SET THEORY FOR CATEGORY THEORY

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Abstract. Questions of set-theoretic size play an essential role in category theory, especially the distinction between sets and proper classes (or small sets and large sets). There are many different ways to formalize this, and which choice is made can have noticeable effects on what categorical constructions are permissible. In this expository paper we summarize and compare a number of such "set-theoretic foundations for category theory," and describe their implications for the everyday use of category theory. We assume the reader has some basic knowledge of category theory, but little or no prior experience with formal logic or set theory.

1. Introduction

Since its earliest days, category theory has had to deal with set-theoretic questions. This is because unlike in most fields of mathematics outside of set theory, questions of *size* play an essential role in category theory.

A good example is Freyd's *Special Adjoint Functor Theorem*: a functor from a complete, locally small, and well-powered category with a cogenerating set to a locally small category has a left adjoint if and only if it preserves small limits. This is always one of the first results I quote when people ask me "are there any real *theorems* in category theory?" So it is all the more striking that it involves in an *essential way* notions like 'locally small', 'small limits', and 'cogenerating set' which refer explicitly to the difference between sets and proper classes (or between small sets and large sets).

Despite this, in my experience there is a certain amount of confusion among users and students of category theory about its foundations, and in particular about what constructions on large categories are or are not possible. Most introductory category theory books and courses quite rightly ignore deeper set-theoretic questions, which will only confuse most beginners. However, intermediate and advanced students of category theory may naturally begin to wonder about these questions.

It turns out that there are several possible foundational choices for category theory, and which choice is made can have noticeable effects on what is possible and what is not. The purpose of this informal paper is to summarize and compare some of these proposed foundations, including both 'set-theoretical' and 'category-theoretical' ones, and describe their implications for the everyday use of category theory. I assume the reader has some basic knowledge of category theory, such as can be obtained from [ML98] or [Awo06], but little or no experience with formal logic or set theory. I found some brief excursions into mathematical logic unavoidable, but I have tried to explain all logical notions as they occur and relegate the more complicated logical discussion to footnotes.

Related papers include [Fef69, Appendix II] (by G. Kreisel) and [Bla84]; a historical survey can be found in [Krö07]. Since my goal is expository, I have tried to restrict citations to those most useful for the student, when such exist. Likewise, this is not a history paper, so I will concentrate on current understanding.

Finally, this is a work in progress, so suggested corrections and improvements, both in mathematical content and exposition, are extremely welcome. Please also let me know if you have suggestions for other references to include. Numerous people have already helped me with editing and feedback; I would like to especially thank Damir Dzhafarov and Kenny Easwaran for useful discussions about set theory and foundations; Antonio Montalban for explaining why the reflection principle doesn't violate the incompleteness theorem; Peter Lumsdaine for pointing out that a transitive set model without full power sets can be countable; and Colin McLarty for directing my attention to the categorical replacement axiom.

2. SIZE DOES MATTER

Before diving into set theory, it's natural to wonder why we need to worry about size issues at all. In this section I'll review two theorems of basic category theory (interestingly, both due to Peter Freyd) which, I think, display the essential nature of size considerations. Since this section is just motivation, I'll be vague about the exact meaning of 'set', 'class', 'small', and 'large', assuming the reader has some familiarity with their use.

First of all, we say that a category is **complete** if it admits all limits indexed by *small categories*; that is, categories with only a set of objects and a set of morphisms. The basic example of a complete category is **Set**: if $\mathbf{A}$ is small and $F\colon \mathbf{A} \to \mathbf{Set}$ is a functor, then the limit set $\lim(F)$ consists of families $(x_a)_{a \in \mathbf{A}}$ such that $x_a \in Fa$ and for all $f\colon a \to b$ in $\mathbf{A}$, $Ff(x_a) = x_b$. There is the dual notion of **cocomplete**.

It is essential in giving this definition that we restrict to *small* limits, since there are many large limits that **Set** does not admit. For example, if X is a set with more than one element, then the A-fold product $\prod_{a \in A} X$ exists if A is any set, but not if A is a proper class. More generally, we have the following, which is our first theorem in which size considerations are essential.

Theorem 2.1. *If a category $\mathbf{A}$ has products indexed by the collection $\mathrm{Arr}(\mathbf{A})$ of arrows in $\mathbf{A}$, then $\mathbf{A}$ is a preorder. In particular, any small complete category is a preorder, and no large category that is not a preorder can admit products indexed by proper classes.*

Proof. Suppose that we had two different arrows $f, g\colon a \to b$, and form the product $\prod_{\mathrm{Arr}(\mathbf{A})} b$. Then f and g give us $2^{|\mathrm{Arr}(\mathbf{A})|}$ different arrows $a \to \prod_{\mathrm{Arr}(\mathbf{A})} b$; but there are only $|\mathrm{Arr}(\mathbf{A})|$ total arrows in $\mathbf{A}$, a contradiction.[1]   □

Thus, in order to capture most interesting examples, the notion of *complete category* must allow *large* categories, but restrict to *small* limits.

However, many large categories *do* admit *some* large limits. For example, most large categories admit an intersection for any family of monomorphisms with common codomain, no matter how large. This is usually, but not always, because

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[1] The use of proof by contradiction in this argument is essential. In intuitionistic logic the theorem can fail; see [Hyl88] or [McL92a, Ch. 24].

the category is **well-powered**, meaning that each object has only a set of iso-
morphism classes of subobjects. Other large limits also often exist; see, for exam-
ple, [Kel86, KK81].

Well-poweredness figures prominently in our second example: the Special Adjoint
Functor Theorem. Recall that a family Q of objects in a category $\mathbf{A}$ is said to be
cogenerating if whenever $f \neq g\colon a \rightrightarrows b$ are unequal parallel arrows, there exists
an arrow $h\colon b \to q$ with $q \in Q$ such that $hf \neq hg$. We are usually only interested
in this when Q is a set. Recall also that a category $\mathbf{A}$ is **locally small** if for any
objects a, b the collection of morphisms $\mathbf{A}(a, b)$ is a set.

Theorem 2.2. *If* $\mathbf{A}$ *is locally small, complete, well-powered and has a cogenerating
set, and* $\mathbf{B}$ *is locally small, then a functor* $G\colon \mathbf{A} \to \mathbf{B}$ *has a left adjoint if and only
if it preserves small limits.*

Proof. It suffices to construct, for each $b \in \mathbf{B}$, an arrow $b \to GFb$, for some object
$Fb \in \mathbf{A}$, which is initial among arrows $b \to Ga$. To define Fb, we first form the
product $p = \prod_{q \in Q} \prod_{\mathbf{B}(b, Gq)} q$ in $\mathbf{A}$. This product exists since Q is a set, $\mathbf{B}$ is locally
small, and $\mathbf{A}$ is complete. Since G preserves products, we have an induced map
$b \to Gp$. Now let Fb be the intersection of all monomorphisms $a \rightarrowtail p$ such that
$b \to Gp$ factors through $Ga \to Gp$. This intersection exists since $\mathbf{A}$ is well-powered
and complete. Since G preserves monomorphisms and intersections, we have an
induced map $b \to GFb$. We leave it to the reader to verify that this has the desired
universal property (or see [ML98, V.8]). $\square$

For example, if $\mathbf{A}$ satisfies the hypotheses of the theorem and $\mathbf{C}$ is any small
category, then the functor category $[\mathbf{C}, \mathbf{A}]$ is locally small and the diagonal functor
$\Delta\colon \mathbf{A} \to [\mathbf{C}, \mathbf{A}]$ preserves limits, hence has a left adjoint. Thus any such $\mathbf{A}$ is also
cocomplete.

Of course, as pointed out in the introduction, size distinctions play an essential
role in this theorem. As stated, it applies to small categories just as well as large
ones, but it becomes somewhat degenerate: any small category is locally small,
well-powered, and has a cogenerating set, so we obtain the following.

Corollary 2.3. *If* $\mathbf{A}$ *is a complete lattice and* $G\colon \mathbf{A} \to \mathbf{B}$ *preserves greatest lower
bounds, then it has a left adjoint.*

While undoubtedly important, this result is only a pale shadow of the full Adjoint
Functor Theorem. Moreover, the Adjoint Functor Theorem is not just a bit of fluff;
there are examples even outside of pure category theory where it is the only known
way to construct an adjoint. So, like it or not, we are forced to deal with the
question of size in category theory.

3. ZFC

With that motivation under our belts, we now turn to a quick summary of set
theory. A natural question to begin with is: what is a *set*? One modern answer is
that *sets* are special sorts of *collections*, which can be manipulated in well-defined
ways that

    (a) suffice for applications in mathematics, but
    (b) are not powerful enough to reproduce the well-known paradoxes.

There are three classical paradoxes of set theory, traditionally known as Russell's, Cantor's, and Burali-Forti's.[2] Russell's paradox is non-categorical in flavor, while Burali-Forti's requires ordinal numbers (see Theorem 7.3), so here we recall only Cantor's.

Theorem 3.1. *There is no set containing all sets as members.*

Proof. Suppose that V were such a set. Then every subset of V, being a set, would be a member of V; thus $\mathcal{P}V \subset V$ and so $|\mathcal{P}V| \leq |V|$, contradicting Cantor's proof by diagonalization that $|A| < |\mathcal{P}A|$ for any A. $\square$

Thus, if we want to manipulate sets in the intuitive ways we are used to, there must be some limitation on what collections we are allowed to call 'sets'. The modern solution is to use a system of axioms which allows us to construct enough sets to do mathematics, but not to construct problematic sets such as V.

The set-theoretic axioms which have come to be accepted as standard are today called **ZFC (Zermelo-Fraenkel set theory with Choice)** and can be found in any book on set theory (I like [Dev93, End77] as introductions, while [Jec03] is encyclopedic) or on the Internet. For later reference, we divide the axioms of ZFC into four types.

  (i) The *basic* axioms: extensionality, foundation (or regularity), pairing, union, empty set, and separation.
  (ii) The *size-increasing* axioms: replacement and power set.
  (iii) The *size-assertion* axiom: infinity.
  (iv) The axiom of *choice*.

Most of these are well-known or obvious. The axiom schema of *separation* states that for any set A and any definable property $\varphi(x)$, the set $\{x \in A \mid \varphi(x)\}$ exists. The axiom schema of *replacement* states that for any set A and any definable property $\varphi(x, y)$ such that for any $x \in A$, there is a unique y with $\varphi(x, y)$, the set $\{y \mid \exists x \in A \; \varphi(x, y)\}$ exists. These are both both axiom *schemas*: there is one separation axiom for each definable property φ and one replacement axiom for each definable and 'functional' property φ. In the presence of the other axioms, the replacement schema implies the separation schema.

To be precise, *definable* here means 'definable in the formal first-order language of set theory'. But don't worry if you don't know any logic; this really just means that it can be described in ordinary mathematical language, referring only to sets or things that can be defined in terms of sets (which includes most of mathematics). For example, 'x is a continuous function from $\mathbb{R}$ to $\mathbb{R}$' is a definable property of x, so separation allows us to form the set of all such functions (taking, A to be, say, the power set of $\mathbb{R} \times \mathbb{R}$).

All the ordinary constructions of mathematics can be performed using these axioms. For instance, we can define the ordered pair (a, b) as the set $\{\{a\}, \{a, b\}\}$, which exists by pairing, and the cartesian product $A \times B$ as

$$A \times B = \left\{ z \in \mathcal{P}\big(\mathcal{P}(A \cup B)\big) \;\middle|\; \exists\, a \in A : \exists\, b \in B : z = (a, b) \right\},$$

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[2]Cesare Burali-Forti (1861–1931) is one of those mathematicians who are easily mistaken for two people by the unwary student. Other distinguished members of this club include Tullio Levi-Civita (1873–1941) and Gösta Mittag-Leffler (1846–1927).

which exists by power-set and separation. Similarly, the set B^A of functions from A to B can be defined by

$$B^A = \left\{ f \in \mathcal{P}(A \times B) \,\middle|\, \forall\, a \in A : \exists\,!\, b \in B : (a,b) \in f \right\}.$$

Let me comment briefly on the axiom schema of replacement, which may seem the strangest one in the list from a categorical point of view. In particular, it may seem odd to call it a size-increasing axiom, since it merely replaces a set by an isomorphic one, or at most a quotient. We will see in later sections that given the other (also non-categorical) axioms of ZFC, replacement in fact allows us to construct much larger sets than would otherwise be possible. But we will also see that above and beyond this, replacement plays a subtle and important role in category theory—so much so that this paper could easily have been subtitled "a tale of the replacement axiom"!

Remark 3.2. The approach of ZFC, and its relatives to be described in later sections, is not the only way to avoid paradoxes in set theory. For example, in *NFU (New Foundations with Urelements)*, any collection of things characterized by a 'stratified' property is a set. This allows for the existence of an actual set of all sets, while still avoiding paradoxes; see [Hol98] for a good introduction. However, NFU is not much good for category theory, since the category **Set** it produces is not cartesian closed [McL92b].

4. ORDINALS AND CARDINALS

We now briefly review the theory of ordinal and cardinal numbers. Succinctly, a **cardinal number** is a canonically chosen representative for a bijection class of sets, while an **ordinal number** is a canonically chosen representative for an isomorphism class of well-ordered sets. Here a *well-ordering* on a set is a total ordering such that every nonempty subset has a least element. The ordinals have the following properties.

(i) Every ordinal α has an immediate successor $\alpha + 1$, obtained by adding an extra element at the end of a well-ordering of type α.
(ii) There is a natural well-ordering on the collection of all ordinals: $\alpha \leq \beta$ iff α is isomorphic to an initial segment of β.
(iii) The induced well-ordering on $\{\beta : \beta < \alpha\}$ is in the isomorphism class represented by α.
(iv) Every set is well-orderable, and hence bijective to some ordinal (this is equivalent to the axiom of choice).

Because of (iii), one definition of an ordinal number (due to von Neumann) is as the set of all smaller ordinals. Thus, the ordinals begin with the natural numbers $0 = \emptyset$, $1 = \{0\}$, $2 = \{0,1\}$, and so on, but continue afterwards with

$$\omega, \omega + 1, \ldots, \omega \cdot 2, \ldots, \omega \cdot 3, \ldots, \omega^2, \ldots, \omega^3, \ldots, \omega^\omega, \ldots, \omega^{\omega^\omega}, \ldots, \omega^{\omega^{\omega^\omega}}, \ldots$$

We note in passing that the replacement axioms are first necessary to construct $\omega \cdot 2 = \{0, 1, 2, \ldots, \omega, \omega + 1, \omega + 2, \ldots\}$. Without replacement, we can construct each ordinal $\omega + n$, but we cannot collect them all as elements of a single set. We can construct well-ordered sets *isomorphic* to $\omega \cdot 2$ without replacement, however,

so the von Neumann definition of an ordinal is only appropriate in the presence of replacement.

All the ordinals listed above are *countable* (bijective with ω). We denote the first uncountable ordinal by ω_1, the next ordinal not bijective with ω_1 by ω_2, and so on. In fact, we can define a cardinal number to be an ordinal not bijective with any smaller ordinal. It follows that the cardinal numbers are also well-ordered, and can be indexed by the ordinal numbers. We write $\aleph_\alpha$ for the αth cardinal number; thus $\aleph_0 = \omega$, $\aleph_1 = \omega_1$, and so on. Every set X is bijective to a unique cardinal $|X|$, called its cardinality. If κ is a cardinal, we write 2^κ for $|\mathcal{P}\kappa|$.

There are two types of ordinals. Those of the form $\alpha + 1$, for some α, are called **successor ordinals**, while the rest are called **limit ordinals**. Every cardinal is a limit ordinal. A cardinal is a **successor cardinal** or a **limit cardinal** just when its indexing ordinal is a successor or limit ordinal.

Just as the well-ordering of $\mathbb{N}$ justifies the usual sort of mathematical induction, the well-ordering of ordinals justifies definition and proof by **transfinite induction**. This involves proving or constructing something in stages, one for each ordinal. The two cases of successor ordinals and limit ordinals are usually dealt with differently; for $\alpha + 1$ we base the construction on α, while for a limit ordinal β we base it on all other ordinals $\alpha < \beta$. We consider several important examples, three from set theory and one from category theory.

Example 4.1. The definition of the alephs can be phrased as a transfinite induction: we set $\aleph_0 = \omega$, let $\aleph_{\alpha+1}$ be the smallest cardinal greater than $\aleph_\alpha$, and for a limit β we let $\aleph_\beta = \lim_{\alpha<\beta} \aleph_\alpha$.

Similarly, we define $\beth_0 = \omega$, $\beth_{\alpha+1} = 2^{\beth_\alpha}$, and $\beth_\beta = \lim_{\alpha<\beta} \beth_\alpha$. The *Generalized Continuum Hypothesis (GCH)* is equivalent to $\aleph_\alpha = \beth_\alpha$ for all α.

Example 4.2. Define a set V_α for each ordinal α by transfinite induction as follows.

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\begin{aligned}
V_0 &= \emptyset \\
V_{\alpha+1} &= \mathcal{P}V_\alpha \\
V_\beta &= \bigcup_{\alpha<\beta} V_\alpha \qquad (\beta \text{ a limit}).
\end{aligned}
$$

The sets V_α are called the **cumulative hierarchy**. The axiom of foundation is equivalent to the assertion that every set is in V_α for some α; this is often phrased as $V = \bigcup V_\alpha$, even though the class V of all sets is not itself a set.

The **rank** of a set X is the smallest α such that $X \in V_\alpha$. For instance, the rank of each ordinal α is $\alpha + 1$. Most ordinary mathematical objects, as usually constructed from sets, have very low rank: the rank of $\mathbb{N} = \omega$ is $\omega + 1$, the rank of $\mathbb{Z}$ is $\omega + 2$, the rank of $\mathbb{Q}$ is $\omega + 5$, and the rank of $\mathbb{R}$ is $\omega + 9$ (or even less, if we are sufficiently clever).

Example 4.3. For any set X, let $\mathrm{Def}(X)$ denote the set of all subsets of X which are *definable* from elements of X. By this I mean all sets of the form $\{x \in X \mid \varphi(x)\}$ for some definable property $\varphi(x)$ which refers only to elements of X—that is, its parameters and quantifiers ("for all y" or "there exists y") range only over elements

of X. We define sets L_α by transfinite induction as follows.

$$L_0 = \emptyset$$
$$L_{\alpha+1} = \mathrm{Def}(L_\alpha)$$
$$L_\beta = \bigcup_{\alpha<\beta} L_\alpha \qquad (\beta \text{ a limit}).$$

The sets L_α are called the *constructible hierarchy* and the class $L = \bigcup L_\alpha$ is called the *constructible universe*. The *axiom of constructibility* is the assertion that $V = L$; that is, that every set is constructible. (Note that this does *not* mean $V_\alpha = L_\alpha$!) Most set theorists do not believe this axiom is 'true', because it is so restrictive about what sets exist, but it cannot be proven or disproven from ZFC alone, though it does contradict some large cardinal axioms (see §9).

Example 4.4. Let $\mathbf{A}$ be a cocomplete category and S be a *pointed endofunctor*, meaning a functor $S\colon \mathbf{A} \to \mathbf{A}$ equipped with a natural transformation $\sigma\colon \mathrm{Id}_{\mathbf{A}} \to S$. For any object $X \in \mathbf{A}$ we define a sequence of objects X_α, together with morphisms $X_\gamma \to X_\alpha$ for $\gamma < \alpha$, by transfinite induction as follows.

$$X_0 = X$$
$$X_{\alpha+1} = SX_\alpha$$
$$X_\beta = \mathrm{colim}_{\alpha<\beta} X_\alpha \qquad (\beta \text{ a limit}).$$

The colimit at limit ordinals is, of course, over the diagram formed by the morphisms $X_\gamma \to X_\alpha$, which we define by a parallel transfinite induction. Namely, at a successor stage each morphism $X_\gamma \to X_{\alpha+1}$ is the composite

$$X_\gamma \longrightarrow X_\alpha \overset{\sigma}{\longrightarrow} SX_\alpha = X_{\alpha+1},$$

while at a limit stage the morphisms $X_\gamma \to X_\beta$ are just the colimit cocone.

If for some ordinal δ, the maps $X_\gamma \to X_\alpha$ are isomorphisms for all $\delta < \gamma < \alpha$, we say that this process *converges*. The intuition is that this happens when SX is defined from X by a 'small amount of data', since then for a large enough limit ordinal α, all the data necessary to construct SX_α will be contained in the objects X_γ for $\gamma < \alpha$, so we will have $SX_\alpha \cong X_\alpha$. Converging sequences of this sort are often used to construct reflections for subcategories and colimits in categories of algebras; an encyclopedic reference is [Kel80, Kel82].

A similar procedure is often followed in homotopy theory, but in this case usually we instead want the maps $X_\gamma \to X_\alpha$ to become 'weak equivalences' in an appropriate sense. For example, if $\mathbf{A}$ is the category of topological spaces and 'weak equivalence' means 'weak homotopy equivalence' (that is, a map inducing isomorphisms on all homotopy groups), then it usually suffices to take $\delta = \omega$. This is because homotopy groups are detected by maps out of spheres, but spheres are compact, and so a map from a sphere into a well-behaved sequential colimit must factor through some finite stage. However, in more complicated arguments, very large values of δ may be necessary. In homotopy theory this is called the *small object argument*, because it relies on the 'smallness' of objects like spheres; see, for instance, [Hov99]. For a version of the small object argument which does converge in the category-theoretic sense, see [Gar].

5. Logic and Incompleteness

A common mistake is to regard the axioms of ZFC as assertions only about 'the real' universe of sets, when in fact they are satisfied by many different 'universes of sets'. This is not a philosophical statement, but a mathematical one. The reader is free to entertain a Platonic belief that a 'real' universe of sets exists (as many set theorists seem to do), but it will still be true that the axioms of ZFC support many different models in addition to this 'real' one. To clarify the situation it is helpful to consider an analogy.

The *axioms of group theory* are the following; they deal with a collection of things and a binary operation "·".

- For all x, y, z we have $x \cdot (y \cdot z) = (x \cdot y) \cdot z$.
- There exists an e such that for all x, we have $x \cdot e = x = e \cdot x$.
- For all x, there exists a y such that $x \cdot y = e = y \cdot x$.

A *model* of these axioms is a collection of things with a binary operation satisfying them. Of course, this is just a group. We can prove theorems from the axioms, which will then be true statements about any group. However, some statements, like "for all x and y, $x \cdot y = y \cdot x$", are neither provable nor disprovable from the axioms; these are true for some groups and false for others.

In fact, *Gödel's Completeness Theorem* says that if a statement is unprovable from the axioms of a theory $\mathcal{T}$, then there exist some models of the theory in which it is true and others in which it is false. We say that a theory is **consistent** if its axioms do not imply a contradiction; the completeness theorem can then be rephrased as "any consistent theory has a model". Conversely, the rather more obvious *Soundness Theorem* says that any theory with a model is consistent.

Now, the axioms of ZFC, which we summarized in §3, deal with a collection of things and a binary relation $\in$. A *model* of the axioms of set theory is a collection of things, which we usually call *sets*[3], together with a binary relation $\in$, usually called *membership*, which satisfy the axioms. Let us call such a model a **universe**.

We can prove many theorems from ZFC (in fact, we can develop most of mathematics), and these theorems will then be true statements about any universe. However, just as for the theory of groups, some statements are neither provable nor disprovable from the axioms; a classic example is the Continuum Hypothesis (CH). In fact, given any universe, we can construct from it both a universe in which CH is true and a universe in which CH is false. The former is easy to describe: the constructible universe $L = \bigcup L_\alpha$ is always a model of ZFC+CH. The latter requires a more involved technique called *forcing* which is irrelevant to us here. Thus, *if* ZFC is consistent, then both ZFC+CH and ZFC + not CH are consistent.

Of course, one is entitled to wonder whether ZFC *is* consistent; that is, whether there are any universes. This is no more or less valid, from a purely logical standpoint, as wondering whether there are any groups. We are used to the existence of lots of groups, but all the groups we are familiar with are constructed within the framework of a stronger theory—namely, set theory. In other words, *assuming* the existence of a universe, we can construct groups, but with only the axioms of group theory, we can't expect to get anywhere. By analogy, we can't expect to be able to

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[3]As we have stated them, the axioms of ZFC do not allow objects which are not sets. It is easy to modify them to allow such 'urelements', but there seems little point to doing so, since experience shows that everything in mathematics can be constructed using sets.

prove the existence of a universe unless we work within the framework of some yet stronger theory.

Gödel's Second Incompleteness Theorem is a formal way of saying this: no reasonable[4] and consistent axiom system $\mathcal{T}$ which includes arithmetic (such as ZFC) can prove its own consistency. It can be found in many books on logic (I like [CL01] for mathematicians, while everyone should read [Hof79]), but the proof is so simple in outline that every mathematician should be exposed to it. First, by coding logical statements and proofs as natural numbers, Gödel enabled $\mathcal{T}$ to talk about provability and consistency. He then constructed a statement G *about natural numbers* which said in effect "this statement is not provable in $\mathcal{T}$". Thus, if G is provable, it is false. Hence, if $\mathcal{T}$ is consistent, it cannot prove G, and thus G is true. So there is a statement which is true, but not provable in $\mathcal{T}$; this is the First Incompleteness Theorem. Note that the completeness theorem then implies that any reasonable and consistent theory has more than one model.

Now, the same coding of statements and proofs produces a statement *about natural numbers* which expresses 'internally' our inability to derive a contradiction from the axioms of $\mathcal{T}$; call this statement $\mathrm{Con}(\mathcal{T})$. By internalizing the proof of the First Incompleteness Theorem, we can prove *in* $\mathcal{T}$ that $\mathrm{Con}(\mathcal{T})$ implies that G is not provable, and hence that G is true. Since $\mathcal{T}$ cannot prove that G is true, it follows that $\mathcal{T}$ cannot prove $\mathrm{Con}(\mathcal{T})$; this is the Second Incompleteness Theorem.[5]

By the completeness theorem, it follows that we cannot prove in $\mathcal{T}$ the existence of a model for $\mathcal{T}$. Moreover, if we can prove the existence of a model for $\mathcal{T}$ in some other theory $\mathcal{T}'$, then $\mathcal{T}'$ implies $\mathrm{Con}(\mathcal{T})$, and therefore $\mathrm{Con}(\mathcal{T})$ does not imply $\mathrm{Con}(\mathcal{T}')$; otherwise it would imply $\mathrm{Con}(\mathrm{Con}(\mathcal{T}))$, contradicting the incompleteness theorem.[6] So if one theory can prove the existence of a model for another, the first theory is irreducibly stronger in this precise sense.

Remark 5.1. Actually, even without knowing the incompleteness theorem, or if we have an 'unreasonable' $\mathcal{T}$ to which it does not apply, it is easy to see that a proof of $\mathrm{Con}(\mathcal{T})$ in $\mathcal{T}$ would be useless anyway. For since anything follows from a contradiction, if $\mathcal{T}$ were inconsistent, it would also prove $\mathrm{Con}(\mathcal{T})$. Thus, a proof of $\mathrm{Con}(\mathcal{T})$ in $\mathcal{T}$ would still not allow us to conclude that $\mathcal{T}$ is *actually* consistent. The incompleteness theorem gives the stronger result that a proof of $\mathrm{Con}(\mathcal{T})$ in $\mathcal{T}$ in fact implies that $\mathcal{T}$ is *in*consistent.

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[4]By 'reasonable' I mean that there is a systematic way to verify whether or not any given statement is an axiom. This excludes, for example, the system whose axioms are 'all true statements about the natural numbers', to which of course the incompleteness theorem does not apply, but which is not much use as an axiom system in practice.

[5]The attentive reader will notice that this implies that if $\mathcal{T}$ is consistent, then there exist models of $\mathcal{T}$ in which $\mathrm{Con}(\mathcal{T})$ is false! To make sense of this, remember that $\mathrm{Con}(\mathcal{T})$ actually says something like "there does not exist a natural number n which codes for a proof of $0 = 1$ in $\mathcal{T}$." Since $\mathcal{T}$ is consistent, there is no such proof, and thus no 'real' natural number can code one, but bizarre models of $\mathcal{T}$ can contain 'nonstandard' natural numbers which satisfy the *arithmetical* property which we interpret as coding for such a proof.

[6]Care is needed, however, when dealing with theories like ZFC that have infinitely many axioms. It is possible to have a consistent theory $\mathcal{T}$ in which one can define a set M and prove, for each axiom ψ of $\mathcal{T}$, that ψ is true in M. Such a theorem-schema does not contradict the incompleteness theorem; that would require proving a single theorem in $\mathcal{T}$ to the effect of "for each axiom ψ of $\mathcal{T}$, ψ is true in M". Ironically, one example of such a theory $\mathcal{T}$ is ZFC + "ZFC is inconsistent", which (by the incompleteness theorem) is consistent if ZFC is; see [Kun80, IV.10]. We will see in §11 that ZFC itself is almost such a theory.

A similar result called *Tarski's undefinability theorem* says that there is no definable property φ of natural numbers such that $\varphi(n)$ is true in some model of $\mathcal{T}$ if and only if n is the Gödel code of a statement which is true in that model. For suppose there were. Let $\psi_1, \psi_2, \ldots$ be an enumeration of all definable properties of natural numbers, and for any n let $\#n(n)$ be the Gödel code of the statement $\psi_n(n)$. Now the statement "$\varphi(\#n(n))$ is false" is a definable property of n, so it is equal to ψ_k for some k. Then we have (in our model)

$$\psi_k(k) \iff \varphi(\#k(k)) \text{ is false} \iff \psi_k(k) \text{ is false}$$

which is absurd; thus φ cannot exist.

On the other hand, since mathematical logic can be formalized in set theory just as most branches of mathematics can, ZFC has no problem talking about truth in a model which is a set. That is, in ZFC there is a definable property φ such that $\varphi(n, x)$ is true if and only if n is the Gödel code of a statement which is true in x, regarded as a model of set theory. This will also be important later on.

Remark 5.2. The axioms of set theory and of group theory do differ in an important philosophical way. The axioms of group theory are chosen because we see many objects 'in nature' which satisfy them (whatever that means), and we want to study all these objects under one heading. On the other hand, we do not see many examples of universes in nature—many people would argue that we see only one. The axioms of set theory are chosen for their usefulness, sufficiency, and consistency in working with sets, rather than claiming to be the 'correct' description of an independently occurring class of models.

6. CLASSES AND LARGE CATEGORIES

With some basic set theory under our belts, we now move on to category theory. By analogy with the theory of groups and the theory of sets discussed in §5, we can consider the *theory of categories*. This theory deals with two types of things, called 'objects' and 'arrows', together with domain, codomain, identity, and composition functions satisfying unit and associativity axioms. A model of this theory is, of course, a category. Note that this abstract notion of 'a category' can be defined without reference to any sort of set theory.

In the context of a universe V, we generally refer to a category whose collections of objects and arrows are sets in V as a **small** category. When working with small categories with respect to some universe, we have all the tools of set theory at our disposal, and everything we might expect to be true, is. For example, given any two small categories $\mathbf{A}$ and $\mathbf{B}$, there is a small category $[\mathbf{A}, \mathbf{B}]$ whose objects are functors $\mathbf{A} \to \mathbf{B}$ and whose arrows are natural transformations.

However, frequently even when working in the context of set theory, we want to consider categories which are not small. The obvious example is the category $\mathbf{Set} = \mathbf{Set}[V]$ whose objects are all sets (that is, all elements of the universe V) and whose arrows are all functions between sets. Cantor's paradox ensures that $\mathbf{Set}$ is not small. Non-small categories are usually called (surprise!) **large**.

If we are content to work with one large category at a time, we can just use the theory of categories described above. It is when we want to construct new large categories and functors that we run into problems, because the powerful tools of set theory are no longer at our disposal for working with collections of objects that are not sets. Our goal is to consider various methods for dealing with this problem.

First of all, there is an approach that remains completely within ZFC. We define a **class** to be a collection of sets specified by some property expressible in the language of set theory. Instead of working directly with classes, for which we have no axioms, we can then work instead with the properties which characterize them. For example, the property "X is a pair $(G, \cdot)$ where G is a set and $\cdot$ is a binary operation on G making it into a group" is expressible in the language of set theory, so there is a class of all groups—by which we mean all groups defined from the universe V. Of course, there is also a class of all sets, which we usually identify with the universe V. Note that classes of this sort are actually implicit in the axioms of ZFC; for example, the axiom of replacement says that the image of a set under any 'class function' is a set. Some classes, of course, are sets; those which are not sets are called **proper classes**.

We then define a **large category** to be one, such as **Set** and **Grp**, whose collections of objects and arrows are classes in this sense. We can now perform some basic constructions on large categories. For example, if P and Q are properties expressible in set theory, then "X is a pair (Y, Z) such that Y satisfies P and Z satisfies Q" is also so expressible. Thus the cartesian product of two classes is a class, and the cartesian product of two large categories is another large category. We can also prove that **Set** and other familiar categories are complete and cocomplete, as defined in §2.

Remark 6.1. Most large categories which arise in applications are also locally small; that is, they have only a set of morphisms between any two objects.. This property is undeniably important, as we saw in the proof of the Adjoint Functor Theorem, but we will mostly ignore it, because it plays almost no role when discussing foundations: locally small categories present exactly the same set-theoretic issues that all large categories do.

However, this approach to large categories has the disadvantage that we have no axioms for manipulating classes; they are not 'things' that ZFC knows about at all. Thus, instead of working with classes directly, we have to work with the logical formulas which characterize their elements, and interpret any construction in these terms.

In particular, the language of ZFC does not include a way to *quantify* over classes. In other words, no theorem containing the phrases "for any large category **A**" or "there exists a large category **A**" can be even *stated*, let alone proven, in ZFC. This includes, for example, the Adjoint Functor Theorem. Usually this is dealt with by proving instead a 'meta-theorem' of the form "for any large category **A**, we can prove in ZFC that (some statement about **A**)", but again in stating such a theorem we have moved beyond ZFC into some sort of 'meta-language'.

Moreover, even this trick cannot handle theorems whose *hypotheses* involve quantification over classes. For example, consider the final statement in Theorem 2.1: if a large category has products indexed by proper classes, then it is a preorder. Even if we fix a large category **A**, the statement "**A** has products indexed by proper classes" is of the form "for any class ...", and thus cannot be stated in ZFC. Hence we cannot prove this result even as a theorem-schema in ZFC.

7. Axioms for classes

To resolve these sorts of issues, we are motivated to extend ZFC by introducing *classes* as a new type of thing, in addition to sets, along with axioms for manipulating them. A good overview of such *class-set theories* can be found in [Lév76]. Probably the most common such theory is **von Neumann-Bernays-Gödel (NBG) set theory**. Its axioms can also be found in books or on the Internet; as with ZFC we divide them into several groups.

(i) *Typing*: only sets can be elements of sets or classes.
(ii) The *basic* axioms: extensionality for sets and classes, the empty set, pairing and union for sets, foundation for sets and classes, and comprehension.
(iii) The *size-increasing* axioms: power sets and limitation of size.
(iv) The *size-assertion* axiom: an infinite set exists.
(v) The axiom of *choice*.

The axiom schema of *comprehension* states that for any property $\varphi(x)$ which does not quantify over classes, there is a *class* $\{x \mid \varphi(x)\}$ of all *sets* x such that $\varphi(x)$. The *limitation of size* axiom says that a class is a set if and only if it is not bijective with the class V of all sets. Thus, sets are precisely the classes which are 'not too big', while all proper classes are the same size.

Comprehension and limitation of size easily imply separation and replacement, so the sets in NBG satisfy ZFC. Moreover, NBG can be shown to be a **conservative** extension of ZFC. That means that any statement *about sets* which is provable in NBG is also provable in ZFC. In fact, if we start with any model of ZFC, then taking the classes to be those defined in §6, we obtain a model of NBG. Thus, using NBG really entails no 'ontological' commitment beyond that of ZFC.

However, unlike ZFC, the language of NBG allows us to quantify over classes (although such quantifications cannot be used in the comprehension axiom). Thus, theorems such as the Adjoint Functor Theorem can be stated and proven formally within NBG. To prove the large-category version of Theorem 2.1, we have to be careful to avoid talking about $2^{|\mathrm{Arr}(\mathbf{A})|}$, which doesn't exist in NBG, but we can instead directly derive a contradiction to the axiom of limitation of size.

Moreover, NBG makes constructions on classes easier to deal with. For example, comprehension proves easily that any two classes have a cartesian product, and thus so do any two large categories. We can also perform more complicated constructions as long as they don't produce things that are 'too big'.

Example 7.1. Let $\mathbf{A}$ be a large category; we construct its *idempotent-splitting* $\overline{\mathbf{A}}$, also called *Karoubian completion* or *Cauchy completion*. The objects of $\overline{\mathbf{A}}$ are the idempotents of $\mathbf{A}$; that is, arrows e with $ee = e$. The property "e is an arrow of $\mathbf{A}$ and $ee = e$" is expressible in NBG and doesn't quantify over classes, so by comprehension, there is a class of all such idempotents. Similarly, the arrows of $\overline{\mathbf{A}}$ from e to e' are the arrows f with $fe = f$ and $e'f = f$; this is likewise expressible without quantification over classes, so there is a class of all such arrows. The rest of the structure of $\overline{\mathbf{A}}$ follows in the same way.

Example 7.2. Let $\mathbf{A}$ be a large monoidal category; we construct a strict monoidal category $\mathbf{A}'$ monoidally equivalent to $\mathbf{A}$. One of the usual constructions is to let the objects of $\mathbf{A}'$ be finite strings of objects of $\mathbf{A}$, with morphisms induced by those of $\mathbf{A}$. Since the property "X is a function from some natural number n to the class of objects of $\mathbf{A}$" does not quantify over classes, the class of all such functions exists;

we take this to be the class of objects of $\mathbf{A}'$. Any such function (as a class of ordered pairs) is bijective with n and thus is a set. The rest of the structure is similar.

Another example of the usefulness of having a good axiomatic system for classes lies in our ability to make a large number of choices. Suppose that $\mathbf{A}$ is a category with finite products; it is usual to make a choice of a product $X \times Y$ for each pair of objects $X, Y \in \mathbf{A}$ in order to define a product functor $\mathbf{A} \times \mathbf{A} \to \mathbf{A}$. In particular cases there is usually a standard choice of $X \times Y$, but to do this in general one needs an axiom of choice for the objects of $\mathbf{A}$. If $\mathbf{A}$ is a large category, then one needs an axiom of choice for *classes*. We call this the **axiom of global choice**; it turns out to have the following equivalent forms.

- (i) We can choose an element from each of any class of nonempty sets.
- (ii) We can choose an element from each of any collection of nonempty classes.[7]
- (iii) The class V of all sets can be well-ordered.

Global choice (and hence ordinary choice as well) is a consequence of the axioms of NBG, by the following observation of von Neumann.

Theorem 7.3. *In* NBG, *there is a well-ordering of V.*

Proof. The class Ω of ordinals is well-ordered. Thus, if it were a set, it would itself be an ordinal; but then it would have a successor, which is absurd. Thus Ω is not a set (this is Burali-Forti's paradox). By limitation of size, Ω is bijective to V, and thus V acquires a well-ordering. $\square$

If one objects to global choice despite the pleasing cleverness of this argument, it is not hard to modify the axioms of NBG so that they no longer imply it. Here we should also mention [Mak96]. However, global choice is implicitly used by many familiar categorical constructions.

Example 7.4. We have already noted that if $\mathbf{A}$ has products, then applying version (ii) of global choice, we can choose a product $X \times Y$ for each pair X, Y and thereby define a functor $\mathbf{A} \times \mathbf{A} \to \mathbf{A}$. Similar remarks, of course, apply to other limits and colimits, and other objects defined by universal properties, such as tensor products.

Example 7.5. If $F \colon \mathbf{A} \to \mathbf{B}$ is a functor which is full, faithful, and essentially surjective, then by choosing for each $b \in \mathbf{B}$ an object $Gb \in \mathbf{A}$ and an isomorphism $FGa \cong b$, we can construct an inverse equivalence $G \colon \mathbf{B} \to \mathbf{A}$.

Example 7.6. By choosing one object in each isomorphism class, we can show that any large category has a skeleton.

Example 7.7. Let W be a class of morphisms in a large category $\mathbf{A}$; we want to construct a 'localization' $\mathbf{A}[W^{-1}]$ by formally adding inverses to the morphisms in W. The objects of $\mathbf{A}[W^{-1}]$ are the same as those of $\mathbf{A}$, while its morphisms are supposed to be equivalence classes of zigzags

$$\cdot \overset{\sim}{\longleftarrow} \cdot \longrightarrow \cdots \cdots \longrightarrow \cdot \overset{\sim}{\longleftarrow}$$

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[7]Note, however, that there is a standard trick in ZFC which enables us to choose an element from each of any *set* of nonempty *classes*. To be precise, if we have a formula $\varphi(x, y)$ such that for any $x \in X$ there is a y with $\varphi(x, y)$, we can define a function f on X such that $\varphi(x, f(x))$ for all $x \in X$. We do this by first considering, for each $x \in X$, the class of all sets y of *least rank* such that $\varphi(x, y)$; this is a set since it is a subset of some V_α. We then apply the ordinary axiom of choice.

of morphisms in $\mathbf{A}$, where the backwards arrows are in W. Comprehension guarantees there is a class of such zigzags. However, we cannot define the quotient of a class by an equivalence relation whose equivalence classes are proper classes—at least not in the usual way, since no class can be an element of another class. But what we can do instead is use global choice to choose one zigzag from each equivalence class, thereby obtaining a class of morphisms for $\mathbf{A}[W^{-1}]$.

Of course, in general, $\mathbf{A}[W^{-1}]$ defined in this way need not be locally small or at all amenable to computation. In practice, there are usually alternate ways to construct $\mathbf{A}[W^{-1}]$, which also show that in those cases it is locally small.

Example 7.8. The *left derived functors* of a functor F evaluated at an object X of some abelian category are given by choosing a projective resolution $\cdots \to P_1 \to P_0 \to X$ and computing the homology of the chain complex $\cdots \to FP_1 \to FP_0$. Of course, defining the whole derived functor requires the global choice of a projective resolution for each object X.

On the other hand, NBG is not quite as comfortable an axiom system as we might like. Consider mathematical induction, which is surely a basic notion in mathematics if ever there was one. In ZFC, we can prove induction for any definable statement $\varphi(n)$, by considering the set $\{n \in \mathbb{N} \mid \text{not } \varphi(n)\}$ and using the fact that $\mathbb{N}$ is well-ordered. In NBG the same argument works only if φ does not involve quantification over class variables, due to the analogous restriction in the comprehension axiom. This includes all statements for which ZFC could prove induction (as it must, since NBG is a conservative extension of ZFC), but not all statements in our 'new language' which can refer to classes.

One might object to this consequence of NBG on the philosophical grounds that mathematical induction 'should' be true for *all* statements φ, without needing technical restrictions on quantification. But the failure of induction also has real consequences for dealing with classes. For instance, we cannot prove by induction on n that every large category $\mathbf{A}$ has an n-fold cartesian product $\mathbf{A}^n$. In this case, we can construct $\mathbf{A}^n$ directly as the class of functions from n into $\mathbf{A}$, but it is troubling that the induction proof is not allowed in NBG.[8] If any reader can think of a natural statement about large categories, normally proven by induction, and not having an obvious alternate proof in NBG, I would be very interested.

We may, of course, strengthen the comprehension axiom of NBG to allow formulas with arbitrary quantification (sometimes called *impredicative comprehension*), thereby recovering full mathematical induction. This gives a variant of NBG usually called **Morse-Kelley (MK) set theory**. However, by doing so we lose conservativity over ZFC. In MK one can prove that the class V of all sets is a model for ZFC and conclude that ZFC is consistent. For example, to prove the separation and replacement axioms, we first apply comprehension to construct the desired set or

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[8]In defense of NBG, I should say that it was originally conceived not to deal with large categories, but to provide a finitely axiomatizable theory equivalent to ZFC, and at that it succeeds. This is not immediately obvious, since we have stated comprehension as an axiom schema, but it turns out that a finite number of its instances suffice to imply the rest. There is no great mystery about this: we simply observe that any definable property φ is built up from a finite number of building blocks like 'and', 'or' and 'there exists', and the corresponding class $\{x \mid \varphi(x)\}$ can be built up by a corresponding finite number of constructions like intersection, union, and projection. However, it does depend on limiting comprehension to properties not quantifying over classes; thus MK (see below) is not finitely axiomatizable in this way.

function as a class, then apply limitation of size to conclude that it is a set. It then follows from the incompleteness theorem that Con(ZFC) does not imply Con(MK).[9] So unlike NBG, MK is a genuinely stronger theory than ZFC.

Even MK, however, is not fully satisfactory in the constructions it allows for large categories. For example, if $\mathbf{A}$ and $\mathbf{B}$ are large categories, nothing we have seen so far allows us to construct a functor category $[\mathbf{A}, \mathbf{B}]$. There is no problem when $\mathbf{A}$ is small, since then each functor $\mathbf{A} \to \mathbf{B}$ is itself a set by replacement, and so we have a class of such—but when $\mathbf{A}$ is large, this argument fails. However, there is no intuitive reason preventing us from making such a construction: the collection of functions from one class to another seems like a perfectly good collection.

We could envision adding more axioms which enable us to perform these and other constructions with classes. In fact, the best possible world would be if classes could be manipulated *just like* sets, and any construction we could do for sets could also be done for classes. This would be easy to achieve: we could just write down another copy of the ZFC axioms, substituting 'class' for 'set' everywhere in the second copy. On the other hand, it seems terribly wasteful to have two copies of every axiom, when all we really want to say is that classes and sets behave in just the same ways, except that sets can't be too large. In the next section we consider a cleaner solution.

8. Inaccessibles and Grothendieck universes

In ZFC, there are three ways to prove the existence of larger and larger sets.

(1) By *fiat*: the axiom of infinity asserts that there exists an infinite set. Without it, only finite sets can be constructed.

(2) By *powers*: the axiom of power sets produces a set $\mathcal{P}A$ larger than a given set A (by Cantor's diagonalization argument).

(3) By *limits*: The axiom of replacement guarantees that the union of any family of sets indexed by a set is also a set.[10] If the family is infinite and increasing in size, its union will be larger than any of its elements. For example, this applies to a family such as $A, \mathcal{P}A, \mathcal{P}\mathcal{P}A, \ldots$.

This is where I got the terminology 'size-assertion' and 'size-increasing' in §3; the axiom of infinity produces a large set by fiat, while the axioms of power set and replacement produce larger sets from existing ones. However, not all cardinalities can be reached by these methods; we introduce special names for those that can't.

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[9]Why, the reader may reasonably wonder, can we not do the same in NBG? We can prove in NBG that V satisfies any *particular* axiom of ZFC, but as in footnote 6, to conclude Con(ZFC) we need instead the single theorem "for all axioms ψ of ZFC, V satisfies ψ". To even state this formally when there are infinitely many axioms, we must encode axioms by their Gödel numbers. Just as ZFC can talk about truth in set models, in NBG we have a definable property φ such that $\varphi(n)$ is true if and only if n is the Gödel code of a true statement involving only sets—but this φ involves quantification over classes. Thus, we require the strength of MK to form the class $\{x \mid \psi(x)\}$ given only the Gödel number of ψ, as is necessary for the above proof of Con(ZFC). This distinction is well explained in [Mos50, Mos51], along with resulting concrete examples of the failure of full mathematical induction and full class comprehension in NBG.

[10]In more detail, let A be a set and F a class function on A. Such an F is characterized by some definable property φ such that for any $x \in A$ there exists a unique y with $\varphi(x, y)$. Then replacement gives the set $\{y \mid \exists x \in A \ \varphi(x, y)\}$, to which we can then apply the union axiom to give the set $\bigcup_{x \in A} F(x)$. It is natural to wonder why I call replacement the culprit here, when the union axiom seems at least equally culpable; one answer is that, as we will see below, small models of ZFC satisfying the union axiom abound, while ones satisfying replacement are quite rare.

(i) A cardinal is **uncountable** if it is larger than the smallest infinite set.

(ii) A cardinal κ is a **strong limit** if for any $\lambda < \kappa$ we have $2^\lambda < \kappa$.

(iii) A cardinal κ is **regular** if it is not the union of a family of sets of size $< \kappa$ indexed by a set of size $< \kappa$.

(iv) A cardinal is **inaccessible** if it is uncountable, a strong limit, and regular.

For example, the first uncountable cardinal $\aleph_1$ is regular, since the countable union of countable sets is countable, but it is not a strong limit, since $2^{\aleph_0} \geq \aleph_1$. On the other hand, the cardinal $\beth_\omega$ (see Example 4.1) is a strong limit, but not regular.

Now, in any universe V, the set V_α with its induced relation of membership is itself a model of many of the axioms of ZFC. It is easy to see that if α is a limit ordinal greater than ω, then V_α satisfies the basic axioms, choice, power set, and infinity—all the axioms of ZFC except replacement. In particular, since $|V_{\omega+\alpha}| = \beth_\alpha$, without replacement we can only construct sets of cardinality $< \beth_\omega$.

It turns out that if α is an inaccessible cardinal, then V_α is also a model of replacement, and hence of all of ZFC. (The converse, however, is false, as we will see in §11.) The proof is the same as the proof in MK that V satisfies replacement, using the inaccessibility of α in place of limitation of size. When α is inaccessible we call V_α a **Grothendieck universe**. One can equivalently define a Grothendieck universe to be a set U which is transitive ($x \in y \in U$ implies $x \in U$) and closed under pairing, power sets, and indexed unions. It turns out that this is equivalent to asserting $U = V_\kappa$ for some inaccessible κ; see [Bou72].

Now suppose there exists an inaccessible, and let κ be the smallest inaccessible. Then V_κ satisfies ZFC and also "there does not exist an inaccessible"; hence it is impossible to prove in ZFC that there exists an inaccessible. But this is not really surprising, since we have essentially *defined* inaccessibles to be those cardinals which are unreachable by all the ways that ZFC knows to build bigger sets!

More that this is true, however. If we write **ZFC+I** for ZFC + "there exists an inaccessible", then even assuming that ZFC is consistent, it is not possible to prove that ZFC+I is consistent. For just as we can prove in MK that V is a model of ZFC, we can prove in ZFC that any Grothendieck universe is a model of ZFC; hence ZFC+I implies Con(ZFC). Thus, by the incompleteness theorem, Con(ZFC) does not imply Con(ZFC+I). In other words, in contrast to the situation for CH, we cannot construct, from an arbitrary universe, another universe satisfying ZFC+I. Note that Con(ZFC), which is provable in ZFC+I but not in ZFC, is a statement not about sets but about *natural numbers*—albeit a rather complicated one.

Laying aside questions of existence and consistency for the moment, we can solve the problems raised at the end of §7 as follows. Working in ZFC+I, we choose an inaccessible κ, and re-define *set* to mean 'element of V_κ' and *class* to mean 'set not necessarily in V_κ'. Thus defined, sets and classes will behave in exactly the same way, except that sets are limited in rank. A more common terminology, however, which we will adopt, is not to redefine 'set' but to refer to elements of V_κ as **small sets** and other sets as **large sets**.

Note that small sets have a limitation on *rank* rather than *cardinality*. No small set can be larger in cardinality than κ, but many sets with small cardinality are not small, such as the singleton $\{\kappa\}$. The class of sets with small cardinality is not a model of ZFC, since it is not closed under unions ($\bigcup\{\kappa\} = \kappa$), nor is it itself a set

(even a large one). Thus, if we want a category **Set** of small sets whose collection of objects is a large set, we need to use rank rather than cardinality.[11]

Of course, any category of sets is determined up to equivalence by the cardinalities of its objects, so in some sense, this doesn't matter. For this reason, it is common to also call sets *small* if their cardinality is small, and in category theory, where we work with objects up to isomorphism anyway, this is usually harmless.

This third approach allows many more large categories than the first two. For instance, for any two large categories **A** and **B**, there is a functor category $[\mathbf{A}, \mathbf{B}]$. Note that unlike in NBG, where all proper classes have the same size, $[\mathbf{A}, \mathbf{B}]$ will generally be larger than **A** or **B**. In particular, the term *large category* in this approach is significantly more inclusive than it is in the previous two.

One should think of the classes in NBG and MK as corresponding to the large sets of rank (or cardinality) $\leq \kappa$. In fact, one can make this precise. If κ is inaccessible, we obtain a model of MK (and hence NBG) by taking V_κ for the sets and $V_{\kappa+1} = \mathcal{P}V_\kappa$ for the classes. (Note that this means we can prove Con(MK) in ZFC+I, so ZFC+I is strictly stronger than MK.) Similarly, we obtain a model of NBG (but not MK) if we take V_κ for the sets and $\mathrm{Def}(V_\kappa)$ for the classes (recall Example 4.3).

In [Str81], sets and categories of cardinality $\leq \kappa$ (which are those appearing, up to isomorphism, in $V_{\kappa+1}$) were called **moderate**. We will refer to sets and categories in $\mathrm{Def}(V_\kappa)$ as **small-definable**. Small-definable categories are those which would exist as classes even in ZFC or NBG; this includes nearly all large categories in which we are actually interested for their own sake. However, the existence of other large categories is frequently quite useful for formal reasons; it is the primary advantage of using inaccessibles. In the following examples we assume an inaccessible κ.

Example 8.1. Any large category **A** has a presheaf category $[\mathbf{A}^{op}, \mathbf{Set}]$, and if **A** is locally small, it has a Yoneda embedding $y\colon \mathbf{A} \hookrightarrow [\mathbf{A}^{op}, \mathbf{Set}]$. Limits and colimits in large functor categories can still be calculated pointwise, so $[\mathbf{A}^{op}, \mathbf{Set}]$ is still complete and cocomplete. Important properties of **A** can be expressed in terms of y; for example, **A** is *total* (see [Kel86]) if y has a left adjoint. Totality can be expressed without reference to $[\mathbf{A}^{op}, \mathbf{Set}]$ (although not without quantifying over classes), but at the price of a certain amount of economy and clarity.

Example 8.2. Any large **A** has an endofunctor category $[\mathbf{A}, \mathbf{A}]$, which is strict monoidal under composition of functors. If **A** is also monoidal, then the functor $\mathbf{A} \to [\mathbf{A}, \mathbf{A}]$ taking $X \in \mathbf{A}$ to $(X \otimes -)$ is strong monoidal. We can thus apply the idea of [ML98, XI.3 Ex. 3] to find a strictification of **A** inside $[\mathbf{A}, \mathbf{A}]$.

Example 8.3. It is well-known in algebraic geometry that the category **Sch** of *schemes* is equivalent to a certain subcategory of $[\mathbf{Ring}, \mathbf{Set}]$. The category **Sch** also has other definitions (for instance, as locally affine locally ringed spaces), showing that it is small-definable (up to equivalence) and locally small. However, identifying it with a subcategory of $[\mathbf{Ring}, \mathbf{Set}]$ is often useful, yet impossible unless the latter category exists.

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[11]There is, however, an approach to universes based on cardinality: for any infinite cardinal κ let H_κ denote the set of sets which are *hereditarily* of cardinality $< \kappa$; that is, their transitive closure has cardinality $< \kappa$. Then $H_\kappa \subset V_\kappa$, and if κ is regular and uncountable, H_κ satisfies all the axioms of ZFC except power set. Moreover, κ is inaccessible if and only if H_κ satisfies all of ZFC, and if and only if $H_\kappa = V_\kappa$.

Example 8.4. Similarly, the category $\mathrm{Ind}(\mathbf{A})$ of *ind-objects* in a large category $\mathbf{A}$ can be identified with the small filtered colimits of representable presheaves in $[\mathbf{A}^{op}, \mathbf{Set}]$. Like $\mathbf{Sch}$, the category $\mathrm{Ind}(\mathbf{A})$ has an alternate description showing that it is small-definable (up to equivalence, as long as $\mathbf{A}$ is), but this description of it is frequently also useful.

Example 8.5. Let $U \colon \mathbf{CptHaus} \to \mathbf{Set}$ be the forgetful functor from compact Hausdorff spaces to sets. A *quasi-topological space* is a set X equipped with a sub-functor of $\mathbf{Set}(U-, X) \colon \mathbf{CptHaus}^{op} \to \mathbf{Set}$ satisfying certain natural conditions; see [Spa63]. The category $\mathbf{QTop}$ of quasi-topological spaces is cartesian closed, and was a contender for a convenient category of spaces before the current ascendancy of compactly generated spaces (for which see [ML98, VII.8] and [May99, Ch. 5]).

Since a single quasi-topological space contains a large amount of data, $\mathbf{QTop}$ is *not* small-definable, though it is locally small. Hence NBG and MK are insufficient to guarantee that $\mathbf{QTop}$ even exists. Also, a fixed set X supports a large number of quasi-topologies, and $\mathbf{QTop}$ is not well-powered or well-copowered. It is, however, complete and cocomplete, and admits intersections of arbitrary families of monomorphisms and cointersections of arbitrary families of epimorphisms.

Example 8.6. In addition to its reassuring psychological effect, moderateness (and small-definability) of a category can have mathematical consequences. This is because a set A is moderate if and only if we can express it as an increasing union of small sets indexed by small ordinals, $A = \bigcup_{\alpha < \kappa} A_\alpha$ where each A_α is small. Thus, we can prove things about moderate sets by a transfinite induction in which every stage of the induction is small. For example, a proof of Freyd given in [Str81] shows that if $\mathbf{A}$ is moderate, total, and the left adjoint of y preserves finite limits, then $\mathbf{A}$ actually has a *small* generating set and is a Grothendieck topos.

Sometimes it is useful to assume more than one inaccessible. For example, when doing formal category theory we may want to form the category (or 2-category) of all large categories, or of all locally small categories. Of course, the class of all large categories is not a set, even a large one. (The class of small-definable categories is a large set, but it is not closed under constructions such as functor categories.) To resolve this issue, we can use the techniques of §§6–7 to introduce *classes* that are larger than large sets, or we can assume a second inaccessible $\lambda > \kappa$, define **large** to mean an element of V_λ, and use **very large** to mean a set not necessarily in V_λ. (Some authors have used 'quasi-category' for what we call a 'very large category', but we eschew that term in view of its quite different recent connotations [Joy02]. The term 'meta-category' is also sometimes used.)

Having a very large category $\mathbf{CAT}$ of large categories allows us to make statements like the following.

- Taking a small category to its presheaf category is a functor from the category $\mathbf{Cat}$ of small categories to the category $\mathbf{CAT}$ of large ones.
- Taking a ring R to the category of R-modules is a functor from $\mathbf{Ring}$ to $\mathbf{CAT}$.
- Taking a monoidal category $\mathbf{V}$ to the category $\mathbf{V}\text{-}\mathbf{Cat}$ of small $\mathbf{V}$-enriched categories is a functor from $\mathbf{MONCAT}$ to $\mathbf{CAT}$.

If we want to have a functor taking $\mathbf{V}$ to the very large category $\mathbf{V}\text{-}\mathbf{CAT}$ of *large* $\mathbf{V}$-enriched categories, its codomain will have to be an *extremely large* category of very large categories, so we need at least *three* inaccessibles. One is unavoidably

reminded of how, in the original category theory paper [EM45], (large) categories were introduced only in order to serve as the domains and codomains of functors.

9. Aside: large cardinals

Notice the strong analogy between the axiom "there exists an inaccessible" and the axiom of infinity: both construct "by fiat" larger sets than can otherwise be shown to exist. Other axioms of this sort, asserting the existence of inaccessibles satisfying various extra properties, are also used in modern set theory. Of course, these are stronger assumptions than the mere existence of an inaccessible. In fact, the existence of even one cardinal with one of these stronger properties usually implies the existence of many smaller inaccessibles.

Let me attempt to give a flavor of just how large such *large cardinals* can get. (The terminological collision between 'large category' and 'large cardinal' is unfortunate, but context usually suffices to distinguish.) A subset $X \subset \kappa$ is said to be **closed unbounded** if $\sup(X) = \kappa$ and whenever $Y \subset X$ and $\sup(Y) < \kappa$, then also $\sup(Y) \in X$. A subset $X \subset \kappa$ is **stationary** if it has nonempty intersection with every closed unbounded set. Evidently any stationary set is unbounded, and hence (if κ is inaccessible) has cardinality κ.

An inaccessible κ is said to be **Mahlo** if the set of inaccessibles less than κ is stationary in κ. This implies that there are κ inaccessibles less than κ, but also that there are κ inaccessibles $\lambda < \kappa$ such that there are λ inaccessibles less than λ, and κ inaccessibles below κ with this property, and so on. This pattern continues for many larger types of cardinals:

- If κ is *weakly compact*, then Mahlo cardinals are stationary in κ.
- If κ is *measurable*, then weakly compact cardinals are stationary in κ.
- If κ is *supercompact*, then measurable cardinals are stationary in κ.
- If κ is *extendible*, then supercompact cardinals are stationary in κ.
- If κ is *superhuge*, then extendible cardinals are stationary in κ.

Unlike for Mahlo cardinals, these stationarity properties are not the definitions of these larger cardinals, but consequences thereof.[12] Their actual definitions can be found in books (for example, [Jec03, Kan03]) or the Internet.

The purpose of large cardinals in set theory is not just to see how large sets can get, but to provide a yardstick to measure the strength of other axioms, and in some cases even to prove new results about 'small' sets. For instance, the existence of a measurable cardinal implies that $V \neq L$, the existence of infinitely many Woodin cardinals implies *projective determinacy*, and at least some set theorists hope that a large cardinal axiom can be found which will settle the continuum hypothesis; see [Mad88a, Mad88b].

Perhaps surprisingly, set theorists currently believe that there is an upper bound to how large large cardinals can get: there are notions of *n-huge* cardinal for all $n < \omega$, but the limiting case of an 'ω-huge' cardinal is known to be inconsistent with ZFC. However, even large-cardinal axioms only slightly weaker than ω-hugeness have so far resisted all efforts to disprove them.

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[12]When comparing large cardinal axioms in set theory, *consistency strength* is usually more important than raw size. Obviously, if there are many X cardinals below any Y cardinal, then the consistency of a Y implies that of an X, but the converse is not always true. For example, a huge cardinal implies the consistency of extendible cardinals, and hence of supercompact ones, but the least huge cardinal is less than the least supercompact cardinal (if both exist).

Returning to category theory, it is natural to wonder whether large cardinal properties could have noticeable effects on **Set** and categories constructed from it. This is indeed the case, although for most category-theoretic purposes one inaccessible is as good as another. For example, it is proven in [AR94, A.5] that $\mathbf{Set}^{op}$ has a small dense full subcategory if and only if there do not exist arbitrarily large measurable cardinals. If there are *no* measurable cardinals, then the single set $\mathbb{N}$ is dense in $\mathbf{Set}^{op}$, and thus $\mathbf{Set}^{op}$ is equivalent to a full subcategory of the category of M-sets, where M is the monoid of endomorphisms of $\mathbb{N}$.[13]

In general, while large cardinal axioms in set theory usually assert the *existence* of one or more cardinals with a certain property, what tends to matter for category theory is the character of the *particular* 'size of the universe' cardinal κ. For example, what matters for the result quoted above is whether the measurable cardinals are unbounded below κ, rather than how many measurable cardinals there are in 'absolute' terms. Moreover, in most cases, the assertion that "the cardinal of the universe has property P" can be phrased in NBG (and sometimes even ZFC), without requiring the existence of any sets larger than the universe.

The most interesting examples of this sort that I know of concern **Vopěnka's principle**. This has many equivalent forms; here are a few categorical ones.

- No locally presentable category has a large discrete full subcategory.
- Every complete or cocomplete category with a small dense full subcategory is locally presentable.
- Every category with a small dense full subcategory is well-copowered.

None of these can even be stated in ZFC, since they all involve quantification over large categories, but there is no problem in NBG. Vopěnka's principle has many other pleasing consequences for the structure of locally presentable categories (see [AR94, Ch. 6]), and also implies the existence of arbitrary cohomological localizations in homotopy theory [CSS05], which are not known to exist in ZFC.

We say that a cardinal κ is **Vopěnka** if Vopěnka's principle holds in ZFC+I with κ as the size of the universe. This is equivalent to saying that Vopěnka's principle holds in V_κ regarded as a model of MK, but stronger than the analogous assertion involving NBG. Since $\mathbf{Set}^{op}$ is not locally presentable, the assertions I have made so far imply that measurable cardinals are unbounded below any Vopěnka cardinal, but more is true: if κ is Vopěnka, then the sets

$$\{\lambda < \kappa \mid \lambda \text{ is measurable}\}$$

$$\{\lambda < \kappa \mid \lambda \text{ is extendible in } V_\kappa\}$$

are stationary in κ. This makes the existence of even one Vopěnka cardinal quite a strong assumption, as large-cardinal axioms go. On the other hand, Vopěnka cardinals are stationary in any 'almost huge' cardinal.

10. Inaccessibles or not?

So, should we assume inaccessibles? As we have seen, the existence of an inaccessible—even the *consistency* of the existence of an inaccessible—is unprovable from ZFC, so such an assumption is a genuine strengthening of the axioms. On the

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[13]I am inclined to regard this as an argument in favor of the existence of measurable cardinals, though of course others may disagree. Note that $\mathbf{Set}^{op}$ is also equivalent to the category of profinite Boolean algebras (by Stone duality), and to the category of algebras for the double-power-set monad on **Set**.

other hand, there are many philosophical and mathematical arguments that can be advanced in its favor; see [Mad88a] for a fascinating discussion. Moreover, we have seen that it is quite weak compared to many large-cardinal axioms commonly used in modern set theory, and from which no contradiction has yet been derived. Thus, it seems unlikely that the existence of inaccessibles can be disproven from ZFC (though it is provably impossible to prove that it can't be!).

However, from the point of view of ordinary category theory, these questions are not as relevant, because the role of inaccessibles in category theory is quite different from their role in set theory. In category theory, inaccessibles mostly play the role of a *convenience* which simplifies the statements and proofs of our theorems, without really entailing any deep ontological commitment. This is because we think of the small sets as the 'real' sets; we only introduced the large ones as a well-behaved model for proper classes. All ordinary mathematical objects, like groups, rings, topological spaces, manifolds, and so on, are small.

Moreover, all categories of ordinary objects which arise in practice, such as **Set** and **Grp**, are small-definable, and would exist as classes even in ZFC or NBG. And we saw that categories such as **Sch** and Ind($\mathbf{A}$) have equivalent forms which are small-definable, even though they can also be usefully characterized with reference to categories that are not.

In a similar way, a statement like "$\mathbf{V} \mapsto \mathbf{V}\text{-}\mathbf{Cat}$ is a functor" can be regarded merely as a 'code' which encapsulates many individual statements in a concise way. For example, it implies that any monoidal functor $\mathbf{V} \to \mathbf{W}$ induces a functor $\mathbf{V}\text{-}\mathbf{Cat} \to \mathbf{W}\text{-}\mathbf{Cat}$, and composition is preserved. However, for any *particular* monoidal functor $\mathbf{V} \to \mathbf{W}$, we could easily check this directly, without needing to assert the existence of the whole functor, and thus its very large codomain $\mathbf{CAT}$.[14] Thus, often in category theory, the assumption of inaccessibles can be regarded as merely a convenience (although a very convenient one!). Thus, it is natural to wonder under what conditions the use of inaccessibles can be eliminated from categorical arguments.

There is another unsatisfactory aspect of ZFC+I. We have asserted that we only consider objects smaller than κ to be 'ordinary mathematical objects', but what if at some later date we discover that there happens to be a group larger than κ which we want to include as a mathematical object? Clearly this means that we chose the wrong κ to define 'small' and 'large' and we should choose a larger one. In particular, we may want to use category theory to study the category of categories, and then apply our results to *large* categories as well as small ones.

To ensure that such switches would always be possible, Grothendieck proposed an axiom that there are arbitrarily large inaccessibles, or equivalently that every set is contained in a Grothendieck universe. (Actually, this was the first axiom using universes to be proposed for category theory; only later did Mac Lane observe that one universe was usually sufficient.) This is still quite a weak large-cardinal axiom. Note that this use of multiple inaccessibles is different from our discussion of very large and extremely large categories in §8; here we are changing the size of the universe, rather than using multiple universes at once.

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[14]One cannot help, however, being reminded of how infinite sets in pre-Cantorian mathematics were only regarded as 'potentialities' rather than completed entities, and how at first even ordinary large categories were viewed with suspicion. It seems that the trend of mathematical development is towards recognizing ever larger entities as having an independent existence.

Now, as long as we haven't made use of any properties of κ beyond its inaccessibility, all our results proven for κ will also be true for our new, larger, κ. This means that all our theorems implicitly begin with "For any inaccessible κ, ...". However, the arbitrariness of κ may make us somewhat uneasy. Furthermore, there is no *a priori* guarantee that the properties of particular objects will be preserved by change of universe. For example, suppose that we prove in ZFC+I, for some property φ, that there exists a small group G such that $\varphi(G, H)$ is true for all small groups H. (The assertion that G is a limit or colimit of some specified diagram in **Grp** is of this sort.) This will then be true whether 'small' is interpreted relative to one inaccessible or another, but there is no *a priori* reason why the group G with this property need be the same in the two cases. Thus we have no way to conclude that there is a G satisfying $\varphi(G, H)$ for *all* groups H. In particular cases this is obvious; for example, we have explicit ways to compute (small) limits and colimits in **Grp** which do not depend on the size of the universe. But the absence of a general truth of this sort means that care is needed when engaging in such 'universe-juggling'.

11. Natural models and reflection principles

In an attempt to remedy these problems, let us investigate more specifically what properties of inaccessibility are really necessary for category theory. The most important consequence of inaccessibility of κ appears to be that V_κ is a model of ZFC which is itself a set in our assumed larger model V. Thus it is natural to look more generally at sets which are models of ZFC.

Now, *a priori* a model of ZFC consists only of a set M and a relation $E \subset M \times M$, to be interpreted as 'membership', such that the axioms of ZFC hold. However, we want the elements of a set in M to be the same as its elements in V, so it is natural to require that E coincides with the actual membership relation $\in$ in V, and that M is **transitive**, meaning that $x \in y \in M$ implies $x \in M$. In fact, any model is isomorphic to a transitive one, called its **(Mostowski) transitive collapse**, via an isomorphism defined inductively by $T(x) = \{T(y) \mid yEx\}$. Thus, nothing essential is lost by considering only transitive models.

However, pathologies still exist among transitive models. In particular, if there exists any model of ZFC, then there exists one which is transitive and *countable*— despite the fact that in ZFC one can prove the existence of uncountable sets! This is known as *Skolem's paradox*. The reason is that a set x which is countable in V may be uncountable to the eyes of M, since the bijection from x to ω may not be in M.

Skolem's paradox follows from a model-theoretic result called the *Löwenheim-Skolem theorem*. Like Gödel's incompleteness theorems and Tarski's undefinability theorem, the Löwenheim-Skolem theorem also has important philosophical implications for any axiomatic foundation of mathematics. For this reason, and because we will re-use the same ideas later, I will now sketch a proof of it.

If φ is any statement and M is any structure (that is, a potential model of some theory, like a group or a universe), let φ^M denote φ **relativized** to M, meaning that all its quantifiers are restricted to range only over elements of M. If $M \subset N$ we say that $\varphi(x_1, \ldots, x_n)$ is **reflected from N to M** if

$$\forall x_1 \in M \cdots \forall x_n \in M \left(\varphi^N(x_1, \ldots, x_n) \iff \varphi^M(x_1, \ldots, x_n) \right).$$

If *all* statements φ are reflected from N to M, we say that M is an **elementary substructure** of N and write $M \prec N$; clearly in this case M is a model of some theory if and only if N is.

The *downward Löwenheim-Skolem theorem* says that given any structure N and any infinite $S \subset N$, there exists an $M \prec N$ such that $S \subset M$ and $|M| = |S|$. To construct such an M, start with $M_0 = S$. Now, for each statement $\varphi(x_1, \ldots, x_n)$ of the form $\exists y : \psi(y, x_1, \ldots, x_n)$ and each $a_1, \ldots, a_n \in M_0$ such that $\varphi^N(a_1, \ldots, a_n)$ is true, choose some $b \in N$ such that $\psi^N(b, a_1, \ldots, a_n)$ is true. Let M_1 be M_0 together with all such 'witnesses' b. Iterate this process to define $M_2, M_3, \ldots$, and let $M = \bigcup_{n \in \omega} M_n$. Since there are only countably many statements φ, the cardinality never increases, so $|M| = |S|$. The only potential difficulty in showing that $M \prec N$ is with quantifiers; but we have dealt with existential quantifiers by construction, while universal ones can be rephrased as the nonexistence of a counterexample (this is called the *Tarski-Vaught test* for elementary submodels).

We now obtain Skolem's paradox by starting with a model N of ZFC, applying this theorem for any countably infinite $S \subset N$ (which exists by the axiom of infinity), then taking the transitive collapse of the resulting model M. Note that in fact we have proven more: if ZFC (or any theory) has a model N, then it has a transitive model with cardinality κ for all $\kappa \leq |N|$. The *upward Löwenheim-Skolem theorem* says that this is also true for $\kappa > |N|$.

To avoid Skolem's paradox, it suffices to require that our transitive model M of ZFC be closed under subsets: $x \subset y \in M$ implies $x \in M$. This ensures that if $A, B \in M$ and A and B are bijective in V, then the bijection is also in M, since it is a subset of $A \times B$. A transitive model of ZFC closed under subsets is called a **natural model**; see [MV59]. It is not difficult to show that any natural model is of the form V_α for some limit ordinal α.

We have seen that V_α is a natural model when α is inaccessible, but in fact inaccessibility is much stronger than necessary. Inaccessibility of α asserts, in particular, that V_α contains the image of any function $f : X \to V_\alpha$ such that $X \in V_\alpha$. But saying that V_α satisfies the replacement schema only asserts this when f is *definable* from V_α; that is, when $f \in \mathrm{Def}(V_\alpha)$. Note that in general any such f is in $V_{\alpha+1} = \mathcal{P}V_\alpha$, which contains $\mathrm{Def}(V_\alpha)$ as a proper subset. Another way of expressing this is to say that V_α models ZFC if and only if $(V_\alpha, \mathrm{Def}(V_\alpha))$ models NBG, while $(V_\alpha, V_{\alpha+1})$ can only model NBG when α is inaccessible (in which case it also models MK). To see how nontrivial this distinction is, observe that since there are only countably many statements φ, we have $|\mathrm{Def}(V_\alpha)| = |V_\alpha|$, while of course $|V_{\alpha+1}| = 2^{|V_\alpha|} > |V_\alpha|$.

Remark 11.1. One can also say that if α is inaccessible, then V_α satisfies the *second-order replacement axiom*. To understand this we need to describe second-order logic. The logic we have discussed so far is called *first-order logic* because variables and quantifiers only range over 'things'; in second-order logic they are also allowed to range over 'sets of things'. This distinction can be confusing, since for a first-order theory like ZFC the 'things' are themselves called 'sets'.

Second-order logic is more powerful than first-order logic, but suffers from an ambiguity of interpretation. If by 'set of things' we intend to mean *any* subset of the model under consideration, then we need an external set theory to define what is meant by this. This is the sense in which a Grothendieck universe satisfies the 'second-order replacement axiom'. On the other hand, if we allow the model itself

to stipulate what 'sets of things' exist, then second-order logic reduces to first-order logic augmented with an extra sort of 'thing' called a 'set of things'. This occurs, for example, with the replacement axiom of NBG. The existence of natural models that are not Grothendieck universes shows that this ambiguity has teeth. For this reason I will continue to stick to first-order logic.

Now, by slightly modifying the proof of the Löwenheim-Skolem theorem, we can prove that for any β and $S \in V_\beta$, then there is an $\alpha \le \beta$ such that $S \in V_\alpha$ and $V_\alpha \prec V_\beta$. Namely, instead of a sequence $M_0 \subset M_1 \subset M_2 \subset \ldots$, we construct a sequence $\alpha_0 < \alpha_1 < \alpha_2 < \ldots$, by letting α_{n+1} be some ordinal such that all the witnesses b for V_{α_n} are contained in $V_{\alpha_{n+1}}$. Setting $\alpha = \lim_{n<\omega} \alpha_n$, it follows that $V_\alpha \prec V_\beta$. In particular, if V_β is a natural model, so is V_α.

Of course, in this case there is no guarantee that $\alpha \ne \beta$. But if β is *inaccessible*, then at the nth stage we need to add only $|V_{\alpha_n}| < \beta$ witnesses, so we can choose $\alpha_{n+1} < \beta$; and since β is regular, we then have $\alpha < \beta$. A slight improvement of this proof shows that if β is inaccessible, then

$$\{\alpha < \beta \mid V_\alpha \text{ is a natural model}\}$$

is stationary in β, and in particular has cardinality β. Thus, the existence of natural models can be regarded as a sort of 'large cardinal axiom' significantly weaker than even a single inaccessible (although α need not actually be a cardinal for V_α to be a natural model).

It makes intuitive sense that we could also carry through the above argument using the whole universe V instead of V_β, and thereby construct a natural model V_α with $V_\alpha \prec V$. Of course, this would prove $\mathrm{Con}(\mathrm{ZFC})$ and violate the incompleteness theorem. The flaw in the argument is that with V in place of V_β, Tarski's undefinability theorem prevents us from defining the set of witnesses b. However, we can rescue the argument if instead of asserting $V_\alpha \prec V$, we only assert that a given *finite* set of statements is reflected from V to V_α. This gives a theorem-schema called the *reflection principle*: for any finite set $\varphi_1, \ldots, \varphi_n$ of statements, we can prove in ZFC that for any set y, there exists an α such that $y \in V_\alpha$ and $\varphi_1, \ldots, \varphi_n$ are all reflected in V_α.

Now, while the single statement "there exists a natural model" implies the consistency of ZFC, the reflection principle suggests a version of it that does not. Let $\mathbb{S}$ be an extra constant symbol, and add to ZFC the axiom "$\mathbb{S}$ is transitive and closed under subsets" along with a *reflection axiom*

$$\forall x_1 \in \mathbb{S} \cdots \forall x_n \in \mathbb{S} \left(\varphi(x_1, \ldots, x_n) \iff \varphi^{\mathbb{S}}(x_1, \ldots, x_n) \right).$$

for each statement φ. We denote the resulting system by by **ZFC/S** ("ZFC with smallness"). It follows that for each axiom φ of ZFC, the relativized version $\varphi^{\mathbb{S}}$ is true in ZFC/S, so $\mathbb{S}$ is a model of ZFC. We can also show $\mathbb{S} = V_{\Bbbk}$ for some ordinal $\Bbbk$. However, since the proof of each axiom $\varphi^{\mathbb{S}}$ uses a different instance of the reflection axiom schema, we cannot prove in ZFC/S the single statement "$\mathbb{S}$ is a natural model".

In fact, we can prove that ZFC/S is *conservative* over ZFC. Suppose we have any theorem which is provable in ZFC/S; we show that it can also be proven in ZFC. The original proof, being only finitely long, can only use the reflection schema for finitely many statements φ. Thus, by the reflection principle for ZFC, we can find an α such that all these statements φ are reflected in V_α. We can then replace $\mathbb{S}$

by V_α and carry out the proof in ZFC. Note, however, that unlike the proof for conservativity of NBG, we do not have a way to construct, from a model of ZFC, a model of ZFC/S with the same (small) sets.

The theory ZFC/S is due to Feferman, who proposed it in [Fef69] as a foundation for category theory. Of course, we now define **small** to mean "element of $\mathbb{S}$" and **large** to mean "set not necessarily in $\mathbb{S}$"; otherwise things go mostly as before. Since the axioms of ZFC are satisfied for all sets, we can manipulate large sets as we wish; so we retain that advantage of ZFC+I. If we want to talk about very large and extremely large objects, it is easy to add multiple symbols $\mathbb{S}_1, \mathbb{S}_2, \ldots$, each satisfying reflection and with $\mathbb{S}_1 \in \mathbb{S}_2 \in \cdots$.

However, because ZFC/S is conservative over ZFC, we have not strengthened our basic set theory. In particular, anything about small objects that we prove with the aid of large categories would still be provable in pure ZFC. Thus, we obtain a precise version of our intuition that the use of inaccessibles in category theory is merely for convenience: since many categorical proofs stated using inaccessibles can be formalized in ZFC/S, any *consequence* of such a theorem not referring explicitly to inaccessibles is also provable purely in ZFC.

ZFC/S also eliminates at least some sources of universe-juggling. For example, because every statement about sets is reflected in $\mathbb{S}$, anything we prove in ZFC/S about small objects is *also* true about *large* objects. In particular, anything we prove about small categories, even making use of the large category **Cat**, will also be true about large categories. Moreover, any property of small objects which refers only to small objects is retained when reinterpreted to refer to large objects. For example, the statement that G is a limit of a specified small diagram in the category **Grp** of small groups can be expressed as $\varphi^{\mathbb{S}}(G)$ for some statement φ. Thus, the reflection principle implies that $\varphi(G)$ is also true, and hence G satisfies the same universal property with respect to large groups.

Whether *all* universe-juggling can be eliminated in this way depends on our ontological position towards ZFC/S. If we believe that ZFC/S is a true representation of reality—that is, that there actually *exists* an $\mathbb{S}$ satisfying the reflection axioms, and when working in ZFC/S we are making statements about that particular $\mathbb{S}$—then of course not all objects are small. The reflection principle gives us a precise sense in which they 'might as well be' small, but if we insist on being able to make them *actually* small, we would need to augment ZFC/S by a Grothendieck-like assumption of many natural models and continue to engage in universe-juggling.

However, if we instead take the position that ZFC is 'true', while ZFC/S is only a convenient fiction made possible by the reflection principle, then we can relegate all the universe-juggling to the 'behind the scenes' interpretation of ZFC/S in ZFC. That is, we prove all our results in ZFC/S, and argue that when we want to apply them to particular objects in the 'real world' of ZFC, we tacitly use the reflection principle to choose some V_α which contains all the objects we happen to be interested in. Thus the theorems of category theory take on the character of a 'meta-theory', which can be applied to any particular set of objects in the real world by choosing a sufficiently large V_α containing that set.

Remark 11.2. This seems an appropriate place to mention **Ackermann set theory**, a theory of sets and classes like NBG and MK which has also been proposed as

a foundation for category theory. Unlike NBG and MK, however, it allows classes to be elements of other classes.[15] Its axioms are the following.

(i) Extensionality, foundation, and choice.
(ii) Any element or subset of a set is a set.
(iii) For any definable property $\varphi(x)$, if all objects x satisfying $\varphi(x)$ are sets, then the class $\{x \mid \varphi(x)\}$ exists.
(iv) If in the previous axiom, in addition $\varphi(x)$ does not refer explicitly to sethood (that is, to whether or not any given class is a set), then the class $\{x \mid \varphi(x)\}$ is a set.

These axioms imply (though not obviously) that the class V of all sets is a model of ZFC. Conversely, any model of ZFC/S satisfies these axioms if 'class' means 'set' and 'set' means 'element of $\mathbb{S}$'; see [Lév59, Rei70]. Thus, like NBG and ZFC/S, Ackermann set theory is a conservative extension of ZFC.[16] But while it implies a limited reflection principle, overall it is strictly weaker than ZFC/S in what it can say about its classes. Most of the subsequent remarks about ZFC/S apply just as well to Ackermann set theory.

On the other hand, ZFC/S (and likewise Ackermann set theory) is not quite the paradise it first appears. In order to enable ourselves to manipulate large objects freely without strengthening set theory, we have been forced to weaken the replacement axiom for small sets. In both NBG and ZFC+I, we have a replacement axiom saying, essentially, that the image of a small set under a large function is small. In ZFC/S, however, we can only assert this if the large function is *small-definable* (that is, in Def($\mathbb{S}$)). This distinction is invisible to ZFC and NBG because there, all functions are (or might as well be) small-definable, while it is invisible to MK and ZFC+I because their stronger axioms guarantee that all functions with small domain are small, even those that are not *a priori* small-definable.

Perhaps surprisingly, it turns out that this weakening of replacement has significantly annoying consequences for category theory. For example, it implies that the category $\mathbf{Set} = \mathbf{Set}[\mathbb{S}]$ of small sets need not admit limits and colimits for all functors $F \colon \mathbf{A} \to \mathbf{Set}$ when $\mathbf{A}$ is small, but only those for which F *is also small*. The same is true for other large categories constructed from $\mathbf{Set}$. Similarly, for a functor $u \colon \mathbf{A} \to \mathbf{B}$ between small categories, the functor

$$[\mathbf{B}, \mathbf{Set}] \xrightarrow{-\circ u} [\mathbf{A}, \mathbf{Set}]$$

will not in general have left and right adjoints (Kan extensions).

On the surface, these restrictions appear quite problematic; the completeness and cocompleteness of $\mathbf{Set}$ is certainly of central importance in category theory. One way to respond is to assert that in ZFC/S, the correct definition of *complete* is

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[15]This is not obvious from the axioms, which only assert directly the existence of classes whose elements are sets. To see that there must be classes containing other classes, we observe first that axiom (iv) implies that sethood cannot be characterized without referring to it explicitly; otherwise the class V of all sets would be a set. This means that since the property "$\exists y : x \in y$" is true of all sets, it must also be true of some classes; otherwise it would characterize sethood.

[16]The intuition behind Ackermann set theory, however, is different from that of ZFC. Ackermann argued that that the elements of a *set* must be 'sharply delimited', while the elements of a *class*, such as V, may depend on how broadly we interpret the concept of 'set'. Thus, only properties which do not refer explicitly to sethood are 'sharply delimited' enough to define sets. It is striking that nevertheless, Ackermann's axioms turned out *a posteriori* to be equivalent to ZFC in what they can say about sets.

"having limits for all small functors" rather than "having limits for all functors with small domain". Similarly, if we write $[\![\mathbf{A}, \mathbf{Set}]\!]$ for the full subcategory of $[\mathbf{A}, \mathbf{Set}]$ determined by the *small* functors, then induced functor

$$[\![\mathbf{B}, \mathbf{Set}]\!] \xrightarrow{-\circ u} [\![\mathbf{A}, \mathbf{Set}]\!]$$

does have both adjoints, and we may assert that in ZFC/S, the correct presheaf category to consider is $[\![\mathbf{A}, \mathbf{Set}]\!]$ rather than $[\mathbf{A}, \mathbf{Set}]$.

Another advantage of $[\![\mathbf{A}, \mathbf{Set}]\!]$ is that unlike $[\mathbf{A}, \mathbf{Set}]$, it is small-definable. This is a good thing, because in ZFC/S, only small-definable categories are at all well-behaved. For instance, let A be a small set of objects in a locally small and small-definable category $\mathbf{B}$. We can then prove that

    (a) the full subcategory $\mathbf{A}$ of $\mathbf{B}$ determined by the objects in A is small;

    (b) the inclusion functor $i\colon \mathbf{A} \to \mathbf{B}$ is small;

    (c) for any object $X \in \mathbf{B}$, the restricted hom-functor $\mathbf{A}(i-, X)\colon \mathbf{A}^{op} \to \mathbf{Set}$ is small; and thus

    (d) the restricted Yoneda embedding $\mathbf{B} \to [\mathbf{A}^{op}, \mathbf{Set}]$ factors through $[\![\mathbf{A}^{op}, \mathbf{Set}]\!]$.

However, there seems to be no way to prove any of these statements if $\mathbf{B}$ is not small-definable. Similarly, theorems like the Adjoint Functor Theorem only seem to work for small-definable categories.

As observed by Feferman, all this makes little difference in most concrete applications, because any particular diagram, functor, or category we are interested in will generally be small-definable (at least, up to equivalence). However, this is not always trivial to verify; we saw in §8 examples of categories that were equivalent to small-definable ones, but not obviously. Moreover, small-definability restrictions are tiresome to keep track of, and some would say unaesthetic as well. In the next two sections, we will explore two ways to deal with this problem.

12. Strong reflection principles

There is an obvious way to 'have our cake and eat it too': we can add to ZFC/S the extra axiom "$\mathbb{S}$ is a Grothendieck universe", or equivalently $\mathbb{S} = V_{\mathbb{k}}$ where $\mathbb{k}$ is inaccessible. For reasons to be explained below, I will call the resulting theory **ZMC/S**. It then follows, as in ZFC+I, that every functor with small domain is small, and all small-definability restrictions vanish.

Of course, this may seem like a step backwards, since we began the previous section by looking for a way to *avoid* inaccessibles. However, along the way we discovered the reflection principle, and we saw that the reflection axiom-schema of ZFC/S is really what resolves many of the problems with ZFC+I and allows us to avoid universe-juggling. Since ZMC/S retains reflection, all of this is still true, so the only disadvantage of ZFC+I which carries over to ZMC/S is that it is not conservative over ZFC.

In fact, ZMC/S is significantly stronger than ZFC+I, since reflection implies that $\mathbb{k}$ is far from the smallest inaccessible. Namely, since there exists an inaccessible, there must exist a small inaccessible; but then there exist two inaccessibles, and so there must exist two small inaccessibles, and so on. By applying reflection to the statement "there exists an inaccessible larger than α", in ZMC/S we can even derive Grothendieck's axiom that there are arbitrarily large inaccessibles.

The same argument that shows ZFC/S to be conservative over ZFC shows that ZMC/S is conservative over ZFC + "any finite set of formulas is reflected in some

Grothendieck universe." This stronger reflection principle turns out to be equivalent to the assertion that any closed unbounded (definable) class of ordinals contains an inaccessible, which essentially says that the cardinal of the universe is Mahlo (see [Lév60]). ZFC augmented by this axiom is sometimes called ZMC (whence the notation ZMC/S). While rather stronger than the existence of a single inaccessible, and stronger even than Grothendieck's axiom, this is still quite weak compared to many large-cardinal axioms, as we saw in §9.

Moreover, the principle incorporated in ZMC is one of the most easily motivated large-cardinal axioms. For instance, it can be argued that it is a straightforward expression of the 'inexhaustibility' of the universe of sets by any finite number of operations. Additionally, it is not difficult to show (see [Lév60]) that in the presence of the basic axioms only, the reflection principle of ZFC implies replacement, power set, and infinity—all the axioms of ZFC which produce larger and larger sets. It follows that ZMC is equivalent to just the basic axioms and choice together with the schema "any finite set of formulas is reflected in some Grothendieck universe." I find this aesthetically quite appealing, because it captures exactly what category theory seems to need from set theory: we may not be able to have a category of *all* sets, but for any particular purpose, we can choose a category of sets large enough that it might as well contain all of them.

13. Toposes and indexed categories

There is another way to deal with the small-definability issues in ZFC/S: we can use *indexed categories*, a tool developed to solve a similar problem in elementary topos theory. Since topos theory is of interest in its own right as a foundation for mathematics in general, and category theory in particular, we start with a summary of it. Good introductions to topos theory can be found in [MLM94, McL92a], while [Joh02] is encyclopedic.

By way of motivation, observe that while ZFC suffices as a foundation for most of mathematics, there is a sense in which it is disconnected from most mathematical practice. In ZFC there is a *global* membership predicate, meaning that if I give you two random sets, it makes sense to ask whether one is an element of the other. However, in actual mathematical practice we usually only speak of *local* set membership, meaning that asking whether $x \in A$ is only meaningful in the context of some fixed set B such that x is known to be an element of B and A is known to be a subset of B. In other words, the way most mathematicians usually think of a set (or a group, or a topological space, etc.) is as a collection of 'abstract' elements which have no 'internal' structure aside from being elements of that set. The only way that elements of two different sets relate to each other is via functions and relations between those sets.

Of course, this is a very categorical way of thinking; it is closely related to the assertion that we only care about objects of a category, such as **Set**, up to isomorphism. Thus, it is natural to try to axiomatize the properties of the category **Set**, instead of axiomatizing a global membership relation. (While useful, this motivation for topos theory is ahistorical; see [McL90].) The appropriate axioms can be classified just as we did for ZFC and NBG.

   (i) The *basic* axioms: **Set** is cartesian closed, has finite limits and colimits, and the terminal object is a generator.

(ii) The *size-increasing* axiom: every object has a 'power object' classifying its subobjects.

(iii) The *size-assertion* axiom: there is a 'natural numbers object'.

(iv) The axiom of *choice*: all epimorphisms split.

An **elementary topos**[17] is a category with finite limits and power objects; this implies cartesian closedness and the existence of finite colimits. A **natural numbers object (NNO)** in a topos is an object satisfying the universal property of definition by recursion, or equivalently proof by induction. A topos is **well-pointed** if the terminal object 1 is a generator; this means that an object X is determined by its 'elements' $x \colon 1 \to X$. Thus, a natural axiomatization of **Set** is that it is a well-pointed elementary topos with a NNO and satisfying the axiom of choice (a **WPTNC**). These axioms for **Set** are also referred to as the **Elementary Theory of the Category of Sets (ETCS)**, after Lawvere's influential paper [Law64, Law05]. Just as with ZFC, it is an empirical observation that much of mathematics can be developed starting from a model of ETCS.

Remark 13.1. In fact, much mathematics can be developed from any elementary topos (perhaps with a NNO), as long as one uses *intuitionistic* or *constructive logic* instead of classical logic. I will return to this point in Remark 15.1.

Notably absent from ETCS is any analogue of the axiom of replacement. This means that if α is any limit ordinal greater than ω, so that V_α satisfies all the axioms of ZFC except replacement, then the category $\mathbf{Set}[V_\alpha]$ of sets and functions in V_α is a WPTNC. As remarked in §4, most mathematical objects outside of set theory have very low rank, living quite comfortably in $V_{\omega \cdot 2}$, so this lends extra credence to the observation that ETCS suffices for much of mathematics.

In fact, all one needs to construct a WPTNC is a model of **Bounded Zermelo set theory with Choice (BZC)**, which is ZFC without replacement and in which the properties in the separation axiom are only allowed to have bounded quantifiers ("for all $x \in A$" rather than just "for all x"). This version of separation is variously called **restricted**, **bounded**, or Δ_0**-separation**. Conversely, from a WPTNC one can construct a model of BZC, although some cleverness is needed to obtain a global membership predicate; see [MLM94, VI.10] or [Joh77, Ch. 9] for two approaches. Thus, ETCS and BZC are *equiconsistent*.

I will discuss the implications of ETCS and BZC for mathematical practice in more detail in §14; for now, let us consider their consequences for category theory. We have seen a hint already of what can go wrong without replacement in our study of ZFC/S, where weakening the replacement axiom created unexpected problems. To see the more drastic problem we are now faced with, consider the meaning of the statement "**A** has small products" when **A** is a large category. Intuitively, this means that any X-indexed family of objects of **A** has a product, for any 'set' X. But what exactly is an "X-indexed family of objects of **A**"?

In NBG this can mean a class which is a function from X to the class of objects of **A**, while in ZFC it can mean a definable property φ such that for any $x \in X$ there is a unique $a \in \mathbf{A}$ with $\varphi(x, a)$. In either case, we can then apply the replacement

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[17]There is some controversy about whether the English plural of *topos* should be *toposes* or *topoi*. The paper by Grothendieck and Verdier which coined the term is in French, where the plural is again *topos*; this seems to tell against the Greek plural. However, in English *topos* also means "a literary theme or motif", and in this case the plural used is always *topoi*. Unfortunately one cannot avoid making one choice or the other!

axiom to obtain a *set* such as $\{a \in \mathbf{A} \mid \exists x \in X \; f(x) = a\}$, and go on to construct the product. However, if X is instead an object of an elementary topos $\mathbf{S}$, it is not at all clear what is even *meant* by an X-indexed family of objects of $\mathbf{A}$.

On the other hand, any large category $\mathbf{A}$ which is *constructed from* $\mathbf{S}$ in any reasonable way will come with a canonical notion of X-indexed family of objects for any $X \in \mathbf{S}$. To start with, $\mathbf{S}$ itself has such a notion: an X-indexed family of objects of $\mathbf{S}$ can be defined to be simply an arrow $p \colon K \to X$. The intuition is that an element $x \in X$ indexes the fiber $p^{-1}(x)$, but of course objects of the abstract category $\mathbf{S}$ have no 'elements' as such. It follows that the *category of X-indexed families of objects of $\mathbf{S}$* should be the slice category $\mathbf{S}/X$. We can then extend this to other categories of 'sets with structure'. For example, if $\mathbf{A}$ is the category of 'small groups', meaning internal group objects in $\mathbf{S}$, then an X-indexed family of objects of $\mathbf{A}$ should be an internal group object in $\mathbf{S}/X$.

Thus, when doing category theory relative to an elementary topos $\mathbf{S}$, it is natural to consider, instead of a single category $\mathbf{A}$, a family of categories $\mathbf{A}^X$ for each object $X \in \mathbf{S}$, where $\mathbf{A}^X$ is thought of as the category of 'X-indexed families of objects of $\mathbf{A}$'. Of course, these categories should be related in some way as X varies, and it turns out that the important property is the ability to *reindex* a family: if K is an X-indexed family and $f \colon Y \to X$, then we have a Y-indexed family f^*K. The intuition is that for $y \in Y$ we have $(f^*K)_y = K_{f(y)}$. For $\mathbf{S}$ and categories constructed from it, this reindexing is given by pullback along f.

With this as motivation, we define an **S-indexed category $\mathbf{A}$** to consist of a category $\mathbf{A}^X$ for each object $X \in \mathbf{S}$, together with a functor $f^* \colon \mathbf{A}^X \to \mathbf{A}^Y$ for each arrow $f \colon Y \to X$ in $\mathbf{S}$, and natural isomorphisms $(gf)^* \cong f^*g^*$ and $\mathrm{Id}_{\mathbf{A}^X} \cong (1_X)^*$ satisfying obvious axioms.[18] An $\mathbf{S}$-indexed category can also be described by assembling the categories $\mathbf{A}^X$ into a single category $\oint \mathbf{A}$ equipped with a functor $\oint \mathbf{A} \to \mathbf{S}$ assigning a family to its indexing object; in this form it is called a (categorical) **fibration** over $\mathbf{S}$. Good introductions to indexed categories and fibrations can be found in [Joh02, Part B] and [Str].

If we now replace our naïve large category $\mathbf{A}$ by an $\mathbf{S}$-indexed category, all problems disappear. For example, we can define $\mathbf{A}$ to have $\mathbf{S}$-*indexed products* if each reindexing functor $f^* \colon \mathbf{A}^X \to \mathbf{A}^Y$ has a right adjoint (plus a commutativity condition). For a more general notion of completeness, we need a notion of 'small category', and the obvious candidate is an *internal category* in $\mathbf{S}$, which consists of objects $C_0, C_1 \in \mathbf{S}$ with arrows $s, t \colon C_1 \rightrightarrows C_0$, $i \colon C_0 \to C_1$, and $c \colon C_1 \times_{C_0} C_1 \to C_1$ satisfying obvious axioms. Any internal category C gives rise to an $\mathbf{S}$-indexed category $\mathbf{C}$ with $\mathbf{C}^X = \mathbf{S}(X, C_\bullet)$, and we can define a C-*diagram* in any $\mathbf{S}$-indexed category $\mathbf{A}$ to be an object $F \in \mathbf{A}^{C_0}$ together with a morphism $s^*F \to t^*F$ in $\mathbf{A}^{C_1}$ satisfying suitable axioms. The appropriate notions of *limit* and *completeness* are then fairly straightforward.[19] Similarly, we can define local smallness, generators, and well-poweredness, and state and prove an Indexed Adjoint Functor Theorem.

We saw that $\mathbf{S}$ itself is represented by the $\mathbf{S}$-indexed category with $\mathbf{S}^X = \mathbf{S}/X$; we call this the **self-indexing** of $\mathbf{S}$. It turns out that the self-indexing is always

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[18]The 2-categorically sophisticated reader may call this a pseudofunctor $\mathbf{S}^{op} \to \mathcal{CAT}$. This is fine as long as we have some external set theory with which to define a 2-category $\mathcal{CAT}$ of large enough categories.

[19]This is not quite true; I am omitting some details in an attempt to give the flavor of the subject without getting bogged down. See the references above for a careful treatment.

complete and cocomplete, in the indexed sense, for *any* elementary topos $\mathbf{S}$. In particular, this applies to $\mathbf{Set}[V_\alpha]$ for any limit ordinal α, even though such toposes are not complete and cocomplete in the naïve 'external' sense. For example, $\mathbf{Set}[V_{\omega\cdot2}]$ fails to have the coproduct $\coprod_{n<\omega}\beth_{\omega+n}$, even though it contains ω and each $\beth_{\omega+n}$. But this does not violate indexed cocompleteness, since $\{\beth_{\omega+n}\}_{n\in\omega}$ is not an ω-indexed family in the self-indexing of $\mathbf{Set}[V_{\omega\cdot2}]$: if there were such a family $K\to\omega$, then K would have to essentially already *be* the desired coproduct.

The situation may be clarified by observing that there is a *different* indexed category over any $\mathbf{S}=\mathbf{Set}[V_\alpha]$ called the **naïve indexing**, for which $\mathbf{S}^X_{\text{naïve}}$ is the category of all functions $X\to V_\alpha$. The self-indexing of $\mathbf{Set}[V_\alpha]$ embeds in its naïve indexing, and the two are equivalent precisely when α is inaccessible. In between the two we have the **definable indexing**, for which $\mathbf{S}^X_{\text{def}}$ is the category of all *definable* functions $X\to V_\alpha$. This agrees with the self-indexing whenever V_α is a natural model. Neither the naïve indexing nor the definable indexing of $\mathbf{Set}[V_{\omega\cdot2}]$ is complete or cocomplete, nor is the naïve indexing of $\mathbf{Set}[V_\alpha]$ when V_α is a natural model that is not a Grothendieck universe; but the self-indexing of either always is.

In fact, if V is *any* model of BZC, then the resulting well-pointed topos $\mathbf{Set}[V]$ has both a self-indexing and a definable indexing, and the assertion that V satisfies replacement (hence is a model of ZFC) is equivalent to the assertion that these two indexings agree. To define a 'naïve indexing', our universe V must live inside a larger universe of sets or classes, such as a model of NBG, MK, or ZFC+I. In each of these cases, the version of replacement asserted implies that the resulting naïve indexing is actually equivalent to the self-indexing as well.

We can now see that when working with large categories in ZFC, we have implicitly been using the definable indexing, while in NBG, MK, and ZFC+I we have been using the naïve indexing. Topos theory suggests that actually, the *self-indexing* is always the 'correct' indexing to use, and the role of the replacement axiom is to ensure that the self-indexing is equivalent to the definable or naïve indexings, which are more intuitive and easier to work with. See [Str, §17] for further discussion of 'internal' versus 'external' completeness.

Now let us return to ZFC/S. Since $\mathbb{S}$ is a natural model[20] but not a Grothendieck universe, the self-indexing of $\mathbf{Set}[\mathbb{S}]$ agrees with its definable indexing, but not its naïve indexing. Moreover, the objects of $\mathbf{S}^X_{\text{self}}=\mathbf{S}^X_{\text{def}}=\mathbf{S}/X$ are essentially the same as the *small* functions $X\to\mathbb{S}$; thus indexed completeness of $\mathbf{Set}[\mathbb{S}]$ agrees with our proposed *ad hoc* redefinition of completeness in §11. More generally, any small-definable category in ZFC/S gives rise to a $\mathbf{Set}[\mathbb{S}]$-indexed category containing only the *small* X-indexed families of objects, and the machinery of indexed categories automatically keeps track of all the restrictions we had to impose by hand in §11. For example, if C is an internal category C in $\mathbf{Set}[\mathbb{S}]$, the category of C-diagrams in $\mathbf{S}_{\text{self}}$ is the well-behaved $[\![C,\mathbf{Set}]\!]$ rather than the poorly-behaved $[C,\mathbf{Set}]$.

Thus, one may say that the problems we encountered with ZFC/S arose due to our trying to use the naïve indexing in a situation where our replacement axiom was only sufficient to deal with the definable indexing. Moreover, Feferman's hypothesis that ZFC/S suffices for basic category theory now follows from the observation that most theorems of basic category theory have indexed analogues.

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[20]However, remember that "$\mathbb{S}$ is a natural model" is not a single theorem of ZFC/S, but a schema consisting of one theorem for each axiom of ZFC.

On the other hand, once we are willing to use indexed categories, we do not necessarily need to to assume a replacement axiom at all in order to do category theory. Thus, we could choose *any* WPTNC **S** which is a set, even $\mathbf{Set}[V_{\omega \cdot 2}]$, define a *small category* to mean an internal category in **S** and a *large category* to mean an **S**-indexed category, and develop category theory that way. This way we can remain completely within ZFC (or even something much weaker), and Grothendieck's axiom of arbitrarily large inaccessibles can be replaced by the simple fact that any set is contained in some V_α.

However, the machinery of indexed categories is admittedly rather complicated, and it seems unreasonable to expect most users of category theory to be familiar with it when there are so many simpler foundational options available. If nothing else, though, indexed categories give a more conceptual understanding of the small-definability restrictions arising in ZFC/S. (Of course, indexed categories are also crucial when working with a general elementary topos, rather than a WPTNC.)

14. ASIDE: THE STRENGTH OF CATEGORICAL SET THEORY

Let us pause here briefly to compare the 'category-theoretic foundation' for mathematics offered by ETCS and its relatives with the 'set-theoretic foundation' offered by ZFC and its cousins. This terminology is common, but one can also argue persuasively (see [Law05]) that ETCS is itself a *set theory*, meaning a theory about the behavior of sets. What distinguishes it from ZFC is not its objects of study, but how it studies them: by taking functions as a basic notion rather than global membership. Perhaps a more correct distinction would be to call ETCS a *categorical*[21] *set theory* and ZFC a *membership set theory*.

I mentioned in §13 that ETCS is equiconsistent with BZC. In fact, if we add axioms to ETCS and BZC saying that every set is contained in a transitive one and that transitive collapses exist (§11), then we can obtain a full equivalence between models of the two theories; see [Joh77, Ch. 9] or [Osi74]. These additional axioms can be proven in ZFC using replacement, but are much weaker than it; in particular, adding them does not change the consistency strength of ETCS and BZC. This altered version of BZC is sometimes called **Mac Lane set theory (MAC)**. Thus, at least in one sense, ETCS and BZC/MAC are completely equivalent.

However, as I argued in §13, ETCS may seem closer to most mathematical practice than BZC, since it discards the usually superfluous notion of global membership. Furthermore, in line with the observed fact that most mathematicians only care about objects up to isomorphism, ETCS can only *characterize* any set up to isomorphism; see [McL93]. Even a lot of notions in set theory, including many large-cardinal axioms, are invariant under isomorphism.

On the other hand, ETCS and BZC are both significantly weaker than ZFC: not only are they missing the replacement axiom, but they only allow separation for formulas with bounded quantification. This implies that just as NBG can only prove mathematical induction for statements not quantifying over classes, BZC can only prove induction for statements without unbounded quantifiers. For example, if **A** is a large category in the style of §6, then a statement such as "for all n, **A** has n-fold

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[21]There is an unfortunate collision between the common and natural use of 'categorical' to mean 'related to categories', and the much older philosophical and logical tradition in which 'categorical' means 'absolute' or 'uniquely determined'. This has led some authors to use *categorial* for the former notion.

products" involves unbounded quantifiers and thus cannot be proven by induction in BZC or ETCS, at least not obviously.

The axiom of replacement also has other uses outside of set theory, usually taking the form of transfinite induction arguments. The classic example is *Borel determinacy* in descriptive set theory, which is known to be unprovable even in Zermelo set theory (that is, ZFC without replacement but with full separation). Closer to home for us are transfinite constructions such as Example 4.4, which also require some form of replacement. For instance, the power-set functor cannot be iterated even ω times without replacement, since $|\mathcal{P}^\omega \omega| = \beth_\omega \notin V_{\omega \cdot 2}$. The same is true of the dual-vector-space functor. For a very detailed study of the strength of BZC and its cousins, see [Mat01].

For these reasons, it is natural to wonder whether ETCS can be strengthened with versions of full separation and replacement to obtain a categorical set theory equivalent in strength to ZFC. In fact, it suffices to consider replacement, since at least with classical logic, replacement implies full separation. There has not been a lot of work in this area, but several sorts of categorical replacement-type axioms have been proposed.

The perspective of §13 suggests that perhaps a categorical replacement axiom should essentially say "the definable indexing of **Set** agrees with the self-indexing". In constructing the definable indexing of **Set**[V] we used the fact that our sets have elements, but in a well-pointed topos we can replace these elements by morphisms $1 \to X$. Thus, we say a well-pointed topos **S** *satisfies replacement* if for any object X and any definable property $\varphi(x, S)$ such that for any 'element' $x\colon 1 \to X$ there exists a object S_x unique up to isomorphism with $\varphi(x, S_x)$, there exists a morphism $S \to X$ such that for any x there is a pullback square

$$
\begin{array}{ccc}
S_x & \longrightarrow & S \\
\downarrow & & \downarrow \\
1 & \xrightarrow{\ x\ } & X.
\end{array}
$$

This version of replacement is from [McL04]; other similar ones can be found in [Osi74, Law05]. These references also prove that ETCS plus replacement is equivalent to ZFC, in the strong sense of an equivalence of models. Thus, if all we want is a categorical set theory equivalent to ZFC, we have it.

On the other hand, these axioms of replacement are not fully satisfactory from a categorical point of view, because they all depend heavily on well-pointedness. As mentioned previously, all the other axioms of ETCS make perfect sense without well-pointedness, and much mathematics can be developed from any elementary topos; thus it would be nice to have a version of the replacement axiom that makes sense without well-pointedness.

In [Tay99, Ch. 9] Paul Taylor proposed an axiom he called the *categorical axiom of iterative replacement*, which asserts directly the possibility of transfinite constructions on functors. This axiom makes sense without well-pointedness, but seemingly has not been investigated in very much detail. I do not know of any more explicitly replacement-like axioms that make sense for non-well-pointed toposes.

15. Algebraic set theory

The categorical set theories we have considered so far are analogues of ZFC; in this section we consider categorical analogues of NBG and ZFC+I. One motivation for this is to find a more categorical way to state the replacement axiom; another is to find an easier way to deal with large categories.

We saw in §13 that one appropriate notion of a large category with respect to an elementary topos is an indexed category. As for naïve large categories in ZFC, the elementary theory of indexed categories suffices when we only need to consider them one at a time. However, if we want to quantify over indexed categories (such as in proving the Adjoint Functor Theorem), perform constructions on them, or assemble them into a larger category (or a 2-category), we need to assume some sort of external set theory. This is automatic if our topos $\mathbf{S}$ is the category of small sets in ZFC/S or ZFC+I—though in the latter case there is little need for indexed categories.

A more categorical approach is to introduce a *category of classes* $\mathbf{C}$ which contains the topos $\mathbf{S}$ of sets and also other large categories. This is a recent area of active research known as *algebraic set theory*; good introductions are [Awo, JM95]. The usual approach is to equip $\mathbf{C}$ with a collection $\mathcal{S}$ of *small morphisms*, the intuition being that a morphism is small when it has small fibers. Different authors choose slightly different axioms on $\mathbf{C}$ and $\mathcal{S}$, but in general they can be classified as follows.

(i) $\mathbf{C}$ has finite limits and colimits, and some additional level of structure allowing at least the interpretation of finitary logic.
(ii) Small maps are closed under composition, pullback, descent, and other basic constructions.
(iii) Every object X of $\mathbf{C}$ has a *powerclass* $\mathcal{P}_s X$ classifying its small subobjects.
(iv) There is a *universal* object U of $\mathbf{C}$ which 'contains all small objects'.

I will refer to some suitable version of these axioms collectively as **Algebraic Set Theory (AST)**. As with the elementary topos axioms, the axioms of AST can be augmented by well-pointedness, choice, and existence of a NNO.

We define an object $X \in \mathbf{C}$ to be *small* if $X \to 1$ is a small map. Any model of NBG gives rise to a model of AST in which the small objects are the sets. Conversely, the axioms of AST imply that the category $\mathbf{S}$ of small objects is an elementary topos, which inherits well-pointedness, choice, and a NNO from $\mathbf{C}$. Moreover, as long as $\mathbf{C}$ is well-pointed, the WPTNC $\mathbf{S}$ also satisfies replacement, in the sense described in §14. We may not quite get a model of NBG by taking the objects of $\mathbf{C}$ to be the classes, since some of them may be too large (violating limitation of size), but in practice, working in AST with well-pointedness, choice, and a (small) NNO is essentially equivalent to working in NBG.

Note that just as in a topos, the internal logic of $\mathbf{C}$ is restricted to bounded quantifiers. However, now we can interpret 'unbounded' quantifiers ranging over all *small* objects by using *bounded* quantifiers ranging over the universal object U. This gives a categorical explanation of why comprehension in NBG is restricted to formulas that only quantify over sets.

If we assume in addition that $\mathbf{C}$ is itself a topos, as in [Str05], then we obtain a theory equivalent to what one might call BZC+I. Adding a replacement schema for $\mathbf{C}$, as described in §14, brings us up to ZFC+I. In all cases, we can define a *large*

category to be an internal category in $\mathbf{C}$ and a *small category* to be one in $\mathbf{S}$, and the development of category theory then mirrors what happens in the corresponding membership set theory. (We could also use $\mathbf{C}$ to perform constructions on $\mathbf{S}$-indexed categories, but since $\mathbf{S}$ satisfies replacement with respect to $\mathbf{C}$ there seems little need for the extra complication.)

There are many other beautiful aspects to algebraic set theory, of which I will only mention one: the cumulative hierarchy $V = \bigcup V_\alpha$ and the class Ω of ordinals can be defined by universal properties. Define a **ZF-algebra** to be a partially ordered class A which has suprema of all *small* families and is equipped with a 'successor' operation $s\colon A \to A$. The class V of all sets, ordered by inclusion and with $s(x) = \{x\}$, can then be proven to be the initial ZF-algebra. The class Ω of ordinals is also a ZF-algebra with $s(\alpha) = \alpha+1$, so we have a unique homomorphism of ZF-algebras $\rho\colon V \to \Omega$; this turns out to be essentially the *rank* function. The function $\Omega \to V$ sending α to V_α can be characterized by an analogous universal property, or as a right adjoint to ρ; see [JM95] for details.

Remark 15.1. So far I have focused exclusively on well-pointed categorical set theories with choice, because they are the most relevant to a mathematician looking for a categorical substitute for ZFC. As noted in Remark 13.1, however, much of the independent interest of topos theory comes from the fact that *any* elementary topos has an internal set-like logic, and in general this internal logic is not classical logic but *constructive logic*. Well-pointedness and choice are each quite special properties of a topos, and both independently imply that its logic is classical.

Much of mathematics can be developed using constructive logic, although many classical definitions and results must be rephrased carefully to obtain a constructively meaningful or useful form. For example, *Tychonoff's theorem* that the product of compact spaces is compact is *true* constructively, without the axiom of choice, but the definition of *space* has to be modified; see [Joh02, Part C]. Classical concepts also often bifurcate into two or more inequivalent constructive ones. For example, there are at least three different kinds of constructive ordinals with slightly different properties; see [JM95, Tay96].

Most relevantly for us, the axiom of replacement loses much of its power constructively: it no longer implies unbounded separation, Borel determinacy, or the usual sort of transfinite induction. Moreover, the categorical replacement axiom from §13 is no longer sensible in the non-well-pointed case, and it is an open question whether it has some more general analogue.

Algebraic set theory, like elementary topos theory, makes perfect sense (and is usually studied) without well-pointedness, but its version of replacement is also much weaker constructively. In fact, *any* elementary topos $\mathbf{S}$ can be embedded as the topos of small objects in a category $\mathbf{C}$ of classes, but the logic of $\mathbf{C}$ will not in general be classical even if that of $\mathbf{S}$ is; thus every topos 'satisfies replacement' in a constructive sense. This is analogous to the use of indexed categories to 'define away' the lack of replacement by considering only small families to begin with. The interested reader can learn more about constructive logic in toposes and AST from the references cited above.

16. Higher categories

One can envision more radically 'categorical' foundations. For instance, it is hard to deny that in everyday mathematics we very rarely care about large sets *as*

sets—we only care about them insofar as they form the class of objects of some large *category*. In particular, we only care about their elements up to isomorphism. Thus, in a sense, the 'large' collections which arise naturally in mathematics behave fundamentally differently than the 'small' ones. This distinction is captured elegantly by the theory of indexed categories, which gives an analogue of large categories without making use of a prior notion of large set.

Analogously, we generally only care about large categories up to equivalence, and thus we should regard them as objects of a 2-category $\mathcal{CAT}$, rather than of a category **CAT**. This seems to suggest that instead of axiomatizing the category of classes, a more categorical generalization of elementary toposes would be to axiomatize the 2-category of large categories. Notable steps have been made in this direction (see [Str74, Web07]), but I think it is fair to say that a fully satisfactory answer has not yet emerged. Since this approach also requires a good deal of familiarity with 2-categories, I will not attempt to explain it further here.

Of course, once we start to study 2-categories, we need to assemble them into 3-categories, and so on *ad infinitum*. Philosophical and mathematical remarks along these lines can be found in [Mak98], among other places.

17. Conclusion

We have explored many possible foundations for category theory, including:

(1) A naïve approach which remains within ZFC.
(2) Introducing classes as objects, as in NBG and MK.
(3) Using an inaccessible to distinguish small sets from large ones (ZFC+I).
(4) Using a reflection principle, perhaps combined with inaccessibles, as in ZFC/S and ZMC/S.
(5) Categorical versions of the above, using toposes (ETCS) or categories of classes (AST).

Each has advantages and disadvantages, and I do not mean to put one forward as *the* correct foundation for category theory; I leave that choice to the reader's aesthetic and mathematical judgment.

Instead, let me end by reiterating that for the *basic* theorems of category theory, the choice of foundation is essentially irrelevant. Each of the above proposals deals with the distinction between small and large in a way which is fully satisfactory for proving results such as the Adjoint Functor Theorem (except that in some cases we have to state it as a meta-theorem, or add small-definability restrictions). However, as we have seen, the choice of foundation does matter for some more elaborate constructions. Thus, I believe it is important for students and users of category theory to have *some* familiarity with its possible foundations, and I hope to have partially addressed that need here.

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