Initial Algebras, Terminal Coalgebras, and the Theory of Fixed Points of Functors

Draft: September 2019

Jiří Adámek Stefan Milius Lawrence S. Moss

Contents

1 Introduction

1.1 Why are initial algebras interesting?

Recursion and induction are important tools in mathematics and computer science. In functional programming, for example, recursion is a definition principle for functions over the (inductive) structure of data types such as natural numbers, lists or trees. And induction is the corresponding proof principle used to prove properties of programs. An important question of theoretical computer science concerns the semantics of such definitions. *Initial Algebra Semantics*, studied since the 1970's, uses the tools of category theory to unify recursion and induction at the appropriate abstract conceptual level. In this approach, the type of data on which one wants to define functions recursively and to prove properties inductively is captured by an *endofunctor* F on the category of sets (or another appropriate base category). This functor describes the signature of the data type constructors. An F*-algebra* is a set A together with a map $\alpha : FA \to A$, and an initial algebra for the functor F provides a canonical minimal model of a data type with the desired constructors.

Let us illustrate this by a concrete example: Consider the endofunctor on sets given by $FX = X + 1$, i.e., the set construction adding a fresh element to the set X. An algebra for F is just a set equipped with a unary operation and a constant, and the initial algebra is the algebra of natural numbers $\mathbb{N}$ with the successor function and the constant 0. The abstract property of initiality of that algebra is precisely the usual principle of *recursion on natural numbers*: given an F-algebra X, i.e., a unary operation on $u : X \to X$ and a constant $x \in X$, there exists precisely one function f from the natural numbers to X with $f(0) = x$ and $f(n+1) = u(f(n))$. The existence of f is the fact that functions from $\mathbb{N}$ to X can be defined by recursion, and the uniqueness yields the proof principle of induction.

As a second example consider the set functor $FX = X \times X + 1$. An algebra $\alpha : A \times A + 1 \to A$ can be considered as a Σ-algebra where the siganture Σ consists of a binary operation and a constant. Then the initial algebra is the algebra of finite binary trees, and initiality yields a tree-recursion principle.

In the present book initial algebras are studied for all categories $\mathscr{A}$ and endofunctors $F : \mathscr{A} \to \mathscr{A}$. It was J. Lambek [119] who first studied *algebras* for F as pairs consisting of an object A of $\mathscr{A}$ and a morphism $a : FA \to A$; the corresponding F*-algebra homomorphisms* from (A, α) to (A', α') are those morphisms $h : A \to A'$ in $\mathscr{A}$ for which the square below commutes:

$$\begin{tikzcd} FA \arrow[r, "\alpha"] \arrow[d, "Fh"'] & A \arrow[d, "h"] \\ FA' \arrow[r, "\alpha'"'] & A' \end{tikzcd}$$

If α is invertible (thus $FA \cong A$) we call the algebra a *fixed point* of F.

The category of algebras is denoted by $\mathsf{Alg}\,F$. By an *initial algebra* μF of F is meant its initial object: this is an algebra such that for every algebra there exists a unique F-algebra homomorphism from μF.

Lambek's Lemma. *If F has an initial algebra, then it is a fixed point.*

We conclude immediately that even for $\mathscr{A} = \mathsf{Set}$, there are important examples of endofunctors that do not have an initial algebra: consider the power-set functor $\mathscr{P}$. A fundamental result of set theory known as Cantor's Theorem says that no set A is in bijective correspondence with $\mathscr{P}A$. (The short proof may be found in Example 2.2.7(1).) So for all sets A, A is not a fixed point of the power-set functor.

In the present book, we study the existence and construction of initial algebras. There is a general procedure for constructing the initial algebra for F starting from the initial object 0 of $\mathscr{A}$, first used by J. Adámek [10]. Denoting by $! : 0 \to F0$ the unique morphism, we form the corresponding ω-chain:

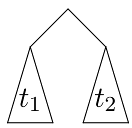
$$0 \xrightarrow{\ !\ } F0 \xrightarrow{F!} FF0 \xrightarrow{FF!} F^3 0 \xrightarrow{F^3!} \cdots$$

If the colimit exists and is preserved by F, then that colimit carries the initial algebra

$$\mu F = \operatorname*{\mathsf{colim}}_{n \in \omega} F^n 0.$$

If F does not preserve that colimit, we iterate further and obtain a transfinite chain $F^i 0$, where i ranges over all ordinal numbers. At successor ordinals $i+1$ we apply F, so that $F^{i+1}0 = FF^i 0$, and at limit ordinals λ we take colimits: $F^\lambda 0 = \mathsf{colim}_{i<\lambda} F^i 0$. If F preserves one of these colimits, we obtain an initial algebra. We go into detail on this construction in Chapter 3 for finite ordinals and in Chapter 6 for general ones.

We illustrate the behaviour of this construction and other methods for obtaining initial algebras by numerous examples. For example the functor $FX = X \times X + 1$ (of binary algebras with a constant) yields the chain with $F^1 0 = 1$ and $F^{n+1}0 = F^n 0 \times F^n 0 + 1$. This recursion allows us to represent $F^n 0$ by the set all binary trees of depth less than n. The initial algebra $\mathsf{colim}_{n<\omega} F^n 0 = \bigcup_{n<\omega} F^n 0$ is the algebra of all finite binary trees. The constant is the trivial single-node tree, and the binary operation is *tree tupling*, i.e. the operation that assigns to a pair (t_1, t_2) of binary trees the binary tree having t_1 and t_2 rooted at the children of the root:

$(t_1, t_2) \longmapsto$ 

Shortly: $\mu F =$ finite binary trees.

1.2 Why are terminal coalgebras interesting?

A *coalgebra* for a functor F is the dual concept of an F-algebra: it consists of an object A and a morphism $\alpha : A \to FA$. Jan Rutten [149] presented a persuasive survey of applications of this idea to the theory of discrete dynamical systems which are ubiquitous in computer science. For example, a deterministic automaton with input alphabet Σ can be described by the set A of its states together with a function

$$\alpha : A \to \{0,1\} \times A^{\Sigma}$$

whose first component

$$\frac{A \to A^{\Sigma}}{A \times \Sigma \to A}$$

describes the next-state function, and the second one

$$A \to \{0,1\}$$

describes the predicate "accepting state". This is a coalgebra for the functor F given by

$$FX = \{0,1\} \times X^{\Sigma}.$$

And a non-deterministic automaton is given by a function from A to $\{0,1\} \times (\mathscr{P}A)^{\Sigma}$ and is thus a coalgebra for the endofunctor $F\mathscr{P}$ composed of the power-set functor and the above functor F.

For another example, consider a dynamic system with states accepting binary input and having also deadlock states (not reacting to inputs). This is given by a set A of states and a function

$$\alpha : A \to A \times A + 1$$

assigning to every deadlock state the element of 1 and to every other state the pair of the possible next states. This is a coalgebra for $FX = X \times X + 1$.

The category $\mathsf{Coalg}\, F$ of coalgebras has as morphisms from (A, α) to (A', α') the F*-coalgebra homomorphisms* which are the morphisms $h : A \to A'$ such that the square below commutes:

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "h"'] & FA \arrow[d, "Fh"] \\ A' \arrow[r, "\alpha'"'] & FA' \end{tikzcd}$$

The *terminal coalgebra* νF is (if it exists) the terminal object of this category: For every coalgebra there exists a unique F-coalgebra homomorphism into νF. If a system is represented by a coalgebra (A, α), then the unique F-coalgebra homomorphism $f : A \to \nu F$ expresses the "abstract behaviour of the states in A", as we demonstrate on numerous examples in Section 2.3 and later.

Example 1.2.1. The functor $FX = \{0,1\} \times X^\Sigma$ has the terminal coalgebra $\mathscr{P}\Sigma^*$, the set of all formal languages over the alphabet Σ:

$$\nu F = \mathscr{P}\Sigma^*.$$

Given an automaton (A, α), the unique F-coalgebra homomorphism $f : A \to \mathscr{P}S^*$ assigns to every state the language this state accepts. For details, see Example 2.5.5.

Example 1.2.2. For the dynamic systems with deadlocks above we have the coalgebra of all (finite and infinite) binary trees as the terminal coalgebra, shortly:

$$\nu F = \text{binary trees}.$$

Given a dynamic system (A, α) the unique F-coalgebra homomorphism $f : A \to \nu F$ assigns to every state the binary tree of all possible future developments in which the names of states are "abstracted away" and only the distinction deadlock/non-deadlock remains, see Example 2.5.11(4).

By dualizing Lambek's Lemma, we see that νF is always a fixed point of F. Thus, we can consider νF also to be an algebra for F. (Example: binary trees form an algebra for $FX = X \times X + 1$, with tree tupling and the single-node tree, analogously to μF). This algebra νF has a strong property of solvability of recursive equations. Let us illustrate this with the functor $FX = X \times X + 1$ of one binary and one nullary operation. Given a system of recursive equations

$$\begin{aligned} x_1 &= t_1 \\ x_2 &= t_2 \\ &\vdots \end{aligned}$$

where each t_i is a Σ-term using the variables $x_1, x_2, \ldots$ (for the above signature of a binary operation and constant there exists a *solution* in νF. This means that to every x_i we can assign a binary tree $x_i^\dagger$ such that the formal equations above become identities when the simultaneous substitution $x_i^\dagger / x_i$ is performed on the left- and right-hand sides of each equation in the system. Moreover, the solution of the system is unique, provided none of the right-hand sides is a bare variable. Algebras with this recursion property are called *completely iterative*. S. Milius [128] proved that whenever νF exists, it is a completely iterative algebra—in fact, it can be characterized as the initial completely iterative algebra. We provide an overview of completely iterativity in Section 7.2.

1.3 Induction and Coinduction

One of the important roles of initial algebras is to enable a very general formulation of induction and recursion; this is discussed in Section 2.3. Jan Rutten also discusses in [149] *corecursion* as an important construction principle dual to recursion, and *coinduction* as an important proof principle dual to induction; we come to this in Section 2.6.

In fact, induction can be formulated abstractly as follows: for a parallel pair $f_1, f_2 \colon \mu F \to A$ of morphisms in the base category $\mathscr{A}$ with domain μF in order to prove $f_1 = f_2$ it is sufficient to present a morphism $\alpha \colon FA \to A$ for which f_1 and f_2 are algebra homomorphisms from $(\mu F, \iota)$ to (A, α). Coinduction is the dual principle which for the terminal coalgebra νF allows us to prove equality of morphisms of the form $f_1, f_2 \colon A \to \nu F$.

It goes without saying that not every functor possesses a terminal coalgebra. This follows from the dual of Lambek's Lemma: the power-set functor does not have a terminal coalgebra. We study the dual of the above initial algebra construction (explicitly used by Michael Barr [50] for the first time): Start with the terminal object 1 of $\mathscr{A}$ and the unique morphism $! \colon F1 \to 1$ and form the ω^{op}-chain

$$1 \xleftarrow{\;!\;} F1 \xleftarrow{\;F!\;} FF1 \xleftarrow{\;FF!\;} F^3 1 \xleftarrow{\;F^3!\;} \cdots . \tag{1.1}$$

If the limit exists and is preserved by F, this is the terminal coalgebra for F:

$$\nu F = \lim_{n<\omega} F^n 1.$$

Example 1.3.1. The above functors $FX = \{0,1\} \times X^S$ and $FX = X \times X + 1$ preserve all limits of ω^{op}-chains (called ω^{op}-limits, for short), and in particular their terminal coalgebras may be obtained by taking the limit of the ω^{op}-chain in (1.1).

If F does not preserve ω^{op}-limits, one may continue to iterate F, obtaining a *transfinite chain*. We discuss this in Chapter 6.

Example 1.3.2. Graphs are nothing else than coalgebras for the power-set functor $\mathscr{P}$: given a graph on the set A of vertices, then consider, for every vertex $x \in A$, the set $\alpha(x) \subseteq A$ of all neighbours of x. This defines a coalgebra $\alpha : A \to \mathscr{P}A$, and conversely, every coalgebra stems from a graph. But be careful: there are fewer coalgebra homomorphisms than graph homomorphisms. Given graphs (A, α) and (B, β) a coalgebra homomorphism $f : A \to B$ does not only fulfill

$$\text{if } x \to x' \text{ in } A \text{ then } f(x) \to f(x') \text{ in } B$$

but also

$$\text{if } f(x) \to y' \text{ in } B \text{ then } x \to y \text{ in } A \text{ for some } y \text{ with } f(y) = y'.$$

In other words, when we consider the graphs A and B as (unlabelled) transition systems, a coalgebra homomorphism is precisely a function whose graph is a bisimulation in the sense of Milner [134, 135] and Park [143].

Lambek's Lemma tells us that there exists no terminal graph. However, for $\mathscr{P}_\mathsf{f}$ the subfunctor $\mathscr{P}_\mathsf{f}$ of $\mathscr{P}$ mapping a set to the set of all its finite subsets, coalgebras are precisely the finitely branching graphs, and a terminal coalgebra exists. We mention this example because the limit of the ω^{op}-chain in (1.1) is *not* preserved by the functor $\mathscr{P}_\mathsf{f}$, and so one needs a more sophisticated construction.

As we shall see in Section 4.5, there are *several different* constructions of the terminal coalgebra for $\mathscr{P}_\mathsf{f}$. For other functors on other categories, there are yet other constructions.

algebra for a functor	coalgebra for a functor
initial algebra	terminal coalgebra
least fixed point	greatest fixed point
congruence relation	bisimulation equivalence relation
equational logic	modal logic
recursion: map out of an initial algebra	corecursion: map into a terminal coalgebra
Foundation Axiom	Anti-Foundation Axiom
iterative conception of set	coiterative conception of set
set with operations	set with transitions and observations
useful in syntax	useful in semantics
bottom-up	top-down

Figure 1.1: The conceptual comparison

1.4 Algebraic versus coalgebraic concepts

Although *algebra* and *coalgebra* are dual terms, and although this duality persists to the level of *initial algebra* and *terminal coalgebra*, it is by no means the case that $\mathsf{Alg}\,F$ is dual to $\mathsf{Coalg}\,F$. There are easy examples of this; here is a very simple one: consider the poset $\{x, y, z\}$ with $x \leq y$, $x \leq z$ as a category (with morphisms given by the order relation $\leq$), and consider the identity map on this poset as a functor. This poset has an initial object x but no terminal one. Thus the identity functor has an initial algebra but no terminal coalgebra.

What is true is the following: every functor $F \colon \mathscr{A} \to \mathscr{A}$ defines a functor $F^{op} \colon \mathscr{A}^{op} \to \mathscr{A}^{op}$, by the same rule as F. The category of algebras for F (in $\mathscr{A}$) is dual to the category of coalgebras for F^{op} (in $\mathscr{A}^{op}$). Shortly,

$$(\mathsf{Alg}\,F)^{op} = \mathsf{Coalg}(F^{op}).$$

We present in Figure 1.1 a comparison between concepts and ideas in algebra and coalgebra. As this indicates, one reason why the coalgebraic concepts on the right of Figure 1.1 are interesting is that they are the structures used in the mathematics of *transition* and *observation*, as opposed to *operations*. Terminal coalgebras in this sense are like the most abstract collections of "transitions" or "observations". We know that this is very vague, and so we hope that the examples throughout this book will explain what we mean.

We also would like to mention other sources, such as Rutten [149], a highly recommendable general source on coalgebra (this is the source of Figure 1.1), Gumm's survey in [96], or Moss [137], which includes much conceptual discussion related to the set-theoretic topics. Last but not least, Jacobs' textbook [98] on coalgebra is a very valuable source.

1.5 The aim of this book

We present general conditions which guarantee the existence of an initial algebra or a terminal coalgebra. We are also interested in *representations* of terminal coalgebras. The reason for this is that the existence theorems themselves frequently are fairly abstract, and so concrete representations make the terminal coalgebras more intuitive. We also study a number of topics related to terminal coalgebras, e.g. interaction between initial algebras and terminal coalgebras (**??**). At this point, we want to mention the main categories and functors of interest in our study.

We begin with Set, the category of sets and functions. We are interested in the *polynomial functors* obtained from the identity functor and the constant functors by products and coproducts (including exponents $X \mapsto X^B$ for a fixed set B).

Another functor which we shall study is the *discrete probability measure functor* $\mathcal{D}$, where $\mathcal{D}X$ is the set of functions from X to $[0,1]$ which have the value 0 except on finitely many points and which sum to 1. $\mathcal{D}$ takes a function $f : X \to Y$ to the function $\mathcal{D}f : \mathcal{D}X \to \mathcal{D}Y$. For each $\mu \in \mathcal{D}X$, $\mathcal{D}f(\mu)$ is given by

$$\mathcal{D}f(\mu)(y) \;=\; \mu(f^{-1}(y)) \;=\; \sum_{x \in f^{-1}(y)} \mu(x).$$

We already mentioned the power-set functor $\mathscr{P}$; this gives the set of subsets of X. There are also a few refinements of $\mathscr{P}$, including $\mathscr{P}_{\mathsf{f}}$, the *finite power-set* functor.

Other categories include (posets and monotone maps), and CPO (complete partial orders and continuous maps). Further, we shall consider MS, the category of metric spaces with distances bounded by 1 and non-expanding maps, and also the full subcategory CMS of complete metric spaces with distances bounded by 1. Both of these categories have a “power-set-like” operation, and this will be of special interest. We are also intersected in the categories of many-sorted sets.

The structure of this book. We have tried to collect interesting results about terminal coalgebras (and initial algebras) scattered throughout the literature. We have found some results not quite complete and we completed them. For technically more involved proofs we usually indicate the idea of the proof, but otherwise we provide references to where proofs can be found.

2 Algebras and Coalgebras

As the title of our book suggests, we are mainly interested in *initial algebras* and *terminal coalgebras*. But to understand what these are, we must discuss the more general concepts of *algebras* and *coalgebras* first. This is the purpose of this chapter.

Contents This chapter has two halves: one devoted to algebras and the other to coalgebras. For algebras, we first discuss them in general, primaily using examples from the categories Set of sets, Set^S of sorted sets, and $\mathsf{CPO}_\perp$ of complete partial orders with a least element (see Example 2.1.6(2)). Then we turn to initial algebras. The main results are detailed constructions of initial algebras, mainly for polynomial endofunctors on these categories. Following this, we examine two concepts closely related to initial algebras, recursion and induction. Then we turn to the parallel topics for coalgebras: examples from the same categories, terminal coalgebras for polynomial functors, and finally corecursion and bisimulation.

In addition to the results which we have mentioned, the point of the chapter is also to present examples and intuitions. But the overall treatment in this chapter is in a sense preliminary: at numerous points we mention discussions later in the book which amplify what we do here.

Background needed for this chapter To read this chapter, you need to know a few of the most basic definitions and concepts from category theory. These include initial and terminal objects (denoted 0 and 1, respectively), products and coproducts, monomorphisms and epimorphisms. Very little else is required. Later chapters in the book require more, of course. And throughout the book, we are mainly writing to readers well-versed in the basics of category theory. The main categories in this chapter are Set and $\mathsf{CPO}_\perp$. It would be useful to have seen *signatures* and Σ-algebras, say as they appear in theoretical computer science or general algebra.

In general, in this book when we say "Recall X", we assume that you have a passing knowledge of X.

2.1 Algebras

To specify algebras, we must have an *underlying* or *base category*, say $\mathscr{A}$, and an endofunctor F, i.e. a functor $F : \mathscr{A} \to \mathscr{A}$. With this in mind, we introduce the concepts of *algebra* and *homomorphism*. They are due to Lambek [119]. Often the action of F on morphisms is quite obvious, so one surpresses it and states only its action on objects. For example, $FX = X \times X + A$ is given for objects X, and it is understood that for every morphism $f : X \to Y$ we have $Ff = f \times f + \mathsf{id}_A$.

Definition 2.1.1. An *algebra* for an endofunctor F (or an *F-algebra*) consists of an object A and a morphism $\alpha : FA \to A$. We sometimes call A the *carrier* and α the structure. So technically the algebra is a pair (A, α). But as usual we shorten this to A whenever we can. Usually we use upper-case letters for algebras and Greek letters for structures.

A *homomorphism* of F-algebras from (A, α) to (B, β) (or a *morphism* of F-algebras) is a morphism $h : A \to B$ of $\mathscr{A}$ such that the square below commutes:

$$\begin{tikzcd} FA \arrow[r, "\alpha"] \arrow[d, "Fh"'] & A \arrow[d, "h"] \\ FB \arrow[r, "\beta"'] & B \end{tikzcd} \tag{2.1}$$

It should be clear that if A, B, and C are algebras, and $g : A \to B$ and $h : B \to C$ are homomorphisms, then $h \cdot g : A \to C$ is an algebra homomorphism. That is, composition of homomorphisms works as in the base category $\mathscr{A}$. Similarly, the identity morphism from the base category is always an algebra homomorphism. This endows the collection of algebras for a fixed endofunctor with a category structure.

Notation 2.1.2. The category of F-algebras and homomorphisms is denoted by $\mathsf{Alg}\, F$. The base category $\mathscr{A}$ is suppressed in our notation.

Examples 2.1.3. Algebras in $\mathscr{A} = \mathsf{Set}$ for various endofunctors.

(1) Algebras for $FX = X + 1$. Recall that 1 denotes a terminal object, here a singleton set. The coproduct operation $+$ is disjoint union. An algebra consists of a set A and a function $\alpha : A + 1 \to A$. That is, on a set A (of data) a unary operation $\alpha_1 : A \to A$ is given together with an "initial datum": an element expressed by $\alpha_0 : 1 \to A$. Indeed, to specify $\alpha : A + 1 \to A$ is the same as to specify α_1 and α_0, the two components of α.

A homomorphism from (A, α) to (B, β) is a function $h : A \to B$ between the data sets which preserves the unary operations, i.e., $h \cdot \alpha_1 = \beta_1 \cdot h$, and the initial data, i.e., $h \cdot \alpha_0 = \beta_0$. Indeed, this is equivalent to the commutativity of the following square

$$\begin{tikzcd} A + 1 \arrow[r, "{[\alpha_1, \alpha_0]}"] \arrow[d, "h+\mathsf{id}"'] & A \arrow[d, "h"] \\ B + 1 \arrow[r, "{[\beta_1, \beta_0]}"'] & B \end{tikzcd}$$

(2) Analogously, algebras for $FX = X \times X + 1$ are sets (of data) with a binary operation and a constant. And homomorphisms are functions commuting with the operation and preserving the constant.

(3) For a given set Σ, the algebras for $FX = \Sigma \times X$ are sets A endowed with unary operations $\alpha_s : A \to A$ where s ranges through Σ. Indeed, this is the same as specifying

a function $\alpha : A \times \Sigma \to A$. Homomorphisms from (A, α) and (B, β) are the functions $h : A \to B$ preserving all the unary operations: $h \cdot \alpha_s = \beta_s \cdot h$ for every $s \in \Sigma$.

(4) The *power-set functor* is defined by $\mathscr{P}X = \{M : M \subseteq X\}$, and for $f : X \to Y$ and $M \subseteq X$, $\mathscr{P}f(M)$ is the image set $f[M]$. For this functor, an algebra consists of a set A and a function $\alpha : \mathscr{P}A \to A$. For example, joins in a complete lattice A form an algebra for $\mathscr{P}$.

A homomorphism from (A, α) to (B, β) is a function $h : A \to B$ such that for every set $M \subseteq A$ we have

$$h(\alpha(M)) = \beta(h[M]).$$

If α and β are both joins (of complete lattices), this states that h is join-preserving.

Example 2.1.4. Next, we generalize Example 2.1.3, parts (1)–(3). A *signature* is a collection $\Sigma = (\Sigma_n)_{n \in \mathbb{N}}$ of sets Σ_n, where Σ_n is called the set of *n-ary operation symbols*, in particular, the elements of Σ_0 are the *nullary* symbols, or *constants*.

A Σ-*algebra* is a set A together with interpretations of the operation symbols; a symbol σ of arity n is interpreted as a function $\sigma^A : A^n \to A$. Moreover, a homomorphism of Σ-algebras is a function $h\colon A \to B$ such that for all $\sigma \in \Sigma_n$ and all $x_0, \ldots, x_{n-1} \in A$,

$$h(\sigma^A(x_0, \ldots, x_{n-1})) = \sigma^B(h(x_0), \ldots, h(x_{n-1})). \tag{2.2}$$

This notion of a Σ-algebra is from general algebra.

We wish to recast (2.2) in the language of algebras for an endofunctor on Set. For this and for many other purposes, it is convenient to associate to the signature Σ the *polynomial functor*

$$H_\Sigma\colon \mathsf{Set} \to \mathsf{Set}$$

given by

$$H_\Sigma X = \coprod_{n \in \mathbb{N}} \Sigma_n \times X^n. \tag{2.3}$$

The operation symbols from the signature are "inscribed in" the functor. This functor assigns to a function $f : X \to Y$ the function

$$H_\Sigma f = \coprod_{n \in \mathbb{N}} \mathsf{id}_{\Sigma_n} \times \underbrace{f \times \cdots \times f}_{n \text{ times}}$$

The point is that Σ-algebras and homomorphisms as we have defined them in (2.2) are the same as algebras and homomorphisms for this functor H_Σ. Given a Σ-algebra A, we define $\alpha\colon H_\Sigma A \to A$ by coproduct components: on the summand $\Sigma_n \times A^n$ of $H_\Sigma A$ we let

$$\alpha(\sigma, (x_0, \ldots, x_{n-1})) = \sigma^A(x_0, \ldots, x_{n-1}). \tag{2.4}$$

In the other direction, given α, we obtain a Σ-algebra A in the following way. For every $\sigma \in \Sigma_n$ define σ^A using (2.4) in the right-to-left direction. This recovers algebras. And as for homomorphisms, $h : A \to B$ is a homomorphism of Σ-algebras iff the square below commutes:

$$\begin{tikzcd}
\coprod_{n\in\mathbb{N}} \Sigma_n \times A^n \arrow[r, "\alpha"] \arrow[d, "H_\Sigma h"'] & A \arrow[d, "h"] \\
\coprod_{n\in\mathbb{N}} \Sigma_n \times B^n \arrow[r, "\beta"'] & B
\end{tikzcd}$$

Again, given a signature Σ, the category of Σ-algebras is the same as the category of H_Σ-algebras.

Example 2.1.5. Many-sorted algebras. Fix a set S. An S-sorted set is just a family of sets $X = (X_s)_{s\in S}$ indexed by S. Given $X = (X_s)_{s\in S}$ and $Y = (Y_s)_{s\in S}$, an S-sorted function is a family $f = (f_s)_{s\in S}$ of functions, where $f_s : A_s \to B_s$. We thus work in the category

$$\mathsf{Set}^S$$

of S-sorted sets and S-sorted functions. An *S-sorted signature* is a set Σ of operation symbols together with arities $ar(s)$ of members s of Σ. An arity has the form $s_0 \times \ldots \times s_{n-1} \to s$ for an operation symbol whose n variables have sorts $s_0, \ldots, s_{n-1}$, respectively, and whose result has sort s.

Incidentally, in all of our discussions of many-sorted sets in this book, S can be arbitrary. Most of the time, taking $S = 2 = \{0,1\}$ illustrates our point: as categories, Set and Set^2 have enough differences to be interesting for us.

A many-sorted algebra is an algebra for a polynomial endofunctor of Set^S. If Σ consists of a single operation symbol of arity $s_0 \times \ldots \times s_{n-1} \to s$, then the corresponding polynomial functor H_Σ has empty sorts $(H_\Sigma X)_t$ for all $t \neq s$ and its sort s is

$$(H_\Sigma X)_s = X_{s_0} \times \ldots \times X_{s_{n-1}}$$

Its action on morphisms is analogous:

$$(H_\Sigma f)_s = f_{s_0} \times \ldots \times f_{s_{n-1}}$$

In the case of general S-sorted signatures the *polynomial functor* $H_\Sigma : \mathsf{Set}^S \to \mathsf{Set}^S$ is defined by

$$(H_\Sigma X)_s = \coprod_{\sigma\in\Sigma} X_{s_0} \times \ldots \times X_{s_{n-1}}$$

where the coproduct ranges over all $\sigma \in \Sigma$ with the result sort s and the arity $s_1 \times \ldots \times s_n \to s$. It is easy to verify that many-sorted Σ-algebras and homomorphisms in the above sense form precisely the category $\mathsf{Alg}\, H_\Sigma$.

Example 2.1.6. *Continuous algebras.* These are algebras with a complete partial order on them and where operations are continuous. A *complete partial order*[1], or *cpo* for short, is a poset $(A, \sqsubseteq)$ in which all ω-chains $a_0 \sqsubseteq a_1 \sqsubseteq a_2 \sqsubseteq \cdots$ have a join. A *continuous function* from a cpo A to a cpo B is a monotone function $f : A \to B$ preserving joins of

[1]This is often called an ω-complete partial order, in contrast to dcpos that we introduce in Example 5.1.2(3).

ω-chains. An operation $\sigma^A : A^n \to A$ is continous if it is a continuous function on the cpo A^n ordered componentwise. Continuous algebras can be represented as F-algebras over one of the following categories:

(1) The category CPO is the category of all cpos and all continuous maps between them. Products of cpos are formed as in Set, i.e., they are cartesian products with the coordinatewise order and joins. Similary, coproducts are as in Set the disjoint union with the order given by the order of the summands and with elements in different summands incomparable. Hence, a number of examples of F-algebras work analogously as for Set:

(a) The algebras for $FX = X + 1$ are cpos A together with a constant and a unary operation that is a continuous function on A.

(b) The algebra for $FX = X \times X + 1$ are cpos A with a constant and a binary operation $A \times A \to A$ that is a continuous function.

(c) For every cpo X we denote by $X_\perp$ its *lifting* obtained by adding a (new) least element $\perp$ to X. This defines an endofunctor $FX = X_\perp$ assigning to every continous function f its extension $f_\perp$ preserving the least element $\perp$. An algebra for this endofunctor is given by a cpo A, a continous unary operation $\alpha : A \to A$ and a constant $c_\perp \in A$ satisfying $c_\perp \sqsubseteq \alpha(x)$ for all $x \in A$.

(2) The picture is more interesting for the (non full sub-)category $\mathsf{CPO}_\perp$ of all cpos with a least element $\perp$, and all *strict* continuous maps (i.e. those preserving $\perp$). This subcategory is closed under products in CPO. But its coproduct are not Set-based: a coproduct in $\mathsf{CPO}_\perp$ is obtained from that in CPO by indentifying in the disjoint union the least elements of all summands to one (least) element. Thus, here $FX = X + 1$ is simply the identity functor on $\mathsf{CPO}_\perp$.

(a) Strict continuous algebras with one unary operation and one constant are the algebras for the functor $FX = X_\perp + 1_\perp$ forming the coproduct of the lifting of X and the 2-chain $1_\perp = \{\perp, *\}$. On morphisms F acts as expected: $Ff = f_\perp + \mathsf{id}_{1_\perp}$. To give an algebra $\alpha : A_\perp + 1_\perp \to A$ in $\mathsf{CPO}_\perp$ is equivalent to giving a cpo A with $\perp$, a constant $\alpha(*) \in A$ and a strict continuous unary operation $A \to A$.

(b) Analogously, algebras for the endofunctor on $\mathsf{CPO}_\perp$ defined by $FX = (X \times X)_\perp + 1_\perp$ are the strict continuous algebras with one binary operation and one constant.

General properties of the category $\mathsf{Alg}\, F$ The foregoing part of this chapter was devoted to examples of functors and algebras. We close this section with a discussion of a few general properties of $\mathsf{Alg}\, F$. These facts hold for all endofunctors on all categories.

Proposition 2.1.7. *If the base category $\mathscr{A}$ has finite products, then a product of algebras $\alpha : FA \to A$ and $\beta : FB \to B$ is formed on the level of $\mathscr{A}$. That is, there exists a unique algebra structure on $A \times B$*

$$\gamma : F(A \times B) \to A \times B$$

for which both projections (in $\mathscr{A}$) become homomorphisms. And $(A \times B, \gamma)$ is a product of the given algebras in $\mathsf{Alg}\, F$.

Proof. Let π_A and π_B denote the projections in $\mathscr{A}$. Given γ for which the following diagram

$$\begin{tikzcd}
& F(A\times B) \arrow[r, "\gamma"] \arrow[dl, "F\pi_A"'] \arrow[dr, "F\pi_B"] & A\times B \arrow[dl, "\pi_A"'] \arrow[dr, "\pi_B"] & \\
FA \arrow[r, "\alpha"'] & A & FB \arrow[r, "\beta"'] & B
\end{tikzcd}$$

commutes, it follows that

$$\gamma = \langle \alpha \cdot F\pi_A, \beta \cdot F_{\pi_B} \rangle.$$

So the uniqueness of γ is clear. Now define γ by the last equality. Then π_A and π_B are both homomorphisms. Consider an algebra $\delta : FD \to D$ and homomorphisms $h_A : D \to A$ and $h_B : D \to B$. Let the $\mathscr{A}$-morphism $h : D \to A \times B$ be $\langle h_A, h_B \rangle$; we check that h is a morphism in $\mathsf{Alg}\, F$ as well. That is, the square below commutes:

$$\begin{tikzcd}
FD \arrow[r, "\delta"] \arrow[d, "Fh"'] & D \arrow[d, "h"] \\
F(A\times B) \arrow[r, "\gamma"'] & A\times B
\end{tikzcd}$$

Indeed, π_A and π_B are collectively monic, thus, we only need to prove that the square above commutes when post-composed with π_A (and π_B – this follows by symmetry). We have

$$\pi_A \cdot (h \cdot d) = h_A \cdot d = a \cdot Fh_A \quad \text{and} \quad \pi_A \cdot (c \cdot Fh) = a \cdot F\pi_A \cdot Fh = a \cdot Fh_A.$$

To complete the proof, we show that h is the unique algebra morphism. Suppose that in $\mathsf{Alg}\, F$, a morphism $h' : D \to A \times B$ satisfies $\pi_A \cdot h' = h_A$ and $\pi_B \cdot h' = h_B$. Then these all hold in the base category, so h' must be $\langle h_A, h_B \rangle = h$. This completes the proof. $\square$

Remark 2.1.8. Similarly terminal objects, pullbacks, in fact, all limits are formed on the level of $\mathscr{A}$. This is discussed in detail in Section 4.1 (see Remark 4.1.2).

Corollary 2.1.9. *If $\mathscr{A}$ has a terminal object 1, then the terminal algebra is $F1 \to 1$.*

So there is not much to say about terminal algebras. Initial algebras, on the contrary, are really interesting.

Corollary 2.1.10. *A homomorphism $h : (A, \alpha) \to (B, \beta)$ is monic in $\mathsf{Alg}\, F$ iff h is monic in $\mathscr{A}$.*

Indeed, recall that h is monic iff the following commutative square is a pullback

$$\begin{tikzcd}
A \arrow[r, "\mathsf{id}"] \arrow[d, "\mathsf{id}"'] & A \arrow[d, "h"] \\
A \arrow[r, "h"'] & B
\end{tikzcd}$$

and use that pullbacks of F-algebras are formed on the level of $\mathscr{A}$.

Remark 2.1.11. For epimorphisms, the situation is not as simple. However, every algebra homomorphism $h : (A, \alpha) \to (B, \beta)$ with h epic in $\mathscr{A}$ is epic in $\mathsf{Alg}\, F$.

Remark 2.1.12. We assume that the reader is familiar with the general notion of a *subobject*. For an object A, a *subobject* is represented by a monomorphism $m_B : B \to A$. If $m_B : B \to A$ and $m_C : C \to A$ are monomorphisms, we write

$$m_B \leq m_C$$

if m_B factorizes through m_C. (That is, for some $k : B \to C$, $m_B = m_C \cdot k$.) Whenever $m_B \leq m_C \leq m_B$, then k above is an isomorphism. In this case, we identify m_B and m_C as subobject representatives. Thus, we pass to the poset of equivalence classes. The category $\mathscr{A}$ is called *wellpowered* if for every object A, only a set of subobjects exist.

Subalgebras of an algebra are understood to be subobjects in the category $\mathsf{Alg}\, F$ of algebras for F.

2.2 Initial algebras

We have motivated initial algebras in Section 1.1. This section presents our first examples, mainly using endofunctors on Set and $\mathsf{CPO}_\perp$. In addition, we present some basic material on initial algebras.

Definition 2.2.1. An algebra for F is *initial* if it admits a unique homomorphism into every F-algebra. Shortly: it is the inital object of $\mathsf{Alg}\, F$.

Example 2.2.2. If F preserves the initial object 0 of $\mathscr{A}$, then $(0, \mathsf{id})$ is the initial algebra.

Notation 2.2.3. μF or $\mu X.FX$ is the usual notation for the (underlying object of) the initial algebra. The algebra structure is often denoted by

$$\iota : F(\mu F) \to \mu F.$$

Examples will follow soon. But even before presenting those, we would like to mention an important result.

Definition 2.2.4. By a *fixed point* of an endofunctor F is meant an object A together with an isomorphism $FA \xrightarrow{\cong} A$.

Lambek's Lemma 2.2.5 [119]**.** *If an initial F-algebra exists, then it is a fixed point of F.*

Proof. Let (A, α) be initial. Since $(FA, F\alpha)$ is an algebra, we have a homomorphism h:

$$\begin{tikzcd}
FA \arrow[r, "\alpha"] \arrow[d, "Fh"'] & A \arrow[d, "h"] \\
FFA \arrow[r, "F\alpha"] \arrow[d, "Fa"'] & FA \arrow[d, "\alpha"] \\
FA \arrow[r, "\alpha"'] & A
\end{tikzcd}$$

The square at the bottom obviously commutes. Consequently $\alpha \cdot h$ is an endomorphism of (A, α) in $\mathsf{Alg}\, F$. Thus $\alpha \cdot h = \mathsf{id}_A$ by initiality. Then $h \cdot \alpha = F\alpha \cdot Fh = \mathsf{id}_{FA}$. Thus $h = \alpha^{-1}$. □

For endofunctors on Set there is a converse:

Theorem 2.2.6 [162, Thm. II.4]**.** *A set functor has an initial algebra iff it has a fixed point.*

We shall discuss this result in more detail and greater generality in Chapter 6 (see Theorem 6.1.22).

Example 2.2.7. (1) The power-set functor $\mathscr{P}\colon \mathsf{Set} \to \mathsf{Set}$ does not have an initial algebra: Cantor's Theorem tells us that for all sets A, there is no map of A onto $\mathscr{P}A$. For the proof, suppose towards a contradiction that $f : A \to \mathscr{P}A$ is a surjection. Let $X = \{s \in S : s \notin f(s)\}$. Since f is surjective, let s_0 be such that $f(s_0) = X$. Then we have a contradiction: $s_0 \in X$ iff $s_0 \in f(s_0)$ iff $s_0 \notin X$. Therefore, there exists no fixed point of $\mathscr{P}$.

(2) Let $(P, \leq)$ be a poset, considered as a category. An endofunctor $F : P \to P$ is just an order-preserving function, and a fixed point is an element $p \in P$ with $F(p) = p$ as usual. An algebra is an element $p \in P$ such that $Fp \leq p$. Lambek's Lemma then says that if p is the least algebra, then $p = Fp$.

(3) The initial algebra of the set functor $FX = X + 1$ is the algebra of natural numbers:

$$\mu X.X + 1 = \mathbb{N}.$$

The algebra structure $a = [a_1, a_0]$, compare Example 2.1.3(1), consists of the unary operation a_1 which is the successor function $s : \mathbb{N} \to \mathbb{N}$, and the constant $a_0 = 0$. Shortly: $\iota = [s, 0]$. Indeed, given an algebra $[\beta_1, \beta_0] : B + 1 \to B$ the unique homomorphism $h : \mathbb{N} \to B$ is determined by

$$\begin{aligned} h(0) &= \beta_0 \\ h(sn) &= \beta_1(h(n)) && \text{for all } n \in \mathbb{N}. \end{aligned}$$

See Example 2.2.9 for a generalization of this example to general base categories.

(4) Let $\mathscr{P}_f$, *the finite power-set functor*, be the subfunctor of $\mathscr{P}$ from Example 2.1.3(4) given by

$$\mathscr{P}_f A = \{M \subseteq A : M \text{ is finite}\}.$$

The initial algebra $\mu\mathscr{P}_f$ can be described as the set of all *hereditarily finite sets.* These are finite sets all elements of which are hereditarily finite sets again.

In concrete terms,

$$V_\omega = \emptyset \cup \mathscr{P}\emptyset \cup \mathscr{P}\mathscr{P}\emptyset \cup \cdots \cup \mathscr{P}^n\emptyset \cup \cdots .$$

Each hereditarily finite set is a finite subset of V_ω, and conversely. Thus $\mathscr{P}_f V_\omega = V_\omega$. The proof that (V_ω, id) is initial for $\mathscr{P}_f$ can be found in Example 3.2.9.

Remark 2.2.8. Although we use notation μF (and speak about *the* initial algebra), initial algebras are only unique up to isomorphism. Indeed, given an isomorphism $i : A \to \mu F$ in the base category, then A with the algebra structure

$$\alpha = (FA \xrightarrow{Fi} F(\mu F) \xrightarrow{\iota} \mu F \xrightarrow{i^{-1}} A)$$

is also initial. This follows from $i : (A, \alpha) \to (\mu F, \iota)$ being an isomorphism in $\mathsf{Alg}\, F$.

Nevertheless, we shall always speak of *the* initial algebra for an endofunctor (whenever it exists).

Frequently, there are different ways to present an initial algebra; we shall see some examples in this chapter.

Example 2.2.9. We now present an example that applies to all of the base categories which we have seen so far in this chapter: Set, Set^S, and CPO. In fact, it applies to every base category that has countable copowers

$$\mathbb{N} \bullet A = A + A + A \dots$$

Then for every object A the endofunctor $FX = X + A$ has the initial algebra

$$\mu X.X + A = \mathbb{N} \bullet A.$$

If $\mathsf{in}_k : A \to \mathbb{N} \bullet A$ are the coproduct injections for $k \in \mathbb{N}$, then the algebra structure is

$$\iota = [\alpha_1, \mathsf{in}_0] : (\mathbb{N} \bullet A) + A \to \mathbb{N} \bullet A.$$

where α_1 is determined by the following commutative triangles

$$\begin{tikzcd} \mathbb{N} \bullet A \arrow[rr, "\alpha_1"] & & \mathbb{N} \bullet A \\ & \mathbb{N} \arrow[ul, "\mathsf{in}_k"] \arrow[ur, "\mathsf{in}_{k+1}"'] & \end{tikzcd} \qquad (k \in \mathbb{N})$$

Indeed, let $[\beta_1, \beta_0] : B + A \to B$ be an algebra. Then a homomorphism h:

$$\begin{tikzcd} \mathbb{N} \bullet A + A \arrow[r, "{[\alpha_1,\alpha_0]}"] \arrow[d, "h+\mathsf{id}"'] & \mathbb{N} \bullet A \arrow[d, "h"] \\ B + A \arrow[r, "{[\beta_1,\beta_0]}"'] & B \end{tikzcd}$$

is uniquely determined by $h \cdot \mathsf{in}_0$ and $h \cdot \mathsf{in}_{k+1} = \beta_1 \cdot (h \cdot \mathsf{in}_k)$: we have

$$h = [\beta_0, \beta_1\beta_0, \beta_1\beta_1\beta_0, \dots].$$

Remark 2.2.10. (1) In the next example, and at many later points, we use trees to describe algebras of special interest. Let us recall that a *tree* is a directed graph with a distinguished node called the *root* from which every other node can be reached by a unique directed path. A tree might well be infinite. We always identify isomorphic trees.

(2) We distinguish between *unordered trees*, defined as above, and *ordered trees*. An ordered tree comes with a linear order on the children of every node. In pictures, this linear order is the left-to-right order. In this chapter, all the trees will be ordered. But later in the book we shall also consider unordered trees.

A node is a *leaf* if it has no children. A tree is *binary* if every node which is not a leaf has precisely two children. The *depth* of a node of a tree is its distance from the root.

(3) A *labelled tree* (ordered or unordered) is a tree together with a function assigning to every node an element of a given set M (of labels). We consider also labelled trees up to (label-preserving) isomorphism.

(4) Let us also recall that the *complete n-ary tree* (in which every node has precisely n children) can be encoded as A^*, the set of words over $A = \{a_0, \dots, a_{n-1}\}$, for any n-element set A. The root is ε, the empty word. And the n children of a word w are the words $wa_0, \dots, wa_{n-1}$.

Example 2.2.11. Initial Σ-algebra. It is well known from general algebra that μH_Σ, the initial algebra of signature Σ, can be described as follows:

$$\mu H_\Sigma = \text{all ground terms, i.e., terms without variables.}$$

This is the smallest set such that

(1) every constant symbol is a term, i.e., $\Sigma_0 \subseteq \mu H_\Sigma$, and

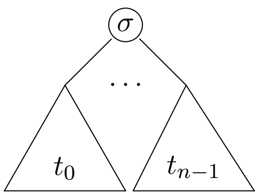
(2) given an n-ary symbol $\sigma \in \Sigma_n$ and n terms $t_0, \dots, t_{n-1}$, then $\sigma(t_0, \dots, t_{n-1})$ is a term.

The operations of the algebra μH_Σ are the obvious ones.

Given a Σ-algebra A, every term computes in A to a (unique) element, and the resulting mapping from μH_Σ to A is the unique homomorphism from μH_Σ to A.

We now provide an alternative description of μH_Σ using trees.

Definition 2.2.12. Given a signature Σ, by a *Σ-tree* is meant an ordered tree labelled in $\coprod_{n\in\mathbb{N}} \Sigma_n$ in such a way that every node of k children is labelled by a k-ary symbol. Every n-ary symbol $\sigma \in \Sigma_n$ defines an n-ary operation of *tree-tupling* on the set of all Σ-trees: it sends an n-tuple $t_0, \cdots, t_{n-1}$ to the following Σ-tree



Remark 2.2.13. In most of the classical literature (e.g.Courcelle's paper [63]), Σ-trees are described in the following equivalent way: a Σ-tree is a parial function $t : \mathbb{N}^* \to \coprod_{n\in\mathbb{N}} \Sigma_n$ such that

(1) the domain of definition of t is nonempty and *prefix-closed*, i.e. whenever a word wi lies in the domain, so does w (this entails that ε lies in the domain), and

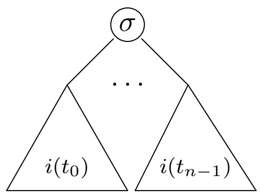
(2) whenever $t(w) \in \Sigma_n$ then $t(wi)$ is defined if and only if $i = 0, \dots, n-1$.

We will use this description of Σ-trees in the proof of Theorem 2.5.9. It is equivalent to the above one: given a partial function t as above, the nodes of the corresponding Σ-tree are the elements of the domain of definition of t, ε is the root, the label of a node w is $t(w)$, and the children of w with an n-ary label are $w0, \dots, w(n-1)$.

Proposition 2.2.14. *μH_Σ is the algebra of all finite Σ-trees.*

Proof. It is sufficient to find an isomorphism i between the above algebras of terms and of finite Σ-trees in $\mathsf{Alg}\, H_\Sigma$. It is defined by structural induction on terms as follows: for all $\sigma \in \Sigma_0$ let $i(\sigma)$ be the single-node tree labelled by σ:

and for all $\sigma \in \Sigma_n$, $n > 0$, use tree-tupling: the term $\sigma(t_0, \dots, t_{n-1})$ is mapped to the following Σ-tree

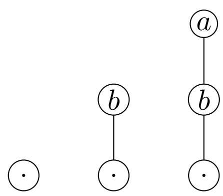


This is a bijection: the inverse function i^{-1} assigns to a tree t with root labelled by $\sigma \in \Sigma_n$ the term $\sigma(t_0, \dots, t_{n-1})$, where t_k is the term such that $i(t_k)$ is the k'th maximum subtree of t.

It is clear that i preserves the operations. $\square$

Example 2.2.15. (1) $\mu X.X \times X + 1$ is the algebra of all finite binary trees. (Labels are not needed since for every arity there is at most one operation symbol.)

(2) For a set B, the set functor $FX = B \times X + 1$ is the polynomial functor of the signature Σ with $\Sigma_1 = B$ and $\Sigma_0 = \{\cdot\}$. That is, the elements of B appear as unary operation symbols, and we have a single nullary symbol, here written as a dot $\cdot$ symbol. Here are three examples of finite Σ-trees:



We thus conclude that

$$\mu X.B \times X + 1 = B^*$$

Indeed, the set of all Σ-trees for this signature is isomorphic to the set B^* of all words over B.

Example 2.2.16. For many-sorted algebras, see Example 2.1.5, we also get

$$\mu H_\Sigma = \text{all ground terms.}$$

The sorted set of all ground terms is defined as the set $T = (T_s)_{s\in S}$ where each T_s (terms of output-sort s) is the smallest set such that

(1) every constant of sort s is a term in T_s, and

(2) given a symbol σ of arity $s_0 \times \ldots \times s_{k-1} \to s$ and k terms $t_0 \in T_{s_0}, \ldots, t_{k-1} \in T_{s_{k-1}}$ then $\sigma(t_0, \ldots, t_{k-1})$ is a term in T_s.

Again, we can instead work with Σ-*trees* which are ordered trees labelled in $S \times \Sigma$, where the label (s, σ) means that the node has sort s and corresponds to the opertation σ. This labelling is such that every node labelled by (s, σ) with a symbol σ of arity $s_1 \times \ldots \times s_n \to s$ has n children, and the i-th child has sort s_i for all $i = 0, \ldots, n-1$. The *sort* of a Σ-tree is defined to be the sort in its root label. We have the following description of μH_Σ: for every sort s,

$$(\mu H_\Sigma)_s = \text{all finite } \Sigma\text{-trees of output sort } s.$$

The proof is completely analogous to that of Proposition 2.2.14.

Example 2.2.17. Let us consider initial algebras in $\mathsf{CPO}_\perp$ (see Example 2.1.6). Recall that coproducts are disjoint unions with bottom elements smashed to one.

(1) The initial algebra for $FX = X_\perp$ is the algebra

$$\mu X.X_\perp = \mathbb{N}^\top$$

of natural numbers extended by a largest element ∞. The unary operation is the successor function $s : \mathbb{N}^\top \to \mathbb{N}^\top$ where we put $s(\infty) = \infty$.

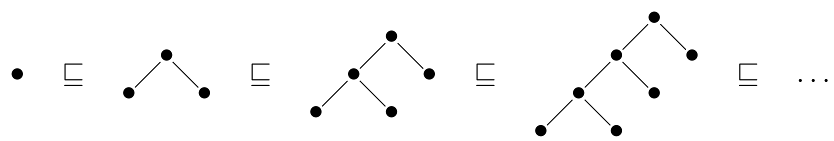
Indeed, given an algebra $\alpha : A_\perp \to A$, the unique homomorphism $h : \mathbb{N}^\top \to A_\perp$ fulfils $h(0) = \perp$ (since h is strict) and $h(sn) = \alpha(h(n))$. This determines h on $\mathbb{N}$, and continuity determines the value

$$h(\infty) = h(\bigsqcup_{n\in\mathbb{N}} n) = \bigsqcup_{n\in\mathbb{N}} h(n).$$

(2) For the functor $FX = (X \times X)_\perp + 1_\perp$ of one binary continuous operation and one constant we have

$$\mu X.(X \times X)_\perp + 1_\perp = \text{all binary trees}$$

(finite and infinite!). The constant is the root-only tree. The binary operation is tree-tupling. And the ordering is the least one for which the root-only tree is the smallest element, and tree-tupling is continuous. Example:



The above is an ω-chain whose join is an (obvious) infinite tree. This explains the presence of infinite trees in μH_Σ. Indeed, eevry infinite binary tree t has the form $t = \bigsqcup_{n\in\mathbb{N}} t_n$, where t_n is the finite tree obtained by cutting t at the level n. Clearly, $t_0 \sqsubseteq t_1 \sqsubseteq t_2 \cdots$.

Initial algebras and free algebras

Remark 2.2.18. Initial algebras are closely related to free algebras. By a *free F-algebra* on an object A (of generators) in $\mathscr{A}$ is meant an algebra

$$\varphi_A : FA^\sharp \to A^\sharp$$

together with a universal arrow $\eta_A : A \to A^\sharp$. Universality means that for every algebra $\beta : FB \to B$ and every morphism $f : A \to B$ in $\mathscr{A}$, there exists a unique homomorphism $\bar{f} : A^\sharp \to B$ extending f, i.e. a unique morphism of $\mathscr{A}$ for which the diagram below commutes:

$$\begin{tikzcd} FA^\sharp \arrow[r, "\varphi_A"] \arrow[d, "F\bar{f}"'] & A^\sharp \arrow[d, "\bar{f}"] & A \arrow[l, "\eta_A"'] \arrow[dl, "f"] \\ FB \arrow[r, "\beta"'] & B & \end{tikzcd} \tag{2.5}$$

In the case where $\mathscr{A}$ has binary coproducts, we can reduce (2.5) to a square as follows:

$$\begin{tikzcd} FA^\sharp + A \arrow[r, "{[\varphi_A,\eta_A]}"] \arrow[d, "F\bar{f}+\mathsf{id}_A"'] & A^\sharp \arrow[d, "\bar{f}"] \\ FB + A \arrow[r, "{[\beta,f]}"'] & B \end{tikzcd} \tag{2.6}$$

Note that for an endofunctor F and fixed object A, we get a new functor $F(-) + A$ defined in the obvious way.

Proposition 2.2.19. *Let $\mathscr{A}$ be a category with finite coproducts. For every endofunctor F, the free F-algebra on A is precisely the initial algebra of $F(-) + A$.*

That is, if $A^\sharp$ is free, then $\mu X.FX + A = A^\sharp$ with the algebra structure $[\varphi_A, \eta_A]$, and vice versa.

Proof. (1) Let $A^\sharp$ together with φ_A and η_A be a free algebra. Given an algebra B for $F(-) + A$, its algebra structure has the form

$$[\beta, f] : FB + A \to B$$

for a unique pair of morphisms as in (2.6) above. And $\bar{f}$ above is the unique homomorphism of algebras for $F(-) + A$.

(2) Let $A^\sharp$ be an initial algebra for $F(-) + A$, and let $\varphi_A : FA^\sharp \to A^\sharp$ and $\eta_A : A \to A^\sharp$ denote the components of its algebra structure. This is a free F-algebra on A. Indeed, given an F-algebra $\beta : FB \to B$ and a morphism $f : B \to A$, we get a unique homomorphism $\bar{f}$ of algebras for $F(-) + A$ from $A^\sharp$ to B. That is, a unique morphism for which the above square commutes. But this is equivalent to the commutativity of the diagram (2.5). Hence, $(A^\sharp, \varphi_A)$ is a free F-algebra on A.

This concludes the proof. □

Examples 2.2.20. (1) The identity functor on Set has free algebras

$$A^\sharp = \mathbb{N} \times A$$

with the algebra structure $\varphi_A : \mathbb{N} \times A \to \mathbb{N} \times A$ given by $\varphi(n,a) = (n+1,a)$ and the universal morphism $\eta_A : A \to \mathbb{N} \times A$ taking a to $(0,a)$.

More generally, if $\mathscr{A}$ is a category with countable coproducts, then a free algebra for Id on an object A is

$$A^\sharp = \mathbb{N} \bullet A,$$

see Example 2.2.9.

(2) Let Σ be a (one-sorted) signature. A free Σ-algebra (in Set) on a set A is precisely an initial Σ_A-algebra, where Σ_A is the signature Σ expanded by nullary operations indexed by A. Indeed, from the formula $H_\Sigma X = \coprod_{n\in\mathbb{N}} \Sigma_n \times X^n$, we see that $H_\Sigma(-) + A = H_{\Sigma_A}$.

We can describe $A^\sharp$ as the algebra of all finite Σ_A-trees. That is, leaves are labelled either by nullary symbols of Σ or by elements of A.

(3) Analogously for S-sorted signatures: the free S-sorted Σ-algebra on a set $A = (A_s)_{s\in S}$ is the algebra of all finite Σ_A-trees. Here Σ_A is the signature Σ with additional nullary operations of sorts s indexed by A_s for every sort s.

Definition 2.2.21. An endofunctor F is called a *varietor* if every object generates a free algebra for F.

Thus for $\mathscr{A} = \mathsf{Set}$ we have seen that H_Σ is a varietor and $\mathscr{P}$ is not. A characterization of varietors on Set will be presented in Corollary 6.1.31 on page 142.

Remark 2.2.22. A functor F is a varietor iff the forgetful functor from $\mathsf{Alg}\, F$ to $\mathscr{A}$ has a left adjoint $A \mapsto A^\sharp$. This defines a monad (M,η,μ) on $\mathscr{A}$ by

$$MA = A^\sharp,$$

For a morphism $f : A \to B$, we have a unique homomorphism of F-algebras $Mf : A^\sharp \to B^\sharp$ with $Mf \cdot \eta_A = \eta_B \cdot f$. The unit $\eta : \mathsf{Id} \to M$ has components the universal arrows $\eta_A : A \to A^\sharp$, and the monad multiplication $\mu : MM \to M$ has as its components the unique F-algebra homomorphism

$$\mu_A : (A^\sharp)^\sharp \to A^\sharp \qquad \text{with} \qquad \mu_A \cdot \eta_{A^\sharp} = \mathsf{id}_{A^\sharp}.$$

This monad is free on F with respect to the universal natural transformation $F \to M$ with components

$$FA \xrightarrow{F\eta_A} FA^\sharp \xrightarrow{\varphi_A} A^\sharp.$$

This was proved by Barr [48]. He also proved the converse which was later strengthened by Kelly [106]:

Theorem 2.2.23. *Given a complete category, an endofunctor F generates a free monad iff it is a varietor.*

Examples 2.2.24. (1) If $\mathscr{A}$ has countable coproducts, then the free monad on Id is $MA = \mathbb{N} \bullet A$; cf. see Example 2.2.9.

(2) The free monad on the set functor $FX = X \times X$ is given by

$$MA = \text{ finite binary ordered trees with leaves labelled in } A.$$

Indeed, the functor $F(-) + A$ is the polynomial functor of the signature of one binary operation and constants indexed by A.

2.3 Recursion and induction

The most basic form of *recursion* concerns functions on natural numbers: we specify a function f by specifying its value at 0, and for all n, deriving the value $f(n+1)$ from $f(n)$. As we have seen, the natural numbers form the initial algebra of $FX = X + 1$, see Example 2.2.7(2). This is a special case of the following

Definition 2.3.1. Let F be an endofunctor with an initial algebra. Given an object A, we say that a morphism $f : \mu F \to A$ is *recursively specified* if there exists an algebra structure $\alpha : FA \to A$ (a specification of f) turning f into a homomorphism.

Example 2.3.2. (1) The depth of a finite tree. Here we work with finite binary trees as an initial algebra

$$T \quad = \quad \mu X.X \times X + 1 \ ,$$

see Example 2.2.15. The function $\mathsf{depth} : T_\Sigma \to \mathbb{N}$ assigns 0 to the root-only tree, and given a tree t with maximum subtrees t_1, t_2, then

$$\mathsf{depth}(t) = 1 + \max(\mathsf{depth}(t_1), \mathsf{depth}(t_2)).$$

In order to specify depth recursively, we need an algebra structure on $\mathbb{N}$ of the form $[\alpha_1, \alpha_0] : \mathbb{N} \times \mathbb{N} + 1 \to \mathbb{N}$. The obvious candidate is

$$\alpha_1(n, m) = 1 + \max(n, m) \qquad \text{and} \qquad \alpha_0 = 0.$$

Indeed, the square below (where ι_1 is the tree-tupling and ι_0 represents the root-only tree)

$$\begin{tikzcd}
T \times T + 1 \arrow[r, "{[\iota_1,\iota_0]}"] \arrow[d, "\mathsf{depth}\times\mathsf{depth}+\mathsf{id}"'] & T \arrow[d, "\mathsf{depth}"] \\
\mathbb{N} \times \mathbb{N} + 1 \arrow[r, "{[\alpha_1,\alpha_0]}"'] & \mathbb{N}
\end{tikzcd}$$

clearly commutes. It is easy to see by structural induction that depth is indeed the unique homomorphism from T to $\mathbb{N}$.

(2) The same example now played with potentially infinite trees: put $\mathscr{A} = \mathsf{CPO}_\perp$ and $FX = (X \times X)_\perp + 1_\perp$, the functor with the initial algebra

$$\bar{T} = \text{(finite and infinite) binary trees;}$$

see Example 2.2.17(2). We want to recursively specify the function

$$\mathsf{depth} : \bar{T} \to \mathbb{N}^\top$$

assigning ∞ to every infinite tree. This requires an algebra structure for F (in $\mathsf{CPO}_\perp$) on the set $\mathbb{N}^\top$. Analogously to the preceding example we put

$$[\overline{\alpha}_1, \overline{\alpha}_0] : \mathbb{N}^\top \times \mathbb{N}^\top + 1 \to \mathbb{N}^\top$$

where $\overline{\alpha}_1(n,m) = 1 + \max(n,m)$, which is understood as expected ($1+\infty = \infty$ and $\max(n,\infty) = \infty$) and $\overline{\alpha}_0 = 0$.

The initial algebra $\mathbb{N} \bullet A$ of $FX = X + A$ of Example 2.2.9 has the following property stronger than initiality:

Proposition 2.3.3 (Primitive Recursion)**.** *The initial algebra $\bar{A} = \mathbb{N} \bullet A$ for $FX = X + A$ has the following property: given an object B and morphisms*

$$\bar{A} \times B \xrightarrow{g} B \xleftarrow{g_0} A$$

there exists a unique morphism $h : \bar{A} \to B$ such that the square below commutes:

$$\begin{tikzcd}
\bar{A} + A \arrow[r, "\iota"] \arrow[d, "\langle \mathsf{id}, h\rangle + \mathsf{id}"'] & \bar{A} \arrow[d, "h"] \\
\bar{A} \times B + A \arrow[r, "{[g, g_0]}"'] & B
\end{tikzcd}$$

Proof. The coproduct injections $i_n : A \to \bar{A}$ define the components $h_n = h \cdot i_n : A \to B$. Since $\iota_0 = i_0$, the right-hand component of the above square (with domain A) tells us that

$$h_0 = g_0.$$

Since $\iota \cdot i_n = i_{n+1}$, the left-hand component tells us that

$$\begin{aligned} h_{n+1} &= g \cdot \langle \mathsf{id}_{\bar{A}}, h\rangle \cdot i_n \\ &= g \cdot \langle i_n, h_n \rangle \end{aligned}$$

We see that h is uniquely determined by these rules. $\square$

Example 2.3.4. The classical formulation of primitive recursion for functions $h : \mathbb{N}^k \to \mathbb{N}$ on k variables is a special case: choose $\mathscr{A} = \mathsf{Set}$ and $A = 1$, thus, $\bar{A} = \mathbb{N}$. Indeed:

(1) For $k = 1$ put $B = \mathbb{N}$. The ingredients of primitive recursion are a function g of two variables and a number g_0. This gives the desired morphism $[g, g_0] : \mathbb{N} \times \mathbb{N} + 1 \to \mathbb{N}$. The above square commutes iff h is defined by

$$h(0) = g_0 \qquad \text{and} \qquad h(n+1) = g(n, h(n)).$$

(2) For $k = 2$ we choose $B = [\mathbb{N}, \mathbb{N}]$, the set of all endofunctions of $\mathbb{N}$. And we look for h in the curried form

$$\mathsf{curry}\, h : \mathbb{N} \to [\mathbb{N}, \mathbb{N}], \qquad n \mapsto h(-, n).$$

The ingredients: a function $\bar{g}$ on three variables and a function $\bar{g}_0$ of one variable. Define

$$g : \mathbb{N} \times [\mathbb{N}, \mathbb{N}] \to [\mathbb{N}, \mathbb{N}]$$

by setting for all $n \in \mathbb{N}$ and $u : \mathbb{N} \to \mathbb{N}$

$$g(n, u) = \bar{g}(n, -, u(n)) : \mathbb{N} \to \mathbb{N}.$$

This together with $g_0 = \mathsf{curry}\, \bar{g}_0 : 1 \to [\mathbb{N}, \mathbb{N}]$ yields the desired morphism $[g, g_0] : \mathbb{N} \times [\mathbb{N}, \mathbb{N}] + 1 \to [\mathbb{N}, \mathbb{N}]$.

The following square

$$\begin{tikzcd}
\mathbb{N} + 1 \arrow[r, "\iota"] \arrow[d, "\langle \mathsf{id}, \mathsf{curry} \rangle + \mathsf{id}"'] & \mathbb{N} \arrow[d, "\mathsf{curry}\, h"] \\
\mathbb{N} \times [\mathbb{N}, \mathbb{N}] + 1 \arrow[r, "{[g, g_0]}"'] & {[\mathbb{N}, \mathbb{N}]}
\end{tikzcd}$$

commutes iff for all m, m we have

$$h(m, 0) = g_0(m) \qquad \text{and} \qquad h(m, n+1) = g(n, m, h(m, n)),$$

as required.

Analogously for $k = 3, 4, \ldots$.

Remark 2.3.5. A classical induction proof takes a subset A of $\mathbb{N} = \mu X.X + 1$, and if $0 \in A$ and $n \in A \Rightarrow sn \in A$, then it concludes $A = \mathbb{N}$. In other words, if A is a subalgebra $(\mathbb{N}, \iota)$, where $\iota = [s, 0]$, then $A = \mathbb{N}$.

We now formulate the corresponding principle for F-algebras. Recall the concept of subalgebra from Remark 2.1.12.

A *proper subalgebra* is one represented by a monomorphism which is not invertible.

Lemma 2.3.6 (Induction Principle [121])**.** *If F is an endofunctor with an initial algebra, then μF has no proper subalgebra.*

Proof. Consider a subalgebra represented by a monomorphism m:

$$\begin{tikzcd}
FB \arrow[r, "\beta"] \arrow[d, "Fm"'] & B \arrow[d, "m"] \\
F(\mu F) \arrow[r, "\iota"'] & \mu F
\end{tikzcd}$$

We have a unique homomorphism $h : (\mu F, \iota) \to (B, \beta)$. Then $m \cdot h$ is an endomorphism of the initial object $\mathsf{Alg}\, F$. But an initial object has only one endomorphism, the identity. Therefore $m \cdot h = \mathsf{id}$. Since m is monic, $m = h^{-1}$. $\square$

In Set, here is how this is used. To show that a given subset $A \subseteq \mu F$ is equal to μF, we need only show that A is a subalgebra of μF.

Examples 2.3.7. (1) The case $\mathbb{N} = \mu X.X + 1$ is the most basic form of mathematical induction. To show that $A \subseteq \mathbb{N}$ is all of $\mathbb{N}$, one shows that $0 \in A$ and that A is closed under the successor function.

(2) For the set $T = \mu X.X \times X + 1$ of all finite binary trees (cf. Example 2.2.15) the "tree-induction principle" states that to prove that a set $A \subseteq T$ contains all trees, one needs only to verify that

(a) A contains the root-only tree, and

(b) with every pair A contains its tree tupling.

(3) For $\Sigma^* = \mu X.X \times \Sigma + 1$ (cf. Example 2.2.15), a set $A \subseteq X^*$ contains all words provided that

(a) A contains ε, and

(b) with every word w it contains all $sw, s \in \Sigma$.

2.4 Coalgebras

The concept of a coalgebra for an endofunctor F is formally just the dual of an algebra: it consists of an object A and a morphism $\alpha : A \to FA$. A number of interesting types of systems can be formalized as coalgebras, thinking of the object A as the collection of states and the structure α as the dynamics of the system.

Definition 2.4.1. A *coalgebra* for an endofunctor F (or just F-coalgebra) consists of an object A and a morphism $\alpha : A \to FA$.

Given another coalgebra $\beta : B \to FB$, a *homomorphism* (or a morphism of F-coalgebras) is a morphism $h : A \to B$ of $\mathscr{A}$ such that the square below commutes:

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "h"'] & FA \arrow[d, "Fh"] \\ B \arrow[r, "\beta"'] & FB \end{tikzcd} \tag{2.7}$$

The category of coalgebras and homomorphisms for F is denoted by $\mathsf{Coalg}\, F$.

Examples 2.4.2. The following examples of coalgebras over Set stem essentially from Rutten's fundamental paper [149].

(1) *Systems with termination* are given by a set A of states; for every state either a unique next state is specified, or the state is terminal. Such systems are precisely coalgebras for the endofunctor

$$FX = X + 1$$

Indeed, define $\alpha : A \to A + 1$ by assigning to a nonterminal state x its next state $\alpha(x)$ (in the left-hand summand) and to a terminal state the unique element of the right-hand summand.

Homomorphisms $h : (A, \alpha) \to (A', \alpha')$ between systems with termination are functions $h : A \to A'$ between their state sets preserving next states: if $\alpha(x) = x'$ in A', then $h \cdot \alpha(x) = \alpha' \cdot h(x)$. Moreover, x is terminal iff $h(x)$ is.

(2) *Binary input and termination.* These are coalgebras for

$$FX = X \times X + 1$$

A coalgebra $\alpha : A \to A \times A + 1$ is given by a set A of states x which are either terminal ($\alpha(x)$ in the right-hand summand) or have, for every input $i = 0, 1$ precisely one next state x_i. Then $\alpha(x) = (x_0, x_1)$.

Coalgebra homomorphisms preserve next states and preserve and reflect termination.

(3) *Deterministic automata.* Let Σ be a set of inputs. A deterministic automaton on a set S of states is given by a state transition function $\delta_s \colon S \to S$ for every $s \in \Sigma$ and a set $A \subseteq S$ of *accepting states.* We can represent the state transition functions in the curried form by

$$\delta \colon S \to S^{\Sigma},$$

and the subset A via the characteristic function

$$\gamma \colon S \to \{0, 1\}, \qquad A = \gamma^{-1}(1).$$

Thus, deterministic automata are precisely the coalgebras for the functor

$$FX = \{0, 1\} \times X^{\Sigma}.$$

Indeed, to specify a coalgebra $\alpha \colon S \to \{0, 1\} \times S^{\Sigma}$ means precisely to specify a pair $\delta \colon S \to S^{\Sigma}$ and $\gamma \colon A \to \{0, 1\}$ and then to set $\alpha = \langle \gamma, \delta \rangle$.

Homomorphisms of coalgebras

$$\begin{tikzcd}
S \arrow[r, "\langle \gamma,\delta \rangle"] \arrow[d, "h"'] & \{0,1\} \times S^{\Sigma} \arrow[d, "\mathsf{id} \times h"] \\
S' \arrow[r, "\langle \gamma',\delta' \rangle"'] & \{0,1\} \times (S')^{\Sigma}
\end{tikzcd}$$

are precisely the functions preserving transitions:

$$h \cdot \delta_s = \delta'_s \cdot h \qquad \text{for all } s \in \Sigma$$

and preserving and reflecting accepting states:

$$\gamma = \gamma' \cdot h$$

We have not mentioned initial states here, but we “recover” them in Example 2.5.5.

(4) *Directed graphs.* A directed graph (V, E) where $E \subseteq V \times V$ is, equivalently, a coalgebra for the power set functor $\mathscr{P}$. More precisely, given a coalgebra $\alpha\colon V \to \mathscr{P}V$, take V as the set of vertices of the graph, and

$$E = \{(u, v) \mid u \in V, v \in \alpha(u)\}$$

as its edges. Conversely, every graph (V, E) defined a coalgebra $\alpha\colon V \to \mathscr{P}V$ with $\alpha(u) = \{v \mid (u, v) \in E\}$.

However, as pointed out in Example 1.3.2, coalgebra homomorphisms are more special than the usual graph morphisms viz. edge-preserving functions. The square

$$\begin{tikzcd} V \arrow[r, "a"] \arrow[d, "h"'] & \mathscr{P}V \arrow[d, "\mathscr{P}h"] \\ V' \arrow[r, "a'"'] & \mathscr{P}V' \end{tikzcd}$$

commutes iff

(a) h preserves edges: $(u, v) \in E$ implies $(h(u), h(v)) \in E'$, and
(b) for every edge $(h(u), v') \in E'$ there exists $v \in V$ with $(u, v) \in E$ and $v' = h(v)$.

(5) *Non-Deterministic automata.* Here transitions are given by relations on the state set S, or, equivalently, by functions

$$\delta_s : S \to \mathscr{P}S \qquad \text{for all } s \in \Sigma,$$

where $\mathscr{P}$ is the power-set functor. These can be represented by a single function

$$\delta : S \to (\mathscr{P}S)^{\Sigma}.$$

We conclude that non-deterministic automata are coalgebras for the endofunctor

$$FX = \{0, 1\} \times (\mathscr{P}X)^{\Sigma}.$$

Homomorphisms of coalgebras

$$\begin{tikzcd} S \arrow[r, "\langle\gamma,\delta\rangle"] \arrow[d, "h"'] & \{0,1\} \times (\mathscr{P}S)^{\Sigma} \arrow[d, "\mathsf{id}\times\mathscr{P}h"] \\ S' \arrow[r, "\langle\gamma',\delta'\rangle"'] & \{0,1\} \times (\mathscr{P}S')^{\Sigma} \end{tikzcd}$$

are precisely the functions that

(a) preserve transitions, i.e. for every $s \in \Sigma$, $y \in \delta_s(x)$ implies $h(y) \in \delta_s(h(x))$,
(b) reflect them, i.e. for every $y' \in \delta_s(h(x))$ there exists a $y \in \delta_s(x)$ with $h(y) = y'$, and
(c) preserve and reflect accepting states, i.e. $\gamma = \gamma' \cdot h$.

(6) *Moore and Mealy automata.* Here a set Σ of inputs and a set Γ of outputs are given. In a Moore automaton, every state emits an output, thus a function $\gamma : S \to \Gamma$ is given (generalizing the case $\Gamma = \{0, 1\}$ in point (3)). We see that Moore automata are coalgebras for the endofunctor

$$FX = \Gamma \times X^\Sigma$$

In a Mealy automaton the output depends not only on the state but also on the input. Thus, the output function has the form $\gamma : \Sigma \times S \to \Gamma$ that we consider curried as $S \to \Gamma^\Sigma$. Thus Mealy automata are coalgebras for the endofunctor

$$FX = (\Gamma \times X)^\Sigma.$$

(7) *Labelled transition systems (LTS).* Given a set Σ of actions, an LTS consists of a set S of states and a binary relation $\xrightarrow{s}$ on S for every action s. We consider, again, relations as functions from S to $\mathscr{P}S$, then the above Σ-tuple of relations forms a function

$$\alpha : S \to (\mathscr{P}S)^\Sigma$$

Thus an LTS is precisely a coalgebra for the functor $FX = (\mathscr{P}X)^\Sigma$ which is composed of $\mathscr{P}$ and the polynomial functor $X \mapsto X^\Sigma$. An equivalent way to present a labelled transition system is by one relation $\to \subseteq S \times \Sigma \times S$ which is equivalent to giving a coalgebra $S \to \mathscr{P}(\Sigma \times S)$. Thus, an LTS is, equivalently, a coalgebra for the endofunctor $FX = \mathscr{P}(\Sigma \times X)$.

Example 2.4.3. Given a signature Σ (see Example 2.1.4), Σ-*coalgebras* are coalgebras for the polynomial endofunctor H_Σ on Set. A coalgebra

$$\alpha : A \to H_\Sigma A = \coprod_{n \in \mathbb{N}} \Sigma_n \times A^n.$$

can be viewed as a system with the state set A and a dynamics consisting of two functions: one, denoted by $\mathsf{head} : A \to \coprod_n \Sigma_n$, states which of the operation symbols is assigned to the state $x \in A$. The other function $\mathsf{body}(x, i)$ is defined for $i = 0, \dots, n-1$ where n is the arity of $\mathsf{head}(x)$. If $\alpha(x) = (\sigma, (y_0, \dots, y_{n-1}))$ and $i < n$, then $\mathsf{body}(x, i) = y_i$.

This example subsumes

(1) systems with termination: Σ has a unary operation and a constant,

(2) systems with binary input and termination: Σ has a binary operation and a constant,

(3) deterministic automata: consider two n-ary operations if the input set Σ has n elements (and observe that $\{0, 1\} \times X^\Sigma \cong X^n + X^n$),

(4) Moore automata (n-ary operations indexed by Γ) and Mealy automata (n-ary operations indexed by Γ^n).

Example 2.4.4. *Multigraphs.* Let us now move from the base category Set to the category $\mathsf{Set} \times \mathsf{Set}$ of two-sorted sets. This is a special case of many-sorted sets; see Example 2.1.5. Recall that a *multigraph*, where multiple edges between a pair of vertices are allowed, can be represented by two sets V (vertices) and E (edges) and two functions $s, t : E \to V$ (source and target of an edge). Define an endofunctor

$$F : \mathsf{Set} \times \mathsf{Set} \to \mathsf{Set} \times \mathsf{Set}$$

by

$$F\langle V, E\rangle \quad = \quad \langle 1, V \times V\rangle.$$

on objects, and $F\langle f, g\rangle = \langle \mathsf{id}, g \times g\rangle$ on morphisms. Then a multigraph is precisely a coalgebra for F: it consists of a two-sorted set $A = \langle V, E\rangle$ and a two-sorted function

$$\langle \alpha_v, \alpha_e\rangle : \langle V, E\rangle \to \langle 1, V \times V\rangle.$$

Indeed, we can ignore α_v. And $\alpha_e : E \to V \times V$ is precisely a pair of functions from E to V, as desired.

This time, coalgebra homomorphisms *are* precisely the usual graph homomorphisms, that is, pairs $h = \langle h_v, h_e\rangle$ of functions $h_v : V \to V'$ and $h_e : E \to E'$ such that for every edge $x \in E$, the edge $h_e(x)$ has the corresponding source and target:

$$h_v \cdot s(x) = s' \cdot h_e(x) \qquad \text{and} \qquad h_v \cdot t(x) = t' \cdot h_e(x). \tag{2.8}$$

Indeed, this condition is equivalent to the commutativity of the following square:

$$\begin{tikzcd}
\langle V, E\rangle \arrow[r, "\alpha"] \arrow[d, "\langle h_v, h_e\rangle"'] & \langle 1, V \times V\rangle \arrow[d, "\langle \mathsf{id}, h_v \times h_v\rangle"] \\
\langle V', E'\rangle \arrow[r, "\alpha'"'] & \langle 1, V \times V\rangle
\end{tikzcd}$$

We can clearly ignore the left-hand component, thus, this commutativity condition is equivalent to the commutativity of the square below:

$$\begin{tikzcd}
E \arrow[r, "\langle s, t\rangle"] \arrow[d, "h_e"'] & V \times V \arrow[d, "h_v \times h_v"] \\
E' \arrow[r, "\langle s', t'\rangle"'] & V' \times V'.
\end{tikzcd}$$

This is precisely (2.8).

Example 2.4.5. Here we consider coalgebras for endofunctors of $\mathsf{CPO}_\perp$, see Example 2.2.17. First, we need a definition. By an *ideal* of a cpo P is meant a nonempty subset closed both downwards and under ω-joins. The ideals of P are exactly the sets of the form $f^{-1}(\perp)$, where $f : P \to 2$ ranges over morphisms of $\mathsf{CPO}_\perp$ with codomain the two-chain $2 = \{\perp, \top\}$ where $\perp \leq \top$.

(1) A coalgebra for $FX = X_\perp$ is a system with termination whose state space A has the structure of a cpo such that

(a) all terminal states form an ideal, and

(b) the next-state function (on non-terminal states) is continuous.

Indeed, this yields a strict and continuous function $\alpha : A \to A_\perp$ sending all terminal states to the (new) $\perp$ element.

(2) A coalgebra for $FX = X_\perp + 1_\perp$ can be viewed as a system with termination and one deadlock state $\perp$. Every state is either a terminal state or $\perp$, or it has a (unique) next state. In the cpo A of states the terminal states and the states with a next state are always incomparable. And conditions and (b) above hold.

Indeed, this yields a strict and continuous function $\alpha : A \to A_\perp + 1_\perp$ mapping deadlock states to the non-bottom element of the right-hand summand.

(3) Continuous deterministic automata. These are automata with a cpo A as a state space and whose transition self-maps are strict and continuous. Moreover, all non-accepting states form an ideal.

Such automata are precisely the coalgebras for the endofunctor of $\mathsf{CPO}_\perp$ defined (as in the case of Set) by

$$FX = 2 \times X^\Sigma.$$

(This is a product of the chain $0 \leq 1$ and of n copies of X if $|\Sigma| = n$.) Indeed, a coalgebra

$$\alpha \colon A \to 2 \times A^\Sigma$$

defines two strict and continuous functions

$$\delta \colon A \to A^\Sigma \qquad \text{and} \qquad \gamma \colon A \to 2.$$

Here δ represents the transitions and γ the accepting states.

General Properties of the category $\mathsf{Coalg}\, F$

Notation 2.4.6. Given an endofunctor F of $\mathscr{A}$ the corresponding endofunctor of $\mathscr{A}^{op}$ is denoted by F^{op}: it is defined by $F^{op}A = FA$ on objects and it sends a morphism $f \colon X \to Y$ of $\mathscr{A}^{op}$ to $Ff \colon FY \to FX$ in $\mathscr{A}$, i.e., to $Ff : FX \to FY$ in $\mathscr{A}^{op}$.

Lemma 2.4.7. *The category of coalgebras for $F : \mathscr{A} \to \mathscr{A}$ is dual to the category of algebras for $F^{op} \colon \mathscr{A}^{op} \to \mathscr{A}^{op}$.*

Proof. Indeed, the objects are the same: an F-coalgebra $\alpha \colon A \to FA$ in $\mathscr{A}$ is precisely an F^{op}-algebra $a : FA \to A$ in $\mathscr{A}^{op}$. Given another coalgebra $b \colon B \to FA$, a homomorphism of coalgebras $h \colon (A, \alpha) \to (B, \beta)$ for F is precisely a homomorphism of algebras $h \colon (B, \beta) \to (A, \alpha)$ for F^{op}. $\square$

Remark 2.4.8. This lemma is trivial, but important: it implies that every statement about algebras has a dual statement about coalgebras. For example, Proposition 2.1.7 yields the following fact:

If $\mathscr{A}$ has finite coproducts, then finite coproducts of coalgebras are formed on the level of $\mathscr{A}$. In particular, the initial coalgebra is simply $0 \to F0$. (Terminal coalgebras are much more interesting!)

LM: I moved the short point below to here, where I think it belongs most. Lemma 2.4.7 does *not* say that the category of coalgebras for $F : \mathscr{A} \to \mathscr{A}$ is dual to the category of algebras for the same functor F. In fact, this will almost always turn out to be false!

However, dually to Corollary 2.1.9 and Corollary 2.1.10 we have the following facts.

Corollary 2.4.9. *If $\mathscr{A}$ has an initial object 0, then initial coalgebra is $0 \to F0$.*

Corollary 2.4.10. *A homomorphism $h : (A, \alpha) \to (B, \beta)$ is epic in $\mathsf{Coalg}\, F$ iff h is epic in $\mathscr{A}$.*

Remark 2.4.11. For monomorphisms, the situation is not as simple in $\mathsf{Coalg}\, F$ as in $\mathsf{Alg}\, F$ (cf. Corollary 2.1.10). However, every coalgebra homomorphism $h : (A, \alpha) \to (B, \beta)$ with h monic in $\mathscr{A}$ is monic in $\mathsf{Coalg}\, F$. Thus it represents a subobject of (B, b) (cf. Remark 2.1.12). When one speaks about *subcoalgebras* of a coalgebra (B, β), we always understands those represented by coalgebra homomorphisms carried by monic morphisms of $\mathscr{A}$.

2.5 Terminal coalgebras

Due to the duality between coalgebras in $\mathscr{A}$ and algebras in $\mathscr{A}^{op}$ which we saw in Lemma 2.4.7, everything we said about initial algebras in Section 2.2 dualizes:

Definition 2.5.1. A coalgebra for F is *terminal* if it admits a unique homomorphism from every coalgebra. We denote it by νF or $\nu X.FX$ and the coalgebra structure by

$$\tau : \nu F \to F(\nu F).$$

Lemma 2.5.2 [119]**.** *If a terminal F-coalgebra exists, then it is a fixed point of F.*

A converse similar to Theorem 2.2.6 for set functors is not known for terminal coalgebras, and it is unlikely that one is possible. In fact, there exist two set functors that coincide on all sets (but not all maps) such that one has a terminal coalgebra and one does not, see [23].

Example 2.5.3. (1) The power-set functor does not have a terminal coalgebra, since it has no fixed points (see Example 2.2.7(1)).

(2) The terminal coalgebra for $FX = X + 1$ (cf. Example 2.4.2(1)) is the set $\mathbb{N}^\top$, with the "predecessor" map as coalgebra structure:

$$* \longleftarrow 0 \longleftarrow 1 \longleftarrow 2 \longleftarrow \cdots \qquad \infty \circlearrowright$$

where $*$ denotes the element of 1. Indeed, for every F-coalgebra, i.e., a system A with termination, the behavior defines a homomorphism $b_A : A \to \mathbb{N}^\top$: If a state x has behavior $n > 0$, then its next state x' has behavior $n - 1 = \tau(n)$. If x has behavior ∞, then so does the next state. Finally, x is terminal iff $b_A(x) = 0$. This shows that the square defining homomorphisms commutes:

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "b_A"'] & A+1 \arrow[d, "b_A+\mathsf{id}"] \\ \mathbb{N}^\top \arrow[r, "\tau"'] & \mathbb{N}^\top + 1 \end{tikzcd}$$

It is further easy to see that b_A is the unique homomorphism.

(3) Coalgebras for the functor

$$FX = \Sigma \times X$$

can be viewed as dynamical systems with outputs in Σ: Given a coalgebra

$$\alpha = \langle \alpha_0, \alpha_1 \rangle : A \to \Sigma \times A,$$

then $\alpha_1 : A \to A$ determines next states and $\alpha_0 : A \to \Sigma$ determines outputs.

Every state $q \in A$ emits an infinite stream of outputs $x_0 = \alpha_0(q), x_1 = \alpha_0\alpha_1(q), x_2 = \alpha_0\alpha_1\alpha_1(q), \ldots$. This defines a function

$$b_A : A \to \Sigma^\omega$$

into the set Σ^ω of all infinite streams. This set carries itself the structure of a coalgebra

$$\tau = \langle \mathsf{head}, \mathsf{tail} \rangle : \Sigma^\omega \to \Sigma \times \Sigma^\omega.$$

Here head assigns to a stream $(x_n)_{n\in\mathbb{N}}$ its head x_0, and tail its tail $(x_{n+1})_{n\in\mathbb{N}}$. The following square commutes:

$$\begin{tikzcd} A \arrow[r, "\langle \alpha_1, \alpha_0 \rangle"] \arrow[d, "b_A"'] & \Sigma \times A^\omega \arrow[d, "\mathsf{id}\times(b_A)^\omega"] \\ \Sigma^\omega \arrow[r, "\langle \mathsf{head}, \mathsf{tail} \rangle"'] & \Sigma \times \Sigma^\omega \end{tikzcd}$$

i.e. b_A is a homomorphism.

It is easy to verify that no other homomorphism exists from A to Σ^ω. We thus proved that Σ^ω is the terminal coalgebra:

$$\nu X.\Sigma \times X = \Sigma^\omega.$$

(4) A slight variation: adding terminal states means that we work with the functor

$$FX = \Sigma \times X + 1.$$

In a coalgebra every states which is not terminal has a next state and emits an output in Σ. The behavior of a state is either a finite word (if the path encounters a terminal state) or an infinite stream. The set of all these streams is denoted by

$$\Sigma^\infty = \Sigma^* + \Sigma^\omega.$$

Again, we have a coalgebra structure

$$\tau : \Sigma^\infty \to \Sigma \times \Sigma^\infty + 1$$

taking the empty word to the right-hand summand, and all nonempty strings to the pair $\langle \mathsf{head}, \mathsf{tail} \rangle$. And again behavior defines a unique homomorphism into (Σ^∞, τ). Thus we proved that

$$\mu X.\Sigma \times X + 1 = \Sigma^\infty$$

Remark 2.5.4. The preceding examples demonstrate the important role that the terminal coalgebra νF has: for systems "of type F" the elements of νF are all possible behaviors of states. Given a system with the state set A, the unique coalgebra homomorphism from A to νF assigns to every state its behavior. The next example is a classical case:

Example 2.5.5. *Terminal automaton.* The terminal coalgebra for the functor $FX = \{0,1\} \times X^\Sigma$ of deterministic automata, see Example 2.4.2(4) is

$$\nu X.\{0,1\} \times X^\Sigma = \mathscr{P}\Sigma^*,$$

the set of all *formal languages* over Σ. Recall that $\mathscr{P}\Sigma^*$ can itself be considered as an automaton: a language $L \subseteq \Sigma^*$ is accepting iff it contains the empty word ε. Given a state (i.e. a language) L and an input $s \in \Sigma$ the next state is defined by the *Brzozowski derivative*

$$s^{-1}L = \{w \in \Sigma^*; sw \in L\}.$$

This yields a coalgebra structure $\tau : \mathscr{P}\Sigma^* \to F(\mathscr{P}\Sigma^*)$.

For every automaton A and every state $x \in A$ we form the language $L(x)$ accepted by x in A. Observe that

(1) x is accepting iff $L(x)$ contains ε, i.e., iff $L(x)$ is accepting in $\mathscr{P}\Sigma^*$, and

(2) for every input $s \in \Sigma$ if y denotes the next state of x, then

$$L(y) = s^{-1}L(x)$$

From (a) and (b) it is easy to see that the above language-function $L : A \to \mathscr{P}\Sigma^*$ is a coalgebra homomorphism, i.e. the square below commutes:

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d, "L"'] & \{0,1\} \times A^\Sigma \arrow[d, "\mathsf{id} \times L^\Sigma"] \\
\mathscr{P}\Sigma^* \arrow[r, "\tau"'] & \{0,1\} \times (\mathscr{P}\Sigma^*)^\Sigma
\end{tikzcd}$$

Here L^Σ is the function $p \mapsto L \cdot p$, for all $p \in A^\Sigma$. It is also easy to verify the uniqueness of L.

Remark 2.5.6. The functor $FX = \{0,1\} \times (\mathscr{P}X)^\Sigma$ used for modelling non-deterministic automata as coalgebra does not have a terminal coalgebra just as $\mathscr{P}$ does not have one. However, since in applications one is usually interested in *finite* non-deterministic automata, one may of course replace $\mathscr{P}$ above by the finite power-set functor $\mathcal{P}_\mathrm{f}$. Then the resulting functor does have a terminal coalgebra, and we will discuss it when we present the terminal coalgebra for $\mathcal{P}_\mathrm{f}$ in Theorem 4.5.7. For the moment let us just note that the terminal semantics does not provide the usual language semantics of non-deterministic automata. This can be achieved by working over the category os sets and relations, see Example 5.1.27.

Example 2.5.7. (1) Terminal Moore automata. Moore automata are coalgebras for $FX = \Gamma \times X^\Sigma$, see Example 2.4.2(6). The terminal coalgebra is the coalgebra Γ^{Σ^*} of all functions from Σ^* to Γ (formal power series). Again Γ^{Σ^*} is a Moore automaton: given a state $f : \Sigma^* \to \Gamma$ and an input $s \in \Sigma$, the corresponding output is $f(\varepsilon)$ and the next state is $\lambda w.f(sw)$.

(2) Analogously, for Mealy automata presented by the functor $FX = (\Gamma \times X)^\Sigma$: here terminal coalgebra can be described as the set of all *causal stream functions* from Σ^ω (the set of all streams in Σ) to Γ^ω. These are functions $f : \Sigma^\omega \to \Gamma^\omega$ such that for every stream $x \in \Sigma^\omega$ the n-th element of $f(x)$ depends only on the first n elements of x. The coalgebra structure assigns to every causal stream function f and every input $a \in \Sigma$

(a) the causal stream function f_a given by $f_a(x) = \mathsf{tail}(f(ax))$ and

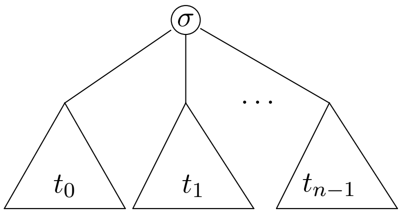
(b) the output $\mathsf{head}(f(ay))$ for any stream y.

(Note that this is well-defined since the head of the causal stream function f only depends on the head of its argument). This is how Rutten [151] described the terminal coalgebra. Another description is presented in Example 2.5.11(6) below.

Example 2.5.8 (Terminal Σ-coalgebra)**.** Generalizing all of the above examples, consider coalgebras for a polynomial endofunctor H_Σ, see Example 2.4.3. The terminal coalgebra is pleasantly analogous to the initial algebra (consisting of all finite Σ-trees, see Example 2.2.11). Recall that trees are always considered up to isomorphism. We shall show that

$$\nu H_\Sigma = \text{all } \Sigma\text{-trees}.$$

The coalgebra structure $\tau^{-1} : \nu H_\Sigma \to H_\Sigma(\nu H_\Sigma)$ is the inverse of the tree tupling map τ from Definition 2.2.12. It assigns to every Σ-tree



the pair $(\sigma, (t_0, \ldots, t_{n-1}))$.

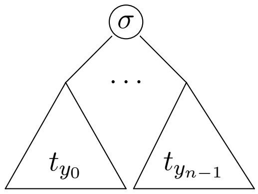
Theorem 2.5.9. *The terminal coalgebra for a polynomial functor H_Σ is the coalgebra of all Σ-trees with coalgebra structure inverse to the tree-tupling.*

Proof. Let us denote by T_Σ the set of all Σ-trees, and let (A, α) be a coalgebra for H_Σ. We are to define a homomorphism h as shown below:

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "h"'] & H_\Sigma A \arrow[d, "H_\Sigma h"] \\ T_\Sigma \arrow[r, "\tau^{-1}"'] & H_\Sigma T_\Sigma \end{tikzcd}$$

Since τ^{-1} is the inverse of tree tupling, the Σ-tree $t_x = h(x)$ must fulfil the following equation

$t_x =$



where $\alpha(x) = (\sigma, y_0, \ldots, y_{n-1})$. (2.9)

This determines the trees t_x for all $x \in A$ uniquely. Indeed, let us use their description as partial functions on $\mathbb{N}^*$ as in Remark 2.2.13. The values $t_x(w)$ are defined simultanously for all x by recursion over $w \in \mathbb{N}^*$ as follows:

$$\begin{aligned} t_x(\varepsilon) &= \sigma \\ t_x(iv) &= \begin{cases} t_{y_i}(v) & \text{if } i < n \\ \text{undefined} & \text{else.} \end{cases} \end{aligned} \tag{2.10}$$

This recursion determines a unique function $h : A \to T_\Sigma$, $h(x) = t_x$ for which (2.9) holds. Thus, h is the desired unique homomorphism. $\square$

Remark 2.5.10. Note that the unique homomorphism $h : (A, \alpha) \to (\nu H_\Sigma, \tau^{-1})$ assigns to every element $x \in A$ its unravelling as a Σ-tree using the coalgebras structure α. In fact, this is precisely the contents of the recursive definition (2.10). We shall therefore speak of $h(x) = t_x$ as the *tree expansion* of $x \in A$.

Examples 2.5.11. (1) $FX = X + 1$ corresponds to one constant and one unary operation. A Σ-tree is either a single path of length $n \in \mathbb{N}$, or a single infinite path. This yields $\nu X.X + 1 = \mathbb{N}^\top$, see Example 2.5.3(2).

(2) $FX = \Sigma \times X$ corresponds to the signature Σ with unary operations only. The corresponding trees have no leaves, thus, they are simply infinite paths labelled in Σ. This yields $\nu X.\Sigma \times X = \Sigma^\omega$, see Example 2.5.3(3).

(3) Analogously $FX = \Sigma \times X + 1$: the additional constant is a (unique) labelling of a leaf. Thus the corresponding tree is a finite or infinite path labelled in Σ, therefore $\nu X.\Sigma \times X + 1 = \Sigma^\infty$ as in Example 2.5.3(4).

(4) The functor $FX = X \times X + 1$ is the polynomial functor associated to the signature with one binary operation symbol and one constant. Thus νF may be identified with the set of all (finite and infinite) binary trees.

(5) Let A and B be fixed sets, and assume that A has size n. Consider $FX = B \times X^A$; this functor F corresponds to H_Σ where σ consists of n-ary operations indexed by B. An F-coalgebra is a dynamic system with n inputs and with outputs using symbols of B. Recall from Remark 2.2.10(4) that A^* represents the complete n-ary tree. Therefore, the set of Σ-trees for this signature is exactly B^{A^*}. (This is the set of functions from words on A into B.) This is the terminal coalgebra; we leave the description of the coalgebra structure to the reader.

(6) As a special case of the last item, consider deterministic automata as coalgebras for $FX = \{0,1\} \times X^A$ from Example 2.4.2(3). As we have just seen, the terminal coalgebra is $\{0,1\}^{\Sigma^*}$. Owing to the general correspondence between subsets of a given sets and maps into $\{0,1\}$, the terminal coalgebra may be described as the set $\mathscr{P}\Sigma^*$ of formal languages on Σ as in Example 2.5.5.

(7) For Mealy automata regarded as coalgebras for $FX = (\Gamma \times X)^\Sigma$ we saw a description of the terminal coalgebra in Example 2.5.7(2). Another description is obtained as follows: the functor $FX \cong \Gamma^\Sigma \times X^\Sigma$ is polynomial for the signature of n-ary operations (for Σ of size n) indexed by Γ^Σ. Thus the terminal coalgebra consists of all labellings of the complete n-ary tree Σ^* by labels in Γ^Σ. Every such tree is given by a function from Σ^* to Γ^Σ, or, equivalently, a function from $\Sigma^+ = \Sigma \times \Sigma^*$ to Γ:

$$\nu X.(\Gamma^\Sigma \times X^\Sigma) = \Gamma^{\Sigma^+}.$$

The coalgebra structure assigns to every function $g : \Sigma^+ \to \Gamma$ and every input $a \in \Sigma$ the function $g(a-) : \Sigma^+ \to \Gamma$ and the output $g(a) \in \Gamma$. This coalgebra Γ^{Σ^+} is indeed isomorphic to that of all causal stream functions (see Example 2.5.7(2)) by the following canonical isomorphism. Let $g : \Sigma^+ \to \Gamma$ be a function and define $\bar{g} : \Sigma^\omega \to \Gamma^\omega$ by assigning to every stream $a_0 a_1 a_2 \cdots$ in Σ the stream

$$g(a_0)\ g(a_0 a_1)\ g(a_0 a_1 a_2)\ \ldots.$$

Then $\bar{g}$ is obviously a causal stream function, and it is easy to verify that this yields the desired isomorphism.

(8) Let A be a set and consider the endofunctor on Set given by $FX = A \times X \times X$. This is the polynomial functor for the signature with a binary operation symbols for every $a \in A$. So we know from Theorem 2.5.9 that νF is carried by all labellings of the infinite binary tree by elements of $\mathscr{A}$.

However, a different (and very useful) isomorphic description of the terminal coalgebra was given by Kupke and Rutten [147] (see also Grabmayer et al. [82]). Consider the set A^ω of streams over A and equip it with the coalgebra structure

$$\langle \mathsf{head}, \mathsf{odd}, \mathsf{even} \rangle : A^\omega \to A \times A^\omega \times A^\omega,$$

where for a stream $s = (s_0, s_1, s_2, s_3, \ldots)$ we have $\mathsf{head}(s) = s_0$, $\mathsf{odd}(s) = (s_1, s_3, s, 5, \ldots)$ and $\mathsf{even}(s) = (s_2, s_4, s_6, \ldots)$. Then, this is a terminal F-coalgebra. To see this consider the bijection $\mathsf{z} : \omega \to \{0,1\}^*$ recursively defined by

$$\mathsf{z}(0) = \varepsilon \quad \mathsf{z}(2n+1) = 0\mathsf{z}(n) \quad \mathsf{z}(2n+2) = 1\mathsf{z}(n).$$

More explicitly, to obtain $\mathsf{z}(n)$ for $n > 0$, take the binary representation of $n+1$, drop the leading 1, and reverse the sequence. This map is Eilenberg's *reversed bijective interpretation*.

Now labellings of the infinite binary tree in A may be identified with functions $\{0,1\}^* \to A$, and streams over A are functions $\omega \to A$. Thus, precomposition with z yields an isomorphism between the sets νF above and A^ω. One then verifies that this yields an isomorphism of coalgebras.

We shall revisit this example in **??** when we study the behaviour of finite coalgebras, which in this case turn out to be the so-called *automatic sequences* in A^ω.

Example 2.5.12. Let us consider terminal coalgebras in $\mathsf{CPO}_\perp$.

(1) The terminal coalgebra for $FX = X_\perp$ is the poset

$$\nu X.X_\perp = \mathbb{N}^\top$$

of natural numbers extended by the top element ∞ with coalgebra structure inverse to the algebra structure of Example 2.2.17(1).

(2) The terminal coalgebra for $FX = (X \times X)_\perp + 1_\perp$ consists of all binary trees. Again, the coalgebra structure is inverse to tree-tupling of Example 2.2.17(2).

Remark 2.5.13. Initial algebras are related to terminal coalgebras in a canonical way: suppose that both exist for an endofunctor F. Then the inverse of the coalgebra structure $\tau : \nu F \to F(\nu F)$ makes νF an algebra. Hence, we have a unique algebra homomorphism $m : (\mu F, \iota) \to (\nu F, \tau^-1)$. (We can play the dual game and consider μF as a coalgebra: that would lead to the same morphism m, which is indeed also a coalgebra homomorphism.)

We will later see that for all set functors

(1) this morphism is monomorphic, and

(2) if νF exists, so does μF.

Thus, an initial algebra is always a subalgebra of the terminal coalgebra (when the latter is considered as an algebra). Observe that this holds in all the examples of the present section.

2.6 Corecursion and bisimulation

We have seen the connection of *initiality, recursion, and induction*: recursion in its most basic form on numbers may be considered as an application of initiality. Proof by induction is thus an application of the minimality of initial algebras.

In this section, we study the dual concepts. Whereas recursion deals with functions out of an initial algebra, corecursion is a definition principle for functions *into* a terminal coalgebra. This is the first topic in this section. After seeing examples, we then turn to *bisimulation principles* allowing us to prove interesting assertions about corecursively defined functions.

Definition 2.6.1. Let F be an endofunctor with a terminal coalgebra. Given an object A, we say that a morphism $f : A \to \nu F$ is *corecursively specified* if there exists a (specification) morphism $\alpha : A \to FA$ such that f is a coalgebra homomorphism, i.e. $\tau \cdot f = Ff \cdot \alpha$.

Example 2.6.2. Corecursive specification of addition [149]. We consider systems with termination as coalgebras for the set functor $FX = X + 1$. The terminal coalgebra is given by natural numbers extended by ∞:

$$\nu F = \mathbb{N}^\top \qquad \text{with coalgebra structure } \tau = \mathsf{pred},$$

the predecessor function, see Example 2.5.3(2). In order to recursively specify addition (extended by $\infty + x = \infty = x + \infty$) we need a coalgebra structure on the set $\mathbb{N}^\top \times \mathbb{N}^\top$. Its terminating states must sum to 0, whence $(0,0)$ is the only terminating state. The next-state function of $\mathbb{N}^\top \times \mathbb{N}^\top$ expresses a move from (x,y) to (x',y') such that $\tau(x+y) = x' + y'$. Thus, the coalgebra structure α has $(0,0)$ as the only terminating state, and the next state of $(x,y) \neq (0,0)$ is $(0, \tau y)$ if $x = 0$, else $(\tau x, y)$.

The corresponding coalgebra homomorphism $\mathsf{add} : \mathbb{N}^\top \times \mathbb{N}^\top \to \mathbb{N}^\top$ is addition since the square below commutes:

$$\begin{tikzcd}
\mathbb{N}^\top \times \mathbb{N}^\top \arrow[r, "\alpha"] \arrow[d, "\mathsf{add}"'] & \mathbb{N}^\top \times \mathbb{N}^\top + 1 \arrow[d, "\mathsf{add}+\mathsf{id}"] \\
\mathbb{N}^\top \arrow[r, "\tau"'] & \mathbb{N}^\top + 1
\end{tikzcd} \tag{2.11}$$

Thus, α is a corecursive specification of addition.

Example 2.6.3. Corecursive specification of streams. Recall from Example 2.5.3(3) that

$$\nu X.\Sigma \times X = \Sigma^\omega$$

(1) To specify corecursively a concrete stream as a function $f : 1 \to \Sigma^\omega$ we need a specification $\alpha : 1 \to F(1) \cong \Sigma$, i.e. an element $x \in \Sigma$. It is eay to see that the corresponding stream is simply $(x, x, x, \ldots)$ as a homomorphism from 1 to Σ^ω.

(2) What about non-constant streams, say,

$$(x, y, x, y, x, y, \ldots)?$$

We can corecursively specify this stream by using pairs of streams, $f : 2 \to \Sigma^\omega$ where $2 = \{0, 1\}$. The specification

$$\alpha : \{0,1\} \to \Sigma \times \{0,1\} \quad \text{with} \quad \alpha(0) = (x, 1) \quad \text{and} \quad \alpha(1) = (y, 0)$$

defines the following coalgebra homomorphism

$$\begin{tikzcd}
\{0,1\} \arrow[r, "\alpha"] \arrow[d, "h"'] & \Sigma \times \{0,1\} \arrow[d, "\mathsf{id} \times h"] \\
\Sigma^\omega \arrow[r, "\langle \mathsf{head}, \mathsf{tail} \rangle"'] & \Sigma \times \Sigma^\omega
\end{tikzcd}$$

such that the streams $h(0)$ and $h(1)$ have heads

$$\mathsf{head} \cdot h(0) = x \qquad \text{and} \qquad \mathsf{head} \cdot h(1) = y$$

and tails that satisfy

$$\mathsf{tail} \cdot h(0) = h(1) \qquad \text{and} \qquad \mathsf{tail} \cdot h(1) = h(0).$$

Obviously, this determines h as follows:

$$h(0) = (x, y, x, y, \ldots) \qquad \text{and} \qquad h(1) = (y, x, y, x, \ldots).$$

(3) We next want to specify the zipping function which assigns to a pair of streams $\overrightarrow{x} = (x_0, x_1, x_2 \ldots)$ and $\overrightarrow{y} = (y_0, y_1, y_2 \ldots)$ the stream

$$\mathsf{zip}(\overrightarrow{x}, \overrightarrow{y}) = (x_0, y_0, x_1, y_1, \ldots)$$

The corecursive specification describes the head and tail of $\mathsf{zip}(\overrightarrow{x}, \overrightarrow{y})$. Obviously

$$\mathsf{head} \cdot \mathsf{zip}(\overrightarrow{x}, \overrightarrow{y}) = x_0 = \mathsf{head}(\overrightarrow{x}). \tag{2.12}$$

Moreover, $\mathsf{tail} \cdot \mathsf{zip}(\overrightarrow{x}, \overrightarrow{y}) = (y_0, x_1, y_1, x_2, y_2 \ldots)$ fulfils

$$\begin{aligned} \mathsf{head}(\mathsf{tail} \cdot \mathsf{zip}(\overrightarrow{x}, \overrightarrow{y})) &= \mathsf{head}(\overrightarrow{y}) \\ \mathsf{tail}(\mathsf{tail} \cdot \mathsf{zip}(\overrightarrow{x}, \overrightarrow{y})) &= \mathsf{zip}(\mathsf{tail}\,\overrightarrow{x}, \mathsf{tail}\,\overrightarrow{y}). \end{aligned} \tag{2.13}$$

The desired coalgebra structure $\alpha : \Sigma^\omega \times \Sigma^\omega \to \Sigma \times \Sigma^\omega \times \Sigma^\omega$ will have the form $\alpha = \langle \alpha_{\mathsf{head}}, \alpha_{\mathsf{tail}} \rangle$ for the following functions

$$\alpha_{\mathsf{head}} : \Sigma^\omega \times \Sigma^\omega \to \Sigma \qquad \text{and} \qquad \alpha_{\mathsf{tail}} : \Sigma^\omega \times \Sigma^\omega \to \Sigma^\omega \times \Sigma^\omega$$

where (2.12) and (2.13) indicate that

$$\alpha_{\mathsf{head}}(\overrightarrow{x}, \overrightarrow{y}) = \mathsf{head}(\overrightarrow{x}) \qquad \text{and} \qquad \alpha_{\mathsf{tail}}(\overrightarrow{x}, \overrightarrow{y}) = (\overrightarrow{y}, \mathsf{tail}(\overrightarrow{x})).$$

Indeed, the function zip clearly makes the following squares commutative:

$$\begin{tikzcd} \Sigma^\omega \times \Sigma^\omega \arrow[r, "\alpha_{\mathsf{head}}"] \arrow[d, "\mathsf{zip}"'] & \Sigma \arrow[d, "\mathsf{id}"] \\ \Sigma^\omega \arrow[r, "\mathsf{head}"'] & \Sigma \end{tikzcd} \qquad \begin{tikzcd} \Sigma^\omega \times \Sigma^\omega \arrow[r, "\alpha_{\mathsf{tail}}"] \arrow[d, "\mathsf{zip}"'] & \Sigma^\omega \times \Sigma^\omega \arrow[d, "\mathsf{zip}"] \\ \Sigma^\omega \arrow[r, "\mathsf{tail}"'] & \Sigma^\omega \end{tikzcd}$$

Thus, zip is uniquely determined by $\alpha = \langle \alpha_{\mathsf{head}}, \alpha_{\mathsf{tail}} \rangle$ since the pairing the above two squares yields the commutative square below:

$$\begin{tikzcd} \Sigma^\omega \times \Sigma^\omega \arrow[r, "\alpha"] \arrow[d, "\mathsf{zip}"'] & \Sigma \times \Sigma^\omega \times \Sigma^\omega \arrow[d, "\mathsf{id} \times \mathsf{zip}"] \\ \Sigma^\omega \arrow[r, "\langle \mathsf{head}, \mathsf{tail} \rangle"'] & \Sigma \times \Sigma^\omega \end{tikzcd}$$

Coinduction Principle By dualizing the Induction Principle (see Lemma 2.3.6) we could formulate a coinduction principle by stating that homomorphisms into the terminal coalgebra exist uniquely. Instead, we introduce the concept of a *bisimulation* here and formulate that principle accordingly.

Bisimulation is an important concept in the theory of coalgebra, somewhat dual to that of congruence for algebras. In the case of set functors, a bisimulation between coalgebras $\alpha : A \to FA$ and $\beta : B \to FB$ is a relation $R \subseteq A \times B$ such that both projections $\pi_A : R \to A$ and $\pi_B : R \to B$ become coalgebra homomorphisms for some coalgebra structure on R.

In general categories, *relations* between two objects A and B are subobjects of $A \times B$. As explained in Remark 2.1.12, they are represented by monomorphisms $r : R \to A \times B$. Or, equivalently, by collectively monic spans of morphisms

$$\begin{tikzcd} & R \arrow[dl, "r_A"'] \arrow[dr, "r_B"] & \\ A & & B \end{tikzcd}$$

Indeed, the span is collectively monic iff $\langle r_A, r_B \rangle : R \to A \times B$ is a monomorphism.

Definition 2.6.4. Let (A, α) and (B, β) be coalgebras for F. By a *bisimulation* between them is meant a relation $\langle r_A, r_B \rangle : R \to A \times B$ for which a coalgebra structure on R exists turning both r_A and r_B into coalgebra homomorphisms. If the two coalgebras are equal, we speak about a *bisimulation on* (A, α).

The importance of this notion is that pairs of states related by some bisimulation intuitively have the same behavior. And continuing with this intuition, the largest bisimulation between two coalgebras should be exactly the relation that relates two states iff they have the same behavior. We want to mention some examples here.

Examples 2.6.5. (1) Given two systems with termination (see Example 2.4.2(1)) $\alpha : A \to A+1$ and $\beta : B \to B+1$, a bisimulation is a relation $R \subseteq A \times B$ such that whenever $x\,R\,y$ then

(a) x is terminal iff y is, and

(b) if x and y are nonterminal, then $x'\,R\,y'$ for the next states x' of x and $y'$of y.

Indeed, if (a) and (b) hold, we can define a coalgebra structure $\gamma : R \to R+1$ by sending a pair (x, y) in R to the right-hand summand of $FR = R+1$ if x is terminal, else $\gamma(x, y) = (x', y')$ for the above next states. The projections of R into A and B are clearly homomorphisms. Conversely, every bisimulation is easily seen to satisfy (a) and (b).

The *behavior* function $b_A : A \to \mathbb{N}^\top$ of Example 2.5.3(2) fulfils for every bisimulation R that

$$x\,R\,y \quad \text{implies} \qquad b_A(x) = b_A(y).$$

Conversely, define a relation between coalgebras A and B by "having the same behavior". This is an example of a bisimulation. Consequently, this is the largest bisimulation between (A, α) and (B, β).

(2) Let A and B be deterministic automata as coalgebras for the set functor $FX = \{0,1\} \times X^{\Sigma}$, see Example 2.4.2(3). A bisimulation $R \subseteq A \times B$ is a relation such that for $x\,R\,y$ we have

(a) x is accepting iff y is, and

(b) $x'\,R\,y'$ whenever for some input $s \in \Sigma$, x' is the next state of x and y' the next state of y under s.

It is easy to see that this implies that the language accepting by x in A is equal to that accepting by y in B.

Conversely, the relation "accepting the same language" is a bisimulation between A and B – thus, it is characterized as the largest bisimulation.

(3) The name bisimulation stems from the theory of labelled transition systems (see Example 2.4.2(6)) where we recover precisely Milner's notion of a strong bisimulation [135]: given a set Σ of actions and two LTS's, A and B over Σ, a relation R between A and B is a bisimulation iff for every $x\,R\,y$ and every action $s \in \Sigma$ we have

(a) given a transition $x \xrightarrow{s} x'$ in A, there exists a transition $y \xrightarrow{s} y'$ in B with $x'\,R\,y'$, and

(b) given a transition $y \xrightarrow{s} y'$ in B, there exists a transition $x \xrightarrow{s} x'$ in A with $x'\,R\,y'$.

It is easy to see that this coincides, for the functor $FX = (\mathscr{P}X)^{\Sigma}$ (equivalently, for $FX = \mathscr{P}(\Sigma \times X)$), with above concept of bisimulation.

(4) Given graphs (V_1, E_1) and (V_2, E_2) as coalgebras for $\mathscr{P}$ (see Example 2.4.4), a bisimulation is a relation $R \subseteq V_1 \times V_2$ with the following properties: whenever $x\,R\,y$,

(a) for every edge $(x, x') \in E_1$ there exists an edge $(y, y') \in E'$ with $y\,R\,y'$, and

(b) for every edge $(y, y') \in E_2$ there exists and edge $(x, x') \in E_1$ with $x\,R\,x'$.

Remark 2.6.6. (1) The *diagonal subobject* of $A \times A$ is the relation $\triangle_A$ represented by $\mathsf{id}_A, \mathsf{id}_A : A \to A$. For every coalgebra (A, α) this relation is clearly a bisimulation on (A, a). Any bisimulation not contained in $\triangle_A$ is called *proper*. Observe that a relation $r_1, r_2 : R \to A$ is proper iff $r_1 \neq r_2$.

(2) Another bisimulation on (A, α) is given by an arbitrary parallel pair of coalgebra homomorphisms

$$h_1, h_2 : (B, \beta) \to (A, \alpha).$$

More precisely, given such a pair in $\mathsf{Coalg}\,F$, where F is a set functor, the relation $R \subseteq A \times A$ of all the pairs $(h_1(x), h_2(x))$, $x \in B$, is a bisimulation.

Indeed, for the following epimorphism $e : B \to R$ given by $e(x) = (h_1(x), h_2(x))$ choose a splitting $m : R \to B$, i.e., we have $e \cdot m = \mathsf{id}$. Since the projections $r_1, r_2 : R \to B$ fulfil $h_i = r_i \cdot e$, thus, $h_i \cdot m = r_i$, the following diagram commutes for $i = 1$ and 2:

$$\begin{tikzcd}
R \arrow[r, "m"] \arrow[dr, "r_i"'] & B \arrow[r, "\beta"] \arrow[d, "h_i"] & FB \arrow[r, "Fe"] \arrow[d, "Fh_i"'] & FR \arrow[dl, "Fr_i"] \\
 & A \arrow[r, "\alpha"'] & FA &
\end{tikzcd}$$

Consequently, r_i are homomorphisms for the coalgebra structure $Fe \cdot \beta \cdot m$ on R.

(3) In the case where a bisimulation $R \subseteq A \times B$ is the graph of a function $f : A \to B$, we call it a *functional bisimulation*. Observe that, for example, coalgebra homomorphisms between automata are precisely the functional bisimulations: see the description of homomorphisms in Example 2.4.2(3). The same holds for non-deterministic automata, systems with termination and graphs (as coalgebras of $\mathscr{P}$). In all these examples coalgebra homomorphisms are precisely the functional bisimulations.

We now formulate the Coinduction Principle by using the concept of proper bisimulation.

Lemma 2.6.7 (Coinduction Principle)**.** *If F is an endofunctor with a terminal coalgebra, then no proper bisimulation on νF exists.*

Proof. Let $r = \langle r_1, r_2 \rangle : R \to \nu F \times \nu F$ be a bisimulation. Then R has a coalgebra structure such that $r_1, r_2 : R \to \nu F$ are coalgebra homomorphisms. Since νF is terminal, this proves $r_1 = r_2$. Equivalently, R is not proper. □

Remark 2.6.8. Lemma 2.6.7 shows that bisimulations are a sound proof principle to establish behavioural equivalence of states. The latter notion is defined for set functors F as follows: given two F-coalgebras (A, α) and (B, β) a pair of element $x \in A$ and $y \in B$ is called *behaviourally equivalent* if there exists a pair of coalgebra homomorphisms $h : (A, \gamma) \to (C, \gamma)$ and $k : (B, \beta) \to (C, \gamma)$ such that $h(x) = k(y)$. Equivalently, x and y are merged by the unique coalgebra homomorphisms into νF (if it exists).

As bisimulations are not a core topic of this book, we do not discuss the completeness of the bisimulation proof method at length. Let us just remark that it fails in general: there exist functors where behaviourally equivalent states need not be related be a bisimulation. However, if the functor F preserves weak pullbacks then every pair of behaviourally equivalent states is related by a bisimulation. For details see [5, 149].

Example 2.6.9. Addition on $\mathbb{N}^\top = \nu X.X + 1$ specified in Example 2.6.2 has the usual properties. We repeat from [149] coinductive proofs of some of them. Recall the coalgebra structure $\tau : \mathbb{N}^\top \to \mathbb{N}^\top + 1$ given by predecessor. We denote by $s : \mathbb{N}^\top \to \mathbb{N}^\top$ the successor function, which is the restriction of the inverse of τ. Let us write $x \oplus y$ in lieu of $\mathsf{add}(x, y)$.

(1) We prove by coinduction that addition $n \oplus k$ satisfies the equations by which addition on $\mathbb{N}$ is usually defined by recursion:

$$
\begin{aligned}
0 \oplus k &= k \\
s(n) \oplus k &= s(n \oplus k).
\end{aligned}
$$

For the first equation we define the relation $R = \{(0 \oplus k, k) \mid k \in \mathbb{N}^\top\}$. If we prove that R is a bisimulation, we are ready: then $R = \triangle$ and our equality is proved. We verify the two conditions in Example 2.6.5(1). Indeed, we have that $0 \oplus k$ is terminating iff k is so (equivalently, $k = 0$). For $k \neq 0$ we have $\tau(0 \oplus k) = 0 \oplus \tau(k) \; R \; \tau(k)$.

To prove the second equation, let R be the relation on $\mathbb{N}^\top$ consisting of all pairs $(s(n) \oplus k, s(n \oplus k))$ and all (n, n). We show that R is a bisimulation. First observe that the desired conditions hold for each pair (n, n) in R. The remaining pairs do not contain any terminal states, and we have

$$\tau(s(n) \oplus k) = \tau(s(n)) \oplus k = (n \oplus k)\; R\; (n \oplus k) = \tau(s(n \oplus k)).$$

This proves that R is a bisimulation.

(2) Let us verify the following equation

$$n \oplus s(k) = s(n) \oplus k.$$

To prove this by coinduction, let R be the relation on $\mathbb{N}^\top$ containing the pairs $(n \oplus s(k), s(n) \oplus k)$ and all (n, n). Again, the two conditions are trivially true for all pairs (n, n), and the remaining ones are non-terminating. We use that $\tau(s(n)) = n$ for all $n \in \mathbb{N}^\top$ and that $s(\tau(n)) = n$ for all $n \neq 0$ to verify that, for $n = 0$ we have

$$\tau(n \oplus s(k)) = n \oplus \tau(s(k)) = n \oplus k\; R\; n \oplus k = \tau(s(n)) \oplus k = \tau(s(n) \oplus k).$$

and similarly, for $n \neq 0$ we have

$$\tau(n \oplus s(k)) = (\tau(n) \oplus s(k))\; R\; (s(\tau(n)) \oplus k) = \tau(s(n)) \oplus k = \tau(s(n) \oplus k).$$

This proves that R is a bisimulation.

(3) Addition is commutative. Indeed, let R be the relation of all pairs $(n \oplus k, k \oplus n)$. To see that R is a bisimulation, note first that the only pair in R containing terminating states is $(0 \oplus 0, 0 \oplus 0)$. For the remaining states we have, for $n = 0$ and $k \neq 0$ that

$$\tau(n \oplus k) = (n \oplus \tau(k))\; R\; (\tau(k) \oplus n) = \tau(k \oplus n),$$

and for $n \neq 0 \neq k$ we have

$$\begin{aligned}
\tau(n \oplus k) &= \tau(n) \oplus k \\
&\quad R\; (k \oplus \tau(n)) \\
&= s(\tau(k)) \oplus \tau(n) = \tau(k) \oplus s(\tau(n)) = \tau(k) \oplus n = \tau(k \oplus n),
\end{aligned}$$

where we use item (2) as a lemma. Thus R is a bisimulation, and the commutativity is proved.

Example 2.6.10 [149]**.** Recall the streams $(x, x, x \ldots)$, $(y, y, y, \ldots)$ and $(x, y, x, y, \ldots)$ corecursively specified in Example 2.6.3 that we now denote by $\bar{x}$, $\bar{y}$ and $\overline{xy}$, respectively, We will prove

$$\mathsf{zip}(\bar{x}, \bar{y}) = \overline{xy}$$

by coinduction. Let R be the relation on $\Sigma^\omega = \nu X.X \times \Sigma$ consisting of just two pairs:

$$\mathsf{zip}(\bar{x}, \bar{y})\; R\; \overline{xy} \qquad \text{and} \qquad \mathsf{zip}(\bar{y}, \bar{x})\; R\; \overline{yx}.$$

We prove that this is a bisimulation. Define a coalgebra structure on R by the following dynamic system $\langle \delta, \gamma \rangle : R \to \Sigma \times R$ with outputs, see Example 2.5.3(3), where the output is denoted by $\mapsto$:

$$\begin{tikzcd}
\langle \mathsf{zip}(\bar{x}, \bar{y}), \overline{xy} \rangle \arrow[r, mapsto, shift left, "\delta"] \arrow[d, mapsto, "\gamma"'] & \langle \mathsf{zip}(\bar{y}, \bar{x}), \overline{yx} \rangle \arrow[l, mapsto, shift left, "\delta"] \arrow[d, mapsto, "\gamma"] \\
x & y
\end{tikzcd}$$

Both projections $r_1, r_2 : R \to \Sigma^\omega$ are easily seen to be coalgebra homomorphisms. Thus, $R \subseteq \triangle$ which proves the desired equality.

Cofree coalgebras At the close of Section 2.2, we saw that free algebras generalize initial algebras, and also that the free algebra construction leads to a monad. We now pursue the same topic, this time on the dual side. Here is the motivation for this. We often view a coalgebra $\alpha : A \to FA$ as a "black box": we cannot observe its states directly and therefore do not have a full information about next states. A terminal coalgebra gives us the observable part: for the given homomorphism $h : A \to \nu F$ we do not know the state x, but we know its behavior $h(x)$. An improved view is provided by an additional output functions $g : A \to \Gamma$. Here Γ is a set of colors, and whenever a cofree coalgebra $\Gamma_\sharp$ (as defined below) exists, then the corresponding homomorphism $\bar{g} : A \to \Gamma_\sharp$ gives us visible information about the colour of current state x.

Definition 2.6.11. By a *cofree F-coalgebra* on an object Γ (of colors) in $\mathscr{A}$ is meant a coalgebra

$$\tau_\Gamma : \Gamma_\sharp \to F\Gamma_\sharp$$

together with a couniversal (colouring) morphism $\varepsilon_\Gamma : \Gamma_\sharp \to \Gamma$. Couniversality means that for every coalgebra $\alpha : A \to FA$ and every morphism $g : A \to \Gamma$ there exists a unique homomorphism $\bar{g} : A \to \Gamma_\sharp$ with $g = \varepsilon_\Gamma \cdot \bar{g}$

$$\begin{tikzcd}
\Gamma & A \arrow[l, "g"'] \arrow[r, "\alpha"] \arrow[d, "\bar{g}"'] & FA \arrow[d, "F\bar{g}"] \\
& \Gamma_\sharp \arrow[ul, "\varepsilon_\Gamma"] \arrow[r, "\tau_\Gamma"'] & F\Gamma_\sharp
\end{tikzcd}$$

Example 2.6.12. Consider dynamical systems as coalgebras of $FX = X + 1$. Let A be a system with an additional output function $g : A \to \{0, 1\}$. Every state $x \in B$ defines a finite or infinite sequence of states: we start in x and continue moving, stopping only when a terminating state is reached. We do not know the sequence $x = x_0, x_1, x_2 \ldots$ of states, but we do see the sequence of colors $g(x_0), g(x_1), g(x_2) \ldots$.

So if our interpretation of $A_\sharp$ above is correct, we should have

$$\{0, 1\}_\sharp = \{0, 1\}^\infty$$

the set of finite and infinite binary streams. This is indeed true, and it is a consequence of the following.

Proposition 2.6.13. *The cofree F-coalgebra on Γ is precisely the terminal coalgebra for the endofunctor $F(-) \times \Gamma$.*

This is dual to Proposition 2.2.19. Applied to $FX = X + 1$ this tells us that the cofree algebra

$$\{0,1\}_\sharp = \nu X.X \times \{0,1\} + \{0,1\}$$

is the terminal Σ-coalgebra, where Σ has two unary operations and two constants. This terminal Σ-coalgebra consists of all Σ-trees. These are unary trees in which all nodes (inner ones as well as leaves) are labelled by 0 or 1. And this is isomorphic to $\{0,1\}^\infty$.

Definition 2.6.14. An endofunctor F is called a *covarietor* if every object generates a cofree coalgebra for F. The corresponding comonad is denoted by $F_\sharp$.

Example 2.6.15. (1) $FX = X + 1$ is a covarietor with

$$\Gamma_\sharp = \Gamma^\infty$$

This follows from $F(-) \times \Gamma = H_\Sigma$ where $\Sigma_0 = \Sigma_1 = \Gamma$ and $\Sigma_n = \emptyset$ for $n > 1$.

(2) $FX = X \times X + 1$ is a covarietor. The coalgebra $\Gamma_\sharp$ consists of all binary trees with nodes labelled in Γ. This follows from $F(-) \times \Gamma = H_\Sigma$ where $\Sigma_0 = \Gamma = \Sigma_2$ and otherwise $\Sigma_n = \emptyset$.

Proposition 2.6.16. *For every covarietor F the cofree comonad on F is $F_\sharp$. Conversely, if $\mathscr{A}$ is a cocomplete category, then every endofunctor generating a cofree comonad is a covarietor.*

This is dual to Theorem 2.2.23.

2.7 Summary of this chapter

As this long chapter draws to a close, here are the points that readers should be most confident of, and let us also raise some general points about the material.

The main theme of this chapter has been the description of initial algebras and terminal coalgebras of endofunctors of Set and $\mathsf{CPO}_\perp$, particularly of polynomial endofunctors. Initial algebras μF are important for a generic form of induction and recursion. Terminal coalgebras νF formalize the concept of behaviour of a state: given a state-based system on a set A of states (whose dynamics is expressed as a coalgebra for F) then the unique homomorphism from A to νF assigns to each state its behaviour. Terminal coalgebras are also important as the codomains for corecursive definitions. For the corresponding proof principle, coinduction, the concept of a bisimulation plays a crucial rôle. These themes will persist through the book.

3 Finitary Iteration

We show how to obtain an initial algebra for an endofunctor by iterating that endofunctor on the initial object. This can be seen as the categorical version of the famous Kleene Fixpoint Theorem for continuous functions on cpos that we recall first. The iterations considered here are countable, the more general case (generalizing the Knaster-Tarski Fixpoint Theorem) is treated in Chapter 6. We concentrate more on the dual construction: the terminal coalgebra as a limit of iterations of the endofunctor on the terminal object. Most of this chapter is devoted to examples. In later chapters of this book, we take up other topics: more on representations of terminal coalgebras, more on constructions that either use uncountable iterations or else use countable iteration in connection with extra order-theoretic or metric structure.

3.1 Initial-algebra chain

We recall the Kleene Fixpoint Theorem and present its categorical version: the initial algebra for an endofunctor as a colimit of iterations on the initial object. We show on concrete examples how this yields initial algebras for "well-behaved" endofunctors. For example, this yields a new view on the initial algebra for a polynomial functor associated to a signature Σ described in Chapter 2 as the algebra of all finite Σ-labelled trees.

Throughout this section we assume that a category $\mathscr{A}$ and an endofunctor F on $\mathscr{A}$ are given. We assume that $\mathscr{A}$ has an initial object 0 and denote by $! : 0 \to X$ the unique morphism to a given object X.

Theorem 3.1.1 (Kleene)**.** *Let A be a cpo with a least element $\perp$. Then every continuous endofunction F has a least fixed point*

$$\mu F = \sup_{n \in \omega} F^n(\perp).$$

Proof. First, an induction on $n < \omega$ shows that $F^n(\perp) \leq F^{n+1}(\perp)$. So $\{F^n(\perp) : n < \omega\}$ is an ω-chain. Write μF for its join. By continuity, $F(\mu F) = \bigvee_n F(F^n(\perp))$. It is easy to check that $\bigvee_n F^n(\perp) = \bigvee_n F^{n+1}(\perp)$, and so $F(\mu F) = \mu F$. Thus, we have a fixed point of F. If $Fx \leq x$, then an easy induction on n shows that $F^n(\perp) \leq x$; hence $\mu F \leq x$ as well. □

Remark 3.1.2. More generally, a *pre-fixed point* of F is an element x with $Fx \leq x$; the argument above proves that μF is also the least pre-fixed point of F.

In this book, we are not really interested in Kleene's Theorem but in generalizations of it, and in dualizations of those generalizations, etc. Figure 3.1 shows how we generalize

order-theoretic concept	category-theoretic generalization
preorder $(A, \leq)$	category $\mathscr{A}$
$x \leq y$ and $y \leq x$	A and B are isomorphic objects
least element 0	initial object 0
monotone $F\colon A \to A$	functor $F\colon \mathscr{A} \to \mathscr{A}$
pre-fixed point: $Fx \leq x$	F-algebra: $f\colon FA \to A$
ω-chain	functor from $(\omega, \leq)$ to $\mathscr{A}$
F is continuous	F preserves ω-colimits
least pre-fixed point: $Fx \leq x$	initial F-algebra: $\iota\colon F(\mu F) \to \mu F$

Figure 3.1: Generalizing Kleene's Theorem to categories

the order-theoretic concepts in Kleene's Theorem to the level of categories. In each line, the order-theoretic concept on the left is a special case of the category-theoretic concept to its right. (To see this, recall that a pre-order $(P, \leq)$ is exactly a category in which every homset is either empty or singleton set.) Of special interest is the generalization of the completeness condition on the poset and the continuity condition on the function.

Definition 3.1.3. By the *initial-algebra ω-chain* of an endofunctor $F : \mathscr{A} \to \mathscr{A}$ is meant the ω-chain

$$0 \xrightarrow{\ !\ } F0 \xrightarrow{F!} F^2 0 \xrightarrow{F^2!} \cdots \xrightarrow{F^{n-1}!} F^n 0 \xrightarrow{F^n!} F^{n+1} 0 \longrightarrow \cdots \tag{3.1}$$

Incidentally, the reason we use a hyphen in *initial-algebra chain* is to avoid the suggestion that this chain is initial in any category of "algebra chains." (In fact, we have no such chains in this book.)

Notation 3.1.4. The above diagram gives a functor from $(\omega, \leq)$ to $\mathscr{A}$. Such functors are in general called *ω-chains* in $\mathscr{A}$ and their colimits are called ω-colimits.

A colimit of (3.1) is denoted by $F^\omega 0$ with the colimit cocone $c_n\colon F^n 0 \to F^\omega 0$, $n < \omega$.

We must mention that later in Chapter 6 we shall need *transfinite* iterations of the initial-algebra chain. But in this section, we only consider the finite iterations as in (3.1).

A *cocone* of the initial-algebra ω-chain is an object A of $\mathscr{A}$ together with a family of morphisms $a_n\colon F^n 0 \to A$ such that the triangles below commute:

$$\begin{tikzcd} F^n 0 \arrow[rr, "F^n!"] \arrow[dr, "a_n"'] & & F^{n+1} 0 \arrow[dl, "a_{n+1}"] \\ & A & \end{tikzcd} \qquad \text{for all } n < \omega.$$

A *colimit* of the initial-algebra ω-chain is a cocone $c_n : F^n 0 \to F^\omega 0$ over it with the universal property that if $a_n : F^n 0 \to A$ is any cocone, then there is a unique *factorizing morphism* $f : C \to A$, i.e., such that for all $n < \omega$, $a_n = f \cdot c_n$.

Construction 3.1.5. Every F-algebra (A, α) induces a canonical cocone $\alpha_n \colon F^n 0 \to A$ of the initial-algebra ω-chain as follows: $\alpha_0 \colon 0 \to A$ is unique (since 0 is initial) and

$$\alpha_{n+1} = (FF^n 0 \xrightarrow{F\alpha_n} FA \xrightarrow{\alpha} A). \tag{3.2}$$

The cocone property, $\alpha_n = \alpha_{n+1} \cdot F^n!$, is easy to verify by induction. We call this cocone the *cocone induced by* (A, α).

Remark 3.1.6. (1) Homomorphisms of algebras preserve the induced cocones: given a homomorphism h from (A, α) to (B, β), i.e. $h \cdot \alpha = \beta \cdot Fh$, it follows that

$$h \cdot \alpha_n = \beta_n.$$

This is trivial for $n = 0$, and the induction step is easy:

$$h \cdot \alpha_{n+1} = (h \cdot \alpha) \cdot F\alpha_n = (\beta \cdot Fh) \cdot F\alpha_n = \beta \cdot F\beta_n = \beta_{n+1}.$$

(2) Let $c_n \colon F^n 0 \to \mu F$ be the colimit of the initial-algebra ω-chain. Applying F to each object and morphism in (3.1) yields another ω-chain

$$F0 \xrightarrow{F!} F^2 0 \xrightarrow{F^2!} F^3 0 \xrightarrow{F^3!} \cdots \xrightarrow{F^n!} F^{n+1} 0 \xrightarrow{F^{n+1}!} F^{n+2} 0 \longrightarrow \cdots \tag{3.3}$$

which obviously has the same colimit as (3.1).

This leads to the following result:

Theorem 3.1.7 [10]**.** *Let $\mathscr{A}$ be a category with initial object* 0 *and with colimits of ω-chains. If $F : \mathscr{A} \to \mathscr{A}$ preserves ω-colimits, then it has an initial algebra*

$$\mu F = \operatorname*{colim}_{n<\omega} F^n 0.$$

Remark 3.1.8. Specifically, we have the colimit cocone $Fc_n \colon F^{n+1} 0 \to F(\mu F)$ of the chain (3.3). Let $\iota \colon F(\mu F) \to \mu F$ be the unique morphism such that the following triangles commute:

$$\begin{tikzcd} F(F^n 0) = F^{n+1} 0 \arrow[d, "Fc_n"'] \arrow[dr, "c_{n+1}"] & \\ F(\mu F) \arrow[r, "\iota"'] & \mu F \end{tikzcd} \qquad (n < \omega). \tag{3.4}$$

Then $(\mu F, \iota)$ is an initial F-algebra. Moreover, for any algebra (A, α), the unique algebra morphism $h \colon \mu F \to A$ is the factorization induced by the canonical cocone:

$$h \cdot c_n = \alpha_n \qquad \text{for all } n < \omega.$$

Proof. In this proof we denote by μF the colimit above. The hypothesis that F preserves ω-colimits implies that $Fc_n \colon F^{n+1} 0 \to F(\mu F)$ is a colimit cocone of (3.3). We have another cocone of (3.3), namely $c_{n+1} : F^{n+1} 0 \to F(\nu F))$. Hence there is a unique morphism $\iota \colon F(\mu F) \to \mu F$ with the property (3.4).

We claim that the F-algebra $(\mu F, \iota)$ is initial. To check this, let (A, α) be any F-algebra. Consider the induced cocone $\alpha_n : F^n 0 \to A$ from Construction 3.1.5. It is easy to check that $F\alpha_n : F^{n+1}0 \to FA$ is a cocone of (3.3). Let $h \colon \mu F \to A$ be the unique factorization morphism i.e. $h \cdot c_n = \alpha_n$ for all $n < \omega$.

(a) We prove that $h : (\mu F, \iota) \to (A, \alpha)$ is a homomorphism. In order to see that $h \cdot \iota = \alpha \cdot Fh$, we check that both are mediating morphisms for the cocone $\alpha_{n+1} : F^{n+1}0 \to A$. That is, we check that for all n,

$$(h \cdot \iota) \cdot Fc_n = \alpha_{n+1} = (\alpha \cdot Fh) \cdot Fc_n.$$

For the first assertion, $h \cdot (\iota \cdot Fc_n) = h \cdot c_{n+1} = \alpha_{n+1}$. For the second, consider the diagram below:

$$\begin{tikzcd}
F^{n+1}0 \arrow[r, "\alpha_{n+1}"] \arrow[d, "Fc_n"'] \arrow[dr, "F\alpha_n"] & A \\
F(\mu F) \arrow[r, "Fh"'] & FA \arrow[u, "\alpha"']
\end{tikzcd}$$

The upper right-hand triangle commutes by (3.2). The other one commutes by definition of h. So the square commutes, showing that indeed $(\alpha \cdot Fh) \cdot Fc_n = \alpha_{n+1}$. This verifies that h is a homomorphism.

(b) To prove that h is unique, suppose that $k \colon \mu F \to A$ is also a homomorphism: $k \cdot \iota = \alpha \cdot Fk$. We show that k is also factorization of the induced cocone, i.e. that $k \cdot c_n = \alpha_n$; then the uniqueness of h implies that $k = h$. For $n = 0$, $k \cdot c_n$ and α_n are both morphisms with domain 0, so they are the same. Assuming that $k \cdot c_n = \alpha_n$, we see that

$$\begin{array}{rcll}
k \cdot c_{n+1} & = & k \cdot \iota \cdot Fc_n & \text{by definition of } \iota \\
& = & \alpha \cdot F(k \cdot c_n) & \text{since } k \text{ is a homomorphism} \\
& = & \alpha \cdot F\alpha_n & \text{by induction hypothesis} \\
& = & \alpha_{n+1} & \text{by (3.2)}
\end{array}$$

This concludes the proof. □

Remark 3.1.9. To obtain an initial algebra for an endofunctor $F : \mathscr{A} \to \mathscr{A}$, it is not really necessary that $\mathscr{A}$ have colimits of all ω-chains or that F preserve all ω-colimits. It is sufficient to assume that the colimit of the initial-algebra ω-chain exists and that F preserve this colimit. That is, these are the only facts about $\mathscr{A}$ and F that are used in the proof of Theorem 3.1.7. In many cases, it is just as easy to verify the stronger requirements that we stated in Theorem 3.1.7 than it is to verify the special cases used in the proof. But this is not always the case, and we shall see examples in topological and measure-theoretic settings where the sufficient conditions hold but the stronger ones do not.

3.2 Examples of initial algebras

We present examples of initial algebras obtained by finite iteration as in Theorem 3.1.7. We discuss these at some length because the same functors will appear throughout this book.

Example 3.2.1. (1) The functor $FX = X + 1$. Here 1 is a terminal object. In Example 2.2.7(3), we considered this as a functor on Set and found that μF is the set of natural numbers.

In an arbitrary category $\mathscr{A}$ with binary coproducts and a terminal object 1 the initial algebra for $FX = X + 1$ was called by Lawvere [120] the *natural numbers object*, NNO for short. Thus an NNO is an object N together with a morphism $\iota : N + 1 \to N$ (or, equivalently, a pair of morphisms $N \to N$ and $1 \to N$) universal among such morphisms. In more detail, given an object A and morphisms $\alpha_0 : 1 \to A$ and $\alpha_1 : A \to A$, there exists a unique morphism $h : N \to A$ such that the square below commutes:

$$\begin{tikzcd}
N + 1 \arrow[r, "\iota"] \arrow[d, "h+\mathsf{id}"'] & N \arrow[d, "h"] \\
A + 1 \arrow[r, "{[\alpha_0,\alpha_1]}"'] & A
\end{tikzcd}$$

The functor $FX = X + 1$ is easily seen to preserve ω-colimits. As such, it has an initial algebra obtained as the colimit of the initial-algebra ω-chain formed by the left-hand coproduct injections

$$0 \longrightarrow 1 \longrightarrow 1 + 1 \longrightarrow (1 + 1) + 1 \longrightarrow \cdots \tag{3.5}$$

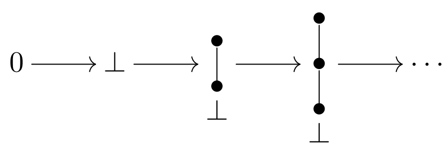
More precisely, $\mathscr{A}$ has an NNO iff a colimit of this ω-chain exists. When $\mathscr{A} = \mathsf{Set}$, we may identify its n-th term with the natural number $n = \{0, \ldots, n-1\}$ and obtain an ω-chain of inclusion maps. The colimit is the set $\mathbb{N}$ of natural numbers, with the colimit morphisms $c_n : n \to \mathbb{N}$ inclusion maps. The algebra structure

$$\iota : \mathbb{N} + 1 \to \mathbb{N} \tag{3.6}$$

is the isomorphism taking 0 in the right-hand summand, 1, to $0 \in \mathbb{N}$, and the left summand $\mathbb{N}$ to itself via the successor function. This follows easily from $\iota \cdot Fc_n = c_{n+1}$ for all $n < \omega$. This is a new explanation of Example 2.2.7(2).

(2) We consider now the category Pos of posets and monotone functions. Here coproducts are disjoints unions of partially ordered sets, i.e., with elements of different coproduct components incomparable. The above endofunctor takes a poset and adds a fresh element incomparable to the elements of X. Then $F^n 0$ can be identified with the set $\{0, \ldots, n-1\}$ discretely ordered, and the NNO is again $\mathbb{N}$ with the same structure as in (3.6) and discretely ordered. Observe that, in accordance to Example 2.2.9 $\mathbb{N} = \mathbb{N} \bullet 1$ in Pos.

There is also the functor $FX = X_\perp$ adding a new least element to a poset. Here the initial chain has the following form



(3.7)

We can describe $F^n 0$ as the linear order $0 < 1 < 2 < \cdots < n-1$ and μF is $\mathbb{N}$ with the usual order and the algebra structure ι with $\iota(\perp) = 0$ and $\iota(n) = n+1$.

Similarly, we have the endofunctor $FX = X^\top$ adding a new top element. Here μF is the set of natural numbers with the reverse of the usual order: $0 > 1 > 2 > \cdots$.

(3) For the analogous endofunctors on CPO see Example 2.2.17. In contrast, the NNO in this category is uninteresting: the functor $FX = X + 1$ is the identity functor with initial algebra $0 = \{\perp\}$.

Notation 3.2.2. (1) Let

$$\mathsf{MS}$$

be the category of 1-bounded metric spaces and non-expanding maps. That is, objects are sets endowed with a *metric* $d : X \times X \to [0, 1]$, i.e., a function which satisfies

(a) $d(x, y) = 0$ iff $x = y$

(b) $d(x, y) = d(y, x)$

(c) $d(x, z) \leq d(x, y) + d(x, z)$ (the triangle inequality)

Morphisms are *non-expanding functions*, i.e., functions $f : X \to X'$ such that for all $x, y \in X$,

$$d'(f(x), f(y)) \leq d(x, y).$$

(2) We also have the full subcategory

$$\mathsf{CMS}$$

of MS of *complete* metric spaces; i.e., such that every Cauchy sequence has a limit.

Example 3.2.3. We now consider $FX = X + 1$ as an endofunctor of MS. Coproducts are disjoint unions with distance 1 between points in different summands. The functor $FX = X + 1$ now has as an initial chain (3.5). That, it has as the initial algebra the set $\mathbb{N}$ of natural numbers, but as a discrete space: distinct points have distance 1. The same is true for $FX = X + 1$ as an endofunctor of CMS. Indeed, in the initial-algebra ω-chain, we see that $F^n 0$ is the discrete space of n elements, and the NNO, which is the colimit, is the discrete space on $\mathbb{N}$. The algebra structure (3.4) is the same as in Example 3.2.1.

Example 3.2.4. The situation changes when we scale the metric by $\frac{1}{2}$ (or by any other constant between 0 and 1). Let $\frac{1}{2} : \mathsf{MS} \to \mathsf{MS}$ scale a space by $\frac{1}{2}$. We now consider $GX = \frac{1}{2}X + 1$.

The initial-algebra ω-chain of G is the following chain of inclusions:

$G\emptyset$ $\quad$ $GG\emptyset$ $\quad$ $GGG\emptyset$

Here 0 represents the unique element of the right-hand summand of

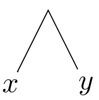
$$G^{n+1}0 = \frac{1}{2}(G^n 0) + 1$$

and the element i of G^n0 is represented by $i+1$ in $G^{n+1}0$. We see that G^n0 is the space $\{0, 1, \ldots, n-1\}$ with the following metric:

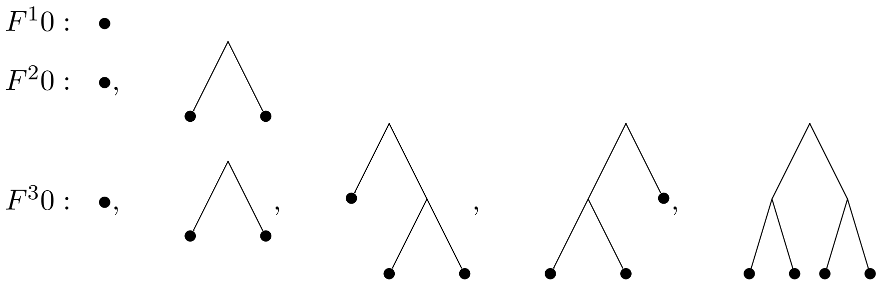
$$d(i,i) = 0 \qquad \text{and} \qquad d(i,j) = 2^{-\min(i,j)} \quad \text{if } i \neq j.$$

The colimit of this chain is $\mu G = \mathbb{N}$ with the above metric.

Example 3.2.5. Let us consider the set functor $FX = X \times X + 1$. When dealing with this functor and related ones, it is often useful to adopt a graphical notation. We shall draw $(x, y) \in X \times X \hookrightarrow FX$ as the ordered tree



We start with $F^0 0 = \emptyset$. Now we picture the elements of F^i0 for $i = 1$, 2, and 3 using $\bullet$ to denote the element of 1:



In general, F^n0 is the set of all binary trees of depth less than n, and the connecting maps $F^n0 \to F^{n+1}0$ are the inclusions.

Then the carrier of the initial algebra μF may be taken to be the union $\bigcup_{n<\omega} F^n0$:

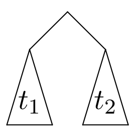
$$\mu F = \text{all finite binary ordered trees.}$$

The structure map

$$\iota : (\mu F \times \mu F) + 1 \to \mu F$$

takes the unique element of 1 to the single-node tree, and the left-hand component is given by *tree tupling*, i.e.

$(t_1, t_2) \longmapsto$



This is a new explanation of Example 2.2.15(1).

Example 3.2.6. (1) We return to the category MS introduced in Notation 3.2.2. In this category, products $X \times X'$ are cartesian products with the *maximum metric*:

$$d((x,x'),(y,y')) \quad = \quad \max\{d(x,y), d'(x',y')\}.$$

Thus, $FX = X \times X + 1$ can be considered as an endofunctor on MS. Its initial-algebra ω-chain is

$$0 \longrightarrow 1 \longrightarrow (1 \times 1) + 1 \longrightarrow ((1 \times 1) + 1) \times ((1 \times 1) + 1) + 1 \longrightarrow \cdots$$

the same as that of Example 3.2.5, but with each set $F^n 0$ taken to be the *discrete space*: all distances between distinct points are 1. The colimit μF is then, not surprisingly, the space of all binary trees equipped with the discrete metric.

(2) The situation changes when we scale the distances by half. This leads to the functor

$$GX = \frac{1}{2}(X \times X) + 1.$$

Here $G^{n+1}0$ consists of pairs of ordered trees $(t_1, t_2) \in G^n 0 \times G^n 0$ and of the single-node tree, whose distance from any (t_1, t_2) is 1. And the distance between (t_1, t_2) and (s_1, s_2) is one half of the maximum of the distances $d(t_i, s_i)$, $i = 1, 2$, in $G^n 0$. From this it follows that $G^n 0$ can be described as the set of all binary trees of depth less than n with the metric

$$d(t,u) = \begin{cases} 2^{-k} & \text{if } t \neq u \\ 0 & \text{if } t = u \end{cases} \tag{3.8}$$

for the least number k such that t and u have different cuttings at level k.

The colimit in MS is the space of all finite binary ordered trees with the above metric (3.8).

(3) When we turn from MS to CMS, the situation changes because for G above the colimit in MS is not complete. In fact, under the metric (3.8) every infinite binary tree t yields a Cauchy sequence t_0, t_1, ..., where t_k cuts t at level k. It turns out that the colimit of the initial-algebra ω-chain $G^n 0$ (of isometric embeddings) in CMS is obtained from the colimit in MS by forming the Cauchy completion. And this Cauchy completion is

$$\mu G = \text{all (finite and infinite) ordered binary trees,}$$

with the metric (3.8) above. We shall see a general reason for this in Section 5.2 below.

Example 3.2.7. Consider the endofunctor $FX = M \times X$, where M is a fixed object in one of our categories.

(1) As an endofunctor on Set, $\mu F = \emptyset$. The same holds in Pos, MS, and CMS.

(2) We next consider the situation in $\mathsf{CPO}_\perp$. For a cpo M the initial algebra ω-chain has the following form:

$$\{\perp\} \longrightarrow M \times \{\perp\} \longrightarrow M \times M \times \{\perp\} \longrightarrow \cdots$$

The tuples in each factor are ordered componentwise, and the connecting maps add to every n-tuple the element $\perp$ in the $n+1$-st coordinate. The colimit in $\mathsf{CPO}_\perp$ can be described as the cpo M^ω of all streams of elements of M ordered componentwise. Shortly:

$$\mu X.(M \times X) = M^\omega.$$

The algebra structure $M \times M^\omega \to M^\omega$ adjoins a new head to a stream in the evident manner.

Example 3.2.8. (1) For finitary signatures Σ, the polynomial set functor H_Σ of Example 2.1.3(4) preserves colimits of ω-chains. We apply Theorem 3.1.7 to get a short proof that the initial chain yields

$$\mu H_\Sigma = \text{the algebra of finite } \Sigma\text{-trees;}$$

cf. Proposition 2.2.14. (For infinitary signatures we describe μH_Σ in Example 6.1.13(2) below.) In fact, we can identify $H^1_\Sigma 0 = H_\Sigma \emptyset = \Sigma_0$ with the set of all Σ trees of depth 0, i.e., one-node trees labelled by an element of Σ_0. And

$$H^2_\Sigma 0 = H_\Sigma \Sigma_0 = \Sigma_0 + \coprod_{k>0} \Sigma_k \times \Sigma_0^k$$

with the set of all Σ-trees of depth at most 1. More generally, $H^n_\Sigma 0$ can be identified with the set of Σ-trees of height less than n. We obtain the chain of inclusion maps

$$H^0_\Sigma 0 \hookrightarrow H^1_\Sigma 0 \hookrightarrow H^2_\Sigma 0 \hookrightarrow \cdots$$

whose colimit (union) is the set of all finite Σ-trees.

The algebra structure $\iota : H_\Sigma(\mu H_\Sigma) \to \mu H_\Sigma$ of (3.4) is tree-tupling: ι maps an element $(\sigma, (t_i)_{i<k})$ of $\Sigma_k \times (\mu H_\Sigma)^k$ to the Σ-tree with root labelled by σ (k-ary) and with the children $t_0, \ldots, t_{k-1}$ from left to right (cf. Definition 2.2.12).

(2) As a concrete example, consider a finite set $V = \{v_1 \ldots v_n\}$ (of boolean variables). Recall the concept of a *binary decision tree*: it is a finite binary ordered tree whose inner nodes are labelled by variables v_i and leaves are labelled by 1 (for “true”) or 0 (for “false”). We see that the polynomial functor

$$FX = \{0,1\} + V \times X \times X$$

has the initial algebra

$$\mu F = \text{all binary decision trees.}$$

For a connection with binary decision diagrams see Example 9.3.14(5).

Example 3.2.9. The finite power set endofunctor $\mathscr{P}_\mathsf{f} : \mathsf{Set} \to \mathsf{Set}$ will be of special interest in this book. It preserves ω-colimits, thus we can apply Theorem 3.1.7.

(1) Its initial sequence is given by the inclusion functions of the following sets

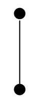
$$\emptyset \longrightarrow \{\emptyset\} \longrightarrow \{\emptyset, \{\emptyset\}\} \longrightarrow \cdots$$

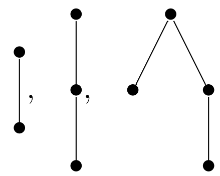
where $V_0 = \emptyset$, and $V_{n+1} = \mathscr{P} V_n$. The colimit of this sequence is the union $V_\omega = \bigcup_{n<\omega} V_n$, see Example 2.2.7(4).

(2) An alternative important representation of $\mu\mathscr{P}_{\mathsf{f}}$ is by finite *extensional trees*. (Recall from Remark 2.2.10 that trees are considered up to isomorphism.) An unordered tree is *extensional* if for every pair of children of a given node the two corresponding subtrees are different. Every unordered tree has an *extensional quotient* obtained by successively identifying children of a given node with the same subtrees. See also Section 4.5 for more on extensional trees.

Observe that if $\mathscr{P}_{\mathsf{f}}\emptyset$ is represented by the singleton tree, then we have a natural representation of the elements of $\mathscr{P}_{\mathsf{f}}^n\emptyset$ by all finite extensional trees of depth less than n: the tree representing of a set $\{\, x_1, \dots, x_n \,\}$ has n children (representing x_i):

$\mathscr{P}_{\mathsf{f}}^1\emptyset \;=\; \{\emptyset\}$ is represented by $\bullet$

$\mathscr{P}_{\mathsf{f}}^2\emptyset \;=\; \{\emptyset, \{\emptyset\}\}$ is represented by $\bullet$, 

$\mathscr{P}_{\mathsf{f}}^3\emptyset \;=\; \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$ is represented by $\bullet$, 

Etc. Therefore, we obtain

$$\mu\mathscr{P}_{\mathsf{f}} = \text{all finite extensional trees}$$

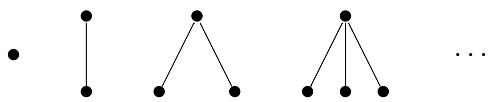
with the algebra structure $\mathscr{P}_{\mathsf{f}}(\mu\mathscr{P}_{\mathsf{f}}) \to \mu\mathscr{P}_{\mathsf{f}}$ given by tree tupling.

Example 3.2.10. A *bag* is a pair (X, b), where X is a set and $b : X \to \mathbb{N}$ is a function such that for all but finitely many x we have $b(x) = 0$. (These are also called *finite multisets* and the number $b(x)$ is called the *multiplicity of* x in the bag.) The *size* of (X, b) is $\sum_{x\in X} b(x)$. A morphism of bags $f : (X, b) \to (Y, c)$ is a function $f\colon X \to Y$ such that the multiplicity of every $y \in Y$ is the sum of multiplicities in $f^{-1}(y)$; in symbols: $c(y) = \sum_{f(x)=y} b(x)$ for all $y \in Y$.

The *bag functor* $\mathscr{B} : \mathsf{Set} \to \mathsf{Set}$ takes a set X to the set of all bags on X, and to every function $f : X \to Y$ it assigns the function $\mathscr{B}f : \mathscr{B}X \to \mathscr{B}Y$ given by

$$\mathscr{B}(f)(b) = \lambda y. \sum_{x\in f^{-1}(y)} b(x).$$

This functor preserves ω-colimits, and so Theorem 3.1.7 applies. The set $\mathscr{B}\emptyset$ has a single element which we represent as the single-node tree. Given a representation of $\mathscr{B}^n\emptyset$ by (unordered) trees, represent $\mathscr{B}^{n+1}\emptyset$ as follows: every bag consisting of trees $t_1, \dots, t_n \in \mathscr{B}^n\emptyset$ with multiplicities $k_1, \dots, k_n$ is represented by the tree having k_i children given by t_i for $i = 1, \dots, n$. Thus, $\mathscr{B}\mathscr{B}\emptyset$ is the set of the following trees



These are all unordered trees of depth at most 1. It is easy to see that $\mathscr{B}^3\emptyset$ are all finite unordered trees of depth at most 2, etc.

The colimit of the initial-algebra ω-chain is again the union. The initial algebra $\mu\mathscr{B}$ can thus be described as follows:

$$\mu\mathscr{B} = \text{all finite unordered trees.}$$

The algebra structure $\iota : \mathscr{B}(\mu\mathscr{B}) \to \mathscr{B}$ is *tree-tupling with multiplicities*, i.e. ι maps a bag $(\{t_1, \dots t_n\}, b)$ of finite unordered trees to the unordered tree that joins $b(t_i)$ copies of t_i, $i = 1, \dots, n$, with a new common root.

Example 3.2.11. Generalizing several of the above examples, we recall the concept of an *analytic functor* introduced by André Joyal [102, 103]: given a group G of permutations on $k = \{0, \dots, k-1\}$, we denote by X^k/G the set of orbits under the action of G on X^k by coordinate interchange, i.e. the set of equivalence classes

$$X^k/G = X^k/{\sim_G}$$

for the least equivalence $\sim_G$ with $(x_1, \dots, x_k) \sim_G (x_{p(1)}, \dots, x_{p(k)})$ for each $p \in G$. This defines a set functor taking $f : X \to Y$ to the obvious function $X^k/G \to Y^k/G$ derived from f^k. The analytic functors are precisely the coproducts of functors of the form X^k/G. More precisely, an analytic functor F is of the form

$$FX = \coprod_{\sigma \in \Sigma} X^k/G_\sigma, \tag{3.9}$$

where Σ is a finitary signature, k the arity of $\sigma \in \Sigma$ and G_σ a group of permutations on k. Thus, every polynomial functor H_Σ is analytic, and an important example of an analytic functor is the bag functor, which can be expressed as follows:

$$\mathscr{B}X = \coprod_{k \in \mathbb{N}} X^k/S_k, \tag{3.10}$$

where S_k is the symmetric group of all permutations on k. In contrast, $\mathscr{P}_f$ is not analytic.

For every analytic functor F the initial-algebra ω-chain yields a quotient of the algebra μH_Σ of Example 3.2.8.

$$\mu F = \mu H_\Sigma/{\sim}$$

where $\sim$ is the least equivalence such that for every tree $t \in \mu H_\Sigma$, every node x of t labelled by $\sigma \in \Sigma$ and every permutation g in the group G_σ we have $t \sim t'$, where t' is obtained from t by permuting the children of x according to g.

In fact, the proof is completely analogous to Example 3.2.8. We identify $F^1 0 = \Sigma_0$ with one-point trees labelled in Σ_0. And since $F^{n+1}0 = \coprod_{\sigma \in \Sigma}(F^n 0)^k/G_\sigma$, where σ is a k-ary symbol, we can identify, for every $n \in \mathbb{N}$, the set $F^n 0$ with the quotient of the set of all Σ-trees of depth less than n modulo the above equivalence $\sim$. We again obtain a chain of inclusion maps whose colimit (union) is $F_\Sigma/{\sim}$. This is illustrated by the case of the bag functor $\mathscr{B}$ in Example 3.2.10.

Example 3.2.12. Here is an example of an endofunctor F on a category with a rather interesting initial algebra. Let BiP be the category of *bipointed* sets: objects are sets with distinguished elements $\perp$ and $\top$ that are required to be different. A morphism in BiP is a function preserving the distinguished points. There is a binary operation $\oplus$ on BiP taking $(X, \perp_X, \top_X)$ and $(Y, \perp_Y, \top_Y)$ to the disjoint union $X + Y$, identifying $\top_X$ and $\perp_Y$, and then using as distinguished points $\perp_X$ and $\top_Y$. Define an endofunctor on BiP by

$$FX = X \oplus X.$$

Its initial-algebra ω-chain starts with the initial object $\mathbf{0} = \{0, 1\}$ where $0 = \perp_{\mathbf{0}}$ and $1 = \top_{\mathbf{0}}$. Then $F\mathbf{0}$ can be represented by as $\{0, \frac{1}{2}, 1\}$ where $\frac{1}{2}$ represents the equivalence class $\{\perp_1, \top_0\}$. In this manner the initial chain is the following chain of injections

$$\underset{\mathbf{0}}{\boxed{\begin{array}{c}\top = 1\\ \perp = 0\end{array}}} \hookrightarrow \underset{F\mathbf{0}}{\boxed{\begin{array}{c}\top = 1\\ 1/2\\ \perp = 0\end{array}}} \hookrightarrow \underset{FF\mathbf{0}}{\boxed{\begin{array}{c}\top = 1\\ 3/4\\ 1/2\\ 1/4\\ \perp = 0\end{array}}} \hookrightarrow \cdots$$

The colimit of this chain is its union

$$D = \text{all dyadic rational numbers in } [0, 1].$$

That means that D consists of all numbers $\frac{k}{2^n}$, $k = 0, \ldots, 2^n$. The algebra structure $\iota : D \otimes D \to D$ takes y in the left-hand part to $\frac{y}{2}$ and in the right-hand one to $\frac{y+1}{2}$. Indeed, F preserves the above colimit, thus D is the initial algebra for F. To see this, observe that the connecting map $j : D \to FD$ from $\mathsf{colim}\, F^n\mathbf{0}$ to $F(\mathsf{colim}\, F^n\mathbf{0})$ is $x \mapsto 2x$ on $[0, \frac{1}{2}]$ and $x \mapsto 2x - 1$ on $[\frac{1}{2}, 1]$. This maps j is bijective, and its inverse $j^{-1} : FD \to D$ is the structure of the initial algebra. Related examples are discussed in **??**.

Example 3.2.13. Let us mention also some negative examples. An endofunctor of Set need not have an initial algebra, as we have seen in Example 2.2.7. The endofunctor $F : X \mapsto X^{\mathbb{N}} + 1$ does have an initial algebra, as we will see in Chapter 4, but its initial-algebra ω-chain does not converge. In a similar manner to what we have seen in Example 3.2.5, $F^n 0$ can be represented by all countably branching ordered trees of depth less than n. But $\bigcup F^n 0$ is not a fixed point of F. We will see in Chapter 6 that an initial-algebra chain of uncountable length is needed for this functor.

Continuing with a discussion of examples of initial algebras obtained by ω-iteration on categories other than Set, we mention a result implying that for some special categories, *every* endofunctor has an initial algebra obtained that way.

Definition 3.2.14 [73]**.** A category is called *algebraically complete* if every endofunctor has an initial algebra.

Theorem 3.2.15 [12]**.** *The categories* Set_c *(countable sets and functions),* Rel_c *(countable sets and relations), and* K-Vec_c *(countably-dimensional vector spaces over a field* K *and* *linear functions) are algebraically complete. Moreover, every endofunctor F has the initial algebra*

$$\mu F = \operatorname*{colim}_{n<\omega} F^n 0.$$

Remark 3.2.16. (1) Every complete lattice is algebraically complete by the classical fixed-point theorem of G. Birkhoff.

(2) Among categories with products, there are essentially no other examples:

Theorem 3.2.17 [21]**.** *Every algebraically complete category with products is a preorder.*

Proof. The following beautiful proof was provided by Peter Freyd [73]: suppose $\mathscr{A}$ has products but is not a preorder. That is, some hom-set $\mathscr{A}(A,B)$ has at least two elements. Consider the endofunctor F obtained by the following composite

$$F = (\mathscr{A} \xrightarrow{\mathscr{A}(A,-)} \mathsf{Set} \xrightarrow{\mathsf{Set}(-,2)} \mathsf{Set}^{op} \xrightarrow{B^{(-)}} \mathscr{A}).$$

(Here $B^{(-)}$ denotes the functor taking a set M to the power B^M.) This functor F can be expressed as $FD = B^{S(D)}$ where $S(D) = 2^{\mathscr{A}(A,D)}$. It does not have fixed points. In fact, assuming $D \cong FD$, we conclude that $\mathscr{A}(A,D)$ is isomorphic to

$$\mathscr{A}(A, FD) \cong \mathscr{A}(A, B^{S(D)}) \cong \mathscr{A}(A,B)^{S(D)}.$$

But the cardinality of the right-hand side is at least

$$2^{S(D)} \cong 2^{2^{\mathscr{A}(A,D)}}$$

a contradiction: for every set X the cardinality of 2^X is larger than that of X – apply this to $X = \mathscr{A}(A,D)$. $\square$

3.3 Terminal-coalgebra chain

The construction of the terminal coalgebra in this section is another categorical version of the Kleene Fixpoint Theorem: a terminal coalgebra for an endofunctor is constructed as a limit of iterations of the functor on the terminal object. An important example is the terminal coalgebra for a polynomial functor which is described as the algebra of all (finite and infinite) Σ-trees. Throughout this section we assume that a category $\mathscr{A}$ with a terminal object 1 is given. We denote by $! : X \to 1$ the unique morphism from an object X.

It is straightforward to dualize the general results of the last section. One dualizes initial objects 0 to terminal objects 1, ω-chains to ω^{op}-chains (that is, functors from ω^{op} to $\mathscr{A}$), colimits to limits, and the initial-algebra ω-chain of Definition 3.1.3 as follows:

Definition 3.3.1. By the *terminal-coalgebra ω^{op}-chain* of an endofunctor $F : \mathscr{A} \to A$ is meant the ω^{op}-chain

$$1 \xleftarrow{\;!\;} F1 \xleftarrow{F!} F^2 1 \xleftarrow{F^2!} \cdots \xleftarrow{F^{n-1}!} F^n 1 \xleftarrow{F^n!} F^{n+1}! \longleftarrow \cdots \tag{3.11}$$

Construction 3.3.2. Dually to Construction 3.1.5 every coalgebra $\alpha : A \to FA$ induces a canonical cone over the terminal-coalgebra ω^{op}-chain of F by induction: $\alpha_0 : A \to 1$ is uniquely determined and $\alpha_{n+1} = F\alpha_n \cdot \alpha : A \to F^{n+1}1$.

Remark 3.3.3. Dually to Remark 3.1.6, homomorphisms $h : (A, \alpha) \to (B, \beta)$ of coalgebras preserve the induced cones:

$$\alpha_n = \beta_n \cdot h \qquad \text{for all } n < \omega.$$

It is worthwhile putting down the dual statement of Theorem 3.1.7. This was first explicitly formulated by Barr [50]:

Theorem 3.3.4. *Let $\mathscr{A}$ be a category with terminal object* 1 *and with limits of ω^{op}-chains. If $F : \mathscr{A} \to \mathscr{A}$ preserves limits of ω^{op}-chains, then it has the terminal coalgebra*

$$\nu F = \lim_{n \in \omega^{op}} F^n 1.$$

Remark 3.3.5. (1) It is sufficient to assume that F preserves the limit of its terminal-coalgebra chain (cf. Remark 3.1.9).
(2) Let $\ell_n : \nu F \to F^n 1 (n < \omega)$ be the limit cone. The coalgebra structure

$$\tau : \nu F \to F(\nu F)$$

is the unique morphism with

$$F\ell_n \cdot \tau = \ell_{n+1} \qquad (n < \omega)$$

This is dual to Remark 8.1.8.

We revisit the examples from Section 3.2.

Examples 3.3.6. The functor $FX = X + 1$ on Set. The terminal-coalgebra ω^{op}-chain is

$$1 \longleftarrow 1 + 1 \longleftarrow 1 + 1 + 1 \longleftarrow \cdots$$

The n-th object may be identified with $n + 1 = \{0, 1, \ldots, n\}$, and the connecting function $F^n! = f_n : n+2 \to n+1$ is then given by $f_n(i) = i$ for $i \leq n$, and $f_n(n+1) = n$. The limit of this chain is the set of all ω-tuples $(x_0, \ldots, x_n, \ldots)$ with $f_n(x_{n+1}) = x_n$ for every n. One such tuple is $\top = (0, 1, 2, \ldots)$. Every other tuple is of the form $(0, 1, \ldots, k, k, \ldots, k, \ldots)$ which we may identify with k. Thus we describe the terminal coalgebra as

$$\nu X.(X + 1) \quad = \quad \mathbb{N}^{\top},$$

the set of natural numbers with an element $\top$ added. The coalgebra structure $\tau : \mathbb{N}^{\top} \to \mathbb{N}^{\top} + 1$ has $\top$ as a fixed point, sends 0 to the point in the right-hand summand of $X + 1$, and is otherwise the predecessor function on $\mathbb{N}$. This is a new explanation of Example 2.5.3(2).

Example 3.3.7. (1) The functor $FX = X + 1$ on Pos. Using the same argument as above, we see that the terminal coalgebra is $\mathbb{N}^\top$ with the discrete order. The structure is the same map as in Example 3.3.6 above.

The functor $FX = X_\perp$ (cf. Example 2.1.6(1)) has $\nu F = \mathbb{N}^\top$ with the usual order $0 < 1 < 2 < \cdots < \infty$ and the same structure as before, the argument is analogous.

(2) The functor $FX = X + 1$ on $\mathsf{CPO}_\perp$ essentially is the identity functor, whence $\nu F = 1 = \{\perp\}$. The functor $FX = X_\perp$ has as terminal coalgebra the same poset as in point (1) above. This is not surprising: $\mathsf{CPO}_\perp$ is closed in Pos under limits (but not colimits).

(3) On the category MS of 1-bounded metric spaces, the above argument shows that the terminal coalgebra of the same functor $FX = X + 1$ is again $\mathbb{N}^\top$, this time with a discrete metric. The structure is the same as we have seen. The same is true for CMS.

When we change the functor to $GX = \frac{1}{2}X + 1$ as in Example 3.2.4, the terminal-coalgebra ω^{op}-chain has essentially the same spaces that we saw in that example since $G\emptyset = 1$, and the connecting map $G^{n+1}1 \to G^n 1$ merges n and $n+1$ and is identity else. We again get $\mathbb{N}^\top$ as the terminal coalgebra but with the metric shown in Example 3.2.4.

Example 3.3.8. (1) Let $\mathscr{A}$ have countable products. The functor $FX = M \times X$ where M is a fixed (but arbitrary) object of $\mathscr{A}$. We identify each object with its product with 1. Thus we can write the terminal-coalgebra chain as

$$1 \longleftarrow M \longleftarrow M \times M \longleftarrow M \times M \times M \longleftarrow \cdots$$

The connecting morphisms are all the projections onto the left-most factors. We obtain as a limit the countable power M^ω with the limit projections $\ell_n : M^\omega \to M^n$ given by the left-hand projections of $M^\omega \cong M^n \times \coprod_{i \geq n} M$. For example, if $\mathscr{A} = \mathsf{Set}$, then M^ω is the set of all *streams* on M, also known as the *infinite words* on M:

$$\nu X.(M \times X) = M^\omega.$$

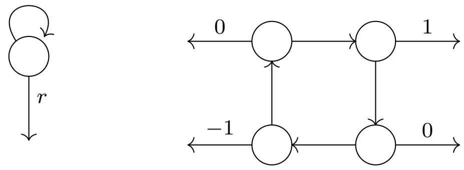
The coalgebra structure $M^\omega \to M \times M^\omega$ is $\langle \mathsf{head}, \mathsf{tail} \rangle$, where

$$\mathsf{head}(a_1, a_2, a_3 \ldots) = a_1 \qquad \text{and} \qquad \mathsf{tail}(a_1, a_2, a_3 \ldots) = (a_2, a_3 \ldots)$$

This is a special case of Theorem 2.5.9 for Σ consisting of unary operations indexed by M: a Σ-tree in an infinite path labelled in M, i.e., a steam on M.

(2) As a concrete example, take $M = \mathbb{R}$, the set of real numbers. A coalgebra for $\mathbb{R} \times X$ is a stream automaton, i.e., a dynamic system with real outputs, see Example 2.5.3(3).

Here are two examples in which the output is represented by a label of the output arrow



One way to represent real valued streams is by taking the set A of *real analytic functions*, see [145]. Here one considers $f : \mathbb{R} \to \mathbb{R}$ such that for every x there is an open interval around 0 on which the n-th derivative $f^{(n)}$ is defined, and f agrees with its Taylor series. The coalgebra structure on A is given by

$$f(x) \mapsto (f(0), f'(x))$$

for every analytic function. This coalgebra is isomorphic to a subcoalgebra of the stream coalgebra that we saw above; to every analytic function $f(x)$ one associates the stream of coefficients of the Taylor series of $f(x)$, i.e., A is isomorphic to the subcoalgebra of those streams σ in $\mathbb{R}^\omega$ such that $\sum_{i=0}^{\infty} \frac{\sigma_i}{i!} x_i < \infty$.

The above two stream automata thus present analytic functions by corecursion. In the coalgebra on the left, we have a function whose value at 0 is r and equal to its own derivative: $f(x) = re^x$. On the right, we obtain four functions:

$$\sin x, \cos x, -\sin x, -\cos x.$$

Example 3.3.9. For every signature Σ, the terminal-coalgebra ω^{op}-chain for the polynomial functor H_Σ of Example 2.1.3 can be described as follows. Denote by $\Sigma_\perp = \Sigma + \{\perp\}$ the signature Σ extended by a new constant symbol $\perp$. For every $n < \omega$ we have:

$$H_\Sigma^n 1 = \text{all } \Sigma_\perp\text{-trees of depth at most } n \text{ with all leaves at depth } n \text{ labelled by } \perp.$$

To see this, let us identify $1 = H_\Sigma^0 1$ with the set consisting of the singleton tree labelled by $\perp$. Then $H_\Sigma^1 1 = \coprod \Sigma_k \times 1^k$ can be identified with the set of all trees of depth ≤ 1 whose root is labelled in Σ and (for a root label of arity k) whose k leaves are labelled by $\perp$. Analogously $H_\Sigma^2 1 = \coprod \Sigma_k \times (H_\Sigma 1)^k$ are trees t with a root labelled in Σ_k and the k subtrees are trees in $H_\Sigma^1 1$; thus all leaves of t of depth 2 have label $\perp$ etc.

Moreover, the connecting map

$$H_\Sigma^{n+1} 1 \longrightarrow H_\Sigma^n 1$$

simply cuts the $\perp$-labelled leaves away and relabels all leaves of depth n by $\perp$. This yields a new proof of Theorem 2.5.9:

Theorem 3.3.10. *The terminal coalgebra for the polynomial functor H_Σ is the set T_Σ of all ordered Σ-trees, with the coalgebra structure given by the inverse to tree tupling.*

Proof. We have described $H_\Sigma^n 1$ above as the set of all $\Sigma_\perp$-trees of depth at most n with leaves of depth n labelled by $\perp$. Since H_Σ preserves limits of ω^{op}-chains, we know that

$$\nu H_\Sigma = \lim_{n<\omega} H_\Sigma^n 1.$$

(1) We prove that the limit of the chain $H_\Sigma^n 1$ (with the connecting maps $v_{n+1,n} : H_\Sigma^{n+1} 1 \to H_\Sigma^n 1$ given by cutting at level n as above) is the set T_Σ with the limit cone $\ell_n : T_\Sigma \to H_\Sigma^n 1$

given, again, by cutting at level n and relabelling all leaves at level n by $\perp$. In other words, we need to show that for every collection of trees $t_n \in H_\Sigma^n 1$ which is compatible:

$$v_{n,m}(t_n) = t_m \qquad \text{for all } n \geq m \text{ finite,}$$

there exists a unique tree $t \in T_\Sigma$ with $\ell_n(t) = t_n$ for all $n < \omega$. Existence: let t be the unique Σ-tree which on levels smaller than n agrees with t_n for every n. More precisely, the root of t has the same label as t_n for all $n \geq 1$, the level one labels are the same as in t_n for all $n \geq 2$, etc. Obviously $\ell_n(t) = t_n$ for all n finite. Uniqueness: if $\ell_n(t) = t_n$, then t and t_n agree on all levels smaller than n.

(2) The coalgebra structure $\tau : T_\Sigma \to H_\Sigma T_\Sigma$ is, following Remark 3.3.5, the unique morphism with $\ell_{m+1} = H_\Sigma \ell_m \cdot \tau$ for all m. Since the tree tupling $\bar{\tau} : H_\Sigma T_\Sigma \to T_\Sigma$ clearly fulfils $\ell_{m+1} \cdot \bar{\tau} = H_\Sigma \ell_m$ (because to cut a tree tupling at level $m+1$ is the same as cutting the maximum subtrees at level m and then performing a tree tupling), we conclude $\tau = \bar{\tau}^{-1}$. □

Example 3.3.11. The functor $FX = \{0, 1\} \times X^A$ whose coalgebras are deterministic automata (see Example 2.5.11) yields

$$F^n 1 \cong \text{sets of words of length} < n \text{ in } A^*.$$

More precisely, let A have k elements and let Σ be the signature with two k-ary operation symbols. Then $F \cong H_\Sigma$. To give a tree in $F^n 1$, i.e., a $\Sigma_\perp$-tree of depth $\leq n$ with leaves labelled in by $\perp$, means precisely to label the nodes of a complete k-ary (see Remark 2.2.10(4)) tree of depth $\leq n - 1$ by 0 or 1. This, is equivalent to giving a set of words of length $< n$. The above connecting maps $v_{n,m}$ take a set M of words of length $< n$ to the set formed by the prefixes of length $< m$ of words from M.

A limit of the resulting terminal-coalgebra ω^{op}-chain is the set of all formal languages $\mathscr{P} A^*$. The limit cone $\ell_n : \mathscr{P} A^* \to F^n 1$ takes a formal language $L \subseteq A^*$ to the set of all prefixes of length $n - 1$ of words in L.

Remark 3.3.12. In Chapter 6, we shall consider a more general notion of *signature* which allows for *infinitary* operations, see Example 6.1.13(2). Everything which we said in Example 3.3.9 holds for this more general notion of a signature.

Example 3.3.13. For every analytic functor F (see Example 3.2.11) the terminal coalgebra is obtained by the terminal-coalgebra ω^{op}-chain. In fact, it is easy to derive from the formula (3.9) that analytic functors preserve limits of ω^{op}-chains. We will see a description of this terminal coalgebra in Example 4.3.23(3) below. The following example is a special case.

Example 3.3.14. The bag functor $\mathscr{B}$ of Example 3.2.10 has the terminal coalgebra

$$\nu\mathscr{B} = \text{all finitely branching unordered trees.}$$

In fact, we can identify $\mathscr{B}^n 1$, analogously as in Example 3.3.9, with all unordered finitely branching trees of depth $\leq n$ whose leaves at level n are labelled by $\perp$. The limit $\nu\mathscr{B}$ is then as stated.

Example 3.3.15. The finite power-set functor $\mathscr{P}_{\mathsf{f}}$ is an example of a set functor which does not preserve limits of ω^{op}-chains, and, moreover, as we demonstrate in Section 4.5 the limit of its terminal-coalgebra ω^{op}-chain is not the terminal $\mathscr{P}_{\mathsf{f}}$-coalgebra.

We will describe the terminal coalgebra for $\mathscr{P}_{\mathsf{f}}$ below in several different ways (see Section 4.5).

Example 3.3.16. Consider the functor $FX = X \oplus X$ on the category of BiP of bipointed sets. Recall from Example 3.2.12 that the initial F-algebra is carried by the set of all dyadic rationals in $[0, 1]$. Freyd [75] proved that the terminal F-coalgebra is the real unit interval $[0, 1]$ with the structure $t : [0, 1] \to [0, 1] \oplus [0, 1]$ given by $x \mapsto 2x$ on $[0, \frac{1}{2}]$ and $x \mapsto 2x - 1$ on $[\frac{1}{2}, 1]$. We discuss this example and related ones in more detail in **??**.

Remark 3.3.17. *Many-sorted sets* are used in many applications. These lead to endofunctors of the category Set^S of S-sorted sets. In Example 2.1.5, we have seen the corresponding polynomial endofunctors on Set^S. Theorem 3.3.10 (and its proof) hold for these more general polynomial functors. Let us illustrate the initial-algebra ω-chain and the terminal-coalgebra ω^{op}-chain for a concrete 2-sorted signature:

Example 3.3.18. Consider finite lists of symbols over (arbitrary) alphabets. Here we work with two sorts a (alphabet) and l (list) and consider the signature Σ with the following operation symbols

$$\begin{array}{ll} \varepsilon : l & \text{(empty list)} \\ \gamma : al \to l & \text{(concatenation)} \\ 0, 1 :\to a & \text{(constants 0 and 1)} \end{array}$$

The corresponding polynomial functor H_Σ has sorts

$$(H_\Sigma X)_a = \{0, 1\} \qquad \text{and} \qquad (H_\Sigma X)_l = X_a \times X_l + \{\varepsilon\}.$$

(1) Here is its initial-algebra chain:

	sort a	sort l
$(\emptyset, \emptyset)$	$\emptyset$	$\emptyset$
$H_\Sigma(\emptyset, \emptyset)$	$0, 1$	ε
$H_\Sigma^2(\emptyset, \emptyset)$	$0, 1$	$0\varepsilon, 1\varepsilon, \varepsilon$
$H_\Sigma^3(\emptyset, \emptyset)$	$0, 1$	$00\varepsilon, 01\varepsilon, 10\varepsilon, 11\varepsilon, \varepsilon$

etc. If we drop the superfluous ε at the ending of words, we see that for $n \geq 1$

$$H_\Sigma^n(\emptyset, \emptyset) = (\{0, 1\}, \coprod_{k<n} \{0, 1\}^k)$$

The connecting maps are inclusions. Thus, the colimit is

$$\mu H_\Sigma = (\{0, 1\}, \{0, 1\}^*).$$

(2) We next describe the terminal-coalgebra ω^{op}-chain:

	sort a	sort l
$(1,1)$	$\perp_a$	$\perp_l$
$H_\Sigma(1,1)$	$0,1$	$\perp_a\perp_l, \varepsilon$
$H_\Sigma^2(1,1)$	$0,1$	$0\perp_a\perp_l, 1\perp_a\perp_l, 0, 1, \varepsilon$
$H_\Sigma^3(1,1)$	$0,1$	$00\perp_a\perp_l, 01\perp_a\perp_l, 10\perp_a\perp_l, 11\perp_a\perp_l,$ $00, 01, 10, 11, 01, 1, \varepsilon$

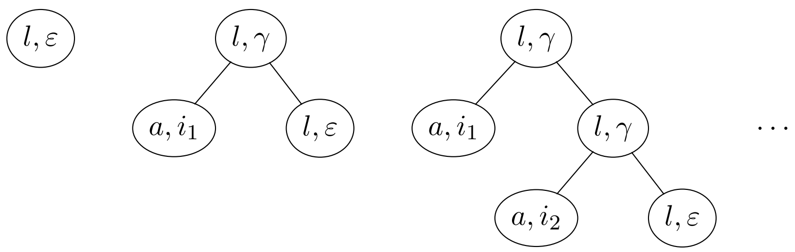
etc. Again dropping ε at the end of words, we see that sort l of $H_\Sigma^n(1,1)$ consists of all words $w\perp_a\perp_l$ where w is a binary word of length $n-1$, and all binary words w of lengths $0, 1, \ldots, n-1$. The connecting map $v_{n+1,n} : H_\Sigma^{n+1}1 \to H_\Sigma^n 1$ maps each word $w\perp_a\perp_l$ by dropping the last letter of w; analogously for binary words $w \neq \varepsilon$, it drops the last letter. The limit of this ω^{op}-chain has the sort l which consists of (a) all $u\perp_a\perp_l$ where u is an infinite binary word and (b) all finite binary words. By dropping the superfluous $\perp_a\perp_l$ here, we get

$$\nu H_\Sigma = (\{0,1\}, \{0,1\}^* + \{0,1\}^\omega).$$

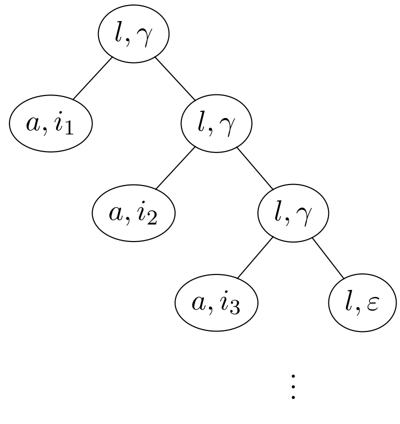
We thus see that a Σ-tree has one of the following forms: for sort a we just have two trees

$$a, 0 \qquad \text{and} \qquad a, 1$$

For sort l we have the finite trees



representing the finite binary words $w = i_1 \cdots i_n$, plus the infinite trees



representing the infinite binary words $i_1 i_2 i_3 \cdots$.

Example 3.3.19. Recursive domain equations involving mutual recursion can often be solved by using polynomial endofunctors of many-sorted sets. For example, suppose that we need sets X and Y satisfying

$$X \cong X \times Y + 1 \qquad \text{and} \qquad Y \cong X + X.$$

We form the polynomial endofunctor F on $\mathsf{Set} \times \mathsf{Set}$ given by

$$F(X, Y) = (X \times Y + 1, X + X).$$

It clearly preserves colimits of ω-chains, thus, it has an initial algebra which is a colimit of $F^n(0, 0)$. By Lambek's Lambek's Lemma 2.2.5 the components of the initial algebra form an (initial) solution of the above recursive equations.

Theorem 3.3.20. *For every many-sorted signature Σ the polynomial endofunctor H_Σ has*

(1) *the initial algebra $\mu H_\Sigma = F_\Sigma$, all finite Σ-trees with the algebra structure given by tree tupling, and*

(2) *the terminal coalgebra $\nu H_\Sigma = T_\Sigma$, all Σ-trees, with the coalgebra structure given by the inverse of tree tupling.*

The proof is completely analogous to that in Proposition 2.2.14 and Theorem 3.3.10.

Remark 3.3.21. Kozen introduces in [115] coalgebras for a special class of polynomial functors F on the category of many-sorted sets that arise from graphs representing type declarations of a programming language with sum and product types. He also presents an interesting equivalent description of the category of F-coalgebras and of the terminal F-coalgebra, in particular. However, the type declarations considered do not allow one to capture all polynomial functors on (many-sorted) sets.

3.4 Summary of this chapter

We started the chapter with a construction of the initial algebra μF of an endofunctor as a colimit of the ω-chain $F^n 0$. In short, $\mu F = \mathsf{colim}\, F^n 0$. This holds whenever F preserves ω-colimits. We saw many examples. To mention one by way of review, for a polynomial endofunctor $F = H_\Sigma$ on Set, $F^n 0$ can be represented by Σ-trees of depth less than or equal to n, and this yields $\mu H_\Sigma =$ all finite Σ-trees.

Dually, a construction of the terminal coalgebra νF as a limit of the ω^{op}-chain of iterated application of F to the terminal object 1 is obtained whenever F preserves ω^{op}-limits. For polynomial functors, this shows that νH_Σ consists of all Σ-trees, and for the functor F representing deterministic automata with input alphabet Σ, we see that νF consists of all formal languages over Σ. Note however, that preservation of ω^{op}-limits is a much more restrictive condition than preservation of ω-colimits; for example, the finite power-set functor satisfies the latter but not the former.

4 Finitary Set Functors

The aim of this chapter is to study terminal coalgebras and weakly terminal ones for finitary set functors in connection with the presentation of these functors by (finitary) signatures and equations. We start in Section 4.2 with the topic of weakly terminal coalgebras: the definition weakens the definition of *terminal* by dropping the uniqueness of homomorphisms. For a construction of weakly terminal coalgebras we introduce in Section 4.3 presentations of set functors by operations and equations. This is followed by a discussion of precongruences. In Section 4.4, we present Worrell's construction of the terminal coalgebra as a limit of "twice" the finitary iterations. The chapter concludes with an examination of coalgebras of the finite power-set functor $\mathscr{P}_{\mathsf{f}}$. We are especially interested in the limit of the terminal-coalgebra ω^{op}-chain of $\mathscr{P}_{\mathsf{f}}$, an object with many descriptions.

We begin with a some background on constructions of algebras and coalgebras in general.

4.1 Limits and Colimits of Algebras and Coalgebras

Before turning to finitary set functors in Section 4.3 we present some basic facts about algebra and coalgebras for arbitrary endofunctors that will be used throughout the rest of this book. We prove that the forgetful functor of the category of F-coalgebras creates (1) all colimits and (2) all limits that are preserved by F. Recall, e.g. from MacLane's book [124], that a functor $F : \mathscr{B} \to \mathscr{A}$ is said to *create colimits* of some diagram scheme (i.e., a small category $\mathscr{D}$) provided that for every diagram $D : \mathscr{D} \to \mathscr{B}$ the following holds: given a colimit cocone of FD in $\mathscr{A}$, say

$$c_d : FDd \to C \qquad (d \in \mathsf{obj}\,\mathscr{D}),$$

then there exists a unique cocone of D in $\mathscr{B}$ that F maps to (c_d), and, moreover, that cocone is a colimit of D in $\mathscr{B}$.

We apply this to the forgetful functor

$$U : \mathsf{Coalg}\,F \to \mathscr{A} \qquad U(A,\alpha) = A$$

That U creates colimits means that colimits of F-coalgebras are formed on the level of $\mathscr{A}$. More precisely, given a diagram $D : \mathscr{D} \to \mathsf{Coalg}\,F$ of coalgebras with $Dd = (A_d, \alpha_d)$, U creates the colimit of D if for a colimit $c_d : A_d \to C$ f UD in $\mathscr{A}$ we a unique coalgebra structure $\gamma : C \to FC$ such that every c_d is a coalgebra homomorphism $c_d : (A_d, \alpha_a) \to (C, \gamma)$. Moreover, the last cocone is a colimit of D in $\mathsf{Coalg}\,F$. Note that creation of colimits does not make any assumptions on the existence of colimits in $\mathscr{A}$, and it holds for all endofunctors and diagrams:

Proposition 4.1.1. *For every endofunctor F on $\mathscr{A}$, the forgetful functor $U : \mathsf{Coalg}\, F \to \mathscr{A}$ creates colimits*

Proof. Given the diagram $D : \mathscr{D} \to \mathsf{Coalg}\, F$ and a colimit cocone $(c_d : A_d \to C)$ of UD in $\mathscr{A}$, we need to find a unique coalgebra structure $\gamma : C \to FC$ such that each of the following squares commute:

$$\begin{tikzcd} A_d \arrow[r, "\alpha_d"] \arrow[d, "c_d"'] & FA_d \arrow[d, "Fc_d"] \\ C \arrow[r, dashed, "\gamma"'] & FC \end{tikzcd}$$

For that it is sufficient to verify that all $\alpha_d \cdot Fc_d$ form a compatible cocone of the diagram UD; then γ is uniquely determined by the universal property of the colimit cocone (c_d). Indeed for every morphism $h : d \to d'$ in $\mathscr{D}$ we have

$$Fc_{d'} \cdot \alpha_{d'} \cdot UDh = Fc_{d'} \cdot FUDh \cdot \alpha_d = Fc_d \cdot \alpha_d,$$

using that $Dh : (A_d, \alpha_d) \to (A_{d'}, \alpha_{d'})$ is a coalgebra homomorphism and that $c_{d'} \cdot UDh = c_d$.

It remains to verify the universal property of the cocone $c_d : (A_d, \alpha_d) \to (C, \gamma)$ in $\mathsf{Coalg}\, F$. Let $f_d : (A_d, \alpha_d) \to (B, \beta)$ be a cocone of D. Then $f_d : A_d \to B$ is a cocone of UD. Thus there exists a unique morphism $f : C \to B$ of $\mathscr{A}$ with $f \cdot c_d = f_d$ for all $d \in \mathsf{obj}\, \mathscr{D}$. To see that f is a homomorphism consider the diagram below:

$$\begin{tikzcd} A_d \arrow[r, "\alpha_d"] \arrow[d, "c_d"'] \arrow[dd, bend right=60, "f_d"'] & FA_d \arrow[d, "Fc_d"] \arrow[dd, bend left=60, "Ff_d"] \\ C \arrow[r, "\gamma"] \arrow[d, "f"'] & FC \arrow[d, "Ff"] \\ B \arrow[r, "\beta"'] & FB \end{tikzcd} \tag{4.1}$$

The left-hand and right-hand parts, the upper square and the outside of the diagram clearly commute. Thus so does the lower square when precomposed by every colimit injection c_d. Since the colimit injections c_d are collectively epimorphic, this proves that f is a coalgebra homomorphism, as desired. □

Remark 4.1.2. Dually, all limits of F-algebras are formed on the level of $\mathscr{A}$. More precisely, creation of limits is the dual concept of creation of colimits, and for every endofunctor $F : \mathscr{A} \to \mathscr{A}$ the forgetful functor $\mathsf{Alg}\, F \to \mathscr{A}$ creates limits.

For limits of coalgebras the situation is more involved, as they are not always formed on the level of $\mathscr{A}$. However, limits which F preserves are:

Proposition 4.1.3. *For every endofunctor F on $\mathscr{A}$ preserving limits of scheme $\mathscr{D}$, the forgetful functor $U : \mathsf{Coalg}\, F \to \mathscr{A}$ creates limits of that scheme.*

Proof. Given is a diagram $D : \mathscr{D} \to \mathsf{Coalg}\, F$ with $Dd = (A_d, \alpha_d)$ and a limit of UD with limit projections

$$\pi_d : L \to A_d \qquad (d \in \mathsf{obj}\, \mathscr{D}).$$

We need to prove that there exists a unique coalgebra structure $\lambda : L \to FL$ making each π_d a homomorphism and that, moreover, $\pi_d : (L, \lambda) \to (A_d, \alpha_d)$ form the limit of D. In order to obtain a unique λ such that all squares

$$\begin{tikzcd}
L \arrow[r, dashed, "\lambda"] \arrow[d, "\pi_d"'] & FL \arrow[d, "F\pi_d"] \\
A_d \arrow[r, "\alpha_d"'] & FA_d
\end{tikzcd}$$

are commutative, we use that, since F preserves the limit of UD, the diagram FUD has the limit cone given by the $F\pi_d : FL \to FA_d$, $d \in \mathsf{obj}\, \mathscr{D}$. Thus, we show that $\alpha_d \cdot \pi_d$ form a compatible cone of that diagram. Indeed, for every morphism $f : d \to d'$ of $\mathscr{D}$ we have:

$$FUDh \cdot \alpha_a \cdot \pi_d = \alpha_{d'} \cdot UDh \cdot \pi_d = \alpha_{d'} \cdot \pi_{d'}.$$

It remains to verify that the cone $\pi_d : (L, \lambda) \to (A_d, \alpha_d)$ is a limit of D in $\mathsf{Coalg}\, F$. Let $f_d : (B, \beta) \to (A_d, \alpha_d)$ be a cone of D. Then $f_d : B \to A_d$ is a cone of UD. Thus there exists a unique morphism $f : B \to L$ such that $\pi_d \cdot f = f_d$ for all $d \in \mathsf{obj}\, \mathscr{D}$. To see that f is a homomorphism, we argue dually as for Diagram (4.1), i.e. we consider the following diagram

$$\begin{tikzcd}
L \arrow[r, "\lambda"] \arrow[d, "f"'] \arrow[dd, bend right=60, "f_d"'] & FL \arrow[d, "Ff"] \arrow[dd, bend left=60, "Ff_d"] \\
B \arrow[r, "\beta"] \arrow[d, "\pi_d"'] & FB \arrow[d, "F\pi_d"] \\
A_d \arrow[r, "\alpha_d"'] & FA_d
\end{tikzcd}$$

and use that the $F\pi_d$ are collectively monomorphic. □

Remark 4.1.4. Dually, colimits of algebras are not always formed on the level of $\mathscr{A}$. However, for every endofunctor F preserving colimits of a scheme $\mathscr{D}$ the forgetful functor $\mathsf{Alg}\, F \to \mathscr{A}$ creates colimits of that scheme.

4.2 Weakly terminal coalgebras

We now discuss weakly terminal coalgebras for a given endofunctor. It turns out that the terminal coalgebra for a functor F is the quotient of (any) weakly terminal coalgebra. A beautiful example: the coalgebra of all finitely branching unordered trees is weakly terminal for $\mathscr{P}_f$.

Definition 4.2.1. Let $F : \mathscr{A} \to \mathscr{A}$ be an endofunctor. An F-coalgebra A is *weakly terminal* if for every coalgebra B there is *at least one* homomorphism of B into A.

Examples 4.2.2. (1) The terminal coalgebra for the identity functor $\mathsf{Id} : \mathsf{Set} \to \mathsf{Set}$ is the singleton coalgebra. A coalgebra $\alpha : A \to A$ is weakly terminal iff α has a fixed point.

(2) If A is a weakly terminal coalgebra and $f : A \to B$ a coalgebra homomorphism, then B is again weakly terminal.

(3) We next consider the finite power-set functor $\mathscr{P}_\mathsf{f} : \mathsf{Set} \to \mathsf{Set}$. The coalgebra D of all unordered finitely branching trees is weakly terminal for $\mathscr{P}_\mathsf{f}$. Recall from Example 2.4.2(4) that coalgebras of $\mathscr{P}_f$ are the finitely branching graphs. The graph structure of D has edges from a tree t to all maximum proper subtrees of t.

Every coalgebra A has a canonical coalgebra homomorphism $h : A \to D$: it assigns to every node x the *tree unfolding* of x. This is the unordered tree obtained by breadth-first search of the graph A starting at x. To see that h is a coalgebra homomorphism, observe that for every neighbor y of x the breadth-first seach $h(y)$ is indeed a maximum subtree of $h(x)$. Conversely, every maximum subtree of $h(x)$ has the form $h(y)$ for a neighbor y of x.

However, D is not terminal: the one-point "loop graph" has many homomorphisms into D; for example we can map it to the infinite binary tree or to the infinite chain. In Example 4.2.11 below, we shall see a quotient of D which is a terminal $\mathscr{P}_\mathsf{f}$-coalgebra.

(4) The power-set functor $\mathscr{P} : \mathsf{Set} \to \mathsf{Set}$ does not have a weakly terminal coalgebra. Indeed, from a weakly terminal coalgebra we can always construct a terminal one, as we now prove.

Remark 4.2.3. (1) Recall that, dually to Remark 2.1.12, quotient objects of an object A of a category $\mathscr{A}$ are represented (uniquely up to isomorphism of the codomain) by epimorphisms $e : A \to \bar{A}$. The quotients of A are ordered by factorization: given quotients $e : A \to \bar{A}$ and $f : A \to A^*$ we write $e \leq f$ iff e factorizes through f:

$$\begin{tikzcd} & A \arrow[dl, "e"'] \arrow[dr, "f"] & \\ \bar{A} & & A^* \arrow[ll, dashed] \end{tikzcd}$$

Thus for $\mathscr{A} = \mathsf{Set}$ the quotients of a set A are ordered by their cardinality as usual, but equivalence relations on A, representing quotients are ordered dually: $A/{\sim}$ is smaller or equal to $A/{\approx}$ iff $\approx$ is contained in $\sim$. (The smallest quotient thus has codomain 1 whenever $A \neq \emptyset$.)

(2) Given a pushout of epimorphisms f_1, f_2:

$$\begin{tikzcd} A \arrow[r, two heads, "f_1"] \arrow[d, two heads, "f_2"'] & A_1 \arrow[d, two heads, "g_1"] \\ A_2 \arrow[r, two heads, "g_2"'] & B \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

it follows that g_1, g_2 are epimorphisms, too. Thus, the quotient of A represented by $e = g_i \cdot f_i : A \twoheadrightarrow B$ is the meet of f_1 and f_2 in the poset of quotients of A.

(3) More generally, a meet of a collection of quotients $f_i : A \to A_i$ $(i \in I)$ is represented by their wide pushout.

Recall that the *kernel equivalence* of a map $f : X \to Y$ in Set is the equivalence relation $\ker(f) = \{(x, x') : f(x) = f(x')\} \subseteq X \times X$. Categorically speaking $\ker(f)$ is the (vertex of the) pullback of f along itself.

Definition 4.2.4. Let $\alpha : A \to FA$ be a coalgebra. A *quotient coalgebra* on it is a quotient $e : A \twoheadrightarrow \bar{A}$ in the base category $\mathscr{A}$ for which there is an (obviously unique) coalgebra structure $\overline{\alpha} : \bar{A} \to F\bar{A}$ making e a coalgebra homomorphism:

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, two heads, "e"'] & FA \arrow[d, "Fe"] \\ \bar{A} \arrow[r, "\overline{\alpha}"'] & F\bar{A} \end{tikzcd}$$

In the case where $\mathscr{A} = \mathsf{Set}$, we often identify the epimorphism e with its kernel equivalence $\sim$ given by $a \sim b$ iff $e(a) = e(b)$; the kernel equivalence of a quotient coalgebra is called a *congruence*. We write $A/{\sim}$ for the above coalgebra $\bar{A}$.

Remark 4.2.5. Quotient coalgebras of (A, α) are ordered as shown in Remark 4.2.3(1). For quotient coalgebras $e : A \to \bar{A}$ and $f : A \to A^*$ we write $e \leq f$ if e factorizes through f, i.e. there is $g : A^* \to \bar{A}$ with $e = g \cdot f$. It follows that g is a coalgebra homomorphism:

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "f"'] \arrow[dd, bend right=60, "e"'] & FA \arrow[d, "Ff"] \arrow[dd, bend left=60, "Fe"] \\ A^* \arrow[r, "\alpha^*"] \arrow[d, "g"'] & FA^* \arrow[d, "Fg"] \\ \bar{A} \arrow[r, "\bar{\alpha}"'] & F\bar{A} \end{tikzcd}$$

Example 4.2.6. Let G be a finitely branching graph, considered as a coalgebra for $\mathscr{P}$. A congruence on G is then precisely an equivalence relation on G which is also a *graph bisimulation*, i.e., a relation $R \subseteq G \times G$ on the nodes such that whenever xRy, then

$$\begin{array}{l} \text{for every child } x' \text{ of } x \text{ there is some child } y' \text{ of } y \text{ such that } x'Ry', \text{ and} \\ \text{for every child } y' \text{ of } y \text{ there is some child } x' \text{ of } x \text{ such that } x'Ry'. \end{array} \tag{4.2}$$

We leave the easy proof to the reader.

Remark 4.2.7. Although we will not be using the concept of bisimulation in this section, we remind readers familiar with it that the largest bisimulation relation on a coalgebra for a set functor which preserves weak pullbacks is always an equivalence relation, and hence a congruence (see Rutten [149, Corollary 5.6]).

Proposition 4.2.8. *Given a coalgebra (A, α) for a set functor, a meet of its quotient coalgebras (in the poset of quotients of A in* Set*) is a quotient coalgebra.*

Indeed, this follows from Proposition 4.1.1.

Theorem 4.2.9. *Let F be an endofunctor on a complete and cowellpowered category. Let (A, α) be a weakly terminal F-coalgebra. Then a smallest quotient coalgebra $e : (A, \alpha) \to (\bar{A}, \bar{\alpha})$ exists, and $(\bar{A}, \bar{\alpha})$ is a terminal coalgebra for F.*

Proof. Since the base category $\mathscr{A}$ is cowellpowered, A has only a set of quotients, and the corresponding lattice is complete, because meets exist via wide pushouts.

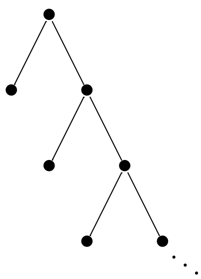
By the Proposition 4.2.8, the meet of all quotient coalgebras (in the poset of quotients of A) is a quotient coalgebra, and this obviously is the smallest quotient coalgebra $e : (A, \alpha) \to (\bar{A}, \bar{\alpha})$. Moreover, $(\bar{A}, \bar{\alpha})$ is obviously, a weakly terminal coalgebra (since (A, α) is, cf. Example 4.2.2(2)). It remains to prove that if $f, g : B \to \bar{A}$ are two coalgebra homomorphisms, then $f = g$. To this end, take the coequalizer k of f and g in $\mathscr{A}$. Then since f and g are coalgebra homomorphisms, so is k by Proposition 4.1.1. Being a coequalizer, k is an epimorphism. Thus, $k \cdot e$ is a quotient coalgebra of A. But the choice of e as the least quotient coalgebra implies that $e \leq k \cdot e$, i.e. $e = x \cdot k \cdot e$ for some x. It follows that $x \cdot k = \mathsf{id}$, thus k is an epi and a split mono, whence an isomorphism. Therefore $f = g$, as desired. □

Remark 4.2.10. Note that the above argument is a special instance of the (dual of) the classical characterization of the existence of an initial object [124, Theorem V.6.1] which is at the heart of Freyd's Adjoint Functor Theorem.

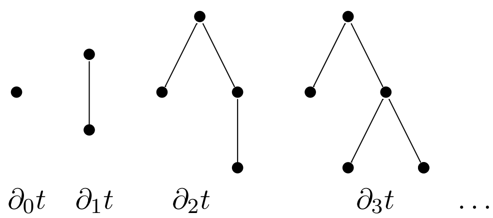
Example 4.2.11. The terminal coalgebra of $\mathscr{P}_f$ is the quotient $D/{\approx}$ of the coalgebra D of all finitely branching trees modulo the greatest congruence. To describe $\approx$, recall from Example 3.2.9 that the terminal chain is given by

$$P_f^n 1 = \text{ all extensional trees of depth } \leq n.$$

Given a tree $t \in D$, considered as a coalgebra (see Example 2.4.4), the canonical cone from t to $\mathscr{P}_f^n 1$ of Construction 3.3.2 assigns to t the following tree $\partial_n t$: we cut t at level n and let $\partial_n t$ be the extensional quotient of this cutting. Example: the following tree t



has cuttings



Formally, the cutting of t at level n is the set of points whose distance from the root of t is at most n, considered as an induced subgraph of t. For example, $\partial_0 t$ is a one-point tree.

We prove later, see Example 4.3.20, that the greatest congruence $\approx$ of D is defined by $t \approx s$ iff $\partial_n t = \partial_n s$ holds for all n.

4.3 Presentation of set functors

Just as quotient objects of an object A are represented by epimorphisms with domain A, *quotient functors* of a functor H are represented by natural *epi-transformations* with domain H. That is, natural transformations with all components epic. An important special case are the set functors that are quotients of the polynomial functors H_Σ: these functors have an equational presentation using the signature Σ.

The main results of this section relate presentations of set functors to initial algebras and terminal coalgebras. If F is a quotient of H_Σ, then the initial algebra μF is a quotient of the initial algebra (of all finite ordered Σ-trees) of H_Σ and the terminal coalgebra of H_Σ (of all ordered Σ-trees) is weakly terminal for F.

Lemma 4.3.1. *Let H be an endofunctor with a terminal coalgebra $\tau : T \to HT$. Then every quotient functor $\varepsilon : H \twoheadrightarrow F$, where every component ε_X is a split epi has the following weakly terminal coalgebra:*

$$T \xrightarrow{\tau} HT \xrightarrow{\varepsilon_T} FT.$$

Proof. Let $\alpha : A \to FA$ be any F-coalgebra. Since $\varepsilon_A : HA \to FA$ is a split epimorphism, we can choose a morphism $m : FA \to HA$ such that $\varepsilon_A \cdot m = \mathsf{id}$. So we obtain an H-coalgebra $m \cdot \alpha : A \to HA$. We thus have an H-coalgebra homomorphism $h : (A, m \cdot \alpha) \to (T, \tau)$, and we show that h is also an F-coalgebra homomorphism from (A, α) to $(T, \varepsilon_T \cdot \tau)$. Indeed, consider the diagram below:

$$\begin{tikzcd}
A \arrow[r, "m\cdot\alpha"] \arrow[rr, bend left, "\alpha"] \arrow[d, "h"'] & HA \arrow[r, "\varepsilon_A"] \arrow[d, "Hh"] & FA \arrow[d, "Fh"] \\
T \arrow[r, "\tau"'] & HT \arrow[r, "\varepsilon_T"'] & FT
\end{tikzcd}$$

The top commutes because $\varepsilon_A \cdot m = \mathsf{id}$, the square on the left by definition of h, and the one on the right by naturality of ε. Thus the outside commutes, showing h to be a morphism of F-coalgebras. $\square$

Definition 4.3.2. By a *presentation* of a set functor F we mean a signature Σ and a natural epi-transformation $\varepsilon : H_\Sigma \twoheadrightarrow F$. If Σ is finitary we call the presentation finitary.

Examples 4.3.3. (1) The functor $\mathscr{P}_3$ takes a set X to the set of subset of X of size less then 3. This functor has a presentation given by the signature Σ with one constant and one binary operation symbol. Hence $H_\Sigma X = 1 + X \times X$, and the natural epi-transformation $\varepsilon : H_\Sigma \twoheadrightarrow F$ is given by

$$\varepsilon(\bullet) = \emptyset \qquad \text{and} \qquad \varepsilon_X(x, y) = \{x, y\} \qquad \text{for every } x, y \in X,$$

where $\bullet$ denotes the element of 1.

(2) A presentation of $\mathscr{P}_\mathsf{f}$ is given by the signature Σ having, for each natural number n, just one function symbol σ_n of arity n. Thus,

$$H_\Sigma X = \coprod_{n<\omega} X^n = X^*.$$

The natural transformation $\varepsilon : H_\Sigma \to \mathscr{P}_\mathsf{f}$ given by

$$\varepsilon_X(x_1, \ldots, x_n) = \{x_1, \ldots, x_n\}.$$

is an epi-transformation.

(3) A presentation for the bag functor $\mathscr{B}$ (see Example 3.2.10) can be easily obtained from (3.10). Take the same Σ as in the previous item and let $\varepsilon_X : H_\Sigma X \to \mathscr{B}X$ be given as the coproduct of the quotients $X^n \to X^n/S_n$, $n \in \mathbb{N}$, where S_n is the symmetric group.

(4) The *Aczel-Mendler functor* [5] is the set functor $(-)^3_2$ given by

$$(X)^3_2 = \{(a, b, c) \in X \times X \times X : (a = b) \text{ or } (a = c) \text{ or } (b = c)\}.$$

Note that if $(a, b, c) \in (X)^3_2$ and $f : X \to Y$ is any function, then $(fa, fb, fc) \in (Y)^3_2$. We therefore take the action of $(-)^3_2$ on morphisms to be pointwise application, as expected. For a presentation of $(-)^3_2$ we take Σ to have three binary symbols, say σ, τ, and ρ. Then we define $\varepsilon : H_\Sigma \to (-)^3_2$ by: $\varepsilon_X(\sigma(a, b)) = (a, a, b)$, $\varepsilon_X(\tau(a, b)) = (a, b, a)$, and $\varepsilon_X(\rho(a, b)) = (b, a, a)$.

(5) The countable power-set functor $\mathscr{P}_c$ (of all countable subsets) has a presentation given by an ω-ary operation σ and a constant c. The natural transformation $\varepsilon : H_\Sigma \to \mathscr{P}_c$ has components $\varepsilon_X : X^{\mathbb{N}} + \{c\}$ given by $(x_i)_{i\in\mathbb{N}} \mapsto \{x_i\}_{i\in\mathbb{N}}$ and $c \mapsto \emptyset$.

From now on, we shall work only with *finitary* endofunctors on Set. Intuitively, a functor on sets is finitary if its behavior is completely determined by its action on *finite* sets and functions. For a general functor, this intuition is captured by requiring that the functor preserves filtered colimits. For a set functor F this is equivalent to being *finitely bounded*, which is the following condition: for each element $x \in FX$ there exists a finite subset $M \subseteq X$ such that $x \in Fi[FM]$, where $i\colon M \hookrightarrow X$ is the inclusion map. For more on this, see **??**, in particular, in **??** it is shown that the finitary set functors are precisely those with a finitary presentation.

Remark 4.3.4. Note that every presentation $\varepsilon : H_\Sigma \to F$ induces a functor from the category of H_Σ-coalgebras into the category of F-coalgebras by post-composition:

$$(\, A \xrightarrow{\alpha} H_\Sigma A \,) \qquad \longmapsto \qquad (\, A \xrightarrow{\alpha} H_\Sigma A \xrightarrow{\varepsilon_A} FA \,).$$

For example, the terminal coalgebra $\nu H_\Sigma = T_\Sigma$ of all ordered Σ-trees can be understood as an F-coalgebra.

Corollary 4.3.5. *If a finitary functor F has a presentation $\varepsilon : H_\Sigma \twoheadrightarrow F$, then F has a terminal coalgebra given by the smallest quotient coalgebra of the F-coalgebra of all Σ-trees.*

Indeed, this follows from Lemma 4.3.1 amd Theorem 4.2.9.

Remark 4.3.6. Corollary 4.3.5 above stems from Gumm and Schröder [86] where a more general situation is considered: an epi-transformation $\varepsilon : G \twoheadrightarrow F$ where G has a terminal coalgebra.

Example 4.3.7. The coalgebra D of all finitely branching nonordered trees is weakly terminal for $\mathscr{P}_\mathsf{f}$. We saw this in Example 4.2.2(3). But we can also apply Lemma 4.3.1 to the presentation of $\mathscr{P}_\mathsf{f}$ in Example 4.3.3(2) to see that the algebra T_Σ of all finitely branching ordered trees is weakly terminal.

Remark 4.3.8. The presented functor F is determined (up to natural isomorphism) by the kernel equivalences of $\varepsilon_X : H_\Sigma X \to FX$ for all sets X (of variables). Moreover, since F is finitary, we can restrict our attention to finite sets X. We can describe the kernel equivalence of ε_X as a set of pairs of elements of $H_\Sigma X = \coprod_{\sigma\in\Sigma} X^n$ (where n is the arity of σ). Each such element is, for some $\sigma \in \Sigma_n$, an n-tuple $(x_0, \dots, x_{n-1})$. We use the notations $\sigma(x_0, \dots, x_{n-1})$, and we write the pairs contained in the kernel equivalence of ε_X as

$$\sigma(x_0, \dots, x_{x-1}) = \tau(y_0, \dots, y_{m-1}),$$

where $\sigma \in \Sigma_n$, $\tau \in \Sigma_m$ and all x_i and y_j are elements of X. These equations have the following special form:

Definition 4.3.9. A term over Σ is called *basic* if it contains precisely one symbol from Σ. That is, it has the form $\sigma(x_0, \dots, x_{n-1})$ for some $\sigma \in \Sigma_n$ and n variables x_i.

A *basic equation* over X is a pair of basic terms with variables from the set X.

Note that $H_\Sigma X$ is thus the set of all basic terms with variables in the set X.

Notation 4.3.10. Let Σ be a signature, and let $\mathcal{E}$ be a set of basic equations over the fixed set X_0 of variables. We defines a quotient functor $F = H_\Sigma/\mathcal{E}$ of H_Σ as follows: given a set X form the smallest equivalence on $H_\Sigma X$ given by all basic equations

$$\sigma(s(x_0), \dots, s(x_{n-1})) = \tau(s(y_0), \dots, s(y_{m-1})),$$

for every $s : X_0 \to X$ and every equation $\sigma(x_0, \dots, x_{n-1}) = \tau(y_0, \dots, y_{m-1})$ in $\mathcal{E}$. Then FX is the quotient of $H_\Sigma X$ modulo that equivalence. To every function $f : X \to Y$, F assigns the function taking the equivalence class of $\sigma(x_0, \dots, x_{n-1})$ to that of $\sigma(f(x_0), \dots, f(x_{n-1})) \in H_\Sigma Y$.

Remark 4.3.11. Note that, the above smallest equivalence on $H_\Sigma X$ consists of all basic equations over X obtained by applying the following deduction rules (here we simply write x for a tuple $(x_0, \dots x_{n-1})$):

$$\text{(axiom)}\ \frac{}{\sigma(x) = \tau(y)} \quad \text{for all } \sigma(x) = \tau(y) \text{ in } \mathcal{E}$$

$$\text{(reflexivity)}\ \frac{}{\sigma(x) = \sigma(x)}$$

$$\text{(symmetry)}\ \frac{\sigma(x) = \tau(y)}{\tau(y) = \sigma(x)}$$

$$\text{(transitivity)}\ \frac{\sigma(x) = \tau(y), \quad \tau(y) = \rho(z)}{\sigma(x) = \rho(z)}$$

$$\text{(substitution)}\ \frac{\sigma(x) = \tau(y)}{\sigma(s(x)) = \tau(s(y))} \quad \text{for all functions } s : X_0 \to X$$

These rules are a restricted form of the standard rules of equational logic of General Algebra; in fact, the substitution rule is restricted to substitutions that replace variables in X_0 by variables in X.

Proposition 4.3.12 [6, Thm. 3.12]**.** *Every finitary set functor F can be presented by a finitary signature Σ and a set $\mathcal{E}$ of basic equations in the sense that F and $H_\Sigma/\mathcal{E}$ are naturally isomorphic functors.*

Examples 4.3.13. (1) The functor $\mathscr{P}_3$ is presented by the signature Σ with one constant c and one binary operation symbol $*$ modulo the equation $x * y = y * x$ expressing commutativity of $*$
(2) $\mathscr{P}_f$ is presented by the signature $\Sigma = \{\sigma_n\}_{n<\omega}$ with σ_n n-ary and all equations

$$\sigma_n(x_0, \dots, x_{n-1}) = \sigma_m(y_0, \dots, y_{m-1})$$

for which the sets $\{x_0, \dots, x_{n-1}\}$ and $\{y_0, \dots, y_{m-1}\}$ are equal. For example $\sigma_2(x, y) = \sigma_3(x, y, x)$ is one of these equations.
(3) Let $FX = \coprod_{\sigma\in\Sigma} X^k/G_\sigma$ be an analytic functor, see Example 3.2.11, where k is the arity of σ and G_σ is the given group of permutations on k. Then F is presented by the signature Σ and the following basic equations:

$$\sigma(x_1, \dots, x_k) = \sigma(x_{p(1)}, \dots, x_{p(k)}) \qquad \text{for all } \sigma \in \Sigma \text{ and all } p \in G_\sigma.$$

(4) We have seen the Aczel-Mendler $(-)^3_2$ functor in Example 4.3.3(4). This functor is presented by the basic equations

$$\sigma(x, x) = \tau(x, x) \qquad \text{and} \qquad \tau(x, x) = \rho(x, x).$$

We now consider algebras of set functors in connection with functor presentations.

Remark 4.3.14. (1) Given a presentation $F \cong H_\Sigma/\mathcal{E}$, the category of F-algebras is equivalent to the variety of all Σ-algebras satisfying $\mathcal{E}$.

(2) Conversely, every variety of Σ-algebras presented by basic equations is equivalent to $\mathsf{Alg}\, F$ for a finitary set functor F. This is proved by Adámek and Trnková [38, Chapter III].

Given a presentation $\varepsilon : H_\Sigma \to F$ of a set functor F, we can consider the category $\mathsf{Alg}\, F$ of all algebras for F as a full subcategory of $\mathsf{Alg}\, H_\Sigma$, following Remark 4.3.14(1). More precisely, we have a full embedding assigning to every algebra $\alpha : FA \to A$ for F the corresponding Σ-algebra $\alpha \cdot \varepsilon_A : H_\Sigma A \to A$ and defined on homomorphisms by $f \mapsto f$.

Analogously to quotient coalgebras we have the concept of *quotient algebra* of $\alpha : FA \to A$: it is an epimorphism $e : A \twoheadrightarrow \bar{A}$ in $\mathscr{A}$ for which an algebra structure $\bar{\alpha}$ exists making it a homomorphism:

$$\begin{tikzcd} FA \arrow[r, "\alpha"] \arrow[d, two heads, "Fe"'] & A \arrow[d, two heads, "e"] \\ F\bar{A} \arrow[r, "\bar{\alpha}"'] & \bar{A} \end{tikzcd}$$

Since all set functors preserve epimorphisms, Fe is an epimorphism, and hence the structure morphism $\bar{\alpha}$ is unique.

Proposition 4.3.15 [27]. *Given a presentation $\varepsilon : H_\Sigma \to F$, the initial algebra μF is the largest quotient of the initial Σ-algebra lying in* $\mathsf{Alg}\, F$.

More concretely,

$$\mu F = F_\Sigma/{\sim},$$

where for two finite Σ-trees s, t we have

$$s \sim t \quad \text{iff } s \text{ and } t \text{ can be equated by finitely many applications of the basic equations for } \varepsilon. \tag{4.3}$$

That means that $s = t$ is derivable using the standard rules of equational logic.

Example 4.3.16. The initial algebra for $\mathscr{P}_3$ is the quotient of $\mu X.(1 + X \times X)$, the algebra of all finite binary ordered tree of Example 2.2.15, modulo the least congruence merging trees $x * y$ and $y * x$. Consequently, the linear ordering of children is simply forgotten:

$$\mu\mathscr{P}_3 = \text{all finite binary unordered trees.}$$

Remark 4.3.17. Given a presentation $\varepsilon : H_\Sigma \to F$ of a finitary set functor, then, as shown in [27], a similar description holds for the terminal coalgebra; we have

$$\nu F = T_\Sigma/{\approx},$$

where $\approx$ is the congruence of finite and *infinite* applications of the basic equations. Of course, we have to make clear what is meant by "infinite application" of basic equations:

Notation 4.3.18. Recall from Theorem 3.3.10 that for every Σ-tree t, the limit projection $\ell_n : T_\Sigma = \nu H_\Sigma \to H_\Sigma^n 1$ assigns to every Σ-tree t the cutting of t at level n, where all leaves of that level are relabelled by $\perp$. Thus $l_n s$ is a finite $\Sigma_\perp$-tree where $\Sigma_\perp$ is the signature Σ with an additional nullary symbol $\perp$. We define an equivalence relation $\approx$ on T_Σ as follows: for two Σ-trees s and t we put

$$s \approx t \qquad \text{iff for all } n < \omega \text{ we have } \ell_n s \sim \ell_n t. \tag{4.4}$$

Here $\sim$ is the congruence of (4.3) for the quotient functor $\varepsilon_X + \{\perp\} \colon H_{\Sigma_\perp} X \to FX + \{\perp\}$.

Recall from Corollary 4.3.5 that the terminal coalgebra $\tau : T_\Sigma \to HT_\Sigma$ (of all ordered Σ-trees) is a weakly terminal coalgebra for F.

Theorem 4.3.19 [27]. *Given a presentation $\varepsilon : H_\Sigma \to F$ of a finitary set functor F, the terminal coalgebra of F is a quotient of the F-algebra of all Σ-trees T_Σ, modulo the above equivalence $\approx$; in symbols:*

$$\nu F = T_\Sigma / {\approx}.$$

Example 4.3.20. For the finite power-set functor $\mathscr{P}_f$ consider the presentation

$$\varepsilon_X : X^* \to \mathscr{P}_f X$$

of Example 4.3.3. Since Σ has one n-ary operation for every n, we know that the terminal coalgebra T_Σ is the coalgebra of all finitely branching ordered trees. For finite trees t and u the equivalence $\sim$ of (4.3) is easily seen to be

$$t \sim u \qquad \text{iff the extensional quotients of } t \text{ and } u \text{ are equal.}$$

(Recall our convention that isomorphic trees are identified.) By applying (4.4) we conclude that for arbitrary trees t and u in T_Σ we have in the notation of Example 4.2.11

$$t \approx u \qquad \text{iff } \partial_n t = \partial_n u \text{ for all } n < \omega.$$

And we have $\nu \mathscr{P}_f = T_\Sigma / {\approx}$.

Remark 4.3.21. The above equivalence ignores the linear order on the children of nodes, therefore, the example above can be formulated using the algebra D of all nonordered finitely branching trees of Example 4.2.2(3): denoting by $t \approx u$ the above equivalence for unordered trees, we have

$$\nu \mathscr{P}_f = D / {\approx}$$

This description of $\nu \mathscr{P}_f$ is due to Barr [49].

Remark 4.3.22. There is an interesting connection of the last result to the terminal-coalgebra ω^{op}-chains. Let $\varepsilon : H_\Sigma \twoheadrightarrow F$ be a presentation. Firstly, ε induces a natural transformation γ from the terminal chain of H_Σ to the terminal-coalgebra ω^{op}-chain of F by induction: $\gamma_0 = \mathsf{id}_1$ and

$$\gamma_{n+1} = (H_\Sigma H_\Sigma^n 1 \xrightarrow{\varepsilon_{H_\Sigma^n 1}} F H_\Sigma^n 1 \xrightarrow{F\gamma_n} FF^n 1).$$

Hence we obtain the following commutative diagram:

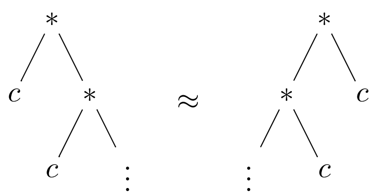
$$\begin{tikzcd}
1 \arrow[dd, equal, "\gamma_0"'] & H_\sigma 1 \arrow[l, "!"'] \arrow[dd, two heads, "\gamma_1=\varepsilon_1"'] & H_\Sigma H_\Sigma 1 \arrow[l, "H_\Sigma !"'] \arrow[d, two heads, "\varepsilon_{H_\Sigma 1}"] \arrow[dd, bend right, two heads, "\gamma_2"'] & \cdots \arrow[l, "H_\Sigma H_\Sigma !"'] \\
 & & FH_\Sigma 1 \arrow[d, two heads, "F\gamma_1"] & \\
1 & F1 \arrow[l, "!"] & FF1 \arrow[l, "F!"] & \cdots \arrow[l, "FF!"]
\end{tikzcd}$$

It is not difficult to prove that $\gamma_n : H_\Sigma^n 1 \twoheadrightarrow F_\Sigma^n 1$ is an epimorphism with the kernel equivalence given by the congruence $\sim$ of finite applications of basic equations from (4.3). Thus, for every $s, t \in T_\Sigma$ we have

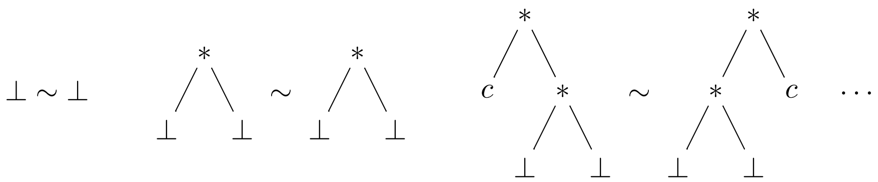
$$s \approx t \qquad \text{iff } \gamma_n \cdot \ell_n(s) = \gamma_n \cdot \ell_n(t) \text{ for all } n < \omega,$$

where ℓ_n is cutting trees at level n and γ_n is the quotient of finite application of basic equations.

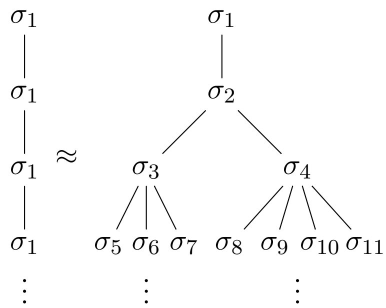
Example 4.3.23. (1) We continue Example 4.3.16 where $F = \mathscr{P}_3$. Here is a pair of trees which are merged by $T_\Sigma \to \nu F$:



Moreover, we can check that this pair is related by $\approx$, since we have



(2) For the finite power-set functor $\mathscr{P}_\mathsf{f}$ recall the presentation $\varepsilon_X : X^* \to \mathscr{P}_f X$ of Example 4.3.3(1). Here we have, for example,



similarly as in (1) above. (Notice that these Σ-trees represent the extensional trees in (4.9).)

(3) We have introduced analytic functors F on Set in Example 3.2.11. By Theorem 4.3.19 we have a direct description of νF: Let $F = \coprod_{\sigma\in\Sigma} X^k/G_\sigma$ be an analytic functor, where k is the arity of σ and G_σ is the given group of permutations on k. Then the terminal coalgebra is the quotient

$$\nu F = T_\Sigma/{\approx}$$

of the Σ-tree coalgebra modulo the equivalence $\approx$ analogous to $\sim$ of (4.3) but allowing infinitely many permutations of children of nodes, i.e., νF is the coalgebra of all Σ-trees modulo permutations of children of any σ-labelled node (using elements of the permutation group asssociated with σ).

(4) In particular, for the bag functor $\mathscr{B}X = \coprod_{n\in\mathbb{N}} X^n/S_n$ (cf. Example 3.2.10), the terminal coalgebra $\nu\mathscr{B}$ is the coalgebra of all finitely branching non-ordered trees. In fact, the corresponding polynomial functor is $FX = X^*$. We know that νF is the coalgebra of all finitely branching trees. And $\nu\mathscr{B}$ is the quotient coalgebra of νF given by allowing arbitrary permutations of children. This means that the carrier consists of all unordered trees.

Remark 4.3.24. We have seen *congruences* in Definition 4.2.4 on page 81. To determine whether or not a given relation is a congruence is often tedious, as is the task of coming up with a congruence that relates two points in a given coalgebra. The point of the definition below is to make both of these tasks easier.

A relation $r \subseteq A \times A$ generates a smallest equivalence relation r^* containing r; let $q : A \twoheadrightarrow A/r^*$ denote the corresponding quotient. Given a coalgebra

$$\alpha : A \to FA$$

for a set functor, a *precongruence* on it is a relation r contained in the kernel equivalence of $Fq \cdot \alpha : A \to F(A/r^*)$.

The following is essentially proved by Sprunger and Moss [157].

Theorem 4.3.25. *Let $\varepsilon : H_\Sigma \to F$ be a presentation. Let (A, α) be an H-coalgebra, so that $(A, \varepsilon_A \cdot \alpha)$ is an F-coalgebra. Let r be a relation on A with the property that $r \sqsubseteq \ker(\varepsilon_{A/q} \cdot Hq \cdot \alpha)$. Then*

(1) *r is a precongruence for F.*

(2) *Assume that r has the property that for every pair (a, b) in r, $\alpha(a) \equiv_{\mathcal{E},r} \alpha(b)$. Then r is contained in the kernel equivalence of the unique coalgebra homomorphism from $(A, \varepsilon_A \cdot \alpha)$ to νF.*

The last part of Theorem 4.3.25 gives a *coinduction principle*: to show that two elements of HA have the same image in the terminal F-coalgebra, it is enough to exhibit a relation r on A so that whenever $r(a, b)$, $\alpha(a) \equiv_{\mathcal{E},r} \alpha(b)$.

Example 4.3.26. We revisit Example 4.3.23. We take Σ to have a binary symbol $*$ (written in infix notation) and a constant c. The set $\mathcal{E}$ has just one basic equation: $x * y = y * x$. The coalgebra is (A, α), where $A = \{a, b, z\}$, $\alpha(a) = a * z$, $\alpha(b) = z * b$, and

$\alpha(z) = c$. Consider the relation $r = \{(a, b)\}$ on A. Let us verify that $\alpha(a) \equiv_{\mathcal{E},r} \alpha(b)$:

$$\begin{array}{rcll} a * z & \equiv_{\mathcal{E},r} & z * a & \text{due to } x * y = y * x \\ & \equiv_{\mathcal{E},r} & z * b & \text{since } r^*(a,b) \text{ and } r^*(z,z) \end{array}$$

So for the $\mathscr{P}_3$-coalgebra $(A, \varepsilon_A \cdot \alpha)$, a and b have the same image in $\nu\mathscr{P}_3$.

Example 4.3.27. We return to analytic functors, see Examples 3.2.11 and 4.3.13(3). Consider any analytic functor F with $G_3 = A_3$, the three cyclic permutations of $\{1, 2, 3\}$. For example, $1 \mapsto 2$, $2 \mapsto 3$, $3 \mapsto 1$ is cyclic, but $1 \mapsto 3$, $2 \mapsto 2$, $3 \mapsto 1$ is not cyclic.

Consider the H_Σ-coalgebra (A, α), where $\alpha(a) = \sigma_2(a, b)$, $\alpha(b) = \sigma_3(c, b, a)$, and $\alpha(c) = \sigma_3(a, b, c)$. We claim that b and c have the same image in the terminal F-coalgebra. We use $r = \{(b, c)\}$, and then we verify:

$$\begin{array}{rcll} \sigma_3(c,b,a) & \equiv_{\mathcal{E},r} & \sigma_3(a,c,b) & \text{using a cyclic permutation} \\ & \equiv_{\mathcal{E},r} & \sigma_3(a,b,c) & \text{since } r^* \text{ contains } (a,a),\ (c,b), \text{ and } (b,c). \end{array}$$

The point again is that this argument is simpler than one which uses the cuttings $\ell_n t$ of the images of b and c in the terminal F-coalgebra.

4.4 Iterating the terminal-coalgebra chain to $\omega + \omega$

As we will see in Section 4.5, for a finitary endofunctor of Set (e.g. $\mathscr{P}_\mathsf{f}$) the limit of the terminal ω^{op}-chain need not be the terminal coalgebra. However, James Worrell [170] provided a construction of the terminal coalgebra that works for every finitary endofunctor of Set. Actually, Worrell proved a more general result about *accessible* endofunctors of Set we present later (cf. **??**). Our proof presented here is simpler than the one given in [170].

Notation 4.4.1. (1) Recall from Remark 3.3.5 the limit of the terminal ω^{op}-chain and denote

$$F^\omega 1 = \lim_{n \in \omega^{op}} F^n 1 \qquad \text{with limit projections } \ell_n : F^\omega 1 \to F^n 1 \text{ for all } n \in \omega^{op}. \tag{4.5}$$

We obtain a unique map $m : F(F^\omega 1) \to F^\omega 1$ having the property that for all n, the triangles below commute:

$$\begin{tikzcd} F(F^\omega 1) \arrow[rr, "m"] \arrow[dr, "F\ell_n"'] & & F^\omega 1 \arrow[dl, "\ell_{n+1}"] \\ & F^{n+1} 1 & \end{tikzcd} \tag{4.6}$$

(2) We put $F^{\omega+1} 1 = F(F^\omega 1)$, $F^{\omega+2} 1 = F(F^{\omega+1} 1)$, etc and form the following ω^{op}-chain

$$F^\omega 1 \xleftarrow{m} F^{\omega+1} \xleftarrow{Fm} F^{\omega+2} \xleftarrow{FFm} \cdots \tag{4.7}$$

Its limit is denoted by $\bar{L} = \lim F^{\omega+n}$ with the limit cone $\bar{l}_n : \bar{L} \to F^{\omega+n} 1$.

We know that, $F^\omega 1$ is the terminal F-coalgebra whenever the limit in (4.5) above is preserved by F (see Theorem 3.3.4). Or, equivalently, whenever m is invertible. We shall soon prove that for every finitary set functor F, m is a split monomorphism. Thus, $\bar{L}$ is the intersection of the chain (4.7) of monomorphisms. Moreover, we obtain that $\bar{L} = \nu F$.

Lemma 4.4.2. *Let $\ell_n : L \to L_n$ be a limit cone of an ω^{op}-chain in Set. For every finite subset $s : S \hookrightarrow L$ there exists n such that $\ell_n \cdot s$ is a monomorphism.*

Proof. Since the limit cone is collectively monic, for every pair of distinct elements in S we have $k \in \omega$ such that l_k distinguishes that pair. Let n be the maximum of all k's (ranging over $S \times S$). $\square$

Lemma 4.4.3 [170]**.** *Let F be a finitary set functor. Then the cone $F\ell_n : F(F^\omega 1) \to F^{n+1}1$ is collectively monic, and $m : F(F^\omega 1) \to F^\omega 1$ in (4.6) is a monomorphism.*

Proof. (1) If F is constantly $\emptyset$, the statement is trivial. If not, then $FX \neq \emptyset$ for every $X \neq \emptyset$ (since for every set Y we have a map from Y to X, thus, one from FY to FX.) Consequently, $F^\omega 1 \neq \emptyset$. Indeed, from $F^\omega 1 = \emptyset$ we conclude $m = \mathsf{id}_\emptyset$ is invertible, thus $\nu F = \emptyset$. This implies that every coalgebra A is empty (due to the homomorphism $A \to \nu F = \emptyset$). In contradiction, since $F1 \neq \emptyset$, we have at least one coalgebra $1 \to F1$.

(2) We now prove that $(F\ell_n)$ is a collectively monomorphic cone: Assuming that for all k, $F\ell_k(x) = F\ell_k(y)$, we show that $x = y$. By the assumption that F is finitary, there is a finite subset $s : S \hookrightarrow F^\omega 1$ and $x', y' \in FS$ such that $x = Fs(x')$ and $y = Fs(y')$. Without loss of generality we may assume that S is nonempty. We see that

$$(F\ell_k \cdot Fs)(x') = F\ell_k(x) = F\ell_k(y) = (F\ell_k \cdot Fs)(y').$$

By Lemma 4.4.2, there exists some n such that $\ell_n \cdot s$ is monic, thus $S \neq \emptyset$ implies that it is a split monomorphism. Thus, $F(\ell_n \cdot f)$ is monic, too. So $x' = y'$, and thus $x = y$.

(3) From the definition of m in (4.5) it now follows that it is a monomorphism. It splits since $F(F^\omega 1) \neq \emptyset$. $\square$

Proposition 4.4.4 [164]**.** *For every set functor F there exists an essentially unique set functor $\bar{F}$ which coincides with F on nonempty sets and functions and preserves finite intersections (whence monomorphisms).*

For the proof see Trnková [164, Propositions III.5 and II.4]; for a more direct proof see Adámek and Trnková [38, Theorem III.4.5]. We call the functor $\bar{F}$ the *Trnková hull* of F. Note that it is the reflection of F into the full subcategory of endofunctors on sets preserving finite intersections. For more on this topic see **??**.

Remark 4.4.5. If F is a finitary set functor, then so is its Trnková hull $\bar{F}$. Furthermore, note that every coalgebra for F is, equivalently, a coalgebra for $\bar{F}$.

Corollary 4.4.6. *The Trnková hull of a finitary set functor preserves all intersections.*

Proof. Let F be a finitary set functor. Since $\bar{F}$ is finitary and preserves finite intersections, for every element $x \in \bar{F}X$, there exists a *least* finite set $m\colon Y \hookrightarrow X$ with x contained in $\bar{F}m$. Preservation of all intersections now follows easily: given subsets $v_i\colon V_i \hookrightarrow X$, $i \in I$, with x contained in the image of $\bar{F}v_i$ for each i, then x also lies in the image of the finite set $v_i \cap m$, hence $m \subseteq v_i$ by minimality. This proves $m \subseteq \bigcap_{i\in I} v_i$, thus, x lies in the image of $\bar{F}(\bigcap_{i\in I} v_i)$, as required. $\square$

Theorem 4.4.7 [170]**.** *Every finitary set functor F has the terminal coalgebra*

$$\nu F = \lim_{n<\omega} F^{\omega+n}1.$$

The coalgebra structure τ is determined by the following commutative triangles

$$\begin{tikzcd}
\nu F \arrow[rr, "\tau"] \arrow[dr, "\bar{l}_{n+1}"'] & & F(\nu F) \arrow[dl, "F\bar{l}_n"] \\
& F(F^{\omega+n}1) &
\end{tikzcd} \qquad (n < \omega) \tag{4.8}$$

Proof. (1) We are going to prove in part (2) that F preserves the limit $\bar{L} = \lim_{n<\omega} F^{\omega+n}1$. It then follows that there is a unique isomorphism τ for which the above triangles commute.

The proof that $(\nu F, \tau)$ is a terminal coalgebra is then analogous to the proof of Theorem 3.3.4. There we used the canonical cone $\alpha_n : A \to F^n 1$ for a given coalgebra $\alpha : A \to FA$ defined in Construction 3.3.2. We extend this to a cone $\alpha_{\omega+n} : A \to F^{\omega+n}1$ as follows. The cone α_n induces a unique morphism $\alpha_\omega : A \to F^\omega 1$ with $\ell_n \cdot \alpha_\omega = \alpha_n$ for all $n < \omega$, and given $\alpha_{\omega+n}$ we define $\alpha_{\omega+n+1} = F\alpha_{\omega+n} \cdot \alpha : A \to F^{\omega+n+1}$. To see that this is a cone for (4.7) it suffices to show that $\alpha_\omega = m \cdot \alpha_{\omega+1}$; the rest then follows by an easy induction. We prove the desired equation by postcomposing it by every limit projection ℓ_{n+1}, $n < \omega$, and then use that those limit projections are collectively monic:

$$\begin{aligned}
\ell_{n+1} \cdot \alpha_\omega &= \alpha_{n+1} && \text{(def. of } \alpha_\omega) \\
&= F\alpha_n \cdot \alpha && \text{(def. of } \alpha_{n+1}) \\
&= F\ell_n \cdot F\alpha_\omega \cdot \alpha && \text{(def. of } \alpha_\omega) \\
&= F\ell_n \cdot \alpha_{\omega+1} && \text{(def. of } \alpha_{\omega+1}) \\
&= \ell_{n+1} \cdot m \cdot \alpha_{\omega+1} && \text{(def. of } m)
\end{aligned}$$

We now obtain a unique morphism $\bar{\alpha} : A \to \bar{L}$ such that $\bar{l}_n \cdot \bar{\alpha} = \alpha_{\omega+n}$ for all $n < \omega$. Then $\bar{\alpha} : A \to \bar{L}$ is a coalgebra homomorphism: in the following diagram

$$\begin{tikzcd}
A \arrow[rr, "\alpha"] \arrow[dd, "\bar{\alpha}"'] & & FA \arrow[dl, "F\alpha_{\omega+n}"] \arrow[dd, "F\bar{\alpha}"] \\
& F(F^{\omega+n}1) & \\
\bar{L} \arrow[rr, "\tau"'] \arrow[ur, "\bar{l}_{n+1}"] & & F\bar{L} \arrow[ul, "F\bar{l}_n"']
\end{tikzcd}$$

all inner parts commute, thus the outside does: recall that all $F\bar{l}_n$ form a limit cone which is thus collectively monic. To verify uniqueness, let $\widetilde{\alpha} : A \to \bar{L}$ be another coalgebra homomorphism. Then the above diagram commutes when $\bar{\alpha}$ is substituted by $\widetilde{\alpha}$ (twice). Thus all $\bar{l}_{n+1}$ merge $\bar{\alpha}$ and $\widetilde{\alpha}$, which proves $\widetilde{\alpha} = \bar{\alpha}$ (since $\bar{l}_{n+1}$ form a limit cone).

(2) The proof that F preserves the limit $\bar{L}$ is trivial in the case where F is constant with value $\emptyset$. Assuming the contrary, we have clearly $F1 \neq \emptyset$, thus there is (at least one) coalgebra $\alpha : 1 \to F1$. This implies $\bar{L} \neq \emptyset$ due to $\bar{\alpha} : 1 \to \bar{L}$, whence $F^{\omega+n}1 \neq \emptyset$ for all n. Let $\bar{F}$ be the Trnková hull of F (see Proposition 4.4.4). Then the terminal-coalgebra ω^{op}-chain of F extended to $F^{\omega+n}1$, $n < \omega$, and to $\bar{L}$ is identical with that of $\bar{F}$. Since F is finitary, and using Corollary 4.4.6, we may assume that F preserves all intersections. Recall that this implies that F preserves monomorphisms.

From Lemma 4.4.3 we know that m is monic, thus so are Fm, FFm, etc. Thus the chain (4.7) is a decreasing chain of subobjects $u_n\colon F^{\omega+n} \to F^\omega 1$ of $F^\omega 1$ where $u_0 = \mathsf{id}_{F^\omega 1}$ and $u_{n+1} = u_n \cdot F^n m$. Therefore, the limit $\bar{L}$ is simply the intersection of these subobjects, more precisely, for the first projection $\bar{l}_0\colon \bar{L} \to F^\omega 1$ we have

$$\bar{l}_0 = \bigcap_{n<\omega} u_n.$$

Consequently, since F preserves intersections, it preserves the limit $\bar{L}$. □

Corollary 4.4.8. *For every finitary set functor F we have a weakly terminal coalgebra $F^\omega 1$ given by any splitting of $m\colon F(F^\omega 1) \to F^\omega 1$.*

Proof. Given $\bar{m} : F^\omega 1 \to F(F^\omega 1)$ with $\bar{m} \cdot m = \mathsf{id}$ we observe that $\bar{l}_0 : (\bar{L}, \tau) \to (F^\omega 1, \bar{m})$ of Notation 4.4.1 is a coalgebra homomorphism:

$$\begin{tikzcd}
\bar{L} \arrow[r, "\tau"] \arrow[d, "\bar{l}_0"'] \arrow[dr, dashed, "\bar{l}_1"] & F\bar{L} \arrow[d, "F\bar{l}_0"] \\
F^\omega 1 \arrow[r, "\bar{m}"'] & F(F^\omega 1)
\end{tikzcd}$$

The upper triangle commutes by definition of τ. For the lower one use $\bar{l}_0 = m \cdot \bar{l}_1$ and multiply this equation by $\bar{m}$. Since $\bar{L}$ is terminal this implies $F^\omega 1$ is weakly terminal by Example 4.2.2(2). □

Remark 4.4.9. The limit $F^\omega 1$ is an algebra for F in a canonical sense: see $m : F(F^\omega 1) \to F^\omega 1$ in (4.6). Theorem 4.4.7 implies that the terminal coalgebra, considered as an algebra, is a subalgebra of $F^\omega 1$. In more detail: for the algebra structure τ^{-1}, the square

$$\begin{tikzcd}
F(\nu F) \arrow[r, "\tau^{-1}"] \arrow[d, "F\bar{l}_1"'] & \nu F \arrow[d, "\bar{l}_1"] \\
F(F^\omega 1) \arrow[r, "m"'] & F^\omega 1
\end{tikzcd}$$

commutes. (Here $\bar{l}_1$ is the first projection of $\nu F = \lim_{n<\omega} F^{\omega+n}1$ which, as we know, is a monomorphism.) Indeed, since $F\bar{l}_0 \cdot \tau = \bar{l}_1$, see (4.8), we have $\bar{l}_1 \cdot \tau^{-1} = F\bar{l}_0 = m \cdot F\bar{l}_1$.

4.5 A Cook's Tour of terminal and weakly terminal coalgebras for $\mathscr{P}_f$

The finite power-set functor $\mathcal{P}_\mathrm{f}$ is fascinating when it comes to

(1) its terminal coalgebra $\nu\mathcal{P}_\mathrm{f}$ as well as

(2) the limit of its terminal ω^{op}-chain which we denote by

$$\mathbb{F} = \lim_{n\in\omega^{op}} P_f^n 1$$

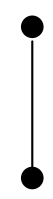
(with limit projections $l_n : F \to P_f^n 1$) which by Corollary 4.4.8 is weakly terminal.

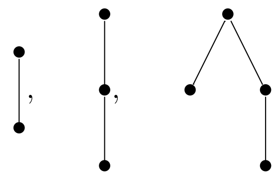
This can be seen from the number of papers devoted to various descriptions of these two algebras (Remark 4.4.9) see e.g. [1, 22, 25, 50, 169, 170]. We have intentionally borrowed the title of Abramsky's paper [1] in which five descriptions of $\mathbb{F}$ are presented – we explain them below. Before doing so, let us recall the initial chain of $\mathcal{P}_\mathrm{f}$ from Example 3.2.9(2):

$$\mathcal{P}_\mathrm{f}^n\emptyset = \text{extensional trees of depth } < n.$$

The terminal ω^{op}-chain is quite analogous. The first members of the chain are:

$1 \quad = \quad \{\emptyset\}$ represented by $\bullet$

$\mathcal{P}_\mathrm{f}^1 1 \quad = \quad \{\emptyset, \{\emptyset\}\}$ represented by $\bullet$, 

$\mathcal{P}_\mathrm{f}^2 1 \quad = \quad \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset\}\}\}$ represented by $\bullet$, 

etc (cf. Example 3.2.9(2)). We thus have

$$\mathscr{P}_f^n 1 = \text{all extensional trees of depth} \le n.$$

The connecting maps

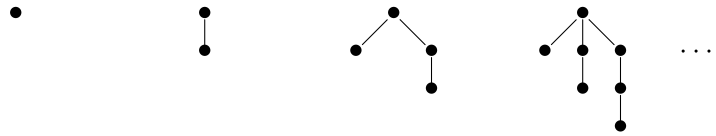
$$v_{n+1,n} : \mathscr{P}_f^{n+1} 1 \to \mathscr{P}_f^n 1, \qquad t \mapsto \partial_n t$$

are given by cutting the trees at level n and forming the extensional quotient, see Example 4.2.11. Indeed, this is trivially true for $n = 0$, and the induction step is easy to See.

A first description: finite-branching trees modulo $\approx$. Recall the congruence $\approx$ merging trees t and u iff $\partial_n(t) = \partial_n(u)$ holds for all $n < \omega$ from Example 3.2.9 and Remark 4.3.21 and recall Barr's description of the terminal coalgebra of $\mathscr{P}_f$ as $D/\approx$.

A second description: sequences of trees. We can describe $\mathbb{F}$ as the set of all sequences $(t_n)_{n<\omega}$ of finite extensional trees, where t_n has depth at most n and is the extensional quotient $\partial_n t_{n+1}$ of the cutting of t_{n+1} (for every n), see Example 4.2.11. This follows from the fact that $\mathbb{F}$ is the limit of the above ω^{op}-chain $(\mathcal{P}_\mathrm{f}^n 1)_{n<\omega}$.

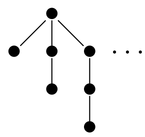
Example 4.5.1. (1) The following sequence of finite trees t_n:



t_0 t_1 t_2 t_3

form an element of $\mathbb{F}$.

(2) Consider the following infinite tree t



Then $\partial_n t$, $n < \omega$, are precisely the above trees t_n. So in a way this single tree t represents the element of $\mathbb{F}$ given by (1) above. (We will use this representation in the subsequent descriptions of $\mathbb{F}$.)

What is the connection of this description of $\mathbb{F}$ with the above description of $\nu\mathcal{P}_\mathrm{f}$? We have an obvious monomorphism from $D/{\approx}$ to $\mathbb{F}$: given a finitely branching tree t, assign to the congruence class $[t]$ modulo $\approx$ the sequence $(\partial_n t)_{n<\omega}$. We will see in Remark 4.5.13 that this monomorphism is precisely the subalgebra $\bar{l}_0 : \nu\mathcal{P}_\mathrm{f} \to \mathbb{F}$ of Remark 4.4.9.

A third description: strongly extensional trees. Here we present a much nicer description of the two objects $\nu\mathscr{P}_f$ and $\mathbb{F}$ due to Worrell [170]. The main concept that Worrell introduced is *tree bisimulation*: this is more special than graph bisimulation, see Example 4.2.6, since the goal is that for every tree by factoring modulo the greatest bisimulation we obtain a tree again.

All trees in this section are considered unordered. We use the notation t_x for the subtree of t rooted in the node x.

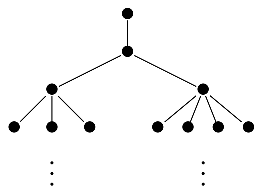
Definition 4.5.2 [170]**.** (1) A *tree bisimulation*between two trees t and u is a graph bisimulation (cf. Example 4.2.6) R such that the roots of t and u are related, and whenever two nodes are related, then they have the same distance from the root.

Two trees are called *tree bisimilar* if there is a tree bisimulation between them.

(2) A tree t is called *strongly extensional* if every tree bisimulation on it is a subrelation of the identity. More explicitly, t is strongly extensional iff distinct children x and y of the same node define subtrees t_x and t_y which are *not* tree bisimilar.

Remark 4.5.3. (1) Every strongly extensional tree is clearly extensional.

(2) Consider the following trees (where the right-hand one has n children for the n-th vertex in the breadth-first search):

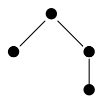


(4.9)

Both are extensional; the left-hand one is strongly extensional, the right-hand one is not, since the relation relating all nodes of the same depth is a tree bisimulation on it.

(3) It is trivial to prove that every composition of tree bisimulations is again a tree bisimulation. In addition, the opposite relation of every tree bisimulation is a tree bisimulation: if R is a tree bisimulation from t to u, then R^{op} is a tree bisimulation from u to t. Consequently, the largest bisimulation on every tree is an equivalence relation.

(4) Observe that the notion of tree bisimulation is different from the usual graph bisimulation. For example, the picture below



(4.10)

depicts a strongly extensional tree but there is a graph bisimulation relating the two leaves.

Proposition 4.5.4. *Let t be a finite tree. Then t is extensional iff it is strongly extensional.*

Proof. Let t be extensional, and let R be a tree bisimulation on it. We claim that if $x \, R \, y$, then the corresponding subtrees t_x and t_y are equal. First notice that every node of t_x must be related by R to some node of t_y (to see this use induction on the distance of nodes from the root) and vice versa. Thus, t_x and t_y have the same height, n say. We now prove $t_x = t_y$ by induction on n. For $n = 0$, the result is obvious because the nodes of height 0 are leaves. Assume our result for n, and let x and y be related by R and of height $n + 1$. Then by the induction hypothesis and extensionality of t, for every child x' of x there is a *unique* child y' of y and $t_{x'} = t_{y'}$; and vice-versa. This implies that $t_x = t_y$.

It now follows that if t is an extensional tree, then t must be strongly extensional. ☐

Definition 4.5.5. The *strongly extensional quotient* $\bar{t}$ of a tree t is the quotient tree of t modulo its largest tree bisimulation.

Observe that every tree t is tree bisimilar to its extensional and strongly extensional quotients; the canonical quotient maps obviously are tree bisimulations.

Lemma 4.5.6. *If t and u are strongly extensional trees related by a tree bisimulation, then they are equal.*

Proof. Suppose we have a tree bisimilation R between t and u. Then by Remark 4.5.3 $R^{op} \cdot R$ is a tree bisimulation on t, whence $R^{op} \cdot R \subseteq \Delta$ by strong extensionality. But every node of t is related to at least one node of u (use induction on the distance of nodes from the root) implying that $R^{op} \cdot R = \Delta$. Similarly, $R \cdot R^{op} = \Delta$. Thus, R (is a function and it) is an isomorphism of trees, and we identify such trees. □

Theorem 4.5.7 [170]**.** *The coalgebra of all finitely branching strongly extensional trees (as a subcoalgebra of D as in Remark 4.3.21) is a terminal coalgebra for $\mathscr{P}_{\mathsf{f}}$.*

Proof. Denote by $\bar{t}$ the strongly extensional quotient of the tree t. The set $D_0 \subseteq D$ of all finitely branching, strongly extensional trees clearly forms a subcoalgebra. We prove that this is isomorphic to the terminal coalgebra $D/{\approx}$. For that we need to verify that given trees $t, u \in D$, then for their strongly extensional quotients we have

$$t \approx u \qquad \text{iff} \qquad \bar{t} = \bar{u}.$$

Then the map $[t] \mapsto \bar{t}$ is then an isomorphism from $D/{\approx}$ to D_0.

($\Rightarrow$) If $t \approx u$ we prove that t and u are tree bisimilar. Then, by Remark 4.5.3, it follows that $\bar{t}$ and $\bar{u}$ are tree bisimilar, which implies that $\bar{t} = \bar{u}$ by the previous Lemma.

Define $R \subseteq t \times u$ by relating nodes $x \in t$ and $y \in u$ iff they have the same depth n and for every $m \geq n$ the node of $\partial_m t$ corresponding to x equals the node of $\partial_m u = \partial_m t$ corresponding to y. Using that t and u are finitely branching, it is not difficult to prove that R is a tree bisimulation. We leave the details to the reader.

($\Leftarrow$) Conversely, suppose we have $\bar{t} = \bar{u}$. Then we get, by composing with the quotient maps, a tree bisimulation $R \subseteq t \times u$. By restricting R to all pairs of nodes of depth at most n we obtain a tree bisimulation R_n between the cuttings of t and u at level n. Then also the extensional quotients $\partial_n t$ and $\partial_n u$ are tree bisimilar. Since $\partial_n t$ and $\partial_n u$ are strongly extensional by Proposition 4.5.4, we see that $\partial_n t = \partial_n u$. Thus, $t \approx u$. □

Example 4.5.8. Similar descriptions arise for the terminal coalgebra of related functors. For example, consider the functor $FX = \{0,1\} \times (\mathscr{P}_{\mathsf{f}} X)^{\Sigma}$, viz. the finitary version of the type functor of non-deterministic automata, see Example 2.4.2(5). In order to describe νF one considers finitely branching trees whose nodes are labelled in $\{0,1\}$ and whose edges are labelled in the input alphabet Σ. The notion of tree bisimulation then needs to be adjusted as follows: it is a bisimulation of labelled transition systems (cf. Example 2.6.5(3)) such that the root are related, and every related nodes have the same distance from the root and the same label in $\{0,1\}$. The notion of a strongly extensional tree is then defined as before, and it is not difficult to see that νF is the coalgebra of all strongly extensional trees.

Note that this means that the unique map from an F-coalgebra into the final one thus provides the behaviour of states modulo bisimilarity, i.e. taking into account the non-deterministic branching of the given automaton. This semantics is therefore different then the usual language semantics of non-deterministic automata. We shall see in Example 5.1.27 that by modelling non-deterministic automata as coalgebras for a functor on the category of sets and relations one obtains their language semantics similarly as for deterministic automata in Example 2.5.5 using the terminal coalgebra.

A fourth description: saturated trees. Here we introduce the concept of a saturated tree and describe $\mathbb{F}$ as

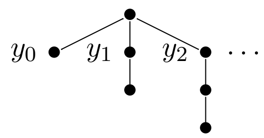
$$\mathbb{F} = \text{ all saturated, strongly extensional trees.}$$

Definition 4.5.9. A tree t is called *saturated* provided that for every node x has the following property: given a tree s for which x has children y_n $(n < \omega)$ such that $\partial_n(s) = \partial_n(t_{y_n})$ it follows that x has a child y with $s \approx t_y$.

Example 4.5.10. (1) Every finitely branching tree is clearly saturated.

(2) The following tree of Example 4.5.1:

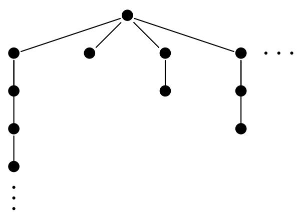
t:



is not saturated. If s denotes the single-path infinite tree, then the root of t has children y_n with $\partial_n(s) = \partial_n(t_{y_n})$, however, no child y fulfils $s \approx t_y$.

In contrast, the following tree is saturated (observe also that $t \approx t'$):

t':



Theorem 4.5.11 [25]**.** *The limit* $\mathbb{F}$ *can be described as follows:*

$$\mathbb{F} = \textit{all saturated, strongly extensional trees}$$

with the limit cone $\partial_n : \mathbb{F} \to \mathscr{P}_f^n 1$*.*

A fifth description: compactly branching trees. Worrell introduced in [170] the following pseudometric d_τ on the class of all strongly extensional trees:

$$d_\tau(z, u) = \inf\{2^{-n}; n < \omega \text{ with } \partial_n t = \partial_n u\}. \tag{4.11}$$

Consequently t and u have distance 0 iff $t \approx u$, that is $\partial_n t = \partial_n u$ for all $n < \omega$. The resulting pseudometric space is compact because for every $\epsilon > 0$ we have a finite number of ϵ-balls covering it. Indeed, choose n with $2^{-n} < \epsilon$ and take the (finite) set A of all extensional trees of depth at most n. Then every tree t satisfies $\partial_n t \in A$ and $d_\tau(t, \partial_n t) \leq 2^{-n} < \epsilon$.

Consequently, a set M of extensional trees is compact iff the corresponding set of $\approx$-classes is closed (in the space of all strongly extensional trees modulo $\approx$). Explicitly: M is compact iff for every collection $t_n \in M$ $(n < \omega)$ and every strongly extensional tree s with $\partial_n s = \partial_n t_n$ $(n < \omega)$ there exists a tree $t \in M$ with $\partial_n s = \partial_n t$ for all $n < \omega$.

We thus see that a strongly extensional tree t is saturated iff it is *compactly branching*, i.e., for every node x the set of all maximum subtrees of t_x is compact.

Corollary 4.5.12 [170]**.** *The limit $\mathbb{F}$ can be described as the set of all compactly branching strongly extensional trees.*

Remark 4.5.13. (1) The pseudometric d_τ is a metric when restricted to $\mathbb{F}$ = all saturated, strongly extensional trees. Indeed, for such trees t and u we have: $t \approx u$ implies $t = u$. (Recall that trees are considered up to isomorphism.)

(2) When $\mathbb{F}$ is described by all saturated, strongly extensional trees and $\nu\mathcal{P}_\mathrm{f}$ by the finitely branching ones, what is the canonical monomorphism $\bar{l}_0 : \nu\mathcal{P}_\mathrm{f} \to \mathbb{F}$ of Remark 4.4.9? This is just the inclusion map. Indeed, the monomorphism m of Notation 4.4.1 represents the subobject of $\mathbb{F}$ of all trees finitely branching at the root. Analogously $\mathcal{P}_\mathrm{f} m : \mathscr{P}_f^2\mathbb{F} \to \mathcal{P}_\mathrm{f}\mathbb{F}$ represents the subobject of all trees finitely branching on levels 0 and 1, etc. And $\bar{l}_0$, which is the intersection of these subobjects, thus represents all finitely branching trees in $\mathbb{F}$.

Other descriptions of $\mathbb{F}$. Let us mention further descriptions due to Abramsky [1]:

(1) Cauchy completion of $\mu\mathcal{P}_\mathrm{f}$. In Section 6.2 we will see that the initial algebra carries a canonical metric whose Cauchy completion is a metric space on the set $\mathbb{F}$.

(2) Ideal completion of $\mu\mathcal{P}_\mathrm{f}$. In Section 6.2 we will also see a canonical partial order on the initial algebra whose ideal (= free CPO) completion is carried by $\mathbb{F}$.

(3) Maximal consistent theories of the modal logic K. We also proved [25, Theorem 5.11] that saturated trees precisely correspond to *modally* saturated trees defined by Fine [72] (when modal logic is taken without atoms). Moreover, we have obtained the following descriptions [25, Proposition 5.7, Theorem 5.9]:

$\mathbb{F}$ = all maximal consistent theories in K, and$\nu\mathcal{P}_\mathrm{f}$ = all hereditarily finite theories in K.

(4) Terminal coalgebra of a modified Hausdorff functor. In Example 5.2.25 we will present an endofunctor on the category CMS of complete metric spaces whose terminal coalgebra is carried by $\mathbb{F}$.

4.6 Summary of this Chapter

In the present chapter we first proved some useful facts about constructions of (co)algebras for general endofunctors F: colimits of coalgebras (and dually limits of algebras) are canonically formed on the level of the base category, and the same is true for those limits of coalgebras that F preserves (and dually for colimits of F-algebras). We then discussed quotient coalgebras and proved that whenever a weakly terminal coalgebra is given, then the smallest quotient coalgebra yields νF.

The main topic of this chapter was a construction of terminal coalgebras for finitary set functors. We presented a new proof of the Worrell'S result [169] that νF is a limit of the terminal-coalgebra chain after $\omega + \omega$ steps. We also observed that the object obtained after the first ω steps of that chain is a weakly terminal coalgebra. For the finite power-set functor $\mathscr{P}_{\mathsf{f}}$ that limit, denoted by $\mathbb{F}$, was studied by a number of authors, and we presented various descriptions of $\nu\mathscr{P}_{\mathsf{f}}$ and $\mathbb{F}$ in the last section.

5 Finitary Iteration in Enriched Settings

In Chapter 3 we saw two constructions: the initial algebra as a colimit of (finitary) iterations and the terminal coalgebra as a limit of (finitary) iterations. Then in Chapter 4 we saw ways to "get our hands on" some of the terminal coalgebras which are obtained by iteration, namely the terminal coalgebras of finitary functors on Set. In this chapter, we strike out in a different direction by considering categories other than Set. In two important types of base categories, namely those enriched over complete partial orders and over complete metric spaces, it turns out that (under mild conditions on the endofunctor) the two constructions from Chapter 3 coincide. Moreover, they yield a fixed point that is *canonical* in the sense of the following definition (recall from Lambek's Lemma that the structure morphism of an initial algebra has an inverse):

Definition 5.0.1. Let $F : \mathscr{A} \to \mathscr{A}$ be an endofunctor on a category $\mathscr{A}$. A *canonical fixed point* of F is an initial algebra $a\colon FA \to A$ such that $a^{-1}\colon A \to FA$ is a terminal coalgebra.

Almost none of the examples of initial algebras and terminal coalgebras in Set are canonical fixed points. (The exceptions are for the constant functors.) But we have seen some examples when we considered $\mathsf{CPO}_\perp$ (complete partial orders with least element) and CMS (complete metric spaces). For example, a canonical fixed point of $FX = X_\perp$ on $\mathsf{CPO}_\perp$ was presented in Examples 2.2.17(1) and 3.3.6. It is $\mathbb{N}^\top$, the natural numbers with an additional largest element $\top$.

What the $\mathsf{CPO}_\perp$ and CMS settings have in common is a nice theory of *approximation.* In the case of $\mathsf{CPO}_\perp$ approximation is given in terms of joins of ω-chains, and in the case of CMS it is limits of Cauchy sequences. In this chapter, we are going to work with functors which preserve this additional approximation structure. We also typically work on a *pointed* category: this means that the initial object 0 is the same as the terminal object 1. We call $0 = 1$ a *zero object.* In such a category we may lay the initial ω-chain on top of the terminal ω^{op}-chain, as follows:

$$0 = 1 \underset{\widehat{e}}{\overset{e}{\rightleftarrows}} F1 \underset{F\widehat{e}}{\overset{Fe}{\rightleftarrows}} F^2 1 \underset{F^2\widehat{e}}{\overset{F^2 e}{\rightleftarrows}} \dots \tag{5.1}$$

Here e is by initiality of 0 and $\widehat{e}$ is by finality of 1. Since clearly $\widehat{e} \cdot e = \mathsf{id}$, the two chains are already related to each other. It turns out that under suitable hypotheses, we have a *limit-colimit coincidence*: the colimit of the initial ω-chain exists and coincides with the limit of the terminal ω^{op}-chain. For $\mathsf{CPO}_\perp$ this is a classical result by Smyth and Plotkin [156] that we shall present in Section 5.1 (see Theorem 5.1.23). The case of CMS was first studied by America and Rutten [42], and we consider it in Section 5.2. In both settings, the "approximation structure" is taken to apply to the morphisms in the category rather than the objects, and this is why we are really working with enriched categories in this chapter.

This chapter has two themes: the canonical fixed points which we just mentioned, but also fixed points of functors involving *mixed variance*. This topic is related to semantic models for the lambda calculus and other calculi. We are not going to discuss these applications in any detail. But we do want to foreshadow some of the ideas that will be prominent in the chapter. The search for interesting models of the lambda calculus essentially becomes a problem of the following form: can we find a cartesian closed category $\mathscr{A}$ and some object X satisfying $X \cong [X, X]$? In this discussion,

$$[X, Y]$$

is the object in the category corresponding to the homset $\mathscr{A}(X, Y)$ which the cartesian closed structure provides. We would especially want constructions in concrete categories, and the methods should be general enough that they allow us to solve related equations such as $X \cong [X, X] + A$ for all objects A. It is clear that $\mathscr{A}$ cannot be Set, since the only way to have $X \cong [X, X]$ in Set is for X to have just one element. So one would want other categories. Indeed, the kinds of categories and functors used in our study are those that have canonical fixed points via the limit-colimit coincidence.

Organization of the chapter. The two themes of this chapter are each interesting and will be presented somewhat separately. Section 5.1 discusses canonical fixed points in categories enriched over complete partial orders. We present full detail in this section because our proofs are shorter than those in the literature, and also because this body of material does not seem to have been presented comprehensively in other books. Following this, we turn in Section 5.2 to canonical fixed points in the metric setting rather than the order-theoretic one. We close the chapter with a discussion of domain equations in Section 5.3. Section 5.2 may be read on its own, but some of the ideas in Section 5.1 are used there. Section 5.3 may be read independently of Section 5.2.

5.1 Canonical fixed points in CPO-enriched categories

Recall the category

$$\mathsf{CPO}$$

of complete partial orders (see Example 2.1.6(1)) and continuous maps. The objects in the category need not have a least element $\perp$.

The category CPO is cartesian closed: the internal hom-objects $[X, Y]$ are the hom-sets $\mathsf{CPO}(X, Y)$ ordered pointwise. We denote by $\bigsqcup f_i$ the join of an ω-chain formed by $f_i \colon X \to Y$, $i < \omega$.

Definition 5.1.1. A category $\mathscr{A}$ is CPO-*enriched* if its hom-sets come with a CPO structure, and composition is continuous in both variables. That is, if $f_1 \sqsubseteq f_2 : A \to B$, then $h \cdot f_1 \cdot g \sqsubseteq h \cdot f_2 \cdot g$ for all h and g composable with f_1 and f_2. And given an ω-chain $f_i : X \to Y$, $h \cdot \left(\bigsqcup_i f_i\right) \cdot g = \bigsqcup_i h \cdot f_i \cdot g$.

Examples 5.1.2. (1) CPO is, of course, CPO-enriched using the above pointwise ordering on hom-sets.

(2) $\mathsf{CPO}_\perp$ is also CPO-enriched (see Example 2.1.6(2)).

(3) Analogously to cpos a poset is called a dcpo (*directed-complete partial order*) if every directed subset has a join. The corresponding morphisms are the Δ-*continuous functions*, i.e. functions preserving directed joins. The category $\mathsf{DCPO}_\perp$ of directed-complete partial orders with least element is again CPO-enriched.

(4) The category Pfn of sets and partial functions is CPO-enriched by inclusion: for partial functions $f_1, f_2 \colon X \to Y$ we put $f_1 \sqsubseteq f_2$ iff whenever $f_1(x)$ is defined so is $f_2(x)$ and $f_1(x) = f_2(x)$, i.e. the graph of f_1 is a subset of the graph of f_2. The least element of $\mathsf{Pfn}(X, Y)$ is the nowhere defined function, and $\bigsqcup f_i$ is the set-theoretic union for every ω-chain $(f_i)_{i<\omega}$.

(5) The category Rel of sets and relations is also CPO-enriched by inclusion.

(6) CLat is the category of complete lattices with maps preserving joins and meets. This category has two CPO-enriched structures. One is where we use the pointwise order in each homset, and the other uses the dual of this order.

(7) If $\mathscr{A}$ is CPO-enriched, then so is $\mathscr{A}^{op}$. We order $\mathscr{A}^{op}(X, Y)$ with the same relation as in $\mathscr{A}(Y, X)$.

(8) A product of CPO-enriched categories is CPO-enriched with respect to the componentwise ordering.

Definition 5.1.3 [154]**.** In a CPO-enriched category, a morphism $e \colon X \to Y$ is called an *embedding* if there exists a morphism $\widehat{e} \colon Y \to X$ with

$$\widehat{e} \cdot e = \mathsf{id}_X \quad \text{and} \quad e \cdot \widehat{e} \sqsubseteq \mathsf{id}_Y. \tag{5.2}$$

Let us check that the morphism $\widehat{e}$ is uniquely determined by (5.2). Suppose that $e^* \cdot e = \mathsf{id}_X$ and $e \cdot e^* \sqsubseteq \mathsf{id}_Y$. Then $\widehat{e} = e^* \cdot e \cdot \widehat{e} \sqsubseteq e^*$. Similarly, $e^* \sqsubseteq \widehat{e}$. Thus $\widehat{e} = e^*$. And $\widehat{e}$ is called the *projection* for e. A pair $(e, \widehat{e})$ is called an *embedding-projection pair*.

Examples 5.1.4. (1) In CPO, the embeddings are precisely those monomorphisms $e \colon X \to Y$ such that for every $y \in Y$ there exists a largest $x \in X$ with $e(x) \sqsubseteq y$. In fact, this condition allows us to define $\widehat{e} \colon Y \to X$ by choosing this largest x as $\widehat{e}(y)$. Then $\widehat{e} \cdot e = \mathsf{id}_X$ (since e is one-to-one), and $e \cdot \widehat{e} \sqsubseteq \mathsf{id}_Y$. The verification that the condition is also necessary is trivial.

(2) It follows from (1) that in the category CLat of complete lattices, every monomorphism is an embedding.

(3) Every monomorphism in Pfn is an embedding: monomorphisms are total injective functions, and for these, $\widehat{e}$ is the (partially defined) inverse function e^{-1}. Identifying functions with their graph relations, $\widehat{e}$ is just the converse of e.

(4) Every split monomorphism in Rel is an embedding: split monomorphisms are precisely those relations $e \colon X \to Y$ which are injective maps, and for these $\widehat{e} = e^{op}$.

Observation 5.1.5. Suppose that $\mathscr{A}$ is a CPO-enriched category in which every hom-set

$\mathscr{A}(X,Y)$ has a least element $\perp_{XY}$.

(1) A general example of an embedding is a coproduct injection v_i of an arbitrary coproduct $X = \coprod_{i\in I} X_i$; indeed, $\widehat{v}_i\colon X \to X_i$ has components id_{X_i} and $\perp_{X_iX_j}$ for $j \neq i$. In particular, the morphisms with domain 0 are always embeddings.

(2) The name "projection" stems from the dual of (1): for every product $X = \prod_{i\in I} X_i$ the projection π_i has the form $\widehat{e}_i$ where $e_i\colon X_i \to X$ has components id_{X_i} and $\perp_{X_iX_j}$.

Notation 5.1.6. The objects of a CPO-enriched category $\mathscr{A}$ together with the embeddings form a category

$$\mathscr{A}^E.$$

The point is that a composite $e \cdot f$ of embeddings is itself an embedding with

$$\widehat{e \cdot f} = \widehat{f} \cdot \widehat{e}. \tag{5.3}$$

Remark 5.1.7. Let $\mathscr{A}$ be a CPO-enriched category in which every hom-set also has meets of decreasing ω-chains. For example, the categories CPO, Pfn, Rel and CLat in Example 5.1.2 have this property. Then $\mathscr{A}^E$ is a CPO-enriched category, since given an ω-chain e_i of embeddings in $\mathscr{A}(X,Y)$, the join $\bigsqcup_i e_i$ is an embedding with projection given by the meet $\bigsqcap_i \widehat{e}_i$.

Observation 5.1.8. If $e : X \to Y$ is an embedding in $\mathscr{A}$, then in $\mathscr{A}^{op}$, $\widehat{e} : X \to Y$ is an embedding. So the dual category of $\mathscr{A}^E$ is the category whose objects are those of $\mathscr{A}$ and whose morphisms are the projections.

The following lemma, due to Smyth and Plotkin [156], could look technical at first glance. Its intend is to consider cocones on chains of embeddings and to connect the global property of a given cocone (being a colimit) and the local property mentioned in (5.5) below. This connection is exploited in all the rest of the results in this section.

Basic Lemma 5.1.9. *Let $\mathscr{A}$ be a CPO-enriched category. Consider an ω-chain of embeddings in $\mathscr{A}$*

$$E_0 \xrightarrow{e_0} E_1 \xrightarrow{e_1} E_2 \xrightarrow{e_2} \cdots \tag{5.4}$$

Let $c_i : E_i \to C$ be a cocone. Then the following are equivalent:

(1) *(c_i) is a colimit cocone.*

(2) *Each c_i is an embedding, the composites $c_i \cdot \widehat{c}_i$ form an ω-chain in $\mathscr{A}(C,C)$, and*

$$\bigsqcup_i c_i \cdot \widehat{c}_i = \mathsf{id}_C. \tag{5.5}$$

Proof. (1) $\Rightarrow$ (2): For every i we have the following cocone of the shortened chain with codomain E_i:

$$
\begin{tikzcd}
E_i \arrow[r, "e_i"] \arrow[dd, equal] & E_{i+1} \arrow[r, "e_{i+1}"] \arrow[d, equal] \arrow[ddl, "\widehat{e}_i"'] & E_{i+2} \arrow[r, "e_{i+2}"] \arrow[dl, "\widehat{e}_{i+1}"] & \cdots \\
& E_{i+1} \arrow[dl, "\widehat{e}_i"] & & \\
E_i & & &
\end{tikzcd}
\tag{5.6}
$$

Since the shortened chain has the same colimit as the original chain, namely C, there exists a unique morphism $\widehat{c}_i\colon C \to E_i$ such that the triangle on the left below commutes for all $k \geq i \geq 0$:

$$
\begin{tikzcd}
E_k \arrow[d, "c_k"'] \arrow[dr, "\widehat{e}_i \cdot \widehat{e}_{i+1} \cdots \widehat{e}_{k-1}"] & \\
C \arrow[r, "\widehat{c}_i"'] & E_i
\end{tikzcd}
\qquad\Bigg|\qquad
\begin{tikzcd}
E_i \arrow[d, "c_i"'] \arrow[dr, "e_k \cdot e_{k-1} \cdot \cdots \cdot e_i"] & \\
C \arrow[r, "\widehat{c}_k"'] & E_k
\end{tikzcd}
\tag{5.7}
$$

In particular
$$\widehat{c}_i \cdot c_i = \mathsf{id}_{E_i}. \tag{5.8}$$

We verify the triangles on the right in (5.7) commute by induction on $n = k - i$. (5.8) is the base case. Assuming the commutativity when the difference is $n-1$, we get it when the difference is n:

$$(e_k \cdot e_{k-1} \cdot \cdots \cdot e_{i+1}) \cdot e_i = \widehat{c}_k \cdot c_{i+1} \cdot e_i = \widehat{c}_k \cdot c_i.$$

Comparing the left-hand triangles in (5.7) for $\widehat{c}_i$ and $\widehat{c}_{i+1}$, we see that given $k \geq i+1$ we have
$$\widehat{c}_i \cdot c_k = \widehat{e}_i \cdot \widehat{c}_{i+1} \cdot c_k.$$
Now $(c_k)_{k \geq i+1}$ is a colimit cocone and therefore a jointly epic family. Thus, we conclude

$$\widehat{c}_i = \widehat{e}_i \cdot \widehat{c}_{i+1}. \tag{5.9}$$

Hence, the morphisms $c_i \cdot \widehat{c}_i$ form a chain in $\mathscr{A}(C, C)$:

$$c_i \cdot \widehat{c}_i = (c_{i+1} \cdot e_i) \cdot (\widehat{e}_i \cdot \widehat{c}_{i+1}) \sqsubseteq c_{i+1} \cdot \widehat{c}_{i+1}$$

due to $e_i \cdot \widehat{e}_i \sqsubseteq \mathsf{id}$. We next prove that the join of this chain is id_C, thus also proving that $c_i \cdot \widehat{c}_i \sqsubseteq \mathsf{id}$; this inequality together with (5.8) establishes that c_i is an embedding. To see that id_C is the desired join it is sufficient to prove that for every k

$$\Big(\bigsqcup_i c_i \cdot \widehat{c}_i\Big) \cdot c_k = c_k.$$

For this, we need only show that for $i \geq k$, $c_i \cdot \widehat{c}_i \cdot c_k = c_k$. Using the triangle on the right in (5.7) and interchanging k and i, we see that

$$c_i \cdot \widehat{c}_i \cdot c_k = c_i \cdot e_i \cdot e_{i-1} \cdot \cdots \cdot e_k = c_k.$$

We have proved (5.5). This verifies all parts of (2) in our lemma.

(2) ⇒ (1): Suppose we are given a cocone $b_i : E_i \to B$ of the given chain:

$$\begin{tikzcd} & C \arrow[dl, "\widehat{c}_0"'] \arrow[d, "\widehat{c}_1"] \arrow[dr, "\widehat{c}_2"] & & \\ E_0 \arrow[r, "e_0"] \arrow[dr, "b_0"'] & E_1 \arrow[r, "e_1"] \arrow[d, "b_1"] & E_2 \arrow[r] \arrow[dl, "b_2"] & \cdots \\ & B & & \end{tikzcd}$$

We first observe that the morphisms $b_i \cdot \widehat{c}_i$ form a chain in $\mathscr{A}(C, B)$. Since the c_i's are embeddings and $c_i = c_{i+1} \cdot e_i$, from (5.3) we get (5.9). Thus

$$b_i \cdot \widehat{c}_i = (b_{i+1} \cdot e_i) \cdot (\widehat{e}_i \cdot \widehat{c}_{i+1}) = b_{i+1} \cdot (e_i \cdot \widehat{e}_i) \cdot \widehat{c}_{i+1} \sqsubseteq b_{i+1} \cdot \widehat{c}_{i+1}.$$

Define

$$b = \bigsqcup_i (b_i \cdot \widehat{c}_i) : C \to B. \tag{5.10}$$

We show that this is the desired factorization: we fix k and show that $b \cdot c_k = b_k$. For this, we may restrict the join to $i \geq k$. So we shall show that

$$b \cdot c_k = \bigsqcup_{i \geq k} (b_i \cdot \widehat{c}_i \cdot c_k) = b_k. \tag{5.11}$$

Recall that (c_i) is a cocone. Thus for $i \geq k$, the top triangle below commutes:

$$\begin{tikzcd} E_k \arrow[r, "e_k"] \arrow[d, "c_k"'] & E_{k+1} \arrow[r, "e_{k+1}"] & \cdots \arrow[r, "e_{i-1}"] & E_i \arrow[llld, "c_i"'] \arrow[llldd, "\mathsf{id}"] \arrow[lllddd, "b_i"] \\ C \arrow[d, "\widehat{c}_i"'] & & & \\ E_i \arrow[d, "b_i"'] & & & \\ B & & & \end{tikzcd}$$

The other triangles clearly commute, and so the outside does too. And as (b_i) is a cocone, we have

$$b_i \cdot \widehat{c}_i \cdot c_k = b_i \cdot (e_{i-1} \cdot \cdots \cdot e_k) = b_k.$$

This for all $i \geq k$ shows (5.11). We have shown that b is a morphism through which the cocone (b_i) factorizes.

The factorization is unique: Given $b' : C \to B$ with $b' \cdot c_i = b_i$ for all $i < \omega$, we have, due to (5.5)

$$b' = b' \cdot \bigsqcup_i c_i \cdot \widehat{c}_i = \bigsqcup_i b_i \cdot \widehat{c}_i = b,$$

which completes the proof. □

Remark 5.1.10. If every b_i, $i < \omega$ is an embedding, then so is the morphism b of (5.10). Indeed, we have $c_i = c_{i+1} \cdot e_i$ and $\widehat{b}_i = \widehat{e}_i \cdot \widehat{b}_{i+1}$. Thus

$$c_i \cdot \widehat{b}_i = c_{i+1} \cdot (e_i \cdot \widehat{e}_i) \cdot \widehat{b}_{i+1} \sqsubseteq c_{i+1} \cdot \widehat{b}_{i+1}.$$

We define $\widehat{b}$ to be $\bigsqcup_i c_i \cdot \widehat{b}_i$. This is a projection for b:

$$b \cdot \widehat{b} = (\bigsqcup_i b_i \cdot \widehat{c}_i) \cdot (\bigsqcup_i c_i \cdot \widehat{b}_i) = \bigsqcup_i b_i \cdot \widehat{c}_i \cdot c_i \cdot \widehat{b}_i = \bigsqcup_i b_i \cdot \widehat{b}_i \sqsubseteq \mathsf{id}_B.$$

and also $\widehat{b} \cdot b = \mathsf{id}_C$, due to (5.5):

$$\widehat{b} \cdot b = \bigsqcup_i c_i \cdot \widehat{b}_i \cdot b_i \cdot \widehat{c}_i = \bigsqcup_i c_i \cdot \widehat{c}_i = \mathsf{id}_C.$$

Remark 5.1.11. Although formulated for ω-chains only, the Basic Lemma holds for λ-chains for all limit ordinals λ (in categories enriched so that hom-sets are posets with joins of all chains and composition preserves joins of chains). This is immediately seen from the proof above.

We draw several corollaries from this Basic Lemma.

Corollary 5.1.12. *Consider an ω-chain of embeddings in a* CPO*-enriched category $\mathscr{A}$*

$$E_0 \xrightarrow{e_0} E_1 \xrightarrow{e_1} E_2 \xrightarrow{e_2} \cdots$$

Let (c_i) be a colimit cocone. Then $(\widehat{c}_i)$ is a limit cone of the ω^{op}-chain of projections $\widehat{e}_i \colon E_{i+1} \to E_i$.

Proof. First, note that the morphisms c_i are embeddings by the Basic Lemma 5.1.9, and so the notation $\widehat{c}_i$ makes sense. We also know that $\bigsqcup_i c_i \cdot \widehat{c}_i = \mathsf{id}_C$. Now recall that the category $\mathscr{A}^{op}$ is also CPO-enriched. In this category, we have a chain of embeddings

$$E_0 \xrightarrow{\widehat{e_0}} E_1 \xrightarrow{\widehat{e_1}} E_2 \xrightarrow{\widehat{e_2}} \cdots$$

Each $\widehat{c}_i$ is also an embedding, $\widehat{c}_i \cdot c_i$ is a chain, and $\bigsqcup_i \widehat{c}_i \cdot c_i = \mathsf{id}_C$. Thus by the Basic Lemma 5.1.9 again, $(\widehat{c}_i)$ is a colimit cocone in $\mathscr{A}^{op}$ of $(\widehat{e}_i)$. Moving from $\mathscr{A}^{op}$ back to $\mathscr{A}$, we see that $(\widehat{c}_i)$ is a limit cone in $\mathscr{A}$ of $(\widehat{e}_i)$, as desired. □

Remark 5.1.13. Thus in a CPO-enriched category we have a limit-colimit coincidence for ω-chains of embeddings (or, equivalently, for ω^{op}-chains of projections in the sense of Definition 5.1.3). A colimit cone of an ω-chain of embeddings yields a limit cone of the corresponding ω^{op}-chain of projections, and vice versa.

Corollary 5.1.14. *If a* CPO*-enriched category has colimits of ω-chains of embeddings, then $\mathscr{A}^E$ has colimits of all ω-chains.*

Proof. Let (e_i) be an ω-chain in $\mathscr{A}^E$ and let (c_i) be a colimit cocone in $\mathscr{A}$ for the underlying ω-chain in $\mathscr{A}$. Then by the Basic Lemma 5.1.9, each colimit morphism c_i is itself an embedding. And given a cocone (b_i) for (e_i) in $\mathscr{A}^E$, the factorization morphism b is an embedding by Remark 5.1.10. □

At this point, we need an important definition in this subject. The concept of local continuity is important because it is typically easier to check than the preservation of ω-colimits.

Definition 5.1.15. A functor $F\colon \mathscr{A} \to \mathscr{B}$ between CPO-enriched categories is called *locally continuous* if it is CPO-enriched, i.e., it preserves the order of hom-sets, as well as joins of ω-chains: given $f_0 \sqsubseteq f_1 \sqsubseteq f_2 \sqsubseteq \cdots$ in $\mathscr{A}(X,Y)$, then $F(\bigsqcup_n f_n) = \bigsqcup_n Ff_n$.

For the following examples recall from Example 2.1.6(1) that coproducts in CPO are disjoint unions, and products are given by cartesian products with coordinate-wise order. More generally, a limit of a diagram in CPO is the limit of the underlying diagram in Set with the largest order making every limit projection monotone (we leave the easy proof of this fact to the reader).

Examples 5.1.16. Here are some examples of locally continuous endofunctors on CPO-enriched categories.

(1) Id is locally continuous, and so is every constant functor.

(2) A composite, product or coproduct of locally continuous functors is locally continuous; this is easy to prove, see also Barr [49].

(3) In analogy to the polynomial endofunctors on Set, given a collection $\Sigma = (\Sigma_n)_{n<\omega}$ of cpos, we denote by H_Σ the endofunctor on CPO given by $H_\Sigma X = \coprod_{n<\omega} \Sigma_n \times X^n$. Its local continuity follows from (1) and (2).

(4) The functor $FX = X_\perp$ on CPO from Example 2.1.6(1) is locally continuous.

Example 5.1.17. Ideal completion is an example of a natural endofunctor that is *not* locally continuous. For every poset P the ideal completion is the free dcpo on it. By an *ideal* is meant a nonempty directed down-set in P. The *ideal completion* $\mathsf{Idl}(P)$ is the poset of all ideals ordered by inclusion. In particular, every element $x \in P$ defines the *prime ideal* $\downarrow x = \{y \in P : y \leq x\}$, and $x \leq y$ holds in P iff $\downarrow x \subseteq \downarrow y$ in $\mathsf{Idl}(P)$. Therefore, P may be identified with the subposet of $\mathsf{Idl}(P)$ given by the prime ideals. The universal property is that every monotone map from P to a cpo has a unique continuous extension to a continuous map from $\mathsf{Idl}(P)$ to that cpo, see Abramsky and Jung [2].

Denote by I the endofunctor on CPO mapping a cpo X to $\mathsf{Idl}(X)$ and a continuous map $f\colon X \to Y$ to the continuous extension $\mathsf{Idl}(X) \to Y$ composed with the embedding $Y \hookrightarrow \mathsf{Idl}(Y)$. This endofunctor preserves order on homsets, but it is not locally continuous. To see this, recall the poset $\mathbb{N}^\top$ from Example 3.3.7(1). Let $f_i\colon \mathbb{N}^\top \to \mathbb{N}^\top$ be $n \mapsto \min(n,i)$. Then $\bigsqcup_i f_i = \mathsf{id}$. But $\mathsf{Idl}(\mathbb{N}^\top)$ is the poset $0 < 1 < 2 < \cdots < \top < \top'$. Clearly, $\mathsf{Idl}(\mathsf{id})(\top') = \mathsf{id}(\top') = \top'$. However, $\mathsf{Idl}(f_i)(\top') = \{0, 1, \ldots, i\}$, and so $(\bigsqcup_i \mathsf{Idl}\ f_i)(\top') = \top$. This example is from [78, Example IV-5.12].

Remark 5.1.18. Most of the functors found in domain theory are locally continuous. Here are some further examples.

(1) Consider

$$F\colon \mathsf{CPO}^{op} \times \mathsf{CPO} \to \mathsf{CPO}$$

defined as follows: $F(D,E) = [D,E]$ is the set of CPO morphisms as a CPO object. On morphisms, $(f,g):(D,E)\to(X,Y)$, define $F(f,g)$ as the function which takes $u:D\to E$ to $f\cdot u\cdot g$. The category $\mathsf{CPO}^{op}\times\mathsf{CPO}$ is CPO-enriched due to Example 5.1.2(7), (8). Moreover, F is easily seen to be locally continuous.

(2) In contrast, the functor $F:\mathsf{Pfn}^{op}\times\mathsf{Pfn}\to\mathsf{Pfn}$ defined by $F(D,E)=\mathsf{Pfn}(D,E)$ and $F(f,g)$ sending u to $f\cdot u\cdot g$ is not locally continuous: $F(f,g)$ is a total function. Thus, if $f\sqsubseteq f'$ but $f'\not\sqsubseteq f$, we have $F(f,g)\not\sqsubseteq F(f',g)$.

(3) On the category of ω-algebraic CPO's, the Plotkin powerdomain operation (i.e. the formation of a free semilattice in that category) extends to a locally continuous functor (see e.g. Knijnenburg [111]).

We have another corollary to the Basic Lemma, connecting local continuity to the preservation of ω-colimits.

Corollary 5.1.19. *Let $\mathscr{A}$ be a* CPO*-enriched category. Then every locally continuous endofunctor on $\mathscr{A}$ preserves colimits of ω-chains of embeddings.*

Proof. Let $F\colon\mathscr{A}\to\mathscr{A}$ be locally continuous. Consider an ω-chain of embeddings in $\mathscr{A}$

$$E_0\xrightarrow{e_0}E_1\xrightarrow{e_1}E_2\xrightarrow{e_2}\cdots$$

Let $c_i\colon E_i\to C$ be a colimit cocone. Since F preserves the order on homsets, each Fc_i is an embedding and $\widehat{Fc_i}=F\widehat{c_i}$. Each composite $Fc_i\cdot\widehat{Fc_i}$ is $F(c_i\cdot\widehat{c_i})$. Thus the composites $Fc_i\cdot F\widehat{c_i}$ form a chain in $\mathscr{A}(FC,FC)$, and their join is id_{FC} using local continuity of F:

$$\bigsqcup_i Fc_i\cdot F\widehat{c_i}=F\left(\bigsqcup_i c_i\cdot\widehat{c_i}\right)=F\mathsf{id}_C=\mathsf{id}_{FC}.$$

By Basic Lemma 5.1.9, (Fc_i) is a colimit cocone. $\square$

We now restrict attention to certain CPO-enriched categories pertaining to least elements. We should point out that our next definition has a certain subtlety: this is not the same thing as being *enriched over* $\mathsf{CPO}_\perp$.

Definition 5.1.20. A *strict* CPO*-enriched* category is a nonempty CPO-enriched category $\mathscr{A}$ with a least element $\perp_{XY}$ in every hom-CPO $\mathscr{A}(X,Y)$ such that for all $f:Y\to Y'$ we have

$$f\cdot\perp_{XY}=\perp_{XY'}\quad\text{and}\quad\perp_{ZY'}\cdot f=\perp_{YZ}.$$

Example 5.1.21. The category $\mathsf{CPO}_\perp$ of complete partial orders with least element and continuous functions that are strict, i.e., preserve that least element, is a strict CPO-enriched category. Also, $\mathsf{DCPO}_\perp$, Pfn and Rel are strict CPO-enriched.

Lemma 5.1.22 [49]**.** *Every strict* CPO*-enriched category with ω-colimits has a zero object* $0=1$.

Proof. Choose any object X and form the ω-chain whose objects are all equal to X and whose connecting morphisms are $\perp_{XX}$. The colimit $c_n\colon X\to C$ then fulfils $c_n=c_{n+1}\cdot\perp_{XX}=\perp_{XC}$ for all n. Thus C is initial: for every object Y the unique morphism from C is $\bot_{CY}$. Indeed, given $f : C \to Y$ then $f \cdot c_n = f \cdot \bot_{CX} = \bot_{XY} = \bot_{CY} \cdot c_n$ for all n, thus, $f = \bot_{CY}$ since the colimit injections are jointly epimorphic. And C is terminal: the unique morphism from Y to C is $\bot_{YC}$ because clearly $\bot_{CC} = \mathsf{id}_C$, hence, given $f : Y \to C$ we have $f = \mathsf{id}_C \cdot f = \bot_{CC} \cdot f = \bot_{YC}$. □

The next result provides a sufficient condition for the existence of canonical fixed points. It is the centerpiece of this section.

Theorem 5.1.23 [156]**.** *Let $\mathscr{A}$ be a strict CPO-enriched category with ω-colimits. Every locally continuous endofunctor $F : \mathscr{A} \to \mathscr{A}$ has a canonical fixed point*

$$\mu F \quad = \quad \nu F \quad = \quad \mathsf{colim}\, F^n 0.$$

Proof. The unique morphism $e : 0 \to F0$ is an embedding with projection $\widehat{e} = \bot_{F0,0}$. Indeed, $\widehat{e} \cdot e = \mathsf{id}$ is clear, and $e \cdot \widehat{e} = e \cdot \bot_{F0,0} = \bot_{F0,F0} \sqsubseteq \mathsf{id}_{F0}$. In (5.1), we have the initial-algebra chain and the terminal-coalgebra chain of F together, and all the pairs of morphisms are embedding-projection pairs. Let $c_i \colon F^i 1 \to C$ be a colimit cocone of the initial-algebra chain. By Corollary 5.1.19, F preserves the colimit of its initial-algebra chain. Moreover, by Corollaries 5.1.12 and 5.1.19, $(\widehat{c}_i)$ is a limit cone of the terminal-coalgebra chain of F, and $(F\widehat{c}_i)$ is a limit cone of the same chain, but again without the first term.

As a result, the initial algebra μF and the terminal coalgebra νF exist. The structure ι of the initial algebra is, by Remark 3.1.8, defined by $\iota \cdot Fc_i = c_{i+1}$. Following equation (5.10) this yields

$$\iota = \bigsqcup_{i \geq 1} c_i \cdot F\widehat{c}_{i-1}.$$

Dually, the structure τ of the terminal coalgebra fulfils

$$\tau = \bigsqcup_{i \geq 1} Fc_{i-1} \cdot \widehat{c}_i.$$

It is our task to prove that ι is inverse to τ:

$$\begin{aligned} \iota \cdot \tau &= (\textstyle\bigsqcup c_i \cdot F\widehat{c}_{i-1}) \cdot (\bigsqcup Fc_{i-1} \cdot \widehat{c}_i) \\ &= \textstyle\bigsqcup c_i \cdot F(\widehat{c}_{i-1} \cdot c_{i-1}) \cdot \widehat{c}_i \\ &= \textstyle\bigsqcup c_i \cdot \widehat{c}_i \\ &= \mathsf{id} \end{aligned} \qquad \begin{aligned} \tau \cdot \iota &= (\textstyle\bigsqcup Fc_i \cdot \widehat{c}_{i-1}) \cdot (\bigsqcup c_{i-1} \cdot F\widehat{c}_i) \\ &= \textstyle\bigsqcup Fc_i \cdot (\widehat{c}_{i-1} \cdot c_{i-1}) \cdot F\widehat{c}_i \\ &= \textstyle\bigsqcup Fc_i \cdot F\widehat{c}_i \\ &= \mathsf{id} \end{aligned}$$

We have used the Basic Lemma once again. This proves $\tau = \iota^{-1}$. □

Remark 5.1.24. Instead of assuming all ω-colimits in $\mathscr{A}$, it is sufficient in Theorem 5.1.23 above to assume a zero object $0 = 1$ and colimits of ω-chains of embeddings. Also, instead of assuming that F be locally continuous, we could assume the weaker condition that it preserve ω-colimits of embeddings.

Example 5.1.25. (1) Continuous algebras with a binary operation and a constant considered in CPO consist of a cpo A, a continuous operation $\alpha_0 \colon A \times A \to A$, and an element of A. As we have previously observed, CPO has limits, coproducts, and furthermore, it has colimits of chains of monomorphisms preserved by the forgetful functor into Set. A continuous algebra is given by a single morphism $A \times A + 1 \to A$, viz. the structure of an algebra for the endofunctor $FX = X \times X + 1$.

This functor preserves colimits of ω-chains of monomorphism, thus $\mu F = F^\omega 0$. As in Example 3.2.5 the underlying set is that of all finite binary trees with the discrete order (since F preserves discrete posets).

The functor F also preserves limits of ω^{op}-chains. Thus, $\nu F = F^\omega 1$. As in Theorem 3.3.10 this is the set of all binary trees, again with the discrete order.

(2) Let us now consider continuous algebras with a binary operation and a constant in $\mathsf{CPO}_\perp$, where operations are required to be continuous (but not necessarily strict), and morphisms are the strict continuous homomorphisms. A non-strict binary operation on a poset A can be expressed by a strict continuous functions from $(A \times A)_\perp$ to A, where $(-)_\perp$ is the lifting, see Example 2.1.6(2). Recall that coproducts in $\mathsf{CPO}_\perp$ are formed by identifying in the disjoint union the bottom elements. Thus, the structure of a continuous algebra is expressed by a single morphism $\alpha\colon (A \times A)_\perp + 1_\perp$, i.e. the structure of an algebra for the endofunctor $FX = (X \times X)_\perp + 1_\perp$ from Example 2.1.6(2). It is easy to see that F is locally continous so the above theorem it has a canonical fixed point $\mu F = \nu F = \mathsf{colim}_{n<\omega} F^n 0$.

Let us represent the initial object of $\mathsf{CPO}_\perp$ by $\{\perp\}$ where $\perp$ is considered as a single node labelled tree. Given a tree representation of $F^n 0$, represent pairs (t_1, t_2) in $F^n 0 \times F^n 0$ as binary trees with maximum subtrees t_1 and t_2. Then it is easy to see that $F^n 0$ is the cpo of all binary trees of depth at most n ordered by

$$t \sqsubseteq t' \quad \text{iff} \quad t \text{ is a cutting of } t' \text{ at some level.}$$

The connecting maps $F^n 0 \hookrightarrow F^{n+1} 0$ are the inclusions. It follows that the canonical fixed point is

$$\mu F = \nu F = \text{all binary trees,}$$

ordered as above.

(3) Generalizing the previous example, for every finitary signature Σ we consider continuous algebras in $\mathsf{CPO}_\perp$. Operations are continuous and homomorphisms are the strict continuous Σ-homomorphisms. They are precisely the F-algebras for the endofunctor given by

$$FX = \coprod_{\sigma \in \Sigma} (X^n)_\perp,$$

where n denotes the arity of $\sigma \in \Sigma$. This functor is locally continous. Its canonical fixed point is T_Σ, the algebra of all Σ-trees, with the following order [34]. Given $t \in T_\Sigma$ as a partial function on $\mathbb{N}^*$ as explained in Remark 2.2.13, let $\hat{t}\colon \mathbb{N}^* \to (\coprod_{n\in\mathbb{N}} \Sigma_n)_\perp$, where the coproduct on the right is discretely ordered, be the total function with $\hat{t}(w) = \perp$ if $t(w)$ is undefined and $\hat{t}(w) = t(w)$ else. Then for two trees $s, t \in T_\Sigma$ put

$$s \sqsubseteq t \quad \text{iff} \quad \hat{s}(w) \sqsubseteq \hat{t}(w) \text{ for all } w \in \mathbb{N}^*.$$

Remark 5.1.26. In our next example we work in the strict CPO-enriched category Rel of sets and relations. Coproducts in this category are disjoint unions (as in Set). Since Rel is self-dual, products are also given by disjoint unions with the projections the opposite relations to the coproduct injections. Moreover, an embedding in Rel is precisely an injective function $e\colon A \to B$ with the projection $\hat{e}$ the opposite relation. Colimits of ω-chains of embeddings are the unions, again as in Set.

Example 5.1.27 [91]**.** We have discussed deterministic automata in Examples 2.4.2(3) and 2.5.5, and we have seen non-deterministic automata as coalgebras over Set in Example 2.4.2(5). Here we consider them as coalgebras over the category Rel. Recall that non-deterministic automata are defined in the same way as deterministic ones, except that the transition functions $\delta_s\colon S \to S$ for every $s \in \Sigma$ are replaced by transition relations $\delta_s \subseteq S \times S$. Thus, in the category Rel an nda is given by an object S of states, a morphism $S \to 1$ relating the accepting states to the single element of 1 and a morphism $\delta\colon S \to \Sigma \times S$ (where $\times$ denotes cartesian product) relating a state $x \in S$ to all pairs (s, y) with $(x, y) \in \delta_s$. We can combine this to obtain a single morphism

$$\alpha\colon S \to 1 + \Sigma \times S,$$

whence ndas are precisely the coalgebras for the functor $F\colon \mathsf{Rel} \to \mathsf{Rel}$ given by the coproduct of 1 and Σ copies of Id, shortly $FX = 1 + \Sigma \times S$. The initial-algebra chain

$$\emptyset \xrightarrow{!} 1 \xrightarrow{F!} 1 + \Sigma \xrightarrow{FF!} 1 + \Sigma + \Sigma \times \Sigma \xrightarrow{F^3!} \cdots$$

coincides with that of the corresponding endofunctor on Set, since the connection relations are simply the inclusion maps. Thus, the colimit is Σ^*. It follows from Theorem 5.1.23 that the canonical fixed point of F is $\mu F = \nu F = \Sigma^*$.

It is easy to see that a coalgebra homomorphism from an nda $\alpha\colon S \to 1 + \Sigma \times S$ to an nda $\alpha'\colon 1 + \Sigma S'$ is a relation $R\colon S \to S'$ such that

(1) a state $x \in S$ is accepting iff $x\, R\, x'$ for some accepting state $x' \in S'$, and

(2) for every triple $(x, s, x') \in S \times \Sigma \times S'$ the following statements are equivalent

- there is a transition $x \xrightarrow{s} y$ in S with $y\, R\, x'$, and
- there is a transition $z \xrightarrow{s} x'$ in S' with $x\, R\, z$.

For example, for every nda (S, α) the unique coalgebra homomorphism $R\colon S \to \Sigma^*$ to the terminal coalgebra relates every state x to the language $L(x)$ it accepts, i.e. we have $x\, R\, w$ iff $w \in L(x)$.

In contrast, we saw in Example 4.5.8 that the semantics of non-deterministic automata considered them as coalgebras over Set yields the behaviour of states modulo bisimilarity rather than the usual language semantics.

We have seen the finite power set functor $\mathcal{P}_\mathrm{f}\colon \mathsf{Set} \to \mathsf{Set}$ at several points, and we gave many descriptions of its terminal coalgebra. So it should be interesting to consider analogs of $\mathcal{P}_\mathrm{f}$ on Rel.

Example 5.1.28. (1) The functor $\mathcal{P}_\mathrm{f}$ has a lifting $P_\mathrm{f}\colon \mathsf{Rel} \to \mathsf{Rel}$ assigning to every morphism $r\colon X \to Y$, i.e. a relation $r \subseteq X \times Y$, the relation $P_\mathrm{f} r \subseteq \mathcal{P}_\mathrm{f} X \times \mathcal{P}_\mathrm{f} Y$ that consists of all pairs (A, B) of finite sets such that

$$\text{for every } a \in A \text{ there exist } b \in B \text{ with } (a, b) \in r. \tag{5.12}$$

It is easy to see that this is a well-defined locally continuous functor on Rel. Indeed, suppose that $r_n\colon X \to Y$, $n < \omega$, is an ω-chain of relations and $r = \bigcup_{n<\omega} r_n$. Then a pair $(A, B) \in \mathcal{P}_\mathrm{f} X \times \mathcal{P}_\mathrm{f} Y$ lies in $P_\mathrm{f} r$ iff there exists an $n < \omega$ such that (A, B) lies in $P_\mathrm{f} r_n$, using the fact that A and B are finite.

Therefore P_f has a canonical fixed point by Theorem 5.1.23 which is a colimit of its initial-algebra chain. Since this is a chain of embeddings and colimits of such chains in Rel are the same as in Set, we conclude that

$$\mu P_\mathrm{f} = \nu P_\mathrm{f} = V_\omega,$$

i.e. the hereditarily finite sets (see Example 2.2.7(1)).

(2) There are other locally continuous liftings of $\mathcal{P}_\mathrm{f}$ to Rel. For example, the lifting P'_f obtained by changing (5.12) to

$$\text{for every } b \in B \text{ there exists } a \in A \text{ with } (a, b) \in r,$$

or P''_f defined such that $P''_\mathrm{f} r$ is the intersection of $P_\mathrm{f} r$ and $P'_\mathrm{f} r$. The canonical fixed points of P'_f and P''_f are given by V_ω, again.

Remark 5.1.29. In some applications one uses categories such as CPO_* whose objects are complete partial orders with $\perp$ and morphisms are continuous maps (not necessarily preserving $\perp$). Observe that this means that composition of morphisms is *left-strict*: $\perp \cdot f = \perp$, but not necessarily right-strict:

Definition 5.1.30. A CPO-enriched category is called *left-strict* if every hom-set has a least element $\perp$ and every morphism f fulfils $\perp \cdot f = \perp$.

Example 5.1.31. Smyth-Plotkin's Theorem 5.1.23 does not extend to left-strict CPO-enriched categories. Indeed, the identity functor on CPO_* does not have an initial algebra although it is locally continuous. (The "natural candidate" $\mathsf{id}_{\{\perp\}}$ is not an initial algebra: consider any algebra with two fixed points.) For that matter, CPO_* does not have an initial object. This shows that Lemma 5.1.22 requires the strict CPO-enrichedness. However, Freyd [74] proved a weaker result:

Proposition 5.1.32. *Let $\mathscr{A}$ be a left-strict CPO-enriched category. If a locally continuous functor F has an initial algebra, then μF is a canonical fixed point.*

Proof. Let $\iota\colon FI \to I$ be an initial algebra. For every coalgebra $\alpha : A \to FA$, we prove that a unique homomorphism into (I, ι^{-1}) exists.

(1) Existence. The endomap ϱ of $\mathscr{A}(A, I)$ given by $h \mapsto \iota \cdot Fh \cdot \alpha$ is obviously continuous, hence, it has a fixed point $h\colon A \to I$ with $\iota^{-1} \cdot h = Fh \cdot \alpha$ by the Kleene Theorem 3.1.1. This is a coalgebra homomorphism. Observe that $h = \bigsqcup \varrho^i(\perp)$ by Kleene's Theorem.

(2) Uniqueness. First notice that for $\mathscr{A}(I,I)$ we have an analogous endomorphism $k \mapsto \iota \cdot Fk \cdot \iota^{-1}$. Since I is initial, the only fixed point is $k = \mathsf{id}_I$. And this is the least fixed point, thus, $\mathsf{id}_I = \bigsqcup_i u_i$ where $u_0 = \perp$ and $u_{i+1} = \iota \cdot Fu_i \cdot \iota^{-1}$. Now suppose that $h' : (A,\alpha) \to (I, \iota^{-1})$ is any coalgebra homomorphism. Next we show that for every i we have

$$u_i \cdot h' = \varrho^i(\perp).$$

We verify this by induction: $u_0 \cdot h' = \perp \cdot h' = \perp = \rho^0(\perp)$ and

$$u_{i+1} \cdot h' = \iota \cdot Fu_i \cdot \iota^{-1} \cdot h' = \iota \cdot Fu_i \cdot Fh' \cdot \alpha = \iota \cdot F\varrho^i(\perp) \cdot \alpha = \varrho^{i+1}(\perp).$$

Consequently,

$$h' = (\bigsqcup_i u_i) \cdot h' = \bigsqcup_i \varrho^i(\perp). \qquad \square$$

One of the themes of this section has been canonical fixed points. We saw in Theorem 5.1.23 conditions on a category and a functor that insure the existence of a canonical fixed point. Looking ahead to Section 5.2, we shall see conditions there that insure *unique* fixed points. So one might ask whether the fixed points given by Theorem 5.1.23 are unique. As it happens, they are not (see Example 5.1.28 below). However if we assume an extra condition on the fixed point, we do have a uniqueness result.

Theorem 5.1.33. *The canonical fixed point of a locally continuous endofunctor F on $\mathsf{CPO}_\perp$ is the only fixed point $FX \cong X$ with the property that id_X is the least algebra endomorphism of X.*

Moreover, it is the only fixed point with the property that the id_X is the only strict algebra endomorphism. For the proof, see Abramsky and Jung [2, Section 5].

We also might note another fact about the canonical fixed points coming from Theorem 5.1.23.

Theorem 5.1.34. *Under the conditions in Theorem 5.1.23, the algebra morphism from the canonical fixed point μF to any fixed point of F is an embedding.*

Proof. Let $\alpha\colon FA \to A$ be an isomorphism. Recall the cocone $\alpha_n\colon F^n 0 \to A$ from (3.2): $\alpha_0\colon 0 \to A$ is given by initiality, and $\alpha_{n+1} = \alpha \cdot F\alpha_n$. The algebra morphism $h\colon \mu F \to A$ is the colimit morphism for this cocone. So our result follows from Corollary 5.1.14 as soon as we show that each α_n is an embedding. We also have a coalgebra $\alpha^{-1}\colon A \to FA$ and hence a canonical cone $\beta_n\colon A \to F^n 1 = F^n 0$. This cone fulfils $\beta_0\colon A \to 1$ by finality, and $\beta_{n+1} = F\beta_n \cdot \alpha^{-1}$. An easy induction shows that $\widehat{\alpha_n} = \beta_n$ for all n. So indeed each α_n is an embedding. $\square$

Remark 5.1.35. A higher-order variation of Theorem 5.1.23 was established in Adámek [13] where cpo's are generalized as follows: A category $\mathscr{A}$ is called *Scott complete* if it has

(1) filtered colimits,

(2) an initial object,

(3) limits of diagrams with a cone, and

(4) a set of finitely presentable objects (i.e. objects X whose hom-functor $\mathscr{A}(X,-)$ is finitary) whose closure under filtered colimits is $\mathscr{A}$.

We obtain a 2-category of Scott-complete categories, using as 1-cells the functors preserving filtered colimits and the initial object, and as 2-cells the natural transformations.

Theorem 5.1.36 [13]**.** *Every locally continuous* 2*-endofunctor of the category of Scott complete categories has a canonical fixed point.*

Remark 5.1.37. We have seen how to obtain initial algebras and terminal coalgebras for locally continuous endofunctors on strict CPO-enriched categories. Surprisingly, for all set functors F with $F\emptyset \neq \emptyset$ and preserving ω^{op}-limits it turns out that the initial-algebra chain and the terminal-coalgebra chain converge, and the terminal coalgebra also carries a structure of a CPO. It is obtained from the initial algebra as a free ideal completion (see Example 5.1.17). Indeed, let $u\colon \emptyset \to 1$ be the unique morphism. Then there exists a unique $\bar{u}\colon F^\omega\emptyset \to F^\omega 1$ such that $F^n u\colon F^n\emptyset \to F^n 1$ has the form $\ell_n\colon \bar{u}\cdot c_n$ (see Notation 3.1.4 and Remark 3.3.5) for every n. Choose a point $p\colon 1 \to F\emptyset$ and put

$$e_n = (F^n 1 \xrightarrow{F^n p} F^{n+1}\emptyset \xrightarrow{c_{n+1}} F^\omega\emptyset \xrightarrow{\bar{u}} \nu F).$$

Then the partial order $\sqsubseteq$ on νF is given as follows for $x \neq y$:

$$x \sqsubseteq y \text{iff } x = e_n \cdot \ell_n(y) \text{ for some } n < \omega.$$

This is a cpo with the properties mentioned above [15].

In Section 6.3 we prove a (non-finitary) generalization of the main result: every stable functor on $\mathsf{DCPO}_\perp$ with a fixed point has a canonical one (see Corollary 6.3.8).

5.2 CMS-enriched categories

Complete partial orders, used in Section 5.1, express the idea of how much information is contained in an approximation of a solution of a recursive specification. But we can also measure the distance of various approximations, and apply the idea of a metric space in lieu of a poset.

For example, recall that the terminal coalgebra for the polynomial functor H_Σ is the set of all Σ-labelled trees, as explained by Theorem 3.3.10. Arnold and Nivat [44] studied this set using the following natural metric: the distance of distinct trees t and s is

$$d(s,t) = 2^{-n} \text{ for the least level } n \text{ at which } t \text{ and } s \text{ differ.} \tag{5.13}$$

This metric space is *complete*, i.e., every Cauchy sequence converges.

In this section we consider categories enriched over the category CMS of complete metric spaces, and prove a result analogous to that of Section 5.1: every locally contracting endofunctor has a canonical fixed point. In fact, a stronger and surprising result is true: such a functor has, up to isomorphism, a *unique* fixed point. Thus every fixed point is canonical.

Complete metric spaces were first applied by de Bakker and Zucker [65], and their method was further developed by America and Rutten in their seminal paper [42] on this topic. In both of these papers, the morphisms are a little different from the non-expansive maps: they are the *retraction-embedding pairs* $(e, \widehat{e})$ consisting of an isometric embedding $e : X \to Y$ together with a non-expanding map $\widehat{e} : Y \to X$, such that $\widehat{e} \cdot e = \mathsf{id}_X$.

Notation 5.2.1. We denote by

$$\mathsf{CMS}$$

the category of complete metric spaces with distances bounded by 1 and *non-expanding* functions as morphisms; these are the functions f with the property that $d(fx, fy) \leq d(x, y)$ for all x, y.

Definition 5.2.2. A category $\mathscr{A}$ is CMS-*enriched* if its hom-sets come equipped with a complete metric, and composition is non-expanding in both variables. Thisq means that given morphisms $f_1, f_2 : A \to B$, then for all h and g composable with f_i,

$$d(h \cdot f_1 \cdot g, h \cdot f_2 \cdot g) \leq d(f_1, f_2). \tag{5.14}$$

Example 5.2.3. (1) CMS is CMS-enriched with respect to the supremum metric

$$d_{X,Y}(f, g) = \sup_{x \in X} d_Y(f(x), g(x)).$$

(2) CMS_*, the category of pointed complete metric spaces, i.e. objects are pairs (X, x) where X is a complete metric space and $x \in X$, and morphisms are and non-expanding maps preserving the distinguished point, is also CMS-enriched using the supremum metric.

(3) If $\mathscr{C}$ is CMS-enriched then so is $\mathscr{C}^{op}$, where the metric of $\mathscr{C}^{op}(A, B)$ is that of $\mathscr{C}(B, A)$.

(4) A product $\mathscr{C}' \times \mathscr{C}''$ of CMS-enriched categories (with metrics $d'_{A',B'}$ and $d''_{A'',B''}$ on hom-sets, respectively) is CMS-enriched using the maximum metric: the distance of (f', f'') and (g', g'') is $\max\{d'(f', g'), d''(f'', g'')\}$.

At the root of our study is the following classical result. In it, recall that a function $f \colon X \to Y$ is *contracting* for some $\varepsilon < 1$, if $d(fx, fy) < \varepsilon \cdot d(x, y)$ for all $x, y \in X$.

Theorem 5.2.4 (Banach Fixed Point Theorem)**.** *Every contracting endofunction on a nonempty complete metric space has a unique fixed point.*

Definition 5.2.5 [42]**.** An endofunctor F of a CMS-enriched category $\mathscr{A}$ is *locally contracting* if there exists $\varepsilon < 1$ such that for all parallel pairs $f, g : X \to Y$ we have

$$d(Ff, Fg) \leq \varepsilon \cdot d(f, g).$$

Example 5.2.6. (1) For every $\varepsilon < 1$, the functor which takes a space (X, d) and shrinks the metric by ε, giving $(X, \varepsilon \cdot d)$ is contracting.

(2) Every composite, coproduct, and finite product of locally contracting endofunctors is locally contracting.

(3) Every polynomial functor H_Σ on Set has a *contracting lifting* H'_Σ to CMS. This means that the following square

$$\begin{tikzcd} \mathsf{CMS} \arrow[r, "H'_\Sigma"] \arrow[d, "U"'] & \mathsf{CMS} \arrow[d, "U"] \\ \mathsf{Set} \arrow[r, "H_\Sigma"'] & \mathsf{Set} \end{tikzcd}$$

commutes, where U is the forgetful functor taking a metric space to its set of points. This follows from (1) and (2) since polynomial functors are formed from Id by finite products and coproducts. Using the constant $\varepsilon = \frac{1}{2}$ as in (1) above, we can describe the terminal coalgebra for H'_Σ on CMS as the set of all Σ-trees equipped with the metric of (5.13) as we will see in Example 5.2.24(3) below. Moreover, if Σ contains at least one constant symbol, then this is a canonical fixed point of H'_Σ.

(4) The Hausdorff Functor. This is the endofunctor $\mathcal{H}$ of CMS assigning to a metric space (X, d) the space $\mathcal{H}(X, d)$ of all nonempty compact subsets. Its metric $\bar{d}$ was introduced by Hausdorff who also proved that it yields a complete metric space. It is defined by

$$\bar{d}(A, B) = \max(\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)).$$

for every pair of compact sets A and B. Here $d(a, B)$ denotes, as usual, $\inf_{b \in B} d(a, b)$. For morphisms $f\colon (X, d) \to (X', d')$ we define $\mathcal{H}f : A \mapsto f[A]$. The resulting functor $\mathcal{H}$ is locally non-expanding, but not locally contracting. However, we will see in **??** that it is finitary.

By modifying the Hausdorff functor by halving all distances as in (3), we obtain a contracting endofunctor.

We are going to present the proof that every contracting endofunctor has a canonical fixed point with all details. As in the CPO-enriched case, this material has not been presented so comprehensively before. Once again, for CMS the main ideas stem from America and Rutten [42]. Their results were extended in [36], and we sharpen them up a bit. First a surprising result about unicity of fixed points:

Theorem 5.2.7. *Let $\mathscr{A}$ be a CMS-enriched category with nonempty hom-sets. Every fixed point of a locally contracting endofunctor on $\mathscr{A}$ is canonical.*

Proof. (1) Let $\alpha\colon FA \to A$ be a fixed point of a locally contracting endofunctor F with contraction factor ε. We prove that this is the initial algebra of F.

Given an algebra $\beta\colon FB \to B$, define an endofunction k of $\mathscr{A}(A, B)$ as follows:

$$k\colon (A \xrightarrow{f} B) \longmapsto (A \xrightarrow{\alpha^{-1}} FA \xrightarrow{Ff} FB \xrightarrow{\beta} B).$$

Then k is ε-contracting: for every parallel pair $f, g : A \to B$ we have

$$\begin{aligned} d(k(f), k(g)) &= d(\beta \cdot Ff \cdot \alpha^{-1}, \beta \cdot Fg \cdot \alpha^{-1}) \\ &\le d(Ff, Fg) \\ &\le \varepsilon \cdot d(f, g) \end{aligned}$$

since composition is non-expanding and F is ε-contracting. Thus k has a fixed point:

$$h\colon A \to B \text{ with } h = \beta \cdot Fh \cdot \alpha^{-1}.$$

This implies that h is an algebra homomorphism. Conversely, every algebra homomorphism ($\alpha \cdot h = \beta \cdot Fh$) is a fixed point of k. By Banach's Theorem 5.2.4 (A, α) is an initial algebra.

(2) The proof that $\alpha^{-1}\colon A \to FA$ is a terminal coalgebra is by duality: if $\mathscr{C}$ is a CMS-enriched category, then so is $\mathscr{C}^{op}$ (using the same metric on hom-sets). And F^{op} is a contracting endofunctor. By applying (1) to it we see that (A, α^{-1}) is an initial algebra for F^{op}, i.e. a terminal coalgebra for F. □

Notation 5.2.8. Let $\mathscr{A}$ be a CMS-enriched category. We denote by $\mathscr{A}^E$ the category of all objects of $\mathscr{A}$, where the morphisms from A to B are all pairs $(e, \widehat{e})$ of morphisms in $\mathscr{A}$, with $e\colon A \to B$, and $\widehat{e}\colon B \to A$, and $\widehat{e} \cdot e = \mathsf{id}_A$:

$$\begin{tikzcd} A \arrow[loop left, "\mathsf{id}"] \arrow[r, shift left, "e"] & B \arrow[l, shift left, "\widehat{e}"] \end{tikzcd}$$

Composition and identity morphisms in $\mathscr{A}^E$ are defined componentwise. If $\mathscr{A} = \mathsf{CMS}$, then $\widehat{e} \cdot e = \mathsf{id}_A$ implies that e is an isometric embedding. Thus CMS^E is precisely the category $\mathscr{C}$ of America and Rutten [42].

One difference between the work in this section and the previous one is that in the case of CPO-enriched categories, each embedding e had a unique projection $\widehat{e}$. The uniqueness is lost in the CMS setting. And so the morphisms in categories $\mathscr{A}^E$ are pairs in the CMS setting, while in the CPO setting we were able to get by with only the embeddings.

Definition 5.2.9. An ω-chain in the category $\mathscr{A}^E$

$$\begin{tikzcd} E_0 \arrow[r, shift left, "e_0"] & E_1 \arrow[l, shift left, "\widehat{e_0}"] \arrow[r, shift left, "e_1"] & E_2 \arrow[l, shift left, "\widehat{e_1}"] \arrow[r, shift left, "e_2"] & \cdots \arrow[l, shift left, "\widehat{e_2}"] \end{tikzcd} \tag{5.15}$$

is called *contracting* if there exists $\varepsilon < 1$ such that for all i we have

$$d(e_{i+1} \cdot \widehat{e}_{i+1}, \mathsf{id}_{E_{i+1}}) \le \varepsilon \cdot d(e_i \cdot \widehat{e}_i, \mathsf{id}_{E_i}).$$

Example 5.2.10. Let F be a locally contracting endofunctor on a CMS-enriched category with a terminal object 1 and such that $\mathscr{A}(1, F1)$ is nonempty. Given $e : 1 \to F1$, the chain

$$\begin{tikzcd} 1 \arrow[r, shift left, "e"] & F1 \arrow[l, shift left, "!"] \arrow[r, shift left, "Fe"] & F^2 1 \arrow[l, shift left, "F!"] \arrow[r, shift left, "F^2 e"] & \cdots \arrow[l, shift left, "F^2 !"] \end{tikzcd}$$

is contracting. Indeed, if ε is the contraction constant for F, then

$$d(F^{i+1}e \cdot F^{i+1}!, \mathsf{id}) = d(F(F^i e \cdot F^i !), F\mathsf{id}) \le \varepsilon \cdot d(F^i e \cdot F^i !, \mathsf{id}).$$

Basic Lemma 5.2.11. *Let $\mathscr{A}$ be a CMS-enriched category. Consider a contracting chain*

$$E_0 \underset{\widehat{e}_0}{\overset{e_0}{\rightleftarrows}} E_1 \underset{\widehat{e}_1}{\overset{e_1}{\rightleftarrows}} E_2 \underset{\widehat{e}_2}{\overset{e_2}{\rightleftarrows}} \cdots \tag{5.16}$$

in $\mathscr{A}^E$. Let $c_i : E_i \to C$ be a cocone of $(e_i)_{i\in\mathbb{N}}$ in $\mathscr{A}$. Then the following are equivalent:

(1) *(c_i) is a colimit cocone in $\mathscr{A}$.*

(2) *Each c_i is a split monomorphism and there exist splittings $\widehat{c}_i$ forming a cone of $(\widehat{e}_i)$ in $\mathscr{A}$ with $\lim_i(c_i \cdot \widehat{c}_i) = \mathsf{id}_C$.*

Proof. (1) First observe that the above condition $d(e_i \cdot \widehat{e}_i, \mathsf{id}) \le \varepsilon \cdot d(e_{i-1} \cdot \widehat{e}_{i-1}, \mathsf{id})$ implies that for all $i < k$ we have

$$d(e_k \cdot \widehat{e}_k, \mathsf{id}) \le \varepsilon^{k-i} \cdot d(e_i \cdot \widehat{e}_i, \mathsf{id}) \le \varepsilon^k$$

The proof is any easy induction on i where in the first step we use $d(e_0 \cdot \widehat{e}_0, \mathsf{id}) \le 1$. (Recall that 1 is our bound on all distances).

(2) To prove (1) $\Rightarrow$ (2) use, for every i, the shortened ω-chain $E_i \xrightarrow{e_i} E_{+1} \xrightarrow{e_{i+1}} E_{i+2} \xrightarrow{e_{i+2}} \cdots$ and define, analogously to the Basic Lemma 5.1.9, morphisms $\widehat{c}_i \colon C \to E_i$ by

$$\widehat{c}_i \cdot \widehat{c}_i = \mathsf{id}_{E_i} \quad \text{and} \quad \widehat{c}_i \cdot c_k = \widehat{e}_i \cdot \dots \cdot \widehat{e}_{k-1} \quad \text{for } k > i.$$

Again, we obtain the formula (5.9) for all i

$$\widehat{c}_i \;=\; \widehat{e}_i \cdot \widehat{c}_{i+1}.$$

The sequence $c_i \cdot \widehat{c}_i : C \to C$ is Cauchy: to prove this, we verify below that

$$d(c_i \cdot \widehat{c}_i, c_{i+1} \cdot \widehat{c}_{i+1}) \;\le\; \varepsilon^i \quad \text{for all } i,$$

from which we conclude that the distance of $c_i \cdot \widehat{c}_i$ and $c_k \cdot \widehat{c}_k$, where $k > i$, is at most

$$\varepsilon^i + \varepsilon^{i+1} + \ldots + \varepsilon^{k-1} \;\le\; \textstyle\sum_{j=i}^{\infty} \varepsilon^j \;=\; \frac{\varepsilon^i}{1-\varepsilon}.$$

This converges to 0 as $i \to \infty$. The desired inequality is clear from $\widehat{e}_i \cdot \widehat{c}_{i+1} = \widehat{c}_i$ and $c_{i+1} \cdot e_i = c_i$:

$$\begin{aligned} d(c_i \cdot \widehat{c}_i, c_{i+1} \cdot \widehat{c}_{i+1}) &= d(c_{i+1} \cdot (e_i \cdot \widehat{e}_i) \cdot \widehat{c}_{i+1}, c_{i+1} \cdot \widehat{c}_{i+1}) \\ &\le d(e_i \cdot \widehat{e}_i, \mathsf{id}) \qquad\qquad \text{by (5.14)} \\ &\le \varepsilon^i \end{aligned}$$

The rest is analogous to the Basic Lemma 5.1.9 again: we know that $\lim_i(c_i \cdot \widehat{c}_i)$ exists, and we derive that it is id_C.

(3) The proof of (2) $\Rightarrow$ (1) is also analogous to the Basic Lemma 5.1.9. Given a cocone $b_i : E_i \to B$ of $(\widehat{e}_i)$, the sequence $b_i \cdot \widehat{c}_i : C \to B$ is Cauchy. This follows from

$$\begin{array}{rcll} d(b_i \cdot \widehat{c}_i, b_{i+1} \cdot \widehat{c}_{i+1}) & = & d(b_{i+1} \cdot (e_i \cdot \widehat{e}_i) \cdot \widehat{c}_{i+1}, b_{i+1} \cdot \widehat{c}_{i+1}) & \\ & \leq & d(e_i \cdot \widehat{e}_i, \mathsf{id}) & \text{by (5.14)} \\ & \leq & \varepsilon^i & \end{array}$$

Then $\lim_i(b_i \cdot \widehat{c}_i)$ is the desired unique factorization maorphism. □

The following corollary yields a limit-colimit coincidence in CMS-enriched categories analogous to the one we have seen in Remark 5.1.13.

Corollary 5.2.12. *Given a chain (5.16) in $\mathscr{A}^E$, a cone $(\widehat{c}_i)$ of the ω^{op}-chain $(\widehat{e}_i)$ is a limit cone in $\mathscr{A}$ iff each $\widehat{e}_i$ is a split epimorphism and there exist splittings c_i forming a cocone of (e_i) with* $\lim c_i \cdot \widehat{c}_i = \mathsf{id}_C$.

By Example 5.2.3(3), we can apply the Basic Lemma 5.2.11 to $\mathscr{A}^{op}$.

Corollary 5.2.13. *If a CMS-enriched category has colimits of ω-chains, then so does $\mathscr{A}^E$.*

Remark 5.2.14. (1) The Basic Lemma also holds for the *converging towers* of America and Rutten [42]: These are sequences (5.15) such that for every $\varepsilon > 0$ there exists N with the distance between id_{E_m} and $e_n \cdot \ldots \cdot e_{m+1} \cdot \widehat{e}_{m+1} \cdot \ldots \cdot \widehat{e}_n$ at most ε for all $m > n \geq N$. The proof is the same.

(2) In analogy to Theorem 5.1.23 we might now expect that contracting endofunctors on CMS-enriched categories have canonical fixed points. Unfortunately, this is not quite true:

(a) A locally contracting endofunctor need not have a terminal coalgebra: take the poset $\mathbb{N}^{op}$ as a category. It is trivially CMS-enriched. Every endofunctor is (trivially) contracting. But not every endofunctor, e.g. $s(x) = x+1$, has a terminal coalgebra.

(b) A locally contracting endofunctor on CMS need not have a canonical fixed point: take $F(X,d) = (X, \frac{1}{2}d)$. Here $\mu F = \emptyset$ and $\nu F = 1$. (We might note in passing that on MS, this functor has other fixed points, such as the open interval $(0,1)$.)

Nevertheless, for categories enriched over CMS_* of Example 5.2.3(2) (which is analogous to the strictness required in the Smyth-Plotkin Theorem 5.1.23) canonical fixed points exist:

Definition 5.2.15. A *strict* CMS*-enriched category* $\mathscr{A}$ is a nonempty CMS-enriched category in which every hom-object $\mathscr{A}(X,Y)$ has a chosen element $\perp_{XY}$ such that for all morphisms $f : A \to B$ we have $f \cdot \perp_{XA} = \perp_{XB}$ and $\perp_{BY} \cdot f = \perp_{AY}$.

Theorem 5.2.16. *Every locally contracting endofunctor of a strict CMS-enriched category with ω-colimits has a canonical fixed point.*

Proof. Every strict CMS-enriched category $\mathscr{A}$ has a zero object, the proof is completely analogous to Lemma 5.1.22. The rest of the proof is analogous to the proof of Theorem 5.1.23, using Basic Basic Lemma 5.2.11 and arguing with $\lim_i$ in lieu of $\bigsqcup_i$. □

Corollary 5.2.17. *Every locally contracting endofunctor F of* CMS *with $F\emptyset \neq \emptyset$ has a canonical fixed point.*

Indeed, we modify F to an endofunctor $\bar{F}$ on CMS_* as follows: choose $a_0 \in F\emptyset$ and for every space X put $a_X = Fh_X(a_0)$, where $h_X \colon \emptyset \to X$ is the unique morphism. Define $\bar{F}$ on objects by

$$\bar{F}(X, x) = (FX, a_X)$$

and on morphisms $f \colon (X, x) \to (Y, y)$ by $\bar{F}f = Ff$. This is well-defined due to $h_Y = f \cdot h_X$. Then $\bar{F}$ is locally contracting, hence, has a canonical fixed point. The same space is the canonical fixed point of F.

Remark 5.2.18. (1) Instead of assuming all ω-colimits in Theorem 5.2.16, it is sufficient to assume a zero object and colimits of ω-chain of split monomorphisms.

(2) For CMS-enriched categories the theorem above does not hold in general, but the following weaker result has a completely analogous proof:

Theorem 5.2.19. *Let $\mathscr{A}$ be a* CMS*-enriched category with ω-colimits. Let $F : \mathscr{A} \to \mathscr{A}$ be locally contracting with $\mathscr{A}(1, F1)$ nonempty. Then F has a terminal coalgebra.*

Remark 5.2.20. In the above proof of the existence of a canonical fixed point we did not fully use the power of F being locally contracting: we only needed this for pairs consisting of id_A and an endomorphism on A. This led America and Rutten [42] to the concept of a *hom-contracting* endofunctor F, i.e., there exists $\varepsilon < 1$ with

$$d(Ff, \mathsf{id}_A) \leq \varepsilon \cdot d(f, \mathsf{id}_A) \qquad \text{for all } f : A \to A.$$

However, this is equivalent to being locally contracting whenever the following type of coproducts exists:

Definition 5.2.21. In a CMS-enriched category an *ε-contracting copower* of an object A is an object $A +_\varepsilon A$ with morphisms $i_1, i_2 : A \to A +_\varepsilon A$ universal is the following sense:

(1) $d(i_1, i_2) \leq \varepsilon$;

(2) given morphisms $f_1, f_2 : A \to B$ with $d(f_1, f_2) \leq \varepsilon$, there exists a unique morphism $[f_1, f_2] : A +_\varepsilon A \to B$ with $f_k = [f_1, f_2] \cdot i_k$ for $k = 1, 2$;

(3) the distance of $\mathsf{id}_{A+_\varepsilon A}$ and $[i_2, i_1]$ is at most ε: $d(id, [i_2, i_1]) \leq \varepsilon$.

Example 5.2.22. CMS has contracting copowers. Indeed, for $\varepsilon \geq 1$ these are just copowers. That is, disjoint unions

$$A + A = A \times \{1, 2\}$$

with the original distance on $A \times \{1\}$ as well as $A \times \{2\}$, and with $d((x, 1), (y, 2)) = 1$ for all $x, y \in A$.

The space $A +_\varepsilon A$ differs only in the last item, and then only for $x = y$: we put $d((x, 1), (x, 2)) = \varepsilon$.

Lemma 5.2.23. *Every hom-contracting endofunctor of a* CMS*-enriched category with contracting coproducts is locally contracting.*

Proof. Let F be hom-contracting with contraction factor ε. Given a parallel pair $f_1, f_2 : A \to B$, we prove $d(Ff_1, Ff_2) \leq \varepsilon \cdot d(f_1, f_2)$. To this end let $r = d(f_1, f_2)$, consider $g = [f_1, f_2] : A +_r A \to B$ and put $\sigma = [i_2, i_1]$, which satisfies $\sigma \cdot i_1 = i_2$. Then we have

$$\begin{array}{rcl} d(Ff_1, Ff_2) & = & d(Fg \cdot Fi_1, Fg \cdot Fi_2) \\ & \leq & d(Fi_1, Fi_2) \\ & = & d(Fi_1, F\sigma \cdot Fi_1) \\ & \leq & d(\mathsf{id}, F\sigma) \\ & \leq & \varepsilon \cdot d(\mathsf{id}, \sigma) \\ & \leq & \varepsilon \cdot r \\ & = & \varepsilon \cdot d(f_1, f_2) \end{array}$$

□

Example 5.2.24. (1) Binary streams. Let us consider the functor $F(X, d) = 2 \times (X, \frac{1}{2}d)$ on CMS. Here, 2 is the two-point space $\{0, 1\}$ with the discrete metric. Since $F\emptyset = \emptyset$, the initial algebra of F is also $\emptyset$. Its terminal coalgebra is carried by the set 2^ω of binary streams with the metric given by

$$d(s_0 s_1 s_2 \cdots, t_0 t_1 t_2 \cdots) = 2^i, \quad \text{where } i \text{ is least with } s_i \neq t_i,$$

for distinct streams $s_0 s_1 s_2 \cdots$ and $t_0 t_1 t_2 \cdots$. Note that the set 2^ω is the terminal coalgebra for $FX = 2 \times X$ on Set and also that every space $F^n 1$ is the set 2^n of binary sequences of length n with the metric given similarly. Notice also that the above metric is a special case of the metric (5.13) as F is (isomorphic to) the lifting H'_Σ of the polynomial functor $H_\Sigma X = X + X$.

(2) We have discussed deterministic automata and formal languages in Example 2.5.5, and now we revisit the topic from the metric point of view. This involves the functor $FA = \{0, 1\} \times X^A$ on Set. We lift this to CMS as in our previous examples: $\{0, 1\}$ is the discrete space and X^A carries the half of the maximum metric. We obtain as a terminal coalgebra the set $\mathscr{P}(A^*)$ of formal languages over A. The structure

$$\langle o, t \rangle : \mathscr{P}(A^*) \to \{0, 1\} \times \mathscr{P}(A^*)^A$$

is given by $t(L)(a) = \{w : aw \in L\}$, and $o(L) = 1$ iff the empty word ε belongs to L. The metric works by assigning to two different languages L and M the number 2^{-n}, where n is least so that there is a word of length n in $(L \setminus M) \cup (M \setminus L)$.

(3) Σ-trees. Recall the contracting lifting H'_Σ of the polynomial endofunctor H_Σ from Example 5.2.6(3). That is, given elements of $H'_\Sigma X$ of the same summand $\sigma \in \Sigma$, say $x = \sigma(t_i)$ and $x' = \sigma(t'_i)$, then $d(x, x') = \frac{1}{2} \max d(t_i, t'_i)$. It follows that the metric space T_Σ of all Σ-trees with the Arnold-Nivat metric (5.13) is a fixed point of H'_Σ: the usual tree-tupling $\alpha : H'_\Sigma T_\Sigma \to T_\Sigma$ is an isomorphism. Indeed, consider two elements of $H' T_\Sigma = \amalg_{\sigma \in \Sigma} T_\Sigma^n : x = (t_i)_{i<n}$ in the summand σ and $x' = (t'_j)_{j<m}$ in the summand σ'. If $\sigma \neq \sigma'$ the distances of x, x' and $\alpha(x), \alpha(x')$ are both 1. If $\sigma = \sigma'$, then $d(x, x') = \frac{1}{2} 2^{-n}$ where $2^{-n} = \max d(t_i, t'_i)$ means that for some i_0 the trees t_{i_0}, t'_{i_0} differ at level n. Then $(x), \alpha(x')$ differ first at level $n + 1$, thus $d(\sigma(x), \sigma(x')) \leq 2^{-(n+1)}$.

In fact, T_Σ is always the terminal coalgebra for H'_Σ. And if Σ contains a constant symbol, T_Σ is the canonical fixed point of H'_Σ.

If Σ consists of two $|A|$-ary operations, we get ((2)) above. Here T_Σ is the complete $|A|$-ary tree without leaves, labeled by $\{0,1\}$. Since that complete tree can be identified with A^*, we again get $T_\Sigma = \mathscr{P}(\Sigma^*)$ with the metric above.

As another concrete example consider Σ of one binary operation and one constant. The functor $H_\Sigma X = X \times X + 1$ with the maximum metric is not contracting: its initial algebra is as in Set with the discrete metric, the same holds about the terminal coalgebra. But the scaled functor $H'_\Sigma X$ has T_Σ, the binary trees (with the metric (5.13)) as the unique fixed point.

Example 5.2.25 [1]**.** Recall the limit $\mathbb{F} = \lim_{n<\omega} \mathscr{P}_f^n 1$ from Section 4.5. We present a "natural" endofunctor on CMS whose terminal coalgebra is $\mathbb{F}$.

Denote by 1 the terminal endofunctor of CMS with value, say, $\{\emptyset\}$, and recall the Hausdorff functor $\mathcal{H}$ from Example 5.2.6(3). Then the coproduct $\mathcal{H}+1$ assigns to every finite metric space (X,d) a metric space on the set $\mathscr{P}_f X$: every nonempty subset is of course compact, and the summand 1 takes care of the empty set. Let

$$\widetilde{\mathcal{H}} \colon \mathsf{CMS} \to \mathsf{CMS}$$

be the functor scaling the metric on $\mathcal{H}+1$ by $\frac{1}{2}$. Then $\widetilde{\mathcal{H}}$ is a locally contracting endofunctor and, again, $\widetilde{\mathcal{H}}(X,d)$ has the underlying set $\mathscr{P}_f X$ for all finite metric spaces (X,d). Consequently, the terminal ω^{op}-chain of $\widetilde{\mathcal{H}}$ is a lifting of that of $\mathscr{P}_f$. Since ω^{op}-limits in CMS are preserved by the forgetful functor to Set, and since $\nu\widetilde{\mathcal{H}} = \lim \widetilde{\mathcal{H}}^n 1$, we conclude that $\nu\widetilde{\mathcal{H}}$ is carried by $\mathbb{F}$ by Theorem 5.2.19,. More precisely, considering $\nu\widetilde{\mathcal{H}}$ as an algebra for $\widetilde{\mathcal{H}}$ (by inverting the coalgebra structure), the underlying $\mathscr{P}_f$-algebra is $\mathbb{F}$.

Later, in **??**, we shall obtain a result that allows us to obtain a terminal coalgebra for the class of functors formed by coproducts, products and compositions of polynomial functors and the Hausdorff functor without any scaling, see **??**.

Example 5.2.26. Every limit $\lim X_n$ of an ω^{op}-chain in Set (with projections ℓ_n) carries a canonical complete metric as follows: given elements $x \neq y$, put

$$d(x,y) = 2^{-n} \quad \text{for the least } n \text{ with } \ell_n(x) \neq \ell_n(y).$$

For every Cauchy sequence (x_k) in the limit it is easy to see that there exists a subsequence (y_m) satisfying $d(y_m, y_{m+1}) \leq 2^{-(m+1)}$ for every m. This means that the connection morphism $f_n \colon X_{n+1} \to X_n$ fulfils $f_n \cdot \ell_{n+1}(y_{m+1}) = \ell_n(y_{n+1}) = \ell_n(y_n)$. Thus, the elements $\ell_n(y_n) \in X_n$ form a compatible sequence of elements of the given diagram. Therefore, there is a unique y in the limit with $\ell_n(y) = \ell_n(y_n)$ for all n, hence $d(y, y_n) \leq 2^{-n}$, which means that $\lim y_m = y$ whence $\lim x_k = y$.

The following result was first proved by Barr [50] under the additional assumption that F also preserves ω-colimits:

Theorem 5.2.27 (Adámek [15])**.** *Let $F : \mathsf{Set} \to \mathsf{Set}$ fulfil $F\emptyset \neq \emptyset$, and assume that F has the terminal coalgebra $\nu F = F^{\omega}1$. Then the above complete metric space νF is a Cauchy completion of the initial algebra.*

Remark 5.2.28. (1) Birkedal, Støvring, and Thamsborg [55] considered a setting which is quite similar to our work in this section. They work with *ultrametric spaces*, where one strengthens the triangle inequality of metric spaces to the condition that $d(x, z) \leq \max\{d(x, y), d(y, z)\}$. For example, the above metric on $\lim X_n$ is always an ultrametric.
(2) Alessi, Baldan and Belle [41] study endofunctors of KMS, the full subcategory of CMS on compact metric spaces. This category is obviously CMS-enriched, but fails to have ω^{op}-limits, which makes the situation more complicated:

Theorem 5.2.19 has an analog. Let us call an endofunctor F on KMS locally contraction if there exists a constant $\varepsilon < 1$ with $d(Ff, Fg) \leq \varepsilon \cdot d(f, g)$ for every parallel pair f, g of morphisms.

Theorem 5.2.29 [41]**.** *Let $F\colon \mathsf{KMS} \to \mathsf{KMS}$ be locally contracting, and assume that $F0 \neq 0$. Then F has a canonical fixed point.*

5.3 Solving domain equations

Theorem 5.1.23 allows us to solve $X \cong FX$ whenever F is a locally continuous endofunctor on a strict CPO-enriched category with ω-colimits. The solutions are not in general unique, but the canonical fixed point gives a workable notion of solution.

We are also interested in solving "equations with mixed variance" such as $D \cong [D, D] + N_\perp$. Indeed, solving this kind of equation was Scott's original motivation for much of the work on embedding-projection pairs, due to its connection with models of the untyped lambda-calculus. So work on this topic is, in effect, the source of embedding-projection pairs in domain theory and also the categorical formulation of the limit-colimit coincidence (see Scott [154] and Smyth and Plotkin [156]). The obstacle is that $[D, D]$ is not a functor of the form $\mathsf{CPO}_\perp \times \mathsf{CPO}_\perp \to \mathsf{CPO}_\perp$. Indeed, it is contravariant in its first argument, so its type is $\mathsf{CPO}_\perp^{op} \times \mathsf{CPO}_\perp \to \mathsf{CPO}_\perp$. But the theory which we have developed up to this point does not allow us to obtain a fixed point of such a functor, due precisely to the contravariance. As it happens, there are two ways to proceed. They are related, and we are going to present both and briefly touch on the relation afterwards. Recall from Notation 5.1.6 the category $\mathscr{A}^E$ of objects and embeddings.

Theorem 5.3.1. *Let $\mathscr{A}$ be a strict CPO-enriched category with ω-colimits of embeddings. Every locally continuous functor $F\colon \mathscr{A}^{op} \times \mathscr{A} \to \mathscr{A}$ defines an endofunctor F^E on $\mathscr{A}^E$ by*

$$F^E X = F(X, X) \quad \text{and} \quad F^E e = F(\widehat{e}, e),$$

which has an initial algebra $X \cong F^E(X)$. It satisfies $F(X, X) \cong X$ in $\mathscr{A}$.

Proof. It is easy check the functoriality of F^E. To verify that $F(\widehat{e}, e)$ is an embedding, one checks that its projection is $F(e, \widehat{e})$.

First, note that $\mathscr{A}^E$ has ω-colimits, by Corollary 5.1.14. Moreover, we show that F^E preserves ω-colimits of embeddings:

$$E_0 \xrightarrow{e_0} E_1 \xrightarrow{e_1} E_2 \xrightarrow{e_2} \cdots$$

Let $c_i \colon E_i \to C$ be a colimit cocone. By the Basic Lemma, $\bigsqcup_i c_i \cdot \widehat{c_i} = \mathsf{id}$. We consider

$$F(E_0, E_0) \xrightarrow{F(\widehat{e_0}, e_0)} F(E_1, E_1) \xrightarrow{F(\widehat{e_1}, e_1)} F(E_2, E_2) \xrightarrow{F(\widehat{e_2}, e_2)} \cdots,$$

and we show that $(F(\widehat{c_i}, c_i))$ is a colimit cocone. By the Basic Lemma 5.1.9 again, we need to show that

$$\bigsqcup_i F(\widehat{c_i}, c_i) \cdot F(c_i, \widehat{c_i}) = \mathsf{id}.$$

For this, we use the local continuity:

$$\begin{aligned}
\bigsqcup_i F(\widehat{c_i}, c_i) \cdot F(c_i, \widehat{c_i}) &= \bigsqcup_i F(\widehat{c_i} \cdot c_i, c_i \cdot \widehat{c_i}) \\
&= F(\bigsqcup_i \widehat{c_i} \cdot c_i, \bigsqcup_i c_i \cdot \widehat{c_i}) \\
&= F(\mathsf{id}, \mathsf{id}) \\
&= \mathsf{id}.
\end{aligned}$$

From Theorem 3.1.7, we conclude that $X = \mathsf{colim}(F^E)^n 0$ is an initial F^E-algebra. By Lambek's Lemma, X is a fixed point of F^E. Clearly, $X \cong F(X, X)$ in $\mathscr{A}$. This completes the proof. $\square$

We now turn to the second solution method.

Theorem 5.3.2 [74]. *Let $\mathscr{A}$ be a strict CPO-enriched category with ω-colimits. Let $F \colon \mathscr{A}^{op} \times \mathscr{A} \to \mathscr{A}$ be locally continuous. Let G be the endofunctor on $\mathscr{A}^{op} \times \mathscr{A}$ defined by*

$$\begin{aligned}
G(X, Y) &= (F(Y, X), F(X, Y)) \\
G(f, g) &= (F(g, f), F(f, g)) \qquad \text{for } f \colon X \to X' \text{ and } g \colon Y \to Y'
\end{aligned}$$

Then G has a fixed point of the form (X, X), and for such X, we have $F(X, X) \cong X$.

Proof. This proof is based on an argument in Glimming and Ghani [80].

We define an isomorphism of categories

$$I \colon \mathsf{Alg}\, G \to (\mathsf{Coalg}\, G)^{op}.$$

Consider a G-algebra $A = ((Y, X), (\alpha, \beta))$, so that $\alpha \colon F(Y, X) \to Y$ and $\beta \colon F(X, Y) \to X$ are morphisms of $\mathscr{A}$. Then IA is defined to be the G-coalgebra

$$(Y, X) \xrightarrow{(\beta, \alpha)} (F(Y, X), F(X, Y)).$$

If $A' = (\alpha', \beta')\colon (F(Y', X'), F(X', Y')) \to (Y', X')$ is also a G-algebra, and if $(f, g)\colon A \to A'$ is an algebra morphism, then I assigns to (f, g) the G-coalgebra morphism $(g, f) : IA' \to IA$.

The category $\mathscr{A}^{op} \times \mathscr{A}$ is strict CPO-enriched and has ω-colimits. G is locally continuous. By Theorem 5.1.23, we have a canonical fixed point

$$(\alpha, \beta)\colon G(X, Y) \to (X, Y).$$

In particular,

$$(\alpha, \beta)^{-1} = (\alpha^{-1}, \beta^{-1})\colon (X, Y) \to G(X, Y)$$

is a terminal coalgebra, and since I takes initial algebras to terminal coalgebras,

$$(\beta, \alpha)\colon (Y, X) \to G(Y, X)$$

is also a terminal coalgebra. Thus, there is an isomorphism of G-coalgebras $(i, j) : (X, Y) \to (Y, X)$. In particular, i and j are invertible. All of the morphisms shown below are isomorphisms:

$$(X,X) \underset{(\mathsf{id},j)}{\overset{(\mathsf{id},j^{-1})}{\rightleftarrows}} (X,Y) \underset{(\alpha,\beta)}{\overset{(\alpha^{-1},\beta^{-1})}{\rightleftarrows}} (F(Y,X),(F(X,Y)) \underset{(F(i,\mathsf{id}),F(\mathsf{id},j^{-1}))}{\overset{(F(i^{-1},\mathsf{id}),F(\mathsf{id},j))}{\rightleftarrows}} (F(X,X),F(X,X)).$$

The composites show that $(X, X) \cong G(X, X)$, and also $X \cong F(X, X)$, as desired. $\square$

Example 5.3.3. This example illustrates two ways to solve a system of domain equations in $\mathsf{CPO}_\perp$. This category has coproducts given by disjoint union with least elements identified. Consider the following system:

$$X \cong [Y, X] \qquad Y \cong Z + D \qquad Z \cong [Z, X], \tag{5.17}$$

where D is a fixed object of $\mathsf{CPO}_\perp$.

First, we solve (5.17) following the method suggested by Theorem 5.3.1. Let

$$A = \mathsf{CPO}_\perp \times \mathsf{CPO}_\perp \times \mathsf{CPO}_\perp.$$

This category is strict CPO-enriched. We also have $\mathscr{A}^E = (\mathsf{CPO}_\perp^E)^3$. We express our system in terms of an endofunctor on $\mathscr{A}^E$:

$$(X, Y, Z) \mapsto ([Y, X], Z + D, [Z, X])$$

For embeddings $f\colon X \to X'$, $g\colon Y \to Y'$, $h\colon Z \to Z'$, the first component of $F(f, g, h)$ takes $p\colon Y \to X$ to $f \cdot p \cdot \widehat{f}$. In more detail, we take the contravariant functor $[Y, X]$ from Remark 5.1.18(1) and then apply the construction in Theorem 5.3.1. The third component is defined similarly, again using a projection $\widehat{g}$. The second component is $h + \mathsf{id}_D$. Then one forms the colimit of the initial sequence of this functor to obtain the canonical fixed point. Call this fixed point (X^*, Y^*, Z^*). Then we get the desired solution $X^* \cong [Y^*, X^*]$, $Y^* \cong Z^* + D$, and $Z^* \cong [Z^*, X^*]$.

Second, let us solve (5.17) in a manner inspired by Theorem 5.3.2. The variables X, Y, and Z become doubled: each gets a $+$ version and a $-$ version. Our equations now become

$$\begin{aligned} X^+ &\cong [Y^-, X^+] & X^- &\cong [Y^+, X^-] \\ Y^+ &\cong Z^+ + D & Y^- &\cong Z^- + D \\ Z^+ &\cong [Z^-, X^+] & Z^- &\cong [Z^+, X^+] \end{aligned}$$

We think of this as a locally continuous endofunctor $G : \mathscr{B} \to \mathscr{B}$, where

$$\mathscr{B} = (\mathsf{CPO}_\perp)^3 \times (\mathsf{CPO}_\perp^{op})^3.$$

and $G(X^+, Y^+, Z^+, X^-, Y^-, Z^-)$ is given by the equations above. Then Theorem 5.3.2 tells us that the canonical fixed point will be of the form $(X^*, Y^*, Z^*, X^*, Y^*, Z^*)$: the solutions for the "positive" and "negative" variables will agree. Moreover, the triple (X^*, Y^*, Z^*) solves our original system (5.17).

Example 5.3.4. Scott's model of the untyped λ-calculus. The formulas t of the λ-calculus have the form

$$t ::= k \mid x \mid tt \mid \lambda x.t$$

where k ranges through a set K of constants, and x through a countable set of variables. The meaning of $t_1 t_2$ is "application": we evaluate t_2 (a function) in t_1. The meaning of $\lambda x.t$ is "λ-abstraction": this function takes a value a and responds with $t[a/x]$, the term t in which x is substituted by a. Thus if D is the set of all closed terms, we obtain an isomorphism

$$D \cong K + D \times D + [D, D].$$

No such set D exists because $|[D, D]| > |D|$ whenever D is not a singleton set.

Scott [154] decided to use a cartesian closed category with products and coproducts, and interpret the above equation in that category. He originally used continuous lattices, but Smyth and Plotkin [156] made it clear that $\mathsf{CPO}_\perp$ is sufficient (and simpler). In fact, consider the flat cpo

$$K_\perp = K + \{\perp\}$$

with all pairs in K incomparable, in place of the set K. The category $\mathsf{CPO}_\perp$ has the usual products $\times$, and coproducts $+$ are disjoint unions with bottom elements identified. Recall from Example 5.1.16 that $D \mapsto D \times D$ is locally continuous. We obtain a locally continuous functor

$$F \colon \mathsf{CPO}^{op} \times \mathsf{CPO} \to \mathsf{CPO} \quad \text{with} \quad F(X, Y) = K_\perp + Y \times Y + [X, Y].$$

If D is the initial algebra for F^E, see Theorem 5.3.1, then

$$D \cong K_\perp + D \times D + [D, D]$$

is a model of λ-calculus. (Observe that the "artificial" $\perp$ of K disappears in the formation of coproduct.)

We conclude this section by mentioning a result which is parallel to Theorem 5.3.1 but in the metric setting. It can be used for getting fixed points of functors of mixed variance.

Theorem 5.3.5 [42]. *Let $\mathscr{A}$ be a strict* CMS*-enriched category with colimits of ω-chains. Every locally contracting endofunctor $F\colon \mathscr{A} \times \mathscr{A}^{op} \to \mathscr{A}$ defines an endofunctor F^E of $\mathscr{A}^E$ by*

$$F^E X = F(X, X) \quad \textit{and} \quad F^E(e, \widehat{e}) = F(\widehat{e}, e),$$

and F^E has an initial algebra $X \simeq F^E(X)$. It satisfies $F(X, X) \simeq X$ in $\mathscr{A}$.

There is also an obvious parallel to Theorem 5.3.2, and with the same proof, mutatis mutandis.

5.4 Summary of this chapter

We have presented two closely related types of categories in which the terminal coalgebra and the initial algebra coincide for every "well-behaved" endofunctor: categories enriched over

- complete partial orders, where "well-behaved" are the locally continuous endofunctors,
- complete metric spaces, where "well-behaved" are the locally contracting endofunctors.

In both cases the main technical result was a coincidence of colimits of ω-chains of embeddings (or of contracting ω-chains of split subobjects, respectively) and limits of ω^{op}-chains of the corresponding projections (or splittings, respectively).

Let us point out some of the sources that inspired the material of this chapter. The famous construction of a model of the untyped-λ-calculus presented by Scott in [154] applied ω^{op}-limits and it is the starting point of our study. Scott used the category of continuous lattices and embedding-projection maps, and we shall see these below. Later, Smyth and Plotkin [156] introduced the concept of a locally continuous endofunctor and noticed that in the category of domains, the finitary constructions of the initial algebra and the terminal coalgebra coincide for these endofunctors, yielding a canonical fixed point. For more on this, see Abramsky and Jung [2]. Taylor proved the same result in the category of domains and adjoint pairs [158], and this was generalized further in the unpublished Ph.D. thesis of Velebil [166]. Adámek [13] proves a related result in which CPOs are generalized to finitely accessible categories and locally continuous functions to finitary functors.

6 Transfinite Iteration

This chapter deals with iterations of the initial-algebra chain and terminal-coalgebra chain which go into the transfinite. We generalize $F^n 0$ for $n \in \mathbb{N}$ to $F^i 0$ for all ordinals i and prove that whenever the connecting map to $F^{i+1}0$ is invertible, then $F^i 0$ is the initial algebra. A dual result holds for $F^i 1$ and terminal coalgebras.

In order to help the reader understand this material, we provide a very short background discussion on ordinal numbers.

Background on ordinal and cardinal numbers. Cardinal numbers are representatives of sets up to isomorphism: to every set X one assigns a cardinal number $|X|$, called the *cardinality* of X, so that a set Y is isomorphic (in Set) to X iff $|X| = |Y|$. The set-theoretic choice of such representatives uses ordinal numbers (or, ordinals, for short). Recall that a *well-ordered set* is a poset with a total order (i.e. for every pair x, y of elements in it, either $x \le y$ or $y \le x$) such that every nonempty subset contains a least element. For example, $\mathbb{N}$ is a well-ordered set but $\mathbb{Z}$ is not. For every well-ordered set, one obtains another well-ordered set by adding a new top element. Thus starting from the smallest (empty) well-ordered set we get the following well-ordered subsets of $\mathbb{N}$:

$$0 = \emptyset, \quad 1 = \{\emptyset\}, \quad 2 = \{\emptyset, \{\emptyset\}\}, \quad 3 = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \quad \ldots \tag{6.1}$$

Recall that, in formal set theory, elements of every set are sets themselves. The proper definition of an ordinal is as follows:

Definition 6.0.1. An *ordinal* is a set which is well-ordered by the following relation

$$x \le y \qquad \text{iff} \qquad x \in y \text{ or } x = y.$$

The first ordinals are the natural numbers in (6.1) above. Note that for every ordinal i we obtain an ordinal $i \cup \{i\}$ denoted by $i + 1$. An ordinal is a *successor ordinal* if it is of the form $i + 1$ for some i. An ordinal which is neither 0 nor a successor ordinal is called a *limit ordinal*. The smallest limit ordinal is denoted by ω; it is the set of natural numbers,

$$\omega = \{0, 1, 2, \ldots\}.$$

The *sum* $i + j$ of two ordinals is the ordinal $i \cup \{i + k : k \in j\}$. (Observe that $i + 1$ above is a special case.) Incidentally, the second limit ordinal $\omega + \omega$ plays an interesting role in work on the finitary set functors (see Example 6.4.2). The first uncountable ordinal is denoted by ω_1.

Moreover, the ordinals themselves form a proper class which is itself well-ordered by the inclusion relation: $i \le j$ iff $i \subseteq j$. Every element of an ordinal is an ordinal, and the union of every set of ordinals is an ordinal. This gives a well-ordered proper class

$$\mathsf{Ord}.$$

We often use proofs by *transfinite induction*: to prove that a class C of ordinals contains all ordinals, it is sufficient to verify that 0 lies in C, if $i \in C$, then $i+1 \in C$, and that a limit ordinal i lies in C provided each ordinal $j < i$ belongs to C.

Cardinals are isomorphism classes of sets. That means that every set is in bijective correspondence with a unique cardinal. The set-theoretic definition is as follows:

Definition 6.0.2. A *cardinal* is an ordinal i such that whenever j is an ordinal which is isomorphic to i in Set, then $i \leq j$.

The smallest cardinals are the natural numbers (the finite cardinals), and then ω itself. The infinite cardinals are listed in a sequence $(\aleph_i)$, where i ranges through ordinals. So $\aleph_0 = \omega$, then $\aleph_1$ is the first uncountable cardinal, etc. For every cardinal λ, the cardinal successor is denoted by λ^+. Thus, if $\lambda = \aleph_i$, then $\lambda^+ = \aleph_{i+1}$.

A function between ordinals $f\colon i \to j$ is called *cofinal* if for every ordinal $k < j$ there is some $i' < i$ such that $k < f(i')$. A cardinal κ is *regular* if it is infinite and there is no cofinal map from a smaller ordinal to κ. A cardinal is *singular* if it is infinite and not regular. The smallest singular cardinal is $\aleph_\omega$. For every ordinal i, the cardinal $\aleph_{i+1}$ is regular.

6.1 The initial-algebra chain

In this section, we pursue the *transfinite* iteration of the initial-algebra chain. The terminal-coalgebra chain is treated later, in Section 6.4. We begin with a famous result on fixed points of order-preserving maps on a dcpo; see Example 5.1.2.

Let P be a dcpo with the least element $\perp$, and $f\colon P \to P$ an order-preserving map. Then we can define an ordinal-indexed sequence of $f^i(\perp)$ as follows:

$$f^0(\perp) = \perp \qquad f^{j+1}(\perp) = f(f^j(\perp)) \qquad \text{and} \qquad f^j(\perp) = \bigsqcup_{i<j} f^i(\perp) \quad \text{for limit ordinals } j. \tag{6.2}$$

It is easy to verify that this is a chain in P; i.e., $f^i(\perp) \leq f^j(\perp)$ if $i \leq j$.

Theorem 6.1.1 (Knaster-Tarski). *Let P be a dcpo. Every order-preserving function $f\colon P \to P$ has a least fixed point μf. Moreover, for some ordinal $j < |P|^+$ we have*

$$\mu f = f^j(\perp).$$

Proof. Given a dcpo P of cardinality less than κ there must be some $j < \kappa$ such that $f^j(\perp) = f^{j+1}(\perp)$; if not, then $\{f^j(\perp) : j < \kappa\}$ is a subset of P of size κ, and this is a contradiction. Thus, $f^j(\perp)$ is a fixed point of f. Let $f(x) = x$. An easy transfinite induction shows that $x \geq f^i(\perp)$ for all i. This shows that $f^j(\perp)$ is the least fixed point of f. $\square$

We should attribute this theorem to Zermelo, since the mathematical content of the result appears in his 1904 paper [171] proving the Wellordering Theorem. Although it is common to refer to it as the Knaster-Tarski Theorem, their result dealt with a slightly different situation, see [110].

Remark 6.1.2. Note that μf is also the least *prefixed point* of f: whenever $x \in P$ fulfils $f(x) \leq x$, then $\mu f \leq x$. The proof remains the same.

A category-theoretic generalization of Theorem 6.1.1 was formulated by Adámek [10]. It was applied there to the functor $F(-) + A$; in other words, the free F-algebra on an object A was considered instead of the initial F-algebra (see Proposition 2.2.19).

Remark 6.1.3. (1) By a *transfinite chain* in a category $\mathscr{A}$ we understand a functor from the ordered class[1] Ord of all ordinals into $\mathscr{A}$. And for an ordinal α, an *α-chain* in $\mathscr{A}$ is a functor from α to $\mathscr{A}$. A category *has colimits of chains* if for every ordinal α it has a colimit of every α-chain. This includes the initial object 0 (the case $\alpha = 0$).

(2) Dually, an *α^{op}-chain* is a functor from α^{op} to $\mathscr{A}$, and $\mathscr{A}$ is said to have limits of op-chains, if for every ordinal α it has a limit of every α^{op}-chain. This included the terminal object 1.

(3) For the proof of Theorem 6.1.1 it is sufficient to assume that P is a poset with joins of all chains.

Definition 6.1.4. Let $\mathscr{A}$ be a category with colimits of chains. For every endofunctor F we define the *initial-algebra chain* $W : \mathsf{Ord} \to \mathscr{A}$. Its objects are denoted by W_i or $F^i 0$, its connecting morphisms by $w_{ij}\colon W_i \to W_j$, $i \leq j \in \mathsf{Ord}$. They are defined by transfinite recursion:

$$\begin{aligned} W_0 &= 0, && \\ W_{j+1} &= FW_j && \text{for all ordinals } i, \\ W_j &= \mathsf{colim}_{i<j} W_i && \text{for all limit ordinals } j, \end{aligned}$$

and

$$\begin{aligned} &w_{0,1}\colon 0 \to W_0 \text{ is unique },\\ &w_{j+1,k+1} = Fw_{j,k}\colon FW_j \to FW_k,\\ &w_{i,j}\ (i<j) \text{ is the colimit cocone for limit ordinals } j. \end{aligned}$$

The above rules define an essentially unique chain $W\colon \mathsf{Ord} \to \mathscr{A}$. In fact, we now verify that any two chains satisfying these rules are naturally isomorphic. We therefore speak of *the* initial-algebra chain of F.

Lemma 6.1.5. *If two transfinite chains satisfy the conditions of Definition 6.1.4, then they are naturally isomorphic.*

Proof. Given another such chain $W'\colon \mathsf{Ord} \to \mathscr{A}$, define the morphisms $\alpha_i\colon W_i \to W'_i$ ($i \in \mathsf{Ord}$) by transfinite recursion as follows

$$\alpha_0 = \mathsf{id}_0 \qquad \text{and} \qquad \alpha_{j+1} = F\alpha_j\colon FW_j \to FW'_j \quad (\text{for all } j \in \mathsf{Ord}).$$

[1]Recall that posets (and, more generally, ordered classes) may be considered as categories.

We now prove that this defines a natural transformation; it follows that α_j need not be defined for limit ordinals: it is uniquely determined by the commutativity of the naturality squares

$$\begin{tikzcd} W_i \arrow[r, "\alpha_i"] \arrow[d, "w_{ij}"'] & W'_i \arrow[d, "w'_{ij}"] \\ W_j \arrow[r, "\alpha_j"] & W'_j \end{tikzcd} \qquad \text{for all } i < j. \tag{6.3}$$

The proof that the above squares commute is by nested transfinite induction. We perform this induction on the ordinal j.

(1) The case $j = 0$ is trivial.

(2) If the above squares commutes for j, then they also commute for $j+1$. This is shown by transfinite induction in i. For $i = 0$ one uses that $W_0 = 0$ is the initial object. The case $i+1$ follows from F applied to the square (6.3). For the limit step, assume that the above squares commute, for all $k < i$, and use that the colimit injections w_{ki}, $k < i$, are collectively epimorphic.

(3) Let j be a limit ordinal. Then the squares (6.3) commute by the definition of α_j.

Analogously, one defines a natural transformation with components $\beta_i \colon W'_i \to W_i$ by $\beta_0 = \mathsf{id}_0$ and $\beta_{j+1} = F\beta_j$. It is easy to see that β is the inverse of α. □

Notice that Definition 6.1.4 generalizes what we saw in (6.2) above for sequences associated with a given order-preserving endofunction $f \colon P \to P$.

Construction 6.1.6. Analogously to Construction 3.1.5 every algebra $\alpha \colon FA \to A$ induces a *canonical cocone*

$$\alpha_i \colon F^i 0 \to A \qquad (i \in \mathsf{Ord})$$

of the initial-algebra chain: this is the unique cocone with

$$\alpha_{i+1} = (F^{i+1}0 = FF^i 0 \xrightarrow{F\alpha_i} FA \xrightarrow{\alpha} A)$$

for all ordinals i.

Remark 6.1.7. Homomorphisms of F-algebras $h \colon (A, \alpha) \to (B, \beta)$ preserve the canonical cocones: for every ordinal i we have

$$\beta_i = h \cdot \alpha_i \colon F^i 0 \to B.$$

This is easy to verify by transfinite induction.

Definition 6.1.8. We say that the initial-algebra chain of a functor F *converges in λ steps* if $w_{\lambda,\lambda+1}$ is an isomorphism. We say that the chain converges in *exactly λ steps* if λ is the least ordinal such that the chain converges in λ steps.

If $w_{\lambda,\lambda+1}$ is an isomorphism, then so is $w_{\lambda,j}$, for all $j > \lambda$. This is easy to prove by transfinite induction. Conversely:

Proposition 6.1.9. *If $i < j$ and $w_{i,j}$ is an isomorphism, then so is $w_{i,i+1}$.*

Proof. The following proof is from Barr [49]. Since $w_{i,j}$ is an isomorphism, so is $Fw_{i,j} = w_{i+1,j+1}$. And as $w_{i+1,j+1} = w_{j,j+1} \cdot w_{i+1,j}$, we see that $w_{i+1,j}$ is a split monomorphism. In addition, $w_{i+1,j}$ is a split epimorphism, since $w_{i,j} = w_{i+1,j} \cdot w_{i,i+1}$. Hence $w_{i+1,j}$ is an isomorphism. Thus so is $w_{i,i+1} = w_{i+1,j}^{-1} \cdot w_{i,j}$. □

Theorem 6.1.10 [10]**.** *Let $\mathscr{A}$ be a category with colimits of chains. If the initial-algebra chain of an endofunctor F converges in λ steps, then $F^\lambda 0$ is the initial algebra with the algebra structure*

$$w_{\lambda,\lambda+1}^{-1} \colon F(F^\lambda 0) \to F^\lambda 0.$$

The proof is analogous to that of Theorem 3.1.7:

Proof. Let (A, α) be any F-algebra. Denote by $h \colon F^\lambda 0 \to A$ the component α_λ of the canonical cocone $\alpha_i \colon F^i 0 \to A$, see Construction 6.1.6.

(1) In order to see that h is a homomorphism, note that the outside of the following diagram commutes:

$$\begin{tikzcd} FW_\lambda \arrow[r, "w_{\lambda,\lambda+1}^{-1}"] \arrow[d, "Fh=F\alpha_\lambda"'] \arrow[dr, "\alpha_{\lambda+1}"] & W_\lambda \arrow[d, "h=\alpha_\lambda"] \\ FA \arrow[r, "\alpha"] & A \end{tikzcd}$$

Indeed, the lower left-hand triangle commutes by the definition of $\alpha_{\lambda+1}$, and the upper right-hand one by the fact that (α_i) is a cocone.

(2) To prove that h is unique, let $k \colon F^\lambda 0 \to A$ be any homomorphism. Then $h \cdot w_{i,\lambda} = k \cdot w_{i,\lambda}$ holds for all $i \le \lambda$; the proof of this fact by transfinite induction is as in Theorem 3.1.7. The case $i = \lambda$ yields $k = h$. □

Theorem 6.1.10 is the raison d'être of this section. The rest of the section studies the following topics:

(1) sufficient conditions for the convergence of the initial-algebra chain;

(2) examples of specific set functors and their convergence ordinals;

(3) an examination of when convergence of the initial-algebra chain is necessary for the existence of the initial algebra.

Corollary 6.1.11. *Let $\mathscr{A}$ be a category with colimits of chains. Let λ be an infinite cardinal, and let $F : \mathscr{A} \to \mathscr{A}$ preserve colimits of λ-chains. Then the initial-algebra chain of F converges in λ steps. Therefore,*

$$\mu F = F^\lambda 0.$$

The proof is completely analogous to that of Theorem 3.1.7: since $W_\lambda = \mathsf{colim}_{i<\lambda} W_i$ implies $FW_\lambda = \mathsf{colim}_{i<\lambda} FW_i$ with the colimit cocone $Fw_{i\lambda} = w_{i+1,\lambda+1}$, we have a unique morphism $\iota \colon FW_\lambda \to W_\lambda$ with $\iota \cdot Fw_{i,\lambda} = w_{i+1,\lambda}$ for all i, and this is the inverse of $w_{\lambda,\lambda+i}$.

Remark 6.1.12. (1) In Example 6.1.13 we use the concept of *height* of a tree (see Remark 2.2.10). For a finite tree t the height $h(t)$ is the length of the longest path from the root to a leaf. If t has maximum proper subtrees $t_0, \ldots, t_{n-1}$, then $h(t) = \max\{h(t_i) + 1 : i = 0, \ldots n-1\}$.

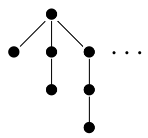
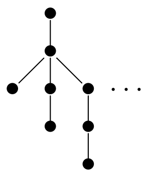
For an infinite tree t the height is an ordinal number or ∞ (larger than all ordinals) defined as follows:

$$h(t) = \infty \qquad \text{iff } t \text{ has an infinite path.}$$

Trees without an infinite path are called *well-founded* (cf. Chapter 7, where well-founded coalgebras are studied more generally). The height of a well-founded tree t is defined by

$$h(t) = \sup\{h(t') + 1 : t' \text{ a maximus proper subtree of } t\}.$$

For example, the following trees

t:  and t': 

have height $h(t) = \omega$ and $h(t') = \omega + 1$, respectively.

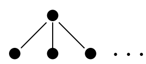
(2) It follows that, for a given ordinal i, a tree has height at most i iff all its proper subtrees have heights less than i.

Example 6.1.13. (1) Let us consider the set functor $FX = 1 + X^{\mathbb{N}}$ whose algebras have a constant and an $\mathbb{N}$-ary operation. We will show that the initial-algebra chain is given by

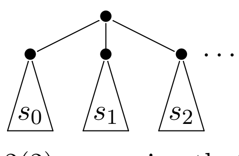
$$W_i = \text{all countably branching trees of height} < i.$$

The connecting maps are given by the inclusions $W_i \hookrightarrow W_j$ for all $i \leq j$. Recall that trees are always considered up to isomorphism.

In the first step, $W_0 = \emptyset$ since no tree has height less than 0, and $W_1 = 1$ is the singleton $\{t_0\}$, where t_0 is the root-only tree (i.e. the unique tree of height 0). Next $W_2 = 1 + W_1^{\mathbb{N}}$ has two elements: the tree t_0 (in the left-hand summand) and the sequence $(t_0, t_0, t_0, \ldots)$ which is represented by the unique countably branching tree of height 1:

t_1: 

In general, the equation $W_{i+1} = 1 + W_i^{\mathbb{N}}$ states that W_{i+1} consists of t_0 and all the trees below:



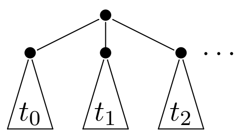
where $s_n \in W_i$ for every $n \in \mathbb{N}$.

By Remark 6.1.12(2), assuming that W_i consists of all countably branching trees of height less than i, it follows that W_{i+1} consists of all such trees of height less than $i + 1$.

The connecting map $w_{01} : \emptyset \to W_1$ is, of course, the (empty) inclusion map. The functor $FX = 1 + X^{\mathbb{N}}$ clearly takes every inclusion map to an inclusion map. Thus, $w_{12} = Fw_{01}$, $w_{23} = Fw_{12}$ etc. are all inclusion maps. Consequently the colimit $W_\omega = \mathsf{colim}_{i<\omega} W_i$ is simply the union:

$$W_\omega = \bigcup_{i<\omega} W_i = \text{all countably branching trees of finite height.}$$

The initial-algebra chain continues: $W_{\omega+1}$ is a proper super-set of W_ω: for example, the following tree



where t_n is the complete countably branching tree of height n, has height ω; thus, it lies in $W_{\omega+1} \setminus W_\omega$.

It is easy to see that, for every well-founded countably branching tree t, the height $h(t)$ is a countable ordinal, i.e. $h(t) < \omega_1$ for the first uncoutable ordinal ω_1. Consequently, the initial-algebra chain converges at this ordinal: $\mu F = W_{\omega_1} = W_{\omega_1+1}$. This set consists of all well-founded countable branching trees.

(2) More generally, we return to polynomial functors H_Σ on sets, see Example 2.1.4. Recall from Definition 2.2.12 and we saw in Proposition 2.2.14 that the initial algebra for H_Σ is formed by all finite Σ-trees. We generalize this to signatures which allow for *infinitary* operations. Such a signature is a set Σ of operation symbols σ, each with an arity $\mathsf{ar}(\sigma)$, which is a cardinal number. The corresponding polynomial functor is defined on a set X by

$$H_\Sigma X = \coprod_{\sigma\in\Sigma} X^{\mathsf{ar}(\sigma)}$$

and similarly on maps.

The concept of a Σ-tree naturally generalizes to infinitary signatures, too: it is a tree labelled in Σ such that every node with k children (where k is a cardinal) is labelled by a k-ary operation symbol.

Analogously to (1), one easily shows that the polynomial functor H_Σ has the initial-algebra chain

$$W_i = \text{all } \Sigma\text{-trees of height} < i$$

and the initial-algebra

$$\mu H_\Sigma = \text{all well-founded } \Sigma\text{-trees.}$$

The algebra structure is given by tree-tupling.

In contrast to Example 6.1.13(1), $1 + X^{\mathbb{N}}$ as an endofunctor on CPO is locally continuous. Thus its initial algebra is, by Theorem 5.1.23, W_ω. It consists of *all* countably branching trees.

Example 6.1.14. (1) The power-set functor has no fixed point, thus, no initial algebra. Nonetheless, it has an interesting initial-algebra chain. It is given by the sets

$$W_0 = \emptyset, \qquad W_{i+1} = \mathscr{P} W_i, \qquad \text{and} \qquad W_i = \bigcup_{j<i} W_j \quad \text{for limit ordinals } i$$

and the inclusion functions as connecting maps $W_i \hookrightarrow W_j$ for $i \leq j$. In set theory these sets are usually written V_i. They constitute a proper hierarchy: every ordinal i belongs to $W_{i+1} \setminus W_i$. Due to the Foundation Axiom, every set x lies in W_{i+1} for some i. The least such i is called the *rank* of the set x.

Of special interest is the set

$$W_\omega = HF$$

of all *hereditarily finite sets.* We have seen in Example 2.2.7 that HF is the initial algebra of $\mathscr{P}_f$. Analogously, denote by

$$\mathscr{P}_c$$

the subfunctor of $\mathscr{P}$ of all countable subsets. The initial-algebra chain of $\mathscr{P}_c$ yields the initial algebra in ω_1 steps. It is the algebra $W_{\omega_1} = HC$ of *hereditarily countable* sets.

(2) Generalizing $\mathscr{P}_f$ and $\mathscr{P}_c$ above, let us define, for every infinite regular cardinal λ, the subfunctor

$$\mathscr{P}_\lambda$$

of $\mathscr{P}$ by taking all subsets of cardinality less than λ. It is easy to see that $\mathscr{P}_\lambda$ preserves colimits of λ-chains. Hence,

$$\mu \mathscr{P}_\lambda = W_\lambda$$

is the λ^{th} set in the cumulative hierarchy.

The initial-algebra chain converges exactly at λ. To see this, assume it converges exactly at $i < \lambda$. Then for all $j < i$ we have $W_j \subsetneqq W_i$ and we can choose $x_j \in W_i - W_j$. The set $X = \{x_j, j < i\}$ has cardinality less than λ, thus, $X \in \mathscr{P}_\lambda W_i = W_{i+1}$. It is easy to see that $X \notin W_i$, thus, $W_i \neq W_{i+1}$, a contradiction.

Remark 6.1.15. We can also define $\mathscr{P}_\lambda$ for singular cardinals λ. These functors do not preserve colimits of λ-chains. But for the cardinal successor λ^+ of λ they preserve colimits of λ^+-chains. The exact convergence of the initial-algebra chain is then λ^+. Indeed, it cannot be less than λ, see the above argument, and it cannot by an ordinal i with $\lambda \leq i < \lambda^+$ due to the following result:

Theorem 6.1.16 [39]**.** *For every set endofunctor F with an initial algebra, the initial-algebra chain of F converges in exactly λ steps either for some regular infinite cardinal λ, or for $\lambda \leq 3$.*

Example 6.1.17. Convergence in exactly λ steps is illustrated by $\mathscr{P}_\lambda$ for regular cardinals λ. Convergence in $i \leq 3$ steps is illustrated by the following functors $F_i \colon F_0 = \mathsf{Id}, F_1 = C_1$, the constant functor of value 1, and $F_2 = C_{01}$ differing from C_1 only by $C_{01}\emptyset = 2 = \{0, 1\}$. Define $F_3\emptyset = 3$ and for $X \neq \emptyset$ put $F_3 X = \{A \subseteq X : |A| = 3 \text{ or } 0\}$. For morphisms $f \colon X \to Y$, Ff is constantly $\emptyset$ if $X = \emptyset$; if $X \neq \emptyset$ put $Ff(A) = f[A]$ whenever $|f[A]| = 3$, else $Ff(A) = \emptyset$.

Are all initial algebras obtainable by iteration? At this point, we know that iteration is a powerful way of obtaining initial algebras. Thus we can ask whether it is *necessary*. In other words, are all initial algebras obtainable by iteration? It turns out that the answer is "No, but in settings that are sufficiently Set-like, Yes." Spelling this out is our next goal.

Definition 6.1.18 [162]**.** A class $\mathcal{M}$ of monomorphisms in a category $\mathscr{A}$ is called *constructive* provided that it is closed under composition, contains all isomorphisms, and for every chain of monomorphisms in $\mathcal{M}$,

(1) a colimit exists and is formed by monomorphisms in $\mathcal{M}$, and

(2) the factorization morphism of every cocone of monomorphisms in $\mathcal{M}$ is again a monomorphism in $\mathcal{M}$.

Remark 6.1.19. (1) In particular, a category with a constructive class of monomorphism has an initial object 0 and the unique morphism $0 \to A$ is a monomorphism for every object A. Indeed, 0 is the colimit of the empty chain of monomorphisms in $\mathcal{M}$.

(2) Note that the notion of a constructive class has nothing to do with constructive mathematics.

Examples 6.1.20. (1) In the categories of sets, graphs, posets, and semigroups the class of all monomorphisms is constructive.

(2) In the category Pfn of sets and partial functions, the classes of all monomorphisms and all the split monomorphisms both form a constructive class. (The latter are precisely the monomorphisms of Set.)

(3) In contrast, the collection of all monomorphisms is not constructive in the category of rings. Its initial object is the ring $\mathbb{Z}$ of integers, and there are non-injective homomorphisms whose domain is $\mathbb{Z}$.

(4) In CPO-enriched categories with colimits of ω-chains of embeddings, the class of all embeddings is constructive: see Basic Lemma 5.1.9 and Corollary 5.1.14. However, the class of all monomorphisms is usually non-constructive. For example, in the category CPO itself consider the cpo $\mathbb{N}^\top$ of natural numbers with a top element $\top$. The subposets $C_n = \{0, \dots, n\} \cup \{\top\}$, $n \in \mathbb{N}$, form an ω-chain in CPO. Its colimit is $\mathbb{N}^\top \cup \{\infty\}$ where $n < \infty < \top$ for all $n \in \mathbb{N}$. The cocone formed by the inclusion maps $C_n \hookrightarrow \mathbb{N}^\top$ is formed by monomorphisms. However the factorizing morphism from $\mathsf{colim}\, C_n$ to $\mathbb{N}^\top$ is not monomorphic, as it merges ∞ and $\top$.

Remark 6.1.21. We recall the general definition of a *subobject* from Remark 2.1.12. Generalizing a bit, for an object A, an $\mathcal{M}$-*subobject* is a subobject represented by a morphism $m_B \colon B \to A$ in $\mathcal{M}$. When the base category is well-powered, we write

$$(\mathsf{Sub}_{\mathcal{M}}(A), \leq)$$

for the poset of $\mathcal{M}$-subobjects of A.

Theorem 6.1.22 (Initial-Algebra Theorem [162])**.** *Let $\mathscr{A}$ be a well-powered category with colimits of chains and a constructive class $\mathcal{M}$ of monomorphisms. Let $F \colon \mathscr{A} \to \mathscr{A}$ preserve monomorphisms in $\mathcal{M}$. The following are equivalent:*

(1) *the initial-algebra chain converges.*

(2) F *has an initial algebra.*

(3) F *has a fixed point, i.e., an object* $A \cong FA$*,*

(4) F *has an* $\mathcal{M}$-prefixed point*, i.e. an object* A *with an* $\mathcal{M}$*-monomorphism* $m\colon FA \to A$*,*

Proof. We know that (1) implies (2), and Lambek's Lemma tells us that (2) implies (3)). Clearly, (3) implies (4).

Here is a proof that (4) implies (1). First we define the function

$$f\colon \mathsf{Sub}_{\mathcal{M}}(A) \to \mathsf{Sub}_{\mathcal{M}}(A)$$

which maps a given subobject $u\colon B \rightarrowtail A$ to

$$f(u) = m \cdot Fu\colon FB \rightarrowtail FA.$$

It is clearly monotone, hence, by Theorem 6.1.1, it has the least fixed point

$$\mu f = f^i(\bot) \qquad \text{for some ordinal } i.$$

The canonical cocone $\alpha_j\colon F^j 0 \to A$ (see Construction 6.1.6) satisfies

$$\alpha_j = f^j(\bot) \qquad \text{for all } j \in Ord,$$

which is easily verified by transfinite induction. Thus, from $f(f^i\bot) = f^i\bot$ we conclude that α_i and α_{i+1} represent the same subobject of A. Since $\alpha_i = \alpha_{i+1} \cdot w_{i,i+1}$, it follows that $w_{i,i+1}$ is invertible. So the initial-algebra chain converges. □

We present several examples which show that none of the hypotheses in Theorem 6.1.22 can be left out.

Example 6.1.23. Consider the category $\mathsf{Ord}^\top$, the ordinals with a new element $\top$ added "on top". Let $F : \mathsf{Ord}^\top \to \mathsf{Ord}^\top$ be given by $F(\lambda) = \lambda + 1$, and $F(\top) = \top$. All of the conditions in Theorem 6.1.22 are satisfied except for the well-poweredness of the category. Clearly, $\top$ is the only fixed point of F, and this is not $F^\lambda 0$ for any λ.

Example 6.1.24. The existence of an initial algebra does not imply convergence of the initial-algebra chain. This example is a small variation on one from [38]. We use the category Gra of graphs and graph homomorphisms (i.e., functions preserving edges). Gra is cocomplete, and the class of all monomorphisms is constructive. The terminal object in the category is a loop on one point; we write this as 1. The initial object, 0, is the empty graph. The *chromatic number* of a (loop-free) graph G, denoted by $\chi(G)$, is the smallest (finite or infinite) cardinal k such that G has a homomorphism into C_k, the (loop-free) clique of size k. For example, the clique on the cardinal $k = \{i : i < k\}$ has $\chi(C_k) = k$. Note that if $f\colon G \to H$ is a homomorphism, then $\chi(H) \leq \chi(G)$. Recall from the beginning of this chapter that for every cardinal k the cardinal successor is denoted by k^+.

Consider $F\colon \mathsf{Gra} \to \mathsf{Gra}$ defined by

$$F(G) = \begin{cases} 1 & \text{if } G \text{ has loops} \\ C_{k^+} & \text{if } G \text{ has no loops, where } k = \chi(G) + \aleph_1. \end{cases}$$

If $f\colon G \to H$ is a homomorphism and neither G nor H has loops, then Ff is the inclusion. If G has loops, then so does H, and $Ff = \mathsf{id}$. If G has no loops but H does, the Ff is the constant. (So F does not preserve monics.) The point of the "+" in the definition of F is that every F-algebra must have loops. Indeed, an F-algebra structure on a graph amounts to a choice of a loop. The graph 1 is a fixed point, and it is easy to see that $\mathsf{id}_1\colon 1 \to 1$ is an initial algebra. But the initial-algebra chain does not converge: $F^i 0 = C_{\aleph_i}$ for every $i > 0$ in Ord.

Example 6.1.25 [39]**.** Fixed points do not always imply initial algebras. Let $F\colon \mathsf{Set} \times \mathsf{Set} \to \mathsf{Set} \times \mathsf{Set}$ be given by

$$F(X,Y) = \begin{cases} (1,1) & \text{if } X \neq \emptyset \\ (\emptyset, \mathscr{P}Y) & \text{if } X = \emptyset. \end{cases}$$

Given a morphism $(f,g)\colon (X,Y) \to (X',Y')$ with $X = \emptyset$, then $F(f,g) = (\mathsf{id}_\emptyset, \mathscr{P}g)$, and F works in the evident way on the other morphisms, using the fact that $(1,1)$ is the terminal object. Although $(1,1)$ is a fixed point, F has no initial algebra: the F-algebras $(\emptyset, \mathscr{P}Y) \to (\emptyset, Y)$ cannot be initial by Lambek's Lemma, and algebras of the form $(1,1) \to (X,Y)$ admit no homomorphisms to those of the form $(\emptyset, \mathscr{P}Y) \to (\emptyset, Y)$. Again, what is going wrong here is that this functor does not preserve monomorphisms.

Example 6.1.26. Pre-fixed points do not in general imply fixed points. Again, we use $\mathsf{Set} \times \mathsf{Set}$. This time, the endofunctor is

$$F(X,Y) \begin{cases} (\emptyset,1) & \text{if } X \neq \emptyset \\ (\emptyset, \mathscr{P}Y) & \text{if } X = \emptyset \end{cases}$$

It is defined on morphisms as expected, using $\mathscr{P}$ in the case where $X = \emptyset$. This functor has no fixed points for two reasons: first, $(\emptyset, Y)$ and $(\emptyset, \mathscr{P}Y)$ are never isomorphic, by Cantor's Theorem (cf. Example 2.2.7(1)). Second, if $X \neq \emptyset$, then there exists no morphism from (X,Y) to $F(X,Y) = (\emptyset, 1)$. But there is a monic $(\emptyset,1) = F(1,1) \to (1,1)$.

For every endofunctor on Set, the four conditions of Theorem 6.1.22 are equivalent. This is clear if F preserves monomorphisms. For general F, see [24]. However, we present here a substantially shorter proof:

Lemma 6.1.27. *Let F be a set functor, and let $\alpha\colon FA \to A$ be a prefixed point with the canonical cocone $\alpha_j\colon F^j 0 \to A$. Then α_ω is a monomorphism.*

Proof. If $F\emptyset = \emptyset$, the statement is trivial. Assuming that $F\emptyset \neq \emptyset$, FA and therefore A are nonempty. Hence, we can choose $\alpha'\colon A \to FA$ with $\alpha' \cdot \alpha = \mathsf{id}_{FA}$. We also define morphisms $u_n\colon A \to F^{n+1}0$ by induction: choose a morphism $u_0\colon A \to F\emptyset$ and let $u_{n+1} = Fu_n \cdot \alpha'$. Then we obtain

$$w_{n,n+1} = u_n \cdot \alpha_n\colon F^n 0 \to F^{n+1}0 \qquad \text{for all } n < \omega$$

by induction: we clearly have $w_{0,1} = u_0 \cdot \alpha_0\colon \emptyset \to F\emptyset$ and for the induction step we compute:

$$w_{n+1,n+2} = Fw_{n,n+1} = Fu_n \cdot F\alpha_n = (Fu_n \cdot \alpha') \cdot (\alpha \cdot F\alpha_n) = u_{n+1} \cdot \alpha_{n+1}.$$

Given elements $x, y \in F^\omega 0$ merged by α_ω, we prove $x = y$. Since $F^\omega\emptyset = \mathsf{colim}_{n<\omega} F^n\emptyset$, we can choose $n < \omega$ and elements x', y' of $F^n 0$ mapped by $w_{n,\omega}$ to the given pair. Then $\alpha_n = \alpha_\omega \cdot w_{n,\omega}$ merges x' and y', thus, these elements are also merged by

$$(w_{n+1,\omega} \cdot u_n) \cdot \alpha_n = w_{n+1,\omega} \cdot w_{n,n+1} = w_{n,\omega},$$

which proves $x = y$. □

Corollary 6.1.28. *For every set functor F, the conditions of the Initial-Algebra Theorem 6.1.22 are equivalent.*

Proof. It it sufficient to verify (3)⇒(4). If $F(\emptyset) = \emptyset$, then the initial-algebra chain converges. So we may assume that $F(\emptyset) \neq \emptyset$. Now we use Lemma 6.1.27. Since α_ω is a monomorphism with nonempty domain, it splits. Thus, $F\alpha_\omega$ is also a monomorphism. This proves that $\alpha_{\omega+1} = \alpha \cdot F\alpha_\omega$ is a monomorphism. In this manner we see that all α_j with $j \geq \omega$ are monomorphisms, and then we argue as in Theorem 6.1.22. □

Remark 6.1.29. (1) The same general argument shows that in the category Rel, prefixed points implies the convergence of the initial-algebra chain.

(2) For many sorted sets, statements (1)–(4) in Theorem 6.1.22 are not equivalent. See Example 6.1.25.

The proof of the following theorem is rather technical. We therefore state it without proof.

Theorem 6.1.30 [39]**.** *For the category Set^S of many sorted sets, whenever an endofunctor has an initial algebra, then the initial-algebra chain converges.*

This concludes our discussion of the question on page 139 concerning the relation between convergence of the initial-algebra chain and the existence of initial algebras. Now we return to a loose end from Chapter 2.

Corollary 6.1.31. *A set functor F is a varietor, i.e., all free F-algebras exist, iff F has arbitrarily large prefixed points.*

Indeed, by Proposition 2.2.19, F is a varietor iff for every set A an initial algebra for $F(-) + A$ exists. Equivalently, a prefixed point $FX + A \rightarrowtail X$ exists. We see that X is a prefixed point of F of cardinality at least $|A|$. Consequently, given a prefixed point $FX \rightarrowtail X$ for F of cardinality at least $|A|$, where A is infinite, then we have

$$FX + A \rightarrowtail X + X \simeq X,$$

That is, X is a prefixed point of $F(-) + A$; i.e. a free algebra on A exists.

Example 6.1.32 [38]**.** An unfortunate fact: the collection of all set functors possessing initial algebras is not well-behaved. There are set functors F_1, F_2 having initial algebras such that neither $F_1 + F_2$ nor $F_1 \times F_2$ has one. We demonstrate this by using the following modification $\mathscr{P}^G$ of the power-set functor, where $G \subseteq \mathsf{Card}$ is a nonempty class of cardinals such that

$$\ell < k \text{ implies } 2^\ell < k \text{ for all } k \in G \text{ and } \ell \in \mathsf{Card} \setminus G. \tag{6.4}$$

(There are many such classes, e.g. the class of all cardinals k_i, $i \in \mathsf{Ord}$, where $k_0 = 0$, $k_{i+1} = 2^{2^{k_i}}$ and $k_i = \sup_{j<i} 2^{k_j}$ for limit ordinals i.) The functor $\mathscr{P}^G$ maps a set X to the set

$$\mathscr{P}^G X = \{M \subseteq X : |M| \notin G\}.$$

On functions $f \colon X \to Y$ put

$$\mathscr{P}^G f(M) = \begin{cases} f[M] & \text{if } f|_M \text{ is injective} \\ \emptyset & \text{else.} \end{cases}$$

It is easy to verify that $\mathscr{P}^G$ is a well-defined functor. Moreover, every set X whose cardinality $|X|$ is in G is a prefixed point. Indeed, the cardinality of $\mathscr{P}^G X$ is

$$|\mathscr{P}^G X| = \sum_{\ell < |X|, \ell \notin G} 2^\ell.$$

By (6.4) we have $2^\ell < |X|$, and since the number of all cardinals $\ell < |X|$ is at most $|X|$, we conclude that $|\mathscr{P}^G X| \le |X|$.

By the Initial-Algebra Theorem 6.1.22, $\mathscr{P}^G$ has an initial algebra. Now take a pair G_1, G_2 of disjoint classes of cardinals satisfying (6.4) and whose union is the class of all cardinal numbers. Then the functors $F_i = \mathscr{P}^{G_i}$ have initial algebras, but $F_1 + F_2$ and $F_1 \times F_2$ have no fixed points.

6.2 Subfunctors and quotient functors

Given a functor $F \colon \mathscr{A} \to \mathscr{B}$, and a class $\mathcal{M}$ of monomorphisms, an $\mathcal{M}$-*subfunctor* of F is represented by a functor $F' \colon \mathscr{A} \to \mathscr{B}$ with a natural transformation $m \colon F' \rightarrowtail F$ whose components m_X belong to $\mathcal{M}$.

If an initial algebra of a functor F exists, do all $\mathcal{M}$-subfunctors have an initial algebra? Not in general.

Example 6.2.1. Let Gra be the category of graphs and graph homomorphisms. Let F be the endofunctor mapping every graph $(X, \emptyset)$ without edges to the graph $(\mathscr{P}X, \{(\emptyset, \emptyset)\})$ on the power set that has just a loop at $\emptyset$. All graphs with edges are mapped to the terminal graph 1. Homomorphisms $f \colon (X, \emptyset) \to (Y, \emptyset)$ are mapped to $\mathscr{P}f$. Then F has an initial algebra carried by the terminal graph:

$$\mu F = 1.$$

But the subfunctor of F given by $(X, \emptyset) \mapsto (\mathscr{P}X, \emptyset)$ and else constantly 1 does not have an initial algebra, since 1 is its only fixed point, and there exist no algebra homomorphism from 1 to the algebras $\alpha\colon (\mathscr{P}X, \emptyset) \to (X, \emptyset)$.

Corollary 6.2.2. *Let $\mathscr{A}$ be a cocomplete and well-powered category with a constructive class $\mathcal{M}$ of monomorphisms. If an endofunctor F on $\mathscr{A}$ that preserves $\mathcal{M}$-monomorphisms has an initial algebra, then so does every $\mathcal{M}$-subfunctor of F preserving $\mathcal{M}$-monomorphisms.*

Proof. Indeed, F' has the prefixed point

$$F'(\mu F) \overset{m_{\mu F}}{\rightarrowtail} F(\mu F) \xrightarrow{\cong} \mu F,$$

where the second morphism is the structure of the initial F-algebra. Thus, F' has an initial algebra by Theorem 6.1.22. □

Observe that the assumption that F' preserves $\mathcal{M}$ is not needed whenever $\mathcal{M}$ is *left cancellative*, i.e. $u \cdot v \in \mathcal{M}$ implies $v \in \mathcal{M}$. Indeed, let $m\colon F' \rightarrowtail F$ be a natural transformation with components in $\mathcal{M}$. Then F' preserves $\mathcal{M}$-monomorphisms: if $u\colon X \rightarrowtail Y$ is an $\mathcal{M}$-monomorphism then $F'u$ is one, too, since $m_Y \cdot F'u = Fu \cdot m_X$ holds by the naturality of m.

Surprisingly, the initial algebra of any $\mathcal{M}$-subfunctor of F has a universal property with respect to all algebras of F:

Proposition 6.2.3. *Let $\mathscr{A}$ have a constructive class $\mathcal{M}$ of monomorphisms preserved by the functor $F\colon \mathscr{A} \to \mathscr{A}$. If an $\mathcal{M}$-subfunctor $m\colon H \rightarrowtail F$ preserves $\mathcal{M}$ and has an initial algebra $(\mu H, \iota)$, then for every F-algebra (A, α) there exists a unique morphism $h\colon \mu H \to A$ such that the square below commutes:*

$$\begin{tikzcd}
H(\mu H) \arrow[r, "\iota"] \arrow[d, tail, "m_{\mu H}"'] & \mu H \arrow[dd, "h"] \\
F(\mu H) \arrow[d, "Fh"'] & \\
FA \arrow[r, "\alpha"'] & A
\end{tikzcd} \tag{6.5}$$

Proof. We know from Theorem 6.1.22 that the initial algebra of H has the form $H^j 0$ for some ordinal j with $w_{j,j+1}$ invertible. Define a cocone $h_i\colon H^i 0 \to A$ of the initial-algebra chain for H by

$$h_{i+1} = (\ H(H^i 0) \overset{m_{H^i 0}}{\rightarrowtail} F(H^i 0) \xrightarrow{Fh_i} FA \xrightarrow{\alpha} A\ ) \tag{6.6}$$

for all ordinals. Then $h_j\colon \mu H \to A$ has the required property: recall from Theorem 6.1.10 that $\iota = w_{j,j+1}^{-1}$ and use the definition of h_{j+1} to show that (6.5) commutes:

$$(\alpha \cdot Fh_j \cdot m_{\mu H}) \cdot \iota^{-1} = h_{j+1} \cdot w_{j,j+1} = h_j.$$

And h is unique, since from (6.5), we derive $h \cdot w_{i,j} = h_i$ for all $i \leq j$ by transfinite induction. At the isolated step we use the commutative diagram

$$\begin{tikzcd}
H(\mu H) \arrow[rr, "\iota=w_{j,j+1}^{-1}"] \arrow[dd, "m_{\mu H}"'] & & \mu H \arrow[dddd, "h"] \\
& H^{i+1}0 \arrow[ul, "Hw_{i,j}"'] \arrow[ur, "w_{i+1,j}"] \arrow[dd, "m_{H^i0}"] & \\
F(\mu H) \arrow[dd, "Fh"'] & & \\
& FH^i0 \arrow[ul, "Fw_{i,j}"'] \arrow[dl, "Fh_i"'] & \\
FA \arrow[rr, "\alpha"'] & & A
\end{tikzcd}$$

Since the outside of the diagram commutes and all the inner parts but the right-hand one do, that right-hand part also commutes and yields $h \cdot w_{i+1} = h_{i+1}$, as desired. Hence, $h = h_j$. $\square$

Remark 6.2.4. The proposition above can be made sharper (and we use this sharper variant in **??** below). For any category $\mathscr{A}$, write $\mathscr{A}_{\mathcal{M}}$ for the category whose objects are those of $\mathscr{A}$ and whose morphisms are the morphisms of $\mathcal{M}$.

Since $\mathscr{A}$ has constructive monomorphisms, so does $\mathscr{A}_{\mathcal{M}}$, obviously. And since F preserves monomorphisms, it restricts to an endofunctor F' of $\mathscr{A}_{\mathcal{M}}$. Now take a subfunctor $m : H \hookrightarrow F'$ of this restriction. Then the statement of Proposition 6.2.3 still holds: if μH exists, then for every algebra (A, a) of F, there exists a unique morphism $h : \mu H \to A$ for which (6.5) commutes. The proof is precisely the same.

Quotients. Just as subfunctors inherit initial algebras, so do quotient functors.

Remark 6.2.5. Recall that a (proper) *factorization system* is a pair $(\mathcal{E}, \mathcal{M})$ where $\mathcal{E}$ is a class of epimorphisms and $\mathcal{M}$ a class of monomorphisms, both containing all isomorphisms, such that every morphism f has an $(\mathcal{E}, \mathcal{M})$-factorization, i.e. $f = m \cdot e$ with m in $\mathcal{M}$ and e and $\mathcal{E}$, unique up to isomorphism. It follows that $\mathcal{M}$ contains all regular monomorphisms. We call a factorization system *constructive* if $\mathcal{M}$ is constructive.

In every factorization system, $\mathcal{M}$ is *left-cancellative*: if $u \cdot v$ lies in $\mathcal{M}$, so does v. Moreover, limits of chains of $\mathcal{M}$-monomorphisms are conies in $\mathcal{M}$. Dual properties hold for $\mathcal{E}$.

Given a factorization system $(\mathcal{E}, \mathcal{M})$, by an $\mathcal{E}$-*quotient* of an endofunctor F is meant an endofunctor G for which there exists a natural transformation $q : F \to G$ whose components belong to $\mathcal{E}$. We have seen such examples in Section 4.4: finitary endofunctors of Set are precisely the quotients of the polynomial functors. The following is a certain dual of Corollary 6.2.2 above:

Proposition 6.2.6. *Let $\mathscr{A}$ be an $\mathcal{E}$-cowellpowered category, and let $(\mathcal{E}, \mathcal{M})$ be a constructive factorization system on $\mathscr{A}$. Let $F : \mathscr{A} \to \mathscr{A}$ preserve $\mathcal{M}$-monomorphisms and $\mathcal{E}$-epimorphisms. If F has an initial algebra, then so does every $\mathcal{E}$-quotient of it.*

Proof. Suppose we have a natural transformation $q : F \to G$ such that each component of q is in $\mathcal{E}$. Then this induces a unique natural transformation t from the initial-algebra chain of F to that of G in the following way:

$$t_0 = \mathsf{id}_0 \qquad \text{and} \qquad t_{i+1} = (FF^i 0 \xrightarrow{Ft_i} FG^i 0 \xrightarrow{q_{G^i 0}} GG^i 0);$$

(note that for limit ordinals j, t_j is uniquely determined). By transfinite induction we see that all t_i are in $\mathcal{E}$, since F is $\mathcal{E}$-preserving and colimits of chains of $\mathcal{E}$-morphisms lie in $\mathcal{E}$

By Theorem 6.1.22, the initial-algebra chain of F converges. Let λ be such that $w_{\lambda,\lambda+1} : F^\lambda 0 \to F^{\lambda+1} 0$ is an isomorphism. Then also all $w_{i,j}$ with $j \geq i \geq \lambda$ are isomorphisms. It follows that the chain $(G^i 0)_{i \geq \lambda}$ consists of $\mathcal{E}$-quotients of the initial algebra $F^\lambda 0$. Since $\mathscr{A}$ is $\mathcal{E}$-cowellpowered, there exist $\lambda'' \geq \lambda' \geq \lambda$ such that the connecting morphism $w^G_{\lambda',\lambda''} : G^{\lambda'} 0 \to G^{\lambda''} 0$ is an isomorphism. By Proposition 6.1.9, $w^G_{\lambda',\lambda'+1}$ is an isomorphism, thus by Theorem 6.1.10, $G^{\lambda'} 0$ is an initial G-algebra. □

Corollary 6.2.7. *Every finitary set functor has an initial algebra and a terminal coalgebra.*

6.3 Canonical fixed points in CPO-enriched categories

We now return to the CPO-enriched categories and consider sufficient conditions for endofunctors to have a canonical fixed point. We know from Theorem 5.1.23 that locally continuous endofunctors have canonical fixed points. We introduce a weaker condition, stability (which is, by Observation 6.3.5 below, more or less equivalent to preservation of embeddings) and prove that whenever a stable functor has a fixed point, it has a canonical one. Recall from Definition 5.1.20 that a category is called *strict* CPO*-enriched* if its hom-sets carry the structure of a cpo with $\perp$, and composition is continuous and preserves $\perp$ (in both variables).

Definition 6.3.1. A functor F between strict CPO-enriched categories is called *stable* if for every idempotent endomorphism $f \colon X \to X$, $f = f \cdot f$, we have:

$$f \sqsubseteq \mathsf{id}_X \quad \text{implies} \quad Ff \sqsubseteq \mathsf{id}_{FX}.$$

Examples 6.3.2. (1) Every locally monotone functor (see Definition 5.1.15) is stable.

(2) A composite, product or coproduct of stable endofunctors on the category $\mathsf{CPO}_\perp$ (see Example 5.1.21) is stable. (Recall from Example 2.1.6(2) that coproducts in $\mathsf{CPO}_\perp$ are given by disjoint unions with least elements merged.)

(3) For every stable functor $F \colon \mathsf{CPO}_\perp \to \mathsf{CPO}_\perp$ the *lifting* given by $F_\perp X = FX \cup \{\perp\}$ ($\perp$ a new bottom element) is stable.

(4) It follows that for every signature Σ, we have two stable endofunctors

$$H_\Sigma X = \coprod_{\sigma\in\Sigma} X^n \qquad \text{and} \qquad H'_\Sigma X = \coprod_{\sigma\in\Sigma} X^n_\perp \qquad n = \mathrm{ar}(\sigma),$$

Remark 6.3.3. Let Σ be a signature. The H_Σ-algebras in $\mathsf{CPO}_\perp$ are the *strict continuous Σ-algebras*: they are given by a cpo, A, which is a Σ-algebra such that every operation $\sigma_A\colon A^n \to A$ is continuous and strict: $\sigma_A(\perp, \dots, \perp) = \perp$.

The H'_Σ-algebras are Σ-algebras whose operations are continuous but not necessarily strict.

Remark 6.3.4. (1) A category is said to have *split idempotents* if every idempotent endomorphism, i.e. a morphism $f\colon X \to X$ with $f \cdot f = f$, has a factorization $f = s \cdot u$ with $u \cdot s = \mathsf{id}_X$.

(2) Every category with equalizers has split idempotents: given f one takes the equalizer u of f and id_X. Dually, a category with coequalizers has split idempotents.

(3) The above factorization is unique up to isomorphism, for suppose $g = e' \cdot u'$ where $u' \cdot e' = \mathsf{id}_X$. Then $i = u' \cdot e$ is an isomorphism with $e' = e \cdot i^{-1}$ and $u' = i \cdot u$.

Observation 6.3.5. (1) Stable functors F preserve embeddings. To see this, let e be an embedding with projection $\widehat{e}$. Observe that $e \cdot \widehat{e}$ is idempotent. Since $e \cdot \widehat{e} \leq \mathsf{id}$, we also have $Fe \cdot F\widehat{e} \leq \mathsf{id}$. Therefore $F\widehat{e}$ is the projection for Fe.

(2) For endofunctors on CPO-enriched categories with split idempotents we have:

$$F \text{ stable} \iff F \text{ preserves embedding-projection pairs.}$$

Indeed, let F preserve these pairs and let $g \cdot g = g \sqsubseteq \mathsf{id}_X$. For the factorization in Remark 6.3.4 we have $e \cdot u = g \sqsubseteq \mathsf{id}_X$, thus, e is an embedding with $u = \widehat{e}$. Therefore $Fu = \widehat{Fe}$ and we conclude that

$$Fg = Fe \cdot \widehat{Fe} \sqsubseteq \mathsf{id}_{FX}.$$

Lemma 6.3.6. *Let $\mathscr{A}$ be a nonempty, strict* CPO*-enriched category with colimits of ω-chains. The initial-algebra chain of every stable endofunctor consists of embeddings.*

Proof. We prove that if F is a stable endofunctor, then $w_{i,j}\colon F^i0 \to F^j0$ is an embedding by transfinite induction. First, $w_{0,1}$ is an embedding since, by Lemma 5.1.22, we have a zero object $0 = 1$, and the unique morphism $F0 \to 0$ is $\widehat{w}_{0,1}$ because we clearly have $\widehat{w}_{0,1} \cdot w_{0,1} = \mathsf{id}_0$ and since $w_{0,1} = \perp\colon 0 \to F0$, we conclude $w_{0,1} \cdot \widehat{w}_{0,1} = \perp \leq \mathsf{id}_{F0}$.

If $w_{i,j}$ is an embedding, so is $w_{i+1,j+1} = Fw_{ij}$ by Observation 6.3.5.

For the limit steps, use the fact that a colimit of a chain of embeddings in formed by embeddings (see Basic Lemma 5.1.9 and Remark 5.1.11). □

Remark 6.3.7. In the proof of Lemma 6.3.6, the projections $\widehat{w}_{i,j}\colon F^j1 \to F^i1$ form the terminal-coalgebra chain of F (cf. Section 6.4). This is easily seen by transfinite induction.

Corollary 6.3.8. *Let $\mathscr{A}$ be a strict* CPO*-enriched category with colimits of chains. Then every stable endofunctor F with a fixed point has a canonical fixed point $\mu F = \nu F$.*

This follows from Theorem 6.1.22 applied to the class $\mathcal{M}$ of all embeddings. This is a constructive class, see Basic Lemma 5.1.9 and Remark 5.1.11. Due to Observation 6.3.5, F preserves embeddings. Wellpoweredness with respect to embeddings (which are split monics) follows from the observation that the number of split subobjects of an object A is bounded by the number of endomorphisms of A. Indeed, given a subobject represented by $m\colon A' \rightarrowtail A$ with a splitting $e\colon A' \twoheadrightarrow A$, so that $e \cdot m = \mathsf{id}$, the endomorphism $m \cdot e$ determines the subobject, since every endomorphism has up to isomorphism at most one such factorization.

Example 6.3.9 [49]**.** The category of sets and partial one-to-one functions with hom-sets ordered by inclusion is $\mathsf{CPO}_\perp$-enriched. An endomorphism f is smaller than the identity iff it is idempotent. Thus, every endofunctor is stable, and consequently, all endofunctors with a fixed point have a canonical one.

Remark 6.3.10. Corollary 6.3.8 stems from Barr [49], where the slightly stronger assumption that F be order-preserving on homsets was made, and the proof was somewhat more technical. Barr also proves a result establishing canonical fixpoint for endofunctors on a (non-strict) CPO-enriched category (see [49, Theorem 5.1]).

Example 6.3.11. The functor is $\mathsf{Idl}\colon \mathsf{DCPO}_\perp \to \mathsf{DCPO}_\perp$ taking an object of $\mathsf{DCPO}_\perp$ to the dcpo of its ideals (see Example 5.1.17) is stable, but it does not have any fixed point (see **??**).

6.4 The terminal-coalgebra chain

Functors from Ord^{op}, the dually ordered class of all ordinals, should be called *transfinite opchains*. We however follow the usual custom of calling them *transfinite chains*. Recall that a category $\mathscr{A}$ is said to *have limits of chains* if for every ordinal α all diagrams $D\colon \alpha^{op} \to \mathscr{A}$ have limits. This includes the existence of a terminal object 1 (the case $\alpha = 0$).

The following definition is nothing else than the dual of the initial-algebra chain of Definition 6.1.4. It was formulated explicitly by Barr [50].

Definition 6.4.1. Let $\mathscr{A}$ be a category with limits of chains. For every endofunctor F the *terminal-coalgebra chain* is the transfinite chain in $\mathscr{A}$ indexed by Ord^{op}, having objects $V_j = F^j 1$ and connecting morphism v_{ji}, $j \geq i$, defined by transfinite recursion as follows:

$$\begin{aligned} V_0 &= 1, \\ V_{j+1} &= FV_j && \text{for all ordinals } j, \\ V_j &= \lim_{i<j} V_i && \text{for all limit ordinals } j, \end{aligned}$$

and

$v_{1,0}\colon F1 \to 1$ is unique,
$v_{k+1,j+1} = Fv_{k,j}\colon FV_k \to FV_j$,
$v_{ji}(j > i)$ is the limit cone for every limit ordinal j.

We say that the terminal-coalgebra chain *converges in at most i steps* if $v_{i+1,i}$ is an isomorphism. It converges in *exactly i steps* if it converges in at most i steps, and for all $j < i$, it does not converge in at most j steps.

Example 6.4.2. For finitary set functors, the chain (4.7) is precisely the part of the chain above from ω to $\omega + \omega$. For example $m = v_{\omega+1,\omega}$. The terminal-coalgebra chain of $\mathscr{P}_\mathsf{f}$ (or indeed of any finitary endofunctor on Set) converges in $\omega + \omega$ steps. The connecting morphisms after $F^\omega 1$ are monomorphisms; this follows from Lemma 4.4.3.

Theorem 6.4.3 (Dual of Theorem 6.1.10)**.** *Let $\mathscr{A}$ be a category with limits of chains. If the terminal-coalgebra chain of a functor F converges in λ steps, then $F^\lambda 1$ is the terminal algebra w.r.t. the coalgebra structure $v^{-1}_{\lambda,\lambda+1}\colon V_\lambda \to FV_\lambda$.*

Corollary 6.4.4 (Dual of Corollary 6.1.11)**.** *Let $\mathscr{A}$ be a category with limits of chains. If an endofunctor F preserves limits of λ^{op}-chains for some infinite ordinal λ, then the terminal-coalgebra chain converges in λ steps, hence,*

$$\nu F = F^\lambda 1.$$

In **??** we will introduce the concept of a λ-accessible functor for an infinite regular cardinal λ. For a set functor F this is equivalent to being λ-*bounded*, which is the following condition: for each element $x \in FX$ there exists a subset $u\colon U \hookrightarrow X$ with $|U| < \lambda$ such that $x \in Fu[FU]$ (see **??**). Note that $\aleph_0$-accessible functors are precisely the finitary ones (cf. Section 4.3). An example, of an $\aleph_1$-accessible functor is the countable power-set functor. Worrell's Theorem 4.4.7 holds more generally for λ-accessible set functors as follows:

Theorem 6.4.5 [170, Thm. 11]**.** *Let λ be an infinite regular cardinal, and let F be a λ-accessible set functor.*

(1) *The terminal-coalgebra chain has monomorphic connecting morphisms from λ onwards.*

(2) *It converges after at most $\lambda + \lambda$ steps.*

Proof. If $F1 = \emptyset$, then F is constant with value $\emptyset$, and the statements hold trivially. So we may assume $F1 \neq \emptyset$. Then all members of the terminal-coalgebra chain are nonempty, for if we had $V_i = \emptyset$, then $V_{i+1} = \emptyset$ by presence of the map $v_{i+1,i}$, which would be invertible, whence $\emptyset$ would be a terminal coalgebra. But this contradicts the existence of a coalgebra $1 \to F1$.

As to (1): It is sufficient to prove that $v_{\lambda+1,\lambda}$ is monic. Indeed then $v_{\lambda+1,\lambda}$ is a split monomorphism, hence $v_{\lambda+2,\lambda+1} = Fv_{\lambda+1,\lambda}$ is monic, *etc.* We obtain by transfinite induction that all $v_{i,\lambda}$ with $i \geq \lambda$ are monic.

We prove that two distinct elements x and y of $V_{\lambda+1} = FV_\lambda$ remain distinct under $v_{\lambda+1,\lambda}$. Since F is λ-bounded, there exists a nonempty subset $u\colon U \hookrightarrow V_\lambda$ such that

$|U| < \lambda$ and x and y lie in the image of Fu. Since $(v_{\lambda,i})_{i<\lambda}$ is a limit cone, thus collectively monic, every pair of distinct elements of U remains distinct under $v_{\lambda,i}$ for some $i < \lambda$. From the fact that $|U \times U| < \lambda$ we conclude that one $i_0 < \lambda$ can be chosen for *all* distinct pairs in U. (This is precisely where we use the regularity of λ.) In other words, $v_{\lambda,i_0} \cdot u$ is a monomorphism. It splits since $U \neq \emptyset$, thus $Fv_{\lambda,i_0} \cdot Fu$ is monic, which implies that Fv_{λ,i_0} keeps x and y distinct. Thus, $v_{\lambda+1,\lambda}$ too keeps them distinct, because

$$Fv_{\lambda,i_0} = v_{\lambda+1,i_0+1} = v_{\lambda,i_0+1} \cdot v_{\lambda+1,\lambda}.$$

As to (2): To prove that $v_{\lambda+\lambda+1,\lambda+\lambda}$ is invertible is to prove that F preserves the limit $V_{\lambda+\lambda}$ of $(V_i)_{i<\lambda+\lambda}$. We can disregard the first λ members of that chain and obtain the same limit:

$$V_{\lambda+\lambda} = \lim_{i<\lambda} V_{\lambda+i}.$$

The connecting morphisms

$$v_i = v_{\lambda+i,\lambda} \colon V_{\lambda+i} \to V_\lambda$$

of that last λ-chain are monic by (1). So the limit is just the intersection $V_{\lambda+\lambda} = \bigcap_{i<\lambda} V_{\lambda+i}$, and we are to prove that F preserves this intersection.

Given an element $x \in FV_\lambda$ lying in the image of Fv_i, for all $i < \lambda$, we have the task to prove that x lies in the image of $Fv_{\lambda+\lambda,\lambda}$. Using that F is λ-bounded, we choose a subset $u \colon U \hookrightarrow V_\lambda$, $|U| < \lambda$, such that x lies in the image of Fu. By Proposition 4.4.4 we may assume that F preserves finite intersections. Then for $u_i = u \cap v_i$ we know that $x \in \mathrm{im}(Fu_i)$ for all $i < \lambda$. However, (u_i) is a decreasing λ-chain of subsets of U. Since $|U| < \lambda$, this chain converges at some $i_0 < \lambda$. It follows that $u_{i_0} = u_\lambda \subseteq v_{\lambda+\lambda,\lambda}$, whence indeed x lies in the image of $Fv_{\lambda+\lambda,\lambda}$. □

Example 6.4.6. For polynomial functors H_Σ (even the non-finitary ones of Example 6.1.13(2)) we conclude that the terminal coalgebra is constructed by the finitary terminal-coalgebra chain:

$$\nu H_\Sigma = H_\Sigma^\omega 1.$$

Indeed, for all cardinals k, the functors $X \mapsto X^k$ preserve limits of ω^{op}-chains. Hence, so do all polynomial functors. Recall that νH_Σ is the coalgebra of all Σ-trees with the inverse of tree tupling as its structure; for finitary signatures we showed this in Theorem 2.5.9, and for non-finitary ones the proof is analogous.

Example 6.4.7 [23]**.** (1) We present a set functor whose terminal-coalgebra converges in exactly $\lambda + \lambda$ steps. A *filter* on a set X is a (possibly empty) set of subsets of X closed under superset and intersection. We define a set functor $\mathscr{F}$ such that $\mathscr{F}X$ is the set of filters on X. For a function $f \colon X \to Y$ put $\mathscr{F}f(\mathbb{F}) = \{M \subseteq Y : f^{-1}[M] \in \mathbb{F}\}$ (see **??** on page **??**). The functor $\mathscr{F}$ has no fixed point, hence no terminal coalgebra. The λ-accessible coreflection $\mathscr{F}_\lambda$ is the functor of all λ-*small filters*; i.e. $\mathscr{F}_\lambda X$ consists of those filters containing some member of size smaller than λ. The terminal-coalgebra chain of $\mathscr{F}_\lambda$ convergest in exactly $\lambda + \lambda$ steps.

(2) As a variation on (1), let α be an infinite regular cardinal. A *base* of a filter $\mathbb{F}$ is a family $\mathbb{F}_0$ of sets such that $\mathbb{F}$ is the smallest filter with $\mathbb{F}_0 \subseteq \mathbb{F}$. Let us call a filter *$\alpha$-based* if it has a *base* whose cardinality is less than α. The subfunctor $\mathscr{F}^\alpha$ of $\mathscr{F}$ given by all α-based filters again has no fixed point. However, its λ-accessible modification $\mathscr{F}^\alpha_\lambda$ (all λ-small, α-based filters) has the terminal-coalgebra chain converging in exactly $\lambda + \alpha$ steps, provided $\alpha \leq \lambda$.

(3) For every regular cardinal λ, the bounded power set functor $\mathscr{P}_\lambda$ is the subfunctor of $\mathscr{P}$ that takes a set X to the set of subsets of X of size $< \lambda$. We present a modification $\overline{\mathscr{P}}_\lambda$ whose terminal-coalgebra chain converges in exactly λ steps. No modification is needed on objects, but for a morphism $f\colon X \to Y$, we put,

$$\overline{\mathscr{P}}_\lambda(M) = \begin{cases} f[M] & \text{if } f|_M \text{ is monic} \\ \emptyset & \text{otherwise} \end{cases} \qquad \text{for } M \in \overline{\mathscr{P}}_\lambda(X).$$

Example 6.4.8. The countable power set functor $\mathscr{P}_c$ has a terminal coalgebra consisting of all countably branching strongly extensional trees (see Definition 4.5.2). The coalgebra structure $\nu\mathscr{P}_c \to \mathscr{P}_c(\nu\mathscr{P}_c)$ is the inverse map of tree-tupling. This was proved by Worrell [170].

The terminal-coalgebra chain of $\mathscr{P}_c$ has a similar description to that of $\mathscr{P}_{\mathsf{f}}$. Recall from Section 4.5 that $\mathscr{P}_{\mathsf{f}}^\omega 1$ is the set of all strongly extensional saturated trees, $\mathscr{P}_{\mathsf{f}}^{\omega+n}1$ is the subset containing the trees in $\mathscr{P}_{\mathsf{f}}^\omega 1$ which are finitely branching below level n, and the terminal coalgebra $\nu\mathscr{P}_{\mathsf{f}}$ is the set of all finitely branching, strongly extensional trees. In order to describe the terminal-coalgebra chain of the functor $\mathscr{P}_c$, we now generalize the concept of a saturated tree to that of an α-saturated tree for all countable ordinals α. Recall that the maximum proper subtrees of a tree t are called the children of t. Analogously, the *children of a node x* are the maximum proper subtrees of the tree with root x.

We first denote by $\sim_\alpha$ the following equivalence relation on trees:

$s \sim_0 t$	holds always
$s \sim_{\alpha+1} t$	iff for every child s' of s there is a child t' of t such that $s' \sim_\alpha t'$ and vice versa
$s \sim_i t$	iff $s \sim_\alpha t$ for all $\alpha < i$, when i is a limit ordinal.

Definition 6.4.9. We define the *α-saturated* trees by transfinite recursion on α as follows:

(1) The only 0-saturated tree is the root-only tree.

(2) A tree is $(\alpha+1)$-saturated all its children are α-saturated.

(3) For limit ordinals i, a tree t is i-saturated iff given a tree s and a node x of t having children t_j with $s \sim_j t_j$ (for all $j < i$), then x has a child t' with $s \sim_i t'$.

Remark 6.4.10. (1) For finite i, the i-saturated trees are precisely the trees of depth at most i. What we are calling ω-saturated trees in this section are precisely the *saturated trees* in the sense of Definition 4.5.9.

(2) The step $V_i = \mathscr{P}_c^i 1$ of the terminal-coalgebra chain was described in [25] as follows:

$$\mathscr{P}_c^i 1 = i\text{-saturated, strongly extensional trees}$$

for all $i \leq \omega_1$, whereas for $i = \omega_1 + n$ we have

$$\mathscr{P}_c^i 1 = i\text{-saturated, strongly extensional trees countably branching at all depths} < n.$$

Theorem 6.4.11 [25, Thm. 3.17]**.** *The terminal-coalgebra chain of $\mathscr{P}_c$ converges in exactly $\omega_1 + \omega$ steps, yielding $\nu\mathscr{P}_c =$ all countably branching, strongly extensional trees.*

Open Question 6.4.12. For which ordinals α does there exist a set functor whose terminal-coalgebra chain converges in exactly α steps? See Example 6.4.7 for ordinals which are exact convergence ordinals for various functors. But not much more seems to be known. However, whenever a terminal coalgebra exists, it can be constructed by the terminal-coalgebra chain. This holds, more generally, for many-sorted sets:

Theorem 6.4.13 (Adámek and Trnková [39])**.** *Whenever an endofunctor of Set^S has a terminal coalgebra, then the terminal-coalgebra chain converges.*

This generalizes the previous result for endofunctors on Set in Adámek and Koubek [22]. Both proofs heavily depend on the theory of algebraized chains developed by Jan Reiterman in his PhD thesis and summarized in Koubek and Reiterman [114].

The expected generalization of Theorem 6.4.13 to, say, all presheaf categories does not hold. Adámek and Trnková [39] constructed an endofunctor on the category of graphs that has a terminal coalgebra although the terminal-coalgebra chain does not converge.

Example 6.4.14 [22]**.** In contrast to what we saw in the Initial-Algebra Theorem 6.1.22, a set functor with a fixed point need not have a terminal coalgebra. In Example 6.1.32 put $\gamma = \{\omega\}$. The set functor

$$\mathscr{P}^{\{\omega\}}X = \text{the set of all subsets of } X \text{ of cardinality different from } \omega$$

has an initial algebra but not a terminal coalgebra. Obviously, $\mathbb{N}$ is a fixed point of $\mathscr{P}^{\{\omega\}}$: the set $\mathscr{P}^{\{\omega\}}\mathbb{N}$ of all finite subsets is countable. Thus, an initial algebra exists due to Theorem 6.1.22.

The reason why $\mathscr{P}^{\{\omega\}}$ does not have a terminal coalgebra follows from the fact that

$$|V_i| \geq 2^i \qquad \text{for all } i \in \mathsf{Ord}.$$

Indeed, this follows from an easy transfinite induction. First, the finitary part of the terminal-coalgebra chain is the same for $\mathscr{P}^{\{\omega\}}$ and $\mathscr{P}_{\mathsf{f}}$. It is not difficult to derive from the description in Theorem 4.5.11 that the limit $\mathscr{P}_{\mathsf{f}}^{\omega}1$ has cardinality 2^ω. (Indeed, for every set A of natural numbers we have a saturated tree t_A obtained by taking an infinite path and adding a leaf at precisely those levels n with $n \in A$.) From this the rest of the transfinite induction is obvious.

The last example illustrates that a set functor with a countable initial algebra need not have a terminal coalgebra. However, it turns out that the cardinaliy $\aleph_0$ is the only exception: every other regular cardinal λ has the property that a set functor with a fixed pooint of cardinality λ has a terminal coalgebra:

Theorem 6.4.15 [16]. *Let F be a set functor with a nonempty fixed point of regular cardinality $\lambda \neq \aleph_0$. Then F has a terminal coalgebra and an initial algebra, and their cardinalities are equal, i.e. $\mu F \cong \nu F$.*

Remark 6.4.16. The concept of constructive monomorphisms does not have a satisfactory dualization, which explains why a set functor with a fixed point need not have a terminal coalgebra. Indeed, there exist ω^{op}-chains of epimorphismsm $a_n \colon A_{n+1} \twoheadrightarrow A_n$, $n < \omega$, in Set such that given a cone of epimorphisms $b_n \colon B \twoheadrightarrow A_n$, $n < \omega$, the factorizing morphism $b \colon B \to \lim_{n<\omega} A_n$ can fail to be epimorphic.

For example, let $A_n = n + 1$ be the ω^{op}-chain of Example 3.3.6, i.e. $A_n = F^n 1$ for $FX = X + 1$. We have $\lim A_n = \mathbb{N}^\top$. The following cone

$$b_n \colon \mathbb{N} \twoheadrightarrow n + 1, \qquad b_n(i) = \min(i, n + 1) \qquad \text{for all } i \in \mathbb{N}$$

consists of epimorphisms. But the factorizing morphism $b \colon B \to A$ is not epimorphic, it is the inclusion map $b \colon \mathbb{N} \hookrightarrow \mathbb{N}^\top$.

Remark 6.4.17. The following were proved by Barr [49].

(1) The initial and terminal-coalgebra chain of a functor F are connected by a unique collection of morphisms $h_i \colon F^i 0 \to F^i 1$ satisfying $h_{i+1} = F h_i$ for every ordinal i.

(2) If some h_i is an isomorphism and at least one algebra-to-coalgebra morphism exists, then F has a canonical fixed point. In more detail, the hypothesis is that there is an algebra $a : FA \to A$ and a coalgebra $b : B \to FB$ and some $f : A \to B$ with $Ff = b \cdot f \cdot a$.

Remark 6.4.18. Returning to $\mathsf{CPO}_\perp$-enriched categories, recall that locally continuous endofunctors have canonical fixed points (see Theorem 5.1.23). The concept of a locally continuous endofunctor F can be generalized to that of a *locally λ-continuous* one, where λ is a given infinite cardinal: This means that for every λ-chain $(f_i)_{i<\lambda}$ in $\mathscr{A}(X, Y)$ we have

$$F(\bigsqcup_{i<\lambda} f_i) = \bigsqcup_{i<\lambda} Ff_i \qquad \text{in } \mathscr{A}(FX, FY).$$

For these endofunctors we have, whenever $\mathscr{A}$ is strict CPO-enriched and has colimits of chains of cofinality at most λ, the following:

$$F \text{ has a canonical fixed point } \mu F = \nu F = \lim_{n<\lambda} F^n 1.$$

This is completely analogous to Theorem 5.1.23.

6.5 Summary of this chapter

We have presented a transfinite construction of initial algebras μF and terminal coalgebras νF for endofunctors on categories with (co)limits of (op-)chains. For a polynomial set functor H_Σ, the initial-algebra chain has the steps

$$W_i = \text{all } \Sigma\text{-trees of height less than } i.$$

It converges in λ steps to μH_Σ, the algebra of all well-founded Σ-trees, where λ is the first regular cardinal larger than all arities of the operation symbols in Σ. In contrast, the terminal-coalgebra chain converges already in ω steps to νH_Σ, the coalgebra of all Σ-trees. For every set functor F, an initial algebra exists iff F has a (pre-)fixed point, and it is always constructed by the initial-algebra chain. In contrast, existence of a fixed point X does not guarantee that a terminal coalgebra exists – but it does in the case of regular cardinal $|X| \neq \aleph_0$. For a number of categories, given an endofunctor F preserving monomorphisms, we proved that the existence of a fixed point implies that μF exists.

All terminal coalgebras of endofunctor on Set^S, the category of S-sorted sets, that exist are obtained by the terminal-coalgebra construction. This does not hold for endofunctors of the (presheaf) category of graphs. We also dicussed the precise length of the construction until convergence of the initial-algebra chain or the terminal coalgebra chain, respectively, happens.

For endofunctors F of strict CPO-enriched categories the concept of stability was introduced. This is the (very weak) condition that if $f\colon X \to X$ is an idempotent morphism which $f \sqsubseteq \mathsf{id}_X$, then $Ff \sqsubseteq \mathsf{id}_{FX}$. For a stable endofunctor we proved that if it has a fixed point, then it has a canonical one, i.e. $\mu F = \nu F$.

7 Terminal Coalgebras as Algebras, Initial Algebras as Coalgebras

A fixed point of a functor can be considered both as an algebra or as a coalgebra. For example, the terminal coalgebra can be viewed as an algebra (by inverting its structure map). The theme of this chapter are characterizations of the terminal coalgebra as the initial object in a category of algebras that allow one to interpret *structured corecursive* specifications. Dually, we are interested in characterizations of the initial algebra as a terminal object in a category of coalgebras that allow one to interpret specifications by *structured recursion*. The chapter is split into sections that present the various notions of (co)algebras. Each of those notions is interesting in its own right. In fact, they were originally introduced and studied for reasons not connected to our theme.

We begin in Section 7.1 with *corecursive algebras*, and we show that the terminal coalgebra is characterized as the initial corecursive algebra. In Section 7.2 we present the stronger notion of a *completely iterative algebra* and again characterize the terminal coalgebra as an initial completely iterative algebra. Dually, we then turn to *recursive* and *parametrically recursive coalgebras*, respectively, in Section 7.3, where the initial algebra is characterized as a terminal object.

Before we turn to Sections 7.1–7.3, let us informally motivate corecursive and completely iterative algebras with an example from general algebra. Consider the set functor $FX = X \times X + 1$ expressing the type of algebras with a binary operation $*$ and constant c. In this example, corecursive specifications take the form of systems of formal recursive equations, and we require that those are uniquely solvable. For example, the recursive equation $x \approx x * x$ has a unique solution in an algebra A iff A has a unique idempotent element $i \in A$. The following system of recursive equations

$$x \approx x * x \qquad y \approx x * (y * c) \tag{7.1}$$

requires, in addition, that a unique element $a \in A$ with $a = i *^A (a *^A c^A)$. Moreover, we would like to have unique solutions of *as many recursive equations as possible*. It will not be possible to allow the right-hand sides to be bare variables because we do not expect equations like $x \approx x$ to have a unique solution. Similarly, "ungrounded" systems like

$$x_1 \approx x_2 \qquad x_2 \approx x_3 \qquad x_3 \approx x_4 \qquad \cdots$$

are not expected to have unique solutions. So our formal definition of an equation is designed to forbid the right-hand sides of systems to be bare variables. In addition, our definition also includes an important simplification. Even if we wish to solve as many recursive equations as possible, it is sufficient to restrict attention to systems whose right-hand sides are *flat terms*; i.e. terms containing precisely one operation symbol. Fortunately, every system of recursive equations where each right-hand side is a non-variable term (or even an infinite Σ-tree) can be flattened by introducing fresh auxiliary variables to represent subterms of non-flat terms. This implies that, as soon as we have solutions for all systems with flat right-hand sides, we also have solutions to systems where the right-hand sides are more complex. For example, for the above system (7.1) we obtain the following flattened system (using fresh variables z and z'):

$$x \approx x * x \quad y \approx x * z \quad z \approx y * z' \quad z' \approx c.$$

Let X denote the set of recursion variables (i.e. the ones on the left-hand sides of the above formal equations). Then a flat system of formal recursive equations assigns to every element of X its right-hand side, viz. an element of $FX = X \times X + 1$. Such assignments are simply coalgebras $e\colon X \to FX$. Given an algebra $\alpha\colon A \times A + 1 \to A$, a *solution* of e assigns to every recursion variable in X an element of A, i.e. we obtain a morphism $e^\dagger\colon X \to A$. The fact that $e^\dagger$ solves e means precisely that the following square commutes:

$$\begin{tikzcd} X \arrow[r, "e^\dagger"] \arrow[d, "e"'] & A \\ FX \arrow[r, "Fe^\dagger"'] & FA \arrow[u, "\alpha"'] \end{tikzcd}$$

This means that a solution $e^\dagger$ is just a coalgebra-to-algebra morphism. The F-algebra (A, α) is called *corecursive* if every coalgebra has a unique solution.

Although this notion is interesting, it is not exactly what we are after in this chapter. We are more interested in a stronger notion that allows a wider class of recursive specifications. For example, suppose that we again fix (A, α) and allow elements of A to appear as *parameters* on the right-hand sides of formal recursive equations such as (7.1). This means that we require that all *flat equation morphisms* $e\colon X \to FX + A$ have a unique solution $e^\dagger$ in the sense that the following square commutes:

$$\begin{tikzcd} X \arrow[r, "e^\dagger"] \arrow[d, "e"'] & A \\ FX + A \arrow[r, "Fe^\dagger + \mathsf{id}_A"'] & FA + A \arrow[u, "{[\alpha, \mathsf{id}_A]}"'] \end{tikzcd}$$

Then we call the algebra A *completely iterative*, or a *cia*, for short. Again, the commutativity of the above square just expresses the fact that formal equations are turned into identities in A. The main result in Section 7.2 is that a terminal coalgebra for F is precisely the same as an initial cia for F.

The two dual notions are notions of coalgebras providing semantics for structured recursion. A coalgebra (A, α) is called *recursive* if for every morphism $e\colon FX \to X$ there is a unique morphism $e^\dagger\colon A \to X$ such that the square below commutes:

$$\begin{tikzcd} A \arrow[r, "e^\dagger"] \arrow[d, "\alpha"'] & X \\ FA \arrow[r, "Fe^\dagger"] & FX \arrow[u, "e"'] \end{tikzcd}$$

The dual of complete iterativity is the notion of *parametric recursiveness*. A coalgebra (A, α) is *parametrically recursive* if for every morphism $e\colon FX \times A \to X$ there is a unique morphism $e^\dagger\colon A \to X$ so that the square below commutes:

$$\begin{tikzcd} A \arrow[r, "e^\dagger"] \arrow[d, "\langle \alpha, A\rangle"'] & X \\ FA \times A \arrow[r, "Fe^\dagger \times A"] & FX \times A \arrow[u, "e"'] \end{tikzcd} \tag{7.2}$$

Finally, we list the four definitions given above in Table 7.1.

type of equation morphism	a ______ algebra has unique solutions to these	coalgebra in the dual setting
$X \to FX$	corecursive	recursive
$X \to FX + A$	completely iterative	parametrically recursive

Table 7.1: Definitions studied in Sections 7.1–7.3.

7.1 Corecursive algebras

Corecursive algebras were first studied by Capretta et al. [60], and they compared that notion to a number of related concepts. Adámek et al. [19] investigated corecursive algebras further, and, in particular, they provided a description and construction of *free* corecursive algebras as well as a study of the ensuing monad of free corecursive algebras and its properties. Formally, a corecursive algebra is one that admits a unique coalgebra-to-algebra morphism for every given coalgebra. We use the terminology from the introduction of this chapter:

Definition 7.1.1. Let F be an endofunctor on a category $\mathscr{A}$. By a *solution* of a coalgebra $X \to FX$ in an F-algebra (A, α) we mean a morphism $e^\dagger$ such that the following square commutes:

$$\begin{tikzcd} X \arrow[r, "e^\dagger"] \arrow[d, "e"'] & A \\ FX \arrow[r, "Fe^\dagger"'] & FA \arrow[u, "\alpha"'] \end{tikzcd}$$

The algebra (A, α) is called *corecursive* if every coalgebra has a unique solution in it.

Remark 7.1.2. (1) Fields like general algebra and abstract data types study Σ-algebras, which are precisely the algebras for the functor H_Σ (see Example 2.1.4). Let X be a set of "recursion variables". A coalgebra $e\colon X \to H_\Sigma X$ can be understood as a system of formal mutually recursive equations

$$x \approx \sigma(x_1, \dots, x_n), \tag{7.3}$$

one for every recursion variable $x \in X$, where σ is an n-ary operation symbol in Σ and $x_1, \dots, x_n \in X$. Indeed, one just writes $e(x) = \sigma(x_1, \dots, x_n)$ in lieu of the above formal equation. For a Σ-algebra A, a solution $e^\dagger\colon X \to A$ assigns to every recursion variable x an element $e^\dagger(x) \in A$ such that the formal equations in (7.3) become actual identities in A when recursion variables are substituted by their solutions, i.e. we have $e^\dagger(x) = \sigma^A(e^\dagger(x_1), \dots, e^\dagger(x_n))$.

A concrete example of a corecurive algebra is the algebra $T_\Sigma = \nu H_\Sigma$ of all Σ-trees, cf. Example 7.1.3(2) below.

(2) In Example 7.1.3, we will present further examples of corecursive Σ-algebras. However, let us observe that classical algebras (such as groups, lattices, or boolean algebras) are almost never corecursive. This is due to the uniqueness requirement for solutions, more precisely, it follows from the fact that a corecursive algebra with a binary operation has a unique idempotent element. Indeed, the formal recursive equation $x \approx x \vee x$ has a unique solution in a lattice or boolean algebra A iff A is trivial. Similarly, the system $x \approx x \cdot y, y \approx 1$ has a unique solution in a group G iff G is trivial.

Example 7.1.3. (1) Capretta et al. [60] provide the following example of a corecursive algebra for the endofunctor $FX = E \times X \times X$ on Set. Consider the set E^ω of all streams over E with the F-algebra structure $\alpha\colon E \times E^\omega \times E^\omega \to E^\omega$ where $\alpha(e, s, t)$ is the stream with head e and continuing by the merge of s and t. This is a corecursive F-algebra.

(2) Every terminal coalgebra is a corecursive algebra. More precisely, by Lambek's Lemma 2.2.5, the structure $\tau\colon \nu F \to F(\nu F)$ of the terminal coalgebra is invertible, and hence $(\nu F, \tau^{-1})$ an F-algebra. This algebra is corecursive because for every $e\colon X \to FX$ a coalgebra homomorphism from (X, e) to $(\nu F, \tau)$ is the same thing as a solution of e in the algebra $(\nu F, \tau^{-1})$.

(3) The set functor $FX = X \times X + 1$ is the polynomial functor associated to the signature Σ with a binary operation symbol $*$ and a constant symbol c. By the previous point, the terminal coalgebra νF consisting of all finite and infinite binary trees (see Example 3.3.9) is corecursive. Let us consider inner nodes of binary trees as being labelled by $*$ and leaves as labelled by c. Let $X = \{x_1, x_2\}$ and consider the following system of mutually recursive equations

$$x_1 \approx c * x_2 \qquad x_2 \approx x_1 * c \tag{7.4}$$

Then its solution in νF consists of two infinite trees, ${x_1}^\dagger$ and ${x_2}^\dagger$, here written as infinite terms:

$${x_1}^\dagger = c * ((c * (\cdots * c)) * c) \quad \text{and} \quad {x_2}^\dagger = (c * ((c * \cdots) * c)) * c.$$

We invite the reader to draw these as pictures; cf. Example 7.2.6(1).

(4) Here is a different example of a corecursive algebra which we shall revisit when we discuss completely iterative algebras. It comes from Capretta et al. [60]. Let $FX = X \times X$ on Set. Let $A = \{0, 1, 2\}$, and let $\alpha\colon A \times A \to A$ be the algebra defined by $\alpha(1, 2) = 1$, and otherwise $\alpha(i, j) = 0$. Then A is corecursive: given any $e\colon X \to X \times X$, the unique solution $e^\dagger\colon X \to A$ is the constant function on 0. Indeed, this is clearly a solution since $\alpha(0, 0) = 0$. Moreover, since 2 cannot be written as $\alpha(i, j)$, we see that for all x, $e^\dagger(x) \neq 2$. And we claim that the same is true for 1. For suppose that $e^\dagger(x) = 1$. Then $e(x)$ must be (y, z) for some y and z such that $e^\dagger(y) = 1$ and $e^\dagger(z) = 2$. However, this contradicts what we have seen about 2. This proves that $e^\dagger$ is the constant function with value 0.

(5) In Section **??**, we shall see a number of corecursive algebras for set functors whose definitions involve elementary mathematics. Here is one example. Consider $FX = \mathbb{N} \times X$. Then the half-open interval $(0, 1]$ is a corecursive algebra for F, via the structure $n, r \mapsto (n + r)/(1 + n + r)$. See Theorem **??** on page **??**.

Corecursive algebras for F may be considered as a full subcategory of $\mathsf{Alg}\, F$ because of the following

Proposition 7.1.4 [19, Lem. 2.9]**.** *Let (A, α) and (B, β) be corecursive algebras. Every homomorphism $h\colon (A, \alpha) \to (B, \beta)$ preserves solutions. This means that for every $e\colon X \to FX$ we have that*

$$X \xrightarrow{e^\dagger} A \xrightarrow{h} B$$

is the unique solution of e in B.

Proof. The following diagram

$$\begin{tikzcd}
X \arrow[r, "e^\dagger"] \arrow[d, "e"'] \arrow[rr, bend left, "h\cdot e^\dagger"] & A \arrow[r, "h"] & B \\
FX \arrow[r, "Fe^\dagger"'] \arrow[rr, bend right, "F(h\cdot e^\dagger)"'] & FA \arrow[u, "\alpha"] \arrow[r, "Fh"'] & FB \arrow[u, "\beta"']
\end{tikzcd}$$

shows that $h \cdot e^\dagger$ is a solution of e in B and therefore the unique one. □

In order to see that an initial corecursive F-algebra is a fixed point of F, a crucial step is the following result by Capretta, Uustalu and Vene.

Proposition 7.1.5 [59, dual of Prop. 6]**.** *If (A, α) is a corecursive algebra, then so is $(FA, F\alpha)$.*

Proof. Suppose that (A, α) is a corecursive algebra for F, let $e\colon X \to FX$ and denote by $e^\dagger\colon X \to A$ its unique solution. We will show that

$$e^\ddagger = (X \xrightarrow{e} FX \xrightarrow{Fe^\dagger} FA)$$

is the unique solution of e in FA. First, the following diagram shows that it is a solution:

$$\begin{tikzcd}
X \arrow[r, "e"] \arrow[d, "e"'] \arrow[rr, bend left, "e^{\ddagger}"] & FX \arrow[r, "Fe^{\dagger}"] \arrow[d, "Fe"] & FA \\
FX \arrow[r, "Fe"'] \arrow[rr, bend right, "Fe^{\ddagger}"'] & FFX \arrow[r, "FFe^{\dagger}"'] & FFA \arrow[u, "F\alpha"']
\end{tikzcd}$$

Indeed, the upper and lower parts commute by the definition of $e^{\ddagger}$, the right-hand square does since $e^{\dagger}$ solves e, and the left-hand square is trivial.

To see that $e^{\ddagger}$ is the unique solution in FA, suppose that $s\colon X \to FA$ solves e. Then it follows that $\alpha \cdot s\colon X \to A$ is a solution of e in A, since the following diagram commutes:

$$\begin{tikzcd}
X \arrow[r, "s"] \arrow[d, "e"'] \arrow[rr, bend left, "\alpha\cdot s"] & FA \arrow[r, "\alpha"] & A \\
FX \arrow[r, "Fs"'] \arrow[rr, bend right, "F(\alpha\cdot s)"'] & FFA \arrow[r, "F\alpha"'] \arrow[u, "F\alpha"'] & FA \arrow[u, "\alpha"']
\end{tikzcd}$$

Indeed, the left-hand square commutes since s solves e in FA, and the remaining parts are trivial. Thus, we have $e^{\dagger} = \alpha \cdot s$, and we conclude that

$$e^{\ddagger} = Fe^{\dagger} \cdot e = F\alpha \cdot Fs \cdot e = s,$$

where the last equation holds since, once again, s solves e in FA. □

From this result one obtains Lambek's Lemma for corecursive algebras.

Corollary 7.1.6. *If an initial corecursive F-algebra exists, it is a fixed point of F.*

Indeed, the proof is the same as the one for Lambek's Lemma 2.2.5, using Proposition 7.1.5 to see that for an initial corecursive algebra (I, ι), $(FI, F\iota)$ is corecursive, too.

Theorem 7.1.7 [59, dual of Prop. 7]**.** *The initial corecursive algebra is precisely the same as the terminal coalgebra.*

In more detail, let $F\colon \mathscr{A} \to \mathscr{A}$ be an endofunctor. Then we have:

(1) If (I, ι) is an initial corecursive algebra, then (I, ι^{-1}) is a terminal coalgebra.

(2) If $(\nu F, \tau)$ is a terminal coalgebra, then $(\nu F, \tau^{-1})$ is an initial corecursive algebra.

Proof. (1) Suppose that $\iota\colon FI \to I$ is an initial corecursive algebra. By Corollary 7.1.6, we have a coalgebra $\iota^{-1}\colon I \to FI$. We need to verify that it is terminal. Let (C, γ) be any coalgebra. The coalgebra homomorphisms from (C, γ) to (I, ι^{-1}) are the same as the solutions of γ in (I, ι). So since (I, ι) is a corecursive algebra, (I, ι^{-1}) is a terminal coalgebra.

(2) Suppose that $\tau\colon \nu F \to F(\nu F)$ is the terminal coalgebra. We have seen in Example 7.1.3(2) that the algebra $(\nu F, \tau^{-1})$ is corecursive. It remains to verify its initiality. So let (A, α) be any corecursive algebra. There is a unique solution of $\tau\colon \nu F \to F(\nu F)$ in (A, α), and this means that there is a unique algebra homomorphism h from $(\nu F, \tau^{-1})$ to (A, α). $\square$

Remark 7.1.8. The only fixed point among the corecursive algebras for a functor is the initial corecursive algebra. Indeed, notice that in part (1) of the above proof we used the initiality of the corecursive algebra (I, ι) only to see that the structure morphism is an isomorphism. Consequently, by the argument above, *every* corecursive algebra with an isomorphic structure map is a terminal coalgebra.

Theorem 7.1.7 connects to the theme of this chapter in that it expresses the terminal coalgebra of a functor as the initial object in an interesting category of algebras.

At this point, we would like to briefly recall how *free* corecursive algebras are constructed [19]. Freeness is an analogous concept as that for ordinary F-algebras (cf. Remark 2.2.18): a free corecursive algebra on the object Y is a corecursive algebra $\gamma_Y\colon FCY \to CY$ together with a universal morphism $\eta_Y\colon Y \to CY$. This means that for every corecursive algebra (A, α) and every morphism $f\colon Y \to A$ there exists a unique algebra homomorphism $f^\sharp\colon (CY, \gamma_Y) \to (A, \alpha)$ satisfying $f^\sharp \cdot \eta_Y = f$:

$$\begin{tikzcd}
Y \arrow[r, "\eta_Y"] \arrow[dr, "\forall f"'] & CY \arrow[d, dashed, "\exists! f^\sharp"] & FCY \arrow[l, "\gamma_Y"'] \arrow[d, "Ff^\sharp"] \\
& A & FA \arrow[l, "\alpha"]
\end{tikzcd}$$

Theorem 7.1.9 [19]**.** *Assuming that a terminal coalgebra νF and a free F-algebra ΦX on X exist, the free corecursive algebra is their coproduct $\nu F \oplus \Phi X$ in the category of algebras for F.*

Example 7.1.10. For a polynomial endofunctor on Set, this means that the free corecursive algebra is formed by all Σ-trees that have only finitely many leaves labelled in X (but possibly infinitely many leaves labelled by constant symbols from Σ).

Furthermore, op. cit. provides the following iterative construction of free corecursive algebras somewhat similar to the initial algebra chain.

Construction 7.1.11 (Free-Corecursive-Algebra Chain)**.** Let $\mathscr{A}$ be a cocomplete category and let $F\colon \mathscr{A} \to \mathscr{A}$ have a terminal coalgebra. We define for every object Y of $\mathscr{A}$ an essentially unique chain $U\colon \mathsf{Ord} \to \mathscr{A}$ by the following transfinite recursion:

$$\begin{aligned}
U_0 &= \nu F \\
U_{i+1} &= FU_i + Y \\
U_j &= \operatorname*{colim}_{i<j} U_i && \text{for limit ordinals } j.
\end{aligned}$$

The connecting morphisms $u_{i,j} : U_i \to U_j$ are defined as follows:

$$u_{0,1} = (\nu F \xrightarrow{\tau} F(\nu F) \xrightarrow{\mathsf{inl}} F(\nu F) + Y),$$
$$u_{i+1,j+1} = F u_{i,j} + \mathsf{id}_Y,$$

and for limit ordinals j, $(u_{i,j})_{i<j}$ is the colimit cocone.

Theorem 7.1.12 [19, Thm. 4.6]**.** *Let $\mathscr{A}$ be a locally presentable category, and let $F\colon \mathscr{A} \to \mathscr{A}$ be accessible and preserving monomorphisms. Then the free-corecursive-algebra chain converges to the free corecursive algebra on Y.*

More precisely, there exists an ordinal λ such that $u_{\lambda,\lambda+1}\colon U_\lambda \to FU_\lambda + Y$ is an isomorphism, and its inverse yields (by composition with the coproduct injections) the structure and universal morphism, respectively, of a free corecursive algebra on Y.

Example 7.1.13 [19]**.** For a polynomial endofunctor H_Σ on Set the free corecursive algebra on a set Y can be described as follows. Consider the Σ-algebra $T_\Sigma Y$ of all Σ-*trees over* Y, i.e. trees for the signature obtained by adding the elements of Y as constant symbols to Σ_0, which means these trees are defined like Σ-trees except. that the leaves are labelled by constant symbols in Σ_0 or generators in Y. The free corecursive Σ-algebra on Y is the subalgebra $C_\Sigma Y$ of all Σ-trees over Y having only finitely many leaves labelled by generators in Y (and the remaining leaves labelled in Σ_0).

7.2 Completely Iterative Algebras

We continue our study by looking at a natural strengthening of the defining property of corecursive algebras and verifying that again, the terminal coalgebra is, equivalently, the initial object in the relevant (smaller) subcategory of algebras.

The idea of algebras with unique solutions of recursive equations stems from work in general algebra by Evelyn Nelson [140] and Jerzy Tiuryn [161]. Nelson introduced iterative algebras for a signature as algebras with unique solutions of finite systems of mutually recursive equations such as (7.1). In these systems, parameters from A are allowed to occur on right-hand sides of equations. Dropping the finiteness assumption one arrives at the notion of a *completely iterative algebra*, introduced by Milius [128]. Op. cit. also investigates the connection to terminal coalgebras.

Although the addition of parameters to formal recursive equations may seem like a small extension, the fact that each flat equation morphism has a unique solution is a strong property of an algebra and has many interesting consequences, as we shall explain in Remark 7.2.18.

Assumption 7.2.1. Throughout this section, we assume that $\mathscr{A}$ is a category with binary coproducts.

Definition 7.2.2 [128]**.** Let F be an endofunctor on $\mathscr{A}$. By a *flat equation morphism* in an object A we mean a morphism $e\colon X \to FX + A$. A *solution* of e in an algebra

(A, α) is a morphism $e^\dagger\colon X \to A$ such that the following square commutes:

$$\begin{tikzcd} X \arrow[r, "e^\dagger"] \arrow[d, "e"'] & A \\ FX + A \arrow[r, "Fe^\dagger + \mathsf{id}_A"'] & FA + A \arrow[u, "{[\alpha, \mathsf{id}_A]}"'] \end{tikzcd} \tag{7.5}$$

The algebra (A, α) is called *completely iterative* (or, a *cia* for short) provided that every flat equation morphism in A has a unique solution.

Remark 7.2.3. (1) Every cia (A, α) is, of course, a corecursive algebra. Indeed, for every coalgebra $e\colon X \to FX$ one forms

$$\bar{e} = (X \xrightarrow{e} FX \xrightarrow{\mathsf{inl}} FX + A),$$

and one readily verifies that the unique solution of $\bar{e}$ in the present sense is a unique solution of e in (A, α) in the sense of Definition 7.1.1.

(2) Thus, classical algebras are seldom cias as they are not corecursive, as explained in Remark 7.1.2(2).

We take cias to be a full subcategory of the category of all F-algebras. This choice is justified by the following result:

Proposition 7.2.4 [128, Prop. 2.3]**.** *Let (A, α) and (B, β) be cias for F. Then a morphism $h : A \to B$ is an algebra homomorphism if and only if it preserves solutions, i.e., for every flat equation morphism $e : X \to FX + A$ we have*

$$(X \xrightarrow{e^\dagger} A \xrightarrow{h} B) = (X \xrightarrow{e} FX + A \xrightarrow{FX+h} FX + B)^\dagger.$$

Sketch of Proof. The 'only if' direction is very similar to what we have seen in Proposition 7.1.4: one proves with a straightforward diagram chase that $h \cdot e^\dagger$ is a solution for $(FX + h) \cdot e$ in B and then obtains the desired result by the uniqueness of solutions in B.

For the 'if' direction one first observes that the morphism $[\alpha, \mathsf{id}_A] : FA + A \to A$ appears as the unique solution of the flat equation morphism $e = F\mathsf{inr} + \mathsf{inr} : FA + A \to F(FA + A) + A$.

Similarly, one proves that $[\beta \cdot Fh, h] : FA + A \to B$ is the unique solution of $(F(FA + A) + h) \cdot e$. Since h preserves solutions, one then obtains

$$h \cdot [\alpha, \mathsf{id}_A] = h \cdot e^\dagger = (F(FA + A) + h) \cdot e)^\dagger = [\beta \cdot Fh, h] : FA + A \to B,$$

whose left-hand coproduct component shows that h is a homomorphism. □

Proposition 7.2.5. *For every endofunctor, the terminal coalgebra is a cia.*

Proof sketch. Let $F\colon \mathscr{A} \to \mathscr{A}$ be an endofunctor for which $(\nu F, \tau)$ exists. Given a flat equation morphism $e : X \to FX + \nu F$, form the following F-coalgebra

$$\overline{e} = (X + \nu F \xrightarrow{[e, \mathsf{inr}]} FX + \nu F \xrightarrow{FX + \tau} FX + F(\nu F) \xrightarrow{\mathsf{can}} F(X + \nu F)),$$

where $\mathsf{can} = [F\mathsf{inl}, F\mathsf{inr}]$. Let $h : X + \nu F \to \nu F$ be the corresponding unique coalgebra homomorphism and define

$$e^\dagger = (\, X \xrightarrow{\mathsf{inl}} X + \nu F \xrightarrow{h} \nu F \,).$$

One readily shows that $h \cdot \mathsf{inr}$ is a coalgebra homomorphism from νF to itself, whence $h \cdot \mathsf{inr} = \mathsf{id}$. Now it is not difficult to prove that $e^\dagger$ is a solution of e iff $[e^\dagger, \mathsf{id}_{\nu F}]$ is a coalgebra homomorphism from $(X + \nu F, \overline{e})$ to the terminal coalgebra $(\nu F, \tau)$. Since the latter exists uniquely, so does the former. For more details, see [128, Example 2.5]. □

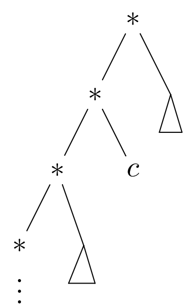
Examples 7.2.6. (1) It follows from Proposition 7.2.5 that the collection of all infinite binary trees is a cia for $FX = X \times X + 1$ (cf. Example 7.1.3(3)). For example, fix an element $t \in \nu F$, and consider the following system of equations:

$$x_1 \approx x_2 * t \qquad x_2 \approx x_1 * c \tag{7.6}$$

We emphasize that t here is a fixed element of the terminal coalgebra (i.e. a finite or infinite binary tree). Then the solution of this equation in νF consists of two infinite trees, ${x_1}^\dagger$ and ${x_2}^\dagger$. Here are two different depictions of these trees. First, we may write them as infinite Σ-terms:

$${x_1}^\dagger = (\cdots(\cdots * c) * t) * c) * t) \qquad \text{and} \qquad {x_2}^\dagger = (\cdots(\cdots * t) * c) * t) * c)$$

Second, we may picture them as infinite Σ-trees. For example, here is ${x_1}^\dagger$ shown as a such a tree. The small triangles represent the infinite tree t.



(2) The collection of all finitely branching strongly extensional trees is a cia for $\mathscr{P}_\mathsf{f}$ (see Theorem 4.5.7). The collection of all unordered finitely branching trees is a terminal coalgebra for the bag functor $\mathscr{B}$ (see Example 3.2.10.)

(3) The algebra of addition on the extended natural numbers $\widetilde{\mathbb{N}} = \{\, 1, 2, 3, \ldots \,\} \cup \{\, \infty \,\}$ is a cia for the functor $FX = X \times X$, see Adámek et al. [30].

(4) As shown by Milius [128] a unary algebra $\alpha : A \to A$ (here we consider $F = \mathsf{Id}$ on Set) is a cia if and only if

(a) there exists a unique fixed point $x_0 \in A$ of α, and

(b) if an infinite sequence $y_0, y_1, y_2, \ldots$ fulfils $\alpha(y_{n+1}) = y_n$ $(n < \omega)$ then $y_n = x_0$ for all n.

(5) Recall the corecursive algebra A from Example 7.1.3(4). This algebra A is not a cia: the system $x \approx (x, y)$, $y \approx 2$ has two solutions, corresponding to the facts that both $\alpha(1,2) = 1$ and $\alpha(0,2) = 0$. This point again comes from Capretta et al. [60].

(6) Free corecursive algebras often fail to be cias. For example, for every signature Σ containing a binary operation symbol $*$ a free corecurive algebra on $Y = \{y\}$ (see Example 7.1.13) does not have a unique solution of recursive equation $x \approx x * y$.

We continue with examples of cias obtained from complete metric spaces. Recall from Notation 3.2.2 that CMS is the category of complete metric spaces with distances at most 1 and non-expanding maps. Recall *locally contracting* endofunctors from Definition 5.2.5.

Proposition 7.2.7 [32, Lem. 2.9]**.** *Let $F\colon \mathsf{CMS} \to \mathsf{CMS}$ be a locally contracting endofunctor. Then every nonempty F-algebra is a cia.*

Proof. Suppose that F has a contraction factor $\varepsilon < 1$. Let $\alpha\colon FA \to A$ be a nonempty F-algebra and let $e\colon X \to FX + A$ be a flat equation morphism. Recall that the hom-set $\mathsf{CMS}(X, A)$ is a complete metric space with the supremum metric $d_{X,A}$. Moreover, $\mathsf{CMS}(X, A)$ is nonempty, since A is nonempty. It is clear from Definition 7.2.2 that a solution of e is equivalently a fixed point of the endomap Φ on $\mathsf{CMS}(X, A)$ assigning to every $s\colon X \to A$ the map

$$\Phi(s) = (X \xrightarrow{e} FX + A \xrightarrow{Fs+A} FA + A \xrightarrow{[\alpha,A]} A).$$

We shall now prove that this function is a contraction on $\mathsf{CMS}(X, A)$. Indeed, for two nonexpanding maps $s, t\colon X \to A$ we have

$$\begin{aligned}
d_{X,A}(\Phi s, \Phi t) &= d_{X,A}([\alpha, A] \cdot (Fs + A) \cdot e, [\alpha, A] \cdot (Ft + A) \cdot e) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(definition of } \Phi) \\
&\le d_{FX+A,FA+A}(Fs + A, Ft + A) \quad \text{(composition is nonexpanding)} \\
&= d_{FX,FA}(Fs, Ft) \\
&\le \varepsilon d_{X,A}(s, t) \qquad\qquad\qquad\qquad (F \text{ is } \varepsilon\text{-contracting}).
\end{aligned}$$

By Banach's Fixed Point Theorem 5.2.4, there exists a unique fixed point of Φ. □

Remark 7.2.8. Note that Banach's Fixed Point Theorem yields the unique solution $e^\dagger$ as the limit of a Cauchy sequence as follows. Choose some element $a \in A$ and define a sequence $(e^\dagger_n)_{n\in\mathbb{N}}$ in $\mathsf{CMS}(X, A)$ inductively as follows: let $e^\dagger_0 = \mathsf{const}_a$, and given $e^\dagger_n$ define $e^\dagger_{n+1}$ by the commutativity of the following square

$$\begin{tikzcd}
X \arrow[r, "e^\dagger_{n+1}"] \arrow[d, "e"'] & A \\
FX + A \arrow[r, "Fe^\dagger_n + \mathsf{id}_A"'] & FA + A \arrow[u, "{[\alpha, \mathsf{id}_A]}"']
\end{tikzcd}$$

Then $(e^\dagger_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathsf{CMS}(X, A)$ whose limit is the unique solution of e:

$$e^\dagger = \lim_{n\to\infty} e^\dagger_n.$$

Examples 7.2.9. (1) Many set functors F have a lifting to a locally contracting endofunctor F' on CMS. We have seen this for polynomial endofunctors in Example 5.2.6. We call an F-algebra $\alpha\colon FA \to A$ *completely metrizable* if there exists a complete metric d on A such that the algebra structure is a non-expanding map $\alpha\colon F'(A,d) \to (A,d)$.

Every nonempty completely metrizable F-algebra is a cia. Indeed, to every equation morphism $e\colon X \to FX + A$ its unique solution is the unique solution of $e\colon (X, d_0) \to F'(X, d_0) + (A, d)$ in the F'-algebra (A, α), where d_0 is the discrete metric.

(2) Let $F\colon \mathsf{CMS} \to \mathsf{CMS}$ be given by

$$F(X,d) = (X, \frac{1}{2}d) + (X, \frac{1}{2}d),$$

i.e. F takes a space into the disjoint union of two copies of itself, each equipped with a metric shrinking the distance by half, and with the distance between the copies set to 1. Let $A = [0,1]$, and let $\alpha\colon FA \to A$ be defined by $x \mapsto \frac{x}{2}$ on the left-hand copy, and $x \mapsto \frac{x}{2} + \frac{1}{2}$ on the right-hand one. Then α is clearly a non-expanding map, and hence (A, α) is a cia.

As an illustration, we show how to obtain every real number $r \in [0,1]$ as $e^\dagger(x)$ for some flat equation morphism $e : X \to FX + A$ and $x \in X$. We take $X = \{x_0, x_1, \ldots, x_i, \ldots\}$ as discrete metric space, and we start with r in binary notation as $0.b_1 b_2 b_3 \cdots b_i \cdots$. Our flat equation morphism e is given by the system

$$x_i \approx \frac{b_i}{2} + \frac{1}{2}x_{i+1} \qquad (i \in \mathbb{N}).$$

It is not hard to see that $e^\dagger(x_i) = 0.b_i b_{i+1} b_{i+2} \cdots$ is the solution. So $e^\dagger(x_0)$ is the real number r with which we started.

(3) Proposition 7.2.7 yields further interesting examples of cias, where solutions of recursive equations are fractals. The most basic of these examples is the following one by Milius and Moss [130].

Let A be the set of closed subsets of the interval $[0,1]$. Then A is a complete metric space equipped with the *Hausdorff metric*:

$$d(S,T) = \max\{\sup_{x\in S}\inf_{y\in T} d(x,y), \sup_{y\in T}\inf_{x\in S} d(x,y)\} \quad \text{for closed subsets } S,T \subseteq [0,1].$$

We also consider the functor $F(X,d) = (X \times X, \frac{1}{3}d_{\mathsf{max}})$ sending the complete metric space (X,d) to $X \times X$ equipped with the maximum metric scaled by $\frac{1}{3}$; this functor is clearly locally contracting. Then A is an algebra for F with structure α given by

$$\alpha(S,T) = \frac{1}{3}S \cup \left(\frac{1}{3}T + \frac{2}{3}\right)$$

with the obvious interpretation of addition and multiplication on the closed subsets $S, T \subseteq [0,1]$, e.g. $\frac{1}{3}S = \{\frac{1}{3}s : s \in S\}$. Now let $X = \{*\}$ be the one-point space and $e\colon X \to FX + A$ be given by $e(*) = (*, *)$. Then $e^\dagger(*)$ is the famous Cantor dust.

We now turn to constructions of cias as a preparation of the proof that initial cias and terminal coalgebras are the same.

Lemma 7.2.10. (1) *If* (A, α) *is a cia, then so is* $(FA, F\alpha)$.

(2) *A limit of cias in* $\mathsf{Alg}\, F$ *is a cia. In particular, if* $\mathscr{A}$ *has a terminal object* 1*, the trivial (terminal) algebra* $F1 \to 1$ *is a cia.*

Proof sketch. (1) Let $e\colon X \to FX + FA$ be a flat equation morphism in FA. Form the equation morphism

$$\overline{e} = (X \xrightarrow{e} FX + FA \xrightarrow{FX+\alpha} FX + A),$$

and obtain $\overline{e}^\dagger\colon X \to A$. Using this, define

$$e^\dagger = (X \xrightarrow{e} FX + FA \xrightarrow{[F\overline{e}^\dagger, FA]} FA).$$

It is not difficult to check that $e^\dagger$ is the unique solution of e in FA. For the details, see Milius [128, Proposition 2.6].

(2) Suppose we have a diagram D of cias (A_i, α_i), $i \in I$, and let (A, α) be a limit in the category of algebras for F with limit projections $p_i\colon (A, \alpha) \to (A_i, \alpha_i)$. Recall from Remark 4.1.2 that this limit is formed on the level of $\mathscr{A}$. We need to show that (A, α) is a cia. Let $e\colon X \to FX + A$ be a flat equation morphism. For every $i \in I$ one forms the flat equation morphism

$$e_i = (X \xrightarrow{e} FX + A \xrightarrow{FX+p_i} FX + A_i)$$

and takes the unique solution $e_i^\dagger\colon X \to A_i$. Then the $e_i^\dagger$ $(i \in I)$ form a cone of D. Indeed, for every connecting morphism $h : (A_i, \alpha_i) \to (A_j, \alpha_j)$ we know from Proposition 7.2.4 that h preserves solutions, thus $h \cdot e_i^\dagger = e_j^\dagger$. We define $e^\dagger\colon X \to A$ to be the unique morphism with $p_i \cdot e^\dagger = e_i^\dagger$ for every $i \in I$. Then $e^\dagger$ is a solution of e: indeed, in order to see that (7.5) commutes, extend this by the limit projections p_i to obtain the following diagram:

$$\begin{tikzcd}
X \arrow[r, "e^\dagger"] \arrow[d, "e"'] \arrow[rr, bend left, "e_i^\dagger"] & A \arrow[r, "p_i"] & A_i \\
FX + A \arrow[r, "Fe^\dagger + A"'] \arrow[d, "FX+p_i"'] & FA + A \arrow[u, "{[\alpha, A]}"'] \arrow[dr, "Fp_i + p_i"] & \\
FX + A_i \arrow[rr, "Fe_i^\dagger + A_i"'] & & FA_i + A_i \arrow[uu, "{[\alpha_i, A_i]}"']
\end{tikzcd}$$

Its outside commutes since $e_i^\dagger$ solves $(FX + p_i) \cdot e$, its right-hand part commutes because p_i is a homomorphism, and the lower and upper parts commute by the definition of $e^\dagger$. Thus, the upper left-hand square commutes when extended by p_i, as desired.

For the uniqueness of $e^\dagger$ suppose that $s\colon X \to A$ is any solution of e in A. Then we see that for every $i \in I$, $p_i \cdot s$ is a solution of $(FX + p_i) \cdot e$ (to see this, repeat the reasoning in the 'only if' direction of the proof of Proposition 7.2.4). Thus $p_i \cdot s = e_i^\dagger$, whence $s = e^\dagger$, by the universal property of the limit A. □

Corollary 7.2.11. *Suppose that $\mathscr{A}$ is a complete category. Then in the terminal-coalgebra ω^{op}-chain of F, all algebras $F^i!\colon F(F^i 1) = F^{i+1}1 \to F^i 1$ are cias.*

This holds even if the terminal coalgebra itself does not exist.

We also obtain a version of Lambek's Lemma for cias:

Corollary 7.2.12. *If an initial cia for F exists, then it is a fixed point of F.*

Indeed, the proof is the same as the one for Lambek's Lemma 2.2.5 but using Lemma 7.2.10(1) to see that for an initial cia (I, ι), $(FI, F\iota)$ is a cia, too.

Theorem 7.2.13 [128, Thm. 2.8]**.** *The initial cia is precisely the same as the terminal coalgebra.*

More precisely, let $F\colon \mathscr{A} \to \mathscr{A}$ be an endofunctor. Then (I, ι) is an initial cia for F iff (I, ι^{-1}) is a terminal coalgebra. The proof is somewhat similar to what we saw in Theorem 7.1.7; we include it for the convenience of the reader.

Proof. (1) For a coalgebra $\gamma\colon C \to FC$ and an algebra $\alpha\colon FA \to A$ consider the following diagram:

$$\begin{tikzcd}
C \arrow[r, "h"] \arrow[d, "\gamma"] \arrow[dd, bend right, "e"'] & A \\
FC \arrow[r, "Fh"'] \arrow[d, "\mathsf{inl}"] & FA \arrow[u, "\alpha"'] \arrow[d, "\mathsf{inl}"'] \\
FC + A \arrow[r, "Fh+\mathsf{id}"'] & FA + A \arrow[uu, bend right, "{[\alpha, A]}"']
\end{tikzcd} \tag{7.7}$$

Let us define $e = \mathsf{inl} \cdot \gamma$ so that the left-hand part of this diagram commutes. Notice also that the right-hand part and the lower square of the diagram obviously commute. Now the outside of the diagram commutes iff the upper square does; in other words, h is a solution of γ in the algebra (A, α) iff it is a solution of the flat equation morphism e in that algebra.

(2) Suppose that $\iota : FI \to I$ is an initial cia. Then ι is an isomorphism by Corollary 7.2.12. So we have the coalgebra $\iota^{-1} : I \to FI$, and we need to verify that it is terminal. Indeed, for every coalgebra (C, γ) replace (A, α) in Diagram (7.7) by the algebra (I, ι). Then since this algebra is a cia we have a unique coalgebra-to-algebra homomorphism, i.e. a unique coalgebra homomorphism from (C, γ) to (I, ι^{-1}).

(3) Conversely, suppose that $\tau : \nu F \to F(\nu F)$ is a terminal coalgebra. Then it is a cia by Proposition 7.2.5, and it remains to verify its initiality. So let (A, α) be any cia and let (C, γ) in Diagram (7.7) be $(\nu F, \tau)$. Since A is a cia, we have a unique solution of $e = \mathsf{inl} \cdot \tau$ in A, equivalently a unique F-algebra homomorphism from the cia $(\nu F, \tau^{-1})$ to the cia (A, α). □

Remark 7.2.14. The only fixed point among the cias for a functor is the initial one. This follows from Theorem 7.2.13 and Remark 7.1.8.

More generally, we can characterize *free* cias (cf. Remark 2.2.18). That is, the forgetful functor from the category of all cias for F to the base category $\mathscr{A}$ has a left adjoint whenever all endofunctors $F(-)+Y$ for object Y of $\mathscr{A}$ have a terminal coalgebra:

Theorem 7.2.15 [128, Cor. 2.11]**.** *A free cia for F on Y is precisely the same as a terminal coalgebra for $F(-)+Y$.*

More precisely, given a terminal coalgebra $\tau_Y\colon TY \to FTY+Y$, then extending τ_Y^{-1} by the two coproduct injections yields an algebra structure $\alpha_Y\colon FTY \to TY$ and a morphism $\eta_Y\colon Y \to TY$ which are the structure and universal morphism, respectively, of a free cia on Y. Conversely, given a free cia (TY, α_Y) with universal morphism η_Y then $[\alpha_Y, \eta_Y]\colon FTY+Y \to TY$ is an isomorphism whose inverse is the structure of a terminal coalgebra for $F(-)+Y$.

Remark 7.2.16. In general, corecursive algebras need not be cias, see Example 7.2.6(5) and (6). But are there interesting functors for which the two notions coincide? The answer is "yes" in the case where the base category $\mathscr{A}$ has "very well-behaved" coproducts (this includes examples such as sets, posets, graphs, and presheaf categories) and the endofunctor has the form $FX = A \times X + B$ for fixed objects A and B [28]. Moreover, among finitary set functors, there are essentially no other functors. In the appendix (see **??**) we recall that that for every set functor F its Trnková hull (cf. Proposition 4.4.4) is a *standard* functor which agrees with F on all nonempty sets and functions. Standard means that the functor preserves finite intersections and inclusions.

Theorem 7.2.17 [28]**.** *Let F be a standard finitary set functor. If every corecursive F-algebra is a cia, then F is naturally isomorphic to a functor given by $X \mapsto A \times X + B$.*

Remark 7.2.18. To conclude this section, let us mention some further work on cia's and related structures in the literature.

(1) As we mention in the introduction of this section, one of the reasons why cias are interesting is that one can uniquely solve much more general recursive equations than the flat ones in Definition 7.2.2 in a cia. For example, [4, 128] contain a solution theorem which, when instantiated for a polynomial set functor, states that mutually recursive systems

$$x_i \approx t_i \qquad (i \in I),$$

where I is any set and the right-hand sides t_i are arbitrary (possibly infinite) non-variable Σ-trees have unique solutions.

(2) Assuming the existence of a free cia TX on every object X of a base category $\mathscr{C}$ (see Theorem 7.2.15), it turns out that T is the object assigment of a monad on $\mathscr{C}$. This monad is characterized by a universal property: it is the free completely iterative monad on F. These results appear in [4, 128] generalizing work on (free) completely iterative theories by Elgot, Bloom and Tindell [70].

(3) In a number of settings, equation morphisms do have a solution albeit not a unique

one. Then one is more interested in taking canonical solutions (or fixed points) than requiring their unique existence, e.g., least fixed points in complete partial orders. The idea of Bloom and Ésik's iteration theories [57] is to study the equational properties that characterize least fixed points in complete partial orders. It turns out that similar ideas are important in connection with cias, and they lead to the notion of a *complete Elgot algebra* [32]. A complete Elgot algebra for a functor F is a triple $(A, \alpha, (\_)^\dagger)$ where $\alpha : FA \to A$ is an algebra for F and $(\_)^\dagger$ is an operation taking a flat equation morphism $e : X \to FX + A$ to a solution $e^\dagger : X \to A$ such that two simple and well-motivated properties are satisfied.

Every cia is a complete Elgot algebra. Further examples are continuous algebras (i.e. algebras for locally continuous functors on CPO) or complete lattices, which are complete Elgot algebras for $FX = X \times X$ on Set.

Theorem 7.2.13 can be augmented to state that the terminal coalgebra is the same as the initial Elgot algebra (in addition, the terminal coalgebra for $F(-) + X$ is equivalently, the free complete Elgot algebra for F on X). The main result [32] is that complete Elgot algebras form precisely the category of Eilenberg-Moore algebras for the monad T of item (2) above.

(4) Completely iterative algebras and complete Elgot algebras as in item (3) play a significant rôle in the category-theoretic semantics of recursive program schemes as presented by Milius and Moss [130]. This approach to the semantics of recursive function definitions is based on terminal coalgebras for functors in lieu of concepts from general algebra like signatures and infinite trees. In the category theoretic setting, cias and Elgot algebras serve as those classes of algebras in which one interprets recursive program schemes – in those algebras one can uniquely solve recursive program schemes.

(5) Milius, Moss, and Schwencke [131] demonstrate how Turi and Plotkin's abstract operational rules [146] give rise to new cia structures on the terminal coalgebra. The latter yield theorems that instantiate to several known unique solution theorems in the literature, for example, Milner's solution theorem for CCS [134], and a unique solution theorem for stream circuits [150], as well as new ones for non-well founded sets (extending previous results of Barwise and Moss [52]) or formal languages. Besides, one obtains a modular framework for the specification of operations by abstract operational rules [131]; modularity here means that unique solution theorems are preserved when adding operations specified by abstract operational rules to a given cia structure on the terminal coalgebra.

7.3 Recursive coalgebras

After having looked at algebras with special properties, we now turn to coalgebras with special properties, and we will see the initial algebra appearing as a terminal object in certain categories of coalgebras. The first notion we mention is that of a *recursive coalgebra*, which is formally dual to the notion of a corecursive algebra, thus formulating a unique definition principle for recursive functions. Recursive coalgebras are closely connected to well-founded coalgebras, which we consider in Chapter 8 and which provide a categorical formulation of well-founded induction. Both notions were first studied by Osius in his work [142] on categorical set theory. In fact, he just considered recursive and well-founded coalgebras for the power-set functor on sets (and, more generally, the power-object functor on an elementary topos) and proved the general recursion theorem, that well-founded and recursive coalgebras are the same.

Taylor [160, 159] took Osius' ideas much further, and in particular he proved the general recursion theorem for every set functor preserving inverse images. Recursive coalgebras were also investigated by Eppendahl [71], who called them algebra-initial coalgebras.

Capretta, Uustalu, and Vene [59] studied recursive coalgebras, and they showed how to construct new ones from given ones by using comonads. They also explained nicely how recursive coalgebras allow for the semantic treatment of (functional) divide-and-conquer programs.

In this section we will just recall a few basic results, which are dual to what we have seen in Section 7.1 and Section 7.2 on corecursive algebras, and some examples. We then move on to well-founded coalgebras in the Chapter 8.

For the dual concept of a solution in a corecursive algebras we stick to the standard terminology of an algebra-to-coalgebra morphism:

Definition 7.3.1. A coalgebra $\gamma\colon C \to FC$ is called *recursive* if for every algebra $\alpha\colon FA \to A$ there exists a unique coalgebra-to-algebra morphism $h\colon C \to A$, i.e. a unique morphism such that the square below commutes:

$$\begin{tikzcd} C \arrow[r, "\gamma"] \arrow[d, "h"'] & FC \arrow[d, "Fh"] \\ A & FA \arrow[l, "\alpha"] \end{tikzcd}$$

Examples 7.3.2. (1) The first examples of recursive coalgebras are well-founded relations.

Recall that a binary relation R on a set X is well-founded if there is no infinite descending sequence

$$\cdots R\, x_3\, R\, x_2\, R\, x_1\, R\, x_0.$$

Now a binary relation $R \subseteq X \times X$ is essentially a graph on X, equivalently the coalgebra structure $\alpha\colon X \to \mathscr{P}X$ with $\alpha(x) = \{y \mid y\, R\, x\}$ (cf. Example 1.3.2). Osius [142] showed that for every well-founded relation the associated $\mathscr{P}$-coalgebra is recursive. Shortly: a graph regarded as a coalgebra for $\mathscr{P}$ is recursive iff it has no infinite path.

(2) If μF exists, then it is a recursive coalgebra. This is dual to Example 7.1.3(2).

(3) The initial coalgebra $0 \to F0$ is recursive.

(4) If (C, γ) is recursive so is $(FC, F\gamma)$. This is dual to Proposition 7.1.5.

(5) Every colimit of recursive coalgebras in $\mathsf{Coalg}\, F$ is recursive. This is easy to prove, using that colimits of coalgebras are formed on the level of the underlying category.

(6) It follows from items (3)–(6) that in the initial-algebra chain $(F^i 0 \to F^{i+1}0)_{i \in \mathsf{Ord}}$ all coalgebras are recursive.

Dually to Corollary 7.1.6, we see that a terminal recursive F-coalgebra is a fixed point of F, and we have

Corollary 7.3.3 [59, Prop. 7]**.** *The initial algebra is precisely the same as the terminal recursive coalgebra.*

The dual notion of a completely iterative algebra (see Definition 7.2.2) is called a *parametrically recursive coalgebra* by Capretta et al. [59] (cf. (7.2)). So the dual statement of Theorem 7.2.13 states that the initial algebra is, equivalently, the terminal parametrically recursive coalgebra. Of course, every parametrically recursive coalgebra is recursive. (To see this, form for a given $e\colon FX \to X$ the morphism $e' = e \cdot \pi$, where $\pi\colon FX \times A \to FX$ is the projection.) However, in general the converse fails:

Example 7.3.4 [7]**.** Let $R\colon \mathsf{Set} \to \mathsf{Set}$ be the functor defined by $RX = \{(x, y) \in X \times X : x \neq y\} + \{d\}$ for sets X and for a function $f\colon X \to Y$ put

$$Rf(d) = d \quad \text{and} \quad Rf(x,y) = \begin{cases} d & \text{if } f(x) \neq f(y) \\ (f(x), f(y)) & \text{else.} \end{cases}$$

Now let $C = \{0, 1\}$, and define $\gamma\colon C \to RC$ by $\gamma(0) = \gamma(1) = (0, 1)$. Then (C, γ) is a recursive coalgebra. Indeed, for every algebra $\alpha\colon RA \to A$ the constant map $h\colon C \to A$ with $h(0) = h(1) = \alpha(d)$ is the unique coalgebra-to-algebra morphism.

However, (C, γ) is not parametrically recursive. To see this, consider any morphism $e\colon RX \times \{0, 1\} \to X$ such that RX contains more than one pair (x_0, x_1), $x_0 \neq x_1$ with $e((x_0, x_1), i) = x_i$ for $i = 0, 1$. Then each such pair yields $h\colon C \to X$ with $h(i) = x_i$ making (7.2) commutative. Thus, (C, γ) is not parametrically recursive.

The situation in Example 7.3.4 is relatively rare and artificial because for functors preserving inverse images, recursive and parametrically recursive coalgebras coincide (see Corollary 8.7.5 and Corollary 8.7.10).

We conclude this section with a few examples explaining how recursive coalgebras capture familiar recursive function definitions as well as functional divide-and-conquer programs.

Examples 7.3.5. (1) The functor $FX = X + 1$ has unary algebras with a constant as algebras, and coalgebras for F may be identified with partial unary algebras. As we know from Example 2.2.7(3), the initial algebra is the set natural numbers $\mathbb{N}$ with the successor function and the constant 0. The inverse of the initial F-algebra is the coalgebra given by the partial unary operation $n \mapsto n - 1$ (defined iff $n > 0$). This coalgebra is parametrically recursive. Hence every function

$$e = [u, v]\colon FX \times \mathbb{N} \cong X \times \mathbb{N} + \mathbb{N} \to X$$

defines a unique sequence $e^\dagger\colon \mathbb{N} \to X$, $e^\dagger(n) = x_n$ such that (7.2) commutes. This means that $x_0 = v(0)$ and $x_{n+1} = u(x_n, n+1)$. For example, the factorial function is then given by the choice $X = \mathbb{N}$; $u(n, m) = n \cdot m$ and $v(0) = 1$.

(2) For the set functor F given by $FX = X \times X + 1$, coalgebras $\gamma\colon C \times C + 1$ are deterministic systems with a state set C, a binary input and with halting states (expressed by $\gamma^{-1}(1)$).

The coalgebra $\mathbb{N}$ of natural numbers with halting states 0 and 1 and input structure $\gamma\colon n \mapsto (n-1, n-2)$ for $n \geq 2$ is parametrically recursive (see Example 8.6.9).

For example, to define the Fibonacci sequence starting with $a_0, a_1 \in \mathbb{N}$, consider the morphism $e\colon F\mathbb{N} \times \mathbb{N} \cong \mathbb{N}^3 + \mathbb{N} \to \mathbb{N}$ with

$$e(i,j,k) = i + j \quad \text{and} \quad e(n) = \begin{cases} a_0 & \text{if } n = 0 \\ a_1 & \text{if } n = 1 \\ 0 & \text{if } n \geq 2. \end{cases}$$

We know that there is a unique sequence $e^\dagger : \mathbb{N} \to \mathbb{N}$ such that the diagram (7.2) commutes, which means $e^\dagger(0) = a_0$, $e^\dagger(1) = a_1$ and $e^\dagger(n+2) = e^\dagger(n+1) + e^\dagger(n)$.

(3) Capretta et al. [60] showed how to obtain Quicksort using parametric recursivity. Let A be any linearly ordered set (of data elements). Then Quicksort is usually defined as the recursive function $q : A^* \to A^*$ given by

$$q(\varepsilon) = \varepsilon \qquad \text{and} \qquad q(aw) = q(w_{\leq a}) \star (aq(w_{>a})),$$

where A^* is the set of all lists on A, ε is the empty list, $\star$ is the concatenation of lists and $w_{\leq a}$ denotes the list of those elements of w which are less than or equal to a; analogously for $w_{>a}$.

Now consider the functor $FX = A \times X \times X + 1$ on Set, where $1 = \{\bullet\}$, and form the coalgebra $s\colon A^* \to A \times A^* \times A^* + 1$ given by

$$s(\varepsilon) = \bullet \qquad \text{and} \qquad s(aw) = (a, w_{\leq a}, w_{>a}) \qquad \text{for } a \in A \text{ and } w \in A^*. \tag{7.8}$$

Again, we shall see that this coalgebra is recursive in Example 8.6.9. Thus, for the F-algebra $m\colon A \times A^* \times A^* + 1 \to A^*$ given by

$$m(\bullet) = \varepsilon \qquad \text{and} \qquad m(a, w, v) = w \star (av)$$

there exists a unique function q on A^* such that $q = m \cdot Fq \cdot s$. Notice that the last equation reflects the idea that Quicksort is a divide-and-conquer algorithm. The coalgebra structure s divides a list into two parts $w_{\leq a}$ and $w_{>a}$. Then Fq sorts these two smaller lists, and finally in the combine- (or conquer-)step, the algebra structure m merges the two sorted parts to obtain the desired whole sorted list.

Similarly, functions defined by parametric recursivity (cf. Diagram (7.2)), can be understood as divide-and-conquer algorithms, where the combine-step is allowed to access the original parameter additionally. For instance, in the current example the divide-step $\langle s, id_{A^*}\rangle$ produces the pair consisting of $(a, w_{\leq a}, w_{>a})$ and the original parameter aw, and the combine-step, which is given by an algebra $FX \times A^* \to X$ will, by the commutativity of (7.2), get aw as its right-hand input.

7.4 Summary of this Chapter

Our theme in this chapter were characterizations of the initial algebra as a terminal object in a category of coalgebras and of the terminal coalgebra as an initial object in a category of algebras.

We have presented corecursive and completely iterative algebras (cias), which allow for the unique solution of recursive equation systems, in other words, they allow for the interpretation of definitions by structured corecursion. The terminal coalgebra turns out to be the initial corecursive algebra as well as the initial cia. Dually, the initial algebra is the initial (parametrically) recursive coalgebra. The latter notions allow for the unique interpretation of definitions by structured recursion.

8 Well-Founded Coalgebras

In the last chapter we have seen recursive coalgebras, which have the property that functions out of them may be specified by structured recursion. This is a desirable property, but it may not be straightforward to establish in general. In this chapter we consider *well-foundedness* of coalgebras, which captures well-founded induction and is usually much easier to establish for a given coalgebra. We will present a proof of Taylor's General Recursion Theorem that every well-founded coalgebra is (parametrically) recursive. For set functors preserving inverse images the converse holds, too. Moreover, if an initial algebra exists, well-foundedness of a coalgebra C is equivalent to the existence of a coalgebra-to-algebra morphism from C to μF (Corollary 8.7.5).

Finally, we shall see that for every set functor, the initial algebra is characterized as the terminal well-founded coalgebra. This is connected to the theme of the previous chapter, where we have seen in Corollary 7.3.3 that the initial algebra is the terminal (parametric) recursive coalgebra.

In Chapter 9 we will see that well-founded coalgebras lead to a surprising description of the initial algebra for a set functor preserving intersections.

On a more technical note, we shall once again make use of *transfinite iteration* that we already studied in Chapter 6. Here we turn to a somewhat different use of it, when prove the General Recursion Theorem 8.6.7 in Section 8.6.

8.1 Well-Founded Coalgebras and Well-Founded Graphs

The concept of well-foundedness is well-known for directed graphs: it means that the graph has no infinite directed paths. Similarly for relations: for example, the elementhood relation $\in$ of set theory is well-founded; this is precisely the Foundation Axiom.

Taylor [160, Def. 6.2.3] gave a more general category theoretic formulation of well-foundedness. His definition can be presented based on the concept of subcoalgebra; recall that this means a coalgebra homomorphism into the given coalgebra carried by a monomorphism in the base category (see Remark 2.4.11). For example, consider a graph as a coalgebra $\alpha\colon A \to \mathscr{P}A$ for the power-set functor (see Example 1.3.2). A subcoalgebra is a subset $m\colon B \hookrightarrow A$ such that with every vertex it contains all the neighbors. The coalgebra structure $\beta\colon B \to \mathscr{P}B$ is then the domain-codomain restriction of α. We call a subcoalgebra $m\colon B \hookrightarrow A$ *cartesian* if the square below expressing that m

is a homomorphism is a pullback:

$$\begin{tikzcd} B \arrow[r, "\beta"] \arrow[d, tail, "m"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathscr{P}B \arrow[d, tail, "\mathscr{P}m"] \\ A \arrow[r, "\alpha"] & \mathscr{P}A \end{tikzcd}$$

(Being a pullback is indicated by the "corner" symbol.) This means that

$$\text{whenever a vertex of } A \text{ has all neighbors in } B\text{, it also lies in } B. \tag{8.1}$$

This implies that a graph is well-founded iff no proper subcoalgebra is cartesian. That is, whenever the above square is a pullback, m is an isomorphism.

Definition 8.1.1. Let $\alpha\colon A \to FA$ be a coalgebra.

(1) A subcoalgebra $m\colon (B,\beta) \rightarrowtail (A,\alpha)$ is *cartesian* provided that the square below is a pullback.

$$\begin{tikzcd} B \arrow[r, "\beta"] \arrow[d, tail, "m"'] \arrow[dr, phantom, "\lrcorner", very near start] & FB \arrow[d, tail, "Fm"] \\ A \arrow[r, "\alpha"] & FA \end{tikzcd} \tag{8.2}$$

(2) (A,α) is called *well-founded* if it has no proper cartesian subcoalgebra.

Lemma 8.1.2 [160, Example 6.3.3]**.** *The well-founded coalgebras for the power-set functor are precisely the graphs without infinite paths.*

Proof. Let $\alpha\colon A \to \mathscr{P}A$ be a graph without any infinite paths. Given a cartesian subcoalgebra $m\colon (B,\beta) \hookrightarrow (A,\alpha)$, we prove that $B = A$ by contradiction: every node $a_0 \in A \setminus B$ has, by (8.1), a neighbor $a_1 \in A \setminus B$. By the same argument, a_1 has neighbor $a_2 \in A \setminus B$, etc. This yields an infinite path $a_0, a_1, a_2, \ldots$, which is a contradiction.

Conversely, if (A,α) is a well-founded coalgebra, then there is no infinite path. Indeed, the subset $B \subseteq A$ of all vertices that do not lie on an infinite path satisfies (8.1). Thus, B is a cartesian subcoalgebra of A, whence $B = A$. $\square$

Examples 8.1.3. (1) If an initial algebra μF exists, then (considered as a coalgebra) it is well-founded. Indeed, in every pullback (8.2), since α is invertible, so is β. The unique algebra homomorphism from μF to the algebra $\beta^{-1} : FB \to B$ is clearly inverse to m.

(2) If a set functor F fulfils $F\emptyset = \emptyset$, then the only well-founded coalgebra is the empty one. Indeed, this follows from the fact that the empty coalgebra is a cartesian subcoalgebra of every coalgebra for F.

For example, a deterministic automaton, as a coalgebra for $FX = \{0,1\} \times X^{\Sigma}$ (see Example 2.4.2(3)), is well-founded iff it is empty.

(3) Non-deterministic automata were discussed in Example 5.1.27 as coalgebras for an endofunctor on Rel. But they can also be considered as coalgebras for the set functor $FX = \{0,1\} \times (\mathscr{P}X)^{\Sigma}$. An F-coalgebra is well-founded iff the state transition graph of the corresponding non-deterministic automaton is well-founded (i.e. has no infinite path). This also follows immediately from Corollary 8.3.10 below.

We next show that to every coalgebra for a set functor F one may associate a graph, in a canonical way. Moreover, if F preserves intersections, then a coalgebra is well-founded if and only if so is its canonical graph.

Notation 8.1.4. Given a set functor F, we define for every set X the map $\tau_X\colon FX \to \mathscr{P}X$ assigning to every element $x \in FX$ the intersection of all subsets $m\colon M \hookrightarrow X$ such that x lies in the image of Fm:

$$\tau_X(x) = \bigcap\{m \mid m\colon M \hookrightarrow X \text{ satisfies } x \in Fm[FM]\}. \tag{8.3}$$

Definition 8.1.5. Let F be a set functor. For every coalgebra $\alpha\colon A \to FA$ its *canonical graph* is the following coalgebra for $\mathscr{P}$:

$$A \xrightarrow{\alpha} FA \xrightarrow{\tau_A} \mathscr{P}A.$$

Examples 8.1.6. (1) Given a graph as a coalgebra $\alpha\colon A \to \mathscr{P}A$, the condition $\alpha(x) \in \mathscr{P}m[\mathscr{P}M]$ states precisely that all successors of x lie in the set M. The least such set is $\alpha(x)$. Therefore, the canonical graph of (A, α) is itself (see [160, Example 6.3.3]).
(2) For the type functor of $FX = \{0,1\} \times X^\Sigma$ of deterministic automata, we have

$$\tau_X(i,t) = \{t(s) : s \in \Sigma\} \qquad \text{for } i = 0,1 \text{ and } t\colon \Sigma \to X.$$

Thus, the canonical graph of a deterministic automaton A is precisely its state transition graph (forgetting the labels of transitions and the finality of states), i.e. we have an edge (a, a') iff $a' = \delta(a, s)$ for some $s \in \Sigma$, where δ is the nextstate function of A.

Similarly, for the type functor $FX = \{0,1\} \times (\mathscr{P}X)^\Sigma$ of non-deterministic automata we have

$$\tau_X(i,g) = \bigcup_{s\in\Sigma} t(s) \qquad \text{for } i = 0,1 \text{ and } t\colon \Sigma \to \mathscr{P}X.$$

(3) For the functor $FX = \mathscr{P}(\Sigma \times X)$ whose coalgebras are labeled transition systems we have

$$\tau_X = (\mathscr{P}(\Sigma \times X) \xrightarrow{\mathscr{P}\pi_X} \mathscr{P}X),$$

where $\pi_X\colon \Sigma \times X \to X$ is the projection. Again, the canonical graph of a labelled transition system is its state transition graph. Thus (a, a') is an edge iff some action leads from state a to a'.

Recall that a functor *preserves intersections* if it preserves (wide) pullbacks of families of monomorphisms. Gumm [88, Theorem 7.3] observed that for a set functor preserving intersections, the maps $\tau_X\colon FX \to \mathscr{P}X$ in (8.3) form a "subnatural" transformation from F to the power-set functor $\mathscr{P}$. Subnaturality means that (although these maps do not form a natural transformation in general) for every monomorphism $i\colon X \to Y$ we have a commutative square:

$$\begin{tikzcd} FX \arrow[r, "\tau_X"] \arrow[d, tail, "Fi"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathscr{P}X \arrow[d, tail, "\mathscr{P}i"] \\ FY \arrow[r, "\tau_Y"] & \mathscr{P}Y \end{tikzcd} \tag{8.4}$$

For many set functors this is even a pullback square:

Theorem 8.1.7 [88, Thm. 7.4] and [159, Prop. 7.5]**.** *A set functor F preserves intersections iff the above squares* (8.4) *are pullbacks.*

Remark 8.1.8. The condition that F preserves intersections is an extremely mild one for set functors. We discuss this further in **??**; see especially **??** on page **??**. Furthermore, we have some additional points:

(1) "Almost" all finitary set functors preserve intersections. In fact, we saw in Corollary 4.4.6 that for a finitary set functor F its Trnková hull $\bar{F}$, which is identical with F on all nonempty sets and functions, preserves all intersections.

(2) The collection of set functors which preserve intersections is closed under products, coproducts, and compositions. A subfunctor $m\colon G \rightarrowtail F$ of an intersection preserving functor F preserves intersections whenever m is a cartesian natural transformation, i.e. all naturality squares are pullbacks:

$$\begin{tikzcd} GX \arrow[r, tail, "m_X"] \arrow[d, "Gf"'] \arrow[dr, phantom, very near start, "\lrcorner"] & FX \arrow[d, "Ff"] \\ GY \arrow[r, tail, "m_Y"] & FY \end{tikzcd}$$

(3) All the functors in Example 8.1.6 preserve intersections.

Theorem 8.1.9 [88, Thm. 8.1]**.** *Let F be a set functor which preserves inverse images and intersections. Then $\tau\colon F \to \mathscr{P}$ is a natural transformation.*

Example 8.1.10. To see that τ is not a natural transformation in general, we use the set functor R from Example 7.3.4. Let $X = \{0, 1\}$, $Y = \{0\}$, and $f\colon X \to Y$ the evident function. Then $(0, 1) \in FX$, and $\tau_X(0, 1) = X$. Further, $\mathscr{P}f(X) = Y$. But $Rf(0, 1) = d$, and $\tau_Y(d) = \emptyset$.

Remark 8.1.11. Taylor [160, Rem. 6.3.4] proved that, for functors preserving intersections and inverse images, a coalgebra is well-founded iff its canonical graph is so. We shall see in Corollary 8.3.10 that the proof only needs preservation of intersections.

Examples 8.1.12. (1) A coalgebra for the identity functor $FX = X$ on Set is a set A equipped with a function $\alpha\colon A \to A$. The canonical graph of (A, α) is the graph of the function α, i.e. the graph with edges $(a, \alpha(a))$ for all $a \in A$. Hence, (A, α) is well-founded iff it is empty (see Example 8.1.3(2)).

(2) For $FX = X + 1$ coalgebras are sets A equipped with a *partial* function $\alpha : A \to A$, and the canonical graph is the graph of α. This functor has many nonempty well-founded coalgebras. For example, the initial F-algebra, considered as the coalgebra on $\mathbb{N}$ with the structure given by the partial function $n \mapsto n - 1$, for $n > 0$ (cf. Example 7.3.5(1)), is well-founded since its canonical graph is so.

(3) Consider the functor $FX = X \times X + 1$ and a coalgebra $\alpha\colon A \to A \times A + 1$. The edges in its canonical graph are all of the pairs (a, a_1) and (a, a_2) such that $a \in A$ and $\alpha(a) = (a_1, a_2)$. For example, the coalgebra $(\mathbb{N}, \gamma)$ from Example 7.3.5(2) has the

canonical graph with edge set $\{(n, n-1), (n, n-2) : n \geq 2\}$, which is clearly well-founded, and therefore so is the coalgebra.

Similarly, for the functor $FX = A \times X \times X + 1$, the coalgebra (A^*, s) in Example 7.3.5(3) is easily seen to be well-founded via its canonical graph. Indeed, this graph has for every list w one outgoing edge to the list $w_{\leq a}$ and one to $w_{>a}$ for every $a \in A$. Hence, this is a well-founded graph, and therefore (A^*, s) is a well-founded coalgebra.

(4) More generally, for a polynomial functor $H_\Sigma \colon \mathsf{Set} \to \mathsf{Set}$ associated to a finitary signature Σ, a coalgebra $\alpha \colon A \to \coprod_{n \in \mathbb{N}} \Sigma_n \times A^n$ has the canonical graph where every vertex $a \in A$ has an outgoing edge (a, a') for every $a' \in A$ occuring in the tuple $\alpha(a) \in \Sigma_n \times A^n$ for some $n < \omega$.

Thus, the coalgebra (A, α) is well-founded iff for every $a \in A$ its tree-unfolding, i.e. its image under the unique homomorphism $h : A \to \nu F$, is a finite Σ-tree.

In particular, if the signature Σ does not contain any constant symbols, then the only well-founded H_Σ-coalgebra is $A = \emptyset$.

For further use we now compare well-founded and recursive coalgebras for a given set functor F with those of its Trnková hull $\bar{F}$ (see Proposition 4.4.4). Since empty coalgebras are (trivially) well-founded and recursive, we can restrict ourselves to the nonempty ones. Observe that $\mathsf{Coalg}\, F$ and $\mathsf{Coalg}\, \bar{F}$ have the same nonempty objects, and these categories are isomorphic.

Lemma 8.1.13. *Let (A, α) be a nonempty coalgebra for a set functor F. If it is well-founded or (parametrically) recursive, then it also has those properties as a coalgebra for the Trnková hull $\bar{F}$.*

Proof. (1) Let (A, α) be well-founded for F. Nonempty subcoalgebras of (A, α) for F and for $\bar{F}$ coincide. Thus, we only need to show that the left-hand square below, where $r_X \colon \emptyset \to X$ denotes the empty map, is not a pullback:

$$
\begin{tikzcd}
\emptyset \arrow[r, "r_{\bar{F}\emptyset}"] \arrow[d, "r_A"'] & \bar{F}\emptyset \arrow[d, "\bar{F}r_A"] \\
A \arrow[r, "\alpha"] & \bar{F}A = FA
\end{tikzcd}
\qquad
\begin{tikzcd}
\emptyset \arrow[r, "r_{F\emptyset}"] \arrow[d, "r_A"'] & F\emptyset \arrow[d, "Fr_A"] \\
A \arrow[r, "\alpha"] & FA
\end{tikzcd}
$$

Since (A, α) is well-founded, we know that the right-hand square is not a pullback. Thus, there exist $a \in A$ and $x \in F\emptyset$ with $\alpha(a) = Fr_A(x)$. For the functor C_{01} of **??**, define a natural transformation $\tau \colon C_{01} \to F$ by $\tau_X = Fr_X(x) \in FX$. Then $\alpha(a) = \tau_A$, and we know that τ lies in $\bar{F}\emptyset$ so that $\tau_A = \bar{F}r_A(\tau)$. Consequently, we have $\alpha(a) = \bar{F}r_A(\tau)$, which proves that the above left-hand square is not a pullback.

(2) Let (A, α) be a nonempty recursive coalgebra for F. Given an algebra $e \colon \bar{F}X \to X$ we know that $X \neq \emptyset$, for otherwise the existence of a unique coalgebra-to-algebra morphism $A \to X$ would force A to be empty. But then the unique coalgebra-to-algebra morphism from (A, α) to (X, e) w.r.t. F is also one for $\bar{F}$. $\square$

8.2 Factorization of Coalgebra Homomorphisms

Before we continue our study of well-founded coalgebras, we shortly digress and consider factorizations of coalgebra homomomorphisms inherited from the factorizations in the base category.

Assumption 8.2.1. Throughout the remainder of this chapter we work with a category $\mathscr{A}$ which is complete and well-powered. We also assume that $F\colon \mathscr{A} \to \mathscr{A}$ preserves monomorphisms. (See Remark 8.2.6 for a weakening of the last assumption.)

Remark 8.2.2. (1) Recall that an epimorphism $e\colon A \to B$ is called *strong* if it satisfies the following *diagonal fill-in property*: given a monomorphism $m\colon C \rightarrowtail D$ and morphisms $f\colon A \to C$ and $g\colon B \to D$ such that $m \cdot f = g \cdot e$ (i.e. the outside of the square below commutes) then there exists a unique $d\colon B \to C$ such that the diagram below commutes:

$$\begin{tikzcd} A \arrow[r, two heads, "e"] \arrow[d, "f"'] & B \arrow[d, "g"] \arrow[dl, dashed, "d"'] \\ C \arrow[r, tail, "m"] & D \end{tikzcd} \tag{8.5}$$

(2) Our category $\mathscr{A}$ has factorizations of morphisms f as $f = m \cdot e$, where e is a strong epimorphism and m is a monomorphism. This follows from Adámek et al. [20, Theorem 14.17 and dual of Exercise 14C(d)]. We call the subobject m the *image* of f.

(3) We indicate monomorphisms by $\rightarrowtail$ and strong epimorphisms by $\twoheadrightarrow$.

It follows from a result in Kurz' thesis [118, Prop. 1.3.6] that factorizations of morphisms lift to coalgebras:

Proposition 8.2.3 ($\mathsf{Coalg}\,F$ inherits factorizations from $\mathscr{A}$)**.** *Since F preserves monomorphisms, the category $\mathsf{Coalg}\,F$ has factorizations of homomorphisms f as $f = m \cdot e$, where e is carried by a strong epimorphism and m by a monomorphism in $\mathscr{A}$.*

Proof. Let $f\colon (A, \alpha) \to (B, \beta)$ be a coalgebra homomorphism and take its (strong epi, mono)-factorization $f = m \cdot e$ in $\mathscr{A}$. Now consider the diagram below:

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, two heads, "e"'] & FA \arrow[d, "Fe"] \\ C \arrow[r, dashed, "\gamma"] \arrow[d, tail, "m"'] & FC \arrow[d, tail, "Fm"] \\ B \arrow[r, "\beta"'] & FB \end{tikzcd} \tag{8.6}$$

Since Fm is monomorphic we can use the diagonal fill-in property to obtain a unique coalgebra structure γ such that e and m are coalgebra homomorphisms. $\square$

Definition 8.2.4. We call (C, γ) in (8.6) above the *image* of (A, α) under f.

Remark 8.2.5. (1) Observe that the diagonal fill-in property in Remark 8.2.2(1) lifts to coalgebra homomorphisms. This shows that we obtain a factorization system on $\mathsf{Coalg}\, F$. In fact, if m and g in (8.5) are coalgebra homomorphisms, then so is d. To see this we use that F preserves monomorphisms and consider the diagram below:

$$\begin{tikzcd}
B \arrow[r, "\beta"] \arrow[d, "d"'] \arrow[dd, bend right=60, "g"'] & FB \arrow[d, "Fd"] \arrow[dd, bend left=60, "Fg"] \\
C \arrow[r, "\gamma"] \arrow[d, tail, "m"'] & FC \arrow[d, tail, "Fm"] \\
D \arrow[r, "\delta"] & FD
\end{tikzcd}$$

Its outside, left-hand, and right-hand parts commute. Thus so does the upper square when extended by the monomorphism Fm, which implies that it commutes.

(2) Note that Proposition 8.2.3 generalizes to an arbitrary factorization system $(\mathcal{E}, \mathcal{M})$ on the base category and endofunctors F such that $m \in \mathcal{M}$ implies $Fm \in \mathcal{M}$. However, we are not going to work in this generality.

Remark 8.2.6. (1) Note that for $\mathscr{A} = \mathsf{Set}$ the condition that F preserve monomorphisms may be dropped in Proposition 8.2.3. In fact, every set functor preserves all nonempty monomorphisms, and whenever A is nonempty, so is C, whence Fm is monomorphic. If A empty, then so is C, thus γ is the empty map.

(2) Milius et al. [133, Lemma 2.5] show that this argument can easily be generalized. Recall [62] that an initial object 0 is called *strict* if every morphism $I \to 0$ is an isomorphism. A nonempty monomorphism in a category $\mathscr{A}$ is a monomorphism whose domain is not a strict initial object. Proposition 8.2.3 (and in fact all the results of this section) then hold for endofunctors F on $\mathscr{A}$ preserving nonempty monomorphisms. Note that in categories not having (strict) initial objects, e.g. the categories of groups or vector spaces over a field, all monomorphisms are nonempty. In such categories, preservation of nonempty monomorphisms is the same as mono-preservation. However, in every category of algebras containing the empty algebra all functors lifted from Set preserve nonempty monomorphisms.

8.3 The Next Time Operator on Coalgebras

A compact characterization of well-foundedness can be given by using Jacobs' next time operator for coalgebras [97]. He defined this operator for so-called Kripke polynomial functors on sets; intuitively it is the semantic counterpart of the well-known next time modality in temporal logic (see e.g. Manna and Pnüeli [126]). We generalize the next time operator to arbitrary functors and investigate its properties. In the next section, we will see that a coalgebra is well-founded iff it has no proper subcoalgebra as a fixed point of the next time operator (Proposition 8.4.5).

Notation 8.3.1. For every object A, we denote by $\mathsf{Sub}(A)$ the poset of subobjects of A (cf. Remark 6.1.21). The top of this poset is represented by id_A, and the bottom $\perp_A$ is the intersection of all subobjects of A.

Remark 8.3.2. Note that $\mathsf{Sub}(A)$ is a complete lattice: it is small since $\mathscr{A}$ is well-powered, and a meet of subobjects $m_i\colon A_i \rightarrowtail A$, $i \in I$, is their intersection, obtained by forming their wide pullback. It follows that $\mathsf{Sub}(A)$ has all joins as well.

We shall need that forming inverse images, i.e. pulling back along a morphism, is a right adjoint.

Notation 8.3.3. For every morphism $f\colon B \to A$ we have two operators:

(1) The inverse image operator

$$\overleftarrow{f}\colon \mathsf{Sub}(A) \to \mathsf{Sub}(B),$$

assigning to every subobject $s\colon S \rightarrowtail A$ its inverse image under f obtained by the following pullback

$$\begin{tikzcd} P \arrow[r] \arrow[d, tail, "\overleftarrow{f}(s)"'] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[d, tail, "s"] \\ B \arrow[r, "f"] & A \end{tikzcd}$$

(2) The (direct) image operator

$$\overrightarrow{f}\colon \mathsf{Sub}(B) \to \mathsf{Sub}(A),$$

assigning to every subobject $t\colon T \rightarrowtail B$ the image of $f \cdot t$:

$$\begin{tikzcd} T \arrow[r, two heads] \arrow[d, tail, "t"'] & S \arrow[d, tail, "\overrightarrow{f}(t)"] \\ B \arrow[r, "f"] & A \end{tikzcd}$$

Remark 8.3.4. (1) A monotone map $r\colon X \to Y$ between posets, regarded as a functor from X to Y considered as categories, is a right adjoint iff there exists a monotone map $\ell\colon Y \to X$ such that

$$\ell(y) \le x \qquad \text{iff} \qquad y \le r(x) \qquad \text{for every } x \in X \text{ and } y \in Y.$$

(2) A monotone map $r\colon \mathsf{Sub}(B) \to \mathsf{Sub}(A)$ is a right adjoint iff it preserves intersections. Indeed, the necessity follows since right adjoints preserve limits. For the sufficiency, suppose that r preserves intersections, and define $\ell\colon \mathsf{Sub}(A) \to \mathsf{Sub}(B)$ by

$$\ell(m) = \bigwedge_{m \le r(m)} m \qquad \text{for every } m \in \mathsf{Sub}(A).$$

Then ℓ is clearly monotone, and for every m' in $\mathsf{Sub}(B)$ we have

$$\ell(m) \le m' \qquad \text{iff} \qquad m \le r(m').$$

Thus, ℓ is the desired left adjoint of r.

Proposition 8.3.5. *For every morphism $f\colon B \to A$ we have an adjoint situation:*

$$\begin{tikzcd} \mathsf{Sub}(A) \arrow[r, phantom, "\perp"] \arrow[r, shift right=1.5ex, "\overleftarrow{f}"'] & \mathsf{Sub}(B). \arrow[l, shift right=1.5ex, "\overrightarrow{f}"'] \end{tikzcd}$$

In other words: $\overrightarrow{f}(t) \le s$ iff $t \le \overleftarrow{f}(s)$ for all subobjects $s\colon S \rightarrowtail A$ and $t\colon T \rightarrowtail B$.

Proof. In order to see this we consider the following diagram:

$$\begin{tikzcd} T \arrow[rrr, two heads, "e"] \arrow[dr, dashed, tail] \arrow[ddr, tail, "t"'] & & & I \arrow[dl, dashed, tail] \arrow[ddl, tail, "\overrightarrow{f}(t)"] \\ & P \arrow[r] \arrow[d, tail, "\overleftarrow{f}(s)"'] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[d, tail, "s"] & \\ & B \arrow[r, "f"] & A & \end{tikzcd}$$

By the universal property of the lower middle pullback square and the diagonal fill-in property, we have the dashed morphism on the left iff we have the one on the right. Thus, $t \le \overleftarrow{f}(s)$ iff $\overrightarrow{f}(t) \le s$, as desired. $\square$

We now come to the main definition in this section. For Kripke polynomial functors on sets this was presented by Jacobs [97].

Definition 8.3.6. Every coalgebra $\alpha\colon A \to FA$ induces an endofunction $\bigcirc$ on $\mathsf{Sub}(A)$, called the *next time operator*

$$\bigcirc\colon \mathsf{Sub}(A) \to \mathsf{Sub}(A), \qquad \bigcirc(s) = \overleftarrow{\alpha}(Fs) \quad \text{for } s \in \mathsf{Sub}(A).$$

In more detail: we define $\bigcirc s$ and $\alpha(s)$ by the following pullack:

$$\begin{tikzcd} \bigcirc S' \arrow[r, "\alpha(s)"] \arrow[d, tail, "\bigcirc s"'] \arrow[dr, phantom, "\lrcorner", very near start] & FS \arrow[d, tail, "Fs"] \\ A \arrow[r, "\alpha"] & FA \end{tikzcd} \tag{8.7}$$

Since Fs is a monomorphism, $\bigcirc s$ is a monomorphism and $\alpha(s)$ is (for every representation $\bigcirc s$ of that subobject of A) uniquely determined.

Example 8.3.7. (1) Let A be a graph, considered as a coalgebra for $\mathscr{P}\colon \mathsf{Set} \to \mathsf{Set}$. If $S \subseteq A$ is a set of vertices, then $\bigcirc S$ is the set of vertices all of whose successors belong to S.

(2) For the set functor $FX = \mathscr{P}(\Sigma \times X)$ expressing labelled transition systems the operator $\bigcirc$ for a coalgebra $\alpha\colon A \to FA$ is the semantic counterpart of the next time operator of classical linear temporal logic, see e.g. Manna and Pnüeli [126]. In fact, for a labelled transition system $\alpha\colon A \to \mathscr{P}(\Sigma \times A)$ and a subset $S \hookrightarrow A$ we have that $\bigcirc S$ consists of those states whose next states lie in S, in symbols:

$$\bigcirc S = \{x \in A \mid (s, y) \in \alpha(x) \text{ implies } y \in S, \text{ for all } s \in \Sigma\}.$$

Lemma 8.3.8. *The next time operator $\bigcirc$ is monotone: if $m \le n$, then $\bigcirc m \le \bigcirc n$.*

Proof. Suppose that $m\colon A' \rightarrowtail A$ and $n\colon A'' \rightarrowtail A$ are subobjects such that $m \le n$, i.e. $n \cdot x = m$ for some $x\colon A' \rightarrowtail A''$. Then we obtain the dashed arrow in the diagram below using that its lower square is a pullback:

$$
\begin{tikzcd}
\bigcirc A' \arrow[r, "\alpha(m)"] \arrow[d, dashed] \arrow[dd, tail, bend right=60, "\bigcirc m"'] & FA' \arrow[d, "Fx"] \arrow[dd, tail, bend left=60, "Fm"] \\
\bigcirc A'' \arrow[r, "\alpha(n)"] \arrow[d, tail, "\bigcirc n"'] \arrow[dr, phantom, "\lrcorner", very near start] & FA'' \arrow[d, tail, "Fn"] \\
A \arrow[r, "\alpha"] & FA
\end{tikzcd}
$$

This shows that $\bigcirc m \le \bigcirc n$. ☐

Lemma 8.3.9. *For every set functor F preserving intersections, the next time operator of a coalgebra (A, α) coincides with that of its canonical graph (see Definition 8.1.5).*

Proof. In the diagram below the outside is a pullback if and only if so is the left-hand square:

$$
\begin{tikzcd}
\bigcirc A' \arrow[r, dashed, "\alpha(m)"] \arrow[d, tail, "\bigcirc m"'] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[rr, bend left, "\tau_{A'} \cdot \alpha(m)"] & FA' \arrow[r, "\tau_{A'}"] \arrow[d, "Fm"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathscr{P}A' \arrow[d, "\mathscr{P}m"] \\
A \arrow[r, "\alpha"] & FA \arrow[r, "\tau_A"] & \mathscr{P}A
\end{tikzcd}
$$

☐

Corollary 8.3.10 [160, Rem. 6.3.4]**.** *A coalgebra for a set functor preserving intersections is well-founded iff its canonical graph is a well-founded graph.*

The following lemma will be useful when we establish the universal property of the well-founded part of a coalgebra in the next section.

Lemma 8.3.11. *For every coalgebra homomorphism $f\colon (B, \beta) \to (A, \alpha)$ we have*

$$\bigcirc_\beta \cdot \overleftarrow{f} \le \overleftarrow{f} \cdot \bigcirc_\alpha,$$

where $\bigcirc_\alpha$ and $\bigcirc_\beta$ denote the next time operators of the coalgebras (A, α) and (B, β), respectively.

Proof. Let $s\colon S \rightarrowtail A$ be a subobject. We see that $\overleftarrow{f}(\bigcirc_\alpha s)$ is obtained by pasting two pullback squares as shown below:

$$
\begin{tikzcd}
T \arrow[r, "t"] \arrow[d, tail, "\overleftarrow{f}(\bigcirc_\alpha s)"'] \arrow[dr, phantom, "\lrcorner", very near start] & \bigcirc_\alpha S \arrow[r, "\alpha(s)"] \arrow[d, tail, "\bigcirc_\alpha s"'] \arrow[dr, phantom, "\lrcorner", very near start] & FS \arrow[d, tail, "Fs"] \\
B \arrow[r, "f"] & A \arrow[r, "\alpha"] & FA
\end{tikzcd}
\tag{8.8}
$$

In order to show that $\bigcirc_\beta(\overleftarrow{f}(s)) \leq \overleftarrow{f}(\bigcirc_\alpha s)$, we consider the following diagram:

$$
\begin{tikzcd}
\bigcirc_\beta U \arrow[r, "\beta(\overleftarrow{f}(s))"] \arrow[dd, tail, "\bigcirc_\beta(\overleftarrow{f}(s))"'] \arrow[dr, phantom, "\lrcorner", very near start] & FU \arrow[r, "Fu"] \arrow[d, tail, "F(\overleftarrow{f}(s))"'] & FS \arrow[dd, tail, "Fs"] \\
 & FB \arrow[dr, "Ff"] & \\
B \arrow[ur, "\beta"] \arrow[r, "f"] & A \arrow[r, "\alpha"] & FA
\end{tikzcd}
\tag{8.9}
$$

The upper left-hand part is the pullback square defining $\bigcirc_\beta(\overleftarrow{f}(s))$, and the upper right-hand one is that defining $\overleftarrow{f}(s)$, with F applied. On the bottom, we use that f is a coalgebra homomorphism. Thus, the outside of the diagram commutes. Since the outside of the diagram in (8.8) is a pullback, we have some $g\colon \bigcirc_\beta U \to T$ such that $\bigcirc_\beta(\overleftarrow{f}(s)) = \overleftarrow{f}(\bigcirc_\alpha(s)) \cdot g$, which proves the desired inequality. $\square$

Corollary 8.3.12. *For every coalgebra homomorphismus $f\colon (B, \beta) \to (A, \alpha)$ we have $\bigcirc_\beta \cdot \overleftarrow{f} = \overleftarrow{f} \cdot \bigcirc_\alpha$ provided that either*

(1) *f is a monomorphism in $\mathscr{A}$ and F preserves finite intersections, or*

(2) *F preserves inverse images.*

Proof. Indeed, under the above conditions the upper right-hand part in Diagram (8.9) is a pullback. Thus, pasting this part with the pullback in the upper left of (8.9) and using that the lower part commutes we see that $\bigcirc_\beta(\overleftarrow{f}(s))$ is obtained by pulling back Fs along $f \cdot \alpha$. This implies the desired equality since this is how $\overleftarrow{f}(\bigcirc_\alpha s)$ is obtained (see (8.8)). $\square$

Lemma 8.3.13. *Let $\alpha\colon A \to FA$ be a coalgebra and $m\colon B \rightarrowtail A$ be a monomorphism.*

(1) *There is a coalgebra structure $\beta\colon B \to FB$ for which m gives a subcoalgebra of (A, α) iff $m \leq \bigcirc m$.*

(2) *There is a coalgebra structure $\beta\colon B \to FB$ for which m gives a cartesian subcoalgebra of (A, α) iff $m = \bigcirc m$.*

Proof. We prove the left-to-right directions of both assertions first, and then the right-to-left ones.

Suppose first that there exists $\beta\colon B \to FB$ such that $m\colon (B, \beta) \rightarrowtail (A, \alpha)$ is a coalgebra morphism. Then the fact that $\bigcirc B$ is given by a pullback yields a morphism $x\colon B \rightarrowtail \bigcirc B$ such that, inter alia, $\bigcirc m \cdot x = m$. It follows that $m \leq \bigcirc m$. If (B, β) is a cartesian subcoalgebra, then we have a pullback square

$$
\begin{tikzcd}
B \arrow[r, "\beta"] \arrow[d, tail, "m"'] \arrow[dr, phantom, "\lrcorner", very near start] & FB \arrow[d, "Fm"] \\
A \arrow[r, "\alpha"] & FA
\end{tikzcd}
$$

So clearly $m = \bigcirc m$ in $\mathsf{Sub}(A)$.

Conversely, suppose that $m \leq \bigcirc m$ via $x\colon B \rightarrowtail \bigcirc B$. Then $\alpha(m) \cdot x\colon B \to FB$ is a coalgebra, and $m\colon B \rightarrowtail A$ is a homomorphism:

$$\begin{tikzcd}
B \arrow[r, "x"] \arrow[dr, tail, "m"'] & \bigcirc B \arrow[r, "\alpha(m)"] \arrow[d, tail, "\bigcirc m"] \arrow[dr, phantom, very near start, "\lrcorner"] & FB \arrow[d, "Fm"] \\
& A \arrow[r, "\alpha"] & FA
\end{tikzcd}$$

If in addition $m = \bigcirc m$, i.e. x is an isomorphism, we see that m is a cartesian subcoalgebra. □

We close this section with a characterization result: F preserves intersections if and only if the following "generalized next time" operators are right adjoints. Given a morphism $f\colon A \to FB$, we have the operator $\bigcirc_f\colon \mathsf{Sub}(B) \to \mathsf{Sub}(A)$ that maps $m\colon B' \rightarrowtail B$ to the pullback of Fm along f:

$$\begin{tikzcd}
\bigcirc_f A' \arrow[r, "f(m)"] \arrow[d, tail, "\bigcirc_f m"'] \arrow[dr, phantom, very near start, "\lrcorner"] & FB' \arrow[d, tail, "Fm"] \\
A \arrow[r, "f"] & FB
\end{tikzcd}$$

Proposition 8.3.14 [168]**.** *The functor F preserves intersections if and only if every generalized next time operator $\bigcirc_f$ is a right adjoint.*

Proof. For the "if"-direction, choose $f = \mathsf{id}_{FY}$. Then $\bigcirc_{\mathsf{id}_{FY}}\colon m \mapsto Fm$ is a right adjoint and so preserves all meets, i.e. F preserves intersections.

The converse follows from the easily established fact that intersections are stable under inverse image, i.e. for every morphism $f\colon X \to Y$ and every family $m_i\colon S_i \rightarrowtail Y$ of subobjects, the intersection $m\colon P \rightarrowtail X$ of the inverse images of the m_i under f yields a pullback

$$\begin{tikzcd}
P \arrow[r] \arrow[d, tail, "m"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \bigcap S_i \arrow[d, tail, "\bigcap m_i"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

Hence, if F preserves intersections, then so does every operator $\bigcirc_f$. Equivalently, $\bigcirc_f$ is a right adjoint. □

We will mention another equivalent characterizaton of intersection preservation in Remark 9.2.26.

8.4 The Well-Founded Part of a Coalgebra

We introduced well-founded coalgebras in Section 8.1. We now discuss the *well-founded part* of a coalgebra, i.e. its largest well-founded subcoalgebra. We prove that this is the least fixed point of the next time operator. We will also see that a coalgebra is well-founded iff it has no proper subcoalgebra as a fixed point of the next time operator (Proposition 8.4.5). Then we prove that the well-founded part is the coreflection of a coalgebra in the category of well-founded coalgebras.

Definition 8.4.1. The *well-founded part* of a coalgebra is its largest well-founded subcoalgebra.

We shall prove that the well-founded part of a coalgebra always exists and is the coreflection in the category of well-founded coalgebras (Proposition 8.4.3 and Proposition 8.4.6). It is obtained by the following:

Construction 8.4.2. Let $\alpha\colon A \to FA$ be a coalgebra. We know that $\mathsf{Sub}(A)$ is a complete lattice and that the next time operator $\bigcirc$ is monotone (see Lemma 8.3.8). Hence, by Theorem 6.1.1, $\bigcirc$ has a least fixed point, which we denote by

$$a^*\colon A^* \rightarrowtail A.$$

Moreover, by Lemma 8.3.13(2), we know that there is a coalgebra structure $\alpha^*\colon A^* \to FA^*$ so that $a^*\colon (A^*, \alpha^*) \rightarrowtail (A, \alpha)$ is the smallest cartesian subcoalgebra of (A, α).

Proposition 8.4.3. *For every coalgebra (A, α), the coalgebra (A^*, α^*) is well-founded.*

Proof. Let $m\colon (B, \beta) \rightarrowtail (A^*, \alpha^*)$ be a cartesian subcoalgebra. By Lemma 8.3.13, $a^* \cdot m\colon B \to A$ is a fixed point of $\bigcirc$. Since a^* is the least fixed point, we have $a^* \leq a^* \cdot m$, i.e. $a^* = a^* \cdot m \cdot x$ for some $x\colon A^* \rightarrowtail B$. Since a^* is monic, we thus have $m \cdot x = \mathsf{id}_{A^*}$. So m is a mono and a split epi, whence an isomorphism. □

Example 8.4.4. Consider the coalgebra G for $\mathscr{P}$ depicted as the following graph:

$$\begin{tikzcd} a \arrow[r] & b & c \arrow[r, bend right] & d \arrow[l, bend right] \end{tikzcd}$$

We list all subcoalgebras below (the structures are the obvious ones given by the picture of G). Those are $\emptyset$, $\{b\}$, $\{a, b\}$, $\{c, d\}$, $\{b, c, d\}$, and $\{a, b, c, d\}$. Of these, the cartesian subcoalgebras of G are $\{a, b\}$, and $\{a, b, c, d\}$. The well-founded part of G is the least cartesian subcoalgebra, namely $\{a, b\}$.

For the next result cf. Taylor [160, Exercise VI.17] or Adámek et al. [29, Corollary 2.19]; the formulation (3) is Taylor's definition of a well-founded coalgebra [160]:

Proposition 8.4.5. *For every coalgebra (A, α), the following are equivalent:*

(1) (A, α) *is well-founded,*

(2) id_A *is the only fixed point of* $\bigcirc$*,*

(3) id_A *is the only prefixed point of* $\bigcirc$*.*

Proof. Since $\bigcirc$ is monotone, (2) and (3) state the same fact, see Remark 6.1.2.

(1) $\Rightarrow$ (2). Let (A, α) be well-founded. Since A is a cartesian subcoalgebra of itself via id_A, we have $\mathsf{id}_A = \bigcirc \mathsf{id}_A$ by Lemma 8.3.13. Consider an arbitrary subobject $m\colon A' \rightarrowtail A$ such that $m = \bigcirc m$. By Lemma 8.3.13, there is a coalgebra structure $\alpha\colon A' \to FA'$ giving a cartesian subcoalgebra. By well-foundedness, m is an isomorphism. Thus it represents the same subobject as id_A.

(3) $\Rightarrow$ (1). By (3), the least fixed point of $\bigcirc$ must be id_A. It follows from Proposition 8.4.3 that $\alpha(\mathsf{id}) = \alpha$ is the structure of a well-founded coalgebra, as desired. $\square$

We know from Proposition 8.4.3 that for every coalgebra (A, α) its subcoalgebra represented by $a^*\colon A^* \rightarrowtail A$ is well-founded. We now prove that, categorically, this subcoalgebra is characterized uniquely up to isomorphism by the following universal property: every homomorphism from a well-founded coalgebra into (A, α) factorizes uniquely through a^*. In particular, this implies that $a^*\colon A^* \to A$ is the largest well-founded subcoalgebra of A, viz. the well-founded part of A.

Proposition 8.4.6. *The full subcategory of* $\mathsf{Coalg}\, F$ *given by well-founded coalgebras is coreflective. In fact, the well-founded coreflection of a coalgebra* (A, α) *is its well-founded part* (A^*, α^*).

Proof. We are to prove that for every coalgebra homomorphism $f\colon (B, \beta) \to (A, \alpha)$, where (B, β) is well-founded, there exists a coalgebra homomorphism $f^\sharp \cdot (B, \beta) \to (A^*, \alpha^*)$ such that $a^* \cdot f^\sharp = f$. It is unique since $a^*\colon A^* \rightarrowtail A$ is a monomorphism. It then follows that $a^*\colon (A^*, \alpha^*) \rightarrowtail (A, \alpha)$ is the largest well-founded subcoalgebra.

For the existence of $f^\sharp$, we first observe that $\overleftarrow{f}(a^*)$ is a prefixed point of $\bigcirc_\beta$: indeed, using Lemma 8.3.11 we have

$$\bigcirc_\beta(\overleftarrow{f}(a^*)) \le \overleftarrow{f}(\bigcirc_\alpha(a^*)) = \overleftarrow{f}(a^*).$$

By Proposition 8.4.5, we therefore have $id_B \;= b^* \le \overleftarrow{f}(a^*)$ in $\mathsf{Sub}(B)$. Using the adjunction in Proposition 8.3.5, we have $\overrightarrow{f}(\mathsf{id}_B) \le a^*$ in $\mathsf{Sub}(A)$. Now let

$$f = (B \overset{e}{\twoheadrightarrow} C \overset{m}{\rightarrowtail} A)$$

be the factorization of f as in Remark 8.2.2(2). This implies that $\overrightarrow{f}(\mathsf{id}_B) = m$. Thus we obtain

$$m = \overrightarrow{f}(\mathsf{id}_B) \le a^*,$$

i.e. there exists a morphism $h\colon C \rightarrowtail A^*$ such that $a^* \cdot h = m$. Thus, $f^\sharp = h \cdot e\colon B \to A^*$ is a morphism satisfying

$$a^* \cdot f^\sharp = a^* \cdot h \cdot e = m \cdot e = f.$$

It follows that $f^\sharp$ is a coalgebra homomorphism from (B, β) to (A^*, α^*) since f and a^* are and F preserves monomorphisms (cf. Remark 8.2.5(1)). $\square$

8.5 Quotients and Subcoalgebras of Well-Founded Coalgebras

In this section we will see that quotients and subcoalgebras of well-founded coalgebras are well-founded again. For subcoalgebras we need to assume more about $\mathscr{A}$ and F (see Corollary 8.5.6).

Remark 8.5.1. Recall from Remark 2.4.11 that *subcoalgebras* of a coalgebra (A, α) are represented by homomorphisms with codomain (A, α) carried by monomorphisms in $\mathscr{A}$. Moreover, according to Definition 4.2.4, quotient coalgebras of (A, α) are represented by homomorphisms with domain (A, α) carried by epimorphisms in $\mathscr{A}$. In this chapter we will consider a stronger notion of quotient coalgebra.

Definition 8.5.2. By a *strong quotient* of a coalgebra (A, α) is meant a quotient coalgebra of (A, α) represented by a strong epimorphism in $\mathscr{A}$ (see Remark 8.2.2(1)).

Later, in Section 9.3, we will make use of the following corollary to Proposition 8.4.6. For endofunctors on sets preserving inverse images this was stated by Taylor [160, Exercise VI.16]:

Corollary 8.5.3. *The subcategory of* $\mathsf{Coalg}\, F$ *formed by all well-founded coalgebras is closed under strong quotients and coproducts in* $\mathsf{Coalg}\, F$*.*

This follows from a general result on coreflective subcategories: the category $\mathsf{Coalg}\, F$ has the factorization system of Proposition 8.2.3 (cf. Remark 8.2.5(1)), and its full subcategory of well-founded coalgebras is coreflective with monomorphic coreflections (see Proposition 8.4.6). Consequently, it is closed under strong quotients and colimits.

Remark 8.5.4. We prove next that, for an endofunctor preserving finite intersections, well-founded coalgebras are closed under subcoalgebras provided that $\mathsf{Sub}(A)$ forms a frame. Recall that $\mathsf{Sub}(A)$ is a frame if for every subobject $m \colon B \rightarrowtail A$ and every family m_i $(i \in I)$ of subobjects of A we have

$$m \wedge \bigvee_{i \in I} m_i = \bigvee_{i \in I} (m \wedge m_i).$$

Equivalently, $\overleftarrow{m} \colon \mathsf{Sub}(A) \to \mathsf{Sub}(B)$ has a right adjoint $m_* \colon \mathsf{Sub}(B) \to \mathsf{Sub}(A)$ (use the dual of Remark 8.3.4).

Examples 8.5.5. (1) Set has the property that all $\mathsf{Sub}(A)$ are frames. In fact, given subsets S and S_i $(i \in I)$ of A the equality $S \cap (\bigcup_{i \in I} S_i) = \bigcup_{i \in I} (S \cap S_i)$ clearly holds.

(2) This property is shared by categories such as posets and monotone maps, graphs and homomorphisms, unary algebras and homomorphisms, and presheaf categories $\mathsf{Set}^{\mathscr{C}^{\mathsf{op}}}$, with $\mathscr{C}$ small. This follows from the fact that joins and meets of subobjects of an object A are formed on the level of subsets of the underlying set of A.

(3) The category CPO does not have the above property: for the cpo $A = \mathbb{N}^\top$ of natural numbers with a top element $\top$ (linearly ordered) the lattice $\mathsf{Sub}(A)$ is not a frame. Consider the subobjects given by inclusion maps $m_i \colon \{0, \dots, i\} \hookrightarrow \mathbb{N}^\top$ for $i \in \mathbb{N}$, with domains linearly ordered. It is easy to see that $\bigvee_{i \in \mathbb{N}} m_i = \mathsf{id}_A$. For the inclusion map $m \colon \{\top\} \hookrightarrow \mathbb{N}^\top$ we have $m \wedge m_i = 0$ $(i \in I)$, the empty subobject. Thus, $\bigvee_{i \in \mathbb{N}} (m \wedge m_i) = 0 \neq m = m \wedge \bigvee_{i \in I} m_i$.

(4) For every Grothendieck topos, the posets $\mathsf{Sub}(A)$ are frames. In fact, it is sufficient for a topos to have all coproducts or intersections to satisfy this requirement.

Corollary 8.5.6. *Suppose that F preserves finite intersections, and let (A, α) be a well-founded coalgebra such that $\mathsf{Sub}(A)$ a frame. Then every subcoalgebra of (A, α) is well-founded.*

Proof. Let $m\colon (B, \beta) \rightarrowtail (A, \alpha)$ be a subcoalgebra. We will show that the only prefixed point of $\bigcirc_\beta$ is id_B (cf. Proposition 8.4.5). Suppose $s\colon S \rightarrowtail B$ fulfils $\bigcirc_\beta(s) \leq s$. Since F preserves finite intersections, we have

$$\overleftarrow{m} \cdot \bigcirc_\alpha = \bigcirc_\beta \cdot \overleftarrow{m}$$

by Corollary 8.3.12(1). The counit of the adjunction $\overleftarrow{m} \dashv m_*$ yields $\overleftarrow{m}(m_*(s)) \leq s$, so that we obtain

$$\overleftarrow{m}(\bigcirc_\alpha(m_*(s))) = \bigcirc_\beta(\overleftarrow{m}(m_*(s))) \leq \bigcirc_\beta(s) \leq s.$$

Using again the adjunction $\overleftarrow{m} \dashv m_*$, we have equivalently that $\bigcirc_\alpha(m_*(s)) \leq m_*(s)$, i.e. $m_*(s)$ is a prefixed point of $\bigcirc_\alpha$. Since (A, α) is well-founded, Corollary 8.3.12(1) implies that $m_*(s) = \mathsf{id}_A$. Since $\overleftarrow{m}$ is also a right adjoint and therefore preserves the top element of $\mathsf{Sub}(B)$, we thus obtain

$$\mathsf{id}_B = \overleftarrow{m}(\mathsf{id}_A) = \overleftarrow{m}(m_*(s)) \leq s,$$

which completes the proof. □

Remark 8.5.7. (1) For a set functor F preserving inverse images, a much better result was proved by Taylor [160, Corollary 6.3.6]: for every coalgebra homomorphism $f\colon (B, \beta) \to (A, \alpha)$ with (A, α) well-founded so is (B, β). In fact, our proof above is essentially Taylor's who (implicitly) uses Corollary 8.3.12(2) instead.

(2) The fact that subcoalgebras of a well-founded coalgebra are well-founded does not necessarily need the assumption that $\mathsf{Sub}(A)$ is a frame. Using the construction of the least fixed point a^* of $\bigcirc$ provided by the (proof of the) Knaster-Tarski fixed point theorem, it is essentially sufficient that $\overleftarrow{m}$ in the proof of Corollary 8.5.6 preserves joins of unions of chains in $\mathsf{Sub}(A)$. For example, this holds for a complete and well-powered category $\mathscr{A}$ whenever it has *universally* constructive monomorphisms (see Definition 8.7.1). Then for endofunctors preserving finite intersections, well-founded coalgebras are closed under taking subcoalgebras (see [8, Thm. 6.12]).

Example 8.5.8. A well-founded coalgebra for a set functor can have non-well-founded subcoalgebras. Let $F\emptyset = 1$ and $FX = 1 + 1$ for all nonempty sets X and $Ff = \mathsf{inl}$ for all maps $f\colon \emptyset \to X$ with X nonempty. The coalgebra $\mathsf{inr}\colon 1 \to F1$ is not well-founded because its empty subcoalgebra is cartesian. However, it is a subcoalgebra of $\mathsf{id}\colon 1 + 1 \to 1 + 1$ (via the embedding inr), and the latter is well-founded.

8.6 The General Recursion Theorem

The main consequence of well-foundedness is parametric recursivity. This is Taylor's General Recursion Theorem [160, Theorem 6.3.13]. Taylor assumed that F preserves inverse images. We present a new proof for which it is sufficient that F preserves monomorphisms, assuming those are constructive. In the next section, we discuss the converse implication in Theorem 8.7.4 and Theorem 8.7.8.

Assumption 8.6.1. For the rest of this section, we assume that in our complete and well-powered category $\mathscr{A}$ the class of all monomorphisms is constructive (see Definition 6.1.18

Remark 8.6.2. (1) In this setting, the bottom element of each poset $\mathsf{Sub}(A)$ is $\bot_A\colon 0 \to A$, where 0 is the initial object of $\mathscr{A}$ (cf. Remark 6.1.19(1)).

(2) Furthermore, a join of a chain in $\mathsf{Sub}(A)$ is obtained by forming a colimit. More precisely, given an ordinal k and an k-chain $m_i\colon A_i \rightarrowtail A$ of subobjects ($i < k$), we have the diagram of objects $(A_i)_{i<k}$, where for all $i \leq j < k$ the connecting morphisms $a_{ij}\colon A_i \rightarrowtail A_j$ are the unique factorizations witnessing $m_i \leq m_j$:

$$\begin{tikzcd} A_i \arrow[rr, "a_{ij}"] \arrow[dr, "m_i"'] & & A_j \arrow[dl, "m_j"] \\ & A & \end{tikzcd}$$

The colimit B of this diagram is formed by monomorphisms $b_i\colon A_i \to B$, $i < k$, and the unique monomorphism $m\colon B \rightarrowtail A$ with $m \cdot b_i = m_i$ for all $i < k$ is the join of all m_i, in symbols: $m = \bigvee_{i<k} m_i$.

Following the proof of Theorem 6.1.1, the well-founded part a^* of a coalgebra (A, α) may be constructed as the join of a transfinite chain of subobjects of A. This will be used in the proof of Theorem 8.6.7.

Construction 8.6.3. Let (A, α) be a coalgebra. We obtain a^*, the least fixed point of $\bigcirc$, as the join of the following transfinite chain of subobjects $a_i\colon A_i \rightarrowtail A$, $i \in \mathsf{Ord}$. First, put $a_0 = \bot_A\colon A_0 = 0 \to A$, the least subobject of A. Given $a_i\colon A_i \rightarrowtail A$, put $a_{i+1} = \bigcirc a_i\colon A_{i+1} = \bigcirc A_i \rightarrowtail A$. For every limit ordinal j, put $a_j = \bigvee_{i<j} a_i$. It follows from the proof of Theorem 6.1.1 that there exists an ordinal i such that $a_i\colon A_i \rightarrowtail A$ is the least fixed point $a^*\colon A^* \rightarrowtail A$ of $\bigcirc$.

Remark 8.6.4. (1) By Remark 8.6.2(2), the above join a_j is obtained as the colimit of the chain of the subobject $a_i\colon A_i \rightarrowtail A$, $i < j$, whenever monomorphisms are constructive.

(2) If F is a finitary functor on a locally finitely presentable category $\mathscr{A}$ (see [37] for the definition), then the least such ordinal is at most ω. Indeed, $\bigcirc$ preserves joins of ω-chains in $\mathsf{Sub}(A)$ because F does, since these joins are obtained as chain colimits (see [37, Prop. 1.62]), and so does $\overleftarrow{\alpha}$ because filtered colimits commute with pullback in $\mathscr{A}$. Indeed, suppose that $c_i\colon C_i \to C$, $i \in I$ is a filtered colimit, and let $f\colon B \to C$ be a

morphism. Form the pullback of every c_i along f:

$$\begin{tikzcd} B_i \arrow[r, "f_i"] \arrow[d, "b_i"'] \arrow[dr, phantom, "\lrcorner", very near start] & C_i \arrow[d, "c_i"] \\ B \arrow[r, "f"] & C \end{tikzcd}$$

Then $b_i\colon B_i \to B$ is a colimit cocone. Indeed, in the category of commmutative squares in $\mathscr{A}$, the filtered diagram formed by the above pullbacks squares has as a colimit a pullback square

$$\begin{tikzcd} \mathsf{colim}\, B_i \arrow[r, "\mathsf{colim}\, f_i"] \arrow[d, "\cong"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{colim}\, C_i = C \arrow[d, equal] \\ B \arrow[r, "f"] & C \end{tikzcd}$$

We conclude that, by Kleene's fixed point theorem, $a^* = \bigvee_{i\in\mathbb{N}} a_i$.

(3) The same holds for a finitary functor on a category with universally constructive monomorphisms (see Definition 8.7.1 below). However, in general one needs transfinite iteration to reach a fixed point (see Remark 8.7.6(2)).

Example 8.6.5. Let (A,α) be a graph regarded as a coalgebra for $\mathscr{P}$ (see Example 1.3.2). Then A_1 is formed by all leaves, i.e. those nodes with no neighbours, A_2 by all leaves and all nodes such that every neighbour is a leaf, etc. We see that a node x lies in A_{i+1} iff every path starting in x has length at most i. Hence $A^* = A_\omega$ is the set of all nodes from which no infinite paths starts.

Notation 8.6.6. For every pair $i \leq j$ or ordinals, we denote by $a_{ij}\colon A_i \rightarrowtail A_j$ the unique morphism witnessing $a_i \leq a_j$, i.e. $a_i = a_j \cdot a_{ij}$. Note that these arise by transfinite recursion as well: a_{0i} is obtained by initiality, at limit steps use the colimit morphisms, and at successor steps one uses the pullback property. That is, in the following diagram (in which all vertical morphisms are monos)

$$\begin{tikzcd} A_{i+1} \arrow[r, "\alpha(a_i)"] \arrow[d, tail, dashed, "a_{i+1,j+1}"] \arrow[dd, bend right=60, tail, "a_{i+1}"'] & FA_i \arrow[d, tail, "Fa_{ij}"'] \arrow[dd, bend left=60, tail, "Fa_i"] \\ A_{j+1} \arrow[r, "\alpha(a_j)"] \arrow[d, tail, "a_{j+1}"] \arrow[dr, phantom, "\lrcorner", very near start] & FA_j \arrow[d, tail, "Fa_j"'] \\ A \arrow[r, "\alpha"'] & FA \end{tikzcd} \tag{8.10}$$

the outside commutes by the definitions of A_{i+1}, a_{i+1}, and $\alpha(a_i)$; also the triangle on the right commutes by induction hypothesis on i. Since the bottom square is a pullback, we obtain $a_{i+1,j+1}$ as desired.

ecall from Definition 6.1.4 the initial-algebra chain for F. If $\mathscr{A}$ has constructive monomorphisms and F preserves monomorphisms, then all $w_{i,j}$ in the initial-algebra chain are monic. This follows from an easy transfinite induction.

Theorem 8.6.7. *Let $\mathscr{A}$ be a complete and wellpowered category with constructive monomorphisms. For $F\colon \mathscr{A} \to \mathscr{A}$ preserving monomorphisms, every well-founded coalgebra is parametrically recursive.*

Proof. (1) Given an arbitrary coalgebra (A, α) we use the chain of subobjects $a_i\colon A_i \rightarrowtail A$ from Construction 8.6.3. We also have the initial-algebra chain $W_i = F^i 0$ with connecting morphisms w_{ji} (see Definition 6.1.4). We obtain a natural transformation

$$h_i\colon A_i \to W_i \qquad i \in \mathsf{Ord},$$

by transfinite recursion as follows: $h_0 = \mathsf{id}_0$, and given $h_i\colon A_i \to W_i$, let

$$h_{i+1} = (A_{i+1} \xrightarrow{\alpha(a_i)} FA_i \xrightarrow{Fh_i} FW_i = W_{i+1}).$$

Finally, for a limit ordinal i, h_i is uniquely determined by the universal property of the colimit A_i.

We must verify that for $j \leq i$ the naturality square below commutes:

$$\begin{tikzcd}
A_j \arrow[r, "h_j"] \arrow[d, tail, "a_{ji}"'] & W_j \arrow[d, tail, "w_{ji}"] \\
A_i \arrow[r, "h_i"'] & W_i
\end{tikzcd} \tag{8.11}$$

The proof is by transfinite induction on i. The base case for $i = 0$ is trivial, and the step when i is a limit ordinal follows from the fact that we use colimits to define both A_i and $F^i 0$. We are left with the successor step $i + 1$. Here we again use induction on j. The verification amounts to assuming (8.11) for i and j and showing the same equation for $j + 1$ and $i + 1$. For this, consider the diagram below:

$$\begin{tikzcd}
A_{j+1} \arrow[rrr, "a_{j+1,i+1}"] \arrow[dr, "\alpha(a_j)"] \arrow[dd, "h_{j+1}"'] & & & A_{i+1} \arrow[dl, "\alpha(a_i)"'] \arrow[dd, "h_{i+1}"] \\
& FA_{j+1} \arrow[r, "Fa_{ji}"] \arrow[dl, "Fh_j"'] & FA_{i+1} \arrow[dr, "Fh_i"] & \\
W_{j+1} = F^{j+1}0 \arrow[rrr, "w_{j+1,i+1}"'] & & & F^{i+1}0 = W_{i+1}
\end{tikzcd} \tag{8.12}$$

The region at the top is also the top square of (8.10), the triangles commute by the definition of (h_i), and the region at the bottom commutes by the induction hypothesis and the fact that $Fw_{ji} = w_{j+1,i+1}$. Thus the outside commutes, as desired.

(2) Now suppose that (A, α) is a well-founded coalgebras. We prove that (A, α) is recursive, i.e. for every algebra $e\colon FX \to X$ we present a coalgebra-to-algebra morphism $e^\dagger$ and prove that it is unique.

For every ordinal i, the coalgebra $w_{i,i+1}\colon W_i \to FW_i$ is recursive (see Example 7.3.2(6)). Hence we have a morphism $f_i\colon W_i \to X$ such that the square on the bottom below commutes:

$$
\begin{tikzcd}
A \arrow[r, "\alpha(a_i)=\alpha"] \arrow[d, "h_i"'] \arrow[dr, "h_{i+1}"] & FA \arrow[d, "Fh_i"] \\
W_i \arrow[r, "w_{i,i+1}"'] \arrow[d, "f_i"'] & FW_i \arrow[d, "Ff_i"] \\
X & FX \arrow[l, "e"]
\end{tikzcd}
\tag{8.13}
$$

Since (A,α) is well-founded, there exists an ordinal i such that $A = A_i = A_{i+1}$ (see Construction 8.6.3). Then we have $\alpha(a_i) = \alpha$, so that the upper triangle commutes by definition of h_{i+1}. Moreover, the lower triangle is an instance of (8.11) using the fact that $a_{i,i+1} = \mathsf{id}$. Thus the outside of the diagram commutes, and so $f_i \cdot h_i$ is the desired coalgebra-to-algebra morphism.

(3) For the uniqueness, suppose that $e^\dagger$ is any coalgebra-to-algebra morphism from α to e, i.e., in the diagram below the lower square commutes:

$$
\begin{tikzcd}
A_{i+1} \arrow[r, "\alpha(a_i)"] \arrow[d, tail, "a_{i+1}"'] & FA_i \arrow[d, tail, "Fa_i"] \\
A \arrow[r, "\alpha"] \arrow[d, "e^\dagger"'] & FA \arrow[d, "Fe^\dagger"] \\
X & FX \arrow[l, "e"]
\end{tikzcd}
\tag{8.14}
$$

Moreover, the upper one is the square defining $\alpha(a_i)$ (see Definition 8.3.6).

We verify by induction on j that $e^\dagger \cdot a_j = f_j \cdot h_j \cdot a_j$. Then for the above ordinal i with $a_i = \mathsf{id}_A$, we have $e^\dagger = f_i \cdot h_i$ as desired. For the base case $j = 0$, the equation trivially holds, and for limit ordinals we use the universal property of the colimit A_j. For the successor step we use that (8.14) and (8.13) commute (with j substituted for i). By pasting (8.13) and the upper square of (8.14) we obtain

$$
e \cdot F(f_j \cdot h_j \cdot a_j) \cdot \alpha(a_j) = f_j \cdot h_j \cdot a_{j+1}. \tag{8.15}
$$

This yields the desired equality:

$$
\begin{aligned}
e^\dagger \cdot a_{j+1} &= e \cdot F(e^\dagger \cdot a_j) \cdot \alpha(a_j) && \text{(by (8.14))} \\
&= e \cdot F(f_j \cdot h_j \cdot a_j) \cdot \alpha(a_j) && \text{(by induction hypothesis)} \\
&= f_j \cdot h_j \cdot a_{j+1} && \text{(by (8.15))}.
\end{aligned}
$$

(4) Finally, we prove that the coalgebra (A,α) is a parametrically recursive.

Consider the coalgebra $\langle \alpha, \mathsf{id}_A\rangle\colon A \to FA \times A$ for $F(-) \times A$. This functor preserves monomorphisms since F does and monomorphisms are closed under products. The next time operator $\bigcirc$ on $\mathsf{Sub}(A)$ is the same for both coalgebras since the square (8.7) is a pullback if and only if the square below is one:

$$\begin{tikzcd} \bigcirc A' \arrow[r, "\langle \alpha(m), \bigcirc(m) \rangle"] \arrow[d, tail, "\bigcirc m"'] \arrow[dr, phantom, "\lrcorner", very near start] & FA' \times A \arrow[d, tail, "Fm \times A"] \\ A \arrow[r, "\langle \alpha, A \rangle"] & FA \times A \end{tikzcd}$$

Since id_A is the unique fixed point of $\bigcirc$ w.r.t. F (see Proposition 8.4.5), it is also the unique fixed point of $\bigcirc$ w.r.t. $F(-) \times A$. Thus, by Proposition 8.4.5, $(A, \langle \alpha, A \rangle)$ is a well-founded coalgebra for $F(-) \times A$. By point (2), it is thus recursive for $F(-) \times A$. This states equivalently that (A, α) is a parametrically recursive coalgebra for F. $\square$

Corollary 8.6.8. *For every endofunctor on sets* Set *or* K-Vec *(the vector spaces over a fixed field, with linear maps), every well-founded coalgebra is parametrically recursive.*

Proof. For Set, we apply Theorem 8.6.7 to the Trnková hull $\bar{F}$ (see Proposition 4.4.4), noting that F and $\bar{F}$ have the same (non-empty) coalgebras. By Lemma 8.1.13 the desired result follows. For K-Vec, observe that monomorphisms split and are therefore preserved by every endofunctor F. $\square$

Example 8.6.9. For the set functor $FX = X \times X + 1$ the coalgebra $(\mathbb{N}, \gamma)$ from Example 8.1.12(3) is well-founded. Hence it is parametrically recursive.

Similarly, we saw that for $FX = A \times X \times X + 1$ the coalgebra (A, s) from Example 7.3.5(3) is well-founded, and therefore it is (parametrically) recursive.

Example 8.6.10. Well-founded coalgebras need not be recursive when F does not preserve monomorphisms. We take $\mathscr{A}$ to be the category of *sets with a predicate*, i.e. pairs (X, A), where $A \subseteq X$. Morphisms $f\colon (X, A) \to (Y, B)$ satisfy $f[A] \subseteq B$. Denote by $\mathbb{1}$ the terminal object $(1, 1)$. We define an endofunctor F by $F(X, \emptyset) = (X + 1, \emptyset)$, and for $A \neq \emptyset$, $F(X, A) = \mathbb{1}$. For a morphism $f\colon (X, A) \to (Y, B)$, put $F = f + \mathsf{id}$ if $A = \emptyset$; if $A \neq \emptyset$, then also $B \neq \emptyset$ and Ff is $\mathsf{id}\colon \mathbb{1} \to \mathbb{1}$.

The terminal coalgebra is $\mathsf{id}\colon \mathbb{1} \to \mathbb{1}$, and it is easy to see that it is well-founded. But it is not recursive: there are no coalgebra-to-algebra morphisms into an algebra of the form $F(X, \emptyset) \to (X, \emptyset)$.

We close this section with a general fact on well-founded parts of fixed points of F:

Theorem 8.6.11. *Let $\mathscr{A}$ be a complete and well-powered category with constructive monomorphisms. For F preserving monomorphism, the well-founded part of every fixed point is an initial algebra. In particular, the only well-founded fixed point is the initial algebra.*

Proof. Let $\alpha\colon A \to FA$ be a fixed point of F. By Theorem 6.1.22 we know that the initial algebra $(\mu F, \iota)$ exists. Now let $a^*\colon (A^*, \alpha^*) \rightarrowtail (A, \alpha)$ be the well-founded part of

A given in Proposition 8.4.3. This is a cartesian subcoalgebra, i.e. we have a pullback square

$$\begin{tikzcd} A^* \arrow[r, "\alpha^*"] \arrow[d, tail, "a^*"'] \arrow[dr, phantom, "\lrcorner", very near start] & FA^* \arrow[d, tail, "Fa^*"] \\ A \arrow[r, "\alpha"] & FA \end{tikzcd}$$

Since α is an isomorphism, so is α^*.

By initiality, we have an algebra homomorphism $h\colon (\mu F, \iota) \to (A^*, (\alpha^*)^{-1})$, i.e. a coalgebra homomorphism

$$\begin{tikzcd} \mu F \arrow[r, "\iota^{-1}"] \arrow[d, "h"'] \arrow[dr, phantom, "\lrcorner", very near start] & F(\mu F) \arrow[d, "Fh"] \\ A^* \arrow[r, "\alpha^*"] & FA^* \end{tikzcd}$$

Since both horizontal morphism are invertible, this square is a pullback. By Theorem 8.6.7, (A^*, α^*) is recursive. Thus, we have a coalgebra homomorphism $k\colon (A^*, \alpha^*) \to (\mu F, \iota^{-1})$ by Corollary 7.3.3. By the universal property of μF, we obtain $k \cdot h = \mathsf{id}_{\mu F}$, whence h is a split monomorphism. Thus the above square exhibits $(\mu F, \iota^{-1})$ as a cartesian subcoalgebra of (A^*, α^*). Since (A^*, α^*) is well-founded, we conclude that h is an isomorphism. $\square$

Corollary 8.6.12. *If F in Theorem 8.6.11 has a terminal coalgebra νF, it also has an initial algebra which is the well-founded part of νF.*

Example 8.6.13. (1) We illustrate that for a set functor F preserving monos, the well-founded part of the terminal coalgebra is the initial algebra. Consider $FX = A \times X + 1$. The terminal coalgebra is the set $A^\infty \cup A^*$ of finite and infinite sequences from the set A. The initial algebra is A^*. It is easy to check that A^* is the well-founded part of $A^\infty \cup A^*$.

(2) For a polynomial set functor H_Σ the final coalgebra νH_Σ is formed by all Σ-trees (see Example 6.4.6). Its well-founded part is given by all well-founded Σ-trees (see Example 6.1.13(2)).

Example 8.6.14. When F does not preserve monomorphisms, the well-founded part of the final coalgebra need not be the initial algebra. Indeed, for the functor F in Example 8.6.10, the initial algebra is carried by $(\mathbb{N}, \emptyset)$, which is not even a subcoalgebra of $\mathbb{1}$.

8.7 The Converse of the General Recursion Theorem

Our last topic is in this chapter is a converse to Theorem 8.6.7, under various hypotheses. This is based on Taylor [159, 160]. It also generalizes results by Adámek et al. [7] and Jeannin et al. [99]. In order to prove the implication

$$\text{recursive} \implies \text{well-founded},$$

one needs to assume more than preservation of finite intersections. In fact, we will assume that F preserves inverse images. But even this is not enough. We additionally assume that either

(1) The underlying category $\mathscr{A}$ has a subobject classifier; or

(2) The underlying category $\mathscr{A}$ has universally constructive monomorphisms and the endofunctor F has a *pre-fixed point*, i.e. an object A with a monomorphism $\alpha\colon FA \rightarrowtail A$.

We start with the second of these possible assumptions.

Definition 8.7.1. Let $\mathscr{A}$ be a category with colimits of chains and pullbacks. We say that $\mathscr{A}$ has *universally constructive monomorphisms* if it has constructive monomorphism, and for every morphism $f\colon X \to \mathsf{colim}\, C$, the functor $\mathscr{A}/\,\mathsf{colim}\, C \to \mathscr{A}/X$ forming pullbacks along f preserves the colimit of C.

Remark 8.7.2. (1) If $\mathscr{A}$ has universally constructive monomorphisms, then the initial object 0 is strict (see Remark 8.2.6(2) Indeed, consider the empty colimit in Definition 8.7.1.

(2) If $\mathscr{A}$ has universally constructive monomorphisms, then for every morphism $f\colon A \to B$, the operator $\overleftarrow{f}\colon \mathsf{Sub}(B) \to \mathsf{Sub}(A)$ preserves unions of chains.

Indeed, suppose that $c\colon C \rightarrowtail A$ is the union of a chain of subobjects $a_i\colon A_i \rightarrowtail A$ in $\mathsf{Sub}(A)$. Then C is the colimit of the (chain of connecting morphisms between the) A_i with colimit injections $c_i\colon A_i \rightarrowtail C$, say. For the morphism $p = \overleftarrow{f}(c)\colon P \rightarrowtail B$ we paste two pullback squares for every i as shown below:

$$\begin{tikzcd}
B_i \arrow[r, tail, "f_i"] \arrow[d, tail, "p_i"'] \arrow[dd, tail, bend right=60, "b_i"'] \arrow[dr, phantom, "\lrcorner", very near start] & A_i \arrow[d, tail, "c_i"] \arrow[dd, tail, bend left=60, "a_i"] \\
P \arrow[r, tail, "f'"] \arrow[d, tail, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, tail, "c"] \\
B \arrow[r, tail, "f"] & A
\end{tikzcd}$$

The outside is then the pullback square stating that $b_i = \overleftarrow{f}(a_i)$. By universality, $P = \mathsf{colim}\, B_i$ with colimit injections p_i. Thus, by the constructivity of monomorphisms p is the union of the subobjects b_i in $\mathsf{Sub}(B)$; in symbols: $\bigvee_i \overleftarrow{f}(a_i) = \overleftarrow{f}\left(\bigvee_i a_i\right)$ as desired.

Example 8.7.3. (1) Set has universally constructive monomorphisms. More generally, every Grothendieck topos does.

(2) K-Vec has constructive monomorphisms, but not universally so because the initial object is not strict (cf. Remark 8.6.2(1)).

(3) Categories in which colimits of chains and pullbacks are formed "set-like" have universally constructive monomorphisms. These include the categories of posets, graphs, topological spaces, presheaf categories, and many varieties, such as monoids, graphs, and unary algebras.

(4) Every locally finitely presentable category $\mathscr{A}$ with a strict initial object (see Remark 8.7.2(1)) has constructive monomorphisms. This follows from [37, Prop. 1.62] and since colimits of chains are universal (cf. Remark 8.6.4(2)).

Unfortunately, the example of rings demonstrates that the assumption of strictness of 0 cannot be lifted.

Theorem 8.7.4. *Let $\mathscr{A}$ be a complete and wellpowered category with universally constructive monos, and suppose that $F\colon \mathscr{A} \to \mathscr{A}$ preserves inverse images and has a pre-fixed point. Then every recursive F-coalgebra is well-founded.*

Proof. (1) We first observe that an initial algebra for F exists. This follows from results by Trnková et al. [162] as we now briefly recall. Recall the initial-algebra chain of F from Definition 6.1.4. Let $\beta\colon FB \rightarrowtail B$ be a pre-fixed point. Then there is a unique cocone $\beta_i\colon W_i \to B$ satisfying $\beta_{i+1} = \beta \cdot F\beta_i$. Also, each β_i is monomorphic. Since B has only a set of subobjects, there is some λ such that for every $i > \lambda$, all morphisms β_i represent the same subobject of B. Consequently, $w_{\lambda,\lambda+1}$ is an isomorphism. Then $\mu F = F^\lambda 0$ with the structure $\alpha = w_{\lambda,\lambda+1}^{-1}\colon F(\mu F) \to \mu F$ is an initial algebra.

(2) Now suppose that (A, α) is a recursive coalgebra. Then there exists a unique coalgebra homomorphism $h\colon (A, \alpha) \to (\mu F, \iota^{-1})$. Let us abbreviate $w_{i\lambda}$ by $c_i\colon F^i 0 \rightarrowtail \mu F$ and recall the subobjects $a_i\colon A_i \rightarrowtail A$ from Construction 8.6.3. We are going to prove by transfinite induction that for every $i \in \mathsf{Ord}$, a_i is the inverse image of c_i, i.e. we have a pullback square

$$\begin{tikzcd} A_i \arrow[r, "h_i"] \arrow[d, tail, "a_i"'] \arrow[dr, phantom, "\lrcorner", very near start] & W_i \arrow[d, tail, "c_i"] \\ A \arrow[r, "h"] & \mu F \end{tikzcd} \qquad \text{for some morphism } h_i\colon A_i \to W_i; \tag{8.16}$$

in symbols: $a_i = \overleftarrow{h}(c_i)$ for all ordinals i. Then it follows that a_λ is an isomorphism, since so is c_λ, whence (A, α) is well-founded. In the base case $i = 0$ the above square clearly is a pullback since $A_0 = W_0 = 0$ is a strict initial object (see Remark 8.6.2(1)).

For the isolated step we compute the pullback of $c_{i+1}\colon W_{i+1} \to \mu F$ along h using the following diagram:

$$\begin{tikzcd} A_{i+1} \arrow[r, "\alpha(a_i)"] \arrow[d, tail, "a_{i+1}"'] \arrow[dr, phantom, "\lrcorner", very near start] & FA_i \arrow[r, "Fh_i"] \arrow[d, tail, "Fa_i"'] \arrow[dr, phantom, "\lrcorner", very near start] & FW_i \arrow[d, tail, "Fc_i"'] \arrow[dr, "c_{i+1}"] & \\ A \arrow[r, "\alpha"] \arrow[rrr, bend right, "h"'] & FA \arrow[r, "Fh"] & F(\mu F) \arrow[r, "\iota"] & \mu F \end{tikzcd}$$

By the induction hypothesis and since F preserves inverse images, the middle square above is a pullback. Since the structure map ι of the initial algebra is an isomorphism, it follows that the middle square pasted with the right-hand triangle is a pullback. Finally, the left-hand square is a pullback by the definition of a_{i+1}. Thus, the outside of the above diagram is a pullback, as required.

For a limit ordinal j, we know that $a_j = \bigvee_{i<j} a_i$ and similarly, $c_j = \bigvee_{i<j} c_i$ since $W_j = \mathsf{colim}_{i<j} W_j$ and monomorphisms are constructive (see Remark 8.6.2(2)). Using Remark 8.7.2(2) and the induction hypothesis we thus obtain

$$\overleftarrow{h}(c_j) = \overleftarrow{h}\Big(\bigvee_{i<j} c_i\Big) = \bigvee_{i<j} \overleftarrow{h}(c_i) = \bigvee_{i<j} a_i = a_j. \qquad \square$$

Corollary 8.7.5. *Let $\mathscr{A}$ and F satisfy the assumptions of Theorem 8.7.4. Then the following properties of a coalgebra are equivalent:*

(1) *well-foundedness,*

(2) *parametric recursiveness,*

(3) *recursiveness,*

(4) *existence of a homomorphism into $(\mu F, \iota^{-1})$,*

(5) *existence of a homomorphism into a well-founded one.*

Proof. We already know (1) ⇒ (2) ⇒ (3). Since F has an initial algebra (as proved in Theorem 8.7.4), the implication (3) ⇒ (4) follows from Corollary 7.3.3. In Theorem 8.7.4 we also proved (4) ⇒ (1). The implication (4) ⇒ (5) follows from Example 8.1.3(1). Finally, it follows from our previous results (see [29, Remark 2.40]) that $(\mu F, \iota^{-1})$ is a terminal well-founded coalgebra. Thus, (5) ⇒ (4), which completes the proof. □

Remark 8.7.6. (1) Observe that one can replace universally constructive monomorphisms in Corollary 8.7.5 by universally constructive class $\mathcal{M}$ of monomorphisms. Taking as $\mathcal{M}$ the class of all strong monomorphisms, this yields examples such as posets, graphs, and topological spaces (where the strong monos represent subposets, subgraphs, and subspaces, respectively).

(2) Coming back to Remark 8.6.4(3), we see from the proof of Theorem 8.7.4 that in general one needs transfinite iteration to obtain the least fixed point $\bigcirc$. Indeed, for $(A, \alpha) = (\mu F, \iota^{-1})$ we have $h = \mathsf{id}$ in (8.16) and therefore $a_i = c_i$. Now we saw in Example 6.1.13(2) that for $FX = 1 + X^{\mathbb{N}}$ on Set the initial-algebra chain converges in exactly ω_1 steps, i.e. $\mu F = W_{\omega_1} = A_{\omega_1}$ but $W_i = A_i$, $i < \omega_1$, are proper subsets of μF.

Our next proof stems from Peter Johnstone's book [100, Prop. 4.1.5], and he informs us that he learned it from Paul Taylor. In Theorem 8.7.4, we assumed that the endofunctor has a prefixed point. For set functors, this assumption may be lifted. Indeed, whenever a category has a subobject classifier, then every recursive coalgebra is well-founded, as we show next.

Remark 8.7.7. (1) Let us recall the definition of a subobject classifier. This is an object Ω with a subobject $t\colon 1 \to \Omega$ such that for every subobject $b\colon B \rightarrowtail A$ there is a unique $\hat{b}\colon C \to \Omega$ such that the square below is a pullback:

$$\begin{tikzcd} B \arrow[r, "!"] \arrow[d, tail, "b"'] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, tail, "t"] \\ A \arrow[r, "\hat{b}"'] & \Omega \end{tikzcd} \tag{8.17}$$

The subjobject classifier is a prominent concept in topos theory, and by definition, every elementary topos has a subjobject classifier; for example, every category $\mathsf{Set}^{\mathscr{C}}$ with $\mathscr{C}$ small.

(2) Set has a subobject classifier given by $\Omega = \{0, 1\}$ with the evident $t\colon 1 \hookrightarrow 2$. Indeed, subsets $b\colon B \hookrightarrow A$ are in one-to-one correspondence with characteristic maps $\hat{b}\colon B \to \{0, 1\}$.

(3) Our standing assumption that $\mathscr{A}$ is a complete and well-powered category is not needed for the next result. One may verify that finite limits are sufficient.

Theorem 8.7.8 (Taylor)**.** *Let F be an endofunctor on a finitely complete category with a subobject classifier preserving inverse images. Then every recursive F-coalgebra is well-founded.*

Proof. Let (A, α) be a recursive coalgebra. Clearly, id_A is a fixed point of $\bigcirc$, and we prove below that it is the unique one. Thus, (A, α) is well-founded.

Let $b\colon B \to FB$ be any fixed point of $\bigcirc$. Consider the following diagram:

$$\begin{tikzcd}
B \arrow[r, "\alpha(b)"] \arrow[d, tail, "b"'] \arrow[dr, phantom, very near start, "\lrcorner"] \arrow[rrr, bend left, "!"] & FB \arrow[r, "F!"] \arrow[d, tail, "Fb"'] \arrow[dr, phantom, very near start, "\lrcorner"] & F1 \arrow[r, "!"] \arrow[d, tail, "F!"'] \arrow[dr, phantom, very near start, "\lrcorner"] & 1 \arrow[d, tail, "t"] \\
A \arrow[r, "\alpha"] \arrow[rrr, bend right, "\widehat{b}"'] & FA \arrow[r, "F\widehat{b}"] & F\Omega \arrow[r, "\widehat{Ft}"] & \Omega
\end{tikzcd}$$

The square on the left is a pullback because $b = \bigcirc b$. The central square is F applied to the pullback square (8.17) for $b\colon B \rightarrowtail A$. The square on the right is the pullback square (8.17) for $Ft\colon F1 \to F\Omega$. The upper morphism is $!\colon B \to 1$, and so the lower one is $\widehat{b}$. Thus the outside rectangle is again a pullback. In particular,

$$\widehat{b} = \widehat{Ft} \cdot F\widehat{b} \cdot \alpha.$$

So we have a coalgebra-to-algebra morphism

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d, "\widehat{b}"'] & FA \arrow[d, "F\widehat{b}"] \\
\Omega & F\Omega \arrow[l, "\widehat{Ft}"]
\end{tikzcd}$$

Since (A, α) is recursive, this means that $\widehat{b}$ is uniquely determined from α, independent of which fixed point b of $\bigcirc$ was used in our argument. Thus $\widehat{b} = \widehat{\mathsf{id}_A}$, as desired. □ □

Corollary 8.7.9. *For every set functor preserving inverse images, recursive coalgebras are well-founded.*

Corollary 8.7.10. *Let $\mathscr{A}$ and F satisfy the assumptions of Theorem 8.7.8. Then we have equivalent concepts for coalgebras:*

$$\textit{well-founded} \iff \textit{recursive} \iff \textit{parametrically recursive.}$$

Example 8.7.11. The hypothesis in Theorem 8.7.4 and Theorem 8.7.8 that the functor preserves inverse images cannot be lifted. In order to see this, we consider the functor $R\colon \mathsf{Set} \to \mathsf{Set}$ of Example 7.3.4. This functor preserves monomorphisms but not inverse images (see **????**). The coalgebra $A = \{0,1\}$ with the structure α constant to $(0,1)$ is recursive: given an algebra $\beta\colon RB \to B$, the unique coalgebra-to-algebra homomorphism $h\colon \{0,1\} \to B$ is given by

$$h(0) = h(1) = \beta(d).$$

But A is not well-founded: $\emptyset$ is a cartesian subcoalgebra.

We have seen that for set functors preserving monomorphisms well-founded coalgebras are recursive, and the converse holds for functors preserving inverse images. Moreover, the latter requirement cannot be lifted as we just saw in Example 8.7.11. Surprisingly, something interesting is true for *all* set functors. Recall that an initial algebra $(\mu F, \iota)$ is also considered as a coalgebra $(\mu F, \iota^{-1})$.

Theorem 8.7.12 [29, Thm. 2.46]**.** *For every set functor, a terminal well-founded coalgebra is precisely an initial algebra.*

The proof is nontrivial, and we are not going to present it. It is based on properties of well-founded coalgebras in locally presentable categories (see **??**). The fact that no assumptions on F are needed seems very special to Set. On the one hand, Theorem 8.7.12 can be proved for every locally finitely presentable base category $\mathscr{A}$ having a strict initial object (cf. Remark 8.2.6(2)) and every endofunctor on $\mathscr{A}$ preserving finite intersections [29, Theorem 2.36]. On the other hand, without this last assumptions, Theorem 8.7.12 does not even generalize from Set to the category Gra or other presheaf categories, as the following example shows.

Example 8.7.13. Let Gra be the category of graphs, i.e. the category of presheaves over the category $\{\bullet \rightrightarrows \bullet\}$ given by two parallel morphisms. Here is a simple endofunctor F on Gra whose initial algebra is infinite and whose terminal well-founded coalgebra is a singleton graph: On objects A put $FA = 1$ (the terminal graph) if A has edges. For a graph A without edges, let FA be the graph without edges whose vertices are those of A plus an additional vertex. The definition of F on morphisms $h\colon A \to B$ is as expected: Fh maps the additional vertex of A to that of B in the case where B has no edges. Then μF is the graph of natural numbers without edges. However, the terminal well-founded coalgebra is $F1 \xrightarrow{\cong} 1$.

8.8 Summary of this Chapter

We have presented well-founded coalgebras whose definition captures the concept of well-founded induction on an abstract level. We have also provided a new proof of Taylor's General Recursion Theorem stating that every well-founded coalgebra is parametrically recursive. This holds for functors preserving monomorphisms on a complete and well-founded category with constructive monomorphisms. In the category of sets, this even holds for every endofunctor, and the converse holds for endofunctors preserving inverse images. Coming back to our theme in this chapter, for every set functor the initial algebra is, equivalently, the terminal well-founded coalgebra.

Finally, we have also provided an iterative construction of the well-founded part of a given coalgebra. It is carried by the least fixed point of Jacobs' next time operator. In addition, the well-founded part yields the coreflection of a coalgebra in the category of well-founded coalgebras.

Finally, let us remark that a dual notion to well-foundedness has been studied by Capretta, Uustalu, and Vene [60].

In the next chapter we will meet well-founded coalgebras again in a different role. There we will describe the initial algebra of set a functor as the algebra consisting of all well-founded, well-pointed coalgebras up to isomorphism.

9 State Minimality and Well-Pointed Coalgebras

In this chapter we will first investigate state minimality of systems on the level of coalgebras. In addition, we will also provide new descriptions of initial algebras and terminal coalgebras. The notions and constructions we present here are inspired by the well-known minimization of deterministic automata. Recall that a minimal deterministic automaton is precisely one that is *reachable* and *observable*, i.e. every state is reachable from the initial state and no two states accept the same language. Both notions turn out to have an equivalent element-free characterization: a deterministic automaton is observable iff it has no proper quotient (via a coalgebra homomorphism, see Example 2.4.2(3)), and it is reachable iff it has no proper subautomaton containing the initial state. Both of these element-free characterizations can be formulated, more generally, for coalgebras. We first turn to the study of *simple* coalgebras, i.e. those having no proper quotient coalgebra, in Section 9.1. In particular, under rather mild assumptions, we show how to construct for every coalgebra its simple quotient. Next, in Section 9.2 we study reachable coalgebras. The coalgebraic formulation of that property requires to consider *pointed* coalgebras These are coalgebras equipped with a *point*, a morphism from the terminal object, modelling an initial state. We show that under mild assumptions every coalgebra has a reachable part, i.e. a unique reachable subcoalgebra. Moreover, we present a construction of the reachable part that resembles the standard breadth-first-search procedure with which one computes the reachable part of a pointed graph.

Finally, in Section 9.3, we treat *well-pointed* coalgebras which are defined to be both simple and reachable and thus model state minimality of coalgebras. We will present a construction of the well-pointed modification of a given coalgebra much in the spirit of the well-known minimization of deterministic automata.

As to the second main aim of this chapter, that last section also presents a new description of the initial algebra and the terminal coalgebra for a set functors F preserving intersections. In fact, we shall see that the set of all well-pointed F-coalgebras, considered up to isomorphism, carries the terminal coalgebra νF, and its subset of all well-founded and well-pointed F-coalgebras carries the initial algebra μF.

9.1 Simple Coalgebras

In this section we study coalgebras having no proper quotient. This property generalizes the notion of an observable deterministic automaton, i.e. one where distinct states accept distinct languages.

Assumption 9.1.1. Throughout this section, $\mathscr{A}$ denotes a cocomplete, wellpowered and cowellpowered category.

Remark 9.1.2. (1) As in Section 8.3, we are going to work with factorizations of morphisms f as $f = m \cdot e$, where e is a strong epimorphism and m is a monomorphism. It follows from Adámek et al. [20, dual of Theorem 14.19 and Exercise 14C(e)] that every cocomplete and cowellpowered category has such factorizations.

(2) Recall from Definition 8.5.2 that we consider strong quotient coalgebras in this chapter.

(3) Note that it follows from Proposition 4.1.1 that for every endofunctor F on $\mathscr{A}$, $\mathsf{Coalg}\, F$ is cocomplete with colimits formed on the level of $\mathscr{A}$ since the forgetful functor of $\mathsf{Coalg}\, F$ creates all colimits.

(4) It is an easy exercise to show that strong epimorphisms are stable under (wide) pushouts. We call a wide pushout of a span of strong epimorphisms a *cointersection*. Thus, our assumptions imply that all cointersections exist in $\mathscr{A}$ and also in $\mathsf{Coalg}\, F$ for every endofunctor F, where they are formed on the level of $\mathscr{A}$.

Definition 9.1.3. A coalgebra (A, α) is called *simple* if it has no proper strong quotient coalgebra, i.e. every strong quotient coalgebra of (A, α) is an isomorphism.

Examples 9.1.4. (1) If F has a terminal coalgebra, then it is simple. Indeed, given any quotient $e : \nu F \twoheadrightarrow A$ then the unique homomorphism $f : A \to \nu F$ satisfies $f \cdot e = \mathsf{id}_{\nu F}$. Thus, e is a split monomorphism, and since it is also an epimorphism, it is an isomorphism.

(2) A deterministic automaton, considered as a coalgebra for $FX = \{0,1\} \times X^\Sigma$ is simple iff it is observable. To see this, let (A, α) be simple and consider the unique coalgebra homomorphism $f : A \to \nu F$ assigning to every state the formal language it accepts (see Example 2.5.5). Take its (strong epi, mono)-factorization $f = m \cdot e$ to obtain its image C in νF. Then C is a quotient of A via e and therefore e is an isomorphism by simplicity. It follows that two states x and y in A accept the same language (i.e. $f(x) = f(y)$) iff $e(x) = e(y)$, and equivalently, $x = y$.

(3) A graph considered as a coalgebra for $\mathscr{P}$ is simple iff no distinct nodes are bisimilar. Indeed, for every graph (A, α), the largest bisimulation $\sim$ on it is a congruence (see Remark 4.2.7), i.e., the quotient $A/{\sim}$ carries the structure of a $\mathscr{P}$-coalgebra such that the canonical quotient map $e : A \twoheadrightarrow A/{\sim}$ is a coalgebra homomorphism. Thus, (A, α) is simple iff e is an isomorphism, which is equivalent to saying that for every bisimilar pair x, y in A we have $x = y$.

Observation 9.1.5. (1) If F preserves monomorphisms, then a coalgebra (A, α) is simple iff every homomorphism with domain (A, α) is monic. Indeed, given a coalgebra homomorphism $f : (A, \alpha) \to (B, \beta)$ with (A, α) simple, take its (strong epi, mono)-factorization $f = m \cdot e$. Then (A, α) must be isomorphic to its image via e, which implies that f is monic.

Conversely, suppose that every coalgebra homomorphism $f : (A, \alpha) \to (B, \beta)$ is monic. Then every quotient $q : (A, \alpha) \twoheadrightarrow (B, \beta)$ is carried by a strong epi, which is also monic, whence an isomorphism.

(2) For every pair of coalgebra homomorphisms $f_1, f_2\colon (A,\alpha) \to (B,\beta)$ with (B,β) simple we have $f_1 = f_2$. Indeed, take the coequalizer e of f_1, f_2 in $\mathsf{Coalg}\, F$. Then, e is an isomorphism by simplicity (since every coequalizer is strongly epic). It follows that $f_1 = f_2$.

Proposition 9.1.6 [96]**.** *Every coalgebra has a unique simple quotient represented by the cointersection*

$$e_{(A,\alpha)}\colon (A,\alpha) \twoheadrightarrow (\bar{A},\bar{\alpha})$$

of all strong quotient coalgebras of (A,α). This is the reflection of (A,α) in the full subcategory of all simple coalgebras.

Proof. (1) Take the cointersection $e_{(A,\alpha)}\colon (A,\alpha) \twoheadrightarrow (\bar{A},\bar{\alpha})$ of all strong quotient coalgebfras of (A,α) in $\mathsf{Coalg}\, F$. Then $e_{(A,\alpha)}$ is a strong quotient coalgebra. To see that $(\bar{A},\bar{\alpha})$ is simple, consider any strong quotient coalgebra $q\colon (\bar{A},\bar{\alpha}) \twoheadrightarrow (B,\beta)$. Then $q \cdot e_{(A,\alpha)}\colon (A,\alpha) \twoheadrightarrow (B,\beta)$ is a strong quotient coalgebra. Hence, by the construction of $e_{(A,\alpha)}$ we have a homomorphism $f\colon (B,\beta) \to (\bar{A},\bar{\alpha})$ such that $f \cdot q \cdot e_{(A,\alpha)} = e_{(A,\alpha)}$. Since $e_{(A,\alpha)}$ is epic, we obtain $f \cdot q = \mathsf{id}_B$. Thus, q is a strong epi which is also monic, whence an isomorphism.

(2) In order to prove that $e_{(A,\alpha)}$ is a reflection, let $h\colon (A,\alpha) \to (B,\beta)$ be any coalgebra homomorphism with (B,β) simple. Form a pushout in $\mathsf{Coalg}\, F$ as shown below:

$$\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, two heads, "e_{(A,\alpha)}"'] & B \arrow[d, two heads, "\bar{e}"] \\
\bar{A} \arrow[r, "\bar{h}"'] & \bar{B} \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

By the stability of strong epimorphisms under pushouts, we obtain that $\bar{e}$ is strongly epic. Since (B,β) is simple, we thus know that $\bar{e}$ is an isomorphism and therefore $k = \bar{e}^{-1} \cdot \bar{h}$ is the desired coalgebra homomorphism satisfying $k \cdot e_{(A,\alpha)} = h$. Uniqueness of k is clear since $e_{(A,\alpha)}$ is epic. □

Remark 9.1.7. If the terminal coalgebra νF exists, then the simple reflection of a coalgebra (A,α) is given by the image of the unique coalgebra homomorphism $A \to \nu F$. In other words, an F-coalgebra (A,α) is simple iff the unique homomorphism $h : A \to \nu F$ is a monomorphism. We leave the easy proof to the reader.

Theorem 9.1.8. *If F preserves monos, then the following are equivalent:*

(1) *F has a terminal coalgebra, and*

(2) *F has only a set of simple coalgebra up to isomorphism (of $\mathsf{Coalg}\, F$).*

Proof. Suppose that a terminal coalgebra νF exists. Since $\mathscr{A}$ is wellpowered, νF has only a set of subobjects in $\mathscr{A}$. Every simple coalgebra A gives such a subobject, since the coalgebra homorphism $m_A\colon A \to \nu F$ is, by Observation 9.1.5(1), monic. And if A and B determine the same subobject, then m_A and m_B factor through each other and are thus isomorphic.

Conversely, if (A_i, α_i) $(i \in I)$ is a representative set of all simple coalgebras for F, then we prove that their coproduct

$$(B, \beta) = \coprod_{i \in I} (A_i, \alpha_i)$$

is weakly terminal. The existence of νF then follows from Theorem 4.2.9. For every coalgebra (A, α) since $(\bar{A}, \bar{\alpha})$ is simple, we have an isomorphism $u\colon (\bar{A}, \bar{\alpha}) \to (A_i, \alpha_i)$ for some $i \in I$. Denote by $v_i\colon (A_i, \alpha_i) \to (B, \beta)$ the coproduct injection. Then $v_i \cdot u \cdot e_{(A,\alpha)}\colon (A, \alpha) \to (B, \beta)$ is the desired homomorphism. □

Proposition 9.1.9. *Every subcoalgebra of a simple coalgebra is simple.*

Proof. Suppose that (A, α) is simple, and let $m\colon (B, \beta) \rightarrowtail (A, \alpha)$ represent a subcoalgebra. Given a quotient $e : (B, \beta) \twoheadrightarrow (C, \gamma)$, we prove that e is an isomorphism. Form the following pushout in $\mathsf{Coalg}\, F$:

$$\begin{tikzcd} B \arrow[r, tail, "m"] \arrow[d, two heads, "e"'] & A \arrow[d, two heads, "\bar{e}"] \\ C \arrow[r, tail, "\bar{m}"'] & \bar{C} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Since e is a strong epi in $\mathscr{A}$, so is $\bar{e}$, i.e. $\bar{e}$ represents a strong quotient coalgebra in $\mathsf{Coalg}\, F$. Hence, $\bar{e}$ is an isomorphism by simplicity. Thus, $\bar{m} \cdot e = \bar{e} \cdot m$ is monic, whence e is monic. Since e is a strong epi which is also a mono, it is an isomorphism, and we are done. □

9.2 Pointed and Reachable Coalgebras

In this section we study reachability on the level of coalgebras. This is the second ingredient of state minimality, besides observability.

Reachability has a simple formulation: a coalgebra with a given distinguished point (thought of as an initial state) is reachable iff it does not have any proper pointed subcoalgebras. We will also see an iterative construction of the reachable part of a given pointed coalgebra, which is reminiscent of the usual breadth-first search algorithm for computing the reachable part of a pointed graph. This construction appears in work by Barlocco et al. [47], and independently by Wißmann et al. [168].

Assumption 9.2.1. Throughout this section $\mathscr{A}$ denotes a complete and well-powered category, and $F\colon \mathscr{A} \to \mathscr{A}$ a functor preserving intersections.

We denote by 1 the terminal object of $\mathscr{A}$. Morphisms $1 \to A$ are called *points* (aka *global elements*) of the object A.

Definition 9.2.2 (Adámek et al. [29])**.** (1) By a *pointed coalgebra* is meant a triple (A, α, x) consisting of a coalgebra $\alpha\colon A \to FA$ and a point $x\colon 1 \to A$. The category

$$\mathsf{Coalg}_{\mathsf{p}}\, F$$

of pointed coalgebras has as morphisms from (A, α, x) to (B, β, y) those coalgebra homomorphisms $f\colon (A, \alpha) \to (B, \beta)$ which preserve the point, i.e. $f \cdot x = y$.

(2) A pointed coalgebra (A, α, x) is called *reachable* if it has no proper pointed subcoalgebra. That is, every homomorphism $m\colon (A', \alpha', x') \rightarrowtail (A, \alpha, x)$ of pointed coalgebras where m is a monomorphism of $\mathscr{A}$ is an isomorphism.

Examples 9.2.3. (1) A deterministic automaton with a given initial state is a pointed coalgebra for the set functor $FX = \{0,1\} \times X^\Sigma$. Reachability means that every state can be reached (in finitely many steps) from the initial state.

(2) For the power-set functor the pointed coalgebras are the pointed directed graphs. Reachability here means that every vertex can be reached by a directed paths from the distinguished one.

(3) A labelled transition systems with a given initial state is a pointed coalgebra for $FX = \mathscr{P}(\Sigma \times X)$ on Set. It is reachable iff every state can be reached from the initial one by a finite sequence of transitions.

Definition 9.2.4. By the *reachable part* of a pointed coalgebra we mean its smallest pointed subcoalgebra.

Example 9.2.5. (1) For a deterministic automaton as a pointed coalgebra for $FX = \{0,1\} \times X^\Sigma$ the reachable part is the subautomaton given by all states reachable (by input words from Σ) from the initial state.

(2) Analogous, a labelled transition system as a pointed coalgebra for $\mathscr{P}(\Sigma \times A)$ has the reachable part given by all states reachable (by a sequence of action from Σ) from the initial state.

Proposition 9.2.6. *Every pointed F-coalgebra has a reachable part.*

Proof. Recall from Proposition 4.1.3 that the forgetful functor $\mathsf{Coalg}\, F \to \mathscr{A}$ creates all limits that F preserves. Hence, since F is assumed to preserve intersections, it follows that intersections (of coalgebra homomorphisms carried by monomorphisms) exist in $\mathsf{Coalg}\, F$ and are formed on the level of $\mathscr{A}$.

Given a pointed F-coalgebra (A, α, x) one forms the intersections $m_0\colon (A, \alpha_0) \rightarrowtail (A, \alpha)$ of all its pointed subcoalgebras in $\mathsf{Coalg}\, F$. Since this intersection is formed on the level of $\mathscr{A}$, we see that m_0 is the intersection of all those $m\colon B \rightarrowtail A$ in $\mathscr{A}$ that carry a subcoalgebra and such that $x\colon 1 \to A$ factorizes through m. Indeed, one uses the universal property of the pullback to obtain the latter factorizations. Hence, we have $x_0\colon 1 \to A_0$ such that $m_0 \cdot x_0 = m$. Then $m_0\colon (A_0, \alpha_0, x_0) \rightarrowtail (A, \alpha, x)$ is the desired reachable part of (A, α, x). ◻

Remark 9.2.7. The reachable part is the unique reachable subcoalgebra of (A, α, x). For given any reachable subcoalgebra $m'\colon (A', \alpha', x') \to (A, \alpha, x)$, its intersection with the reachable part m_0 is a pointed subcoalgebra of both (A_0, α_0, x_0) and (A', α', x') and so m_0 and m' represent the same subobject.

Remark 9.2.8. (1) Since our base category $\mathscr{A}$ is complete and well-powered, it has (strong epi, mono)-factorizations of morphisms (see [20, Theorem 14.17]).

(2) Furthermore, recall from Remark 8.3.2 that for every object A of $\mathscr{A}$ the poset $\mathsf{Sub}(A)$ of subobjects of A is a complete lattice with meets obtained by forming intersections, i.e. (wide) pullbacks of monomorphisms.

We are ready to proceed to an iterative construction of the reachable part. Recall that under our assumption that the functor $F\colon \mathscr{A} \to \mathscr{A}$ preserves intersections we know that for every F-coalgebra (A, α) the next time operator $\bigcirc$ has a left adjoint by Proposition 8.3.14.

Definition 9.2.9. For every F-coalgebra (A, α) the *previous time* operator is the left adjoint of the next time operator. We denote it by

$$\ominus\colon \mathsf{Sub}(A) \to \mathsf{Sub}(A).$$

Remark 9.2.10. (1) More explicitly, $\ominus$ assigns to every subobject $s\colon S \rightarrowtail A$ in $\mathscr{A}$ the intersection of all subobjects m with $s \le \bigcirc m$. This follows from the formula for the left adjoint given in Remark 8.3.4:

$$\ominus s = \bigwedge_{s \le \bigcirc m} m.$$

(2) Thus, for the least subobject $\perp$ of A, we have $\ominus \perp = \perp$.

(3) It also follows that $\alpha \cdot s$ factorizes through $F(\ominus s)$:

$$\begin{tikzcd} S \arrow[r, dashed, "\exists"] \arrow[d, tail, "s"'] & F(\ominus S) \arrow[d, tail, "F(\ominus s)"] \\ A \arrow[r, "\alpha"] & FA \end{tikzcd}$$

Indeed, since F preserves intersections, $F(\ominus s) = \bigwedge_{s \le \bigcirc m} Fm$. Thus, we just need to verify that for every $m\colon M \rightarrowtail A$ with $s \le \bigcirc m$, $\alpha \cdot s$ factorizes through Fm:

$$\begin{tikzcd} S \arrow[r, dashed, tail] \arrow[dr, tail, "s"'] & \bigcirc M \arrow[r, "\alpha(m)"] \arrow[d, tail, "\bigcirc m"] \arrow[dr, phantom, "\lrcorner", very near start] & FM \arrow[d, tail, "Fm"] \\ & A \arrow[r, "\alpha"] & FA \end{tikzcd}$$

Proposition 9.2.11. *The previous time operator assigns to every subobject s of the carrier of a coalgebra (A, α) the least subobject m of A for which $\alpha \cdot s$ factorizes through Fm.*

Proof. Given such a factorization $g\colon S \to FM$, i.e. we have $\alpha \cdot s = Fm \cdot g$, it is our task to prove that $\ominus s \le m$. One uses the universal property of the pullback to obtain the dashed arrow in the diagram below:

$$\begin{tikzcd} S \arrow[r, dashed, tail] \arrow[rr, bend left, "g"] \arrow[dr, tail, "s"'] & \bigcirc M \arrow[r, "\alpha(m)"] \arrow[d, tail, "\bigcirc m"] \arrow[dr, phantom, "\lrcorner", very near start] & FM \arrow[d, tail, "Fm"] \\ & A \arrow[r, "\alpha"] & FA \end{tikzcd}$$

This proves that $s \leq \bigcirc m$, and equivalently, $\ominus s \leq m$. $\square$

The name of the operator $\ominus$ comes from the fact that it is a generalized semantic counterpart of the previous time operator of classical linear temporal logic (see e.g. Manna and Pnüeli [126]) as we will now illustrate.

Examples 9.2.12. (1) For the functor $FX = \mathscr{P}(\Sigma \times X)$ on Set, consider a coalgebra $\alpha\colon A \to \mathscr{P}(\Sigma \times A)$, i.e. a labelled transition system. Then for every subset $S \subseteq A$, the set $\ominus S$ consists of those states which are reachable from S by a single transition:

$$\ominus S = \{y \in A \mid y \in \alpha(s, x) \text{ for some } s \in \Sigma \text{ and } x \in S\}.$$

Cf. Example 8.3.7(2) on the "next time" operator $\bigcirc$.

(2) Analogously for graphs as coalgebras for $\mathscr{P}$: for a given set S of vertices, $\ominus S$ consists of all successor vertices of S.

(3) Finally, deterministic automata A considered as coalgebras for $FX = \{0, 1\} \times X^{\Sigma}$ have an analogous description of the previous time operator: for a set $S \subset A$ of states, $\ominus S$ are the states reachable from a state in S by a single transition.

In the case where $\mathscr{A} = \mathsf{Set}$, the canonical graph (see Definition 8.1.5) of a given pointed F-coalgebra is a pointed graph. Moreover, the operator $\ominus$ can be computed on the canonical graph:

Corollary 9.2.13. *The previous time operator of a coalgebra and its canonical pointed graph are the same.*

Indeed, this follows from Lemma 8.3.88.3.9 and the fact that left adjoints are unique.

Corollary 9.2.14. *A pointed coalgebra for an intersection preserving set functor is reachable iff so is its canonical pointed graph.*

This result can be used to prove a coalgebraic version of the well-known consequence of Kőnig's lemma [108] that every finitely branching, well-founded and reachable graph (in the sense of Definition 9.2.2(2)) is finite. This statement holds more generally for every finitary endofunctor on sets.

Lemma 9.2.15. *Every well-founded, reachable coalgebra for a finitary set functor is finite.*

Proof. Let (A, α, x) be a well-founded, reachable coalgebra for the finitary set funtor F. Suppose first that F preserves intersections. Then we construct the canonical graph

$G = (A, \tau_A \cdot \alpha)$. Since F is finitary, one can show that G is a finitely-branching graph (use **??**). Moreover, G is well-founded by Corollary 8.3.10 and reachable (from the given point $x \in A$) by Corollary 9.2.14. It follows from Kőnig's Lemma that G has only finitely many vertices.

In general, a finitary functor F need not preserve intersections. In this case, consider the Trnková hull $\bar{F}$ (see Proposition 4.4.4), whose category of coalgebras is clearly isomorphic to that of F-coalgebras. From Corollary 4.4.6 we know that $\bar{F}$ preserves intersections.

Thus, the only fact we need to prove is that the given well-founded and reachable coalgebra (A, α, x) is well-founded and reachable as an $\bar{F}$-coalgebra. Recall that $FA = \bar{F}A$ since A is nonempty because $x \in A$ is given.

For reachability this is clear because pointed subcoalgebras of (A, α, x) are nonempty, and therefore they are the same for F and $\bar{F}$. For well-foundedness, see the proof of Corollary 8.6.8. □

Barlocco et al. [47] proved the following fact. Here we obtain it as a corollary of Lemma 8.3.13(1).

Corollary 9.2.16. *Let (A, α) be an coalgebra. A subobject $m\colon A' \rightarrowtail A$ carries a subcoalgebra of (A, α) if and only if m is a prefixed point of $\ominus$, i.e. $\ominus m \leq m$.*

Indeed, the above inequality is equivalent to $m \leq \bigcirc m$.

Remark 9.2.17. For every pointed coalgebra (A, α, x) we have, besides the previous time operator $\ominus$ the constant endomap on $\mathsf{Sub}(A)$ with value $x\colon 1 \to A$. Indeed, since 1 is a terminal object, x is a monomorphism. We can then form their (pointwise) join

$$x \vee \ominus(-)\colon \mathsf{Sub}(A) \to \mathsf{Sub}(A),$$

which assigns to every subobject $m\colon S \rightarrowtail A$ the join of x and $\ominus m$ in the complete lattice $\mathsf{Sub}(A)$.

Theorem 9.2.18 [47]**.** *A pointed coalgebra (A, α, x) is reachable if and only if the operator $x \vee \ominus(-)$ on $\mathsf{Sub}(A)$ has the unique fixed point $\top = \mathsf{id}_A$.*

Proof. Suppose that (A, α, x) is reachable, and let $r\colon R \rightarrowtail A$ be the least fixed point of $x \vee \ominus(-)$. Then we have $\ominus(r) \leq x \vee \ominus(r) = r$. Hence, by Corollary 9.2.16, R carries a subcoalgebra of (A, α) via r. Moreover, we have $x \leq x \vee \ominus(r) = r$, i.e. we have a morphism $x_0\colon 1 \to R$ such that $r \cdot x_0 = x$. Thus R is a pointed subcoalgebra of A via r, which implies that r is an isomorphism.

Conversely, suppose that id_A is the only fixed point of $x \vee \ominus(-)$. Let $m\colon (A', \alpha', x') \rightarrowtail (A, \alpha, x)$ be any pointed subcoalgebra. By Corollary 9.2.16, we know that $\ominus m \leq m$. Since $m \cdot x' = x$, we also know $x \leq m$. Thus $x \vee \ominus m \leq m$, i.e. m is a prefixed point of $x \vee \ominus(-)$. By the proof of Theorem 6.1.1, the least fixed point is also the least prefixed point. Thus, we obtain $\mathsf{id}_A \leq m$, which implies that m is an isomorphism. □

Corollary 9.2.19. *For every pointed coalgebra (A, α, x) its reachable part is the least fixed point of $x \vee \ominus(-)$.*

A related construction of the reachable part was given independently and almost at the same time by Wißmann et al. [168]. It can be obtained as a consequence of Corollary 9.2.19 using the following general fact from lattice theory.

Remark 9.2.20. For every join-preserving map $\varphi\colon L \to L$ on a complete lattice L, and every $\ell \in L$ the least fixed point of $\ell \vee \varphi(-)$ is given by the join

$$\bigvee_{i<\omega} \varphi^i(\ell). \tag{9.1}$$

To see this, note that $\ell \vee \varphi(-)$ preserves joins. Hence, by the proof of Kleene's Fixed Point Theorem 3.1.1, the least fixed point of $\ell \vee \varphi(-)$ is the join of the ω-chain given by $x_0 = \bot$, the least element of L, and $x_{n+1} = \ell \vee \varphi(x_n)$. Since φ preserves joins, we know that $\varphi(\bot) = \bot$ and furthermore that this ω-chain is the following one:

$$\bot, \quad \ell \vee \varphi(\bot) = \ell, \quad \ell \vee \varphi(\ell), \quad x \vee \varphi(\ell) \vee \varphi^2(\ell), \quad \dots,$$

whose join clearly is the one in (9.1).

Since the left adjoint $\ominus$ is a join-preserving map on the complete lattice $\mathsf{Sub}(A)$, we obtain:

Corollary 9.2.21 [168]**.** *For every pointed coalgebra (A, α, x) its reachable part is carried by the following union of subobjects of A:*

$$\bigvee_{i<\omega} \ominus^i(x) = x \vee \ominus(x) \vee \ominus(\ominus x)) \vee \cdots . \tag{9.2}$$

More explicitly, one defines subobjects $m_i\colon A_i \rightarrowtail A$ as follows: $m_0 = x$, and given $m_i\colon A_i \rightarrowtail A$, $m_{i+1} = \ominus m_i\colon A_{i+1} = \ominus A_i \rightarrowtail A$ is the least subobject such that $\alpha \cdot m_i$ factorizes through Fm_{i+1}:

$$\begin{tikzcd}
A_i \arrow[r] \arrow[d, tail, "m_i"'] & A_{i+1} = \ominus A_i \arrow[d, tail, "m_{i+1} = \ominus m_i"] \\
A \arrow[r, "\alpha"] & FA
\end{tikzcd}$$

The reachable part is then the union of all $m_i\colon A_i \rightarrowtail A$ for $i < \omega$.

Remark 9.2.22. Note that the above subobjects m_i yield a precise connection to standard algorithms for computing reachability. Indeed, we may compute the subsets $m_i\colon A_i \hookrightarrow A$ as subgraphs of the canonical graph of (A, α, x). This follows by an easy induction from Corollary 9.2.13. Furthermore, observe that in this case the i^{th} subset $m_i\colon A_i \hookrightarrow A$ consists of precisely those states of A that are reachable by a directed path of length precisely i from the initial state x. Consequently, one can compute the reachable part of a given pointed coalgebra by a standard graph algorithm such as breadth-first search.

Theorem 9.2.23. *Suppose that F preserves inverse images. Then the full subcategory of $\mathsf{Coalg}_{\mathsf{p}} F$ given by all reachable coalgebras is coreflective in $\mathsf{Alg}\, F$.*

Proof. Let (A, α, x) be a pointed coalgebra and $m\colon (A_0, \alpha_0, x_0) \rightarrowtail (A, \alpha, x)$ its reachable part. We will show that this is a coreflection.

Given a homomorphism $h\colon (B, \beta, y) \to (A, \alpha, x)$ where (B, β, y) is reachable, we need to prove that h factorizes uniquely through m. Uniqueness is clear since m is monic. For the existence, we form the inverse image of m under h:

$$\begin{tikzcd}
P \arrow[r, "h'"] \arrow[d, tail, "m'"'] \arrow[dr, phantom, "\lrcorner", very near start] & R \arrow[d, tail, "m"] \\
B \arrow[r, "h"'] & A
\end{tikzcd} \tag{9.3}$$

Since $m \cdot x_0 = x = h \cdot y$, we obtain a point $z\colon 1 \to P$, and since F preserves inverse images we also obtain a coalgebra structure π:

$$\begin{tikzcd}
 & 1 \arrow[d, dashed, "\exists! z"] \arrow[ddl, bend right, "y"'] \arrow[ddr, bend left, "x_0"] \arrow[dddl, phantom] \arrow[ddd, bend right=120, "x"'] \arrow[ddd, bend left=120, "x"] & \\
 & P \arrow[dl, tail, "m'"'] \arrow[dr, "h'"] & \\
B \arrow[dr, "h"'] & & A_0 \arrow[dl, tail, "m"] \\
 & A &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & P \arrow[dl, tail, "m'"'] \arrow[d, dashed, "\exists! \pi"] \arrow[dr, "h'"] & \\
B \arrow[d, "\beta"] \arrow[dd, bend right=60, "h"'] & FP \arrow[dl, tail, "Fm'"'] \arrow[dr, "Fh'"] & A_0 \arrow[d, "\alpha_0"'] \arrow[dd, tail, bend left=60, "m"] \\
FB \arrow[dr, "Fh"'] & & FA_0 \arrow[dl, tail, "Fm"] \\
A \arrow[r, "\alpha"'] & FA & A \arrow[l, "\alpha"]
\end{tikzcd}$$

This defines a pointed coalgebra (P, π, z) making m' and h' pointed coalgebra homomorphisms. Since (B, β, y) is reachable and m' is monic, the latter must be an isomorphism. Thus, $h' \cdot (m')^{-1}$ is the desired factorization of h through m, cf. (9.3). □

Corollary 9.2.24. *If F preserves inverse images, then all quotients of a reachable F-coalgebra in $\mathsf{Coalg_p}\, F$ are reachable.*

Proof. Suppose we have a reachable coalgebra (A, α, x) and a quotient $e\colon (A, \alpha, x) \twoheadrightarrow (B, \beta, y)$ in $\mathsf{Coalg_p}\, F$. Denote by $m\colon (B_0, \beta_0, y_0) \rightarrowtail (B, \beta, y)$ the reachable part of (B, β, y). By Theorem 9.2.23, e factorizes through m:

$$\begin{tikzcd}
(B_0, \beta_0, y_0) \arrow[r, tail, "m"] & (B, \beta, y) \\
(A, \alpha, x) \arrow[u, dashed, "h"] \arrow[ur, two heads, "e"'] &
\end{tikzcd}$$

Using the diagonal fill-in property, we obtain a unique coalgebra homomorphism $d\colon (B, \beta, y) \to (B_0, \beta_0, y_0)$ with $h = d \cdot e$ and $m \cdot d = \mathsf{id}$. This implies that m is a split epimorphism and a monomorphism. Thus, it is an isomorphism. □

Example 9.2.25. For functors not preserving inverse images, reachable coalgebras need not be closed under quotients. For example, recall the functor $R\colon \mathsf{Set} \to \mathsf{Set}$

from Example 7.3.4, which preserves intersections but not inverse images. Consider the coalgebras $\gamma\colon C \to RC$ with $C = \{x, y, z\}$ and $\gamma(x) = (y, z)$ and $\gamma(y) = \gamma(z) = *$ and $\delta\colon D \to RD$ with $D = \{x', y'\}$ and $\delta(x') = \delta(y') = *$. Then (D, δ) is a quotient of (C, γ) via the coalgebra homomorphism q with $q(x) = x'$ and $q(y) = q(z) = y'$. However, (C, γ, x) is reachable, whereas (D, δ, x') is not.

Note that, in the light of the proof of Corollary 9.2.24, this example also shows that reachable coalgebras need not form a coreflective subcategory if F does not preserve inverse images.

We have seen that intersection preservation by F plays an important role in the results above on the reachable part of a pointed coalgebra. Let us remark that this property has an equivalent characterization in terms of the least subobjects that the previous time operator $\ominus$ delivers (cf. Proposition 9.2.11):

Remark 9.2.26. (1) Let $F\colon \mathscr{A} \to A$ preserve monomorphisms. Then F preserves intersections if and only if

for every morphism $f\colon X \to FY$ there is a *least* subobject $m\colon Z \rightarrowtail Y$ such that f factorizes through Fm. (9.4)

This means that there exists some $g\colon X \to FZ$ with

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g"'] & FY \\ & FZ \arrow[u, "Fm"'] \end{tikzcd}$$

and satisfying the following universal property: For every subobject $m'\colon Z' \rightarrowtail Y$ and every morphism $g'\colon X \to FZ'$ with $Fm' \cdot g' = f$ there exists a (necessarily unique) $h\colon Z \rightarrowtail Z'$ with $m' \cdot h = m$.

For $\mathscr{A} = \mathsf{Set}$, the above equivalence was established by Gumm [88, Corollary 4.8]. For a proof for the current setting of a complete and well-powered category see Wißmann at al. [168, Proposition 5.9].

(2) For a morphism $f\colon X \to FY$, the above factor g provides the part of Y used by f. More concretely, consider a polynomial functor $F = H_\Sigma$ on Set. Then for every $f\colon A \to H_\Sigma B$ the bound of f consists of all those elements occuring in the flat terms $f(a)$, $a \in A$.

We finish this section with a few remarks on generalizations of the results on reachability presented here.

Remark 9.2.27. (1) The results presented in this section hold more generally in a category $\mathscr{A}$ equipped with a class $\mathcal{M}$ of monomorphisms. Indeed, $\mathsf{Sub}(A)$ is then replaced by the class of subobjects of A represented by $\mathcal{M}$-morphisms $m\colon S \rightarrowtail A$. One requires that $\mathsf{Sub}(A)$ is a complete lattice for every object A, that inverse images (i.e. pullbacks along monomorphisms in $\mathcal{M}$) exist, and that for every morphism $f\colon A \to B$ the map $f^*\colon \mathsf{Sub}(B) \to \mathsf{Sub}(A)$ given by forming inverse images is a right adjoint.

(2) Of course, these requirements hold in our present setting (see Assumption 9.2.1) where $\mathcal{M}$ is the class of all monomorphisms. Another class of examples are well-powered categories $\mathscr{A}$ having coproducts and an $(\mathcal{E}, \mathcal{M})$-factorization system where $\mathcal{M}$ is a class of monomorphisms. It is then easy to see that inverse images exist (see Wißmann et al. [168, Remark 5.2(3)]). Moreover, for every $f\colon X \to Y$, the left adjoint to f^* is $f_*\colon \mathsf{Sub}(X) \to \mathsf{Sub}(Y)$, mapping an $\mathcal{M}$-subobject $s\colon S \rightarrowtail X$ to the *image* m of $f \cdot s$, i.e. m is given by taking an $(\mathcal{E}, \mathcal{M})$ factorization $m \cdot e$ of $f \cdot s$.

As explained in *loc. cit.*, this setting allows one to construct the reachable part of coalgebras over categories which are not complete, such as Rel, the category of sets and relations (viz. the Kleisli category for the power-set monad $\mathscr{P}$), and other Kleisli categories, e.g. the one for the distribution monad.

(3) In some applications, and notably in the setting outlined in point (2), it is desirable to replace the terminal object in a (generalized) element $1 \to A$ by a different object I. This leads to the notion of an *I-pointed F-coalgebra*, i.e. an F-coalgebra (A, α) equipped with a morphism $i\colon I \to A$. In fact, Wißmann et al. [168] formulated and proved the results of this section for I-pointed F-coalgebras over a category satisfying the assumptions in (2) above.

(4) Finally, let us come back to Kleisli categories for a monad. They are used as base categories when one wants to obtain the finite trace or language semantics of state-based systems modelled as coalgebras (cf. Example 5.1.27, and for further related examples see **??**).

Wißmann et al. [168, Section 6] show how to obtain the reachable part of a coalgebra over a Kleisli category, more generally than in point (2), even if this category is not equipped with an $(\mathcal{E}, \mathcal{M})$-factorization system. In fact, one works under an additional assumptions on the base category $\mathscr{A}$, and one assumes that the monad T on $\mathscr{A}$, and the coalgebraic type functor $F\colon \mathscr{A} \to \mathscr{A}$ lifting to $\bar{F}$ on the Kleisli category $\mathscr{A}_T$, preserve finite intersection. One then obtains that the reachable part of a pointed $\bar{F}$-coalgebra is the reachable part of a related coalgebra for F in $\mathscr{A}$.

9.3 Well-pointed Coalgebras

In this section we turn to the study of well-pointed coalgebras which capture the notion of minimality (e.g. of deterministic automata). We also present a new description of the initial algebra and the terminal coalgebra for set functors preserving intersections (cf. **??**): we describe νF as the set of all well-pointed F-coalgebras up to isomorphism, and μF consists of all well-pointed, well-founded F-coalgebras.

We restrict our presentation of these descriptions to endofunctors on Set. However, the same description holds for endofunctors on concrete categories $\mathscr{A}$ satisfying a number of natural assumptions (see Adámek et al. [29]), in particular for endofunctors on varieties, as we mention at the end of this section.

Assumption 9.3.1. Throughout this section F denotes a set functor preserving intersections.

Definition 9.3.2 [29]**.** A pointed coalgebra is called *well-pointed* if it is reachable and simple (i.e. it has no proper strong quotient and no proper subobject in $\mathsf{Coalg_p}\, F$).

Examples 9.3.3. We combine Example 9.1.4 and Example 9.2.3:

(1) A deterministic automaton with a given initial state is a well-pointed coalgebra for $FX = \{0,1\} \times X^\Sigma$ iff it is reachable and observable (= simple), i.e. iff it is a minimal automaton.

(2) A well-pointed $\mathscr{P}$-coalgebra is a pointed graph which is reachable in the usual sense (see Example 9.2.3(2)) and simple, i.e. it has no distinct bisimilar vertices (see Example 9.1.4(3)).

(3) A labelled transition systems with a given initial state is a well-pointed coalgebra for $FX = (\mathscr{P}X)^\Sigma$ iff it is reachable in the usual sense (see Example 9.2.3(3)) and no distinct states are bisimilar.

Notation 9.3.4. There is a canonical construction for turning an arbitrary pointed coalgebra (A, α, x) into a well-pointed one: first form the simple quotient $(\bar{A}, \bar{\alpha})$ (see Proposition 9.1.6) pointed by $e_{(A,\alpha)}(x) \in \bar{A}$, and then form the reachable part:

$$\begin{tikzcd}
 & & \bar{A}_0 \arrow[d, "m"] \\
1 \arrow[urr, dashed, "x_0"] \arrow[r, "x"'] & A \arrow[r, "e_{(A,\alpha)}"'] & \bar{A}
\end{tikzcd}
\qquad
\begin{tikzcd}
\bar{A}_0 \arrow[r, "\bar{\alpha}_0"] \arrow[d, tail, "m"'] & F\bar{A}_0 \arrow[d, "Fm"] \\
\bar{A} \arrow[r, "\bar{\alpha}"'] & F\bar{A}
\end{tikzcd}$$

Then $(\bar{A}_0, \bar{\alpha}_0, x_0)$ is well-pointed by Proposition 9.1.9 and Proposition 9.2.6. We denote the well-pointed coalgebra $(\bar{A}_0, \bar{\alpha}_0, x_0)$ (which is unique up to isomorphism) by

$$\mathsf{wp}(A, \alpha, x)$$

and call it the *well-pointed modification* of (A, α, x).

Remark 9.3.5. As in the case of deterministic automata, well-pointed coalgebras are rather rare in general. In fact, for many functors there exists, up to isomorphism of $\mathsf{Coalg_p}\, F$, only a set of well-pointed coalgebras. In such a case, we can choose a set T of representatives of the isomorphism classes of well-pointed coalgebras. Then we can assume, without loss of generality, that all well-pointed modifications are chosen to be elements of T. For every coalgebra $\alpha\colon A \to FA$ we have a function

$$\alpha^+\colon A \to T \qquad \text{defined by} \qquad \alpha^+(x) = \mathsf{wp}(A, \alpha, x).$$

Definition 9.3.6. For each coalgebra $\alpha\colon A \to FA$ and each $x \in FA$, there is a least subcoalgebra $B_x \subseteq A$ such that $x \in FB$. Indeed,

$$B_x = \bigcap \{A_0 \subseteq A : A_0 \text{ is a subcoalgebra, and } x \in FA_0\}$$

We call B_x the *subcoalgebra generated by x*.

Lemma 9.3.7 [29]**.** *For every coalgebra homomorphism $h : (A,\alpha) \to (B,\beta)$ the following triangle commutes:*

$$\begin{tikzcd} A \arrow[rr, "h"] \arrow[dr, "\alpha^+"'] & & B \arrow[dl, "\beta^+"] \\ & T & \end{tikzcd}$$

Proof. (1) Assume that first both coalgebras above are simple. In particular, h is a monomorphism by simplicity of (B,β) (see Observation 9.1.5(1)). For every element $x \in A$ we know that $\alpha^+(x)$ is the subcoalgebra $m : (A_0,\alpha_0) \rightarrowtail (A,\alpha)$ generated by x. Therefore $h \cdot m : (A_0,\alpha_0) \to (B,\beta)$ is a subcoalgebra of $(B,\beta,h(x))$, and since (A_0,α_0,x_0), with $m(x_0) = x$, is well-pointed, we conclude that it is isomorphic to $\beta^+(h(x))$. Now T contains just one representative of every well-pointed coalgebra up to isomorphism, consequently, $\beta^+(h(x)) = \alpha^+(x)$.

(2) If the two coalgebras are arbitrary, form their simple reflections $\bar{h}$ of h (see Proposition 9.1.6):

$$\begin{tikzcd} (A,\alpha) \arrow[r, "h"] \arrow[d, two heads, "e_{(A,\alpha)}"'] & (B,\beta) \arrow[d, two heads, "e_{(B,\beta)}"] \\ (\bar{A},\bar{\alpha}) \arrow[r, "\bar{h}"'] & (\bar{B},\bar{\beta}) \end{tikzcd}$$

Then for every element $x \in A$ we have that $\alpha^+(x)$ is the subcoalgebra of $(\bar{A},\bar{\alpha})$ generated by $\bar{x} = e_{(A,\alpha)}(x)$, thus $\alpha^+(x) = \bar{\alpha}^+(\bar{x})$; analogously for $\beta^+(h(x))$. By applying (1) to $\bar{h}$ in lieu of h we conclude $\alpha^+(x) = \bar{\alpha}^+(\bar{x}) = \bar{\beta}^+(\bar{h}(\bar{x})) = \beta^+(h(x))$. ☐

Theorem 9.3.8. *A set functor F preserving intersections has a terminal coalgebra iff it has only a set T of well-pointed coalgebras up to isomorphism. Moreover, T carries the terminal coalgebra with the structure $\tau\colon T \to FT$ assigning to every element (A,α,x) of T the following element of FT:*

$$\tau(A,\alpha,x) = (1 \xrightarrow{x} A \xrightarrow{\alpha} FA \xrightarrow{F\alpha^+} FT). \tag{9.5}$$

Proof. Necessity follows from Theorem 9.1.8. Let us prove sufficiency. Assume that T above is given. For every coalgebra (A,α) we prove that there is a unique homomorphism into (T,τ).

(1) Let (A,α) be simple. Given an element $x \in A$ we have $\mathsf{wp}(A,\alpha,x) = (A_0,\alpha_0,x)$, where $m : (A_0,\alpha_0) \rightarrowtail (A,\alpha)$ is the subcoalgebra generated by x. We prove the equation

$$F\alpha^+ \cdot \alpha(x) = F\alpha_0^+ \cdot \alpha_0(x). \tag{9.6}$$

Indeed, $\alpha_0^+ \cdot m = \alpha^+$ by Lemma 9.3.7. Since m is a coalgebra homomorphism, we have $Fm \cdot \alpha_0 = \alpha \cdot m$, which implies

$$F\alpha_0^+ \cdot \alpha_0(x) = F\alpha^+ \cdot Fm \cdot \alpha_0(x) = F\alpha^+ \cdot \alpha \cdot m(x).$$

This is the desired equality, since $m(x) = x$. It follows that $\alpha^+\colon (A,\alpha) \to (T,\tau)$ is a homomorphism. Indeed, given $x \in A$ we need to show that $F\alpha^+ \cdot \alpha(x) = \tau \cdot \alpha^+(x)$. Take

(A_0, α_0) as in part (1). Then we have $\alpha^+(x) = (A_0, \alpha_0, x)$, and therefore $\tau \cdot \alpha^+(x) = \tau(A_0, \alpha_0, x) = F\alpha_0^+ \cdot \alpha_0(x)$. Now use (9.6).

(2) For an arbitrary coalgebra (A, α), let $e_{(A,\alpha)} : (A, \alpha) \twoheadrightarrow (\bar{A}, \bar{\alpha})$ be the simple reflection. Then, by Lemma 9.3.7, we have $\alpha^+ = \bar{\alpha}^+ \cdot e_{(A,\alpha)} : (A, \alpha) \to (T, \tau)$, which is a homomorphism since so is $\bar{\alpha}^+$.

(3) Finally, we prove that for every homomorphism $h : (A, \alpha) \to (T, \tau)$ we have $\alpha^+ = h$. Indeed, by Lemma 9.3.7 we have $\tau^+ \cdot h = \alpha^+$, and moreover we show that $\tau^+ = \mathsf{id}_T$. This holds because for every element (A, α, x) of T we have $\alpha^+(x) = (A, \alpha, x)$. By Lemma 9.3.7, $\alpha^+ = \tau^+ \cdot \alpha^+$, which proves that

$$\tau^+(A, \alpha, x) = \tau^+ \cdot \alpha^+(A, \alpha, x) = \alpha^+(A, \alpha, x) = (A, \alpha, x). \qquad \square$$

Remark 9.3.9. We have seen that if T is a *set* which contains representatives of all well-pointed coalgebras, then T is a terminal coalgebra. Even if T is a proper class (of representatives), it is a terminal coalgebra for a functor related to F. More precisely, consider Set as the full subcategory of the category Class of all classes and functions, then F has an essentially unique extension to an endofunctor $\widehat{F}$ on the latter (see Adámek et al. [31]). And T is a terminal coalgebra for $\widehat{F}$. (The proof is completely analogous to that of Theorem 9.3.8.)

Remark 9.3.10. An analogous decription of νF is possible for all endofunctors on a variety $\mathscr{A}$ preserving intersections. Here $\mathscr{A}$ is a category of algebras specified by a (finitary) signature Σ and a set of equations. The concept of a pointed and a well-pointed coalgebra is defined analogously to the case $\mathscr{A} = \mathsf{Set}$: a pointed coalgebra is (A, α, x) where x is an element of the coalgebra (A, α). Thus, we work here with the generalization mentioned in Remark 9.2.27(3): we choose I to be the free algebra on one generator in the variety $\mathscr{A}$. A pointed coalgebra (A, α, x) is well-pointed if (A, α) is simple and has no proper subcoalgebra containing x. Since F preserves intersections, it preserves monomorphisms. Thus, by the proof of Theorem 9.1.8, νF exists iff F has only a set of simple coalgebras (up to isomorphism). Now let T be a representative set of all well-pointed F-coalgebras. Then T carries a canonical structure of an object of $\mathscr{A}$. For example, let σ be a binary operation symbol in Σ. Then the canonical binary operation $\sigma^T : T \times T \to T$ is defined as follows: given a pair of well-pointed coalgebras (A_i, α_i, x_i), $i = 1, 2$, form their coproduct

$$(A, \alpha) = (A_1, \alpha_1) + (A_2, \alpha_2) \qquad \text{in } \mathsf{Coalg}\, F$$

(recall that this is formed on the level of $\mathscr{A}$, see Proposition 4.1.1), and apply $\sigma^A : A \times A \to A$ to obtain $x = \sigma^A(\mathsf{inl}(x_1), \mathsf{inr}(x_2))$ in A. Then (A, α, x) is the result of σ^T in T (see [29] for details). Moreover, it is shown in *loc. cit.* that the Σ-algebra thus defined on T satisfies the equations specifying $\mathscr{A}$, that the map $\tau : T \to FT$ defined in Theorem 9.3.8 is a morphism of $\mathscr{A}$, and that the resulting coalgebra (T, τ) is terminal.

Initial algebras and well-founded well-pointed coalgebras We saw in Theorem 9.3.8 that the terminal coalgebra νF for a set functor preserving intersections is carried by a set which contains all well-pointed coalgebras (up to isomorphism). We now show that the initial algebra μF can be described similarly as the algebra of all well-founded, well-pointed coalgebras (up to isomorphism). For this result we use the fact that μF is, equivalently, the terminal well-founded coalgebra (see Theorem 8.7.12).

Notation 9.3.11. Suppose that F has only a set of well-founded, well-pointed coalgebras up to isomorphism, and choose a set W of representatives. Then for every well-founded (A, α) coalgebra and $x \in A$, the well-pointed modification $\mathsf{wp}(A, \alpha, x)$ can be chosen as an element of W since well-founded coalgebras are closed under subcoalgebras and quotients by Corollary 8.5.6 and Corollary 8.5.3. Thus, analogously to Remark 9.3.5, we have a function

$$\alpha^+ \colon A \to W \qquad \text{defined by} \qquad \alpha^+(x) = \mathsf{wp}(A, \alpha, x).$$

We obtain a coalgebra structure $\tau_0 \colon W \to FW$ analogously to τ in Theorem 9.3.8: τ_0 assigns to (A, α, x) in W the element $F\alpha^+ \cdot \alpha \cdot x \colon 1 \to FW$.

Theorem 9.3.12 [29, Thm. 3.48]**.** *A set functor preserving intersections has an initial algebra iff it has only a set W of well-founded, well-pointed coalgebra up to isomorphism. Moreover, W carries the initial algebra with the structure $\tau_0^{-1} \colon FW \to W$.*

Proof. For the necessity, suppose that μF exists. By Theorem 8.7.12, $(\mu F, \iota^{-1})$ is the terminal well-founded coalgebra. For every well-founded, pointed coalgebra (A, α, x) the unique homomorphism $f \colon (A, \alpha) \to (\mu F, \iota^{-1})$ is injective (see Observation 9.1.5(1)). Since μF has only a set of subsets, this implies that there is only a set of well-founded coalgebras up to isomorphism. Thus, the same holds for well-founded, well-pointed coalgebras.

For the sufficiency, suppose that the set W is given. By Theorem 8.7.12, we only need to prove that (W, τ_0) is a terminal well-founded coalgebra. This is completely analogous to the proof of Theorem 9.3.8. One first proves that for every simple, well-founded coalgebra (A, α), the map α^+ is the unique coalgebra homomorphism from (A, α) to (W, τ_0). Secondly, for an arbitrary well-founded coalgebra one uses that simple reflection (see Proposition 9.1.6), which is well-founded by Corollary 8.5.3. □

Remark 9.3.13. Note that for a finitary set functor F, the elements of $\mu F \cong W$ in Theorem 9.3.12 are finite coalgebras. This follows from Lemma 9.2.15.

We conclude this section with a number of examples of concrete descriptions of initial algebras and terminal coalgebras we obtain from Theorem 9.3.8 and Theorem 9.3.12.

Examples 9.3.14. (1) For the functor $FX = \{0, 1\} \times X^\Sigma$ the terminal coalgebra T is carried by the set of (isomorphism classes of) all minimal deterministic automata with initial states. The coalgebra structure $\tau \colon T \to FT$, interpreted as an automaton on T, has as accepting states those minimal automata A whose initial state is accepting in A, and the next-state function maps a pair consisting of a minimal automaton (A, δ, a_0, F) and an input symbol $s \in \Sigma$ to the minimization of the automaton $(A, \delta, \delta(a_0, s), F)$, i.e. one shifts the initial state along its s-transition and then minimizes the result.

The unique coalgebra homomorphism $\alpha^+ \colon A \to T$ assigns to a state a of the automaton A the minimization of A with that initial state.

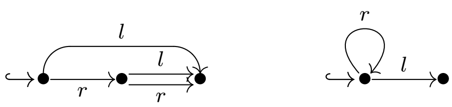
As there are no well-founded pointed coalgebras for F, the initial algebra W in Theorem 9.3.12 is empty.

(2) For the power-set functor $\mathscr{P}$, T is the collection of pointed, reachable and simple graphs (cf. Example 9.3.3(2)), which is *not* a set. However, for the finite power-set functor $\mathscr{P}_\mathsf{f}$ the well-pointed coalgebras are the pointed, reachable and simple finitely branching graphs. Every such graph must be countable, which can be seen using that it is finitely branching and every vertex is reachable from the chosen vertex. Thus we obtain

$$\nu\mathscr{P}_\mathsf{f} \cong \text{all pointed, reachable and simple finitely branching graphs,}$$
$$\mu\mathscr{P}_\mathsf{f} \cong \text{all pointed, reachable, simple and well-founded finitely branching graphs}$$

(up to isomorphism).

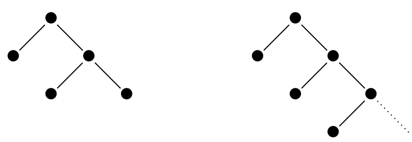
(3) We next consider the functor $FX = X \times X + 1$ on Set. Here we view coalgebras for F (differently than earlier in this chapter) as directed graphs with edges labelled in $\{l, r\}$ such that every node either has exactly one successor node reachable by an l-labelled edge and one by an r-labelled edge, respectively, or no successor node. We call pointed such graphs *binary term graphs.* Here are two examples:



(9.7)

Recall from Remark 2.5.10 that to every node x of a binary term graph A the unique homomorphism $h : A \to \nu F$ assigns its tree expansion: the root is x and a node is either a leaf, if it is a terminating state, or has the two next states as children (left-hand for input l, right-hand for input r).

It follows from Remark 9.1.7 that a binary term graph is a simple coalgebra iff different nodes have different tree expansions. Furthermore, a well-pointed F-coalgebra is a *minimal* binary term graph, i.e. one which is reachable and simple. Note that tree expansion of the given point is a bijection between well-pointed coalgebras and binary trees. Its inverse is given by identifying in a given binary tree systematically all shared subtrees of the nodes. This explains why the following descriptions of μF and νF are isomorphic to the ones in Examples 3.2.5 and 2.5.11(4). For example the two binary trees



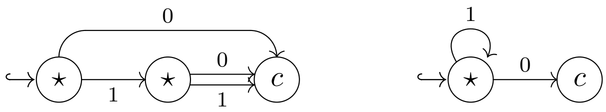
yield the two minimal binary term graphs in (9.7).

From Theorem 9.3.8 and Theorem 9.3.12 we obtain:

$$\nu F \cong \text{all minimal binary term graphs,}$$
$$\mu F \cong \text{all well-founded, minimal binary term graphs}$$

(up to isomorphism). Indeed, the well-founded coalgebras are precisely those yielding a finite tree expansion.

(4) **Σ-term graphs.** More generally, consider a polynomial functor H_Σ (see Example 2.1.4). Analogously to the previous example, well-pointed H_Σ-coalgebras may be identified with minimal Σ-term graphs. A Σ-*term graph* is a pointed directed graph such that every node with n successors is labelled by some $\sigma \in \Sigma_n$, and moreover, the successor nodes are reachable by an edge labelled in $\{0, \dots, n-1\}$ so that each edge label appears precisely once. The previous example is the special case where Σ is a signature containing a binary operations symbol $\star$ and a constant symbol c, and the two binary term graphs in (9.7) above can be presented as Σ-term graphs as follows:



Minimality of a Σ-term graph means that every node is reachable from the given point and that the Σ-tree expansions of distinct nodes are always different. We thus obtain

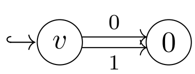
$$\nu H_\Sigma \cong \text{all minimal } \Sigma\text{-term graphs},$$
$$\mu H_\Sigma \cong \text{all well-founded minimal } \Sigma\text{-term graphs}$$

(up to isomorphism).

(5) Binary decision diagram (BDD). Given a set V (of boolean variables), we have seen in Example 3.2.8(2) that all binary decision *trees* form the initial algebra for $FX = \{0,1\} + V \times X \times X$. Now a BDD is given by a pointed directed acyclic graph whose nodes are labelled by variables or by 0 or 1, and every node labelled by a variable has a 0-successor and a 1-successor, whereas nodes labelled by 0 or 1 are leaves (i.e. they do not have successors). Then BDDs are precisely the finite, pointed, well-founded F-coalgebras. In practice, one is usually more interested in *reduced* BDDs, i.e. reachable BDDs that are obtained as the result of a reduction process applying the following two rules:

- merge any two isomorphic subgraphs;
- eliminate any node whose 0- and 1-successors yield isomorphic subgraphs.

Reduced BDDs are well-founded, well-pointed F-coalgebras, but not conversely; for example, the following well-founded well-pointed F-coalgebra is not a reduced BDD:



9.4 Summary of this chapter

We have presented simple and reachable coalgebras, and we have seen constructions of the simple quotient of a coalgebra and the reachable part of a pointed one. In addition, well-pointed coalgebras are the ones which are both reachable and simple, and those yield a coalgebraic formulation of minimality of state-based systems.

For set functors preserving intersections, we also saw new descriptions of the initial algebra and the terminal coalgebra. In fact, the terminal coalgebra is formed by all well-pointed coalgebras (considered up to isomorphism), and the initial algebra is formed by all well-founded, well-pointed coalgebras.

Bibliography

[1] Samson Abramsky. A Cook's tour of the finitary non-well-founded sets. In *We Will Show Them: Essays in honour of Dov Gabbay*, volume 1, pages 1–18. College Publications, 2005.

[2] Samson Abramsky and Achim Jung. *Handbook of Logic in Computer Science*, volume III, chapter Domain Theory, pages 1–168. Clarendon Press, 1994.

[3] Peter Aczel. *Non-Well-Founded Sets*, volume 14 of *CLSI Lecture Notes*. CLSI Publications, Stanford, 1988.

[4] Peter Aczel, Jiří Adámek, Stefan Milius, and Jiří Velebil. Infinite trees and completely iterative theories: A coalgebraic view. *Theoret. Comput. Sci.*, 300:1–45, 2003.

[5] Peter Aczel and Nax Mendler. A final coalgebra theorem. In *Proc. Category Theory and Computer Science (CTCS)*, volume 389 of *Lecture Notes Comput. Sci.*, pages 357–365. Springer, 1989.

[6] J. Adámek, Heinz-Peter Gumm, and Věra Trnková. Presentation of set functors: a coalgebraic perspective. *J. Logic Comput.*, 20(5):991–1015, 2010.

[7] Jirí Adámek, Dominik Lücke, and Stefan Milius. Recursive coalgebras of finitary functors. *Informatique Théorique et Applications*, 41(4):447–462, 2007.

[8] Jirí Adámek, Stefan Milius, and Lawrence S. Moss. On well-founded and recursive coalgebras. draft, 2019.

[9] Jiri Adámek and H-E Porst. On tree coalgebras and coalgebra presentations. *Theoretical Computer Science*, 311(1):257–283, 2004.

[10] Jiří Adámek. Free algebras and automata realizations in the language of categories. *Comment. Math. Univ. Carolin.*, 15:589–602, 1974.

[11] Jiří Adámek. *Theory of Mathematical Structures*. Springer, 1983.

[12] Jiří Adámek. Recursive data types in algebraically ω-complete categories. *Inform. and Comput.*, 118(2):181–190, 1995.

[13] Jiří Adámek. A categorical generalization of scott domains. *Math. Structures Comput. Sci.*, 7:419–443, 1997.

[14] Jiří Adámek. Final coalgebras are ideal completions of initial algebras. *J. Logic Comput.*, 12:217–242, 2002.

[15] Jiří Adámek. On final coalgebras of continuous functors. *Theoret. Comput. Sci.*, 294(1–2):3–29, 2003.

[16] Jiří Adámek. Terminal coalgebras derived from initial algebras. preprint, available online at `https://arxiv.org/abs/1903.02438`, 2019.

[17] Jiří Adámek, Stephen L. Bloom, and Stefan Milius. On algebras with iteration. *J. Logic. Comput.*, 18(6):1047–1085, 2008.

[18] Jiří Adámek, Nathan Bowler, Paul B. Levy, and Stefan Milius. Coproducts of monads on Set. In *Proc. 27th Annual Symposium on Logic in Computer Science (LICS'12)*. IEEE Computer Society, 2012.

[19] Jiří Adámek, Mahdieh Haddadi, and Stefan Milius. Corecursive algebras, corecursive monads and Bloom monads. *Log. Methods Comput. Sci.*, 10(3:19):51 pp., 2014.

[20] Jiří Adámek, Horst Herrlich, and George E. Strecker. *Abstract and Concrete Categories: The Joy of Cats*. Dover Publications, 3rd edition, 2009.

[21] Jiří Adámek and Vaclav Koubek. On the least fixed point of a set functor. *J. Comput. Syst. Sci.*, 19:163–178, 1979.

[22] Jiří Adámek and Vaclav Koubek. On the greatest fixed point of a set functor. *Theoret. Comput. Sci*, 150:57–75, 1995.

[23] Jiří Adámek, Vaclav Koubek, and Thorsten Palm. Fixed points of set functors: how many iterations are needed? *Appl. Categ. Structures*, 24(5):649–661, October 2016.

[24] Jiří Adámek, Vaclav Koubek, and Věra Pohlová. The colimits in the generalized algebraic categories. *Acta Univ. Carolinæ—Math. et Phys.*, 13(2):29–40, 1972.

[25] Jiří Adámek, Paul B. Levy, Lawrence S. Moss, Stefan Milius, and Lurdes Sousa. On final coalgebras of power-set functors and saturated trees. *Appl. Categ. Structures*, 23(4):609–641, August 2015. Available online; `http://dx.doi.org/10.1007/s10485-014-9372-9`.

[26] Jiří Adámek, Dominik Lücke, and Stefan Milius. Recursive coalgebras of finitary functors. *Theor. Inform. Appl.*, 41(4):447–462, 2007.

[27] Jiří Adámek and Stefan Milius. Terminal coalgebras and free iterative theories. *Inform. and Comput.*, 204:1139–1172, 2006.

[28] Jiří Adámek and Stefan Milius. On functors preserving coproducts and algebras with iterartivity. *Theoret. Comput. Sci.*, 763:66–87, 2019.

[29] Jiří Adámek, Stefan Milius, Lawrence S. Moss, and Lurdes Sousa. Well-pointed coalgebras. *Log. Methods Comput. Sci.*, 9(2):1–51, 2014.

[30] Jiří Adámek, Stefan Milius, and Jiří Velebil. Free iterative theories: a coalgebraic view. *Math. Structures Comput. Sci.*, 13(2):259–320, 2003.

[31] Jiří Adámek, Stefan Milius, and Jiří Velebil. On coalgebras based on classes. *Theoret. Comput. Sci.*, 316:3–23, 2004.

[32] Jiří Adámek, Stefan Milius, and Jiří Velebil. Elgot algebras. *Log. Methods Comput. Sci.*, 2(5:4):1–31, 2006.

[33] Jiří Adámek, Stefan Milius, and Jiří Velebil. Iterative algebras at work. *Math. Structures Comput. Sci.*, 16(6):1085–1131, 2006.

[34] Jiří Adámek, Evelyn Nelson, and Jan Reiterman. Tree constructions of free continuous algebras. *J. Comput. System Sci.*, 24:114–146, 1982.

[35] Jiří Adámek and Hans-Eberhardt Porst. On tree coalgebras and coalgebra presentations. *Theoret. Comput. Sci.*, 311:257–283, 2004.

[36] Jiří Adámek and Jan Reitermann. Banach's fixed-point theorem as a base for data-type equations. *Appl. Categ. Structures*, 2:77–90, 1994.

[37] Jiří Adámek and Jiří Rosický. *Locally Presentable and Accessible Categories*. Cambridge University Press, 1994.

[38] Jiří Adámek and Věra Trnková. *Automata and Algebras in Categories*, volume 37 of *Mathematics and its Applications*. Kluwer Academic Publishers, 1990.

[39] Jiří Adámek and Věra Trnková. Initial algebras and terminal coalgebras in many-sorted sets. *Math. Structures Comput. Sci.*, 21:481–509, 2011.

[40] Jiří Adámek and Věra Trnková. Relatively terminal coalgebras. *J. Pure Appl. Algebra*, 216(8–9):1887–1895, 2012.

[41] F. Alessi, P. Baldan, and G. Bellé. A fixed point theorem in a category of compact metric spaces. *Theoret. Comput. Sci.*, 146:311–320, 1995.

[42] Pierre America and Jan J. M. M. Rutten. Solving reflexive domain equations in a category of complete metric spaces. *J. Comput. System Sci.*, 39:343–375, 1989.

[43] H. Applegate. *Acyclic models and resolvent functors*. PhD thesis, Columbia University, 1965.

[44] André Arnold and Maurice Nivat. The metric space of infinite trees. algebraic and topological properties. *Fund. Inform.*, 3(4):445–476, 1980.

[45] F. Baader and C. Tinelli. Deciding the word problem in the union of algebraic theories. *Inf. Comput.*, 13:201–232, 2002.

[46] Adriana Balan and Alexander Kurz. A.: Finitary functors: from set to preord and poset. In *Proc. CALCO 2011*, volume 6859, pages 85–99. Springer LNCS, 2011.

[47] Simone Barlocco, Clemens Kupke, and Jurriaan Rot. Coalgebra learning via duality. In *Proceedings of the 22nd International Conference on Foundations of Software Science and Computation Structures (FoSSaCS'19)*, To appear.

[48] Michael Barr. Coequalizers and free triples. *Math. Z.*, 116:307–322, 1970.

[49] Michael Barr. Algebraically compact functors. *J. Pure. Appl. Algebra*, 82:211–231, 1992.

[50] Michael Barr. Terminal coalgebras in well-founded set theory. *Theoret. Comput. Sci.*, 114(2):299–315, 1993.

[51] Falk Bartels. *On generalized coinduction and probabilistic specification formats.* PhD thesis, Vrije Universiteit Amsterdam, 2004.

[52] Jon Barwise and Lawrence S. Moss. *Vicious Circles.* Number 60 in CSLI Lecture Notes. CSLI Publications, Stanford, 1996.

[53] G. Bergman. On lattices and their ideal lattices, and posets and their ideal posets. *Tbilisi Mathematical Journal*, 1:89–103, 2008.

[54] P. Bhattacharya, L. S. Moss, J. Ratnayake, and R. Rose. Fractal sets as final coalgebras obtained by completing an initial algebra. In *Horizons of the Mind. A Tribute to Prakash Panangaden*, volume 8464 of *Lecture Notes Comput. Sci.*, pages 146–167. Springer, 2014.

[55] L. Birkedal, K. Støvring, and J. Thamsborg. The category-theoretic solution of recursive metric-space equations. *Theoret. Comput. Sci.*, 411:4102–4122, 2010.

[56] Bard Bloom, Sorin Istrail, and Albert R. Meyer. Bisimulation can't be traced. *J. ACM*, 42(1):232–268, 1995.

[57] Stephen L. Bloom and Zoltán Ésik. *Iteration Theories: the equational logic of iterative processes.* EATCS Monographs on Theoretical Computer Science. Springer, 1993.

[58] Henno Brandsma. Ask a topologist: Shrinking map on a compact set has a fixed point. `http://at.yorku.ca/cgi-bin/bbqa?forum=ask_a_topologist_2004&task=show_msg&msg=1113.0001`, 2004. Accessed November 19, 2018.

[59] Venanzio Capretta, Tarmo Uustalu, and Varmo Vene. Recursive coalgebras from comonads. *Inform. and Comput.*, 204:437–468, 2006.

[60] Venanzio Capretta, Tarmo Uustalu, and Varmo Vene. Corecursive algebras: A study of general structured corecursion. In M. Oliveira and J. Woodcock, editors, *Formal Methods: Foundations and Applications*, volume 5902 of *Lecture Notes in Computer Science*, pages 84–100. Springer Berlin Heidelberg, 2009.

[61] Aurelio Carboni, G. M. Kelly, and R. J. Wood. A 2-categorical approach to change of base and geometric morphisms i. *Cahiers de Topologie et Géometrie Différentielle Catégoriques*, 32:47?95, 1991.

[62] Aurelio Carboni, Steve Lack, and Robert F. C. Walters. Introduction to extensive and distributive categories. *Journal of Pure and Applied Algebra*, 84(2):145–158, 1993.

[63] Bruno Courcelle. Fundamental properties of infinite trees. *Theoret. Comput. Sci.*, 25:95–169, 1983.

[64] R. O. Davies, A. Hayes, and G. Rousseau. Complete lattices and the generalized Cantor theorem. *Proc. Amer. Math. Soc.*, 27(2):253–258, 1971.

[65] J. W. de Bakker and J. I. Zucker. Processes and the denotational semantics of concurrency. *Inform. and Control*, 54(1–2):70–120, 1982.

[66] Erik P de Vink and Jan JMM Rutten. Bisimulation for probabilistic transition systems: a coalgebraic approach. *Theoretical Computer Science*, 221(1):271–293, 1999.

[67] J.I. den Hartog and Erik de Vink. Building metric structures with the meas-functor. In Frank de Boer, Marlin van der Heijden, Paul Klint, and Jan Rutten, editors, *Liber Americorum Jaco de Bakker*, pages 93–108. CWI Amsterdam, 2002.

[68] Ernst-Erich Doberkat. *Stochastic Relations: Foundations for Markov Transition Systems.* Chapman Hall & CRC Press, Boca Raton, 2007.

[69] Calvin C. Elgot. Monadic computation and iterative algebraic theories. In H. E. Rose and J. C. Sheperdson, editors, *Logic Colloquium '73*, volume 80, pages 175–230, Amsterdam, 1975. North-Holland Publishers.

[70] Calvin C. Elgot, Stephen L. Bloom, and Ralph Tindell. On the algebraic structure of rooted trees. *J. Comput. System Sci.*, 16:362–399, 1978.

[71] Adam Eppendahl. Coalgebra-to-algebra morphisms. In *Proc. Category Theory and Computer Science (CTCS)*, volume 29 of *Electron. Notes Theor. Comput. Sci.*, pages 42–49, 1999.

[72] K. Fine. Normal forms in modal logic. *Notre Dame J. Form. Log.*, 16:229–237, 1975.

[73] Peter Freyd. Algebraically complete categories. In Aurelio Carboni, Maria Cristina Pedicchio, and Guiseppe Rosolini, editors, *Category Theory: Proceedings of the International Conference held in Como*, volume 1488 of *Lecture Notes in Math.*, pages 95–104. Springer, 1991.

[74] Peter Freyd. Remarks on algebraically compact categories. In M. P. Fourman, P. T. Johnstone, and A. M. Pitts, editors, *Applications of category theory in computer science: Proceedings of the London Mathematical Society Symposium, Durham 1991*, volume 177 of *London Mathematical Society Lecture Note Series*, pages 95–106. Cambridge University Press, 1992.

[75] Peter Freyd. Algebraic real analysis. *Theory Appl. Categ.*, 20(10):215–306, 2008.

[76] Peter Gabriel and Friedrich Ulmer. *Lokal Präsentierbare Kategorien*, volume 221 of *Lecture Notes Math.* Springer-Verlag, 1971.

[77] Neil Ghani and Tarmo Uustalu. Coproducts of ideal monads. *Theor. Inform. Appl.*, 38(4):321–342, 2004.

[78] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, and D. S. Scott. *Continuous Lattices and Domains.* Cambridge University Press, 2003.

[79] A. M. Gleason and R. P. Dilworth. A generalized Cantor theorem. *Proc. Amer. Math. Soc.*, 13(5):704–705, 1962.

[80] J. Glimming and N. Ghani. Difunctorial semantics of object calculus. *Electron. Notes Theor. Comput. Sci.*, 138:79–94, 2005.

[81] J. A. Goguen, J. W. Thatcher, E. G. Wagner, and J. B. Wright. Initial algebra semantics and continuous algebras. *J. ACM*, 24(1):68–95, January 1977.

[82] C. Grabmayer, J. Endrullis, D. Hendriks, J. W. Klop, and L. S. Moss. Automatic Sequences and Zip-Specifications. In *Proc. Symp. on Logic in Computer Science (LICS 2012)*, pages 335–344. IEEE Computer Society, 2012.

[83] H. Peter Gumm. Functors for coalgebras. *Algebra Universalis*, 45:135–147, 2001.

[84] H. Peter Gumm and Tobias Schröder. Coalgebraic structure from weak limit preserving functors. *Electronic Notes in Theoretical Computer Science*, 33(0):111 – 131, 2000. CMCS'2000, Coalgebraic Methods in Computer Science.

[85] H. Peter Gumm and Tobias Schröder. Monoid-labelled transition systems. *Electronic Notes in Theoretical Computer Science*, 44(1), 2001.

[86] H. Peter Gumm and Tobias Schröder. Coalgebras of bounded type. *Math. Structures Comput. Sci.*, 12(5):565–578, 2002.

[87] H. Peter Gumm and Tobias Schröder. Types and coalgebraic structure. *Algebra Universalis*, 53:229–252, 2005.

[88] H.Peter Gumm. From t-coalgebras to filter structures and transition systems. In José Luiz Fiadeiro, Neil Harman, Markus Roggenbach, and Jan Rutten, editors, *Algebra and Coalgebra in Computer Science*, volume 3629 of *Lecture Notes in Computer Science*, pages 194–212. Springer Berlin Heidelberg, 2005.

[89] Helle Hvid Hansen, Clemens Kupke, and Eric Pacuit. Neighbourhood structures: Bisimilarity and basic model theory. *Log. Methods Comput. Sci.*, 5, 2009.

[90] G. H. Hardy and E. M. Wright. *An introduction to the theory of numbers.* Oxford University Press, Oxford, fourth edition, 1960.

[91] Ichiro Hasuo and Bart Jacobs. Context-free languages via coalgebraic trace semantics. In J. Fiadeiro et al., editor, *Proc. Algebra and Coalgebra in Computer Science: First International Conference (CALCO'05)*, volume 3629 of *Lecture Notes in Comput. Sci.*, pages 213–231. Springer, 2005.

[92] Ichiro Hasuo, Bart Jacobs, and Ana Sokolova. Generic trace semantics via coinduction. *Log. Methods Comput. Sci.*, 3(4:11):1–36, 2007.

[93] A. Heifetz and D. Samet. Topology-free typology of beliefs. *J. Econom. Theory*, 82(2):324–341, 1998.

[94] Claudio Hermida and Bart Jacobs. Structural induction and coinduction in a fibrational setting. *Inform. and Comput.*, 145:107–152, 1998.

[95] Martin Hyland, Gordon D. Plotkin, and A. John Power. Combining effects: sums and tensor. *Theoret. Comput. Sci.*, 357:70–99, 2006.

[96] Thomas Ihringer. *Algemeine Algebra. Mit einem Anhang über Universelle Coalgebra von H. P. Gumm*, volume 10 of *Berliner Studienreihe zur Mathematik*. Heldermann Verlag, 2003.

[97] Bart Jacobs. The temporal logic of coalgebras via galois algebras. *Math. Structures Comput. Sci.*, 12(6):875–903, 2002.

[98] Bart Jacobs. *Introduction to coalgebra. Towards Mathematics of States and Observation.* Cambridge University Press, 2016.

[99] Jean-Baptiste Jeannin, Dexter Kozen, and Alexandra Silva. Well-founded coalgebras, revisited. *Math. Structures Comput. Sci.*, 27:1111–1131, 2017.

[100] Peter T. Johnstone. Sketches of an elephant. in preparation.

[101] Peter T. Johnstone. Adjoint lifting theorems for categories of algebras. *Bull. London Math. Soc.*, 7:294–297, 1975.

[102] André Joyal. Une théorie combinatoire des séries formelles. *Adv. Math.*, 42:1–82, 1981.

[103] André Joyal. Foncteurs analytiques et espèces de structures. *Lecture Notes in Math.*, 1234:126–159, 1986.

[104] Y. Kawahara and M. Mori. A small final coalgebra theorem. *Theoret. Comput. Sci.*, 233(1–2):129–145, 2000.

[105] A. Kechris. *Classical Descriptive Set Theory*, volume 156. Springer, 1995.

[106] G. Max Kelly. A unified treatment of transfinite constructions for free algebras, free monoids, colimits, associated sheaves, and so on. *Bull. Austral. Math. Soc.*, 22:1–83, 1980.

[107] A. Ya Khinchin. *Continued Fractions.* University of Chicago Press, 1964.

[108] Dénes Kőnig. Über eine schlussweise aus dem endlichen ins unendliche. *Acta Sci. Math.*, 3(2–3):121–130, 1927.

[109] Bartek Klin. Bialgebras for structural operational semantics: An introduction. *Theoret. Comput. Sci.*, 412(38):5043–5069, 2011.

[110] Bronislav Knaster. Un theéorème sur les fonctions d'ensembles. *Ann. Soc. Pol. Math.*, 6:133–134, 1928.

[111] P. Knijnburg. Algebraic domains, chain completion, and the plotkin powerdomain contruction. *Technical Report RUU-CS-93-03*, 1993.

[112] Anders Kock. Monads on symmetric monoidal closed categories. *Arch. Math. (Basel)*, 21:1–10, 1970.

[113] Anders Kock. Strong functors and monoidal monads. *Arch. Math. (Basel)*, 23:113–120, 1972.

[114] V. Koubek and J. Reiterman. Categorical constructions of free algebras, colimits, and completions of partial algebras. *J. Pure Appl. Algebra*, 14(2):195–231, 1979.

[115] Dexter Kozen. Realization of coinductive types. In *Proc. Mathematical Foundations of Programming Semantincs (MFPS XXVII)*, volume 276 of *Electon. Notes Theor. Comput. Sci.*, pages 237–246, 2011.

[116] C. Kupke, A. Kurz, and Y. Venema. Stone coalgebras. *Theoret. Comput. Sci.*, 327(1–2):109–132, 2004.

[117] Clemens Kupke. *Finitary Coalgebraic Logics.* PhD thesis, Institute for Logic, Language and Computation, Universiteit van Amsterdam, 2006.

[118] Alexander Kurz. *Logics for Coalgebras and Applications to Computer Science.* PhD thesis, Ludwig-Maximilians-Universität München, 2000.

[119] Joachim Lambek. A fixpoint theorem for complete categories. *Math. Z.*, 103:151–161, 1968.

[120] F. William Lawvere. An elementary thoery of the category of sets. *Reprints in Theor. Appl. Categ.*, 11:1–35, 2005. an abstract appear in Proc. Nat. Acad. Sci. U.S.A. 51 (1964), 1506–1511.

[121] Daniel J. Lehmann and Michael B. Smyth. Algebraic specification of data types: A synthetic approach. *Mathematical Systems Theory*, 14(1):97–139, 1981.

[122] T. Leinster. A general theory of self-similarity. *Adv. Math.*, 226, 2011.

[123] Paul Loya. *Amazing and aesthetic aspects of analysis.* Springer, New York, 2017.

[124] Saunders MacLane. *Categories for the working mathematician.* Springer, 2nd edition, 1998.

[125] Michael Makkai and Robert Paré. *Accessible categories: the foundation of categorical model theory*, volume 104 of *Contemporary Math.* Amer. Math. Soc., Providence, RI, 1989.

[126] Zohar Manna and Amir Pnüeli. *The Temporal Logic of Reactive and Concurrent Systems: Specification.* Springer-Verlag, 1992.

[127] E. Michael. Topologies on spaces of subsets. *Trans. Amer. Math. Soc.*, 1(71):152–182, 1951.

[128] Stefan Milius. Completely iterative algebras and completely iterative monads. *Inform. and Comput.*, 196:1–41, 2005.

[129] Stefan Milius. A sound and complete calculus for finite stream circuits. In *Proc. 25th Annual Symposium on Logic in Computer Science (LICS'10)*, pages 449–458. IEEE Computer Society, 2010.

[130] Stefan Milius and Lawrence S. Moss. The category theoretic solution of recursive program schemes. *Theoret. Comput. Sci.*, 366(1–2):3–59, 2006. Fundamental study.

[131] Stefan Milius, Lawrence S. Moss, and Daniel Schwencke. CIA structures and the semantics of recursion. In C.-H. Luke Ong, editor, *Proc. Foundations of Software Science and Computation Structures (FoSSaCS)*, volume 6014 of *Lecture Notes Comput. Sci.*, pages 312–327. Springer, 2010.

[132] Stefan Milius, Thorsten Palm, and Daniel Schwencke. Complete iterativity for algebras with effects. In Alexander Kurz, Marina Lenisa, and Andrzej Tarlecki, editors, *Proc. Coalgebraic and Algebraic Methods in Computer Science (CALCO'09)*, volume 5728 of *Lecture Notes Comput. Sci.*, pages 34–48. Springer, 2009.

[133] Stefan Milius, Dirk Pattinson, and Thorsten Wißmann. A new foundation for finitary corecursion and iterative algebras. *Inform. and Comput.*, 2019. Accepted for publication.

[134] Robin Milner. *A calculus of communication systems*, volume 92 of *Lecture Notes Comput. Sci.* Springer, 1980.

[135] Robin Milner. *Communication and Concurrency.* International Series in Computer Science. Prentice Hall, 1989.

[136] E. Moggi. Notions of computations and monads. *Inform. and Comput.*, 93(1):55–92, 1991.

[137] L. S. Moss. Non-wellfounded set theory. In Edward N. Zalta, editor, *Stanford Encyclopedia of Philosophy*, page 42 pp. Stanford University, 2008. available at `http://plato.stanford.edu/entries/nonwellfounded-set-theory/`.

[138] Lawrence S. Moss. Coalgebraic logic. *Annals of Pure and Applied Logic*, 96(1):277–317, 1999.

[139] Philip S. Mulry. Lifting theorems for Kleisli categories. In S. Brookes, M. Main, A. Melton, M. Mislove, and D. Schmidt, editors, *Proc. Mathematical Foundations of Programming Semantics (MFPS'93)*, volume 802 of *Lecture Notes Comput. Sci.*, pages 304–319. Springer, 1994.

[140] Evelyn Nelson. Iterative algebras. *Theoret. Comput. Sci.*, 25:67–94, 1983.

[141] Ivan Niven, Herbert S. Zuckerman, and Hugh L. Montgomery. *An introduction to the theory of numbers.* John Wiley & Sons, Inc., New York, fifth edition, 1991.

[142] G. Osius. Categorical set theory: a characterization of the category of sets. *J. Pure Appl. Algebra*, 4(79–119), 1974.

[143] David Park. Concurrency on automata and infinite sequences. In P. Deussen, editor, *Proc. Conf. on Theoretical Computer Science*, volume 104 of *Lecture Notes Comput. Sci.*, pages 167–183, 1981.

[144] D. Pavlović and V. Pratt. The continuum as a final coalgebra. *Theoret. Comput. Sci.*, 280(1–2):105–122, 2002.

[145] Dusko Pavlović and Martín H. Escardó. Calculus in coinductive form. In *Proc. Thirteenth Annual IEEE Symposium on Logic in Computer Science*, pages 408–417. IEEE Computer Society, 1998.

[146] Gordon D. Plotkin and Daniele Turi. Towards a mathematical operational semantics. In *Proc. Logic in Computer Science (LICS)*, 1997.

[147] C. Kupke and J. J. M. M. Rutten. On the Final Coalgebra of Automatic Sequences. In *Logic and Program Semantics - Essays Dedicated to Dexter Kozen on the Occasion of his 60th Birthday*, volume 7230 of *Lecture Notes in Computer Science*, pages 149–164. Springer, 2012.

[148] Jan J. M. M. Rutten. Relators and metric bisimulations. *Electr. Notes Theor. Comput. Sci.*, 11:252–258, 1998.

[149] Jan J. M. M. Rutten. Universal coalgebra: a theory of systems. *Theoret. Comput. Sci.*, 249(1):3–80, 2000.

[150] Jan J. M. M. Rutten. A coinductive calculus of streams. *Math. Structures Comput. Sci.*, 15(1):93–147, 2005.

[151] Jan J. M. M. Rutten. Algebraic specification and coalgebraic synthesis of mealy automata. *Electron. Notes. Theor. Comput. Sci.*, 160:305–319, 2006.

[152] T. Schröder. *Coalgebren und Funktoren.* PhD thesis, Philipps-Universität Marburg, 2001.

[153] Christoph Schubert. Terminal coalgebras for measure-polynomial functors. In Jianer Chen and S. Barry Cooper, editors, *Proc. Conference on Theory and Applications of Models of Computation (TAMC'2009)*, volume 5532 of *Lecture Notes Comput. Sci.*, pages 325–334. Springer, 2009.

[154] D. Scott. Continuous lattices. *Lecture Notes in Math.*, 274:97–136, 1972.

[155] S. Sirota. The spectral representation of spaces of closed subsets of bicompacta. (russian). *Dokl. Akad. Nauk SSSR*, 181:1069?1072, 1968.

[156] M. B. Smyth and G. D. Plotkin. The category-theoretic solution of recursive domain equations. *SIAM J. Comput.*, 11(4):761–783, 1982.

[157] David Sprunger and Lawrence S. Moss. Precongruences and parametrized coinduction for logics for behavioral equivalence. In Filippo Bonchi and Barbara König, editors, *Proc. Coalgebraic Methods in Computer Science (CALCO'17)*, Leibniz International Proceedings in Informatics, pages 23:1–23:15. Dagstuhl Publishing, 2017.

[158] Paul Taylor. The limit-colimit coincidence for categories. available at `http://www.paultaylor.eu/domains/limccc.pdf`, 1994.

[159] Paul Taylor. Towards a unified treatment of induction i: the general recursion theorem. preprint, available at `www.paultaylor.eu/ordinals/#towuti`, 1995–6.

[160] Paul Taylor. *Practical Foundations of Mathematics.* Cambridge University Press, 1999.

[161] Jerzy Tiuryn. Unique fixed points vs. least fixed points. *Theoret. Comput. Sci.*, 12:229–254, 1980.

[162] V. Trnková, J. Adámek, V. Koubek, and J. Reiterman. Free algebras, input processes and free monads. *Comment. Math. Univ. Carolin.*, 16:339–351, 1975.

[163] Věra Trnková. Some properties of set functors. *Comment. Math. Univ. Carolin.*, 10:323–352, 1969.

[164] Věra Trnková. On a descriptive classification of set functors I. *Comment. Math. Univ. Carolin.*, 12:143–174, 1971.

[165] Franck van Breugel. De Bakker-Zucker processes revisited. *Inform. and Comput.*, 188(1):68–76, 2004.

[166] Jiří Velebil. *Categorical domain theory.* PhD thesis, Czech Technical University Prague, 1998. available at `http://math.feld.cvut.cz/velebil/research/papers.html`.

[167] Ignacio Viglizzo. *Coalgebras on Measurable Spaces.* PhD thesis, Indiana University, Bloomington, 2005.

[168] Thorsten Wißmann, Stefan Milius, Shin ya Katsumata, and Jérémy Dubut. A coalgebraic view on reachability. submitted; available online at `https://arxiv.org/abs/1901.10717`.

[169] James Worrell. Terminal sequences for accessible endofunctors. In *Proc. Coalgebraic Methods in Computer Science (CMCS'99)*, volume 19 of *Electron. Notes Theor. Comput. Sci.*, pages 24–38, 1999.

[170] James Worrell. On the final sequence of a finitary set functor. *Theoret. Comput. Sci.*, 338:184–199, 2005.

[171] Ernst Zermelo. Beweis, daß jede Menge wohlgeordnet werden kann. *Math. Ann.*, 59:514–516, 1904.