THE COALGEBRAIC ENRICHMENT OF ALGEBRAS IN HIGHER CATEGORIES

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Abstract. We prove that given $\mathcal{C}$ a presentably symmetric monoidal ∞-category, and any essentially small ∞-operad $\mathcal{O}$, the ∞-category of $\mathcal{O}$-algebras in $\mathcal{C}$ is enriched, tensored and cotensored over the presentably symmetric monoidal ∞-category of $\mathcal{O}$-coalgebras in $\mathcal{C}$. We provide a higher categorical analogue of the universal measuring coalgebra. For categories in the usual sense, the result was proved by Hyland, López Franco, and Vasilakopoulou.

1. Introduction

The dual of a coalgebra is always an algebra. However, unless we require the algebra to be finite dimensional, the dual of an algebra is not a coalgebra. The universal measuring coalgebra was introduced in [Swe69] as a way to balance this issue. In ordinary categories, the measuring provides an enrichment for algebras over coalgebras: this was established in [HLV17, 5.2] and [Vas19, 2.18]. We provide here, in Theorem 3.16, its ∞-categorical analogue. In any presentably symmetric monoidal ∞-category, the algebras objects are enriched, tensored and cotensored over coalgebras. Therefore the space of morphisms of algebras is endowed with a rich structure. We use the notion of enriched ∞-categories following [GH15] and [Hin18].

Algebras in ∞-categories formalize the notion of homotopy coherent associative and unital algebras, see [Lur17]. Following [Lur18a], we provide a general dual definition of coalgebras in ∞-categories. These are objects with a comultiplication that is coassociative up to higher homotopies. We show, in Proposition 2.7, that if an ∞-operad $\mathcal{O}$ is essentially small, the ∞-category of $\mathcal{O}$-coalgebras in a presentable ∞-category remains presentable.

A similar result would be very challenging to prove in model categories. Let M be a combinatorial symmetric monoidal model category. Suppose we have a model structure for algebras $\mathsf{Alg}(\mathsf{M})$ in M and a model structure for coalgebras $\mathsf{CoAlg}(\mathsf{M})$ in M, where the weak equivalences in both of these models are created by their underlying functor. One analogous result would be to show that $\mathsf{Alg}(\mathsf{M})$ is a $\mathsf{CoAlg}(\mathsf{M})$-model category, in the sense of [Hov99, 4.2.18]. There are several issues with that. A left-induced model structure on $\mathsf{CoAlg}(\mathsf{M})$ may not always exists, and when it does, M may have been replaced by a Quillen equivalent model category that is not a monoidal model category, see [HKRS17]. Even in cases where we can left-induce from a monoidal model category, the homotopy theory associated to $\mathsf{CoAlg}(\mathsf{M})$ may not be the correct one, see [PS19] and [Pér20a].

2010 *Mathematics Subject Classification.* 16T15, 18C35, 18D10, 18D20, 18N70, 55P43.

Key words and phrases. algebra, coalgebra, enrichment, operads, ∞-categories, presentable.

Acknowledgement. The results here are part of my PhD thesis [Pér20b], and as such, I would like to express my gratitude to my advisor Brooke Shipley for her help and guidance throughout the years. I would also like to thank Rune Haugseng for clarifying and answering many of my questions. I am also thankful for many fruitful conversations with Shaul Barkan and tslil clingman that sparked results in this paper.

2. Presentability of Coalgebras

We present here the formal definition of coalgebras in ∞-categories, generalizing [Lur18a, Section 3.1], which was for the case of $\mathbb{E}_\infty$-coalgebras. We define and extend the results for coalgebras over any ∞-operad. Our main result in this section is that coalgebras of a presentably symmetric monoidal ∞-category form also a presentable ∞-category, see Corollary 2.8.

We invite the reader to look for the definition of a symmetric monoidal ∞-category in [Lur17, 2.0.0.7]. More generally, for any ∞-operad $\mathcal{O}$ (see [Lur17, 2.1.1.10]), we will consider the notion of a *$\mathcal{O}$-monoidal ∞-category* as in [Lur17, 2.1.2.15]. If we choose $\mathcal{O}$ to be the commutative ∞-operad ([Lur17, 2.1.1.18]), then $\mathcal{O}$-monoidal ∞-categories are precisely symmetric monoidal ∞-categories.

Definition 2.1. Let $\mathcal{O}$ be an ∞-operad. Let $\mathcal{C}$ be an $\mathcal{O}$-monoidal ∞-category. An *$\mathcal{O}$-coalgebra object in $\mathcal{C}$* is an $\mathcal{O}$-algebra object in $\mathcal{C}^{\mathsf{op}}$. The ∞-category of $\mathcal{O}$-coalgebra objects in $\mathcal{C}$ is defined as the ∞-category $\mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C}) := (\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C}^{\mathsf{op}}))^{\mathsf{op}}$. More generally, given any map $\mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ of ∞-operads , we define the ∞-category of $\mathcal{O}'$-coalgebra in $\mathcal{C}$ as $\mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}) = (\mathcal{A}\ell g_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}^{\mathsf{op}}))^{\mathsf{op}}$.

Remark 2.2. If $\mathcal{C}$ is an $\mathcal{O}$-monoidal ∞-category, then $\mathcal{C}^{\mathsf{op}}$ can be given an $\mathcal{O}$-monoidal structure uniquely up to contractible choice, as in [Lur17, 2.4.2.7]. One can use the work of [BGN18] to give an explicit choice of the coCartesian fibration for $\mathcal{C}^{\mathsf{op}}$. For instance, let $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be the coCartesian fibration associated to the symmetric monoidal structure of $\mathcal{C}$. Then straightening of the coCartesian fibration gives a functor:

$$F : \mathcal{O}^{\otimes} \longrightarrow \widehat{\mathcal{C}at}_\infty,$$

where $\widehat{\mathcal{C}at}_\infty$ is the ∞-category of ∞-categories, as in [Lur17, 3.0.0.5]. Then, by [BGN18, 1.5], the functor F also classifies a Cartesian fibration:

$$p^{\vee} : (\mathcal{C}^{\otimes})^{\vee} \longrightarrow (\mathcal{O}^{\otimes})^{\mathsf{op}}.$$

An explicit construction is given in [BGN18, 1.7]. The opposite map:

$$(p^{\vee})^{\mathsf{op}} : ((\mathcal{C}^{\otimes})^{\vee})^{\mathsf{op}} \longrightarrow \mathcal{O}^{\otimes},$$

is a coCartesian fibration that is classified by:

$$\mathcal{O}^{\otimes} \xrightarrow{F} \widehat{\mathcal{C}at}_\infty \xrightarrow{\mathsf{op}} \widehat{\mathcal{C}at}_\infty.$$

One can check that the fiber of $(p^{\vee})^{\mathsf{op}}$ over X in $\mathcal{O}$ is equivalent to $(\mathcal{C}_X)^{\mathsf{op}}$, and thus gives $\mathcal{C}^{\mathsf{op}}$ a $\mathcal{O}$-monoidal structure. We see that $\mathcal{O}$-coalgebras are sections of the Cartesian fibration $p^{\vee} : \mathcal{C}^{\otimes} \to (\mathcal{O}^{\otimes})^{\mathsf{op}}$ that sends inert morphisms in $(\mathcal{O}^{\otimes})^{\mathsf{op}}$ to $p^{\vee}$-Cartesian morphisms in $\mathcal{C}^{\otimes}$.

Remark 2.3. Recall from [Lur17, 2.0.0.1] that to any symmetric monoidal (ordinary) category C, one can define a category $\mathsf{C}^{\otimes}$, such that the nerve $\mathscr{N}(\mathsf{C}^{\otimes})$ is a symmetric monoidal ∞-category whose underlying ∞-category is $\mathscr{N}(\mathsf{C})$, see

[Lur17, 2.1.2.21]. If we denote $\mathsf{CoAlg}(\mathsf{C})$ the category of coassociative and counital coalgebras in C, then, dually from [Gro15, 4.21], we obtain:

$$\mathcal{CoAlg}_{\mathbb{A}_\infty}(\mathscr{N}(\mathsf{C})) \simeq \mathscr{N}(\mathsf{CoAlg}(\mathsf{C})).$$

Similarly, if we denote $\mathsf{CoCAlg}(\mathsf{C})$ the category of cocommutative coalgebras in C, we obtain:

$$\mathcal{CoAlg}_{\mathbb{E}_\infty}(\mathscr{N}(\mathsf{C})) \simeq \mathscr{N}(\mathsf{CoCAlg}(\mathsf{C})).$$

Proposition 2.4 ([Lur17, 3.2.4.4])**.** *Let $\mathcal{O}$ be an ∞-operad. Let $\mathcal{C}$ be an $\mathcal{O}$-monoidal ∞-category. Then the ∞-category $\mathcal{Alg}_{\mathcal{O}}(\mathcal{C})$ inherits a $\mathcal{O}$-monoidal structure, given by pointwise tensor product. Dually, the ∞-category $\mathcal{CoAlg}_{\mathcal{O}}(\mathcal{C})$ inherits a $\mathcal{O}$-monoidal structure, given by pointwise tensor product.*

Proposition 2.5. *Let $\mathcal{C}$ be a $\mathcal{O}$-monoidal ∞-category and let K be a simplicial set. If, for each X in $\mathcal{O}$, the fiber $\mathcal{C}_X$ admits K-indexed colimits, then the ∞-category $\mathcal{CoAlg}_{\mathcal{O}}(\mathcal{C})$ admits K-indexed colimits, and the forgetful functor $U : \mathcal{CoAlg}_{\mathcal{O}}(\mathcal{C}) \to \mathcal{C}$ preserves K-indexed colimits.*

Proof. Apply [Lur17, 3.2.2.5] to the coCartesian $(p^\vee)^{\mathsf{op}} : ((\mathcal{C}^\otimes)^\vee)^{\mathsf{op}} \longrightarrow \mathcal{O}^\otimes$ defined in Remark 2.2. □

Recall the definition [Lur09, 5.5.0.1] of a presentable ∞-category. Denote $\mathcal{Pr}^L$ the ∞-category of presentable ∞-categories with small colimit preserving functors. It is endowed with a symmetric monoidal structure ([Lur17, 4.8.1.15]).

Definition 2.6. An ∞-category $\mathcal{C}$ is said to be *presentably $\mathcal{O}$-monoidal* if it is an $\mathcal{O}$-algebra in $\mathcal{Pr}^L$, i.e., $\mathcal{C}$ is both presentable and $\mathcal{O}$-monoidal such that the tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves small colimits in each variable.

The following dualizes the result on algebras in [Lur17, 3.2.3.5].

Proposition 2.7. *Let $\mathcal{O}$ be an essentially small ∞-operad. Let $\mathcal{C}$ be a presentably $\mathcal{O}$-monoidal ∞-category. Then $\mathcal{CoAlg}_{\mathcal{O}}(\mathcal{C})$ is a presentably $\mathcal{O}$-monoidal ∞-category.*

Proof. Suppose the $\mathcal{O}$-monoidal structure of $\mathcal{C}$ is defined via a coCartesian fibration $p : \mathcal{C}^\otimes \to \mathcal{O}^\otimes$. We apply [Lur09, 5.4.7.11] to the Cartesian fibration $p^\vee : (\mathcal{C}^\otimes)^\vee \to \mathcal{O}^{\mathsf{op}}$ described in Remark 2.2. For any object X in $\mathcal{O}^\otimes$, the fiber of $p^\vee$ over X is equivalent to the fiber $\mathcal{C}_X$ of p over X. By [Lur17, 3.2.3.4], these fibers are accessible and $\mathcal{C}_X \to \mathcal{C}_{X'}$ are accessible maps. Thus the induced maps $\mathcal{C}_{X'}^\vee \to \mathcal{C}_X^\vee$ are also accessible by [BGN18, 1.3]. □

Corollary 2.8. *Let $\mathcal{O}$ be an essentially small ∞-operad. If $\mathcal{C}$ is a presentably symmetric monoidal ∞-category, then $\mathcal{CoAlg}_{\mathcal{O}}(\mathcal{C})$ is a presentably symmetric monoidal ∞-category.*

Remark 2.9. In general, if $\mathcal{C}$ is compactly generated ([Lur09, 5.5.7.1]), there is no guarantee that $\mathcal{CoAlg}_{\mathcal{O}}(\mathcal{C})$ is also compactly generated. However, the *fundamental theorem of coalgebras* (see [Swe69, II.2.2.1] or [GG99, 1.6]) states that if $\mathcal{C}$ is (the nerve of) vector spaces, or chain complexes over a field, then $\mathcal{CoAlg}_{\mathbb{A}_\infty}(\mathcal{C})$ is compactly generated and the forgetful functor $U : \mathcal{CoAlg}_{\mathbb{A}_\infty}(\mathcal{C}) \to \mathcal{C}$ preserves and reflects compact objects. From [AP04, 4.2], if κ is an uncountable regular cardinal, we conjecture that the fundamental theorem of coalgebra can be expended in the following sense. If $\mathcal{C}$ is κ-compactly generated then $\mathcal{CoAlg}_{\mathcal{O}}(\mathcal{C})$ is κ-compactly generated and the forgetful functor preserves and reflects κ-compact objects.

In some cases, the ∞-category $\mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})$ is not mysterious. We recall the following result from Lurie. Let $\mathcal{C}$ be a symmetric monoidal ∞-category, and denote by $\mathcal{C}_{\mathsf{fd}}$ the full subcategory spanned by the dualizable objects, see [Lur17, 4.6.1]. It inherits a symmetric monoidal structure. For each dualizable object X, we denote $X^\vee$ its dual and this defines a contravariant endofunctor on $\mathcal{C}_{\mathsf{fd}}$.

Proposition 2.10 ([Lur18a, 3.2.4])**.** *Let $\mathcal{C}$ be a symmetric monoidal ∞-category. Then taking dual objects assigns an equivalence of symmetric monoidal ∞-categories $(\mathcal{C}_{\mathsf{fd}})^{\mathsf{op}} \xrightarrow{\simeq} \mathcal{C}_{\mathsf{fd}}$. In particular, for any ∞-operad $\mathcal{O}$, we obtain an equivalence $\mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C}_{\mathsf{fd}})^{\mathsf{op}} \simeq \mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C}_{\mathsf{fd}})$ of symmetric monoidal ∞-categories.*

3. The Universal Measuring Coalgebra

Classically, in any presentable symmetric monoidal closed ordinary category, the category of monoids is enriched, tensored and cotensored in the symmetric monoidal category of comonoids. This was proven in [HLV17, 5.2] and [Vas19, 2.18]. See also the example of the differential graded case in [AJ13]. We show here in Theorem 3.16 an equivalent statement in ∞-categories.

An ∞-category shall be defined to be *enriched* over a symmetric monoidal ∞-category in the sense of [Hin18, 3.1.2], or in the sense of [GH15]. By [Hin18, 3.4.4] they are equivalent. An ∞-category is *tensored* or *cotensored* over a monoidal ∞-category in the sense of [Lur17, 4.2.1.19] or [Lur17, 4.2.1.28] respectively. Our desired enrichment in Theorem 3.16 will also be enriched in the sense of [Lur17, 4.2.1.28], see Remark 3.17 below. It is conjectured in [GH15] that the definitions of enrichment of Lurie and Gepner-Haugseng are equivalent.

Throughout this section, let $\mathcal{C}$ be a presentably symmetric monoidal ∞-category. It is in particular closed, and thus the strong symmetric monoidal functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ induces a lax symmetric monoidal functor $[-,-] : \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathcal{C}$ characterized by the universal mapping property $\mathcal{C}(X \otimes Y, Z) \simeq \mathcal{C}(X, [Y, Z])$, for all X, Y, and Z in $\mathcal{C}$. In other words, the functor $- \otimes Y : \mathcal{C} \to \mathcal{C}$ is a left adjoint to $[Y, -] : \mathcal{C} \to \mathcal{C}$.

Let $\mathcal{O}$ be an essentially small ∞-operad. From the lax symmetric monoidal structure of $[-,-] : \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathcal{C}$, we obtain a functor:

$$[-,-] : \mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C}^{\mathsf{op}}) \times \mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C}) \longrightarrow \mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C}).$$

By definition of $\mathcal{O}$-coalgebras, we identify $\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C}^{\mathsf{op}})$ simply as $\mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})^{\mathsf{op}}$, and thus obtain the following definition.

Definition 3.1. Let $\mathcal{C}$ and $\mathcal{O}$ be as above. We call the induced functor:

$$[-,-] : \mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})^{\mathsf{op}} \times \mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C}) \longrightarrow \mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C}),$$

the *Sweedler cotensor*. In the literature, it is sometimes called the *convolution algebra* or the *convolution product*, see [Swe69, 4.0] and [AJ13].

Remark 3.2. The term convolution product stems from the algebra structure that generalizes the usual convolution product in representation theory. See [HGK10, 2.12.3]. It also generalizes the classical convolutions of real functions of compact support, see [HGK10, 2.14.4].

Example 3.3. The Sweedler cotensor in the case where $\mathcal{O} = \mathbb{E}_\infty$ and $\mathcal{C}$ is the ∞-category of R-modules in a symmetric monoidal ∞-category, where R is an $\mathbb{E}_\infty$-algebra, was presented in [Lur18b, Section 1.3.1]. See also [Nik16, 6.6].

Example 3.4. Let $\mathbb{I}$ be the unit of the symmetric monoidal structure of $\mathcal{C}$. Let C be any $\mathcal{O}$-coalgebra, then the Sweedler cotensor $[C, \mathbb{I}]$ is simply the *linear dual* C^*, which is always an $\mathcal{O}$-algebra. Thus the linear dual functor $(-)^* : \mathcal{C}^{\mathsf{op}} \to \mathcal{C}$ lifts to the particular Sweedler cotensor $(-)^* = [-, \mathbb{I}] : \mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})^{\mathsf{op}} \to \mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})$. Here we recover the classical result that the dual of a coalgebra is always an algebra, see [Swe69, 1.1.1].

Remark 3.5. In a presentably symmetric monoidal ∞-category $\mathcal{C}$, an object X is dualizable precisely if X is equivalent to its linear dual X^*. Thus, the above defined functor $(-)^* : \mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})^{\mathsf{op}} \to \mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})$ coincides with the equivalence of Proposition 2.10 $(-)^\vee : \mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C}_{\mathsf{fd}})^{\mathsf{op}} \xrightarrow{\simeq} \mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C}_{\mathsf{fd}})$, when we restrict $(-)^*$ to the subcategory $\mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C}_{\mathsf{fd}})^{\mathsf{op}}$.

Since $[-,-] : \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathcal{C}$ is a continuous functor, and limits in $\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})$ are computed in $\mathcal{C}$, we get that the Sweedler cotensor is a continuous functor. Fix C an $\mathcal{O}$-coalgebra in $\mathcal{C}$. Then the continuous functor

$$[C, -] : \mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C}) \to \mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C}),$$

is accessible (as filtered colimits in $\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})$ are computed in $\mathcal{C}$) and is between presentable ∞-categories. Therefore, by the adjoint functor theorem [Lur09, 5.5.2.9], the functor $[C, -]$ admits a left adjoint denoted $C \triangleright - : \mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C}) \to \mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})$.

Definition 3.6. Let $\mathcal{C}$ and $\mathcal{O}$ be as above. We call the induced functor:

$$- \triangleright - : \mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C}) \times \mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C}) \to \mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C}),$$

the *Sweedler tensor*. Previously, it was called the *Sweedler product* in [AJ13] and later in [Vas19]. For C a fixed $\mathcal{O}$-coalgebra, the functor $C \triangleright -$ is left adjoint to $[C, -]$ and we have in particular the equivalence of spaces:

$$\mathcal{A}\ell g_{\mathcal{O}}(C \triangleright A, B) \simeq \mathcal{A}\ell g_{\mathcal{O}}(A, [C, B]),$$

for any $\mathcal{O}$-algebras A and B.

Example 3.7. In [AJ13, 3.4.1], an explicit formula of the Sweedler tensor was given in the discrete differential graded case.

Fix now A an $\mathcal{O}$-algebra in $\mathcal{C}$. The continuous functor:

$$[-, A] : (\mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C}))^{\mathsf{op}} \to \mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C}),$$

induces a cocontinuous functor on its opposites:

$$[-, A]^{\mathsf{op}} : \mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C}) \to (\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C}))^{\mathsf{op}}.$$

The cocontinuous functor is from a presentable ∞-category to an essentially locally small ∞-category: as the opposite of an essentially locally small ∞-category is also essentially locally small, and presentable ∞-categories are always essentially locally small. Thus, by the adjoint functor theorem [Lur09, 5.5.2.9, 5.5.2.10], the functor $[-, A]^{\mathsf{op}}$ admits a right adjoint $\{-, A\} : \mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})^{\mathsf{op}} \to \mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})$.

Definition 3.8. Let $\mathcal{C}$ and $\mathcal{O}$ be as above. We call the induced functor:

$$\{-, -\} : \mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})^{\mathsf{op}} \times \mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C}) \to \mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C}),$$

the *Sweedler hom*. For A and B any $\mathcal{O}$-algebra in $\mathcal{C}$, the $\mathcal{O}$-coalgebra $\{A, B\}$ is called the *universal measuring coalgebra in $\mathcal{C}$ of A and B*. See [Swe69, 7.0] for the discrete case in vector spaces. In particular, if we fix A, we obtain that $\{-, A\}$ is the right adjoint of $[-, A]^{\mathsf{op}}$ and we have the equivalence of spaces:

$$\mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})(C, \{A, B\}) \simeq \mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})(B, [C, A]),$$

for any $\mathcal{O}$-coalgebra C.

Example 3.9. Let $\mathbb{I}$ be the unit of the symmetric monoidal structure of $\mathcal{C}$. Then, for any $\mathcal{O}$-algebra A in $\mathcal{C}$, define A° to be the measuring coalgebra $\{A, \mathbb{I}\}$. It is called the *Sweedler dual* or *finite dual* of the $\mathcal{O}$-algebra A in $\mathcal{C}$. In particular, we obtain a functor $(-)^\circ = \{-, \mathbb{I}\}^{\mathsf{op}} : \mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C}) \to \mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})^{\mathsf{op}}$, which is the left adjoint of the linear dual functor $(-)^* : \mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})^{\mathsf{op}} \to \mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})$ defined in Example 3.4. In particular, we have the equivalence of spaces:

$$\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})(A, C^*) \simeq \mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})(C, A^\circ),$$

for any $\mathcal{O}$-coalgebra C and any $\mathcal{O}$-algebra A. This was proven in the discrete classical case of vector spaces in [Swe69, 6.0.5]. By Remark 3.5, when the $\mathcal{O}$-algebra A is dualizable in $\mathcal{C}$, then $A^\circ \simeq A^*$ as an object in $\mathcal{C}$.

Example 3.10. The origin of the term *measure* could be due to the following example. Let X be a compact Hausdorff space. Then the Sweedler dual of the algebra of continuous real functions $\mathsf{Map}(X, \mathbb{R})$ is equivalent to finitely supported measures on X, see [HGK10, 2.12.10].

We shall explain where the term *universal* measuring is coming from. Recall that the internal hom property of $\mathcal{C}$ implies that, for any X, Y and Z objects in $\mathcal{C}$, there is an equivalence of spaces: $\mathcal{C}(X \otimes Y, Z) \simeq \mathcal{C}(Y, [X, Z])$. The Sweedler cotensor guarantees conditions for an $\mathcal{O}$-algebra structure on $[X, Z]$. The following is a generalization of [Swe69, 7.0.1] and [AJ13, 3.3.1].

Definition 3.11. Let $\mathcal{C}$ and $\mathcal{O}$ be as above. Let C be an $\mathcal{O}$-coalgebra in $\mathcal{C}$, and A and B be $\mathcal{O}$-algebras in $\mathcal{C}$. Let $\psi : C \otimes A \to B$ be a map in $\mathcal{C}$. We say that (C, ψ) *measures* A *to* B (or (C, ψ) *is a measuring of* A *to* B) if the adjoint map $A \to [C, B]$ is a map of $\mathcal{O}$-algebras in $\mathcal{C}$.

We give examples generalized from [AJ13].

Example 3.12 ([AJ13, 3.3.3])**.** If $\mathbb{I}$ is the unit of the symmetric monoidal structure of $\mathcal{C}$, then a map $\mathbb{I} \otimes A \to B$ in $\mathcal{C}$ is a measuring of A to B if and only if it is a map in $\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})$.

Example 3.13 ([AJ13, 3.3.4])**.** The adjoint of the identity map on $[C, A]$ is a map $C \otimes [C, A] \to A$ and is always a measuring. In particular, the evaluation $C \otimes C^* \to \mathbb{I}$ is always a measuring of C^* to $\mathbb{I}$. Similarly $A^\circ \otimes A \to \mathbb{I}$ is a measuring of A to $\mathbb{I}$.

By definition of the Sweedler hom, as we have:

$$\mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})(C, \{A, B\}) \simeq \mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})(B, [C, A]),$$

we see that the $\mathcal{O}$-coalgebra $\{A, B\}$, together with the natural map $\{A, B\} \otimes A \to B$ (adjoint of the identity over $\{A, B\}$), is indeed the universal measuring algebra of A to B, in the following sense. Given any other measuring (C, ψ) of A to B, there exists a unique (up to contractible choice) map $C \to \{A, B\}$ of $\mathcal{O}$-coalgebras in $\mathcal{C}$

such that the following diagram commutes in $\mathcal{C}$:

$$\begin{tikzcd} C \otimes A \arrow[d, dashed] \arrow[dr, "\psi"] & \\ \{A, B\} \otimes A \arrow[r] & B. \end{tikzcd}$$

Remark 3.14. Following [AJ13, 3.3.6], we see that, given maps $A' \to A$ and $B \to B'$ in $\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})$, a map $C' \to C$ in $\mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})$, together with a map $A \to [C, B]$ in $\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})$, we obtain the following map in $\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})$:

$$A' \longrightarrow A \longrightarrow [C, B] \longrightarrow [C', B'].$$

This shows that the space of measurings provides a functor:

$$\mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})^{\mathsf{op}} \times \mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})^{\mathsf{op}} \times \mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C}) \longrightarrow \mathcal{S},$$

that is representable in each variable with respect to the Sweedler hom, tensor and cotensor.

Let $\mathcal{D}^{\otimes}$ be a monoidal ∞-category. Its *reverse*, denoted $(\mathcal{D}^{\otimes})^{\mathsf{rev}}$ or simply $\mathcal{D}^{\mathsf{rev}}$, is defined in [Hin18, 2.13.1]. Essentially, $\mathcal{D}$ and $\mathcal{D}^{\mathsf{rev}}$ have the same underlying ∞-category but the tensor $X \otimes Y$ in $\mathcal{D}^{\mathsf{rev}}$ corresponds precisely to $Y \otimes X$ in $\mathcal{D}$. Left modules over $\mathcal{D}$ corresponds to right modules over $\mathcal{D}^{\mathsf{rev}}$. If $\mathcal{D}$ is symmetric, then $\mathcal{D}^{\mathsf{rev}} = \mathcal{D}$ by [Hin18, 2.13.4]. We shall be interested with the *reverse opposite*, denoted $\mathcal{D}^{\mathsf{rop}} = (\mathcal{D}^{\mathsf{op}})^{\mathsf{rev}}$, of a monoidal ∞-category $\mathcal{D}$. The following is a generalization of the discrete ordinary case [HLV17, 5.1].

Lemma 3.15. *Let $\mathcal{C}$ and $\mathcal{O}$ be as above. Then the Sweedler cotensor endows the ∞-category $\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})$ the structure of a right module over the reverse opposite of the (symmetric) monoidal ∞-category $\mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})$.*

Proof. Notice first that $\mathcal{C}$ is a right module over its reverse opposite $\mathcal{C}^{\mathsf{rop}}$ via its internal hom $[-,-] : \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathcal{C}$, as it is lax symmetric monoidal. Therefore, by Proposition 2.4, the ∞-category $\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})$ is a right module over $\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C}^{\mathsf{rop}})$ via the Sweedler cotensor. Since $\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C}^{\mathsf{rev}}) \simeq \mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})^{\mathsf{rev}}$, then $\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C}^{\mathsf{rop}}) \simeq \mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})^{\mathsf{rop}}$. $\square$

Since $\mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})$ is a presentably symmetric monoidal ∞-category, it is enriched over itself by [GH15, 7.4.10]. We denote $\underline{\mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})}(D, E)$ the $\mathcal{O}$-coalgebra in $\mathcal{C}$ which classifies coalgebra maps from D to E, characterized by the universal mapping property:

$$\mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})\Big(C \otimes D, E\Big) \simeq \mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})\left(C, \underline{\mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})}(D, E)\right).$$

Theorem 3.16. *Let $\mathcal{C}$ be a presentably symmetric monoidal ∞-category. Let $\mathcal{O}$ be an essentially small ∞-operad. The ∞-category of $\mathcal{O}$-algebras $\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})$ is enriched over the symmetric monoidal ∞-category $\mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})$, via the Sweedler hom. Moreover it is tensored and cotensored respectively using the Sweedler tensor and Sweedler cotensor. In particular, we have an equivalence of $\mathcal{O}$-coalgebras:*

$$\underline{\mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})}\Big(C, \{A, B\}\Big) \simeq \Big\{A, [C, B]\Big\} \simeq \Big\{C \triangleright A, B\Big\},$$

for any $\mathcal{O}$-coalgebra C in $\mathcal{C}$ and any $\mathcal{O}$-algebras A and B in $\mathcal{C}$.

Proof. By Lemma 3.15, the ∞-category $\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})^{\mathsf{op}}$ is a left module over the symmetric monoidal ∞-category $\mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})$, via $[-,-]^{\mathsf{op}}$ the opposite of the Sweedler cotensor, such that $[-,A]^{\mathsf{op}} : \mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C}) \to \mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})^{\mathsf{op}}$ admits a right adjoint $\{-,A\}$ for all A in $\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})$. By [Hin18, 6.3.1, 7.2.1] (see also [GH15, 7.4.9]) this shows that $\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})^{\mathsf{op}}$ is enriched over $\mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})$, with tensor $[-,-]^{\mathsf{op}}$. Thus, by [Hin18, 6.2.1], we get that $\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})$ is enriched over $\mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})$, with cotensor $[-,-]$. □

Remark 3.17. We could have applied [Lur17, 4.2.1.33] in the proof of Theorem 3.16 to show that $\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})$ is enriched over $\mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})$ in the sense of Lurie, see [Lur17, 4.2.1.28]. It is conjectured that the definitions of enrichment are equivalent in [GH15].

Remark 3.18. The previous theorem shows that we can enrich the equivalence in Example 3.9 to an equivalence of $\mathcal{O}$-coalgebras in $\mathcal{C}$:

$$\underline{\mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})}\Big(C, A^{\circ}\Big) \simeq \Big\{A, C^*\Big\} \simeq \Big(C \triangleright A\Big)^{\circ},$$

for any $\mathcal{O}$-coalgebra C and any $\mathcal{O}$-algebra A.

A particular consequence of the theorem gives the following adjunction which was shown in [AJ13, 5.3.14] to generalize the algebraic cobar-bar adjunction.

Corollary 3.19. *Let $\mathcal{C}$ be a presentably symmetric monoidal ∞-category. Let $\mathcal{O}$ be an essentially small ∞-category. Let A be an $\mathcal{O}$-algebra in $\mathcal{C}$. Then there is an adjunction of enriched ∞-categories over $\mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})$:*

$$- \triangleright A : \mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C}) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C}) : \{A,-\}.$$

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