REDUCED TENSOR PRODUCT ON THE DRINFELD CENTER

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Abstract. The annulus comes with a "stacking" operation which glues two annuli into one. This provides a tensor product structure on the category of boundary values $Z_{\text{CY}}(\text{Ann})$ associated to the annulus in an extended Crane-Yetter TQFT. It is known that $Z_{\text{CY}}(\text{Ann}) \simeq \mathcal{Z}(\mathcal{A})$, the Drinfeld center of the premodular category $\mathcal{A}$ from which Z_{CY} is constructed. We give an explicit formula for the tensor product on $\mathcal{Z}(\mathcal{A})$ that corresponds to the stacking operation on $Z_{\text{CY}}(\text{Ann})$.

1. Introduction

The Crane-Yetter state-sum Z_{CY}, as defined in [CY1993] and further generalized in [CKY2], is a 4D TQFT that depends on a choice of premodular category $\mathcal{A}$, that is, a braided spherical fusion category. When $\mathcal{A}$ is modular, it is known that Z_{CY} is an invertible theory that detects only the signature and Euler characteristic of a closed 4-manifold [CKY1]. In [KT], we studied a certain category-valued invariant $Z_{\text{CY}}(\Sigma)$ of surfaces Σ, which is supposed to be a codimension-2 extension of the Crane-Yetter TQFT. Briefly, $Z_{\text{CY}}(\Sigma)$ has as objects configurations of $\mathcal{A}$-labelled marked points in Σ, and morphisms are $\mathcal{A}$-colored graphs in $\Sigma \times [0,1]$ (see Section 4). We showed that Z_{CY} satisfies an excision property: when a surface Σ is built from two smaller surfaces Σ_1 and Σ_2 along a circular boundary, we have that

$$Z_{\text{CY}}(\Sigma) \simeq Z_{\text{CY}}(\Sigma_1) \boxtimes_{Z_{\text{CY}}(\text{Ann})} Z_{\text{CY}}(\Sigma_2)$$

(one can also consider surfaces glued along segments of boundaries; see [KT, Theorem 7.5]). Here $Z_{\text{CY}}(\text{Ann})$ is endowed with the "stacking" tensor product, which comes from the operation of gluing two annuli along a boundary to get a new annulus, and $Z_{\text{CY}}(\Sigma_1), Z_{\text{CY}}(\Sigma_2)$ have $Z_{\text{CY}}(\text{Ann})$-module structures by virtue of their circular boundary component. We note that this excision property was also proved independently in [Co].

It is known that

$$\mathcal{Z}(\mathcal{A}) \simeq Z_{\text{CY}}(\text{Ann})$$

where $\mathcal{Z}(\mathcal{A})$ is the Drinfeld center of $\mathcal{A}$ (see for example [KT, Example 8.2]). Thus, we were motivated to describe this stacking tensor product on $\mathcal{Z}(\mathcal{A})$ intrinsically, and the result is the main construction of the paper, the *reduced tensor product*, denoted $\overline{\otimes}$ (see Definition 2.7).

The main result of this paper is Theorem 4.3, which says that the *reduced tensor product* on $\mathcal{Z}(\mathcal{A})$ encodes the *"stacking" tensor product* on $Z_{\text{CY}}(\text{Ann})$, expressed in the following 2-commutative diagram of tensor functors:

$$\begin{tikzcd}
(\mathcal{A}, \otimes) \arrow[r, "\text{hTr}"] \arrow[d, "I"'] & (\hat{\mathcal{A}}, \hat{\otimes}) \arrow[r, "H", "\simeq"'] \arrow[d, "\text{Kar}"] \arrow[dl, "G"'] & (\hat{Z}_{\text{CY}}(\text{Ann}), \overline{\otimes}_{\text{st}}) \arrow[d, "\text{Kar}"] \\
(\mathcal{Z}(\mathcal{A}), \overline{\otimes}) & (\text{Kar}(\hat{\mathcal{A}}), \hat{\otimes}) \arrow[l, "\simeq", "\text{Kar}(G)"'] \arrow[r, "\simeq", "\text{Kar}(H)"'] & (Z_{\text{CY}}(\text{Ann}), \overline{\otimes}_{\text{st}})
\end{tikzcd} \tag{1.1}$$

where

- $(\mathcal{Z}(\mathcal{A}), \overline{\otimes})$: the Drinfeld center with the reduced tensor product, see Section 2, in particular Definition 2.7;
- I: see Proposition 2.2;
- $\hat{\mathcal{A}}$, hTr: see Definition 3.1,Definition 3.2;
- G: see Proposition 3.3;
- $\hat{Z}_{\text{CY}}(\text{Ann}), Z_{\text{CY}}(\text{Ann})$: category of boundary values, with its "stacking" tensor product, see Definition 4.1;

- H: see (4.1);
- Kar refers to Karoubi envelope.

We also describe a $\mathbb{Z}/2$-action on each category, and show that the functors above are $\mathbb{Z}/2$-equivariant.

As an application of Theorem 4.3, we describe $Z_{\mathrm{CY}}(\mathbf{T}^2)$ purely algebraically in terms of $\mathcal{A}$ (see Corollary 4.5).

Finally in the last two sections, we consider these constructions in light of additional assumptions on $\mathcal{A}$, in particular the extreme cases of when $\mathcal{A}$ is modular (Section 5) and when $\mathcal{A}$ is symmetric (Section 6). In these cases, the reduced tensor product has simpler descriptions:

- When $\mathcal{A}$ is modular, one has the equivalence $\mathcal{Z}(\mathcal{A}) \simeq_{\otimes,br} \mathcal{A} \boxtimes \mathcal{A}^{\mathrm{bop}}$ [Mu], which has simple objects $X_i \boxtimes X_j^*$, where $X_i, X_j \in \mathcal{A}$ are simple. Then the reduced tensor product on $\mathcal{Z}(\mathcal{A})$ induces the following tensor product on $\mathcal{A} \boxtimes \mathcal{A}$ (see (5.1)):
$$(X_i \boxtimes X_j^*) \,\overline{\otimes}\, (X_k \boxtimes X_l^*) \simeq \delta_{j,k} X_i \boxtimes X_l^*$$
- When $\mathcal{A}$ is symmetric, in particular when $\mathcal{A} = \mathrm{Rep}(G)$ for a finite group G, $\mathcal{Z}(\mathcal{A})$ can be described as the category of G-equivariant bundles over G, where G acts by conjugation on the base (see e.g. [BK, Section 3.2]). Then the reduced tensor product on $\mathcal{Z}(\mathcal{A})$ corresponds to the fibrewise tensor product of bundles.

We are certainly not the first to consider the stacking product. In [BZBJ1], they show that the value of factorization homology $\int_{\mathrm{Ann}} \mathcal{A}$, as introduced in [AFR], on the annulus is equivalent to the category of modules over a certain algebra $\mathfrak{F}_{\mathcal{A}}$ in $\mathcal{A}$. In a subsequent paper [BZBJ2], they show that the stacking product on $\int_{\mathrm{Ann}} \mathcal{A}$ corresponds to the relative tensor product $M \otimes_{\mathfrak{F}_{\mathcal{A}}} N$ for two modules $M, N \in \mathfrak{F}_{\mathcal{A}} - \mathrm{mod}$. Since Z_{CY} satisfies excision, it must be the same as $\int_{-} \mathcal{A}$ (see [AF]), thus our definitions must agree (but we have not worked out an explicit correspondence).

We also note that Wasserman [Was] defined a "symmetric tensor product" on $\mathcal{Z}(\mathcal{A})$ when $\mathcal{A}$ is symmetric, and coincides with the reduced tensor product defined in this paper. He shows in [Was2] that this symmetric tensor product, together with the usual tensor product, makes $\mathcal{Z}(\mathcal{A})$ a "bilax 2-fold tensor category", which is of interest to topologists as they are closely related to iterated loop spaces (see [Was2] and references therein). The reduced tensor product defined in this paper should fit into a similar structure, but we will not pursue this connection in much detail in this paper, merely commenting on it briefly in Remark 2.14 and Remark 4.9.

Acknowledgements: The author thanks David Ben-Zvi, David Jordan, Louis Kauffman, Alexander Kirillov, Pavel Safranov, Dennis Sullivan, and Thomas Wasserman for many helpful discussions.

Notations, Conventions, and Lemmas. We adopt notations and conventions from [Kir], to which we refer the reader for details and proofs. Throughout the paper, we fix an algebraically closed field $\mathbf{k}$ of characteristic zero. All vector spaces and linear maps will be considered over $\mathbf{k}$. $\mathcal{A}$ will denote a premodular category over $\mathbf{k}$. In particular, $\mathcal{A}$ is semisimple with finitely many isomorphism classes of simple objects. We will denote by $\mathrm{Irr}(\mathcal{A})$ the set of isomorphism classes of simple objects. We fix a representative X_i for each isomorphism class $i \in \mathrm{Irr}(\mathcal{A})$; abusing language, we will frequently use the same letter i for denoting both the isomorphism class and the choice of representative X_i. We will also denote by $\mathbf{1} = X_1$ the unit object in $\mathcal{A}$ (which is simple).

Rigidity gives us an involution $-^*$ on $\mathrm{Irr}(\mathcal{A})$; X_i's are chosen so that $X_{i^*} = X_i^*$. When there is little cause for confusion, we will suppress the associativity, unit, and pivotal morphisms $\delta : X \simeq X^{**}$.

We denote the categorical dimensions of simple objects by $d_i = \dim X_i$. For each i, we will fix a choice of square root $\sqrt{d_i}$ so that $\sqrt{d_1} = 1$ and $\sqrt{d_i} = \sqrt{d_{i^*}}$. We also denote the dimension of $\mathcal{A}$ by $\mathcal{D} = \sum_{i \in \mathrm{Irr}(\mathcal{A})} d_i^2$, and fix a square root $\sqrt{\mathcal{D}}$. Note that by results of [ENO2005], $\mathcal{D} \neq 0$.

Concatenation of objects will mean tensor product; this will always be the "standard" one, e.g. the tensor product in $\mathcal{A}$ or tensor product of vector spaces. When we are constructing a non-standard tensor product, we use a different symbol like $\overline{\otimes}$, and duals are represented by $X^\vee$ or ${}^\vee X$.

We define the functor $\langle\rangle : \mathcal{A}^{\boxtimes n} \to \mathrm{Vec}$ by

$$\langle V_1, \dots, V_n \rangle = \mathrm{Hom}_{\mathcal{A}}(\mathbf{1}, V_1 \cdots V_n) \tag{1.2}$$

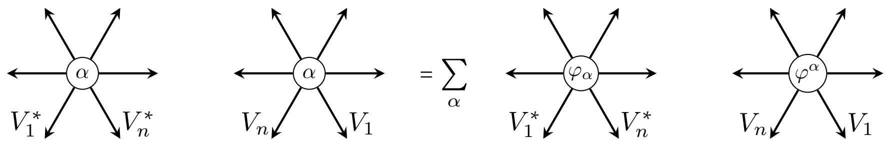
The pivotal structure gives functorial isomorphisms

$$z : \langle V_1, \dots, V_n\rangle \simeq \langle V_n, V_1, \dots, V_{n-1}\rangle \tag{1.3}$$

such that $z^n = \mathrm{id}$ (see [BK, Section 5.3]); so up to canonical isomorphism, $\langle V_1, \dots, V_n\rangle$ only depends on the cyclic order of $V_1, \dots, V_n$.

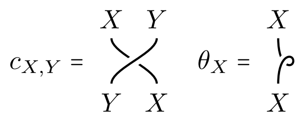
There is a non-degenerate pairing $\mathrm{ev} : \langle V_1, \dots, V_n\rangle \otimes \langle V_n^*, \dots, V_1^*\rangle \to \mathbf{k}$ obtained by post-composing with evaluation maps. When two nodes in a graph are labelled by the same greek letter, say α, it stands for a summation over a pair of dual bases:

(1.4)



where $\varphi_\alpha \in \langle V_1, \dots, V_n\rangle$, $\varphi^\alpha \in \langle V_n^*, \dots, V_1^*\rangle$ are dual bases with respect to the pairing ev.

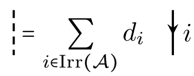
Diagrams represent morphisms from top to bottom. The braiding c and twist operators θ are depicted by right-hand twists:



We will denote $\overline{\theta} := \theta^{-1}$.

A dashed line stands for the regular coloring, i.e. the sum of all colorings by simple objects i, each taken with coefficient d_i:

(1.5)

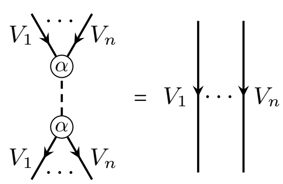


An oriented edge labelled X is the same as the oppositely oriented edge labelled X^*.

Note that although we will be discussing various pivotal multifusion categories, the morphisms are described in terms of morphisms in $\mathcal{A}$, in particular all morphisms depicted graphically are of morphisms in $\mathcal{A}$, unless specified otherwise.

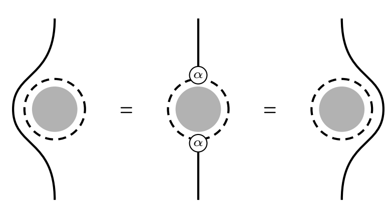
Finally, we record some facts and lemmas that are useful for computations. We give no proofs, referring readers to [Kir],[KT].

(1.6)



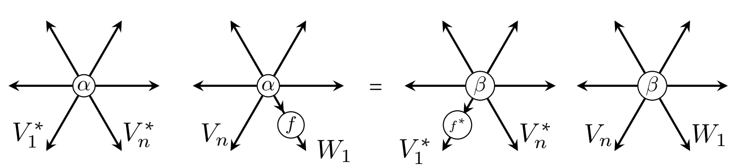
Sliding lemma:

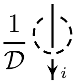
(1.7)



where the shaded region can contain anything.

(1.8) For $f : V_1 \to W_1$,



(1.9) When $\mathcal{A}$ is modular, $\frac{1}{\mathcal{D}}$  $= \delta_{i,1}\,\mathrm{id}_{X_i}$

2. Reduced Tensor Product on $\mathcal{Z}(\mathcal{A})$

Recall that $\mathcal{Z}(\mathcal{A})$, the Drinfeld center of $\mathcal{A}$, is the category with

Objects: pairs (X,γ), where $X \in \mathcal{A}$ and γ is a half-braiding, i.e. a natural isomorphism of functors $\gamma_A : A \otimes X \to X \otimes A$, $A \in \mathcal{C}$ satisfying natural compatibility conditions.

Morphisms: $\mathrm{Hom}_{\mathcal{Z}(\mathcal{A})}((X,\gamma),(X',\gamma')) = \{f \in \mathrm{Hom}_{\mathcal{A}}(X,X') \mid f\gamma = \gamma' f\}$.

We recall some well-known properties of $\mathcal{Z}(\mathcal{A})$. These do not require the braiding on $\mathcal{A}$, only its spherical fusion structure.

The following is standard (see e.g. [EGNO, Corollary 8.20.14]):

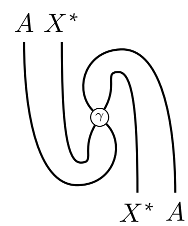
Definition 2.1. $\mathcal{Z}(\mathcal{A})$ is modular, with tensor product

$$(X,\gamma)\otimes(Y,\mu) := (X\otimes Y, \gamma\otimes\mu) \tag{2.1}$$

where

$$(\gamma\otimes\mu)_A := (\mathrm{id}_X \otimes \mu_A)\circ(\gamma_A \otimes \mathrm{id}_Y) \tag{2.2}$$

and left dual given by

(2.3) $(X,\gamma)^* = (X^*,\gamma^*)$, where $(\gamma^*)_A = (\gamma_{^*A})^* =$ 

and similarly the right dual is $^*(X,\gamma) = (^*X, {}^*\gamma)$, where $(^*\gamma)_A = {}^*(\gamma_{A^*})$.

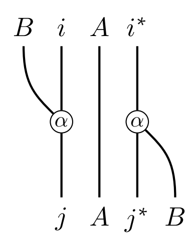
The pivotal structure is given by that of $\mathcal{A}$. △

The following is taken from [Kir, Theorem 8.2]:

Proposition 2.2. *Let $F : \mathcal{Z}(\mathcal{A}) \to \mathcal{A}$ be the natural forgetful functor $F : (X,\gamma) \mapsto X$. Then it has a two-sided adjoint functor $I : \mathcal{A} \to \mathcal{Z}(\mathcal{A})$, given by*

$$I(A) = \Big(\bigoplus_{i\in\mathrm{Irr}(\mathcal{A})} X_i \otimes A \otimes X_i^*, \Gamma\Big) \tag{2.4}$$

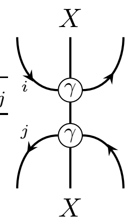
where Γ is the half-braiding given by

(2.5) $\Gamma_B = \sum_{i,j\in\mathrm{Irr}(\mathcal{A})} \sqrt{d_i}\sqrt{d_j}$ 

We refer the reader to [Kir] *for more details.*

Proposition 2.3. *The adjoint functor $I : \mathcal{A} \to \mathcal{Z}(\mathcal{A})$ above is dominant. More explicitly, the object (X,γ) is a direct summand of $I(X)$, given by the projection $P_{(X,\gamma)}$, described below:*

(2.6) $$P_{(X,\gamma)} := \sum_{i,j\in\mathrm{Irr}(\mathcal{A})} \frac{\sqrt{d_i}\sqrt{d_j}}{\mathcal{D}}$$



2.1. The Reduced Tensor Product, $\overline{\otimes}$.

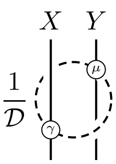
From here on, we will assume that $\mathcal{A}$ is premodular. We define a different monoidal structure on $\mathcal{Z}(\mathcal{A})$, which we call the *reduced tensor product*, denoted by $\overline{\otimes}$; we emphasize that *braiding* is required to define this monoidal structure. We will see in the coming sections that the definitions and results proved here have topological origin, but we first present them purely algebraically.

We note that many of the definitions and formulas here can be found in [Tham, Section 3], where we used them to define a tensor product on a different category, $\mathcal{Z}^{\mathrm{el}}(\mathcal{A})$. It is no coincidence that there is overlap, as $\mathcal{Z}^{\mathrm{el}}(\mathcal{A})$ is the category associated to the once-punctured torus, $Z_{\mathrm{CY}}(\mathbf{T}^2_0)$ (see [KT, Proposition 9.5]), and both the tensor product defined here and the one in [Tham] have similar topological origins (compare [Tham, Remark 3.21] and Remark 4.8). We also note again that [Was] defines a similar tensor product that coincides with ours when $\mathcal{A}$ is symmetric.

The definition of $\overline{\otimes}$ will be given in several steps.

Definition 2.4. Let X, Y be objects in $\mathcal{A}$, and let γ, μ be half-braidings on X, Y respectively. The *reduced tensor product of* X *and* Y *with respect to* γ, μ is defined as the image of the projection $Q_{\gamma,\mu} : X \otimes Y \to X \otimes Y$ defined as follows:

(2.7) $$X\,{}_{\gamma}\overline{\otimes}_{\mu} Y := \mathrm{im}(Q_{\gamma,\mu}) \quad , \quad Q_{\gamma,\mu} := \frac{1}{\mathcal{D}}$$



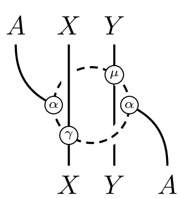
It is easy to check that $Q^2_{\gamma,\mu} = Q_{\gamma,\mu}$. △

(essentially [Tham, Definition 3.14], compare [Was, Equation (11)])

There is an accompanying definition of $\overline{\otimes}$ for half-braidings:

Definition 2.5. Let γ, μ be half-braidings on X, Y respectively. Define $\gamma \overline{\otimes} \mu$ to be natural transformation $- \otimes XY \to XY \otimes -$ given by

$$(\gamma \overline{\otimes} \mu)_A = \frac{1}{\mathcal{D}}$$



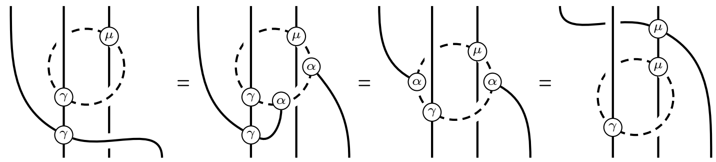
△

$\gamma \overline{\otimes} \mu$ generally fails to be a half-braiding, only insofar as it is not an isomorphism. Observe that $\gamma \overline{\otimes} \mu$ commutes with $Q_{\gamma,\mu}$, so it descends to a natural transformation $- \otimes (X\,{}_{\gamma}\overline{\otimes}_{\mu} Y) \to (X\,{}_{\gamma}\overline{\otimes}_{\mu} Y) \otimes -$. This is in fact a half-braiding on $X\,{}_{\gamma}\overline{\otimes}_{\mu} Y$:

Lemma 2.6. *Let γ, μ be half-braidings on X, Y respectively. $\gamma \otimes c^{-1}$ and $c \otimes \mu$ define half-braidings on $X \otimes Y$, where recall c is the braiding on $\mathcal{A}$. Observe that the projection $Q_{\gamma,\mu}$ intertwines both $\gamma \otimes c^{-1}$ and $c \otimes \mu$, thus they restrict to half-braiding on $X\,{}_{\gamma}\overline{\otimes}_{\mu} Y$. Then as half-braidings on $X\,{}_{\gamma}\overline{\otimes}_{\mu} Y$, we have*

$$\gamma \overline{\otimes} \mu = \gamma \otimes c^{-1} = c \otimes \mu$$

Proof. This follows from the following computation:

(2.8) 

□

(essentially [Tham, Lemma 3.16], compare [Was, Lemma 10])

Definition 2.7. Let $(X,\gamma),(Y,\mu)$ be objects in $\mathcal{Z}(\mathcal{A})$. Their *reduced tensor product* is defined as follows:

$$(X,\gamma)\,\overline{\otimes}\,(Y,\mu) := (X\,{}_{\gamma}\overline{\otimes}_{\mu}\,Y, \gamma\,\overline{\otimes}\,\mu)$$

For $f:(X,\gamma)\to(X,\gamma'), g:(Y,\mu)\to(Y',\mu')$, their reduced tensor product is

$$f\,\overline{\otimes}\,g := Q_{\gamma',\mu'}\circ(f\otimes f')\circ Q_{\gamma,\mu}$$

or more simply, it is $f\otimes f'$ restricted to $X\,{}_{\gamma}\overline{\otimes}_{\mu}\,Y$. △

Lemma 2.8. *The reduced tensor product of Definition 2.7 is associative. More precisely, if* $a:(X_1\otimes X_2)\otimes X_3\simeq X_1\otimes(X_2\otimes X_3)$ *is the associativity constraint of* $\mathcal{A}$*, and* $\gamma_1,\gamma_2,\gamma_3$ *are half-braidings on* X_1,X_2,X_3 *respectively, then* a *restricts to an isomorphism*

$$a:(X_1\,{}_{\gamma_1}\overline{\otimes}_{\gamma_2}\,X_2)\,{}_{\gamma_1\overline{\otimes}\gamma_2}\overline{\otimes}_{\gamma_3}\,X_3\simeq X_1\,{}_{\gamma_1}\overline{\otimes}_{\gamma_2\overline{\otimes}\gamma_3}\,(X_2\,{}_{\gamma_2}\overline{\otimes}_{\gamma_3}\,X_3)$$

and hence

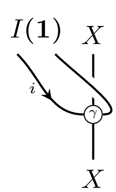
$$a:\big((X_1,\gamma_1)\,\overline{\otimes}\,(X_2,\gamma_2)\big)\,\overline{\otimes}\,(X_3,\gamma_3)\simeq(X_1,\gamma_1)\,\overline{\otimes}\,\big((X_2,\gamma_2)\,\overline{\otimes}\,(X_3,\gamma_3)\big)$$

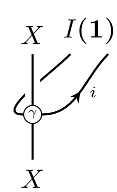
Furthermore, a *is natural in* (X_i,γ_i)*, and satisfies the pentagon equation.*

Proof. Follows easily from Lemma 2.6. See also [Tham, Corollary 3.17], and compare [Was, Lemma 17]. □

Proposition 2.9. $(\mathcal{Z}(\mathcal{A}),\overline{\otimes})$ *is a pivotal multifusion category. More precisely,*

- *the associativity constraint is given by the associativity constraint of* $\mathcal{A}$ *(see Lemma 2.8);*
- *the unit object, denoted* $\overline{\mathbf{1}}$*, is* $I(\mathbf{1})$ *(see Proposition 2.2), with left and right unit constraints given by*

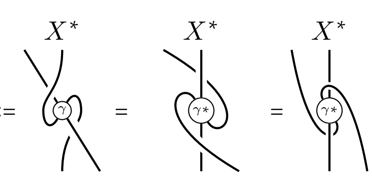
(2.9) $$l_{(X,\gamma)} := \sum_i \frac{\sqrt{d_i}}{\sqrt{\mathcal{D}}}$$ 

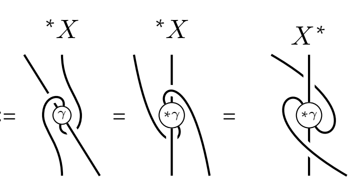
$$r_{(X,\gamma)} := \sum_i \frac{\sqrt{d_i}}{\sqrt{\mathcal{D}}}$$ 

- *the duals are given by*

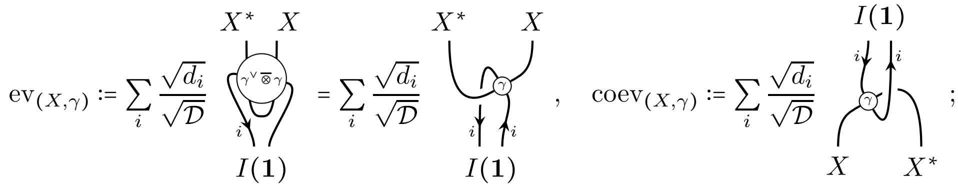
$$(X,\gamma)^\vee := (X^*,\gamma^\vee),\quad {}^\vee(X,\gamma) := ({}^*X,{}^\vee\gamma)$$

where

(2.10) $\gamma^\vee :=$ 

, ${}^\vee\gamma :=$ 

with evaluation and coevaluation maps given by (the projections $Q_{\gamma^\vee,\gamma}$ *are implicit)*

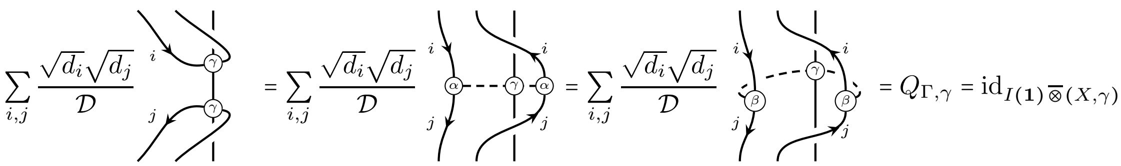
(2.11) 

and similarly for right duals.

- *the pivotal structure is given by that of $\mathcal{A}$.*

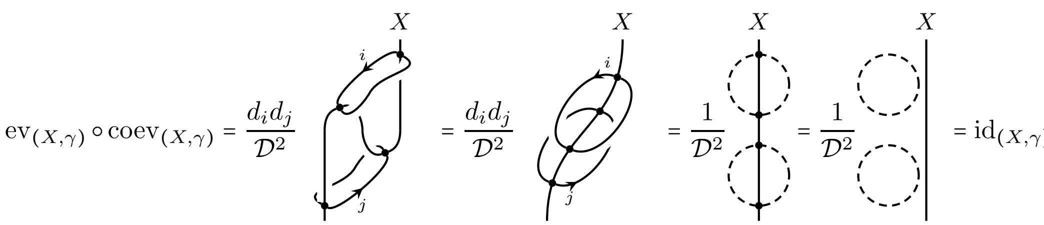
(compare [Was, Theorem 24])

Proof. The inverses of the unit constraints are given by reflecting the diagrams vertically. For example, we have



The last two forms of $\gamma^\vee$ are equivalent from the following consideration: pulling the diagonal strands across the γ^* morphism accumulates a (left) right-hand twist, i.e. (inverse) Drinfeld morphism, and these cancel out by the naturality of half-braidings.

It is also easy to check that the (co)evalution maps described have the desired properties. The only potentially confusing thing is that the "empty strand" in graphical calculus is no longer the unit of $\mathcal{Z}(\mathcal{A})$, so the unit object and unit constraints have to be explicitly included. For example, (all dot nodes represent γ, and sum over $i, j \in \mathrm{Irr}(\mathcal{A})$ is implicit)



As for the pivotal structure, it is straightforward to check that $\gamma^{\vee\vee} = \gamma^{**}$. □

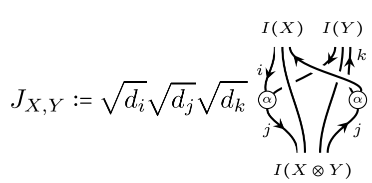
Remark 2.10. $\overline{\otimes}$ is not braided; indeed, we will see later (e.g. (5.2)) that there can be objects $(X,\gamma),(Y,\mu) \in \mathcal{Z}(\mathcal{A})$ such that $(X,\gamma)\overline{\otimes}(Y,\mu) \not\simeq 0$ and $(Y,\mu)\overline{\otimes}(X,\gamma) \simeq 0$. In particular, this necessitates the "multi" in multifusion. △

When considering the usual monoidal structure on $\mathcal{Z}(\mathcal{A})$, the forgetful functor $F : \mathcal{Z}(\mathcal{A}) \to \mathcal{A}$ is naturally a tensor functor, but its adjoint, $I : \mathcal{A} \to \mathcal{Z}(\mathcal{A})$ is not. With the reduced tensor product, however, F is clearly not tensor, but I is:

Proposition 2.11. *The functor I from Proposition 2.2 is a tensor functor. More precisely, for $X, Y \in \mathcal{A}$, define the morphism*

$$J_{X,Y} : I(X)\overline{\otimes} I(Y) \to I(X \otimes Y)$$

by



Then

$$(I,J):(\mathcal{A},\otimes)\to(\mathcal{Z}(\mathcal{A}),\overline{\otimes})$$

is a pivotal tensor functor.

Proof. Straightforward computations; we refer the reader to [Tham, Prop 3.19], where we prove essentially the same result (the only thing new here is the pivotal structure). □

Observe that for a half-braiding γ on X, $({}^*\gamma)^\vee$ is also a half-braiding on X. This gives us an anti-tensor automorphism of $\mathcal{Z}(\mathcal{A})$:

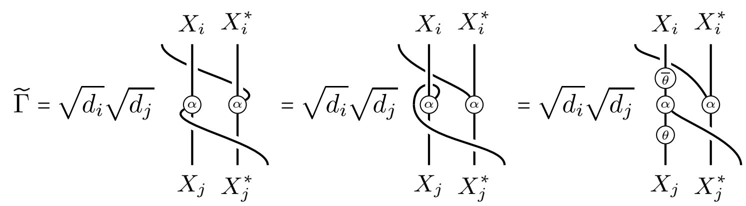
Proposition 2.12. *There is a tensor equivalence*

$$\begin{gathered}U:(\mathcal{Z}(\mathcal{A}),\overline{\otimes})\simeq_{\otimes}(\mathcal{Z}(\mathcal{A}),\overline{\otimes}^{\,op})\\(X,\gamma)\mapsto(X,({}^*\gamma)^\vee)\end{gathered}$$

Furthermore, $\overline{\theta}:U^2\simeq\mathrm{id}$, so U generates a $\mathbb{Z}/2$-action on $\mathcal{Z}(\mathcal{A})$ (but not tensor action).

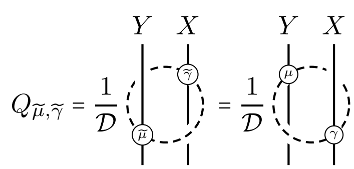
Proof. Denote $\widetilde{\gamma}:=({}^*\gamma)^\vee$.

First we check that $U(\overline{\mathbf{1}})\simeq\overline{\mathbf{1}}$. Recall $\overline{\mathbf{1}}=(\bigoplus X_iX_i^*,\Gamma)$. We have (sum over $i,j\in\mathrm{Irr}(\mathcal{A})$ is implicit)

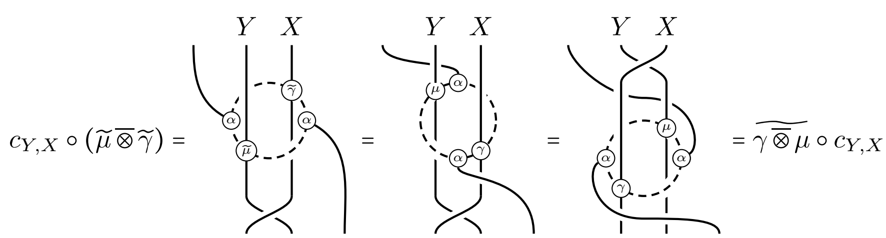


where recall θ is the twist operator, and $\overline{\theta}$ is its inverse. Then it is easy to see that $c^{-1}\circ(\overline{\theta}\otimes\mathrm{id}):U(\overline{\mathbf{1}})\simeq\overline{\mathbf{1}}$ is an isomorphism.

Next we construct the tensor structure on U. Let us compute $U((X,\gamma))\overline{\otimes}^{\mathrm{op}}U((Y,\mu))$. $Y{}_{\widetilde{\mu}}\overline{\otimes}_{\widetilde{\gamma}}X$ is the image of the projection $Q_{\widetilde{\mu},\widetilde{\gamma}}$, which equals



Observe that the braiding gives an isomorphism $c_{Y,X}:Y{}_{\widetilde{\mu}}\overline{\otimes}_{\widetilde{\gamma}}X\simeq X{}_{\gamma}\overline{\otimes}_{\mu}Y$, and in fact intertwines the half-braidings $\widetilde{\mu}\overline{\otimes}\widetilde{\gamma}$ and $\widetilde{\gamma\overline{\otimes}\mu}$ (see (2.8)):



It follows easily that $W_{(X,\gamma),(Y,\mu)}:=c_{Y,X}$ is a tensor structure on U.

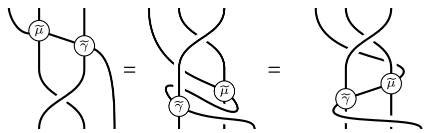
Finally, observe that $\widetilde{\widetilde{\gamma}}=(\theta_X\otimes\mathrm{id})\circ\gamma\circ(\mathrm{id}\otimes\overline{\theta}_X)$, as already hinted at in the computation of $\widetilde{\Gamma}$ above. So $\overline{\theta}_X:U^2((X,\gamma))=(X,\widetilde{\widetilde{\gamma}})\simeq(X,\gamma)$ is our desired natural isomorphism $U^2\simeq\mathrm{id}$. This is in fact a natural isomorphism of tensor functors, due to the identity $\overline{\theta}_{X\otimes Y}\circ(c_{Y,X}\circ c_{X,Y})=\overline{\theta}_X\otimes\overline{\theta}_Y$. □

We also note that the above proposition holds true for $\mathcal{Z}(\mathcal{A})$ with the standard tensor product. We will not use this fact in the rest of the paper, but we state it anyway for completeness:

Proposition 2.13. *There is a tensor equivalence*

$$U : (\mathcal{Z}(\mathcal{A}), \otimes) \simeq_{\otimes} (\mathcal{Z}(\mathcal{A}), \otimes^{op})$$
$$(X, \gamma) \mapsto (X, ({}^*\gamma)^{\vee})$$

Proof. Essentially the same as before; even the tensor structure and natural isomorphism are the same. We just show one computation, checking $c_{Y,X}$ intertwines $\widetilde{\mu} \otimes \widetilde{\gamma}$ and $\widetilde{\gamma \otimes \mu}$:

$c_{Y,X} \circ (\widetilde{\mu} \otimes \widetilde{\gamma}) =$  $= \widetilde{\gamma \otimes \mu} \circ c_{Y,X}$

□

Remark 2.14. In [Was2], Wasserman showed that the Drinfeld center of a symmetric $\mathcal{A}$ is a "bilax 2-fold tensor category", which loosely means a category with two monoidal structures that almost commute. More precisely, it consists of a pair of natural transformations

$$\eta : ((X,\gamma) \otimes (X',\gamma'))\,\overline{\otimes}\,((Y,\mu) \otimes (Y',\mu')) \rightleftharpoons ((X,\gamma)\,\overline{\otimes}\,(Y,\mu)) \otimes ((X',\gamma')\,\overline{\otimes}\,(Y',\mu')) : \zeta$$

such that $\eta \circ \zeta = \mathrm{id}$, together with several morphisms relating the units $\mathbf{1}$ and $\overline{\mathbf{1}}$, and they satisfy a cocktail of compatibility axioms. We claim that $\mathcal{Z}(\mathcal{A})$ also has such a structure when $\mathcal{A}$ is not symmetric. The only difference to the structure maps is where we have to distinguish the braiding c from its inverse: $\eta = \mathrm{id}_X \otimes c^{-1}_{X',Y} \otimes \mathrm{id}_{Y'}$ and $\zeta = \mathrm{id}_X \otimes c_{Y,X'} \otimes \mathrm{id}_{Y'}$ (with the various projections implicit). The proof that they satisfy the various compatibility axioms is a lengthy calculation that we do not share here. There is a more topological approach, see Remark 4.9. We do not prove this claim in this paper, as it will take us too far afield. △

3. Horizontal Trace of $\mathcal{A}$

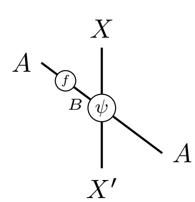
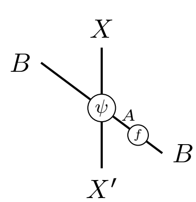
In this section, we will consider the *horizontal trace* of $\mathcal{A}$, as defined in [BHLZ, Section 2.4], which is a generalization of Ocneanu's tube algebra [O]. We follow the exposition in [KT], where we also considered a minor generalization to bimodule categories $\mathcal{M}$; here we only consider $\mathcal{M} = \mathcal{A}$.

Definition 3.1. Consider $\mathcal{A}$ as a bimodule category over itself by left and right multiplication. Its *horizontal trace*, denoted $\mathrm{hTr}(\mathcal{A})$ or simply $\hat{\mathcal{A}}$, is the category with the following objects and morphisms:

Objects: same as in $\mathcal{A}$

Morphisms: $\mathrm{Hom}_{\hat{\mathcal{A}}}(X,X') := \bigoplus_A \mathrm{Hom}^A_{\mathcal{A}}(X,X')/\sim$, where $\mathrm{Hom}^A_{\mathcal{A}}(X,X') := \mathrm{Hom}_{\mathcal{A}}(A \otimes X, X' \otimes A)$, the sum is over all objects $A \in \mathcal{A}$, and $\sim$ is the equivalence relation generated by the following:

For any $\psi \in \mathrm{Hom}^{B,A}_{\mathcal{A}}(X,X') := \mathrm{Hom}_{\mathcal{A}}(B \otimes X, X' \otimes A)$ and $f \in \mathrm{Hom}_{\mathcal{C}}(A,B)$, we have

(3.1) $\mathrm{Hom}^A_{\mathcal{A}}(X,X') \ni$  $\sim$  $\in \mathrm{Hom}^B_{\mathcal{A}}(X,X')$

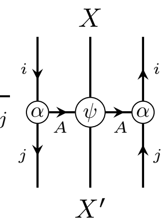
In other words, $\mathrm{Hom}_{\hat{\mathcal{A}}}(X,X') = \int^A \mathrm{Hom}^A_{\mathcal{A}}(X,X')$. △

Definition 3.2. Let $\mathrm{hTr} : \mathcal{A} \to \hat{\mathcal{A}}$ be the natural inclusion functor that is identity on objects, and on morphisms is the natural map $\mathrm{Hom}_{\mathcal{A}}(X,X') = \mathrm{Hom}^{\mathbf{1}}_{\mathcal{A}}(X,X') \to \mathrm{Hom}_{\hat{\mathcal{A}}}(X,X')$. △

The adjective "inclusion" further justified by the following proposition (along with the fact that I is faithful), which implies hTr is faithful also.

Proposition 3.3. [KT, Theorem 3.9] *Let $G : \hat{\mathcal{A}} \to \mathcal{Z}(\mathcal{A})$ be defined as follows: on objects, $G(X) = I(X)$, and on morphisms, for $\psi \in \mathrm{Hom}^A_{\mathcal{M}}(X, X')$,*

$$G(\psi) = \sum_{i,j\ \mathrm{Irr}(\mathcal{A})} \sqrt{d_i}\sqrt{d_j} \tag{3.2}$$



Then the extension to the Karoubi envelope is an equivalence of abelian categories:

$$\mathrm{Kar}(G) : \mathrm{Kar}(\hat{\mathcal{A}}) \simeq \mathcal{Z}(\mathcal{A})$$

Under this equivalence, the natural functor $\mathrm{hTr} : \mathcal{A} \to \hat{\mathcal{A}}$ is identified with $I : \mathcal{A} \to \mathcal{Z}(\mathcal{A})$, i.e. we have the commutative diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\mathrm{hTr}"] \arrow[d, "I"'] & \hat{\mathcal{A}} \arrow[dl, "G"'] \arrow[d, "\mathrm{Kar}"] \\ \mathcal{Z}(\mathcal{A}) & \mathrm{Kar}(\hat{\mathcal{A}}) \arrow[l, "\simeq", "\mathrm{Kar}(G)"'] \end{tikzcd} \tag{3.3}$$

We note that in [KT], the proposition would establish an equivalence $\mathrm{Kar}(\mathrm{hTr}(\mathcal{M})) \simeq \mathcal{Z}(\mathcal{M})$, where $\mathrm{hTr}(\mathcal{M})$ is the horizontal trace of an $\mathcal{A}$-bimodule category, and $\mathcal{Z}(\mathcal{M})$ is the center of $\mathcal{M}$ [GNN, Definition 2.1] which is analogous to the Drinfeld center.

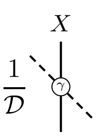
It is useful to construct an inverse to $\mathrm{Kar}(G)$:

Proposition 3.4. *An inverse to $\mathrm{Kar}(G)$ is given by:*

$$\mathrm{Kar}(G)^{-1} : \mathcal{Z}(\mathcal{A}) \simeq \mathrm{Kar}(\hat{\mathcal{A}})$$
$$(X, \gamma) \mapsto (X, \hat{P}_\gamma)$$

where

$$\hat{P}_\gamma := \sum_{i \in \mathrm{Irr}(\mathcal{A})} \frac{d_i}{\mathcal{D}} \gamma_{X_i} = \frac{1}{\mathcal{D}}$$



and on morphisms, for $f \in \mathrm{Hom}_{\mathcal{Z}(\mathcal{A})}((X, \gamma), (Y, \mu))$,

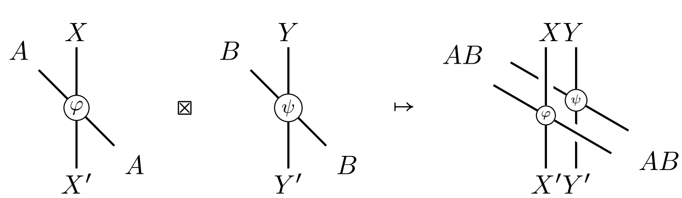
$$f \mapsto \hat{P}_\mu \circ f = f \circ \hat{P}_\gamma$$

Proof. Straightforward. □

When $\mathcal{A}$ is premodular, in particular when $\mathcal{A}$ is *braided*, $\hat{\mathcal{A}}$ has a natural monoidal structure. This was discussed in [KT, Example 8.2]; here we spell it out more explicitly.

Proposition 3.5. *There is a tensor product on $\hat{\mathcal{A}}$, denoted $\hat{\otimes}$: on objects, it is simply the same as $\mathcal{A}$, and on morphisms,*

$$\hat{\otimes} : \ \mathrm{Hom}^A_{\mathcal{A}}(X, X') \ \boxtimes \ \mathrm{Hom}^B_{\mathcal{A}}(Y, Y') \ \longrightarrow \ \mathrm{Hom}^{AB}_{\mathcal{A}}(XY, X'Y')$$



The unit object is the same as that of $\mathcal{A}$.

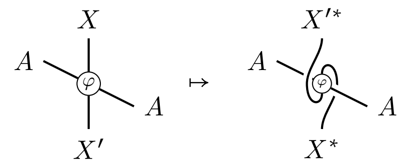
Furthermore, this tensor product structure is compatible with the rigid and pivotal structures of $\mathcal{A}$.

In other words, this is an extension the tensor product on $\mathcal{A}$, so that the inclusion functor $\mathrm{hTr} : \mathcal{A} \to \hat{\mathcal{A}}$ *is a pivotal tensor functor.*

Proof. Straightforward. □

It is useful to work out the left dual of a morphism (right dual being similar):

$$\mathrm{Hom}^A_{\mathcal{A}}(X, X') \;\to\; \mathrm{Hom}^A_{\mathcal{A}}(X'^*, X^*)$$

(3.4) 

In particular, when $X' = X$, and $\varphi = \gamma$ is a half-braiding on X, the right side is nothing but $\gamma^\vee$.

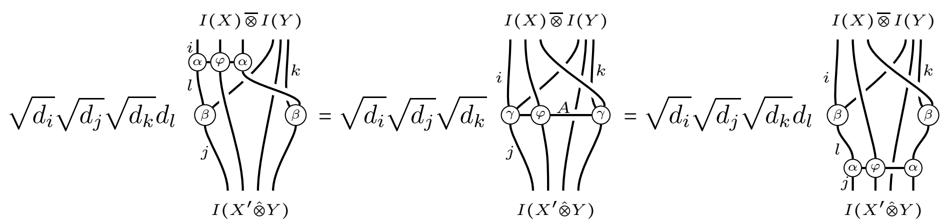
The following proposition is an upgrade of Proposition 3.3:

Proposition 3.6. *When $\mathcal{Z}(\mathcal{A})$ is endowed with the reduced tensor product, the equivalence in Proposition 3.3 is an equivalence of pivotal multifusion categories. More precisely, let J be the tensor structure on I from Proposition 2.11. Then*

$$(G, J) : (\hat{\mathcal{A}}, \hat{\otimes}) \to (\mathcal{Z}(\mathcal{A}), \overline{\otimes})$$

is a pivotal tensor functor, and hence its completion to $\mathrm{Kar}(\hat{\mathcal{A}})$ *is a pivotal tensor equivalence.*

Proof. As Proposition 2.11 takes care of objects, it remains to check naturality of J with respect to morphisms of $\hat{\mathcal{A}}$. This is easy to do: for example, for $\varphi \in \mathrm{Hom}^A_{\mathcal{A}}(X, X')$, we see that $J_{X',Y} \circ (I(\varphi) \overline{\otimes} I(\mathrm{id}_Y)) = I(\varphi \otimes \mathrm{id}_Y) \circ J_{X,Y}$ by



using (1.8) and (1.6). □

Proposition 3.7. *The inverse functor* $\mathrm{Kar}(G)^{-1}$ *defined in Proposition 3.4 is naturally a tensor functor.*

Proof. For the tensor structure on $\mathrm{Kar}(G)^{-1}$, there is a natural isomorphism $(X, \hat{P}_\gamma) \otimes (Y, \hat{P}_\mu) = (XY, \hat{P}_\gamma \otimes \hat{P}_\mu) \simeq (X\,_\gamma\overline{\otimes}_\mu Y, \hat{P}_{\gamma\overline{\otimes}\mu})$, which is given by the natural projection $Q_{\gamma,\mu} : XY \to X\,_\gamma\overline{\otimes}_\mu Y$. This follows from two simple observations:

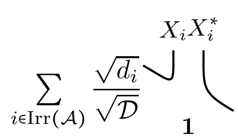
$$\hat{P}_\gamma \otimes \hat{P}_\mu = \hat{P}_{\gamma\overline{\otimes}\mu} \in \mathrm{End}_{\hat{\mathcal{A}}}(XY)$$

(using (1.6)), which implies $(XY, \hat{P}_\gamma \otimes \hat{P}_\mu) = (XY, \hat{P}_{\gamma\overline{\otimes}\mu})$, and

$$\hat{P}_{\gamma\overline{\otimes}\mu} = \hat{P}_\gamma \circ Q_{\gamma,\mu} = \hat{P}_\gamma \circ Q_{\gamma,\mu} \circ Q_{\gamma,\mu} = \hat{P}_{\gamma\overline{\otimes}\mu} \circ Q_{\gamma,\mu} \in \mathrm{End}_{\hat{\mathcal{A}}}(XY)$$

(see (2.8)) which implies $Q_{\gamma,\mu} : (XY, \hat{P}_{\gamma\overline{\otimes}\mu}) \to (X\,_\gamma\overline{\otimes}_{,\mu} Y, \hat{P}_{\gamma\overline{\otimes}\mu})$ is an isomorphism. It is easy to check that this satisfies the hexagon axiom for tensor structure.

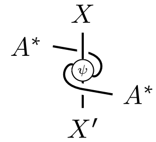
The isomorphism $\mathrm{Kar}(G)^{-1}(\overline{\mathbf{1}}) \simeq \mathbf{1}$ is given by

(3.5) 

□

Next we consider a $\mathbb{Z}/2$-action, related to Proposition 2.12. Let $\widetilde{\cdot} : \mathrm{Hom}^A_{\mathcal{A}}(X, X') \to \mathrm{Hom}^{A^*}_{\mathcal{A}}(X, X')$ be the map

$$\psi \mapsto \widetilde{\psi} :=$$



This is very similar to the definition of $\widetilde{\gamma}$ in proof of Proposition 2.12. Indeed, if γ is a half-braiding on A, then $\gamma_A \in \mathrm{Hom}^A_{\mathcal{A}}(X, X)$, and $\widetilde{\gamma_A} = (\widetilde{\gamma})_{A^*}$.

This defines an endomorphism $\widetilde{\cdot} : \mathrm{Hom}_{\hat{\mathcal{A}}}(X, X') \simeq \mathrm{Hom}_{\hat{\mathcal{A}}}(X, X')$, and in fact is part of an automorphism of $\hat{\mathcal{A}}$: (compare Proposition 2.12)

Proposition 3.8. *Let $\hat{U} : \hat{\mathcal{A}} \to \hat{\mathcal{A}}$ be the functor that is identity on objects, and acts by $\widetilde{\cdot}$ on morphisms. Then there is a tensor equivalence*

$$(\hat{U}, c) : (\hat{\mathcal{A}}, \hat{\otimes}) \simeq_{\otimes} (\hat{\mathcal{A}}, \hat{\otimes}^{op})$$

Furthermore, $\overline{\theta} : \hat{U}^2 \simeq \mathrm{id}$*, so $\hat{U}$ generates a $\mathbb{Z}/2$-action on $\hat{\mathcal{A}}$. (but not tensor action).*

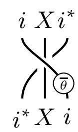
The tensor equivalence and $\mathbb{Z}/2$-action naturally extends to the Karoubi envelope.

Proof. Similar to Proposition 2.12. We note again $\overline{\theta} : \hat{U}^2 \simeq \mathrm{id}$ is a natural isomorphism of tensor functors. It is also useful to observe that $\mathrm{Kar}(\hat{U})((X, \hat{P}_{\gamma})) = (X, \hat{P}_{\widetilde{\gamma}})$. ☐

Proposition 3.9. $\mathrm{Kar}(G) : \mathrm{Kar}(\hat{\mathcal{A}}) \simeq \mathcal{Z}(\mathcal{A})$ *respects the $\mathbb{Z}/2$-actions. More precisely, there is a natural isomorphism $u : U \circ \mathrm{Kar}(G) \simeq \mathrm{Kar}(G) \circ \hat{U}$ that makes $\mathrm{Kar}(G)$ a $\mathbb{Z}/2$-equivariant functor.*

Proof. The natural isomorphism is given as follows: for $(X, p) \in \mathrm{Kar}(\hat{\mathcal{A}})$, $u_{(X,p)} : \mathrm{im}(G(p)) \to \mathrm{im}(G(\widetilde{p}))$ is given by

$$u_{(X,p)} =$$



where the i^* strand is at the bottom, with the projections $G(p), G(\widetilde{p})$ implicit. It is easy to check $u_{(X,p)} \circ G(p) = G(\widetilde{p}) \circ u_{(X,p)}$ - u "drags" the middle strand around, introducing a half-twist to turn p into $\widetilde{p}$ - so this is an isomorphism. It is also easy to check that it intertwines the half-braidings $\widetilde{\Gamma}$ and Γ (essentially the same computation for $U(\overline{\mathbf{1}}) \simeq \overline{\mathbf{1}}$ in the proof of Proposition 2.12).

Next we need to check the following diagram of natural isomorphisms commutes:

$$\begin{tikzcd} U^2 G \arrow[r, "Uu"] \arrow[d, "\overline{\theta} G"] & UG\hat{U} \arrow[r, "u\hat{U}"] & G\hat{U}^2 \arrow[d, "G(\overline{\theta})"] \\ G \arrow[rr, equal] & & G \end{tikzcd}$$

The left vertical arrow is a full inverse twist on three strands, and commutativity follows from the three strand version of the identity relating braiding and twist. ☐

4. Crane-Yetter theory on the Annulus

As mentioned in the introduction, we were motivated from the study of an extended Crane-Yetter theory Z_{CY}, in particular its value on surfaces. There the annulus played an important role, so this section is devoted to relating the previous sections to $Z_{\mathrm{CY}}(\mathrm{Ann})$.

We review the definition and some properties about $Z_{\mathrm{CY}}(\Sigma)$ from [KT], to which we refer the reader for more details. For most of this section, we will take $\Sigma = \mathrm{Ann} = S^1 \times (0, 1)$, the annulus.

A *boundary value* on a surface Σ is a configuration B of finitely many framed points, where each point $b \in B$ is labelled with an object $V_b \in \mathcal{A}$ (here framed point means a trivialization of its normal bundle, or simply a tangent vector). We denote such a boundary value by $(B, \{V_b\})$.

An $\mathcal{A}$*-colored ribbon graph* Γ in an oriented 3-manifold M is a ribbon graph in M where each oriented edge $\mathbf{e}$ is assigned an object $V_{\mathbf{e}} \in \mathcal{A}$ (with opposite orientations of the same edge assigned mutually dual objects), and each vertex is assigned some morphism in $\mathrm{Hom}(\mathbf{1}, V_{\mathbf{e}_1} \otimes \cdots \otimes V_{\mathbf{e}_k})$, $\mathbf{e}_i$'s taken with outgoing orientation, and edges ordered arbitrarily. (Some care should be taken when identifying Hom's for different choices of orderings, since $\mathcal{A}$ may not be symmetric. See [KT, Definition 5.1] for a more careful approach using the Reshetikhin-Turaev invariant [RT].) Such graphs are allowed to meet the boundary transversally, in which case it leaves a "trace", $\partial\Gamma = \Gamma \cap \partial M$. Note that the coloring and framing of Γ induce colorings and framings on $\partial\Gamma$, so we call $\partial\Gamma$ the boundary value of Γ.

Given a boundary value $(B, \{V_b\})$ on ∂M, one can consider the vector space generated by $\mathcal{A}$-colored ribbon graphs Γ in M with boundary value $\partial\Gamma = (B, \{V_b\})$, modulo isotopy and certain local relations. These local relations essentially arise from taking a small ball D in M, "evaluating" $\Gamma \cap D$ using the Reshetikhin-Turaev invariant [RT], and replacing $\Gamma \cap D$ with another graph with the same evaluation. The resulting quotient vector space is called the *skein module of M with boundary value* $(B, \{V_b\})$, denoted $\mathrm{Skein}(M; (B, \{V_b\}))$. We will mostly consider $M = \Sigma \times [0,1]$, where $\Sigma \times \{0\}$ is thought of as the incoming boundary, and $\Sigma \times \{1\}$ the outgoing boundary.

Definition 4.1. $\hat{Z}_{\mathrm{CY}}(\Sigma)$ is the category with the following objects and morphisms:
Objects: boundary values on Σ,
Morphisms: for boundary values $\mathbf{V} = (B, \{V_b\}), \mathbf{V}'$,

$$\mathrm{Hom}_{\hat{Z}_{\mathrm{CY}}(\Sigma)}(\mathbf{V}, \mathbf{V}') := \mathrm{Skein}(\Sigma \times [0,1]; \overline{\mathbf{V}}, \mathbf{V}')$$

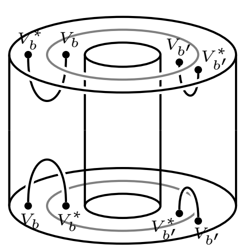
is the skein module of $\Sigma \times [0,1]$ with boundary value $\overline{\mathbf{V}} = (B, \{V_b^*\})$ on the incoming boundary $\Sigma \times \{0\}$, and boundary value $\mathbf{V}'$ on the outgoing boundary $\Sigma \times \{1\}$.
$Z_{\mathrm{CY}}(\Sigma)$ is defined to be the Karoubi envelope of $\hat{Z}_{\mathrm{CY}}(\Sigma)$. △

Ann, as the product of something with the interval, has a "stacking" structure, in that we can include two annuli $S^1 \times (0, 1/2)$ and $S^1 \times (1/2, 1)$ in Ann. In particular, two objects $\mathbf{V}_1, \mathbf{V}_2$ can be included side by side in a bigger annulus, and similarly morphisms $f : \mathbf{V}_1 \to \mathbf{V}_1'$, $g : \mathbf{V}_2 \to \mathbf{V}_2'$ can be placed side by side. The unit object with respect to this tensor product is the empty configuration $\overline{\mathbf{1}}_{\mathrm{st}} = (\varnothing, \{\})$. This gives us a "stacking" tensor product structure on $\hat{Z}_{\mathrm{CY}}(\mathrm{Ann})$ that is in fact rigid and pivotal:

Proposition 4.2 ([KT, Proposition 6.7]). *The stacking tensor product described above, denoted by $\overline{\otimes}_{st}$, makes $\hat{Z}_{CY}(Ann)$ a pivotal multifusion category:*

- *For an object $\mathbf{V} = (B, \{V_b\})$, its left dual is the object $\mathbf{V}^\vee := (\theta(B), \{V_b^\vee\})$, where θ is the operation on Ann that flips $(0,1)$, and has the following evaluation and coevaluation maps:*

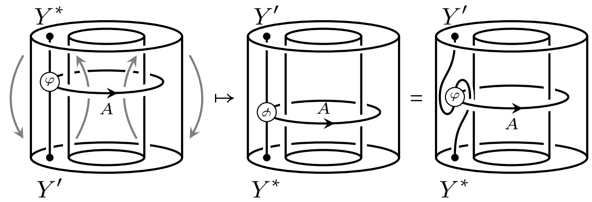
$$\begin{array}{c} \mathbf{V}^\vee \overline{\otimes}_{st} \mathbf{V} \\ \mathrm{ev}_{\mathbf{V}} \big\downarrow \\ \overline{\mathbf{1}}_{st} \\ \mathrm{coev}_{\mathbf{V}} \big\downarrow \\ \mathbf{V} \overline{\otimes}_{st} \mathbf{V}^\vee \end{array}$$



(The gray lines indicate $S^1 \times \{1/2\}$ which separates $\mathbf{V}^\vee$ from $\mathbf{V}$ in the tensor product, and play no role in defining these morphisms.) We think of the outside half of the annulus as the left side in the tensor product. The right dual is ${}^\vee\mathbf{V} := (\theta(B), \{{}^*V_b\})$, with essentially the same (co)evaluation maps.
- *The pivotal map $\delta_{\mathbf{V}} : \mathbf{V} \to \mathbf{V}^{\vee\vee}$ is simply the graph with vertical strands running down, and one node on each strand labelled with δ_{V_b}.*

$Z_{CY}(Ann)$, as the Karoubi envelope of $\hat{Z}_{CY}(Ann)$, naturally inherits these structures.

It is not hard to see that the left dual of a morphism is obtained by turning the solid annulus "inside-out"; for example,

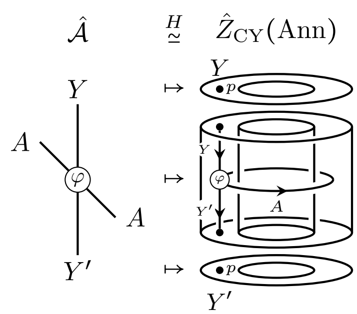


The gray arrows indicate the "inside-out" operation. [1] Note the upside-down 'φ' in the second diagram; the last diagram can be turned into the second by pulling on the upward and downward strands, forcing the φ node to turn upside-down.

The last diagram is reminiscent of duals in $\hat{\mathcal{A}}$ (see (3.4)). In particular, when $Y' = Y$, and φ is a half-braiding on Y, the extra bending in the last diagram can be incorporated into the node to become $\varphi^\vee$ (see Proposition 2.9).

In [KT], in Example 8.2, we provided an explicit equivalence $H = H_p : \hat{\mathcal{A}} \simeq \hat{Z}_{\mathrm{CY}}(\mathrm{Ann})$, where $p \in \mathrm{Ann}$, given as follows: [2]

(4.1)



It is not hard to see that this equivalence is in fact tensor and pivotal. For example, for $X, Y \in \mathrm{Obj}\,\hat{\mathcal{A}}$, the tensor product in $\hat{Z}_{\mathrm{CY}}(\mathrm{Ann})$, $H(X)\,\overline{\otimes}_{\mathrm{st}}\,H(Y)$, is an object with two marked points, labelled with X and Y. The tensor structure on H would be a trivalent graph connecting $H(X)\,\overline{\otimes}_{\mathrm{st}}\,H(Y)$ to $H(XY)$, with the unique vertex labelled by id_{XY} (which is naturally identified with $\mathrm{coev} \in \mathrm{Hom}_{\mathcal{A}}(\mathbf{1}, XY(XY)^*)$). The unit object $\overline{\mathbf{1}}_{\mathrm{st}}$ in $\hat{Z}_{\mathrm{CY}}(\mathrm{Ann})$, i.e. the empty configuration, is isomorphic to the object $H(\mathbf{1}) = (\{p\}, \{\mathbf{1}\})$. Hence we have the following:

Theorem 4.3. *We have the following commutative diagram, where all functors are pivotal tensor functors:*

$$\begin{tikzcd}
(\mathcal{A}, \otimes) \arrow[r, "\mathrm{hTr}"] \arrow[d, "I"'] & (\hat{\mathcal{A}}, \hat{\otimes}) \arrow[r, "H", "\simeq"'] \arrow[d, "\mathrm{Kar}"] \arrow[dl, "G"'] & (\hat{Z}_{CY}(Ann), \overline{\otimes}_{st}) \arrow[d, "\mathrm{Kar}"] \\
(\mathcal{Z}(\mathcal{A}), \overline{\otimes}) & (\mathrm{Kar}(\hat{\mathcal{A}}), \hat{\otimes}) \arrow[l, "\simeq", "\mathrm{Kar}(G)"'] \arrow[r, "\simeq", "\mathrm{Kar}(H)"'] & (Z_{CY}(Ann), \overline{\otimes}_{st})
\end{tikzcd}$$

In other words, the reduced tensor product $\overline{\otimes}$ on $\mathcal{Z}(\mathcal{A})$ encodes the stacking tensor product on $Z_{CY}(Ann)$.

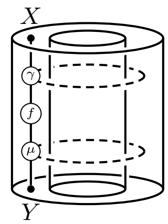
Together with Proposition 3.4, the pivotal tensor equivalence $(\mathcal{Z}(\mathcal{A}), \overline{\otimes}) \simeq Z_{\mathrm{CY}}(\mathrm{Ann})$ is given by

[1] Imagine pulling your hand out of the sleeve of a tight sweater - the second diagram is inside-out the same way the sleeve is.

[2] H_p is dependent on the choice of a point $p \in \mathrm{Ann}$, but all H_p are naturally isomorphic (by non-unique natural isomorphism). One can consider the full subcategory with objects of the form $(\{p\}, \{X\})$. This is strictly speaking not a tensor subcategory, since the tensor product of such objects would have two marked points. However, one can put a different tensor product, $(\{p\}, \{X\})\,\overline{\otimes}_{\mathrm{st}}{}'(\{p\}, \{Y\}) = (\{p\}, \{XY\})$, and the inclusion can be made a pivotal tensor equivalence.

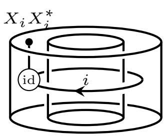
(4.2) $$\mathrm{Kar}(H)\circ\mathrm{Kar}(G)^{-1} : (\mathcal{Z}(\mathcal{A}), \overline{\otimes}\,) \simeq Z_{\mathrm{CY}}(\mathrm{Ann})$$

(4.3)
$$\begin{array}{ccl} (X,\gamma) & \mapsto & \mathrm{im}\Big\{ \\ \big\downarrow f & \mapsto & \\ (Y,\mu) & \mapsto & \mathrm{im}\Big\{ \end{array}$$



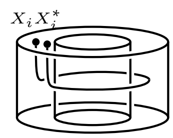
Once again this functor doesn't send unit to unit; the isomorphism is given by (compare (3.5))

$$\sum_{i\in\mathrm{Irr}(\mathcal{A})} \frac{\sqrt{d_i}}{\sqrt{\mathcal{D}}}$$



, or more intuitively,

$$\sum_{i\in\mathrm{Irr}(\mathcal{A})} \frac{\sqrt{d_i}}{\sqrt{\mathcal{D}}}$$



As an application of Theorem 4.3, we describe $Z_{\mathrm{CY}}(\mathbf{T}^2)$ purely algebraically in terms of $\mathcal{A}$. We can produce $\mathbf{T}^2$ from Ann by gluing (neighbourhoods of) $S^1\times\{0\}$ and $S^1\times\{1\}$. The excision property of Z_{CY} as stated in the introduction doesn't work as is, but [KT, Theorem 7.5] actually proves an apparently slightly more general but ultimately equivalent form of excision, which allows Σ to be obtained by gluing two boundaries of a single surface Σ'; the balanced tensor product is then replaced by the center of $Z_{\mathrm{CY}}(\Sigma')$ as a $Z_{\mathrm{CY}}(\mathrm{Ann})$-bimodule (as defined in [GNN, Definition 2.1], repeated in [KT, Definition 3.1]; for applications here, it suffices to know that this notion of center for a monoidal category as a bimodule over itself coincides with the Drinfeld center.)

We can view $Z_{\mathrm{CY}}(\mathrm{Ann})$ as a bimodule category over itself, thinking of the left and right actions as "insertions" from the left ($S^1\times\{0\}$) and right ($S^1\times\{0\}$). Thus we have the following corollary of [KT, Theorem 7.5]:

Proposition 4.4.
$$Z_{CY}(\mathbf{T}^2) \simeq \mathcal{Z}((Z_{CY}(Ann), \overline{\otimes}_{st}\,))$$

Proof. Take $X' = S^1\times(0,3)$ in [KT, Theorem 7.5]. ☐

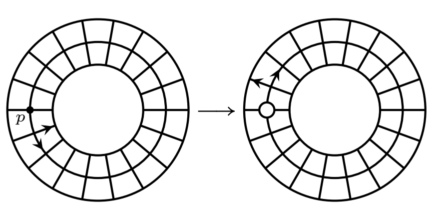
As an immediate corollary of this and Theorem 4.3, we have

Corollary 4.5.
$$Z_{CY}(\mathbf{T}^2) \simeq \mathcal{Z}((\mathcal{Z}(\mathcal{A}), \overline{\otimes}\,))$$
as abelian categories.

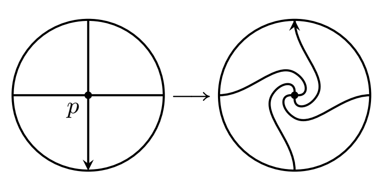
We will study $Z_{\mathrm{CY}}(\mathbf{T}^2)$ for $\mathcal{A} = \mathrm{Rep}(G)$ in Section 6.1.

Now we can also give a topological interpretation of the $\mathbb{Z}/2$-actions discussed in previous sections (see Proposition 2.12,Proposition 3.8). An orientation-preserving diffeomorphism from the annulus to itself naturally gives an automorphism of $Z_{\mathrm{CY}}(\mathrm{Ann})$. In particular, consider the following diffeomorphism U_{CY}:

U_{CY} :



, where near p,



where we recall that p was some fixed based point used to define H. U_{CY} acts on objects of $Z_{\mathrm{CY}}(\mathrm{Ann})$ by applying U_{CY} to the marked points, and on morphisms, it acts by applying $U_{\mathrm{CY}}\times\mathrm{id}_{[0,1]}$ to graphs.

Proposition 4.6. *U_{CY} generates a $\mathbb{Z}/2$-action on $\hat{Z}_{CY}(Ann)$ and $Z_{CY}(Ann)$.*

Proof. It suffices to consider $\hat{Z}_{\mathrm{CY}}(\mathrm{Ann})$. U^2_{CY} is isotopic to the identity by "untwisting" around p. This gives us a natural isomorphism $\overline{\Theta} : U^2_{\mathrm{CY}} \simeq \mathrm{id}_{\hat{Z}_{\mathrm{CY}}(\mathrm{Ann})}$ as follows: for a marked point b, the isotopy takes $U^2_{\mathrm{CY}}(b)$ back to b; thinking of the interval direction of the cylinder $\mathrm{Ann} \times [0,1]$ as time, this traces out a curve interpolating $(U^2_{\mathrm{CY}}(b)), 0)$ and $(b, 1)$.

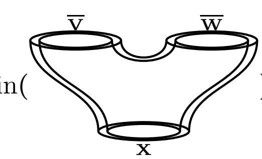
It is a simple topology exercise to check that $\overline{\Theta}$ is a natural isomorphism; it is helpful to think of $\overline{\Theta}$ as relating $\mathrm{Ann} \times [0,1]$ to the mapping cylinder of $(U^2_{\mathrm{CY}})^{-1}$. □

In particular, when the object is $H(X)$, that is, p labelled with X, the natural isomorphism is the vertical graph with a full negative twist, i.e. $\overline{\Theta}_{H(X)}$ is simply $\overline{\theta}_X$, hence the suggestive name for $\overline{\Theta}$.

Proposition 4.7. *$H : \hat{\mathcal{A}} \simeq \hat{Z}_{CY}(Ann)$ and $\mathrm{Kar}(H) : \mathrm{Kar}(\hat{\mathcal{A}}) \simeq Z_{CY}(Ann)$ are $\mathbb{Z}/2$-equivariant.*

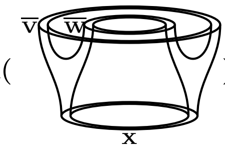
Proof. It is easy to check that H commutes with U's on the nose, i.e. $U_{\mathrm{CY}} \circ H = H \circ \hat{U}$, and likewise for Karoubi envelopes. (Indeed, U_{CY} was cooked up to have this property.) The observation above shows $\overline{\Theta} H = H \overline{\theta}$ as natural isomorphisms $U^2_{\mathrm{CY}} \circ H = H \circ \hat{U}^2 \simeq H$, and likewise for Karoubi envelopes. Thus, H is $\mathbb{Z}/2$-equivariant. □

Remark 4.8. Since $\overline{\otimes}$ on $\mathcal{Z}(\mathcal{A})$ has a nice topological interpretation as $\overline{\otimes}_{\mathrm{st}}$ on $Z_{\mathrm{CY}}(\mathrm{Ann})$, it is also nice to have a topological interpretation for the standard tensor product on $\mathcal{Z}(\mathcal{A})$. This comes from the (thickened) pair of pants, denoted M_{POP}, in the following manner. For $(X,\gamma), (Y,\mu) \in \mathcal{Z}(\mathcal{A})$, with corresponding objects $\mathbf{V} = (H(X), H(\hat{P}_\gamma)), \mathbf{W} = (H(Y), H(\hat{P}_\mu)) \in Z_{\mathrm{CY}}(\mathrm{Ann})$, the object $\mathbf{V} \otimes \mathbf{W}$ which corresponds to $(X,\gamma) \otimes (Y,\mu)$ is characterized by a natural isomorphism

$$\mathrm{Hom}_{Z_{\mathrm{CY}}(\mathrm{Ann})}(\mathbf{V} \otimes \mathbf{W}, \mathbf{X}) \simeq \mathrm{Skein}(M_{\mathrm{POP}}; \overline{\mathbf{V}}, \overline{\mathbf{W}}, \mathbf{X}) = \mathrm{Skein}($$


$)$

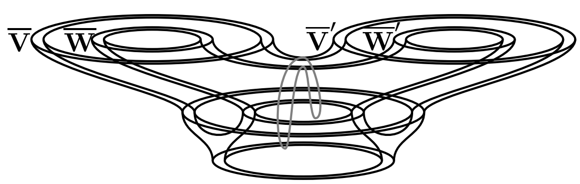
for all $\mathbf{X} \in Z_{\mathrm{CY}}(\mathrm{Ann})$. (This is well-known for the extended Turaev-Viro theory [TV], where given spherical fusion $\mathcal{A}$, one has $Z_{\mathrm{TV}}(S^1) = \mathcal{Z}(\mathcal{A})$ [Kir], and the standard tensor product on $\mathcal{Z}(\mathcal{A})$ is given by the pair of pants just as above.) The stacking product can also be described this way, but instead of the usual pair of pants M_{POP}, we use a different cobordism, M_{Y}:

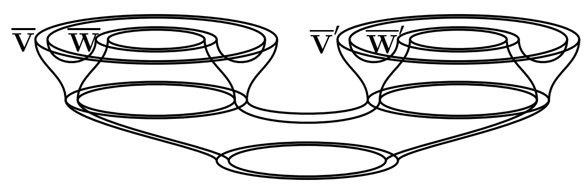
$$\mathrm{Hom}_{Z_{\mathrm{CY}}(\mathrm{Ann})}(\mathbf{V} \overline{\otimes}_{\mathrm{st}} \mathbf{W}, \mathbf{X}) \simeq \mathrm{Skein}(M_{\mathrm{Y}}; \overline{\mathbf{V}}, \overline{\mathbf{W}}, \mathbf{X}) = \mathrm{Skein}($$


$)$

M_{Y} is a thickened 'Y' crossed with S^1. We do not prove these claims here, which are not hard to prove after all the work in this section. △

Remark 4.9. The topological interpretations of $\otimes$ and $\overline{\otimes}_{\mathrm{st}}$ above can also elucidate the structure morphisms mentioned in Remark 2.14. Consider the two cobordisms below:





The cobordism on the left (ignoring the gray curve) corresponds to $(\mathbf{V} \otimes \mathbf{V}') \overline{\otimes}_{\mathrm{st}} (\mathbf{W} \otimes \mathbf{W}')$, or $((X,\gamma) \otimes (X',\gamma')) \overline{\otimes} ((Y,\mu) \otimes (Y',\mu'))$, while the cobordism on the right corresponds to $(\mathbf{V} \overline{\otimes}_{\mathrm{st}} \mathbf{W}) \otimes (\mathbf{V}' \overline{\otimes}_{\mathrm{st}} \mathbf{W}')$, or $((X,\gamma) \overline{\otimes} (Y,\mu)) \otimes ((X',\gamma') \overline{\otimes} (Y',\mu'))$. They are different, but not by much: the right can be obtained from the left by surgery, specifically, by attaching a 2-handle along the gray curve (the gray curve starts in the outside M_{POP} for $\mathbf{V}$'s, goes down into the bottom M_{Y}, back up into the M_{POP} for $\mathbf{W}$'s, then goes down again into M_{Y}, and closes up, all the while staying close to the "inner boundary", i.e. keeping as tight as possible like a rubberband). One way to visualize this is to consider the reverse process of removing a

2-handle: start with the right side, push the two "troughs" - between $\mathbf{V}$ and $\mathbf{W}$ and between $\mathbf{V'}$ and $\mathbf{W'}$ - downward and into the bottom pair of pants, and when they are about to meet in the middle, drill a hole through the wall.

On the level of skein modules, this surgery is the same as adding the gray curve colored by the regular coloring (up to a factor, see (1.7)); this is the topological interpretation of η. Conversely, graphs in the right cobordism can be lifted to a graph in the left cobordism plus the gray curve with regular coloring; this is the topological interpretation of ζ.

The other structure morphisms can also be described very easily. For example, one of them is a morphism $v_2 : \overline{\mathbf{1}} \otimes \overline{\mathbf{1}} \to \overline{\mathbf{1}}$, which is simply the empty graph in M_{POP} (up to a factor) interpreted as a morphism $\overline{\mathbf{1}}_{\mathrm{st}} \overline{\otimes}_{\mathrm{st}} \overline{\mathbf{1}}_{\mathrm{st}} \to \overline{\mathbf{1}}_{\mathrm{st}}$.

Thus, checking the compatibility axioms becomes an exercise in topology. Once again, we do not show the work in this paper.

As mentioned in the introduction, such 2-fold monoidal structures are related to iterated loop spaces. It may be interesting to see if these two topological aspects of $(\mathcal{Z}(\mathcal{A}), \otimes, \overline{\otimes})$ are directly related. △

5. $\mathcal{A}$ Modular

When $\mathcal{A}$ is modular, we have (see for example [EGNO, Proposition 8.20.12]):

$$\begin{aligned} \mathcal{A} \boxtimes \mathcal{A}^{\mathrm{bop}} &\simeq_{\otimes,\mathrm{br}} (\mathcal{Z}(\mathcal{A}), \otimes) \\ X \boxtimes Y &\mapsto (XY, cc^{-1}) = (X, c) \otimes (Y, c^{-1}) \end{aligned}$$

where $\mathcal{A}^{\mathrm{bop}}$ is $\mathcal{A}$ with the opposite braiding, and c is the braiding on $\mathcal{A}$. Here the monoidal structure on $\mathcal{A} \boxtimes \mathcal{A}^{\mathrm{bop}}$ is defined component-wise. In particular, duals are given by $(X \boxtimes Y)^* = X^* \boxtimes Y^*$.

It is natural to ask: what is the reduced tensor product on $\mathcal{A} \boxtimes \mathcal{A}^{\mathrm{bop}}$ under this equivalence? We claim (proven below in Theorem 5.3) that the following definition is the answer, which justifies the repeated use of the name "reduced" and notation like $\overline{\otimes}$ and $\overline{\mathbf{1}}$. (The reduced tensor product on $\mathcal{Z}(\mathcal{A})$ cannot in general be braided, as we shall see soon, so we will ignore the difference in braiding on $\mathcal{A}$.)

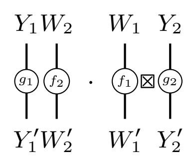
Definition 5.1. Let $W_1 \boxtimes Y_1, W_2 \boxtimes Y_2 \in \mathcal{A} \boxtimes \mathcal{A}$. Define their *reduced tensor product* to be

$$(W_1 \boxtimes Y_1) \overline{\otimes} (W_2 \boxtimes Y_2) := \langle Y_1, W_2 \rangle \cdot W_1 \boxtimes Y_2$$

where recall $\langle Y_1, W_2 \rangle := \mathrm{Hom}_{\mathcal{A}}(\mathbf{1}, Y_1 W_2)$. $\overline{\otimes}$ naturally extends to direct sums, and is clearly associative.

For morphisms $f_1 \boxtimes g_1 : W_1 \boxtimes Y_1 \to W_1' \boxtimes Y_1'$, $f_2 \boxtimes g_2 : W_2 \boxtimes Y_2 \to W_2' \boxtimes Y_2'$, their *reduced tensor product* is given by

$$(f_1 \boxtimes g_1) \overline{\otimes} (f_2 \boxtimes g_2) := \langle g_1, f_2 \rangle \cdot f_1 \boxtimes g_2 =$$



where the left side, the "coefficient" $\langle g_1, f_2 \rangle$, is to be interpreted as a linear map $\langle Y_1, W_2 \rangle \to \langle Y_1', W_2' \rangle$ by composition. △

For example,

$$(X_i \boxtimes X_j^*) \overline{\otimes} (X_k \boxtimes X_l^*) \simeq \delta_{j,k} X_i \boxtimes X_l^* \tag{5.1}$$

In particular, when $i \neq j$,

$$\begin{aligned} (X_i \boxtimes X_i^*) \overline{\otimes} (X_i \boxtimes X_j^*) &\simeq X_i \boxtimes X_j^* \\ (X_i \boxtimes X_j^*) \overline{\otimes} (X_i \boxtimes X_i^*) &\simeq 0 \end{aligned} \tag{5.2}$$

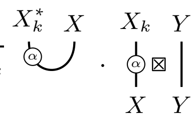
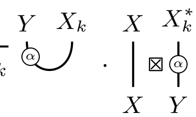
so $\overline{\otimes}$ cannot be braided.

Proposition 5.2. $(\mathcal{A} \boxtimes \mathcal{A}, \overline{\otimes})$ *is a pivotal multifusion category. More precisely,*

- *The unit object, denoted $\overline{\mathbf{1}}$, is $\bigoplus_{i \in \mathrm{Irr}(\mathcal{A})} X_i \boxtimes X_i^*$;*
- $(X \boxtimes Y)^\vee = Y^* \boxtimes X^*$, ${}^\vee(X \boxtimes Y) = {}^\vee Y \boxtimes {}^* X$, *the (co)evaluation maps are described in the proof;*

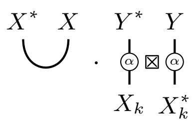
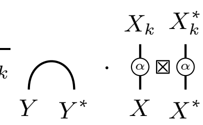
- *The pivotal structure is defined component-wise: $\delta_{X\boxtimes Y} = \delta_X \boxtimes \delta_Y$.*

Proof. Fix $X \boxtimes Y \in \mathcal{A} \boxtimes \mathcal{A}$. The left and right unit constraint are given by

$$l_{X\boxtimes Y} := \sum_{k\in\mathrm{Irr}(\mathcal{A})} \sqrt{d_k}$$  ; $$r_{X\boxtimes Y} := \sum_{k\in\mathrm{Irr}(\mathcal{A})} \sqrt{d_k}$$ 

where we recall that α is a sum over a pair of dual bases (see (1.4)). Their inverses are given by flipping the diagram upside down.

The left (co)evaluation maps are given by

$$\mathrm{ev}_{X\boxtimes Y} := \sum_{k\in\mathrm{Irr}(\mathcal{A})} \sqrt{d_k}$$  ; $$\mathrm{coev}_{X\boxtimes Y} := \sum_{k\in\mathrm{Irr}(\mathcal{A})} \sqrt{d_k}$$ 

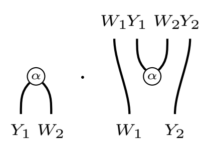
The right (co)evaluation maps are given by similar diagrams. It is straightforward to check that these have the right properties. $\square$

Theorem 5.3. *There is an equivalence of pivotal multifusion categories*

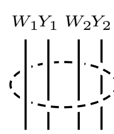
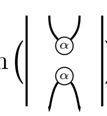
$$K : \mathcal{A} \boxtimes \mathcal{A} \simeq_{\otimes} (\mathcal{Z}(\mathcal{A}), \overline{\otimes})$$
$$X \boxtimes Y \mapsto (XY, cc^{-1})$$

Proof. The tensor structure L on K is given as follows: for $W_1 \boxtimes Y_1, W_2 \boxtimes Y_2$, the isomorphism $L : K(W_1 \boxtimes Y_1) \overline{\otimes} K(W_2 \boxtimes Y_2) \simeq K((W_1 \boxtimes Y_1) \overline{\otimes} (W_2 \boxtimes Y_2))$ is given by

$$L : (W_1Y_1, cc^{-1}) \overline{\otimes} (W_2Y_2, cc^{-1}) \simeq \langle Y_1, W_2 \rangle \cdot (W_1Y_2, cc^{-1})$$

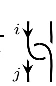


The inverse to L is given by flipping the diagram upside down. The following observation is helpful: for $(W_1Y_1, cc^{-1}), (W_2Y_2, cc^{-1}) \in \mathcal{Z}(\mathcal{A})$, we have

$$W_1Y_1 {}_{cc^{-1}} \overline{\otimes}_{cc^{-1}} W_2Y_2 = \mathrm{im}\Big(\frac{1}{\mathcal{D}}$$  $$\Big) = \mathrm{im}\Big($$  $$\Big)$$

It is easy to check that L satisfies the hexagon axiom.

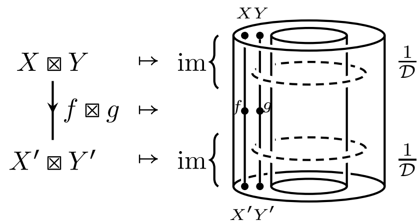
Note that K does not send unit to unit - the half-braiding on $K(\overline{\mathbf{1}}) = (\bigoplus X_i X_i^*, cc^{-1})$ is not the same as $\overline{\mathbf{1}} = (\bigoplus X_i X_i^*, \Gamma)$; the isomorphism $K(\overline{\mathbf{1}}) \simeq \overline{\mathbf{1}}$ is essentially given by the S-matrix:

$$S = \sum_{i,j\in\mathrm{Irr}(\mathcal{A})} \sqrt{d_i}\sqrt{d_j}$$ 

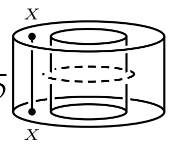
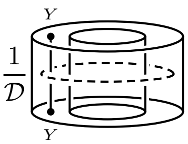
Clearly the pivotal structures agree. $\square$

The equivalence given above has a nice interpretation in $Z_{\mathrm{CY}}(\mathrm{Ann})$. Namely, the composition $\mathrm{Kar}(H)\circ \mathrm{Kar}(G)^{-1}\circ K : \mathcal{A}\boxtimes\mathcal{A}\simeq Z_{\mathrm{CY}}(\mathrm{Ann})$ is naturally isomorphic to the following functor:

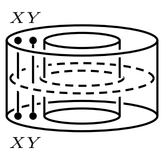
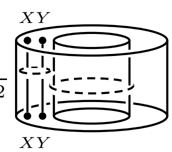
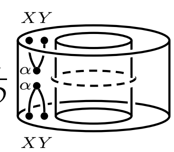
$$\mathcal{A}\boxtimes\mathcal{A}\simeq Z_{\mathrm{CY}}(\mathrm{Ann})$$



The composition $\mathrm{Kar}(H)\circ\mathrm{Kar}(G)^{-1}\circ K$ itself only hits objects with one marked point p. The functor presented above is more intuitive from the following perspective. Restricting to the first factor, i.e. setting $Y=\mathbf{1}$, this is like including $\mathcal{A}$ into $Z_{\mathrm{CY}}(\mathrm{Ann})$ along the outer boundary; likewise, the second factor is including $\mathcal{A}$ into $Z_{\mathrm{CY}}(\mathrm{Ann})$ along the inner boundary:

$X\mapsto (X,c)\mapsto \mathrm{im}\big(\frac{1}{\mathcal{D}}$  $\big)$; $Y\mapsto (Y,c^{-1})\mapsto \mathrm{im}\big(\frac{1}{\mathcal{D}}$  $\big)$

The functor presented before is the $\overline{\otimes}_{\mathrm{st}}$-tensor product of these two. This picture also elucidates the definition of $\overline{\otimes}$ on $\mathcal{A}\boxtimes\mathcal{A}$: if we take the tensor product of the two functors above in opposite order, essentially looking at $(\mathbf{1}\boxtimes X)\,\overline{\otimes}\,(Y\boxtimes\mathbf{1})$, we get

$K(\mathbf{1}\boxtimes X)\,\overline{\otimes}_{\mathrm{st}}\,K(Y\boxtimes\mathbf{1}) = \mathrm{im}\big(\frac{1}{\mathcal{D}^2}$  $\big) = \mathrm{im}\big(\frac{1}{\mathcal{D}^2}$  $\big) = \mathrm{im}\big(\frac{1}{\mathcal{D}}$  $\big) \simeq \langle X,Y\rangle\cdot\overline{\mathbf{1}}_{\mathrm{st}}$

where we used (1.7) and (1.9).

Note that the equivalence $\mathcal{A}\boxtimes\mathcal{A}^{\mathrm{bop}}\simeq_{\otimes,\mathrm{br}}(\mathcal{Z}(\mathcal{A}),\otimes)$ mentioned in the beginning of this section is also built by tensoring the same two functors together; it just happens that [3]

$$(X,c)\otimes(Y,c^{-1}) = (X,c)\,\overline{\otimes}\,(Y,c^{-1})$$

Remark 5.4. One can consider a similar tensor product on $\mathcal{A}\boxtimes\mathcal{A}$ when $\mathcal{A}$ is not modular. Definition 5.1 of $\overline{\otimes}$ will look the same, but $\langle Y_1,W_2\rangle$ would be replaced by the symmetric part of Y_1W_2, that is, the direct summands of Y_1W_2 that belong to the symmetric center of $\mathcal{A}$. The functor $K:\mathcal{A}\boxtimes\mathcal{A}\to\mathcal{Z}(\mathcal{A})$ will still respect $\overline{\otimes}$, but K will not be an equivalence. Furthermore, it is not clear whether $\overline{\otimes}$ on $\mathcal{A}\boxtimes\mathcal{A}$ possesses a unit. As evidence suggestive of this, recall that in the modular case, even showing $K(\overline{\mathbf{1}})\simeq\overline{\mathbf{1}}$ required the non-degeneracy of the S-matrix. △

The $\mathbb{Z}/2$-action is particularly simple on $\mathcal{A}\boxtimes\mathcal{A}$.

Proposition 5.5. *There is a tensor equivalence*

$$U_M:(\mathcal{A}\boxtimes\mathcal{A},\overline{\otimes})\simeq_{\otimes}(\mathcal{A}\boxtimes\mathcal{A},\overline{\otimes}^{\,op})$$
$$X\boxtimes Y\mapsto Y\boxtimes X$$

$U_M^2=\mathrm{id}$*, so* U_M *generates a* $\mathbb{Z}/2$*-action on* $\mathcal{A}\boxtimes\mathcal{A}$ *(but not tensor action).*

Proof. Let $W_1\boxtimes Y_1, W_2\boxtimes Y_2\in\mathcal{A}\boxtimes\mathcal{A}$. We have

$$U_M(W_1\boxtimes Y_1)\,\overline{\otimes}^{\mathrm{op}}U_M(W_2\boxtimes Y_2) = \langle W_2,Y_1\rangle\cdot Y_2\boxtimes W_1$$

[3]Coincidence? I think NOT!

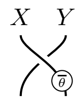
and

$$U_M(W_1 \boxtimes Y_1 \mathbin{\overline{\otimes}} W_2 \boxtimes Y_2) = \langle Y_1, W_2\rangle \cdot Y_2 \boxtimes W_1$$

We have the natural isomorphism $z : \langle W_2, Y_1\rangle \simeq \langle Y_1, W_2\rangle$ (see (1.3)). The tensor structure is given by $z \cdot \mathrm{id}_{Y_2 \boxtimes W_1}$. $\square$

Proposition 5.6. *The natural isomorphism $u_M : U \circ K \simeq K \circ U_M$ given by*

$$(u_M)_{X\boxtimes Y} = (\mathrm{id}_Y \otimes \overline{\theta}_X) \circ (c^{-1})_{X,Y} =$$



makes $K : \mathcal{A} \boxtimes \mathcal{A} \simeq \mathcal{Z}(\mathcal{A})$ a $\mathbb{Z}/2$-equivariant equivalence.

Proof. Straightforward. (The $\overline{\theta}$ is needed so that $u_M^2 = \overline{\theta}_{XY}$, because $\overline{\theta} : U^2 \simeq \mathrm{id}$.) $\square$

There is a more topological reason why $\overline{\theta}$ is needed. In the equivalence $\mathcal{A} \boxtimes \mathcal{A} \simeq Z_{\mathrm{CY}}(\mathrm{Ann})$ shown above, $X \boxtimes Y$ is sent to an object with two marked points b_1 and b_2, labelled X and Y. We can choose to make the marked points very close to p so that they are fixed by U_{CY}. Then u_M would be a graph in $Z_{\mathrm{CY}}(\mathrm{Ann})$ between the images of $X \boxtimes Y$ and $Y \boxtimes X$, and u_M^2 is a full inverse twist.

Finally, we compare $(\mathcal{A} \boxtimes \mathcal{A}, \overline{\otimes})$ to $\mathrm{Mat}(\mathrm{Vec})$, the category of Vec-valued matrices (see [EGNO, Example 4.1.3]). Let $\mathcal{S}$ be some finite set. The objects of $\mathrm{Mat}_{\mathcal{S}}(\mathrm{Vec})$ are bigraded vector spaces $V = \bigoplus_{i,j\in\mathcal{S}} V_i^j$, and the tensor product is given by

$$(V \otimes W)_i^j = \bigoplus_{k\in\mathcal{S}} V_i^k \otimes W_k^j$$

Let $\mathbf{k}_i^j$ be $\mathbf{k}$ with bigrading i, j. Then the unit is $\bigoplus_{i\in\mathcal{S}} \mathbf{k}_i^i$. Duals are given by transposing the matrix and then taking duals componentwise, i.e. $(V^*)_i^j = (V_j^i)^*$.

Proposition 5.7. *There is a tensor equivalence*

$$\mathrm{Mat}_{\mathrm{Irr}(\mathcal{A})}(\mathrm{Vec}) \simeq_{\otimes} (\mathcal{A} \boxtimes \mathcal{A}, \overline{\otimes})$$
$$\mathbf{k}_i^j \mapsto X_i \boxtimes X_j^*$$

Proof. Clearly the functor above is an equivalence of abelian categories, and sends unit to unit.

Denote $X_i^j \coloneqq X_i \boxtimes X_j^*$, so $(X_i^j)^\vee = X_j^i$. One has $X_i^j \mathbin{\overline{\otimes}} X_j^l = \langle X_j^*, X_k\rangle X_i^l$, which is 0 if $j \neq k$. When $j = k$, define J_k to be the map

$$(\mathrm{ev}_{X_k} \circ -) \cdot \mathrm{id}_{X_i^l} : X_i^k \mathbin{\overline{\otimes}} X_k^l \simeq X_i^l$$

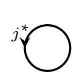
Then it is easy to check that

$$J_{\mathbf{k}_i^j, \mathbf{k}_k^l} = \delta_{j,k} J_k$$

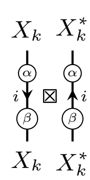
is a tensor structure on the functor above. $\square$

While $(\mathcal{A} \boxtimes \mathcal{A}, \overline{\otimes})$ and $\mathrm{Mat}_{\mathrm{Irr}(\mathcal{A})}(\mathrm{Vec})$ are tensor equivalent, they are not pivotal tensor equivalent (assuming the natural pivotal structure on $\mathrm{Mat}_{\mathrm{Irr}(\mathcal{A})}(\mathrm{Vec})$ coming from Vec). This can be seen from computing dimensions. For example, the left trace of $\mathrm{id}_{X_i^j}$ is

$$\sum_{k\in\mathrm{Irr}(\mathcal{A})} d_k$$



$\cdot$



$$= \sum_k \delta_{i,k} \frac{d_j}{d_k} \mathrm{id}_{X_k^k}$$

so the left dimension of X_i^j is $d^L_{X_i^j} = \frac{d_j}{d_i}$, which cannot always be 1 for all pairs i, j. Its right dimension is $d^R_{X_i^j} = d^L_{X_j^i} = \frac{d_i}{d_j}$. Thus $(\mathcal{A} \boxtimes \mathcal{A}, \overline{\otimes})$ can be thought of as a non-pivotal deformation of $\mathrm{Mat}(\mathrm{Vec})$.

6. $\mathcal{A}$ Symmetric

Next we consider the other extreme case, when $\mathcal{A}$ is symmetric. The main contribution of this section is to give a more explicit description of $Z_{\mathrm{CY}}(\mathbf{T}^2)$ when $\mathcal{A}$ is symmetric.

As mentioned before, when $\mathcal{A}$ is symmetric, our reduced tensor product construction coincides with Wasserman's symmetric tensor product in [Was]. While we arrived at these results independently, many of the results here can be found there, which we include for completeness; we also add several new results, e.g. Theorem 6.13 and Theorem 6.19. We thank Thomas Wasserman for pointing out the subtlety of the braidings that we missed (see Remark 6.14).

In this section, all tensor products (not including $\overline{\otimes}$) and diagrams are considered in the category of vector spaces over $\mathbf{k}$.

We start with taking $\mathcal{A} = \mathrm{Rep}(G)$ for some finite group G, leaving the supergroup case as a slight modification. Of course, $\mathrm{Rep}(G)$ can be treated by the supergroup case by taking $z = e$, but we find exposition this way to be clearer. We will also show (Theorem 6.13) that the supergroup case is essentially the same as the non-super case (but see Remark 6.14), so one may choose to ignore the supergroup case and its complexities for the purposes of this paper.

Let us first recall the following (see e.g. [BK, Section 3.2], [EGNO, Section 7.14]):

Proposition 6.1. *We have an equivalence of modular categories*

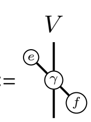
$$T : \mathcal{Z}(\mathrm{Rep}(G)) \simeq_{\otimes, br} \mathcal{D}(G) - \mathrm{mod}$$

where $\mathcal{D}(G)$ is the Drinfeld double of G.

Proof. Recall that $\mathcal{D}(G) \cong \mathbf{k}[G] \otimes F(G)$ as coalgebras, multiplication is defined by $(g_1 \otimes \delta_{h_1}) \cdot (g_2 \otimes \delta_{h_2}) = g_1 g_2 \otimes \delta_{g_2^{-1} h_1 g_2} \delta_{h_2}$, and the R-matrix is given by $\sum_{g \in G} g \otimes \delta_g \in \mathcal{D}(G) \otimes \mathcal{D}(G)$.

Given $(V, \gamma) \in \mathcal{Z}(\mathrm{Rep}(G))$, we put a $F(G)$ action on V as follows: for $f \in F(G), v \in V$,

$$f \cdot v := \langle \mathrm{id}_V \otimes f, \gamma_{\mathbf{k}[G]}(e \otimes v) \rangle \quad \text{or} \quad f \cdot (-) :=$$



where e is the identity element of G, here thought of as the (not G-)linear map $\mathbf{k} \to \mathbf{k}[G]$ sending $1 \mapsto e$; the diagram on the right is not a morphism in $\mathcal{A}$, it is just a linear map of vector spaces.

Conversely, given a $\mathcal{D}(G)$-module V, one has the half-braiding γ on V defined as follows: for $A \in \mathcal{Z}(\mathrm{Rep}(G))$, and $a \in A, v \in V$,

$$\gamma_A : A \otimes V \simeq V \otimes A$$
$$a \otimes v \mapsto \sum_{g \in G} \delta_g \cdot v \otimes g \cdot a$$

In other words, $\gamma = P \circ R$, where P is the canonical swap in vector spaces over $\mathbf{k}$.

$\mathcal{D}(G) - \mathrm{mod}$ can be thought of as the category of G-equivariant bundles over G, where the action of G on the base G is conjugation. Namely, $V = \bigoplus_{g \in G} V_g$, where $V_g = \delta_g \cdot V$. The commutation relation between $\mathbf{k}[G]$ and $F(G)$ inside $\mathcal{D}(G)$ means that the action of $g \in G$ gives an isomorphism $g : V_h \simeq V_{ghg^{-1}}$. The orbit of g in the base is the conjugacy class $\bar{g}$, which determines a central idempotent $\delta_{\bar{g}} := \sum_{h \in \bar{g}} \delta_h$. This decomposes $\mathcal{D}(G) - \mathrm{mod}$ (as an abelian category) as a direct sum

$$\mathcal{D}(G) - \mathrm{mod} \sim \bigoplus_{\bar{g} \in G/G} \mathcal{D}(G) \delta_{\bar{g}} - \mathrm{mod}$$

There is a *convolution* tensor product on $\mathcal{D}(G) - \mathrm{mod}$ given by

$$(V \otimes W)_g = \bigoplus_h V_{gh} \otimes W_{h^{-1}}$$

which arises from the coproduct structure on $\mathcal{D}(G)$,

$$g \mapsto g \otimes g \ , \ \ \delta_g \mapsto \sum_h \delta_{gh} \otimes \delta_{h^{-1}}$$

and it is easy to check that it agrees with the usual tensor product on $\mathcal{Z}(\mathrm{Rep}(G))$ under this equivalence. The dual is given by

$$(V^*)_g = (V_{g^{-1}})^*$$

and the pivotal structure is the natural $\delta : V_g \simeq (V_g)^{**}$. □

In particular, T is an equivalence of abelian categories, so we want to interpret the reduced tensor product $\overline{\otimes}$ as a tensor product on $\mathcal{D}(G)-\mathrm{mod}$. As in the modular case, we use the same symbol $\overline{\otimes}$ and name, which will be justified when we prove their equivalence (Theorem 6.4).

Definition 6.2. Let $V, W \in \mathcal{D}(G)-\mathrm{mod}$. Their *reduced tensor product*, or *fibrewise tensor product*, denoted $V \overline{\otimes} W$, is defined first as a G-graded vector space by

$$(V \overline{\otimes} W)_g = V_g \otimes W_g$$

and then as a G-module, $h \in G$ acts diagonally,

$$h = h \otimes h : V_g \otimes W_g \simeq V_{hgh^{-1}} \otimes W_{hgh^{-1}}$$

For morphisms $\varphi : V \to V', \psi : W \to W'$,

$$(\varphi \overline{\otimes} \psi)_g = \varphi_g \otimes \psi_g : V_g \otimes W_g \to V'_g \otimes W'_g$$

△

Proposition 6.3. $(\mathcal{D}(G)-\mathrm{mod}, \overline{\otimes})$ *is a pivotal multifusion category, with*

- *Unit* $\overline{\mathbf{1}} = \bigoplus_{g\in G} \mathbf{k}_g$*, i.e.* $(\overline{\mathbf{1}})_g = \mathbf{k}$ *for all* $g \in G$*, where* $\mathbf{k}$ *are "trivial" (see proof);*
- *Duals:* $(V^\vee)_g = (V_g)^*$*, so* $V^\vee = {}^\vee V$*;*
- *Pivotal structure is the natural one* $\delta : V_g \simeq (V_g)^{**} = (V^{\vee\vee})_g$

Proof. For the unit, the fibres are "trivial" in the sense that if $h \in G$ fixes g, then h acts trivially on $(\overline{\mathbf{1}})_g$. More concisely, $\overline{\mathbf{1}}$ has a global non-vanishing G-equivariant section. The rest is straightforward. □

Note that the unit is $\mathbf{k}[G]$ with conjugation action, in other words $\bigoplus_i \mathrm{End}_{\mathbf{k}}(X_i)$, and not the regular representation $\mathbf{k}[G]$. This should be reminiscent of the unit for $(\mathcal{Z}(\mathcal{A}), \overline{\otimes})$, $\bigoplus X_i X_i^*$. In fact, we have the following result (see also [Was, Theorem 44]):

Theorem 6.4. *The equivalence T of Proposition 6.1 is a pivotal tensor equivalence*

$$T : (\mathcal{Z}(\mathcal{A}), \overline{\otimes}) \simeq_{\otimes} (\mathcal{D}(G)-\mathrm{mod}, \overline{\otimes})$$

Proof. Observe that the dashed line is simply the identity on the regular representation:

$$\sum_{i\in\mathrm{Irr}(\mathrm{Rep}(G))} d_i \cdot \mathrm{id}_{X_i} = \mathrm{id}_{\mathbf{k}[G]}$$

Consider objects $(V,\gamma),(W,\mu) \in \mathcal{Z}(\mathrm{Rep}(G))$, their reduced tensor product is the image of the projector $Q_{\gamma,\mu}$. We compute the projection $T(Q_{\gamma,\mu})$ on $V \otimes W \in \mathcal{D}(G)-\mathrm{mod}$ (all sums are over G):

$$\begin{aligned} v \otimes w &\mapsto \frac{1}{|G|} \sum_g g \otimes v \otimes \delta_g \otimes w \\ &\mapsto \frac{1}{|G|} \sum_{g,h_1,h_2} \delta_{h_1} \cdot v \otimes h_1 g \otimes \delta_{h_2} \cdot w \otimes \delta_{h_2 g} \\ &\mapsto \frac{1}{|G|} \sum_{g,h_1,h_2} \langle h_1 g, \delta_{h_2 g} \rangle \delta_{h_1} \cdot v \otimes \delta_{h_2} \cdot w \\ &= \sum_h \delta_h \cdot v \otimes \delta_h \cdot w \end{aligned}$$

In particular,

$$T(Q_{\gamma,\mu})|_{V_g \otimes W_h} = \delta_{g,h} \, \mathrm{id}_{V_g \otimes W_h}$$

It follows easily that

$$\mathrm{im}(T(Q_{\gamma,\mu})) = V \overline{\otimes} W$$

□

Remark 6.5. We note that both $(\mathcal{Z}(\mathrm{Rep}(G)), \overline{\otimes})$ and $(\mathcal{D}(G)-\mathrm{mod}, \overline{\otimes})$ are symmetric: the braiding on $\mathrm{Rep}(G)$, being symmetric, is naturally a symmetric braiding for $(\mathcal{Z}(\mathrm{Rep}(G)), \overline{\otimes})$, and $(\mathcal{D}(G)-\mathrm{mod}, \overline{\otimes})$ inherits the braiding on vector spaces $P : V_g \otimes W_g \simeq W_g \otimes V_g$. Then the above equivalence T is in fact braided (see [Was, Theorem 44]). △

It is possible to describe $\overline{\otimes}$ in terms of a coproduct structure on $\mathcal{D}(G)$, although we need to slightly modify the usual correspondence between tensor products and coproducts. This perspective will be particularly useful in the supergroup case.

Proposition 6.6. *Consider the map*

$$\begin{aligned} \overline{\Delta} : \mathcal{D}(G) &\mapsto \mathcal{D}(G) \otimes \mathcal{D}(G) \\ g &\mapsto g \otimes g \\ \delta_g &\mapsto \delta_g \otimes \delta_g \end{aligned}$$

for all $g \in G$. This is an algebra homomorphisms, and for $V, W \in \mathrm{Rep}(G, z)$,

$$V \overline{\otimes} W = \overline{\Delta}(1) \cdot (V \otimes W)$$

where $1 = e \otimes \delta_* = e \otimes \sum_h \delta_h$ is the unit of $\mathcal{D}(G)$.

Proof. Straightforward. □

The methods above can be adapted for finite supergroups. Recall that $\mathrm{Rep}(G, z)$ is the same as $\mathrm{Rep}(G)$ as a pivotal fusion category, but has a $\mathbb{Z}/2$-grading determined by the action of the order 2 central element z, and the braiding on $\mathrm{Rep}(G, z)$ is twisted by the Koszul sign convention.

For $V \in \mathrm{Rep}(G, z)$, the actions of $e^0 := \frac{1+z}{2}$ and $e^1 := \frac{1-z}{2}$ are projections to its even and odd part, which we denote by V^0 and V^1, respectively. Similarly, for $v \in V$, we denote by v^0 and v^1 its even and odd part. In particular, for elements of $\mathcal{D}(G)$ as a G-module by left multiplication, $g^\sigma = e^\sigma g = (g + (-1)^\sigma gz)/2$ and $\delta_g^\sigma = e^\sigma \delta_g$. We warn the reader not to confuse the left multiplication action of G on $\mathcal{D}(G)$ with the G-action on $\mathbf{k}[G]^* = F(G) \subseteq \mathcal{D}(G)$, which is by precomposition; in the latter, $\delta_g^\sigma = (\delta_g + (-1)^\sigma \delta_{gz})/2$, which, under the inclusion $F(G) \hookrightarrow \mathcal{D}(G)$, does not agree with $e^\sigma \delta_g$.

Letting $\mathcal{A} = \mathrm{Rep}(G, z)$, there is no change in the proof of Proposition 6.1, so that we still have an equivalence of abelian categories $\mathcal{Z}(\mathrm{Rep}(G, z)) \simeq \mathcal{D}(G)-\mathrm{mod}$. However, the different braiding on $\mathrm{Rep}(G, z)$ affects $\overline{\otimes}$ on $\mathcal{Z}(\mathrm{Rep}(G, z))$, so we will need to modify the definition of the reduced/fibrewise tensor product on $\mathcal{D}(G)-\mathrm{mod}$ (see also [Was, Definition 49]):

Definition 6.7. Let $V, W \in \mathcal{D}(G)-\mathrm{mod}$. Their *reduced tensor product with respect to* z, or *z-twisted fibrewise tensor product*, denoted $V \overline{\otimes}_z W$, is defined first as a G-graded vector space by

$$(V \overline{\otimes}_z W)_g := (V_g^0 \otimes W_g^0) \oplus (V_{gz}^1 \otimes W_{gz}^1) \oplus (V_{gz}^0 \otimes W_g^1) \oplus (V_g^1 \otimes W_{gz}^0) = \bigoplus_{\sigma, \tau \in \mathbb{Z}/2} V_{gz^\tau}^\sigma \otimes W_{gz^\sigma}^\tau$$

and then as a G-module, $h \in G$ acts diagonally, i.e.

$$h = h \otimes h : V_{gz^\tau}^\sigma \otimes W_{gz^\sigma}^\tau \simeq V_{hgz^\tau h^{-1}}^\sigma \otimes W_{hgz^\sigma h^{-1}}^\tau$$

For morphisms, it is just defined by the tensor product of the appropriate restrictions. △

The definition of $\overline{\otimes}_z$ looks complicated. It is useful to observe the following: the tensor product mixes fibres over g and gz together, and of the terms in $(\bigoplus_{\sigma,\tau} V_{gz^\tau}^\sigma) \otimes (\bigoplus_{\sigma',\tau'} W_{gz^{\tau'}}^{\sigma'})$, only those with $\sigma + \tau + \sigma' + \tau' = 0$ appear in $V \overline{\otimes}_z W$; such a term will have G-grading $gz^{\sigma+\tau'}$, and have parity $\sigma + \sigma'$.

Proposition 6.8. *$(\mathcal{D}(G)-\mathrm{mod}, \overline{\otimes}_z)$ is a pivotal multifusion category, with*

- *Unit $\overline{\mathbf{1}} = \bigoplus_{g \in G} \mathbf{k}_g$ is the same as for $(\mathcal{D}(G)-\mathrm{mod}, \overline{\otimes})$.*
- *Duals: $(V^\vee)_g = (V_g)^*$, so $V^\vee = {}^\vee V$; the evaluation map is 0 on the odd part of $V^\vee \overline{\otimes}_z V$, and the obvious one on the even part; similarly for the coevaluation (see proof for details);*
- *Pivotal structure is the natural one $\delta : V_g \simeq (V_g)^{**} = (V^{\vee\vee})_g$*

Proof. This is more or less straightforward, if not a little tricky for one not accustomed to the world of supervector spaces. Let us check $\mathrm{ev}\circ\mathrm{coev} = \mathrm{id} : V_g \to (V \,\overline{\otimes}_z\, V^\vee \,\overline{\otimes}_z\, V)_g \to V_g$. Let $\{v_i\}, \{\bar{v}_i\}$ be bases for V_g^0, V_g^1 respectively, and $\{w_i\}, \{\bar{w}_i\}$ bases for V_{gz}^0, V_{gz}^1 respectively. We raise indices for their dual bases (not to be confused with parity).

Let $x \in V_g^0, \bar{x} \in V_g^1$. The coevaluation is given by (sums over bases are implicit)

$$1 \mapsto v_i \otimes v^i + \bar{w}_i \otimes \bar{w}^i + w_i \otimes w^i + \bar{v}_i \otimes \bar{v}^i + \dots$$

where we only show the terms relevant to x^0 and x^1 (i.e. the summands appearing in $(V \,\overline{\otimes}_z\, V^\vee)_g$ (first two terms) and $(V \,\overline{\otimes}_z\, V^\vee)_{gz}$ (next two terms)). Then

$$x \mapsto v_i \otimes v^i \otimes x + \bar{w}_i \otimes \bar{w}^i \otimes x \mapsto v_i \langle v^i, x\rangle + 0 = x$$
$$\bar{x} \mapsto w_i \otimes w^i \otimes \bar{x} + \bar{v}_i \otimes \bar{v}^i \otimes \bar{x} \mapsto 0 + \bar{v}_i \langle \bar{v}^i, \bar{x}\rangle = \bar{x}$$

□

Then we have the analog of Theorem 6.4 (see also [Was, Theorem 51]):

Theorem 6.9. *The equivalence T of Proposition 6.1 is a pivotal tensor equivalence*

$$T : (\mathcal{Z}(\mathrm{Rep}(G,z)), \overline{\otimes}) \simeq_\otimes (\mathcal{D}(G) - \mathrm{mod}, \overline{\otimes}_z\,)$$

Proof. Essentially the same as before, but again slightly tricky. The dashed line is still the regular representation. Recall that the actions of $(1+z)/2$ and $(1-z)/2$ are projections onto even and odd parts respectively. Then the braiding $c : \mathbf{k}[G] \otimes V^\sigma \simeq V^\sigma \otimes \mathbf{k}[G]$ for $\sigma \in \mathbb{Z}/2$ is given by

$$g \otimes v = (\frac{g+gz}{2} + \frac{g-gz}{2}) \otimes v \overset{c}{\mapsto} v \otimes (\frac{g+gz}{2} + (-1)^\sigma \frac{g-gz}{2}) = v \otimes gz^\sigma$$

Similarly, $c : F(G) \otimes V^\sigma \simeq V^\sigma \otimes F(G)$ is given by

$$\delta_g \otimes v = (\frac{\delta_g+\delta_{gz}}{2} + \frac{\delta_g-\delta_{gz}}{2}) \otimes v \overset{c}{\mapsto} v \otimes (\frac{\delta_g+\delta_{gz}}{2} + (-1)^\sigma \frac{\delta_g-\delta_{gz}}{2}) = v \otimes \delta_{gz^\sigma}$$

We perform the same computation, that is, $T(Q_{\gamma,\mu})$ on $V \otimes W$, For $v \in V_g^\sigma, w \in W_h^\tau$, (sum over $a \in G$ is implicit)

$$\begin{aligned} v \otimes w \mapsto \frac{1}{|G|} v \otimes w &\overset{\mathrm{coev}_{\mathbf{k}[G]}}{\mapsto} \frac{1}{|G|} a \otimes \delta_a \otimes v \otimes w \\ &\overset{c}{\mapsto} \frac{1}{|G|} a \otimes v \otimes \delta_{az^\sigma} \otimes w \\ &\overset{\gamma,\mu}{\mapsto} \frac{1}{|G|} v \otimes ga \otimes w \otimes h \cdot \delta_{az^\sigma} \\ &\overset{c}{\mapsto} \frac{1}{|G|} v \otimes w \otimes gaz^\tau \otimes \delta_{haz^\sigma} \\ &\overset{\mathrm{ev}_{\mathbf{k}[G]}}{\mapsto} \delta_{gz^\tau, hz^\sigma} v \otimes w \end{aligned}$$

The action of δ_b on $v \otimes w$, i.e. projection to its b-graded component, is the composition

$$v \otimes w \mapsto e \otimes v \otimes w \overset{\gamma}{\mapsto} v \otimes g \otimes w \overset{c}{\mapsto} v \otimes w \otimes gz^\tau \overset{\delta_b}{\mapsto} \delta_{gz^\tau, b} v \otimes w$$

Thus, the projection of $T(Q_{\gamma,\mu})$ restricted to $V_g^\sigma \otimes W_h^\tau$ has G-grading gz^τ, so the b-graded component of $\mathrm{im}\, T(Q_{\gamma,\mu})$ is $\bigoplus V_g^\sigma \otimes W_h^\tau$, where the sum is over all g, h with $gz^\tau = hz^\sigma = b$, which agrees with Definition 6.7. □

Remark 6.10. Similar to Remark 6.10, we note that both $(\mathcal{Z}(\mathrm{Rep}(G,z)), \overline{\otimes})$ and $(\mathcal{D}(G)\text{-}\mathrm{mod}, \overline{\otimes}_z\,)$ are symmetric: the braiding on $\mathrm{Rep}(G,z)$, being symmetric, is naturally a symmetric braiding for $(\mathcal{Z}(\mathrm{Rep}(G,z)), \overline{\otimes})$, and $(\mathcal{D}(G) - \mathrm{mod}, \overline{\otimes}_z\,)$ inherits the braiding of supervector spaces $(-1)^{\sigma\cdot\tau} P : V_{gz^\tau}^\sigma \otimes W_{gz^\sigma}^\tau \simeq W_{gz^\sigma}^\tau \otimes V_{gz^\tau}^\sigma$. Then the above equivalence T is in fact symmetric (see [Was, Theorem 51]). △

We have an analogue of Proposition 6.6:

Proposition 6.11. *Consider the map*

$$\overline{\Delta}_z : \mathcal{D}(G) \mapsto \mathcal{D}(G) \otimes \mathcal{D}(G)$$
$$g \mapsto g \otimes g$$
$$\delta_g \mapsto \bigoplus_{\sigma,\tau} \delta^{\sigma}_{gz^{\tau}} \otimes \delta^{\tau}_{gz^{\sigma}}$$

for all $g \in G$, where $\delta^{\sigma}_g = e^{\sigma}\delta_g$ is projection to the even/odd g-graded component. This is an algebra homomorphisms, and for $V, W \in \mathrm{Rep}(G,z)$,

$$V \mathbin{\overline{\otimes}} W = \overline{\Delta}_z(1) \cdot (V \otimes W)$$

where $1 = e \otimes \delta_* = e \otimes \sum_h \delta_h$ is the unit of $\mathcal{D}(G)$.

Proof. Straightforward. We note that by parity consideration,

$$\overline{\Delta}_z(\delta^0_g) = \delta^0_g \otimes \delta^0_g + \delta^1_{gz} \otimes \delta^1_{gz}$$
$$\overline{\Delta}_z(\delta^1_g) = \delta^1_g \otimes \delta^0_{gz} + \delta^0_{gz} \otimes \delta^1_g$$

which reflects the observation made after Definition 6.7. □

It turns out that these coproducts are the same up to an algebra automorphism of $\mathcal{D}(G)$:

Proposition 6.12. *The map*

$$\lambda : \mathcal{D}(G) \simeq \mathcal{D}(G)$$
$$g \mapsto g$$
$$\delta_g \mapsto \delta^0_g + \delta^1_{gz}$$

defines an isomorphism of algebras. Furthermore, this isomorphism intertwines the two coproducts $\overline{\Delta}$ and $\overline{\Delta}_z$, i.e. the following diagram commutes:

$$\begin{tikzcd} \mathcal{D}(G) \arrow[r, "\lambda"] \arrow[d, "\overline{\Delta}"] & \mathcal{D}(G) \arrow[d, "\overline{\Delta}_z"] \\ \mathcal{D}(G) \otimes \mathcal{D}(G) \arrow[r, "\lambda\otimes\lambda"] & \mathcal{D}(G) \otimes \mathcal{D}(G) \end{tikzcd}$$

Proof. Straightforward. The computation simplifies greatly following the observation in the proof of Proposition 6.11. □

The following result is the corresponding categorical statement for Proposition 6.12, which says that the two tensor products $\overline{\otimes}$ and $\overline{\otimes}_z$ are essentially the same:

Theorem 6.13. *Let $\Lambda : \mathcal{D}(G) - \mathrm{mod} \simeq \mathcal{D}(G) - \mathrm{mod}$ be the pullback functor induced from $\lambda : \mathcal{D}(G) \simeq \mathcal{D}(G)$ as defined in Proposition 6.12. More explicitly, for $V \in \mathcal{D}(G) - \mathrm{mod}$, $\Lambda(V)$ is the same as V as G-module, but the $F(G)$-action is changed so gradings are mixed:*

$$\Lambda(V)_g = V^0_g \oplus V^1_{gz}$$

Then Λ is a pivotal tensor equivalence

$$\Lambda : (\mathcal{D}(G) - \mathrm{mod}, \overline{\otimes}_z) \simeq_{\otimes} (\mathcal{D}(G) - \mathrm{mod}, \overline{\otimes})$$

Proof. Clearly the unit $\overline{\mathbf{1}} = \bigoplus_{g\in G} \mathbf{k}_g$ is preserved (all $\mathbf{k}_g$ are even). By Proposition 6.6 and Proposition 6.11, the tensor products $\overline{\otimes}$ and $\overline{\otimes}_z$ arise from the coproducts $\overline{\Delta}$ and $\overline{\Delta}_z$. Since λ intertwines these coproducts, we see that Λ respects tensor products:

$$\begin{aligned}(\Lambda(V) \mathbin{\overline{\otimes}} \Lambda(W))_g &= \Lambda(V)_g \otimes \Lambda(W)_g \\ &= \overline{\Delta}(\delta_g) \cdot (\Lambda(V) \otimes \Lambda(W)) \\ &= ((\lambda \otimes \lambda) \circ \overline{\Delta})(\delta_g) \cdot (V \otimes W) \\ &= (\overline{\Delta}_z \circ \lambda)(\delta_g) \cdot (V \otimes W) \\ &= \Lambda(V \mathbin{\overline{\otimes}_z} W)_g\end{aligned}$$

Of course, a direct computation works too. The pivotal structure is obviously respected. $\square$

Remark 6.14. Thomas Wasserman pointed out that the equivalence Λ above does not respect the symmetric braidings described in Remarks 6.5 and 6.10, hence $(\mathcal{D}(G)-\mathrm{mod}, \overline{\otimes})$ and $(\mathcal{D}(G)-\mathrm{mod}, \overline{\otimes}_z)$ are not braided tensor equivalent. $\triangle$

Next we study the $\mathbb{Z}/2$-action on $\mathcal{D}(G)-\mathrm{mod}$. Here $\mathcal{A} = \mathrm{Rep}(G,z)$, where we treat the non-super case as setting $z = e$.

Proposition 6.15. *Let $U_S : \mathcal{D}(G)-\mathrm{mod} \to \mathcal{D}(G)-\mathrm{mod}$ that changes the grading by the inverse map on G. That is,*

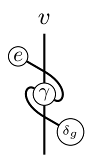
$$(U_S(V))_g = V_{g^{-1}}$$

Then U_S is naturally a tensor automorphism, and $U_S^2 = \mathrm{id}$, so U_S generates a $\mathbb{Z}/2$-action.

Proof. Trivial. $\square$

Proposition 6.16. *The equivalence T of Proposition 6.1 is $\mathbb{Z}/2$-equivariant.*

Proof. Let $(V,\gamma) \in \mathcal{Z}(\mathrm{Rep}(G,z))$. U on $\mathcal{Z}(\mathrm{Rep}(G,z))$ only affects the half-braiding, so $T((V,\gamma)) = V = T((V,\widetilde{\gamma}))$ are the same as G-modules. It suffices to check that the action of δ_g on $T((V,\widetilde{\gamma}))$ is the same as the action of $\delta_{g^{-1}}$ on $T((V,\gamma))$. For $v \in T((V,\widetilde{\gamma}))^\sigma$, the action of δ_g is



$$\begin{aligned} v &\mapsto e \otimes v \\ &\mapsto \sum_h h \otimes \delta_h \otimes v \otimes z^\sigma \\ &\mapsto \sum_{h,h'} h \otimes \delta_{h'} \otimes h'\delta_h \cdot v \otimes z^\sigma \;\; \text{(using } \delta_{h'} \text{ action of } T((V,\gamma))) \\ &\mapsto \sum_{h,h'} \delta_g(hz^\sigma)\delta_{hh'}(z^\sigma)\delta_{h'} \cdot v = \delta_{g^{-1}} \cdot v \end{aligned}$$

Thus T commutes with the $\mathbb{Z}/2$-actions. It remains to note that for simple (V,γ), V is either all even or odd, so the twist operator is ± 1, so $\widetilde{\widetilde{\gamma}} = \gamma$ (see end of proof of Proposition 2.12) and hence $U^2 = \mathrm{id}$. $\square$

6.1. $Z_{\mathbf{CY}}(\mathbf{T}^2)$ for $\mathcal{A}$ symmetric.

Here we describe $Z_{\mathrm{CY}}(\mathbf{T}^2)$ when $\mathcal{A}$ is symmetric. Recall that we had Corollary 4.5, which stated

$$Z_{\mathrm{CY}}(\mathbf{T}^2) \simeq \mathcal{Z}((\mathcal{Z}(\mathcal{A}), \overline{\otimes}))$$

By Theorem 6.13, it suffices to deal with the non-super case $\mathcal{A} = \mathrm{Rep}(G)$. The following is taken from [Tham, Definition-Proposition 5.2]:

Definition 6.17. The *elliptic double* of G is the algebra $\mathcal{D}^{\mathrm{el}}(G)$ whose underlying vector space is

$$\mathcal{D}^{\mathrm{el}}(G) = \mathbf{k}[G] \otimes F(G) \otimes F(G)$$

with algebra structure

$$g \otimes \delta_{h_1} \otimes \delta_{h_2} \cdot g' \otimes \delta_{h'_1} \otimes \delta_{h'_2} = gg' \otimes \delta_{h_1}\delta_{g'^{-1}h'_1g'} \otimes \delta_{h_2}\delta_{g'^{-1}h'_2g'}$$

Let $\delta_* = \sum_{h\in G} \delta_h$ be the unit of $F(G)$. We will denote $g \otimes \delta_* \otimes \delta_*$ simply by g, $e \otimes \delta_h \otimes \delta_*$ by δ_h^1, and $e \otimes \delta_* \otimes \delta_h$ by δ_h^2. We may also denote $\delta_h^1\delta_h^2$ by $\delta_{(h,h')}$ $\triangle$

The $*$ notation in the subscript will be used often, and will generally mean to sum over all possible entries.

Observe that any orbit c of the diagonal action of G on $G \times G$ (acting by conjugation on each factor) gives rise to a central idempotent $\sum_{(h,h')\in c} \delta_{(h,h')}$. In particular, the set of commuting pairs

$$\Omega = \{(h,h') \in G \times G | [h,h'] = e\}$$

is a union of conjugacy classes, so gives rise to the following subalgebra:

Definition 6.18. $\mathcal{D}_{\mathbf{T}^2}(G)$ is the subalgebra of $\mathcal{D}^{\mathrm{el}}(G)$

$$\mathcal{D}_{\mathbf{T}^2}(G) := \{g\delta_{(h,h')} | (h,h') \in \Omega\} = \mathcal{D}^{\mathrm{el}}(G) \cdot \delta_\Omega$$

where $\delta_\Omega = \sum_{(h,h')\in\Omega} \delta_{(h,h')}$. We will use the same notation as in Definition 6.17, so for example δ_h^1 stands for $\delta_h^1\delta_\Omega = \sum_{h' : [h,h'] = e} \delta_{(h,h')}$ $\triangle$

It is easy to see that $\mathcal{D}_{\mathbf{T}^2}(G)-\mathrm{mod}$ is the category of G-equivariant bundles over Ω.

Theorem 6.19. *As abelian categories.*

$$\mathcal{Z}((\mathcal{D}(G)-\mathrm{mod},\overline{\otimes}))\simeq\mathcal{D}_{\mathbf{T}^2}(G)-\mathrm{mod}$$

Proof. Observe that a G-equivariant bundle over G, $V\in\mathcal{D}(G)-\mathrm{mod}$, naturally decomposes into bundles where each is supported on a single conjugacy class:

$$V=\bigoplus_{\kappa\in G/G}V_\kappa,\ \ V_\kappa:=\bigoplus_{g\in\kappa}V_g$$

Furthermore, V_κ is determined by V_g for some $g\in\kappa$ together with its restricted $Z(g)$-action. To recover V_κ from V_g, simply take the induction $\mathbf{k}[G]\otimes_{Z(g)}V_g$, and specify that the grading of $h\otimes v$ is hgh^{-1}. Moreover, the fibrewise tensor product restricts to the standard tensor product of $Z(g)$-modules. Thus we have, for choices $g\in\kappa$ for each conjugacy class $\kappa\in G/G$,

$$(\mathcal{D}(G)-\mathrm{mod},\overline{\otimes})\simeq_\otimes\bigoplus_{\kappa\in G/G}\mathrm{Rep}(Z(g))$$

and hence,

$$\mathcal{Z}((\mathcal{D}(G)-\mathrm{mod},\overline{\otimes}))\simeq\bigoplus_{\kappa\in G/G}\mathcal{Z}(\mathrm{Rep}(Z(g))$$

As we've seen before, $\mathcal{Z}(\mathrm{Rep}(Z(g))\simeq\mathcal{D}(Z(g))-\mathrm{mod}$, or the category of $Z(g)$-equivariant bundles over $Z(g)$.

Now consider $W\in\mathcal{D}_{\mathbf{T}^2}(G)-\mathrm{mod}$. It can also be decomposed

$$W=\bigoplus_{\kappa\in G/G}W_{(\kappa,*)},\ \ W_{(\kappa,*)}:=\bigoplus_{h\in\kappa,(h,h')\in\Omega}W_{(h,h')}$$

Similarly, $W_{(\kappa,*)}$ is determined by its restriction to $W_{(h,*)}=\oplus_{h':(h,h')\in\Omega}W_{(h,h')}$ for some $h\in\kappa$, together with its $Z(h)$-action. Once again, induction $\mathbf{k}[G]\otimes_{Z(h)}W_{(h,*)}$ recovers $W_{(\kappa,*)}$, with grading of $g\otimes v$ for $v\in W_{(h,h')}$ given by $g\cdot(h,h')=(ghg^{-1},gh'g^{-1})$. Note that $W_{(h,*)}$ is naturally a $Z(h)$-equivariant bundle over $Z(h)$, and thus we have, for choices $g\in\kappa$ for each conjugacy class $\kappa\in G/G$,

$$\mathcal{D}_{\mathbf{T}^2}(G)-\mathrm{mod}\simeq\bigoplus_{\kappa\in G/G}\mathcal{D}(Z(g))-\mathrm{mod}$$

and hence we are done. □

Remark 6.20. Let us discuss a more topological approach to/insight on $\mathcal{D}_{\mathbf{T}^2}(G)$. In [Tham], we defined a category called the "elliptic Drinfeld center" of $\mathcal{A}$, which we showed in [KT] to be equivalent to $Z_{\mathrm{CY}}(\mathbf{T}_0^2)$, where $\mathbf{T}_0^2$ is the once-punctured torus. When $\mathcal{A}=\mathrm{Rep}(G)$, this category is equivalent to the category of G-equivariant bundles over $G\times G$; the equivalence depends on a choice of meridian and longitude on $\mathbf{T}_0^2$, so that each G factor in $G\times G$ corresponds to one of the curves. As we seal up the puncture in $\mathbf{T}_0^2$ to get $\mathbf{T}^2$, the two G factors are forced to commute, thus $Z_{\mathrm{CY}}(\mathrm{Ann})$ consists of bundles supported only on Ω.

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