CONTINUOUS QUIVERS OF TYPE A (IV) CONTINUOUS MUTATION AND GEOMETRIC MODELS OF E-CLUSTERS

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Abstract. We continue the work of parts (I), (II), and (III) of this series ([15], [23], and [16], respectively). In this final paper of the series we begin with a continuous generalization of mutation used to unify some existing types of mutation as well as allow for new types of mutation. We introduce the space of mutations which generalizes the exchange graph of a cluster structure. Then we complete the relationships of known type A cluster theories (introduced in part (III)). We define an abstract cluster structure and relate some known type A cluster structures which coincide with their previous definitions. We conclude by constructing geometric models of **E**-clusters (the new continuous clusters from part (III)) which generalize the existing geometric models of type A clusters. Along the way we pose open questions about classifications of the space of mutations and continuous type A cluster theories.

Contents

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Introduction

History. Cluster algebras were first introduced by Fomin and Zelevinksy in [9, 10, 11, 12]. In particle physics they can be used to study scattering diagrams (see work of Golden, Goncharov, Spradlin, Vergud, and Volovicha in [13]). The structure of cluster algebras was first categoricalized independently by two teams in 2006: Buan, Marsh, Reineke, Reiten, and Todorov in [6] and Caldero, Chapaton, and Schiffler in [8]. The first team's construction provided a way to construct a cluster category from the category of finitely generated representations of a Dynkin quiver and the second team's construction related the category to a geometric model. This geometric model was extended to the infinity-gon by Holm and Jørgensen and the completed infinity-gon by Baur and Graz in [14] and [4], respectively. A continuous construction, both categorically and geometrically, was introduced by Igusa and Todorov in [18]. Structures relating to clusters are still activity studied ([2, 24, 21, 22]). In particular continuous structures were studied by Arkani-Hamed, He, Salvatori, and Thomas in [1].

In Part (I) [15] of this series the author, with Igusa and Todorov, introduced continuous quivers of type A, denoted $A_{\mathbb{R}}$, which generalize quivers of type A. Results about decomposition of pointwise finite-dimensional representations of such a quiver and the category of finitely-generated representations (denoted $\mathrm{rep}_k(A_{\mathbb{R}})$) were proven. In Part (II) [23] the author generalized the Auslander-Reiten quiver for finitely-generated representations of an A_n quiver and its bounded derived category to the Auslander-Reiten space for $\mathrm{rep}_k(A_{\mathbb{R}})$ and its bounded derived category, denoted $\mathcal{D}^b(A_{\mathbb{R}})$. Results were proven about constructions of extensions in $\mathrm{rep}_k(A_{\mathbb{R}})$ and distinguished triangles in $\mathcal{D}^b(A_{\mathbb{R}})$ in relation to the Auslander-Reiten space. In Part (III) [16] the author, with Igusa and Todorov, classified which continuous quivers of type A are derived equivalent, constructed the new continuous cluster category (denoted $\mathcal{C}(A_{\mathbb{R}})$) with $\mathbf{E}$-clusters, and generalized the notion of cluster structures to cluster theories. It was shown that each element in an $\mathbf{E}$-cluster has none or one choice of mutation and the result of mutation yielded another $\mathbf{E}$-cluster. It was also shown that some type A cluster theories (recovered from existing cluster structures) can be embedded in this new cluster theory.

Contributions. In this final part of the series we begin with a continuous generalization of mutation (Definition 2.1.2) with two key motivations. The first is to unify various ways of describing a sequence (possibly infinite as in [4]) of mutations. In [16, Examples 3.2.2 and 4.3.1] the authors show that the indecomposable objects that were projective in $\mathrm{rep}_k(A_{\mathbb{R}})$ form an $\mathbf{E}$-cluster but many of the elements are not $\mathbf{E}$-mutable. The second motivation for continuous mutation is to work around this obstruction so that we may mutate the cluster of projectives into the cluster of injectives as one usually does for type A_n.

We describe mutation paths (Definition 2.3.2) and generalize the exchange graph of a cluster structure to the space of mutations for a cluster theory (Definition 2.4.2). In Definition 2.4.7 we define what it means for one cluster to be (strongly) reachable from another. We then show we have achieved the goal of working around the afore-mentioned obstruction.

Theorem A (Theorem 2.4.8). *Consider the* $\mathbf{E}$*-cluster theory of* $\mathcal{C}(A_{\mathbb{R}})$ *where* $A_{\mathbb{R}}$ *has the straight descending orientation. The cluster of injectives is strongly reachable from the cluster of projectives.*

We then provide a commutative diagram of embeddings of cluster theories to show how many existing type A cluster structures are related:

Theorem B (Theorem 3.2.13)**.** *For any $0 < m < n \in \mathbb{Z}$ there is a commutative diagram of embeddings of cluster theories*

$$\begin{tikzcd}
\mathscr{T}_{\mathbf{N}_m}(\mathcal{C}(A_m)) \arrow[r] \arrow[drr] & \mathscr{T}_{\mathbf{N}_n}(\mathcal{C}(A_n)) \arrow[r] \arrow[dr] & \mathscr{T}_{\mathbf{N}_\infty}(\mathcal{C}(A_\infty)) \arrow[r] \arrow[d] & \mathscr{T}_{\mathbf{N}_{\overline{\infty}}}(\mathcal{C}(A_{\overline{\infty}})) \arrow[r] \arrow[dl] & \mathscr{T}_{\mathbf{N}_\mathbb{R}}(\mathcal{C}_\pi) \arrow[dll] \\
 & & \mathscr{T}_{\mathbf{E}}(\mathcal{C}(A_\mathbb{R})) & &
\end{tikzcd}$$

The cluster theories cross the top are from [8], [14], [4], and [18]. The one on the bottom is from Part (III) of this series. We define an abstract cluster structure (Definition 3.3.1) and show there is a commutative diagram of cluster theories that almost completely restricts to a commutative diagram of cluster structures:

Theorem C (Theorem 3.3.16)**.** *The commutative diagram of cluster theories on the left restricts to the commutative diagram of cluster structures on the right (without the vertical arrow).*

$$\begin{tikzcd}
 & \mathscr{T}_{\mathbf{N}_\infty}(\mathcal{C}(A_\infty)) \arrow[dd] & \\
\mathscr{T}_{\mathbf{N}_m}(\mathcal{C}(A_m)) \arrow[ur, bend left] \arrow[rr] \arrow[dr, bend right] & & \mathscr{T}_{\mathbf{N}_n}(\mathcal{C}(A_n)) \arrow[ul, bend right] \arrow[dl, bend left] \\
 & \mathscr{T}_{\mathbf{N}_\mathbb{R}}(\mathcal{C}_\pi) &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathscr{S}_{\mathbf{N}_\infty}(\mathcal{C}(A_\infty)) & \\
\mathscr{S}_{\mathbf{N}_m}(\mathcal{C}(A_m)) \arrow[ur, bend left] \arrow[rr] \arrow[dr, bend right] & & \mathscr{S}_{\mathbf{N}_n}(\mathcal{C}(A_n)) \arrow[ul, bend right] \arrow[dl, bend left] \\
 & \mathscr{S}_{\mathbf{N}_\mathbb{R}}(\mathcal{C}_\pi) &
\end{tikzcd}$$

We conclude with geometric models of $\mathbf{E}$-clusters for any orientation of $A_\mathbb{R}$ using arcs (Definitions 4.1.2 and 4.2.2). These generalize the triangulations of polygons in the discrete case [8, 14, 4] and discrete laminations of the hyperbolic plane in the original continuous case [18]. By constructing additive categories $\mathcal{C}_{A_\mathbb{R}}$ (Definitions 4.1.7 and 4.2.15) whose indecomposable objects are arcs we define noncrossing conditions (Rules 4.1.3, 4.2.9, 4.2.11, and 4.2.13) on the arcs, denoted $\mathbf{N}_{\overline{\mathbb{R}}}$. We also define an isomorphism of cluster theories (Definition 4.0.1). We justify calling these models of $\mathbf{E}$-clusters by creating cluster theories from them which are isomorphic to the $\mathbf{E}$-cluster theory we have already defined.

Theorem D (Theorems 4.1.10 and 4.2.18)**.** *Let $A_\mathbb{R}$ be a continuous quiver of type A. The pairwise compatibility condition $\mathbf{N}_{\overline{\mathbb{R}}}$ induces the $\mathbf{N}_{\overline{\mathbb{R}}}$-cluster theory of $\mathcal{C}_{A_\mathbb{R}}$ and there is an isomorphism of cluster theories $(F, \eta) : \mathscr{T}_{\mathbf{N}_{\overline{\mathbb{R}}}}(\mathcal{C}_{A_\mathbb{R}}) \to \mathscr{T}_{\mathbf{E}}(\mathcal{C}(A_\mathbb{R}))$.*

In the final section we show how mutations and continuous mutations can be interpreted with these geometric models.

Future Work. There is still much to study about the space of mutations, including its topology. In the case of type A, while the exchange graph of an A_n cluster structure is well-understood but this general setting for $\mathbf{E}$-clusters poses difficult question due to continuous mutations. We ask some questions specifically about $\mathbf{E}$-clusters at the end of Section 2.4.

Also because of the continuum, it is not clear exactly which $\mathbf{E}$-cluster theories for continuous type A quivers are equivalent. There are some theories which can be shown to be isomorphic (see Remarks 4.3.2 and 4.3.3) but the exact classification is still open. We provide two ideas for future work at the end of Section 4.3.

Finally, as this is the last part of the series entitled "Continuous Quivers of Type A," the obvious question is ask is, "What about other continuous types?" The next natural steps are continuous type $\widetilde{A}$ and D; each present their own complications to our constructions. If one performs a similar constructions for continuous type D then the resulting cluster theory should be similar to Igusa and Todorov's construction in [17].

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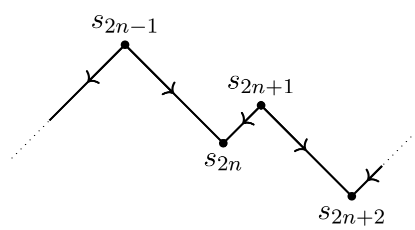
1. Parts (I), (II), and (III)

In this section we recall the most relevant definitions and theorems from the the first three parts of this series. We divide this section into three subsections, each dedicated to one paper of this series. Fix a field k for the remainder of this paper.

1.1. Continuous Quivers of Type A and Their Representations.

In this subsection we recall relevant definitions and theorems from part (I) of this series. In particular, we provide a definition of a continuous quiver of type A, its representations, and its indecomposables. The reader may use the following picture for intuition when reading the definition of a continuous quiver of type A.



Definition 1.1.1. A quiver of continuous type A, denoted by $A_{\mathbb{R}}$, is a triple $(\mathbb{R}, S, \preceq)$, where:

(1) (a) $S \subset \mathbb{R}$ is a discrete subset, possibly empty, with no accumulation points.
    (b) Order on $S \cup \{\pm\infty\}$ is induced by the order of $\mathbb{R}$, and $-\infty < s < +\infty$ for $\forall s \in S$.
    (c) Elements of $S \cup \{\pm\infty\}$ are indexed by a subset of $\mathbb{Z} \cup \{\pm\infty\}$ so that s_n denotes the element of $S \cup \{\pm\infty\}$ with index n. The indexing must adhere to the following two conditions:
        i1 There exists $s_0 \in S \cup \{\pm\infty\}$.
        i2 If $m \leq n \in \mathbb{Z} \cup \{\pm\infty\}$ and $s_m, s_n \in S \cup \{\pm\infty\}$ then for all $p \in \mathbb{Z} \cup \{\pm\infty\}$ such that $m \leq p \leq n$ the element s_p is in $S \cup \{\pm\infty\}$.

(2) New partial order $\preceq$ on $\mathbb{R}$, which we call the orientation of $A_{\mathbb{R}}$, is defined as:
    p1 The $\preceq$ order between consecutive elements of $S \cup \{\pm\infty\}$ does not change.
    p2 Order reverses at each element of S.
    p3 If n is even s_n is a sink.
    p3' If n is odd s_n is a source.

Definition 1.1.2. Let $A_{\mathbb{R}} = (\mathbb{R}, S \preceq)$ be a continuous quiver of type A. A representation V of $A_{\mathbb{R}}$ is the following data:

- A vector space $V(x)$ for each $x \in \mathbb{R}$.
- For every pair $y \preceq x$ in $A_{\mathbb{R}}$ a linear map $V(x, y) : V(x) \to V(y)$ such that if $z \preceq y \preceq x$ then $V(x, z) = V(y, z) \circ V(x, y)$.

We say V is pointwise finite-dimensional if $\dim V(x) < \infty$ for all $x \in \mathbb{R}$.

Definition 1.1.3. Let $A_{\mathbb{R}}$ be a continuous quiver of type A and $I \subset \mathbb{R}$ be an interval. We denote by M_I the representation of $A_{\mathbb{R}}$ where

$$M_I(x) = \begin{cases} k & x \in I \\ 0 & \text{otherwise} \end{cases} \qquad M_I(x, y) = \begin{cases} 1_k & y \preceq x \in I \\ 0 & \text{otherwise.} \end{cases}$$

We call M_I an <u>interval indecomposable</u>.

Notation 1.1.4. Let $a < b \in \mathbb{R} \cup \{\pm\infty\}$. By the notation $|a, b|$ we mean an interval subset of $\mathbb{R}$ whose endpoints are a and b. The $|$'s indicate that a and b may or may not be in the interval. In practice this is (i) clear from context, (ii) does not matter in its context, or (iii) intentionally left as an unknown. There is one exception: if a or b is $-\infty$ or $+\infty$, respectively, then the $|$ always means (or), respectively.

We require the two following results from [15] (the first recovers a result from [5]).

Theorem 1.1.5 (Theorems 2.3.2 and 2.4.13 in [15])**.** *Let $A_{\mathbb{R}}$ be a continuous quiver of type A. For any interval $I \subset \mathbb{R}$, the representation M_I of $A_{\mathbb{R}}$ is indecomposable. Any indecomposable pointwise finite-dimensional representation of $A_{\mathbb{R}}$ is isomorphic to M_I for some interval I. Finally, any pointwise finite-dimensional representation V of $A_{\mathbb{R}}$ is the direct sum of interval indecomposables.*

Theorem 1.1.6 (Theorem 2.1.6 and Remark 2.4.16 in [15])**.** *Let P be a projective indecomposable in the category of pointwise finite-dimensional representations of a continuous quiver $A_{\mathbb{R}}$. Then there exists $a \in \mathbb{R} \cup \{\pm\infty\}$ such that P is isomorphic to one of the following indecomposables: P_a, $P_{(a}$, or $P_{a)}$:*

$$P_a(x) = \begin{cases} k & x \preceq a \\ 0 & \textit{otherwise} \end{cases} \qquad P_a(x,y) = \begin{cases} 1_k & y \preceq x \preceq a \\ 0 & \textit{otherwise} \end{cases}$$

$$P_{(a}(x) = \begin{cases} k & x \preceq a \textit{ and } x > a \textit{ in } \mathbb{R} \\ 0 & \textit{otherwise} \end{cases} \qquad P_{(a}(x,y) = \begin{cases} 1_k & y \preceq x \preceq a \textit{ and } x, y > a \\ 0 & \textit{otherwise} \end{cases}$$

$$P_{a)}(x) = \begin{cases} k & x \preceq a \textit{ and } x < a \textit{ in } \mathbb{R} \\ 0 & \textit{otherwise} \end{cases} \qquad P_{a)}(x,y) = \begin{cases} 1_k & y \preceq x \preceq a \textit{ and } x, y < a \\ 0 & \textit{otherwise} \end{cases}$$

These allow us to define the category of finitely-generated representations:

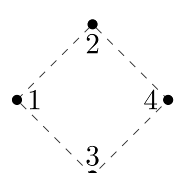
Definition 1.1.7. Let $A_{\mathbb{R}}$ be a continuous quiver of type A. By $\operatorname{rep}_k(A_{\mathbb{R}})$ we denote the full subcategory of pointwise finite-dimensional representations whose objects are finitely generated by the indecomposable projectives in Theorem 1.1.6. By [15, Theorem 3.0.1] this category is Krull-Schmidt with global dimension 1.

1.2. **The Auslander-Reiten Space of $\mathcal{D}^b(A_{\mathbb{R}})$.**

In this subsection we recall the Auslander-Reiten space, or AR-space of the bounded derives category of $\operatorname{rep}_k(A_{\mathbb{R}})$, denoted $\mathcal{D}^b(A_{\mathbb{R}})$. Before doing so we recall the following:

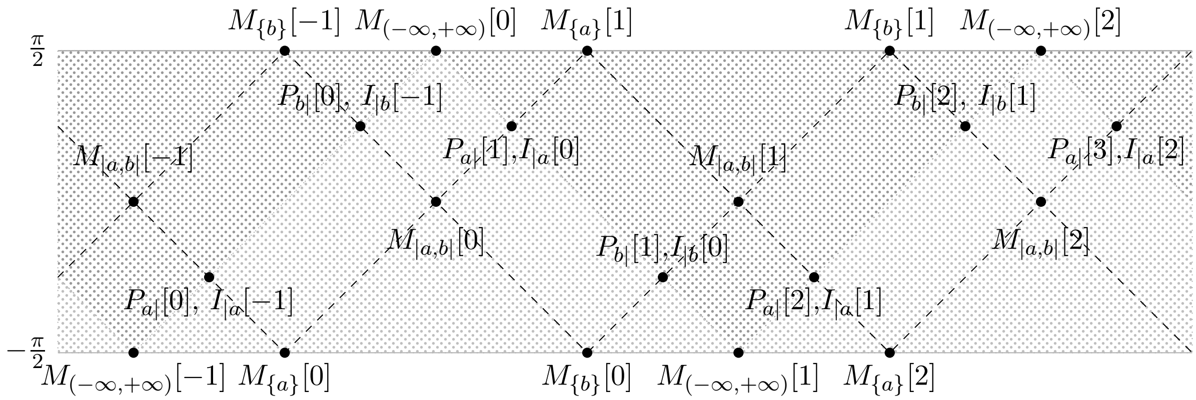
Proposition 1.2.1 (Proposition 5.1.2 in [23])**.** *The category $\mathcal{D}^b(A_{\mathbb{R}})$ is Krull-Schmidt. The indecomposable objects are shifts of indecomposables in $\operatorname{rep}_k(A_{\mathbb{R}})$.*

In [23] the author defined a function $\mathbf{\Gamma}^b : \operatorname{Ind}(\mathcal{D}^b(A_{\mathbb{R}})) \to \mathbb{R} \times [-\frac{\pi}{2}, \frac{\pi}{2}]$. This map is used to define the AR-space or $\mathcal{D}^b(A_{\mathbb{R}})$. There is also a function that sends isomorphism classes of indecomposables to the set $\{1, 2, 3, 4\}$. The value assigned to an indecomposable is called its position. The values are to be thought of as occupying the points on a diamond:



A pair of indecomposables V and W in $\mathcal{D}^b(A_{\mathbb{R}})$ are isomorphic if and only if $\mathbf{\Gamma}^b V = \mathbf{\Gamma}^b W$ and their positions are the same.

Example 1.2.2. Let $A_{\mathbb{R}}$ have the straight descending orientation. Then (part of) the AR-space of $\mathcal{D}^b(A_{\mathbb{R}})$ appears as



In [23] the author defined an extra generalized metric in order to define lines, slopes, and rectangles in this new space. The author also defined almost-complete rectangles, which one may think of as a rectangle without one of its corner points. In the following result, "good slopes" are analogous to 45° angles. The phrase "nontrivial triangle" means a distinguished triangle that is not of the form $(A \to A \to 0 \to)$, $(A \to 0 \to A[1] \to)$, or $(0 \to A \to A \to)$. Furthermore, we consider a triangle to be distinct from any of its rotations for the statement.

Theorem 1.2.3 (Theorem 5.2.10 in [23])**.** *Let $V = M_{|a,b|}[m]$ and $W = M_{|c,d|}[n]$ be indecomposables in $\mathcal{D}^b(A_{\mathbb{R}})$ such that $V \ncong W$. Then there is a nontrivial distinguished triangle $V \to U \to W \to$ if and only if there exists a rectangle or almost complete rectangle in the AR-space of $\mathcal{D}^b(A_{\mathbb{R}})$ whose corners are the indecomposables in the triangle with V as the left-most corner and W as the right-most corner.*

- *If the rectangle is complete E is a direct sum of two indecomposables.*
- *If the rectangle is almost complete E is indecomposable.*

Furthermore, there is a bijection

$$\begin{array}{c} \{\textit{rectangles and almost complete rectangles with "good" slopes of sides in AR-space of } \mathcal{D}^b(A_{\mathbb{R}})\} \\ \updownarrow \cong \\ \{\textit{nontrivial triangles with first and third term indecomposable up to scaling and isomorphisms}\} \end{array}$$

1.3. **Embeddings of Cluster Theories.** In thus subsection we recall the definitions, results, and examples we need about cluster theories from Part (III) [16].

Definition 1.3.1. The category $\mathcal{C}(A_{\mathbb{R}})$ is the orbit category of the doubling of $\mathcal{D}^b(A_{\mathbb{R}})$ via almost-shift as in [18].

Importantly, the isomorphism classes of indecomposables in $\mathcal{C}(A_{\mathbb{R}})$ are the same as if we had took the orbit of $\mathcal{D}^b(A_{\mathbb{R}})$ by shift; i.e., $V \cong V[1]$ for all indecomposables V in $\mathcal{C}(A_{\mathbb{R}})$. However we obtain a triangulated structure. Thus we have distinguished triangles of the form $Q_V \to P_V \to V \to$ where $Q_V \to P_V \to V \to 0$ is the minimal projective resolution of V in $\text{rep}_k(A_{\mathbb{R}})$. Furthermore, for indecomposables V and W in $\mathcal{C}(A_{\mathbb{R}})$, either $\text{Hom}_{\mathcal{C}(A_{\mathbb{R}})}(V,W) \cong k$ or $\text{Hom}_{\mathcal{C}(A_{\mathbb{R}})}(V,W) = 0$ [16, Proposition 3.1.2]. The authors of [16] then defined g-vectors following Jørgensen and Yakimov in [20].

Definition 1.3.2. Let V be an indecomposable in $\mathcal{C}(A_{\mathbb{R}})$. The g-vector of V is the element $[P_V] - [Q_V]$ in $K_0^{\text{split}}(\mathcal{C}(A_{\mathbb{R}}))$ where $Q_V \to P_V \to V \to 0$ is the minimal projective resolution of V in $\text{rep}_k(A_{\mathbb{R}})$.

The authors also defined an Euler form on $K_0^{\text{split}}(\mathcal{C}(A_{\mathbb{R}}))$.

Definition 1.3.3. Let $[A] = \sum_i m_i[A_i]$ and $[B] = \sum_j n_j[B_j]$ be elements in $K_0^{\text{split}}(\mathcal{C}(A_{\mathbb{R}}))$ where each A_i and B_j are indecomposable. For A_i and B_j we define

$$\langle m_i[A_i]\,,\, n_j[B_j]\rangle := (m_i \cdot n_j)(\dim \operatorname{Hom}_{\mathcal{C}(A_{\mathbb{R}})}(A_i, B_j)).$$

The form is defined to be

$$\langle [A]\,,\,[B]\rangle := \sum_i \sum_j \langle m_i[A_i]\,,\, n_j[B_j]\rangle.$$

From here the authors defined **E**-compatibility and **E**-clusters.

Definition 1.3.4.

- Let V and W be two indecomposables in $\mathcal{C}(A_{\mathbb{R}})$ with g-vectors $[P_V]-[Q_V]$ and $[P_W]-[Q_W]$. We say $\{V, W\}$ is <u>**E**-compatible</u> if
$$\langle [P_V]-[Q_V]\,,\,[P_W]-[Q_W]\rangle \geq 0 \qquad \text{and} \qquad \langle [P_W]-[Q_W]\,,\,[P_V]-[Q_V]\rangle \geq 0.$$
- A set T is called <u>**E**-compatible</u> if for every $V, W \in T$ the set $\{V, W\}$ is **E**-compatible. If T is maximally **E**-compatible then we call T an <u>**E**-cluster</u>.
- Let T be an **E**-cluster and $V \in T$ such that there exists $W \notin T$ where $\{V, W\}$ is not **E**-compatible but $(T \setminus \{V\}) \cup \{W\}$ is **E**-compatible. Then we say V is <u>**E**-mutable</u>. The bijection (see Theorem 1.3.5) $T \to (T \setminus \{V\}) \cup \{W\})$ given by $V \mapsto W$ and $X \mapsto X$ if $X \neq V$ is called an <u>**E**-mutation</u> or <u>**E**-mutation at V</u>.

The words **E**-cluster and **E**-mutation are justified with the following theorem.

Theorem 1.3.5 (Theorem 3.2.8 in [16])**.** *Let T be an **E**-cluster and $V \in T$ **E**-mutable with choice W. Then $(T \setminus \{V\}) \cup \{W\}$ is an **E**-cluster and any other choice W' for V is isomorphic to W.*

The following proposition will be quite useful in Section 4.2.

Proposition 1.3.6 (Proposition 3.1.8 in [16])**.** *Let $V[m]$ and $W[n]$ be indecomposable objects in $\mathcal{C}(A_{\mathbb{R}})$ where V and W are indecomposables in the 0th degree. Then $\{V, W\}$ is not **E**-compatible if and only if there is a rectangle or almost complete rectangle in the AR-space of $\mathcal{D}^b(A_{\mathbb{R}})$, whose sides have slopes $\pm(1,1)$ and whose left and right corners are V and W (not necessarily respectively).*

The key difference between **E**-clusters and the usual cluster structures is that not all V in an **E**-cluster T need be mutable. The authors only require there be none or one choice. This is generalized to the abstract notion of cluster theories.

Definition 1.3.7. Let $\mathcal{C}$ be a skeletally small Krull-Schmidt additive category and **P** a pairwise compatibility condition on its (isomorphism classes of) indecomposable objects. Suppose that for each (isomorphism class of) indecomposable X in a maximally **P**-compatible set T there exists none or one (isomorphism class of) indecomposable Y such that $\{X, Y\}$ is not **P**-compatible but $(T \setminus \{X\}) \cup \{Y\}$ is maximally **P**-compatible. Then

- We call the maximally **P**-compatible sets <u>**P**-clusters</u>.
- We call a function of the form $\mu : T \to (T \setminus \{X\}) \cup \{Y\}$ such that $\mu Z = Z$ when $Z \neq X$ and $\mu X = Y$ a <u>**P**-mutation</u> or <u>**P**-mutation at X</u>.
- If there exists a **P**-mutation $\mu : T \to (T \setminus \{X\}) \cup \{Y\}$ we say $X \in T$ is <u>**P**-mutable</u>.
- The subcategory $\mathscr{T}_{\mathbf{P}}(\mathcal{C})$ of $\mathcal{S}$et whose objects are **P**-clusters and whose morphisms are generated by **P**-mutations (and identity functions) is called <u>the **P**-cluster theory of $\mathcal{C}$</u>.
- The functor $I_{\mathbf{P},\mathcal{C}} : \mathscr{T}_{\mathbf{P}}(\mathcal{C}) \to \mathcal{S}$et is the inclusion of the subcategory.

From now on, when we say a "Krull-Schmidt category" we mean a "skeletally small Krull-Schmidt additive category." In Definition 1.3.7 we see that **P** determines the cluster theory. Thus we may say that **P** induces the cluster theory.

Definition 1.3.8. Let $\mathcal{C}$ be a Krull-Schmidt category and $\mathbf{P}$ a pairwise compatibility condition such that $\mathbf{P}$ induces the $\mathbf{P}$-cluster theory of $\mathcal{C}$. If for every $\mathbf{P}$-cluster T and $X \in T$ there is a $\mathbf{P}$-mutation at X then we call $\mathscr{T}_{\mathbf{P}}(\mathcal{C})$ the tilting $\mathbf{P}$-cluster theory.

We provide known examples of cluster theories:

Example 1.3.9.

- The triangulations of the $(n+3)$-gon cluster category introduced by Caldero, Chapaton, and Schiffler in [8] form a tilting cluster theory. Indecomposable objects are diagonals in the $(n+3)$-gon and compatibility, denoted $\mathbf{N}_n$ is non-crossing. Maximal $\mathbf{N}_n$-compatible sets are triangulations of the $(n+3)$-gon. This is the cluster structure associated to type A_n and we denote the category by $\mathcal{C}(A_n)$. This is equivalent to the construction by Buan, Marsh, Reineke, Reiten, and Todorov in [6] that was developed independently the same year.
- Holm and Jørgensen describe in [14] the cluster theory given by triangulations of the infinity-gon. The vertices of the infinity-gon are indexed on $\mathbb{Z}$ and diagonals have endpoints i and j where $j - i \geq 2$. Compatibility is given by non-crossing and we denote it by $\mathbf{N}_\infty$. Maximal sets of non-crossing diagonals, i.e. triangulations of the infinity-gon, are the maximal compatible sets. The triangulated category in [14] has precisely diagonals as indecomposables; we denote it by $\mathcal{C}(A_\infty)$. The authors do not examine all such triangulations; the $\mathbf{N}_\infty$-cluster theory is not tilting.
- We may take $\mathbf{E}$-compatibility on $\mathcal{C}(A_{\mathbb{R}})$ and see that we obtain a cluster theory also. However, by [16, Example 4.3.1], we see the cluster theory is not tilting.

And now we recall the authors' definition of an embedding of cluster theories.

Definition 1.3.10. Let $\mathcal{C}$ and $\mathcal{D}$ be two Krull-Schmidt categories with respective pairwise compatibility conditions $\mathbf{P}$ and $\mathbf{Q}$. Suppose these compatibility conditions induce the $\mathbf{P}$-cluster theory and $\mathbf{Q}$-cluster theory of $\mathcal{C}$ and $\mathcal{D}$, respectively.

Suppose there exists a functor $F : \mathscr{T}_{\mathbf{P}}(\mathcal{C}) \to \mathscr{T}_{\mathbf{Q}}(\mathcal{D})$ such that F is an injection on objects and an injection from $\mathbf{P}$-mutations to $\mathbf{Q}$-mutations. Suppose also there is a natural transformation $\eta : I_{\mathbf{P},\mathcal{C}} \to I_{\mathbf{Q},\mathcal{D}} \circ F$ whose morphisms $\eta_T : I_{\mathbf{P},\mathcal{C}}(T) \to I_{\mathbf{Q},\mathcal{D}} \circ F(T)$ are all injections. Then we call $(F, \eta) : \mathscr{T}_{\mathbf{P}}(\mathcal{C}) \to \mathscr{T}_{\mathbf{Q}}(\mathcal{D})$ an embedding of cluster theories.

Let $A_{\mathbb{R}}$ be the continuous quiver of type A with straight descending orientation.

Definition 1.3.11. Let $\{a_i\}_{i\in\mathbb{Z}}$ be a collection of real numbers such that

- $a_i < a_{i+1}$ for all $i \in \mathbb{Z}$ and
- $\lim_{i\to-\infty} a_i, \lim_{i\to+\infty} a_i \in \mathbb{Z}$.

Let $a_{-\infty} = \lim_{i\to-\infty} a_i$ and $a_{+\infty} = \lim_{i\to+\infty} a_i$. For each $i, j, \ell \in \mathbb{Z}$ such that $l \geq 0$ and $0 \leq j \leq 2^\ell$ define

$$a_{i,j,\ell} := a_i + \left(\frac{j}{2^\ell}\right)(a_{i+1} - a_i).$$

For each a_i, we define the following $\mathbf{E}$-compatible set:

$$\begin{aligned} T_{a_i} := & \left\{ M_{(a_{i,j,\ell},\, a_{i,j+1,\ell})} : j, \ell \in \mathbb{Z},\ \ell \geq 0,\ 0 \leq j < 2^\ell \right\} \\ & \cup \left\{ M_{\{x\}} : x \in (a_i, a_i + 1),\ x \neq a_{i,j,l},\ j, \ell \in \mathbb{Z},\ \ell \geq 0,\ 0 \leq j < 2^\ell \right\}, \end{aligned}$$

Note that for any a_i and a_j then $T_{a_i} \cup T_{a_j}$ is $\mathbf{E}$-compatible. Now, for each $i \in \mathbb{Z}$ such that $i < a_{-\infty}$ or $i \geq a_{+\infty}$ define a similar type of $\mathbf{E}$-compatible set:

$$\begin{aligned} T_i := & \left\{ M_{(i+j/2^\ell,\, i+(j+1)/2^\ell)} : j, \ell \in \mathbb{Z},\ \ell \geq 0,\ 0 \leq j < 2^\ell \right\} \cup \{P_{i+1}\} \\ & \cup \left\{ M_{\{x\}} : x \in (i, i+1),\ x \neq i + j/2^\ell,\ j, \ell \in \mathbb{Z},\ \ell \geq 0,\ 0 \leq j < 2^\ell \right\} \end{aligned}$$

The $\mathbf{E}$-compatible set we want is

$$T_\infty := \left(\bigcup_{i\in\mathbb{Z}} T_{a_i}\right) \cup \left(\bigcup_{i<a_{-\infty} \text{ or } i\geq a_{+\infty}} T_i\right) \cup \{M_{(a_{-\infty},a_{+\infty})}, P_{+\infty}\} \cup \{P_i : i \leq a_{-\infty} \text{ or } i \geq a_{+\infty}\}.$$

For an $(n+3)$-gon (whose vertices are indexed 1–$(n+3)$) or the infinity-gon (whose vertices are labeled by $\mathbb{Z}$) we denote diagonals by $i\!\!\frown\!\! j$ where $j-i\geq 2$.

Definition 1.3.12. Let $i\!\!\frown\!\! j$ be a diagonal in an $(n+3)$-gon or infinity-gon and $A_\mathbb{R}$ the continuous quiver of type A with straight descending orientation. We define $M_{i\frown j}$ to be the indecomposable in $\mathcal{C}(A_\mathbb{R})$ from the indecomposable $M_{(a_i,a_j)}$ in $\operatorname{rep}_k(A_\mathbb{R})$.

Using Definition 1.3.12 we define the functors used for the embeddings of cluster theories for the first two examples in Example 1.3.9 into the third example.

Definition 1.3.13. Let $A_\mathbb{R}$ be the continuous quiver of type A with straight descending orientation. Let $n\geq 1\in\mathbb{Z}$ and define the set T_n:

$$T_n := T_\infty \cup \{M_{(a_i,a_1)} : i<0\} \cup \{M_{(a_1,a_j)} : j\geq n+3\} \cup \{M_{(a_{-\infty},a_1)}, M_{(a_1,a_{+\infty})}\}$$

This allows us to define a functor $F_n^\mathbb{R} : \mathscr{T}_{\mathbf{N}_n}(\mathcal{C}(A_n)) \to \mathscr{T}_{\mathbf{E}}(\mathcal{C}(A_\mathbb{R}))$. For an $\mathbf{N}_n$-cluster $T_{\mathbf{N}_n}$ we define

$$F_n^\mathbb{R}(T_{\mathbf{N}_n}) := T_n \cup \{M_{i\frown j} : i\!\!\frown\!\! j \in T_{\mathbf{N}_n}\}$$

By [16, Lemma 5.2.6], for an $\mathbf{N}_n$-mutation $\mu : T\to T'$ we get an $\mathbf{E}$-mutation $F_n^\mathbb{R}\mu : F_n^\mathbb{R}T \to F_n^\mathbb{R}T'$. Since mutations generate all the morphisms we have a functor of groupoids.

The authors of [14] show that a triangulation $T_{\mathbf{N}_\infty}$ of the infinity-gon is either locally finite (each vertex is the endpoint of only finitely-many arcs) or has a unique left- and unique right-fountain (infinitely-many arcs with the same upper endpoint or same lower endpoint, respectively). When the left- and right-fountains are at the same vertex we call the union a fountain.

Definition 1.3.14. Let $A_\mathbb{R}$ be the continuous quiver of type A with straight descending orientation. We define a functor $F_\infty^\mathbb{R} : \mathscr{T}_{\mathbf{N}_\infty}(\mathcal{C}(A_\infty)) \to \mathscr{T}_{\mathbf{E}}(\mathcal{C}(A_\mathbb{R}))$. For a locally finite $\mathbf{N}_\infty$-cluster $T_{\mathbf{N}_\infty}$ we define

$$F_\infty^\mathbb{R} T_{\mathbf{N}_\infty} = T_\infty \cup \{M_{i\frown j} : i\!\!\frown\!\! j \in T_{\mathbf{N}_\infty}\}.$$

If $T_{\mathbf{N}_\infty}$ is an $\mathbf{N}_\infty$-cluster with a left-fountain at m and a right fountain at n (where $m<n$) then we define

$$F_\infty^\mathbb{R} T_{\mathbf{N}_\infty} = T_\infty \cup \{M_{i\frown j} : i\!\!\frown\!\! j \in T_{\mathbf{N}_\infty}\} \cup \{M_{(a_{-\infty},a_m)}, M_{(a_{-\infty},a_n)}, M_{(a_n,a_{+\infty})}\}.$$

Again, this time using [16, Lemma 5.3.9] we have a functor of groupoids.

The final functor we need requires an auxiliary function $\mathfrak{f}$, which we will also use in Section 3.2.

Definition 1.3.15. Let $\mathcal{C}A$, $\mathcal{C}B$, and $\mathcal{C}C$ be the sets

$$\mathcal{C}A = \{(x,y)\in\mathbb{R}^2 : |x-y|<\pi, x\geq 0, y<\pi\}$$
$$\mathcal{C}B = \left\{(\alpha,\beta)\in\mathbb{R}^2 : -\frac{\pi}{2}<\beta<\frac{\pi}{2} \text{ and } \beta\leq\alpha<\pi-\beta\right\}$$
$$\mathcal{C}C = \{(a,b)\in(\mathbb{R}\cup\{-\infty\})\times\mathbb{R} : -\infty\leq a<b<+\infty\}.$$

Define $\mathfrak{g}:\mathcal{C}A\to\mathcal{C}B$ and $\mathfrak{h}:\mathcal{C}B\to\mathcal{C}C$ by

$$\mathfrak{g}(x,y) := \left(\frac{y+x}{2}, \frac{y-x}{2}\right)$$
$$\mathfrak{h}(\alpha,\beta) := \left(\tan\left(\frac{\alpha-\beta-\pi}{2}\right), \tan\left(\frac{\alpha+\beta}{2}\right)\right)$$

We define $\mathfrak{f} : \mathcal{C}A \to \mathcal{C}C$ by $\mathfrak{f}(x, y) = \mathfrak{h} \circ \mathfrak{g}(x, y)$. In [16, Proposition 5.4.3] the authors show $\mathfrak{f}$ is a bijection.

In [18] the authors construct the category $\mathcal{C}_\pi$ which is triangulated and whose indecomposable objects are of the form $M(x, y)$ for $x, y \in \mathcal{C}A$ from Definition 1.3.15. We'll call their compatibility condition $\mathbf{N}_\mathbb{R}$. The embedding $F_\pi^\mathbb{R} : \mathscr{T}_{\mathbf{N}_\mathbb{R}}(\mathcal{C}_\pi) \to \mathscr{T}_\mathbf{E}(\mathcal{C}(A_\mathbb{R}))$ is quite complicated to desribe and so we refer the reader to [16, Section 5.4] for details. However, for an $\mathbf{N}_\mathbb{R}$-cluster $T_{\mathbf{N}_\mathbb{R}}$ we have the following inclusion

$$\{M_{\mathfrak{f}(x,y)} : M(x, y) \in T_{\mathbf{N}_\mathbb{R}}\} \subset F_\pi^\mathbb{R} T_{\mathbf{N}_\mathbb{R}}.$$

Each of the functors above come with a natural transformation to complete the definition of an embedding of cluster theories. We conclude our recollection of Parts (I), (II), and (III) with the following theorem

Theorem 1.3.16 (Theorems 5.2.7, 5.3.10, and 5.4.9 in [16])**.**

(i) *There exists an embedding of cluster theories* $(F_n^\mathbb{R}, \eta_n^\mathbb{R}) : \mathscr{T}_{\mathbf{N}_n}(\mathcal{C}(A_n)) \to \mathscr{T}_\mathbf{E}(\mathcal{C}(A_\mathbb{R}))$.

(ii) *There exists an embedding of cluster theories* $(F_\infty^\mathbb{R}, \eta_\infty^\mathbb{R}) : \mathscr{T}_{\mathbf{N}_\infty}(\mathcal{C}(A_\infty)) \to \mathscr{T}_\mathbf{E}(\mathcal{C}(A_\mathbb{R}))$.

(iii) *There exists an embedding of cluster theories* $(F_\pi^\mathbb{R}, \eta_\pi^\mathbb{R}) : \mathscr{T}_{\mathbf{N}_\mathbb{R}}(\mathcal{C}_\pi) \to \mathscr{T}_\mathbf{E}(\mathcal{C}(A_\mathbb{R}))$.

2. Continuous Mutation

This subsection is dedicated to the definition of a continuous mutation and the basic properties of continuous mutations. These generalize the familiar notion of mutation in a cluster structure. We define this new type of mutation for all cluster theories (Definition 1.3.7) though we will use type A cluster theories for our examples. Notably, any $\mathbf{P}$-mutation can be thought of as a continuous $\mathbf{P}$-mutation (see Example 2.2.1). The final subsection of this section is dedicated to the space of mutations (Definition 2.4.2), which generalizes the exchange graph of a cluster structure. We pose questions related specifically to the $\mathbf{E}$-cluster theory of an arbitrary $A_\mathbb{R}$ quiver (Example 1.3.9) at the very end of the section.

2.1. The Basics.

Recall the definition of a $\mathbf{P}$-cluster theory of $\mathcal{C}$ (Definition 1.3.7).

Definition 2.1.1. Let $\mathcal{C}$ be a Krull-Schmidt category and $\mathbf{P}$ a pairwise compatibility condition on the indecomposables in $\mathcal{C}$ such that $\mathbf{P}$ induces the $\mathbf{P}$-cluster theory of $\mathcal{C}$. For any $\mathbf{P}$-cluster T, we call the identity function $\mathrm{id}_T : T \to T$ a <u>trivial **P**-mutation</u>.

Definition 2.1.2. Let $\mathcal{C}$ be a Krull-Schmidt category and $\mathbf{P}$ a pairwise compatibility condition on the indecomposables in $\mathcal{C}$ such that $\mathbf{P}$ induces the $\mathbf{P}$-cluster theory of $\mathcal{C}$.

Let T and T' be $\mathbf{P}$-clusters. Let $S \subset T$ such that there exists a bijection $\mu : T \to T'$ where $\mu X = X$ if and only if $X \notin S$. Let $S' = \mu(S)$ and further assume that for all $\mu X \in S'$ (i) $\mu X \notin T$ and (ii) $\{X, \mu X\}$ is not $\mathbf{P}$-compatible. Finally, suppose there exist injections $f_\mu : S \to [0, 1]$ and $g_\mu : S' \to [0, 1]$ such that

- The following equality holds: $g_\mu \circ \mu|_S = f_\mu$,

$$\begin{tikzcd} S \arrow[rr, "\mu|_S"] \arrow[dr, "f_\mu"'] & & S' \arrow[dl, "g_\mu"] \\ & {[0,1].} & \end{tikzcd}$$

- For any subinterval $J \subset [0, 1]$ where $0 \in J$ and $1 \notin J$ the following is a $\mathbf{P}$-cluster:

$$(T \setminus f_\mu^{-1}(J)) \cup g_\mu^{-1}(J) = (T' \setminus g_\mu^{-1}(\bar{J})) \cup f_\mu(\bar{J}),$$

where $\bar{J} = [0, 1] \setminus J$.

If all this holds we call μ a <u>continuous **P**-mutation</u>.

We need to justify the word 'mutation.' We do this with Propositions 2.1.3 and 2.1.4. The first of the two propositions shows that every continuous mutation can be reversed and the second that we can consider a continuous mutation as being a mutation at time t for all $t \in [0,1]$.

Proposition 2.1.3. *Let $\mu : T \to T'$ be a continuous* $\mathbf{P}$*-mutation. Then $\mu^{-1} : T' \to T$ is also a continuous* $\mathbf{P}$*-mutation.*

Proof. By the definition of a continuous $\mathbf{P}$-mutation, we have

- a bijection $T' \to T$ such that $X \mapsto X$ if and only if $X \notin S'$,
- $\{X, \mu^{-1}X\}$ is not $\mathbf{P}$-compatible for each $X \in S'$,
- for all $X \in S'$, $\mu^{-1}X \notin T$.

Let $\rho : [0,1] \to [0,1]$ be given by $t \mapsto 1-t$. Set $\hat{f} = \rho \circ g$ and $\hat{g} = \rho \circ f$. Then we have $\hat{f} : S' \hookrightarrow [0,1]$ and $\hat{g} : S \hookrightarrow [0,1]$ such that the following diagram commutes

$$\begin{tikzcd} S' \arrow[rr, "\mu^{-1}|_{S'}"] \arrow[dr, "\hat{f}"'] & & S \arrow[dl, "\hat{g}"] \\ & {[0,1],} & \end{tikzcd}$$

and for each $\bar{J} \subset [0,1]$ where $0 \in \bar{J}$

$$(T' \setminus \hat{f}_\mu^{-1}(\bar{J})) \cup \hat{g}_\mu(\bar{J}) = (T \setminus \hat{g}_\mu^{-1}(J)) \cup \hat{f}_\mu^{-1}(J),$$

is a $\mathbf{P}$-cluster, where $J = [0,1] \setminus \bar{J}$. This is exactly the requirement for a continuous mutation. □

Proposition 2.1.4. *Let $\mu : T \to T'$ be a continuous* $\mathbf{P}$*-mutation. For every $t \in [0,1]$, the bijection $(T \setminus f_\mu^{-1}([0,t))) \cup g_\mu^{-1}([0,t)) \to (T \setminus f_\mu^{-1}([0,t])) \cup g_\mu^{-1}([0,t])$ given by*

$$X \mapsto \begin{cases} X & X \neq f^{-1}(t) \\ g^{-1}(t) & X = f^{-1}(t) \end{cases}$$

is a $\mathbf{P}$*-mutation.*

Proof. In the case $(T \setminus f_\mu^{-1}([0,t))) \cup g_\mu^{-1}([0,t)) = (T \setminus f_\mu^{-1}([0,t])) \cup g_\mu^{-1}([0,t])$ we have a trivial $\mathbf{P}$-mutation. Suppose $(T \setminus f_\mu^{-1}([0,t))) \cup g_\mu^{-1}([0,t)) \neq (T \setminus f_\mu^{-1}([0,t])) \cup g_\mu^{-1}([0,t])$. Since f_μ and g_μ are injections, $f_\mu^{-1}([0,t))$ differs from $f_\mu^{-1}([0,t])$ by at most one element and by assumption they differ by at least one element; thus differing by exactly one element. This is similarly true for $g_\mu^{-1}([0,t))$ and $g_\mu^{-1}([0,t])$. By definition, $\mu(f_\mu^{-1}(t)) = g_\mu^{-1}(t)$ and $\{f_\mu^{-1}(t), g_\mu^{-1}(t)\}$ is not $\mathbf{P}$-compatible. Therefore we have a $\mathbf{P}$-mutation. □

We conclude with this final definition that will be useful in asking questions about the classification of $\mathbf{E}$-clusters in $\mathcal{C}(A_{\mathbb{R}})$ (Definition 1.3.4). We ask these questions at the end of Section 2.4.2.

Definition 2.1.5. Let $\mathcal{C}$ be a Krull-Schmidt category and $\mathbf{P}$ a pairwise compatibility condition on its indecomposables such that $\mathbf{P}$ induces the $\mathbf{P}$-cluster theory of $\mathcal{C}$. Let $Z = \{1, \ldots, n\}$ or $Z = \mathbb{Z}_{>0}$. For each $i \in Z$ let μ_i be a continuous $\mathbf{P}$-mutation such that the target of μ_i is the source of μ_{i+1} when $i, i+1 \in Z$. We call $\{\mu_i\}_{i \in Z}$ a <u>sequence of continuous $\mathbf{P}$-mutations</u>. If each μ_i mutates only one element of T_i we may also say that $\{\mu_i\}$ is a <u>sequence of $\mathbf{P}$-mutations</u>.

2.2. **Examples.** In this subsection we highlight two existing examples of continuous mutations that do not feel so continuous followed by a new example. The first (Example 2.2.1) shows that a mutation, in the traditional sense, can be thought of as a continuous mutation. The second (Example 2.2.5) describes an infinite sequence of mutations. While these both exist in the literature, the contribution is that continuous mutation unifies the way to describe these types of mutations.

We conclude with Proposition 2.2.7, which, as far as the author knows, does not exist anywhere in the literature.

Example 2.2.1. Let $\mathcal{C}$ be a Krull-Schmidt category with pairwise compatibility condition $\mathbf{P}$ on indecomposables such that $\mathbf{P}$ induces the $\mathbf{P}$-cluster theory of $\mathcal{C}$. Let $\mu : T \to (T \setminus \{X\}) \cup \{Y\}$ be a $\mathbf{P}$-mutation. Furthermore, let $S = \{X\}$, $S' = \{Y\}$, and $T' = (T \setminus \{X\}) \cup \{Y\}$. Finally, let $f : \{X\} \to [0,1]$ and $g : \{Y\} \to [0,1]$ each send X and Y to $\frac{1}{2}$, respectively. This meets the requirements for the definition of a continuous mutation.

The second example requires some definitions (Definition 2.2.2) from [4] as well as a category with a pairwise compatibility to work in (Definitions 2.2.3 and 2.2.4). Where we say "diagonal" Baur and Graz say "arc" in [4].

Definition 2.2.2 (From [4])**.** We denote by $\overline{\mathcal{C}_\infty}$ the <u>completed infinity-gon</u>. The vertices of $\overline{\mathcal{C}_\infty}$ are $\mathbb{Z} \cup \{\pm\infty\}$ with total order inherited by $\mathbb{Z}$ and $-\infty < i < +\infty$ for all $i \in \mathbb{Z}$. The sides of $\overline{\mathcal{C}_\infty}$ are pairs $(i, i+1)$ for $i \in \mathbb{Z}$ and $(-\infty, +\infty)$, called the <u>generic diagonal</u>. <u>Diagonals</u> are pairs i, j written $i \frown j$ such that $j - i \geq 2$ and i or j may respectively be $-\infty$ or $+\infty$, but not both. We say two diagonals $\theta = i \frown j$ and $\theta' = i' \frown j'$ <u>cross</u> if $i < i' < j < j'$ or $i' < i < j' < j$.

Definition 2.2.3. Let $\mathcal{C}(A_{\overline{\infty}})$ be the k-linear additive category whose objects are the 0 object and direct sums of diagonals θ of $\overline{\mathcal{C}_\infty}$ and whose morphisms on diagonals are given by

$$\mathrm{Hom}_{\mathcal{C}(A_{\overline{\infty}})}(\theta, \theta') = \begin{cases} k & (\theta = \theta') \text{ or } (\theta \text{ crosses } \theta') \\ 0 & \text{otherwise.} \end{cases}$$

Let $f : \theta \to \theta'$ and $g : \theta' \to \theta''$ be morphisms on diagonals in $\mathcal{C}(A_{\overline{\infty}})$. Note f and g are also considered as scalars in k. Define composition as

$$g \circ f = \begin{cases} g \cdot f \in \mathrm{Hom}_{\mathcal{C}(A_{\overline{\infty}})}(\theta, \theta'') & (\theta = \theta' \text{ or } \theta' = \theta'') \text{ and } ((\theta \text{ crosses } \theta'') \text{ or } \theta = \theta'') \\ 0 & \text{otherwise.} \end{cases}$$

Note that by definition each diagonal θ is in its own isomorphism class and $\mathcal{C}(A_{\overline{\infty}})$ is Krull-Schmidt.

Definition 2.2.4. Let $\mathbf{N}_{\overline{\infty}}$ be the following pairwise condition on indecomposables in $\mathcal{C}(A_{\overline{\infty}})$. We say $\{\theta, \theta'\}$ is $\mathbf{N}_{\overline{\infty}}$-compatible if $\mathrm{Hom}(\theta, \theta') = \mathrm{Hom}(\theta', \theta) = 0$ or $\theta = \theta'$. This coincides with compatibility of θ and θ' as diagonals in [4] by definition. Immediately we see that a maximally $\mathbf{N}_{\overline{\infty}}$-compatible set of indecomposables in $\mathcal{C}(A_{\overline{\infty}})$ is a triangulation $\overline{\mathcal{C}_\infty}$.

It is remarked in [4] that if a diagonal θ is $\mathbf{N}_{\overline{\infty}}$-mutable then its replacement is unique. Thus $\mathbf{N}_{\overline{\infty}}$ induces the $\mathbf{N}_{\overline{\infty}}$-cluster theory of $\mathcal{C}(A_{\overline{\infty}})$. Let T be an $\mathbf{N}_{\overline{\infty}}$-cluster in $\mathcal{C}(A_{\overline{\infty}})$. Following [4], a <u>$T$-admissible sequence of diagonals</u> $\{\theta_i\}$ is one where θ_1 is $\mathbf{N}_{\overline{\infty}}$-mutable in $T_1 = T$ and each T_i for $i > 1$ is obtained by mutating θ_{i-1} which must be mutable in T_{i-1}. Note this sequence may be infinite so long as there is a first diagonal in the sequence.

The authors in [4] note that mutating along a T-admissible sequence does not always result in a $\mathbf{N}_{\overline{\infty}}$-cluster. I.e., the colimit of such a sequence of mutations may not be a $\mathbf{N}_{\overline{\infty}}$-cluster.

Example 2.2.5. Let T be an $\mathbf{N}_{\overline{\infty}}$-cluster in $\mathcal{C}(A_{\overline{\infty}})$ and $\{\theta_i\}$ a T-admissible sequence of diagonals. Since each $\mathbf{N}_{\overline{\infty}}$-mutation $\mu_i : T_i \to T_{i+1}$ is also a continuous $\mathbf{N}_{\overline{\infty}}$-mutation any admissible sequence of diagonals yields a sequence of continuous $\mathbf{N}_{\overline{\infty}}$-mutations.

Now suppose $\{\theta_i\} \subset T$ and the result of mutating along $\{\theta_i\}$ yields an $\mathbf{N}_{\overline{\infty}}$-cluster T'. Then we let $S = \{\theta_i\}$ and let $f : S \to [0,1]$ be given by $\theta_i \mapsto 1 - \frac{1}{i+1}$. Let $S' = \{\mu_i(\theta_i)\}$ and let $g : S' \to [0,1]$ be given by $\mu_i(\theta_i) \mapsto 1 - \frac{1}{i+1}$. We now have a continuous $\mathbf{N}_{\overline{\infty}}$-mutation.

In general, a T-admissible sequence of diagonals can be "grouped" into intervals of diagonals which each belong to the first cluster of the group. This yields a sequence of $\mathbf{N}_{\overline{\infty}}$-mutations in a somewhat minimal way. Of course, this doesn't work if $\{\theta_i\} \subset T$ and mutation along $\{\theta_i\}$ does not result in an $\mathbf{N}_{\overline{\infty}}$-cluster.

Remark 2.2.6. Let $\mathcal{C}$ be a Krull-Schmidt category with pairwise compatibility condition $\mathbf{P}$ on indecomposables such that $\mathbf{P}$ induces the $\mathbf{P}$-cluster theory of $\mathcal{C}$. As seen in Example 2.2.5 it might be possible to construct a sequence of (continuous) $\mathbf{P}$-mutations that does not yield a $\mathbf{P}$-cluster. The authors of [4] provide a way to complete their compatible sets for their cluster theory. Later we will use a similar technique to complete an $\mathbf{E}$-compatible set given $A_{\mathbb{R}}$ has the straight descending orientation. This technique does not generalize to all orientations of a $A_{\mathbb{R}}$ quiver but is useful to prove Theorem 3.2.13.

Proposition 2.2.7. *Let $A_{\mathbb{R}}$ have the straight descending orientation, $\mathcal{P}roj$ be the $\mathbf{E}$-cluster containing all the projectives from $\operatorname{rep}_k(A_{\mathbb{R}})$, and $\mathcal{I}nj$ be the $\mathbf{E}$-cluster containing the injectives from $\operatorname{rep}_k(A_{\mathbb{R}})$. There is a sequence of continuous mutations $\{\mu_1, \mu_2\}$ that starts at $\mathcal{P}roj$ and ends at $\mathcal{I}nj$.*

Proof. Recall that every indecomposable in $\mathcal{C}(A_{\mathbb{R}})$ comes from an indecomposable M_I in $\operatorname{rep}_k(A_{\mathbb{R}})$ (Definition 1.1.3, Theorem 1.1.6, Proposition 1.2.1, and [16, Proposition 3.1.3]). Recall also that $|a, b|$ means the inclusion of a or b is either indeterminate or clear from context (Notation 1.1.4 and Theorem 1.1.5). Note that $\mathcal{P}roj \cap \mathcal{I}nj = \{M_{(-\infty,+\infty)}\}$.

We will construct two continuous $\mathbf{E}$-mutations to mutate $\mathcal{P}roj$ to $\mathcal{I}nj$. First, let $S_1 = \mathcal{P}roj$ and define $f_1 : \mathcal{P}roj \to [0, 1]$ in two parts. For $M_{(-\infty,x)} \in \mathcal{P}roj$ we let

$$f_1\left(M_{(-\infty,x)}\right) = \frac{1}{2} - \left(\frac{\tan^{-1} x}{2\pi} + \frac{1}{4}\right).$$

For $M_{(-\infty,x]} \in \mathcal{P}roj$ we let

$$f_1\left(M_{(-\infty,x]}\right) = 1 - \left(\frac{\tan^{-1} x}{2\pi} + \frac{3}{4}\right).$$

The "middle" $\mathbf{E}$-cluster is

$$T_2 := \left\{M_{(-\infty,+\infty)}\right\} \cup \left\{M_{[x,+\infty)}, M_{\{x\}} : x \in \mathbb{R}\right\}.$$

We then define $g_1 : T_2 \to [0, 1]$ to match with f_1:

$$g_1\left(M_{\{x\}}\right) = \frac{1}{2} - \left(\frac{\tan^{-1} x}{2\pi} + \frac{1}{4}\right)$$
$$g_1\left(M_{[x,+\infty)}\right) = 1 - \left(\frac{\tan^{-1} x}{2\pi} + \frac{3}{4}\right)$$

Both f_1 and g_1 are injections and we may define $\mu_1(M) = g^{-1}(f(M))$ and obtain the continuous $\mathbf{E}$-mutation $\mu_1 : \mathcal{P}roj \to T_2$.

Now let $S_2 = \{M_{\{x\}} : x \in \mathbb{R}\} \subset T_2$ and $S_2' = \{M_{(x,+\infty)} : x \in \mathbb{R}\} \subset \mathcal{I}nj$. We define $f_2 : T_2 \to [0, 1]$ and $g_2 : \mathcal{I}nj \to [0, 1]$ by

$$f_2\left(M_{\{x\}}\right) = \frac{\tan^{-1} x}{\pi} + \frac{1}{2} = g_2\left(M_{(x,+\infty)}\right).$$

We define $\mu_2(M)$ to be M if $M \notin S_2$ and $g^{-1}(f(M))$ if $M \in S_2$. This gives the continuous $\mathbf{E}$-mutation $\mu_2 : T_2 \to \mathcal{I}nj$. Thus we have a sequence of continuous $\mathbf{E}$-mutations $\{\mu_1, \mu_2\}$ to mutate the projectives into the injectives. □

2.3. **Mutation Paths.** In this subsection we define mutation paths, which should be thought of as a generalization of a sequence of mutations. At first we formally define a long sequence of continuous mutations (Definition 2.3.1) and then move on to mutation paths in general (Definition 2.3.2). Note also that a continuous mutation is an example of a mutation path (Example 2.3.5) just as a mutation is an example of a continuous mutation.

A mutation path should be thought of as a generalization of a path of mutations in the exchange graph of a cluster structure. This is formalized in Section 2.4. As before our definitions are for any cluster theory but our interest is in **E**-cluster theories of $A_{\mathbb{R}}$ quivers.

Definition 2.3.1. Let $\mathcal{C}$ be a Krull-Schmidt category and $\mathbf{P}$ a pairwise compatibility condition on its indecomposables such that $\mathbf{P}$ induces the $\mathbf{P}$-cluster theory of $\mathcal{C}$. Let

$$\overline{\mu} = \{{}_i\mu : {}_iT_0 \to {}_iT_1\}_{i\in\mathbb{Z}}$$

be a collection of continuous mutations such that ${}_iT_1 = {}_{i+1}T_0$. This yields a diagram in $\mathscr{T}_{\mathbf{P}}(\mathcal{C})$:

$$\cdots \xrightarrow{{}_{i-1}\mu} {}_{i-1}T_1 = {}_iT_0 \xrightarrow{{}_i\mu} {}_iT_1 = {}_{i+1}T_0 \xrightarrow{{}_{i+1}\mu} {}_{i+1}T_1 = {}_{i+2}T_0 \xrightarrow{{}_{i+2}\mu} \cdots$$

If this diagram has a limit and colimit we call $\overline{\mu}$ a long sequence of continuous mutations and we call the limit and colimit the source and target of $\overline{\mu}$, respectively.

Definition 2.3.2. Define a category $\mathcal{I}$ whose objects are pairs $(x, i) \in [0, 1] \times \{0, 1\}$. Consider $[0, 1]$ and $\{0, 1\}$ with their respective usual total ordering. Morphisms in $\mathcal{I}$ are defined by

$$\mathrm{Hom}_{\mathcal{I}}\left((s, i), (t, j)\right) := \begin{cases} \{*\} & s < t \text{ or } (s = t \text{ and } i \leq j) \\ \emptyset & \text{otherwise.} \end{cases}$$

Let $\mathcal{C}$ be a Krull-Schmidt category and $\mathbf{P}$ a pairwise compatibility condition on its indecomposables such that $\mathbf{P}$ induces the $\mathbf{P}$-cluster theory of $\mathcal{C}$. Let $\overline{\mu} : \mathcal{I} \to \mathcal{S}ets$ be a functor such that $\overline{\mu}* : \overline{\mu}(s, 0) \to \overline{\mu}(s, 1)$ is a (possibly trivial) $\mathbf{P}$-mutation in $\mathscr{T}_{\mathbf{P}}(\mathcal{C})$. Then we call $\overline{\mu}$ a $\mathbf{P}$-mutation path.

Remark 2.3.3. The reader may notice that the target of the functor is not $\mathscr{T}_{\mathbf{P}}(\mathcal{C})$, but just $\mathcal{S}ets$. This is because we have not defined $\mathscr{T}_{\mathbf{P}}(\mathcal{C})$ (in Definition 1.3.7) to be closed under any kind of transfinite composition. However, transfinite composition is indeed sometimes defined in $\mathcal{S}ets$. For example, if every set in a diagram has the same cardinality and every morphism is a bijection, the transfinite composition is well-defined (and in this case will also be a bijection). We only ensure the smallest morphisms $(s, 0) \to (s, 1)$ are in $\mathscr{T}_{\mathbf{P}}(\mathcal{C})$.

Proposition 2.3.4. *Let $\mathcal{C}$ be a Krull-Schmidt category and $\mathbf{P}$ a pairwise compatibility condition on its indecomposables such that $\mathbf{P}$ induces the $\mathbf{P}$-cluster theory of $\mathcal{C}$. Let $\overline{\mu} : \mathcal{I} \to \mathcal{S}et$ be $\mathbf{P}$-mutation path.*

Let $\overline{\mu}^{-1} : \mathcal{I} \to \mathcal{S}et$ be a functor given by

$$\overline{\mu}^{-1}(s, i) := \overline{\mu}(1 - s, 1 - i)$$
$$\overline{\mu}^{-1}((s_i) \to (t, j)) := \overline{\mu}((1 - t, 1 - j) \to (1 - s, 1 - i)).$$

Then $\overline{\mu}^{-1}$ is also a $\mathbf{P}$-mutation path.

Proof. Since $\mathscr{T}_{\mathbf{P}}(\mathcal{C})$ is a groupoid inside $\mathcal{S}et$ the definition of $\overline{\mu}^{-1}$ amounts to reversing the order of the objects and taking the inverse morphism between each pair of objects in the image. $\square$

Example 2.3.5. Let $\mathcal{C}$ be a Krull-Schmidt category and $\mathbf{P}$ a pairwise compatibility condition on its indecomposables such that $\mathbf{P}$ induces the $\mathbf{P}$-cluster theory of $\mathcal{C}$. Let $\mu : T \to T'$ be a continuous $\mathbf{P}$-mutation. Let $\bar{\mu} : \mathcal{I} \to \mathcal{S}et$ be defined in the following way. On objects,

$$\overline{\mu}(s, 0) = (T \setminus f^{-1}[0, s)) \cup g^{-1}[0, s)$$
$$\overline{\mu}(s, 1) = (T \setminus f^{-1}[0, s]) \cup g^{-1}[0, s].$$

By Proposition 2.1.4, for each $s \in [0, 1]$, μ defines a $\mathbf{P}$-mutation $\overline{\mu}(s, 0) \to \overline{\mu}(s, 1)$. Define $\overline{\mu} : \overline{\mu}(s, 0) \to \overline{\mu}(s, 1)$ to be precisely this $\mathbf{P}$-mutation. Thus each continuous $\mathbf{P}$-mutation is a $\mathbf{P}$-mutation path.

Below we construct some variables i_s, a_s, b_s, and t_s for each $s \in [0,1]$. We will use these to show how a long sequence of continuous mutations can be considered as a mutation path.

Construction 2.3.6. Let $\mathcal{C}$ be a Krull-Schmidt category and $\mathbf{P}$ a pairwise compatibility condition on its indecomposables such that $\mathbf{P}$ induces the $\mathbf{P}$-cluster theory of $\mathcal{C}$. Let $\overline{\mu}$ be a long sequence of continuous mutations and fix $0 < \varepsilon << 1$. For each $s \in (0,1)$, there exists $i \in \mathbb{Z}$ such that

$$\left(\frac{\tan^{-1} i}{\pi} + \frac{1}{2}\right) \leq s < \left(\frac{\tan^{-1}(i+1)}{\pi} + \frac{1}{2}\right).$$

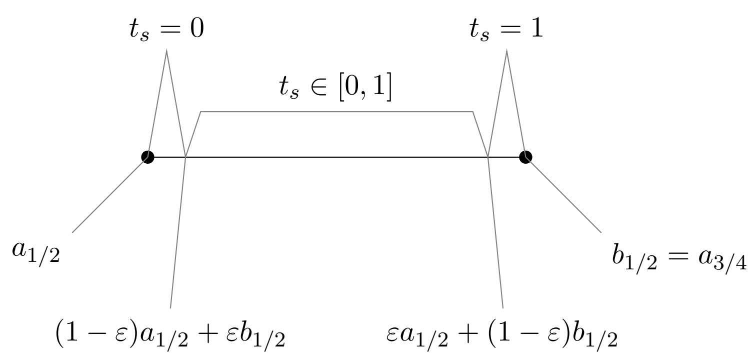
Note that since the right inequality is strict, there is a unique such i for each $s \in (0,1)$. Denote it by i_s. Let

$$\begin{aligned} a_s &:= \left(\frac{\tan^{-1} i_s}{\pi} + \frac{1}{2}\right) \\ b_s &:= \left(\frac{\tan^{-1}(i_s+1)}{\pi} + \frac{1}{2}\right). \end{aligned}$$

Note that if $i_s = i_{s'}$ for s and s' then $a_s = a_{s'}$ and $b_s = b_{s'}$. We now define t_s:

$$t_s := \begin{cases} 0 & s \in [a_s,\ (1-\varepsilon)a_s + \varepsilon b_s] \\ (s - (1-\varepsilon)a_s - \varepsilon b_s)/((1-2\varepsilon)(b_s - a_s)) & s \in [(1-\varepsilon)a_s + \varepsilon b_s,\ \varepsilon a_s + (1-\varepsilon)b_s] \\ 1 & s \in (\varepsilon a_s + (1-\varepsilon)b_s,\ b_s). \end{cases}$$

We provide a picture to make the variables easier to understand for $s \in [\frac{1}{2}, \frac{3}{4}]$:



Example 2.3.7. Let $\mathcal{C}$ be a Krull-Schmidt category and $\mathbf{P}$ a pairwise compatibility condition on its indecomposables such that $\mathbf{P}$ induces the $\mathbf{P}$-cluster theory of $\mathcal{C}$. Let $\overline{\mu}$ be a long sequence of continuous mutations. We may also consider $\overline{\mu}$ as a functor $\mathcal{I} \to \mathcal{S}et$ in the following way.

We now make our assignment on objects:

$$(s,0) \mapsto \begin{cases} {}_iT_0 = {}_{i-1}T_1 & s \in [a_s, (1-\varepsilon)a_s + \varepsilon b_s) \\ ({}_{i_s}T \setminus {}_if^{-1}[0,t_s)) \cup {}_ig^{-1}[0,t_s) & s \in [(1-\varepsilon)a_s + \varepsilon b_s, \varepsilon a_s + (1-\varepsilon)b_s] \\ {}_iT_1 = {}_{i+1}T_0 & s \in (\varepsilon a_s + (1-\varepsilon)b_s, b_s) \end{cases}$$

$$(s,1) \mapsto \begin{cases} {}_iT_0 = {}_{i-1}T_1 & s \in [a_s, (1-\varepsilon)a_s + \varepsilon b_s) \\ ({}_{i_s}T \setminus {}_if^{-1}[0,t_s]) \cup {}_ig^{-1}[0,t_s] & s \in [(1-\varepsilon)a_s + \varepsilon b_s, \varepsilon a_s + (1-\varepsilon)b_s] \\ {}_iT_1 = {}_{i+1}T_0 & s \in (\varepsilon a_s + (1-\varepsilon)b_s, b_s). \end{cases}$$

When $s \in [(1-\varepsilon)a_s + \varepsilon b_s, \varepsilon a_s + (1-\varepsilon)b_s]$ we see by Proposition 2.1.4 that the morphism $* : (s,0) \to (s,1)$ is sent to a (possibly trivial) $\mathbf{P}$-mutation. When $s \in [a_s, (1-\varepsilon)a_s + \varepsilon b_s) \cup (\varepsilon a_s + (1-\varepsilon)b_s, b_s)$ the morphism $* : (s_0) \to (s_1)$ is sent to the trivial $\mathbf{P}$-mutation on $\bar{\mu}(s,0)$. This defines a mutation path. The ε "padding" in Construction 2.3.6 is necessary or else we'd attempt to assign two $\mathbf{P}$-mutations, or their composition, to morphisms such as $* : (\frac{3}{4}, 0) \to (\frac{3}{4}, 1)$.

Remark 2.3.8. Let $\mathcal{C}$ be a Krull-Schmidt category and $\mathbf{P}$ a pairwise compatibility condition on its indecomposables such that $\mathbf{P}$ induces the $\mathbf{P}$-cluster theory of $\mathcal{C}$. Let $\overline{\mu}$ be a long sequence of continuous $\mathbf{P}$-mutations. We see in Example 2.3.7 that for a fixed ε the the inverse *path* $\overline{\mu}^{-1}$ agrees with the inverse *sequence* $\{{}_{-i}\mu\}_{i\in\mathbb{Z}}$. Thus when working with a long sequence of continuous mutations we need not be specific about which inverse we take as long as an ε has been chosen.

Definition 2.3.9. Let $\mathcal{C}$ be a Krull-Schmidt category and $\mathbf{P}$ a pairwise compatibility condition on its indecomposables such that $\mathbf{P}$ induces the $\mathbf{P}$-cluster theory of $\mathcal{C}$. Let $\overline{\mu}_1, \overline{\mu}_2 : \mathcal{I} \to \mathcal{S}et$ be two $\mathbf{P}$-mutation paths and suppose $\overline{\mu}_1(1,0) = \overline{\mu}_2(0,0)$ and $\overline{\mu}_1(1,1) = \overline{\mu}_2(0,1)$.

We define the composition of $\mathbf{P}$-mutation paths, denoted $\overline{\mu}_1 \cdot \overline{\mu}_2$ in the following way:

$$\overline{\mu}_1 \cdot \overline{\mu}_2(s,i) := \begin{cases} \overline{\mu}_1(2s,i) & 0 \le s \le \frac{1}{2} \\ \overline{\mu}_2(2s-1,i) & \frac{1}{2} \le s \le 1. \end{cases}$$

$$\overline{\mu}_1 \cdot \overline{\mu}_2((s,0) \to (s,1)) := \begin{cases} \overline{\mu}_1((2s,0) \to (2s,1)) & 0 \le s \le \frac{1}{2} \\ \overline{\mu}_2((2s-1,0) \to (2s-1,1)) & \frac{1}{2} \le s \le 1. \end{cases}$$

Proposition 2.3.10. *Let $\overline{\mu}_1$ and $\overline{\mu}_2$ be $\mathbf{P}$-mutation paths such that*

$$\overline{\mu}_1(1,0) = \overline{\mu}_2(0,0) \qquad \text{and} \qquad \overline{\mu}_1(1,1) = \overline{\mu}_2(0,1).$$

Then $\overline{\mu}_1 \cdot \overline{\mu}_2$ is a $\mathbf{P}$-mutation path.

Proof. By assumption the definitions agree at $\frac{1}{2}$. For $0 \le s < \frac{1}{2}$ and $\frac{1}{2} < t \le 1$, the morphism $\overline{\mu}_1 \cdot \overline{\mu}_2 * : \overline{\mu}_1 \cdot \overline{\mu}_2(s,i) \to \overline{\mu}_1 \cdot \overline{\mu}_2(t,j)$ is the composition

$$\overline{\mu}_1 \cdot \overline{\mu}_2(s,i) \to \overline{\mu}_1 \cdot \overline{\mu}_2\left(\frac{1}{2},0\right) \to \overline{\mu}_1 \cdot \overline{\mu}_2\left(\frac{1}{2},1\right) \to \overline{\mu}_1 \cdot \overline{\mu}_2(t,j). \qquad \square$$

.

Remark 2.3.11. The composition of two long sequences of continuous mutations as in Definition 2.3.9 will not be a long sequence of continuous mutations as in Example 2.3.7.

Notation 2.3.12. We use $\overline{\mu}$ for both sequences of continuous mutations and mutation paths. In the case that the mutation path of study happens to be a long sequence of continuous mutations we will explicitly define $\overline{\mu}$ to be a long sequence of continuous mutations. If it is unknown whether or not $\overline{\mu}$ can be realized as a long sequence of continuous mutations, or it is clear that this is not possible, we will define $\overline{\mu}$ to be a mutation path.

2.4. **Space of Mutations.** In this subsection we define the space of mutations (Definition 2.4.2) which generalizes the exchange graph of a cluster structure. The intent is to view mutation paths (Definition 2.3.2) as paths in a topological space just as a sequence of mutations of a cluster structure forms a path in the exchange graph. This majority of this subsection is for cluster theories in general. However, its purpose is to study $\mathbf{E}$-clusters in the future and so we return our attention to $\mathbf{E}$-clusters at the end of the subsection.

Definition 2.4.1. Let $\mathcal{C}$ be a Krull-Schmidt category and $\mathbf{P}$ a pairwise compatibility condition on its indecomposables such that $\mathbf{P}$ induces the $\mathbf{P}$-cluster theory of $\mathcal{C}$. Suppose the morphisms of $\mathscr{T}_{\mathbf{P}}(\mathcal{C})$, denoted $(\mathscr{T}_{\mathbf{P}}(\mathcal{C}))_1$, form a set. Then any $\mathbf{P}$-mutation path $\overline{\mu}$ induces a function $p_{\overline{\mu}} : [0,1] \to (\mathscr{T}_{\mathbf{P}}(\mathcal{C}))_1$.

Definition 2.4.2. Let $\mathcal{C}$ be a Krull-Schmidt category and $\mathbf{P}$ a pairwise compatibility condition on $\mathrm{Ind}(\mathcal{C})$ such that $\mathbf{P}$ induces the $\mathbf{P}$-cluster theory of $\mathcal{C}$. Assume the morphisms of $\mathscr{T}_{\mathbf{P}}(\mathcal{C})$, denoted $(\mathscr{T}_{\mathbf{P}}(\mathcal{C}))_1$, form a set. We define the set $\mathbf{P}(\mathcal{C}) \subset (\mathscr{T}_{\mathbf{P}}(\mathcal{C}))_1$ to be the set containing all (trivial) $\mathbf{P}$-mutations.

We give the set of $\mathbf{P}$-mutations a topology in the following way. Consider $[0,1]$ with the usual topology. A set $U \subset \mathbf{P}(\mathcal{C})$ is called open if, for all $p_{\overline{\mu}} : [0,1] \to \mathbf{P}(\mathcal{C})$ induced by a $\mathbf{P}$-mutation $\overline{\mu}$, $p_{\overline{\mu}}^{-1}(U)$ is open in $[0,1]$. We call $\mathbf{P}(\mathcal{C})$ the space of $\mathbf{P}$-mutations in $\mathcal{C}$.

Proposition 2.4.3. *Let $\mathcal{C}$ be a Krull-Schmidt category and $\mathbf{P}$ a pairwise compatibility condition on Ind($\mathcal{C}$) such that $\mathbf{P}$ induces the $\mathbf{P}$-cluster theory of $\mathcal{C}$ such that $\mathbf{P}(\mathcal{C})$ is a set. Then the open sets in Definition 2.4.2 form a topology on $\mathbf{P}(\mathcal{C})$.*

Proof. Trivially, both $\emptyset$ and $\mathbf{P}(\mathcal{C})$ are open. Suppose $p_{\overline{\mu}} : [0,1] \to \mathbf{P}(\mathcal{C})$ is induced by a $\mathbf{P}$-mutation path $\overline{\mu}$.

Let $\{U_1, \dots, U_n\}$ be open in $\mathbf{P}(\mathcal{C})$. Since

$$\bigcap_{i=1}^{n} p_{\overline{\mu}}^{-1}(U_i) = p_{\overline{\mu}}^{-1}\left(\bigcap_{i=1}^{n} U_i\right)$$

we see that $\bigcap_{i=1}^{n} U_i$ is open in $\mathbf{P}(\mathcal{C})$.

Now consider a collection $\{U_\alpha\}$ of open sets in $\mathbf{P}(\mathcal{C})$. Since

$$\bigcup_{\alpha} p_{\overline{\mu}}^{-1}(U_\alpha) = p_{\overline{\mu}}^{-1}\left(\bigcup_{\alpha} U_\alpha\right)$$

we see that $\bigcup_\alpha U_\alpha$ is open in $\mathbf{P}(\mathcal{C})$. This concludes the proof. □

Remark 2.4.4. Let $\mathcal{C}$ be a Krull-Schmidt category and $\mathbf{P}$ a pairwise compatibility condition on Ind($\mathcal{C}$) such that $\mathbf{P}$ induces the $\mathbf{P}$-cluster theory of $\mathcal{C}$ such that $\mathbf{P}(\mathcal{C})$ is a set. We consider a $\mathbf{P}$-cluster T to be the trivial mutation $T \to T$ in $\mathbf{P}(\mathcal{C})$. We wish to consider paths that start and end at clusters rather than starting or ending at mutations (see Proposition 2.4.6).

Proposition 2.4.5. *Let $\mathcal{C}$ be a Krull-Schmidt category and $\mathbf{P}$ a pairwise compatibility condition on Ind($\mathcal{C}$) such that $\mathbf{P}$ induces the $\mathbf{P}$-cluster theory of $\mathcal{C}$ such that $\mathbf{P}(\mathcal{C})$ is a set. Then $\mathbf{P}(\mathcal{C})$ is not Hausdorff.*

Proof. Let $\mu : T \to (T \setminus \{X\}) \cup \{Y\}$ be a $\mathbf{P}$-muation. Let $\overline{\mu}$ be the $\mathbf{P}$-mutation path that induces the path $p_{\overline{\mu}}$ given by

$$p_{\overline{\mu}}(t) = \begin{cases} T & t < 1 \\ \mu & t = 1 \end{cases}$$

Let U be an open set that contains μ. If $T \notin U$ then $p_{\overline{\mu}}^{-1}(U)$ is not open. This would be a contradiction and so $T \in U$. Thus, for any $\mathbf{P}$-mutation $\mu : T \to T'$ and open set U containing μ, $T, T' \in U$ as well. Therefore, $\mathbf{P}(\mathcal{C})$ is not Hausdorff. □

Proposition 2.4.6. *Let $\mathcal{C}$ be a Krull-Schmidt category and $\mathbf{P}$ a pairwise compatibility condition on Ind($\mathcal{C}$) such that $\mathbf{P}$ induces the $\mathbf{P}$-cluster theory of $\mathcal{C}$ such that $\mathbf{P}(\mathcal{C})$ is a set. Let $p : [0,1] \to \mathbf{P}(\mathcal{C})$ be a path in $\mathbf{P}(\mathcal{C})$. Then there is a path $q : [0,1] \to \mathbf{P}(\mathcal{C})$ whose endpoints are clusters (see Remark 2.4.4) such that p and q are homotopic.*

Proof. Let $p : [0,1] \to \mathbf{P}(\mathcal{C})$ be a path in $\mathbf{P}(\mathcal{C})$, let T_0 be the source of $p(0)$, and let T_1 the target of $p(1)$.

For any $0 < \varepsilon << \frac{1}{2}$, let $q_\varepsilon : [0,1] \to \mathbf{P}(\mathcal{C})$ be the path given by:

$$q_\varepsilon(t) = \begin{cases} T_0 & \text{if } t < \varepsilon \\ T_1 & \text{if } (1-\varepsilon) < t \\ p\left((t - \frac{1}{2})(1-2\varepsilon) + \frac{1}{2}\right) & \text{if } \varepsilon \le t \le (1-\varepsilon) \end{cases}$$

We see that q_ε is homotopic to the composition of three paths. The first is constant at T_0 except the last point is $p(0)$. The second is p. The third is constant at T_1 except the first point is $p(1)$. In particular, the first and third path are induced by $\mathbf{P}$-mutation paths. Thus, q_ε is indeed a path. We just say $q_0 = p$.

Fix a $0 < \varepsilon << \frac{1}{2}$. Let $H : [0,1] \times [0,1] \to \mathbf{P}(\mathcal{C})$ be given by:

$$H(t, s) := q_{s\varepsilon}(t).$$

Let U be open in $\mathbf{P}(\mathcal{C})$. If the inverse image of U does not contain $p(0)$ or $p(1)$ then $H^{-1}(U)$ is open in $[0,1]\times[0,1]$.

Now suppose U contains $p(0)$. By the proof of Proposition 2.4.5 we see that $T_0 \in U$ as well. Similarly, if $p(1) \in U$ then $T_1 \in U$. Therefore, if U is open in $\mathbf{P}(\mathcal{C})$ then $H^{-1}(U)$ is open in $[0,1]\times[0,1]$, completing the proof. $\square$

Definition 2.4.7. Let $\mathcal{C}$ be a Krull-Schmidt category and $\mathbf{P}$ a pairwise compatibility condition on $\mathrm{Ind}(\mathcal{C})$ such that $\mathbf{P}$ induces the $\mathbf{P}$-cluster theory of $\mathcal{C}$ such that $\mathbf{P}(\mathcal{C})$ is a set. Let T_1 and T_2 be $\mathbf{P}$-clusters of $\mathcal{C}$.

(1) We say T_2 is <u>reachable</u> from T_1 if there is a path $p:[0,1]\to\mathbf{P}(\mathcal{C})$ such that $p(0)=T_1$ and $p(1)=T_2$.
(2) We say T_2 is <u>strongly reachable</u> from T_1 if there is a $\mathbf{P}$-mutation path $\overline{\mu}$ that (i) comes from a long sequence of continuous $\mathbf{P}$-mutations and (ii) indues a path $p_{\overline{\mu}}:[0,1]\to\mathbf{P}(\mathcal{C})$ such that $p_{\overline{\mu}}(0)=T_1$ and $p_{\overline{\mu}}(1)=T_2$.

Theorem 2.4.8. *Consider the $\mathbf{E}$-cluster theory of $\mathcal{C}(A_\mathbb{R})$ where $A_\mathbb{R}$ has the straight descending orientation. The cluster of injectives, $\mathcal{I}nj$ is strongly reachable from the cluster of projectives, $\mathcal{P}roj$.*

Proof. In Proposition 2.2.7 we see there is a sequence of $\mathbf{E}$-mutations $\{\mu_1,\mu_2\}$ to mutate $\mathcal{P}roj$ to $\mathcal{I}nj$. Choose some $0<\varepsilon<<\frac{1}{2}$ and note that a sequence of $\mathbf{E}$-mutations is also a long sequence of $\mathbf{E}$-mutations. Then, as in Example 2.3.7, we have a $\mathbf{E}$-mutation path $\overline{\mu}$ with source $\mathcal{P}roj$ and target $\mathcal{I}nj$. $\square$

Open Questions. Let $A_\mathbb{R}$ be a continuous quiver of type A and $\mathscr{T}_{\mathbf{E}}(\mathcal{C}(A_\mathbb{R}))$ the $\mathbf{E}$-cluster theory of $\mathcal{C}$ (Definition 1.3.4 and Example 1.3.9). Since the objects of $\mathscr{T}_{\mathbf{E}}(\mathcal{C}(A_\mathbb{R}))\subset ets$ form a set so do the morphisms. Thus, we may ask the following questions.

- Is the space $\mathbf{E}(\mathcal{C}(A_\mathbb{R}))$ path connected?
- If $\mathbf{E}(\mathcal{C}(A_\mathbb{R}))$ is not path-connected, what do its path components look like? What does the path component containing the cluster of projectives look like?
- If $\mathbf{E}(\mathcal{C}(A_\mathbb{R}))$ is path connected, which clusters are strongly reachable from the cluster of projectives?

3. Composable Embeddings of Cluster Theories

In this section we further explore how the cluster theories of type A are related, introduce the abstract notion of cluster structures, and show how some of the type A cluster structures are related. Recall that in all of our embeddings of cluster theories defined thus far, $A_\mathbb{R}$ has the straight descending orientation. In Section 3.1 we define the embedding $\mathscr{T}_{\mathbf{N}_{\overline{\infty}}}(\mathcal{C}(A_{\overline{\infty}}))\to\mathscr{T}_{\mathbf{E}}(\mathcal{C}(A_\mathbb{R}))$.

In this Section 3.2 we will show that all the embeddings of cluster theories described in part (III) of this series and in Section 3.1 can be decomposed into a sequence of embeddings of cluster theories (Theorem 3.2.13). I.e., for any $0<m<n\in\mathbb{Z}$ there is a commutative diagram

$$\begin{tikzcd}
\mathscr{T}_{\mathbf{N}_m}(\mathcal{C}(A_m)) \arrow[r] \arrow[drr, "\text{1.3.16(i)}"'] & \mathscr{T}_{\mathbf{N}_n}(\mathcal{C}(A_n)) \arrow[r] \arrow[dr, "\text{1.3.16(i)}"] & \mathscr{T}_{\mathbf{N}_\infty}(\mathcal{C}(A_\infty)) \arrow[r] \arrow[d, "\text{1.3.16(ii)}"] & \mathscr{T}_{\mathbf{N}_{\overline{\infty}}}(\mathcal{C}(A_{\overline{\infty}})) \arrow[r] \arrow[dl, "\text{3.1.7}"] & \mathscr{T}_{\mathbf{N}_\mathbb{R}}(\mathcal{C}_\pi) \arrow[dll, "\text{1.3.16(iii)}"] \\
 & & \mathscr{T}_{\mathbf{E}}(\mathcal{C}(A_\mathbb{R})) & &
\end{tikzcd}$$

where the arrows to $\mathscr{T}_{\mathbf{E}}(\mathcal{C}(A_\mathbb{R}))$ are the embeddings from part (III) of this series and Section 3.1.

In Section 3.3 we define the abstract notion of cluster structures (Definition 3.3.1). We also show that we have a commutative diagram of embeddings of cluster theories, most of which restricts to an embedding of cluster structures (Theorem 3.3.16):

$$\begin{tikzcd}
& \mathscr{T}_{\mathbf{N}_\infty}(\mathcal{C}(A_\infty)) \arrow[dd] & \\
\mathscr{T}_{\mathbf{N}_m}(\mathcal{C}(A_m)) \arrow[ur] \arrow[rr] \arrow[dr] & & \mathscr{T}_{\mathbf{N}_n}(\mathcal{C}(A_n)) \arrow[ul] \arrow[dl] \\
& \mathscr{T}_{\mathbf{N}_{\mathbb{R}}}(\mathcal{C}_\pi) &
\end{tikzcd}$$

$$\begin{tikzcd}
& \mathscr{S}_{\mathbf{N}_\infty}(\mathcal{C}(A_\infty)) & \\
\mathscr{S}_{\mathbf{N}_m}(\mathcal{C}(A_m)) \arrow[ur] \arrow[rr] \arrow[dr] & & \mathscr{S}_{\mathbf{N}_n}(\mathcal{C}(A_n)) \arrow[ul] \arrow[dl] \\
& \mathscr{S}_{\mathbf{N}_{\mathbb{R}}}(\mathcal{C}_\pi) &
\end{tikzcd}$$

3.1. The embedding $\mathscr{T}_{\mathbf{N}_{\overline{\infty}}}(\mathcal{C}(A_{\overline{\infty}})) \to \mathscr{T}_{\mathbf{E}}(\mathcal{C}(A_{\mathbb{R}}))$.

In this section we define the embedding $\mathscr{T}_{\mathbf{N}_{\overline{\infty}}}(\mathcal{C}(A_{\overline{\infty}})) \to \mathscr{T}_{\mathbf{E}}(\mathcal{C}(A_{\mathbb{R}}))$, where $\mathscr{T}_{\mathbf{N}_{\overline{\infty}}}(\mathcal{C}(A_{\overline{\infty}}))$ is the cluster theory of the completed infinity-gon (Definitions 2.2.2, 2.2.3, and 2.2.4). We take $A_{\mathbb{R}}$ to have the straight descending orientation and $\mathscr{T}_{\mathbf{E}}(\mathcal{C}(A_{\mathbb{R}}))$ to be the $\mathbf{E}$-cluster theory of $\mathcal{C}(A_{\mathbb{R}})$.

Recall the $\mathbf{N}_\infty$-cluster theory where clusters are triangulated of the *uncompleted* infinity-gon (Theorem 1.3.16(ii)). Recall T_∞ from Definition 1.3.11 along with the sequence $\{a_i\}_{i\in\mathbb{Z}}$ converging to $a_{-\infty}$ below and $a_{+\infty}$ above, both in $\mathbb{Z}$. Recall also from Definition 1.3.11 the values

$$a_{i,j,\ell} = a_i + \left(\frac{j}{2^\ell}\right)(a_{i+1} - a_i)$$

where $i, j, \ell \in \mathbb{Z}$, $l \geq 0$, and $0 \leq j \leq 2^\ell$.

Definition 3.1.1. For a diagonal $\theta = i \frown j$ in the completed infinity-gon, we denote by M_θ the indecomposable representation $M_{(a_i,a_j)}$ embedded in $\mathcal{C}(A_{\mathbb{R}})$ corresponding to the interval $(a_i, a_j) \subset \mathbb{R}$. The generic diagonal in [4] corresponds to $M_{(a_{-\infty},a_{+\infty})}$.

Proposition 3.1.2. *Let θ and θ' be diagonals in $\mathcal{C}(A_{\overline{\infty}})$. Then $\{\theta, \theta'\}$ is $\mathbf{N}_{\overline{\infty}}$-compatible if and only if $\{M_\theta, M_{\theta'}\}$ is $\mathbf{E}$-compatible.*

Proof. Let $i \frown j = \theta$ and $i' \frown j' = \theta'$, where one or both of i and i' may be $-\infty$ and one or both of j and j' may be $+\infty$. Recall $\{\theta, \theta'\}$ is not $\mathbf{N}_{\overline{\infty}}$-compatible if and only if $i < i' < j < j'$ or $i' < i < j' < j$. Recall also that if $i < i' < j < j'$ or $i' < i < j' < j$ then $\{M_{(a_i,a_j)}, M_{(a_{i'},a_{j'})}\}$ is not $\mathbf{E}$-compatible. This concludes the proof. □

While the generic diagonal $-\infty \frown +\infty$ isn't technically in any $\mathbf{N}_{\overline{\infty}}$-cluster, it plays the role of $P_{+\infty}$ in a $\mathbf{E}$-cluster since $A_{\mathbb{R}}$ has the straight descending orientation. Recall T_∞ from Definition 1.3.11 and note that $P_{+\infty} \in T_\infty$.

Definition 3.1.3. Let $T_{\mathbf{N}_{\overline{\infty}}}$ be an $\mathbf{N}_{\overline{\infty}}$-cluster in $\mathscr{T}_{\mathbf{N}_{\overline{\infty}}}(\mathcal{C}(A_{\overline{\infty}}))$. Define $T_{\mathbf{E}_{\overline{\infty}}}$ to be

$$T_{\mathbf{E}_{\overline{\infty}}} := T_\infty \cup \{M_\theta : \theta \in T_{\mathbf{N}_{\overline{\infty}}}\}.$$

Remark 3.1.4. Using Proposition 3.1.2 it is straightforward to check that given an $\mathbf{N}_{\overline{\infty}}$-cluster $T_{\mathbf{N}_{\overline{\infty}}}$ the set $T_{\mathbf{E}_{\overline{\infty}}}$ is $\mathbf{E}$-compatible.

Proposition 3.1.5. *Let $T_{\mathbf{N}_{\overline{\infty}}}$ be an $\mathbf{N}_{\overline{\infty}}$-cluster. Then $T_{\mathbf{E}_{\overline{\infty}}}$ is an $\mathbf{E}$-cluster.*

Proof. We've noted in Remark 3.1.4 that $T_{\mathbf{E}_{\overline{\infty}}}$ is $\mathbf{E}$-compatible. It remains to show that it is maximally so. Let $M_{|c,d|}$ be an indecomposable in $\mathcal{C}(A_{\mathbb{R}})$ corresponding to the interval $|c,d| \subset \mathbb{R}$. Recall the |s in our notation mean we not assuming whether or not c or d is in the interval $|c,d|$ (Notation 1.1.4). Assume $\{M_{|c,d|}\} \cup T_{\mathbf{E}_{\overline{\infty}}}$ is $\mathbf{E}$-compatible. We will check the various possibilities for c and show that $M_{|c,d|} \in T_{\mathbf{E}_{\overline{\infty}}}$. Using the same technique as [16, Proposition 5.2.5], if $c < a_{-\infty}$ or $c \geq a_{+\infty}$ then $M_{|c,d|} \in T_{\mathbf{E}_{\overline{\infty}}}$.

Suppose $a_{-\infty} < c < a_{+\infty}$ but $c \neq a_i$ for all $i \in \mathbb{Z}$. Then either c is of the form $a_{i,j,\ell}$ for $i \in \mathbb{Z}$, $\ell \geq 0$, and $0j < 2^{\ell}$ or not. If not, then $d = c$ otherwise there exists such a $a_{i,j,\ell}$ between c and d. Then $M_{|c,d|} = M_{\{c\}}$. If c is of the form $a_{i,j,\ell}$ then $c \notin |c,d|$ since there is a $M_{(a,c)} \in T_{\infty} \subset T_{\mathbf{E}_{\overline{\infty}}}$. Furthere, there exists j' and ℓ' such that $\frac{j}{2^{\ell}} = \frac{j'}{2^{\ell'}}$ and $d = a_{i,j'+1,\ell'}$. If d was not of this form $\{M_{|c,d|}\} \cup T_{\infty}$ would not be $\mathbf{E}$-compatible.

If $c \in a_i$ for some $i \in \mathbb{Z}$ then $c \notin |c,d|$ again. Suppose $c < d < a_{i+1}$. Then we apply the same argument as the previous paragraph to $\{M_{|c,d|}\} \cup T_c$, where T_c is from Definition 1.3.11. Now suppose $d \geq a_{i+1}$. Then $d = a_j$ for $j \in \mathbb{Z}$ and $d \notin |c,d|$ since otherwise $\{M_{|c,d|}\} \cup T_{\infty}$ would not be $\mathbf{E}$-compatible. Suppose $d \geq a_{i+1}$ and consider the diagonal $\theta = (c,d)$ such that $\{\theta\} \cup T_{\mathbf{N}_{\overline{\infty}}}$ is $\mathbf{N}_{\overline{\infty}}$-compatible. Then $\theta \in T_{\mathbf{N}_{\overline{\infty}}}$ and so $M_{|c,d|} = M_{\theta} \in T_{\mathbf{E}_{\overline{\infty}}}$.

Now suppose $c = a_{-\infty}$. If $d = a_{+\infty}$ then $M_{|c,d|} \in T_{\mathbf{E}_{\overline{\infty}}}$ by definition; so, suppose $d < a_{+\infty}$. Then our argument in the previous paragraph, $d = a_j$ for $j \in \mathbb{Z}$. Take $\theta = (a_{-\infty}, a_j)$ and suppose $\{\theta\} \cup T_{\mathbf{N}_{\overline{\infty}}}$ is $\mathbf{N}_{\overline{\infty}}$-compatible. Then $\theta \in T_{\mathbf{N}_{\overline{\infty}}}$ and so $M_{|c,d|} = M_{\theta} \in T_{\mathbf{E}_{\overline{\infty}}}$. Therefore, $T_{\mathbf{E}_{\overline{\infty}}}$ is an $\mathbf{E}$-cluster. $\square$

Lemma 3.1.6. *Let $T_{\mathbf{N}_{\overline{\infty}}}$ be an $\mathbf{N}_{\overline{\infty}}$-cluster and $T_{\mathbf{N}_{\overline{\infty}}} \to (T_{\mathbf{N}_{\overline{\infty}}} \setminus \{\theta\}) \cup \{\theta'\}$ an $\mathbf{N}_{\overline{\infty}}$-mutation. Then $T_{\mathbf{E}_{\overline{\infty}}} \to (T_{\mathbf{E}_{\overline{\infty}}} \setminus \{M_{\theta}\}) \cup \{M_{\theta'}\}$ an $\mathbf{E}$-mutation.*

Proof. By Proposition 3.1.2 we know that since $\{\theta, \theta'\}$ is not $\mathbf{N}_{\overline{\infty}}$-compatible the set $\{M_{\theta}, M_{\theta'}\}$ is not $\mathbf{E}$-compatible. Let $T'_{\mathbf{N}_{\overline{\infty}}} = (T_{\mathbf{N}_{\overline{\infty}}} \setminus \{\theta\}) \cup \{\theta'\}$. Then $T'_{\mathbf{E}_{\overline{\infty}}}$ is precisely $(T_{\mathbf{E}_{\overline{\infty}}} \setminus \{M_{\theta}\}) \cup \{M_{\theta'}\}$. Therefore $T_{\mathbf{E}_{\overline{\infty}}} \to T'_{\mathbf{E}_{\overline{\infty}}}$ is an $\mathbf{E}$-mutation. $\square$

Theorem 3.1.7. *There exists an embedding of cluster theories $(F^{\mathbb{R}}_{\overline{\infty}}, \eta^{\mathbb{R}}_{\overline{\infty}}) : \mathscr{T}_{\mathbf{N}_{\overline{\infty}}}(\mathcal{C}(A_{\overline{\infty}})) \to \mathscr{T}_{\mathbf{E}}(\mathcal{C}(A_{\mathbb{R}}))$.*

Proof. By Proposition 3.1.5 and Lemma 3.1.6 we see that defining $F^{\mathbb{R}}_{\overline{\infty}}(T_{\mathbf{N}_{\overline{\infty}}}) := T_{\mathbf{E}_{\overline{\infty}}}$ and sending $\mathbf{N}_{\overline{\infty}}$-mutations to the corresponding $\mathbf{E}$-mutations yields an embedding.

Define $\eta^{\mathbb{R}}_{\overline{\infty} \mathbf{N}_{\overline{\infty}}} : T_{\mathbf{N}_{\overline{\infty}}} \to T_{\mathbf{E}_{\overline{\infty}}}$ by $\eta^{\mathbb{R}}_{\overline{\infty} \mathbf{N}_{\overline{\infty}}}(\theta) = M_{\theta}$. This is injective and again by Lemma 3.1.6 commutes with mutation. Therefore $(F^{\mathbb{R}}_{\overline{\infty}}, \eta^{\mathbb{R}}_{\overline{\infty}})$ is an embedding of cluster theories. $\square$

3.2. **The Sequence of Type A Embeddings.** In this subsection we prove the first diagram from the beginning fo the section commutes:

$$\begin{tikzcd}
\mathscr{T}_{\mathbf{N}_m}(\mathcal{C}(A_m)) \arrow[r, "A"] \arrow[drr, "1.3.16(\mathrm{i})"'] & \mathscr{T}_{\mathbf{N}_n}(\mathcal{C}(A_n)) \arrow[r, "B"] \arrow[dr, "1.3.16(\mathrm{i})"] & \mathscr{T}_{\mathbf{N}_\infty}(\mathcal{C}(A_\infty)) \arrow[r, "C"] \arrow[d, "1.3.16(\mathrm{ii})"] & \mathscr{T}_{\mathbf{N}_{\overline{\infty}}}(\mathcal{C}(A_{\overline{\infty}})) \arrow[r, "D"] \arrow[dl, "3.1.7"'] & \mathscr{T}_{\mathbf{N}_{\mathbb{R}}}(\mathcal{C}_\pi) \arrow[dll, "1.3.16(\mathrm{iii})"] \\
 & & \mathscr{T}_{\mathbf{E}}(\mathcal{C}(A_{\mathbb{R}})) & &
\end{tikzcd}$$

Our strategy will be to define each of the arrows A, B, C, and D in the diagram and show that the respective triangles commutes, thereby making the entire diagram commute.

3.2.1. *Arrow A.* Recall the triangulation model from [8] that we use for $\mathscr{T}_{\mathbf{N}_n}(\mathcal{C}(A_n))$ (Example 1.3.9 and Theorem 1.3.16(i)). We will define the embedding of cluster theories $(F_m^{m+1}, \eta_m^{m+1}) : \mathscr{T}_{\mathbf{N}_m}(\mathcal{C}(A_m)) \to \mathscr{T}_{\mathbf{N}_{m+1}}(\mathcal{C}(A_{m+1}))$. When $n \geq m+2$ we will simply use the finite composition

$$(F_m^n, \eta_m^n) := (F_{n-1}^n, \eta_{n-1}^n) \circ (F_{n-2}^{n-1}, \eta_{n-2}^{n-1}) \circ \cdots \circ (F_{m+1}^{m+2}, \eta_{m+1}^{m+2}) \circ (F_m^{m+1}, \eta_m^{m+1})$$

as arrow A. Denote by $(F_m^{\mathbb{R}}, \eta_m^{\mathbb{R}})$ the embedding of cluster theories $\mathscr{T}_{\mathbf{N}_m}(\mathcal{C}(A_m)) \to \mathscr{T}_{\mathbf{E}}(\mathcal{C}(A_{\mathbb{R}}))$ from Theorem 1.3.16.

Definition 3.2.1. Let $T_{\mathbf{N}_m}$ be an $\mathbf{N}_m$-cluster and thus a triangulation of the $(m+3)$-gon. Recall from Theorem 1.3.16(i) that the vertices of the $(m+3)$-gon are enumerated counterclockwise and we denote a diagonal from vertex i to vertex j by $i \!\frown\! j$, where $2 \leq j - i \leq m+2$. We define $F_m^{m+1} T_{\mathbf{N}_m}$ to be the triangulation of the $(m+4)$-gon given $\{i \!\frown\! j : i \!\frown\! j \in T_{\mathbf{N}_m}\} \cup \{1 \!\frown\! (m+3)\}$. In the $(m+3)$-gon, $1 \!\frown\! (m+3)$ is just an edge on the polygon; however, in the $(m+4)$-gon this is now a diagonal that creates a triangle with $(m+3) \!\frown\! (m+4)$ and $1 \!\frown\! (m+4)$. Define F_m^{m+1} on a mutation $T_{\mathbf{N}_m} \to (T_{\mathbf{N}_m} \setminus \{i \!\frown\! j\}) \cup \{i' \!\frown\! j'\}$ to be precisely

$$F_m^{m+1} T_{\mathbf{N}_m} \to (F_m^{m+1} T_{\mathbf{N}_m} \setminus \{i \!\frown\! j\}) \cup \{i' \!\frown\! j'\}.$$

Finally, define $(\eta_m^{m+1})_{T_{\mathbf{N}_m}} : T_{\mathbf{N}_m} \to F_m^{m+1} T_{\mathbf{N}_m}$ to ge given by $i \!\frown\! j \mapsto i \!\frown\! j$.

Lemma 3.2.2 (Lemma A)**.** *The pair (F_m^{m+1}, η^m) just defined is an embedding of cluster theories $\mathscr{T}_{\mathbf{N}_m}(\mathcal{C}(A_m)) \to \mathscr{T}_{\mathbf{N}_{m+1}}(\mathcal{C}(A_{m+1}))$. Furthermore,*

$$(F_{m+1}^{\mathbb{R}}, \eta_{m+1}^{\mathbb{R}}) \circ (F_m^{m+1}, \eta^m) = (F_m^{\mathbb{R}}, \eta_m^{\mathbb{R}}).$$

Proof. Note that for $j, j' \leq m+3$ the set $\{i \!\frown\! j, i' \!\frown\! j'\}$ is $\mathbf{N}_m$-compatible if and only if it is $\mathbf{N}_{m+1}$-compatible. Next suppose $T_{\mathbf{N}_m}$ is an $\mathbf{N}_m$-cluster. It is straightforward to check that $F_m^{m+1} T_{\mathbf{N}_m}$ is $\mathbf{N}_{m+1}$-compatible.

Suppose $i \!\frown\! j$ is a diagonal of the $(m+4)$-gon and $\{i \!\frown\! j\} \cup F_m^{m+1} T_{\mathbf{N}_m}$ is $\mathbf{N}_{m+1}$-compatible. We know by construction neither i nor j may be $m+4$. Then either $i \!\frown\! j = 1 \!\frown\! (m+3)$ or $i \!\frown\! j$ is compatible with $T_{\mathbf{N}_m}$. In either case, $i \!\frown\! j \in T_{\mathbf{N}_m}$. These first two paragraphs shows that $(F_m^{m+1}, \eta_m^{m+1})$ is an embedding of cluster theories.

Note that $M_{(a_1, a_{m+3})}, M_{(a_1, a_{m+4})} \in T_m$ from Definition 1.3.13. Note also that $T_{m+1} = T_m \setminus \{M_{(a_1, a_{m+3})}\}$ from the same definition. Let

$$\begin{aligned} T_{\mathbf{E}_m} &= F_m^{\mathbb{R}} T_{\mathbf{N}_m} \\ T_{\mathbf{E}_{m+1}} &= F_{m+1}^{\mathbb{R}} F_m^{m+1} T_{\mathbf{N}_m}. \end{aligned}$$

Since $1 \!\frown\! (m+3) \in F_m^{m+1} T_{\mathbf{N}_m}$ for every $\mathbf{N}_m$-cluster $T_{\mathbf{N}_m}$ we see that $T_{\mathbf{E}_m} = T_{\mathbf{E}_{m+1}}$ and so $F_m^{\mathbb{R}} = F_{m+1}^{\mathbb{R}} \circ F_m^{m+1}$. Similarly, we see $\eta_m^{\mathbb{R}} = \eta_{m+1}^{\mathbb{R}} \circ \eta_m^{m+1}$. This completes the proof. □

3.2.2. *Arrow B.* Denote by $(F_\infty^{\mathbb{R}}, \eta_\infty^{\mathbb{R}})$ the embedding of cluster theories form Theorem 1.3.16. Recall the triangulations of the infinity-gon from [14] and the corresponding cluster theory $\mathscr{T}_{\mathbf{N}_\infty}(\mathcal{C}(A_\infty))$ from Theorem 1.3.16(ii). In particular, diagonals are now of the form $i \!\frown\! j$ for any $i, j \in \mathbb{Z}$ where $j - i \geq 2$. We now define the embedding of cluster theories $(F_m^\infty, \eta_m^\infty) : \mathscr{T}_{\mathbf{N}_n}(\mathcal{C}(A_n)) \to \mathscr{T}_{\mathbf{N}_\infty}(\mathcal{C}(A_\infty))$.

Definition 3.2.3. Let $T_{\mathbf{N}_n}$ be an $\mathbf{N}_n$-cluster and

$$F_n^\infty T_{\mathbf{N}_n} := \{i \!\frown\! j : i \!\frown\! j \in T_{\mathbf{N}_n}\} \cup \{i \!\frown\! 1 : i < 0\} \cup \{1 \!\frown\! j : j \geq n+3\}.$$

We define F_n^∞ on mutations similar to Definition 3.2.1:

$$\left(T_{\mathbf{N}_n} \to ((T_{\mathbf{N}_n} \setminus \{i \!\frown\! j\}) \cup \{i' \!\frown\! j'\})\right) \mapsto \left(F_n^\infty T_{\mathbf{N}_n} \to (F_n^\infty T_{\mathbf{N}_n} \setminus \{i \!\frown\! j\}) \cup \{i' \!\frown\! j'\}\right).$$

Finally, define $(\eta_n^\infty)_{T_{\mathbf{N}_n}} : T_{\mathbf{N}_n} \to T_{\mathbf{N}_\infty}$ by $i \!\frown\! j \mapsto i \!\frown\! j$.

Lemma 3.2.4 (Lemma B)**.** *The pair $(F_n^\infty, \eta_n^\infty)$ is an embedding of cluster theories $\mathscr{T}_{\mathbf{N}_n}(\mathcal{C}(A_n)) \to \mathscr{T}_{\mathbf{N}_\infty}(\mathcal{C}(A_\infty))$. Furuthermore,*

$$(F_\infty^{\mathbb{R}}, \eta_\infty^{\mathbb{R}}) \circ (F_n^\infty, \eta_n^\infty) = (F_n^{\mathbb{R}}, F_n^{\mathbb{R}}).$$

Proof. That $(F_n^\infty, \eta_n^\infty)$ is an embedding of cluster theories follows the same argument as Lemma A. Let

$$T_{\mathbf{E}_n} = F_n^{\mathbb{R}} T_{\mathbf{N}_n}$$
$$T_{\mathbf{E}_\infty} = F_\infty^{\mathbb{R}} F_n^\infty T_{\mathbf{N}_n}.$$

The $\mathbf{E}$-cluster $T_{\mathbf{E}_n}$ contains $M_{i\!\bullet\!\!-\!\!\bullet 1}$ for each $i < 0$ and $M_{1\bullet\!\!-\!\!\bullet j}$ for each $j \geq n+3$. It also contains $M_{1\bullet\!\!-\!\!\bullet(n+3)}$. These are precisely the image of $i\bullet\!\!-\!\!\bullet 1$, $1\bullet\!\!-\!\!\bullet j$, and $1\bullet\!\!-\!\!\bullet(n+3)$ under $\eta_\infty^{\mathbb{R}}$ and so $T_{\mathbf{E}_n} = T_{\mathbf{E}_\infty}$. Thus $F_\infty^{\mathbb{R}} \circ F_n^\infty = F_n^{\mathbb{R}}$ and $\eta_\infty^{\mathbb{R}} \circ \eta_n^\infty = \eta_n^{\mathbb{R}}$. Therefore the lemma holds. □

3.2.3. *Arrow C.* Recall the $\mathbf{N}_{\overline{\infty}}$ cluster theory from Section 2.2 and the embedding of cluster theories $(F_{\overline{\infty}}^{\mathbb{R}}, \eta_{\overline{\infty}}^{\mathbb{R}})$ from Theorem 3.1.7. In particular we now allow $i - \infty$ or $j = +\infty$ (but not both) for a diagonal $i\bullet\!\!-\!\!\bullet j$. Since $-\infty\bullet\!\!-\!\!\bullet+\infty$ is in every $\mathbf{N}_{\overline{\infty}}$-cluster we consider it to be a side of the completed infinity-gon as before. We now define the embedding of cluster theories $(F_\infty^{\overline{\infty}}, \eta_\infty^{\overline{\infty}}) : \mathscr{T}_{\mathbf{N}_\infty}(\mathcal{C}(A_\infty)) \to T_{\mathbf{N}_{\overline{\infty}}}(\mathcal{C}(A_{\overline{\infty}}))$.

We adapt the following definitions from [4].

Definition 3.2.5. Let $i\bullet\!\!-\!\!\bullet j$ be a diagonal in the completed infinity-gon. The first two definitions are from [4, Section 1]

- If $i = -\infty$ we call $i\bullet\!\!-\!\!\bullet j$ an <u>adic</u> diagonal. We may denote it by α_j.
- If $j = +\infty$ we call $i\bullet\!\!-\!\!\bullet j$ a <u>Prüfer</u> diagonal. We may denote it by π_i.

Note that the index is the endpoint of the diagonal that is contained in $\mathbb{Z}$.

Let T be an $\mathbf{N}_{\overline{\infty}}$-compatible set. The following are from [4, Definition 5.2]. Define

$$\mathcal{A}(T) := T \cup \{\alpha_i : T \cup \{\alpha_i\} \text{ is } \mathbf{N}_\infty\text{-compatible}\}$$
$$\mathcal{P}(T) := T \cup \{\pi_i : T \cup \{\pi_i\} \text{ is } \mathbf{N}_\infty\text{-compatible}\}.$$

We call $\mathcal{P}(\mathcal{A}(T))$ the <u>adic completion</u> of T and $\mathcal{A}(\mathcal{P}(T))$ the <u>Prüfer completion</u> of T.

In [4], the authors do not include T in $\mathcal{A}(T)$ or $\mathcal{P}(T)$. However, we include T for our convenience. The authors also prove in [4, Theorem 5.4] that $\mathcal{A}(\mathcal{P}(T))$ is a triangulation of the completed infinity-gon and state that by symmetry so is $\mathcal{P}(\mathcal{A}(T))$.

Definition 3.2.6. Let $T_{\mathbf{N}_\infty}$ be an $\mathbf{N}_\infty$-cluster. Define $T^\circ_{\mathbf{N}_{\overline{\infty}}} := \{i\bullet\!\!-\!\!\bullet j : i\bullet\!\!-\!\!\bullet j \in T_{\mathbf{N}_\infty}\}$. Then

$$F_\infty^{\overline{\infty}}(T_{\mathbf{N}_\infty}) := \mathcal{P}(\mathcal{A}(T^\circ_{\mathbf{N}_{\overline{\infty}}}))$$
$$\eta^{\overline{\infty}}_{\infty\, T_{\mathbf{N}_\infty}}(i\bullet\!\!-\!\!\bullet j) := i\bullet\!\!-\!\!\bullet j.$$

Proposition 3.2.7. *Let $T_{\mathbf{N}_\infty}$ be an $\mathbf{N}_\infty$-cluster. Then $F_\infty^{\overline{\infty}} T_{\mathbf{N}_\infty}$ contains at most 2 adic diagonals and at most 1 Prüfer diagonal. Furthermore, $F_\infty^{\overline{\infty}} T_{\mathbf{N}_\infty}$ contains an adic diagonal if and only if it contains a Prüfer diagonal.*

Proof. For contradiction, suppose there are more than 2 adic diagonals. Since adic diagonals are totally ordered by their index consider three consecutive adic diagonals $\alpha_i, \alpha_\ell, \alpha_j$. I.e., there is no $\alpha_{\ell'}$ such that $i < \ell' < \ell$ or $\ell < \ell' < j$. But then $\{i\bullet\!\!-\!\!\bullet j\} \cup T_{\mathbf{N}_\infty}$ is $\mathbf{N}_\infty$-compatible but $\{i\bullet\!\!-\!\!\bullet j, \alpha_\ell\}$ is not $\mathbf{N}_{\overline{\infty}}$-compatible. This contradicts our assumption that $T_{\mathbf{N}_\infty}$ is an $\mathbf{N}_\infty$-cluster. By symmetry there exist no more than 2 Prüfer diagonals in $F_\infty^{\overline{\infty}} T_{\mathbf{N}_\infty}$.

Suppose $F_\infty^{\overline{\infty}} T_{\mathbf{N}_\infty}$ has an adic diagonal. By the previous paragraph there are at most 2 adic diagonals so let α_ℓ be the adic diagonal such that if α_i is also an adic diagonal, $i \leq \ell$. Then there is no $i \bullet\!\!-\!\!\bullet j \in T_{\mathbf{N}_\infty}$ such that $i < \ell < j$ and so $\{\pi_\ell\} \cup F_\infty^{\overline{\infty}} T_{\mathbf{N}_\infty}$ is $\mathbf{N}_{\overline{\infty}}$-compatible. By [4, Theorem 5.4] $F_\infty^{\overline{\infty}} T_{\mathbf{N}_\infty}$ is an $\mathbf{N}_{\overline{\infty}}$-cluster and so $\pi_\ell \in F_\infty^{\overline{\infty}} T_{\mathbf{N}_\infty}$. Dually, if there is a Prüfer diagonal there is an adic diagonal.

Finally suppose, for contradiction, that $F_\infty^{\overline{\infty}} T_{\mathbf{N}_\infty}$ has 2 Prüfer diagonals, π_ℓ and π_j for $\ell < j$. Then it also has at least 1 adic diagonal; let α_i be the higher-indexed of the possible 2. If $i < \ell < j$ we have a contradiction similar to the previou paragraph. If $i = \ell$ then $\{\alpha_j\} \cup T^\circ_{\mathbf{N}_{\overline{\infty}}}$ is $\mathbf{N}_{\overline{\infty}}$-compatible, contradicting our definition of $F_\infty^{\overline{\infty}} T_{\mathbf{N}_\infty}$. □

Lemma 3.2.8 (Lemma C)**.** *As defined,* $(F_\infty^{\overline{\infty}}, \eta_\infty^{\overline{\infty}}) : \mathscr{T}_{\mathbf{N}_\infty}(\mathcal{C}(A_\infty)) \to \mathscr{T}_{\mathbf{N}_{\overline{\infty}}}(\mathcal{C}(A_{\overline{\infty}}))$ *is an embedding of cluster theories. Furuthermore,*

$$(F_{\overline{\infty}}^{\mathbb{R}}, \eta_{\overline{\infty}}^{\mathbb{R}}) \circ (F_\infty^{\overline{\infty}}, \eta_\infty^{\overline{\infty}}) = (F_\infty^{\mathbb{R}}, F_\infty^{\mathbb{R}}).$$

Proof. Again by [4, Theorem 5.4] we know $F_\infty^{\overline{\infty}}$ takes $\mathbf{N}_\infty$-clusters to $\mathbf{N}_{\overline{\infty}}$-clusters. Let $T_{\mathbf{N}_\infty}$ be an $\mathbf{N}_\infty$ cluster and $\mu : T_{\mathbf{N}_\infty} \to (T_{\mathbf{N}_\infty} \setminus \{i \bullet\!\!-\!\!\bullet j\}) \cup \{i' \bullet\!\!-\!\!\bullet j'\}$ an $\mathbf{N}_\infty$-mutation. Set $T_{\mathbf{N}_{\overline{\infty}}} = F_\infty^{\overline{\infty}} T_{\mathbf{N}_\infty}$. We will show that

$$F_\infty^{\overline{\infty}} T_{\mathbf{N}_\infty} \to \left(F_\infty^{\overline{\infty}} T_{\mathbf{N}_\infty} \setminus \{i \bullet\!\!-\!\!\bullet j\}\right) \cup \{i' \bullet\!\!-\!\!\bullet j'\}.$$

is an $\mathbf{N}_{\overline{\infty}}$-mutation.

By symmetry assume $i < i' < j < j'$ as we know $\{i \bullet\!\!-\!\!\bullet j, i' \bullet\!\!-\!\!\bullet j'\}$ is not $\mathbf{N}_\infty$-compatible. Since μ is an $\mathbf{N}_\infty$-mutation, the following diagonals are all in $T_{\mathbf{N}_\infty}$:

$$i \bullet\!\!-\!\!\bullet i' \qquad i' \bullet\!\!-\!\!\bullet j \qquad j \bullet\!\!-\!\!\bullet j' \qquad i \bullet\!\!-\!\!\bullet j'.$$

Then they are all in $F_\infty^{\overline{\infty}} T_{\mathbf{N}_\infty}$ and form a quadrilateral inside the completed infinity-gon. Noting [4, Remark 2.2] we see that $i \bullet\!\!-\!\!\bullet j$ in $F_\infty^{\overline{\infty}} T_{\mathbf{N}_\infty}$ is $\mathbf{N}_{\overline{\infty}}$-mutable and the desired map is indeed an $\mathbf{N}_{\overline{\infty}}$-mutation.

Note that $\eta_{\infty T_{\mathbf{N}_\infty}}^{\overline{\infty}}$ is injective and the $\eta_\infty^{\overline{\infty}}$'s commute with mutation by definition. Therefore $(F_\infty^{\overline{\infty}}, \eta_\infty^{\overline{\infty}})$ is an embedding of cluster theories. Using Proposition 3.2.7 it is straightforward to check that $T_{\mathbf{E}_\infty}$ in Definition 1.3.14 is precisely $F_{\overline{\infty}}^{\mathbb{R}} \circ F_\infty^{\overline{\infty}}(T_{\mathbf{N}_\infty})$ and that $(\eta_{\overline{\infty} T_{\mathbf{N}_{\overline{\infty}}}}^{\mathbb{R}}) \circ (\eta_{\infty T_{\mathbf{N}_\infty}}^{\overline{\infty}}) = \eta_{\infty T_{\mathbf{N}_\infty}}^{\mathbb{R}}$. Therefore $(F_{\overline{\infty}}^{\mathbb{R}}, \eta_{\overline{\infty}}^{\mathbb{R}}) \circ (F_\infty^{\overline{\infty}}, \eta_\infty^{\overline{\infty}}) = (F_\infty^{\mathbb{R}}, F_\infty^{\mathbb{R}})$. □

3.2.4. *Arrow D and Conclusion.* Denote by $(F_\pi^{\mathbb{R}}, \eta_\pi^{\mathbb{R}})$ the embedding of cluster theories from Theorem 1.3.16(iii). We're using π for the $\mathbf{N}_{\mathbb{R}}$-cluster theory $\mathscr{T}_{\mathbf{N}_{\mathbb{R}}}(\mathcal{C}_\pi)$ instead of $\mathbb{R}$ as we're already using $\mathbb{R}$ for the $\mathbf{E}$-cluster theory $\mathscr{T}_{\mathbf{E}}(\mathcal{C}(A_{\mathbb{R}}))$. Additionally, π speaks to the origin of $\mathcal{C}_\pi$ and as an irrational number speaks to the cardinality of the set of isomorphism classes of indecomposables in $\mathcal{C}_\pi$. We define $(F_{\overline{\infty}}^{\pi}, \eta_{\overline{\infty}}^{\pi})$ deliberately so that $(F_\pi^{\mathbb{R}}, \eta_\pi^{\mathbb{R}}) \circ (F_{\overline{\infty}}^{\pi}, \eta_{\overline{\infty}}^{\pi}) = (F_{\overline{\infty}}^{\mathbb{R}}, \eta_{\overline{\infty}}^{\mathbb{R}})$.

Recall from Definition 1.3.11 the sequence $\{a_i\}_{i\in\mathbb{Z}}$ converging to $a_{-\infty}$ below and $a_{+\infty}$ above, both in $\mathbb{Z}$. From the same definition recall $a_{i,j,\ell} = a_i + \left(\frac{j}{2^\ell}\right)(a_{i+1} - a_i)$ for $i, j, \ell \in \mathbb{Z}$, $\ell \geq 0$, $0 \leq j \leq 2^\ell$. Recall also $\mathfrak{f}$ from Definition 1.3.15 that is a bijection from the points of indecomposables in the fundamental domain of $\mathcal{C}_\pi$ to the image under $\mathbf{\Gamma}^b$ of indecomposables in the fundamental domain of $\mathcal{C}(A_{\mathbb{R}})$.

Definition 3.2.9. For a diagonal $\theta = i \bullet\!\!-\!\!\bullet j$ in the completed infinity-gon define $M(\theta) := M(\mathfrak{f}^{-1}(a_i, a_j))$. For a_i in the sequence $\{a_i\}_{i\in\mathbb{Z}}$ and $i \in \{j \in \mathbb{Z} : j < a_{-\infty}$ or $j \geq a_{+\infty}\}$ we define

$$\begin{aligned}
T_{\pi,a_i} &:= \left\{M(\mathfrak{f}^{-1}(a_{i,j,\ell},\, a_{i,j+1,\ell})) : j, \ell \in \mathbb{Z},\, \ell \geq 0,\, 0 \leq j < 2^\ell\right\} \\
T_{\pi,i} &:= \left\{M(\mathfrak{f}^{-1}(i + j/2^\ell,\, i + (j+1)/2^\ell)) : j, \ell \in \mathbb{Z},\, \ell \geq 0,\, 0 \leq j < 2^\ell\right\} \\
&\quad \cup \{M(\mathfrak{f}^{-1}(-\infty,\, i+1))\} \\
T_\pi &:= \left(\bigcup_{i\in\mathbb{Z}} T_{\pi,a_i}\right) \cup \left(\bigcup_{i<a_{-\infty} \text{ or } i \geq a_{+\infty}} T_{\pi,i}\right) \cup \{M(\mathfrak{f}^{-1}(-\infty, a_{+\infty})), M(\mathfrak{f}^{-1}(a_{-\infty}, a_{+\infty}))\}.
\end{aligned}$$

Let $T_{\mathbf{N}_{\overline{\infty}}}$ be an $\mathbf{N}_{\overline{\infty}}$-cluster. Define

$$\begin{aligned} F^{\pi}_{\overline{\infty}} T_{\mathbf{N}_{\overline{\infty}}} &:= T_\pi \cup \{M(\theta) : \theta \in T_{\mathbf{N}_{\overline{\infty}}}\} \\ \eta^{\pi}_{\overline{\infty} T_{\mathbf{N}_{\overline{\infty}}}}(\theta) &:= M(\theta). \end{aligned}$$

Proposition 3.2.10. *The set $T_{\mathbf{N}_{\mathbb{R}}} = F^{\pi}_{\overline{\infty}} T_{\mathbf{N}_{\overline{\infty}}}$ is an $\mathbf{N}_{\mathbb{R}}$-cluster.*

Proof. Suppose $\{M(x,y)\} \cup T_{\mathbf{N}_{\mathbb{R}}}$ is $\mathbf{N}_{\mathbb{R}}$-compatible. Exploiting the fact that $\mathfrak{f}$ is a bijection, by [16, Proposition 5.4.3], we will consider various possibilities of $\mathfrak{f}_1(x,y)$. Suppose $\mathfrak{f}_1(x,y) < a_{-\infty}$ or $> a_{+\infty}$. Then $\mathfrak{f}_1(x,y)$ must be of the form $i + j/2^\ell$ from Definition 3.2.9 or $\mathfrak{f}_1(x,y) = -\infty$. Then, respectively, $\mathfrak{f}_2(x,y)$ is $i + (j+1)/2^\ell$ (assuming we've chosen the minimal ℓ) or $\mathfrak{f}_2(x,y) \in \mathbb{Z}$ (where $\mathfrak{f}_2(x,y) < a_{-\infty}$ or $\mathfrak{f}_2(x,y) \geq a_{+\infty}$). Thus, $M(x,y) \in T_{\mathbf{N}_{\mathbb{R}}}$. If $\mathfrak{f}_1(x,y) \geq a_{+\infty}$ then $\mathfrak{f}_2(x,y) = a_{+\infty}+1$ or $\mathfrak{f}_2(x,y)$ is of the form $i + (j+1)/2^\ell$. If $a_i < \mathfrak{f}_1(x,y) < a_{i+1}$ for some i then $\mathfrak{f}_2(x,y)$ is of the form $a_{i+(j+1)/2^\ell}$.

Now suppose $\mathfrak{f}_1(x,y) = a_i$ for some $i \in \mathbb{Z}$ or $\mathfrak{f}_1(x,y) = a_{-\infty}$. First, $\mathfrak{f}_1(x,y) = a_i$. If $\mathfrak{f}_2(x,y) < a_{i+1}$ then it must be of the form $a_{i+(j+1)/2^\ell}$. If $\mathfrak{f}_2(x,y) \geq a_{i+1}$ then it must be a_j for $j \geq i+1$ or $j = +\infty$.

If $\mathfrak{f}_2(x,y) = a_{i+1}$ then $M(x,y) \in T_{\mathbf{N}_{\mathbb{R}}}$. If $\mathfrak{f}_2(x,y) = a_j$ for $j > i+1$ then there must be a diagonal $i \!\frown\! j$ in $T_{\mathbf{N}_{\overline{\infty}}}$ and so is in $T_{\mathbf{N}_{\mathbb{R}}}$. If $\mathfrak{f}_1(x,y) = a_{-\infty}$ then $\mathfrak{f}_2(x,y) = a_j$ for $j \in \mathbb{Z}$ or $j = +\infty$. Again $M(x,y)$ must be in $T_{\mathbf{N}_{\mathbb{R}}}$. We've checked all possible values for $\mathfrak{f}_1(x,y)$ and so $T_{\mathbf{N}_{\mathbb{R}}}$ is an $\mathbf{N}_{\mathbb{R}}$-cluster. $\square$

Remark 3.2.11. Note that for any $M \in T_\pi \subset T_{\mathbf{N}_{\mathbb{R}}}$, where $T_{\mathbf{N}_{\mathbb{R}}}$ is an $\mathbf{N}_{\mathbb{R}}$-cluster, $\eta^{\mathbb{R}}_{\pi}(M) \in T_\infty$ from Definition 1.3.11. In fact, for any $\mathbf{N}_{\overline{\infty}}$-cluster $T_{\mathbf{N}_{\overline{\infty}}}$,

$$T_\infty = \left(\eta^{\mathbb{R}}_{\pi\, T_{\mathbf{N}_{\overline{\infty}}}}(T_\pi)\right) \cup \left\{ M_{\{x\}} : \{M_{\{x\}}, \eta^{\mathbb{R}}_{\pi\, T_{\mathbf{N}_{\overline{\infty}}}}(T_\pi)\} \text{ is } \mathbf{E}\text{-compatible} \right\}$$

Lemma 3.2.12 (Lemma D)**.** *As defined, $(F^{\pi}_{\overline{\infty}}, \eta^{\pi}_{\overline{\infty}}) : \mathscr{T}_{\mathbf{N}_{\overline{\infty}}}(\mathcal{C}(A_{\overline{\infty}})) \to \mathscr{T}_{\mathbf{N}_{\mathbb{R}}}(\mathcal{C}_\pi)$ is an embedding of cluster theories. Furuthermore,*

$$(F^{\mathbb{R}}_{\pi}, \eta^{\mathbb{R}}_{\pi}) \circ (F^{\pi}_{\overline{\infty}}, \eta^{\pi}_{\overline{\infty}}) = (F^{\mathbb{R}}_{\overline{\infty}}, \eta^{\mathbb{R}}_{\overline{\infty}}).$$

Proof. By Proposition 3.2.10 we know $F^{\pi}_{\overline{\infty}} T_{\mathbf{N}_{\overline{\infty}}}$ is an $\mathbf{N}_{\mathbb{R}}$-cluster. Let $\mu : T_{\mathbf{N}_{\overline{\infty}}} \to (T_{\mathbf{N}_{\overline{\infty}}} \setminus \{\theta\}) \cup \{\theta'\}$ be an $\mathbf{N}_{\overline{\infty}}$-mutation. We'll show $T_{\mathbf{N}_{\mathbb{R}}} \to (T_{\mathbf{N}_{\mathbb{R}}} \setminus \{M(\theta)\}) \cup \{M(\theta')\}$ is an $\mathbf{N}_{\mathbb{R}}$-mutation.

Let $i \!\frown\! j = \theta$ and $i' \!\frown\! j' = \theta'$. By symmetry, assume $i < i' < j < j'$. Then $\{M(\theta), M(\theta')\}$ is not $\mathbf{N}_{\mathbb{R}}$-compatible. Since $T'_{\mathbf{N}_{\overline{\infty}}} = (T_{\mathbf{N}_{\overline{\infty}}} \setminus \{\theta\}) \cup \{\theta'\}$ is an $\mathbf{N}_{\overline{\infty}}$-cluster we know $F^{\pi}_{\overline{\infty}} T'_{\mathbf{N}_{\overline{\infty}}}$ is an $\mathbf{N}_{\mathbb{R}}$-cluster. However, $F^{\pi}_{\overline{\infty}} T'_{\mathbf{N}_{\overline{\infty}}} = (T_{\mathbf{N}_{\mathbb{R}}} \setminus \{M(\theta)\}) \cup \{M(\theta')\}$ and so we have a $\mathbf{N}_{\mathbb{R}}$-mutation. By definition we see that the the $\eta^{\pi}_{\overline{\infty}}$'s commute with mutation. Finally, noting Remark 3.2.11, we see $(F^{\mathbb{R}}_{\pi}, \eta^{\mathbb{R}}_{\pi}) \circ (F^{\pi}_{\overline{\infty}}, \eta^{\pi}_{\overline{\infty}}) = (F^{\mathbb{R}}_{\overline{\infty}}, \eta^{\mathbb{R}}_{\overline{\infty}})$. $\square$

Theorem 3.2.13. *The following diagram of embeddings of cluster theories commutes:*

$$\begin{tikzcd}
\mathscr{T}_{\mathbf{N}_m}(\mathcal{C}(A_m)) \arrow[r, "(F^n_m, \eta^n_m)"] \arrow[drr, "(F^{\mathbb{R}}_m, \eta^{\mathbb{R}}_m)"'] & \mathscr{T}_{\mathbf{N}_n}(\mathcal{C}(A_n)) \arrow[r, "(F^\infty_n, \eta^\infty_n)"] \arrow[dr, "(F^{\mathbb{R}}_n, \eta^{\mathbb{R}}_n)"'] & \mathscr{T}_{\mathbf{N}_\infty}(\mathcal{C}(A_\infty)) \arrow[r, "(F^{\overline{\infty}}_{\infty}, \eta^{\overline{\infty}}_{\infty})"] \arrow[d, "(F^{\mathbb{R}}_\infty, \eta^{\mathbb{R}}_\infty)"] & \mathscr{T}_{\mathbf{N}_{\overline{\infty}}}(\mathcal{C}(A_{\overline{\infty}})) \arrow[r, "(F^{\pi}_{\overline{\infty}}, \eta^{\pi}_{\overline{\infty}})"] \arrow[dl, "(F^{\mathbb{R}}_{\overline{\infty}}, \eta^{\mathbb{R}}_{\overline{\infty}})"] & \mathscr{T}_{\mathbf{N}_{\mathbb{R}}}(\mathcal{C}_\pi) \arrow[dll, "(F^{\mathbb{R}}_\pi, \eta^{\mathbb{R}}_\pi)"] \\
& & \mathscr{T}_{\mathbf{E}}(\mathcal{C}(A_{\mathbb{R}})). & &
\end{tikzcd}$$

Proof. Apply Lemmas A, B, C, and D. $\square$

3.3. **Cluster Structures in Cluster Theories.** In this subsection we define the abstract notion of a cluster structure as a subcategory of the cluster theory. We prove how some cluster structures of type A are related (Theorem 3.3.16). Along the we then relate it to the embeddings in Theorem 3.2.13. Note that Definition 3.3.1 requires a much more mild property compared to the usual requirement of functorial finiteness (the definition of which we will omit).

Definition 3.3.1. Let $\mathcal{C}$ be a Krull-Schmidt category and $\mathbf{P}$ a pairwise compatibility condition on the indecomposables of $\mathcal{C}$ such that $\mathbf{P}$ induces the $\mathbf{P}$-cluster theory $\mathscr{T}_{\mathbf{P}}(\mathcal{C})$ of $\mathcal{C}$. Suppose there exists a subcategory $\mathscr{S}_{\mathbf{P}}(\mathcal{C}) \subset \mathscr{T}_{\mathbf{P}}(\mathcal{C})$ with the following properties:

- For each $\mathbf{P}$-cluster T in $\mathscr{S}_{\mathbf{P}}(\mathcal{C})$ every $X \in T$ is $\mathbf{P}$-mutable and its $\mathbf{P}$-mutation $(T \setminus \{X\}) \cup \{Y\}$ is also in $\mathscr{S}_{\mathbf{P}}(\mathcal{C})$.
- For each $\mathbf{P}$-cluster T in $\mathscr{S}_{\mathbf{P}}(\mathcal{C})$ the category $\mathcal{C}/\operatorname{add} T$ is abelian.
- If T is a $\mathbf{P}$-cluster such that all $X \in T$ are $\mathbf{P}$-mutable and $\mathcal{C}/\operatorname{add} T$ is abelian then T is in $\mathscr{S}_{\mathbf{P}}(\mathcal{C})$.

We call such a $\mathscr{S}_{\mathbf{P}}(\mathcal{C})$ the <u>$\mathbf{P}$-cluster structure of $\mathcal{C}$</u>.

Remark 3.3.2. Given a pairwise compatibility condition $\mathbf{P}$ there is no assumption that $\mathbf{P}$ induces a cluster structure even if $\mathbf{P}$ induces a cluster theory. Suppose $\mathcal{C}$ is a Krull-Schmidt category and $\mathbf{P}$ a pairwise compatibility condition on the indecomposables of $\mathcal{C}$ such that $\mathbf{P}$ induces the tilting $\mathbf{P}$-cluster theory $\mathscr{T}_{\mathbf{P}}(\mathcal{C})$ of $\mathcal{C}$. Then one would only need to check that for each $\mathbf{P}$-cluster T the category $\mathcal{C}/\operatorname{add} T$ is abelian.

Example 3.3.3. Recall the tilting $\mathbf{N}_n$-cluster theory of $\mathcal{C}(A_n)$ given by triangulations of the $(n+3)$-gon. It is well known (see for a more general result [7], for example) that for any $\mathbf{N}_n$-cluster T, $\mathcal{C}(A_n)/\operatorname{add} T$ is abelian. Thus, $\mathscr{T}_{\mathbf{N}_n}(\mathcal{C}(A_n)) = \mathscr{S}_{\mathbf{N}_n}(\mathcal{C}(A_n))$.

Example 3.3.4. In [14] the authors describe the $\mathbf{N}_\infty$ cluster structure of $\mathcal{C}(A_\infty)$ as those triangulations of the infinity-gon that are functorially finite. In their setting, those triangulations T which are not functorially finite do not yield abelian quotient categories. The authors prove that the functorially finite triangulations indeed yield abelian quotient categories and that every diagonal in a such a triangulation is $\mathbf{N}_\infty$-mutable. Therefore, $\mathscr{S}_{\mathbf{N}_\infty}(\mathcal{C}(A_\infty))$ is precisely the cluster structure in [14].

Definition 3.3.5. Let $(F,\eta) : \mathscr{T}_{\mathbf{P}}(\mathcal{C}) \to \mathscr{T}_{\mathbf{Q}}(\mathcal{D})$ be an embedding of cluster theories. We say (F,η) <u>restricts to an embedding of cluster structures</u> if the following hold.

- The pairwise compatibility conditions $\mathbf{P}$ and $\mathbf{Q}$ induce the cluster structures $\mathscr{S}_{\mathbf{P}}(\mathcal{C})$ and $\mathscr{S}_{\mathbf{Q}}(\mathcal{D})$, respectively.
- If T is a $\mathbf{P}$-cluster in $\mathscr{S}_{\mathbf{P}}(\mathcal{C})$ then $F(T)$ is in $\mathscr{S}_{\mathbf{Q}}(\mathcal{D})$.

Example 3.3.6. The triangulations of the completed infinity-gon do *not* form a $\mathbf{N}_{\overline{\infty}}$-cluster structure. In particular, consider the image $F_\infty^{\overline{\infty}} T$ of an $\mathbf{N}_\infty$-cluster T. If $F_\infty^{\overline{\infty}} T$ has an adic and Prüfer diagonal then T has a left- and right-fountain. Even if T belongs to $\mathscr{S}_{\mathbf{N}_\infty}(\mathcal{C}(A_\infty))$ (thus the left- and right-fountains form one fountain together) the adic and Prüfer diagonals are not $\mathbf{N}_{\overline{\infty}}$-mutable. As the categories $\mathcal{C}(A_\infty)$ and $\mathcal{C}(A_{\overline{\infty}})$ are both discrete we do not have any "wiggle room" to make meaningful adjustments to the embedding $(F_\infty^{\overline{\infty}}, \eta_\infty^{\overline{\infty}})$. Therefore, in the main theorem of this section we omit $\mathscr{T}_{\mathbf{N}_{\overline{\infty}}}(\mathcal{C}(A_{\overline{\infty}}))$.

Example 3.3.7. In [18] the authors show that any indecomposable X in a discrete $\mathbf{N}_\mathbb{R}$ cluster T (in the sense of the points in the fundamental domain) without accumulation is $\mathbf{N}_\mathbb{R}$-mutable. In [19] the authors prove that $\mathcal{C}_\pi/\operatorname{add} T$, for such a cluster T, is abelian. It is noted in [18] that if an $\mathbf{N}_\mathbb{R}$-cluster T is not discrete then there will be $X \in T$ that are not $\mathbf{N}_\mathbb{R}$-mutable. Thus the cluster structure described in [18, 19] is precisely $\mathscr{S}_{\mathbf{N}_\mathbb{R}}(\mathcal{C}_\pi)$.

Remark 3.3.8. The embedding of cluster theories $(F^{\pi}_{\infty}, \eta^{\pi}_{\infty}) = (F^{\pi}_{\overline{\infty}}, \eta^{\pi}_{\overline{\infty}}) \circ (F^{\overline{\infty}}_{\infty}, \eta^{\overline{\infty}}_{\infty})$ does not take $\mathscr{S}_{\mathbf{N}_\infty}(\mathcal{C}(A_\infty))$ to $\mathscr{S}_{\mathbf{N}_\mathbb{R}}(\mathcal{C}_\pi)$.

We define new embeddings of cluster theories $(G^{\pi}_{\infty}, \xi^{\pi}_{\infty}) : \mathscr{T}_{\mathbf{N}_\infty}(\mathcal{C}(A_\infty)) \to \mathscr{T}_{\mathbf{N}_\mathbb{R}}(\mathcal{C}_\pi)$.

Definition 3.3.9. Recall $T_{\pi,i}$ from Definition 3.2.9. For each $i \in \mathbb{Z}$ let $S_{\pi,i} = T_{\pi,i} \setminus \{M(\mathfrak{f}^{-1}(-\infty, i+1))\}$.

Definition 3.3.10. Let $m \geq 1$ be an integer. If m is even define $(G^{m+1}_m, \xi^{m+1}_m) := (F^{m+1}_m, \eta^{m+1}_m)$ (Definition 3.2.1). If m is odd we define (G^{m+1}_m, ξ^{m+1}_m) in a similar, but distinct, way. Let $T_{\mathbf{N}_m}$ be an $\mathbf{N}_m$ cluster for odd m. Define

$$G^{m+1}_m(T_{\mathbf{N}_m}) = \{(i+1) \frac{\quad}{} (j+1) : i \frac{\quad}{} j \in T_{\mathbf{N}_m}\} \cup \{2 \frac{\quad}{} (m+4)\}$$
$${\eta^{m+1}_m}_{T_{\mathbf{N}_m}}(i \frac{\quad}{} j) = (i+1) \frac{\quad}{} (j+1).$$

for $n \geq m+2$ we define (G^n_m, ξ^n_m) to be the composition as before.

Remark 3.3.11. A nearly identical proof to the that of the first half of Lemma A shows that (G^n_m, ξ^n_m) is an embedding of cluster theories for any $1 \leq m \leq n$.

Definition 3.3.12. Let $T_{\mathbf{N}_n}$ be an $\mathbf{N}_n$-cluster. If n is even define the set T_{n+} to be

$$\left\{ -i \frac{\quad}{} i : i \geq \frac{n+2}{2} \right\} \cup \left\{ -i+1 \frac{\quad}{} i : i > \frac{n+2}{2} \right\}$$

and if n is odd define the set T_{n+} to be

$$\left\{ -i \frac{\quad}{} i : i \geq \frac{n+3}{2} \right\} \cup \left\{ -i+1 \frac{\quad}{} i : i \geq \frac{n+3}{2} \right\}.$$

We define $(G^{\infty}_n, \xi^{\infty}_n)$ based on whether n is even or odd. Similarly to $(F^{\infty}_n, \eta^{\infty}_n)$, we define $(G^{\infty}_n, \xi^{\infty}_n)$ by defining what happens to diagonals in the $(n+3)$-gon.

Let $i \frac{\quad}{} j$ be such a diagonal.

- If n is even ${\xi^{\infty}_n}_{T_{\mathbf{N}_n}}(i \frac{\quad}{} j) = (i - \frac{n+2}{2}) \frac{\quad}{} (j - \frac{n+2}{2})$.
- If n is odd ${\xi^{\infty}_n}_{T_{\mathbf{N}_n}}(i \frac{\quad}{} j) = (i - \frac{n+3}{2}) \frac{\quad}{} (j - \frac{i+3}{2})$.

Now we define

$$G^{\infty}_n T_{\mathbf{N}_n} = \{{\xi^{\infty}_n}_{T_{\mathbf{N}_n}}(i \frac{\quad}{} j) : i \frac{\quad}{} j \in T_{\mathbf{N}_n}\} \cup T_{n+}.$$

Proposition 3.3.13. *The image of the embedding of cluster theories $(G^{\infty}_n, \xi^{\infty}_n)$ is contained in $\mathscr{S}_{\mathbf{N}_\infty}(\mathcal{C}(A_\infty))$.*

Proof. Based on our remarks in Example 3.3.4, it suffices to show that given any $\mathbf{N}_n$-cluster $T_{\mathbf{N}_n}$ the category add $F^{\infty}_n T_{\mathbf{N}_n}$ is functorially finite in $\mathcal{C}(A_\infty)$. The authors of [14] prove this is equivalent to $F^{\infty}_n T_{\mathbf{N}_n}$ being either locally finite or having its left- and right-fountain at the same vertex i. However in Definition 3.2.3 we ensure that this is exactly the case. $\square$

Definition 3.3.14. Let $T_{\mathbf{N}_\infty}$ be an $\mathbf{N}_\infty$-cluster. Define

$$T^{\circ}_{\mathbf{N}_\mathbb{R}} = \{M(\mathfrak{f}^{-1}(i,j)) : i \frac{\quad}{} j \in T_{\mathbf{N}_\infty}\} \cup \left(\bigcup_{i \in \mathbb{Z}} T_{\pi,i} \right)$$
$${\xi^{\pi}_{\infty}}_{S_{\mathbf{N}_\infty}}(i \frac{\quad}{} j) = M(\mathfrak{f}^{-1}(i,j)).$$

If $T_{\mathbf{N}_\infty}$ is locally finite $G^{\pi}_{\infty} T_{\mathbf{N}_\infty} = T^{\circ}_{\mathbf{N}_\mathbb{R}}$. If $T_{\mathbf{N}_\infty}$ has a left- and right-fountain at i and j, respectively, $G^{\pi}_{\infty} T_{\mathbf{N}_\infty} = T^{\circ}_{\mathbf{N}_\mathbb{R}} \cup \{M(\mathfrak{f}^{-1}(-\infty, i)), M(\mathfrak{f}^{-1}(-\infty, j))\}$.

Lemma 3.3.15. *As defined, (G^{π}_n, ξ^{π}_n) is an embedding of cluster theories.*

Proof. Let $T_{\mathbf{N}_\infty}$ be an $\mathbf{N}_\infty$-cluster and $T_{\mathbf{N}_\mathbb{R}} = G^\pi_\infty T_{\mathbf{N}_\infty}$. Let $M(x,y)$ be an indecomposable in (the fundamental domain of) $\mathcal{C}_\pi$ and suppose $M(x,y) \cup T_{\mathbf{N}_\mathbb{R}}$ is $\mathbf{N}_\mathbb{R}$-compatible. As always we check various values for $\mathfrak{f}_1(x,y)$.

If $\mathfrak{f}_1(x,y) \notin \mathbb{Z}$ or $\mathfrak{f}_2(x,y) - \mathfrak{f}_1(x,y) < 1$ then $f_1(x,y)$ is of the form $i + \frac{j}{2^\ell}$ for $i \in \mathbb{Z}$, $\ell \geq 0$, and $0 \leq j < 2\ell$ and $\mathfrak{f}_2(x,y)$ is of the form $i + \frac{j+1}{2^\ell}$. If $\mathfrak{f}_1(x,y) \in \mathbb{Z}$ and $\mathfrak{f}_2(x,y) \geq \mathfrak{f}_1(x,y) + 1$ then $\mathfrak{f}_2(x,y) \in \mathbb{Z}$.

Set $i = \mathfrak{f}_1(x,y)$ and $j = \mathfrak{f}_2(x,y)$. Note that for any diagonal $i' \!\bullet\!\!\!\longrightarrow\!\!\bullet j'$ of the infinity-gon, the set $\{i \bullet\!\!\longrightarrow\!\!\bullet j, i' \bullet\!\!\longrightarrow\!\!\bullet j'\}$ is $\mathbf{N}_\infty$-compatible if and only if $\{M(\mathfrak{f}^{-1}(i,j)), M(\mathfrak{f}^{-1}(i',j'))\}$ is $\mathbf{N}_\mathbb{R}$-compatible. Therefore, $T_{\mathbf{N}_\mathbb{R}}$ is an $\mathbf{N}_\mathbb{R}$-cluster.

Note that

$$G^\pi_\infty \left[(T_{\mathbf{N}_\infty} \setminus \{i \bullet\!\!\longrightarrow\!\!\bullet j\}) \cup \{i' \bullet\!\!\longrightarrow\!\!\bullet j'\}\right] = \left(T_{\mathbf{N}_\mathbb{R}} \setminus \{M(\mathfrak{f}^{-1}(i,j))\}\right) \cup \{M(\mathfrak{f}^{-1}(i',j'))\}.$$

With our note on the two different compatibilities in the previous paragraph we see that G^π_∞ is indeed a functor. Furthermore, we have carefully defined the ξ^π_∞s to commute with mutation and so $(G^\pi_\infty, \xi^\pi_\infty)$ is an embedding of cluster theories.

Finally, consider an indecomposable $M(x,y)$ in $T_{\mathbf{N}_\mathbb{R}}$. If $M(x,y) = M(\mathfrak{f}^{-1}(i,j))$ for $i,j \in \mathbb{Z}$ and $j - i \geq 2$ then $M(x,y)$ is $\mathbf{N}_\mathbb{R}$-mutable as we have seen. If $j - i = 1$ then it is one side of a triangle

$$\begin{aligned} R =& \{M(\mathfrak{f}^{-1}(\ell,i)), M(\mathfrak{f}^{-1}(i,i+1)), M(\mathfrak{f}^{-1}(\ell,i+1))\} \\ &\text{or} \\ R =& \{M(\mathfrak{f}^{-1}(i,\ell)), M(\mathfrak{f}^{-1}(i,i+1)), M(\mathfrak{f}^{-1}(i+1,\ell))\}. \end{aligned}$$

depending on whether or not $\ell < i$ or $i < \ell$, respectively.

Then $R \cup \{M(\mathfrak{f}^{-1}(i, i+\frac{1}{2})), M(\mathfrak{f}^{-1}(i+\frac{1}{2}, i+1))\}$ is a quadrilateral with diagonal $M(\mathfrak{f}^{-1}(i,i+1))$. It's replacement is $M(\mathfrak{f}^{-1}(\ell, i+\frac{1}{2}))$ or $M(\mathfrak{f}^{-1}(i+\frac{1}{2}, \ell))$ if $\ell < i$ or $i < \ell$, respectively. For $M(x,y)$ where $\mathfrak{f}(x,y) \notin \mathbb{Z}$ we can perform the same trick by observing that each $T_{\pi,i}$ is fractal-like in nature. Therefore each $M(x,y)$ in $T_{\mathbf{N}_\mathbb{R}}$ is $\mathbf{N}_\mathbb{R}$-mutable and by [18] must discrete. Then apply [19] and see $T_{\mathbf{N}_\mathbb{R}}$ is in $\mathscr{S}_{\mathbf{N}_\mathbb{R}}(\mathcal{C}_\pi)$. $\square$

Theorem 3.3.16. *The top diagram of embeddings of cluster theories restricts to the bottom diagram of embeddings of cluster structures (without the vertical arrow). Both commute.*

$$\begin{tikzcd} & \mathscr{T}_{\mathbf{N}_\infty}(\mathcal{C}(A_\infty)) \arrow[dd, "(G^\pi_\infty{,}\xi^\pi_\infty)"] & \\ \mathscr{T}_{\mathbf{N}_m}(\mathcal{C}(A_m)) \arrow[ur, bend left, "(G^\infty_m{,}\xi^\infty_m)"] \arrow[rr, "(G^n_m{,}\xi^n_m)"] \arrow[dr, bend right, "(G^\pi_m{,}\xi^\pi_m)"'] & & \mathscr{T}_{\mathbf{N}_n}(\mathcal{C}(A_n)) \arrow[ul, bend right, "(G^\infty_n{,}\xi^\infty_n)"'] \arrow[dl, bend left, "(G^\pi_n{,}\xi^\pi_n)"] \\ & \mathscr{T}_{\mathbf{N}_\mathbb{R}}(\mathcal{C}_\pi) & \end{tikzcd}$$

$$\begin{tikzcd} & \mathscr{S}_{\mathbf{N}_\infty}(\mathcal{C}(A_\infty)) & \\ \mathscr{S}_{\mathbf{N}_m}(\mathcal{C}(A_m)) \arrow[ur, bend left, "(G^\infty_m{,}\xi^\infty_m)"] \arrow[rr, "(G^n_m{,}\xi^n_m)"] \arrow[dr, bend right, "(G^\pi_m{,}\xi^\pi_m)"'] & & \mathscr{S}_{\mathbf{N}_n}(\mathcal{C}(A_n)) \arrow[ul, bend right, "(G^\infty_n{,}\xi^\infty_n)"'] \arrow[dl, bend left, "(G^\pi_n{,}\xi^\pi_n)"] \\ & \mathscr{S}_{\mathbf{N}_\mathbb{R}}(\mathcal{C}_\pi) & \end{tikzcd}$$

Proof. The image of $\mathscr{T}_{\mathbf{N}_\infty}(\mathcal{C}(A_\infty))$ in $\mathscr{T}_{\mathbf{N}_\mathbb{R}}(\mathcal{C}_\pi)$ under $(G^\pi_\infty, \xi^\pi_\infty)$ includes only discrete clusters. However, even when the $\mathbf{N}_\infty$-cluster is functorially finite its image may may have accumulation indecomposables. For example $G^\pi_\infty\{-i\bullet\!\!-\!\!\bullet 0, 0\bullet\!\!-\!\!\bullet i : i \geq 2 \in \mathbb{Z}\}$ has limiting indecomposables which are not mutable. And so $G^\pi_\infty \mathscr{S}_{\mathbf{N}_\infty}(\mathcal{C}(A_\infty))$ is not contained in $\mathscr{S}_{\mathbf{N}_\infty}(\mathcal{C}_\pi)$.

By contrast, each image of an $\mathbf{N}_n$ cluster in $\mathscr{T}_{\mathbf{N}_\infty}(\mathcal{C}(A_\infty))$ is locally finite. Thus there are no accumulation indecomposables in $G^\pi_\infty G^\infty_n T_{\mathbf{N}_n}$ and so it is in $\mathscr{S}_{\mathbf{N}_\infty}(\mathcal{C}_\pi)$. A straightforward computation similar those for the (F^*_*, η^*_*)'s shows that both diagrams are commutative. □

4. Geometric Models of E-clusters

In this section we generalize the geometric models of previously existing cluster structures to the **E**-cluster theory. In particular we want to use the cluster theory $\mathscr{T}_{\mathbf{E}}(\mathcal{C}(A_{\mathbb{R}}))$ to construct a new cluster theory from a category whose objects are arcs of some kind and where compatibility is equivalent to not crossing. In each case, this new cluster theory should be equivalent to the previous cluster theory.

In Section 4.1 we address the straight descending orientation of $A_{\mathbb{R}}$ and in Section 4.2 we address the rest of the orientations. We discuss the classification of cluster theories of continuous type A in Section 4.3 conclude with Sections 4.4 and 4.5 where we show pictures of how (continuous) mutations can be interpreted geometrically.

To accomplish all this we need the Definition 4.0.1 and Lemma 4.0.3. Recall that an isomorphism of categories $F : \mathcal{C} \to \mathcal{D}$ has an inverse functor $G : \mathcal{D} \to \mathcal{C}$ such that $GF = 1_{\mathcal{C}}$ and $FG = 1_{\mathcal{D}}$; the compositions are *equal* to the identity.

Definition 4.0.1. Let $\mathcal{C}$ and $\mathcal{D}$ be a Krull-Schmidt categories. Let **P** and **Q** be pairwise compatibility conditions in $\mathcal{C}$ and $\mathcal{D}$ such that they, respectively, induce the cluster theories $\mathscr{T}_{\mathbf{P}}(\mathcal{C})$ and $\mathscr{T}_{\mathbf{Q}}(\mathcal{D})$. A <u>weak equivalence of cluster theories</u> is an embedding of cluster theories $(F, \eta) : \mathscr{T}_{\mathbf{P}}(\mathcal{C}) \to \mathscr{T}_{\mathbf{Q}}(\mathcal{D})$ such that F is an isomorphism of categories. We instead say (F, η) is an <u>isomorphism of cluster theories</u> if additionally each η_T is an isomorphism.

Remark 4.0.2. An isomorphism of categories is ordinarily a very stringent requirement. However, since every cluster theory is a groupoid the only real control we really have over comparing the "size" of each category is to insist they be identically the same via an isomorphism on objects. And, since clusters in a cluster theory are sets of *isomorphism classes of* objects in $\mathcal{C}$ and $\mathcal{D}$, respectively, we are already accounting for the type of equivalence with which we are familiar.

We will use the following lemma in Sections 4.1 and 4.2.

Lemma 4.0.3. *Let $\mathcal{C}$ and $\mathcal{D}$ be Krull-Schmidt categories. Let* **P** *be a pairwise compatibility condition in $\mathcal{C}$ such that* **P** *induces the cluster theory $\mathscr{T}_{\mathbf{P}}(\mathcal{C})$ and let* **Q** *be a pairwise compatibility condition in $\mathcal{D}$. Suppose*

- *there is a bijection $\Phi : Ind(\mathcal{C}) \to Ind(\mathcal{D})$ and*
- *for indecomposables A and B in $\mathcal{C}$, $\{A, B\}$ is* **P***-compatible if and only if $\{\Phi(A), \Phi(B)\}$ is* **Q***-compatible.*

Then **Q** *induces the cluster theory $\mathscr{T}_{\mathbf{Q}}(\mathcal{D})$ and Φ induces an isomorphism of cluster theories $(F, \eta) : \mathscr{T}_{\mathbf{Q}}(\mathcal{D}) \to \mathscr{T}_{\mathbf{P}}(\mathcal{C})$.*

Proof. Let $T_{\mathbf{Q}}$ be a maximally **Q**-compatible set of $\mathcal{D}$-indecomposables. Let

$$T_{\mathbf{P}} = \{\Phi^{-1}(A) : A \in T_{\mathbf{Q}}\}.$$

First we show $T_{\mathbf{P}}$ is an **P**-cluster. Suppose $\{X\} \cup T_{\mathbf{P}}$ is **P**-compatible. Then $\{\Phi(X)\} \cup T_{\mathbf{Q}}$ is **Q**-compatible; however, $T_{\mathbf{Q}}$ is maximally **Q**-compatible and so $\Phi(X) \in T_{\mathbf{Q}}$. Then $X \in T_{\mathbf{P}}$.

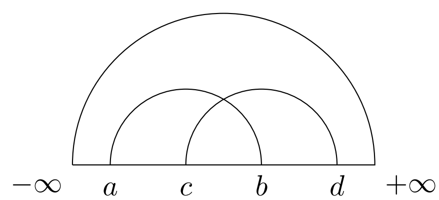
Suppose there is $A \in T_{\mathbf{Q}}$ and $B \notin T_{\mathbf{Q}}$ such that $(T_{\mathbf{Q}} \setminus \{A\}) \cup \{B\}$ is **Q**-compatible. Then $\{A, B\}$ is not **Q**-compatible since $T_{\mathbf{Q}}$ is maximally **Q**-compatible. So $\{\Phi^{-1}(A), \Phi^{-1}(B)\}$ is not **P**-compatible but $(T_{\mathbf{P}} \setminus \{\Phi^{-1}(A)\}) \cup \{\Phi^{-1}(B)\}$ is **P**-compatible. This is a **P**-mutation and so $(T_{\mathbf{P}} \setminus \{\Phi^{-1}(A)\}) \cup \{\Phi^{-1}(B)\}$ is a **P**-cluster. Then by a similar argument to beginning of this proof,

$(T_{\mathbf{Q}}\setminus\{A\})\cup\{B\}$ is maximally $\mathbf{Q}$-compatible. Suppose there is $C\notin T_{\mathbf{Q}}$ such that $(T_{\mathbf{Q}}\setminus\{A\})\cup\{C\}$ is $\mathbf{Q}$-compatible. Again, $\{A,C\}$ is not $\mathbf{Q}$-compatible and $(T_{\mathbf{Q}}\setminus\{A\})\cup\{C\}$ is maximally $\mathbf{Q}$-compatible. However, this means $\Phi^{-1}(B)=\Phi^{-1}(C)$ and so $C=B$. Therefore, $\mathbf{Q}$ induces the cluster theory $\mathscr{T}_{\mathbf{Q}}(\mathcal{D})$.

Let $FT_{\mathbf{Q}}:=T_{\mathbf{P}}$. We have already shown F is a functor. Suppose $T_{\mathbf{Q}}\neq T'_{\mathbf{Q}}$. Then $T_{\mathbf{Q}}\cap T'_{\mathbf{Q}}\subsetneq T_{\mathbf{Q}}$ and $T_{\mathbf{Q}}\cap T'_{\mathbf{Q}}\subsetneq T'_{\mathbf{Q}}$ Using Φ^{-1} we see $T_{\mathbf{P}}\cap T'_{\mathbf{P}}\subsetneq T_{\mathbf{P}}$ and $T_{\mathbf{P}}\cap T'_{\mathbf{P}}\subsetneq T'_{\mathbf{P}}$ which means $T_{\mathbf{P}}\neq T'_{\mathbf{P}}$. Suppose T is a $\mathbf{P}$-cluster. Then $\tilde{T}=\{\Phi(X):X\in T\}$ is a $\mathbf{Q}$-cluster by a similar argument to that at the beginning of the proof. Therefore, F is an isomorphism of categories. Finally, we define $\eta_{T_{\mathbf{Q}}}:T_{\mathbf{Q}}\to T_{\mathbf{P}}$ by $A\mapsto\Phi^{-1}(A)$. This is an isomorphism as desired. □

4.1. **Straight $A_{\mathbb{R}}$.** In this section we construct a geometric model for the cluster theory $\mathscr{T}_{\mathbf{E}}(\mathcal{C}(A_{\mathbb{R}}))$ when $A_{\mathbb{R}}$ has the straight descending orientation. With this orientaiton there is a single frozen indecomposable in every $\mathbf{E}$-cluster (Definition 1.3.4): $P_{+\infty}$. The geometric model for $\mathbf{E}$-clusters of this orientation is a generalization of the models in [14, 4]. The generic diagonal in [4] is very similar to $P_{+\infty}$.

It is straightforward to check that for $M_{|a,b|}$ and $M_{|c,d|}$ where a,b,c,d are all distinct the set $\{M_{|a,b|},M_{|c,d|}\}$ is not $\mathbf{E}$-compatible if and only if $a<c<b<d$ or $c<a<d<b$. Thus for the "macroscopic" perspective we draw $M_{|a,b|}$ as an arc from a to b. If $a<c<b<d$ we can draw the following crossing arcs, each of which is always $\mathbf{E}$-compatible with $P_{+\infty}=M_{(-\infty,+\infty)}$:



However, on the "microscopic" scale things are different. Because we allow all types of intervals, we need two possible arc endpoints per $x\in\mathbb{R}$, but only one endpoint at each $-\infty$ and $+\infty$.

Definition 4.1.1. Let $A_{\mathbb{R}}$ have the straight descending orientation. In the set $\{-,+\}$ we consider $-<+$ and denote an arbitrary element by ε, ε', etc. We give the set $\mathcal{E}:=(\mathbb{R}\times\{-,+\})\cup\{\pm\infty\}$ the total ordering where

- $-\infty<(x,\pm)<+\infty$ for all $x\in\mathbb{R}$ and
- $(x,\varepsilon)<(y,\varepsilon')$ if either $x<y$ or $x=y$ and $\varepsilon<\varepsilon'$.

We call $\mathcal{E}$ the set of <u>endpoints</u>. For ease of notation we write $(-\infty,+)$ for $-\infty$ and $(+\infty,-)$ for $+\infty$. We also write $\overline{a}$ to mean (a,ε) for arbitrary $\varepsilon\in\{-,+\}$.

Definition 4.1.2. An <u>arc with endpoints in $\mathcal{E}$</u> is a pair of endpoints $\theta=(\overline{a},\overline{b})\in\mathcal{E}\times\mathcal{E}$ such that $\overline{a}<\overline{b}$. Let $M_{|a,b|}$ be the indecomposable in $\mathcal{C}(A_{\mathbb{R}})$ that is the image of the indecomposable with the same name in $\text{rep}_k(A_{\mathbb{R}})$. The <u>arc associated to $M_{|a,b|}$</u> is the arc whose endpoints are

- $(a,-)$ if $a\in|a,b|$ and $(a,+)$ if $a\notin|a,b|$, and
- $(b,-)$ if $b\notin|a,b|$ and $(b,+)$ if $b\in|a,b|$.

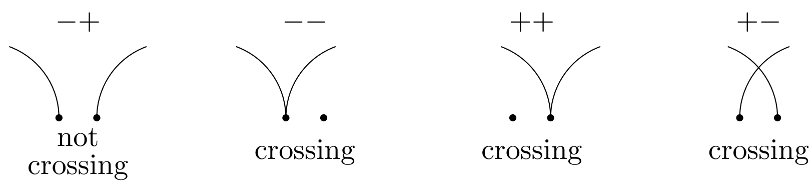
Note that if $M_{|a,b|}=M_{\{a\}}$ then the arc associated to $M_{|a,b|}$ is $((a,-),(a,+))$.

We impose the following rule on our arcs that will generalize in Section 4.2.

Rule 4.1.3. Let $\theta=(\overline{a},\overline{b})$ and $\theta'=(\overline{c},\overline{d})$ be arcs with endpoints in $\mathcal{E}$. We say θ and θ' cross if and only if

- $\overline{a}<\overline{c}\leq\overline{b}<\overline{d}$ or
- $\overline{c}<\overline{a}\leq\overline{d}<\overline{b}$.

!! Notice the difference from the usual convention in the middle. If two arcs meet from opposing sides we still consider them to cross. This only happens on the "microscopic" scale. I.e., for $a < b < d$, $(\overline{a}, (b, -))$ and $((b, +), \overline{d})$ do not cross but any other combination of $+$ and $-$ for $\overline{b}$ will cross.



Definition 4.1.4. Let $\Phi : \mathrm{Ind}(\mathcal{C}(A_{\mathbb{R}})) \to \{\text{arcs with endpoints in } \mathcal{E}\}$ be given by sending $M_{|a,b|}$ to the arc associated to $M_{|a,b|}$. It follows from Definition 4.1.2 that Φ is well-defined.

For the following proposition, recall the definition of a g-vector of an indecomposable V in $\mathcal{C}(A_{\mathbb{R}})$ (Definition 1.3.2).

Proposition 4.1.5. *The map Φ in Definition 4.1.4 is a bijection.*

Proof. Suppose $M_{|a,b|} \ncong M_{|c,d|}$. Then $|a, b| \neq |c, d|$ and so one of the endpoints of the intervals must differ. I.e., even if $a = c$ and $b = d$ then $a \notin |a, b|$ or $a \notin |c, d|$ or $b \notin |a, b|$ or $b \notin |c, d|$. Then the differing endpoints of the arcs associated to $M_{|a,b|}$ and $M_{|c,d|}$ will be different. Let $\theta = (\overline{a}, \overline{b})$ be an arc. Then θ is the arc associated to $M_{|a,b|}$ where $a \in |a, b|$ if and only if $\bar{a} = (a, -)$ and $b \in |a, b|$ if and only if $\overline{b} = (b, +)$. Therefore Φ is both injective and surjective and so bijective. $\square$

Lemma 4.1.6. *Let $M_{|a,b|}$ and $M_{|c,d|}$ be indecomposables in $\mathcal{C}(A_{\mathbb{R}})$. Let θ and θ' be the arcs associated to the respective indecomposables. Then $\{M_{|a,b|}, M_{|c,d|}\}$ is $\mathbf{E}$-compatible if and only if θ and θ' do not cross.*

Proof. Suppose $\{M_{|a,b|}, M_{|c,d|}\}$ is not $\mathbf{E}$-compatible. As we have discussed, if a, b, c, d are all distinct then $a < c < b < d$ or $c < a < d < b$. In either case it follows that θ and θ' cross. Suppose $a = c$. Since the g-vectors of $M_{|a,b|}$ and $M_{|c,d|}$ are not $\mathbf{E}$-compatible, (Definitions 1.3.4 and 1.3.4), we must have $a \notin |a, b|$ and $c \in |c, d|$ or vice versa.

Without loss of generality suppose $a \notin |a, b|$ and $c \in |c, d|$. Then either $d < b$ or if $d = b$ then $d \notin |c, d|$ and $b \in |a, b|$. In either case the arcs θ and θ' cross. We can perform a similar argument starting with $b = d$ and see that θ and θ' cross.

Now suppose θ and θ' cross. Then $\overline{a} < \overline{c} \leq \overline{b} < \overline{d}$ or $\overline{c} < \overline{a} \leq \overline{d} < \overline{b}$. Without loss of generality assume the first. Then if $a = c$, $a \in |a, b|$ but $c \notin |c, d|$. Similarly if $b = d$ then $b \notin |a, b|$ and $d \in |c, d|$. In all cases we see that the g-vectors of $M_{|a,b|}$ and $M_{|c,d|}$ are not $\mathbf{E}$-compatible and so the set $\{M_{|a,b|}, M_{|c,d|}\}$ is not $\mathbf{E}$-compatible. $\square$

Definition 4.1.7. We now define the k-linear Krull-Schmidt category $\mathcal{C}_{A_{\mathbb{R}}}$ similarly to $\mathcal{C}(A_{\overline{\infty}})$ in Definition 2.2.3. The objects of $\mathcal{C}_{A_{\mathbb{R}}}$ are the 0 object and direct suns of arcs with endpoints in $\mathcal{E}$. For a pair of arcs θ and θ' we define hom sets as

$$\mathrm{Hom}_{\mathcal{C}_{A_{\mathbb{R}}}}(\theta, \theta') = \begin{cases} k & (\theta \text{ and } \theta' \text{ cross}) \text{ or } (\theta = \theta') \\ 0 & \text{otherwise.} \end{cases}$$

Let $f : \theta \to \theta'$ and $g : \theta' \to \theta''$ be morphisms (also scalars in k). We define composition as

$$g \circ f = \begin{cases} g \cdot f \in \mathrm{Hom}_{\mathcal{C}_{A_{\mathbb{R}}}}(\theta, \theta'') & (\theta = \theta' \text{ or } \theta' = \theta'') \text{ and } ((\theta \text{ crosses } \theta'') \text{ or } \theta = \theta'') \\ 0 & \text{otherwise.} \end{cases}$$

We say $\{\theta, \theta'\}$ is $\mathbf{N}_{\overline{\mathbb{R}}}$-compatible if and only if $\mathrm{Hom}_{\mathcal{C}_{A_{\mathbb{R}}}}(\theta, \theta') = 0$ or $\theta = \theta'$.

Again by definition each θ is in its own isomorphism class and $\mathcal{C}_{A_{\mathbb{R}}}$ is Krull-Schmidt.

The notation for the categories $\mathcal{C}(A_{\mathbb{R}})$ and $\mathcal{C}_{A_{\mathbb{R}}}$ may look quite similar. However, as we are in the process of showing that we have a geometric model for the clusters in $\mathcal{C}(A_{\mathbb{R}})$ based on arcs, which are the indecomposables in $\mathcal{C}_{A_{\mathbb{R}}}$, this is actually a feature.

Remark 4.1.8. We do not claim that $\mathcal{C}_{A_{\mathbb{R}}}$ has any structure beyond being k-linear (and thus additive) and Krull-Schmidt with the prescribed composition. It is worth noting, however, that each θ is in its own isomorphism class as there are no invertible morphisms with distinct source and target in $\mathcal{C}_{A_{\mathbb{R}}}$. If $f : \theta \to \theta'$ and $g : \theta' \to \theta$ for $\theta \neq \theta'$ then $gf = fg = 0$. In particular, $\operatorname{Ind}(\mathcal{C}_{A_{\mathbb{R}}}) = \{\text{arcs with endpoints in } \mathcal{E}\}$ and $\{\theta, \theta'\}$ is $\mathbf{N}_{\overline{\mathbb{R}}}$-compatible if and only if $\operatorname{Hom}(\theta, \theta') \neq 0$.

Corollary 4.1.9 (to Lemma 4.1.6). *Let $M_{|a,b|}$ and $M_{|c,d|}$ be indecomposables in $\mathcal{C}(A_{\mathbb{R}})$. Let θ and θ' be the arcs associated to the respective indecomposables. Then $\{\theta, \theta'\}$ is $\mathbf{N}_{\overline{\mathbb{R}}}$-compatible if and only if $\{M_{|a,b|}, M_{|c,d|}\}$ is $\mathbf{E}$-compatible.*

Proof. The Corollary is immediately true if $\theta = \theta'$ or $M_{|a,b|} \cong M_{|c,d|}$. Suppose $\theta \neq \theta'$. Then $\{\theta, \theta'\}$ is $\mathbf{N}_{\overline{\mathbb{R}}}$-compatible if and only if $\operatorname{Hom}(\theta, \theta') = 0$ if and only if θ crosses θ'. Now apply Lemma 4.1.6. □

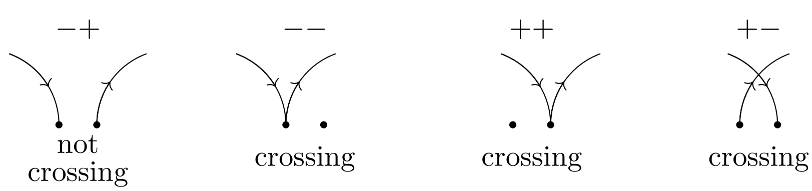
Theorem 4.1.10. *The pairwise compatibility condition $\mathbf{N}_{\overline{\mathbb{R}}}$ induces the $\mathbf{N}_{\overline{\mathbb{R}}}$-cluster theory of $\mathcal{C}_{A_{\mathbb{R}}}$ and Φ induces the isomorphism of cluster theories $(F, \eta) : \mathscr{T}_{\mathbf{N}_{\overline{\mathbb{R}}}}(\mathcal{C}_{A_{\mathbb{R}}}) \to \mathscr{T}_{\mathbf{E}}(\mathcal{C}(A_{\mathbb{R}}))$.*

Proof. Recalling Remark 4.1.8, we have shown there is a bijection $\Phi : \operatorname{Ind}(\mathcal{C}(A_{\mathbb{R}})) \to \operatorname{Ind}(\mathcal{C}_{A_{\mathbb{R}}})$ (Proposition 4.1.5) and that $\{M_{|a,b|}, M_{|c,d|}\}$ is $\mathbf{E}$-compatible if and only if $\{\Phi(M_{|a,b|}), \Phi(M_{|c,d|})\}$ is $\mathbf{N}_{\overline{\mathbb{R}}}$-compatible (Corollary 4.1.9). By Lemma 4.0.3 $\mathbf{N}_{\overline{\mathbb{R}}}$ induces the cluster theory $\mathscr{T}_{\mathbf{N}_{\overline{\mathbb{R}}}}(\mathcal{C}_{A_{\mathbb{R}}})$ and we have the isomorphism of cluster theories given by

$$\begin{aligned} FT_{\mathbf{N}_{\overline{\mathbb{R}}}} &:= \{\Phi^{-1}(\theta) : \theta \in T_{\mathbf{N}_{\overline{\mathbb{R}}}}\} \\ \eta_{T_{\mathbf{N}_{\overline{\mathbb{R}}}}}(\theta) &:= \Phi^{-1}(\theta). \end{aligned}$$

□

4.2. **Other orientations of $A_{\mathbb{R}}$.** We now construct a geometric model for orientations of $A_{\mathbb{R}}$. This model is inspired by the model of representations in [3]. In the case of straight $A_{\mathbb{R}}$, we can think of all the arcs with endpoints in $\mathcal{E}$ as originating at the lower point and ending at the upper point. Our pictures from Rule 4.1.3 can be slightly updated:



All of the arcs are pointing in the same direction when they come infinitesimally close because $\mathcal{E}$ is totally ordered. This only happens when $A_{\mathbb{R}}$ has a straight orientation.

Now suppose $A_{\mathbb{R}}$ has some orientation other than straight descending or straight ascending. We construct $\mathcal{E}_{A_{\mathbb{R}}}$ as the union of two sets: $\mathcal{E}^{\downarrow}_{A_{\mathbb{R}}}$ and $\mathcal{E}^{\uparrow}_{A_{\mathbb{R}}}$. Recall in the definition of a continuous quiver of type A (Definition 1.1.1) that sinks have even index, s_{2n}, and sources have odd index, s_{2n+1}. Recall also that if the sinks and sources of $A_{\mathbb{R}}$ are bounded below then $-\infty$ is assigned the next available index below and similarly for $+\infty$ when the sinks and sources are bounded above. When the sinks and sources are not bounded below (above) we assign the index $-\infty$ to $-\infty$ ($+\infty$ to $+\infty$).

Definition 4.2.1. The sets $\mathcal{E}^{\downarrow}_{A_{\mathbb{R}}}$ and $\mathcal{E}^{\uparrow}_{A_{\mathbb{R}}}$ are defined as follows, where each s_m in the notation is a sink or source in $A_{\mathbb{R}}$ or one of $\pm\infty$ where appropriate:

$$\mathcal{E}^{\downarrow}_{A_{\mathbb{R}}} := (\{x \in \mathbb{R} : \exists \text{ a sink and source } s_{2m} < x < s_{2m+1}\} \cup \{|s_{2n-1}, s_{2n}|\}) \times \{-,+\}$$

$$\mathcal{E}^{\uparrow}_{A_{\mathbb{R}}} := (\{x \in \mathbb{R} : \exists \text{ a source and sink } s_{2m-1} < x < s_{2m}\} \cup \{|s_{2n}, s_{2n+1}|\}) \times \{-,+\}$$

We write $\bar{a}$ to mean (a, ε). That is, we are writing the first coordinate with an overline.

Recall $- < +$ in $\{-,+\}$. We define a total order on $\mathcal{E}^{\downarrow}_{A_{\mathbb{R}}}$:

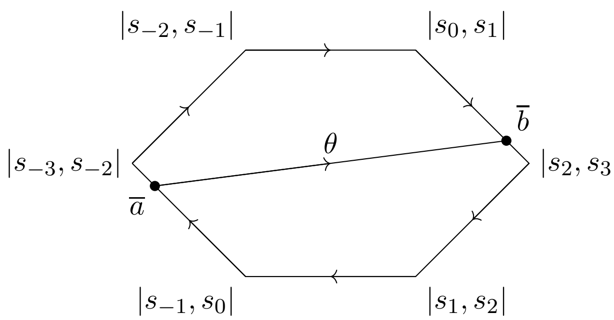
- We say $(x,\varepsilon) < (y,\varepsilon')$ if and only if $x < y$ or $x = y$ and $\varepsilon < \varepsilon'$.
- We say $(|s_m, s_{m+1}|, \varepsilon) < (|s_n, s_{n+1}|, \varepsilon')$ if and only if $s_m < s_n$ or $s_m = s_n$ and $\varepsilon < \varepsilon'$.
- We say $(x,\varepsilon) < (|s_m, s_{m+1}|, \varepsilon')$ if and only if $x < s_m$.
- We say $(|s_m, s_{m+1}|, \varepsilon) < (y, \varepsilon')$ if and only if $s_{m+1} < y$.

We define a total order on $\mathcal{E}^{\uparrow}_{A_{\mathbb{R}}}$ the same way. The set $\mathcal{E}^{\downarrow}_{A_{\mathbb{R}}}$ has a maximal (respectively minimal) element if and only if $+\infty$ has an odd index (respectively $-\infty$ has an even index). Dually, $\mathcal{E}^{\uparrow}_{A_{\mathbb{R}}}$ has a maximal (respectively minimal) element if and only if $+\infty$ has an even index (respectively $-\infty$ has an odd index). Let $\mathcal{E}_{A_{\mathbb{R}}} = \mathcal{E}^{\downarrow}_{A_{\mathbb{R}}} \cup \mathcal{E}^{\uparrow}_{A_{\mathbb{R}}}$.

Definition 4.2.2. Let $A_{\mathbb{R}}$ be a continuous quiver of type A and $\mathcal{E}_{A_{\mathbb{R}}}$ as we have defined. An arc with endpoints in $\mathcal{E}_{A_{\mathbb{R}}}$ is an unordered pair $\{\overline{a}, \overline{b}\} \subset \mathcal{E}_{A_{\mathbb{R}}}$ where $\overline{a} \neq \overline{b}$.

To better visualize why we call these unordered pairs arcs let us use an example.

Example 4.2.3. Let $A_{\mathbb{R}}$ have sinks $s_{-2} = -2, s_0 = 0, s_2 = 2$ and sources $s_{-1} = -1, s_1 = 1$. Then $-\infty = s_{-3}$ and $+\infty = s_3$. The set $\mathcal{E}^{\downarrow}_{A_{\mathbb{R}}}$ has a maximum element and $\mathcal{E}^{\uparrow}_{A_{\mathbb{R}}}$ has a minimum element. We can draw $\mathcal{E}^{\downarrow}_{A_{\mathbb{R}}}$ and $\mathcal{E}^{\uparrow}_{A_{\mathbb{R}}}$ using piece-wise linear curves in the plane and draw arcs on the "macroscopic" scale as lines between two points in $\mathcal{E}_{A_{\mathbb{R}}}$. For example, let $\theta = (\overline{a}, \overline{b})$ where $s_{-2} < a < s_{-1}$ and $s_1 < b < s_2$.



The orientation we have given θ above is according to the following definitions. Definition 4.2.4 handles endpoints that are both in $\mathcal{E}^{\downarrow}_{A_{\mathbb{R}}}$ or $\mathcal{E}^{\uparrow}_{A_{\mathbb{R}}}$ or where both endpoints are in $\mathbb{R} \times \{-,+\}$. Definition 4.2.5 handles when one endpoint is of the form $(|s_n, s_{n+1}|, \varepsilon)$ and the other is in $\mathbb{R} \times \{-,+\}$. Finally, definition 4.2.6 handles when both endpoints are of the form $(|s_n, s_{n+1}|, \varepsilon)$.

Definition 4.2.4. Let $\theta = \{\overline{a}, \overline{b}\}$ be an arc with endpoints in $\mathcal{E}_{A_{\mathbb{R}}}$. The orientation of θ is a bijection $\{\overline{a}, \overline{b}\} \to \{\text{source}, \text{target}\}$ which we define the following way.

If $\overline{a} < \overline{b} \in \mathcal{E}^{\downarrow}_{A_{\mathbb{R}}}$ or $\in \mathcal{E}^{\uparrow}_{A_{\mathbb{R}}}$ then we say $\overline{a}$ is the source and $\overline{b}$ is the target. First suppose both a and b are real numbers (as opposed to one of the $|s_n, s_{n+1}|$'s). If $a = b$ we have covered this and if $a < b$ in $\mathbb{R}$ we say $\overline{a}$ is the source and $\overline{b}$ is the target.

Definition 4.2.5. Let $\theta = \{\overline{a}, \overline{b}\}$ be an arc with endpoints in $\mathcal{E}_{A_\mathbb{R}}$. Now suppose one endpoint of θ is of the form $(|s_n, s_{n+1}|, \varepsilon)$ and the other is (x, ε') for $x \in \mathbb{R}$. Since our pairs are unordered assume that $\overline{a} = (|s_n, s_{n+1}|, \varepsilon)$ and $\overline{b} = (x, \varepsilon')$. Note $x \neq s_m$ for any sink or source (including $\pm\infty$) in $A_\mathbb{R}$.

- If $x < s_n$ then set $\overline{b}$ as the source and $\overline{a}$ as the target.
- If $x > s_{n+1}$ then set $\overline{a}$ as the source and $\overline{b}$ as the target.
- If $s_n < x < s_{n+1}$ then we check ε in $(|s_n, s_{n+1}|, \varepsilon)$. (i) If $\varepsilon = -$ then we set $\overline{a}$ as the source and $\overline{b}$ as the target. (ii) If $\varepsilon = +$ then we set $\overline{b}$ as the source and $\overline{a}$ as the target.

Definition 4.2.6. Let $\theta = \{\overline{a}, \overline{b}\}$ be an arc with endpoints in $\mathcal{E}_{A_\mathbb{R}}$. Suppose $\overline{a} = (|s_m, s_{m+1}|, \varepsilon)$ and $\overline{b} = (|s_n, s_{n+1}|, \varepsilon')$. Since we have covered $\overline{a}, \overline{b} \in \mathcal{E}^{\downarrow}_{A_\mathbb{R}}$ or $\in \mathcal{E}^{\uparrow}_{A_\mathbb{R}}$ in Definition 4.2.4 we may assume by symmetry that m is odd and n is even. If $\varepsilon = -$ let $s_a = s_m$ and if $\varepsilon = +$ let $s_a = s_{m+1}$. If $\varepsilon' = -$ let $s_b = s_n$ and if $\varepsilon' = +$ let $s_b = s_{n+1}$. Then

- If $s_a < s_b$ we say $\overline{a}$ is the source and $\overline{b}$ is the target.
- If $s_b < s_a$ we say $\overline{a}$ is the target and $\overline{b}$ is the source.
- If $s_b = s_a$ then $\{\varepsilon, \varepsilon'\} = \{-, +\}$. (i) If $\varepsilon = -$ then we say $\overline{a}$ is the source and $\overline{b}$ is the target. (ii) If $\varepsilon' = -$ then we say $\overline{b}$ is the source and $\overline{a}$ is the target.

Recall that if the sinks and sources of $A_\mathbb{R}$ are unbounded below (respectively above) then no indecomposable in $\operatorname{rep}_k(A_\mathbb{R})$ may have $-\infty$ as a lower endpoint (respectively $+\infty$ as an upper endpoint) of its support. Thus if we have $M_{|a,b|}$ and $a = -\infty$ (respectively $b = +\infty$) then we know the sinks and sources of $A_\mathbb{R}$ are bounded below (respectively above).

Definition 4.2.7. We now define $\Phi : \operatorname{Ind}(\mathcal{C}(A_\mathbb{R})) \to \{\text{arcs with endpoints in } \mathcal{E}_{A_\mathbb{R}}\}$. Let $M_{|a,b|}$ be an indecomposable in $\mathcal{C}(A_\mathbb{R})$. We will define $\overline{a}$ and $\overline{b}$ in $\mathcal{E}_{A_\mathbb{R}}$. First, $\overline{a}$.

- If $a \in \mathbb{R}$ is neither a sink nor a source then $\overline{a} = (a, \varepsilon)$ where $\varepsilon = -$ if and only if $a \in |a, b|$.
- If $a = -\infty = s_m$ then $\overline{a} = (|s_m, s_{m+1}|, -)$.
- If $-\infty < a = s_m$ and $a \in |a, b|$ then $\overline{a} = (|s_m, s_{m+1}|, -)$.
- If $-\infty < a = s_m$ and $a \notin |a, b|$ then $\overline{a} = (|s_{m-1}, s_m|, +)$.

Now, $\overline{b}$.

- If $b \in \mathbb{R}$ is neither a sink nor a source then $\overline{b} = (b, \varepsilon)$ where $\varepsilon = +$ if and only if $b \in |a, b|$.
- If $b = +\infty = s_n$ then $\overline{b} = (|s_{n-1}, s_n|, +)$.
- If $+\infty > b = s_n$ and $b \in |a, b|$ then $\overline{b} = (|s_{n-1}, s_n|, +)$.
- If $+\infty > b = s_n$ and $b \notin |a, b|$ then $\overline{b} = (|s_n, s_{n+1}|, -)$.

Proposition 4.2.8. *The function Φ in Definition 4.2.7 is a bijection.*

Proof. Let $M_{|a,b|} \not\cong M_{|c,d|}$ be indecomposables in $\mathcal{C}(A_\mathbb{R})$. Using the definition it is straightforward to check that if $a \neq c$ or $b \neq d$ then $\Phi(M_{|a,b|}) \neq \Phi(M_{|c,d|})$. Now suppose $a = c$ and $b = d$. Since $M_{|a,b|} \not\cong M_{|c,d|}$ the endpoints of $|a, b|$ and $|c, d|$ must differ by at least one point. By symmetry and possibly reversing the roles of $M_{|a,b|}$ and $M_{|c,d|}$, assume $a \in |a, b|$ and $c \notin |a, b|$. Then $\overline{a} \neq \overline{c}$ and so $\Phi(M_{|a,b|}) \neq \Phi(M_{|c,d|})$. Thus, Φ is injective.

Let $\theta = \{\overline{a}, \overline{b}\}$ be an arc with endpoints in $\mathcal{E}_{A_\mathbb{R}}$ such that $\overline{a} = (a, \varepsilon)$ is the source and $\overline{b} = (b, \varepsilon')$ is the target. We will not construct an interval $|c, d|$ such that $\Phi(M_{|c,d|}) = \theta$. If $a \in \mathbb{R}$ is neither a sink nor a source then we let $c = a$ and $c \in |c, d|$ if and only if $\varepsilon = -$. If $b \in \mathbb{R}$ is neither a sink nor a source then we let $d = b$ and $d \in |c, d|$ if and only if $\varepsilon' = +$.

Suppose $a = |s_m, s_{m+1}|$. If $\varepsilon = +$ then, since $\overline{a}$ is the source, either $b \in \mathbb{R}$ is greater than s_{m+1} or $b = |s_n, s_{n+1}|$ where $n \geq m$; if $n = m$ then $\overline{b} = (|s_m, s_{m+1}|, -)$. In this case we let $c = s_{m+1}$ and $c \notin |c, d|$. If $\varepsilon = -$ then, since $\overline{a}$ is the source, either $b \in \mathbb{R}$ is greater than s_m or $b = |s_n, s_{n+1}|$ where $n + 1 \geq m$; if $n + 1 = m$ then $\overline{b} = (|s_{m-1}, s_m|, +)$. In this case if $s_m = -\infty$ then we let $c = -\infty$ and note $c \notin |c, d|$. If $s_m > -\infty$ then we let $c = s_m$ and $c \in |c, d|$. We perform the dual constructions for $\overline{b}$ and d as well.

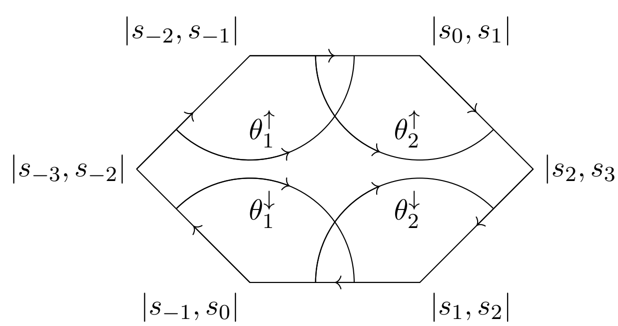
Now we have $c \leq d$ and the requirements for $|c, d|$ to contain either c or d. We need to check $c = d$ to ensure that in this case $c, d \in |c, d|$ by our construction. If $c = d \in \mathbb{R}$ is neither a sink nor a source then $\theta = \{(c, -), (c, +)\}$ and so $|c, d| = \{c\}$. If $c = d \in \mathbb{R}$ is a sink or a source let $s_n = c = d$. By Definition 4.2.6 we see that $|c, d| = \{s_n\}$. Thus Φ is surjective. Therefore, Φ is bijective. □

Our rules for crossing are slightly more complicated than before. The order of stating rules will be: straightforward cases (Rule 4.2.9), then “macroscopic” cases (Rule 4.2.11), then “microscopic” cases (Rule 4.2.13).

Rule 4.2.9. Let θ and θ' be arcs with endpoints in $\mathcal{E}_{A_\mathbb{R}}$.

- If both θ and θ' have endpoints in $\mathcal{E}^{\downarrow}_{A_\mathbb{R}}$ then we follow Rule 4.1.3.
- If both θ and θ' have endpoints in $\mathcal{E}^{\uparrow}_{A_\mathbb{R}}$ then we follow Rule 4.1.3.
- If θ has endpoints in $\mathcal{E}^{\downarrow}_{A_\mathbb{R}}$ and θ' has endpoints in $\mathcal{E}^{\uparrow}_{A_\mathbb{R}}$ then we say θ and θ' do not cross.

Example 4.2.10 (Example of Rule 4.2.9)**.** Let $A_\mathbb{R}$ have sinks $s_{-2} = -2, s_0 = 0, s_2 = 2$ and sources $s_{-1} = -1, s_1 = 1$ with $-\infty = s_{-3}$ and $+\infty = s_3$ as in Example 4.2.3. Let $\theta^{\downarrow}_1$ and $\theta^{\downarrow}_2$ be crossing arcs with endpoints in $\mathcal{E}^{\downarrow}_{A_\mathbb{R}}$. Let $\theta^{\uparrow}_1$ and $\theta^{\uparrow}_2$ be crossing arcs with endpoints in $\mathcal{E}^{\uparrow}_{A_\mathbb{R}}$.



We see that the $\theta^{\downarrow}_i$'s do not cross the $\theta^{\uparrow}_j$'s.

Rule 4.2.11. Let θ and θ' be arcs with endpoints in $\mathcal{E}_{A_\mathbb{R}}$. Suppose θ has one endpoint in $\mathcal{E}^{\downarrow}_{A_\mathbb{R}}$ and the other in $\mathcal{E}^{\uparrow}_{A_\mathbb{R}}$.

(1) Suppose $\theta' = \{\overline{a}, \overline{b}\}$ has both endpoints in $\mathcal{E}^{\downarrow}_{A_\mathbb{R}}$ where $\overline{a} < \overline{b}$. Let $\overline{x}$ be the endpoint of θ in $\mathcal{E}^{\downarrow}_{A_\mathbb{R}}$. If $\overline{a} < \overline{x} < \overline{b}$ we say θ and θ' cross.

(2) Suppose $\theta' = \{\overline{a}, \overline{b}\}$ has both endpoints in $\mathcal{E}^{\uparrow}_{A_\mathbb{R}}$ where $\overline{a} < \overline{b}$. Let $\overline{x}$ be the endpoint of θ in $\mathcal{E}^{\uparrow}_{A_\mathbb{R}}$. If $\overline{a} < \overline{x} < \overline{b}$ we say θ and θ' cross.

(3) Now suppose θ' also has one endpoint in $\mathcal{E}^{\downarrow}_{A_\mathbb{R}}$ and the other in $\mathcal{E}^{\uparrow}_{A_\mathbb{R}}$. Let $\{\overline{a}, \overline{b}\} = \theta$ and $\{\overline{c}, \overline{d}\} = \theta'$ where $\overline{a}, \overline{c} \in \mathcal{E}^{\downarrow}_{A_\mathbb{R}}$ and $\overline{b}, \overline{d} \in \mathcal{E}^{\uparrow}_{A_\mathbb{R}}$. If

$$(\overline{a} < \overline{c} \text{ and } \overline{d} < \overline{b}) \text{ or } (\overline{c} < \overline{a} \text{ and } \overline{b} < \overline{d})$$

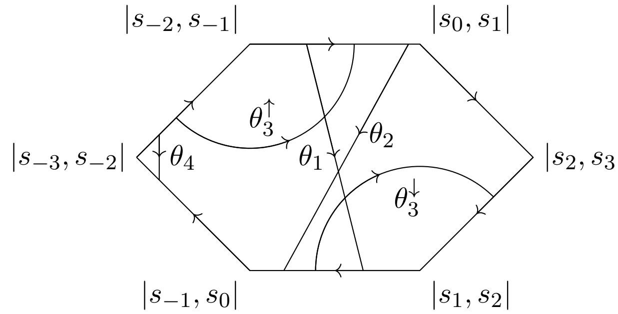
then we say θ and θ' cross. If

$$(\overline{a} < \overline{c} \text{ and } \overline{b} < \overline{d}) \text{ or } (\overline{c} < \overline{a} \text{ and } \overline{d} < \overline{b})$$

then we say θ and θ' do not cross.

Example 4.2.12 (Example of Rule 4.2.11)**.** Let $A_\mathbb{R}$ have sinks $s_{-2} = -2, s_0 = 0, s_2 = 2$ and sources $s_{-1} = -1, s_1 = 1$ with $-\infty = s_{-3}$ and $+\infty = s_3$ as in Example 4.2.3. We will have θ_1 and θ_2 each have one endpoint in $\mathcal{E}^{\downarrow}_{A_\mathbb{R}}$ and the other in $\mathcal{E}^{\uparrow}_{A_\mathbb{R}}$. We'll also have $\theta^{\downarrow}_3$ with endpoints in $\mathcal{E}^{\downarrow}_{A_\mathbb{R}}$

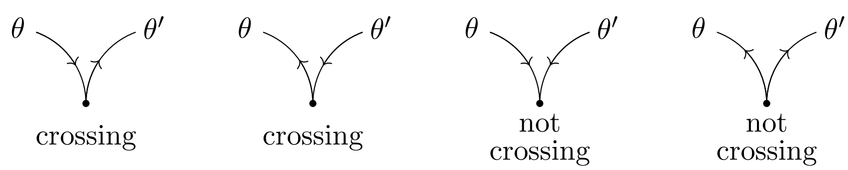
and $\theta_3^{\uparrow}$ with endpoints in $\mathcal{E}^{\uparrow}_{A_{\mathbb{R}}}$. Finally we'll have θ_4 off to the side. We'll have θ_1 cross all the other arcs (none of which cross each other) except θ_4.



Rule 4.2.13. Let $\theta = (\overline{a}, \overline{b})$ and $\theta' = (\overline{c}, \overline{d})$ be arcs with endpoints in $\mathcal{E}_{A_{\mathbb{R}}}$. The only case not covered by Rules 4.2.9 and 4.2.11 is when θ and θ' share an endpoint in $\mathcal{E}^{\downarrow}_{A_{\mathbb{R}}}$ or $\mathcal{E}^{\uparrow}_{A_{\mathbb{R}}}$. Let the shared endpoint be $\overline{a} = \overline{d}$.

- If $\overline{b} = \overline{d}$ then $\theta = \theta'$ and we say they are not crossing.
- Now $\overline{b} \neq \overline{d}$.
  - (i) If $\overline{a}$ is the source of θ but the target of θ', or vice versa, we say θ and θ' are crossing.
  - (ii) If $\overline{a}$ is the source of both θ and θ', or the target of both θ and θ', then we say θ and θ' are not crossing.

We illustrate Rule 4.2.13:



Our definition of arcs, definition of the bijection Φ, and crossing rules have been carefully contrived to yield the following lemma.

Lemma 4.2.14. *Let $M_{|a,b|}$ and $M_{|c,d|}$ be indecomposables in $\mathcal{C}(A_{\mathbb{R}})$. Then $\{M_{|a,b|}, M_{|c,d|}\}$ is* **E**-*compatible if and only if $\Phi(M_{|a,b|})$ and $\Phi(M_{|c,d|})$ do not cross.*

Proof. Setup. Let $\theta = \{\overline{x}, \overline{y}\} = \Phi(M_{|a,b|})$ and $\theta' = \{\overline{z}, \overline{w}\} = \Phi(M_{|c,d|})$. We note that Rules 4.2.9, 4.2.11, and 4.2.13 cover all possible combinations of endpoints for θ and θ'. Our proof will be by assessment of the possible crossings and non-crossings for θ and θ'. That is, we will show that if θ and θ' cross then $\{M_{|a,b|}, M_{|c,d|}\}$ is not **E**-compatible and if θ and θ' do not cross then $\{M_{|a,b|}, M_{|c,d|}\}$ is **E**-compatible. We follow the order in which the rules were stated.

Rule 4.2.9. If the endpoints of θ and θ' are all contained in $\mathcal{E}^{\downarrow}_{A_{\mathbb{R}}}$ or all contained in $\mathcal{E}^{\uparrow}_{A_{\mathbb{R}}}$ then our if and only if statement follows from arguments similar to those in the proof of Lemma 4.1.6. Suppose θ has endpoints in $\mathcal{E}^{\downarrow}_{A_{\mathbb{R}}}$ and θ' has endpoints in $\mathcal{E}^{\uparrow}_{A_{\mathbb{R}}}$. If $a = s_m$ then a is a source and if $b = s_n$ then b is a sink. Dual statements for c and d are true as well. Again using Definition 1.3.4 we see that $\{M_{|a,b|}, M_{|c,d|}\}$ is **E**-compatible.

Rule 4.2.11. Suppose θ has both endpoints in $\mathcal{E}^{\downarrow}_{A_{\mathbb{R}}}$ and θ' has one endpoint each in $\mathcal{E}^{\downarrow}_{A_{\mathbb{R}}}$ and $\mathcal{E}^{\uparrow}_{A_{\mathbb{R}}}$. For now we will assume all four endpoints of θ and θ' are distinct. Suppose $\overline{x} < \overline{y}$, $\overline{z} \in \mathcal{E}^{\downarrow}_{A_{\mathbb{R}}}$, and $\overline{w} \in \mathcal{E}^{\uparrow}_{A_{\mathbb{R}}}$. If $\overline{x} < \overline{z} < \overline{y}$ and $\overline{z}$ is the source of θ' then one verifies there exists a distinguished triangle

$$M_{|a,b|} \to M_{|a,d|} \oplus M_{|c,b|} \to M_{|c,d|} \to$$

in $\mathcal{C}(A_{\mathbb{R}})$. By Proposition 1.3.6 the g-vectors of $M_{|a,b|}$ and $M_{|c,d|}$ are not compatible and so $\{M_{|a,b|}, M_{|c,d|}\}$ is not **E**-compatible. If $\overline{x} < \overline{z} < \overline{y}$ and $\overline{z}$ is the target of θ' then one verifies there exists a distinguished triangle

$$M_{|c,d|} \to M_{|c,b|} \oplus M_{|a,d|} \to M_{|a,b|} \to$$

in $\mathcal{C}(A_{\mathbb{R}})$ and by the same proposition $\{M_{|a,b|}, M_{|c,d|}\}$ is not **E**-compatible. If $\overline{z} < \overline{x}$ or $\overline{y} < \overline{z}$ it is straightforward to check that the g-vectors of $M_{|a,b|}$ and $M_{|c,d|}$ are **E**-compatible and so $\{M_{|a,b|}, M_{|c,d|}\}$ is **E**-compatible.

Now we check when θ and θ' each have one endpoint in $\mathcal{E}^{\downarrow}_{A_{\mathbb{R}}}$ and the other in $\mathcal{E}^{\uparrow}_{A_{\mathbb{R}}}$. Suppsoe θ and θ' cross. Let $\overline{x}, \overline{z} \in \mathcal{E}^{\downarrow}_{A_{\mathbb{R}}}$ and $\overline{y}, \overline{w} \in \mathcal{E}^{\uparrow}_{A_{\mathbb{R}}}$. If $\overline{x} < \overline{z}$ and $\overline{w} < \overline{y}$ this means $a < c \leq d < b$. One then verifies there exists a distinguished triange

$$M_{|a,b|} \to M_{|a,d|} \oplus M_{|c,b|} \to M_{|c,d|} \to$$

in $\mathcal{C}(A_{\mathbb{R}})$. Again using Proposition 1.3.6 we see $\{M_{|a,b|}, M_{|c,d|}\}$ is not **E**-compatible. If $\overline{z} < \overline{x}$ and $\overline{y} < \overline{w}$ one verifies there exists a distinguished triangle

$$M_{|c,d|} \to M_{|c,b|} \oplus M_{|a,d} \to M_{|a,b|} \to$$

in $\mathcal{C}(A_{\mathbb{R}})$. Again $\{M_{|a,b|}, M_{|c,d|}\}$ is not **E**-compatible.

Now suppose θ and θ' do not cross. If $\overline{x} < \overline{z}$ and $\overline{y} < \overline{w}$ one verifies the g-vectors of $M_{|a,b|}$ and $M_{|c,d|}$ are **E**-compatible. If $\overline{z} < \overline{x}$ and $\overline{w} < \overline{y}$ this is true again. Thus, in these last two cases $\{M_{|a,b|}, M_{|c,d|}\}$ is **E**-compatible.

Rule 4.2.13. Now we assume θ and θ' share an endpoint. By symmetries suppose the shared endpoint is $\overline{x} = \overline{z}$ in $\mathcal{E}^{\downarrow}_{A_{\mathbb{R}}}$. If $\overline{y} = \overline{w}$ as well then $\theta = \theta'$ and by Proposition 4.2.8 $M|a,b| \cong M_{|c,d|}$. Thus $\{M_{|a,b|}, M_{|c,d|}\}$ is **E**-compatible.

If $\overline{x} = \overline{z}$ is the source of both θ and θ'. A straightforward calculation shows the g-vectors of $M_{|a,b|}$ and $M_{|c,d|}$ are **E**-compatible. Thus $\{M_{|a,b|}, M_{|c,d|}\}$ is **E**-compatible. Dually if $\overline{x} = \overline{z}$ is the target of both θ and θ' then $\{M_{|a,b|}, M_{|c,d|}\}$ is **E**-compatible.

Finally, suppose $\overline{x} = \overline{z}$ is the source of θ but the target of θ'. First suppose $\overline{x} = \overline{z} = (e, \varepsilon)$ for $e \in \mathbb{R}$. Then $M_{|a,b|} = M_{|e,b|}$ and $M_{|c,d|} = M_{|c,e|}$. In particular, $e \in |e,b|$ if and only if $e \notin |c,e|$. Then one verifies the following is a distinguished triangle in $\mathcal{C}(A_{\mathbb{R}})$:

$$M_{|c,d|} \to M_{|c,b|} \to M_{|a,b|} \to .$$

By Proposition 1.3.6 again we see $\{M_{|a,b|}, M_{|c,d|}\}$ is not **E**-compatible.

Now suppose $\overline{x} = \overline{z} = (|s_n, s_{n+1}|, \varepsilon)$. Since $\overline{z}$ is the target of θ' we know $s_n > -\infty$. Since $\overline{x}$ is the source of θ we know $s_{n+1} < +\infty$. If $\varepsilon = -$ then $|a,b| = [s_n, b|$ and $|c,d| = |c, s_n)$. If $\varepsilon = +$ then $|a,b| = (s_{n+1}, b|$ and $|c,d| = |c, s_{n+1}]$. In either case, one verifies we have the following distinguished triangle in $\mathcal{C}(A_{\mathbb{R}})$:

$$M_{|c,d|} \to M_{|c,b|} \to M_{|a,b|} \to .$$

By Proposition 1.3.6 again we see $\{M_{|a,b|}, M_{|c,d|}\}$ is not **E**-compatible.

Conclusion. For each of Rules 4.2.9, 4.2.11, and 4.2.13 we have shown (i) if θ and θ' cross then $\{M_{|a,b|}, M_{|c,d|}\}$ is not **E**-compatible and (ii) if θ and θ' do not cross then $\{M_{|a,b|}, M_{|c,d|}\}$ is **E**-compatible. Therefore, the lemma holds. $\square$

Definition 4.2.15. For any continuous quiver $A_{\mathbb{R}}$ of type A we define $\mathcal{C}_{A_{\mathbb{R}}}$ just as in Definition 4.1.7 except we replace $\mathcal{E}$ with $\mathcal{E}_{A_{\mathbb{R}}}$. Note again that each θ is in its own isomorphism class. We again say $\{\theta, \theta'\}$ is $\mathbf{N}_{\overline{\mathbb{R}}}$-compatible if and only if $\mathrm{Hom}(\theta, \theta') = 0$ or $\theta = \theta'$.

We can now think of Φ as being a function $\mathrm{Ind}(\mathcal{C}(A_{\mathbb{R}})) \to \mathrm{Ind}(\mathcal{C}_{A_{\mathbb{R}}})$.

Remark 4.2.16. Interestingly, $\mathbf{N}_{\overline{\mathbb{R}}}$-compatible is equivalent to Hom-orthogonal.

Corollary 4.2.17 (to Lemma 4.2.14)**.** *Let $M_{|a,b|}$ and $M_{|c,d|}$ be indecomposables in $\mathcal{C}(A_{\mathbb{R}})$. Then $\{\Phi(M_{|a,b|}), \Phi(M_{|c,d|})\}$ is $\mathbf{N}_{\overline{\mathbb{R}}}$-compatible if and only if $\{M_{|a,b|}, M_{|c,d|}\}$ is $\mathbf{E}$-compatible.*

Proof. The set $\{\Phi(M_{|a,b|}), \Phi(M_{|c,d|})\}$ is $\mathbf{N}_{\overline{\mathbb{R}}}$-compatible if and only if $\operatorname{Hom}(\Phi(M_{|a,b|}), \Phi(M_{|c,d|}))$ is 0 if and only if $\Phi(M_{|a,b|})$ and $\Phi(M_{|c,d|})$ do not cross. Now apply Lemma 4.2.14. □

Theorem 4.2.18. *Let $A_{\mathbb{R}}$ be a continuous quiver of type A. The pairwise compatibility condition $\mathbf{N}_{\overline{\mathbb{R}}}$ induces the $\mathbf{N}_{\overline{\mathbb{R}}}$-cluster theory of $\mathcal{C}_{A_{\mathbb{R}}}$ and Φ induces an isomorphism of cluster theories (F,η) : $\mathscr{T}_{\mathbf{N}_{\overline{\mathbb{R}}}}(\mathcal{C}_{A_{\mathbb{R}}}) \to \mathscr{T}_{\mathbf{E}}(\mathcal{C}(A_{\mathbb{R}}))$.*

Proof. By Proposition 4.2.8 and Definition 4.2.15 we have a bijection $\Phi : \operatorname{Ind}(\mathcal{C}(A_{\mathbb{R}})) \to \operatorname{Ind}(\mathcal{C}_{A_{\mathbb{R}}})$. The set $\{M_{|a,b|}, M_{|c,d|}\}$ is $\mathbf{E}$-compatible if and only if $\{\Phi(M_{|a,b|}), \Phi(M_{|c,d|})\}$ is $\mathbf{N}_{\overline{\mathbb{R}}}$-compatible, by Corollary 4.2.17. Thus by Lemma 4.0.3 $\mathbf{N}_{\overline{\mathbb{R}}}$ induces the cluster theory $\mathscr{T}_{\mathbf{N}_{\overline{\mathbb{R}}}}(\mathcal{C}_{A_{\mathbb{R}}})$ and we have the isomorphism of cluster theories given by

$$\begin{aligned} FT_{\mathbf{N}_{\overline{\mathbb{R}}}} &:= \{\Phi^{-1}(\theta) : \theta \in T_{\mathbf{N}_{\overline{\mathbb{R}}}}\} \\ \eta_{T_{\mathbf{N}_{\overline{\mathbb{R}}}}}(\theta) &:= \Phi^{-1}(\theta). \end{aligned}$$

□

4.3. **On the Classification of Cluster Theories of Continuous Type** A**.** In this section we identity some cluster theories of continuous type A which are isomorphic. We show there are at least four isomorphism classes of such cluster theories. The following notation will be useful.

Notation 4.3.1. Let $\mathscr{T}_{\mathbf{P}}(\mathcal{C})$ and $\mathscr{T}_{\mathbf{Q}}(\mathcal{D})$ be two cluster theories. If there is an isomorphism of cluster theories $(F,\eta) : \mathscr{T}_{\mathbf{P}}(\mathcal{C}) \to \mathscr{T}_{\mathbf{Q}}(\mathcal{D})$ then we say $\mathscr{T}_{\mathbf{P}}(\mathcal{C})$ is isomorphic to $\mathscr{T}_{\mathbf{Q}}(\mathcal{D})$ and write $\mathscr{T}_{\mathbf{P}}(\mathcal{C}) \cong \mathscr{T}_{\mathbf{Q}}(\mathcal{D})$. If we simply write $\mathscr{T}_{\mathbf{P}}(\mathcal{C}) \cong \mathscr{T}_{\mathbf{Q}}(\mathcal{D})$ we are asserting such an isomorphism of cluster theories exists.

Let us begin our discussion proper with the following remark.

Remark 4.3.2. Let $A_{\mathbb{R}}$ and $A'_{\mathbb{R}}$ be two continuous quivers of type A such that $A'_{\mathbb{R}} = A_{\mathbb{R}}^{-1}$, the opposite quiver of $A_{\mathbb{R}}$. Then there exist order preserving bijections $f_{\downarrow\uparrow} : \mathcal{E}^{\downarrow}_{A_{\mathbb{R}}} \overset{\cong}{\to} \mathcal{E}^{\uparrow}_{A'_{\mathbb{R}}}$ and $f_{\uparrow\downarrow}$: $\mathcal{E}^{\uparrow}_{A_{\mathbb{R}}} \overset{\cong}{\to} \mathcal{E}^{\downarrow}_{A'_{\mathbb{R}}}$. Let $f : \mathcal{E}_{A_{\mathbb{R}}} \overset{\cong}{\to} \mathcal{E}_{A'_{\mathbb{R}}}$ be the bijection on the unions. Note our rules of crossing are symmetric with respect to $\mathcal{E}^{\uparrow}_{A_{\mathbb{R}}}$ and $\mathcal{E}^{\downarrow}_{A_{\mathbb{R}}}$ except at sinks and sources. Let θ_1 and θ_2 be arcs with endpoints in $\mathcal{E}_{A_{\mathbb{R}}}$. Let θ'_1 and θ'_2 be arcs whose endpoints in $\mathcal{E}_{A'_{\mathbb{R}}}$ are the image under f of the endpoints of θ_1 and θ_2. Then θ_1 and θ_2 cross if and only if θ'_1 and θ'_2 cross, since we have reversed the roles of sinks and sources in $A_{\mathbb{R}}$ to $A'_{\mathbb{R}}$. Using Lemma 4.0.3 yields that $\mathscr{T}_{\mathbf{N}_{\overline{\mathbb{R}}}}(\mathcal{C}_{A_{\mathbb{R}}}) \cong \mathscr{T}_{\mathbf{N}_{\overline{\mathbb{R}}}}(\mathcal{C}_{A'_{\mathbb{R}}})$. Note we have not changed the number of sinks and sources from $A_{\mathbb{R}}$ to $A_{\mathbb{R}}^{-1}$.

Remark 4.3.3. Let $A_{\mathbb{R}}$ and $A'_{\mathbb{R}}$ be two continuous quivers of type A such that $A'_{\mathbb{R}}$ is the quiver where each sink $s'_{2n} \in \mathbb{R}$ is equal to the sink $-s_{-2n}$ in $A_{\mathbb{R}}$ and similarly with the sources. Then we have order *reversing* bijections $f^{\downarrow\downarrow} : \mathcal{E}^{\downarrow}_{A_{\mathbb{R}}} \overset{to}{\cong} \mathcal{E}^{\downarrow}_{A'_{\mathbb{R}}}$ and $f^{\uparrow\uparrow} : \mathcal{E}^{\uparrow}_{A_{\mathbb{R}}} \overset{\cong}{\to} \mathcal{E}^{\uparrow}_{A'_{\mathbb{R}}}$. Then as in Remark 4.3.2 we have $\mathscr{T}_{\mathbf{N}_{\overline{\mathbb{R}}}}(\mathcal{C}_{A_{\mathbb{R}}}) \cong \mathscr{T}_{\mathbf{N}_{\overline{\mathbb{R}}}}(\mathcal{C}_{A'_{\mathbb{R}}})$. Again, this doesn't change the number of sinks or the number of sources. Instead this reverses their spacing throughout $\mathbb{R}$. Denote this quiver by $-A_{\mathbb{R}}$.

Using Remarks 4.3.1 and 4.3.3 we have, for any continuous quiver $A_{\mathbb{R}}$ of type A with at least one sink or source in $\mathbb{R}$, a commutative diagram of isomorphisms of cluster theories:

$$\begin{tikzcd} \mathscr{T}_{\mathbf{N}_{\overline{\mathbb{R}}}}(\mathcal{C}_{A_{\mathbb{R}}}) \arrow[r, leftrightarrow, "\cong"] \arrow[d, leftrightarrow, "\cong"'] & \mathscr{T}_{\mathbf{N}_{\overline{\mathbb{R}}}}(\mathcal{C}_{-A_{\mathbb{R}}}) \arrow[d, leftrightarrow, "\cong"] \\ \mathscr{T}_{\mathbf{N}_{\overline{\mathbb{R}}}}(\mathcal{C}_{A_{\mathbb{R}}^{-1}}) \arrow[r, leftrightarrow, "\cong"'] & \mathscr{T}_{\mathbf{N}_{\overline{\mathbb{R}}}}(\mathcal{C}_{-A_{\mathbb{R}}^{-1}}). \end{tikzcd}$$

Since the continuum is "stretchy," if $A_{\mathbb{R}}$ and $A'_{\mathbb{R}}$ have the same number of sinks and sources in $\mathbb{R}$ then their cluster theories will be isomorphic. This includes half-bounded and unbounded sinks and sources.

We have two remaining isomorphsisms of cluster theories we would like:

(1) Any isomorphism between $\mathscr{T}_{\mathbf{N}_{\overline{\mathbb{R}}}}(\mathcal{C}_{A_{\mathbb{R}}})$ and $\mathscr{T}_{\mathbf{N}_{\overline{\mathbb{R}}}}(\mathcal{C}_{A'_{\mathbb{R}}})$ where $A_{\mathbb{R}}$ has an even number ≥ 2 of sinks and sources in $\mathbb{R}$ and $A'_{\mathbb{R}}$ has an odd number of sinks and sources in $\mathbb{R}$.
(2) An isomorphism between $\mathscr{T}_{\mathbf{N}_{\overline{\mathbb{R}}}}(\mathcal{C}_{A_{\mathbb{R}}})$ and $\mathscr{T}_{\mathbf{N}_{\overline{\mathbb{R}}}}(\mathcal{C}_{A'_{\mathbb{R}}})$ where $A_{\mathbb{R}}$ has no sinks or sources in $\mathbb{R}$ and $A'_{\mathbb{R}}$ has an even number ≥ 2 of sinks and sources in $\mathbb{R}$.

We immediately share the unfortunate news:

Proposition 4.3.4. *Let $A_{\mathbb{R}}$ be a continuous quiver of type A with straight descending or straight ascending orientation. Let $A'_{\mathbb{R}}$ be a continuous quiver of type A with at least one sink or source in $\mathbb{R}$. Then there is no isomorphism of cluster theories $\mathscr{T}_{\mathbf{N}_{\overline{\mathbb{R}}}}(\mathcal{C}_{A_{\mathbb{R}}}) \to \mathscr{T}_{\mathbf{N}_{\overline{\mathbb{R}}}}(\mathcal{C}_{A'_{\mathbb{R}}})$.*

Proof. The arc θ corresponding to the indecomposable $M_{(-\infty,+\infty)}$ in $\mathcal{C}(A_{\mathbb{R}})$ is in every $\mathbf{N}_{\overline{\mathbb{R}}}$-cluster of $\mathscr{T}_{\mathbf{N}_{\overline{\mathbb{R}}}}(\mathcal{C}_{A_{\mathbb{R}}})$. The arcs corresponding to the projectives from $\operatorname{rep}_k(A'_{\mathbb{R}})$ form an $\mathbf{N}_{\overline{\mathbb{R}}}$-cluster; this is similarly true for the arcs corresponding to the invectives from $\operatorname{rep}_k(A'_{\mathbb{R}})$. However, there are not projective-injective objects in $\operatorname{rep}_k(A'_{\mathbb{R}})$ and so these two clusters share no elements. Therefore, there cannot be such an isomorphism of cluster theories. $\square$

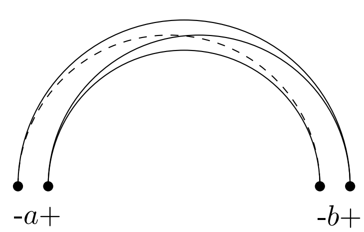
This leaves us with at least four isomorphism classes of cluster theories of continuous type A: (i) no sinks or sources in $\mathbb{R}$, (ii) finitely-many sinks and sources in $\mathbb{R}$, (iii) half-bounded sinks and sources in $\mathbb{R}$, and (iv) unbounded sinks and sources in $\mathbb{R}$. However, it is not clear whether (ii) is just one class, separate classes for even and odd numbers, or a separate class for all numbers.

Open Questions:

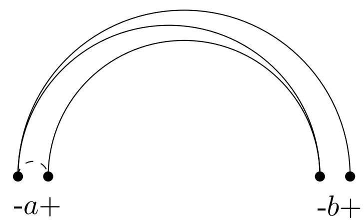
- Does there exist a weak equivalence of cluster theories $\mathscr{T}_{\mathbf{N}_{\overline{\mathbb{R}}}}(\mathcal{C}_{A_{\mathbb{R}}}) \to \mathscr{T}_{\mathbf{N}_{\overline{\mathbb{R}}}}(\mathcal{C}_{A'_{\mathbb{R}}})$ or $\mathscr{T}_{\mathbf{N}_{\overline{\mathbb{R}}}}(\mathcal{C}_{A'_{\mathbb{R}}}) \to \mathscr{T}_{\mathbf{N}_{\overline{\mathbb{R}}}}(\mathcal{C}_{A_{\mathbb{R}}})$ where $A_{\mathbb{R}}$ has no sinks or sources in $\mathbb{R}$ and $A'_{\mathbb{R}}$ has an even number ≥ 2 of sinks and sources in $\mathbb{R}$?
- Does there exist an isomorphism of cluster theories or weak equivalence of cluster theories $\mathscr{T}_{\mathbf{N}_{\overline{\mathbb{R}}}}(\mathcal{C}_{A_{\mathbb{R}}}) \to \mathscr{T}_{\mathbf{N}_{\overline{\mathbb{R}}}}(\mathcal{C}_{A'_{\mathbb{R}}})$ or $\mathscr{T}_{\mathbf{N}_{\overline{\mathbb{R}}}}(\mathcal{C}_{A'_{\mathbb{R}}}) \to \mathscr{T}_{\mathbf{N}_{\overline{\mathbb{R}}}}(\mathcal{C}_{A_{\mathbb{R}}})$ where $A_{\mathbb{R}}$ has an odd number n of sinks and sources in $\mathbb{R}$ and $A'_{\mathbb{R}}$ has $n+1$ sinks and sources in $\mathbb{R}$?

4.4. **Connection to E-Mutations.** In this section we use our geometric models to show how one may picture an $\mathbf{E}$-mutation by drawing the corresponding $\mathbf{N}_{\overline{\mathbb{R}}}$-mutation. Because of our rules on crossing, mutation is not as clearly described as swapping diagonals of a quadrilateral. However, we can make similar descriptions. Let us begin with the "microscopic" scale. Let $A_{\mathbb{R}}$ be a continuous quiver of type A with at least one sink or source in $\mathbb{R}$. Let $a < b \in \mathbb{R}$ such that neither a nor b is a sink or source and $(a,\varepsilon),(b,\varepsilon) \in \mathcal{E}^{\downarrow}_{A_{\mathbb{R}}}$ for any ε.

Let T be an $\mathbf{N}_{\overline{\mathbb{R}}}$-cluster that contains the arcs $\{(a,-),(b,+)\}$, $\{(a,+),(b,+)\}$, and $\{(a,+),(b,-)\}$. These correspond to the indecomposables $M_{[a,b]}$, $M_{(a,b]}$, and $M_{(a,b)}$, respectively, in $\mathcal{C}(A_{\mathbb{R}})$. We can mutate at $\{(a,+),(b,+)\}$ to obtain $(T \setminus \{\{(a,+),(b,+)\}\}) \cup \{\{(a,-),(b,-)\}\}$. The picture one should have in mind is this:



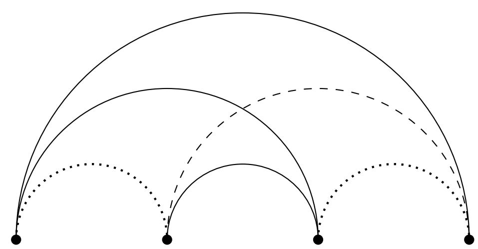
We are exchanging the middle solid arc for the dashed arc. As we can see it is not quite a quadrilateral. We can further exchange $\{(a,+),(b,-)\}$ for $\{(a,-),(a,+)\}$:



We now move to the "macroscopic" scale. In $\mathcal{C}(A_{\mathbb{R}})$, we know that if $\{M_{|a,b|}M_{|c,d|}\}$ is not **E**-compatible then, up to reversing the roles of the indecomposables, we have the following distinguished triangle in $\mathcal{C}(A_{\mathbb{R}})$:

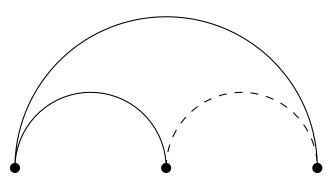
$$M_{|a,b|} \to M_{|a,d|} \oplus M_{|c,b|} \to M_{|c,d|} \to$$

where one of $M_{|a,d|}$ or $M_{|c,b|}$ may be 0. Now suppose we are **E**-mutating in some cluster at $M_{|a,b|}$ and obtain $M_{|c,d|}$. If the middle object in the triangle is not indecomposable then we have two of the four sides of the quadrilateral we are used to seeing.

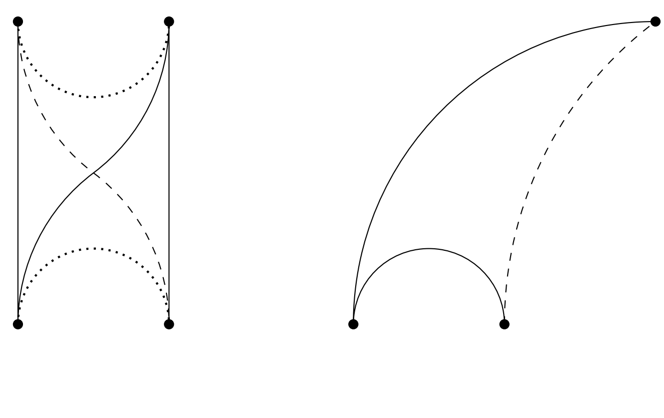


However, we do not know if we have the two dotted arcs that complete the quadrilateral. The dotted arcs may be incompatible with $M_{|a,b|}$ and/or $M_{|c,d|}$. For example, if $b \in |a,b|$ then there is no arc with b as a lower endpoint that is compatible with $M_{|a,b|}$ or $M_{|c,b|}$.

In the case where one of $M_{|c,b|}$ or $M_{|a,d|}$ is 0, we instead get the following picture:

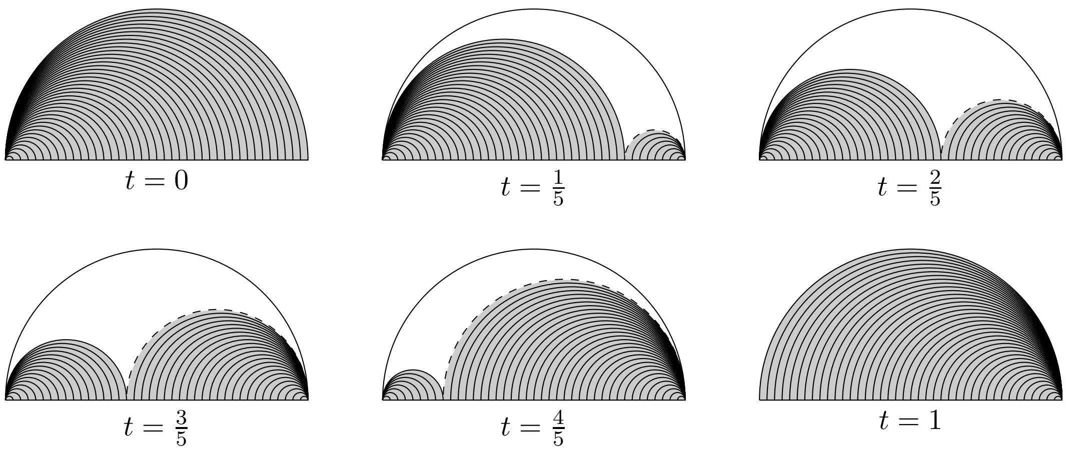


We cannot have both the smaller arc and the dashed arc in this case because there would be an extension. We can draw similar pictures if some of the endpoints happen to be on $\mathcal{E}^{\downarrow}_{A_{\mathbb{R}}}$ and others in $\mathcal{E}^{\uparrow}_{A_{\mathbb{R}}}$:



4.5. **Connection to Continuous E-mutations.** The more interesting pictures are those of continuous mutations. In this section we use our geometric models to show how one may picture a continuous $\mathbf{E}$-mutation by drawing the corresponding continuous $\mathbf{N}_{\overline{\mathbb{R}}}$-mutation. In particular, those continuous mutations that cannot be described as any type of sequence of mutations. Consider $A_{\mathbb{R}}$ with straight descending orientation. Let T be $\{P_x, M_{\{x\}} : x \in \mathbb{R}\} \cup \{P_{+\infty}\}$ and $\phi : \mathbb{R}to(0,1)$ be some order *reversing* bijection. Let $f : \{P_x : x \in \mathbb{R}\} \to [0,1]$ be given by $P_x \mapsto \phi(x)$. Let $g : \{I_x : x \in \mathbb{R}\} \to [0,1]$ be given by $I_x \mapsto \phi(x)$ and let $T' = \{I_x, M_{\{x\}} : x \in \mathbb{R}\} \cup \{P_{+\infty}\}$. Then we have a continuous mutation $T \to T'$. (Something similar was done in Proposition 2.2.7, which is more robust.)

We would like to show what this looks like in terms of arcs. Of course, we can't depict each of the mutations at time t for all $t \in (0,1)$ as we do not have uncountably-many pages. However, we can think of the process as an animation and take a few select frames so that we have the general idea. One could make a proper animation at a sufficiently high frame rate to get the full effect. However, we will just show 6 frames. The first and sixth frames will be T and T', respectively. The other four frames will be at time $\frac{i}{5}$ for $i \in \{1,2,3,4\}$. We include ≈ 40 arcs of the uncountably many in the same way one includes level curves in a topographical map.



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