Exercise 1.10

Given $f(0) :\equiv \Phi_0$ where $\Phi_0 : C$, $C : \mathcal{U}$, and Φ_s may contain "$f(n)$", we can form

$f(\mathrm{succ}(n)) :\equiv \Phi_s$

$$f :\equiv \mathrm{rec}_N(C, \Phi_0, \lambda n. \lambda r. \Phi_s[r/f(n)]) : N \to C$$

Now, let $g : N \to N$, $C :\equiv N$, and consider

$$\mathrm{Iter}_g(0) :\equiv g(1)$$
$$\mathrm{Iter}_g(\mathrm{succ}(n)) :\equiv g(\mathrm{Iter}_g(n))$$

using the above rules,

$$\mathrm{Iter}_g :\equiv \mathrm{rec}_N(N, g(1), \lambda n. \lambda q. g(q))$$

so

$$\mathrm{Iter} : (N \to N) \to N \to N$$
$$\mathrm{Iter} :\equiv \lambda g. \mathrm{rec}_N(N, g(1), \lambda n. \lambda q. g(q))$$

Now, define

$$\mathrm{Ack} :\equiv \mathrm{rec}_N(N \to N, \mathrm{succ}, \lambda n. \lambda g. \mathrm{Iter}(g)) : N \to N \to N$$

Thus,

$$\mathrm{Ack}\ 0 \equiv \mathrm{succ}$$
$$\star \quad \mathrm{Ack}\ (\mathrm{succ}\ m) \equiv \mathrm{rec}_N(N \to N, \mathrm{succ}, \lambda n. \lambda g. \mathrm{Iter}(g), \mathrm{succ}\ m)$$
$$\equiv (\lambda g. \mathrm{Iter}(g))(\mathrm{rec}_N(N \to N, \mathrm{succ}, \lambda n. \lambda g. \mathrm{Iter}(g), m))$$
$$\equiv \mathrm{Iter}\ (\mathrm{Ack}\ m)$$

so $\mathrm{Ack}\ (\mathrm{succ}(m), 0) \equiv \mathrm{Iter}\ (\mathrm{Ack}\ m)\ 0$ by $\star$

$$\equiv \mathrm{rec}_N(N, \mathrm{Ack}(m, 1), \lambda n. \lambda q. \mathrm{Ack}\ q, 0)$$
$$\equiv \mathrm{Ack}(m, 1)$$
$$\equiv \mathrm{Ack}\ m\ 1$$

$\mathrm{ack}\ (S\ m)\ n \equiv$ ~~Ack~~ $\mathrm{Iter}\ (\mathrm{Ack}\ m)\ n$

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Exercise 1.10 (continued)

and

$$\begin{aligned}
\text{Ack } (\text{succ } m)\ (\text{succ } n) &\equiv (\text{Iter } (\text{Ack } m))\ (\text{succ } n) && \text{by } * \\
&\equiv \text{rec}_N(N, \text{Ack } m\ 1, \lambda n. \lambda q. \text{Ack } m\ q, \text{succ } n) \\
&\equiv (\lambda q. \text{Ack } m\ q)\ (\text{rec}_N(N, \text{Ack } m\ 1, \lambda n. \lambda q. \text{Ack } m\ q, n)) \\
&\equiv (\lambda q. \text{Ack } m\ q)\ (\text{Iter } (\text{Ack } m)\ n) \\
&\equiv (\lambda q. \text{Ack } m\ q)\ (\text{Ack } (\text{succ } m)\ n) && \text{by } * \\
&\equiv \text{Ack } m\ (\text{Ack } (\text{succ } m)\ n)
\end{aligned}$$