Coherence in Category Theory and the Church-Rosser Property *

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Abstract

Szabo's derivation systems on sequent calculi with exchange and product are not Church-Rosser. Thus his coherence results for categories having a symmetric product (either monoidal or cartesian) are false.

1 Introduction

Gentzen's sequent calculi [9] have been applied extensively in category theory, e.g [2, 3, 4, 6, 7, 8]. Sequents correspond to morphisms of a category, and the rules of the calculus correspond to categorical structures (e.g. having an associative tensor product). Cut-elimination was then used to put bounds on the complexity of these structures, e.g. to produce exhaustive lists (perhaps with duplications) of the canonical natural transformations between given functors. For symmetric, monoidal closed categories it was shown in [12] how to decide in principle whether two such transformations are equal, while an effective, linear-time decision procedure was given in [1].

Derivation systems (reduction rules) can be used to eliminate some duplicates in the list of cut-free proofs (e.g. [8]). However, in Algebra of Proofs [11] and its

forerunner [10], Szabo claims to have produced derivation systems in which all duplicates have been eliminated, so that every proof has a unique normal form. In fact, none of the systems there which include a symmetry for the product are Curch-Rosser (confluent).

The two major applications of this work were complete coherence theorems for symmetric, monoidal closed categories (mentioned above) and for cartesian closed categories. The former claim leads to self-contradictory assertions about the size of homsets in free categories. The latter problem is solved in [5] using similar methods, but avoids adopting symmetry as a primitive by exploiting the universal property of the cartesian product.

I would like to thank Professor S. Mac Lane for raising this issue and exhorting me to resolve it.

2   The Counter-example

Here is Szabo's sequent calculus for symmetric, monoidal categories [11]

$$\frac{\Gamma\to\gamma \quad \Delta\gamma\Lambda\to\phi}{\Delta\Gamma\Lambda\to\phi}\ (Cut) \qquad \frac{\Gamma\Delta\to\phi}{\Gamma I\Delta\to\phi}\ (LI) \qquad \frac{\Gamma\alpha\beta\Delta\to\phi}{\Gamma\beta\alpha\Delta\to\phi}\ (Ex)$$

$$\frac{\Gamma\to\alpha \quad \Delta\to\beta}{\Gamma\Delta\to\alpha\otimes\beta}\ (R\otimes) \qquad \frac{\Gamma\alpha\beta\Delta\to\phi}{\Gamma\alpha\otimes\beta\Delta\to\phi}\ (L\otimes)$$

which are there labelled (R1), (R2),(R4),(R8) and (R9), respectively. Here is a proof in the calculus:

$$\frac{\dfrac{\dfrac{\alpha\to\alpha \quad \beta\to\beta}{\alpha\beta\to\alpha\otimes\beta}\ (R\otimes) \quad \gamma\to\gamma}{\dfrac{\alpha\beta\gamma\to(\alpha\otimes\beta)\otimes\gamma(=\ \phi)}{\dfrac{\alpha\gamma\beta\to\phi}{\dfrac{\alpha\beta\gamma\to\phi}{\dfrac{\beta\alpha\gamma\to\phi}{\dfrac{\beta(\alpha\otimes\gamma)\to\phi}{\dfrac{(\alpha\otimes\gamma)\beta\to\phi}{(\alpha\otimes\gamma)\otimes\beta\to\phi}\ (L\otimes)}\ (Ex)}\ (L\otimes)}\ (Ex)}\ (Ex)}\ (Ex)}}{}\ (R\otimes)}{}\ (Ex)$$

Two derivations (reductions) applicable to this proof are:

(D.24.1) "For any permutation π of the integers $1,\ldots,n$, and any successions (σ) and (τ) of instances of R4,

$$\frac{\Gamma\to\Phi}{\Delta\to\Phi}\ (\sigma) \qquad \geq \qquad \frac{\Gamma\to\Phi}{\Delta\to\Phi}\ (\tau)$$

provided that $\Gamma=\alpha_1\ldots\alpha_n$, $\Delta=\beta_1\ldots\beta_n$, $\beta_i=\alpha_{\pi(i)}$ for $1\leq i\leq n$, and (τ) is the unique string of interchanges which first moves $\alpha_{\pi(1)}$ to β_1, then $\alpha_{\pi(2)}$

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to β_2, etc. if π is the identity permutation, the right-hand side denotes [the null proof]."

(D.27.3)

$$\cfrac{\cfrac{\cfrac{\Gamma\alpha\beta\gamma\Delta\to\Phi}{\Gamma\alpha\gamma\beta\Delta\to\Phi}}{\Gamma\gamma\alpha\beta\Delta\to\Phi}}{\Gamma\gamma(\alpha\otimes\beta)\Delta\to\Phi} \quad\geq\quad \cfrac{\cfrac{\Gamma\alpha\beta\gamma\Delta\to\Phi}{\Gamma(\alpha\otimes\beta)\gamma\Delta\to\Phi}}{\Gamma\gamma(\alpha\otimes\beta)\Delta\to\Phi}$$

The first two applications of (Ex) in this proof can be deleted by (D.24.1) to yield an irreducible proof. However, (D.27.3) can also be applied, followed by (D.24.1) to obtain a distinct irreducible proof. Thus the system is not Church-Rosser. Categorically, the two irreducible proofs correspond to the morphisms

$$ca(c\otimes 1) = a^{-1}(1\otimes c)a : (\alpha\otimes\gamma)\otimes\beta\to(\alpha\otimes\beta)\otimes\gamma$$

whose proof of equality requires an identity morphism to be expanded to c^2.

The loss of confluence leads to incorrect combinatorial calculations of the size of homsets. Let $\mathcal{V}$ be a free symmetric monoidal closed category on a set of objects. For objects A and B of $\mathcal{V}$ let $\| A, B \|$ denote the cardinality of $\mathcal{V}(A, B)$. Further, let $A^{(1)} = [A, I]$ denote the dual of A and $A^{(n)}$ its nth dual. Claim (2) of Corollary 8.6.13 is that if A is not isomorphic to I we have

$$\| A^{(2n)}, A^{(2m)} \| = \left(\begin{array}{c} n + m - 1 \\ n \end{array} \right) \| A, A \|$$

This formula is self-contradictory since it implies that

$$\| A^{(4)}, A^{(4)} \| = \left(\begin{array}{c} 3 \\ 2 \end{array} \right) \| A, A \| = 3 \| A, A \|$$

while applying it twice shows that

$$\| A^{(4)}, A^{(4)} \| = \| A^{(2)}, A^{(2)} \| = \| A, A \|$$

Similar calculations show that claims (3),(4) and (5) are also false.

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