On a simplicial monoid whose underlying simplicial set is not a quasi-category

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1 Introduction

It is well known that the underlying simplicial set of any simplicial group is a Kan complex (also known as ∞-groupoid or $(\infty, 0)$-category). Informally speaking Kan complex is the infinite-dimensional analogue of groupoid, and the relation between groupoids and categories resembles that between groups and monoids. Thus one may ask if the underlying simplicial set of each simplicial monoid is a quasi-category (a.k.a. ∞-category or $(\infty, 1)$-category or weak Kan complex). In this short note, we construct a simplicial monoid whose underlying simplicial set is not a quasi-category.

Again it is well known that for any commutative group G and any $n \in \mathbb{N}$ one can construct a simplicial group $\mathrm{K}(G, n)$ called the n-th Eilenberg-MacLane space of G, which is thus a Kan complex. This assignment gives rise to a functor from commutative groups to spectra, which again gives rise to a functor from rings to ring spectra that is crucial for the theory of higher algebra.

We apply this Eilenberg-MacLane construction to commutative monoids to construct a simplicial monoid $\mathrm{K}(\mathbb{N}, 2)$ and show this doe not satisfies a certain condition that any quasi-category must do.

2 Theorem

Eilenberg-MacLane construction for commutative monoids is already established. For example, see 3.1 in [1]. We simply apply such construction to commutative monoids instead of commutative groups. Let us first briefly recall the construction.

For a commutative monoid M and a natural number $n \geq 1$, we have a simplicial monoid $M(S^n)$ whose k-simplices is the reduced free monoid $M(S^n[k])$ generated by the set $S^n[k]$ over M, where S^n denotes the simplicial set $\Delta[n]/\partial\Delta[n]$. And the simplicial structure on $M(S^n)$ is induced from that on S^n. We denote this simplical monoid $M(S^n)$ by $\mathrm{K}(M, n)$. We may write $\mathrm{K}(M, 0)$ for the discrete simplicial monoid of M.

Example 2.1. Let $\mathbb{N}$ be the commutative monoid of natural numbers under the usual addition and 0. We write $\alpha(0)\alpha(1)...\alpha(m)$ for any map $\alpha : [m] \to [n]$

in Δ. Then the sets of low dimensional simplicies of $S^2 = \Delta[2]/\partial\Delta[2]$ are the following:

$$S^2[0] = S^2[1] = \{*\}, S^2[2] = \{*, 012\}, S^2[3] = \{*, 0012, 0112, 0122\}.$$

Therefore $\mathrm{K}(\mathbb{N}, s)[0] = \mathrm{K}(\mathbb{N}, s)[1] = *$ is the trivial monoid, and

$$\mathrm{K}(\mathbb{N}, s)[2] = \mathbb{N}\langle 012\rangle,$$

$$\mathrm{K}(\mathbb{N}, s)[3] = \mathbb{N}\langle 0012\rangle \oplus \mathbb{N}\langle 0112\rangle \oplus \mathbb{N}\langle 0122\rangle,$$

where, $\mathbb{N}\langle x\rangle$ denotes the free monoid generated by x. Moreover face maps $d_i : \mathrm{K}(\mathbb{N}, s)[3] \to \mathrm{K}(\mathbb{N}, s)[2]$ for $i \in \{0, 1, 2, 3\}$ are given by

$$d_0(a, b, c) = a,\ d_1(a, b, c) = a + b,\ d_2(a, b, c) = b + c,\ d_3(a, b, c) = c$$

for any $(a, b, c) \in \mathbb{N}\langle 0012\rangle \oplus \mathbb{N}\langle 0112\rangle \oplus \mathbb{N}\langle 0122\rangle$

We assume that the reader knows the definitions of the simplicial sets the k-th horn $\Lambda^k[n]$ of the standard n-simplex $\Delta[n]$ for every $[n] \in \Delta$ and $k \in [n]$.

Definition 2.2 ([2])**.** *A simplicial set X is a quasi-category if it satisfies the following:*

for any map

$$\Lambda^k[n] \to X$$

with $[n] \in \Delta$ and $k \in [n] \setminus \{0, n\}$, there exists a lift

$$\Delta[n] \to X.$$

Theorem 2.3. *The Eilenberg-MacLane simplicial set $\mathrm{K}(M, n)$ for a commutative monoid M is not necessary a quasi-category.*

Proof. Take M to be $\mathbb{N}$ and n to be 2. Consider a map

$$\Lambda^1[3] \to \mathrm{K}(\mathbb{N}, 2)$$

such that the 0-face maps to 3, the 2-face to 1 and the 3-face to an arbitrary $n \in \mathbb{N}$. Assume there is a lift

$$\Delta[3] \to \mathrm{K}(\mathbb{N}, 2)$$

and we denote the image of the unique non-degenerate 3-simplex by $(a, b, c) \in \mathbb{N}\langle 0012\rangle \oplus \mathbb{N}\langle 0112\rangle \oplus \mathbb{N}\langle 0122\rangle$. As we have observed in the example above, its 0-face would be a, 2-face $b + c$, and 3-face c. Thus we should have $3 + b = 1$ with $b \in \mathbb{N}$. Therefore such lift can not exist. Hence $\mathrm{K}(\mathbb{N}, 2)$ is not a quasi-category. $\square$

The existence of a simplicial monoid whose underlying simplicial set is not a quasi-category is probably known to experts. But the author could not find any written proof. Note that, for any monoid M, $\mathrm{K}(M, 1)$ is the nerve of M which is understood as the single object 1-category. Thus by [2] it is a quasi-category. Moreover, since $\mathrm{K}(M, 0)$ is discrete, it is again a quasi-category. Thus $\mathrm{K}(\mathbb{N}, 2)$ is the lowest dimensional example of Eilenberg-MacLane spaces whose underlying simplicial sets are not quasi-categories.

Note also that Street defined at example 1.3 in [3], for any monoid (resp. commutative monoid) M and each $n \in \{0, 1\}$ (resp. $n \in \mathbb{N}$), the Eilenberg-MacLane strict ω-category $\mathrm{K}^{\omega}(M, n)$ which is actually a strict n-category. Thus our main theorem is not surprising. In addition, since $\mathrm{K}(\mathbb{N}, 2)$ is not a quasi-category, example 57 in [4] does not show whether the functor $(-)^{e}$ maps it to a weak complicial set or not. It may be possible to construct Eilenberg-MacLane spaces as (∞, n)-categories directly, but the author is not familiar with such theories. At least we know that the ω-nerve of $\mathrm{K}^{\omega}(M, n)$ is a (weak) complicial set by theorem 266 of [5].

References

[1] Bjørn Ian Dundas, Marc Levine, Paul Arne Østvær, Oliver Röndigs, Vladimir Voevodsky, Motivic Homotopy Theory: Lectures at a Summer School in Nordfjordeid, Norway, August 2002, Springer, Universitext, (2006)

[2] A. Joyal, Theory of quasi-categories I. In preparation.

[3] Ross Street, The algebra of oriented simplices, J. Pure Appl. Algebra 49 (1987) 283-335

[4] D. Verity, Weak complicial sets. I. Basic homotopy theory, Adv. Math. 219 (2008), no. 4, 1081-1149,

[5] D. Verity, Complicial Sets Characterising the Simplicial Nerves of Strict ω-Categories, Memoirs of the American Mathematical Society Volume 193, Number 905 (2008)