Graphical Regular Logic

Brendan Fong and David I. Spivak*

Abstract

Regular logic can be regarded as the *internal language* of regular categories, but the logic itself is generally not given a categorical treatment. In this paper, we understand the syntax and proof rules of regular logic in terms of the free regular category $\mathrm{FRg(T)}$ on a set T. From this point of view, regular theories are certain monoidal 2-functors from a suitable 2-category of contexts—the 2-category of relations in $\mathrm{FRg(T)}$—to that of posets. Such functors assign to each context the set of formulas in that context, ordered by entailment. We refer to such a 2-functor as a *regular calculus* because it naturally gives rise to a graphical string diagram calculus in the spirit of Joyal and Street. Our key aim to prove that the category of regular categories is essentially reflective in that of regular calculi. Along the way, we demonstrate how to use this graphical calculus.

Keywords: regular logic, category theory, primitive positive formula

1 Introduction

Regular logic is the fragment of first order logic generated by equality ($=$), true (`true`), conjunction ($\wedge$), and existential quantification ($\exists$). A defining feature of this fragment is that it is expressive enough to define *functions* and *composition* of functions, or more generally of relations: given relations $R \subseteq X \times Y$ and $S \subseteq Y \times Z$, their composite is given by the formula

$$R \,\mathbin{\fatsemi}\, S = \{(x, z) \mid \exists y. R(x, y) \wedge S(y, z)\}.$$

Indeed, regular logic is the internal language of regular categories, which may in turn be understood as a categorical characterization of the minimal structure needed to have a well-behaved notion of relation.

While regular categories put emphasis on the notion of *binary* relation, the existence of finite products allows them to handle n-ary relations—that is, subobjects of n-fold products—and their composition. To organize more complicated multi-way composites of relations, many fields have developed some notion of wiring diagram. A good amount of recent work, including but not limited to control theory [BSZ14; BE15; FSR16], database

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theory and knowledge representation [BSS18; Pat17], electrical engineering [BF18], and chemistry [BP17], all serve to demonstrate the link between these languages and categories for which the morphisms are relations.

A first goal of this paper is to clarify the link between regular logic and these various graphical languages. In doing so, we provide a new diagrammatic syntax for regular logic, the titular *graphical regular logic*. Rather than pursue a direct translation with the classical syntax for first order logic, we demonstrate a tight connection between graphical regular logic and the notion of *regular category*. A second goal, then, is to repackage the structure of a regular category into terms that cleanly reflect its underlying logical theory. We call the resulting categorical structure a *regular calculus*. Regular calculi are based on free regular categories, so let's begin there.

We will show that the *free regular category* FRg on a singleton set can be obtained by freely adding a fresh terminal object to FinSet$^{\mathrm{op}}$. Here is a depiction of a few objects in FRg:

$$0 \longleftarrow\!\!\!\prec s \ll\!\!\!\longleftarrow 1 \overset{\longleftarrow}{\underset{\longleftarrow}{\Longrightarrow}} 2 \qquad \cdots \tag{1}$$

The object s is the coequalizer of the two distinct maps $2 \rightrightarrows 1$, so in a sense it prevents the unique map $1 \to 0$ from being a regular epimorphism. Thus one may think of s as representing the *support* of an abstract object in a regular category. In Set, the support of any object is either empty or singleton, but in general the concept is more refined. For example, the topos of sheaves on a space X is regular, and the support of a sheaf r is the union $U \subseteq X$ of all open sets on which $r(U)$ is nonempty.

For any small set T of *types* (also known as *sorts*), the free regular category on T is then the T-fold coproduct of regular categories $\mathsf{FRg}(\mathrm{T}) \coloneqq \bigsqcup_{\mathrm{T}} \mathsf{FRg}$. That is, we have an adjunction

$$\mathsf{Set} \underset{\mathrm{Ob}}{\overset{\mathsf{FRg}}{\underset{\Leftarrow}{\rightleftarrows}}} \mathsf{RgCat} \tag{2}$$

which we will construct explicitly in Theorem 4.11. For any regular category $\mathcal{R}$, the counit provides a canonical regular functor, which we denote $\ulcorner - \urcorner \colon \mathsf{FRg}(\mathrm{Ob}\,\mathcal{R}) \to \mathcal{R}$. Note also that this extends to a 2-functor $\ulcorner - \urcorner \colon \mathbb{Rel}_{\mathsf{FRg}(\mathrm{Ob}\,\mathcal{R})} \to \mathbb{Rel}_{\mathcal{R}}$ between the associated relation bicategories.

Write $\mathbb{FRg}(\mathrm{T}) \coloneqq \mathbb{Rel}_{\mathsf{FRg}(\mathrm{T})}$ for this bicategory of relations. Just as FRg is closely related to the opposite of the category of finite sets (see (1)), the objects in $\mathbb{FRg}(\mathrm{T})$ are, at a first approximation, much like finite sets $\underline{n}$ equipped with a function $\underline{n} \to \mathrm{T}$, and morphisms are much like corelations: equivalence relations on some coproduct $\underline{n + n'}$. We draw objects and morphisms as on the left and right below:

The left-hand circle, equipped with its labeled ports and white dot, represents an object in $\mathbb{FRg}(\mathrm{T})$; we call this picture a *shell*. Here each port represents an element of the associated

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finite set $\underline{3}$, the white dot captures aspects related to the support object s of FRg, and the labels x, y etc. are elements of T. In the right-hand morphism, the inner shell represents the domain, outer shell represents the codomain, and the things between them—the connected components of the wires and the white dots—represent the equivalence classes of the aforementioned equivalence relation.

A regular calculus lets us think of each object $\Gamma \in \mathbb{F}Rg(T)$—each shell—as a context for formulas in some regular theory, and of each morphism, i.e. each wiring diagram $\Gamma \twoheadrightarrow \Gamma'$, as a method for converting Γ-formulas to Γ'-formulas, using $=$, $\mathtt{true}, \wedge$, and $\exists$. We next want to think about how regular categories fit into this picture.

If $\mathcal{R}$ is a regular category, formulas in the associated regular theory are given by relations $x \subseteq r_1 \times \cdots \times r_n$, where x and the r_i are objects in $\mathcal{R}$, i.e. $r_\bullet \colon \underline{n} \to \mathcal{R}$. Thus we could consider $\Gamma \coloneqq r_\bullet$ as a context, and this brings us back to the free regular category $\mathbb{F}Rg(\mathrm{Ob}\,\mathcal{R})$. The counit functor $\ulcorner - \urcorner \colon \mathbb{F}Rg(\mathrm{Ob}\,\mathcal{R}) \to \mathcal{R}$ sends Γ to $\ulcorner \Gamma \urcorner \coloneqq r_1 \times \cdots \times r_n$. A key feature of regular categories is that the subobjects $\mathsf{Sub}_\mathcal{R}(r_1 \times \cdots \times r_n)$ form a meet-semilattice, elements of which we call *predicates* in context Γ. As we shall see, the collection of all these semilattices, when related by the structure of $\mathbb{F}Rg(\mathrm{Ob}\,\mathcal{R})$, includes enough data to recover the regular category $\mathcal{R}$ itself.

Indeed, consider the commutative diagram

$$
\begin{array}{ccc}
\mathsf{FRg}(\mathrm{Ob}\,\mathcal{R}) & \xrightarrow{\ulcorner - \urcorner} & \mathcal{R} \\
\downarrow & & \downarrow \\
\mathbb{F}Rg(\mathrm{Ob}\,\mathscr{R}) & \xrightarrow[\ulcorner - \urcorner]{} & \mathscr{R} \xrightarrow[\mathscr{R}(I,-)]{} \mathbb{P}\mathsf{oset}
\end{array}
$$

where the vertical maps represent inclusions of a regular 1-category into its bicategory of relations, and the hom-2-functor $\mathscr{R}(I, -)$ sends each object $r \in \mathrm{Ob}\,\mathcal{R} = \mathrm{Ob}\,\mathscr{R}$ to the subobject lattice $\mathsf{Sub}_\mathcal{R}(r) = \mathscr{R}(I, r)$. We can denote the composite of the bottom maps as

$$
\mathsf{Sub}_\mathcal{R} \ulcorner - \urcorner \colon \mathbb{F}Rg(\mathrm{Ob}\,\mathcal{R}) \longrightarrow \mathbb{P}\mathsf{oset}. \tag{3}
$$

The domain $\mathbb{F}Rg(\mathrm{Ob}\,\mathcal{R})$ is a category of contexts and the functor $\mathsf{Sub}_\mathcal{R} \ulcorner \Gamma \urcorner$ assigns the poset of predicates to each context Γ.

As mentioned, we will show how to reconstruct $\mathcal{R}$—up to equivalence—from the contexts $\Gamma \in \mathsf{FRg}(\mathrm{Ob}\,\mathcal{R})$ and their predicate posets $\mathsf{Sub}_\mathcal{R} \ulcorner \Gamma \urcorner$ as in Eq. (3), once we give the abstract structure by which they hang together. The question is, given any set T, what extra structure do we need on a functor

$$
P \colon \mathbb{F}Rg(T) \longrightarrow \mathbb{P}\mathsf{oset}
$$

in order to construct a regular category from it?

Whatever the required structure on P is, of course $\mathsf{Sub}_\mathcal{R} \ulcorner - \urcorner$ needs to have that structure. First of all, $\mathsf{Sub}_\mathcal{R} \ulcorner - \urcorner$ is a 2-functor, and it happens to be the composite of $\mathbb{R}\mathsf{el} \ulcorner - \urcorner$ and $\mathsf{Sub}_\mathcal{R}$. It is not hard to check that the 2-functor $\ulcorner - \urcorner$ is strong monoidal, whereas the 2-functor $\mathscr{R}(I, -)$ is only lax monoidal: given objects $r_1, r_2 \in \mathcal{R}$ the induced monotone map

$\times\colon \mathsf{Sub}_{\mathcal{R}}(r_1) \times \mathsf{Sub}_{\mathcal{R}}(r_2) \to \mathsf{Sub}_{\mathcal{R}}(r_1 \times r_2)$ is not an isomorphism. However, $\mathsf{Sub}_{\mathcal{R}}\ulcorner - \urcorner$ has a bit more structure than merely being a lax functor: each laxator has a left adjoint

$$1 \underset{!}{\overset{\texttt{true}}{\underset{\Leftarrow}{\rightleftarrows}}} \mathsf{Sub}_{\mathcal{R}}(1) \qquad \mathsf{Sub}_{\mathcal{R}}(r_1) \times \mathsf{Sub}_{\mathcal{R}}(r_2) \underset{\langle \mathrm{im}_1,\ \mathrm{im}_2 \rangle}{\overset{\times}{\underset{\Leftarrow}{\rightleftarrows}}} \mathsf{Sub}_{\mathcal{R}}(r_1 \times r_2).$$

Abstractly, if $\mathscr{R}$ and $\mathscr{P}$ are monoidal 2-categories, we say that a lax monoidal functor $\mathscr{R} \to \mathscr{P}$ is *ajax* ("adjoint-lax") if its laxators ρ and $\rho_{v,v'}$ are right adjoints in $\mathscr{P}$. Thus we have seen that $\mathsf{Sub}_{\mathcal{R}}\ulcorner - \urcorner \colon \mathbb{F}\mathsf{Rg}(\mathrm{Ob}\,\mathcal{R}) \longrightarrow \mathbb{P}\mathsf{oset}$ is ajax. This is precisely the structure required to reconstruct a regular category.

Ajax functors have the important property that they preserve adjoint monoids, a notion we introduce. An *adjoint monoid* is an object with both monoid and comonoid structures, such that the monoid maps are right adjoint to their corresponding comonoid maps. In particular, we will see that each object in $\mathbb{F}\mathsf{Rg}(\mathrm{T})$ has a canonical adjoint monoid structure, and that adjoint monoids in $\mathbb{P}\mathsf{oset}$ are exactly meet-semilattices. This guarantees that ajax functors $\mathbb{F}\mathsf{Rg}(\mathrm{T}) \to \mathbb{P}\mathsf{oset}$ send objects in $\mathbb{F}\mathsf{Rg}(\mathrm{T})$—contexts—to meet-semilattices.

We now come to our main definition.

Definition 1.1. A *regular calculus* is a pair (T, P) where T is a set and $P\colon \mathbb{F}\mathsf{Rg}(\mathrm{T}) \to \mathbb{P}\mathsf{oset}$ is an ajax 2-functor.

A *morphism* $(\mathrm{T}, P) \to (\mathrm{T}', P')$ of regular calculi is a pair $(F, F^\sharp)$ where $F\colon \mathrm{T} \to \mathrm{T}'$ is a function and $F^\sharp$ is a monoidal natural transformation

$$
\begin{array}{ccc}
\mathrm{T} & \mathbb{F}\mathsf{Rg}(\mathrm{T}) \overset{P}{\searrow} & \\
F\downarrow & \mathbb{F}\mathsf{Rg}(F)\downarrow \quad {}^{F^\sharp}\!\Downarrow & \mathbb{P}\mathsf{oset} \\
\mathrm{T}' & \mathbb{F}\mathsf{Rg}(\mathrm{T}') \underset{P'}{\nearrow} &
\end{array}
$$

that is strict in every respect: all the required coherence diagrams of posets commute on the nose. We denote the category of regular calculi by RgCalc.

The goal of this paper is to prove that RgCat is *essentially reflective in* RgCalc (see Theorem 8.5). More precisely, this means:

Theorem 1.2. *The "predicates" mapping in Eq.* (3) *extends to a fully faithful functor*

$$
\begin{aligned}
\mathbf{prd}\colon \mathsf{RgCat} &\to \mathsf{RgCalc} \\
\mathcal{R} &\mapsto (\mathrm{Ob}\,\mathcal{R}, \mathsf{Sub}_{\mathcal{R}}\ulcorner - \urcorner),
\end{aligned}
\tag{4}
$$

and this functor has a left adjoint, the "syntactic category,"

$$\mathsf{RgCalc} \underset{\mathbf{prd}}{\overset{\mathbf{syn}}{\underset{\Rightarrow}{\rightleftarrows}}} \mathsf{RgCat}.$$

Moreover, for any regular category $\mathcal{R}$, the counit functor $\mathbf{syn}(\mathbf{prd}(\mathcal{R})) \to \mathcal{R}$ is an equivalence.

In order to prove this result, we will also show that each object $(\mathrm{T}, P) \in \mathsf{RgCalc}$ can be understood as a graphical language for a theory in regular logic. Indeed, the usual syntactic category for that theory will be the regular category $\mathbf{syn}(\mathrm{T}, P)$.

Related work

Regular categories were first defined by Barr [Bar71], as a way to elucidate the structure present in abelian categories. Shortly thereafter, Freyd and Scedrov were the first to make the connection to regular logic. Similarly to the present work, they focused on the structure of the bicategory of relations, seeking an axiomatization through the notion of an allegory, a poset-enriched category (a *po-category*) with an identity-on-objects involution, such that every hom-poset is a meet-semilattice, and such that the modular law holds [FS90].

Carboni and Walters also sought to axiomatize these objects, defining functionally complete cartesian bicategories of relations [CW87]. A cartesian bicategory is a monoidal po-category in which every object is equipped with an adjoint monoid in a coherent way. Functionally complete bicategories of relations further require that these monoids and comonoids obey the Frobenius law, and that a sensible notion of image factorization exists.

Both allegories and bicategories of relations take the structure of a regular category, and decompress it into a (locally posetal) 2-categorical expression. While regular calculi have similar features to both allegories and cartesian bicategories, such as emphasizing that the hom-posets are meet-semilattices or that there are adjoint monoid structures on each object, they represent this data in terms of a *functor* rather than a category.

In the world of databases, regular formulas correspond to conjunctive queries, and entailment corresponds to query containment. A well-known theorem of Chandra and Merlin states that (conjunctive) query containment is decidable; their proof translates logical expressions into graphical representations [CM77]. In more recent work, Bonchi, Seeber, and Sobociński show that the Chandra–Merlin approach permits an elegant formalization in terms of the Carboni–Walters axioms for bicategories of relations [BSS18]. Patterson has also considered bicategories of relations, and their Joyal-Street string calculus [JS91], as a graphical way of capturing the regular logical aspects knowledge representation [Pat17].

Presenting regular categories using monoidal maps $\mathbb{F}\mathsf{Rg}(T) \to \mathbb{P}\mathsf{oset}$ fits into an emerging pattern. In [SSR16] it was shown that lax monoidal functors $1\text{–}\mathsf{Cob}_T \to \mathsf{Set}$ present traced monoidal categories, and in [FS18] it was shown that lax monoidal functors $\mathsf{Cospan}_T \to \mathsf{Set}$ present hypergraph categories. But now in all three cases, the domain of the functor represents a particular language of string diagrams, and the codomain represents a choice of enriching category. The present paper can be seen as an extension of that work, showing that regular categories are something like poset-enriched hypergraph categories.

Outline

We begin in Section 2 with a section reviewing the definition and basic properties of regular categories $\mathcal{R}$, emphasizing in particular the construction of the symmetric monoidal po-category $\mathbb{R}\mathsf{el}_\mathcal{R}$ of relations in $\mathcal{R}$. In fact, we will say that a po-category $\mathscr{R}$ is a *regular po-category* if it is isomorphic to the relations po-category of some regular category $\mathscr{R} \cong \mathbb{R}\mathsf{el}_\mathcal{R}$.

In Section 3 we introduce the notion of adjoint monoid. We show the category of

adjoint monoids in a po-category $\mathscr{C}$ is given by the category of ajax monoidal functors $1 \to \mathscr{C}$, that adjoint monoids in $\mathbb{P}$oset are meet-semilattices, that every object in a relations po-category has a canonical adjoint monoid structure, and that the subobject functor of a regular po-category is ajax.

In Section 4 we turn our attention to free regular categories and free regular po-categories on a set. In particular, we give an explicit construction of the free regular category on a set T as the opposite of the comma category $\mathsf{FinSet} \downarrow \mathcal{P}_f(T)$; the free regular po-category on T is its relations po-category. At this point we can give our main definition: a regular calculus is an ajax functor from a free regular po-category to that of posets. We then give a fully faithful functor $\mathbf{prd}\colon \mathsf{RgCat} \to \mathsf{RgCalc}$, from regular categories to regular calculi.

In Section 5, we introduce graphical regular logic. First, we give an explicit, graphical description of the objects, morphisms, and order in a free regular po-category. We then define the *graphical terms* of a regular calculus. Given a regular calculus $P\colon \mathbb{F}\mathsf{Rg}(T) \to \mathbb{P}$oset, a graphical term is a morphism $\omega\colon \Gamma_1 \times \cdots \times \Gamma_k \twoheadrightarrow \Gamma_{\mathrm{out}}$ in $\mathbb{F}\mathsf{Rg}(T)$ together with elements $\theta_i \in P(\Gamma_i)$ for each $i = 1, \ldots, k$. We give rules for composing and reasoning with these. Having set up our language, we now proceed towards the construction of a regular category from a regular calculus.

In Section 6, we define the po-category of internal relations of an regular calculus. This construction is a relational version of the standard *syntactic category* constructions: an object is a context–predicate pair (Γ, φ), where Γ is an object of $\mathbb{F}\mathsf{Rg}(T)$ and $\varphi \in P(\Gamma)$, and a morphism $(\Gamma, \varphi) \to (\Gamma', \varphi')$ is a predicate θ in the joint context $\Gamma \times \Gamma'$ that entails φ and φ'.

In Section 7, we show that the category of left adjoints in the po-category of internal relations, which we call the category of internal functions, is a regular category. We explicitly construct limits and image factorizations using graphical regular logic.

Finally, in Section 8, we construct the functor $\mathbf{syn}\colon \mathsf{RgCalc} \to \mathsf{RgCat}$ adjoint to $\mathbf{prd}$, and show that the two form an essential reflection.

Notation and 2-categorical background

Let us fix some notation. Most is standard, but we highlight in particular our use of $\mathbin{\raisebox{0.2ex}{\tiny$\mathsection$}}$ for composition, of the term *po-category* for locally posetal 2-category, and of an arrow $\Rightarrow$ pointing in the direction of the left adjoint to signify an adjunction.

- We typically denote composition in diagrammatic order, so the composite of $f\colon A \to B$ and $g\colon B \to C$ is $f \mathbin{\raisebox{0.2ex}{\tiny$\mathsection$}} g\colon A \to C$. We often denote the identity morphism $\mathrm{id}_c\colon c \to c$ on an object $c \in \mathcal{C}$ simply by the name of the object, c. Thus if $f\colon c \to d$, we have $(c \mathbin{\raisebox{0.2ex}{\tiny$\mathsection$}} f) = f = (f \mathbin{\raisebox{0.2ex}{\tiny$\mathsection$}} d)$.
- We may denote the terminal object of any category by $\star$, and the associated map from an object c as $!\colon c \to \star$, but we denote the top element of any poset P by $\mathtt{true} \in P$.
- We denote the universal map into a product by $\langle f, g \rangle$ and the universal map out of a coproduct by $[f, g]$.

- Given a natural number $n \in \mathbb{N}$, define $\underline{n} := \{1, 2, \ldots, n\} \in \mathsf{FinSet}$; in particular $\underline{0} = \varnothing$.
- Given a lax monoidal functor $F \colon \mathcal{C} \to \mathcal{D}$, we denote the laxators by $\rho \colon I \to F(I)$ and $\rho_{c,c'} \colon F(c) \otimes F(c') \to F(c \otimes c')$ for any $c, c' \in \mathcal{C}$. We use the same notation for longer lists, e.g. we write $\rho_{c,c',c''}$ for the canonical map $F(c) \otimes F(c') \otimes F(c'') \to F(c \otimes c' \otimes c'')$.

Symmetric monoidal po-categories. We use the term *po-category* to mean locally posetal 2-category, i.e. a category enriched in partially ordered sets (posets). *Po-functors* are, of course, poset-enriched functors (functors that preserve the local order). The set of po-functors $\mathcal{C} \to \mathcal{D}$ itself has a natural order, where $F \leq G$ iff $F(c) \leq G(c)$ for all $c \in \mathcal{C}$. We define $\mathbb{P}\mathsf{ocat}$ to be the po-category of po-categories and po-functors.

We use $\mathbb{X}\mathsf{yz}$—with first character made blackboard bold—to denote named po-categories and Xyz for named 1-categories. We rely fairly heavily on this; for example our notations for the free regular category and the free regular po-category on a set T differ only in this way: $\mathsf{FRg}(\mathrm{T})$ vs. $\mathbb{F}\mathsf{Rg}(\mathrm{T})$.

A po-category is, in particular, a (strict) 2-category, and po-functors are (strict) 2-functors. As such there is a forgetful functor $\mathbb{P}\mathsf{ocat} \to \mathsf{Cat}$ sending each po-category and po-functor to its underlying 1-category and 1-functor. A *symmetric monoidal po-category* is a po-category whose underlying 1-category is equipped with a symmetric monoidal structure.

The symmetric monoidal po-category $\mathbb{P}\mathsf{oset}$ has posets P as objects, monotone maps $f \colon P \to Q$ as morphisms, and order given by $f \leq g$ iff $f(p) \leq g(p)$ for all p. Its monoidal structure is given by cartesian product $P \times Q$, with the terminal poset 1 the monoidal unit.

Adjunctions in a 2-category. Recall that, given a 2-category $\mathcal{C}$, an *adjunction in $\mathcal{C}$* consists of a pair of objects $c, d \in \mathrm{Ob}\,\mathcal{C}$, a pair of morphisms $L \colon c \to d$ and $R \colon d \to c$, and a pair of 2-morphisms $\eta \colon d \Rightarrow (L \mathbin{\fatsemi} R)$ and $\epsilon \colon (R \mathbin{\fatsemi} L) \Rightarrow c$ such that a pair of equations hold:

$$
\mathrm{id}_L =
\begin{array}{c}
c \xrightarrow{\;L\;} \\
\stackrel{\eta}{\Longrightarrow} \\
c \xrightarrow{R} d \\
\stackrel{\epsilon}{\Longrightarrow} \\
L \searrow d
\end{array}
\qquad \text{and} \qquad
\mathrm{id}_R =
\begin{array}{c}
R \nearrow d \\
c \stackrel{\epsilon}{\Longrightarrow} \\
\stackrel{\eta}{\Longrightarrow} L \searrow d \\
c \xleftarrow{R}
\end{array}
$$

Noting that both η and ϵ always point in the direction of the left adjoint L, we write

$$
c \; \underset{R}{\overset{L}{\underset{\Longleftarrow}{\overset{\Rightarrow}{\rightleftarrows}}}} \; d
$$

to denote an adjunction, where the 2-arrow points in the direction of the left adjoint. We sometimes write $L \dashv R$ inline, but are careful to avoid the $\vdash$ symbol in this context; *the symbol $\vdash$ always means entailment*. We denote the category with the same objects and with left adjoints as morphisms as $\mathsf{LAdj}(\mathcal{C})$.

2 Background on regular categories

Regular categories are, roughly speaking, categories that have a good notion of relations. Relations, which we sometimes call *predicates*, are subobjects of products, and composites of relations are formed using pullbacks and image factorizations; regular categories are categories that have suitably interoperable finite limits and image factorizations. We now proceed to make this precise.

2.1 Definition of regular categories and functors

Regular categories were first defined by Barr [Bar71] to isolate important aspects of abelian categories. The reader who is unacquainted with regular categories and/or regular logic may see [But98].

Definition 2.1 (Barr). A *regular category* is a category $\mathcal{R}$ with the following properties:
1. it has all finite limits;
2. the kernel pair of any morphism $f\colon r \to s$ admits a coequalizer $r \times_s r \rightrightarrows r \to \mathrm{coeq}(f)$, which we denote $\mathrm{im}(f) := \mathrm{coeq}(f)$ and call the *image* of f; and
3. the pullback—along any map—of a regular epimorphism (a coequalizer of any parallel pair) is again a regular epimorphism.

A *regular functor* is a functor between regular categories that preserves finite limits and regular epis. We write RgCat for the category of regular categories.

Lemma 2.2. *For any $f\colon r \to r'$, the universal map $\mathrm{im}(f) \to r'$ is monic. Thus every map f can be factored into a regular epimorphism followed by a monomorphism: $r \twoheadrightarrow \mathrm{im}(f) \rightarrowtail r'$, and this constitutes an orthogonal factorization system. In particular, image factorization is unique up to isomorphism.*

Proof. This is [But98, Proposition 2.4]. $\qquad\square$

Definition 2.3. The *support* of an object r in a regular category is the image $r \twoheadrightarrow \mathrm{Supp}(r) \rightarrowtail \star$ of its unique map to the terminal object.

Definition 2.4. A *subobject* of an object r in a category is an isomorphism class of monomorphisms $r' \rightarrowtail r$, where morphisms between monomorphisms are as in the slice category over r. This defines a partially ordered set $\mathsf{Sub}(r)$. We write $r' \subseteq r$ to denote the equivalence class represented by $r' \rightarrowtail r$.

Proposition 2.5. *Any morphism $f\colon r \to s$ in a regular category $\mathcal{R}$ induces an adjunction*

$$\mathsf{Sub}(r) \underset{f^*}{\overset{f_!}{\underset{\Leftarrow}{\rightrightarrows}}} \mathsf{Sub}(s). \tag{5}$$

This extends to a functor $\mathsf{Sub}\colon \mathcal{R} \to \mathsf{LAdj}(\mathbb{P}\mathsf{oset})$.

Proof. Given a subobject $r' \subseteq r$ or $s' \subseteq s$, define $f_!(r') \subseteq s$ and $f^*(s') \subseteq r$ as follows:

$$
\begin{array}{ccc}
r' & \rightarrowtail & r \\
\downarrow & & \downarrow f \\
f_!(r') & \rightarrowtail & s
\end{array}
\qquad\qquad
\begin{array}{ccc}
f^*(s') & \rightarrowtail & r \\
\downarrow & \lrcorner & \downarrow f \\
s' & \rightarrowtail & s
\end{array}
$$

The fact that these are adjoint follows from the orthogonality of the factorization system in Lemma 2.2, and the constructions are functorial. $\qquad\square$

The following proposition discusses some well-known properties of subobjects in a regular category. In Remark 3.19 we explain how these properties are 1-categorical reflections of a more elementary 2-categorical story.

Proposition 2.6. *Let $\mathcal{R}$ be a regular category. The functor* $\mathsf{Sub}\colon \mathcal{R} \to \mathsf{LAdj}(\mathbb{P}\mathrm{oset})$ *satisfies the following:*

1. $\mathsf{Sub}(r)$ *is a meet-semilattice for each $r \in \mathcal{R}$,*
2. *for each cospan $f\colon r' \to r \leftarrow s : g$, the Beck-Chevalley condition (right) holds for the pullback square (left):*

$$
\begin{array}{ccc}
r' \times_r s & \xrightarrow{\pi_2} & s \\
\pi_1 \downarrow & \lrcorner & \downarrow g \\
r' & \xrightarrow{f} & r
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathsf{Sub}(r' \times_r s) & \xleftarrow{\pi_2^*} & \mathsf{Sub}(s) \\
\pi_{1!} \downarrow & & \downarrow g_! \\
\mathsf{Sub}(r') & \xleftarrow{f^*} & \mathsf{Sub}(r)
\end{array}
$$

3. *for each regular epimorphism $f\colon r' \twoheadrightarrow r$ and $\varphi \in \mathsf{Sub}(r)$, the following holds:*

$$
f_!\big(f^*(\varphi)\big) = \varphi.
$$

4. *for each $f\colon r' \to r$, and $\varphi \in \mathsf{Sub}(r)$ and $\varphi' \in \mathsf{Sub}(r')$, Frobenius reciprocity holds:*

$$
f_!(\varphi \wedge f^*(\varphi')) = f_!(\varphi) \wedge \varphi'
$$

A regular functor $\mathcal{F}\colon \mathcal{R} \to \mathcal{R}'$ induces a natural transformation $\alpha\colon \mathsf{Sub}_{\mathcal{R}} \to \mathsf{Sub}_{\mathcal{R}'}(\mathcal{F}-)$ such that

1. α *is natural with respect to both adjoints, $f_!$ and f^*, for each $f\colon r' \to r$, and*
2. α_r *is a meet-semilattice map for each $r \in \mathcal{R}$.*

Proof. For the properties of the functor Sub, (1) can be easily verified by checking that that binary meets are given by pullback and the top element is given by the identity map, (2) is [But98, Lemma 2.9], (3) follows from pullback stability of regular epis and uniqueness of factorizations (Lemma 2.2), and (4) is [But98, Lemma 2.6].

The properties of $\alpha\colon \mathsf{Sub} \to \mathsf{Sub}(\mathcal{F}-)$ are found in/above [But98, Lemma 2.10]. $\qquad\square$

2.2 The relations po-category construction

A regular category $\mathcal{R}$ has exactly the structure and properties necessary to construct a po-category of relations, or *relations po-category*.

Definition 2.7. Let $\mathcal{R}$ be a regular category; its *relations po-category* $\mathbb{Rel}_\mathcal{R}$ is the po-category with the same objects as $\mathcal{R}$ but whose morphisms, written $x\colon r \twoheadrightarrow s$, are relations $x \subseteq r \times s$ in $\mathcal{R}$ equipped with the subobject ordering $x \leq x'$ iff $x \subseteq x'$. The composite $x \mathbin{\mathring{,}} y$ with a relation $y\colon s \twoheadrightarrow t$ is obtained by pulling back over s and image factorizing the map to $r \times t$:

$$
\begin{array}{ccccc}
 & & x \times_s y & & \\
 & \swarrow & \downarrow & \searrow & \\
x & & x \mathbin{\mathring{,}} y & & y \\
\downarrow & & \downarrow & & \downarrow \\
r \times s & & r \times t & & s \times t \\
\swarrow & \searrow & & \swarrow & \searrow \\
r & & s & & t
\end{array}
\tag{6}
$$

$\mathbb{Rel}_R$ also inherits a symmetric monoidal structure $I := 1$ and $r_1 \otimes r_2 := r_1 \times r_2$ from the cartesian monoidal structure on $\mathcal{R}$.

Given a regular functor $\mathcal{F}\colon \mathcal{R} \to \mathcal{R}'$, mapping a relation $x \subseteq r \times s$ to its factorization $\mathcal{F}(x) \twoheadrightarrow \mathbb{Rel}_\mathcal{F}(x) \rightarrowtail \mathcal{F}(r \times s) \cong \mathcal{F}(r) \times \mathcal{F}(s)$ induces a (strong) symmetric monoidal po-functor $\mathbb{Rel}_\mathcal{F}\colon \mathbb{Rel}_\mathcal{R} \to \mathbb{Rel}_{\mathcal{R}'}$. We refer to this po-functor as the *relations po-functor* of $\mathcal{F}$.

It is straightforward to check that the composition rule Eq. (6) is unital and associative using the pullback stability of factorizations, and to check that $\mathbb{Rel}_\mathcal{F}$ is indeed a symmetric monoidal po-functor using the fact that a regular functor $\mathcal{F}\colon \mathcal{R} \to \mathcal{R}'$ preserves pullbacks and image factorizations. Direct proofs in the literature of these two facts seem difficult to find, but see for example [JW00, Theorem 2.3] and [Fon18, Proposition 4.1] respectively.

The relations po-category is just a repackaging of the data of the regular category: any regular category can be recovered, at least up to isomorphism, by looking at the adjunctions in its relations po-category.

Lemma 2.8 (Fundamental lemma of regular categories)**.** *Let $\mathcal{R}$ be a regular category. Then there is an identity-on-objects isomorphism*

$$
\mathcal{R} \to \mathsf{LAdj}(\mathbb{Rel}_\mathcal{R}).
$$

In particular, a relation $x\colon r \twoheadrightarrow s$ is a left adjoint iff it is the graph $x = \langle \mathrm{id}_r, f \rangle$ of a morphism $f\colon r \to s$ in $\mathcal{R}$.

Proof. This fact is well known, but since it is crucial to what follows, we provide a proof here. We shall show that there is an identity-on-objects, full, and faithful functor from $\mathcal{R}$ to its relations po-category $\mathbb{Rel}_\mathcal{R}$, which maps a morphism $f\colon r \to s$ to its graph $\langle \mathrm{id}_r, f \rangle \subseteq r \times s$.

Indeed, it is straightforward to check that any pair of the form $\langle \mathrm{id}_r, f\rangle \dashv \langle f, \mathrm{id}_s\rangle$ is an adjunction, and subsequently that the proposed map is functorial.

To show that it is full and faithful, we characterize the adjunctions $x \dashv x'$ in $\mathbb{Rel}_{\mathcal{R}}$. Suppose we have $x \overset{\langle g, f\rangle}{\rightarrowtail} r \times s$ and $x' \overset{\langle f', g'\rangle}{\rightarrowtail} s \times r$ with unit $i \colon r \rightarrowtail (x \mathbin{\fatsemi} x')$ and counit $j \colon (x' \mathbin{\fatsemi} x) \to s$. This gives rise to the following diagram (equations shown right):

$$i \mathbin{\fatsemi} \pi_s \mathbin{\fatsemi} g = \mathrm{id}_r = i \mathbin{\fatsemi} \pi'_s \mathbin{\fatsemi} g'$$
$$\pi_r \mathbin{\fatsemi} f = j = \pi'_r \mathbin{\fatsemi} f'$$

We shall show that g and g' are isomorphisms, and that $f' = g' \mathbin{\fatsemi} g^{-1} \mathbin{\fatsemi} f$.

We first show that $i \mathbin{\fatsemi} \pi_s$ is inverse to g. Since the unit already gives that $i \mathbin{\fatsemi} \pi_s \mathbin{\fatsemi} g = \mathrm{id}_r$, it suffices to show that $g \mathbin{\fatsemi} i \mathbin{\fatsemi} \pi_s = \mathrm{id}_x$. Moreover, since $\langle g, f\rangle \colon x \to r \times s$ is monic and $g = (g \mathbin{\fatsemi} i \mathbin{\fatsemi} \pi_s) \mathbin{\fatsemi} g$, it suffices to show that $f = (g \mathbin{\fatsemi} i \mathbin{\fatsemi} \pi_s) \mathbin{\fatsemi} f$. This is a diagram chase: since $g = g \mathbin{\fatsemi} i \mathbin{\fatsemi} \pi'_s \mathbin{\fatsemi} g'$, we can define a morphism $q \coloneqq \langle \mathrm{id}_x, g \mathbin{\fatsemi} i \mathbin{\fatsemi} \pi'_s\rangle \colon x \to x \times_r x'$, and we conclude

$$f = q \mathbin{\fatsemi} \pi_r \mathbin{\fatsemi} f = q \mathbin{\fatsemi} \pi'_r \mathbin{\fatsemi} f' = g \mathbin{\fatsemi} i \mathbin{\fatsemi} \pi'_s \mathbin{\fatsemi} f' = g \mathbin{\fatsemi} i \mathbin{\fatsemi} \pi_s \mathbin{\fatsemi} f.$$

Similarly, we see that $i \mathbin{\fatsemi} \pi'_s$ is inverse to g', and hence obtain $f' = g' \mathbin{\fatsemi} g^{-1} \mathbin{\fatsemi} f$.

Note that this implies the adjunction $x \dashv x'$ is isomorphic to the adjunction $\langle 1_r, (g^{-1} \mathbin{\fatsemi} f)\rangle \dashv \langle (g^{-1} \mathbin{\fatsemi} f), \mathrm{id}_s\rangle$. Thus the proposed functor is full. Faithfulness amounts to the fact that the existence of a morphism $\langle 1_r, f\rangle \to \langle 1_r, f'\rangle$ implies $f = f'$. This proves the lemma. $\square$

Remark 2.9. It follows from the proof of Lemma 2.8 that $x \colon r \rightarrowtail s$ is a right adjoint iff it is the co-graph $\langle f, \mathrm{id}_s\rangle$ of a morphism $f \colon s \to r$. Furthermore, since any morphism $x = \langle g, f\rangle \colon r \rightarrowtail s$ in $\mathcal{R}$ can be written as $x = \langle g, \mathrm{id}_x\rangle \mathbin{\fatsemi} \langle \mathrm{id}_x, f\rangle$, it follows that every morphism in $\mathcal{R}$ can be written as the composite of a right adjoint followed by a left adjoint.

The fundamental lemma says that regular categories can be recovered from their relations po-categories. Similarly, any regular functor can be recovered as the action of its relations po-functor on left adjoints. Before expressing this as a categorical equivalence in Eq. (7), we first make the following observation.

Proposition 2.10. *For any regular functor $\mathcal{F} \colon \mathcal{R} \to \mathcal{R}'$, the relations po-functor $\mathbb{Rel}_{\mathcal{F}} \colon \mathbb{Rel}_{\mathcal{R}} \to \mathbb{Rel}_{\mathcal{R}'}$ is strong symmetric monoidal.*

Proof. The functor $\mathcal{F}$ and its relations po-functor $\mathbb{Rel}_{\mathcal{F}}$ act the same on objects, so since $\mathcal{F}$ is product preserving, $\mathbb{Rel}_{\mathcal{F}}$ is strong monoidal. $\square$

Although we do not assume it below, it is a result of Carboni and Walters that every strong symmetric monoidal functor $\mathbb{Rel}_{\mathcal{R}} \to \mathbb{Rel}_{\mathcal{R}'}$ is the relations po-functor associated to

a regular functor $\mathcal{F}\colon \mathcal{R} \to \mathcal{R}'$ [CW87]. Indeed, this foreshadows the rephrasing of regular structure in terms of monoidal structure, which runs through this paper.

In any case, this motivates the following definition.

Definition 2.11. A po-category is called a *regular po-category* if it is isomorphic to the relations po-category $\mathbb{Rel}_{\mathcal{R}}$ of some regular category $\mathcal{R}$.

A strong symmetric monoidal po-functor between regular po-categories is called a *regular po-functor* if it is isomorphic to the relations po-functor $\mathbb{Rel}_{\mathcal{F}}$ associated to a regular functor $\mathcal{F}$. We write RgPocat for the category of regular po-categories.

By the fundamental lemma (2.8), we now have an equivalence of categories:

$$\mathsf{RgCat} \; \overset{\mathbb{Rel}_-}{\underset{\mathsf{LAdj}}{\overset{\cong}{\rightleftarrows}}} \; \mathsf{RgPocat}. \tag{7}$$

3 Adjoint monoids and adjoint-lax functors

The poset of subobjects of an object in a regular category is always a meet-semilattice. We characterize these as precisely the adjoint monoids in $\mathbb{Poset}$. The seemingly new notion of adjoint monoid makes sense in any monoidal po-category (and more generally): an *adjoint monoid* is an object equipped with commutative monoid and comonoid structures such that the multiplication and unit are right adjoint to the comultiplication and counit. Every regular po-category $\mathcal{R}$ is isomorphic to its own po-category of adjoint monoids $\mathcal{R} \cong \mathbb{AdjMon}(\mathcal{R})$. Finally, the subobjects functor preserves adjoint monoids.

All these ideas are founded on the notion of adjoint-lax monoidal (ajax) po-functor.

3.1 Definition and motivation

In this section we introduce the notions of ajax functor and adjoint monoid.

Definition 3.1. Let $\mathscr{C}$ and $\mathscr{D}$ be monoidal po-categories. An *adjoint-lax* or *ajax* po-functor $F\colon \mathscr{C} \to \mathscr{D}$ is a lax symmetric monoidal po-functor for which the laxators are right adjoints.

We denote the laxators by ρ and their left adjoints by λ:

$$I \; \overset{\rho}{\underset{\lambda}{\rightleftarrows}} \; F(I) \qquad \text{and} \qquad F(c) \otimes F(c') \; \overset{\rho_{c,c'}}{\underset{\lambda_{c,c'}}{\rightleftarrows}} \; F(c \otimes c')$$

Warning 3.2. The notion of ajax functor has a dual notion of op-ajax functor: an oplax functor $\mathscr{C} \to \mathscr{D}$ for which the op-laxators are left adjoints. These two notions *do not coincide*! The laxator naturality squares are asked to strictly commute in an ajax functor, and this property only implies that their mate squares, the corresponding oplaxator naturality squares weakly commute.

Here is a obvious, but useful, consequence of the definition.

Lemma 3.3. *Every strong monoidal functor between monoidal po-categories is ajax. The composite of ajax po-functors is ajax.*

Recall that 1 is the terminal monoidal po-category.

Proposition 3.4. *Let $(\mathscr{C}, I, \otimes)$ be a monoidal po-category. There is a bijection between:*
1. *The set of ajax functors $1 \to \mathscr{C}$,*
2. *The set of commutative monoid objects (c, μ, η) such that μ and η are right adjoints, and*
3. *The set of cocommutative comonoid objects (c, δ, ϵ) such that δ and ϵ are left adjoints.*

Proof. $(1) \Leftrightarrow (2)$**:** The set $\mathsf{Lax}(1, \mathscr{C})$ of lax symmetric monoidal functors $1 \to \mathscr{C}$ is well-known to be in bijection with the set of commutative monoid objects (c, μ, η) in $\mathscr{C}$. Indeed, η and unit μ come from the 0-ary and 2-ary laxators respectively: $\eta = \rho$ and $\mu = \rho_{1,1}$. Hence the added condition that η and μ have left adjoints is precisely the ajax condition.

$(2) \Leftrightarrow (3)$**:** Suppose given an object $c \in \mathscr{C}$ and two adjunctions

$$ I \underset{\epsilon}{\overset{\eta}{\underset{\Leftarrow}{\Longrightarrow}}} c \ . \qquad \text{and} \qquad c \otimes c \underset{\delta}{\overset{\mu}{\underset{\Leftarrow}{\Longrightarrow}}} c \tag{8} $$

Then μ, η satisfy the commutative monoid laws iff δ, ϵ satisfy the cocommutative comonoid laws. $\square$

To summarize, if $(c, \rho, \lambda) \colon 1 \to \mathscr{C}$ is an ajax functor then the corresponding monoid and comonoid structures on c are given by

$$ \eta = \rho \qquad \mu = \rho_{1,1} \qquad \text{and} \qquad \epsilon = \lambda \qquad \delta = \lambda_{1,1} \tag{9} $$

Proposition 3.4 motivates the following definition.

Definition 3.5. Let $(\mathscr{C}, I, \otimes)$ be a monoidal po-category. An *adjoint commutative monoid* (or simply *adjoint monoid*) in $\mathscr{C}$ is a commutative monoid object (c, μ, η) in $\mathscr{C}$ such that μ and η are right adjoints.

Adjoint monoids are a slight weakening of the *internal meet semi-lattice* notion from theoretical computer science; see **[schalk1994algebras]** and references therein.

Proposition 3.6. *Ajax functors send adjoint monoids to adjoint monoids.*

Proof. The composite of ajax functors is ajax, so the result follows from Proposition 3.4. $\square$

We give examples of adjoint monoids after recalling the proof of a well-known lemma.

Lemma 3.7. *Let $\mathscr{C}$ be a monoidal po-category. If the monoidal structure is cartesian (given by finite products in the underlying 1-category) then every object has a unique comonoid structure, and it is commutative.*

Proof. Since the unit object is terminal, the maps $c \times \epsilon$ and $\epsilon \times c$ are forced to be the projections $c \times c \to c$, so δ is forced to be the diagonal. $\qquad\square$

Proposition 3.8. *A poset $P \in \mathbb{P}\text{oset}$ is an adjoint monoid iff it is a meet-semilattice, in which case $\eta = \texttt{true}$ and $\mu = \wedge$.*

Proof. By Lemma 3.7, P has a unique comonoid structure given by the terminal and diagonal maps $\epsilon \colon P \to 1$ and $\delta \colon P \to P \times P$. Thus P is an adjoint monoid iff these maps have adjoints as in Eq. (8), which holds iff η is a top element and μ is a meet. $\qquad\square$

Proposition 3.9. *Let $\mathcal{R}$ be a regular category. Every object $r \in \mathcal{R}$ in its relations po-category has a unique adjoint monoid structure.*

Proof. Since $\mathcal{R}$ is cartesian monoidal, there is a unique cocommutative comonoid structure on every object by Lemma 3.7. By the fundamental lemma (2.8), we have an isomorphism $\mathcal{R} \cong \mathsf{LAdj}(\mathcal{R})$, and we are done by Proposition 3.4 (2) $\Leftrightarrow$ (3). $\qquad\square$

3.2 Notions of morphism between ajax functors

Given two ajax functors $F, F' \colon \mathscr{C} \rightrightarrows \mathscr{D}$, we will consider two sorts of (strong) natural transformations $\alpha \colon F \to F'$ between them, differing in terms of the strength of their *laxator naturality*. The first sort only demands that the laxator naturality squares for any $c \in \mathscr{C}$ be *mate squares* in $\mathscr{D}$:

$$
\begin{array}{ccc}
F(I) \xrightarrow{\alpha_I} F'(I) & & F(c \otimes c') \xrightarrow{\alpha_{c \otimes c'}} F'(c \otimes c') \\
\rho {\Downarrow} \lambda \quad \Downarrow \quad \rho' {\Downarrow} \lambda' & \text{and} & \rho_{c,c'} {\Downarrow} \lambda_{c,c'} \quad \Downarrow \quad \rho'_{c,c'} {\Downarrow} \lambda'_{c,c'} \\
I = I & & F(c) \otimes F(c') \xrightarrow{\alpha_c \otimes \alpha_{c'}} F'(c) \otimes F'(c')
\end{array}
\tag{10}
$$

The meaning of each diagram in (10) is that any (and all) of the following four equivalent conditions hold (dropping subscripts and writing $\alpha^{\otimes 2} := (\alpha \otimes \alpha)$):

$$
\begin{array}{ccccc}
\rho \, \mathbin{\fatsemi} \, \alpha \le \rho' & \Leftrightarrow & \rho \, \mathbin{\fatsemi} \, \alpha \, \mathbin{\fatsemi} \, \lambda' \le I & & \rho \, \mathbin{\fatsemi} \, \alpha \le \alpha^{\otimes 2} \, \mathbin{\fatsemi} \, \rho' \quad \Leftrightarrow \quad \rho \, \mathbin{\fatsemi} \, \alpha \, \mathbin{\fatsemi} \, \lambda' \le \alpha^{\otimes 2} \\
\Updownarrow & & \Updownarrow & \text{and} & \Updownarrow \qquad\qquad\qquad\qquad \Updownarrow \\
\alpha \, \mathbin{\fatsemi} \, \lambda' \le \lambda & \Leftrightarrow & \alpha \le \lambda \, \mathbin{\fatsemi} \, \rho' & & \alpha \, \mathbin{\fatsemi} \, \lambda' \le \lambda \, \mathbin{\fatsemi} \, \alpha^{\otimes 2} \quad \Leftrightarrow \quad \alpha \le \lambda \, \mathbin{\fatsemi} \, \alpha^{\otimes 2} \, \mathbin{\fatsemi} \, \rho'
\end{array}
\tag{11}
$$

The second sort demands further that some of these inequalities be equalities.

Definition 3.10. Let $F, F' \colon \mathscr{C} \rightrightarrows \mathscr{D}$ be ajax functors. A *mate morphism* between them is a natural transformation $\alpha \colon F \to F'$ with mate squares as in Eq. (10). We say that α is *strong* if the monoid part of the diagram strictly commutes (for all $c, c' \in \mathscr{C}$):

$$
\rho \, \mathbin{\fatsemi} \, \alpha_I = \rho' \qquad \text{and} \qquad \rho_{c,c'} \, \mathbin{\fatsemi} \, \alpha_{c \otimes c'} = (\alpha_c \otimes \alpha_{c'}) \, \mathbin{\fatsemi} \, \rho'_{c,c'}.
\tag{12}
$$

We denote the corresponding categories by $\mathsf{Ajax}(\mathscr{C}, \mathscr{D})$ and $\mathsf{Ajax}^{\mathtt{str}}(\mathscr{C}, \mathscr{D})$ respectively.

Suppose $\alpha, \beta \colon F \to F'$ are mate morphisms (possibly strong). We write $\alpha \leq \beta$ if for all $c \in \mathscr{C}$ we have $\alpha_c \leq \beta_c$ in the poset $\mathscr{C}(F(c), F'(c))$. We denote the corresponding po-categories as

$$\mathbb{A}\mathsf{jax}(\mathscr{C}, \mathscr{D}) \qquad \text{and} \qquad \mathbb{A}\mathsf{jax}^{\mathtt{str}}(\mathscr{C}, \mathscr{D}).$$

Clearly, the po-category structure of $\mathbb{A}\mathsf{jax}(\mathscr{C}, \mathscr{D})$ is inherited from $\mathbb{P}\mathsf{ocat}(\mathscr{C}, \mathscr{D})$; we record this fact in the following obvious lemma.

Lemma 3.11. *The map* $\mathbb{A}\mathsf{jax}(\mathscr{C}, \mathscr{D}) \to \mathbb{P}\mathsf{ocat}(\mathscr{C}, \mathscr{D})$ *is locally fully faithful.*

Definition 3.12. Let $\mathscr{C}$ be a monoidal po-category. Define po-categories

$$\mathbb{A}\mathsf{djMon}(\mathscr{C}) \coloneqq \mathbb{A}\mathsf{jax}(1, \mathscr{C}) \qquad \text{and} \qquad \mathbb{A}\mathsf{djMon}^{\mathtt{str}}(\mathscr{C}) \coloneqq \mathbb{A}\mathsf{jax}^{\mathtt{str}}(1, \mathscr{C}),$$

and refer to them as the po-category of *adjoint monoids* and the po-category of *adjoint monoids and strong maps* respectively.

Let $\mathbb{P}\mathsf{oset}|_{\wedge\text{-}\mathsf{SL}}$ denote the full sub-po-category of $\mathbb{P}\mathsf{oset}$ spanned by the meet-semilattices, and let $\wedge\text{-}\mathsf{SL}$ denote the po-category of meet-semilattices and meet-preserving maps.

Proposition 3.13. *There are isomorphisms of po-categories*

$$\mathbb{A}\mathsf{djMon}(\mathbb{P}\mathsf{oset}) \cong \mathbb{P}\mathsf{oset}|_{\wedge\text{-}\mathsf{SL}} \qquad and \qquad \mathbb{A}\mathsf{djMon}^{\mathtt{str}}(\mathbb{P}\mathsf{oset}) \cong \wedge\text{-}\mathsf{SL}.$$

Proof. By Proposition 3.8 and Eq. (9) we have the desired isomorphisms on objects, and $\rho = \eta = \mathtt{true}$ and $\rho_{1,1} = \mu = \wedge$. Since every poset map $\alpha \colon P \to P'$ is a comonoid homomorphism, we have mate diagrams as in Eq. (10), giving $\mathbb{A}\mathsf{djMon}(\mathbb{P}\mathsf{oset}) \cong \mathbb{P}\mathsf{oset}|_{\wedge\text{-}\mathsf{SL}}$.

To see the isomorphism $\mathbb{A}\mathsf{djMon}^{\mathtt{str}}(\mathbb{P}\mathsf{oset}) \cong \wedge\text{-}\mathsf{SL}$, note that by (9), the equations in (12) precisely say $\alpha(\mathtt{true}) = \mathtt{true}$ and $\alpha(\wedge) = \wedge(\alpha, \alpha)$. $\qquad\square$

Proposition 3.14. *Let* $\mathscr{R}$ *be a regular po-category. There are isomorphisms*

$$\mathbb{A}\mathsf{djMon}(\mathscr{R}) \cong \mathscr{R} \qquad and \qquad \mathbb{A}\mathsf{djMon}^{\mathtt{str}}(\mathscr{R}) \cong \mathbb{R}\mathsf{Adj}(\mathscr{R}).$$

Proof. In Proposition 3.9 we gave an isomorphism $\mathrm{Ob}\,\mathscr{R} \cong \mathrm{Ob}\,\mathbb{A}\mathsf{djMon}(\mathscr{R})$ coming from the fact that every object $r \in \mathscr{R} = \mathsf{LAdj}(\mathscr{R})$ has a unique comonoid structure. Suppose $r \leftarrow \alpha \to r'$ is a morphism in $\mathscr{R}$. Then there exist unique maps e, d making following diagrams commute:

$$
\begin{array}{ccccc}
r & \xleftarrow{\;f\;} & \alpha & \xrightarrow{\;f'\;} & r' \\
\| & & \vdots e & & \| \\
r & & & & r' \\
\downarrow & & \downarrow & & \downarrow \\
1 & =\!\!= & 1 & =\!\!= & 1
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccccc}
r & \xleftarrow{\;f\;} & \alpha & \xrightarrow{\;f'\;} & r' \\
\| & & \vdots d & & \| \\
r & & & & r' \\
\downarrow & & \downarrow & & \downarrow \\
r \times r & \longleftarrow & \alpha \times \alpha & \longrightarrow & r' \times r'
\end{array}
$$

Thus there is an isomorphism between the posets $\mathscr{R}(r, r')$ and $\mathbb{A}\mathsf{djMon}(r, r')$.

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Unwinding the definition of strong morphisms between the ajax maps $r, r' \colon 1 \to \mathscr{R}$ the equation $\eta \mathbin{\fatsemi} \alpha = \eta$ implies that the map $\mathrm{im}(f') \to r'$ is an iso, i.e. f' is a regular epi; similarly the equation $\mu \mathbin{\fatsemi} \alpha = (\alpha \otimes \alpha) \mathbin{\fatsemi} \mu$ implies that the map $\alpha \to \alpha \times_{r'} \alpha$ is iso, i.e. f' is a mono. In other words, α is strong iff f' is iso, and this holds iff α is a right adjoint (see Remark 2.9). $\qquad\square$

In passing we note the following connection to hypergraph categories, which are well known for their own graphical language, and may help some readers contextualize our main result. This is a corollary of [Fon18, Theorem 3.1].

Proposition 3.15. *Given a regular category $\mathcal{R}$, the monoidal category underlying $\mathbb{R}\mathrm{el}_\mathcal{R}$ has a hypergraph structure, where the symmetric monoidal structure is given by the product in $\mathcal{R}$, and where for any object x in $\mathbb{R}\mathrm{el}_\mathcal{R}$ we have μ_x and δ_x given by the diagonal subobject $x \subseteq x \times x \times x$, and η_x and ϵ_x given by the maximal subobject $x \subseteq x$.*

Loosely speaking, one might think of a regular po-category (Definition 2.11) as a poset-enriched hypergraph category in which the hom-posets are meet-semilattices.

3.3 The subobjects-functor is ajax

Let $\mathcal{R}$ be a regular category and recall the subobjects functor $\mathsf{Sub}\colon \mathcal{R} \to \mathsf{LAdj}(\mathbb{P}\mathrm{oset})$ from Proposition 2.5. It extends to a po-functor $\mathsf{Sub}\colon \mathscr{R} \to \mathbb{P}\mathrm{oset}$, where $\mathscr{R} = \mathbb{R}\mathrm{el}_\mathcal{R}$ is the relations po-category. To be explicit, write a relation $A \subseteq r \times r'$ as a span $r \xleftarrow{f} A \xrightarrow{f'} r'$. Then the map $\mathsf{Sub}(A)\colon \mathsf{Sub}(r) \to \mathsf{Sub}(r')$ applied to a subobject $\varphi \subseteq r$ is given pulling back and then taking the image:

$$
\begin{array}{ccccc}
\varphi & \longleftarrow & \cdot & \twoheadrightarrow & \cdot \\
\Big\updownarrow & & \ulcorner\ \Big\updownarrow & & \Big\updownarrow \\
\downarrow & & \downarrow & & \downarrow \\
r & \longleftarrow & A & \longrightarrow & r'
\end{array}
$$

That is, $\mathsf{Sub}(A) = f_! \mathbin{\fatsemi} g^*$. This po-functor is representable: $\mathsf{Sub}(-) = \mathscr{R}(I, -)$, where I is the terminal object in $\mathcal{R}$. We now show this po-functor is ajax.

Theorem 3.16. *The po-functor $\mathsf{Sub}_\mathscr{R}\colon \mathscr{R} \longrightarrow \mathbb{P}\mathrm{oset}$ is ajax for any regular po-category $\mathscr{R}$.*

Proof. The functor $\mathsf{Sub}_\mathscr{R}(-) = \mathscr{R}(I, -)$ has a canonical lax monoidal structure since $I \otimes I \cong I$. We need to show the laxators $\otimes$ and id_I have left adjoints in $\mathbb{P}\mathrm{oset}$. The first is easy: id_I is the top element in $\mathscr{R}(I, I)$ and thus a right adjoint since there is a unique map $\mathscr{R}(I, I) \to 1$.

Now suppose given $r_1, r_2 \in \mathscr{R}$, and consider the morphisms $\pi_i \colon r_1 \otimes r_2 \to r_i$ and $\delta \colon r \to r \otimes r$ corresponding to the ith projection and the diagonal in $\mathcal{R}$. Composition with the π_i induces a monotone map $\lambda_{r_1, r_2} \colon \mathscr{R}(I, r_1 \otimes r_2) \to \mathscr{R}(I, r_1) \times \mathscr{R}(I, r_2)$, natural in r_1, r_2. It remains to show that each λ_{r_1, r_2} is indeed a left adjoint,

$$
\mathscr{R}(I, r_1 \otimes r_2) \mathrel{\substack{\xrightarrow{\lambda_{r_1, r_2}} \\ \xLeftarrow{} \\ \otimes}} \mathscr{R}(I, r_1) \times \mathscr{R}(I, r_2) \ .
$$

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For the unit, given $g\colon I \to r_1 \otimes r_2$, we have

$$
\begin{aligned}
g &= g \,\mathring{,}\, \delta_{r_1 \otimes r_2} \,\mathring{,}\, ((r_1 \otimes r_2) \otimes \epsilon_{r_1 \otimes r_2}) \\
&= g \,\mathring{,}\, \delta_{r_1 \otimes r_2} \,\mathring{,}\, ((r_1 \otimes \epsilon_{r_2}) \otimes (\epsilon_{r_1} \otimes r_2)) \\
&\leq \delta_I \,\mathring{,}\, (g \otimes g) \,\mathring{,}\, ((r_1 \otimes \epsilon_{r_2})) \otimes (\epsilon_{r_1} \otimes r_2) \\
&= (g \,\mathring{,}\, (r_1 \otimes \epsilon_{r_2})) \otimes (g \,\mathring{,}\, (\epsilon_{r_1} \otimes r_2)).
\end{aligned}
$$

For the counit, given $f_1\colon I \to r_1$ and $f_2\colon I \to r_2$, it suffices to show that $(f_1 \otimes f_2) \,\mathring{,}\, (r_1 \otimes \epsilon_{r_2}) \leq f_1$, since the other projection is similar. And this holds because

$$
(f_1 \otimes f_2) \,\mathring{,}\, (r_1 \otimes \epsilon_{r_2}) = f_1 \otimes (f_2 \,\mathring{,}\, \epsilon_{r_2}) \leq f_1 \otimes \epsilon_I = f_1. \qquad \square
$$

Corollary 3.17. *The po-functor* $\mathsf{Sub}_{\mathscr{R}}\colon \mathscr{R} \to \mathbb{P}\mathsf{oset}$ *sends each object* $r \in \mathscr{R}$ *to a meet-semilattice.*

Proof. This follows from the fact that ajax functors send adjoint monoids to adjoint monoids; see Theorem 3.16 and Propositions 3.6, 3.13, and 3.14. $\qquad \square$

Remark 3.18. In fact, we can get a bit more from Propositions 3.13 and 3.14. If $x\colon r \nrightarrow r'$ is an arbitrary map in $\mathscr{R}$ then the monotone map $\mathsf{Sub}(x)\colon \mathsf{Sub}_{\mathscr{R}}(r) \to \mathsf{Sub}_{\mathscr{R}}(r')$ is *not necessarily meet-preserving*. However, if x is the image of a morphism in $\mathscr{R}^{\mathrm{op}} = \mathsf{RAdj}(\mathscr{R})$ then $\mathsf{Sub}(x)$ is meet-preserving. That is, $\mathscr{R}^{\mathrm{op}}$ is isomorphic to the underlying 1-category of $\mathbb{A}\mathsf{djMon}^{\mathtt{str}}(\mathscr{R})$.

Remark 3.19. In Proposition 2.6 we discussed four properties of the subobjects functor $\mathsf{Sub}\colon \mathscr{R} \to \mathsf{LAdj}(\mathbb{P}\mathsf{oset})$: a meet-semilattice on each object, Beck-Chevalley for pullbacks, Beck-Chevalley for pushouts of effective epimorphisms, and Frobenius reciprocity. These same four properties in fact hold for any ajax po-functor $F\colon \mathbb{R}\mathsf{el}_{\mathscr{R}} \to \mathbb{P}\mathsf{oset}$. Moreover, this construction is invertible: given any functor $\mathscr{R} \to \mathsf{LAdj}(\mathbb{P}\mathsf{oset})$, there is an induced ajax po-functor $\mathbb{R}\mathsf{el}_{\mathscr{R}} \to \mathbb{P}\mathsf{oset}$, and the two constructions are mutually inverse

$$
\mathsf{Ajax}(\mathbb{R}\mathsf{el}_{\mathscr{R}}, \mathbb{P}\mathsf{oset}) \cong \{S\colon \mathscr{R} \to \mathsf{LAdj}(\mathbb{P}\mathsf{oset}) \mid \text{four properties in 2.6}\}.
$$

We do not need this result for our work, so we omit the details. However, it is interesting to see these four well-known—though slightly mysterious—properties fall out of the more elementary definition of adjoint-lax functors to $\mathbb{P}\mathsf{oset}$.

4 Free regular categories and regular calculi

We now construct the free regular category $\mathsf{FRg}(\mathrm{T})$—as well as the free regular po-category $\mathbb{F}\mathsf{Rg}(\mathrm{T})$—on a set T. This allows us to define, in Section 4.2, a *regular calculus* to be an ajax po-functor $\mathbb{F}\mathsf{Rg}(\mathrm{T}) \to \mathbb{P}\mathsf{oset}$. Eventually, in Theorem 8.5, we will see that RgCat is essentially a reflective subcategory of the category RgCalc of regular calculi, in the sense that there is an adjunction $\mathsf{RgCat} \leftrightarrows \mathsf{RgCalc}$ such that for any regular category the counit map is an equivalence of categories. Towards that end, we conclude this section in Proposition 4.15 by defining the right adjoint part, $\mathbf{prd}\colon \mathsf{RgCat} \to \mathsf{RgCalc}$.

4.1 The free regular category on a set

We will propose a graphical calculus based on regular categories $\mathsf{FRg}(T)$ free on a set T. We define $\mathsf{FRg}(T)$ in Definition 4.1 and show that it is free in Theorem 4.11.

Write $\mathbb{P}_f(T)$ for the poset of finite subsets of T; this, or equally its opposite category $\mathbb{P}_f(T)^{\mathrm{op}}$, is a free $\wedge$-semilattice on T. Write also FinSet for the category of finite sets and functions. Note that $\mathsf{FinSet}^{\mathrm{op}}$ is the free category with finite limits on one object. The free regular category on T arises when these two structures interact.

Note that for any T there is an inclusion of categories $\mathrm{inc}\colon \mathbb{P}_f(T) \to \mathsf{FinSet}$.

Definition 4.1. Define $\mathsf{FRg}(T) := (\mathbb{P}_f(T)^{\mathrm{op}} \downarrow \mathsf{FinSet}^{\mathrm{op}})$ to be the comma category

$$
\begin{array}{ccc}
 & \mathsf{FRg}(T) & \\
\xleftarrow{\mathrm{Supp}} & \overset{\mathrm{Tp}}{\Longrightarrow} & \xrightarrow{\mathrm{Vars}} \\
\mathbb{P}_f(T)^{\mathrm{op}} & & \mathsf{FinSet}^{\mathrm{op}} \\
\xrightarrow{\mathrm{inc}} & \mathsf{FinSet}^{\mathrm{op}} & \xleftarrow{\mathrm{id}}
\end{array}
$$

for any set T. We refer to objects $\Gamma \in \mathsf{FRg}(T)$ as *contexts*.

We can unpack a context Γ into a quasi-traditional form, e.g. as

$$
\Gamma \quad = \quad x_1 : \tau_1, \ldots, x_n : \tau_n \mid \tau_1', \ldots \tau_m'
$$

which has a finite set of *variables*, $\mathrm{Vars}(\Gamma) = \{x_1, \ldots, x_n\}$, whose *support* set is $\mathrm{Supp}(\Gamma) = \{\tau_1, \ldots, \tau_n, \tau_1' \ldots, \tau_m'\}$, and which has the *typing* function $\mathrm{Tp}(x_i) = \tau_i$. The notion of support does not typically have a place in traditional logical contexts, but we include it because $\mathrm{Supp}(\Gamma)$ has a definite place in objects of the free regular category.

Working in the skeleton of $\mathsf{FRg}(T)$, we can assume that each cardinality has a unique set of variables, e.g. $\underline{n} = \{1, \ldots, n\}$. Here is an equivalent but more concrete description of the free regular category on T:

$$
\mathrm{Ob}\,\mathsf{FRg}(T) := \big\{ (n, S, \tau) \mid n \in \mathbb{N}, S \subseteq T \text{ finite}, \tau : \underline{n} \to S \big\}
$$

$$
\mathsf{FRg}(T)\big(\Gamma, \Gamma'\big) := \left\{ f : \underline{n}' \to \underline{n} \;\middle|\;
\begin{array}{c}
\underline{n} \xrightarrow{\tau} S \\
f \uparrow \quad \cup\mid \overset{\subseteq}{\underset{\subseteq}{}} T \\
\underline{n}' \xrightarrow{\tau'} S'
\end{array}
\right\} \tag{13}
$$

Given a map $f \colon \Gamma \to \Gamma'$, we denote the corresponding function as $\underline{f} \colon \underline{n}' \to \underline{n}$. Say that a context $\Gamma = (n, S, \tau)$:

- is a *unary context* if it is of the form $(1, \{s\}, !)$, i.e. if it has arity $n = 1$ and full support $|S| = 1$; we denote it simply as $\langle\, s\, \rangle$.
- is a *unary support context* if it is of the form $(0, \{s\}, !)$; i.e. if it has $n = 0$ and $|S| = 1$; we abuse notation to denote this $\mathrm{Supp}(s)$.

Example 4.2. Suppose $T = \{s\}$ is unary. When $n = 0$, the map τ is unique, and we either have $S = \varnothing$ or $S = \{s\}$. Thus we recover the description from Eq. (1), though in the present terms it looks like this:

$$(0, \varnothing) \longleftarrow\!\!\!\prec (0, \{s\}) \twoheadleftarrow\!\!\!- (1, \{s\}) \underset{\longrightarrow}{\overset{\longleftarrow}{\rightrightarrows}} (2, \{s\}) \qquad \cdots$$

Example 4.3. For any set T, the poset of subobjects of 0 in $\mathsf{FRg}(T)$ is the free meet-semilattice on T, i.e. the finite powerset $\mathbb{P}_f(T)$. This will follow from Corollary 4.6.

In Theorem 4.11 we will show that $\mathsf{FRg}(T)$ is indeed the free regular category on T. The following is straightforward.

Lemma 4.4. *Suppose $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ have I-shaped limits, for some small category I, and suppose that $f\colon \mathcal{C} \to \mathcal{E}$ and $g\colon \mathcal{D} \to \mathcal{E}$ preserve I-shaped limits. Then the comma category $\mathcal{B} := (\mathcal{C} \downarrow \mathcal{D})$ has I-shaped limits, and they are preserved and reflected by the projection $(\pi_1, \pi_2)\colon \mathcal{B} \to \mathcal{C} \times \mathcal{D}$.*

Proposition 4.5. *Let $\mathcal{R} \to \mathcal{T} \leftarrow \mathcal{S}$ be regular functors. Then the comma category $\mathcal{B} := (\mathcal{R} \downarrow \mathcal{S})$ is regular, and the projection $\mathcal{B} \to \mathcal{R} \times \mathcal{S}$ preserves and reflects finite limits and regular epimorphisms. In particular, $\mathsf{FRg}(T)$ is regular for any T.*

Proof. It is well-known that $\mathsf{FinSet}^{\mathrm{op}}$ is regular, and the finite powerset $\mathbb{P}_f(T)$ is regular because it has finite meets and, because it is a poset, regular epis are equalities. Hence the second statement follows from the first. Since the opposite of a comma category is the comma category of the opposites of its defining data, Lemma 4.4 shows that $\mathcal{B}$ has finite limits and coequalizers of kernel pairs, and that regular epis are stable under pullback. $\square$

Corollary 4.6. *It will be useful to be have the following explicit computations in $\mathsf{FRg}(T)$.*

terminal: $0 \overset{!}{\to} \varnothing \subseteq T$ *is terminal. We denote it 0.*

product: *The product of $\Gamma = (n, S, \tau)$ and $\Gamma' = (n', S', \tau')$ is $(n + n', S \cup S', [\tau, \tau'])$. We denote it $\Gamma \oplus \Gamma'$.*

pullback: *The pullback of a diagram $(n_1, S_1, \tau_1) \to (n, S, \tau) \leftarrow (n_2, S_2, \tau_2)$ is obtained as a pushout (and union) in FinSet:*

monos: *A map $f\colon (n_1, S_1, \tau_1) \to (n_2, S_2, \tau_2)$ is monic iff the function $\underline{f}\colon \underline{n_2} \to \underline{n_1}$ is surjective.*

regular epis: *A map $f\colon (n_1, S_1, \tau_1) \to (n_2, S_2, \tau_2)$ is regular epic iff both: the corresponding function $\underline{f}\colon \underline{n_2} \to \underline{n_1}$ is injective and $S_2 = S_1$.*

Remark 4.7. As mentioned in Corollary 4.6, we denote the product of Γ_1 and Γ_2 by $\Gamma_1 \oplus \Gamma_2$. This is reminiscent of the notation for products in an abelian category. However, it is not quite analogous: in an abelian category the product $V \oplus W$ is a biproduct—i.e. also a coproduct—and this is not the case in $\mathsf{FRg}(T)$. We use the $\oplus$ notation to remind us that

$$(n, S, \tau) \oplus (n', S', \tau') \cong (n + n', S \cup S', [\tau, \tau']).$$

Remark 4.8. Note that one should think of the support $S = \mathrm{Supp}(\Gamma)$ of a context Γ as a kind of constraint, because the larger S is, the smaller Γ is. Indeed, for any $n \in \mathbb{N}$ and context $\tau \colon \underline{n} \to S$, if one composes with an inclusion $S \subseteq S' \subseteq T$ on the level of support, the result is a monic map in $\mathsf{FRg}(T)$ *going the other way*,

$$(\underline{n} \xrightarrow{\tau} S \subseteq S') \rightarrowtail (\underline{n} \xrightarrow{\tau} S).$$

Recall from Definition 2.3 that the support of an object in a regular category is the image of its unique map to the terminal object.

Corollary 4.9. *Every unary support context is the support of a unary context.*

Proof. Given any unary support context $\mathrm{Supp}(s)$, the explicit descriptions in Corollary 4.6 make it easy to check that $\langle s \rangle \twoheadrightarrow \mathrm{Supp}(s) \rightarrowtail 0$ is the image factorization of the unique map $\langle s \rangle \to 0$. $\square$

Corollary 4.10. *Every object $\Gamma = (n, S, \tau) \in \mathsf{FRg}(T)$ can be written as the product of n-many unary contexts and $|S|$-many unary support contexts, and morphisms in $\mathsf{FRg}(T)$ correspond to projections and diagonals.*

Proof. It follows directly from Corollary 4.6 that $\Gamma = \prod_{i \in \underline{n}} \langle \tau(i) \rangle \times \prod_{s \in S} \mathrm{Supp}(s)$. In particular, it will be useful to note the idempotence of support contexts:

$$\mathrm{Supp}(s) \times \mathrm{Supp}(s) = \mathrm{Supp}(s). \tag{14}$$

If $f \colon \Gamma \to \Gamma'$ is a morphism as in Eq. (13), then the corresponding map

$$\prod_{i \in \underline{n}} \langle \tau(i) \rangle \times \prod_{s \in S} \mathrm{Supp}(s)$$
$$\downarrow f$$
$$\prod_{i' \in \underline{n}'} \langle \tau'(i') \rangle \times \prod_{s' \in S'} \mathrm{Supp}(s')$$

acts coordinatewise according to $\underline{f} \colon \underline{n}' \to \underline{n}$ and $S' \subseteq S$. $\square$

The following theorem establishes the adjunction from Eq. (2).

Theorem 4.11. *The category $\mathsf{FRg}(T)$ is the free regular category on T, i.e. there is an adjunction*

$$\mathsf{Set} \xrightarrow[\mathrm{Ob}]{\overset{\mathsf{FRg}(-)}{\underset{\Rightarrow}{\rightleftarrows}}} \mathsf{RgCat}.$$

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Proof. We denote the unit component for a set T by $\langle - \rangle \colon T \to \mathrm{Ob}\,\mathsf{FRg}(T)$; it is given by unary contexts, $\langle t \rangle = (1, \{t\}, !)$. We denote the counit component $\ulcorner - \urcorner \colon \mathsf{FRg}(\mathrm{Ob}\,\mathcal{R}) \to \mathcal{R}$ for a regular category $\mathcal{R}$; it is roughly-speaking given by products and supports in $\mathcal{R}$ (see Definition 2.3). More precisely, given a context $\Gamma = (n, S, \tau) \in \mathsf{FRg}(\mathrm{Ob}\,\mathcal{R})$, we put

$$\ulcorner \Gamma \urcorner := \prod_{i \in \underline{n}} \tau(i) \times \prod_{s \in S} \mathrm{Supp}(s),$$

By the universal property of products, a morphism $f \colon \Gamma \to \Gamma'$, i.e. a function $\underline{f} \colon \underline{n}' \to \underline{n}$ as in Eq. (13) naturally induces a map $\ulcorner f \urcorner \colon \ulcorner \Gamma \urcorner \to \ulcorner \Gamma' \urcorner$, so $\ulcorner - \urcorner$ is a functor. We need to check that it is regular and for this we use Corollary 4.6.

For preservation of finite limits, first observe that $\ulcorner - \urcorner$ preserves the terminal object because the empty product in $\mathcal{R}$ is terminal. For pullbacks we need to check that for every pushout diagram as to the left, the diagram to the right is a pullback:

$$
\begin{array}{ccc}
\underline{n} & \longrightarrow & \underline{n}_2 \\
\downarrow & \quad \ulcorner \quad \downarrow & \\
\underline{n}_1 & \longrightarrow & \underline{n}' \xrightarrow{\tau'} T
\end{array}
\qquad\qquad
\begin{array}{ccc}
\prod_{i \in \underline{n}} \langle \tau'(i) \rangle & \longleftarrow & \prod_{i_2 \in \underline{n}_2} \langle \tau'(i_2) \rangle \\
\uparrow & \quad \ulcorner \quad \uparrow & \\
\prod_{i_1 \in \underline{n}_1} \langle \tau'(i_1) \rangle & \longleftarrow & \prod_{i' \in \underline{n}'} \langle \tau'(i') \rangle
\end{array}
$$

where $\underline{n}' \cong \underline{n}_1 \sqcup_{\underline{n}} \underline{n}_2$ and $\tau' \colon \underline{n}' \to T$ is the induced map. This follows from the well-known fact that FinSet is the free finite-colimit completion of a singleton [MM92], and fact that the slice category $\mathsf{FinSet}_{/T}$ is the free finite-colimit completion of the set T.

Finally, suppose $f \colon (n_1, S_1, \tau_1) \to (n_2, S_2, \tau_2)$ is a regular epi in $\mathsf{FRg}(T)$; i.e. the corresponding function $\underline{f} \colon \underline{n}_2 \to \underline{n}_1$ is monic and $S_1 = S_2$. Letting $n' := n_1 - n_2$, we use Corollary 4.10 and Eq. (14) to write $\ulcorner f \urcorner$ as follows:

$$\prod_{i \in \underline{n}'} \langle \tau_1(i) \rangle \times \prod_{i \in \underline{n}_2} \langle \tau_2(i) \rangle \times \prod_{s \in S} \mathrm{Supp}(s)$$

$$\downarrow \ulcorner f \urcorner$$

$$\prod_{i \in \underline{n}'} \mathrm{Supp}(\tau_1(i)) \times \prod_{i \in \underline{n}_2} \langle \tau_2(i) \rangle \times \prod_{s \in S} \mathrm{Supp}(s).$$

Since for each $i \in \underline{n}'$ the map $\tau_1(i) \to \mathrm{Supp}(\tau_1(i))$ is a regular epi and regular epis are closed under finite products in a regular category, this shows that $\ulcorner f \urcorner$ is again a regular epi. Hence $\ulcorner - \urcorner$ is a regular functor. The triangle identities are straightforward: the first is that for any $r \in \mathrm{Ob}\,\mathcal{R}$, the product of a unary context $\langle r \rangle$ is just r. The second follows from Corollary 4.10. $\qquad\square$

Given a function $f \colon T \to T'$, we can use Theorem 4.11 and the idempotence of support contexts to see that the induced regular functor $\mathsf{FRg}(f) \colon \mathsf{FRg}(T) \to \mathsf{FRg}(T')$ sends $\Gamma = (\underline{n} \xrightarrow{\tau} S \subseteq T)$ to the composite

$$\mathsf{FRg}(f)(\Gamma) = (\underline{n} \xrightarrow{\tau} S \xrightarrow{f|_S} f(S) \subseteq T'),$$

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where $S \twoheadrightarrow f(S) \rightarrowtail T'$ is the image factorization of f restricted to S.

Remark 4.12. The free finite limit category on a single generator is $\mathsf{FinSet}^{\mathrm{op}}$, and there the unique map $\underline{n} \to \varnothing$ is a regular epimorphism for every object $\underline{n}$. Consequently, $\mathsf{FinSet}^{\mathrm{op}}$ has another universal property: it is the free regular category *in which every object is inhabited*. Of course the same holds for any set T: the free finite limit category is also the free "fully inhabited" regular category. It is equivalent to the result of inverting the map $(\varnothing, S, !) \to (\varnothing, \varnothing, !)$ in $\mathsf{FRg}(T)$ for every $S \in \mathbb{P}_f(T)$.

Because $(\mathsf{FinSet}_{/T})^{\mathrm{op}}$ is very similar to—but far more familiar than—$\mathsf{FRg}(T)$, it can be useful for intuition to replace $\mathsf{FRg}(T)$ with $\mathsf{FinSet}^{\mathrm{op}}$ throughout this story; the only cost is the assumption of inhabitedness, which is a common assumption in classical logic.

For any regular category $\mathcal{R}$, the counit map of the adjunction in Theorem 4.13 gives a regular functor that we have been denoting

$$\ulcorner - \urcorner \colon \mathsf{FRg}(\mathrm{Ob}\,\mathcal{R}) \longrightarrow \mathcal{R}. \tag{15}$$

It sends a context $\Gamma = (n, S, \tau)$ to the product

$$\ulcorner \Gamma \urcorner := \prod_{i \in \underline{n}} \langle\, \tau(i) \,\rangle \times \prod_{s \in S} \mathrm{Supp}(s). \tag{16}$$

The free regular po-category on a set

Since $\mathsf{FRg}(T)$ is a regular category, we may construct its po-category of relations

$$\mathbb{FRg}(T) := \mathbb{Rel}_{\mathsf{FRg}(T)}.$$

It should be no surprise that these are the free regular po-categories. Free regular po-categories will form the foundation of our graphical calculus for regular logic; we give an explicit description in Section 5.

Theorem 4.13. *The po-category* $\mathbb{FRg}(T) := \mathbb{Rel}_{\mathsf{FRg}(T)}$ *is the free regular po-category on the set* T. *That is, there is an adjunction*

$$\mathsf{Set} \;\; \substack{\xrightarrow{\;\mathbb{FRg}(-)\;} \\ \Rightarrow \\ \xleftarrow[\;\mathrm{Ob}\;]{}} \;\; \mathsf{RgPocat}.$$

Proof. This is immediate from Theorem 4.11, which says that $\mathsf{FRg}(T)$ is the free regular category on T, and the fact that the category of regular po-categories is the essential image of RgCat under the po-category of relations construction, Definition 2.11. $\qquad \square$

For any regular po-category $\mathcal{R}$, the counit map of the adjunction in Theorem 4.13 gives a morphism of regular po-categories that we again denote

$$\ulcorner - \urcorner \colon \mathbb{FRg}(\mathrm{Ob}\,\mathcal{R}) \longrightarrow \mathcal{R}. \tag{17}$$

It is a strong monoidal functor, basically because the functor in Eq. (15) in particular preserves finite products.

4.2 Regular calculi

In this section we introduce the notion of a regular calculus. This is a new category-theoretic way to look at the kinds of logical moves—and the relationships between them—found in regular logic.

Definition of regular calculi

The following was given as Definition 1.1, but we repeat it here for convenience. Recall from Theorem 4.13 that $\mathbb{FRg}(\mathrm{T}) := \mathbb{Rel}_{\mathsf{FRg}(\mathrm{T})}$ is the free regular po-category on T.

Definition 4.14. A *regular calculus* is a pair (T, P) where T is a set and $P \colon \mathbb{FRg}(\mathrm{T}) \to \mathbb{Poset}$ is an ajax po-functor. For any object $\Gamma \in \mathbb{FRg}(\mathrm{T})$, we denote the order in the poset $P(\Gamma)$ using the $\vdash_\Gamma$ or $\vdash$ symbol (rather than $\leq$).

A *morphism* $(\mathrm{T}, P) \to (\mathrm{T}', P')$ of regular calculi is a pair $(F, F^\sharp)$ where $F \colon \mathrm{T} \to \mathrm{T}'$ is a function and $F^\sharp$ is a monoidal natural transformation

$$
\begin{array}{ccc}
\mathrm{T} & \mathbb{FRg}(\mathrm{T}) \xrightarrow{\ P\ } & \\
F\downarrow & \mathbb{FRg}(F)\downarrow \quad F^\sharp\Downarrow & \mathbb{Poset} \\
\mathrm{T}' & \mathbb{FRg}(\mathrm{T}') \xrightarrow[P']{} &
\end{array}
$$

that is strict in every respect: all the required coherence diagrams of posets commute on the nose. We denote the category of regular calculi by RgCalc.

Adjoint notation ($f_!$ and f^*) in regular calculi

It will be convenient to define notation mimicking that in Eq. (5) for P's action on adjoints in $\mathbb{Rel}_\mathcal{R}$. Given an ajax po-functor $P \colon \mathbb{Rel}_\mathcal{R} \to \mathbb{Poset}$, we can take adjoints and use the fundamental lemma (Lemma 2.8) to obtain the diagram below:

$$
\begin{array}{ccc}
\mathcal{R} \xrightarrow[\cong]{} \mathsf{LAdj}(\mathbb{Rel}_\mathcal{R}) \xrightarrow[\mathsf{LAdj}(P)]{f \mapsto f_!} & \mathsf{LAdj}(\mathbb{Poset}) \\
\| \qquad\qquad \downarrow\cong & \qquad \downarrow\cong \\
\mathcal{R} \xrightarrow[\cong]{} \mathsf{RAdj}(\mathbb{Rel}_\mathcal{R})^{\mathrm{op}} \xrightarrow[\mathsf{RAdj}(P)^{\mathrm{op}}]{} \mathsf{RAdj}(\mathbb{Poset})^{\mathrm{op}} \\
f \mapsto f^*
\end{array}
$$

That is, for any $f \colon r \to r'$ in $\mathcal{R}$ we have an adjunction between posets:

$$
P(r) \xrightarrow[f^*]{\underset{\Rightarrow}{f_!}} P(r').
$$

In particular, since $\mathsf{FRg}(\mathrm{T})$ has finite products (denoted using 0 and $\oplus$), we will speak of projection maps $\pi_i \colon (\Gamma_1 \oplus \Gamma_2) \to \Gamma_i$, for $i = 1, 2$, diagonal maps $\delta_r \colon r \to r \oplus r$, and the

unique map $\epsilon_r\colon r \to 0$. Each determines an adjunction as above, e.g.

$$P(r_1 \times r_2) \underset{(\pi_i)^*}{\overset{(\pi_i)_!}{\underset{\Leftarrow}{\Rightarrow}}} P(r_i), \qquad P(r) \underset{(\delta_r)^*}{\overset{(\delta_r)_!}{\underset{\Leftarrow}{\Rightarrow}}} P(r \times r), \qquad P(r) \underset{(\epsilon_r)^*}{\overset{(\epsilon_r)_!}{\underset{\Leftarrow}{\Rightarrow}}} P(0).$$

Regular calculi send objects to meet-semilattices

If $P\colon \mathbb{FRg}(\mathrm{T}) \to \mathbb{P}\mathsf{oset}$ is a regular calculus, i.e. an ajax po-functor, then by Corollary 3.17 the poset $P(\Gamma)$ is a meet-semilattice for each object $\Gamma \in \mathscr{R}$. Its top element and meet are given by the composites of right adjoints shown here:

$$1 \underset{!}{\overset{\rho}{\underset{\Leftarrow}{\Rightarrow}}} P(0) \underset{\epsilon_{\Gamma!}}{\overset{\epsilon_\Gamma^*}{\underset{\Leftarrow}{\Rightarrow}}} P(\Gamma) \quad \text{and} \quad P(\Gamma) \times P(\Gamma) \underset{\lambda_{\Gamma,\Gamma}}{\overset{\rho_{\Gamma,\Gamma}}{\underset{\Leftarrow}{\Rightarrow}}} P(\Gamma \oplus \Gamma) \underset{\delta_{\Gamma!}}{\overset{\delta_\Gamma^*}{\underset{\Leftarrow}{\Rightarrow}}} P(\Gamma). \quad (18)$$

4.3 The predicates functor $\mathbf{prd}\colon \mathsf{RgCat} \to \mathsf{RgCalc}$

Let $\mathcal{R}$ be a regular category and let $\mathscr{R} \coloneqq \mathbb{Rel}_\mathcal{R}$ denote its relations po-category; note that $\mathrm{Ob}\,\mathcal{R} = \mathrm{Ob}\,\mathscr{R}$. We have a counit map $\ulcorner-\urcorner\colon \mathbb{FRg}(\mathrm{Ob}\,\mathscr{R}) \to \mathscr{R}$ from Eq. (17), and it is a strong monoidal functor. We can compose it with the "subobjects" functor $\mathsf{Sub}_\mathcal{R} \coloneqq \mathscr{R}(I,-)\colon \mathscr{R} \to \mathbb{P}\mathsf{oset}$. The result is a po-functor

$$\mathsf{Sub}_\mathcal{R}\ulcorner-\urcorner\colon \mathbb{FRg}(\mathrm{Ob}\,\mathscr{R}) \to \mathbb{P}\mathsf{oset} \quad (19)$$

which assigns to each context Γ the poset of *predicates* in Γ. By Lemma 3.3 and Theorem 3.16, the po-functor $\mathsf{Sub}_\mathcal{R}\ulcorner-\urcorner$, is ajax, so $(\mathrm{Ob}\,\mathcal{R}, \mathsf{Sub}_\mathcal{R}\ulcorner-\urcorner)$ is a regular calculus.

Proposition 4.15. *The mapping from Eq. (19) extends to a faithful functor*

$$\mathbf{prd}\colon \mathsf{RgCat} \to \mathsf{RgCalc}.$$

Proof. Given an object $\mathcal{R}$ of RgCat—that is, given a regular category—we define $\mathbf{prd}(\mathcal{R}) \coloneqq (\mathrm{Ob}\,\mathcal{R}, \mathsf{Sub}_\mathcal{R}\ulcorner-\urcorner)$. As mentioned above, $\mathsf{Sub}_\mathcal{R}\ulcorner-\urcorner$ is ajax, so $\mathbf{prd}(\mathcal{R})$ is a regular calculus. We need to say how $\mathbf{prd}$ behaves on morphisms.

A regular functor $\mathcal{F}\colon \mathcal{R} \to \mathcal{R}'$ induces a function $\mathrm{Ob}\,\mathcal{F}\colon \mathrm{Ob}\,\mathcal{R} \to \mathrm{Ob}\,\mathcal{R}'$ and hence a morphism $\overline{\mathcal{F}} \coloneqq \mathbb{FRg}(\mathrm{Ob}\,\mathcal{F})\colon \mathbb{FRg}(\mathrm{Ob}\,\mathcal{R}) \to \mathbb{FRg}(\mathrm{Ob}\,\mathcal{R}')$. We need to construct a (strict) monoidal natural transformation $\mathcal{F}^\sharp\colon \mathsf{Sub}_\mathcal{R}\ulcorner-\urcorner \longrightarrow (\overline{\mathcal{F}} \,\mathring{,}\, \mathsf{Sub}_{\mathcal{R}'}\ulcorner-\urcorner)$.

Let $\Gamma \in \mathbb{FRg}(\mathrm{Ob}\,\mathcal{R})$ be a context. The left-hand square in the following diagram commutes by the naturality of the counit $\ulcorner-\urcorner$, and we have a map $\mathbb{Rel}_\mathcal{F}(I,-)\colon \mathbb{Rel}_\mathcal{R}(I,-) \longrightarrow \mathbb{Rel}_{\mathcal{R}'}(I,-)$ because $\mathcal{F}(I) = I$. We define $\mathcal{F}^\sharp$ to be the composite 2-cell, which we denote $\mathsf{Sub}_\mathcal{F}\ulcorner-\urcorner$:

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Thus we define **prd** on morphisms by $\mathbf{prd}(\mathcal{F}) = (\mathrm{Ob}\,\mathcal{F}, \mathsf{Sub}_{\mathcal{F}}\ulcorner - \urcorner)$; it is easy to check that **prd** preserves identities and compositions. It remains to check that it is faithful, so let $\mathcal{F}, \mathcal{G}\colon \mathcal{R} \to \mathcal{R}'$ be regular functors and suppose $\mathbf{prd}(\mathcal{F}) = \mathbf{prd}(\mathcal{G})$. There is agreement on objects $\mathrm{Ob}\,\mathcal{F} = \mathrm{Ob}\,\mathcal{G}$, so let $f\colon r_1 \to r_2$ be a morphism in $\mathcal{R}$ and consider the its graph $\hat{f} := \langle \mathrm{id}_r, f \rangle \subseteq r_1 \times r_2$. Write $(r_1, r_2) := (2, \{r_1, r_2\}, \cong) \in \mathbb{F}\mathrm{Rg}(\mathrm{T})$. From the fact that $\mathsf{Sub}_{\mathcal{F}}\ulcorner r_1, r_2 \urcorner(\hat{f}) = \mathsf{Sub}_{\mathcal{G}}\ulcorner r_1, r_2 \urcorner(\hat{f})$ it follows that $\mathcal{F}(f) = \mathcal{G}(f)$, completing the proof. $\qquad\square$

In Corollary 8.4, we will show that in fact **prd** is also full.

The goal for the rest of this paper is to construct a left adjoint to **prd** and prove the essential reflection. Our proof will rely on some properties of regular calculi, in particular that they can be incarnated as a sort of *graphical calculus* for regular logic reasoning.

5 Graphical regular logic

A key advantage of the regular calculus perspective on regular categories and regular logic is that it suggests a graphical notation for relations in regular categories, as well as how they behave under base-change and co-base-change. This is the promised graphical regular logic.

In this section we develop this graphical formalism, first by giving a graphical description of the free regular po-category on a set, and then by defining the notion of graphical term, showing how these represent elements of posets, and explaining how to reason with them. In subsequent sections, we'll use this graphical regular logic to prove the main theorem.

5.1 Depicting free regular po-categories $\mathbb{F}\mathrm{Rg}(\mathrm{T})$

Since the po-categories $\mathbb{F}\mathrm{Rg}(\mathrm{T})$ form the foundation of our diagrammatic language for regular logic, we begin our exploration of graphical regular logic by giving an explicit description of the objects, morphisms, 2-cells, and composition in $\mathbb{F}\mathrm{Rg}(\mathrm{T})$ in terms of wiring diagrams.

Notation 5.1. By definition, an object of $\mathbb{F}\mathrm{Rg}(\mathrm{T})$ is simply a context $\Gamma = (\underline{n} \xrightarrow{\tau} S \subseteq \mathrm{T})$ of $\mathrm{FRg}(\mathrm{T})$. We represent a context graphically by a circle with n ports around the exterior, with ith port annotated by the value $\tau(i)$, and with a white dot at the base annotated by the remaining elements of the support $S \setminus \mathrm{im}\,\tau$.[1]

$$\tag{20}$$

Our convention will be for the ports to be numbered clockwise from the left of the circle, unless otherwise indicated, and to omit the white dot if $S = \mathrm{im}\,\tau$. We refer to such an annotated circle as a *shell*.

---

[1] By the idempotence of support contexts Eq. (14), one may equivalently include the whole support, S.

As a syntactic shorthand for the shell in (20), we may combine all the ports and the white dot into a single wire labeled with the context $\Gamma \in \mathbb{F}\mathsf{Rg}(\mathrm{T})$: $\quad \Gamma$ ⊸○.

Example 5.2. Let $\Gamma = (n, S, \tau)$ be the context with arity $n = 3$, support $S = \{w, x, y, z\} \subseteq \mathrm{T}$, and typing $\tau \colon \underline{3} \to S$ given by $\tau(1) = \tau(3) = y$, $\tau(2) = z$. It can be depicted by the shell

$$
\begin{array}{c}
z \\
y \,\text{⊸}\!\!\bigcirc\!\!\text{⊸}\, y \\
w, x
\end{array}
$$

The hom-posets of $\mathbb{F}\mathsf{Rg}(\mathrm{T}) = \mathbb{R}\mathsf{el}_{\mathsf{FRg}(\mathrm{T})}$ are the subobject posets $\mathbb{F}\mathsf{Rg}(\mathrm{T})(\Gamma, \Gamma') = \mathsf{Sub}(\Gamma \oplus \Gamma')$. Explicitly, a morphism $\omega \colon \Gamma_1 \twoheadrightarrow \Gamma_{\mathrm{out}}$ is a represented by monomorphism

$$
\Gamma_\omega = (n_\omega \xrightarrow{\tau_\omega} S_\omega \subseteq \mathrm{T}) \rightarrowtail \Gamma_1 \oplus \Gamma_{\mathrm{out}}
$$

in $\mathsf{FRg}(\mathrm{T})$, and hence specified by a surjection ω such that

$$
\begin{array}{ccc}
\underline{n}_\omega & \xrightarrow{\ \tau_\omega\ } & S_\omega \\
\omega \uparrow & & \cap \\
\underline{n}_1 + \underline{n}_{\mathrm{out}} & \xrightarrow[\tau_1 + \tau_{\mathrm{out}}]{} & S_1 \cup S_{\mathrm{out}}
\end{array}
$$

commutes. We depict ω using a *wiring diagram*. More generally, wiring diagrams will give graphical representations of morphisms $\omega \colon \Gamma_1 \oplus \cdots \oplus \Gamma_k \twoheadrightarrow \Gamma_{\mathrm{out}}$.

Notation 5.3. Suppose we have a morphism $\omega \colon \Gamma_1 \oplus \cdots \oplus \Gamma_k \twoheadrightarrow \Gamma_{\mathrm{out}}$ in $\mathbb{F}\mathsf{Rg}(\mathrm{T})$. We depict ω as follows.
   1. Draw the shell for Γ_{out}.
   2. Draw the object Γ_i, for $i = 1, \ldots, k$, as non-overlapping shells inside the Γ_{out} shell.
   3. For each $i \in \underline{n}_\omega$, draw a black dot anywhere in the region interior to the Γ_{out} shell but exterior to all the Γ_i shells, and annotate it by the value $\tau_\omega(i)$.
   4. Draw a white dot in the same region, annotated by all elements of S_ω not already present in the diagram.
   5. For each element $(i, j) \in \sum_{i=1,\ldots,k,\mathrm{out}} \underline{n}_i$, draw a wire connecting the jth port on the object Γ_i to the black dot $\omega(i, j)$.
Just as for objects, we may neglect to draw a white dot when $\operatorname{im}\tau = S$.

For a more compact notation, we may also neglect to explicitly draw the object Γ_{out}, leaving it implicit as comprising the wires left dangling on the boundary of the diagram.

Example 5.4. Here is the set-theoretic data of a morphism $\omega \colon \Gamma_1 \oplus \Gamma_2 \oplus \Gamma_3 \twoheadrightarrow \Gamma_{\mathrm{out}}$, together with its wiring diagram depiction:

$\Gamma_1 = (3, \{x, y\}, \tau_1)$ where $\tau_1(1) = x, \tau_1(2) = \tau_1(2) = y$;
$\Gamma_2 = (3, \{w, x, y\}, \tau_3)$ where $\tau_3(1) = \tau_3(2) = \tau_3(3) = x$;
$\Gamma_3 = (4, \{x, y\}, \tau_2)$ where $\tau_2(1) = \tau_2(2) = y, \tau_2(3) = \tau_2(4) = x$;
$\Gamma_{\mathrm{out}} = (6, \{w, x, y, z\}, \tau_{\mathrm{out}})$ where $\tau_{\mathrm{out}}(1) = y$,
$\quad \tau_{\mathrm{out}}(2) = \tau_{\mathrm{out}}(3) = \tau_{\mathrm{out}}(6) = z, \tau_{\mathrm{out}}(4) = \tau_{\mathrm{out}}(5) = x$;
$\Gamma_\omega = (7, \{v, w, x, y, z\}, \tau_r)$ where $\tau_\omega(1) = \tau_\omega(2) = y$,
$\quad \tau_\omega(3) = \tau_\omega(7) = z, \tau_\omega(4) = \tau_\omega(5) = \tau_\omega(6) = x$;

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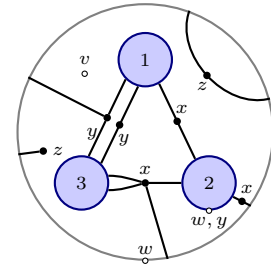
$f(1,1) = 4$, $f(1,2) = 2$, $f(1,3) = 1$, $f(2,1) = 6$, $f(2,2) = 4$,
$\quad f(2,3) = 5$, $f(3,1) = 1$, $f(3,2) = 2$, $f(3,3) = f(3,4) = 6$,
$\quad f(\text{out}, 1) = 1$, $f(\text{out}, 2) = 3$, $f(\text{out}, 3) = 3$, $f(\text{out}, 4) = 5$
$\quad f(\text{out}, 5) = 6$, $f(\text{out}, 6) = 7$.

Example 5.5. Note that we may have $k = 0$, in which case there are no inner shells. For example, the following has $\Gamma_\omega = (2, \{x, y, z, w\}, 1 \mapsto x, 2 \mapsto y)$.
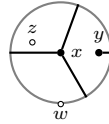
Remark 5.6. When multiple wires meet at a point, our convention will be to draw a dot iff the number of wires is different from two.

When wires intersect and we do not draw a black dot, the intended interpretation is that the wires are *not connected*: $\oplus \neq \oplus$. Of course this is bound to happen when the graph is non-planar.

The following examples give a flavor of how composition, monoidal product, and 2-cells are represented using this graphical notation.

Example 5.7 (Composition as substitution). Composition of morphisms is described by *nesting* of wiring diagrams. Let $\omega' \colon \Gamma' \nrightarrow \Gamma_1$ and $\omega \colon \Gamma_1 \nrightarrow \Gamma_\text{out}$ be morphisms in $\mathbb{F}\text{Rg}(\text{T})$. Then the composite relation $\omega' \, \mathbin{\fatsemi} \, \omega \colon \Gamma' \nrightarrow \Gamma_\text{out}$ is given by

1. drawing the wiring diagram for ω' inside the inner circle of the diagram for ω,
2. erasing the object Γ_1,
3. amalgamating any connected black dots into a single black dot, and
4. removing all components not connected to the objects Γ' or Γ_out, and adding a single white dot annotated by the set containing all elements of T present in these components, but not present elsewhere in the diagram.

Note that step 3 corresponds to taking pullbacks in $\text{FRg}(\text{T})$ (pushouts in FinSet), while step 4 corresponds to epi-mono factorization.

As a shorthand for composition, we simply draw one wiring diagram directly substituted into another, as per step 1. For example, we have
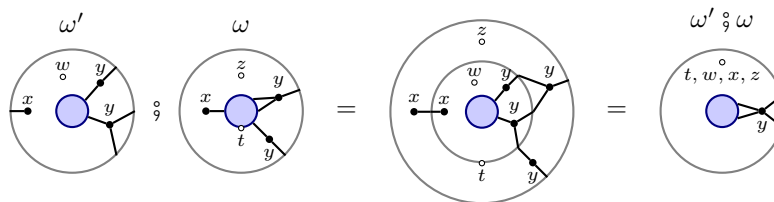
For the more general k-ary or operadic case, we may obtain the composite

$$(\Gamma_1 \oplus \cdots \oplus \Gamma_{i-1} \oplus \omega' \oplus \Gamma_{i+1} \oplus \cdots \oplus \Gamma_k) \, \mathring{,} \, \omega$$

of any two morphisms $\omega' \colon \Gamma_1' \oplus \cdots \oplus \Gamma_k' \twoheadrightarrow \Gamma_i$ and $\omega \colon \Gamma_1 \oplus \cdots \oplus \Gamma_k \twoheadrightarrow \Gamma_{\mathrm{out}}$ by substituting the wiring diagram for ω' into the ith inner circle of the diagram for ω, and following a procedure similar to that in Example 5.7.

Example 5.8 (Monoidal product as juxtaposition). The monoidal product of two morphisms in $\mathbb{FRg}(\mathrm{T})$ is simply their juxtaposition, merging the labels on the floating white dots as appropriate. For example, leaving off labels, we might have:
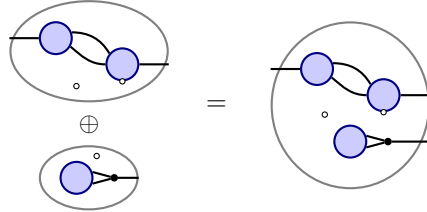
Example 5.9 (2-cells as breaking wires and removing white dots). Let $\omega, \omega' \colon \Gamma_1 \oplus \cdots \oplus \Gamma_k \twoheadrightarrow \Gamma_{\mathrm{out}}$ be morphisms in $\mathbb{FRg}(\mathrm{T}) = \mathbb{R}\mathrm{el}_{\mathsf{FRg}(\mathrm{T})}$. By definition, there exists a 2-cell $\omega \le \omega'$ if there is a monomorphism $m \colon \Gamma_\omega \rightarrowtail \Gamma_{\omega'}$ in $\mathsf{FRg}(\mathrm{T})$ such that $m \, \mathring{,} \, \omega' = \omega$ holds in $\mathsf{FRg}(\mathrm{T})$. By Corollary 4.6, this data consists of a surjection of finite sets $m \colon \underline{n}'_\omega \to \underline{n}_\omega$ and an inclusion $S_{\omega'} \subseteq S_\omega$. In diagrams, the former means 2-cells may break wires, and the latter means they may remove annotations from the inner white dot (or remove it completely). For example, we have 2-cells: ⊖ $\le$ ⊖ and ⊙ $\le$ ○ .

5.2 Graphical terms

Given a regular calculus $P \colon \mathbb{FRg}(\mathrm{T}) \to \mathbb{P}\mathrm{oset}$, we give a graphical representations of its predicates, i.e. the elements in $P(\Gamma)$ for various contexts $\Gamma \in \mathbb{FRg}(\mathrm{T})$. Here's how it works.

Definition 5.10. A *P-graphical term* $(\theta_1, \ldots, \theta_k; \omega)$ in an ajax po-functor $P \colon \mathbb{FRg}(\mathrm{T}) \to \mathbb{P}\mathrm{oset}$ is a morphism $\omega \colon \Gamma_1 \oplus \cdots \oplus \Gamma_k \twoheadrightarrow \Gamma_{\mathrm{out}}$ in $\mathbb{FRg}(\mathrm{T})$ together with, for each $i = 1, \ldots, k$, an element $\theta_i \in P(\Gamma_i)$.

We say that the graphical term $t = (\theta_1, \ldots, \theta_k; \omega)$ *represents* the poset element

$$\llbracket t \rrbracket := (P(\omega) \, \mathring{,} \, \rho)(\theta_1, \ldots, \theta_k) \in P(\Gamma_{\mathrm{out}})$$

where ρ is the k-ary laxator. If t and t' are graphical terms, we write $t \vdash t'$ when $\llbracket t \rrbracket \vdash \llbracket t' \rrbracket$, and $t = t'$ when $\llbracket t \rrbracket = \llbracket t' \rrbracket$.

Notation 5.11. We draw a graphical term $(\theta_1, \ldots, \theta_k; \omega)$ by annotating the ith inner shell with its corresponding poset element θ_i. In the case that $k = 1$ and ω is the identity morphism, we may simply draw the object Γ_1 annotated by θ_1:

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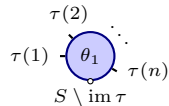
Example 5.12. Recall that we have a diagonal map $\delta\colon \Gamma \to \Gamma \oplus \Gamma$ in $\mathsf{FRg}(\mathrm{T}) \subseteq \mathbb{FRg}(\mathrm{T})$. Given $\theta \in P(\Gamma)$, the element $(\delta)_!(\varphi) \in P(\Gamma \oplus \Gamma)$ is represented by the graphical term
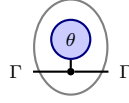
Example 5.13. When $\mathrm{T} = \varnothing$ is empty, $\mathsf{FRg}(\varnothing)$ is the terminal category. By Proposition 3.4, an ajax po-functor $P\colon \mathbb{FRg}(\mathrm{T}) \to \mathbb{P}\mathrm{oset}$ then simply chooses a $\wedge$-semilattice $P(0)$. The po-category $\mathbb{IntRel}_P$ is that $\wedge$-semilattice considered as a one object po-category: it has a unique object whose poset of endomorphisms is $P(0)$. The diagrammatic language has no wires, since there is only the monoidal unit in $\mathsf{FRg}(\varnothing)$. The semantics of an arbitrary graphical term $(\theta_1, \dots, \theta_k; \mathrm{id})$ is simply the meet $\theta_1 \wedge \cdots \wedge \theta_k$.

Remark 5.14. Graphical terms are an alternate syntax for regular logic. While we will not dwell on the translation, a graphical term $(\theta_1, \dots, \theta_k; \omega)$ represents the regular formula

$$\underset{\substack{i \in \underline{n}_j \\ j \in \{1, \dots, k, \omega\}}}{\exists} x_{ij}. \underset{j \in \{1, \dots k\}}{\bigwedge} \theta_k(x_{ij}) \quad \wedge \quad \underset{\substack{i \in \underline{n}_j \\ j \in \{1, \dots, k, \mathrm{out}\}}}{\bigwedge} \left(x_{ij} = x_{\omega(i)j} \right).$$

This formula creates a variable of each element of $\underline{n}_j$, where $j \in \{1, \dots, k, \mathrm{out}, \omega\}$, equates any two variables with the same image under ω, takes the conjunction with all the formulas θ_j, and the existentially quantifies over all variables except those in Γ_{out}. In particular, if we were to take $\omega\colon \Gamma_1 \oplus \Gamma_2 \oplus \Gamma_3 \twoheadrightarrow \Gamma_{\mathrm{out}}$ as in Example 5.4, the resulting graphical term would simplify to the formula

$$\psi(y, z, z', x, x', z'') = \exists \tilde{x}, \tilde{y}. \theta_1(\tilde{x}, \tilde{y}, y) \wedge \theta_2(\tilde{x}, x, x') \wedge \theta_3(y, \tilde{y}, x', x') \wedge (z = z') \wedge (z'' = z'').$$

Remark 5.15. Note that $\mathbb{P}\mathrm{oset}$ is a subcategory of $\mathbb{C}\mathrm{at}$. This allows us to take the monoidal Grothendieck construction $\int P$ of $P\colon \mathbb{FRg}(\mathrm{T}) \to \mathbb{P}\mathrm{oset}$, [MV18]. A P-graphical term is an object in the comma category $\int P {\downarrow} \mathbb{FRg}(\mathrm{T})$. This perspective lends structure to the various operations on diagrams belows; we, however, pursue it no further here.

5.3 Reasoning with graphical terms

The following basic rules for reasoning with diagrams express the (2-)functoriality and monoidality of P.

Proposition 5.16. *Let* $(\theta_1, \dots, \theta_k; \omega)$ *be a graphical term, where* $\theta_i \in P(\Gamma_i)$.
  (i) *(Monotonicity) Suppose* $\theta_i \vdash \theta_i'$ *for some* i. *Then*

$$[\![(\theta_1, \dots, \theta_i, \dots, \theta_k; \omega)]\!] \vdash [\![(\theta_1, \dots, \theta_i', \dots, \theta_k; \omega)]\!].$$

  (ii) *(Breaking) Suppose* $\omega \leq \omega'$ *in* $\mathbb{FRg}(\mathrm{T})$. *Then*

$$[\![(\theta_1, \dots, \theta_k; \omega)]\!] \vdash [\![(\theta_1, \dots, \theta_k; \omega')]\!].$$

(iii) *(Nesting) Suppose $\theta_i = [\![(\theta'_1, \ldots, \theta'_\ell; \omega')]\!]$ for some i. Then*

$$[\![(\theta_1, \ldots, \theta_k; \omega)]\!] = [\![(\theta_1, \ldots, \theta_{i-1}, \theta'_1, \ldots, \theta'_\ell, \theta_{i+1}, \ldots, \theta_k;$$
$$(\Gamma_1 \oplus \cdots \oplus \Gamma_{i-1} \oplus \omega' \oplus \Gamma_{i+1} \oplus \cdots \oplus \Gamma_k) \,\mathbin{\S}\, \omega)]\!].$$

Proof. (i) This is the monotonicity of the map $P(\omega) \,\mathbin{\S}\, \rho$.

(ii) This is the 2-functoriality of P.

(iii) This follows from the monoidality and 1-functoriality of P. In particular, it is the commutativity of the following diagram. Using the braiding we can assume without loss of generality that $i = k$.

The upper triangle commutes by coherence laws for ρ, the square commutes by naturality of ρ, and the right hand triangle commutes by functoriality of P. $\square$

Example 5.17. Proposition 5.16 is perhaps more quickly grasped through a graphical example of these facts in action. Suppose we have the entailment
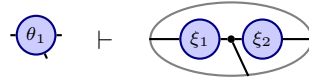
Then using monotonicity, nesting, and then breaking we can deduce the entailment
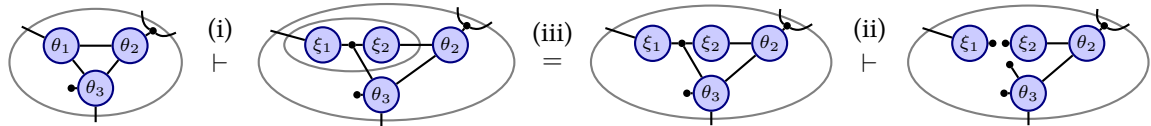
We'll see many further examples of such reasoning in the subsequent sections of this paper, as we prove that we can construct a regular category from a regular calculus.

Example 5.18. The nesting rule in Proposition 5.16 has two particularly important cases. The first occurs when we consider wiring diagrams themselves as poset elements. More precisely, if $f \colon \Gamma_1 \to \Gamma_{\text{out}}$ is a morphism in $\mathsf{FRg}(\mathrm{T})$, and $\hat{f} := \langle \mathrm{id}_{\Gamma_1}, f \rangle$ is its graph, then taking $i = k = 1$, $\ell = 0$, $\theta = [\![(; \hat{f})]\!]$, $\omega = \Gamma_{\text{out}}$ (the identity) and $\omega' = \hat{f}$ in (iii) gives $[\![(\theta; \Gamma_{\text{out}})]\!] = [\![(; \hat{f})]\!]$. Note that this equates a graphical term with inner object Γ_{out} and annotation θ with a term that has no inner object at all; see e.g. Example 5.5.

The second important case is that of 'exterior AND'. If we take $i = k = 1$, $\ell = 2$, and $\omega = \omega' = \Gamma_1 \oplus \Gamma_2$, then $[\![(\theta_1', \theta_2'; \Gamma_1 \oplus \Gamma_2)]\!] = [\![(\rho(\theta_1', \theta_2'); \Gamma_1 \oplus \Gamma_2)]\!]$. In pictures, this means we can take any two circles, say $\theta_1 \in P(\Gamma_1)$ and $\theta_2 \in P(\Gamma_2)$, and merge them, labelling the merged circle with $\rho_{\Gamma_1, \Gamma_2}(\theta_1, \theta_2)$:
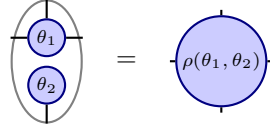
The meet-semilattice structure permits an intuitive graphical interpretation. In the following proposition, the graphical terms on right are illustrative examples of the equalities stated on the left.

Proposition 5.19. *For all contexts Γ in $\mathbb{F}\mathsf{Rg}(\mathrm{T})$ and $\theta, \theta' \in P(\Gamma)$, we have*

(i) *(True is removable)* $[\![(\mathtt{true}_\Gamma; \Gamma)]\!] = [\![(; \epsilon_\Gamma)]\!]$
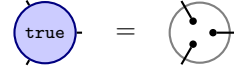
(ii) *(Meets are merges)* $[\![(\theta_1 \wedge \theta_2; \Gamma)]\!] = [\![(\theta_1, \theta_2; \delta_\Gamma)]\!]$.
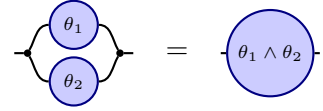
Proof. These equations are simply the definitions of $\mathtt{true}$ and meet; see Eq. (18). $\square$

Example 5.20 (Discarding). Note that Proposition 5.19(i) and the monotonicity of diagrams (Proposition 5.16(i)) further imply that for all $\theta \in P(\Gamma)$ we have $[\![(\theta; \Gamma)]\!] \vdash [\![(; \epsilon_\Gamma)]\!]$:
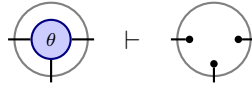
6   Internal relations in a regular calculus

Having set up our proof language, we now return to describing the relationship between regular categories and regular calculi. In this section, we'll see that to every regular calculus we can construct a certain po-category, called its internal relations po-category. Although we shall not prove it directly, this internal relations po-category is in fact a regular po-category. We'll also get to see our graphical logic in action.

6.1   The internal relations po-category

Definition 6.1. Given objects Γ_1, Γ_2 and $\varphi_1 \in P(\Gamma_1)$ and $\varphi_2 \in P(\Gamma_2)$, we define the poset $\mathbb{I}\mathsf{ntRel}_P(\varphi_1, \varphi_2)$ of *P-internal relations from φ_1 to φ_2* to be the subposet

$$\mathbb{I}\mathsf{ntRel}_P(\varphi_1, \varphi_2) := \big\{ \theta \in P(\Gamma_1 \oplus \Gamma_2) \,\big|\, (\pi_1)_! \theta \vdash_{\Gamma_1} \varphi_1 \text{ and } (\pi_2)_! \theta \vdash_{\Gamma_2} \varphi_2 \big\} \subseteq P(\Gamma_1 \oplus \Gamma_2).$$

An internal relation θ may be represented by the graphical term $\Gamma_1\!-\!\theta\!-\!\Gamma_2$ together with the two entailments

$$-\theta-\bullet\;\vdash_{\Gamma_1}\;\varphi_1\qquad\qquad\bullet-\theta-\;\vdash_{\Gamma_2}\;\varphi_2$$

We check that when this definition is applied to the regular calculus $\mathbf{prd}(\mathcal{R})$ associated to a regular category $\mathcal{R}$, it recovers the usual notion of relation between objects in $\mathcal{R}$.

Proposition 6.2. *Let $\mathcal{R}$ be a regular category, let $\Gamma_1, \Gamma_2 \in \mathsf{FRg}(\mathsf{Ob}\,\mathcal{R})$ be contexts, and suppose given $r_1 \in \mathsf{Sub}_\mathcal{R}\ulcorner\Gamma_1\urcorner$ and $r_2 \in \mathsf{Sub}_\mathcal{R}\ulcorner\Gamma_2\urcorner$. There is a natural isomorphism*

$$\mathbb{IntRel}_{\mathbf{prd}(\mathcal{R})}\big((\Gamma_1, r_1), (\Gamma_2, r_2)\big) \cong \mathbb{Rel}_\mathcal{R}(r_1, r_2).$$

Proof. Let $g_1 := \ulcorner\Gamma_1\urcorner$ and $g_2 := \ulcorner\Gamma_2\urcorner$ so we have $r_1 \subseteq g_1$ and $r_2 \subseteq g_2$; see Eq. (16). By Definition 6.1 and Proposition 4.15, a $\mathbf{prd}(\mathcal{R})$-internal relation between them is an element $t \subseteq g_1 \times g_2$ such that there exist dotted arrows making the following diagram commute:

$$
\begin{array}{ccccc}
r_1 & \longleftarrow\!\cdots\!\cdots & t & \cdots\!\cdots\!\longrightarrow & r_2 \\
\downarrow & & \downarrow & & \downarrow \\
g_1 & \longleftarrow & g_1 \times g_2 & \longrightarrow & g_2
\end{array}
$$

The composite $t \to r_1 \times r_2 \to g_1 \times g_2$ is monic, so we have that $t \subseteq r_1 \times r_2$. The result follows. $\qquad\square$

We shall now present some technical lemmas with the goal of proving the following theorem, that internal relations form a po-category. The proof is completed on page 33.

Theorem 6.3. *Let $P\colon \mathbb{FRg}(\mathrm{T}) \to \mathbb{Poset}$ be a regular calculus. Then there exists a po-category $\mathbb{IntRel}_P$ whose objects are pairs (Γ, φ), where Γ is an object of $\mathbb{FRg}(\mathrm{T})$ and $\varphi \in P(\Gamma)$, and with hom-posets $(\Gamma_1, \varphi_1) \to (\Gamma_2, \varphi_2)$ given by $\mathbb{IntRel}_P(\varphi_1, \varphi_2)$.*

We begin by specifying the composition rule. For objects $\Gamma_1, \Gamma_2, \Gamma_3$ in $\mathbb{FRg}(\mathrm{T})$, let

$$\mathsf{comp}_{\Gamma_1,\Gamma_2,\Gamma_3} := {}^{\Gamma_1}\!\!\left(\!\bigcirc\!-\!{}^{\Gamma_2}\!-\!\bigcirc\!\right)\!{}^{\Gamma_3}$$

It is a morphism $(\Gamma_1 \oplus \Gamma_2 \oplus \Gamma_2 \oplus \Gamma_3) \twoheadrightarrow (\Gamma_1 \oplus \Gamma_3)$ in $\mathsf{FRg}(\mathrm{T})$. We then define

$$(-)\,\mathring{,}\,(-)\colon P(\Gamma_1 \oplus \Gamma_2) \times P(\Gamma_2 \oplus \Gamma_3) \xrightarrow{\rho} P(\Gamma_1 \oplus \Gamma_2 \oplus \Gamma_2 \oplus \Gamma_3) \xrightarrow{P(\mathsf{comp})} P(\Gamma_1 \oplus \Gamma_3). \quad (21)$$

Remark 6.4. Note that this construction is reminiscent of the composition map defined in the construction of a hypergraph category from a cospan algebra in [FS18].

Lemma 6.5. *The composite of internal relations is an internal relation. That is, let $\varphi_1 \in P(\Gamma_1)$, $\varphi_2 \in P(\Gamma_2)$, and $\varphi_3 \in P(\Gamma_3)$. Then given $\theta_{12} \in \mathbb{IntRel}_P(\varphi_1, \varphi_2)$ and $\theta_{23} \in \mathbb{IntRel}_P(\varphi_2, \varphi_3)$, the element $(\theta_{12}\,\mathring{,}\,\theta_{23}) \in P(\Gamma_1 \oplus \Gamma_3)$ is in $\mathbb{IntRel}_P(\varphi_1, \varphi_3)$.*

Proof. We must prove $(\pi_1)_!(\theta_{12} \,\fatsemi\, \theta_{23}) \vdash \varphi_1$ and $(\pi_2)_!(\theta_{12} \,\fatsemi\, \theta_{23}) \vdash \varphi_3$. We prove the first; the second follows similarly. This is not hard, we simply use Example 5.20 and then that θ_{12} obeys Definition 6.1:

$$\boxed{\theta_{12}}\!-\!\boxed{\theta_{23}}\!-\!\bullet \quad \vdash_{\Gamma_1} \quad \boxed{\theta_{12}}\!-\!\bullet \quad \vdash_{\Gamma_1} \quad \boxed{\varphi_1} \qquad\qquad \square$$

Given an object $\Gamma \in \mathbb{F}\mathrm{Rg}(\mathrm{T})$ and $\varphi \in P(\Gamma)$, define $\mathrm{id}_\varphi := (\delta_\Gamma)_!(\varphi)$ in $P(\Gamma \oplus \Gamma)$. Here it is graphically.

$$\mathrm{id}_\varphi := \;_\Gamma \overset{\boxed{\varphi}}{\underset{}{\rule{0pt}{0pt}}}\!\!\!\;_\Gamma \tag{22}$$

Lemma 6.6. *For any $\Gamma \in \mathbb{F}\mathrm{Rg}(\mathrm{T})$ and $\varphi \in P(\Gamma)$, the element $\mathrm{id}_\varphi \in P(\Gamma \oplus \Gamma)$ is an element of $\mathbb{I}\mathrm{ntRel}_P(\varphi, \varphi)$.*

Proof. By Proposition 5.16(iii), composing the nested graphical term on the left is precisely the graphical term on the right (and similarly for the codomain):

$$\vdash_\Gamma \quad \boxed{\varphi} \qquad\qquad \square$$

In what follows, we often elide details about—and graphical notation that indicates—nesting and contexts.

Lemma 6.7. *The map $\fatsemi$ from Eq. (21) is unital with respect to id, i.e. $\theta \,\fatsemi\, \mathrm{id} = \theta = \mathrm{id} \,\fatsemi\, \theta$.*

Proof. We prove that $(\theta \,\fatsemi\, \mathrm{id}) = \theta$; the other unitality axiom is similar. The inequality $(\theta \,\fatsemi\, \mathrm{id}) \vdash \theta$ follows from Example 5.20 and Proposition 5.16:

$$\overset{\boxed{\varphi}}{\boxed{\theta}} \quad \vdash \quad \boxed{\theta}\!-\!\bullet \quad = \quad \boxed{\theta}$$

The reverse inequality $\theta \vdash (\theta \,\fatsemi\, \mathrm{id})$ uses Proposition 5.19, Example 5.9, and Definition 6.1:

$$\boxed{\theta} \quad = \quad \overset{\boxed{\theta}}{\underset{\boxed{\theta}}{\rule{0pt}{0pt}}} \quad \vdash \quad \overset{\boxed{\theta}}{\boxed{\theta}} \quad \vdash \quad \overset{\boxed{\varphi}}{\boxed{\theta}} \qquad\qquad \square$$

Lemma 6.8. *The map $\fatsemi$ from Eq. (21) is associative, i.e. $(\theta_1 \,\fatsemi\, \theta_2) \,\fatsemi\, \theta_3 = \theta_1 \,\fatsemi\, (\theta_2 \,\fatsemi\, \theta_3)$.*

Proof. This is immediate from Proposition 5.16(iii). Both sides can be represented by (nested versions of) the graphical term $\boxed{\theta_1}\!-\!\boxed{\theta_2}\!-\!\boxed{\theta_3}$. $\qquad\square$

Proof of Theorem 6.3. Lemmas 6.7 and 6.8 show that we have a 1-category. Each homset $\mathbb{I}\mathrm{ntRel}_P(\varphi_1, \varphi_2) \subseteq P(\Gamma_1, \Gamma_2)$ inherits a partial order from the poset $P(\Gamma_1, \Gamma_2)$. Moreover, composition is given by the monotone map

$$\mathbb{I}\mathrm{ntRel}_P(\varphi_1, \varphi_2) \times \mathbb{I}\mathrm{ntRel}_P(\varphi_2, \varphi_3) \xrightarrow{\;\rho\;} \mathbb{I}\mathrm{ntRel}_P(\varphi_1, \varphi_2, \varphi_2, \varphi_3) \xrightarrow{\;P(\mathrm{comp})\;} \mathbb{I}\mathrm{ntRel}_P(\varphi_1, \varphi_3).$$

We thus have a po-category. $\qquad\qquad\square$

Remark 6.9. Note that although each homset is a $\wedge$-semilattice, composition does *not* preserve meets, and so $\mathbb{I}\mathrm{nt}\mathrm{Rel}_P$ is not $\wedge$-semilattice enriched; see Remark 3.18.

To conclude this section, we mention a useful characterization of internal relations.

Proposition 6.10. *Let $\theta \in P(\Gamma_1 \oplus \Gamma_2)$, $\varphi_i \in P(\Gamma_i)$. Then θ is a relation $\varphi_1 \to \varphi_2$ if and only if*

$$\begin{array}{cc} \text{（圖）} & = & \text{（圖）} \end{array} \tag{23}$$

Proof. Any internal relation $\varphi_1 \to \varphi_2$ obeys the identity Eq. (23) by unitality, Lemma 6.7. Conversely, if θ obeys Eq. (23), then by Example 5.20

$$\text{（圖）} = \text{（圖）} \vdash \text{（圖）}$$

and similarly for φ_2, proving that $\theta \in \mathbb{I}\mathrm{nt}\mathrm{Rel}_P(\varphi_1, \varphi_2)$. $\qquad \square$

Definition 6.11. Write $\sigma_{\Gamma_1, \Gamma_2} \colon \Gamma_1 \oplus \Gamma_2 \longrightarrow \Gamma_2 \oplus \Gamma_1$ for the braiding in $\mathsf{FRg}(T)$, and define the map $(-)^\dagger := \sigma_{\Gamma_1, \Gamma_2!} \colon P(\Gamma_1 \oplus \Gamma_2) \to P(\Gamma_2 \oplus \Gamma_1)$. We say that the *transpose* of a graphical term $(\theta; \Gamma_1 \oplus \Gamma_2)$ is the graphical term $(\theta^\dagger; \Gamma_2 \oplus \Gamma_1)$.

Remark 6.12. Note that transposes are given by "rotating the shell":

$$\Gamma_2 \ \text{（圖）}\ \Gamma_1 \quad = \quad \text{（圖）}$$

In particular, for $\varphi \in P(\Gamma)$, we have $[\![(\varphi^\dagger; \Gamma)]\!] = [\![(\varphi; \Gamma)]\!]$. That is, both φ and $\varphi^\dagger$ can be represented by the diagram （圖）.

We briefly note the following connection to hypergraph categories.

Proposition 6.13. *The monoidal category underlying $\mathbb{I}\mathrm{nt}\mathrm{Rel}_P$ is a hypergraph category.*

More precisely, recall that we write ρ for the laxators of P. We may equip $\mathbb{I}\mathrm{nt}\mathrm{Rel}_P$ with the symmetric strict monoidal product given on objects by $(\Gamma_1, \varphi_1) \otimes (\Gamma_2, \varphi_2) = (\Gamma_1 \oplus \Gamma_2, \rho(\varphi_1, \varphi_2))$, and on morphisms by the restriction to $\mathbb{I}\mathrm{nt}\mathrm{Rel}_P(\varphi_1, \varphi_2) \times \mathbb{I}\mathrm{nt}\mathrm{Rel}_P(\varphi_3, \varphi_4)$ of the map

$$\rho \colon P(\Gamma_1 \oplus \Gamma_2) \times P(\Gamma_3 \oplus \Gamma_4) \to P(\Gamma_1 \oplus \Gamma_3 \oplus \Gamma_2 \oplus \Gamma_4).$$

The braiding $\sigma_{\varphi_1, \varphi_2}$ on objects (Γ_1, φ_1), (Γ_2, φ_2) is given as below. Given this monoidal structure, we may the equip $\mathbb{I}\mathrm{nt}\mathrm{Rel}_P$ with the hypergraph structure given on each object (Γ, φ) by the internal relations below.

$$\sigma_{\varphi_1, \varphi_2} \qquad \mu_\varphi \qquad \eta_\varphi \qquad \delta_\varphi \qquad \epsilon_\varphi$$
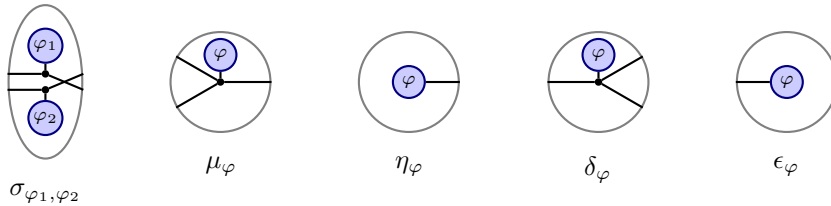
Proof. Recall that we have already shown, in Theorem 6.3, that $\mathbb{IntRel}_P$ is a po-category. To prove this theorem then, it remains to check that the proposed monoidal products and structure maps are always well-defined internal relations, and then that the coherence laws for symmetric monoidal categories and hypergraph categories hold. These facts are all straightforward to verify using the logic of graphical terms. $\qquad\square$

6.2 The Carboni-Walters theorem

In [CW87], Carboni and Walters defined the notions of *cartesian bicategory* and *functionally complete bicategory of relations*. The first of these falls out of our work so far.

Definition 6.14 (Carboni-Walters). A *cartesian bicategory* is a po-category $\mathscr{C}$ with a unique adjoint monoid structure on each object c, such that each map $\alpha \colon c \to c'$ induces a lax comonoid homomorphism,

$$\alpha \mathbin{\fatsemi} \epsilon_{c'} \leq \epsilon_c \qquad \text{and} \qquad \alpha \mathbin{\fatsemi} \delta_{c'} \leq \delta_c \mathbin{\fatsemi} (\alpha \otimes \alpha).$$

Now for any po-category $\mathscr{C}$, there is a po-functor $U \colon \mathbb{AdjMon}(\mathscr{C}) \to \mathscr{C}$ sending an ajax functor $1 \to \mathscr{C}$ to the image of 1.

Theorem 6.15. *A po-category $\mathscr{C}$ is a cartesian bicategory iff $U \colon \mathbb{AdjMon}(\mathscr{C}) \to \mathscr{C}$ is an isomorphism of po-categories.*

Proof. This follows from Eqs. (9) and (11). $\qquad\square$

Our goal is to convert any regular calculus $P \colon \mathbb{FRg}(T) \to \mathbb{Poset}$ into a regular category $\mathbf{syn}(P)$. One approach is to show this directly; we do so in Sections 7 and 8. Another approach would be to use the Carboni-Walters theorem. While seemingly more direct, the latter approach has two drawbacks. First, it would make our paper less self-contained. Second, [CW87] seem not to describe functors between "functionally complete bicategories of relations" precisely enough for our needs. Thus we recall their theorem here and proceed to the direct approach, where we really see the graphical calculus in action. We will not see cartesian bicatgories again in this paper.

Theorem 6.16 (Carboni-Walters). *Let $\mathscr{C}$ be a cartesian bicategory. It is equivalent to $\mathbb{Rel}_{\mathcal{R}}$ for some regular category $\mathcal{R}$ if and only if*
- *(Frobenius) $\mu_c \mathbin{\fatsemi} \delta_c = (c \otimes \delta_c) \mathbin{\fatsemi} (\mu_c \otimes c)$, and*
- *(Images) For every $f \colon b \to I$ there exists an object $\mathrm{im}(f)$ and a left adjoint $i \colon \mathrm{im}(f) \to b$ such that:*
$$i \mathbin{\fatsemi} i^{\dagger} = \mathrm{id}_{\mathrm{im}(b)} \qquad \text{and} \qquad i^{\dagger} \mathbin{\fatsemi} \epsilon_{\mathrm{im}(b)} = f.$$

Proof. This is [CW87, Theorem 3.5]. $\qquad\square$

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7 Internal functions and the syntactic category construction

Internal functions are defined to be the left adjoints in the po-category $\mathbb{IntRel}_P$ of internal relations (see Theorem 6.3).

Definition 7.1. Given a regular calculus (T, P), where $P\colon \mathbb{FRg}(T) \to \mathbb{Poset}$, we define the category $\mathcal{R}_P$ of P-internal functions to be the category of left adjoints in $\mathbb{IntRel}_P$:

$$\mathcal{R}_P := \mathsf{LAdj}(\mathbb{IntRel}_P). \tag{24}$$

In more detail, suppose given elements $\varphi_1 \in P(\Gamma_1)$ and $\varphi_2 \in P(\Gamma_2)$. We say that an internal relation $\theta \in \mathbb{IntRel}_P(\varphi_1, \varphi_2) \subseteq P(\Gamma_1 \oplus \Gamma_2)$ is an *internal function* if there exists an internal relation ξ such that

$$\boxed{\varphi_1} \quad \vdash \quad \boxed{\theta}\!-\!\boxed{\xi} \qquad \text{and} \qquad \boxed{\xi}\!-\!\boxed{\theta} \quad \vdash \quad \boxed{\varphi_2}.$$

The category $\mathcal{R}_P$ has the same objects (Γ, φ) as $\mathbb{IntRel}_P$, and morphisms given by internal functions.

Notation 7.2. Graphically, we'll sometimes denote an internal function $\theta \in P(\Gamma_1, \Gamma_2)$ by the shape $\Gamma_1 \, \boxed{\theta}\!\succ \, \Gamma_2$.

Our aim in this section is to prove that the internal functions form a regular category.

Theorem 7.3. *For any regular calculus $P\colon \mathbb{FRg}(T) \to \mathbb{Poset}$, the category $\mathcal{R}_P$ of internal functions in $\mathbb{IntRel}_P$ is regular.*

The proof, found on page 42 is divided into three parts. In Section 7.1 we'll explore properties of internal functions, in Section 7.2 we'll show $\mathcal{R}_P$ has finite limits, and in Section 7.3 we'll show it has pullback stable image factorizations and conclude with the theorem.

7.1 Properties and examples of internal functions

Before we embark on the theorem, let's get to know the category of internal functions a bit. We'll first characterize functions in two ways: they're the relations that have their own transposes as right adjoints, and they're the relations that are total and deterministic. We'll then note that the order inherited by functions as a subposet of the poset of relations is just the discrete order, and give two important examples of functions: bijections and projections.

To obtain our characterizations of functions, we'll need definitions of deterministic and total.

Definition 7.4. Let $\theta \in \mathbb{IntRel}_P(\varphi_1, \varphi_2)$. We say that θ is

- *total* if $\varphi_1 \vdash \theta$, and

- *deterministic* if $\theta \vdash \theta$.

Remark 7.5. Note that by the domain of θ and discarding (Example 5.20) we always have

$$\theta = \varphi_1 \vdash \varphi_1$$

and that by meets (Proposition 5.19(ii)) and breaking (Proposition 5.16(ii)) we always have

$$\theta = \theta \vdash \theta$$

This means that in Definition 7.4 the two entailments are in fact equalities.

In what follows, we'll often omit the transpose symbol † (see Definition 6.11) from our diagrams when it can be deduced from the ambient contextual information.

Theorem 7.6. *Let* $\theta \in \mathbb{I}\mathrm{ntRel}_P(\varphi_1, \varphi_2)$. *Then the following are equivalent.*

  (i) $\theta \in \mathcal{R}_P$ *is an internal function in the sense of Definition 7.1.*

  (ii) θ *has right adjoint* $\theta^\dagger$. *That is,* $\varphi_1 \vdash \theta - \theta$ *and* $\theta - \theta \vdash \varphi_2$.

  (iii) θ *is total and deterministic in the sense of Definition 7.4.*

Proof. (i) $\Leftrightarrow$ (ii): Clearly (ii) $\Rightarrow$ (i). Conversely, assume θ has a right adjoint ξ. Note that the unit axiom implies $^{\tau_1}\varphi_1 \vdash \theta - \xi \vdash \theta$. Then using meets and breaking we have

$$\Gamma_2 - \xi - \Gamma_1 = \frac{\theta}{\xi} = \frac{\theta\ \theta}{\xi} \vdash \xi - \theta - \theta \vdash \Gamma_2 - \theta^\dagger - \Gamma_1$$

Similarly we can show $\theta \vdash \xi^\dagger$, and hence $\xi = \theta^\dagger$.

(ii) $\Leftrightarrow$ (iii): We shall prove a stronger statement, that θ has a unit if and only if it is total, and that it has a counit if and only if it is deterministic.

First, (ii)-units iff (iii)-totalness. Using the unit of the adjunction we have

$$\Gamma_1 - \varphi_1 \vdash \frac{\theta}{\theta} = \theta$$

Conversely, using totalness, meets, and breaking we have

$$\varphi_1 = \theta = \theta\ \theta \vdash \theta - \Gamma_2 - \theta$$

Next, (ii)-counits iff (iii)-determinism. We can use the counit of the adjunction to give

$$\frac{\theta}{\theta} = \frac{\theta}{\theta\ \theta} \vdash \frac{\theta}{\theta\ \theta} \vdash \theta = \theta$$

Conversely, assuming determinism we get the counit, which concludes the proof:

$$\theta - \Gamma_1 - \theta = \theta \vdash \varphi_2 \qquad \qquad \square$$

Next, we describe how the order on relations restricts to the functions.

Proposition 7.7. *The order on functions is discrete.*

Proof. Suppose $\boxed{\theta} \vdash \boxed{\theta'}$. Then using the unit of θ and counit of θ' we have

$$\boxed{\theta'} \;\vdash\; \boxed{\theta}-\!(\theta)\!-\!\boxed{\theta'} \;\vdash\; \boxed{\theta}-\!(\theta')\!-\!\boxed{\theta'} \;\vdash\; \boxed{\theta} \,.$$

$\square$

Finally, we note that bijections and projections are examples of functions.

Example 7.8. A *P-internal bijection* is an invertible P-internal relation. Note that every bijection is a function. We can also characterise bijections as the adjunctions whose unit and counit are the identity.

Proposition 7.9. *Suppose given $\varphi_1 \in P(\Gamma_1)$ and $\varphi_2 \in P(\Gamma_2)$ and a relation $\theta \in \mathbb{IntRel}_P(\varphi_1, \varphi_2) \subseteq P(\Gamma_1 \oplus \Gamma_2)$. Define*

$$\pi_1 := (\delta_{\Gamma_1} \oplus \Gamma_2)_!(\theta) \quad and \quad \pi_2 := (\Gamma_2 \oplus \delta_{\Gamma_2})_!(\theta).$$

Then $\pi_i \in P(\Gamma_1 \oplus \Gamma_i \oplus \Gamma_2)$ are internal functions for $i = 1, 2$, i.e. $\pi_i \in \mathcal{R}_P(\theta, \varphi_i)$

Proof. We prove π_1 is a function; the argument for π_2 is similar. Note that π_1 is depicted by the graphical term
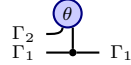
By Proposition 5.19 and the fact that $\theta \in \mathbb{IntRel}_P(\varphi_1, \varphi_2)$ we have

and hence by Proposition 6.10, $\pi_1 \in \mathbb{IntRel}_P(\theta, \varphi_1)$.

Proving that π_1 is an adjunction in $\mathbb{IntRel}_P(\theta, \varphi_1)$ again uses Proposition 5.19 and that $\theta \in \mathbb{IntRel}_P(\varphi_1, \varphi_2)$, as well as Example 5.9:

$\square$

Definition 7.10. With $\varphi_1, \varphi_2, \theta$ as in Proposition 7.9, we refer to the map $(\delta_{\Gamma_1} \oplus \Gamma_2)_! \in \mathcal{R}_P(\theta, \varphi_1)$ as the *left projection* and similarly to $(\Gamma_1 \oplus \delta_{\Gamma_2})_! \in \mathcal{R}_P(\theta, \varphi_2)$ as the *right projection*.

7.2 Finite limits in $\mathcal{R}_P$

We now show how to construct finite limits in the category $\mathcal{R}_P$ of internal functions in P.

Lemma 7.11 (Terminal object). *The object $(0, \mathtt{true}) \in \mathcal{R}_P$ is terminal.*

Proof. For any context Γ and element $\varphi \in P(\Gamma)$ we shall show $\varphi \in \mathcal{R}_P((\Gamma, \varphi), (0, \mathtt{true})) \subseteq P(\Gamma \oplus 0)$ is the unique element. Note first that φ is indeed an internal function: it's an internal relation because $\varphi \vdash \varphi$ and $\pi_{2!}(\varphi) \vdash \mathtt{true}$, and is an adjunction with counit given by the fact that $\mathtt{true}$ is the top element, and unit given by meets and breaking as follows

It remains to show uniqueness. If θ is an internal function then $\theta \vdash \varphi$, so it remains to show that $\varphi \vdash \theta$. But it is easy to verify: . $\qquad\square$

Lemma 7.12 (Pullbacks). *Let* $\theta_1 \colon (\Gamma_1, \varphi_1) \to (\Gamma, \varphi)$ *and* $\theta_2 \colon (\Gamma_2, \varphi_2) \to (\Gamma, \varphi)$ *be morphisms in* $\mathcal{R}_P$*. Let* $\theta_{12} := (\theta_1 \mathbin{\overset{\circ}{,}} \theta_2^\dagger)$*. Then the following is a pullback square in* $\mathcal{R}_P$*:*

$$
\begin{array}{ccc}
((\Gamma_1 \oplus \Gamma_2), \theta_{12}) & \xrightarrow{\;(\Gamma_1 \oplus \delta_{\Gamma_2})_!(\theta_{12})\;} & (\Gamma_2, \varphi_2) \\[2pt]
{\scriptstyle (\delta_{\Gamma_1} \oplus \Gamma_2)_!(\theta_{12})} \downarrow & & \downarrow {\scriptstyle \theta_2} \\[6pt]
(\Gamma_1, \varphi_1) & \xrightarrow[\;\;\theta_1\;\;]{} & (\Gamma, \varphi)
\end{array}
$$

Proof. The graphical term for the proposed pullback $((\Gamma_1 \oplus \Gamma_2), \theta_{12})$ is shown left, and its proposed projection maps are shown middle and right:

Both projections are internal functions by Proposition 7.9. The necessary diagram commutes, i.e. we have equalities

 $\qquad(25)$

because functions are deterministic (Theorem 7.6).

Now we come to the universal property. Suppose given an object (Γ', φ') and morphisms $\theta_1' \colon (\Gamma', \varphi') \to (\Gamma_1, \varphi_1)$ and $\theta_2' \colon (\Gamma', \varphi') \to (\Gamma_2, \varphi_2)$ in $\mathcal{R}_P$, such that the $\theta_1' \mathbin{\overset{\circ}{,}} \theta_1 = \theta_2' \mathbin{\overset{\circ}{,}} \theta_2$. Let $\langle \theta_1', \theta_2' \rangle$ denote the following graphical term:

 $\qquad(26)$

We give one half of the proof that $\langle \theta_1', \theta_2' \rangle \in \mathbb{IntRel}_P(\varphi', \theta_{12})$, the other half being easier.

Moreover, applying Theorem 7.6, a similarly straightforward diagrammatic argument shows $\langle \theta_1', \theta_2' \rangle \in \mathcal{R}_P(\varphi', \theta_{12})$. We next need to show that $\langle \theta_1', \theta_2' \rangle \mathbin{\overset{\circ}{,}} (\delta_{\Gamma_1} \oplus \Gamma_2)_!(\theta_{12}) = \theta_1'$ and similarly for θ_2'. This follows easily from Proposition 7.7 and the diagram

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It only remains to show that this is unique. So suppose given $\theta' \in \mathcal{R}_P(\varphi', \theta_{12})$ with

$$\Gamma' \boxed{\theta_1'} \Gamma_1 \;=\; \Gamma' \boxed{\theta'}^{\Gamma_1} \quad \text{and} \quad \Gamma' \boxed{\theta_2'} \Gamma_2 \;=\; \Gamma' \boxed{\theta'}_{\Gamma_2} \;.\; \text{Then by basic diagram manipu-}$$

lations, one shows that θ' must equal the graphical term in Eq. (26), as desired. $\qquad\square$

Proposition 7.13. *Suppose that* $\boxed{\theta} \in \mathcal{R}_P(\varphi_1, \varphi_2)$ *is an internal function. It is a monomorphism iff it satisfies* $\boxed{\varphi_1} \;=\; \boxed{\theta}\!\!-\!\!\boxed{\theta} \;.$

Proof. Recall that a morphism is a monomorphism iff the projection maps of its pullbacks along itself are the identity maps. Using the characterization of the projection maps of the pullback of θ along itself (Lemma 7.12) and the graphical logic, the proposition is immediate. $\qquad\square$

Corollary 7.14 (Monomorphisms). *If* $\varphi \vdash_\Gamma \varphi'$, *then* $\mathrm{id}_\varphi \in P(\Gamma \oplus \Gamma)$ *as in Eq. (22) is an element of* $\mathcal{R}_P((\Gamma, \varphi), (\Gamma, \varphi'))$ *and it is a monomorphism.*

Proof. Since meets merge circles, we have the equality

$$\underbrace{\boxed{\varphi} \quad \boxed{\varphi}}_{} \;=\; \boxed{\varphi} \tag{27}$$

and it follows easily that $\mathrm{id}_\varphi \in \mathcal{R}_P(\varphi, \varphi')$. But this also proves that id_φ is a monomorphism, by Proposition 7.13. $\qquad\square$

Remark 7.15 (Equalizers). Given parallel arrows $\theta, \theta' \colon (\Gamma_1, \varphi_1) \to (\Gamma_2, \varphi_2)$, their equalizing object (Γ_1, e) is the following graphical term:

$$\Gamma_1 \boxed{e} \;=\; \begin{array}{c}\boxed{\theta}\\\boxed{\theta'}\end{array}$$

7.3   Image factorizations

We next discuss image factorizations, and show that they are stable under pullback.

Definition 7.16. Suppose that $\theta \in \mathcal{R}_P(\varphi_1, \varphi_2)$ is an internal function. Define its *image*, denoted $\mathrm{im}(\theta) \in P(\Gamma_2)$ to be the graphical term $\bullet\!\!-\!\!\boxed{\theta}\!\!-$ or, in symbols, $\epsilon^*_{\Gamma_1} \, \fatsemi \, \theta$.

We will now show that this has the usual properties of images, for example that θ is a regular epimorphism in $\mathcal{R}_P$ iff it satisfies $\boxed{\varphi_2}\!\!-\; \vdash \; \bullet\!\!-\!\!\boxed{\theta}\!\!-$.

Proposition 7.17. *Consider an element* $\theta \in \mathcal{R}_P(\varphi_1, \varphi_2)$. *The following are equivalent:*
1. θ, *considered as a morphism in* $\mathcal{R}_P$, *is a regular epimorphism,*
2. $\varphi_2 \vdash_{\Gamma_2} \mathrm{im}(\theta)$,
3. $\varphi_2 = \mathrm{im}(\theta)$, *and*

4. φ_2 diagram $=$ diagram with θ — θ .

Proof. $(1 \Rightarrow 2)$: It is straightforward to show that $\theta \in P(\Gamma_1, \Gamma_2)$ is an element of $\mathcal{R}_P(\varphi_1, \mathrm{im}(\theta))$. Now supposing that θ is a regular epi, i.e. that the kernel pair diagram

$$\varphi_1 \times_{\varphi_2} \varphi_1 \rightrightarrows \varphi_1 \longrightarrow \varphi_2$$

is a coequalizer, it suffices to show that $\mathrm{im}(\theta)$ also coequalizes the parallel pair:

$$\text{(diagram)} = \text{(diagram)} \qquad (28)$$

This follows directly from determinism.

$(2 \Rightarrow 3)$: For any relation $\theta \in \mathbb{IntRel}_P(\varphi_1, \varphi_2)$ we always have the converse $\mathrm{im}(\theta) \vdash \varphi_2$.

$(3 \Rightarrow 4)$: By determinism of θ, we have $\text{(diagram)} = \text{(diagram)} = \text{(diagram)}$

$(4 \Rightarrow 1)$: Assuming 4, we need to show that $\varphi_1 \times_{\varphi_2} \varphi_1 \rightrightarrows \varphi_1 \rightarrow \varphi_2$ is a coequalizer. It is easy to show that φ_2 coequalizes the parallel pair; this is basically Eq. (28) again. So let $\theta' \colon \varphi_1 \rightarrow \varphi_2'$ coequalize the parallel pair, and define $\xi \in \mathbb{IntRel}_P(\varphi_2, \varphi_2')$ by $\xi := \theta^\dagger \,\mathring{;}\, \theta'$. We need to show that ξ is a function and that $\theta \,\mathring{;}\, \xi = \theta'$.

We obtain $\mathrm{id}_{\varphi_2} \vdash \xi \,\mathring{;}\, \xi^\dagger$ using (4) and the fact that θ' is a function:

$$\varphi_2 \text{ diagram} = \text{(diagram)} \vdash \text{(diagram)} .$$

We obtain $\xi^\dagger \,\mathring{;}\, \xi \vdash \mathrm{id}_{\varphi_2'}$ as follows:

$$\text{(diagram)} = \text{(diagram)} \vdash \text{(diagram)}$$

where the first equality comes from the fact that θ' coequalizes the parallel pair, and the second is discarding and determinism of θ'. Finally, $\theta' \vdash \theta \,\mathring{;}\, \theta^\dagger \,\mathring{;}\, \theta = \theta \,\mathring{;}\, \xi$ follows easily from θ being a function. The converse $\theta \,\mathring{;}\, \xi \vdash \theta'$ follows from the fact that θ' coequalizes the parallel pair:

$$\text{(diagram)} = \text{(diagram)} \vdash \text{(diagram)}$$

$\square$

Lemma 7.18 (Image factorizations). *Any morphism $\theta \colon (\Gamma', \varphi') \rightarrow (\Gamma, \varphi)$ can be factored into a regular epimorphism followed by a monomorphism; the image object is $(\Gamma, \epsilon_\Gamma^* \,\mathring{;}\, \theta)$.*

Proof. The image factorization of θ is given by

$$\Gamma' \text{—} \theta \text{—} \Gamma = \text{(diagram)}$$

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The graphical representation of the image object $(\Gamma, \epsilon_{\Gamma'}^* \mathbin{\fatsemi} \theta)$ is ⟨●─$\boxed{\theta}$─⟩ . It is immediate from Proposition 7.17 that θ is a regular epimorphism $(\Gamma', \varphi') \to (\Gamma, \epsilon_{\Gamma'}^* \mathbin{\fatsemi} \theta)$, and from Corollary 7.14 that $(\delta_\Gamma)_!(\epsilon_{\Gamma'}^* \mathbin{\fatsemi} \theta)$ is a monomorphism $(\Gamma, \epsilon_{\Gamma'}^* \mathbin{\fatsemi} \theta) \to (\Gamma, \varphi)$. $\qquad\square$

Lemma 7.19 (Pullback stability of image factorizations). *The pullback of a regular epimorphism along any morphism is again a regular epimorphism in $\mathcal{R}_P$.*

Proof. Suppose that $\xi \colon \varphi_1 \to \varphi$ is a regular epimorphism and that $\theta \colon \varphi_2 \to \varphi$ is any morphism. Then the pullback $\theta \times_\varphi \xi \to \varphi_2$ is a regular epimorphism by Proposition 7.17 and the following reasoning: ⟨φ_2⟩ $\vdash$ ⟨$\boxed{\theta}$●⟩ $\vdash$ ⟨$\boxed{\theta}$─$\boxed{\xi}$●⟩ . $\qquad\square$

Proof of Theorem 7.3. By Lemmas 7.11 and 7.12, $\mathcal{R}_P$ has all finite limits, and by Lemmas 7.18 and 7.19, it has pullback-stable image factorizations. $\qquad\square$

We record the following for future use.

Proposition 7.20. *Let (T, P) be a regular calculus, let $\Gamma \in \mathsf{FRg}(\mathrm{T})$ be a context, and let* $\mathtt{true} \in P(\Gamma)$ *be the top element. There is an isomorphism of posets*

$$P(\Gamma) \cong \mathsf{Sub}_{\mathcal{R}_P}(\Gamma, \mathtt{true}).$$

Proof. The map $P(\Gamma) \to \mathsf{Sub}_{\mathcal{R}_P}(\Gamma, \mathtt{true})$ sends φ to id_φ as an element of $\mathcal{R}_P((\Gamma, \varphi), (\Gamma, \mathtt{true}))$ and it is a monomorphism by Corollary 7.14.

So suppose given a context Γ' and an element $\theta \in P(\Gamma' \oplus \Gamma)$ which is an internal function and moreover a monomorphism. Let $\varphi := \mathrm{im}\, \theta \in P(\Gamma)$ be its image; we want to show that there is an isomorphism $(\Gamma, \varphi) \cong (\Gamma' \oplus \Gamma, \theta)$ over $(\Gamma, \mathtt{true})$. It is not hard to show this graphically. $\qquad\square$

8 RgCat **is essentially a reflective subcategory of** RgCalc

We have now proved Theorem 7.3, which constructs a regular category $\mathcal{R}_P$ from any regular calculus (T, P). We call $\mathcal{R}_P$ the *syntactic category* corresponding to P. In this section we show that this construction is functorial, and that there is an adjunction

$$\mathsf{RgCalc} \; \overset{\textbf{syn}}{\underset{\textbf{prd}}{\overset{\Rightarrow}{\longleftrightarrow}}} \; \mathsf{RgCat}.$$

Moreover, RgCat is essentially a reflective subcategory of RgCalc, in the sense that for any regular category $\mathcal{R}$, the counit map $\textbf{syn}(\textbf{prd}(\mathcal{R})) \to \mathcal{R}$ is an equivalence of categories. In future work we plan to show that there is 2-dimensional structure throughout, such that the above adjunction extends to a 2-adjunction in which RgCat is 2-reflective.

8.1 The functor syn: RgCalc → RgCat

We want to define a functor $\mathbf{syn}\colon \mathsf{RgCalc} \to \mathsf{RgCat}$ that is adjoint to $\mathbf{prd}$ from Proposition 4.15. On objects, this is now easy: given a regular calculus $(\mathrm{T}, P) \in \mathsf{RgCalc}$, define $\mathbf{syn}(\mathrm{T}, P) := \mathcal{R}_P$ as in Eq. (24); objects are pairs (Γ, φ) where $\Gamma \in \mathsf{FRg}(\mathrm{T})$ and $\varphi \in P(\Gamma)$, and morphisms are internal functions θ as in Theorem 7.6.

For morphisms, suppose given $(F, F^\sharp)\colon (\mathrm{T}, P) \to (\mathrm{T}', P')$:

$$
\begin{array}{ccc}
\mathrm{T} & \mathbb{FRg}(\mathrm{T}) \xrightarrow{\quad P \quad} \\
\downarrow{\scriptstyle F} & \quad\downarrow{\scriptstyle \overline{F}} \quad \Big\Vert{\scriptstyle F^\sharp} \quad \mathbb{P}\mathsf{oset} \\
\mathrm{T}' & \mathbb{FRg}(\mathrm{T}') \xrightarrow[\quad P' \quad]{}
\end{array}
$$

where again $\overline{F} := \mathbb{FRg}(F)$. We define $\mathcal{F} := \mathbf{syn}(F, F^\sharp)\colon \mathcal{R}_P \to \mathcal{R}_{P'}$ on an object $(\Gamma, \varphi) \in \mathcal{R}_P$ by

$$
\mathcal{F}(\Gamma, \varphi) := \left(\overline{F}(\Gamma), F_\Gamma^\sharp(\varphi) \right) \in \mathcal{R}_{P'}. \tag{29}
$$

and on a morphism $\theta\colon (\Gamma_1, \varphi_1) \to (\Gamma_2, \varphi_2)$ by

$$
\mathcal{F}(\theta) := F_{\Gamma_1 \oplus \Gamma_2}^\sharp(\theta). \tag{30}
$$

Theorem 8.1. *The assignment* $\mathbf{syn}(\mathrm{T}, P) := \mathcal{R}_P$ *on objects, and Eqs.* (29) *and* (30) *on morphisms, constitutes a functor* $\mathbf{syn}\colon \mathsf{RgCalc} \to \mathsf{RgCat}$.

Theorem 8.1 is proved on page 44; first we need the following lemma.

Lemma 8.2. $F^\sharp$ *preserves semantics of graphical terms.*

More precisely, given any P-graphical term $(\theta_1, \ldots, \theta_k; \omega)$, the morphism $(F, F^\sharp)$ induces a P'-graphical term $(F^\sharp\theta_1, \ldots, F^\sharp\theta_k; \overline{F}(\omega))$; we call this its image *under* $F^\sharp$. *The image obeys*

$$
F^\sharp[\![(\theta_1, \ldots, \theta_k; \omega)]\!] = [\![(F^\sharp\theta_1, \ldots, F^\sharp\theta_k; \overline{F}(\omega))]\!].
$$

Furthermore, given the entailment $(\theta_1, \ldots, \theta_k; \omega) \vdash (\theta'_1, \ldots, \theta'_{k'}; \omega')$, it follows that

$$
(F^\sharp\theta_1, \ldots, F^\sharp\theta_k; \overline{F}(\omega)) \vdash (F^\sharp\theta'_1, \ldots, F^\sharp\theta'_{k'}; \overline{F}(\omega')).
$$

Proof. The naturality and monoidality of $(F, F^\sharp)$ imply:

$$
\begin{aligned}
F^\sharp[\![(\theta_1, \ldots, \theta_k; \omega)]\!] &= F^\sharp(P(\omega))(\rho(\theta_1, \ldots, \theta_k))) \\
&= P'(\overline{F}(\omega))(F^\sharp(\rho(\theta_1, \ldots, \theta_k))) \\
&= P'(\overline{F}(\omega))(\rho(F^\sharp\theta_1, \ldots, F^\sharp\theta_k)) \\
&= [\![(F^\sharp\theta_1, \ldots, F^\sharp\theta_k; \overline{F}(\omega))]\!].
\end{aligned}
$$

The second claim then follows from the monotonicity of components in $F^\sharp$. $\qquad\square$

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Proof of Theorem 8.1. First we must check that our data type-checks. We have already shown that $\mathcal{R}_P$ is a regular category, so it remains to show that $\mathcal{F}$ is a regular functor. This is a consequence of Lemma 8.2.

In particular, recall from Definition 7.1 that morphisms in $\mathcal{R}_P$ can be represented by P-graphical terms obeying certain entailments. It was shown in Sections 7.2 and 7.3 that composition, identities, finite limits, and regular epis can also be described in this way. Lemma 8.2 implies that given a P-graphical term, its image under $F^\sharp$ preserves entailments and equalities. Thus $\mathcal{F}$ sends internal functions to internal functions of the required domain and codomain, preserves composition, identities, finite limits, and regular epis, and hence is a regular functor.

It is then immediate from the definition (Eqs. (29) and (30)) that **syn** preserves identity morphisms and composition, and so **syn** is indeed a functor. $\qquad\square$

8.2 The essential reflection

Recall that $\mathbf{prd}(\mathcal{R}) = (\mathrm{Ob}\,\mathcal{R}, \mathsf{Sub}_{\mathcal{R}}{}^{\ulcorner} -{}^{\urcorner})$ and $\mathbf{syn}(P) = \mathsf{LAdj}(\mathbb{IntRel}_P)$; see Eq. (19) and Theorem 6.3.

Proposition 8.3. *For any regular category $\mathcal{R}$, there is a natural equivalence of categories*

$$\epsilon \colon \mathbf{syn}(\mathbf{prd}(\mathcal{R})) \xrightarrow{\simeq} \mathcal{R}.$$

Proof. We will define functors $\epsilon \colon \mathcal{R}_{\mathbf{prd}(\mathcal{R})} \leftrightarrows \mathcal{R} \colon \epsilon'$ and show that they constitute an equivalence. We have $\mathrm{Ob}(\mathcal{R}_{\mathbf{prd}(\mathcal{R})}) = \{(\Gamma, r) \mid \Gamma \in \mathsf{FRg}(\mathrm{Ob}\,\mathcal{R}),\ r \in \mathsf{Sub}_{\mathcal{R}}{}^{\ulcorner}\Gamma{}^{\urcorner}\}$, so put

$$\epsilon(\Gamma, r) := r, \qquad \text{and} \qquad \epsilon'(r) := (\langle r \rangle, r),$$

where $\langle r \rangle$ is the unary context on r and $r \subseteq r = {}^{\ulcorner}\langle r \rangle{}^{\urcorner}$ is the top element. Given also (Γ', r'), we have an isomorphism of hom-sets

$$\mathcal{R}_{\mathbf{prd}(\mathcal{R})}\left((\Gamma, r), (\Gamma', r')\right) \cong \mathsf{LAdj}(\mathbb{Rel}_{\mathcal{R}})(r, r') \cong \mathcal{R}(r, r'),$$

by Definition 7.1, Proposition 6.2, , and Lemma 2.8. Hence, we define ϵ and ϵ' on morphisms to be the corresponding mutually-inverse maps. Obviously, ϵ and ϵ' are fully faithful functors, and $\epsilon' \,\mathring{,}\, \epsilon = \mathrm{id}_{\mathcal{R}}$, so ϵ is essentially surjective. $\qquad\square$

We next prove that $\mathbf{prd} \colon \mathsf{RgCat} \to \mathsf{RgCalc}$ is full, fulfilling a promise made after Proposition 4.15, where $\mathbf{prd}$ was first defined. Recall that $\mathbf{prd}(\mathcal{R}) = (\mathrm{Ob}(R), \mathsf{Sub}_{\mathcal{R}}{}^{\ulcorner} -{}^{\urcorner})$.

Corollary 8.4. *The functor $\mathbf{prd} \colon \mathsf{RgCat} \to \mathsf{RgCalc}$ is full.*

Proof. Let $\mathcal{R}, \mathcal{R}'$ be regular categories, and suppose given a map $(F, F^\sharp) \colon \mathbf{prd}(\mathcal{R}) \to \mathbf{prd}(\mathcal{R}')$; we need to show there exists a functor $\mathcal{F} \colon \mathcal{R} \to \mathcal{R}'$ such that $\mathbf{prd}(\mathcal{F}) = (F, F^\sharp)$. The key idea is that $(F, F^\sharp)$ specifies the action of the desired functor $\mathcal{F}$ on subobject semilattices, which is enough, since every morphism in $\mathcal{R}$ can be recovered from its graph.

Applying **syn** to $(F, F^\sharp)$, we obtain a regular functor $\mathbf{syn}(F, F^\sharp)\colon \mathbf{syn}(\mathbf{prd}(\mathcal{R}')) \to \mathbf{syn}(\mathbf{prd}(\mathcal{R}))$. Pre- and post-composing this with the equivalences $\epsilon'_R\colon \mathcal{R} \to \mathbf{syn}(\mathbf{prd}(\mathcal{R}))$ and $\epsilon_{\mathcal{R}'}\colon \mathbf{syn}(\mathbf{prd}(\mathcal{R}')) \to \mathcal{R}'$ from Proposition 8.3, we obtain a regular functor $\mathcal{F}\colon \mathcal{R} \to \mathcal{R}'$. It is routine to check that the image of this functor is $\mathbf{prd}(\mathcal{F}) = (F, F^\sharp)$. $\square$

Theorem 8.5. *The functors **prd** and **syn** are adjoint:*

$$\mathsf{RgCalc} \underset{\mathbf{prd}}{\overset{\mathbf{syn}}{\underset{\Leftarrow}{\rightrightarrows}}} \mathsf{RgCat}.$$

*Moreover, **prd** is fully faithful, and for any regular category $\mathcal{R}$, the counit map $\mathbf{syn}(\mathbf{prd}(\mathcal{R})) \to \mathcal{R}$ is an equivalence.*

Proof. We showed that **prd** is fully faithful in Proposition 4.15 and Corollary 8.4 and that there is a natural transformation $\epsilon\colon \mathbf{prd} \mathbin{\fatsemi} \mathbf{syn} \to \mathrm{id}_{\mathsf{RgCat}}$ with the property that $\epsilon_{\mathcal{R}}$ is an equivalence for any $\mathcal{R}$. It remains to construct $\eta\colon \mathrm{id}_{\mathsf{RgCalc}} \to \mathbf{syn} \mathbin{\fatsemi} \mathbf{prd}$ and check that ϵ and η satisfy the triangle identities.

Given a regular calculus (T, P), we have $\mathbf{prd}(\mathbf{syn}(\mathrm{T}, P)) = (\mathrm{Ob}\,\mathcal{R}_P, \mathsf{Sub}_{\mathcal{R}_P}\ulcorner{-}\urcorner)$, where $\mathrm{Ob}\,\mathcal{R}_P = \{(\Gamma, \varphi) \mid \Gamma \in \mathsf{FRg}(\mathrm{T}), \varphi \in P(\Gamma)\}$. There is an obvious function $e\colon \mathrm{T} \to \mathrm{Ob}\,\mathcal{R}_P$ sending $\tau \mapsto (\langle\tau\rangle, \mathtt{true})$, where as usual, $\langle\tau\rangle$ is the unary context and $\mathtt{true} \in P(\langle\tau\rangle)$ is its top element. We will define $\eta := (e, e^\sharp)$, where $e^\sharp(\Gamma)\colon P(\Gamma) \to \mathsf{Sub}_{\mathcal{R}_P}\ulcorner\overline{e}(\Gamma)\urcorner = \mathsf{Sub}_{\mathcal{R}_P}(\Gamma, \mathtt{true})$ is the natural isomorphism given in Proposition 7.20:

$$
\begin{array}{ccc}
\mathrm{T} & \mathbb{FRg}(\mathrm{T}) \xrightarrow{\ \ P\ \ } & \\
e\big\downarrow & \overline{e}\big\downarrow \quad e^\sharp\big\Vert & \mathbb{P}\mathsf{oset} \\
\mathrm{Ob}\,\mathcal{R}_P & \mathbb{FRg}(\mathrm{Ob}\,\mathcal{R}_P) \xrightarrow[\mathsf{Sub}_{\mathcal{R}_P}\ulcorner{-}\urcorner]{} &
\end{array}
$$

The fact that $\epsilon_{\mathcal{R}}$ is an equivalence and that $e^\sharp$ is a natural isomorphism make the triangle identities particularly easy (if tedious) to verify. This completes the proof. $\square$

Acknowledgments

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