MODEL CATEGORIES OF DIAGRAM SPECTRA

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In [16], which we shall refer to as [MMSS], we described various categories of diagram spectra and their rings and modules. In this continuation, we define and compare model structures on these categories. We make free use of the definitions and results of [MMSS]. The relevant categories are displayed in the following "Main Diagram":

$$\begin{tikzcd}
& \mathscr{P} \arrow[dl, shift right, "\mathbb{P}"'] \arrow[dr, shift left, "\mathbb{P}"] & \\
\Sigma\mathscr{S} \arrow[ur, shift right, "\mathbb{U}"'] \arrow[rr, shift left, "\mathbb{P}"] \arrow[drr, shift left, "\mathbb{P}"] & & \mathscr{I}\mathscr{S} \arrow[ul, shift left, "\mathbb{U}"] \arrow[ll, shift left, "\mathbb{U}"] \arrow[d, shift left, "\mathbb{P}"] \\
\mathscr{F}\mathscr{T} \arrow[rr, shift left, "\mathbb{P}"] & & \mathscr{W}\mathscr{T} \arrow[ll, shift left, "\mathbb{U}"] \arrow[u, shift left, "\mathbb{U}"] \arrow[ull, shift left, "\mathbb{U}"]
\end{tikzcd}$$

We have the dictionary:

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$\mathscr{P}$ is the category of $\mathscr{N}$-spectra, or prespectra.
$\Sigma\mathscr{S}$ is the category of Σ-spectra, or symmetric spectra.
$\mathscr{I}\mathscr{S}$ is the category of $\mathscr{I}$-spectra, or orthogonal spectra.
$\mathscr{F}\mathscr{T}$ is the category of $\mathscr{F}$-spaces, or Γ-spaces.
$\mathscr{W}\mathscr{T}$ is the category of $\mathscr{W}$-spaces.

As we made precise in [MMSS, §8], $\mathscr{N}$ is the category of non-negative integers, Σ is the category of symmetric groups, $\mathscr{I}$ is the category of orthogonal groups, $\mathscr{F}$ is the category of finite based sets, and $\mathscr{W}$ is the category of based spaces homeomorphic to finite CW complexes. We often use $\mathscr{D}$ generically to denote such a domain category for diagram spectra. When $\mathscr{D} = \mathscr{F}$ or $\mathscr{D} = \mathscr{W}$, there is no distinction between $\mathscr{D}$-spaces and $\mathscr{D}$-spectra, $\mathscr{D}\mathscr{T} = \mathscr{D}\mathscr{S}$. The functors $\mathbb{U}$ are forgetful functors, the functors $\mathbb{P}$ are prolongation functors, and in each case $\mathbb{P}$ is left adjoint to $\mathbb{U}$. All of these categories except $\mathscr{P}$ are symmetric monoidal. The functors $\mathbb{U}$ between symmetric monoidal categories are all lax symmetric monoidal, the functors $\mathbb{P}$ between symmetric monoidal categories are all strong symmetric monoidal, and these functors $\mathbb{P}$ and $\mathbb{U}$ restrict to adjoint pairs relating the various categories of rings, commutative rings, and modules over rings.

We give some preliminaries about "compactly generated" topological model categories and their equivalences in Sections 1 and 2. We show in Section 3 that, for any domain category $\mathscr{D}$, the category of $\mathscr{D}$-spaces has a "level model structure" in which the weak equivalences and fibrations are the maps that evaluate to weak equivalences or fibrations at each object of $\mathscr{D}$. This structure has been studied in more detail by Piacenza [23], [19, Ch. VI]. There is a relative variant in which we restrict attention to those objects in some subcategory $\mathscr{C}$ of $\mathscr{D}$.

In preparation for the study of "stable model structures", we recall some homotopical facts about prespectra in Section 4; we use the terms "prespectrum" and "$\mathscr{N}$-spectrum" interchangeably, using the former when we are thinking in classical homotopical terms and using the latter when thinking about the relationship with other categories of diagram spectra.

We define and study "stable weak equivalences" in Section 5, and we give the categories of $\mathscr{N}$-spectra, symmetric spectra, orthogonal spectra, and $\mathscr{W}$-spaces a "stable model structure" in Section 6. The cofibrations are the same as in the level model structure relative to $\mathscr{N}$, and the weak equivalences are the stable weak equivalences. In all cases except that of symmetric spectra, whose homotopy theory is intrinsically more subtle, the stable weak equivalences are just the maps whose underlying maps of $\mathscr{N}$-spectra are π_*-isomorphisms. We give a single self-contained proof of the model axioms that applies to all four of these categories. In the case of symmetric spectra, our model structure recovers that of Hovey, Smith, and Shipley [8]. Although their work inspired and provided a model for ours, our work is logically independent of theirs and makes no use of simplicial techniques.

Turning to comparisons between our various categories of diagram spectra, we prove the following theorem in Section 7.

Theorem 0.1. *The categories of $\mathscr{N}$-spectra, symmetric spectra, orthogonal spectra, and $\mathscr{W}$-spaces are Quillen equivalent.*

In fact, we prove that the categories of $\mathscr{N}$-spectra and orthogonal spectra are Quillen equivalent and that the categories of symmetric spectra, orthogonal spectra, and $\mathscr{W}$-spaces are Quillen equivalent. These comparisons between $\mathscr{N}$-spectra and orthogonal spectra and between symmetric spectra and orthogonal spectra imply that the categories of $\mathscr{N}$-spectra and symmetric spectra are Quillen equivalent. This reproves a result of Hovey, Shipley, and Smith [8, 4.3.2]. The new proof leads to a new perspective on the stable weak equivalences of symmetric spectra.

Corollary 0.2. *A map f of cofibrant symmetric spectra is a stable weak equivalence if and only if $\mathbb{P}f$ is a π_*-isomorphism of orthogonal spectra.*

An analogous characterization of stable weak equivalences in terms of an endofunctor $\mathbb{D}$ on the category of symmetric spectra is given in [29, 3.1.2]. The functor $\mathbb{D}$ was the starting point for a perfectly rigorous but now obsolete approach to the present comparison theorems; see [20].

Of course, the point of introducing categories of diagram spectra is to obtain point-set level models for the classical stable homotopy category that are symmetric monoidal under their smash product. On passage to homotopy categories, the derived smash product must agree with the classical (naive) smash product of prespectra. That is the essential content of the following addendum to Theorem 0.1, which we also prove in Section 7.

Theorem 0.3. *The equivalences of homotopy categories induced by the Quillen equivalences of Theorem 0.1 preserve smash products.*

Here again, we compare $\mathscr{N}$-spectra and symmetric spectra to orthogonal spectra and then deduce the comparison between $\mathscr{N}$-spectra and symmetric spectra; a partial result in this direction was given in [8, 4.3.12].

In Section 8, we prove that the categories of symmetric spectra, orthogonal spectra, and $\mathscr{W}$-spaces satisfy the pushout-product and monoid axioms of [27]. This answers the question of whether or not the monoid axiom holds for (topological) symmetric spectra, which was posed in [8]. Therefore, in these cases, the category of $\mathscr{D}$-ring spectra and the category of modules over a $\mathscr{D}$-ring spectrum inherit model structures from the underlying category of $\mathscr{D}$-spectra. Using these model structures, we obtain the following comparison theorems for categories of diagram ring and module spectra.

Theorem 0.4. *The categories of symmetric ring spectra, orthogonal ring spectra, and $\mathscr{W}$-ring spaces are Quillen equivalent model categories.*

Theorem 0.5. *For a symmetric ring spectrum R, the categories of R-modules and of $\mathbb{P}R$-modules (of orthogonal spectra) are Quillen equivalent model categories. For an orthogonal ring spectrum R, the categories of R-modules and of $\mathbb{P}R$-modules (of $\mathscr{W}$-spaces) are Quillen equivalent model categories.*

We would like the category of commutative $\mathscr{D}$-ring spectra to inherit a model structure from the underlying category of $\mathscr{D}$-spectra, but a familiar argument due to Lewis [9] shows that this fails if the sphere $\mathscr{D}$-spectrum is cofibrant. In the context of symmetric spectra, Jeff Smith explained[1] the mechanism of this failure: if the zeroth term of a symmetric spectrum X is non-trivial, the symmetric powers of X will not behave well homotopically. As Smith saw, it is easy to get around this by replacing the level model structure relative to $\mathscr{N}$ by the relative model structure relative to $\mathscr{N} - \{0\}$. As we show in Section 9, this leads to a "positive stable model structure" that is Quillen equivalent to the stable model structure but has fewer cofibrations. Its cofibrant objects have trivial zeroth terms. All of the

[1] Private communication

results above work equally well starting from the positive stable model structures. Using these model structures, we prove the following theorem in Section 10.

Theorem 0.6. *The categories of commutative symmetric ring spectra and commutative orthogonal ring spectra are Quillen equivalent model categories.*

Smith first proved that the category of commutative symmetric ring spectra of simplicial sets is a model category. Our proof of the model axioms is largely analogous to the proof of the model axioms in the category of commutative S-algebras in [5, VII§§3,5]. We do not know whether or not the category of commutative $\mathscr{W}$-ring spaces is a model category; some of us suspect that it is not.

We return to $\mathscr{W}$-spaces in Section 11. We prove that the category of $\mathscr{W}$-spaces has a second, "absolute", stable model structure that also satisfies the pushout-product and monoid axioms. In the first stable model structure, we start from the level model structure *relative to* $\mathscr{N}$. In the second, we start from the *absolute* level model structure. The weak equivalences in both stable model structures are the π_*-isomorphisms. The cofibrations in the absolute stable structure are the same as those in the absolute level model structure, and there are more of them. A simplicial analogue of the absolute structure has been studied by Lydakis [12].

In the literature [3, 11, 25], $\mathscr{F}$-spaces and $\mathscr{F}$-ring spaces are usually called Gamma spaces (or Γ-spaces) and Gamma-rings. The category of $\mathscr{F}$-spaces has a stable model structure in which the cofibrations are those of the level model structure and the weak equivalences between cofibrant objects are the π_*-isomorphisms. It is studied in [25]; it has fewer cofibrations and more fibrations than the simplicial analogue that was originally studied by Bousfield and Friedlander [3].

Since $\mathscr{F}$-spaces only model connective ($= (-1)$-connected) prespectra and the category of connective $\mathscr{W}$-spaces is not a model category (it fails to have limits), we cannot expect a Quillen equivalence between the categories of $\mathscr{F}$-spaces and connective $\mathscr{W}$-spaces. However, we have nearly that much. We prove the following theorems in Section 12. For the last two, we observe that the pushout-product and monoid axioms for $\mathscr{F}$-spaces follow from these axioms in the absolute stable model structure on $\mathscr{W}$-spaces.

Theorem 0.7. *The functors $\mathbb{P}$ and $\mathbb{U}$ between $\mathscr{F}\mathscr{T}$ and $\mathscr{W}\mathscr{T}$ are a Quillen adjoint pair, and they induce an equivalence between the homotopy categories of $\mathscr{F}$-spaces and of connective $\mathscr{W}$-spaces. Moreover, the equivalence preserves smash products.*

Theorem 0.8. *The functors $\mathbb{P}$ and $\mathbb{U}$ between $\mathscr{F}$-ring spaces and $\mathscr{W}$-ring spaces are a Quillen adjoint pair, and they induce an equivalence between the homotopy categories of $\mathscr{F}$-ring spaces and of connective $\mathscr{W}$-ring spaces.*

Theorem 0.9. *For an $\mathscr{F}$-ring space R, the functors $\mathbb{P}$ and $\mathbb{U}$ between R-modules and $\mathbb{P}R$-modules are a Quillen adjoint pair, and they induce an equivalence between the homotopy categories of R-modules and of connective $\mathbb{P}R$-modules.*

We do not know whether or not the homotopy categories of commutative $\mathscr{F}$-ring spaces and connective commutative $\mathscr{W}$-ring spaces are equivalent. The following remark provides a stopgap for the study of commutativity in these cases.

Remark 0.10. There is a definition of an action of an operad on a $\mathscr{D}$-spectrum. Restricting to an E_∞ operad, this gives the notion of an E_∞-$\mathscr{D}$-ring spectrum. See [20, §5]. It is an easy consequence of results in this paper (especially Lemma 10.5) that the homotopy categories of E_∞ symmetric ring spectra and commutative symmetric ring spectra are equivalent, as was first noted by Smith in the simplicial context, and that the homotopy categories of E_∞ orthogonal ring spectra and commutative orthogonal ring spectra are equivalent. We do not know whether or not the analogues for $\mathscr{W}$-spaces and $\mathscr{F}$-spaces hold, and here the homotopy theory of E_∞-rings seems more tractable than that of commutative rings. It is also an easy consequence of the methods of this paper that the homotopy categories of E_∞ symmetric ring spectra, E_∞ orthogonal ring spectra, and E_∞-$\mathscr{W}$-ring spaces are equivalent and that the homotopy categories of E_∞-$\mathscr{F}$-ring spaces and connective E_∞-$\mathscr{W}$-ring spaces are equivalent.

We review the comparison between diagram categories of spaces and diagram categories of simplicial sets in Section 13. The comparison between $\mathscr{F}$-spaces and $\mathscr{F}$-simplicial sets is used in the proofs of Theorems 0.7–0.9.

A comparison between symmetric spectra and the S-modules of Elmendorf, Kriz, Mandell, and May [5] is given in [26]. Further, it is shown in [14] that the equivalence in [26] is the composite of the Quillen equivalence here between symmetric spectra and orthogonal spectra (in the positive stable model structures) and a Quillen equivalence between orthogonal spectra and S-modules. Moreover, the latter Quillen equivalence also restricts to Quillen equivalences on the various categories of structured ring and module spectra. Thus the category of orthogonal spectra appears intrinsically as an intermediary between the categories of symmetric spectra and of S-modules.

Roughly speaking then, and with some caveats, our comparisons show that all of the known approaches to highly structured ring and module spectra are essentially equivalent. We refer the reader to papers on particular categories [5, 11, 12, 25, 8, 29] and to [20], which gives a largely obsolete summary of our work, for discussions of the history, philosophy, advantages and disadvantages of the various approaches.

1. Preliminaries about topological model categories

As in [MMSS], we understand spaces to be compactly generated and weak Hausdorff. We let $\mathscr{T}$ be the category of based spaces and let $\mathscr{D}$ be a skeletally small (based) topological category. A $\mathscr{D}$-space is a (based) continuous functor $\mathscr{D} \longrightarrow \mathscr{T}$, and we have the topological category $\mathscr{D}\mathscr{T}$ of $\mathscr{D}$-spaces. This category is topologically complete and cocomplete. That is, it has all limits and colimits and is also tensored and cotensored. Colimits and limits of $\mathscr{D}$-spaces are constructed levelwise. The tensor $X \wedge A$ and cotensor $F(A, X)$ of a $\mathscr{D}$-space X and a based space A are given by levelwise smash products and function spaces. Homotopies between maps of $\mathscr{D}$-spaces are defined using the cylinders $X \wedge I_+$.

As explained in [MMSS], diagram spectra are special kinds of diagram spaces. We will first construct model structures on categories of diagram spectra and will then use a general procedure to lift them to model structures on categories of structured diagram spectra. The weak equivalences and fibrations in the lifted model structures will be *created* in the underlying category of diagram spaces. That is, the underlying diagram spectrum functor will preserve and reflect the weak equivalences and fibrations: a map of structured diagram spectra will be a weak equivalence or fibration if and only if its underlying map of diagram spectra is a weak equivalence or fibration. We here describe the kind of model structures that we will encounter and explain the lifting procedure.

While we have the example of diagram spectra in mind, the considerations of this section apply more generally. Thus let $\mathscr{C}$ be any topologically complete and cocomplete category with cotensors denoted $X \wedge A$ and homotopies defined in terms of $X \wedge I_+$. We let $\mathscr{A}$ be a topological category with a continuous and faithful forgetful functor $\mathscr{A} \longrightarrow \mathscr{C}$. We assume that $\mathscr{A}$ is topologically complete and cocomplete. This holds in all of the categories that occur in our work by the following pair of results. The first is [5, VII.2.10] and the second is [5, I.7.2].

Proposition 1.1. *Let $\mathscr{C}$ be a topologically complete and cocomplete category and let $\mathbb{T} : \mathscr{C} \longrightarrow \mathscr{C}$ be a continuous monad that preserves reflexive coequalizers. Then the category $\mathscr{C}[\mathbb{T}]$ of $\mathbb{T}$-algebras is topologically complete and cocomplete, with limits created in $\mathscr{C}$.*

The hypothesis on $\mathbb{T}$ holds trivially when $\mathscr{C}$ is closed symmetric monoidal with product $\wedge$ and $\mathbb{T}$ is the monad $\mathbb{T}X = R \wedge X$ that defines left modules over some monoid R in $\mathscr{C}$, since $\mathbb{T} : \mathscr{C} \longrightarrow \mathscr{C}$ is then a left adjoint. The following analogue is more substantial.

Proposition 1.2. *Let $\mathscr{C}$ be a cocomplete closed symmetric monoidal category. Then the monads that define monoids and commutative monoids in $\mathscr{C}$ preserve reflexive coequalizers.*

As in [5], we write q-cofibration and q-fibration for model cofibrations and fibrations, but we write cofibrant and fibrant rather than q-cofibrant and q-fibrant. We write h-cofibration for a map $i : A \longrightarrow X$ that satisfies the homotopy extension property (HEP) in $\mathscr{C}$. It is equivalent that the (reduced) mapping cylinder $Mi = X \cup_i (A \wedge I_+)$ be a retract of $X \wedge I_+$.

In particular, an h-cofibration of $\mathscr{D}$-spaces is a level h-cofibration and thus a level closed inclusion. For some purposes, we could just as well use level h-cofibrations where we use h-cofibrations, but the stronger condition plays a central role when working in the level model categories of $\mathscr{D}$-spaces and is the most natural condition to verify. The theory of cofibration sequences works in exactly the same way for h-cofibrations of $\mathscr{D}$-spaces as for h-cofibrations of based spaces; we will be more explicit later.

Most work on model categories has been done simplicially rather than topologically. As observed in [5], it is convenient in topological contexts to require some form of "Cofibration Hypothesis". We shall incorporate this in our definition of what it means for $\mathscr{A}$ to be a "compactly generated model category".

Cofibration Hypothesis 1.3. Let I be a set of maps in $\mathscr{A}$. We say that I satisfies the Cofibration Hypothesis if it satisfies the following two conditions.

(i) Let $i : A \longrightarrow B$ be a coproduct of maps in I. In any pushout

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & E \arrow[d, "j"] \\ B \arrow[r] & F \end{tikzcd}$$

in $\mathscr{A}$, the cobase change j is an h-cofibration in $\mathscr{C}$.

(ii) Viewed as an object of $\mathscr{C}$, the colimit of a sequence of maps in $\mathscr{A}$ that are h-cofibrations in $\mathscr{C}$ is their colimit as a sequence of maps in $\mathscr{C}$.

We can use the maps in such a set I as the analogues of (cell, sphere) pairs in the theory of cell complexes, and the following definition and result will imply that q-cofibrations are h-cofibrations in compactly generated model categories.

Definition 1.4. Let I be a set of maps in $\mathscr{A}$. A map $f : X \longrightarrow Y$ is a *relative I-cell complex* if Y is the colimit of a sequence of maps $Y_n \longrightarrow Y_{n+1}$ such that $Y_0 = X$ and $Y_n \longrightarrow Y_{n+1}$ is obtained by cobase change from a coproduct of maps in I.

Lemma 1.5. *Let I satisfy the Cofibration Hypothesis. Then any retract of a relative I-cell complex is an h-cofibration in $\mathscr{C}$.*

Proofs of the model axioms generally depend on one or another version of Quillen's small objects argument [24]. In most of our examples, we can use compactness arguments to avoid the need for set-theoretic elaborations of the original version in terms of sequential colimits.

Definition 1.6. An object X of $\mathscr{A}$ is *compact* if

$$\mathscr{A}(X, Y) \cong \operatorname{colim} \mathscr{A}(X, Y_n)$$

whenever Y is the colimit of a sequence of maps $Y_n \longrightarrow Y_{n+1}$ in $\mathscr{A}$ that are h-cofibrations in $\mathscr{C}$.

Of course, for spaces, we understand compactness in the usual sense. Compact spaces give rise to compact $\mathscr{D}$-spaces via the shift desuspension functors F_d of [MMSS, 2.3]. Explicitly, the evaluation functor Ev_d that sends a $\mathscr{D}$-space X to the space $X(d)$ has the left adjoint $F_d = F_d^{\mathscr{D}}$ specified by $(F_dA)(e) = \mathscr{D}(d, e) \wedge A$; remember that, if $\mathscr{D}$ is given as unbased, we implicitly add a disjoint basepoint to its morphism spaces. The functors F_d will be central to our work. They preserve colimits, smash products with spaces, and h-cofibrations.

Lemma 1.7. *If A is a compact space, then F_dA is a compact $\mathscr{D}$-space. If X is a compact $\mathscr{D}$-space and A is a compact space, then $X \wedge A$ is a compact $\mathscr{D}$-space. If $Y \cup_X Z$ is the pushout of a level closed inclusion $i : X \longrightarrow Y$ and a map $f : X \longrightarrow Z$, where X, Y, and Z are compact $\mathscr{D}$-spaces, then $Y \cup_X Z$ is a compact $\mathscr{D}$-space.*

Abbreviate RLP and LLP for the right lifting property and left lifting property.

Lemma 1.8 (The small objects argument)**.** *Let I be a set of objects of $\mathscr{A}$ such that each map in I has compact domain and I satisfies the Cofibration Hypothesis. Then maps $f : X \longrightarrow Y$ in $\mathscr{A}$ factor functorially as composites*

$$X \xrightarrow{i} X' \xrightarrow{p} Y$$

such that p satisfies the RLP with respect to any map in I and i satisfies the LLP with respect to any map that satisfies the RLP with respect to each map in I. Moreover, $i : X \longrightarrow X'$ is a relative I-cell complex.

Proof. This is standard. Details in a special case are given in [5, VII.5.2]. $\square$

This motivates the following definition.

Definition 1.9. Let $\mathscr{A}$ be a model category. We say that $\mathscr{A}$ is *compactly generated* if there are sets I and J of maps in $\mathscr{A}$ such that the domain of each map in I and each map in J is compact, I and J satisfy the Cofibration Hypothesis, the q-fibrations are the maps that satisfy the RLP with respect to the maps in J and the acyclic q-fibrations are the maps that satisfy the RLP with respect to the maps in I. Note that the maps in I must be q-cofibrations and h-cofibrations and the maps in J must be acyclic q-cofibrations and h-cofibrations. We call the maps in I the *generating q-cofibrations* and the maps in J the *generating acyclic q-cofibrations*.

Remark 1.10. There is a definition in terms of transfinite colimits of what it means for a set of maps to be small relative to a subcategory of $\mathscr{A}$. The more general notion of a *cofibrantly generated* model category $\mathscr{A}$ replaces the compactness condition with the requirement that I be small relative to the q-cofibrations and J be small relative to the acyclic q-cofibrations. See e.g. [6, §12.4] or [7, §2.1]. The Cofibration Hypothesis does not appear in the model theoretic literature, but it is almost always appropriate in topological settings.

Our model categories will all be "topological" in the following sense. For maps $i : A \longrightarrow X$ and $p : E \longrightarrow B$ in $\mathscr{A}$, let

$$\mathscr{A}(i^*, p_*) : \mathscr{A}(X, E) \longrightarrow \mathscr{A}(A, E) \times_{\mathscr{A}(A,B)} \mathscr{A}(X, B) \tag{1.11}$$

be the map of spaces induced by $\mathscr{A}(i, \mathrm{id})$ and $\mathscr{A}(\mathrm{id}, p)$ by passage to pullbacks.

Definition 1.12. A model category $\mathscr{A}$ is *topological* provided that $\mathscr{A}(i^*, p_*)$ is a Serre fibration if i is a q-cofibration and p is a q-fibration and is acyclic if, in addition, either i or p is acyclic.

The following observation helps explain the significance of the map $\mathscr{A}(i^*, p_*)$.

Lemma 1.13. *The pair (i, p) has the lifting property if and only if $\mathscr{A}(i^*, p_*)$ is surjective.*

The following result on lifting model structures is immediate by inspection of the proofs in [5, VII§5] or by combination of our version of the small objects argument with the proof of [27, A.3]. Of course, viewed as a functor $\mathscr{C} \longrightarrow \mathscr{C}[\mathbb{T}]$, a monad $\mathbb{T}$ is the free functor left adjoint to the forgetful functor $\mathscr{C}[\mathbb{T}] \longrightarrow \mathscr{C}$. This implies that $\mathbb{T}$ preserves compact objects.

Proposition 1.14. *Let $\mathscr{C}$ be a topologically complete and cocomplete category and let $\mathbb{T} : \mathscr{C} \longrightarrow \mathscr{C}$ be a continuous monad that preserves reflexive coequalizers. Assume that $\mathscr{C}$ is a compactly generated topological model category with generating sets I of cofibrations and J of acyclic cofibrations. Then $\mathscr{C}[\mathbb{T}]$ is a compactly generated topological model category with weak equivalences and fibrations created in $\mathscr{C}$ and generating sets $\mathbb{T}I$ of cofibrations and $\mathbb{T}J$ of acyclic cofibrations provided that*

(i) *$\mathbb{T}I$ and $\mathbb{T}J$ satisfy the Cofibration Hypothesis and*
(ii) *every relative $\mathbb{T}J$-cell complex is a weak equivalence.*

We will need two pairs of analogues of the maps $\mathscr{A}(i^*, p_*)$. For a map $i : A \longrightarrow B$ of based spaces and a map $j : X \longrightarrow Y$ in $\mathscr{A}$, passage to pushouts gives a map

$$i \square j : (A \wedge Y) \cup_{A \wedge X} (B \wedge X) \longrightarrow B \wedge Y \tag{1.15}$$

and passage to pullbacks gives a map

$$F_{\square}(i, j) : F(B, X) \longrightarrow F(A, X) \times_{F(A,Y)} F(B, Y), \tag{1.16}$$

where $\wedge$ and F denote the tensor and cotensor in $\mathscr{A}$.

Inspection of definitions gives adjunctions relating (1.11), (1.15) and (1.16). Formally, these imply that the category of maps in $\mathscr{A}$ is tensored and cotensored over the category of maps in $\mathscr{T}$.

Lemma 1.17. *let $i : A \longrightarrow B$ be a map of based spaces and let $j : X \longrightarrow Y$ and $p : E \longrightarrow F$ be maps in $\mathscr{A}$. Then there are natural isomorphisms of maps*

$$\mathscr{A}((i\Box j)^*, p_*) \cong \mathscr{T}(i^*, \mathscr{A}(j^*, p_*)_*) \cong \mathscr{A}(j^*, F_\Box(i, p)).$$

Therefore $(i\Box j, p)$ has the lifting property in $\mathscr{A}$ if and only if $(i, \mathscr{A}(j^*, p_*))$ has the lifting property in $\mathscr{T}$.

Now assume that $\mathscr{A}$ is a closed symmetric monoidal category with product $\wedge_{\mathscr{A}}$ and internal function objects $F_{\mathscr{A}}$. For maps $i : X \longrightarrow Y$ and $j : W \longrightarrow Z$ in $\mathscr{A}$, passage to pushouts gives a map

$$i\Box j : (Y \wedge_{\mathscr{A}} W) \cup_{X\wedge_{\mathscr{A}} W} (X \wedge_{\mathscr{A}} Z) \longrightarrow Y \wedge_{\mathscr{A}} Z, \tag{1.18}$$

and passage to pullbacks gives a map

$$F_\Box(i, j) : F_{\mathscr{A}}(Y, W) \longrightarrow F_{\mathscr{A}}(X, W) \times_{F_{\mathscr{A}}(X,Z)} F_{\mathscr{A}}(Y, Z). \tag{1.19}$$

For maps i, j, and k in $\mathscr{A}$, these are related by a natural isomorphism of maps

$$\mathscr{A}((i\Box j)^*, k_*) \cong \mathscr{A}(i^*, F_\Box(j, k)_*). \tag{1.20}$$

2. Preliminaries about equivalences of model categories

We need some basic general facts about adjoint functors and adjoint equivalences between model categories.

Definition 2.1. Let $\mathbb{P} : \mathscr{A} \longrightarrow \mathscr{B}$ and $\mathbb{U} : \mathscr{B} \longrightarrow \mathscr{A}$ be left and right adjoints between model categories $\mathscr{A}$ and $\mathscr{B}$. The functors $\mathbb{P}$ and $\mathbb{U}$ are a *Quillen adjoint pair* if $\mathbb{U}$ preserves q-fibrations and acyclic q-fibrations or, equivalently, if $\mathbb{P}$ preserves q-cofibrations and acyclic q-cofibrations. A Quillen adjoint pair is a *Quillen equivalence* if, for all cofibrant $A \in \mathscr{A}$ and all fibrant $B \in \mathscr{B}$, a map $\mathbb{P}A \longrightarrow B$ is a weak equivalence if and only if its adjoint $A \longrightarrow \mathbb{U}B$ is a weak equivalence.

Quillen [24] proved part of the following result. Its second statement, which is folklore, shows that the question of whether or not a Quillen adjoint pair is a Quillen equivalence is a purely homotopical one, rather than a model theoretic one. Often the functor $\mathbb{U} : \mathscr{B} \longrightarrow \mathscr{A}$ creates the weak equivalences and q-fibrations in $\mathscr{B}$. The third statement then gives a convenient criterion for $(\mathbb{P}, \mathbb{U})$ to be a Quillen equivalence.

Lemma 2.2. *Let $\mathbb{P} : \mathscr{A} \longrightarrow \mathscr{B}$ and $\mathbb{U} : \mathscr{B} \longrightarrow \mathscr{A}$ be a Quillen adjoint pair.*

(i) *The total derived functors*

$$\mathbb{LP} : Ho(\mathscr{A}) \longrightarrow Ho(\mathscr{B}) \quad \textit{and} \quad \mathbb{RU} : Ho(\mathscr{B}) \longrightarrow Ho(\mathscr{A})$$

exist and are adjoint.

(ii) $(\mathbb{P}, \mathbb{U})$ *is a Quillen equivalence if and only if* $\mathbb{RU}$ *or, equivalently,* $\mathbb{LP}$ *is an equivalence of categories.*

(iii) *If $\mathbb{U}$ creates the weak equivalences of $\mathscr{B}$ and $\eta : A \longrightarrow \mathbb{UP}A$ is a weak equivalence for all cofibrant objects A, then $(\mathbb{P}, \mathbb{U})$ is a Quillen equivalence.*

Proof. Quillen proved (i) and the forward implication of (ii). By [13, p. 91], $\mathbb{RU}$ is an equivalence of categories if and only if $\mathbb{LP}$ is an equivalence of categories, and then the unit and counit of the adjunction are isomorphisms. Let A be a cofibrant object of $\mathscr{A}$ and B be a fibrant object of $\mathscr{B}$ and let $f : \mathbb{P}A \longrightarrow B$ be any map. Factor f as the composite of an acyclic q-cofibration $\iota : \mathbb{P}A \longrightarrow C$ and a q-fibration $p : C \longrightarrow B$. Obviously f is a weak equivalence if and only if p is a weak equivalence, and then $\mathbb{U}p$ is a weak equivalence. The adjoint $\tilde{f}: A \longrightarrow \mathbb{U}B$ of f is the composite of $\mathbb{U}p$ and the adjoint $\tilde{\imath}: A \longrightarrow \mathbb{U}C$. Since A is cofibrant and since C and thus $\mathbb{U}C$ are fibrant, the unit map $\eta: A \longrightarrow \mathbb{RULP}A$ in the derived adjunction is represented (up to isomorphism) by $\tilde{\imath}$, which is thus a weak equivalence if $\mathbb{RU}$ is an equivalence of categories. This proves that if $\mathbb{RU}$ is an equivalence of categories and f is a weak equivalence, then $\tilde{f}$ is a weak equivalence, and the dual argument proves that if $\mathbb{LP}$ is an equivalence of categories and $\tilde{f}$ is a weak equivalence, then f is a weak equivalence. For (iii), note that $\tilde{f}$ is the composite of $\eta: A \longrightarrow \mathbb{UP}A$ and $\mathbb{U}f$. If $\mathbb{U}$ creates the weak equivalences of A and η is a weak equivalence, then f is a weak equivalence if and only if $\tilde{f}$ is a weak equivalence. □

The following observation [7, 4.3.3] will be used in the proof of Theorem 0.3.

Lemma 2.3. *Let $\mathbb{P}: \mathscr{A} \longrightarrow \mathscr{B}$ and $\mathbb{U}: \mathscr{B} \longrightarrow \mathscr{A}$ be a Quillen equivalence, where $\mathbb{P}$ is a strong monoidal functor between monoidal categories (under products $\wedge$). Then the given natural isomorphism $\mathbb{P}X \wedge \mathbb{P}Y \longrightarrow \mathbb{P}(X \wedge Y)$ induces a natural isomorphism*

$$\mathbb{LP}X \wedge^{\mathbb{L}} \mathbb{LP}Y \longrightarrow \mathbb{LP}(X \wedge^{\mathbb{L}} Y).$$

3. The level model structure on $\mathscr{D}$-spaces

We give the category of $\mathscr{D}$-spaces a "level model structure". We shall be brief, since this material is well-known. An exposition that makes clear just how close this theory is to CW-theory in the category of spaces has been given by Piacenza [23], [19, Ch. VI].

If $\mathscr{D}$ is symmetric monoidal and R is a $\mathscr{D}$-ring space (= monoid in $\mathscr{D}\mathscr{T}$), then the category $\mathscr{D}\mathscr{S}_R$ of right R-modules (= $\mathscr{D}$-spectra over R) is isomorphic to the category of $\mathscr{D}_R$-spaces [MMSS, §6]. Therefore the category $\mathscr{D}\mathscr{S}_R$ obtains a level model structure by specialization of the following general construction. Recall that $\mathscr{D}_R$ has the same objects as $\mathscr{D}$.

Definition 3.1. We define five properties of maps $f: X \longrightarrow Y$ of $\mathscr{D}$-spaces.

(i) f is a *level weak equivalence* if each $f(d): X(d) \longrightarrow Y(d)$ is a weak equivalence.
(ii) f is a *level fibration* if each $f(d): X(d) \longrightarrow Y(d)$ is a Serre fibration.
(iii) f is a *level acyclic fibration* if it is both a level weak equivalence and a level fibration.
(iv) f is a *q-cofibration* if it satisfies the LLP with respect to the level acyclic fibrations.
(v) f is a *level acyclic q-cofibration* if it is both a level weak equivalence and a q-cofibration.

Definition 3.2. Let I be the set of h-cofibrations $S^{n-1}_+ \longrightarrow D^n_+$, $n \geq 0$ (where S^{-1} is empty). Let J be the set of h-cofibrations $i_0: D^n_+ \longrightarrow (D^n \times I)_+$ and observe that each such map is the inclusion of a deformation retract. Define FI to be the set of all maps $F_d i$ with d in a skeleton of $\mathscr{D}$ and $i \in I$. Define FJ to be the set of all maps $F_d j$ with d in a skeleton of $\mathscr{D}$ and $j \in J$, and observe that each map in FJ is the inclusion of a deformation retract.

We recall the following result of Quillen [24, II§3]; see also [7, Ch.2§2.4].

Proposition 3.3. *$\mathscr{T}$ is a compactly generated proper topological model category with respect to the weak equivalences, Serre fibrations, and retracts of relative I-cell complexes. The sets I and J are the generating q-cofibrations and the generating acyclic q-cofibrations.*

Of course, there is a notion of a level q-cofibration, defined as in Definition 3.1(ii), but we shall make no use of it. Observe that every space is fibrant. To ensure that weak equivalences behave well with respect to standard constructions, we must restrict to spaces with nondegenerate basepoints, meaning that the inclusion of the basepoint is an unbased h-cofibration. Recall that a based h-cofibration between nondegenerately based spaces is an unbased h-cofibration (satisfies the HEP in unbased spaces) [30, Prop. 9].

Definition 3.4. The category $\mathscr{D}$ is *nondegenerately based* if each of its morphism spaces is nondegenerately based. For any $\mathscr{D}$, a $\mathscr{D}$-space X is *nondegenerately based* if each $X(d)$ is nondegenerately based.

All of the categories $\mathscr{D}$ that we consider are nondegenerately based.

Theorem 3.5. *The category of $\mathscr{D}$-spaces is a compactly generated topological model category with respect to the level weak equivalences, level fibrations, and q-cofibrations. It is right proper, and it is left proper if $\mathscr{D}$ is nondegenerately based. The sets FI and FJ are the generating q-cofibrations and generating acyclic q-cofibrations, and the following identifications hold.*

(i) *The level fibrations are the maps that satisfy the RLP with respect to FJ or, equivalently, with respect to retracts of relative FJ-cell complexes, and all $\mathscr{D}$-spaces are level fibrant.*
(ii) *The level acyclic fibrations are the maps that satisfy the RLP with respect to FI or, equivalently, with respect to retracts of relative FI-cell complexes.*
(iii) *The q-cofibrations are the retracts of relative FI-cell complexes.*
(iv) *The level acyclic q-cofibrations are the retracts of relative FJ-cell complexes.*

Proof. The only model axioms that are not obvious from the definitions are the lifting property that is not given by the definition of a q-cofibration and the two factorization properties. Since the generating q-cofibrations and acyclic q-cofibrations are all h-cofibrations between compact $\mathscr{D}$-spaces, we may apply the small objects argument to both FI and FJ. Applying it to FJ, we obtain a factorization of any map as the composite of a relative FJ-cell complex i and a map p that satisfies the RLP with respect to FJ; moreover, i satisfies the LLP with respect to each map that satisfies the RLP with respect to FJ. The map i is a level weak equivalence since it is the inclusion of a deformation retract. Applying the small objects argument to FI, we obtain a factorization of any map as the composite of a relative FI-cell complex i and a map p that satisfies the RLP with respect to FI; moreover, i satisfies the LLP with respect to each map that satisfies the RLP with respect to FI. Now statements (i) through (iv) are proven by adjunction from their space level analogues, together with standard and elementary formal arguments. This gives the model axioms.

To show that $\mathscr{D}\mathscr{T}$ is topological, we must show that if $i : A \longrightarrow X$ is a q-cofibration and $p : E \longrightarrow B$ is a level fibration, then the map $\mathscr{D}\mathscr{T}(i^*, p_*)$ of (1.11) is a Serre fibration which is a weak equivalence if i or p is a level weak equivalence. As in [24, SM7(a), p. II.2.3], this reduces to showing that $\mathscr{D}\mathscr{T}(i^*, p_*)$ is a Serre fibration when i is in FI and an acyclic Serre fibration when i is in FJ. By adjunction, these conclusions follow from their space level analogues.

To show that $\mathscr{D}\mathscr{T}$ is right proper, we must show that the pullback of a level weak equivalence along a level fibration is a level weak equivalence, and this is immediate from its space level analogue.

To show that $\mathscr{D}\mathscr{T}$ is left proper, we must show that the pushout of a level weak equivalence along a q-cofibration is a level weak equivalence. The functors $F_d : \mathscr{T} \longrightarrow \mathscr{D}\mathscr{T}$ preserve h-cofibrations. Since $\mathscr{D}$ is nondegenerately based, F_dA is nondegenerately based for any based CW complex A. Moreover, wedges of nondegenerately based spaces are nondegenerately based. Thus a relative FI-cell complex $i : X \longrightarrow Y$ is obtained by passage to pushouts and sequential colimits from based maps that are unbased h-cofibrations. Although X need not be nondegenerately based, i is a level unbased h-cofibration since pushouts and sequential colimits of unbased h-cofibrations are unbased h-cofibrations. Therefore any q-cofibration is a level unbased h-cofibration. Now the conclusion follows from the space level analogue that the pushout of a weak equivalence along an unbased h-cofibration is a weak equivalence. □

The last step of the proof has the following consequence.

Lemma 3.6. *If $\mathscr{D}$ is nondegenerately based, then any cofibrant $\mathscr{D}$-space is nondegenerately based.*

Now assume for a moment that $\mathscr{D}$ and therefore $\mathscr{D}\mathscr{T}$ are symmetric monoidal categories. We then have the following observation about the maps $i\Box j$ of (1.18).

Lemma 3.7. *If i and j are q-cofibrations, then $i\Box j$ is a q-cofibration which is level acyclic if either i or j is level acyclic. In particular, if Y is cofibrant, then $i \wedge \mathrm{id} : A \wedge Y \longrightarrow X \wedge Y$ is a q-cofibration, and the smash product of cofibrant $\mathscr{D}$-spaces is cofibrant.*

Proof. By [MMSS, 3.8], $F_dA \wedge F_eB \cong F_{d\Box e}(A \wedge B)$ for objects d and e of $\mathscr{D}$ and based spaces A and B. Therefore, writing $i_n : S^{n-1}_+ \longrightarrow D^n_+$, it follows that

$$F_d i_m \Box F_e i_n \cong F_{d\Box e}(i_{m+n}).$$

From here, an easy formal argument using the adjunction (1.20) and the defining lifting property of q-cofibrations gives that $i\Box j$ is a q-cofibration; see [8, 5.3.4]. The acyclicity in the first statement follows by adjointness arguments from the fact that $\mathscr{D}\mathscr{T}$ is topological; compare [24, p. II.2.3] or [7]. □

Remark 3.8. The monoid axiom of [27] would require that any map obtained by cobase change and composition from maps of the form $i \wedge Y$, where i is a level acyclic q-cofibration and Y is arbitrary, be a level weak equivalence. Without nondegenerate basepoint hypotheses, this fails in general. Nevertheless, we shall later prove the monoid axiom for some of our stable model structures.

The level homotopy category $\mathrm{Ho}_\ell\mathscr{D}\mathscr{T}$ behaves exactly like its analogue for based spaces. Let $[X, Y]$ denote the set of maps $X \longrightarrow Y$ in $\mathrm{Ho}_\ell\mathscr{D}\mathscr{T}$ and let $\pi(X, Y)$ denote the set of homotopy classes of maps $X \longrightarrow Y$, where homotopies are defined in terms of cylinders $X \wedge I_+$. Cofibrant $\mathscr{D}$-spaces behave just like CW complexes in $\mathscr{T}$. In fact, giving a cell $F_dD^n_+$ dimension n, we can define an FI-CW complex to be an FI-cell complex whose cells are attached only to cells of lower dimension, and we then have the cellular approximation theorem in its usual form. Moreover, any cofibrant $\mathscr{D}$-space is homotopy equivalent to an FI-CW complex. Whitehead's theorem holds: a level weak equivalence between cofibrant $\mathscr{D}$-spaces is a homotopy equivalence. See Piacenza [23]. Of course $[X, Y] = \pi(X, Y)$ if X is cofibrant. In general, $[X, Y] \cong \pi(\Gamma X, Y)$, where $\Gamma X \longrightarrow X$ is a cofibrant approximation of X.

Similarly, fiber and cofiber sequences of $\mathscr{D}$-spaces behave the same way as for based spaces, starting from the usual definitions of homotopy cofibers and fibers.

Definition 3.9. Let $f : X \longrightarrow Y$ be a map of $\mathscr{D}$-spaces. The *homotopy cofiber* $Cf = Y \cup_f CX$ of f is the pushout along f of the cone h-cofibration $i : X \longrightarrow CX$; here $CX = X \wedge I$, where I has basepoint 1. The homotopy fiber $Ff = X \times_f PY$ of f is the pullback along f of the path fibration $p : PY \longrightarrow Y$; here $PY = F(I, Y)$, where I has basepoint 0. Equivalently, these are the levelwise homotopy cofiber and fiber of f.

We shall indicate the proofs of the following (duplicative) lemmas together, at the end of the section. They are elementary precursors of more sophisticated analogues that will play an important role in our work. We assume that $\mathscr{D}$ is nondegenerately based.

Lemma 3.10. *The suspension of a level weak equivalence of nondegenerately based $\mathscr{D}$-spaces is a level weak equivalence. For a nondegenerately based X and any Y, ΣX is nondegenerately based and*

$$[\Sigma X, Y] \cong [X, \Omega Y].$$

Lemma 3.11. *A wedge of level weak equivalences of nondegenerately based $\mathscr{D}$-spaces is a level weak equivalence. For nondegenerately based X_i and any Y, $\bigvee_i X_i$ is nondegenerately based and*

$$[\bigvee_i X_i, Y] \cong \prod_i [X_i, Y].$$

Lemma 3.12. *If $i : A \longrightarrow X$ is an h-cofibration and level weak equivalence of $\mathscr{D}$-spaces and $f : A \longrightarrow Y$ is a map of $\mathscr{D}$-spaces, where A, X, and Y are nondegenerately based, then $X \cup_A Y$ is nondegenerately based and the cobase change $j : Y \longrightarrow X \cup_A Y$ is an h-cofibration and level weak equivalence.*

Lemma 3.13. *If i and i' are h-cofibrations and the vertical arrows are level weak equivalences in the following commutative diagram of nondegenerately based $\mathscr{D}$-spaces, then the induced map of pushouts is a level weak equivalence.*

$$\begin{tikzcd}
X \arrow[d] & A \arrow[l, "i"'] \arrow[r] \arrow[d] & Y \arrow[d] \\
X' & A' \arrow[l, "i'"] \arrow[r] & Y'
\end{tikzcd}$$

Lemma 3.14. *Let X be the colimit of a sequence of level inclusions $i_n : X_n \longrightarrow X_{n+1}$. If each i_n is a level weak equivalence, then the map from the initial term X_0 into X is a level weak equivalence. If the X_n are nondegenerately based, then X is nondegenerately based and, for any Y, there is a $\lim^1$ exact sequence of pointed sets*

$$0 \longrightarrow \lim{}^1[\Sigma X_n, Y] \longrightarrow [X, Y] \longrightarrow \lim[X_n, Y] \longrightarrow 0.$$

Lemma 3.15. *If $f : X \longrightarrow Y$ is a level weak equivalence of nondegenerately based $\mathscr{D}$-spaces and A is a based CW complex, then $f \wedge \mathrm{id} : X \wedge A \longrightarrow Y \wedge A$ is a level weak equivalence. For a nondegenerately based X and any Y,*

$$[X \wedge A, Y] \cong [X, F(A, Y)].$$

Lemma 3.16. *Let $f : X \longrightarrow Y$ be a map of nondegenerately based $\mathscr{D}$-spaces. Then Cf is nondegenerately based and, for any Z, there is a natural long exact sequence*

$$\cdots \longrightarrow [\Sigma^{n+1}X, Z] \longrightarrow [\Sigma^n Cf, Z] \longrightarrow [\Sigma^n Y, Z] \longrightarrow [\Sigma^n X, Z] \longrightarrow \cdots \longrightarrow [X, Z].$$

Proofs. The statements about level weak equivalences are immediate from their analogues for weak equivalences of based spaces. Using Lemma 3.6, the statements about $[-, Y]$ follow by first passing to cofibrant approximations and then applying the analogue with $[-, Y]$ replaced by $\pi(-, Y)$. The latter results are proven exactly as on the space level. For example, by the naturality of the classical space level argument, cofiber sequences give rise to long exact sequences upon application of the functor $\pi(-, Y)$. The essential point is that if $i : A \longrightarrow X$ is an h-cofibration, then the canonical map $Ci \longrightarrow X/A$ is a homotopy equivalence. Again, if X is the colimit of a sequence of level inclusions $X_n \longrightarrow X_{n+1}$, then the telescope of the X_n maps by a weak equivalence to the colimit of the X_n, and there results a $\lim^1$ exact sequence for the computation of $\pi(X, Y)$. □

We shall need several relative variants of the absolute level model structure that we have been discussing.

Variant 3.17. Let $\mathscr{C}$ be a subcategory of $\mathscr{D}$. We define the *level model structure relative to* $\mathscr{C}$ on the category of $\mathscr{D}$-spaces by restricting attention to those levels in $\mathscr{C}$. That is, we define the level weak equivalences and level fibrations relative to $\mathscr{C}$ to be those maps of $\mathscr{D}$-spaces that are level weak equivalences or level fibrations when regarded as maps of $\mathscr{C}$-spaces. We restrict to maps $F_c(-)$ with $c \in \mathscr{C}$ when defining the generating q-cofibrations and generating acyclic q-cofibrations. The proofs of the model axioms and of all other results in this section go through equally well in the relative context. Clearly, when $\mathscr{C}$ contains all objects of a skeleton of $\mathscr{D}$, the relative level model structure coincides with the absolute level model structure.

4. Preliminaries about π_*-isomorphisms of prespectra

Recall that we are using the terms prespectrum and $\mathscr{N}$-spectrum interchangeably; compare [MMSS, 8.1]. We are following [5, 10] (and earlier sources) in calling a sequence of spaces X_n and maps $\sigma : \Sigma X_n \longrightarrow X_{n+1}$ a "prespectrum", reserving the term "spectrum" for prespectra whose adjoint structure maps $\tilde{\sigma} : X_n \longrightarrow \Omega X_{n+1}$ are homeomorphisms. However, we shall make no use of such spectra in this paper. In fact, the following remark shows that, in a sense, the theory of such spectra is disjoint from the present theory of diagram spectra.

Remark 4.1. If the underlying prespectrum of a symmetric spectrum X is a spectrum, then X is trivial, and similarly for orthogonal spectra and $\mathscr{W}$-spaces. Indeed, the iterated adjoint structure map $X(\mathbf{n}) \longrightarrow \Omega^2 X(\mathbf{n}+\mathbf{2})$ takes image in the subspace of points fixed under the conjugation action of Σ_2, where Σ_2 acts on S^2 by permuting coordinates and acts on $X(\mathbf{n}+\mathbf{2})$ through the embedding of Σ_2 in Σ_{n+2} as the subgroup fixing the first n coordinates. This is a proper subspace unless $\Omega^2 X(\mathbf{n}+\mathbf{2})$ is a point.

We bring the abstract theory down to earth with a brief discussion of CW prespectra. In fact, we now have two notions of such an object in sight. Classically, a *CW prespectrum* is a sequence of based CW complexes X_n and cellular inclusions (= inclusions of subcomplexes) $\Sigma X_n \longrightarrow X_{n+1}$. We define a *CW $\mathscr{N}$-spectrum* to be an FI-cell complex whose cells are attached only to cells of lower dimension, where we here redefine the dimension of a cell $F_n D^m_+$ to be $m-n$. Of course, a CW $\mathscr{N}$-spectrum is cofibrant. The following description of $\mathscr{N}$-spectra, which is implied by [MMSS, 2.6], makes it easy to relate these two notions. Recall from [MMSS, 8.1] that, for a based space A, $(F_nA)_q = A \wedge S^{q-n}$, where $S^m = *$ if $m < 0$.

Definition 4.2. For a based space A, define $\lambda_n : F_{n+1}\Sigma A \longrightarrow F_n A$ to be adjoint to the identity map $\Sigma A \longrightarrow (F_nA)_{n+1}$. For an $\mathscr{N}$-spectrum X, let $X\langle n\rangle$ be the evident $\mathscr{N}$-spectrum such that

$$X\langle n\rangle_q = \begin{cases} X_q & \text{if } q \le n \\ \Sigma^{q-n} X_n & \text{if } q > n \end{cases}$$

and observe that $X\langle 0\rangle = F_0 X_0$.

Lemma 4.3. *An $\mathscr{N}$-spectrum X is isomorphic to the colimit of the right vertical arrows in the inductively constructed pushout diagrams*

(4.4)
$$\begin{tikzcd} F_{n+1}\Sigma X_n \arrow[r, "\lambda_n"] \arrow[d, "F_{n+1}\sigma_n"'] & F_n X_n \arrow[r] & X\langle n\rangle \arrow[d] \\ F_{n+1}X_{n+1} \arrow[rr] & & X\langle n+1\rangle. \end{tikzcd}$$

Lemma 4.5. *A CW prespectrum X is a CW $\mathscr{N}$-spectrum and is thus cofibrant.*

Proof. For a CW complex A, F_nA is easily checked to be a CW $\mathscr{N}$-spectrum, naturally in cellular maps of A; moreover, $\lambda_n : F_{n+1}\Sigma A \longrightarrow F_nA$ is cellular. Just as for spaces, a base change of a cellular inclusion of CW $\mathscr{N}$-spectra along a cellular map is a cellular inclusion of CW $\mathscr{N}$-spectra, and a colimit of cellular inclusions of CW $\mathscr{N}$-spectra is a CW $\mathscr{N}$-spectrum. □

We will need an up to homotopy analogue.

Lemma 4.6. *Let X be a prespectrum such that each X_n has the homotopy type of a CW complex and each $\sigma_n : \Sigma X_n \longrightarrow X_{n+1}$ is an h-cofibration. Then X has the homotopy type of a CW prespectrum and thus of a CW $\mathscr{N}$-spectrum. Conversely, if X is a cofibrant $\mathscr{N}$-spectrum, then the X_n have the homotopy types of CW complexes and the σ_n are h-cofibrations.*

Proof. The first statement is classical, but we give a proof in our context. Since the maps $F_{n+1}\sigma_n$ in (4.4) are h-cofibrations, so are the right vertical arrows in (4.4). Therefore the colimit X is homotopy equivalent to the corresponding telescope. We can construct based CW complexes Y_n, homotopy equivalences $f_n : Y_n \longrightarrow X_n$, and cellular inclusions $\tau_n : \Sigma Y_n \longrightarrow Y_{n+1}$ such that $\sigma_n\Sigma \circ f_n \simeq f_{n+1}\circ \tau_n$. Then $Y \cong \operatorname{colim} Y\langle n\rangle$ is a CW prespectrum, and $Y \simeq \operatorname{tel} Y\langle n\rangle \simeq \operatorname{tel} X\langle n\rangle$. The second statement is a direct levelwise inspection of definitions when X is an FI-cell $\mathscr{N}$-spectrum, and the general case follows. □

We think in classical terms in the rest of this section.

Definition 4.7. The *homotopy groups* of a prespectrum X are defined by

$$\pi_q(X) = \mathrm{colim}\ \pi_{q+n}(X_n).$$

A map of prespectra is called a π_**-isomorphism* if it induces an isomorphism on homotopy groups. A prespectrum X is an Ω*-spectrum* (more logically, Ω-prespectrum) if its adjoint structure maps $\tilde{\sigma} : X_n \longrightarrow \Omega X_{n+1}$ are weak equivalences.

The previous section gives $\mathscr{P}$ a level model structure. We here record some results about π_*-isomorphisms that we shall need in our study of stable model structures on $\mathscr{P}$ and our other categories of diagram spectra. The following observation is trivial, but important.

Lemma 4.8. *A level weak equivalence of prespectra is a π_*-isomorphism. A π_*-isomorphism between Ω-spectra is a level weak equivalence.*

We shall shortly give elementary proofs of the following lemmas. It should be observed that these results are significantly stronger technically than their analogues in the previous section in that no hypotheses about nondegenerate basepoints are required. There is no contradiction since the suspension prespectrum functor does not convert weak equivalences of spaces to π_*-isomorphisms of prespectra in general.

Lemma 4.9. (i) *A map of prespectra is a π_*-isomorphism if and only if its suspension is a π_*-isomorphism.*

(ii) *For all prespectra X, the natural map $\eta : X \longrightarrow \Omega\Sigma X$ is a π_*-isomorphism.*

Lemma 4.10. *The homotopy groups of a wedge of prespectra are the direct sums of the homotopy groups of the wedge summands, hence a wedge of π_*-isomorphisms of prespectra is a π_*-isomorphism.*

Lemma 4.11. *If $i : A \longrightarrow X$ is an h-cofibration and π_*-isomorphism of prespectra and $f : A \longrightarrow Y$ is a map of prespectra, then the cobase change $j : Y \longrightarrow X \cup_A Y$ is an h-cofibration and π_*-isomorphism.*

Lemma 4.12. *If i and i' are h-cofibrations and the vertical arrows are π_*-isomorphisms in the following commutative diagram of prespectra, then the induced map of pushouts is a π_*-isomorphism.*

$$\begin{tikzcd} X \arrow[d] & A \arrow[l, "i"'] \arrow[r] \arrow[d] & Y \arrow[d] \\ X' & A' \arrow[l, "i'"] \arrow[r] & Y' \end{tikzcd}$$

Lemma 4.13. *If X is the colimit of a sequence of h-cofibrations $X_n \longrightarrow X_{n+1}$, each of which is a π_*-isomorphism, then the map from the initial term X_0 into X is a π_*-isomorphism.*

Lemma 4.14. *If $f : X \longrightarrow Y$ is a π_*-isomorphism of prespectra and A is a based CW complex, then $f \wedge \mathrm{id} : X \wedge A \longrightarrow Y \wedge A$ is a π_*-isomorphism.*

Lemma 4.15. *Let $f : X \longrightarrow Y$ be a map of prespectra.*

(i) *There is a natural long exact sequence*

$$\cdots \longrightarrow \pi_q(Ff) \longrightarrow \pi_q(X) \longrightarrow \pi_q(Y) \longrightarrow \pi_{q-1}(Ff) \longrightarrow \cdots .$$

(ii) *There is a natural long exact sequence*

$$\cdots \longrightarrow \pi_q(X) \longrightarrow \pi_q(Y) \longrightarrow \pi_q(Cf) \longrightarrow \pi_{q-1}(X) \longrightarrow \cdots .$$

(iii) *The natural map $\eta : Ff \longrightarrow \Omega Cf$ is a π_*-isomorphism.*

Proofs. Lemma 4.9 is clear since a quick inspection of colimits shows that $\pi_q(X)$ is naturally isomorphic to $\pi_{q+1}(\Sigma X)$, with the isomorphism realized by η_*. Lemma 4.13 is also clear. For (i) of Lemma 4.15, passage to colimits from the level long exact sequences of homotopy groups gives the conclusion. For (ii), we see from Lemma 4.9 that it suffices to prove the exactness of $\pi_n(X) \longrightarrow \pi_n(Y) \longrightarrow \pi_n(Cf)$. Clearly this composite is zero. Let $y \in \mathrm{Ker}(\pi_n(Y) \longrightarrow \pi_n(Cf))$, and also write $y : S^{m+n} \longrightarrow Y_m$ for a map that represents this element. Taking m large enough, we can find a null homotopy $z : CS^{m+n} \longrightarrow Cf$. Consider the following diagram:

$$\begin{tikzcd}
S^{m+n} \arrow[r, "\mathrm{id}"] & S^{m+n} \arrow[r, "i"] \arrow[d, "y"] & CS^{m+n} \arrow[r, "j"] \arrow[d, "z"] & Ci \arrow[r, "\pi"] \arrow[d, "w"] & S^{m+n+1} \arrow[r, "\mathrm{id}"] \arrow[d, "x"] & S^{m+n+1} \arrow[d, "\Sigma y"] \\
X_m \arrow[r, "f"] & Y_m \arrow[r, "i'"] & (Cf)_m \arrow[r, "j'"] & Ci' \arrow[r, "\rho"] & \Sigma X_m \arrow[r, "-\Sigma f"] & \Sigma Y_m
\end{tikzcd}$$

Here Ci is a copy of S^{m+n+1} viewed as the double cone $CS^{m+n} \cup_{\mathrm{id}} CS^{m+n}$, and π is the homotopy equivalence given by collapsing the second cone to a point. Similarly, Ci' is the double cone $(Cf)_m \cup_i CY_m$ and ρ is the homotopy equivalence given by collapsing CY_m to a point. Since $zi = i'y$, we obtain a map $w : Ci \longrightarrow Ci'$ by using z on the first cone and Cy on the second cone. The map x is induced from w by passage to quotients, so that the first three squares all commute. To see that the fourth square commutes up to homotopy, consider the composites

$$\begin{tikzcd}
Ci \arrow[r, "w"] & Ci' \arrow[r, "Cf \cup \mathrm{id}"] & CY_m \cup CY_m \arrow[r, shift left, "\rho_1"] \arrow[r, shift right, "\rho_2"'] & \Sigma Y_m
\end{tikzcd} ,$$

where ρ_1 and ρ_2 are the homotopy equivalences that collapse the first or the second cone to a point. The first composite is homotopic to Σy and the second is homotopic to $(-\Sigma f) \circ x \circ \pi$. This gives the required exactness. Part (iii) of Lemma 4.15 follows from parts (i) and (ii) and Lemma 4.9 by a standard diagram chase (e.g. [10, p. 130]). For finite wedges, Lemma 4.10 follows by inductive use of split cofiber sequences, and passage to colimits gives the general case. Lemma 4.11 follows by a comparison of cofiber sequences. One way of seeing this is to suspend once and obtain a cofiber sequence $\Sigma A \longrightarrow \Sigma X \vee \Sigma Y \longrightarrow \Sigma(X \cup_A Y)$. This construction gives a Mayer-Vietoris sequence for π_* from which Lemma 4.12 follows by the five lemma. The suspension of a based CW complex A is homotopy equivalent to a CW complex with based attaching maps. Now Lemma 4.14 follows from Lemmas 4.9, 4.10, 4.12, and 4.13. □

For later comparisons, we fix a choice of a naive or "handicrafted" smash product of prespectra [1].

Definition 4.16. Define the (naive) smash product of prespectra X and Y by

$$(X \wedge Y)_{2n} = X_n \wedge Y_n \quad \text{and} \quad (X \wedge Y)_{2n+1} = \Sigma(X_n \wedge Y_n),$$

with the evident structure maps.

Proposition 4.17. *For any cofibrant prespectrum X, the functor $X \wedge Y$ of Y preserves π_*-isomorphisms.*

Proof. Each X_n has the homotopy type of a CW complex, hence each functor $X_n \wedge Y$ preserves π_*-isomorphisms by Lemma 4.14. The groups $\pi_*(X \wedge Y)$ are

$$\begin{aligned}
\pi_q(X \wedge Y) &\cong \operatorname{colim}_n \pi_{2n+q}(X_n \wedge Y_n) \\
&\cong \operatorname{colim}_{m,n} \pi_{m+n+q}(X_m \wedge Y_n) \\
&\cong \operatorname{colim}_m \operatorname{colim}_n \pi_{m+n+q}(X_m \wedge Y_n) \\
&= \operatorname{colim}_m \pi_{m+q}(X_m \wedge Y),
\end{aligned}$$

and the conclusion follows. □

5. Stable weak equivalences of $\mathscr{D}$-spectra

In this section and the next, we assume given a nondegenerately based symmetric monoidal domain category $\mathscr{D}$ with a faithful strong symmetric monoidal functor $\iota : \mathscr{N} \longrightarrow \mathscr{D}$ and a sphere $\mathscr{D}$-monoid $S = S_{\mathscr{D}}$ that restricts along ι to the sphere prespectrum $S_{\mathscr{N}}$. We think of ι as an inclusion of categories. We let $\mathscr{D}\mathscr{S}$ be the category of $\mathscr{D}$-spectra over S, or right S-modules in the category of $\mathscr{D}$-spaces. We are thinking of the following examples from [MMSS, §8], but there are surely other examples of interest.

— The category $\mathscr{P}$ of $\mathscr{N}$-spectra
— The category $\Sigma\mathscr{S}$ of symmetric spectra
— The category $\mathscr{I}\mathscr{S}$ of orthogonal spectra
— The category $\mathscr{W}\mathscr{T}$ of $\mathscr{W}$-spaces

The respective categories $\mathscr{D}$ are

— The category $\mathscr{N}$ of non-negative integers
— The category Σ of symmetric groups
— The category $\mathscr{I}$ of finite dimensional real inner product spaces
— The category $\mathscr{W}$ of spaces homeomorphic to finite based CW complexes.

We have strong symmetric monoidal inclusions of categories

$$\mathscr{N} \subset \Sigma \subset \mathscr{I} \subset \mathscr{W} \tag{5.1}$$

that send n to $\mathbf{n}$, $\mathbf{n}$ to $\mathbb{R}^n$ and $\mathbb{R}^n$ to S^n. The respective sphere $\mathscr{D}$-monoids are given by

$$S_{\mathscr{N}}(n) = S^n, \quad S_\Sigma(\mathbf{n}) = S^n, \quad S_{\mathscr{I}}(V) = S^V, \quad \text{and} \quad S_{\mathscr{W}}(X) = X.$$

In the last three cases, but not the first, S is commutative. The sphere spectra for the smaller categories are the restrictions of the sphere spectra for the larger categories. To mesh notations, we write n for its image in any of the $\mathscr{D}$, and we let F_n denote the left adjoint to the nth space evaluation functor Ev_n; for a $\mathscr{D}$-spectrum X, we write $X(n) = Ev_n X = X_n$ interchangeably.

The general theory of [MMSS, §§4,6] gives the adjoint pairs $(\mathbb{P}, \mathbb{U})$ relating these categories that are displayed in the Main Diagram.

Convention 5.2. Until Section 11, we understand the level model structure on $\mathscr{D}$-spectra to mean the level model structure *relative to* $\mathscr{N}$, as defined in Variant 3.17. Since $\mathscr{N}$ contains all of the objects of a skeleton of Σ or $\mathscr{I}$, this is the same as the absolute level model structure in all cases above except the case of $\mathscr{W}$-spaces. We let $[X, Y]$ denote the set of maps $X \longrightarrow Y$ in the homotopy category with respect to the level model structure relative to $\mathscr{N}$. Recall that all of the results of Section 3 apply to this relative model structure.

Definition 5.3. Consider $\mathscr{D}$-spectra E and maps of $\mathscr{D}$-spectra $f : X \longrightarrow Y$.

(i) E is a *$\mathscr{D}$-Ω-spectrum* if $\mathbb{U}X$ is an Ω-spectrum.
(ii) f is a *π_*-isomorphism* if $\mathbb{U}f$ is a π_*-isomorphism.
(iii) f is a *stable weak equivalence* if $f^* : [Y, E] \longrightarrow [X, E]$ is a bijection for all Ω-spectra E.

Certain stable weak equivalences play a central role in the theory.

Definition 5.4. Let X be a $\mathscr{D}$-spectrum and consider the adjoint structure maps $\tilde{\sigma} : X_n \longrightarrow \Omega X_{n+1}$ of its underlying prespectrum. We have homeomorphisms

$$X_n = \mathscr{T}(S^0, X_n) \cong \mathscr{D}\mathscr{S}(F_n S^0, X)$$

and

$$\Omega X_{n+1} = \mathscr{T}(S^1, X_{n+1}) \cong \mathscr{D}\mathscr{S}(F_{n+1} S^1, X).$$

Take $X = F_n S^0$. Then $\tilde{\sigma}$ may be identified with a map

$$\tilde{\sigma} : \mathscr{D}\mathscr{S}(F_n S^0, F_n S^0) \longrightarrow \mathscr{D}\mathscr{S}(F_{n+1} S^1, F_n S^0).$$

Let $\lambda_n : F_{n+1} S^1 \longrightarrow F_n S^0$ be the image of the identity map under $\tilde{\sigma}$. It is adjoint to a canonical map $S^1 \longrightarrow (F_n S^0)_{n+1}$. Then

$$\lambda_n^* : \mathscr{D}\mathscr{S}(F_n S^0, X) \longrightarrow \mathscr{D}\mathscr{S}(F_{n+1} S^1, X)$$

may be identified with $\tilde{\sigma} : X_n \longrightarrow \Omega X_{n+1}$ for all X.

The following lemma is crucial. Due to it, the homotopy theory of symmetric spectra is significantly different, and considerably less intuitive at first sight, than the homotopy theories of $\mathscr{N}$-spectra, orthogonal spectra, and $\mathscr{W}$-spaces.

Lemma 5.5. *In all categories $\mathscr{D}\mathscr{S}$, the maps λ_n are stable weak equivalences. In $\mathscr{P}$, $\mathscr{I}\mathscr{S}$, and $\mathscr{W}\mathscr{T}$, the λ_n are π_*-isomorphisms. In $\Sigma\mathscr{S}$, the λ_n are* not *π_*-isomorphisms.*

Proof. The first statement is immediate from the definitions of a stable weak equivalence and of the maps λ_n. We prove that the λ_n are or are not π_*-isomorphisms separately in the four cases. Let $S^n = *$ if $n < 0$.

$\mathscr{N}$-spectra. Here $(F_n A)(q) = A \wedge S^{q-n}$. Thus $F_n A$ is essentially a reindexing of the suspension $\mathscr{N}$-spectrum of A. The map $\lambda_n(q)$ is the identity unless $q = n$, when it is the inclusion $* \longrightarrow S^0$. Thus λ_n is a π_*-isomorphism.

Orthogonal spectra. We find from [MMSS, 8.4] that

$$(F_n A)(q) = O(q)_+ \wedge_{O(q-n)} A \wedge S^{q-n}.$$

For $q \geq n+1$, $\lambda_n(q)$ is the canonical quotient map

$$O(q)_+ \wedge_{O(q-n-1)} S^1 \wedge S^{q-n-1} = O(q)_+ \wedge_{O(q-n-1)} S^{q-n} \longrightarrow O(q)_+ \wedge_{O(q-n)} S^{q-n}.$$

By Lemma 4.9, it suffices to prove that $\Sigma^n \lambda_n$ is a π_*-isomorphism, and $(\Sigma^n \lambda_n)(q)$ takes the form

$$O(q)_+ \wedge_{O(q-n-1)} S^q \longrightarrow O(q)_+ \wedge_{O(q-n)} S^q.$$

For a closed subgroup H of a Lie group G and a G-space X, we have a canonical homeomorphism of G-spaces

$$G_+ \wedge_H X \cong G/H_+ \wedge X.$$

For based spaces Y and Z and $q \geq 1$ we have a canonical homotopy equivalence

$$Y_+ \wedge \Sigma^q Z \simeq \Sigma^q(Y \wedge Z) \vee \Sigma^q Z.$$

Under such equivalences, $(\Sigma^n \lambda_n)(q)$ corresponds to the map

$$\pi \vee \mathrm{id} : \Sigma^q(O(q)/O(q-n-1)) \vee S^q \longrightarrow \Sigma^q(O(q)/O(q-n)) \vee S^q,$$

where π is the evident quotient map. When we pass to π_{q+r} and then to colimits over q, the first wedge summand contributes the zero group to the colimit. Therefore $\Sigma^n \lambda_n$ induces the identity map on the stable homotopy groups of S^0.

Symmetric spectra. The description of the maps λ_n is the same as for orthogonal spectra, except that orthogonal groups are replaced by symmetric groups. However, in contrast to the quotients $O(q)/O(q-n)$, the quotients Σ_q/Σ_{q-n} do not become highly connected as q increases. In fact, $\pi_*(F_n S^n)$, $n \geq 1$, is the sum of countably many copies of the stable homotopy groups of S^0; compare [8, 3.1.10].

$\mathscr{W}$-spaces. Using [MMSS, 8.6], we see that the qth map of λ_n can be identified with the evaluation map

$$\Sigma\Omega(\Omega^n S^q) \longrightarrow \Omega^n S^q.$$

Applying π_{q+r} and passing to colimits over q, these maps induce an isomorphism with target the stable homotopy groups of spheres, reindexed by n. □

We shall prove the following result in the next section.

Proposition 5.6. *A map of $\mathscr{N}$-spectra, orthogonal spectra, or $\mathscr{W}$-spaces is a π_*-isomorphism if and only if it is a stable weak equivalence.*

For this reason, there is no need to mention stable weak equivalences when setting up the stable model structures in $\mathscr{P}$, $\mathscr{I}\mathscr{S}$ and $\mathscr{W}\mathscr{T}$: everything can be done more simply in terms of π_*-isomorphisms. At the price of introducing an unneccessary additional level of complexity in these cases, we have chosen to work with stable weak equivalences in order to give a uniform general treatment. As suggested by Lemma 5.5, the forward implication of Proposition 5.6 does hold in all cases.

Proposition 5.7. *A π_*-isomorphism in $\mathscr{D}\mathscr{S}$ is a stable weak equivalence. Therefore a level weak equivalence is a stable weak equivalence.*

Proof. Following the analogous argument in [8], define $RX = F_S(F_1 S^1, X)$, where F_S is the function $\mathscr{D}$-spectrum functor. Since $F_n S^n$ is isomorphic to the nth smash power of $F_1 S^1$, by [MMSS, 2.10], the n-fold iterate $R^n X$ is isomorphic to $F_S(F_n S^n, X)$. The map $\lambda = \lambda_1 : F_1 S^1 \longrightarrow F_0 S^0 = S$ induces a map $\lambda^* : X \longrightarrow RX$ and thus a map $R^n \lambda^* : R^n X \longrightarrow R^{n+1} X$. Define QX to be the homotopy colimit (or telescope) of the $R^n X$ and let $\iota : X \longrightarrow QX$ be the natural map. The defining adjunctions of the functors F_S and F_n, together with the isomorphism

$$F_m A \wedge_S F_n B \cong F_{m+n}(A \wedge B)$$

for based spaces A and B [MMSS, 2.10], imply that

$$\mathscr{T}(A, Ev_m F_S(F_n S^n, X)) \cong \mathscr{T}(A, \Omega^n X(m+n)).$$

Therefore $(R^n X)(m) \cong \Omega^n X(m+n)$. Since λ corresponds to $\tilde{\sigma}$ under adjunction, a quick inspection of colimits shows that

$$\pi_q((QX)(m)) \cong \pi_{q-m}(X).$$

Nevertheless, QX need not be a $\mathscr{D}$-Ω-spectrum in general. However, if E is a $\mathscr{D}$-Ω-spectrum, then $\lambda^* : E \longrightarrow RE$ is a level weak equivalence, hence so is $\iota : E \longrightarrow QE$, and QE is a $\mathscr{D}$-Ω-spectrum. Morever, $\iota_* : [X, E] \longrightarrow [X, QE]$ is an isomorphism for any X. By the naturality of ι, ι_* is the composite of $Q : [X, E] \longrightarrow [QX, QE]$ and $\iota^* : [QX, QE] \longrightarrow [X, QE]$. Since $[QX, E] \cong [QX, QE]$, this shows that $[X, E]$ is naturally a retract of $[QX, E]$. If $f : X \longrightarrow Y$ is a π_*-isomorphism, then $Qf : QX \longrightarrow QY$ is a level weak equivalence. Thus $f^* : [Y, E] \longrightarrow [X, E]$ is a retract of the isomorphism $(Qf)^* : [QY, E] \longrightarrow [QX, E]$ and is therefore an isomorphism. □

The proof has the following useful corollary.

Corollary 5.8. *Any $\mathscr{D}$-Ω-spectrum E is level weak equivalent to $\Omega E'$ for another $\mathscr{D}$-Ω-spectrum E'.*

Proof. E is level weak equivalent to RE and RE is easily checked to be the loop spectrum of a $\mathscr{D}$-Ω-spectrum. □

Colimits, h-fibrations, smash products with spaces, and fiber and cofiber sequences are preserved by $\mathbb{U}$, since they are specified in terms of levelwise constructions. This implies the following result about the π_*-isomorphisms of $\mathscr{D}$-spectra.

Proposition 5.9. *Lemmas 4.8 through 4.15 remain valid with $\mathscr{P}$ replaced by $\mathscr{D}\mathscr{S}$.*

We shall prove the analogue for stable weak equivalences in the rest of the section.

Lemma 5.10. *A stable weak equivalence between $\mathscr{D}$-Ω-spectra is a level weak equivalence.*

Proof. This is formal. If $f : E \longrightarrow E'$ is a stable weak equivalence of $\mathscr{D}$-Ω-spectra, then $f^* : [E', E] \longrightarrow [E, E]$ is an isomorphism. A map $g : E' \longrightarrow E$ such that $g \circ f = f^* g = \mathrm{id}$ is an inverse isomorphism to f in the level homotopy category. □

We shall prove the following results together at the end of the section.

Lemma 5.11. *A map of $\mathscr{D}$-spectra is a stable weak equivalence if and only if its suspension is a stable weak equivalence.*

Lemma 5.12. *A wedge of stable weak equivalences of $\mathscr{D}$-spectra is a stable weak equivalence.*

Lemma 5.13. *let $i : A \longrightarrow X$ be an h-cofibration of $\mathscr{D}$-spectra and let $f : A \longrightarrow Y$ be any map of $\mathscr{D}$-spectra. If i is a stable weak equivalence, then the cobase change $j : Y \longrightarrow X \cup_f Y$ is a stable weak equivalence.*

Lemma 5.14. *Assume given a commutative diagram*

$$\begin{tikzcd} X \arrow[d] & A \arrow[l, "i"'] \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X' & A' \arrow[l, "i'"] \arrow[r, "f'"'] & Y' \end{tikzcd}$$

of $\mathscr{D}$-spectra in which i and i' are h-cofibrations. If the vertical arrows are stable weak equivalences, then the induced map of pushouts is a stable weak equivalence.

Lemma 5.15. *Let X be the colimit of a sequence of h-cofibrations $i_n : X_n \longrightarrow X_{n+1}$. If each i_n is a stable weak equivalence, then the map from the initial term X_0 into X is a stable weak equivalence.*

Lemma 5.16. *If $f : X \longrightarrow Y$ is a stable weak equivalence of $\mathscr{D}$-spectra and A is a based CW complex, then $f \wedge \mathrm{id} : X \wedge A \longrightarrow Y \wedge A$ is a stable weak equivalence.*

Lemma 5.17. *Let $f : X \longrightarrow Y$ be a map of $\mathscr{D}$-spectra and let E be an Ω-spectrum.*

(i) *There is a natural long exact sequence*

$$\cdots \longrightarrow [\Sigma X, E] \longrightarrow [Cf, E] \longrightarrow [Y, E] \longrightarrow [X, E] \longrightarrow [\Omega Cf, E] \longrightarrow \cdots .$$

(ii) *There is a natural long exact sequence*

$$\cdots \longrightarrow [\Sigma X, E] \longrightarrow [\Sigma Ff, E] \longrightarrow [Y, E] \longrightarrow [X, E] \longrightarrow [Ff, E] \longrightarrow \cdots .$$

Proofs. Under nondegenerate basepoint hypotheses, most of these results follow directly from the elementary results about the level homotopy category in Section 3. To obtain them in full generality, we make use of Proposition 5.7 and the results on π_*-isomorphisms of Section 4. Cofibrant $\mathscr{D}$-spectra are nondegenerately based by Lemma 3.6, and cofibrant approximations of general $\mathscr{D}$-spectra are level weak equivalences, hence π_*-isomorphisms, hence stable weak equivalences. Thus we can first use cofibrant approximation and the results of Section 4 to reduce each statement to a statement about cofibrant $\mathscr{D}$-spectra and then quote the results of Section 3. The upshot is that statements about $[X, Y]$ that hold for nondegenerately based X and general Y also hold for general X and $\mathscr{D}$-Ω-spectra Y.

For example, we now see from Lemmas 4.9 and 3.10 that $[\Sigma X, E] \cong [X, \Omega E]$ for all X when E is an Ω-spectrum; Lemma 5.11 follows in view of Corollary 5.8. For Lemma 5.12, Lemmas 4.10 and 3.11 imply that the functor $[-, E]$ converts wedges to products when E is an Ω-spectrum. Lemma 5.17 results from Lemmas 4.15 and 3.16, and we use it to prove Lemmas 5.13 and 5.14. For Lemma 5.13, cofibrant approximation gives a commutative diagram

$$\begin{tikzcd}
X' \arrow[d] & A' \arrow[l, "i'"'] \arrow[r, "f'"] \arrow[d] & Y' \arrow[d] \\
X & A \arrow[l, "i"] \arrow[r, "f"'] & Y
\end{tikzcd}$$

in which X', A', and Y' are cofibrant, the vertical arrows are level acyclic fibrations, and the maps i' and f' are h-cofibrations. By Lemma 4.12, the induced map of pushouts is a π_*-isomorphism and thus a stable weak equivalence. By the diagram, i' is a stable weak equivalence, and it suffices to prove that the cobase change $j' : Y' \longrightarrow X' \cup_{f'} Y'$ is a stable weak equivalence. Thus we may assume without loss of generality that the given A, X, and Y are cofibrant. We first deduce from the cofiber sequence $A \longrightarrow X \longrightarrow X/A$ that $[X/A, E] = 0$. Since $X \cup_f Y/Y \cong X/A$, we then deduce that $[Y, E] \longrightarrow [X \cup_f Y, E]$ is a bijection.

Similarly, applying cofibrant approximation to the diagram of Lemma 5.14, we see that we may assume without loss of generality that it is a diagram of cofibrant $\mathscr{D}$-spectra. A comparison of cofiber sequences gives that $X/A \longrightarrow X'/A'$ is a stable weak equivalence, and then another comparison of cofiber sequences gives that $X \cup_f Y \longrightarrow X' \cup_{f'} Y'$ is a stable weak equivalence. For Lemma 5.15, we apply cofibrant approximation to obtain a sequence of h-cofibrations $j_n : Y_n \longrightarrow Y_{n+1}$

between cofibrant $\mathscr{D}$-spectra together with level acyclic fibrations $p_n : Y_n \longrightarrow X_n$ such that $p_{n+1} \circ j_n = i_n \circ p_n$. Since the i_n and p_n are stable weak equivalences, so are the j_n. Let $Y = \operatorname{colim} Y_n$. The map $p : Y \longrightarrow X$ induced by the p_n is a level weak equivalence, and the $\lim^1$-exact sequence of Lemma 3.14 implies that $[Y, E] \longrightarrow [Y_0, E]$ is an isomorphism and thus $[X, E] \longrightarrow [X_0, E]$ is an isomorphism. Finally, for Lemma 5.16, we see from Lemmas 4.14 and 3.15 that $[X \wedge A, E]$ is naturally isomorphic to $[X, F(A, E)]$ when E is a $\mathscr{D}$-Ω-spectrum, in which case $F(A, E)$ is also a $\mathscr{D}$-Ω-spectrum. Thus $f \wedge \mathrm{id}$ is a stable weak equivalence. $\square$

6. The stable model structure on $\mathscr{D}$-spectra

Again, we consider the category $\mathscr{D}\mathscr{S}$ of $\mathscr{D}$-spectra, where $\mathscr{D}$ is a nondegenerately based symmetric monoidal category under $\mathscr{N}$ together with a sphere $\mathscr{D}$-monoid S that restricts to the sphere $\mathscr{N}$-spectrum. We are thinking of $\mathscr{N}$-spectra, symmetric spectra, orthogonal spectra, and $\mathscr{W}$-spaces.

Definition 3.1 specifies the level weak equivalences, level fibrations, level acyclic fibrations, q-cofibrations, and level acyclic q-cofibrations in the category $\mathscr{D}\mathscr{S}$. Definition 5.3 specifies the stable weak equivalences. The class of stable weak equivalences is closed under retracts and is saturated (satisfies the two out of three property for composites).

Definition 6.1. Let $f : X \longrightarrow Y$ be a map of $\mathscr{D}$-spectra.

(i) f is an *acyclic q-cofibration* if it is a stable weak equivalence and a q-cofibration.
(ii) f is a *q-fibration* if it satisfies the RLP with respect to the acyclic q-cofibrations.
(iii) f is an *acyclic q-fibration* if it is a stable weak equivalence and a q-fibration.

We shall prove the following result. In outline, its proof follows that of Hovey, Shipley, and Smith [8] for symmetric spectra of simplicial sets, but there are significant differences of detail.

Theorem 6.2. *The category $\mathscr{D}\mathscr{S}$ is a compactly generated proper topological model category with respect to the stable weak equivalences, q-fibrations, and q-cofibrations.*

The set of generating q-cofibrations is the set FI specified in Definition 3.2. The set K of generating acyclic q-cofibrations properly contains the set FJ specified there. The idea is that level weak equivalences and stable weak equivalences coincide on $\mathscr{D}$-Ω-spectra, by Lemma 5.10, and the model structure is arranged so that the fibrant spectra turn out to be exactly the $\mathscr{D}$-Ω-spectra. We add enough generating acyclic q-cofibrations to FJ to ensure that the RLP with respect to the K-cell complexes forces the adjoint structure maps of fibrant spectra to be weak equivalences. Recall the maps λ_n from Definition 5.4.

Definition 6.3. Let $M\lambda_n$ be the mapping cylinder of λ_n. Then λ_n factors as the composite of a q-cofibration $k_n : F_{n+1}S^1 \longrightarrow M\lambda_n$ and a deformation retraction $r_n : M\lambda_n \longrightarrow F_n S^0$. For $n \geq 0$, let K_n be the set of maps of the form $k_n \Box i$, $i \in I$. Let K be the union of FJ and the sets K_n for $n \geq 0$.

We need a characterization of the maps that satisfy the RLP with respect to K.

Definition 6.4. A commutative diagram of based spaces

$$\begin{tikzcd} D \arrow[r, "g"] \arrow[d, "p"'] & E \arrow[d, "q"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

in which p and q are Serre fibrations is a homotopy pullback if the induced map $D \longrightarrow A \times_B E$ is a weak equivalence or, equivalently, if $g : p^{-1}(a) \longrightarrow q^{-1}(f(a))$ is a weak equivalence for all $a \in A$.

Proposition 6.5. *A map $p : E \longrightarrow B$ satisfies the RLP with respect to K if and only if p is a level fibration and the diagram*

$$\begin{tikzcd} E_n \arrow[r, "\tilde{\sigma}"] \arrow[d, "p_n"'] & \Omega E_{n+1} \arrow[d, "\Omega p_{n+1}"] \\ B_n \arrow[r, "\tilde{\sigma}"'] & \Omega B_{n+1} \end{tikzcd} \tag{6.6}$$

is a homotopy pullback for each $n \geq 0$.

Proof. Clearly p satisfies the RLP with respect to K if and only if p satisfies the RLP with respect to FJ and the K_n for $n \geq 0$. The maps that satisfy the RLP with respect to FJ are the level fibrations. By the definition of K_n, p has the RLP with respect to K_n if and only if p has the RLP with respect to $k_n \Box I$. By Lemma 1.17, this holds if and only if $\mathscr{D}\mathscr{S}(k_n^*, p_*)$ has the RLP with respect to I, which means that $\mathscr{D}\mathscr{S}(k_n^*, p_*)$ is an acyclic Serre fibration. Since k_n is a q-cofibration and p is a level fibration, $\mathscr{D}\mathscr{S}(k_n^*, p_*)$ is a Serre fibration because the level model structure is topological. We conclude that p satisfies the RLP with respect to K if and only if p is a level fibration and $\mathscr{D}\mathscr{S}(k_n^*, p_*)$ is a weak equivalence for $n \geq 0$. Let $j_n : F_nS^0 \longrightarrow M\lambda_n$ be the evident homotopy inverse of $r_n : M\lambda_n \longrightarrow F_nS^0$. Then $\mathscr{D}\mathscr{S}(k_n^*, p_*) \simeq \mathscr{D}\mathscr{S}((j_n\lambda_n)^*, p_*)$. This is a weak equivalence if and only if

$$\mathscr{D}\mathscr{S}(\lambda_n^*, p_*) : \mathscr{D}\mathscr{S}(F_nS^0, E) \longrightarrow \mathscr{D}\mathscr{S}(F_nS^0, B) \times_{\mathscr{D}\mathscr{S}(F_{n+1}S^1, B)} \mathscr{D}\mathscr{S}(F_{n+1}S^1, E))$$

is a weak equivalence. But this is isomorphic to the map

$$E_n \longrightarrow B_n \times_{\Omega B_{n+1}} \Omega E_{n+1}$$

and is thus a weak equivalence if and only if (6.6) is a homotopy pullback. $\square$

Corollary 6.7. *The trivial map $F \longrightarrow *$ satisfies the RLP with respect to K if and only if F is a $\mathscr{D}$-Ω-spectrum.*

Corollary 6.8. *If $p : E \longrightarrow B$ is a stable weak equivalence that satisfies the RLP with respect to K, then p is a level acyclic fibration.*

Proof. Certainly $p : E \longrightarrow B$ is a level fibration. We must prove that p is a level weak equivalence. Let $F = p^{-1}(*)$ be the fiber (defined levelwise) over the basepoint. Since $F \longrightarrow *$ is a pullback of p, it satisfies the RLP with respect to K and is therefore a $\mathscr{D}$-Ω-spectrum. Since p is acyclic, so is F. Therefore, by Lemma 5.10, F is level acyclic. By the level long exact sequences, each $p_n : E_n \longrightarrow B_n$ induces an isomorphism of homotopy groups in positive degrees. However, we must deal with the possibility that the B_n are not path connected. Let $\tilde{p} : \tilde{E} \longrightarrow \tilde{B}$ be the map obtained from p by restricting levelwise to maps between path components of basepoints. Certainly $\tilde{p}$ is a level equivalence. In the homotopy pullback (6.6), the map Ωp_{n+1} depends only on basepoint components and is a weak equivalence, hence p_n is a weak equivalence as required. □

The q-cofibrations are the same for the stable as for the level model structure. The essential part of the proof of the model axioms for the stable model structure is to characterize the acyclic q-cofibrations, the q-fibrations, and the acyclic q-fibrations. Observe that the small objects argument applies to K since the domains of the maps in K are compact by Lemma 1.7.

Proposition 6.9. *Let $f : X \longrightarrow Y$ be a map in $\mathscr{D}\mathscr{S}$.*

(i) *f is an acyclic q-cofibration if and only if it is a retract of a relative K-cell complex.*
(iii) *f is a q-fibration if and only if it satisfies the RLP with respect to K, and X is fibrant if and only if it is a $\mathscr{D}$-Ω-spectrum.*
(iii) *f is an acyclic q-fibration if and only if it is a level acyclic fibration.*

Proof. (i) Let f be a retract of a relative K-cell complex. Since the maps in K are acyclic q-cofibrations, f is an acyclic q-cofibration by the closure properties of the class of q-cofibrations given by the level model structure and the closure properties of the class of stable weak equivalences given by Lemmas 5.12, 5.13, and 5.15. Conversely, let $f : X \longrightarrow Y$ be an acyclic q-cofibration. Using the small objects argument, factor f as the composite of a relative K-cell complex $i : X \longrightarrow X'$ and a map $p : X' \longrightarrow Y$ that satisfies the RLP with respect to K. We have just seen that i is a stable weak equivalence. Since f is a stable weak equivalence, so is p. By Corollary 6.8, p is a level acyclic fibration. Since f is a q-cofibration, it has the LLP with respect to p. Now a standard retract argument applies. There is a map $g : Y \longrightarrow X'$ such that $g \circ f = i$ and $p \circ g = \mathrm{id}$. Thus g and p are maps under X and f is a retract of the relative K-cell complex i.
(ii) Since f satisfies the RLP with respect to K if and only if it satisfies the RLP with respect to all retracts of relative K-cell complexes, this follows from (i) and the definition of a q-fibration.
(iii). By the level model structure, a map is a level acyclic fibration if and only if it satisfies the RLP with respect to the q-cofibrations. By Lemma 5.10, a level weak equivalence is a stable weak equivalence. Thus a level acyclic fibration is an acyclic q-fibration. Conversely, an acyclic q-fibration satisfies the RLP with respect to K, by (ii), and is thus a level acyclic fibration by Corollary 6.8. □

The proof that $\mathscr{D}\mathscr{S}$ is a model category. The definition of a q-fibration gives one of the lifting axioms. The identification of the acyclic q-fibrations as the level acyclic fibrations gives the other lifting axiom and the factorization of a map as a composite of a q-cofibration and an acyclic q-fibration, via the level model structure. It remains to prove that a map $f : X \longrightarrow Y$ factors as the composite of an acyclic q-cofibration and a q-fibration. Applying the small objects argument to K, we obtain a factorization of f as the composite of a relative K-cell complex $i : X \longrightarrow X'$ and a map $p : X' \longrightarrow Y$ that satisfies the RLP with respect to K. By the previous proposition, i is an acyclic q-cofibration and p is a q-fibration. □

The proof that $\mathscr{D}\mathscr{S}$ is topological. Let $i : A \longrightarrow X$ be a q-cofibration and $p : E \longrightarrow B$ be a q-fibration. Since p is a level fibration, the map

$$\mathscr{D}\mathscr{S}(i^*, p_*) : \mathscr{D}\mathscr{S}(X, E) \longrightarrow \mathscr{D}\mathscr{S}(A, E) \times_{\mathscr{D}\mathscr{S}(A,B)} \mathscr{D}\mathscr{S}(X, B)$$

is a Serre fibration because the level model structure is topological. Similarly, if p is acyclic, then p is level acyclic and $\mathscr{D}\mathscr{S}(i^*, p_*)$ is a weak equivalence. We must show that $\mathscr{D}\mathscr{S}(i^*, p_*)$ is a weak equivalence if i is acyclic, and it suffices to show this when $i \in K$. If $i \in FJ$, this again holds by the result for the level model structure. Thus suppose that $i \in k_n \Box I$, say $i = k_n \Box j$. We have seen in the proof of Proposition 6.5 that $\mathscr{D}\mathscr{S}(k_n^*, p_*)$ is a weak equivalence. Thus, since $\mathscr{T}$ is a topological model category, $\mathscr{T}(j^*, \mathscr{D}\mathscr{S}(k_n^*, p_*)_*)$ is a weak equivalence. By Lemma 1.17, this implies that $\mathscr{D}\mathscr{S}(i^*, p_*)$ is a weak equivalence. □

The proof that $\mathscr{D}\mathscr{S}$ is proper. Since q-cofibrations are h-cofibrations and q-fibrations are level fibrations, the following lemma generalizes the claim. □

Lemma 6.10. *(i). If i is an h-cofibration and f is a stable weak equivalence in the pushout diagram*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & B \arrow[d, "j"] \\ X \arrow[r, "g"'] & Y, \end{tikzcd}$$

then g is a stable weak equivalence.
(ii) If p is a level fibration and f is a stable weak equivalence in the pullback diagram

$$\begin{tikzcd} D \arrow[r, "g"] \arrow[d, "q"'] & E \arrow[d, "p"] \\ A \arrow[r, "f"'] & B, \end{tikzcd}$$

then g is a stable weak equivalence.

Proof. (i) The induced map $X/A \longrightarrow Y/B$ is an isomorphism. We compare the cofibration sequences $[-, E]$ of Lemma 5.17(i) for the cofibration sequences $A \longrightarrow X \longrightarrow X/A$ and $B \longrightarrow Y \longrightarrow Y/B$ and apply the five lemma.
(ii) Dually, the induced map from the fiber of q to the fiber of p is an isomorphism. We compare fibration sequences using Lemma 5.17(ii). □

We have left one unfinished piece of business from the previous section.

The proof of Proposition 5.6. By Proposition 5.7, we need only show that a stable weak equivalence f in $\mathscr{D}\mathscr{S}$ is a π_*-isomorphism when $\mathscr{D}\mathscr{S}$ is $\mathscr{P}$, $\mathscr{I}\mathscr{S}$, or $\mathscr{W}\mathscr{T}$. Factor f as the composite of an acyclic q-cofibration and an acyclic q-fibration. Since an acyclic q-fibration is a level acyclic fibration, it is a level weak equivalence and therefore a π_*-isomorphim. We must show that an acyclic q-cofibration is a π_*-isomorphism. We first show that the maps in K are π_*-isomorphisms. The maps in FJ are inclusions of deformation retracts and are therefore π_*-isomorphisms. The maps $k_n \Box i$ with $i \in FI$ specified in Definition 6.3 are also π_*-isomorphisms. Indeed, by Lemma 5.5, the maps λ_n and therefore the maps k_n are π_*-isomorphisms. By Lemma 4.14, so are their smash products with based CW complexes. By passage to pushouts and a little diagram chase, this implies that the maps $k_n \Box i$ are π_*-isomorphisms. By Lemmas 4.10, 4.11, and 4.13, it follows that any relative K-cell complex is a π_*-isomorphism. Since the acyclic q-cofibrations are the retracts of the relative K-cell complexes, the conclusion follows. □

In fact, we now see that, in our development of the stable model structure on $\mathscr{D}\mathscr{S}$ in these three cases, we can start out by defining the weak equivalences to be either the stable weak equivalences or the π_*-isomorphisms. We arrive at the same acyclic q-cofibrations and acyclic q-fibrations either way.

Remark 6.11. By [8, 6.3.3], the stable model structure on (topological) symmetric spectra given by Hovey, Shipley, and Smith has the same q-fibrations and acyclic q-fibrations as does ours, hence it is the same model structure.

Remark 6.12. We can carry out the theory of prespectra and of orthogonal spectra in a "coordinate-free" way. All that is needed is to replace $\mathscr{N}$ by the category $\mathscr{V}$ whose objects are the finite dimensional real inner product spaces and which has only identity morphisms. That is, we work with all objects V of $\mathscr{I}$ rather than just the objects $\mathbb{R}^n$. In the case of $\mathscr{W}$, where the S^n do not give the objects of a skeleton, the analogue of the coordinate-free approach gives a different model structure than the one already obtained. It will be studied in Section 11.

7. Comparisons among $\mathscr{P}$, $\Sigma\mathscr{S}$, $\mathscr{I}\mathscr{S}$, and $\mathscr{W}\mathscr{T}$

We now turn to the proofs that our various adjoint pairs are Quillen equivalences. Write $\mathbb{U} : \mathscr{D}\mathscr{S} \longrightarrow \mathscr{C}\mathscr{S}$ generically for the forgetful functor associated to any of the inclusions $\mathscr{C} \subset \mathscr{D}$ of (5.1). For clarity, we sometimes write $F_n^{\mathscr{D}}$ for the left adjoint to evaluation at n, denoting the latter by $Ev_n^{\mathscr{D}}$. For each of the inclusions $\mathscr{C} \subset \mathscr{D}$, the following diagram commutes up to natural isomorphism since both composites are left adjoint to $Ev_n^{\mathscr{D}}$.

(7.1)
$$\begin{tikzcd} & \mathscr{T} \arrow[dl, "F_n^{\mathscr{C}}"'] \arrow[dr, "F_n^{\mathscr{D}}"] & \\ \mathscr{C}\mathscr{S} \arrow[rr, "\mathbb{P}"'] & & \mathscr{D}\mathscr{S}. \end{tikzcd}$$

The characterizations of the q-fibrations and acyclic q-fibrations given by Propositions 6.5 and 6.9 directly imply the following lemma. Recall Definition 2.1.

Lemma 7.2. *Each of the forgetful functors $\mathbb{U} : \mathscr{D}\mathscr{S} \longrightarrow \mathscr{C}\mathscr{S}$ preserves q-fibrations and acyclic q-fibrations. Therefore each $(\mathbb{P}, \mathbb{U})$ is a Quillen adjoint pair.*

We wish to apply Lemma 2.2(iii) to demonstrate that these pairs are Quillen equivalences. For that, we need to know that $\mathbb{U}$ creates the stable weak equivalences in its domain category. This is false for $\mathbb{U} : \Sigma\mathscr{S} \longrightarrow \mathscr{P}$ because the λ_n are stable weak equivalences of symmetric spectra but the $\mathbb{U}\lambda_n$ are not stable weak equivalences ($= \pi_*$-isomorphisms) of $\mathscr{N}$-spectra. This makes a direct proof of the Quillen equivalence between $\mathscr{M}$-spectra and symmetric spectra fairly difficult; compare [8, §4]. However, this is the only case in which the condition fails.

Lemma 7.3. *The forgetful functors*

$$\mathbb{U} : \mathscr{I}\mathscr{S} \longrightarrow \mathscr{P}, \quad \mathbb{U} : \mathscr{I}\mathscr{S} \longrightarrow \Sigma\mathscr{S}, \quad \textit{and} \quad \mathbb{U} : \mathscr{W}\mathscr{T} \longrightarrow \mathscr{I}\mathscr{S}$$

and their composites create the stable weak equivalences in their domain categories.

Proof. This is immediate in the first and third case, since there the stable weak equivalences coincide with the π_*-isomorphisms in both the domain and codomain categories. To prove that $\mathbb{U} : \mathscr{I}\mathscr{S} \longrightarrow \Sigma\mathscr{S}$ creates the weak equivalences of orthogonal spectra, let $f : X \longrightarrow Y$ be a map of orthogonal spectra such that $\mathbb{U}f$

is a stable weak equivalence and let $f' : X' \longrightarrow Y'$ be a fibrant approximation of f. Then $\mathbb{U}f'$ is a stable weak equivalence between symmetric Ω-spectra and thus a π_*-isomorphism, and it follows that f is a π_*-isomorphism. $\square$

Thus $\mathbb{U} : \mathscr{D}\mathscr{S} \longrightarrow \mathscr{C}\mathscr{S}$ creates the stable weak equivalences in $\mathscr{D}\mathscr{S}$ whenever the stable weak equivalences and π_*-isomorphisms coincide in $\mathscr{D}\mathscr{S}$. In these cases, we also have the following result about the unit $\eta : \mathrm{Id} \longrightarrow \mathbb{U}\mathbb{P}$ of the adjunction.

Lemma 7.4. *Consider $\mathbb{U} : \mathscr{D}\mathscr{S} \longrightarrow \mathscr{C}\mathscr{S}$ and $\mathbb{P} : \mathscr{C}\mathscr{S} \longrightarrow \mathscr{D}\mathscr{S}$. If the stable weak equivalences and π_*-isomorphisms coincide in $\mathscr{D}\mathscr{S}$, then $\eta : X \longrightarrow \mathbb{U}\mathbb{P}X$ is a stable weak equivalence for all cofibrant $\mathscr{C}$-spectra X.*

Proof. The functors $\mathbb{P}$ and $\mathbb{U}$ preserve colimits since $\mathbb{P}$ is a left adjoint and since colimits are constructed levelwise. Since cofibrant $\mathscr{C}$-spectra are retracts of FI-cell $\mathscr{C}$-spectra, we see from Lemmas 5.12, 5.13, 5.14, and 5.15 that it suffices to prove the result when X is the source or target of a map in FI. It more than suffices to prove that

$$\eta : F_n^{\mathscr{C}} A \longrightarrow \mathbb{U}\mathbb{P}F_n^{\mathscr{C}} A = \mathbb{U}F_n^{\mathscr{D}} A$$

is a stable weak equivalence of $\mathscr{C}$-spectra for $n \geq 0$ and any based CW complex A. By Lemma 5.11, it suffices to prove that $\Sigma^n \eta$ is a stable weak equivalence. The functors $F_n^{\mathscr{C}}$ and $\mathbb{U}$ commute with smash products with based spaces and thus with suspension. Moreover, by Lemmas 4.14 and 5.16, smashing with based CW complexes preserves both π_*-isomorphisms and stable weak equivalences. Let $\gamma_n^{\mathscr{C}} : F_n^{\mathscr{C}} S^n \longrightarrow F_n^{\mathscr{C}} S^0$ be adjoint to the identity map $S^n \longrightarrow S^n = (F_n S^0)(n)$. Then $\gamma_n^{\mathscr{C}}$ is the composite of the maps $\Sigma^m \lambda_m$, $0 \leq m < n$. These maps are stable weak equivalences by Lemma 5.5; moreover, with $\mathscr{C}$ replaced by $\mathscr{D}$, they are π_*-isomorphisms. Since $\mathbb{U}$ preserves π_*-isomorphisms and π_*-isomorphisms in $\mathscr{C}\mathscr{S}$ are stable weak equivalences, the conclusion follows from the commutative diagram

$$\begin{tikzcd}
F_n^{\mathscr{C}} A \wedge S^n \arrow[r, "\cong"] \arrow[d, "\eta\wedge\mathrm{id}"'] & F_n^{\mathscr{C}} S^n \wedge A \arrow[r, "\gamma_n^{\mathscr{C}}\wedge\mathrm{id}"] \arrow[d, "\eta\wedge\mathrm{id}"'] & F_0^{\mathscr{C}} S^0 \wedge A \arrow[d, "\eta\wedge\mathrm{id}"', "\cong"] \\
\mathbb{U}F_n^{\mathscr{D}} A \wedge S^n \arrow[r, "\cong"'] & \mathbb{U}F_n^{\mathscr{D}} S^n \wedge A \arrow[r, "\mathbb{U}\gamma_n^{\mathscr{D}}\wedge\mathrm{id}"'] & \mathbb{U}F_0^{\mathscr{D}} S^0 \wedge A,
\end{tikzcd}$$

in which the right vertical arrow is easily checked to be an isomorphism. $\square$

Theorem 7.5. *The categories of $\mathscr{N}$-spectra and orthogonal spectra, of symmetric spectra and orthogonal spectra, and of orthogonal spectra and $\mathscr{W}$-spaces are Quillen equivalent.*

Proof. This is immediate from Lemmas 2.2(iii), 7.2, 7.3, and 7.4. $\square$

Corollary 7.6. *The categories of $\mathscr{N}$-spectra and symmetric spectra are Quillen equivalent.*

Proof. We have the following pair of adjoint pairs:

$$\mathscr{P} \underset{\mathbb{U}}{\overset{\mathbb{P}}{\rightleftarrows}} \Sigma\mathscr{S} \underset{\mathbb{U}}{\overset{\mathbb{P}}{\rightleftarrows}} \mathscr{I}\mathscr{S}.$$

By Lemma 2.2, the fact that the composite pair $(\mathbb{P}\mathbb{P}, \mathbb{U}\mathbb{U})$ and the second pair $(\mathbb{P}, \mathbb{U})$ are Quillen equivalences implies that the first pair $(\mathbb{P}, \mathbb{U})$ is a Quillen equivalence. $\square$

Proof of Corollary 0.2. The result asserts that a map $f : X \longrightarrow Y$ of cofibrant symmetric spectra is a stable weak equivalence if and only if $\mathbb{P}f$ is a π_*-isomorphism of orthogonal spectra. By the naturality of η, Lemma 7.4 implies that f is a stable weak equivalence if and only if $\mathbb{UP}f$ is a stable weak equivalence. Since $\mathbb{U}$ creates the stable weak equivalences of orthogonal spectra, this gives the conclusion. $\square$

We now turn to the proof of Theorem 0.3, which asserts that our induced equivalences of homotopy categories preserve smash products. Since the functors $\mathbb{P} : \Sigma\mathscr{S} \longrightarrow \mathscr{I}\mathscr{S}$ and $\mathbb{P} : \mathscr{I}\mathscr{S} \longrightarrow \mathscr{W}\mathscr{S}$ are strong symmetric monoidal, the conclusion is immediate from Lemma 2.3 in these cases. Of course, since the equivalence of homotopy categories induced by $\mathbb{P}$ preserves smash products, so does the inverse equivalence induced by $\mathbb{U}$. It remains to bring $\mathscr{N}$-spectra into the picture. We begin with the following observation, which is unexpected from a model theoretic point of view.

Proposition 7.7. *Let X be a cofibrant $\mathscr{D}$-spectrum, where $\mathscr{D} = \Sigma$, $\mathscr{I}$, or $\mathscr{W}$. Then the underlying $\mathscr{N}$-spectrum $\mathbb{U}X$ has the homotopy type of a CW prespectrum and thus of a cofibrant $\mathscr{N}$-spectrum.*

Proof. For a finite CW complex A, the spaces $(F^{\mathscr{D}}_m A)_n$ have the homotopy types of CW complexes. Therefore, for an FI-cell spectrum X and thus for any cofibrant $\mathscr{D}$-spectrum X, each X_n has the homotopy type of a CW complex. The conclusion follows from Lemma 4.6. $\square$

Recall the naive smash product of prespectra from Definition 4.16.

Definition 7.8. The definition of the smash product of $\mathscr{D}$-spectra given in [MMSS, §§3, 5] implies that there are canonical maps $X_m \wedge Y_n \longrightarrow (X \wedge_S Y)_{m+n}$. These maps for $m = n$ and the structure maps of the prespectrum $\mathbb{U}(X \wedge_S Y)$ give a natural map of prespectra

$$\phi : \mathbb{U}X \wedge \mathbb{U}Y \longrightarrow \mathbb{U}(X \wedge_S Y).$$

Proposition 7.9. *Let $\mathscr{D} = \mathscr{I}$ or $\mathscr{D} = \mathscr{W}$. For a cofibrant $\mathscr{D}$-spectrum X and any $\mathscr{D}$-spectrum Y, $\phi : \mathbb{U}X \wedge \mathbb{U}Y \longrightarrow \mathbb{U}(X \wedge_S Y)$ is a π_*-isomorphism. The analogue for symmetric spectra is false.*

Proof. We shall prove in Proposition 8.3 below that the functor $X \wedge_S Y$ of Y preserves π_*-isomorphisms. Applying this to a cofibrant approximation of Y and using Propositions 4.17 and 7.7, we see that we may assume that both X and Y are cofibrant. Passing to retracts, we see that we may assume that X and Y are FI-cell complexes. By double induction and passage to suspensions, wedges, pushouts, and colimits, it suffices to prove the result when $X = F^{\mathscr{D}}_m S^0$ and $Y = F^{\mathscr{D}}_n S^0$. Here $X \wedge_S Y \cong F^{\mathscr{D}}_{m+n} S^0$ by [MMSS, 3.8] and $F^{\mathscr{D}} \cong \mathbb{P}F^{\mathscr{P}}$. We have an evident π_*-isomorphism

$$\phi : F^{\mathscr{P}}_m S^0 \wedge F^{\mathscr{P}}_n S^0 \longrightarrow F^{\mathscr{P}}_{m+n} S^0;$$

it sends $S^{q-m} \wedge S^{q-n}$ to S^{2q-m-n} at level $2q$. The following diagram commutes:

$$\begin{tikzcd}
F^{\mathscr{P}}_m S^0 \wedge F^{\mathscr{P}}_n S^0 \arrow[r, "\phi"] \arrow[d, "\eta\wedge\eta"'] & F^{\mathscr{P}}_{m+n} S^0 \arrow[d, "\eta"] \\
\mathbb{UP}F^{\mathscr{P}}_m S^0 \wedge \mathbb{UP}F^{\mathscr{P}}_n S^0 \arrow[r, "\phi"'] & \mathbb{UP}F^{\mathscr{P}}_{m+n} S^0.
\end{tikzcd}$$

The maps η and therefore $\eta\wedge\eta$ are π_*-isomorphisms, by Lemma 7.4 and Proposition 4.17, hence the bottom map ϕ is a π_*-isomorphism. In the case of symmetric spectra, this argument does not apply and the bottom map ϕ is not a π_*-isomorphism, by inspection of definitions as in Lemma 5.5. □

Proof of Theorem 0.3. It follows directly that the equivalence induced on homotopy categories by $\mathbb{U} : \mathscr{D}\mathscr{S} \longrightarrow \mathscr{P}$ preserves smash products when $\mathscr{D} = \mathscr{I}$ or $\mathscr{D} = \mathscr{W}$. Because the equivalence induced by $\mathbb{U} : \mathscr{I}\mathscr{S} \longrightarrow \Sigma\mathscr{S}$ also preserves smash products, it follows formally that the equivalence induced by $\mathbb{U} : \Sigma\mathscr{S} \longrightarrow \mathscr{P}$ preserves smash products. Proposition 7.9 just indicates that the equivalence is not given in the most naive fashion. □

8. Model categories of ring and module spectra

So far in our work, we have resolutely ignored the main point of the introduction of categories of diagram spectra, namely the fact that the category of $\mathscr{D}$-spectra is symmetric monoidal under its smash product $\wedge_S$ when the sphere $\mathscr{D}$-space S is a commutative $\mathscr{D}$-monoid. This holds for all of the categories except $\mathscr{P}$ displayed in the Main Diagram in the introduction. We are writing $\wedge_S$ to avoid confusion with smash products with spaces and as a reminder that the category $\mathscr{D}\mathscr{S}$ of $\mathscr{D}$-spectra coincides with the category of S-modules.

It is now an easy matter to obtain model structures on categories of $\mathscr{D}$-ring and module spectra when $\mathscr{D}$ is Σ, $\mathscr{I}$, or $\mathscr{W}$; we write $\mathscr{D}$ generically for any of these categories. We will quote the general theory of Schwede and Shipley [27], but we are thinking of a variant of that theory based on Proposition 1.14. In particular, use of the Cofibration Hypothesis 1.3, which is verified by exact mimicry of details in [5, VII], allows us to avoid transfinite compositions.

For maps $i : X \longrightarrow Y$ and $j : W \longrightarrow Z$ of $\mathscr{D}$-spectra, we have the map

$$i\Box j : (Y \wedge_S W) \cup_{X\wedge_S W} (X \wedge_S Z) \longrightarrow Y \wedge_S Z$$

of (1.18). By Lemma 3.7, if i and j are q-cofibrations, then so is $i\Box j$. In the language of [27], we will prove the pushout-product and monoid axioms for $\mathscr{D}$-spectra. We make repeated use of the following observation. Recall that, by Lemma 1.5, a q-cofibration is an h-cofibration.

Lemma 8.1. *If $i : X \longrightarrow Y$ is an h-cofibration of $\mathscr{D}$-spectra and W is any $\mathscr{D}$-spectrum, then $i \wedge_S \mathrm{id} : X \wedge_S W \longrightarrow Y \wedge_S W$ is an h-cofibration.*

Proof. Smashing with W preserves colimits and smash products with spaces and so preserves the relevant retraction. □

Proposition 8.2 (Pushout-product axiom)**.** *If $i : X \longrightarrow Y$ and $j : W \longrightarrow Z$ are q-cofibrations of $\mathscr{D}$-spectra and i is a stable weak equivalence, then the q-cofibration $i\Box j$ is a stable weak equivalence.*

Proof. By Proposition 8.5 below, $i \wedge_S \mathrm{id} : X \wedge_S Z \longrightarrow Y \wedge_S Z$ is a stable weak equivalence for any Z. By Lemma 5.13 (or 4.11), any cobase change of an h-cofibration that is a stable weak equivalence is a stable weak equivalence. It is immediate from the definition of $i\Box j$ that its composite with the cobase change of $i \wedge_S \mathrm{id}_W$ along $\mathrm{id}_X \wedge_S j$ is $i \wedge_S \mathrm{id}_Z$. Therefore $i\Box j$ is a stable weak equivalence. □

Proposition 8.3. *For any cofibrant $\mathscr{D}$-spectrum X, the functor $X\wedge_S(-)$ preserves π_*-isomorphisms and stable weak equivalences.*

Proof. Of course, when $\mathscr{D} = \mathscr{I}$ or $\mathscr{D} = \mathscr{W}$, π_*-isomorphisms are the same as stable weak equivalences. We shall prove the result when $X = F_n S^n$ shortly. Using the fact that $F_n A \cong (F_n S^0) \wedge A$ together with Lemmas 4.9, 4.14, 4.10, 4.11, 4.12, and 4.13 for π_*-isomorphisms and Lemmas 5.11, 5.16, 5.12, 5.13, 5.14, and 5.15 for stable weak equivalences, we deduce in order that the conclusion holds when $X = F_n S^0$, when $X = F_n A$ for a finite CW-complex A, and when X is an FI-cell complex. Passage to retracts gives the general case. We treat the case $X = F_n S^n$ separately in the three cases. The proofs make use of explicit descriptions of the functors at hand that are implied by [MMSS, 8.2 and 8.4]. The arguments are similar to the proof of Lemma 5.5.

Symmetric spectra. Here, for $q \geq n$, we have

$$\begin{aligned} (F_n S^n \wedge_S Y)(q) &= \Sigma_{q+} \wedge_{\Sigma_{q-n}} (S^n \wedge Y(q-n)) \\ &\cong (\Sigma_q/\Sigma_{q-n})_+ \wedge (S^n \wedge Y(q-n)). \end{aligned}$$

We find by passage to colimits that $\pi_*(F_n S^n \wedge_S Y)$ is naturally the sum of countably many copies of $\pi_*(Y)$. Thus the functor $F_n S^n \wedge_S Y$ preserves π_*-isomorphisms. To show that it preserves stable weak equivalences, we now see by application of functorial cofibrant approximation in the level model structure that the conclusion will hold for stable weak equivalences in general if it holds for stable weak equivalences between cofibrant symmetric spectra. For cofibrant Y and any E, we have

$$[F_n S^n \wedge_S Y, E] \cong [Y, F_S(F_n S^n, E)],$$

naturally in Y. Since $F_S(F_n S^n, E) = R^n E$ is a Σ-Ω-spectrum by the proof of Proposition 5.7, the conclusion follows.

Orthogonal spectra. Here, for $q \geq n$, we have

$$\begin{aligned} (F_n S^n \wedge_S Y)(q) &= O(q)_+ \wedge_{O(q-n)} (S^n \wedge Y(q-n)) \\ &\cong (O(q)/O(q-n))_+ \wedge (S^n \wedge Y(q-n)). \end{aligned}$$

We find by passage to colimits that $\pi_*(F_n S^n \wedge_S Y)$ is naturally isomorphic to $\pi_*(Y)$, hence the functor $F_n S^n \wedge Y$ preserves π_*-isomorphisms.

Orthogonal spectra and $\mathscr{W}$-spaces. We give a second proof for orthogonal spectra that will imply a proof for $\mathscr{W}$-spaces that does not require explicit examination of the homotopy groups of $F_n S^n \wedge_S Y$. To begin with, we use either orthogonal spectra or $\mathscr{W}$-spaces. By Lemmas 4.15 and 8.1, it suffices to prove that $\pi_*(F_n S^n \wedge_S Y) = 0$ if $\pi_*(Y) = 0$. Let $\gamma_n : F_n S^n \longrightarrow F_0 S^0 = S$ be adjoint to the identity map $S^n \longrightarrow S_n = S^n$ at level n. Let $x \in \pi_q(F_n S^n \wedge Y)$, and also write x for a representative map $S^{q+r} \longrightarrow (F_n S^n \wedge_S Y)_r$. Let

$$y = (\gamma_n \wedge \mathrm{id}) \circ x : S^{q+r} \longrightarrow (S \wedge_S Y)_r = Y_r.$$

We may choose r large enough that y is null homotopic, and then its adjoint $\tilde{y} : F_r S^{q+r} \longrightarrow Y$ is also null homotopic. Let

$$z = \mathrm{id} \wedge \tilde{y} : F_{n+r}(S^{n+q+r}) \cong F_n S^n \wedge_S F_r S^{q+r} \longrightarrow F_n S^n \wedge_S Y$$

and let $x' : S^{n+q+r} \longrightarrow (F_n S^n \wedge_S Y)_{n+r}$ be the adjoint of z. Then z and x' are null homotopic. We claim that x' is homotopic to the composite

$$\sigma \circ (\Sigma^n x) : \Sigma^n S^{q+r} \longrightarrow \Sigma^n (F_n S^n \wedge_S Y)_r \longrightarrow (F_n S^n \wedge_S Y)_{r+n},$$

so that the given element $x \in \pi_q(F_nS^n \wedge_S Y)$ is zero. Let $\tilde{x} : F_rS^{q+r} \longrightarrow F_nS^n \wedge_S Y$ be the adjoint of x. The adjoints $F_{r+n}S^{q+r+n} \cong F_rS^{q+r} \wedge F_nS^n \longrightarrow F_nS^n \wedge_S Y$ of the two maps in question are the composites of

$$\tilde{x} \wedge \mathrm{id} : F_rS^{q+r} \wedge F_nS^n \longrightarrow F_nS^n \wedge_S Y \wedge_S F_nS^n$$

with the two maps obtained by applying γ_n to one or the other factor F_nS^n. Thus it suffices to prove that

$$\gamma_n \wedge \mathrm{id} \simeq \mathrm{id} \wedge \gamma_n : F_{2n}S^{2n} \cong F_nS^n \wedge_S F_nS^n \longrightarrow F_nS^n.$$

So far the argument has been identical for orthogonal spectra and for $\mathscr{W}$-spaces. We prove this last step for orthogonal spectra. The conclusion for $\mathscr{W}$-spaces will follow upon application of the functor $\mathbb{P}$. For orthogonal spectra, the adjoints

$$S^{2n} \longrightarrow (F_nS^n)_{2n} = O(2n)_+ \wedge_{O(n)} S^{2n} \cong O(2n)/O(n)_+ \wedge S^{2n}$$

of the two maps send s to $1 \wedge s$ and to $\tau \wedge s$, where $\tau \in O(2n)$ is the evident transposition on $\mathbb{R}^n \times \mathbb{R}^n$. These maps are homotopic since $O(2n)/O(n)$ is connected. □

We shall later need the following consequence of this result.

Corollary 8.4. *When $\mathscr{D} = \mathscr{I}$ or $\mathscr{D} = \mathscr{W}$, $\gamma_k \wedge \mathrm{id} : F_kS^k \wedge_S Y \longrightarrow Y$ is a π_*-isomorphism for any $\mathscr{D}$-spectrum Y.*

Proof. Let $q : X \longrightarrow Y$ be a π_*-isomorphism, where X is cofibrant. Proposition 8.3 gives that $\gamma_k \wedge \mathrm{id}_X$ and $\mathrm{id} \wedge q : F_kS^k \wedge_S X \longrightarrow F_kS^k \wedge_S Y$ are π_*-isomorphisms. Since $q \circ (\gamma_k \wedge \mathrm{id}_X) = (\gamma_k \wedge \mathrm{id}_Y) \circ (\mathrm{id} \wedge q)$, this gives the conclusion. □

Proposition 8.5 (Monoid axiom)**.** *For any acyclic q-cofibration $i : A \longrightarrow X$ of $\mathscr{D}$-spectra and any $\mathscr{D}$-spectrum Y, the map $i \wedge_S \mathrm{id} : A \wedge_S Y \longrightarrow X \wedge_S Y$ is a stable weak equivalence and an h-cofibration. Moreover, cobase changes and colimits of such maps are also weak equivalences and h-cofibrations.*

Proof. Let $Z = X/A$ and note that Z is homotopy equivalent to the cofiber Ci. Then Z is an acyclic cofibrant $\mathscr{D}$-space. Since the functor $- \wedge_S Y$ preserves cofiber sequences, Lemma 5.17 implies that it suffices to prove that $Z \wedge_S Y$ is acyclic. Let $j : Y' \to Y$ be a cofibrant approximation of Y. By Proposition 8.3, $\mathrm{id}_Z \wedge_S j$ is a stable weak equivalence. Thus, we may assume that Y as well as Z is cofibrant. Here Proposition 8.3 gives the conclusion since $* \longrightarrow Z$ is a stable weak equivalence and $* \wedge_S Y = *$. The last statement holds since cobase changes and colimits of maps that are h-cofibrations and stable weak equivalences are h-cofibrations and stable weak equivalences, by Lemmas 5.13 and 5.15. □

By the methods and results of [27], these results have the following consequences. The essential point is that the pushout-product and monoid axioms allow verification of (i) and (ii) of Proposition 1.14. The hardest case is that of R-algebras, where the verifications require combinatorial analysis of pushouts (= amalgamated free products) of R-algebras; see [5, VII.6.1] or [27, 5.2]. Observe that the unit S of the smash product of $\mathscr{D}$-spectra is cofibrant.

Theorem 8.6. *Let R be a $\mathscr{D}$-ring spectrum, where $\mathscr{D} = \Sigma$, $\mathscr{I}$, or $\mathscr{W}$.*

(i) *The category of left R-modules is a compactly generated model category with weak equivalences and q-fibrations created in $\mathscr{D}\mathscr{S}$. Moreover, if R is cofibrant as a $\mathscr{D}$-spectrum, then every cofibrant R-module is cofibrant as a $\mathscr{D}$-spectrum.*

(ii) *If R is commutative, the symmetric monoidal category $\mathscr{D}\mathscr{S}_R$ of R-modules also satisfies the pushout-product and monoid axioms.*
(iii) *If R is commutative, the category of R-algebras is a compactly generated model category with weak equivalences and q-fibrations created in $\mathscr{D}\mathscr{S}$. Moreover, every q-cofibration of R-algebras whose source is cofibrant as an R-module is a q-cofibration of R-modules; in particular, every cofibrant R-algebra is cofibrant as an R-module.*
(iv) *If $f : Q \longrightarrow R$ is a weak equivalence of $\mathscr{D}$-ring spectra, then restriction and extension of scalars define a Quillen equivalence between the categories of Q-modules and of R-modules.*
(v) *If $f : Q \longrightarrow R$ is a weak equivalence of commutative $\mathscr{D}$-ring spectra, then restriction and extension of scalars define a Quillen equivalence between the categories of Q-algebras and of R-algebras.*

Proof. We focus on (i) and (iii). In both, a map is a weak equivalence or q-fibration if and only if it is a weak equivalence or q-fibration of underlying $\mathscr{D}$-spectra. The generating q-cofibrations and acyclic q-cofibrations are obtained by applying the free R-module functor $R \wedge_S (-)$ or the free R-algebra functor $\mathbb{T}$ to the generating q-cofibrations and acyclic q-cofibrations of $\mathscr{D}$-spectra. Here $\mathbb{T}X = \bigvee_{i\geq 0} X^{(i)}$. The defining adjunctions for the functors $R \wedge (-)$ and $\mathbb{T}$ imply that, if A is a compact $\mathscr{D}$-spectrum, then $R \wedge_S A$ is a compact R-module and $\mathbb{T}A$ is a compact R-algebra, in the sense of Definition 1.6.

We apply Proposition 1.14 to prove the model axioms. Thus we must verify the Cofibration Hypothesis 1.3 for the cited sets of generating q-cofibrations and generating acyclic q-cofibrations, and we must show that relative cell complexes generated by the latter are stable weak equivalences. As in Lemma 1.5, a relative $(R \wedge_S FI)$-cell or $(R \wedge_S K)$-cell R-module is an h-cofibration of R-modules and therefore an h-cofibration of $\mathscr{D}$-spectra. Arguing as in [5, VII.3.9 and 3.10], with a slight elaboration to deal with the maps of K not in FJ, the same is true for relative $\mathbb{T}FI$-cell or $\mathbb{T}K$-cell R-algebras. This verifies the Cofibration Hypothesis 1.3. The monoid axiom implies directly that a relative $R \wedge_S K$-cell complex is a stable weak equivalence, and the second statement of (i) follows easily from Lemma 3.7. The proof that relative $\mathbb{T}K$-cell complexes are stable weak equivalences and the proof of the second statement of (iii) require the combinatorial analysis of pushouts in the category of R-algebras given in [5, VII.6.1] and [27, 5.2]. For (iv) and (v), the following result verifies a hypothesis required in [27]. □

Proposition 8.7. *For a cofibrant right R-module M, the functor $M \wedge_R N$ of N preserves π_*-isomorphisms and stable weak equivalences.*

Proof. It suffices to prove the result when M is an $(FI \wedge_S R)$-cell R-module. As in the proof of Proposition 8.3, we see by induction up the cell filtration that it suffices to prove the result when $M = F_n A \wedge_S R$ for a based CW-complex A. Then $M \wedge_R N \cong F_n A \wedge_S N$ and the result holds by Proposition 8.3. □

We now prove Theorems 0.4 and 0.5, which compare various categories of ring and module diagram spectra.

Proof of Theorem 0.4. The functors $\mathbb{P}$ and $\mathbb{U}$ between symmetric and orthogonal spectra preserve ring spectra, and they restrict to an adjoint pair relating the categories of symmetric and orthogonal ring spectra. This is clearly a Quillen adjoint pair since, in both cases, the forgetful functor to $\mathscr{D}$-spectra creates the weak equivalences and q-fibrations. Since the underlying symmetric spectrum of a cofibrant symmetric ring spectrum is cofibrant, by Theorem 8.6(iii), the restricted pair is a Quillen equivalence by Lemma 2.2(iii). The comparison between orthogonal ring spectra and $\mathscr{W}$-spaces is proven the same way. □

Proof of Theorem 0.5. Similarly, for a symmetric ring spectrum R, $(\mathbb{P}, \mathbb{U})$ restricts to a Quillen adjoint pair between the categories of R-modules and of $\mathbb{P}R$-modules, by [MMSS, 7.2]. By Theorem 8.6(iv), we may assume without loss of generality that R is cofibrant as an R-algebra. Then R and all cofibrant R-modules are cofibrant as symmetric spectra, by Theorem 8.6(i) and (iii), and the restricted pair is a Quillen equivalence by Lemma 2.2(iii). The comparison between modules over an orthogonal ring spectrum R and over the $\mathscr{W}$-ring space $\mathbb{P}R$ works the same way. □

9. The positive stable model structure on $\mathscr{D}$-spectra

We return to the context of Section 5, letting $\mathscr{D}$ be any of $\mathscr{P}$, Σ, $\mathscr{I}$, or $\mathscr{W}$. In the last three cases, we seek a model category of commutative $\mathscr{D}$-ring spectra. However, because the sphere $\mathscr{D}$-spectrum is cofibrant, the stable model structure cannot create a model structure on the category of commutative $\mathscr{D}$-ring spectra. A fibrant approximation of S as a commutative $\mathscr{D}$-ring spectrum would be an Ω-spectrum with zeroth space a commutative topological monoid weakly equivalent to QS^0. That would imply that QS^0 is weakly equivalent to a product of Eilenberg-Mac Lane spaces. This is a manifestation of Lewis's observation [9] that one cannot have an ideal category of spectra that is ideally related to the category of spaces.

Thus, following an idea of Jeff Smith, we modify the stable model structure in such a way that $S_{\mathscr{D}}$ is no longer cofibrant. This is very easy to do. Basically, we just modify the arguments of Sections 3, 5, and 6 by starting with the level model structure relative to $\mathscr{N} - \{0\}$ rather than relative to $\mathscr{N}$.

Definition 9.1. Let $f : X \longrightarrow Y$ be a map of $\mathscr{D}$-spectra.

- (i) f is a *positive level weak equivalence* if f_n is a weak equivalence for $n > 0$.
- (ii) f is a *positive level fibration* if f_n is a Serre fibration for $n > 0$.
- (iii) f is a *positive level acyclic fibration* if it is both a positive weak equivalence and a positive level fibration.
- (iv) f is a *positive q-cofibration* if it satisfies the LLP with respect to the positive level acyclic fibrations.
- (v) f is a *positive level acyclic q-cofibration* if it is both a positive level weak equivalence and a positive q-cofibration.

Definition 9.2. Let F^+I be the set of standard cells $F_n i : F_n S^{q-1}_+ \longrightarrow F_n D^q_+$ with $n > 0$ and $q \geq 0$. Let F^+J be the set of maps $F_n i_0 : F_n D^q_+ \longrightarrow F_n (D^q \times I)_+$ with $n > 0$ and $q \geq 0$. Let K^+ be the union of F^+J and the K_n for $n > 0$.

Theorem 9.3. *The category of $\mathscr{D}$-spectra is a compactly generated proper topological model category with respect to the positive level weak equivalences, positive level fibrations, and positive level q-cofibrations. The sets F^+I and F^+J are the generating sets of positive q-cofibrations and positive level acyclic q-cofibrations. The positive q-cofibrations are the q-cofibrations that are homeomorphisms at level* 0.

Proof. Since the model structure we have specified is the level model structure relative to $\mathscr{N} - \{0\}$, only the last statement is not part of the relative version of

Theorem 3.4. The last statement follows from the fact that a map is a positive q-cofibration if and only if it is a retract of a relative F^+I-cell complex and the observation that a relative FI-cell complex is a homeomorphism at level 0 if and only if no standard cells $F_0 i$ occur in its construction. $\square$

Definition 9.4. A prespectrum X is a positive Ω-spectrum if $\tilde{\sigma} : X_n \longrightarrow \Omega X_{n+1}$ is a weak equivalence for $n > 0$. A $\mathscr{D}$-spectrum X is a positive $\mathscr{D}$-Ω-spectrum if $\mathbb{U}X$ is a positive Ω-spectrum.

Definition 9.5. Let $f : X \longrightarrow Y$ be a map of $\mathscr{D}$-spectra.

(i) f is an *acyclic positive q-cofibration* if it is a stable weak equivalence and a positive q-cofibration.
(ii) f is a *positive q-fibration* if it satisfies the RLP with respect to the acyclic positive q-cofibrations.
(iii) f is an *acyclic positive q-fibration* if it is a stable weak equivalence and a positive q-fibration.

Theorem 9.6. *The category $\mathscr{D}\mathscr{S}$ is a compactly generated proper topological model category with respect to the stable weak equivalences, positive q-fibrations, and positive q-cofibrations. The sets F^+I and K^+ are the generating positive q-cofibrations and generating acyclic positive q-cofibrations. The positive fibrant $\mathscr{D}$-spectra are the positive $\mathscr{D}$-Ω-spectra. The pushout-product and monoid axioms are satisfied.*

For the proof, we need a characterization of the stable weak equivalences in terms of the positive level model structure. Let $[X, Y]^+$ denote the set of maps $X \longrightarrow Y$ in the homotopy category associated to the positive level model structure.

Lemma 9.7. *For $\mathscr{D}$-Ω-spectra E, $[X, E]^+$ is naturally isomorphic to $[X, E]$.*

Proof. Let $q : X' \longrightarrow X$ be a cofibrant approximation to X in the positive level model structure. Then $q^* : [X, E]^+ \longrightarrow [X', E]^+$ is an isomorphism. Since q is a π_*-isomorphism and thus a stable weak equivalence by Proposition 5.7, $q^* : [X, E] \longrightarrow [X', E]$ is also an isomorphism. However, since X' is cofibrant in both model structures, $[X', E] = \pi(X', E) = [X', E]^+$. $\square$

Proposition 9.8. *A map $f : X \longrightarrow Y$ is a stable weak equivalence if and only if $f^* : [Y, E]^+ \longrightarrow [X, E]^+$ is a bijection for all positive $\mathscr{D}$-Ω-spectra E.*

Proof. First, let f be a stable weak equivalence and E be a positive $\mathscr{D}$-Ω-spectrum. Construct RE as in the proof of Proposition 5.7. Then RE is a $\mathscr{D}$-Ω-spectrum and the natural map $E \longrightarrow RE$ is a positive level equivalence. By application of Lemma 9.7 to RE, $f^* : [Y, E]^+ \longrightarrow [X, E]^+$ is a bijection since $f^* : [Y, RE] \longrightarrow [X, RE]$ is a bijection. Since a $\mathscr{D}$-Ω-spectrum is a positive $\mathscr{D}$-Ω-spectrum, the converse implication is immediate from Lemma 9.7. $\square$

From here, Theorem 9.6 is proven by the same arguments as for the stable model structure, with everything restricted to positive levels. Its last statement implies the following analogue of Theorem 8.6 for the positive stable model structure.

Theorem 9.9. *Let R be a $\mathscr{D}$-ring spectrum, where $\mathscr{D} = \Sigma$, $\mathscr{I}$, or $\mathscr{W}$.*

(i) *The category of left R-modules is a compactly generated model category with weak equivalences and q-fibrations created in $\mathscr{D}\mathscr{S}$ with its positive stable model structure.*

(ii) *If R is commutative, the symmetric monoidal category $\mathscr{D}\mathscr{S}_R$ of R-modules satisfies the pushout-product and monoid axioms.*
(iii) *If R is commutative, the category of R-algebras is a compactly generated model category with weak equivalences and q-fibrations created in $\mathscr{D}\mathscr{S}$.*
(iv) *If $f : Q \longrightarrow R$ is a weak equivalence of $\mathscr{D}$-ring spectra, then restriction and extension of scalars define a Quillen equivalence between the categories of Q-modules and of R-modules.*
(v) *If $f : Q \longrightarrow R$ is a weak equivalence of commutative $\mathscr{D}$-ring spectra, then restriction and extension of scalars define a Quillen equivalence between the categories of Q-algebras and of R-algebras.*

The statements about cofibrant objects in (i) and (iii) of Theorem 8.6 are no longer valid since S is not cofibrant. However, since we have both model structures on hand, this is not a serious defect. For example, parts (iv) and (v) of the previous theorem no longer follow directly from [27]. Rather, they follow from parts (iv) and (v) of Theorem 8.6 for the stable model structures and the following comparison result, whose proof is immediate.

Proposition 9.10. *The identity functor from $\mathscr{D}\mathscr{S}$ with its positive stable model structure to $\mathscr{D}\mathscr{S}$ with its stable model structure is the left adjoint of a Quillen equivalence. It restricts to a Quillen equivalence on the category of $\mathscr{D}$-ring spectra, on the category of left modules over a $\mathscr{D}$-ring spectrum, and on the category of algebras over a commutative $\mathscr{D}$-ring spectrum.*

Remark 9.11. The proofs in the previous section show that Theorems 0.1, 0.4, and 0.5 remain valid when reinterpreted in terms of the positive stable model structures. The essential point is that, since these structures have fewer cofibrant objects, verification of the hypothesis of Lemma 2.2(iii) for the stable model structures is more than enough to verify the hypothesis for the positive stable model structures.

10. The model structure on commutative $\mathscr{D}$-ring spectra

We here prove the following two theorems, which together imply Theorem 0.6. Let $\mathbb{A}$ be the monad on $\mathscr{D}$-spectra that defines commutative $\mathscr{D}$-ring spectra. Thus $\mathbb{A}X = \bigvee_{i\geq 0} X^{(i)}/\Sigma_i$, where $X^{(i)}$ denotes the ith smash power, with $X^{(0)} = S$.

Theorem 10.1. *Let $\mathscr{D} = \Sigma$ or $\mathscr{I}$. The category of commutative $\mathscr{D}$-ring spectra is a compactly generated topological model category with q-fibrations and weak equivalences created in the positive stable model category of $\mathscr{D}$-spectra. The sets $\mathbb{A}F^+I$ and $\mathbb{A}K^+$ are the generating sets of q-cofibrations and acyclic q-cofibrations.*

Theorem 10.2. *The pair $(\mathbb{P}, \mathbb{U})$ restricts to a Quillen equivalence between the categories of commutative symmetric ring spectra and commutative orthogonal ring spectra.*

By Propositions 1.1, 1.2, and 1.14, Theorem 10.1 is a consequence of the following two results.

Lemma 10.3. *The sets $\mathbb{A}F^+I$ and $\mathbb{A}K^+$ satisfy the Cofibration Hypothesis 1.3.*

Lemma 10.4. *Every relative $\mathbb{A}K^+$-cell complex is a stable weak equivalence.*

We single out for emphasis the key step of the proof of Lemma 10.4. It is the analogue for symmetric spectra and orthogonal spectra of [5, III.5.1] for the S-modules of Elmendorf, Kriz, Mandell, and May. We do not know whether or not the analogue for $\mathscr{W}$-spaces or $\mathscr{F}$-spaces holds, and it is for this reason that we do not have results for commutative rings in those cases. We assume henceforward in this section that $\mathscr{D} = \Sigma$ or $\mathscr{D} = \mathscr{I}$. It is an insight of Smith that restriction to positive cofibrant symmetric spectra suffices to obtain the following conclusion.

Lemma 10.5. *Let K be a based CW complex, X be a $\mathscr{D}$-spectrum, and $n > 0$. Then the quotient map*

$$q : (E\Sigma_{i+} \wedge_{\Sigma_i} (F_n K)^{(i)}) \wedge_S X \longrightarrow ((F_n K)^{(i)}/\Sigma_i) \wedge_S X$$

is a level homotopy equivalence. For any positive cofibrant $\mathscr{D}$-spectrum X,

$$q : E\Sigma_{i+} \wedge_{\Sigma_i} X^{(i)} \longrightarrow X^{(i)}/\Sigma_i$$

is a π_*-isomorphism.

Proof. We give the details for $\mathscr{D} = \mathscr{I}$. The result for $\mathscr{D} = \Sigma$ is proven by the same argument, but with orthogonal groups replaced by symmetric groups. By [MMSS, 8.4 and 3.8] and inspection of coequalizers,

$$((F_n K)^{(i)} \wedge_S X)(q) \cong O(q)_+ \wedge_{O(q-ni)} (K^{(i)} \wedge X(q-ni)).$$

The action of $\sigma \in \Sigma_i$ is to permute the factors in $K^{(i)}$ and to act through $\sigma \oplus \mathrm{id}_{q-ni}$ on $O(q)$, where $\sigma \in O(ni)$ permutes the summands of $\mathbb{R}^{ni} = (\mathbb{R}^n)^i$. Since Σ_i acts on $O(q)$ as a subgroup of $O(ni)$, the action commutes with the action of $O(q-ni)$. Therefore, passing to orbits over Σ_i,

$$((F_n K)^{(i)}/\Sigma_i \wedge_S X)(q) \cong O(q)_+ \wedge_{\Sigma_i \times O(q-ni)} (K^{(i)} \wedge X(q-ni)).$$

Similarly,

$$((E\Sigma_{i+} \wedge_{\Sigma_i} (F_n K)^{(i)}) \wedge_S X)(q) \cong (E\Sigma_i \times O(q))_+ \wedge_{\Sigma_i \times O(q-ni)} (K^{(i)} \wedge X(q-ni)).$$

The quotient map $E\Sigma_i \times O(q) \longrightarrow O(q)$ is a $(\Sigma_i \times O(q-ni))$-equivariant homotopy equivalence since $O(q)$ is a free $(\Sigma_i \times O(q-ni))$-space that can be triangulated as a finite $(\Sigma_i \times O(q-ni))$-CW complex. The first statement follows. For the second statement, we may assume that X is an F^+I-cell spectrum, and the proof then is the same induction up the cellular filtration as in the proof of [5, III.5.1]. $\square$

The first statement has the following consequence.

Lemma 10.6. *Let K be a based CW complex and let $n > 0$. Then the functor $\mathbb{A}F_n K \wedge_S (-)$ of $\mathscr{D}$-spectra preserves stable weak equivalences.*

Proof. By induction up the cellular filtration of $E\Sigma_{i+}$, the successive subquotients of which are wedges of copies of $\Sigma_{i+} \wedge S^n$, and use of results in Section 5, the functor $E\Sigma_{i+} \wedge_{\Sigma_i} (-)$ preserves stable weak equivalences. $\square$

Similarly, the second statement implies the following result.

Lemma 10.7. *The functor $\mathbb{A}$ preserves stable weak equivalences between positive cofibrant $\mathscr{D}$-spectra. In particular, each map in $\mathbb{A}K^+$ is a stable weak equivalence.*

The rest of the proofs in this section are analogues of proofs of corresponding results about S-modules in [5]. We shall not give details of those arguments that are essentially identical. For the Cofibration Hypothesis 1.3, we first record the following result, whose proof is the same formal argument as in [5, XII.2.3].

Lemma 10.8. *The functor $\mathbb{A} : \mathscr{D}\mathscr{S} \longrightarrow \mathscr{D}\mathscr{S}$ preserves h-cofibrations.*

Since $\mathbb{A}$ commutes with colimits and thus converts wedges of $\mathscr{D}$-spectra to coproducts, Cofibration Hypothesis 1.3(i) for the set $\mathbb{A}F^+I$ is equivalent to the following lemma. As for commutative monoids in any symmetric monoidal category, the pushout of a diagram $R' \longleftarrow R \longrightarrow R''$ of commutative $\mathscr{D}$-ring spectra is the smash product $R' \wedge_R R''$.

Lemma 10.9. *Let $X \longrightarrow Y$ be a wedge of maps in F^+I and let $f : \mathbb{A}X \longrightarrow R$ be a map of commutative R-algebras. Then the cobase change $j : R \longrightarrow R \wedge_{\mathbb{A}X} \mathbb{A}Y$ is an h-cofibration.*

Proof. The proof is similar to that of the analogous result for commutative S-algebras in [5, VII§3]. We use the geometric realization of simplicial $\mathscr{D}$-spaces. This is constructed levelwise and has properties just like the geometric realization of simplicial spaces and of simplicial spectra; see [18, §11] and [5, X§1]. We also use the two-sided bar construction; see [18, §9] and [5, XII].

We first give a convenient, although rather baroque, model for the inclusion $i : S^{q-1}_+ \longrightarrow D^q_+$. Think of the unit interval I as the geometric realization of the standard simplicial 1-simplex $\Delta[1]$. For any space A, $(A \times I)_+ \cong A_+ \wedge I_+$ is homeomorphic to the geometric realization of the simplicial space $A_+ \wedge \Delta[1]_+$. Since $\Delta[1]$ is discrete, the space of q-simplices of $A_+ \wedge \Delta[1]_+$ is the wedge of one copy of A_+ for each simplex of $\Delta[1]$. An explicit examination of the faces and degeneracies of $\Delta[1]$ [17, p.14] shows that $A_+ \wedge \Delta[1]_+$ can be identified with the simplicial bar construction $B_*(A_+, A_+, A_+)$, whose space of q-simplices is the wedge of $q+2$ copies of A_+. The faces and degeneracies are given by successive applications of the folding map $\nabla : A_+ \vee A_+ \longrightarrow A_+$ and inclusions of wedge summands, and all q-simplices with $q > 1$ are degenerate. The inclusion of the zeroth and last wedge summands A_+ in each simplicial degree induce the inclusions i_0 and i_1 of A_+ in $A_+ \wedge I_+$ on passage to realization. Write $B(-)$ for the geometric realization of simplicial bar constructions $B_*(-)$ and let CA be the unreduced cone on A. The quotient map $A_+ \wedge I_+ \longrightarrow (CA)_+$ is isomorphic to the map

$$B(A_+, A_+, A_+) \longrightarrow B(A_+, A_+, S^0)$$

induced by the evident map $A_+ \longrightarrow pt_+ = S^0$, and the inclusion $i_0 : A_+ \longrightarrow (CA)_+$ is isomorphic to the map $\iota : A_+ \longrightarrow B(A_+, A_+, S^0)$ induced from the inclusion of A_+ in the space of zero simplices. Taking $A = S^{q-1}$ and identifying $i : S^{q-1}_+ \longrightarrow D^q_+$ with $i_0 : S^{q-1}_+ \longrightarrow (CS^{q-1})_+$, we can identify i with $\iota : S^{q-1}_+ \longrightarrow B(S^{q-1}_+, S^{q-1}_+, S^0)$.

The functor F_n commutes with colimits and with smash products with based spaces, hence commutes with geometric realization and the bar construction. We can apply wedges to the construction to obtain a similar description of a wedge of a set of standard cells. Explicitly, if $X = \vee_i F_{n_i} S^{q_i-1}_+$ and $Y = \vee_i F_{n_i} D^{q_i}_+$, then $Y \cong B(X, X, T)$ under X, where $T = \vee_i F_{n_i} S^0$. Here $B(X, X, T)$ is the geometric realization of the evident simplicial $\mathscr{D}$-spectrum whose $\mathscr{D}$-spectrum of q-simplices is the wedge of $q + 1$ copies of X and a copy of T.

By Proposition 1.1, the category of $\mathscr{D}$-ring spectra is tensored over the category of unbased spaces; an explicit construction is given in [5, VII.2.10]. The functor $\mathbb{A}$ from $\mathscr{D}$-spectra to commutative $\mathscr{D}$-ring spectra commutes with colimits and converts smash products $X \wedge A_+$ to tensors $\mathbb{A}X \otimes A$, where X is a $\mathscr{D}$-spectrum and A is an unbased space. As is discussed in an analogous situation in [5, VII§3], it follows that $\mathbb{A}$ converts geometric realizations and bar constructions to similar constructions defined in terms of the category of simplicial commutative $\mathscr{D}$-ring spectra.

Exactly as in [5, VII.3.3], the geometric realization of a simplicial commutative $\mathscr{D}$-ring spectrum R_* can be computed by forgetting the ring structure on each R_q, taking geometric realization as a simplicial $\mathscr{D}$-spectrum, and giving this geometric realization the structure of commutative $\mathscr{D}$-ring spectrum that it inherits from R_*. With the notation above, we have the identification

$$R \wedge_{\mathbb{A}X} \mathbb{A}Y \cong R \wedge_{\mathbb{A}X} B(\mathbb{A}X, \mathbb{A}X, \mathbb{A}T) \cong B(R, \mathbb{A}X, \mathbb{A}T) \tag{10.10}$$

under R. It follows as in [5, VII.3.9] that $j : R \longrightarrow R \wedge_{\mathbb{A}X} \mathbb{A}Y$ is an h-cofibration. In summary, the degeneracy operators of the simplicial $\mathscr{D}$-spectrum $B_*(R, \mathbb{A}X, \mathbb{A}T)$ are inclusions of wedge summands, hence $B_*(R, \mathbb{A}X, \mathbb{A}T)$ is *proper*, in the sense that its degenerate q-simplices map by an h-cofibration into its q-simplices; compare [5, X.2.2]. This implies that the map from the $\mathscr{D}$-spectrum of zero simplices into the realization is an h-cofibration, and the map from R into the $\mathscr{D}$-spectrum $R \wedge_S \mathbb{A}T$ is the inclusion of a wedge summand and thus also an h-cofibration. □

Since the maps in $\mathbb{A}K^+$ are relative $\mathbb{A}F^+I$-cell complexes, the previous lemma and Lemma 1.2 imply Cofibration Hypothesis 1.3(i) for $\mathbb{A}K^+$. Cofibration Hypothesis 1.3(ii) for both $\mathbb{A}F^+I$ and $\mathbb{A}K^+$ is implied by the following analogue of [5, VII.3.10], which admits the same easy proof.

Lemma 10.11. *Let $\{R_i \longrightarrow R_{i+1}\}$ be a sequence of maps of commutative $\mathscr{D}$-ring spectra that are h-cofibrations of $\mathscr{D}$-spectra. Then the underlying $\mathscr{D}$-spectrum of the colimit of the sequence computed in the category of commutative $\mathscr{D}$-ring spectra is the colimit of the sequence computed in the category of $\mathscr{D}$-spectra.*

Using Lemma 10.6, the proof of Lemma 10.9 leads to the following analogue of the monoid axiom.

Lemma 10.12. *Let $i : R \longrightarrow R'$ be a relative $\mathbb{A}F^+I$-cell complex. Then the functor $(-) \wedge_R R'$ on commutative R-algebras preserves stable weak equivalences.*

Proof. First let i be the map $\mathbb{A}X \longrightarrow \mathbb{A}Y$ obtained by applying $\mathbb{A}$ to a wedge $X \longrightarrow Y$ of maps in $\mathbb{A}F^+I$. By (10.10), the functor $(-) \wedge_{\mathbb{A}X} \mathbb{A}Y$ is isomorphic to the bar construction $B(-, \mathbb{A}X, \mathbb{A}T)$. In each simplicial degree, the functor $B_q(-, \mathbb{A}X, \mathbb{A}T)$ preserves stable weak equivalences by inductive use of Lemma 10.6. By the $\mathscr{D}$-spectrum analogue of [5, X.2.4], it follows that the functor $B(-, \mathbb{A}X, \mathbb{A}T)$ preserves stable weak equivalences. Given a pushout diagram of commutative $\mathscr{D}$-ring spectra

$$\begin{tikzcd} \mathbb{A}X \arrow[r] \arrow[d] & R \arrow[d] \\ \mathbb{A}Y \arrow[r] & R', \end{tikzcd}$$

we have $R' \cong R \wedge_{\mathbb{A}X} \mathbb{A}Y$ and thus $(-) \wedge_R R' \cong (-) \wedge_{\mathbb{A}X} \mathbb{A}Y$. Therefore the conclusion holds in this case, and the general case follows by passage to colimits, using Lemma 10.11. □

Proof of Lemma 10.4. By passage to pushouts and colimits, it suffices to prove that if $i : X \longrightarrow Y$ is a wedge of maps in K^+ and $f : \mathbb{A}X \longrightarrow R$ is a map of commutative $\mathscr{D}$-ring spectra, then the cobase change $j : R \longrightarrow R \wedge_{\mathbb{A}X} \mathbb{A}Y$ is a stable weak equivalence. Applying the small objects argument, factor f as the composite of a relative $\mathbb{A}F^+I$-cell complex $f' : \mathbb{A}X \longrightarrow R'$ and a map $p : R' \longrightarrow R$ that satisfies the RLP with respect to $\mathbb{A}F^+I$. By adjunction, p regarded as a map of $\mathscr{D}$-spectra satisfies the RLP with respect to F^+I. Thus p is an acyclic positive q-fibration of $\mathscr{D}$-spectra. Consider the commutative diagram

$$\begin{tikzcd} R' \arrow[r, "j'"] \arrow[d, "p"'] & R' \wedge_{\mathbb{A}X} \mathbb{A}Y \arrow[d, "p\wedge \mathrm{id}"] \\ R \arrow[r, "j"'] & R \wedge_{\mathbb{A}X} \mathbb{A}Y. \end{tikzcd}$$

Since p is a stable weak equivalence, $p \wedge \mathrm{id}$ is a stable weak equivalence by Lemma 10.12. Using $R' \cong R' \wedge_{\mathbb{A}X} \mathbb{A}X$, Lemma 10.12 also gives that the cobase change j' is a stable weak equivalence. Therefore j is a stable weak equivalence. $\square$

The proof of Theorem 10.2. The functors $\mathbb{P} : \Sigma\mathscr{S} \longrightarrow \mathscr{I}\mathscr{S}$ and $\mathbb{U} : \mathscr{I}\mathscr{S} \longrightarrow \Sigma\mathscr{S}$ restrict to an adjoint pair between the respective categories of commutative ring spectra. Since weak equivalences and q-fibrations of commutative ring spectra are created in the positive stable model categories of underlying spectra, $\mathbb{U}$ preserves weak equivalences and q-fibrations. Thus we have a Quillen adjoint pair. By Lemma 2.2, it suffices to prove that the unit map $\eta : R \longrightarrow \mathbb{U}\mathbb{P}R$ is a stable weak equivalence for every cofibrant commutative symmetric ring spectrum R.

We may assume that R is an $\mathbb{A}F^+I$-cell complex. We claim first that η is a stable weak equivalence when $R = \mathbb{A}X$ for a positive cofibrant symmetric spectrum X, and it suffices to prove that $\eta : X^{(i)}/\Sigma_i \longrightarrow \mathbb{U}\mathbb{P}(X^{(i)}/\Sigma_i)$ is a stable weak equivalence for $i \geq 1$. On the right, $\mathbb{P}(X^{(i)}/\Sigma_i) \cong (\mathbb{P}X)^{(i)}/\Sigma_i$, and $\mathbb{P}X$ is a positive cofibrant orthogonal spectrum. Applying the second statement of Lemma 10.5 to X and to $\mathbb{P}X$, a quick diagram chase shows that the claim is equivalent to the claim that

$$\eta : E\Sigma_{i+} \wedge_{\Sigma_i} X^{(i)} \longrightarrow \mathbb{U}\mathbb{P}(E\Sigma_{i+} \wedge_{\Sigma_i} X^{(i)})$$

is a stable weak equivalence. Using Lemma 7.4 and the fact that suspensions of $X^{(i)}$ are positive cofibrant, this holds by induction up the skeletal filtration of $E\Sigma_i$. By passage to colimits, the result for general R will follow from the result for an $\mathbb{A}F^+I$-cell complex that is constructed in finitely many stages. We have proven the result when R requires only a single stage, and we assume the result when R is constructed in n stages. Thus suppose that R is constructed in $n+1$ stages. Then R is a pushout $R_n \wedge_{\mathbb{A}X} \mathbb{A}Y$, where R_n is constructed in n-stages and $X \longrightarrow Y$ is a wedge of maps in F^+I. By (10.10), $R \cong B(R_n, \mathbb{A}X, \mathbb{A}T)$. Since the simplicial bar construction is proper and since $\mathbb{U}$ and $\mathbb{P}$ commute with colimits and smash products with spaces and thus with geometric realization, the analogue of [5, X.2.4] shows that it suffices to prove that η is a stable weak equivalence on the $\mathscr{D}$-spectrum

$$R_n \wedge_S (\mathbb{A}X)^{(q)} \wedge \mathbb{A}T \cong R_n \wedge_S \mathbb{A}(X \vee \cdots \vee X \vee T)$$

of q-simplices for each q. By the definition of $\mathbb{A}F^+I$-cell complexes, we see that this smash product (= pushout) of commutative $\mathscr{D}$-ring spectra can be constructed in n-stages, hence the conclusion follows from the induction hypothesis. $\square$

11. The absolute stable model structure on $\mathscr{W}$-spaces

The stable model structure on $\mathscr{W}$-spaces studied so far was based on the level model structure relative to $\mathscr{N}$. That is, the level weak equivalences and level fibrations of $\mathscr{W}$-spaces were only required to be weak equivalences or fibrations when evaluated at S^n for $n \geq 0$. The objects of $\mathscr{F} \subset \mathscr{W}$ are the discrete based spaces

$\mathbf{n}^+ = \{0, 1, \cdots, n\}$, and these are not spheres. We need a stable model structure based on the absolute level model structure in order to make a comparison.

Convention 11.1. From now on, the level model structure on $\mathscr{W}\mathscr{T}$ means the absolute level model structure.

Definition 3.1 specifies the absolute level weak equivalences, level fibrations, level acyclic fibrations, q-cofibrations, and level acyclic q-cofibrations in the category of $\mathscr{W}$-spaces. We omit the distinguishing adjective "absolute" henceforward, but it must always be kept in mind that these classes of maps are different from those of the same name used in the previous sections. For a finite based CW-complex A, let $F_A : \mathscr{T} \longrightarrow \mathscr{W}\mathscr{T}$ denote the left adjoint to evaluation at A. We restrict attention to objects A in a skeleton of $\mathscr{W}$. All of these functors F_A are used in Definition 3.2, which specifies the sets FI and FJ of generating q-cofibrations and generating level acyclic q-cofibrations of the level model structure. Recall that $\mathscr{W}$-spaces and $\mathscr{W}$-spectra coincide [MMSS, 8.9], so that a $\mathscr{W}$-space X has a natural pairing

$$\sigma : X(A) \wedge B \longrightarrow X(A \wedge B).$$

With B fixed, these define a map of $\mathscr{W}$-spaces $X \wedge B \longrightarrow X(- \wedge B)$.

Remark 11.2. In view of $\sigma : X(A) \wedge I_+ \longrightarrow X(A \wedge I_+)$, we see that any $\mathscr{W}$-space X is a homotopy-preserving functor. Of course, a weak equivalence in $\mathscr{W}$ is a homotopy equivalence, by Whitehead's theorem. Thus any X is a "homotopy functor", in the sense that it preserves weak equivalences.

We have maps $\sigma : \Sigma X(A) \longrightarrow X(\Sigma A)$ with adjoints $\tilde{\sigma} : X(A) \longrightarrow \Omega X(\Sigma A)$.

Definition 11.3. Let X be a $\mathscr{W}$-space and A be a finite based CW-complex. Define a prespectrum $X[A]$ by setting $X[A]_n = X(S^n \wedge A)$, with structure maps given by instances of σ. Note that $X[S^0] = \mathbb{U}X$, $\mathbb{U} : \mathscr{W}\mathscr{T} \longrightarrow \mathscr{P}$. We also have the prespectrum $X[S^0] \wedge A$. The maps

$$\sigma : X(S^n) \wedge A \longrightarrow X(S^n \wedge A)$$

specify a map of prespectra

$$\sigma[A] : X[S^0] \wedge A \longrightarrow X[A].$$

The homotopy groups $\pi_*(X[S^0] \wedge A)$ are the homology groups of A with respect to the homology theory represented by the prespectrum $X[S^0]$. The insight that the following result should be true is due to Lydakis, who proved an analoue in the simplicial setting [12, 11.7].

Proposition 11.4. *For every $\mathscr{W}$-space X and finite based CW-complex A, $\sigma[A]$ is a π_*-isomorphism. Therefore, if $f : X \longrightarrow Y$ is a π_*-isomorphism, in the sense that $f[S^0] : X[S^0] \longrightarrow Y[S^0]$ is a π_*-isomorphism, then $f[A] : X[A] \longrightarrow Y[A]$ is a π_*-isomorphism for every A.*

Proof. The second statement follows directly from the first and Lemma 4.14. We prove the first statement in stages. First suppose that $X = F_B S^0$, where B is a finite based CW-complex. Then, on nth-spaces, $\sigma[A]$ is the canonical map

$$F(B, S^n) \wedge A \longrightarrow F(B, S^n \wedge A).$$

It is easy to check directly that this map is a π_*-isomorphism. This is just an explicit prespectrum level precursor of a standard result about Spanier-Whitehead duality.

Since $F_B C \cong (F_B S^0) \wedge C$, Lemma 4.14 implies that $\sigma[A]$ is a π_*-isomorphism when $X = F_B C$ for any based CW-complex C. Using Lemmas 4.10, 4.11, 4.12, and 4.13, it follows that $\sigma[A]$ is a π_*-isomorphism when X is a cell FI-complex. For a general X, we factor the trivial map $* \longrightarrow X$ as the composite of a cell FI-complex $* \longrightarrow X'$ and a level acyclic fibration $p : X' \longrightarrow X$. Since $\sigma[A]$ is a π_*-isomorphism for X', it is a π_*-isomorphism for X. □

The following definitions and lemma will turn out to describe the fibrant $\mathscr{W}$-spaces in the new stable model structure that we are about to construct.

Definition 11.5. Consider a commutative diagram of based spaces

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & B \arrow[d, "j"] \\ X \arrow[r, "g"'] & Y. \end{tikzcd}$$

The diagram is a homotopy cocartesian square if the induced map from the homotopy pushout $M(i, f)$ to Y is a weak equivalence. The diagram is homotopy cartesian if the induced map from A to the homotopy pullback $P(g, j)$ is a weak equivalence.

Definition 11.6. A $\mathscr{W}$-space E is *linear* if it converts homotopy cocartesian squares to homotopy cartesian squares.

Lemma 11.7. *The following properties of a $\mathscr{W}$-space are equivalent.*

(i) *E is linear.*
(ii) *Each $E[A]$ is an Ω-spectrum.*
(iii) *Each $\tilde{\sigma} : E(A) \longrightarrow \Omega E(\Sigma A)$ is a weak equivalence.*

Proof. Recall that our functors are assumed to be based, so that $E(*) = *$. If E is linear, then $E(A)$ is weakly equivalent to the homotopy pullback $\Omega E(\Sigma A)$ of the diagram $* \longrightarrow E(\Sigma A) \longleftarrow *$. This weak equivalence is homotopic to the adjoint structure map $\tilde{\sigma}$, hence E satisfies (iii). Conversely, if E satisfies (iii), then the map $\pi_q(E(A)) \longrightarrow \pi_q(E[A]) = \operatorname{colim} \pi_{q+n}(E(S^n \wedge A))$ is an isomorphism for $q \geq 0$, and these $\pi_q(E(A))$ form part of a homology theory. By the five lemma, this implies that, for a cofiber sequence $A \longrightarrow B \xrightarrow{f} C$, the induced map from $E(A)$ to the homotopy fiber of $E(f)$ is a weak equivalence. In turn, this implies that E is linear. The equivalence of (ii) and (iii) is elementary. □

We insert some observations about connectivity for use in the next section.

Definition 11.8. A prespectrum T is *n-connected* if $\pi_q(E) = 0$ for $q \leq n$; T is *connective* if it is (-1)-connected. A $\mathscr{W}$-space X is connective if its underlying prespectrum $X[S^0]$ is connective; X is *strictly connective* if $X(A)$ is n-connected when A is n-connected.

Observe that, on passage to the homotopy groups $\pi_q(X(A))$ of its spaces, a connective linear $\mathscr{W}$-space X defines a homology theory in all degrees.

Lemma 11.9. *A connective linear $\mathscr{W}$-space is strictly connective. The following conditions on a map $f : X \longrightarrow Y$ between connective linear $\mathscr{W}$-spaces are equivalent.*

(i) *f is a π_*-isomorphism.*

(ii) $f : X(S^0) \longrightarrow Y(S^0)$ *is a weak equivalence.*
(iii) f *is a level weak equivalence.*

Proof. If T is an n-connected Ω-spectrum, then its zeroth space is n-connected. If X is connective and linear and A is n-connected, then $X[A]$ is n-connected because its homotopy groups are the homology groups of A with respect to a connective homology theory. Since $X[A]$ is an Ω-spectrum with zeroth space $X(A)$, $X(A)$ is n-connected and X is strictly connective. In the second statement, the equivalence of conditions (i) and (ii) and the implication (iii) implies (ii) are clear, and the implication (ii) implies (iii) follows from Lemma 11.7 and the first statement. □

We now turn to the promised absolute stable model structure on $\mathscr{W}\mathscr{T}$. Its q-cofibrations are those of the level model structure.

Definition 11.10. Let $f : X \longrightarrow Y$ be a map of $\mathscr{W}$-spaces.

(i) f is an acyclic q-cofibration if it is a π_*-isomorphism and a q-cofibration.
(ii) f is a q-fibration if it satisfies the RLP with respect to the acyclic q-cofibrations.
(iii) f is an acyclic q-fibration if it is a π_*-isomorphism and a q-fibration.

Theorem 11.11. *The category of $\mathscr{W}$-spaces is a compactly generated proper topological model category with respect to the π_*-isomorphisms, q-fibrations, and q-cofibrations.*

The comparison of our two stable model structures takes the following form.

Proposition 11.12. *The identity functor from $\mathscr{W}\mathscr{T}$ with its original stable model structure to $\mathscr{W}\mathscr{T}$ with its absolute stable model structure is the left adjoint of a Quillen equivalence.*

The proof of Theorem 11.11 begins with the following definition and lemma, which generalize Definition 5.4 and part of Lemma 5.5.

Definition 11.13. Exactly as in Definition 5.4, define $\lambda_A : F_{\Sigma A}S^1 \longrightarrow F_A S^0$ to be that map of $\mathscr{W}$-spaces such that

$$\lambda_A^* : \mathscr{W}\mathscr{T}(F_A S^0, X) \longrightarrow \mathscr{W}\mathscr{T}(F_{\Sigma A}S^1, X)$$

corresponds under adjunction to $\tilde{\sigma} : X(A) \longrightarrow \Omega X(\Sigma A)$ for all $\mathscr{W}$-spaces X.

Lemma 11.14. *The maps λ_A are π_*-isomorphisms.*

Proof. Using [MMSS, 8.6], we identify $\lambda_A(S^q)$ as the evaluation map

$$\Sigma\Omega F(A, S^q) \longrightarrow F(A, S^q).$$

Applying π_{q+r} and passing to colimits over q, we obtain an isomorphism. This is clear when A is a sphere and follows in general by induction on the number of cells of A, using Lemma 4.15. □

Definition 11.15. The set of generating q-cofibrations for the stable model structure of Theorem 11.11 is the set FI specified in Definition 3.2. The set K of generating acyclic q-cofibrations is the union of the set FJ defined there with the sets $k_A \square I$ defined as in Definition 6.3, where $k_A : F_{\Sigma A}S^1 \longrightarrow M\lambda_A$ is the acyclic q-cofibration given in terms of the mapping cylinder of λ_A.

From here, the proof of Theorem 11.11 is exactly the same as the proof of Theorem 6.2, but with the stable weak equivalences there replaced by the π_*-isomorphisms here; see also the proof of Proposition 5.6 in Section 6. We use Proposition 11.4 repeatedly, and we apply the results on π_*-isomorphisms of Section 4 to the restricted maps $f[A]$ of prespectra associated to maps f of $\mathscr{W}$-spaces. We record the main steps of the proof since they give useful characterizations of the classes of maps that enter into the model structure.

Proposition 11.16. *A map $p : E \longrightarrow B$ satisfies the RLP with respect to K if and only if p is a level fibration and the diagram*

$$\begin{tikzcd} E(A) \arrow[r, "\tilde{\sigma}"] \arrow[d, "p(A)"'] & \Omega E(\Sigma A) \arrow[d, "\Omega p(\Sigma A)"] \\ B(A) \arrow[r, "\tilde{\sigma}"'] & \Omega B(\Sigma A) \end{tikzcd} \tag{11.17}$$

is a homotopy pullback for each finite based CW-complex A.

Using the third criterion in Lemma 11.7, this gives the following result.

Corollary 11.18. *The trivial map $F \longrightarrow *$ satisfies the RLP with respect to K if and only if F is linear.*

Corollary 11.19. *If $p : E \longrightarrow B$ is a π_*-isomorphism that satisfies the RLP with respect to K, then p is a level acyclic fibration.*

Proposition 11.20. *Let $f : X \longrightarrow Y$ be a map of $\mathscr{W}$-spaces.*

(i) *f is an acyclic q-cofibration if and only if it is a retract of a relative K-cell complex.*
(iii) *f is a q-fibration if and only if it satisfies the RLP with respect to K, and X is fibrant if and only if it is linear.*
(iii) *f is an acyclic q-fibration if and only if it is a level acyclic fibration.*

For the study of $\mathscr{W}$-ring and module spaces, we have the following result, which implies that Theorem 8.6 applies to $\mathscr{W}$-spaces under the absolute as well as the original stable model structure.

Proposition 11.21. *Under the absolute stable model structure, the category of $\mathscr{W}$-spaces satisfies the pushout-product and monoid axioms.*

Exactly as in the proofs of Propositions 8.2 and 8.5, this is a consequence of the following analogue of Proposition 8.3.

Proposition 11.22. *For any cofibrant $\mathscr{W}$-space X, the functor $X \wedge_S (-)$ preserves π_*-isomorphisms.*

Proof. Arguing as in the proof of Proposition 8.3, but taking into account that there are more cofibrant objects to deal with, it suffices to prove that $\pi_*(F_A S^0 \wedge_S Y) = 0$ if $\pi_*(Y) = 0$, where A is any finite based CW complex. Let Z be a Spanier-Whitehead k-dual to A, with duality maps $\eta : S^k \longrightarrow A \wedge Z$ and $\varepsilon : Z \wedge A \longrightarrow S^k$. By adjunction, η gives rise to a map $\tilde{\eta} : F_A S^k \longrightarrow F_0 Z$ and the adjoint

$$Z \longrightarrow F(A, S^k) = (F_A S^0)(S^k)$$

of ε gives rise to a map $\tilde{\varepsilon} : F_k Z \longrightarrow F_A S^0$. Consider the composites

$$\alpha : F_{\Sigma^k A} S^k \cong F_k S^0 \wedge_S F_A S^k \xrightarrow{\mathrm{id} \wedge \tilde{\eta}} F_k S^0 \wedge_S F_0 Z \cong F_k Z \xrightarrow{\tilde{\varepsilon}} F_A S^0$$

and

$$\beta : F_{\Sigma^k A} S^k \cong F_k S^k \wedge F_A S^0 \xrightarrow{\gamma_k \wedge \mathrm{id}} F_0 S^0 \wedge_S F_A S^0 \cong F_A S^0.$$

These maps have adjoints $S^k \longrightarrow (F_A S^0)(\Sigma^k A) = F(A, \Sigma^k A)$, which in turn have adjoints $\overline{\alpha} : \Sigma^k A \longrightarrow \Sigma^k A$ and $\overline{\beta} : \Sigma^k A \longrightarrow \Sigma^k A$. Inspecting definitions, we see that $\overline{\alpha}$ is the composite

$$\Sigma^k A \cong S^k \wedge A \xrightarrow{\eta \wedge \mathrm{id}} A \wedge Z \wedge A \xrightarrow{\mathrm{id} \wedge \varepsilon} A \wedge S^k = \Sigma^k A,$$

which is homotopic to the identity by the definition of a k-duality, and $\overline{\beta}$ is the identity map. Thus $\alpha \simeq \beta$. Since $\pi_*(Y) = 0$, $\pi_*(F_k Z \wedge_S Y) = 0$ by Lemma 4.14 and Proposition 8.3. Therefore $\alpha \wedge_S \mathrm{id}_Y$ induces the zero map on π_*. By Corollary 8.4, $\beta \wedge_S \mathrm{id}_Y$ induces an isomorphism on π_*. Therefore $\pi_*(F_A S^0 \wedge_S Y) = 0$. □

12. The comparison between $\mathscr{F}$-spaces and $\mathscr{W}$-spaces

It remains to relate $\mathscr{F}$-spaces to $\mathscr{W}$-spaces. It is important to keep in mind the two quite different forgetful functors defined on $\mathscr{W}$-spaces, namely

$$\mathbb{U}_{\mathscr{F}} : \mathscr{W}\mathscr{T} \longrightarrow \mathscr{F}\mathscr{T} \quad \text{and} \quad \mathbb{U}_{\mathscr{P}} : \mathscr{W}\mathscr{T} \longrightarrow \mathscr{P}.$$

We write $\mathbb{U}$ for the former and write $\mathbb{P}$ for its left adjoint $\mathscr{F}\mathscr{T} \longrightarrow \mathscr{W}\mathscr{T}$.

We have the level model structure on the category of $\mathscr{F}$-spaces given by the level weak equivalences, level fibrations, and q-cofibrations. We recall what we need about the stable model structure from [25, App B]. Actually, it will require a little proof to show that our definition of stable weak equivalence agrees with that given in [25, App B]; see Remark 13.10 below.

Definition 12.1. Let $f : X \longrightarrow Y$ be a map of $\mathscr{F}$-spaces.

(i) f is a *π_*-isomorphism* if $\mathbb{U}_{\mathscr{P}}\mathbb{P}f$ is a π_*-isomorphism of prespectra.
(ii) f is a *stable weak equivalence* if a cofibrant approximation $f' : X' \longrightarrow Y'$ of f (in the level model structure) is a π_*-isomorphism.
(iii) f is an *acyclic q-cofibration* if it is a stable weak equivalence and a q-cofibration.
(iv) f is a *q-fibration* if it satisfies the RLP with respect to the acyclic q-cofibrations.
(iv) f is an *acyclic q-fibration* if it is a stable weak equivalence and a q-fibration.

We have made the distinction between π_*-isomorphisms and stable weak equivalences because we have not proven that $\mathbb{P}$ preserves level weak equivalences or even carries level weak equivalences to π_*-isomorphisms in general. However, by Lemmas 12.13 and 12.14 below, $\mathbb{P}$ does preserve level weak equivalences between cofibrant $\mathscr{F}$-spaces.

For an $\mathscr{F}$-space X, we write $X_n = X(\mathbf{n}^+)$; recall that $X_0 = *$. Let $\delta_i : \mathbf{n}^+ \longrightarrow \mathbf{1}^+$ be the projection given by $\delta_i(i) = 1$ and $\delta_i(j) = 0$ for $j \neq i$. Let $\phi : \mathbf{2}^+ \longrightarrow \mathbf{1}^+$ be the based map such that $\phi(1) = 1 = \phi(2)$.

Definition 12.2. An $\mathscr{F}$-space X is *special* if the map $X_n \longrightarrow X_1^n$ induced by the n projections $\delta_i : \mathbf{n}^+ \longrightarrow \mathbf{1}^+$ is a weak equivalence. If X is special, then $\pi_0(X_1)$ is an abelian monoid with product $\pi_0(X_1) \times \pi_0(X_1) \cong \pi_0(X_2) \longrightarrow \pi_0(X_1)$ induced by ϕ. A special $\mathscr{F}$-space X is *very special* if $\pi_0(X_1)$ is an abelian group.

Theorem 12.3. *The category $\mathscr{F}\mathscr{T}$ is a cofibrantly generated model category with respect to the stable weak equivalences, q-fibrations, and q-cofibrations. An $\mathscr{F}$-space is fibrant if and only if it is very special.*

We refer the reader to [25] for the proof. While the result is deduced there from its simplicial analogue, a topological argument works just as well. However, it is not known and seems unlikely to be true that the stable model structure on $\mathscr{F}\mathscr{T}$ is compactly generated, so that a more general version of the small objects argument than Lemma 1.8 is needed. The set of generating q-cofibrations is FI, and of course its elements have compact domains. However, there does not seem to be a canonical choice of a set of generating acyclic q-cofibrations, and the elements of the set chosen in [25, App B] do not all have compact domains. All elements of the set are π_*-isomorphisms, and this has the following consequences.

Lemma 12.4. *All acyclic q-cofibrations are π_*-isomorphisms.*

Lemma 12.5. *The pair $(\mathbb{P}, \mathbb{U})$ is a Quillen adjoint pair.*

Proof. Since $\mathbb{U} : \mathscr{W}\mathscr{T} \longrightarrow \mathscr{F}\mathscr{T}$ preserves level weak equivalences and level fibrations, it is immediate by adjunction that $\mathbb{P}$ preserves q-cofibrations. Now the previous lemma gives that $\mathbb{P}$ preserves acyclic q-cofibrations. □

In particular, $\mathbb{U}$ preserves fibrant objects, as could easily be checked directly.

Lemma 12.6. *If Y is a linear $\mathscr{W}$-space, then $\mathbb{U}Y$ is a very special $\mathscr{F}$-space.*

The following result, which was left open in [25], is a consequence of its counterpart, Proposition 11.21, for $\mathscr{W}$-spaces. It implies that Theorem 8.6 applies to $\mathscr{F}$-spaces.

Proposition 12.7. *The stable model structure on the category of $\mathscr{F}$-spaces satisfies the pushout-product and monoid axioms.*

Proof. This is an exercise in the use of cofibrant approximation of maps. The essential points are that smash products and pushouts of cofibrant approximations are cofibrant approximations and that the functor $\mathbb{P}$ preserves colimits and smash products and creates the stable weak equivalences between cofibrant objects. □

Although direct topological proofs from Construction 12.12 and Lemma 12.14 below are possible, we will derive the following two results from their simplicial analogues in the next section. The second goes back to Segal [28] and is at the heart of his infinite loop space machine.

Lemma 12.8. *For a cofibrant $\mathscr{F}$-space X, $\mathbb{P}X$ is a strictly connective $\mathscr{W}$-space.*

Proposition 12.9. *If X is a cofibrant very special $\mathscr{F}$-space, then $\mathbb{P}X$ is a cofibrant linear $\mathscr{W}$-space. That is, the functor $\mathbb{P}$ preserves cofibrant-fibrant objects.*

Granting these results, Lemma 11.9 and the fact that $\mathbb{U}\mathbb{P} \cong \mathrm{Id}$ immediately give the following consequences.

Lemma 12.10. *The following conditions on a map $f : X \longrightarrow X'$ between cofibrant very special $\mathscr{F}$-spaces are equivalent.*

(i) *f is a π_*-isomorphism.*
(ii) *$f : X_1 \longrightarrow X'_1$ is a weak equivalence.*
(iii) *f is a level weak equivalence.*
(iv) *$\mathbb{P}f$ is a level weak equivalence of $\mathscr{W}$-spaces.*

Lemma 12.11. *If Y is a linear $\mathscr{W}$-space and $f : X \longrightarrow \mathbb{U}Y$ is a cofibrant-fibrant approximation of the $\mathscr{F}$-space $\mathbb{U}Y$, then the composite*

$$\varepsilon \circ \mathbb{P}f : \mathbb{P}X \longrightarrow \mathbb{P}\mathbb{U}Y \longrightarrow Y$$

is a level weak equivalence of $\mathscr{W}$-spaces.

The results above directly imply Theorems 0.7, 0.8, and 0.9.

We have quoted results from the simplicial literature because the topological prolongation functor $\mathbb{P}$ is not as easy to analyze as its simplicial analogue. The following explicit description of $\mathbb{P}$ is a direct consequence of its definition in terms of coends [MMSS, 4.2]. A similar analysis of coends is given in [22, §5], and relevant lemmas on the decomposition of morphisms of $\mathscr{F}$ are given there.

Construction 12.12. Let X be an $\mathscr{F}$-space and A be a space in $\mathscr{W}$. We define an expanding sequence of spaces $(\mathbb{P}X)_n(A)$ inductively, starting with $(\mathbb{P}X)_0(A) = *$. Let $F(A,n) \subset A^n$ be the configuration space of n-tuples of distinct points of $A - \{*\}$. For $1 \le j \le n$, let $\sigma_j : F(A,n) \longrightarrow F(A,n-1)$ be the restriction of the projection $A^n \longrightarrow A^{n-1}$ that deletes the jth coordinate, let s_j be the ordered injection $(\mathbf{n-1})^+ \longrightarrow \mathbf{n}^+$ with image $\mathbf{n}^+ - \{j\}$, and also write s_j for the induced map $X_{n-1} \longrightarrow X_n$. Define $(\mathbb{P}X)_n(A)$ to be the pushout displayed in the diagram

$$\begin{tikzcd}
F(A,n)_+ \wedge_{\Sigma_n} sX_{n-1} \arrow[r, "g"] \arrow[d, "i"'] & (\mathbb{P}X)_{n-1}(A) \arrow[d] \\
F(A,n)_+ \wedge_{\Sigma_n} X_n \arrow[r] & (\mathbb{P}X)_n(A).
\end{tikzcd}$$

Here sX_{n-1} is the union of the subspaces $s_j(X_{n-1})$ of X_n, i is induced by the inclusion $sX_{n-1} \subset X_n$, and g is defined by $g(\mathbf{a}, s_j x) = (\sigma_j \mathbf{a}, x)$. By definition,

$$(\mathbb{P}X)(A) = \int^{\mathbf{n}^+ \in \mathscr{F}} \mathscr{W}(\mathbf{n}^+, A) \wedge X_n.$$

The mapping space $\mathscr{W}(\mathbf{n}^+, A)$ is just A^n, and examination of colimits shows that

$$(\mathbb{P}X)(A) \cong \operatorname{colim} (\mathbb{P}X)_n(A).$$

For an injection $\phi : \mathbf{m}^+ \longrightarrow \mathbf{n}^+$, let $\Sigma_\phi \subset \Sigma_n$ be the subgroup of permutations σ such that $\sigma(\operatorname{Im}\phi) = \operatorname{Im}\phi$. If the X_n are nondegenerately based and the induced map $\phi : X_m \longrightarrow X_n$ is a Σ_ϕ-cofibration for each injection ϕ, then the vertical arrows in the diagram are h-cofibrations (in both the based and unbased senses), by [2, App.2.7]. When this holds, we say that X is a *proper* $\mathscr{F}$-space.

As in [22, 5.6] or [4, 2.7] (modulo an erratum in [21, App]), Construction 12.12 implies the following invariance statement.

Lemma 12.13. *If f is a level weak equivalence between proper $\mathscr{F}$-spaces, then $\mathbb{P}f$ is a level weak equivalence of $\mathscr{W}$-spaces.*

Inspection of the $\mathscr{F}$-spaces F_nA and passage to pushouts and colimits gives the following observation.

Lemma 12.14. *If X is an FI-cell complex, then X is a proper $\mathscr{F}$-space.*

13. Simplicial and topological diagram spectra

Let $\mathscr{S}_*$ denote the category of pointed simplicial sets, abbreviated ssets, and let $\mathbb{T} : \mathscr{S}_* \longrightarrow \mathscr{T}$ and $\mathbb{S} : \mathscr{T} \longrightarrow \mathscr{S}_*$ be the geometric realization and total singular complex functors. Both are strong symmetric monoidal. Let $\nu : \mathrm{Id} \longrightarrow \mathbb{ST}$ and $\rho : \mathbb{TS} \longrightarrow \mathrm{Id}$ be the unit and counit of the $(\mathbb{S}, \mathbb{T})$ adjunction. Both are monoidal natural weak equivalences. Recall that a map f of spaces is a weak equivalence or Serre fibration if and only if $\mathbb{S}f$ is a weak equivalence or Kan fibration of simplicial sets.

For a discrete category $\mathscr{D}$, a $\mathscr{D}$-sset is a functor $Y : \mathscr{D} \longrightarrow \mathscr{S}_*$, and we have the category $\mathscr{D}\mathscr{S}_*$ of $\mathscr{D}$-ssets. When we are given a canonical symmetric monoidal functor $S_{\mathscr{D}} : \mathscr{D} \longrightarrow \mathscr{S}_*$, we define $\mathscr{D}$-spectra over $S_{\mathscr{D}}$ in the evident fashion. Let us write $\mathscr{D}\mathscr{S}[\mathscr{S}_*]$ and $\mathscr{D}\mathscr{S}[\mathscr{T}]$ for the categories of $\mathscr{D}$-spectra of ssets over $S_{\mathscr{D}}$ and $\mathscr{D}$-spectra of spaces over $\mathbb{T}S_{\mathscr{D}}$. Both are symmetric monoidal categories. Levelwise application of $\mathbb{S}$ gives a lax symmetric monoidal functor $\mathbb{S} : \mathscr{D}\mathscr{S}[\mathscr{T}] \longrightarrow \mathscr{D}\mathscr{S}[\mathscr{S}_*]$ with unit map $\nu : S_{\mathscr{D}} \longrightarrow \mathbb{ST}S_{\mathscr{D}}$. Levelwise application of $\mathbb{T}$ gives a strong symmetric monoidal functor $\mathbb{T} : \mathscr{D}\mathscr{S}[\mathscr{S}_*] \longrightarrow \mathscr{D}\mathscr{S}[\mathscr{T}]$. These functors are right and left adjoint, and they induce adjoint functors when restricted to categories of rings, commutative rings, and modules over rings.

Warning 13.1. The functor $\mathbb{TS} : \mathscr{T} \longrightarrow \mathscr{T}$ is *not* continuous. Therefore we do not have a functor $\mathbb{TS} : \mathscr{D}\mathscr{T} \longrightarrow \mathscr{D}\mathscr{T}$ when the topological category $\mathscr{D}$ is not discrete.

As far as the relevant homotopy categories go, we can work interchangeably with $\mathscr{D}$-spectra of ssets and $\mathscr{D}$-spectra of spaces.

Proposition 13.2. *Let $\mathscr{D}$ be discrete and suppose that the category of $\mathscr{D}$-spectra of ssets is a model category such that every level weak equivalence is a weak equivalence. Define a weak equivalence of $\mathscr{D}$-spectra of spaces to be a map f such that $\mathbb{S}f$ is a weak equivalence. Then $\mathbb{S}$ and $\mathbb{T}$ induce adjoint equivalences of homotopy categories that induce adjoint equivalences between the respective homotopy categories of rings, commutative rings, and modules over rings.*

Proof. Since $\eta : Y \longrightarrow \mathbb{ST}Y$ is a level weak equivalence for all $\mathscr{D}$-spectra of ssets, an argument much like the proof of Lemma 2.2 applies. □

The proposition applies to symmetric spectra [8] and to $\mathscr{F}$-spectra [25]. In the latter case, just as for $\mathscr{F}$-spaces, $\mathscr{F}$-spectra of ssets are the same as $\mathscr{F}$-ssets. As noted in [8, 6.3.8] and [25], Lemma 2.2 applies to give the following stronger conclusion in these cases.

Theorem 13.3. *Let $\mathscr{D} = \Sigma$ or $\mathscr{D} = \mathscr{F}$. The pair $(\mathbb{T}, \mathbb{S})$ is a Quillen equivalence between the categories $\mathscr{D}[\mathscr{S}_*]$ and $\mathscr{D}[\mathscr{T}]$.*

The cited sources give the pushout-product and monoid axioms in $\mathscr{D}[\mathscr{S}_*]$. Since we have proven these axioms in $\mathscr{D}[\mathscr{T}]$, we are entitled to the following multiplicative elaborations.

Theorem 13.4. *Let $\mathscr{D} = \Sigma$ or $\mathscr{D} = \mathscr{F}$. The functors $\mathbb{T}$ and $\mathbb{S}$ induce a Quillen equivalence between the categories of $\mathscr{D}$-ring spectra of simplicial sets and $\mathscr{D}$-ring spectra of spaces.*

Theorem 13.5. *Let $\mathscr{D} = \Sigma$ or $\mathscr{D} = \mathscr{F}$. For a $\mathscr{D}$-ring R of simplicial sets, the functors $\mathbb{T}$ and $\mathbb{S}$ induce a Quillen equivalence between the categories of R-module spectra (of simplicial sets) and $\mathbb{T}R$-module spectra (of spaces).*

By Smith's result[2] that the category of commutative symmetric ring spectra of simplicial sets is a Quillen model category with definitions parallel to those in Section 10, we also have the commutative analogue of Theorem 13.4 in this case.

Theorem 13.6. *The functors $\mathbb{T}$ and $\mathbb{S}$ induce a Quillen equivalence between the categories of commutative symmetric ring spectra of simplicial sets and commutative symmetric ring spectra of spaces.*

Now focus on $\mathscr{F}$-ssets and $\mathscr{F}$-spaces. We deduce Lemma 12.8 and Proposition 12.9 from their simplicial analogues. There is a prolongation functor $\mathbb{P}^{\mathscr{S}_*}$ from $\mathscr{F}$-ssets to the category $\mathscr{S}_*^{\mathscr{S}_*}$ of simplicial functors $\mathscr{S}_* \longrightarrow \mathscr{S}_*$. We can use it to study the topological prolongation functor $\mathbb{P} = \mathbb{P}^{\mathscr{T}}$ from $\mathscr{F}$-spaces to the category $\mathscr{T}^{\mathscr{T}}$ of continuous functors $\mathscr{T} \longrightarrow \mathscr{T}$. The advantage of $\mathbb{P}^{\mathscr{S}_*}$ is that, although it is characterized as the left adjoint to the forgetful functor, it admits two equivalent explicit descriptions. First, in analogy with $\mathbb{P}^{\mathscr{T}}$, for a functor $Y : \mathscr{F} \longrightarrow \mathscr{S}_*$ and a simplicial set K,

$$(\mathbb{P}^{\mathscr{S}_*}Y)(K) = \int^{\mathbf{n}^+ \in \mathscr{F}} K^n \wedge Y_n. \tag{13.7}$$

Since $\mathbb{T}$ commutes with colimits and finite products, this description implies the relation

$$\mathbb{T}((\mathbb{P}^{\mathscr{S}_*}Y)(K)) \cong (\mathbb{P}^{\mathscr{T}}\mathbb{T}Y)(\mathbb{T}K). \tag{13.8}$$

Of course, this relationship requires us to begin with an $\mathscr{F}$-sset, but we have the following result. Its first statement is an easy observation and its second statement is then immediate from Lemmas 12.13 and 12.14.

Lemma 13.9. *For any $\mathscr{F}$-space X, $\mathbb{TS}X$ is a proper $\mathscr{F}$-space. Therefore, if X is cofibrant, then $\mathbb{P}\rho : \mathbb{PTS}X \longrightarrow \mathbb{P}X$ is a level weak equivalence.*

Remark 13.10. In [25, App B], a map $f : X \longrightarrow X'$ of $\mathscr{F}$-spaces is defined to be a stable weak equivalence if $\mathbb{S}f$, or equivalently $\mathbb{TS}f$, is a π_*-isomorphism. Using Lemmas 13.9 and 12.13 and a little diagram chase with cofibrant approximations, we now see that this agrees with Definition 12.1(ii).

The other description of $\mathbb{P}^{\mathscr{S}_*}$ is given as follows. A pointed set E can be identified with the colimit of its pointed finite ordered subsets, and these can be identified with the pointed injections $\mathbf{n}^+ \longrightarrow E$ for $n \geq 0$. We extend Y to a functor from pointed sets to simplicial sets by defining $Y(E)$ to be the colimit of the simplicial sets $Y(\mathbf{n}^+)$, where the colimit is taken over the pointed functions $\mathbf{n}^+ \longrightarrow E$ or, equivalently, over the pointed injections $\mathbf{n}^+ \longrightarrow E$. We then define $(\mathbb{P}^{\mathscr{S}_*}Y)(K)$ to be the diagonal of the bisimplicial set obtained by applying Y to the set K_q of q-simplices of K for all q. This description is exploited by Bousfield and Friedlander [3] and Lydakis [11] to study the homotopical properties of prolongation. The definitions of special and very special $\mathscr{F}$-ssets are the same as for $\mathscr{F}$-spaces. The definitions of strictly connective and linear $\mathscr{S}_*$-ssets are the same as for $\mathscr{W}$-spaces.

Lemma 13.11. *For an $\mathscr{F}$-sset Y, $\mathbb{P}^{\mathscr{S}_*}Y$ is a strictly connective $\mathscr{S}_*$-sset.*

Proof. This is [3, 4.10]. □

[2] private communication

Proof of Lemma 12.8. Since any finite CW complex is homotopy equivalent to $\mathbb{T}K$ for some finite simplicial complex K, we may as well restrict attention to spaces of the form $\mathbb{T}K$ in $\mathscr{W}$. Thus we must prove that $(\mathbb{P}X)(\mathbb{T}K)$ is n-connected if $\mathbb{T}K$ is n-connected. Since $\mathbb{P}X$ is level weak equivalent to $\mathbb{P}\mathbb{T}\mathbb{S}X$, (13.8) and Lemma 13.11 give the conclusion. □

Proposition 13.12. *If Y is a very special $\mathscr{F}$-sset, then $\mathbb{P}^{\mathscr{S}_*}Y$ is a linear $\mathscr{S}_*$-sset.*

Proof. This is implied by [3, 4.3]. □

Proof of Proposition 12.9. Let X be a cofibrant very special $\mathscr{F}$-space and let $Y = \mathbb{S}X$. Then Y is a very special $\mathscr{F}$-sset. By Lemma 11.7, it suffices to prove that $\tilde{\sigma} : (\mathbb{P}X)(A) \longrightarrow \Omega(\mathbb{P}X)(\Sigma A)$ is a weak equivalence for all finite CW complexes A. Again, we may restrict attention to $A = \mathbb{T}K$, where K is a finite simplicial set. We may replace the target of $\tilde{\sigma}$ by the homotopy fiber of the evident map $(\mathbb{P}X)(C\mathbb{T}K) \longrightarrow (\mathbb{P}X)(\Sigma\mathbb{T}K)$, and we note that the functor $\mathbb{T}$ preserves homotopy fibers. By [3, 4.3], the canonical map from $(\mathbb{P}^{\mathscr{S}_*}Y)(K)$ to the homotopy fiber of $(\mathbb{P}^{\mathscr{S}_*}Y)(CK) \longrightarrow (\mathbb{P}^{\mathscr{S}_*}Y)(\Sigma K)$ is a weak equivalence. Now the conclusion follows from (13.8), evident commutation relations, and use of the level weak equivalence $\mathbb{P}\rho : \mathbb{P}\mathbb{T}Y \longrightarrow \mathbb{P}X$. □

References

[1] J. F. Adams. Stable homotopy and generalized homology. The University of Chicago Press. 1974. (Reprinted 1995.)
[2] J. M. Boardman and R. M. Vogt. Homotopy invariant algebraic structures. Springer Lecture Notes in Mathematics, Vol 347. 1973.
[3] A. K. Bousfield and E. M. Friedlander. Homotopy theory of Γ-spaces, spectra, and bisimplicial sets. Springer Lecture Notes in Mathematics Vol. 658. 1978, 80-130.
[4] F. R. Cohen, J. P. May, and L. R. Taylor. Splitting of certain spaces CX. Math. Proc. Camb. Phil. Soc. 84(1978), 465-496.
[5] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May (with an appendix by M. Cole). Rings, modules, and algebras in stable homotopy theory. Surveys and Monographs in Mathematics Vol. 47. 1997. American Mathematical Society.
[6] P.S. Hirschhorn. Localization, cellularization, and homotopy colimits. Preprint. 1997.
[7] M. Hovey. Model categories. Preprint. 1997.
[8] M. Hovey, B. Shipley, and J. Smith. Symmetric spectra. Preprint. 1998.
[9] L. G. Lewis, Jr. Is there a convenient category of spectra? J. Pure and Applied Algebra 73(1991), 233-246.
[10] L. G. Lewis, Jr., J. P. May, and M. Steinberger (with contributions by J. E. McClure). Equivariant stable homotopy theory. Springer Lecture Notes in Mathematics Vol. 1213. 1986.
[11] M. Lydakis. Smash-products and Γ-spaces. Math. Proc. Camb. Phil. Soc. To appear.
[12] M. Lydakis. Simplicial functors and stable homotopy theory. Preprint. 1998.
[13] S. Mac Lane. Categories for the working mathematician. Springer-Verlag. 1971.
[14] M. A. Mandell and J. P. May. Orthogonal spectra and S-modules. Preprint. 1998.
[15] M. A. Mandell and J. P. May. Equivariant orthogonal spectra. In preparation.
[16] M. A. Mandell, J. P. May, S. Schwede, and B. Shipley. Diagram spaces, diagram spectra, and FSP's. Preprint. 1998.
[17] J. P. May. Simplicial objects in algebraic topology. D. van Nostrand. 1967. (Reprinted by the University of Chicago Press.)
[18] J. P. May. The Geometry of Iterated Loop Spaces, Springer Lecture Notes in Mathematics Vol. 271. 1972.
[19] J. P. May, et al. Equivariant homotopy and cohomology theory. CBMS Regional Conference Series in Mathematics, Number 91. American Mathematical Society. 1996.
[20] J. P. May. Brave new worlds in stable homotopy theory. In Homotopy theory via algebraic geometry and group representations. Contemporary Mathematics Vol 220, 1998, 193-212.

[21] J. P. May and L. R. Taylor. Generalized splitting theorems. Math. Proc. Camb. Phil. Soc. 93(1983), 73-86.
[22] J. P. May and R. Thomason. The uniqueness of infinite loop space machines. Topology 17(1978), 205-224.
[23] R. J. Piacenza. Homotopy theory of diagrams and CW-complexes over a category. Canadian J. Math. 43(1991), 814-824.
[24] D.G. Quillen. Homotopical algebra. Springer Lecture Notes in Mathematics Vol. 43. 1967.
[25] S. Schwede. Stable homotopical algebra and Γ-spaces. Math. Proc. Camb. Phil. Soc. To appear.
[26] S. Schwede. S-modules and symmetric spectra. Preprint, 1998.
[27] S. Schwede and B. Shipley. Algebras and modules in monoidal model categories. Preprint, 1997.
[28] G. Segal. Categories and cohomology theories. Topology 13(1974), 293-312.
[29] B. Shipley. Symmetric ring spectra and topological Hochschild homology. Preprint, 1997.
[30] A. Strom. The homotopy category is a homotopy category. Arch. Math. 23(1972), 435-441.