Unitary pseudonatural transformations

Dominic Verdon

School of Mathematics
University of Bristol

May 5, 2020

Abstract

We define a notion of unitarity for pseudonatural transformations between unitary pseudofunctors on pivotal dagger 2-categories. We prove that the category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ of unitary pseudofunctors $\mathcal{C} \to \mathcal{D}$, unitary pseudonatural transformations and modifications is dagger with left and right duals, and furthermore pivotal dagger upon restriction to pivotal functors.

1 Introduction

1.1 Overview

Natural transformations between functors are a crucial element of category theory. We recall the basic definition. Let $\mathcal{C}, \mathcal{D}$ and $F, G : \mathcal{C} \to \mathcal{D}$ be functors. Then a *natural transformation* $\alpha : F \to G$ is a set of morphisms $\{\alpha_X : F(X) \to G(X)\}_{X \in \mathrm{Obj}(\mathcal{C})}$ such that for any $f : X \to Y$ in $\mathcal{D}$ the following diagram commutes:

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\alpha_X"'] & F(Y) \arrow[d, "\alpha_Y"] \\ G(X) \arrow[r, "G(f)"'] & G(Y) \end{tikzcd}$$

We say that a natural transformation is *invertible* if its components $\{\alpha_X\}$ are invertible in $\mathcal{D}$. If $\mathcal{D}$ is a dagger category, then we say that an invertible natural transformation is *unitary* if its components are additionally unitary in $\mathcal{D}$.

Perhaps more naturally, these notions of invertibility may be defined with respect to the category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ of functors and natural transformations. An invertible natural transformation is just an invertible morphism in this category. If $\mathcal{C}, \mathcal{D}$ are dagger and the functors unitary, the category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ inherits a dagger structure; a unitary natural transformation is a unitary morphism in this dagger category.

Just as natural transformations between functors are a fundamental part of category theory, pseudonatural transformations between pseudofunctors (Definition 3.1) are an important part of 2-category theory, which includes monoidal category theory. This short paper makes the elementary step of generalising the above-mentioned notions of invertibility to pseudonatural transformations.

Let $\mathcal{C}, \mathcal{D}$ be 2-categories, and let $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ be the 2-category of pseudofunctors $\mathcal{C} \to \mathcal{D}$, pseudonatural transformations and modifications. We first consider invertibility. The most general notion of an invertible 1-morphism in a 2-category is duality, or adjunction; the right (resp. left) 'inverse' 1-morphism is called a right (resp. left) dual. A 2-category is said to 'have right (resp. left) duals' when every 1-morphism has a right (resp. left) dual. A coherent choice of left and right duals for every object is called a pivotal structure; a 2-category with a pivotal structure is called pivotal.

Here we unpack the notion of duality for pseudonatural transformations (Definition 4.1) and show the following facts.

- If $\mathcal{C}$ has left (resp. right) duals and $\mathcal{D}$ has right (resp. left) duals, then $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ has right (resp. left) duals (Corollary 4.5).
- If $\mathcal{C},\mathcal{D}$ are pivotal, then $\mathrm{Fun}_p(\mathcal{C},\mathcal{D})$ is also pivotal, where the subscript p represents restriction to pivotal functors. (Theorem 4.8).

If the 2-categories $\mathcal{C},\mathcal{D}$ additionally have a dagger structure, we restrict $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ to unitary pseudofunctors. At this point we need a notion of unitarity for pseudonatural transformations. This requirement arises either physically, by the desire that the components of the transformation should be unitary in $\mathcal{D}$; or categorically, by the desire that the 2-category $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ should itself inherit a dagger structure (for general pseudonatural transformations, there is no obvious dagger structure on $\mathrm{Fun}(\mathcal{C},\mathcal{D})$).

We could say that a pseudonatural transformation is unitary when all its 2-morphism components are unitary in $\mathcal{D}$. However, the more categorically natural way of specifying unitarity of a pseudonatural transformation is to say that its dagger is equal to its inverse — i.e. its right dual. When $\mathcal{C},\mathcal{D}$ are pivotal dagger (i.e. possessing compatible pivotal and dagger structures), we observe that there is a notion of the *dagger* of a pseudonatural transformation such that this definition makes sense and gives the same result. Indeed, for $\mathcal{C},\mathcal{D}$ pivotal dagger, we have the following:

- The following definitions of *unitary pseudonatural transformation* are equivalent (Proposition 5.2):
  - All 2-morphism components of a pseudonatural transformation are unitary.
  - The dual of a pseudonatural transformation is equal to its dagger.
- Upon restriction to unitary pseudonatural transformations, the 2-category $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ inherits a dagger structure. Moreover, $\mathrm{Fun}_p(\mathcal{C},\mathcal{D})$ inherits a pivotal dagger structure (Theorem 5.5).

Our main motivation for this work is the study of unitary pseudonatural transformations between fibre functors on representation categories of compact quantum groups, which is the subject of a companion paper [10]. As a 2-categorical example, we remark, but do not show here, that Jones' biunitaries [3, §2.11] can be understood as examples of unitary pseudonatural transformations between pseudofunctors embedding a planar subalgebra.

1.2 Acknowledgements

The author thanks Ashley Montanaro, David Reutter, Changpeng Shao and Jamie Vicary for useful discussions related to this work. The work was supported by EPSRC.

1.3 Structure

In Section 2 we introduce necessary background material for the rest of this paper. In Section 3 we recall the basic theory of pseudonatural transformations. In Section 4 we discuss dualisability of pseudonatural transformations. In Section 5 we introduce unitary pseudonatural transformations.

2 Background: Pivotal dagger 2-categories

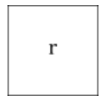
2.1 Diagrams for 2-categories

A 2-category is a generalisation of a category. While a category has objects, morphisms, and composition laws, a 2-category has objects, morphisms, and morphisms between the morphisms, called 2-morphisms, obeying composition laws. The general 'weak' definition of 2-category can be found in e.g. [4]. Roughly, a 2-category $\mathcal{C}$ is defined by a set of objects of objects $r,s,\dots$, together with a category of morphisms $\mathcal{C}(r,s)$ for every pair of objects, and functors $\mathcal{C}(r,s)\times\mathcal{C}(s,t)\to\mathcal{C}(r,t)$ defining composition of these Hom-categories, with various coherence data.

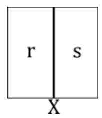
Fortunately, 2-categories are much more manageable than the general definition might suggest. Recall that every monoidal category is equivalent to a strict monoidal category [5]. This allows us to assume our monoidal categories are strict, allowing the use of a convenient and well-known diagrammatic calculus [8]. In 2-category theory, a similar strictification result holds — every weak 2-category is equivalent to a strict 2-category [4]. We can therefore also use a diagrammatic calculus in this case.

A monoidal category is precisely a 2-category with a single object, where 1-morphisms are the 'objects' of the monoidal category, 2-morphisms are the 'morphisms', and composition of 1-morphisms is the 'monoidal product'. The 2-categorical diagrammatic calculus is nothing more than the diagrammatic calculus for monoidal categories enhanced with region labels. We briefly summarise this calculus now, closely following the exposition in [6]. More information can be found in e.g. [2].

Objects $r, s, \cdots$ of a 2-category are represented by labelled regions:

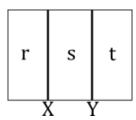


1-morphisms $X : r \to s$ are represented by edges, separating the region r on the left from the region s on the right:

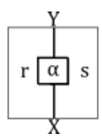


Edges corresponding to identity 1-morphisms $\mathrm{id}_r : r \to r$ are invisible in the diagrammatic calculus.

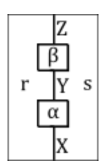
1-morphisms compose from left to right. That is, for 1-morphisms $X : r \to s, Y : s \to t$, the composite[1] $X \circ Y : r \to t$ is represented as:



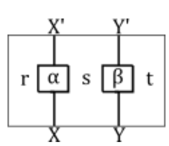
For two parallel 1-morphisms $X, Y : r \to s$, a 2-morphism $\alpha : X \to Y$ is represented by a vertex in the diagram, drawn as a box:



2-morphisms can compose in two ways, depending on their type. For parallel 1-morphisms $X, Y, Z : r \to s$, 2-morphisms $\alpha : X \to Y, \beta : Y \to Z$ can be composed 'vertically' to obtain a 2-morphism $\alpha \circ_V \beta : X \to Z$. This is represented by vertical juxtaposition in the diagram:



For 1-morphisms $X, X' : r \to S$ and $Y, Y' : s \to t$, 2-morphisms $\alpha : X \to X'$ and $\beta : Y \to Y'$ can be composed 'horizontally' to obtain a 2-morphism $\alpha \circ_H \beta : X \circ Y \to X' \circ Y'$. This is represented by horizontal juxtaposition in the diagram:



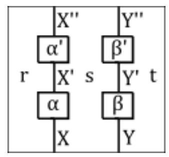
[1]For 1-morphisms, $X \circ Y$ is 'X followed by Y' rather than 'Y followed by X'.

As with 1-morphisms, the identity 2-morphisms $\mathrm{id}_X : X \to X$ are invisible in the diagrammatic calculus.

2-categories satisfy the *interchange law*. For any 1-morphisms $X, X', X'' : r \to s$ and $Y, Y', Y'' : s \to t$, and 2-morphisms $\alpha : X \to X'$, $\alpha' : X' \to X''$, $\beta : Y \to Y'$, $\beta' : Y' \to Y''$:

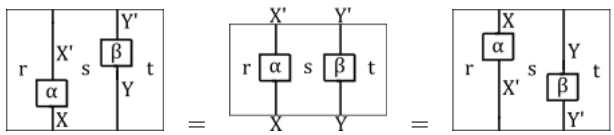
$$(\alpha \circ_V \alpha') \circ_H (\beta \circ_V \beta') = (\alpha \circ_H \beta) \circ_V (\alpha' \circ_H \beta')$$

This corresponds to well-definition of the following diagram:



(1)

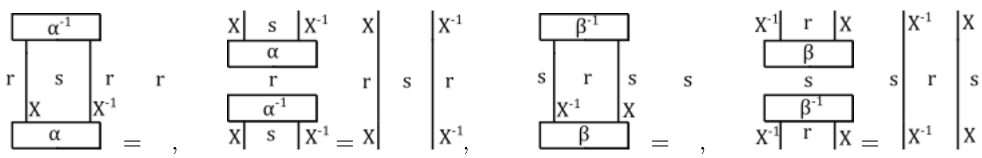
We also have the following *sliding equalities*, which may be obtained by taking some morphisms to be the identity in (1):



These equalities allow us to move 2-morphism boxes past each other provided there are no obstructions.

Before moving onto pseudofunctors, we give a first definition from 2-category theory. Equivalence is a strong notion of invertibility of a 1-morphism in a 2-category. From now on we will not draw an enclosing box around diagrams.

Definition 2.1. Let $\mathcal{C}$ be a 2-category and let $X : r \to s$ be a 1-morphism in $\mathcal{C}$. We say that X is an *equivalence* if there exists a 1-morphism $X^{-1} : s \to r$, and invertible[2] 2-morphisms $\alpha : \mathrm{id}_r \to X \circ X^{-1}$ and $\beta : \mathrm{id}_s \to X^{-1} \circ X$. In diagrams, the equations for invertibility of α, β are as follows, where α^{-1}, β^{-1} are the inverse 2-morphisms:



(2)

If there exists an equivalence $X : r \to s$ we say that the objects r and s are *equivalent* in $\mathcal{C}$.

2.2 Diagrams for pseudofunctors

While our 2-categories are strictified, allowing us to use the diagrammatic calculus, we will consider functors between them which are not strict. For this, we use a graphical calculus of *functorial boxes* previously applied in the special case of monoidal functors [7].

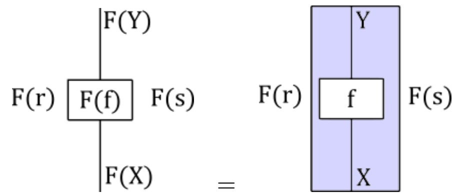
Definition 2.2. Let $\mathcal{C}, \mathcal{D}$, be 2-categories. A *pseudofunctor* $F : \mathcal{C} \to \mathcal{D}$ consists of the following data.

- For each object r of $\mathcal{C}$, an object $F(r)$ of $\mathcal{D}$.
- For each hom-category $\mathcal{C}(r, s)$ of $\mathcal{C}$, a functor $F_{r,s} : \mathcal{C}(r, s) \to \mathcal{D}(F(r), F(s))$.

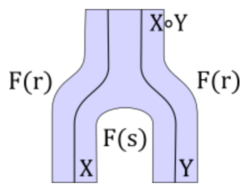
  In the graphical calculus, we represent the effect of the functor $F_{r,s}$ by drawing a shaded box around 1- and 2-morphisms in $\mathcal{C}(r, s)$. For example, $X, Y : r \to s$ be 1-morphisms and

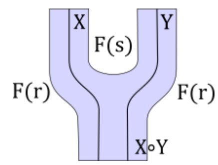
[2] I.e. invertible in the Hom-categories $\mathcal{C}(r, s)$ and $\mathcal{C}(s, r)$. We sometimes call an invertible 2-morphism a *2-isomorphism*.

$f : X \to Y$ a 2-morphism in $\mathcal{C}$. Then the 2-morphism $F(f) : F(X) \to F(Y)$ in $\mathcal{D}(F(r), F(s))$ is represented as:



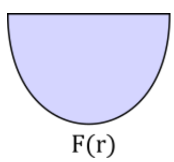
- For every pair of composable 1-morphisms $X : r \to s$, $Y : s \to t$ of $\mathcal{C}$, an invertible *multiplicator* 2-morphism $m_{X,Y} : F(X) \circ_D F(Y) \to F(X \circ_C Y)$. In the graphical calculus, these 2-morphisms and their inverses are represented as follows:

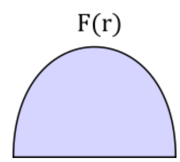




$$m_{X,Y} : F(X) \circ_{\mathcal{D}} F(Y) \to F(X \circ_{\mathcal{C}} Y) \qquad m^{-1}_{X,Y} : F(X \circ_{\mathcal{C}} Y) \to F(X) \circ_{\mathcal{D}} F(Y) \tag{3}$$

- For every object r of $\mathcal{C}$, an invertible 'unitor' 2-morphism $u_r : \mathrm{id}_{F(r)} \to F(\mathrm{id}_r)$. In the diagrammatic calculus, these 2-morphism and their inverses are represented as follows (recall that identity 1-morphisms are invisible):

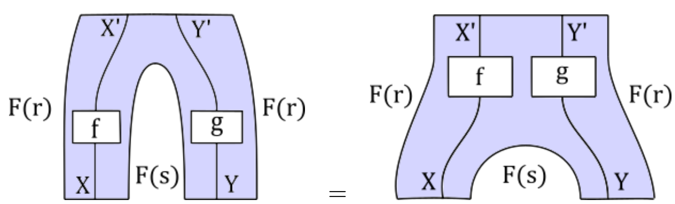




$$u_r : \mathrm{id}_{F(r)} \to F(\mathrm{id}_r) \qquad u^{-1} : F(\mathrm{id}_r) \to \mathrm{id}_{F(r)} \tag{4}$$

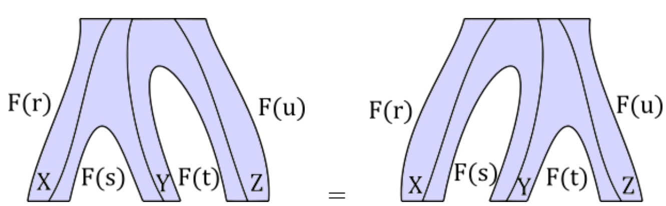
The multiplicators and unitor must obey the following coherence equations:

- *Naturality.* For any objects r, s, t, 1-morphisms $X, X' : r \to s$, $Y, Y' : s \to t$, and 2-morphisms $f : X \to X', g : Y \to Y'$ in $\mathcal{C}$:



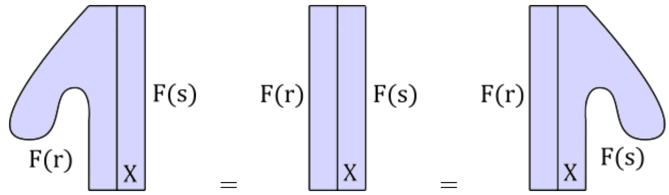
(5)

- *Associativity.* For any objects r, s, t, u and 1-morphisms $X : r \to s$, $Y : s \to t$, $Z : t \to u$ of $\mathcal{C}$:



(6)

- *Unitality.* For any objects r, s and 1-morphism $X : r \to s$ of $\mathcal{C}$:



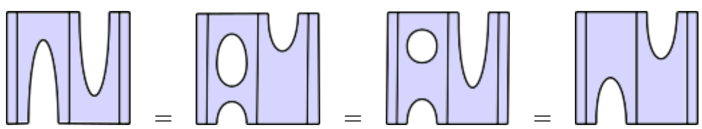
(7)

We say that a pseudofunctor $F : \mathcal{C} \to \mathcal{D}$ is an *equivalence* if every object in $\mathcal{D}$ is equivalent to an object in the image of F and the functors $F_{r,s} : \mathcal{C}(r,s) \to \mathcal{D}(r,s)$ are equivalences.

We observe that the analogous *conaturality*, *coassociativity* and *counitality* equations for the inverses $\{m_{X,Y}^{-1}\}, \{u_r\}^{-1}$, obtained by reflecting (5-7) in a horizontal axis, are already implied by (5-7). To give some idea of the calculus of functorial boxes, we explicitly prove the following lemma and proposition. From now on we will unclutter the diagrams by omitting region and 1-morphism labels, unless adding the labels seems to significantly aid comprehension.

Lemma 2.3. *For any objects r, s, t, u and 1-morphisms $X : r \to s$, $Y : s \to t$, $Z : t \to u$, the following equations are satisfied:*

Proof. We prove the left equation; the right equation is proved similarly.



Here the first and third equalities are by invertibility of $m_{X,Y}$, and the second is by coassociativity. □

With Lemma 2.3, the equations (5-7) are sufficient to deform functorial boxes topologically as required. From now on we will do this mostly without comment.

2.3 Pivotal 2-categories

In a 2-category the most general notion of invertibility of a 1-morphism is *duality*, also known as *adjunction*.

Definition 2.4. Let $X : r \to s$ be a 1-morphism in a 2-category. A *right dual* $[X^*, \eta, \epsilon]$ for X is:

- A 1-morphism $X^* : s \to r$.
- Two 2-morphisms $\eta : \mathrm{id}_s \to X^* \circ X$ and $\epsilon : X \circ X^* \to \mathrm{id}_r$ satisfying the following *snake equations*:

(8)

A *left dual* $[{}^*X, \eta, \epsilon]$ is defined similarly, with 2-morphisms $\eta : \mathrm{id}_s \to X \circ {}^*X$ and $\epsilon : {}^*X \circ X \to \mathrm{id}_r$ satisfying the analogues of (8).

We say that a 2-category $\mathcal{C}$ *has right duals* (resp. *has left duals*) if every 1-morphism X in $\mathcal{C}$ has a chosen right dual $[X^*, \eta, \epsilon]$ (resp. a chosen left dual).

To represent duals in the graphical calculus, we draw an upward-facing arrow on the X-wire and a downward-facing arrow on the X^*- or ${}^*X$-wire, and write η and ϵ as a cup and a cap, respectively. Then the equations (8) become purely topological:

right dual left dual

Since the graphical calculus for 2-categories is just a 'region-labelled' version of the graphical calculus for monoidal categories, various statements about duals in monoidal categories immediately generalise to duals in 2-categories. We recall some of these statements now.

Proposition 2.5 ([1, Lemmas 3.6, 3.7]). *If $[X^*, \eta_X, \epsilon_X]$ and $[Y^*, \eta_Y, \epsilon_Y]$ are right duals for $X : r \to s$ and $Y : s \to t$ respectively, then $[Y^* \circ X^*, \eta_{X\circ Y}, \epsilon_{X\circ Y}]$ is right dual to $X \circ Y$, where $\eta_{X\circ Y}$ and $\epsilon_{X\circ Y}$ are defined by:*

$\eta_{X\circ Y}$ $\epsilon_{X\circ Y}$ (9)

Moreover, for any object r, $[\mathrm{id}_r, \mathrm{id}_{\mathrm{id}_r}, \mathrm{id}_{\mathrm{id}_r}]$ is right dual to id_r. Analogous statements hold for left duals.

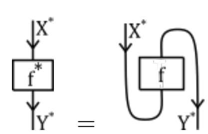
Proposition 2.6 ([1, Lemma 3.4]). *Let $X : r \to s$ be a 1-morphism, and let $[X^*, \eta, \epsilon], [X^{*\prime}, \eta', \epsilon']$ be right duals. Then there is a unique 2-isomorphism $\alpha : X^* \to X^{*\prime}$ such that*

(10)

An analogous statement holds for left duals.

In a 2-category with duals, we can define a notion of transposition for 2-morphisms.

Definition 2.7. Let $X, Y : r \to s$ be 1-morphisms with chosen right duals $[X^*, \eta_X, \epsilon_X]$ and $[Y^*, \eta_Y, \epsilon_Y]$. For any 2-morphism $f : X \to Y$, we define its *right transpose* $f^* : Y^* \to X^*$ as follows:



(11)

For left duals ${}^*X, {}^*Y$, a *left transpose* may be defined analogously.

In this work we are mostly interested in categories with compatible left and right duals. Such categories are called *pivotal*. The definition of pivotality requires a notion of monoidal natural isomorphism between pseudofunctors, which we will not introduce until Definition 3.1. However, we will not need the full definition until after that point; for now we will only require its consequences.

Let $\mathcal{C}$ be a 2-category with right duals. It is straightforward to check that the following defines an identity-on-objects pseudofunctor $\mathcal{C} \to \mathcal{C}$, which we call the *double duals* pseudofunctor:

- 1-morphisms $X : r \to s$ are taken to the double dual $X^{**} := (X^*)^*$.
- 2-morphisms $f : X \to Y$ are taken to the double transpose $f^{**} := (f^*)^*$.
- The multiplicators $m_{X,Y}$ and unitors u_r are defined using the isomorphisms of Proposition 2.6.

Definition 2.8. We say that a 2-category $\mathcal{C}$ with right duals is *pivotal* if the double duals pseudofunctor is monoidally naturally isomorphic to the identity pseudofunctor.

Roughly, the existence of a monoidal natural isomorphism in Definition 2.8 comes down to the following statement:

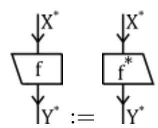
- For every 1-morphism $X : r \to s$, there is a 2-isomorphism $\iota_X : X^{**} \to X$.
- These $\{\iota_X\}$ can be chosen compatibly with composition in $\mathcal{C}$.

In a pivotal 2-category, for any $X : r \to s$ the right dual X^* is also a left dual for X by the following cup and cap (here we have drawn a double upwards arrow on the double dual):

(12)

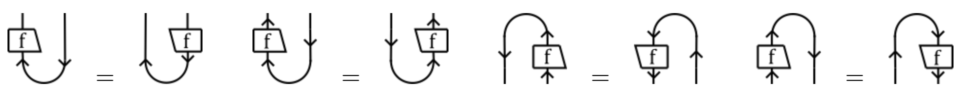
With these left duals, the left transpose of a 2-morphism is equal to the right transpose. Whenever we refer to a pivotal 2-category from now on, we suppose that the left duals are chosen in this way.

There is a very useful graphical calculus for these compatible dualities in a pivotal 2-category. To represent the transpose, we make our 2-morphism boxes asymmetric by tilting the right vertical edge. We now write the transpose by rotating the boxes, as though we had 'yanked' both ends of the wire in the RHS of (11):



Using this notation, 2-morphisms now freely slide around cups and caps.

Proposition 2.9 ([1, Lemma 3.12, Lemma 3.26])**.** *Let $\mathcal{C}$ be a pivotal 2-category and $f : X \to Y$ a 2-morphism. Then:*

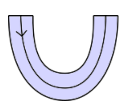


The diagrammatic calculus is summarised by the following theorem, which to our knowledge has only been proved in special cases but is almost certainly true.

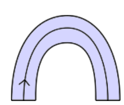
Theorem 2.10 ([8, Theorem 4.14])**.** *Two diagrams for a 2-morphism in a pivotal 2-category represent the same 2-morphism if there is a planar isotopy between them, which may include sliding of 2-morphisms as in Proposition 2.9.*

Pivotal functors. We now consider pseudofunctors between pivotal 2-categories. We first observe that the duals in $\mathcal{C}$ induce duals in $\mathcal{D}$ under a pseudofunctor $F : \mathcal{C} \to \mathcal{D}$.

Proposition 2.11 (Induced duals)**.** *Let $X : r \to s$ be a 1-morphism in $\mathcal{C}$ and $[X^*, \eta, \epsilon]$ a right dual. Then $F(X^*)$ is a right dual of $F(X)$ in $\mathcal{D}$ with the following cup and cap:*



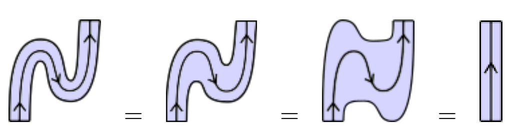
$F(\eta)$



$F(\epsilon)$

The analogous statement holds for left duals.

Proof. We show one of the snake equations (8) in the case of right duals; the others are all proved similarly.



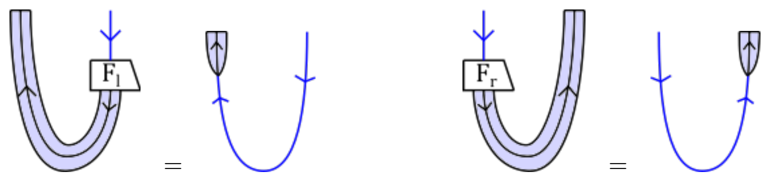
Here the first equality is by Lemma 2.3, the second by (5) and the third by (7). $\square$

For any 1-morphism X of $\mathcal{C}$, then, we have two sets of left and right duals on $F(X)$; the first from the pivotal structure in $\mathcal{C}$ by Proposition 2.11, and the second from the pivotal structure in $\mathcal{D}$. To depict both dualities in the graphical calculus, we here introduce elements of the graphical syntax which allow us to 'zoom in' and 'zoom out', representing $F(X)$ as a directed coloured wire rather than as a boxed wire:

(13)

We emphasise that these elements of the graphical calculus are semantically empty, simply switching between two ways of representing $F(X)$. We can now represent the duality corresponding to the pivotal structure in $\mathcal{D}$ in the usual way on the directed coloured wire, writing $F(X)^*$ and $F(X)^{**}$ with a downwards and a double upwards arrow respectively, as usual.

We now define a pivotal pseudofunctor. Let $\mathcal{C}, \mathcal{D}$ be pivotal dagger 2-categories, and let $F : \mathcal{C} \to \mathcal{D}$ be a pseudofunctor. By Proposition 2.6, for every 1-morphism $X : r \to s$ in $\mathcal{C}$ we obtain two 2-isomorphisms $F_l, F_r : F(X^*) \to F(X)^*$, the first from the left duality and the second from the right duality:



(14)

Definition 2.12 ([9, §1.7.5])**.** Let $\mathcal{C}, \mathcal{D}$ be pivotal dagger 2-categories, and let $F : \mathcal{C} \to \mathcal{D}$ be a pseudofunctor, and let $F_l, F_r : F(X^*) \to F(X)^*$ be the isomorphisms (14). We say that F is *pivotal* if $F_l = F_r =: P$.

In the graphical calculus we again here write these isomorphisms P and their inverses as 'zoom ins' and 'zoom outs', which this time are not semantically empty:

2.4 Pivotal dagger 2-categories

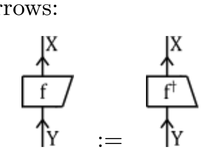
The final structure we will consider on a 2-category is a *dagger*. In this section we define a dagger 2-category and discuss compatibility with the various notions already introduced.

Definition 2.13. A 2-category $\mathcal{C}$ is *dagger* if:

- For each pair of objects r, s there is a contravariant identity-on-objects functor $\dagger_{r,s} : \mathcal{C}(r, s) \to \mathcal{C}(r, s)$, which is *involutive*: for any morphism $f : X \to Y$ in $\mathcal{C}(r, s)$, $\dagger_{r,s}(\dagger_{r,s}(f)) = f$. (This is to say that $\mathcal{C}(r, s)$ is a *dagger category*.)
- The dagger is compatible with composition of 1-morphisms: for any 1-morphisms $X, X' : r \to s$ and $Y, Y' : s \to t$, and 2-morphisms $\alpha : X \to X'$ and $\beta : Y \to Y'$ we have $(\alpha \circ_H \beta)^{\dagger_{r,t}} = \alpha^{\dagger_{r,s}} \circ_H \beta^{\dagger_{s,t}}$.

We call the image of a 2-morphism $f : X \to Y$ under $\dagger_{r,s}$ its *dagger*, and write it as $f^{\dagger_{r,s}}$.

In the graphical calculus, we represent the dagger of a 2-morphism by reflection in a horizontal axis, preserving the direction of any arrows:



(15)

Definition 2.14. Let $\mathcal{C}$ be a dagger 2-category. We say that a 2-morphism $\alpha : X \to Y$ in $\mathcal{C}(r,s)$ is an *isometry* if $\alpha \circ_V \alpha^{\dagger_{r,s}} = \mathrm{id}_X$. We say that it is *unitary* if it is an isometry and additionally $\alpha^{\dagger_{r,s}} \circ_V \alpha = \mathrm{id}_Y$.

Definition 2.15. Let $\mathcal{C}$ be a dagger 2-category and let r, s be objects. We say that a 1-morphism $X : r \to s$ is a *dagger equivalence* if it is an equivalence (Definition 2.1) and the invertible 2-morphisms $\alpha : \mathrm{id}_r \to X \circ X^{-1}$ and $\beta : \mathrm{id}_s \to X^{-1} \circ X$ are unitary.

We now give the condition for compatibility of dagger and pivotal structure.

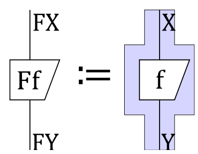
Definition 2.16. Let $\mathcal{C}$ be a pivotal 2-category which is also a dagger 2-category. We say that $\mathcal{C}$ is *pivotal dagger* when, for all 1-morphisms $X : r \to s$:

Remark 2.17. Clearly Definition 2.16 implies compatibility between the graphical calculus of the duality and the graphical calculus of the dagger.

Finally, we consider the right notion of a pseudofunctor between dagger 2-categories.

Definition 2.18. Let $\mathcal{C}, \mathcal{D}$ be dagger 2-categories and let $F : \mathcal{C} \to \mathcal{D}$ be a pseudofunctor. We say that F is *unitary* if the following hold:

- For any 2-morphism f, $F(f^\dagger) = F(f)^\dagger$:



- The multiplicators $\{m_{X,Y}\}$ and unitors $\{u_r\}$ are all unitary 2-morphisms in $\mathcal{D}$.

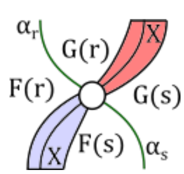
Remark 2.19. The latter condition implies that our depiction of the inverses $\{m_{X,Y}^{-1}\}$ and $\{u_r^{-1}\}$ by reflection in a horizontal axis (3, 4) is consistent with the diagrammatic calculus of the dagger in $\mathcal{D}$.

3 Pseudonatural transformations

Having run through the necessary background on 2-category theory, we recall the definition of a pseudonatural transformation between pseudofunctors [4].

Definition 3.1. Let $\mathcal{C}, \mathcal{D}$ be 2-categories, and let $F, G : \mathcal{C} \to \mathcal{D}$ be pseudofunctors (depicted by blue and red boxes respectively). A *pseudonatural transformation* $\alpha : F \to G$ is defined by the following data:

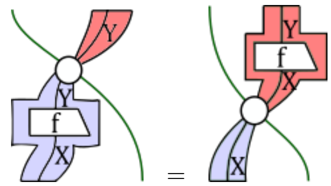
- For every object r of $\mathcal{C}$, a 1-morphism $\alpha_r : F(r) \to G(r)$ of $\mathcal{D}$ (drawn as a green wire).
- For every 1-morphism $X : r \to s$ of $\mathcal{C}$, a 2-morphism $\alpha_X : F(X) \circ \alpha_s \to \alpha_r \circ G(X)$ (drawn as a white vertex):



(16)

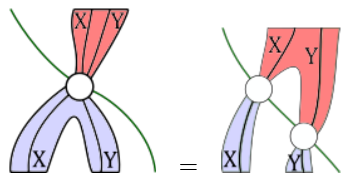
The 2-morphisms α_X must satisfy the following conditions:

- *Naturality.* For every 2-morphism $f : X \to Y$ in $\mathcal{C}$:



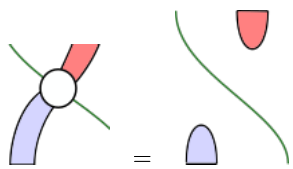
(17)

- *Monoidality.*
  - For every pair of 1-morphisms $X : r \to s, Y : s \to t$ in $\mathcal{C}$:



(18)

  - For every object r of $\mathcal{C}$:



(19)

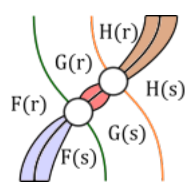
(Equation (18) already implies the analogous pullthroughs for the comultiplicators $\{m_{X,Y}^{-1}\}$.)

If $\alpha_r = \mathrm{id}_{F(r)}$ for every object r of $\mathcal{C}$, we say that α is a *monoidal natural transformation.* (Definition 2.8 is now complete.)

Remark 3.2. The diagrammatic calculus shows that pseudonatural transformation is a planar notion. The $\{\alpha_r\}$-labelled wire (the 'α-wire') forms a boundary between two regions of the $\mathcal{D}$-plane, one in the image of F and the other in the image of G. By pulling through the α-wire, 2-morphisms from $\mathcal{C}$ can move between the two regions (17).

Pseudonatural transformations $\alpha : F \to G$ and $\beta : G \to H$ can be composed associatively. We define $\alpha \circ \beta : F \to H$ as follows.

- For every object r of $\mathcal{C}$, $(\alpha \circ \beta)_r := \alpha_r \circ \beta_r$.
- For any 1-morphism $X : r \to s$ of $\mathcal{C}$, $(\alpha \circ \beta)_X$ is defined as the following composite (we colour the β-wire orange, and the H-box brown):

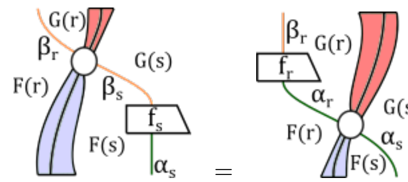


(20)

There are also morphisms between pseudonatural transformations, known as *modifications* [4].

Definition 3.3. Let $\alpha, \beta : F \Rightarrow G$ be pseudonatural transformations between pseudofunctors $F, G : \mathcal{C} \to \mathcal{D}$. (We colour the α-wire green and the β-wire orange.) A *modification* $f : \alpha \to \beta$ is defined by the following data:

- For every object r of $\mathcal{C}$, a 2-morphism $f_r : \alpha_r \to \beta_r$ in $\mathcal{D}$, such that the 2-morphisms $\{f_r\}$ satisfy the following equation for all 1-morphisms $X : r \to s$ in $\mathcal{C}$:



(21)

Modifications can themselves be composed horizontally and vertically in an obvious way. Altogether, this compositional structure is again a 2-category.

Definition 3.4. Let $\mathcal{C}, \mathcal{D}$ be 2-categories. The 2-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is defined as follows:

- Objects: monoidal functors $F, G, \ldots, \cdot : \mathcal{C} \to \mathcal{D}$.
- 1-morphisms: pseudonatural transformations $\alpha, \beta, \cdots : F \to G$.
- 2-morphisms: modifications $f, g, \cdots : \alpha \to \beta$.

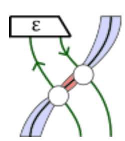
Because we are able to assume that $\mathcal{C}$ and $\mathcal{D}$ are strict, $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is also strict.

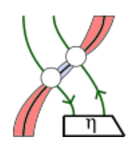
4 Dualisable pseudonatural transformations

4.1 Duals

Pseudonatural transformations categorify natural transformations. We now consider the categorification of natural isomorphisms. As we saw in Definition 2.4, the most general notion of invertibility in a 2-category is dualisability. This unpacks as follows in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

Definition 4.1. Let $F, G : \mathcal{C} \to \mathcal{D}$ be pseudofunctors and $\alpha : F \to G$ a pseudonatural transformation. A *right dual* for α is a triple $[\alpha^*, \eta, \epsilon]$, where $\alpha^* : G \to F$ is a pseudonatural transformation and η, ϵ are modifications

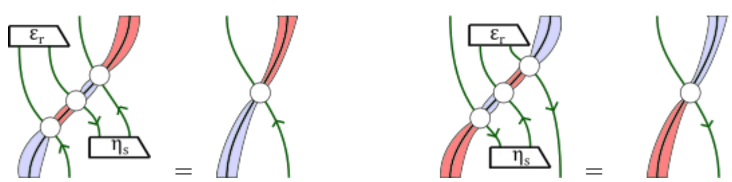




(22)

$$\epsilon : \alpha \circ \alpha^* \to \mathrm{id}_F \qquad\qquad \eta : \mathrm{id}_G \to \alpha^* \circ \alpha \tag{23}$$

such that the following equations hold for any 1-morphism $X : r \to s$ in $\mathcal{C}$:

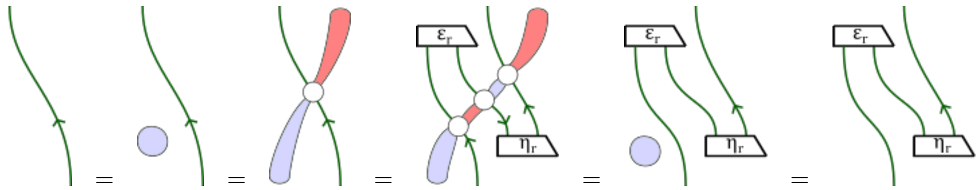


(24)

In the above equations we have drawn the α-wire in green with an upwards-facing arrow and the α^*-wire in green with a downwards-facing arrow, as though α_r and α_r^* were dual 1-morphisms. This will be justified by Lemma 4.2. A *left dual* is defined analogously.

Lemma 4.2. *Let $F, G : \mathcal{C} \to \mathcal{D}$ be pseudofunctors and $\alpha : F \to G$ a pseudonatural transformation with right dual $[\alpha^*, \eta, \epsilon]$. Then for each object r of $\mathcal{C}$, $[\alpha_r^*, \eta_r, \epsilon_r]$ is a right dual for α_r in $\mathcal{D}$. The analogous statement holds for left duals.*

Proof. We prove the right snake equation for right duals; everything else may be proved similarly. For any object r of $\mathcal{C}$:



(25)

Here the first equation is by invertibility of the unitor u_r (4) for F; the second by monoidality (19) of the pseudonatural transformation α on the 1-morphism $\mathrm{id}_r : r \to r$ and invertibility of the unitor for G; the third by (24); the fourth by monoidality of α and α^* on id_r; and the last by invertibility of the unitor u_r. $\square$

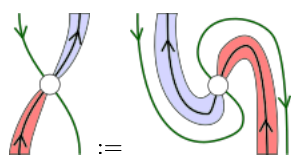
From this point forward, therefore, we will draw η_r and ϵ_r as a cup and cap.

Remark 4.3. From the perspective of the graphical calculus, dualisability of a pseudonatural transformation α corresponds to topological deformability of the α-wire boundary between the F- and $G-$ regions of the $\mathcal{D}$-plane.

If $\mathcal{C}$ has duals, we obtain explicit expressions for the left and right duals in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ whenever they exist.

Theorem 4.4. *Let $F, G : \mathcal{C} \to \mathcal{D}$ be pseudofunctors, and suppose that $\mathcal{C}$ has left duals. A pseudonatural transformation $\alpha : F \to G$ has a right dual in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ precisely when α_r has some right dual $[\alpha_r^*, \eta_r, \epsilon_r]$ in $\mathcal{D}$ for each object r of $\mathcal{C}$. The right dual α^* is defined as follows:*

- *For each object r of $\mathcal{C}$, $(\alpha^*)_r = (\alpha_r)^*$ and the components of the modifications η, ϵ are $[\eta_r, \epsilon_r]$.*
- *For each 1-morphism $X : r \to s$ of $\mathcal{C}$, $(\alpha^*)_X$ is:*



(26)

This statement also holds with 'left' and 'right' swapped, in which case the left dual ${}^*\alpha$ is defined as follows:

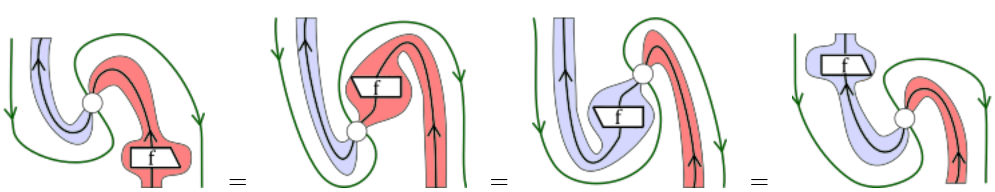
- *For each object r of $\mathcal{C}$, $({}^*\alpha)_r = {}^*(\alpha_r)$ and the components of the modifications η, ϵ are $[\eta_r, \epsilon_r]$.*
- *For each 1-morphism $X : r \to s$ of $\mathcal{C}$, $({}^*\alpha)_X$ is defined as in (26), but with the opposite transposition.*

Proof. We consider the case of the right dual α^*; the argument for the left dual is similar.

If some α_r has no right dual, then nor can α by Lemma 4.2.

If every α_r has some right dual, then we must show firstly that α^* as defined is a pseudonatural transformation, and secondly that η, ϵ as defined are modifications satisfying the snake equations (8).

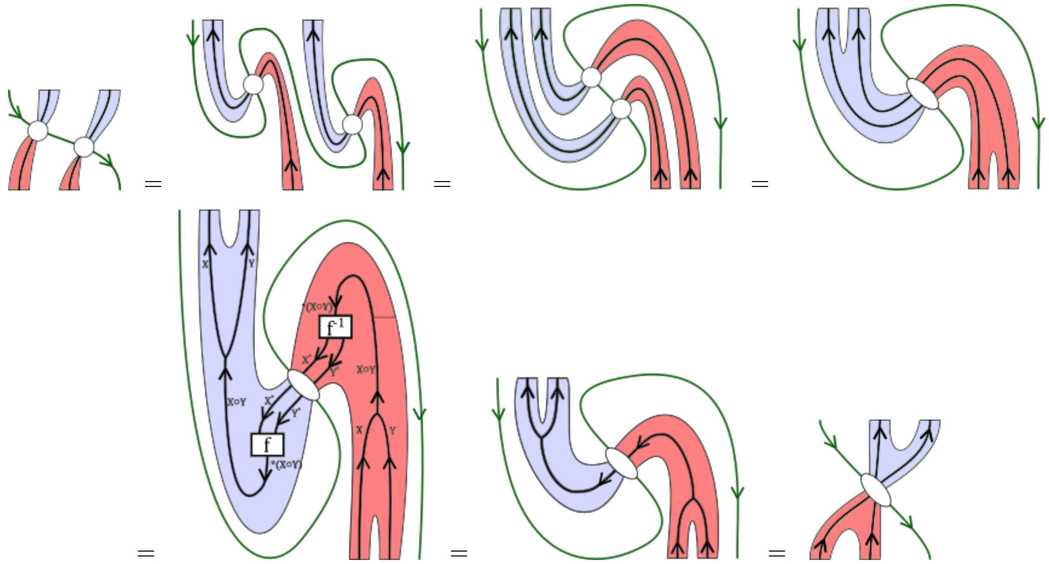
1. *Naturality of α^*.* (17) For all 2-morphisms $f : X \to Y$ in $\mathcal{C}$:



(27)

Here the first and third equalities use the sliding notation of Proposition 2.9 for the left transpose; the second equality is by naturality of α on $f^T : {}^*Y \to {}^*X$.

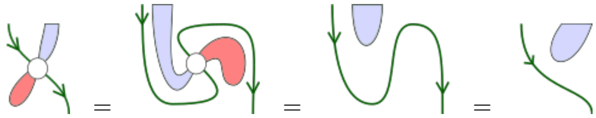
2. *Monoidality of α^*.* (18-19)
   - For every pair of 1-morphisms $X : r \to s, Y : s \to t$ in $\mathcal{C}$:



(28)

Here the first equality is by definition; the second by a snake equation for α_s; the third by monoidality of α and some manipulation of functorial boxes; the fourth by Propositions 2.5 and 2.6, where f is the isomorphism between ${}^*Y \circ {}^*X$ and the chosen left dual ${}^*(X \circ Y)$ in $\mathcal{C}$; the fifth by naturality of α; and the sixth by definition.

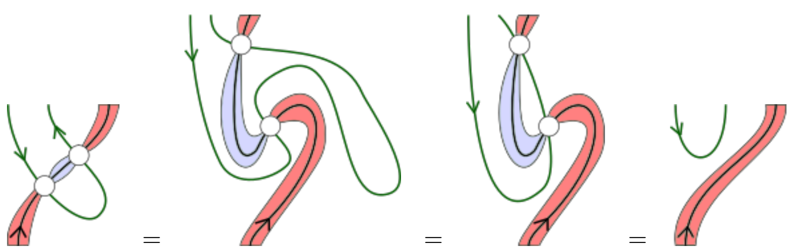
   - For every object r of $\mathcal{C}$:



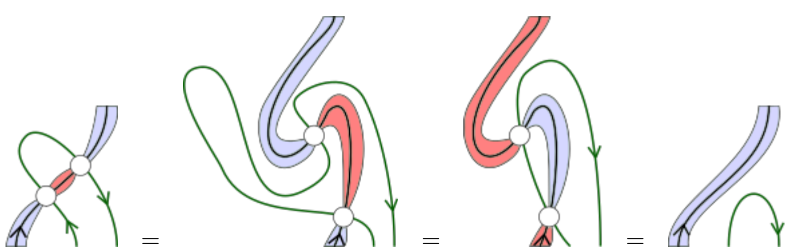
(29)

Here the first equality is by definition, the second by monoidality of α, and the third by a snake equation for α_r. We have assumed for that the chosen left dual of id_r is $[\mathrm{id}_r, \mathrm{id}_{\mathrm{id}_r}, \mathrm{id}_{\mathrm{id}_r}]$; in general one can use Proposition 2.6 and naturality of α as in (28).

3. Since η_r, ϵ_r already satisfy the snake equations for every r by assumption, we need only show that η, ϵ are modifications. For all $X : r \to s$ in $\mathcal{C}$:



(30)



(31)

Here, the first equalities are by definition, the second are by a snake equation for α_r^* or α_s^*, and the third are by naturality and monoidality of α.

$\square$

Corollary 4.5. *If $\mathcal{C}$ has left duals, and $\mathcal{D}$ has right duals, then* $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ *has right duals. This statement also holds with 'left' and 'right' swapped.*

Remark 4.6. It is well-known that a monoidal natural transformation between monoidal functors from a monoidal category with duals is invertible. Theorem 4.4 generalises this result. Indeed, if the objects α_r are all identity morphisms, then the cup and cap are trivial and the dual is simply a strict inverse.

4.2 Pivotality

We have seen that, for a pseudonatural transformation $\alpha : F \to G$, the α-wire forms a boundary between a region in the image of F and a region in the image of G, and dualisability corresponds to topological deformation of this boundary. To freely deform the boundary in a coherent way, we would like $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ to be pivotal. We recall that a 2-category with right duals is *pivotal* (Definition 2.8) if there is a monoidal natural isomorphism (Definition 3.1) from the double duals pseudofunctor to the identity pseudofunctor.

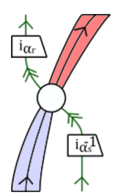
We now show that $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ inherits pivotality from $\mathcal{C}$ and $\mathcal{D}$ upon restriction to pivotal pseudofunctors.

Definition 4.7. When $\mathcal{C},\mathcal{D}$ are pivotal we define $\mathrm{Fun}_p(\mathcal{C},\mathcal{D}) \subset \mathrm{Fun}(\mathcal{C},\mathcal{D})$ to be the subcategory whose objects are pivotal pseudofunctors.

Theorem 4.8. *Let $\mathcal{C},\mathcal{D}$ be pivotal 2-categories, and let $\iota : **_{\mathcal{D}} \to \mathrm{id}_{\mathcal{D}}$ be the pivotal structure on $\mathcal{D}$. Then the 2-category* $\mathrm{Fun}_p(\mathcal{C},\mathcal{D})$ *is itself a pivotal 2-category.*

The monoidal natural transformation $\hat{\iota} : **_{\mathrm{Fun}(\mathcal{C},\mathcal{D})} \to \mathrm{id}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}$ assigns to every pseudonatural transformation $\alpha^{**} : F \to G$ the invertible modification $\hat{\iota}_\alpha : \alpha^{**} \to \alpha$ whose components are the 2-isomorphisms $\iota_{\alpha_r} : \alpha_r^{**} \to \alpha_r$ from the pivotal structure on $\mathcal{D}$.

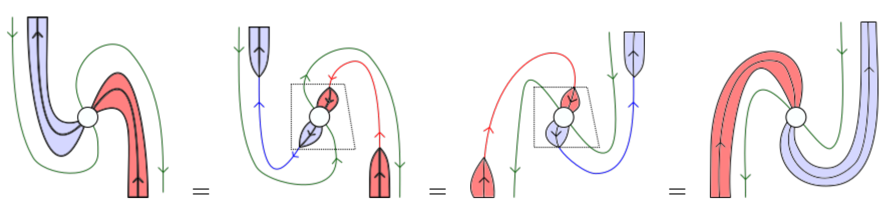
Proof. First we show that the $\hat{\iota}_\alpha$ are really modifications. Since $\{\iota_{\alpha_r}\}$ are 2-isomorphisms it is immediate that the $\hat{\iota}_\alpha$-*conjugate* $(\alpha^{**})^{\hat{\iota}_\alpha}$ of α^{**} is a pseudonatural transformation $F \to G$, where $(\alpha^{**})_r^{\hat{\iota}_\alpha} = \alpha_r$ for all objects r of $\mathcal{C}$, and $(\alpha^{**})_X^{\hat{\iota}_\alpha}$ is defined as follows for all $X : r \to s$:



(32)

It is also clear that $\hat{\iota}_\alpha$ is a modification $\alpha^{**} \to (\alpha^{**})^{\hat{\iota}_\alpha}$.

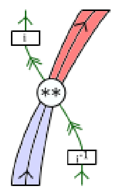
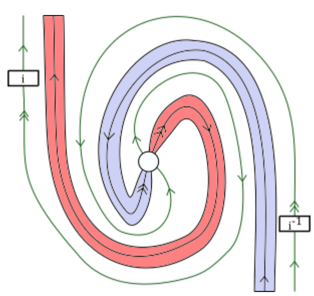
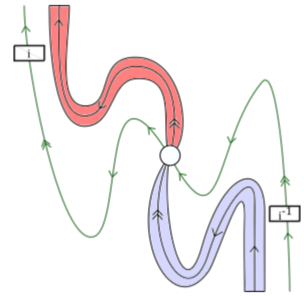
We now show that $\hat{\iota}_\alpha$ has the right target, i.e. $(\alpha^{**})^{\hat{\iota}_\alpha} = \alpha$. We first observe that the left dual of a pseudonatural transformation between pivotal functors is identical to its right dual:

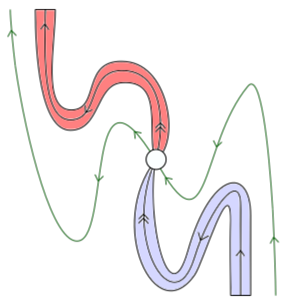
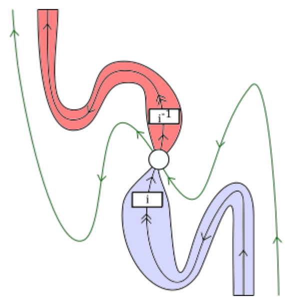
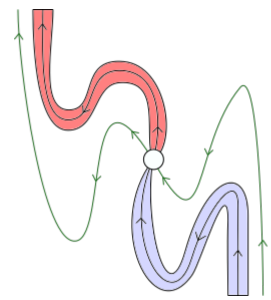


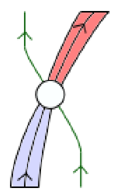
(33)

Here for the first and third equalities we used Proposition 2.6 and the 'zoom out' notation (13) to relate the duals in $\mathcal{C}$ and $\mathcal{D}$, and for the second we used the graphical calculus of the pivotal 2-category $\mathcal{D}$ (Theorem 2.10) to deform the diagram around the morphism in the dashed box. For the third equality we require that the pseudofunctors are pivotal.

Now for any $\alpha : F \to G$ and $X : r \to s$ in $\mathcal{C}$ we have:


$=$

$=$

(34)

$=$

$=$

$=$

(35)

$=$

(36)

Here the first equality is by definition; the second uses (33); the third uses the definition (12) of the left duality in the pivotal 2-category $\mathcal{D}$; the fourth uses naturality of α to insert $\iota\iota^{-1}$, where $\iota : X^{**} \to X$ is the isomorphism from the pivotal structure in $\mathcal{C}$; the fifth uses the definition (12) of the left duality in $\mathcal{C}$; and the last uses the snake equations in $\mathcal{C}$ and $\mathcal{D}$.

Finally, we need to show that $\hat{\iota}$ is a monoidal natural transformation $**_{\mathrm{Fun}(\mathcal{C},\mathcal{D})} \to \mathrm{id}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}$.

- *Monoidality*: For every pair of pseudonatural transformations $\alpha : F \to G$, $\beta : G \to H$, we need $\hat{\iota}_{\alpha\circ\beta} = \hat{\iota}_\alpha \circ_H \hat{\iota}_\beta$. For each $X : r \to s$ this is implied by monoidality of $\iota : **_{\mathcal{D}} \to \mathrm{id}_{\mathcal{D}}$.
- *Naturality*: We need that, for every modification $f : \alpha \to \beta$, $\hat{\iota}_\beta \circ_V f^{**} = f \circ \hat{\iota}_\alpha$. For each $X : r \to s$ this is implied by naturality of $\iota : **_{\mathcal{D}} \to \mathrm{id}_{\mathcal{D}}$.

□

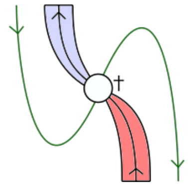
5 Unitary pseudonatural transformations

We have considered the case where $\mathcal{C}, \mathcal{D}$ are pivotal. We now consider the case where $\mathcal{C}, \mathcal{D}$ are pivotal dagger and the pseudofunctors are unitary.

In this case, we get a new contravariant operation on pseudonatural transformations.

Proposition 5.1. *Let $F, G : \mathcal{C} \to \mathcal{D}$ be unitary pseudofunctors between pivotal dagger 2-categories. Then for any pseudonatural transformation $\alpha : F \to G$, its* dagger *$\alpha^\dagger : G \to F$, defined compo-*

nentwise for each $X : r \to s$ in $\mathcal{C}$ as

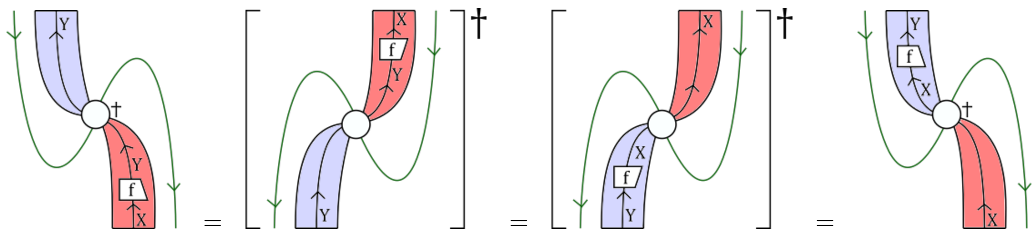


(37)

is also a pseudonatural transformation.

Proof. We must show naturality and monoidality.

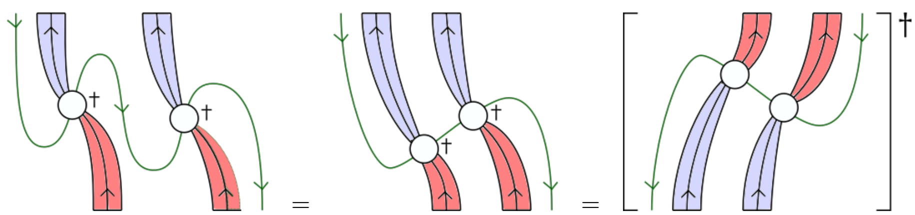
- *Naturality.* For any $f : X \to Y$ in $\mathcal{C}$:



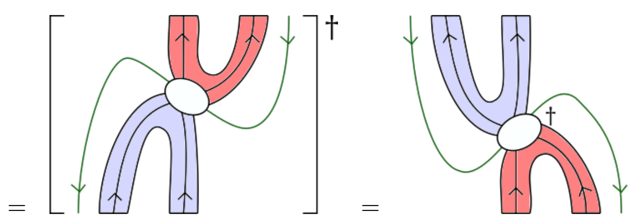
(38)

Here the first equality is by unitarity of G, the second equality is by naturality of α, and the third equality is by unitarity of F.

- *Monoidality.* For any $X : r \to s, Y : s \to t$ in $\mathcal{C}$:



(39)



(40)

Here the first and second equalities are by dagger pivotality of $\mathcal{D}$, the third equality is by monoidality of α, and the fourth equality is by unitarity of F, G and dagger pivotality of $\mathcal{D}$. We leave the other monoidality condition (19) to the reader.

$\square$

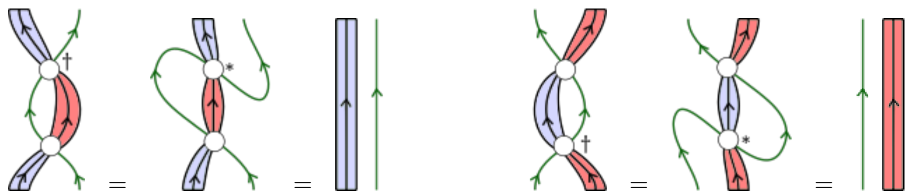
We would like $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ to inherit the structure of a dagger 2-category. In general, however, there is no reason why the componentwise dagger of a modification $f : \alpha \to \beta$ — the only reasonable candidate for a dagger on $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ — should yield a modification $f^\dagger : \beta \to \alpha$.

This problem is resolved by restriction to 'unitary' pseudonatural transformations. There are two obvious ways to define unitarity. First, given that the dual is the 'inverse' of a pseudonatural transformation, we could ask that the dagger (37) of the transformation should be equal to the right dual (26). Alternatively, by analogy with the definition of unitary monoidal natural transformations, and motivated by physicality in quantum mechanics [11], we might demand that the components of the transformation be individually unitary in $\mathcal{D}$. In fact, these definitions are equivalent.

Proposition 5.2. *Let $\mathcal{C}, \mathcal{D}$ be pivotal dagger 2-categories and let $\alpha : F \to G$ be a pseudonatural transformation between functors $F, G : \mathcal{C} \to \mathcal{D}$. The following are equivalent:*

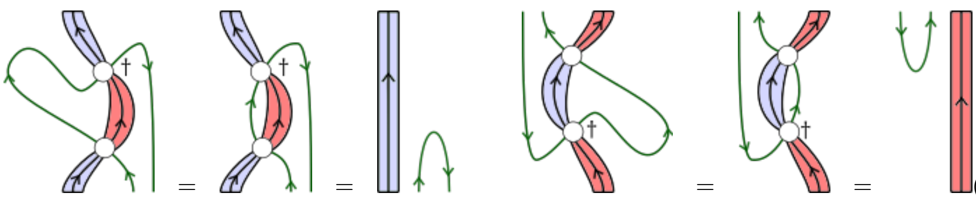
1. *There is an equality of pseudonatural transformations $\alpha^* = \alpha^\dagger$.*
2. *For all 1-morphisms $X : r \to s$ in $\mathcal{C}$, the component $\alpha_X : F(X) \circ \alpha_r \to \alpha_s \circ G(X)$ is unitary.*

Proof. (i) $\Rightarrow$ (ii): For all $X : r \to s$ in $\mathcal{C}$, unitarity of α_X follows from right duality:



(41)

(ii) $\Rightarrow$ (i): Unitarity of the components implies that $[\alpha^\dagger, \eta, \epsilon]$ is a right dual, where η, ϵ are the cup and cap of the right dual $[\alpha^*, \eta, \epsilon]$, since for each component:



(42)

But this implies equality $\alpha^\dagger = \alpha^*$ for all, as, since the cup and cap modifications are identical, the unique 2-isomorphism of Proposition 2.6 relating the two right duals in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ must be the identity. □

We therefore make the following definition.

Definition 5.3. Let $\mathcal{C}, \mathcal{D}$ be pivotal dagger 2-categories and let $F, G : \mathcal{C} \to \mathcal{D}$ be unitary pseudofunctors. Then a *unitary pseudonatural transformation (UPT)* $\alpha : F \to G$ is a pseudonatural transformation such that either of the following equivalent conditions are satisfied:

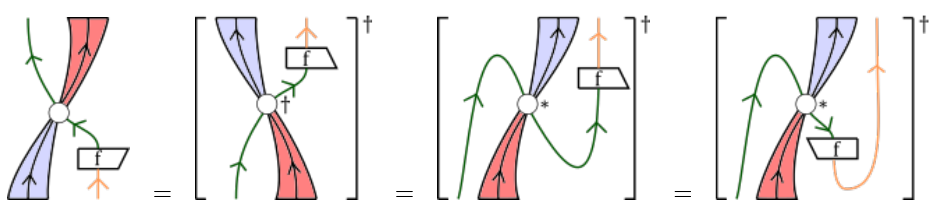
- There is an equality of pseudonatural transformations $\alpha^* = \alpha^\dagger$.
- For all 1-morphisms $X : r \to s$ in $\mathcal{C}$, the component $\alpha_X : F(X) \circ \alpha_r \to \alpha_s \circ G(X)$ is unitary.

Definition 5.4. When $\mathcal{C}, \mathcal{D}$ are pivotal dagger we restrict the 1-morphisms of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ and $\mathrm{Fun}_p(\mathcal{C}, \mathcal{D})$ to UPTs.

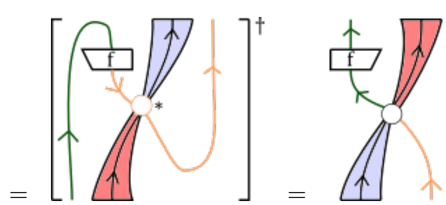
Following this restriction, $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ indeed becomes a dagger 2-category.

Theorem 5.5. *Let $\mathcal{C}, \mathcal{D}$ be pivotal dagger 2-categories. Then the 2-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is dagger, where the dagger of a modification $f : \alpha \to \beta$ is defined on components as $(f^\dagger)_r = (f_r)^\dagger$. Moreover, $\mathrm{Fun}_p(\mathcal{C}, \mathcal{D})$ is pivotal dagger.*

Proof. We first show that $f^\dagger$ is a modification $\beta \to \alpha$:



(43)



(44)

Here the second equality is by unitarity of α, and the fourth equality is by transposition in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

For the last statement we must show that the duals of $\mathrm{Fun}_p(\mathcal{C}, \mathcal{D})$ are dagger duals. This follows from the fact that the dagger of a modification is taken componentwise, and the cup and cap for each component come from the pivotal dagger structure in $\mathcal{D}$. □

References

[1] C. Heunen and J. Vicary. *Categories for Quantum Theory: An Introduction*. Oxford Graduate Texts in Mathematics Series. Oxford University Press, 2019. URL: `https://books.google.co.uk/books?id=PdG8DwAAQBAJ`.

[2] Benjamin Taylor Hummon. *Surface diagrams for gray-categories*. PhD thesis, UC San Diego, 2012. URL: `https://escholarship.org/uc/item/5b24s9cc`.

[3] Vaughan FR Jones. Planar algebras, I. 1999. `arXiv:math/9909027`.

[4] Tom Leinster. Basic bicategories. 1998. `arXiv:math/9810017`.

[5] Saunders MacLane. Natural associativity and commutativity. *Rice Institute Pamphlet-Rice University Studies*, 49(4), 1963.

[6] Daniel Marsden. Category theory using string diagrams. 2014. `arXiv:1401.7220`.

[7] Paul-André Melliès. Functorial boxes in string diagrams. In *International Workshop on Computer Science Logic*, pages 1–30. Springer, 2006. URL: `https://www.irif.fr/~mellies/mpri/mpri-ens/articles/mellies-functorial-boxes.pdf`.

[8] Peter Selinger. A survey of graphical languages for monoidal categories. In *New Structures for Physics*, pages 289–355. Springer, 2010. `arXiv:0908.3347`.

[9] V. Turaev and A. Virelizier. *Monoidal Categories and Topological Field Theory*. Progress in Mathematics. Springer International Publishing, 2017. URL: `https://books.google.co.uk/books?id=sjUqDwAAQBAJ`.

[10] Dominic Verdon. Unitary transformations of fibre functors. 2020.

[11] Jamie Vicary. Higher quantum theory. 2012. `arXiv:arXiv:1207.4563`.