf{rank}_{W}(a)$. The choice of the function Θ_* and the well-orders

W, W' ensures that the function $\dot{G}: \mathsf{Ob}_{\mathcal{C}} \to \mathsf{Ob}_{\mathcal{C}'}$ is bijective and $\dot{\Phi} \subseteq \dot{G}$. Consider the function $\ddot{G}: \mathsf{Mor}_{\mathcal{C}} \to \mathsf{Mor}_{\mathcal{C}'}$ assigning to any \mathcal{C} -objects a, b and morphism $f \in \mathsf{Mor}_{\mathcal{C}}(a, b)$ a unique morphism $g \in \mathsf{Mor}_{\mathcal{C}'}(\dot{G}(a), \dot{G}(b))$ such that $R'g = \Phi Rf$. It can be shown that the function \ddot{G} is bijective and hence $G: \mathcal{C} \to \mathcal{C}'$ is a full faithful functor whose object part \dot{G} is bijective. By Theorem 56.6, the categories $\mathcal{C}, \mathcal{C}'$ are isomorphic.

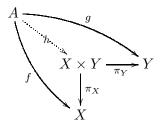
57. Limits and colimits

In this section we discuss the notions of limit and colimit of a diagram in a category. Limits and colimits are general categorial notions whose partial cases are products and coproducts, pullbacks and pushouts, equalizers and coequalizes.

57.1. Products and coproducts. Let \mathcal{C} be a category.

Definition 57.1. For two C-objects X, Y, their C-product is a C-object $X \times Y$ endowed with two C-morphisms $\pi_X \in \mathsf{Mor}_{\mathcal{C}}(X \times Y, X)$ and $\pi_Y \in \mathsf{Mor}_{\mathcal{C}}(X \times Y, Y)$, called the coordinate projections, such that the triple $(X \times Y, \pi_X, \pi_Y)$ has the following universal property: for any C-object A and C-morphisms $f \in \mathsf{Mor}_{\mathcal{C}}(A, X)$ and $g \in \mathsf{Mor}_{\mathcal{C}}(A, Y)$ there exists a unique C-morphism $h \in \mathsf{Mor}_{\mathcal{C}}(A, X \times Y)$ such that $f = \pi_X \circ h$ and $g = \pi_Y \circ h$.

This definition is illustrated by the diagram:



The uniqueness of the morphism h implies that a C-product $X \times Y$ if exists, then is unique up to a C-isomorphism.

Exercise 57.2. Prove that for any sets X, Y, their \mathcal{C} -product $X \times Y$ endowed with the projections $\pi_X = \mathsf{dom} \upharpoonright_{X \times Y}$ and $\pi_Y = \mathsf{rng} \upharpoonright_{X \times Y}$ is a products of X and Y is the category **Set**.

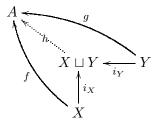
Exercise 57.3. Identify products of objects in the categories considered in Examples 53.10–53.14.

Definition 57.4. A category C is defined to have binary products if for any C-objects X, Y there exists a product $X \times Y$ in C.

The dual notion to a product is that of a coproduct.

Definition 57.5. For two C-objects X, Y their C-coproduct is any C-object $X \sqcup Y$ endowed with two C-morphisms $i_X : X \to X \sqcup Y$ and $i_Y : Y \to X \sqcup Y$, called the *coordinate coprojections*, such that the triple $(X \sqcup Y, i_X, i_Y)$ has the following universal property: for any C-object A and C-morphisms $f : X \to A$ and $g : Y \to A$ there exists a unique C-morphism $h : X \sqcup Y \to A$ such that $f = i_X \circ h$ and $g = i_Y \circ h$.

This definition is illustrated by the diagram:



The uniqueness of the morphism h implies that a C-coproduct $X \sqcup Y$ if exists, then is unique up to a C-isomorphism.

Exercise 57.6. Prove that for any disjoint sets X, Y, their union $X \cup Y$ endowed with the identity embeddings $i_X = \mathbf{Id}|_X : X \to X \cup Y$ and $i_Y = \mathbf{Id}_Y|_Y : X \to X \cup Y$ is a **Set**-coproduct of X and Y.

Exercise 57.7. Prove that for any sets X, Y the set $(X, Y) = (\{0\} \times X) \cup (\{1\} \times Y)$ endowed with natural injective functions $i_X : X \to (X, Y)$ and $i_Y : Y \to (X, Y)$ is a **Set**-coproduct of X and Y.

Exercise 57.8. Identify coproducts of objects in the categories considered in Examples 53.10–53.14.

In fact, products and coproducts can be defined for any indexed families of objects.

Definition 57.9. Let \mathcal{C} be a category. For an indexed family of \mathcal{C} -objects $(X_i)_{i\in I}$ its

- C-product $\prod_{i \in I} X_i$ is any C-object X endowed with a family of C-morphisms $(\pi_i)_{i \in I} \in \prod_{i \in I} \mathsf{Mor}_{\mathcal{C}}(X, X_j)$, which has the following universal property: for any C-object Y and a family of C-morphisms $(f_i)_{i \in I} \in \prod_{i \in I} \mathsf{Mor}_{\mathcal{C}}(Y, X_i)$ there exists a unique C-morphism $h \in \mathsf{Mor}_{\mathcal{C}}(A, X)$ such that $\forall i \in I$ $f_i = \pi_i \circ h$;
- C-coproduct $\coprod_{i \in I} X_i$ is any C-object X endowed with a family of C-morphisms $(e_i)_{i \in I} \in \prod_{i \in I} \mathsf{Mor}_{\mathcal{C}}(X_i, X)$, which has the following universal property: for any C-object Y and a family of C-morphisms $(f_i)_{i \in I} \in \prod_{i \in I} \mathsf{Mor}_{\mathcal{C}}(X_i, Y)$ there exists a unique C-morphism $h \in \mathsf{Mor}_{\mathcal{C}}(X, Y)$ such that $\forall i \in I$ $f_i = h \circ e_i$.

Definition 57.10. A category \mathcal{C} is defined

- to have finite products if for any finite set I and indexed family of C-objects $(X_i)_{i \in I}$, the category C contains a product $\prod_{i \in I} X_i$ of this family;
- to have arbitrary products if for any set I and indexed family of C-objects $(X_i)_{i \in I}$, the category C contains a product $\prod_{i \in I} X_i$ of this family;
- to have finite coproducts if for any finite set I and indexed family of C-objects $(X_i)_{i \in I}$, the category C contains a coproduct $\coprod_{i \in I} X_i$ of this family;
- to have arbitrary coproducts if for any set I and indexed family of C-objects $(X_i)_{i \in I}$, the category C contains a coproduct $\coprod_{i \in I} X_i$ of this family.

Exercise 57.11. Prove that a category has finite products if and only if it has binary products.

Exercise 57.12. Prove that for any indexed family of sets $(X_i)_{i \in I}$ their Cartesian product $\prod_{i \in I} X_i$ endowed with the natural projections is a **Set**-product of the family $(X_i)_{i \in I}$.

Exercise 57.13. Prove that for any indexed family of pairwise disjoint sets $(X_i)_{i \in I}$ their union $\bigcup_{i \in I} X_i$ endowed with the identity inclusions of the sets X_i is a **Set**-coproduct of the family $(X_i)_{i \in I}$.

Exercise 57.14. Prove that for any indexed family of sets $(X_i)_{i \in I}$ the set $\bigcup_{i \in I} (\{i\} \times X_i)$ endowed with the natural embeddings of the sets X_i is a **Set**-coproduct of the family $(X_i)_{i \in I}$.

Exercise 57.15. Identify products of objects in the categories considered in Examples 53.10–53.14.

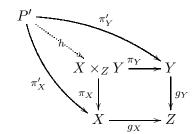
Exercise 57.16. Prove that the category Cat of small categories has arbitrary products and coproducts.

57.2. **Pullbacks and pushouts.** In this subsection we consider pullbacks and pushouts, called also fibered products and coproducts.

Given a category \mathcal{C} , consider the following diagram consisting of three \mathcal{C} -objects X,Y,Z and two \mathcal{C} -morphisms $g_X:X\to Z$ and $g_Y:Y\to Z$.

$$X \xrightarrow{g_X} Z$$

The pullback $X \times_Z Y$ of this diagram is any \mathcal{C} -object P endowed with two \mathcal{C} -morphisms $\pi_X LP \to X$ and $\pi_Y : P \to Y$ such that $g_X \circ \pi_X = g_Y \circ \pi_Y$ and the triple (P, π_X, π_Y) has the following universality property: for any \mathcal{C} -object P' and \mathcal{C} -morphisms $f_X : P' \to X$, $f_Y : P' \to Y$ with $g_X \circ f_X = g_Y \circ f_Y$, there exists a unique \mathcal{C} -morphism $h : P' \to P$ such that $f_X = \pi_X \circ h$ and $f_Y = \pi_Y \circ h$. This definition is better seen at the diagram:



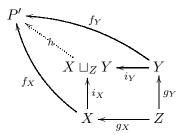
If the category C has a terminal object 1, then the product $X \times Y$ is a pullback $X \times_1 Y$ of the diagram

$$\begin{array}{c} Y \\ \downarrow \\ X \longrightarrow 1 \end{array}$$

The notion of a pushout is dual to that of pullback.

Given a category \mathcal{C} , consider the following diagram consisting of three \mathcal{C} -objects X,Y,Z and two \mathcal{C} -morphisms $g_X:Z\to X$ and $g_Y:Z\to Y$:

The pushout $X \sqcup_Z Y$ of this diagram is any \mathcal{C} -object P endowed with two \mathcal{C} -morphisms $i_X: X \to P$ and $i_Y: Y \to P$ such that $i_X \circ g_X = i_Y \circ g_Y$ and the triple (P, i_X, i_Y) has the following universal property: for any \mathcal{C} -object P' and \mathcal{C} -morphisms $f_X: X \to P'$, $f_Y: Y \to P'$ with $f_X \circ g_X = f_Y \circ g_Y$, there exists a unique \mathcal{C} -morphism $h: P' \to P$ such that $f_X = h \circ i_X$ and $f_Y = h \circ i_Y$. This definition is better seen at the diagram:



If the category $\mathcal C$ has an initial object 0, then the coproduct $X \sqcup Y$ is a pullback $X \sqcup_0 Y$ of the diagram

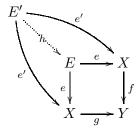


A category \mathcal{C} is defined

- to have pullbacks if any diagram consisting of two C-morphisms $g_X: X \to Z$ and $g_Y: Y \to Z$ has a pullback.
- to have pushouts if any diagram consisting of two C-morphisms $g_X: Z \to X$ and $g_Y: Z \to X$ has a pushout.

It is clear that a category has pullbacks if and only the dual category has pushouts.

57.3. Equalizers and coequalizers. For two objects X,Y of a category \mathcal{C} and two morphisms $f,g \in \mathsf{Mor}_{\mathcal{C}}(X,Y)$, an equalizer of the pair of (f,g) is a \mathcal{C} -object E endowed with a \mathcal{C} -morphism $e:E \to X$ such that $f \circ e = g \circ e$ and for any \mathcal{C} -object E' and \mathcal{C} -morphism $e':E' \to X$ with $f \circ e' = g \circ e'$ there exists a unique \mathcal{C} -morphism $h:E' \to E$ such that $e' = e \circ h$. This definition is better seen on the commutative diagram:



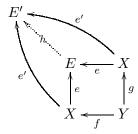
Example 57.17. In the category **Set** an equalizer of two functions $f, g: X \to Y$ is the set $E = \{x \in X : f(x) = g(x)\}$ endowed with the identity embedding $e: E \to X$.

Exercise 57.18. Prove that a category \mathcal{C} has pullbacks if it has equalizers and binary products.

Hint: Observe that for morphisms $f: X \to Z$ and $g: Y \to Z$ the pullback $X \times_Z Y$ is isomorphic to the equalizer E of the pair $(f \circ \operatorname{pr}_X, g \circ \operatorname{pr}_Y)$ where $\operatorname{pr}_X, \operatorname{pr}_Y$ are coordinate projections of the product $X \times Y$.

Coequalizers are defined dually.

For two objects X, Y of a category \mathcal{C} and two morphisms $f, g \in \mathsf{Mor}_{\mathcal{C}}(X, Y)$, a coequalizer of the pair (f, g) is a \mathcal{C} -object E endowed with a \mathcal{C} -morphism $e: Y \to E$ such that $e \circ f = e \circ g$ and for any \mathcal{C} -object E' and \mathcal{C} -morphism $e': X \to E'$ with $e' \circ f = e' \circ g$ there exists a unique \mathcal{C} -morphism $h: E \to E'$ such that $e' = h \circ e$. On a diagram this definition looks as follows.



Exercise 57.19. Find a coequalizer of two functions in the category of sets.

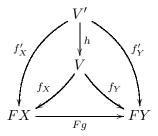
A category \mathcal{C} is defined to have (co)equalizers if any pair of \mathcal{C} -morphisms $f, g: X \to Y$ has a (co)equalizer.

57.4. Limits and colimits of diagrams. Products, pullbacks, and equalizers are particular cases of limits of diagrams in a category.

By definition, a diagram in a category \mathcal{C} is any functor $D: \mathcal{D} \to \mathcal{C}$ defined on a small category \mathcal{D} . For a fixed small category \mathcal{D} , a functor $D: \mathcal{D} \to \mathcal{C}$ is called a \mathcal{D} -diagram in \mathcal{C} .

Definition 57.20. Let \mathcal{D} be a small category and $D: \mathcal{D} \to \mathcal{C}$ be a \mathcal{D} -diagram in a category \mathcal{C} .

- A cone over the \mathcal{D} -diagram D is a pair (V, f) consisting of a \mathcal{C} -object V and a function $f: \mathsf{Ob}_{\mathcal{D}} \to \mathsf{Mor}_{\mathcal{C}}$ assigning to each \mathcal{D} -object $X \in \mathsf{Ob}_{\mathcal{D}}$ a \mathcal{C} -morphism $f_X \in \mathsf{Mor}_{\mathcal{C}}(V, FX)$ such that for any \mathcal{D} -morphism $g: X \to Y$ we have $f_Y = Fg \circ f_X$.
- A limit of the \mathcal{D} -diagram D is any cone (V, f) over F that has the following universal property: for any cone (V', f') over F there exists a unique \mathcal{C} -morphism $h: V' \to V$ such that $\forall X \in \mathsf{Ob}_{\mathcal{D}} \ f'_X = f_X \circ h$.



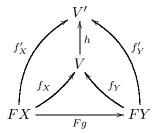
The uniqueness of the morphism h in Definition 57.20 implies the following useful fact.

Proposition 57.21. Let (V, f) be a limit of a \mathcal{D} -diagram in a category \mathcal{C} . A \mathcal{C} -morphism $h: V \to V$ is equal to 1_V if and only if $\forall X \in \mathsf{Ob}(\mathcal{D})$ $f_X \circ h = f_X$.

The notion of a colimit is dual to the notion of a limit.

Definition 57.22. Let \mathcal{D} be a small diagram and $D: \mathcal{D} \to \mathcal{C}$ be a \mathcal{D} -diagram in a category \mathcal{C} .

- A cocone over the \mathcal{D} -diagram D is a pair (V, f) consisting of an object V of the category \mathcal{C} and a function $f: \mathsf{Ob}_{\mathcal{D}} \to \mathsf{Mor}_{\mathcal{C}}$ assigning to each \mathcal{D} -object $X \in \mathsf{Ob}_{\mathcal{D}}$ a \mathcal{C} -morphism $f_X: FX \to V$ such that for any \mathcal{D} -morphism $g: X \to Y$ we have $f_X = f_Y \circ Fg$.
- A colimit of the \mathcal{D} -diagram D is any cocone (V, f) over F that has the following universal property: for any cocone (V', f') over F there exists a unique \mathcal{C} -morphism $h: V \to V'$ such that $\forall X \in \mathsf{Ob}_{\mathcal{D}} \ f'_X = h \circ f_X$.



The uniqueness of the morphism h in Definition 57.22 implies the following useful fact that will be used in the proof of Lemma 58.5.

Proposition 57.23. Let (V, f) be a colimit of a \mathcal{D} -diagram in a category \mathcal{C} . A \mathcal{C} -morphism $h \in \mathsf{Mor}_{\mathcal{C}}(V, V)$ is equal to 1_V if and only if $\forall X \in \mathsf{Ob}_{\mathcal{D}}$ $(h \circ f_X = f_X)$.

Limits and colimits of diagrams are unique up to (a properly defined notion of) an isomorphism.

Remark 57.24. (Co)products are (co)limit of \mathcal{D} -liagram over discrete categories \mathcal{D} .

Exercise 57.25. Which diagrams \mathcal{D} correspond to pullbacks and equalizers?

Exercise 57.26. Investigate the existence and structure of limits and colimits in your favorable category.

Exercise* 57.27. Prove that a category has limits of finite diagrams if and only if it has binary products and equalizers.

58. Characterizations of the category Set

We adjoin eight first-order axioms to the usual first-order theory of an abstract Eilenberg-Mac Lane category to obtain an elementary theory with the following properties:

(a) There is essentially only one category which satisfies these eight axioms together with the additional (nonelementary) axiom of completeness, namely, the category \mathbf{Set} of sets and mappings. Thus our theory distinguishes \mathbf{Set} structurally from other complete categories, such as those of topological spaces, groups, rings, partially ordered sets, etc. (b) The theory provides a foundation for number theory, analysis, ... algebra and topology even though no relation \in with the traditional properties can be defined. Thus we seem to have partially demonstrated that even in foundations not Substance but invariant Form is the carrier of the relevant mathematical information. William Lawvere, 1964

In this section we characterize categories which are equivalent or isomorphic to the category **Set**. The principal results are Theorems 58.13 and 58.14 which are simplified versions of the characterization of the category of sets, proved by Lawvere in 1964 (before he created the theory of elementary topoi).

A distinguishing property of the category of sets is that its terminal object is a generator for this category.

Definition 58.1. Let \mathcal{C} be a category. A \mathcal{C} -object Γ is called a *generator* (more precisely, a \mathcal{C} -generator) if for any \mathcal{C} -objects X, Y and distinct \mathcal{C} -morphisms $f, g \in \mathsf{Mor}(X, Y)$ there exists a \mathcal{C} -morphism $h \in \mathsf{Mor}(\Gamma, X)$ such that $f \circ h \neq g \circ h$.

Exercise 58.2. Observe that in the categories Set and Top any non-initial object is a generator.

Exercise 58.3. Prove that the additive group of integers $(\mathbb{Z}, +)$ is a generator in the category of (commutative) groups.

Definition 58.4. A category \mathcal{C} with a terminal object 1 is defined to be *element-separating* if for any global element $x: 1 \to X$ there exist a \mathcal{C} -object Ω and \mathcal{C} -morphisms $\chi_x: X \to \Omega$ and false: $1 \to \Omega$ such that $\chi_x \circ x \neq \mathsf{false}$ and $\forall x' \in \mathsf{Mor}(1,X) \setminus \{x\} \ (\chi_x \circ x' = \mathsf{false})$.

We recall that a category C is *balanced* if a C-morphism is an isomorphism if and only if it is a bimorphism (i.e., mono and epi).

Lemma 58.5. A skeletal category C is equivalent to the category **Set** if and only if it satisfies the following properties:

- (1) C is locally small;
- (2) C is balanced;
- (3) C has equalizers;
- (4) C has arbitrary coproducts;
- (5) C has a terminal object 1;
- (6) 1 is a C-generator;
- (7) C is element-separating.

Proof. The "only if" part it trivial. To prove the "if" part, assume that a skeletal category \mathcal{C} has the properties (1)–(7). Then \mathcal{C} has a terminal object 1, which is unique by the skeletality of \mathcal{C} .

Consider the functor $G: \mathcal{C} \to \mathbf{Set}$ assigning to every \mathcal{C} -object X the set $\mathsf{Mor}(1,X)$. Elements of the set $\mathsf{Mor}(1,X)$ are called *global elements* of X. To every \mathcal{C} -morphism $f: X \to Y$ the functor G assigns the function

$$Gf: \mathsf{Mor}(\mathsf{1},X) \to \mathsf{Mor}(\mathsf{1},Y), \quad Gf: x \mapsto f \circ x.$$

Therefore, the functor G assigns to each \mathcal{C} -object X the set $\mathsf{Mor}(1,X)$ of its global element. Now we describe a functor $F: \mathbf{Set} \to \mathcal{C}$ acting in the opposite direction. The functor F assigns to every set X the coproduct $\sqcup_{x \in X} 1$ of X many copies of the terminal object 1. Since the category has arbitrary coproducts and is skeletal, the coproduct $\sqcup_{x \in X} 1$ exists and is unique. Let $\eta_X: X \to \mathsf{Mor}(1, FX)$ be the function assigning to every element $x \in X$ the coordinate coprojection $\eta_X(x): 1 \to FX$. By definition of a coproduct, the function η_X has the following universal property: for every \mathcal{C} -object Y and function $g: X \to \mathsf{Mor}(1,Y)$ there exists a unique \mathcal{C} -morphism $h: FX \to Y$ such that $h \circ \eta_X(x) = g(x)$ for every $x \in X$. In particular, for every function $f: X \to X'$ between sets, there exists a unique \mathcal{C} -morphism $Ff: FX \to FX'$ such that

(58.1)
$$Ff \circ \eta_X(x) = \eta_{X'} \circ f(x) \text{ for every } x \in X.$$

This formula defines the action of the functor F on the morphisms of the category **Set**.

Consider the natural transformation $\eta: \mathbf{1_{Set}} \to GF$ whose components are the the **Set**-morphisms $\eta_X: X \to \mathsf{Mor}(\mathbf{1}, FX) = GFX$.

Next, we define a natural transformation $\varepsilon: FG \to 1_{\mathcal{C}}$ witnessing that $F \dashv G$. For every \mathcal{C} -object Y, consider the set $GY = \mathsf{Mor}(1,Y)$ and the coproduct $FGY = \bigcup_{y \in \mathsf{Mor}(1,Y)} 1$. By the universal property of the coproduct, there exists a unique \mathcal{C} -morphism $\varepsilon_Y: FGY \to Y$ such that

$$(58.2) y = \varepsilon_Y \circ \eta_{GY}(y) \text{for every } y \in \mathsf{Mor}(1,Y).$$

The morphism ε_Y is a component of the natural transformation $\varepsilon: FG \to 1_{\mathcal{C}}$.

The following two claims witness that F and G is a pair of adjoint functors.

Claim 58.6.
$$1_F = \varepsilon_F \circ F \eta$$
.

Proof. We should prove that for every set X the composition $\varepsilon_{FX} \circ F \eta_X$ is equal to the identity morphism 1_{FX} of FX. Applying the equality (58.1) to the \mathcal{C} -morphism $\eta_X : X \to GFX$, we obtain the equality

$$F\eta_X \circ \eta_X(x) = \eta_{GFX} \circ \eta_X(x) \quad x \in X.$$

Combining this equality with the equality (58.2) applyied to the C-object Y = FX, we obtain

$$\mathbf{1}_{FX} \circ \eta_X(x) = \eta_X(x) = \varepsilon_{FX} \circ \eta_{GFX} \circ \eta_X(x) = \varepsilon_{FX} \circ F\eta_X \circ \eta_X(x) \quad \text{for all } x \in X.$$

Applying Proposition 57.23, we conclude that $\varepsilon_{FX} \circ F \eta_X = 1_{FX}$.

Claim 58.7.
$$1_G = G\varepsilon \circ \eta_G$$
.

Proof. To check this equality, we should prove that for every C-object Y, the composition $G\varepsilon_Y \circ \eta_{GY} = 1_{GY}$. We recall that $\eta_{GY} : GY \to \mathsf{Mor}(1, FGY) = GFGY$ is a function (i.e., a morphism in the category **Set**). Applying the functor G to the morphism $\varepsilon_Y \in \mathsf{Mor}(FGY, Y)$

we obtain the function $G\varepsilon_Y \in \mathsf{Mor}_{\mathbf{Set}}(GFGY, GY) = \mathsf{Mor}_{\mathbf{Set}}(\mathsf{Mor}(1, FGY), \mathsf{Mor}(1, Y))$ such that $G\varepsilon_Y(f) = \varepsilon_Y \circ f$ for any $f \in \mathsf{Mor}(1, FGY)$. Then for every $x \in GY$ we have

$$G\varepsilon_Y \circ \eta_{GY}(x) = G\varepsilon_Y(\eta_{GY}(x)) = \varepsilon_Y \circ \eta_{GY}(x) = x$$

by (58.2). This yields the desired equality $G\varepsilon_Y \circ \eta_{GY} = 1_{GY}$.

Claim 58.8. The function $\eta_X: X \to GFX$ is injective.

Proof. To prove that the function $\eta_X: X \to GFX = \mathsf{Mor}(1, FX)$ is injective, fix any distinct elements $x, x' \in X$. Since \mathcal{C} is element-separating, there exists a \mathcal{C} -object Ω such that $\mathsf{Mor}(1,\Omega)$ contains at least two distinct morphisms. Then we can choose a function $\chi: X \to \mathsf{Mor}(1,\Omega)$ such that $\chi(x) \neq \chi(x')$. By the definition of the natural transformation η_X , there exists a unique \mathcal{C} -morphism $u: FX \to \Omega$ such that $u \circ \eta_X(z) = \chi(z)$ for every $z \in X$. In particular,

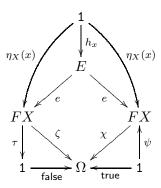
$$u \circ \eta_X(x) = \chi(x) \neq \chi(x') = u \circ \eta_X(x'),$$

which implies that $\eta_X(x) \neq \eta_X(y)$.

Claim 58.9. The function $\eta_X: X \to GFX$ is surjective.

Proof. Assuming that η_X is not surjective, we can find a morphism $\psi \in GFX = \mathsf{Mor}(1, FX)$ such that $\psi \neq \eta_X(x)$ for any $x \in X$. Since the category $\mathcal C$ is element-separating, for the morphism $\psi : \mathbf 1 \to FX$ there exist a $\mathcal C$ -object Ω and $\mathcal C$ -morphisms $\chi : FX \to \Omega$ and false, true : $\mathbf 1 \to \Omega$ such that $\chi \circ \psi = \mathsf{true}$ and for any global element $\varphi \in \mathsf{Mor}(\mathbf 1, FX) \setminus \{\psi\}$ we have $\chi \circ \varphi = \mathsf{false} \neq \mathsf{true}$.

Let $\tau: FX \to 1$ be the unique \mathcal{C} -morphism and $\zeta = \mathsf{false} \circ \tau$. Since the category \mathcal{C} has equalizers, there exist a \mathcal{C} -object E and a \mathcal{C} -morphism $e \in \mathsf{Mor}(E, FX)$ such that $\chi \circ e = \zeta \circ e$ and for any \mathcal{C} -object E' and morphism $e' \in \mathsf{Mor}(E', FX)$ with $\chi \circ e' = \zeta \circ e'$ there exists a unique morphism $h \in \mathsf{Mor}(E', E)$ such that $e \circ h = e'$. We apply this universal property of (E, e) to the pair $(E', e') = (1, \eta_X(x))$ where $x \in X$ is any element.



By the choice of χ , the inequality $\psi \neq \eta_X(x)$, implies

$$\chi \circ \eta_X(x) = \mathsf{false} = \mathsf{false} \circ \mathsf{1}_1 = \mathsf{false} \circ (\tau \circ \eta_X(x)) = \zeta \circ \eta_X(f).$$

By the universal property of the equalizer morphism $e: E \to FX$, there exists a unique morphism $h_f \in \mathsf{Mor}(1,E)$ such that $\eta_X(x) = e \circ h_f$. By the universal property of the coproduct (FX,η_X) , there exists a unique \mathcal{C} -morphism $e': FX \to E$ such that $h_x = e' \circ \eta_X(x)$ for all $f \in \mathsf{Mor}_{\mathcal{C}}(1,X)$. Then $\eta_X(x) = e \circ h_x = e \circ e' \circ \eta_X(x)$ for all $x \in X$ and hence $e \circ e' = 1_{FX}$, see Proposition 57.23.

We claim that $e' \circ e = 1_E$. Assuming that $e' \circ e \neq 1_E$, we can use the generator property of 1 and find a \mathcal{C} -morphism $u: 1 \to E$ such that $e' \circ e \circ u \neq 1_E \circ u = u$. Let $u' = e' \circ e \circ u$ and observe that $e \circ u' = e \circ e' \circ e \circ u = 1_{FX} \circ e \circ u = e \circ u$. Then u and u' are two distinct morphisms such that $e \circ u = e \circ u'$, which contradicts the uniqueness condition in the definition of an equalizer. This contradiction show that $e' \circ e = 1_E$. Together with $e \circ e' = 1_{FX}$ this implies that e is an isomorphism and e' is its inverse.

Now the equality $\chi \circ e = \zeta \circ e$ implies

$$\chi = \chi \circ 1_{FX} = \chi \circ (e \circ e') = (\chi \circ e) \circ e' = (\zeta \circ e) \circ e; = \zeta \circ (e \circ e') = \zeta \circ 1_{FX} = \zeta$$

and

true =
$$\chi \circ \psi = \zeta \circ \psi = (\mathsf{false} \circ \tau) \circ \psi = \mathsf{false} \circ (\tau \circ \psi) = \mathsf{false} \circ 1_1 = \mathsf{false},$$

which contradicts the choice of the morphisms true and false.

Claims 58.8 and 58.9 imply that the function η_X is bijective and hence is an isomorphism of the category **Set**,

Next, we show that for every C-object Y the morphism $\varepsilon_Y : FGY \to Y$ is an isomorphism in the category C. By definition, ε_Y is the unique C-morphism such that $\varepsilon_Y \circ \eta_{GY}(y) = y$ for any $y \in GY = \mathsf{Mor}(1,Y)$.

Claim 58.10. ε_Y is an epimorphism.

Proof. Assuming that ε_Y is not epi, we can find a \mathcal{C} -object Y' and two distinct morphisms $g, g' \in \mathsf{Mor}(Y, Y')$ such that $g \circ \varepsilon_Y = g' \circ \varepsilon_Y$. Since 1 is a generator, there exists a morphism $y \in \mathsf{Mor}(1, Y)$ such that $g \circ y \neq g' \circ y$. Consider the morphism $\eta_{GY}(y) \in \mathsf{Mor}(1, FGY)$ and observe that $\varepsilon_Y \circ \eta_{GY}(y) = y$, see the equation (58.2). Then $g \circ y = g \circ \varepsilon_Y \circ \eta_{GY}(y) = g' \circ \varepsilon_Y \circ \eta_{GY}(y) = g' \circ y$, which contradicts the choice of the morphisms g, g'.

Claim 58.11. ε_Y is a monomorphism.

Proof. By Claims 58.8, 58.9, the function $\eta_{GY}: \mathsf{Mor}(1,Y) \to \mathsf{Mor}(FGY)$ is bijective. Assuming that ε_Y is not a monomorphism and taking into account that 1 is a generator, we can find two distinct morphisms $\phi, \psi \in \mathsf{Mor}(1, FGY)$ such that $\varepsilon_Y \circ \phi = \varepsilon_Y \circ \psi$. By the bijectivity of the function η_{GY} , there are distinct morphisms $\phi', \psi' \in \mathsf{Mor}(1,Y)$ such that $\eta_{GY}(\phi') = \phi$ and $\eta_{GY}(\psi') = \psi$. The definition of the morphism ε_Y guarantees that

$$\phi' = \varepsilon_Y \circ \eta_{GY}(\phi') = \varepsilon_Y \circ \phi = \varepsilon_Y \circ \psi = \varepsilon_Y \circ \eta_{GY}(\psi') = \psi',$$

which is a desired contradiction.

By Claims 58.10, 58.11, the morphism ε_Y is a bimorphism. Since the category \mathcal{C} is balanced, the morphism ε_Y is an isomorphism.

Therefore we proved that the natural transformations $\eta: \mathbf{1_{Set}} \to GF$ and $\varepsilon: FG \to \mathbf{1}_{\mathcal{C}}$ are functor isomorphisms, witnessing that the categories \mathcal{C} and **Set** are equivalent.

Lemma 58.5 implies the following characterizations of the full subcategory $\mathbf{Card} \subseteq \mathbf{Set}$ whose objects are cardinals.

Theorem 58.12. Under (AGC), a category C is isomorphic to the category C and if and only if it satisfies the following properties:

- (0) C is skeletal;
- (1) C is locally small;
- (2) C is balanced;

- (3) C has equalizers;
- (4) C has arbitrary coproducts;
- (5) C has a terminal object 1;
- (6) 1 is a C-generator;
- (7) C is element-separating.

Proof. The "only if" part is trivial and holds without (AGC). To prove the "if" part, assume that the Axiom of Global Choice holds and a category \mathcal{C} has properties (0)–(7). By Lemma 58.5, the skeletal category \mathcal{C} is equivalent to the category **Set**. By Theorem 56.5, under (AGC), the category **Set** is equivalent to its skeleton **Card**. Consequently, the skeletal categories \mathcal{C} and **Card** are equivalent, and by Proposition 56.7, these categories are isomorphic.

Theorem 58.13. Under (EC), a category C is equivalent to the category **Set** if and only if it satisfies the following properties:

- (1) C is locally small;
- (2) C is balanced;
- (3) C has equalizers;
- (4) C has arbitrary coproducts;
- (5) C has a terminal object 1;
- (6) 1 is a C-generator;
- (7) C is element-separating.

Proof. The "only if" part is trivial and holds without (EC). To prove the "if" part, assume that the principle (EC) holds and a category \mathcal{C} has properties (1)–(7).

By Theorem 56.2, under (EC), the category \mathcal{C} has a skeleton \mathcal{S} , which is equivalent to \mathcal{C} . Since the properties (1)–(7) are preserved by the equivalence of categories, the category \mathcal{S} has respective properties (1)–(7) and by Lemma 58.5, the category \mathcal{S} is equivalent to the category **Set**. Then $\mathcal{C} \simeq \mathcal{S} \simeq \mathbf{Set}$.

Theorem 58.14. Under (GWO), a category C is isomorphic to the category **Set** if and only if it satisfies the following properties:

- (1) C is locally small;
- (2) C is balanced;
- (3) C has equalizers;
- (4) C has arbitrary coproducts;
- (5) C has a terminal object 1;
- (6) 1 is a C-generator;
- (7) C is element-separating;
- (8) C has a unique initial object 0;
- (9) for any non-initial C-object x the class $\{y \in \mathsf{Ob}_{\mathcal{C}} : y \cong_{\mathcal{C}} x\}$ is a proper class.

Proof. The "only if" part is trivial and holds without (EC). To prove the "if" part, assume that the principle (GWO) holds and a category \mathcal{C} has properties (1)–(9). Since (GWO) \Rightarrow (EC), we can apply Theorem 58.13 and conclude that the categories \mathcal{C} and **Set** are equivalent. By Theorem 56.8, there exists a full faithful functor $F: \mathcal{C} \to \mathbf{Set}$, which is essentially surjective on objects.

By the condition (8), the category \mathcal{C} contains a unique initial object 0. Since the functor F is essentially surjective on objects, for the empty set $\emptyset \in \mathsf{Ob}_{\mathsf{Set}}$, there exists a \mathcal{C} -object Z

such that $FZ \cong \emptyset$. Since \emptyset is an initial object of the category **Set**, the object Z is initial in the category \mathcal{C} and hence Z=0 by the uniqueness of the initial object 0 in \mathcal{C} . Then

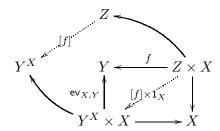
$$|\{x \in \mathsf{Ob}_{\mathcal{C}} : x \cong \mathsf{O}\}| = |1| = |\{y \in \mathsf{Ob}_{\mathbf{Set}} : y \cong \emptyset\}|.$$

On the other hand, for any C-object $x \neq 0$, the uniqueness of an initial object in C implies that x is not initial in \mathcal{C} and hence Fx is not initial in **Set**. The latter means that the set Fx is not empty and then $\{z \in \mathbf{U} : |z| = |Fx|\}$ is a proper class. By the condition (9), the class $\{y \in \mathsf{Ob}_{\mathcal{C}} : y \cong x\}$ is proper, too. By the principle (GWO) the proper classes $\{y \in \mathsf{Ob}_{\mathcal{C}} : y \cong x\}$ and $\{z \in \mathbf{U} : |z| = |Fx|\}$ are well-orderable. By Theorem 23.6 these classes admit a bijective function onto the class **On**, which implies that $|\{y \in \mathsf{Ob}_{\mathcal{C}} : y \cong Fx\}| = |\{z \in \mathbf{U} : |z| = |x|\}|$. Applying Theorem 56.9, we conclude that the categories \mathcal{C} and **Set** are isomorphic.

Remark 58.15. Among conditions characterizing the category Set there are two conditions that have non-finitary nature, namely, the local smallness and the existence of arbitrary colimits. Attempts to give a finitary definition of a category that resembles the category of sets lead Lawvere and Tierney to discovering the notion of an elementary topos: this is a cartesian closed category with a subobject classifier. We shall briefly discuss these notions in the next three sections.

59. Cartesian closed categories

Definition 59.1. A category \mathcal{C} with binary products is called *cartesian closed* if for any \mathcal{C} -objects X,Y there is an exponential object $Y^X \in \mathsf{Ob}_{\mathcal{C}}$ and an evaluation morphism $\mathsf{ev}_{X,Y}$: $Y^X \times X \to Y$ with the universal property that for every C-object Z and C-morphism f: $Z \times X \to Y$ there exist unique C-morphisms $[f]: Z \to Y^X$ and $[f] \times 1_X: Z \times X \to Y^X \times X$ making the following diagram commutative.



In this diagram by arrows without labels we denote the coordinate projections.

Example 59.2. The category **Set** is cartesian closed: for any sets X, Y the exponential object Y^X is the set of all functions $f: X \to Y$, and the evaluation morphism $ev_{X,Y}: Y^X \times X \to Y$ assigns to every ordered pair $\langle \varphi, x \rangle \in Y^X \times X$ the value $\varphi(x)$ of φ at x. For every set Zand function $f: Z \times X \to Y$ the function $[f]: Z \to Y^X$ assigns to every element $z \in Z$ the function $[f]_z: X \to Y, [f]_z: x \mapsto f(z, y).$

Exercise 59.3. Let \mathcal{C} be a cartesian closed category. Prove that for any \mathcal{C} -objects X,Y,Zwe have C-isomorphisms:

- $\bullet \ (Y^X)^Z \cong Y^{X \times Y};$ $\bullet \ Y^X \times Z^X \cong (Y \times Z)^X;$
- $X \cong X^1$ where 1 is a terminal object in \mathcal{C} .

60. Subobject classifiers

In category theory subobjects correspond to subsets in the category of sets. Since the category theory does not "see" the inner structure of objects, subobjects should be defined via morphisms. The idea is to identify subobjects of a given object A with equivalence classes of monomorphisms into A.

We say that two C-morphisms $f: X \to A$ and $g: Y \to A$ of a category C are isomorphic if there exists a C-isomorphism $h: X \to Y$ such that $f = g \circ h$. For a C-morphism f by $[f]_{\cong}$ we denote the class of C-morphisms, which are isomorphic to f.

By definition, a *subobject* of a C-object A is the equivalence class $[i]_{\cong}$ of some monomorphism $i: X \to A$.

Such definition of a subobject is not very convenient to work with because very often subobjects are proper classes. So, it is not even possible to define the class of all subobjects of a given object of a category. In the category of sets subobjects of a given set A can be identified with subsets of A. In its turn, using characteristic functions, we can identify each subset $X \subseteq A$ with the characteristic function $\chi_X : A \to 2$. So, function into the doubleton $2 = \{0, 1\}$ classify subobjects in the category of sets. This property of the doubleton motivates the following definition.

Definition 60.1. Let \mathcal{C} be a category that has a terminal object 1. A *subobject classifier* is a \mathcal{C} -object Ω endowed with a \mathcal{C} -morphism true : $1 \to \Omega$ such that the following two properties are satisfied:

- 1) for any \mathcal{C} -morphism $\chi:A\to\Omega$ the diagram 1 $\stackrel{\mathsf{true}}{\longrightarrow}\Omega\xleftarrow{\chi}A$ has a pullback, and
- 2) for any monomorphism $i: X \to A$ in the category \mathcal{C} there exists a unique \mathcal{C} -morphism $\chi_i: A \to \Omega$, called the characteristic morphism for the monomorphism i, such that for the unique \mathcal{C} -morphism $X \to 1$, the square

$$(60.1) X \xrightarrow{i} A \downarrow \chi_i 1 \xrightarrow{\text{true}} \Omega$$

is a pullback, which means that for any \mathcal{C} -object Y and \mathcal{C} -morphisms $f:Y\to A$ and $g:Y\to 1$ with $\chi_i\circ f=\mathsf{true}\circ g$ there exists a unique \mathcal{C} -morphism $h:Y\to X$ such that $i\circ h=f$.

The uniqueness of the morphism χ_i and the pullback property of the square (60.1) imply that for a \mathcal{C} -object A, two monomorphisms $i: X \to A$ and $j: y \to A$ are isomorphic if and only if $\chi_i = \chi_j$ if and only if $[i]_{\cong} = [j]_{\cong}$. This means that subobjects of a \mathcal{C} -object A are in the bijective correspondence with \mathcal{C} -morphism from A to Ω . The surjectivity of this correspondence follows from the following property of pullbacks.

Exercise 60.2. Prove that for any pullback square

$$X \xrightarrow{i} Y$$

$$\downarrow \qquad \qquad \downarrow$$

$$\downarrow \qquad \qquad \downarrow$$

$$\downarrow \qquad \qquad \downarrow$$

$$\downarrow \qquad \qquad \downarrow$$

the morphism i is always mono.

Proposition 60.3. If a category C has a subobject classifier true : $1 \to \Omega$, then it is unique up to an isomorphism.

Proof. Assume that true: $1 \to \Omega$ and true': $1 \to \Omega'$ are two subobject classifiers. Since 1 is a terminal object, the morphisms true and true' are monomorphisms. Then there are unique characteristic functions $\chi: \Omega' \to \Omega$ and $\chi': \Omega \to \Omega'$ such that the upper and lower squares of the following diagram are pullbacks:

$$\begin{array}{ccc}
1 & \xrightarrow{\text{true}} & \Omega \\
\downarrow & & \downarrow \chi' \\
1 & \xrightarrow{\text{true}'} & \Omega' \\
\downarrow & & \downarrow \chi \\
1 & \xrightarrow{\text{true}} & \Omega
\end{array}$$

Then the external square also is a pullback and then $\chi \circ \chi'$ is the identity morphism of Ω by the definition of a pullback. By analogy we can prove that $\chi' \circ \chi = 1_{\Omega'}$. This means that the morphism $\chi : \Omega' \to \Omega$ is an isomorphism.

Example 60.4. In the category of sets, a subobject classifier exists: it is the function true = $\{\langle 0, 1 \rangle\} : 1 \to 2$.

The morphism true: $1 \to 2$ can be defined in any category with a terminal object 1 and finite coproducts. Namely, let $2 = 1 \sqcup 1$ be a coproduct of two copies of 1 and false: $1 \to 2$, true: $1 \to 2$ be the first and second coordinate coprojections, respectively.

Proposition 60.5. The doubleton 2 endowed with the morphism true: $1 \rightarrow 2$ is a subobject classifier in the category of sets.

Proof. In the category of sets the terminal object 1 is isomorphic to the natural number $1 = \{0\}$ and the coproduct $2 = 1 \sqcup 1$ is isomorphic to the natural number $2 = \{0, 1\}$. Then the morphism true: $1 \to 2$ can be identified with the function $\{\langle 0, 1 \rangle\} : 1 \mapsto \{1\} \subseteq 2$. Given any injective function $i: X \to Y$ between sets, consider the characteristic function $\chi: Y \to 2$ of the subset i[X] of Y. By definition, χ is a unique function such that

$$\chi(y) = \begin{cases} 1 & \text{if } y \in i[X]; \\ 0 & \text{if } y \in Y \setminus i[X]. \end{cases}$$

We should prove that χ is a unique function making the square

$$X \xrightarrow{i} Y$$

$$u \downarrow \qquad \qquad \downarrow \chi$$

$$1 \xrightarrow{\text{true}} 2$$

a pullback. The definition of the function χ ensures that this square is commutative. To prove that it is a pullback, take any set Z and functions $f:Z\to Y$ and $g:Z\to 1$ such that $\chi\circ f=\mathsf{true}\circ g$. The latter equality implies that $f[Z]\subseteq i[X]$. The injectivity of the function $i:X\to Y$ ensures that there exists a unique function $h:Z\to X$ such that $f=i\circ h$. The uniqueness of functions into 1 guarantees that $g=u\circ h$. This means that the above square is indeed a pullback.

To prove the uniqueness of the function χ , take any function $\chi': Y \to 2$ for which the square

$$X \xrightarrow{i} Y$$

$$u \downarrow \qquad \qquad \downarrow \chi'$$

$$1 \xrightarrow{\text{true}} 2$$

is a pullback. The commutativity of this square implies that $\chi'[i[X]] \subseteq \{1\}$. Assuming that $\chi' \neq \chi$, we could find an element $y \in Y \setminus i[X]$ such that $\chi(y) = 1$. Consider the function $f: 1 \to Y$ with f(0) = y and observe that $\chi' \circ f = \mathsf{true}$. The pullback property of the square yields a unique function $h: 1 \to X$ such that $f = i \circ h$. Then $y = f(0) = i(h(0)) \in i[X]$, which contradicts the choice of y.

Exercise 60.6. Let $\mathcal{C}, \mathcal{C}'$ be categories possessing subobject classifiers true : $1 \to \Omega$ and true' : $1' \to \Omega$. Prove that $\langle \text{true}, \text{true}' \rangle$ is a subobject classifier of the product category $\mathcal{C} \times \mathcal{C}'$.

Exercise 60.7. Prove that the category **Set** \times **Set** has a subobject classifier true : $1 \to \Omega$ with $|\mathsf{Mor}(1,\Omega)| = 4$.

Exercise 60.8. Prove that the category of functions $\mathbf{Set}^{\rightarrow}$ has a subobject classifier true : $1 \rightarrow \Omega$ with $|\mathsf{Mor}(1,\Omega)| = 3$.

The existence of subobject classifiers impose some restrictions on a category. We recall that a category is *balanced* if each bimorphism (=mono+epi) is an isomorphism.

Proposition 60.9. If a category C has a subobject classifier true : $1 \to \Omega$, then C is a balanced category.

Proof. Given an bimorphism $f: X \to Y$ in the category \mathcal{C} , find a unique \mathcal{C} -morphism $\chi: Y \to \Omega$ into the classifying object Ω making the square

$$X \xrightarrow{f} Y$$

$$\downarrow \qquad \qquad \downarrow \chi$$

$$1 \xrightarrow{\text{true}} \Omega$$

a pullback. Then for the identity morphism $1_Y: Y \to Y$ and the unique morphism $Y \to 1$, the pullback property of this square implies the existence of a unique \mathcal{C} -morphism $h: Y \to X$ such that $f \circ h = 1_Y$. Since f is a monomorphism, the equality $f \circ (h \circ f) = (f \circ h) \circ f = 1_Y \circ f = f \circ 1_X$ implies $h \circ f = 1_X$. Therefore, f is an isomorphism with $f^{-1} = h$.

Proposition 60.10. Assume that a category C has a subobject classifier true: $1 \to \Omega$. If C has binary squares, then it has limits of finite diagrams. It particular, it has equalizers and pullbacks.

Proof. To show that the category \mathcal{C} has equalizers, fix any \mathcal{C} -morphisms $f, g: X \to Y$. Since \mathcal{C} has binary products, it has a product $Y \times Y$. By definition of the product, there exists a unique \mathcal{C} -morphism $\delta: Y \to Y \times Y$ such that $\operatorname{pr}_1 \circ \delta = 1_Y = \operatorname{pr}_2 \circ \delta$, where $\operatorname{pr}_1, \operatorname{pr}_2: Y \times Y \to Y$ are the coordinate projections of the product $Y \times Y$. The latter equalities imply that δ is a

monomorphism. Then there exists a C-morphism $\chi: Y \times Y \to \Omega$ such that the square

$$Y \xrightarrow{\delta} Y \times Y$$

$$\downarrow \chi$$

$$\downarrow \chi$$

$$\downarrow \chi$$

$$\downarrow \chi$$

$$\downarrow \chi$$

$$\downarrow \chi$$

is a pullback.

By definition of the product $Y \times Y$, there exists a unique \mathcal{C} -morphism $(f,g): X \to Y \times Y$ such that $f = \operatorname{pr}_1 \circ (f,g)$ and $g = \operatorname{pr}_2 \circ (f,g)$. Now consider the morphism $h = \chi \circ (f,g): X \to \Omega$. By definition of a subobject classifier, there exists a pullback

$$E \xrightarrow{e} X$$

$$\downarrow h$$

$$1 \xrightarrow{h} \Omega.$$

It can be shown that the morphism $e: E \to X$ is an equalizer of the pair (f,g).

The existence of binary products and equalizers implies the existence of pullbacks and limits of all finite diagrams, see Exercise 57.18 and 57.27.

61. Elementary Topoi

Lawvere's axioms for elementary topos helped many people outside the community of specialists to enter into this field and make a fruitful research in it.

Everyone who learns today the topos theory begins with Lawvere's axioms for elementary topos.

This makes Lawvere's axiomatization of topos theory a true success story of Axiomatic Method in the twentieth century mathematics.

Andrei Rodin, "Axiomatic Method and Category Theory", 2014

Elementary topoi were introduced by Lawvere and Tierney in 1968-69. Now the theory of elementary topoi is well-developed and is considered as a foundation of mathematics (alternative to Set Theory). The modern definition of an elementary topos is very short.

Definition 61.1. An *elementary topos* if C is a cartesian closed category with a subobject classifier.

A standard example of an elementary topos is the category of sets. On the other hand, the categories $\mathbf{Set} \times \mathbf{Set}$ and $\mathbf{Set}^{\rightarrow}$ are elementary topoi, which are not equivalent to the category \mathbf{Set} (because their subobject classifiers have more than two elements).

For any C-object X of an elementary topos C with a subject classifier Ω , we can consider the exponential object Ω^X , called the *power object* of X. The power object Ω^X indexes all subobjects of X. Using the evaluation morphism $\operatorname{ev}_{X,\Omega}:\Omega^X\times X\to\Omega$, for any global elements $s:1\to\Omega^X$ and $x:1\to X$, we can consider the morphism $\operatorname{ev}_{X,\Omega}\circ(s,x):1\to\Omega$ and compare it with the morphism $\operatorname{true}:1\to\Omega$. The equality $\operatorname{ev}_{X,\Omega}\circ(s,x)=\operatorname{true}$ can be interpreted as the indication that the global element x "belongs" to the subobject s of X. This allows to apply element-based arguments resembling those practiced in the classical Set Theory and Logics.

The subject of Categorial Logic is very extensive and we will not develop it here referring the reader to the monographs [6], [13], [22].

In this section we characterize elementary topoi, which are equivalent or isomorphic to the category of sets.

Definition 61.2. An elementary topos is called *well-pointed* if its terminal object 1 is a generator and its subobject classifier Ω is not a terminal object.

Exercise 61.3. Show that the category of finite sets FinSet is a well-pointed elementary topos.

Proposition 61.4. If an elementary topos C is well-pointed, then its subobject classifier Ω is two-valued in the sense that $|\mathsf{Mor}(1,\Omega)| = 2$.

Proof. Assume that an elementary topos $\mathcal C$ is well-pointed. Then its subobject classifier Ω is not a terminal object of the category $\mathcal C$. Because of the morphism true : $\mathbf 1 \to \Omega$, every $\mathcal C$ -object X has a morphism $X \to \Omega$. Since Ω is not terminal, there exists a $\mathcal C$ -object X admitting two distinct morphisms $f,g:X\to\Omega$. Since 1 is a generator, there exists a morphism $h:1\to X$ such that $f\circ h\neq g\circ h$. Consequently, $|\mathsf{Mor}(1,\Omega)|\geq 2$. Since the set $\mathsf{Mor}(1,\Omega)$ classifies subobjects of 1, the equality $|\mathsf{Mor}(1,\Omega)|=2$ will follow as soon as we show that 1 has exactly two subobjects.

Let $i: X \to 1$ be any monomorphism. If there exists a morphism $x: 1 \to X$, then $i \circ x = 1_1$ and hence i is an epimorphism. By Proposition 60.9, the catgeory \mathcal{C} is balanced, which implies that i is an isomorphism. Assuming that 1 has more than two subobjects, we can find two nonisomorphic monomorphisms $u: U \to 1$ and $v: V \to 1$ such that $\mathsf{Mor}(1, U) = \emptyset = \mathsf{Mor}(1, V)$. Now consider the pullback

$$W \xrightarrow{\phi} U$$

$$\downarrow u$$

$$V \xrightarrow{u} 1$$

which exists as the category \mathcal{C} has binary products and equalizers according to Proposition 60.10. We claim that the morphism $\Phi:W\to U$ is a monomorphism. In the opposite case we could find a \mathcal{C} -object Z and distinct morphisms $f,g:Z\to W$ such that $\phi\circ f=\phi\circ g$. Since 1 is a generator, there exists a \mathcal{C} -morphism $h:1\to Z$ such that $f\circ h\neq g\circ h$. Then the composition $\phi\circ f\circ h$ belongs to the class $\mathsf{Mor}(1,U)=\emptyset$, which is a desired contradiction showing that ϕ is a monomorphism. By analogy we can prove that ψ is a monomorphism. Since the morphisms u,v are not isomorphic, either ϕ or ψ is not an isomorphism. We lose no generality assuming that ϕ is not an isomorphism. Then U has two non-isomorphic monomorphisms: $1_U:U\to U$ and $\phi:W\to U$, which are classified by two distinct morphisms $\chi,\chi':U\to\Omega$. Since 1 is a generator, there a morphism $\varphi:1\to U$ such that $\chi\circ\varphi\neq\chi'\circ\varphi$. But φ cannot exist as $\mathsf{Mor}(1,U)=\emptyset$. This contradiction completes the proof of the equality $|\mathsf{Mor}(1,\Omega)|=2$.

Proposition 61.5. Each well-pointed elementary topos C is element-separating.

Proof. Given any C-morphism $x: 1 \to X$, observe that x is a monomorphism (by the terminal property of 1). By definition of the subobject classifier, there exists a unique C-morphism

 $\chi_x: X \to \Omega$ such that the commutative square

$$\begin{array}{ccc}
1 & \xrightarrow{x} X \\
1_1 & & \downarrow \chi_x \\
1 & \xrightarrow{\text{true}} \Omega
\end{array}$$

is a pullback.

By Proposition 61.4, $|\mathsf{Mor}(\mathtt{1},\Omega)| = 2$. Let false be the unique element of the set $\mathsf{Mor}(\mathtt{1},\Omega) \setminus \{\mathsf{true}\}$. The pullback property of the above square ensures that for any $y \in \mathsf{Mor}(\mathtt{1},X) \setminus \{x\}$ we have $\chi_x \circ y \neq \mathsf{true}$ and hence $\chi_x \circ y = \mathsf{false}$. Now we see that the morphisms $\chi_x : X \to \Omega$ and $\mathsf{false} : \mathtt{1} \to \Omega$ witness that the category $\mathcal C$ is element-separating. \square

Propositions 60.9, 60.10, 61.5 and Theorems 58.13, 58.14 imply the following characterizations of the category **Set** (for the global choice principles (EC) and (GWO), see Section 28).

Theorem 61.6. Under (EC), a category C is equivalent to the category **Set** if and only if C is a locally small well-pointed elementary topos that has arbitrary coproducts.

Theorem 61.7. Under (GWO), a category C is isomorphic to the category **Set** if and only if

- 1) $\mathcal C$ is a well-pointed elementary topos;
- 2) C is locally small;
- 3) C has arbitrary coproducts;
- 4) C has a unique initial object;
- 5) for any non-initial C-object X the class of C-objects that are isomorphic to X is proper.

Epilogue

The material presented in this book is a reasonable minimum which a good (post-graduate) student in Mathematics should know about foundations of this science.

There are many nice textbooks that elaborate in details selected topics that were only briefly touched in this textbook.

In particular, in Mathematical Logic a classical textbook is that of Mendelson [18]; there is also a new book of Kunen [12].

In Set Theory and Forcing recommended textbooks are those of Jech [5] and Kunen [11]. For surreal numbers we refer the interested reader to the original books of Conway [3] and Knuth [10].

Mathematical Structures (of algebraic origin) and their relation to Category Theory are well-elaborated in the lecture notes of Bergman [1]; Model Theory can be further studied via the classical textbook Chang and Keisler [2].

Category Theory can be studied using the classical textbook of Mac Lane [17], and Topos Theory via the "Elephant" of Johnstone [6]. A short and readable introduction to Category Theory and Categorial Logic is that of Streicher [22].

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