The Hom-Long dimodule category and nonlinear equations

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ABSTRACT

In this paper, we construct a kind of new braided monoidal category over two Hom-Hopf algerbas (H,α) and (B,β) and associate it with two nonlinear equations. We first introduce the notion of an (H,B)-Hom-Long dimodule and show that the Hom-Long dimodule category ${}^{B}_{H}\mathbb{L}$ is an autonomous category. Second, we prove that the category ${}^{B}_{H}\mathbb{L}$ is a braided monoidal category if (H,α) is quasitriangular and (B,β) is coquasitriangular and get a solution of the quantum Yang-Baxter equation. Also, we show that the category ${}^{B}_{H}\mathbb{L}$ can be viewed as a subcategory of the Hom-Yetter-Drinfeld category ${}^{H\otimes B}_{H\otimes B}\mathbb{HYD}$. Finally, we obtain a solution of the Hom-Long equation from the Hom-Long dimodules.

Key words: Hom-Long dimodule; Hom-Yetter-Drinfeld category; Yang-Baxter equation; Hom-Long equation.

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INTRODUCTION

The study of Hom-algebras can be traced back to Hartwig, Larsson and Silvestrov's work in [8], where the notion of Hom-Lie algebra in the context of q-deformation theory of Witt and Virasoro algebras [9] was introduced, which plays an important role in physics,

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mainly in conformal field theory. Hom-algebras and Hom-coalgebras were introduced by Makhlouf and Silvestrov [19] as generalizations of ordinary algebras and coalgebras in the following sense: the associativity of the multiplication is replaced by the Hom-associativity and similar for Hom-coassociativity. They also defined the structures of Hom-bialgebras and Hom-Hopf algebras, and described some of their properties extending properties of ordinary bialgebras and Hopf algebras in [20, 21]. In [1], Caenepeel and Goyvaerts studied Hom-bialgebras and Hom-Hopf algebras from a categorical view point, and called them monoidal Hom-bialgebras and monoidal Hom-Hopf algebras respectively, which are different from the normal Hom-bialgebras and Hom-Hopf algebras in [20]. Many more properties and structures of Hom-Hopf algebras have been developed, see [5, 7, 16, 36] and references cited therein.

Later, Yau [31, 33] proposed the definition of quasitriangular Hom-Hopf algebras and showed that each quasitriangular Hom-Hopf algebra yields a solution of the Hom-Yang-Baxter equation. The Hom-Yang-Baxter equation reduces to the usual Yang-Baxter equation when the twist map is trivial. Several classes of solutions of the Hom-Yang-Baxter equation were constructed from different respects, including those associated to Hom-Lie algebras [6, 28, 31, 32], Drinfelds (co)doubles [3, 37, 38], and Hom-Yetter-Drinfeld modules [4, 13, 17, 18, 22, 29, 34].

It is well-known that classical nonlinear equations in Hopf algebra theory including the quantum Yang-Baxter equation, the Hopf equation, the pentagon equation, and the Long equation. In [23], Militaru proved that each Long dimodule gave rise to a solution for the Long equation. Long dimodules are the building stones of the Brauer-Long group. In the case where H is commutative, cocommutative and faithfully projective, the Yetter-Drinfeld category ${}_H^H\mathbb{YD}$ is precisely the Long dimodule category ${}_H^H\mathbb{L}$. Of course, for an arbitrary H, the categories ${}_H^H\mathbb{YD}$ and ${}_H^H\mathbb{L}$ are basically different. In [2], Chen et al. introduced the concept of Long dimodules over a monoidal Hom-bialgebra and discussed its relation with Hom-Long equations. Later, we [27] extended Chen's work to generalized Hom-Long dimodules over monoidal Hom-Hopf algebras and obtained a kind solution for the quantum Yang-Baxter equation. For more details about Long dimodules, see [14, 15, 26, 35] and references cited therein.

The main purpose of this paper is to construct a new braided monoidal category and present solutions for two kinds of nonlinear equations. Different to our previous work in [27], in the present paper we do all the work over Hom-Hopf algebras, which is more unpredictable than the monoidal version. Since Hom-Hopf algebras and monoidal Hom-Hopf algebras are different concepts, it turns out that our definitions, formulas and results are also different from the ones in [27]. Most important, we associate quantum Yang-Baxter equations and Hom-Long equations to the Hom-Long dimodule categories.

This paper is organized as follows. In Section 1, we recall some basic definitions about

Hom-(co)modules and (co)quasitriangular Hom-Hopf algebras .

In Section 2, we first introduce the notion of (H,B)-Hom-Long dimodules over Hom-bialgebras (H,α) and (B,β), then we show that the Hom-Long dimodule category ${}^{B}_{H}\mathbb{L}$ forms an autonomous category (see Theorem 2.6) and prove that the category is equivalent to the category of left $B^{*op}\otimes H$-Hom-modules (see Theorem 2.7).

In Section 3, for a quasitriangular Hom-Hopf algebra (H,R,α) and a coquasitriangular Hom-Hopf algebra $(B,\langle|\rangle,\beta)$, we prove that the Hom-Long dimodule category ${}^{B}_{H}\mathbb{L}$ is a subcategory of the Hom-Yetter-Drinfeld category ${}^{H\otimes B}_{H\otimes B}\mathbb{HYD}$ (see Theorem 3.5), and show that the braiding yields a solution for the quantum Yang-Baxter equation (see Corollary 3.2).

In Section 4, we prove that the category ${}_{H}\mathbb{M}$ over a triangular Hom-Hopf algebra (resp., ${}^{H}\mathbb{M}$ over a cotriangular Hom-Hopf algebra) is a Hom-Long dimodule subcategory of ${}^{B}_{H}\mathbb{L}$ (see Propositions 4.1 and 4.2). We also show that the Hom-Long dimodule category ${}^{B}_{H}\mathbb{L}$ is symmetric in case (H,R,α) is triangular and $(B,\langle|\rangle,\beta)$ is cotriangular (see Theorem 4.3).

In Section 5, we introduce the notion of (H,α)-Hom-Long dimodules and obtain a solution for the Hom-Long equation (see Theorem 5.10).

1 PRELIMINARIES

Throughout this paper, k is a fixed field. Unless otherwise stated, all vector spaces, algebras, modules, maps and unadorned tensor products are over k. For a coalgebra C, the coproduct will be denoted by Δ. We adopt a Sweedler's notation $\triangle(c)=c_1\otimes c_2$, for any $c\in C$, where the summation is understood. We refer to [24, 25] for the Hopf algebra theory and terminology.

We now recall some useful definitions in [12, 19, 20, 21, 30, 33].

Definition 1.1. A Hom-algebra is a quadruple $(A,\mu,1_A,\alpha)$ (abbr. (A,α)), where A is a k-linear space, $\mu: A\otimes A\longrightarrow A$ is a k-linear map, $1_A\in A$ and α is an automorphism of A, such that

$$(HA1)\quad \alpha(aa')=\alpha(a)\alpha(a');\quad \alpha(1_A)=1_A,$$
$$(HA2)\quad \alpha(a)(a'a'')=(aa')\alpha(a'');\quad a1_A=1_Aa=\alpha(a)$$

are satisfied for $a,a',a''\in A$. Here we use the notation $\mu(a\otimes a')=aa'$.

Definition 1.2. Let (A,α) be a Hom-algebra. A left (A,α)-Hom-module is a triple $(M,\triangleright,\nu)$, where M is a linear space, $\triangleright: A\otimes M\longrightarrow M$ is a linear map, and ν is an automorphism of M, such that

$$(HM1)\quad \nu(a\triangleright m)=\alpha(a)\triangleright\nu(m),$$
$$(HM2)\quad \alpha(a)\triangleright(a'\triangleright m)=(aa')\triangleright\nu(m);\quad 1_A\triangleright m=\nu(m)$$

are satisfied for $a, a' \in A$ and $m \in M$.

Let $(M, \triangleright_M, \nu_M)$ and $(N, \triangleright_N, \nu_N)$ be two left (A, α)-Hom-modules. Then a linear morphism $f : M \longrightarrow N$ is called a morphism of left (A, α)-Hom-modules if $f(h \triangleright_M m) = h \triangleright_N f(m)$ and $\nu_N \circ f = f \circ \nu_M$.

Definition 1.3. A Hom-coalgebra is a quadruple $(C, \Delta, \epsilon, \beta)$ (abbr. (C, β)), where C is a k-linear space, $\Delta : C \longrightarrow C \otimes C$, $\epsilon : C \longrightarrow k$ are k-linear maps, and β is an automorphism of C, such that

$$(HC1) \quad \beta(c)_1 \otimes \beta(c)_2 = \beta(c_1) \otimes \beta(c_2); \quad \epsilon \circ \beta = \epsilon;$$
$$(HC2) \quad \beta(c_1) \otimes c_{21} \otimes c_{22} = c_{11} \otimes c_{12} \otimes \beta(c_2); \quad \epsilon(c_1)c_2 = c_1\epsilon(c_2) = \beta(c)$$

are satisfied for $c \in C$.

Definition 1.4. Let (C, β) be a Hom-coalgebra. A left (C, β)-Hom-comodule is a triple (M, ρ, μ), where M is a linear space, $\rho : M \longrightarrow C \otimes M$ (write $\rho(m) = m_{(-1)} \otimes m_{(0)}$, $\forall m \in M$) is a linear map, and μ is an automorphism of M, such that

$$(HCM1) \quad \mu(m)_{(-1)} \otimes \mu(m)_{(0)} = \beta(m_{(-1)}) \otimes \mu(m_{(0)}), \ \epsilon(m_{(-1)})m_{(0)} = \mu(m);$$
$$(HCM2) \quad \beta(m_{(-1)}) \otimes m_{(0)(-1)} \otimes m_{(0)(0)} = m_{(-1)1} \otimes m_{(-1)2} \otimes \mu(m_{(0)})$$

are satisfied for all $m \in M$.

Let (M, ρ^M, μ_M) and (N, ρ^N, μ_N) be two left (C, β)-Hom-comodules. Then a linear map $f : M \longrightarrow N$ is called a map of left (C, β)-Hom-comodules if $f(m)_{(-1)} \otimes f(m)_{(0)} = m_{(-1)} \otimes f(m_{(0)})$ and $\mu_N \circ f = f \circ \mu_M$.

Definition 1.5. A Hom-bialgebra is a sextuple $(H, \mu, 1_H, \Delta, \epsilon, \gamma)$ (abbr. (H, γ)), where $(H, \mu, 1_H, \gamma)$ is a Hom-algebra and $(H, \Delta, \epsilon, \gamma)$ is a Hom-coalgebra, such that Δ and ϵ are morphisms of Hom-algebras, i.e.

$$\Delta(hh') = \Delta(h)\Delta(h'); \ \Delta(1_H) = 1_H \otimes 1_H; \ \epsilon(hh') = \epsilon(h)\epsilon(h'); \ \epsilon(1_H) = 1.$$

Furthermore, if there exists a linear map $S : H \longrightarrow H$ such that

$$S(h_1)h_2 = h_1S(h_2) = \epsilon(h)1_H \text{ and } S(\gamma(h)) = \gamma(S(h)),$$

then we call $(H, \mu, 1_H, \Delta, \epsilon, \gamma, S)$ (abbr. (H, γ, S)) a Hom-Hopf algebra.

Definition 1.6. ([12]) Let (H, β) be a Hom-bialgebra, $(M, \triangleright, \mu)$ a left (H, β)-module with action $\triangleright : H \otimes M \longrightarrow M, h \otimes m \mapsto h \triangleright m$ and (M, ρ, μ) a left (H, β)-comodule with coaction $\rho : M \longrightarrow H \otimes M, m \mapsto m_{(-1)} \otimes m_{(0)}$. Then we call $(M, \triangleright, \rho, \mu)$ a (left-left) Hom-Yetter-Drinfeld module over (H, β) if the following condition holds:

$$(HYD) \quad h_1\beta(m_{(-1)}) \otimes (\beta^3(h_2) \triangleright m_{(0)} = (\beta^2(h_1) \triangleright m)_{(-1)}h_2 \otimes (\beta^2(h_1) \triangleright m)_{(0)},$$

where $h \in H$ and $m \in M$.

When H is a Hom-Hopf algebra, then the condition (HYD) is equivalent to

$$(HYD)' \quad \rho(\beta^4(h) \triangleright m) = \beta^{-2}(h_{11}\beta(m_{(-1)}))S(h_2) \otimes (\beta^3(h_{12}) \triangleright m_0).$$

Definition 1.7. ([12]) Let (H, β) be a Hom-bialgebra. A Hom-Yetter-Drinfeld category ${}^H_H\mathbb{YD}$ is a pre-braided monoidal category whose objects are left-left Hom-Yetter-Drinfeld modules, morphisms are both left (H, β)-linear and (H, β)-colinear maps, and its pre-braiding $C_{-,-}$ is given by

$$C_{M,N}(m \otimes n) = \beta^2(m_{(-1)}) \triangleright \nu^{-1}(n) \otimes \mu^{-1}(m_0), \tag{1.1}$$

for all $m \in (M, \mu) \in {}^H_H\mathbb{YD}$ and $n \in (N, \nu) \in {}^H_H\mathbb{YD}$.

Definition 1.8. A quasitriangular Hom-Hopf algebra is a octuple $(H, \mu, 1_H, \Delta, \epsilon, S, \beta, R)$ (abbr. (H, β, R)) in which $(H, \mu, 1_H, \Delta, \epsilon, S, \beta)$ is a Hom-Hopf algebra and $R = R^{(1)} \otimes R^{(2)} \in H \otimes H$, satisfying the following axioms (for all $h \in H$ and $R = r$):

$$\begin{aligned}
&(QHA1)\ \epsilon(R^{(1)})R^{(2)} = R^{(1)}\epsilon(R^{(2)}) = 1_H,\\
&(QHA2)\ \Delta(R^{(1)}) \otimes \beta(R^{(2)}) = \beta(R^{(1)}) \otimes \beta(r^{(1)}) \otimes R^{(2)}r^{(2)},\\
&(QHA3)\ \beta(R^{(1)}) \otimes \Delta(R^{(2)}) = R^{(1)}r^{(1)} \otimes \beta(r^{(2)}) \otimes \beta(R^{(2)}),\\
&(QHA4)\ \Delta^{cop}(h)R = R\Delta(h),\\
&(QHA5)\ \beta(R^{(1)}) \otimes \beta(R^{(2)}) = R^{(1)} \otimes R^{(2)},
\end{aligned}$$

where $\Delta^{cop}(h) = h_2 \otimes h_1$ for all $h \in H$. A quasitriangular Hom-Hopf algebra (H, R, β) is called triangular if $R^{-1} = R^{(2)} \otimes R^{(1)}$.

Definition 1.9. A coquasitriangular Hom-Hopf algebra is a Hom-Hopf algebra (H, β) together with a bilinear form $\langle | \rangle$ on (H, β) (i.e. $\langle | \rangle \in \mathrm{Hom}(H \otimes H, k)$) such that the following axioms hold:

$$\begin{aligned}
&(CHA1)\ \langle hg | \beta(l) \rangle = \langle \beta(h) | l_2 \rangle \langle \beta(g) | l_1 \rangle,\\
&(CHA2)\ \langle \beta(h) | gl \rangle = \langle h_1 | \beta(g) \rangle \langle h_2 | \beta(l) \rangle,\\
&(CHA3)\ \langle h_1 | g_1 \rangle g_2 h_2 = h_1 g_1 \langle h_2 | g_2 \rangle,\\
&(CHA4)\ \langle 1 | h \rangle = \langle h | 1 \rangle = \epsilon(h),\\
&(CHA5)\ \langle \beta(h) | \beta(g) \rangle = \langle h | g \rangle
\end{aligned}$$

for all $h, g, l \in H$. A coquasitriangular Hom-Hopf algebra $(H, \langle | \rangle, \beta)$ is called cotriangular if $\langle | \rangle$ is convolution invertible in the sense of $\langle h_1 | g_1 \rangle \langle g_2 | h_2 \rangle = \epsilon(h)\epsilon(g)$, for all $h, g \in H$.

2 Hom-Long dimodules over Hom-bialgebras

In this section, we will introduce the notion of Hom-Long dimodules and prove that the Hom-Long dimodule category is an autonomous category.

Definition 2.1. Let (H, α) and (B, β) be two Hom-bialgebras. A left-left (H, B)-Hom-Long dimodule is a quadrupl $(M, \cdot, \rho, \mu)$, where $(M, \cdot, \mu)$ is a left (H, α)-Hom-module and (M, ρ, μ) is a left (B, β)-Hom-comodule such that

$$\rho(h \cdot m) = \beta(m_{(-1)}) \otimes \alpha(h) \cdot m_{(0)}, \tag{2.1}$$

for all $h \in H$ and $m \in M$. We denote by ${}^{B}_{H}\mathbb{L}$ the category of left-left (H, B)-Hom-Long dimodules, morphisms being H-linear B-colinear maps.

Example 2.2. Let (H, α) and (B, β) be two Hom-bialgebras. Then $(H \otimes B, \alpha \otimes \beta)$ is an (H, B)-Hom-Long dimodule with left (H, α)-action $h \cdot (g \otimes x) = hg \otimes \beta(x)$ and left (B, β)-coaction $\rho(g \otimes x) = x_1 \otimes (\alpha(g) \otimes x_2)$, where $h, g \in H, x \in B$.

Proposition 2.3. Let $(M, \mu), (N, \nu)$ be two (H, B)-Hom-Long dimodules, then $(M \otimes N, \mu \otimes \nu)$ is an (H, B)-Hom-Long dimodule with structures:

$$\begin{aligned}
&h \cdot (m \otimes n) = h_1 \cdot m \otimes h_2 \cdot n,\\
&\rho(m \otimes n) = \beta^{-2}(m_{(-1)}n_{(-1)}) \otimes m_{(0)} \otimes n_{(0)},
\end{aligned}$$

for all $m \in M, n \in N$ and $h \in H$.

Proof. From Theorem 4.8 in [17], $(M \otimes N, \mu \otimes \nu)$ is both a left (H, α)-Hom-module and a left (B, β)-Hom-comodule. It remains to check that the compatibility condition (2.1) holds. For any $m \in M, n \in N$ and $h \in H$, we have

$$\begin{aligned}
\rho(h \cdot (m \otimes n)) &= \beta((h_1 \cdot m)_{(-1)}(h_2 \cdot n)_{(-1)}) \otimes (h_1 \cdot m)_{(0)} \otimes (h_2 \cdot n)_{(0)}\\
&= \beta^{-1}(m_{(-1)}n_{(-1)}) \otimes \alpha(h_1) \cdot m_{(0)} \otimes \alpha(h_2) \cdot n_{(0)}\\
&= \beta((m \otimes n)_{(-1)}) \otimes \alpha(h) \cdot ((m \otimes n)_{(0)}),
\end{aligned}$$

as desired. This completes the proof. □

Proposition 2.4. The Hom-Long dimodule category ${}^{B}_{H}\mathbb{L}$ is a monoidal category, where the tensor product is given in Proposition 2.3, the unit $I = (k, id)$, the associator and the constraints are given as follows:

$$\begin{aligned}
&a_{U,V,W} : (U \otimes V) \otimes W \to U \otimes (V \otimes W), (u \otimes v) \otimes w \to \mu^{-1}(u) \otimes (v \otimes \omega(w)),\\
&l_V : k \otimes V \to V, k \otimes v \to k\nu(v), r_V : V \otimes k \to V, v \otimes k \to k\nu(v),
\end{aligned}$$

for $u \in (U, \mu) \in {}^{B}_{H}\mathbb{L}, v \in (V, \nu) \in {}^{B}_{H}\mathbb{L}, w \in (W, \omega) \in {}^{B}_{H}\mathbb{L}$.

Proof. Straightforward. $\square$

Proposition 2.5. Let H and B be two Hom-Hopf algebras with bijective antipodes. For any Hom-Long dimodule (M,μ) in ${}^B_H\mathbb{L}$, set $M^* = Hom_k(M,k)$, with the (H,α)-Hom-module and the (B,β)-Hom-comodule structures:

$$\theta_{M^*}: H\otimes M^* \longrightarrow M^*, \quad (h\cdot f)(m) = f(S_H\alpha^{-1}(h)\cdot \mu^{-2}(m)),$$
$$\rho_{M^*}: M^* \longrightarrow B\otimes M^*, \quad f_{(-1)}\otimes f_{(0)}(m) = S_B^{-1}\beta^{-1}(m_{(-1)})\otimes f(\mu^{-2}(m_{(0)})),$$

and the Hom-structure map μ^* of M^* is $\mu^*(f)(m) = f(\mu^{-1}(m))$. Then M^* is an object in ${}^B_H\mathbb{L}$. Moreover, ${}^B_H\mathbb{L}$ is a left autonomous category.

Proof. It is not hard to check that $(M^*, \theta_{M^*}, \mu^*)$ is an (H,α)-Hom-module and (M^*, ρ_{M^*}, μ^*) is a (B,β)-Hom-comodule. Further, for any $f\in M^*$, $m\in M$, $h\in H$, we have

$$\begin{aligned}
(h\cdot f)_{(-1)}\otimes (h\cdot f)_{(0)}(m) &= S_B^{-1}\beta^{-1}(m_{(-1)})\otimes (h\cdot f)(\mu^{-2}(m_{(0)}))\\
&= S_B^{-1}\beta^{-1}(m_{(-1)})\otimes f(S_H\alpha^{-1}(h)\cdot \mu^{-4}(m_{(0)})),\\
\beta(f_{(-1)})\otimes (\alpha(h)\cdot f_{(0)})(m) &= \beta(f_{(-1)})\otimes f_{(0)}(S_H(h)\cdot \mu^{-2}(m))\\
&= \beta(S_B^{-1}\beta^{-2}(m_{(-1)}))\otimes f(\mu^{-2}(S_H\alpha(h)\cdot \mu^{-2}(m_{(0)})))\\
&= S_B^{-1}\beta^{-1}(m_{(-1)})\otimes f(S_H\alpha^{-1}(h)\cdot \mu^{-4}(m_{(0)})).
\end{aligned}$$

Thus $M^*\in {}^B_H\mathbb{L}$.

Moreover, for any $f\in M^*$ and $m\in M$, one can define the left evaluation map and the left coevaluation map by

$$ev_M: f\otimes m \longmapsto f(m), \; coev_M: 1_k \longmapsto \sum e_i\otimes e^i,$$

where e_i and e^i are dual bases in M and M^* respectively. Next, we will show $(M^*, ev_M, coev_M)$ is the left dual of M.

It is easy to see that ev_M and $coev_M$ are morphisms in ${}^B_H\mathbb{L}$. For this, we need the following computation

$$\begin{aligned}
&(r_M\circ (id_M\otimes ev_M)\circ a_{M,M^*,M}\circ (coev_M\otimes id_M)\circ l_M^{-1})(m)\\
&= (r_M\circ (id_M\otimes ev_M)\circ a_{M,M^*,M})(\sum_i (e_i\otimes e^i)\otimes \mu^{-1}(m))\\
&= (r_M\circ (id_M\otimes ev_M))(\sum_i \mu^{-1}(e_i)\otimes (e^i\otimes m))\\
&= r_M(\sum_i \mu^{-1}(e_i)\otimes e^i(m))\\
&= r_M(\mu^{-1}(m)\otimes 1_k) = m.
\end{aligned}$$

Similarly, we get

$$(l_{M^*}\circ (ev_M\otimes id_{M^*})\circ a^{-1}_{M^*,M,M^*}\circ (id_{M^*}\otimes coev_M)\circ r^{-1}_{M^*})(f)$$

$$= (l_{M^*}\circ(ev_M\otimes id_{M^*})\circ a^{-1}_{M^*,M,M^*})(\sum_i \mu^{*-1}(f)\otimes(e_i\otimes e^i))$$
$$= (l_{M^*}\circ(ev_M\otimes id_{M^*}))(\sum_i f\otimes e_i)\otimes\mu^{*-1}(e^i))$$
$$= l_{M^*}(\sum_i f(e_i)\otimes\mu^{*-1}(e^i))$$
$$= l_{M^*}(1_k\otimes\mu^{*-1}(f)) = f.$$

So ${}^B_H\mathbb{L}$ admits the left duality. The proof is finished. □

Theorem 2.6. The Hom-Long dimodule category ${}^B_H\mathbb{L}$ is an autonomous category.

Proof By Proposition 2.5, it is sufficient to show that ${}^B_H\mathbb{L}$ is also a right autonomous category. In fact, for any $(M,\mu)\in {}^B_H\mathbb{L}$, its right dual $({}^*M,\widetilde{coev}_M,\widetilde{ev}_M)$ is defined as follows:

• ${}^*M = Hom_k(M,k)$ as k-modules, with the Hom-module and Hom-comodule structures:

$$(h\cdot f)(m) = f(S_H^{-1}\alpha^{-1}(h)\cdot\mu^{-2}(m)),$$
$$f_{(-1)}\otimes f_{(0)}(m) = S_B\beta^{-1}(m_{(-1)})\otimes f(\mu^{-2}(m_{(0)})),$$

where $f\in {}^*M$, $m\in M$, and the Hom-structure map μ^* of ${}^*M$ is $\mu^*(f)(m) = f(\mu^{-1}(m))$;

• The right evaluation map and the right coevaluation map are given by

$$\widetilde{ev}_M: m\otimes f\longmapsto f(m),\ \ \widetilde{coev}_M: 1_k\longmapsto \sum a^i\otimes a_i.$$

where a_i and a^i are dual bases of M and ${}^*M$ respectively. By similar verification in Proposition 2.5, one may check that ${}^B_H\mathbb{L}$ is a right autonomous category, as required. This completes the proof. □

Recall from [37] that for any finite dimensional Hom-Hopf algebra B, B^* is also a Hom-Hopf algebra with the following structures

$$(f*g)(y) := f(\beta^{-2}(y_1))g(\beta^{-2}(y_2)),\ \ \Delta_{B^*}(f)(xy) := f(\beta^{-2}(xy)),$$
$$1_{B^*} := \epsilon,\ \ \epsilon_{B^*}(f) := f(1_H),\ \ S_{B^*} := S^*,\ \ \alpha_{B^*}(f) := f\circ\beta^{-1},$$

where $x,y\in H,\ f,g\in B^*$.

Theorem 2.7. If B is a finite dimensional Hom-Hopf algebra, then the Hom-Long dimodule category ${}^B_H\mathbb{L}$ is identified to the category of left $B^{*op}\otimes H$-Hom-modules, where $B^{*op}\otimes H$ means the usual tensor product Hom-Hopf algebra.

Proof Define the functor Ψ from ${}_{B^{*op}\otimes H}\mathbb{M}$ to ${}^B_H\mathbb{L}$ by

$$\Psi(M) := M \text{ as } k\text{-module},\quad \Psi(f) := f,$$

where $(M, \mu, \rightharpoondown)$ is a $B^{*op} \otimes H$-Hom-module, $f : M \to N$ is a morphism of $B^{*op} \otimes H$-Hom-modules. Further, the H-action on M is defined by

$$h \cdot m := (\epsilon_B \otimes h) \rightharpoondown m, \qquad \text{for all } m \in M, \quad h \in H,$$

and the B-coaction on M is given by

$$m_{(-1)} \otimes m_{(0)} := \sum e_i \otimes (e^i \otimes 1_H) \rightharpoondown m,$$

where e_i and e^i are dual bases of B and B^* respectively.

First, we will show $(M, \mu, \cdot)$ is a left (H, α)-Hom-module. Actually, for any $m \in M$, $h, g \in H$, we have $1_H \cdot m = (\epsilon_B \otimes 1_H) \rightharpoondown m = \mu(m)$, and

$$\begin{aligned} \alpha(h) \cdot (g \cdot m) &= (\epsilon_B \otimes \alpha(h)) \rightharpoondown ((\epsilon_B \otimes g) \rightharpoondown m) \\ &= (\epsilon_B \otimes hg) \rightharpoondown \mu(m) = (hg) \cdot \mu(m), \end{aligned}$$

which implies $(M, \mu, \cdot) \in {}_H\mathbb{M}$.

Second, one can show that $(M, \mu) \in {}^B\mathbb{M}$ in a similar way.

At last, for any $m \in M$, $h \in H$, we have

$$\begin{aligned} (h \cdot m)_{(-1)} \otimes (h \cdot m)_{(0)} &= \sum e_i \otimes (e^i \otimes 1_H) \rightharpoondown (h \cdot m) \\ &= \sum e_i \otimes (e^i \otimes \alpha(h)) \rightharpoondown \mu(m) \\ &= \sum \beta(e_i) \otimes ((\epsilon_B \otimes 1_H)(e^i \otimes h) \rightharpoondown \mu(m) \\ &= \sum \beta(e_i) \otimes ((\epsilon_B \otimes h)(e^i \otimes 1_H) \rightharpoondown \mu(m) \\ &= \sum \beta(e_i) \otimes \alpha(h) \cdot ((e^i \otimes 1_H) \rightharpoondown \mu(m)) \\ &= \beta(m_{(-1)}) \otimes \alpha(h) \cdot m_{(0)}, \end{aligned}$$

which implies $(M, \mu) \in {}^B_H\mathbb{L}$.

Conversely, for any object (M, μ), (N, ν), and morphism $f : U \to V$ in ${}^B_H\mathbb{L}$, one can define a functor Φ from ${}^B_H\mathbb{L}$ to ${}_{B^{*op} \otimes H}\mathbb{M}$

$$\Phi(M) := M \text{ as } k\text{-modules} , \qquad \Phi(f) := f,$$

where the $(B^{*op} \otimes H, \beta^* \otimes \alpha)$-Hom-module structure on M is given by

$$(p \otimes h) \rightharpoondown m = p(m_{(-1)})h \cdot \mu^{-1}(m_{(0)}),$$

for all $p \in B^*$, $h \otimes H$, $m \in M$. It is straightforward to check that $(M, \mu, \rightharpoondown)$ is an object in ${}^B_H\mathbb{L}$ to ${}_{B^{*op} \otimes H}\mathbb{M}$, and hence Φ is well defined.

Note that Φ and Ψ are inverse with each other. Hence the conclusion holds.

3 New braided momoidal categories over Hom-Long dimodules

In this section, we will prove that the Hom-Long dimodule category ${}^{B}_{H}\mathbb{L}$ over a quasitriangular Hom-Hopf algebra (H,R,α) and a coquasitriangular Hom-Hopf algebra $(B,\langle|\rangle,\beta)$ is a braided monoidal subcategory of the Hom-Yetter-Drinfeld category ${}^{H\otimes B}_{H\otimes B}\mathbb{HYD}$.

Theorem 3.1. Let (H,R,α) be a quasitriangular Hom-Hopf algebra and $(B,\langle|\rangle,\beta)$ a coquasitriangular Hom-Hopf algebra. Then the category ${}^{B}_{H}\mathbb{L}$ is a braided monoidal category with braiding

$$C_{M,N}:M\otimes N\to N\otimes M, m\otimes n\to \langle m_{(-1)}|n_{(-1)}\rangle R^{(2)}\cdot\nu^{-2}(n_{(0)})\otimes R^{(1)}\cdot\mu^{-2}(m_{(0)}),\quad(3.1)$$

for all $m\in(M,\mu)\in{}^{B}_{H}\mathbb{L}$ and $n\in(N,\nu)\in{}^{B}_{H}\mathbb{L}$.

Proof. We will first show that the braiding $C_{M,N}$ is a morphism in ${}^{B}_{H}\mathbb{L}$. In fact, for any $m\in M,n\in N$ and $h\in H$, we have

$$\begin{aligned}
&C_{M,N}(h_1\cdot m\otimes h_2\cdot n)\\
=\ &\langle(h_1\cdot m)_{(-1)}|(h_2\cdot n)_{(-1)}\rangle R^{(2)}\cdot\nu^{-2}(h_2\cdot n)_{(0)}\otimes R^{(1)}\cdot\mu^{-2}(h_1\cdot m)_{(0)}\\
\overset{(2.1)}{=}\ &\langle\beta(m_{(-1)})|\beta(n_{(-1)})\rangle R^{(2)}\cdot\nu^{-2}(\alpha(h_2)\cdot n_{(0)})\otimes R^{(1)}\cdot\mu^{-2}(\alpha(h_1)\cdot m_{(0)})\\
\overset{(HM2)}{=}\ &\langle m_{(-1)}|n_{(-1)}\rangle\alpha^{-1}(R^{(2)}h_2)\cdot\nu^{-1}(n_{(0)})\otimes\alpha^{-1}(R^{(1)}h_1)\cdot\mu^{-1}(m_{(0)}),\\
&h\cdot C_{M,N}(m\otimes n)\\
=\ &\langle m_{(-1)}|n_{(-1)}\rangle h\cdot(R^{(2)}\cdot\nu^{-2}(n_{(0)})\otimes R^{(1)}\cdot\mu^{-2}(m_{(0)}))\\
=\ &\langle m_{(-1)}|n_{(-1)}\rangle h_1\cdot(\alpha^{-1}(R^{(2)})\cdot\nu^{-2}(n_{(0)}))\otimes h_2\cdot(\alpha^{-1}(R^{(1)})\cdot\mu^{-2}(m_{(0)}))\\
\overset{(HM2)}{=}\ &\langle m_{(-1)}|n_{(-1)}\rangle\alpha^{-1}(h_1R^{(2)})\cdot\nu^{-1}(n_{(0)})\otimes\alpha^{-1}(h_2R^{(1)})\cdot\mu^{-1}(m_{(0)})\\
\overset{(QHA4)}{=}\ &\langle m_{(-1)}|n_{(-1)}\rangle\alpha^{-1}(R^{(2)}h_2)\cdot\nu^{-1}(n_{(0)})\otimes\alpha^{-1}(R^{(1)}h_1)\cdot\mu^{-1}(m_{(0)}).
\end{aligned}$$

The third equality holds since $\langle|\rangle$ is β-invariant and the fifth equality holds since R is α-invariant. So $C_{M,N}$ is left (H,α)-linear. Similarly, one may check that $C_{M,N}$ is left (B,β)-colinear.

Now we prove that the braiding $C_{M,N}$ is natural. For any $(M,\mu),(M',\mu'),(N,\nu),(N',\nu')$ $\in{}^{B}_{H}\mathbb{L}$, let $f:M\to M'$ and $g:N\to N'$ be two morpshisms in ${}^{B}_{H}\mathbb{L}$, it is sufficient to verify the identity $(g\otimes f)\circ C_{M,N}=C_{M',N'}\circ(f\otimes g)$. For this purpose, we take $m\in M,n\in N$ and do the following calculation:

$$\begin{aligned}
(g\otimes f)\circ C_{M,N}(m\otimes n)\ &=\ \langle m_{(-1)}|n_{(-1)}\rangle(g\otimes f)(R^{(2)}\cdot\nu^{-2}(n_{(0)})\otimes R^{(1)}\cdot\mu^{-2}(m_{(0)}))\\
&=\ \langle m_{(-1)}|n_{(-1)}\rangle g(R^{(2)}\cdot\nu^{-2}(n_{(0)}))\otimes f(R^{(1)}\cdot\mu^{-2}(m_{(0)}))\\
&=\ \langle m_{(-1)}|n_{(-1)}\rangle R^{(2)}\cdot g(\nu^{-2}(n_{(0)}))\otimes R^{(1)}\cdot f(\mu^{-2}(m_{(0)})),
\end{aligned}$$

$$
\begin{aligned}
C_{M',N'}\circ(f\otimes g)(m\otimes n) &= C_{M',N'}(f(m)\otimes g(n))\\
&= \langle f(m)_{(-1)}|g(n)_{(-1)}\rangle R^{(2)}\cdot\nu^{-2}(g(n)_{(0)})\otimes(R^{(1)}\cdot\mu^{-2}(f(m)_{(0)})\\
&= \langle m_{(-1)}|n_{(-1)}\rangle R^{(2)}\cdot\nu^{-2}(g(n_{(0)}))\otimes R^{(1)}\cdot\mu^{-2}(f(m_{(0)}))\\
&= \langle m_{(-1)}|n_{(-1)}\rangle R^{(2)}\cdot g(\nu^{-2}(n_{(0)}))\otimes R^{(1)}\cdot f(\mu^{-2}(m_{(0)})).
\end{aligned}
$$

The sixth equality holds since f,g are left (B,β)-colinear. So the braiding $C_{M,N}$ is natural, as needed.

Next, we will show that the braiding $C_{M,N}$ is an isomorphsim with inverse map

$$C_{M,N}^{-1}:N\otimes M\to M\otimes N, n\otimes m\to\langle S^{-1}(m_{(-1)})|n_{(-1)}\rangle S(R^{(1)})\cdot\mu^{-2}(m_{(0)})\otimes R^{(2)}\cdot\nu^{-2}(n_{(0)}).$$

For any $m\in M,n\in N$, we have

$$
\begin{aligned}
& C_{M,N}^{-1}\circ C_{M,N}(m\otimes n)\\
= \; & \langle m_{(-1)}|n_{(-1)}\rangle C_{M,N}^{-1}(R^{(2)}\cdot\nu^{-2}(n_{(0)})\otimes R^{(1)}\cdot\mu^{-2}(m_{(0)}))\\
= \; & \langle m_{(-1)}|n_{(-1)}\rangle\langle S^{-1}(\beta^{-1}(m_{(0)(-1)}))|\beta^{-1}(n_{(0)(-1)})\rangle\\
& \quad S(r^{(1)})\cdot\mu^{-2}(\alpha(R^{(2)})\cdot\mu^{-2}(m_{(0)(0)}))\otimes r^{(2)}\cdot\nu^{-2}(\alpha(R^{(1)})\cdot\nu^{-2}(n_{(0)(0)}))\\
\overset{(HCM2)}{=} \; & \langle\beta^{-1}(m_{(-1)1})|\beta^{-1}(n_{(-1)1})\rangle\langle S^{-1}(\beta^{-1}(m_{(-1)2}))|\beta^{-1}(n_{(-1)2})\rangle\\
& \quad S(r^{(1)})\cdot(\alpha^{-1}(R^{(2)})\cdot\mu^{-3}(m_{(0)}))\otimes r^{(2)}\cdot(\alpha^{-1}(R^{(1)})\cdot\nu^{-3}(n_{(0)}))\\
\overset{(HM2)}{=} \; & \langle m_{(-1)1}|n_{(-1)1}\rangle\langle S^{-1}(m_{(-1)2})|n_{(-1)2}\rangle\\
& \quad \alpha^{-1}(S(r^{(1)})R^{(2)})\cdot\mu^{-2}(m_{(0)})\otimes\alpha^{-1}(r^{(2)}R^{(1)})\cdot\nu^{-2}(n_{(0)})\\
\overset{(CHA1)}{=} \; & \langle S^{-1}(\beta^{-1}(m_{(-1)2}))\beta^{-1}(m_{(-1)1})|\beta(n_{(-1)})\rangle 1_H\cdot\mu^{-2}(m_{(0)})\otimes 1_H\cdot\nu^{-2}(n_{(0)})\\
= \; & \langle\beta^{-2}(S^{-1}(m_{(-1)2})m_{(-1)1})|n_{(-1)}\rangle 1_H\cdot\mu^{-2}(m_{(0)})\otimes 1_H\cdot\nu^{-2}(n_{(0)})\\
= \; & \langle\epsilon(m_{(-1)})1_H|n_{(-1)}\rangle\mu^{-1}(m_{(0)})\otimes\nu^{-1}(n_{(0)})\\
= \; & \epsilon(m_{(-1)})\epsilon(n_{(-1)})\mu^{-1}(m_{(0)})\otimes\nu^{-1}(n_{(0)})\\
= \; & m\otimes n.
\end{aligned}
$$

The second equality holds since $\rho(R^{(2)}\cdot\nu^{-2}(n_{(0)}))=\beta^{-1}(n_{(0)(-1)})\otimes\alpha(R^{(2)})\cdot n_{(0)(0)}$ and the fifth equality holds since $R^{-1}=S(r^{(1)})\otimes r^{(2)}$.

Now let us verify the hexagon axioms (H_1,H_2) from Section XIII. 1.1 of [11]. We need to show that the following diagram (H_1) commutes for any $(U,\mu),(V,\nu),(W,\omega)\in{}^B_H\mathbb{L}$:

$$
\begin{tikzcd}
(U\otimes V)\otimes W \arrow[r, "a_{U,V,W}"] \arrow[d, "C_{U,V}\otimes id_W"'] & U\otimes(V\otimes W) \arrow[r, "C_{U,V\otimes W}"] & (V\otimes W)\otimes U \arrow[d, "a_{V,W,U}"] \\
(V\otimes U)\otimes W \arrow[r, "a_{V,U,W}"] & V\otimes(U\otimes W) \arrow[r, "id_V\otimes C_{U,W}"] & V\otimes(W\otimes U),
\end{tikzcd}
$$

For this purpose, let $u\in U,v\in V,w\in W$, then we have

$$a_{V,U,W}\circ C_{U,V\otimes W}\circ a_{U,V,W}((u\otimes v)\otimes w)$$

$$
\begin{aligned}
&= a_{V,U,W}\circ C_{U,V\otimes W}(\mu^{-1}(u)\otimes(v\otimes\omega(w)))\\
&= \langle\beta^{-1}(u_{(-1)})|\beta^{-2}(v_{(-1)})\beta^{-1}(w_{(-1)})\rangle a_{V,U,W}\\
&\qquad (R^{(2)}\cdot(\nu^{-2}\otimes\omega^{-2})(v_{(0)}\otimes\omega(w_{(0)}))\otimes R^{(1)}\cdot\mu^{-3}(u_{(0)}))\\
&= \langle\beta(u_{(-1)})|v_{(-1)}\beta(w_{(-1)})\rangle a_{V,U,W}\\
&\qquad (R^{(2)}\cdot(\nu^{-2}(v_{(0)})\otimes\omega^{-1}(w_{(0)}))\otimes R^{(1)}\cdot\mu^{-3}(u_{(0)}))\\
&= \langle\beta(u_{(-1)})|v_{(-1)}\beta(w_{(-1)})\rangle\\
&\qquad \alpha^{-1}(R^{(2)}_1)\cdot\nu^{-3}(v_{(0)})\otimes(R^{(2)}_2\cdot\omega^{-1}(w_{(0)})\otimes\alpha(R^{(1)})\cdot\mu^{-2}(u_{(0)}))\\
&\overset{(QHA3)}{=} \langle\beta(u_{(-1)})|v_{(-1)}\beta(w_{(-1)})\rangle\\
&\qquad r^{(2)}\cdot\nu^{-3}(v_{(0)})\otimes(\alpha(R^{(2)})\cdot\omega^{-1}(w_{(0)})\otimes(R^{(1)}r^{(1)})\cdot\mu^{-2}(u_{(0)}))
\end{aligned}
$$

and

$$
\begin{aligned}
&(id_V\otimes C_{U,W})\circ a_{V,U,W}\circ(C_{U,V}\otimes id_W)((u\otimes v)\otimes w)\\
&= \langle u_{(-1)}|v_{(-1)}\rangle(id_V\otimes C_{U,W})\circ a_{V,U,W}((R^{(2)}\cdot\nu^{-2}(v_{(0)})\otimes R^{(1)}\cdot\mu^{-2}(u_{(0)}))\otimes w)\\
&= \langle u_{(-1)}|v_{(-1)}\rangle(id_V\otimes C_{U,W})\alpha^{-1}(R^{(2)})\cdot\nu^{-3}(v_{(0)})\otimes(R^{(1)}\cdot\mu^{-2}(u_{(0)})\otimes\omega(w))\\
&= \langle u_{(-1)}|v_{(-1)}\rangle\langle\beta^{-1}(u_{(0)(-1)})|\beta(w_{(-1)})\rangle\\
&\qquad \alpha^{-1}(R^{(2)})\cdot\nu^{-3}(v_{(0)})\otimes(r^{(2)}\cdot\omega^{-1}(w_{(0)})\otimes r^{(1)}\cdot\mu^{-2}(\alpha(R^{(1)})\cdot\mu^{-2}(u_{(0)(0)})))\\
&\overset{(HCM2)}{=} \langle\beta^{-1}(u_{(-1)1})|v_{(-1)}\rangle\langle\beta^{-1}(u_{(-1)2})|\beta(w_{(-1)})\rangle\\
&\qquad \alpha^{-1}(R^{(2)})\cdot\nu^{-3}(v_{(0)})\otimes(r^{(2)}\cdot\omega^{-1}(w_{(0)})\otimes\alpha^{-1}(r^{(1)}R^{(1)})\cdot\mu^{-2}(u_{(0)}))\\
&\overset{(CHA2)}{=} \langle u_{(-1)}|\beta^{-1}(v_{(-1)})w_{(-1)}\rangle\\
&\qquad \alpha^{-1}(R^{(2)})\cdot\nu^{-3}(v_{(0)})\otimes(r^{(2)}\cdot\omega^{-1}(w_{(0)})\otimes\alpha^{-1}(r^{(1)}R^{(1)})\cdot\mu^{-2}(u_{(0)}))\\
&= \langle\beta(u_{(-1)})|v_{(-1)}\beta(w_{(-1)})\rangle\\
&\qquad R^{(2)}\cdot\nu^{-3}(v_{(0)})\otimes(\alpha(r^{(2)})\cdot\omega^{-1}(w_{(0)})\otimes(r^{(1)}R^{(1)})\cdot\mu^{-2}(u_{(0)}))
\end{aligned}
$$

Since $r=R$, it follows that $a_{V,U,W}\circ C_{U,V\otimes W}\circ a_{U,V,W}=(id_V\otimes C_{U,W})\circ a_{V,U,W}\circ(C_{U,V}\otimes id_W)$, that is, the diagram (H_1) commutes.

Now we check that the diagram (H_2) commutes for any $(U,\mu),(V,\nu),(W,\omega)\in {}_H^B\mathbb{L}$:

$$
\begin{tikzcd}
U\otimes(V\otimes W) \arrow[r, "a^{-1}_{U,V,W}"] \arrow[d, "id_U\otimes C_{V,W}"'] & (U\otimes V)\otimes W \arrow[r, "C_{U\otimes V,W}"] & W\otimes(U\otimes V) \arrow[d, "a^{-1}_{W,U,V}"] \\
U\otimes(W\otimes V) \arrow[r, "a^{-1}_{U,W,V}"] & (U\otimes W)\otimes V \arrow[r, "C_{U,W}\otimes id_V"] & (W\otimes U)\otimes V.
\end{tikzcd}
$$

In fact, for any $u\in U, v\in V, w\in W$, we obtain

$$
\begin{aligned}
&a^{-1}_{W,U,V}\circ C_{U\otimes V,W}\circ a^{-1}_{U,V,W}(u\otimes(v\otimes w))\\
&= a^{-1}_{W,U,V}\circ C_{U\otimes V,W}((\mu(u)\otimes v)\otimes\omega^{-1}(w))
\end{aligned}
$$

$$
\begin{aligned}
&= \langle \beta^{-1}(u_{(-1)})\beta^{-1}(v_{(-2)})|\beta^{-1}(w_{(-1)})\rangle a^{-1}_{W,U,V} \\
&\qquad (R^{(2)}\cdot \omega^{-3}(w_{(0)})\otimes R^{(1)}\cdot(\mu^{-1}(u_{(0)})\otimes \nu^{-2}(v_{(0)}))) \\
&= \langle \beta(u_{(-1)})v_{(-1)}|\beta(w_{(-1)})\rangle a^{-1}_{W,U,V} \\
&\qquad (R^{(2)}\cdot \omega^{-3}(w_{(0)})\otimes (R^{(1)}_1\cdot \mu^{-1}(u_{(0)})\otimes R^{(1)}_2\cdot \nu^{-2}(v_{(0)}))) \\
&= \langle \beta(u_{(-1)})v_{(-1)}|\beta(w_{(-1)})\rangle \\
&\qquad (\omega(R^{(2)}\cdot \omega^{-2}(w_{(0)}))\otimes R^{(1)}_1\cdot \mu^{-1}(u_{(0)}))\otimes \alpha^{-1}(R^{(1)}_2)\cdot \nu^{-3}(v_{(0)}) \\
&= \langle \beta(u_{(-1)})v_{(-1)}|\beta(w_{(-1)})\rangle \\
&\qquad (\alpha^{-1}(R^{(2)})\cdot \omega^{-2}(w_{(0)})\otimes R^{(1)}_1\cdot \mu^{-1}(u_{(0)}))\otimes \alpha(R^{(1)}_2)\cdot \nu(v_{(0)}) \\
&\overset{(QHA2)}{=} \langle \beta(u_{(-1)})v_{(-1)}|\beta(w_{(-1)})\rangle \\
&\qquad (\alpha^{-1}(R^{(2)}r^{(2)})\cdot \omega^{-2}(w_{(0)})\otimes R^{(1)}\cdot \mu^{-1}(u_{(0)}))\otimes \alpha^{-1}(r^{(1)})\cdot \nu^{-3}(v_{(0)}).
\end{aligned}
$$

Also we can get

$$
\begin{aligned}
&(C_{U,W}\otimes id_V)\circ a^{-1}_{U,W,V}\circ(id_U\otimes C_{V,W})(u\otimes(v\otimes w)) \\
&= \langle v_{(-1)})|w_{(-1)}\rangle (C_{U,W}\otimes id_V)\circ a^{-1}_{U,W,V}(u\otimes(R^{(2)}\cdot \omega^{-2}(w_{(0)})\otimes R^{(1)}\cdot \nu^{-2}(v_{(0)}))) \\
&= \langle v_{(-1)})|w_{(-1)}\rangle (C_{U,W}\otimes id_V)((\mu(u)\otimes R^{(2)}\cdot \omega^{-2}(w_{(0)}))\otimes \alpha^{-1}(R^{(1)})\cdot \nu^{-3}(v_{(0)})) \\
&= \langle v_{(-1)})|w_{(-1)}\rangle \langle \beta(u_{(-1)})|\beta^{-1}(w_{(0)(-1)})\rangle \\
&\qquad (r^{(2)}\cdot \omega^{-2}(\alpha(R^{(2)})\cdot \omega^{-2}(w_{(0)(0)}))\otimes r^{(1)}\cdot \mu^{-1}(u_{(0)}))\otimes \alpha^{-1}(R^{(1)})\cdot \nu^{-3}(v_{(0)}) \\
&\overset{(HCM2)}{=} \langle v_{(-1)})|\beta^{-1}(w_{(-1)1})\rangle \langle \beta(u_{(-1)})|\beta^{-1}(w_{(-1)2})\rangle \\
&\qquad (r^{(2)}\cdot(\alpha^{-1}(R^{(2)})\cdot \omega^{-3}(w_{(0)}))\otimes r^{(1)}\cdot \mu^{-1}(u_{(0)}))\otimes \alpha^{-1}(R^{(1)})\cdot \nu^{-3}(v_{(0)}) \\
&\overset{(CHA1)}{=} \langle u_{(-1)}\beta^{-1}(v_{(-1)})|w_{(-1)}\rangle \\
&\qquad (\alpha^{-1}(r^{(2)}R^{(2)})\cdot \omega^{-2}(w_{(0)})\otimes r^{(1)}\cdot \mu^{-1}(u_{(0)}))\otimes \alpha^{-1}(R^{(1)})\cdot \nu^{-3}(v_{(0)}).
\end{aligned}
$$

So the diagram (H_2) commutes since $r = R$. This ends the proof.

Corollary 3.2. Under the hypotheses of the Theorem 3.1, the braiding C is a solution of the quantum Yang-Baxter equation

$$
\begin{aligned}
&(id_W\otimes C_{U,V})\circ a_{W,U,V}\circ(C_{U,W}\otimes id_V)\circ a^{-1}_{W,V,U}\circ(id_U\otimes C_{V,W})\circ a_{U,V,W} \\
&= a_{W,V,U}\circ(C_{W,V}\otimes id_U)\circ a^{-1}_{W,V,U}\circ(id_V\otimes C_{U,W})\circ a_{V,U,W}\circ(C_{U,V}\otimes id_W).
\end{aligned}
$$

Proof. Straightforward.

Lemma 3.3. Let (H,R,α) be a quasitriangular Hom-Hopf algebra and $(B,\langle|\rangle,\beta)$ a coquasitriangular Hom-Hopf algebra. Define a linear map

$$
(H\otimes B)\otimes M\to M, (h\otimes x)\rightharpoonup m = \langle x|m_{(-1)}\rangle \alpha^{-3}(h)\cdot \mu^{-1}(m_{(0)}),
$$

for any $h\in H, x\in B$ and $m\in (M,\mu)\in {}^B_H\mathbb{L}$. Then (M,μ) becomes a left $(H\otimes B)$-Hom-module.

Proof. It is sufficient to show that the Hom-module action defined above satisfies Definition 1.2. For any $h,g\in H, x,y\in B$ and $m\in M$, we have

$$(1_H\otimes 1_B)\rightharpoonup m=\langle 1_B|m_{(-1)}\rangle 1_H\cdot \mu^{-1}(m_{(0)})=\epsilon(m_{(-1)})m_{(0)}=\mu(m).$$

That is, $(1_H\otimes 1_B)\rightharpoonup m=\mu(m)$. For the equality $\mu((h\otimes x)\rightharpoonup m)=(\alpha(h)\otimes\beta(x))\rightharpoonup \mu(m)$, we have

$$\begin{aligned}(\alpha(h)\otimes\beta(x))\rightharpoonup \mu(m) &= \langle \beta(x)|\beta(m_{(-1)})\rangle \alpha^{-2}(h)\cdot m_{(0)}\\ &= \langle x|m_{(-1)}\rangle\alpha^{-2}(h)\cdot m_{(0)}=\mu((h\otimes x)\rightharpoonup m),\end{aligned}$$

as required. Finally, we check the expression $((h\otimes x)(g\otimes y))\rightharpoonup\mu(m)=(\alpha(h)\otimes\beta(x))\rightharpoonup ((g\otimes y)\rightharpoonup m)$. For this, we calculate

$$\begin{aligned}&(\alpha(h)\otimes\beta(x))\rightharpoonup((g\otimes y)\rightharpoonup m)\\ =\;&\langle y|m_{(-1)}\rangle(\alpha(h)\otimes\beta(x))\cdot(\alpha^{-3}(g)\cdot\mu^{-1}(m_{(0)}))\\ =\;&\langle y|m_{(-1)}\rangle\langle\beta(x)|m_{(0)(-1)}\rangle\alpha^{-2}(h)\cdot(\alpha^{-3}(g)\cdot\mu^{-2}(m_{(0)(0)}))\\ \overset{(HCM2)}{=}\;&\langle y|\beta^{-1}(m_{(-1)1})\rangle\langle x|\beta^{-1}(m_{(-1)2})\rangle\alpha^{-3}(hg)\cdot m_{(0)}\\ \overset{(CHA1)}{=}\;&\langle xy|\beta(m_{(-1)})\rangle\alpha^{-3}(hg)\cdot m_{(0)}\\ =\;&((h\otimes x)(g\otimes y))\rightharpoonup\mu(m).\end{aligned}$$

So (M,μ) is a left $(H\otimes B)$-Hom-module. The proof is completed.

Lemma 3.4. Let (H,R,α) be a quasitriangular Hom-Hopf algebra and $(B,\langle|\rangle,\beta)$ a coquasitriangular Hom-Hopf algebra. Define a linear map

$$\overline{\rho}:M\to(H\otimes B)\otimes M,\ \overline{\rho}(m)=m_{[-1]}\otimes m_{[0]}=R^{(2)}\otimes\beta^{-3}(m_{(-1)})\otimes R^{(1)}\cdot\mu^{-1}(m_{(0)}),$$

for any $m\in(M,\mu)$. Then (M,μ) becomes a left $(H\otimes B)$-Hom-comodule.

Proof. We first show that $\overline{\rho}$ satisfies Eq. (HCM2). On the one side, we have

$$\begin{aligned}&\Delta(m_{[-1]})\otimes\mu(m_{[0]})\\ =\;&(R^{(2)}_1\otimes\beta^{-3}(m_{(-1)1}))\otimes(R^{(2)}_2\otimes\beta^{-3}(m_{(-1)2}))\otimes\alpha(R^{(1)})\cdot m_{(0)}\\ =\;&(\alpha(r^{(2)})\otimes\beta^{-2}(m_{(-1)}))\otimes(\alpha(R^{(2)})\otimes\beta^{-3}(m_{(0)(-1)}))\otimes\alpha(R^{(1)})(r^{(1)}\cdot\mu^{-2}(m_{(0)(0)})).\end{aligned}$$

On the other side, we have

$$(\alpha\otimes\beta)(m_{[-1]})\otimes\overline{\rho}(m_{[0]})$$

$$= (\alpha(r^{(2)}) \otimes \beta^{-2}(m_{(-1)})) \otimes (R^{(2)} \otimes \beta^{-3}((r^{(1)} \cdot \mu^{-1}(m_{(0)}))_{(-1)}) \otimes R^{(1)}$$
$$\cdot \mu^{-1}((r^{(1)} \cdot \mu^{-1}(m_{(0)}))_{(0)})$$
$$= (\alpha(r^{(2)}) \otimes \beta^{-2}(m_{(-1)})) \otimes (R^{(2)} \otimes \beta^{-3}(m_{(0)(-1)})) \otimes R^{(1)} \cdot (r^{(1)} \cdot \mu^{-2}(m_{(0)(0)})).$$

Since R is α-invariant, we have $\Delta(m_{[-1]}) \otimes \mu(m_{[0]}) = (\alpha \otimes \beta)(m_{[-1]}) \otimes \overline{\rho}(m_{[0]})$, as needed.

For Eq. (HCM1), we have

$$\begin{aligned}
(\epsilon_H \otimes \epsilon_B)(m_{[-1]})m_{[0]} &= \epsilon_H(R^{(2)})\epsilon_B(m_{(-1)})R^{(1)} \cdot \mu^{-1}(m_{(0)}) \\
&= 1_H \cdot m = \mu(m), \\
(\alpha \otimes \beta)(m_{[-1]}) \otimes \mu(m_{[0]}) &= (\alpha(R^{(2)}) \otimes \beta^{-2}(m_{(-1)})) \otimes \mu(R^{(1)} \cdot \mu^{-1}(m_{(0)})) \\
&= R^{(2)} \otimes \beta^{-3}(\beta(m_{(-1)})) \otimes R^{(1)} \cdot \mu^{-1}(\mu(m_{(0)})) \\
&= \overline{\rho}(\mu(m)),
\end{aligned}$$

as desired. And this finishes the proof.

Theorem 3.5. Let (H, R, α) be a quasitriangular Hom-Hopf algebra and $(B, \langle|\rangle, \beta)$ a coquasitriangular Hom-Hopf algebra. Then the Hom-Long dimodules category ${}^{B}_{H}\mathbb{L}$ is a monoidal subcategory of Hom-Yetter-Drinfeld category ${}^{H\otimes B}_{H\otimes B}\mathbb{YD}$.

Proof. Let $m \in (M, \mu) \in {}^{B}_{H}\mathcal{L}$ and $h \in H$. Here we first note that $\rho(h \cdot \mu^{-1}(m_{(0)})) = m_{(0)(-1)} \otimes \alpha(h) \cdot \mu^{-1}(m_{(0)(0)})$. It is sufficient to show that the left $(H \otimes B)$-Hom-module action in Lemma 3.3 and the left $(H \otimes B)$-Hom-comodule structure in Lemma 3.4 satisfy the compatible condition Eq. (HYD). Indeed, for any $h \in H$, $x \in B$, $m \in M$, we have

$$\begin{aligned}
& (h_1 \otimes x_1)(\alpha \otimes \beta)(m_{[-1]}) \otimes (\alpha^3(h_2) \otimes \beta^3(x_2)) \rightharpoonup m_{[0]} \\
= \;& h_1\alpha(R^{(2)}) \otimes x_1\beta^{-2}(m_{(-1)}) \otimes \langle \beta^3(x_2)|(R^{(1)} \cdot \mu^{-1}(m_{(0)}))_{(-1)}\rangle h_2 \cdot \mu^{-1}((R^{(1)} \cdot \mu^{-1}(m_{(0)}))_{(0)}) \\
= \;& h_1\alpha(R^{(2)}) \otimes x_1\beta^{-3}(m_{(-1)1}) \otimes \langle \beta^3(x_2)|m_{(-1)2}\rangle h_2 \cdot (R^{(1)} \cdot \mu^{-1}(m_{(0)})) \\
= \;& h_1\alpha(R^{(2)}) \otimes x_1\beta^{-3}(m_{(-1)1}) \otimes \langle x_2|\beta^{-3}(m_{(-1)2})\rangle \alpha^{-1}(h_2\alpha(R^{(1)})) \cdot m_{(0)} \\
= \;& R^{(2)}h_2 \otimes \beta^{-3}(m_{(-1)2})x_2\langle x_1|\beta^{-3}(m_{(-1)1})\rangle \otimes (\alpha^{-1}(R^{(1)})\alpha^{-1}(h_1)) \cdot m_{(0)} \\
= \;& \langle \alpha^2(x_1)|m_{(-1)}\rangle R^{(2)}h_2 \otimes \beta^{-3}(m_{(0)(-1)})x_2 \otimes (\alpha^{-1}(R^{(1)})\alpha^{-1}(h_1)) \cdot \mu^{-1}(m_{(0)(0)}) \\
= \;& \langle \alpha^2(x_1)|m_{(-1)}\rangle (R^{(2)} \otimes \beta^{-3}(\alpha^{-1}(h_1) \cdot \mu^{-1}(m_{(0)})_{(-1)}))(h_2 \otimes x_2) \\
& \qquad \otimes R^{(1)} \cdot \mu^{-1}(\alpha^{-1}(h_1) \cdot \mu^{-1}(m_{(0)})_{(0)}) \\
= \;& (\alpha^2(h_1) \otimes \beta^2(x_1)) \rightharpoonup m_{[-1]}(h_2 \otimes x_2) \otimes (\alpha^2(h_1) \otimes \beta^2(x_1)) \rightharpoonup m_{[0]}.
\end{aligned}$$

So $(M, \mu) \in {}^{H\otimes B}_{H\otimes B}\mathbb{HYD}$. The proof is completed.

Proposition 3.6. Under the hypotheses of the Theorem 3.5, ${}^{B}_{H}\mathbb{L}$ is a braided monoidal subcategory of ${}^{H\otimes B}_{H\otimes B}\mathbb{HYD}$.

Proof. It is sufficient to show that the braiding in the category ${}^B_H\mathbb{L}$ is compatible to the braiding in ${}^{H\otimes B}_{H\otimes B}\mathbb{HYD}$. In fact, for any $m \in (M,\mu)$ and $n \in (N,\nu)$, we have

$$\begin{aligned}
C_{M,N}(m\otimes n) &= (\alpha^2(R^{(2)})\otimes\beta^{-1}(m_{(-1)})) \rightharpoonup \nu^{(-1)}(n)\otimes\alpha^{-1}(R^{(1)})\cdot\mu^{-2}(m_{(0)}) \\
&= \langle\beta^{-1}(m_{(-1)})|\beta^{-1}(n_{(-1)})\rangle\alpha^{-1}(R^{(2)})\cdot\nu^{-2}(n_{(0)})\otimes\alpha^{-1}(R^{(1)})\cdot\mu^{-2}(m_{(0)}) \\
&= \langle m_{(-1)}|n_{(-1)}\rangle R^{(2)}\cdot\nu^{-2}(n_{(0)})\otimes R^{(1)}\cdot\mu^{-2}(m_{(0)}),
\end{aligned}$$

as desired.This finishes the proof.

4 Symmetries in Hom-Long dimodule categories

In this section, we obtain a sufficient condition for the Hom-Long dimodule category ${}^B_H\mathbb{L}$ to be symmetric.

Let $\mathcal{C}$ be a monoidal category and C a braiding on $\mathcal{C}$. The braiding C is called a symmetry [10, 11] if $C_{Y,X}\circ C_{X,Y} = id_{X\otimes Y}$ for all $X,Y\in\mathcal{C}$, and the category $\mathcal{C}$ is called symmetric.

Proposition 4.1. Let (H,R,α) be a triangular Hom-Hopf algebra and (B,β) a Hom-Hopf algebra. Then the category ${}_H\mathbb{M}$ of left (H,α)-Hom-modules is a symmetric subcategory of ${}^B_H\mathbb{L}$ under the left (B,β)-comodule structure $\rho(m) = 1_B\otimes\mu(m)$, where $m\in(M,\mu)\in{}_H\mathbb{M}$, and the braiding is defined as

$$C_{M,N}: M\otimes N\to N\otimes M, m\otimes n\to R^{(2)}\cdot\nu^{-1}(n)\otimes R^{(1)}\cdot\mu^{-1}(m),$$

for all $m\in(M,\mu)\in{}_H\mathbb{M}, n\in(N,\nu)\in{}_H\mathbb{M}$.

Proof. It is clear that (M,ρ,μ) is a left (B,β)-Hom-comodule under the left (B,β)-comodule structure given above. Now we check that the left (B,β)-comodule structure satisfies the compatible condition Eq. (2.1). For this purpose, we take $h\in H, m\in(M,\mu)\in{}_H\mathbb{M}$, and calculate

$$\rho(h\cdot m) = 1_B\otimes\mu(h\cdot m) = 1_B\otimes\alpha(h)\cdot\mu(m) = \beta(m_{(-1)})\otimes\alpha(h)\cdot m_{(0)}.$$

So, Eq. (2.1) holds. That is, (M,ρ,μ) is an (H,B)-Hom-Long dimodule.

Next we verify that any morphism in ${}_H\mathbb{M}$ is left (B,β)-colinear, too. Indeed, for any $m\in(M,\mu)\in{}_H\mathbb{M}$ and $n\in(N,\nu)\in{}_H\mathbb{M}$. Assume that $f:(M,\mu)\to(N,\nu)$ is a morphism in ${}_H\mathbb{M}$, then

$$(id_B\otimes f)\rho(m) = 1_B\otimes f(\mu(m)) = 1_B\otimes\nu(f(m)) = \rho(f(m)).$$

So f is left (B,β)-colinear, as desired. Therefore, ${}_H\mathbb{M}$ is a subcategory of ${}^B_H\mathbb{L}$.

Finally, we prove that ${}_H\mathbb{M}$ is a symmetric subcategory of ${}_H^B\mathbb{L}$. Since $C_{M,N}(m\otimes n) = R^{(2)}\cdot\nu^{-1}(n)\otimes R^{(1)}\cdot\mu^{-1}(m)$, for all $m\in(M,\mu)\in{}_H\mathbb{M}$ and $n\in(N,\nu)\in{}_H\mathbb{M}$, we have

$$\begin{aligned}
C_{N,M}\circ C_{M,N}(m\otimes n) &= C_{N,M}(R^{(2)}\cdot\nu^{-1}(n)\otimes R^{(1)}\cdot\mu^{-1}(m))\\
&= r^{(2)}\cdot\mu^{-1}(R^{(1)}\cdot\mu^{-1}(m))\otimes r^{(1)}\cdot\nu^{-1}(R^{(2)}\cdot\nu^{-1}(n))\\
&= r^{(2)}\cdot(\alpha^{-1}(R^{(1)})\cdot\mu^{-2}(m))\otimes r^{(1)}\cdot(\alpha^{-1}(R^{(2)})\cdot\nu^{-2}(n))\\
&= \alpha^{-1}(r^{(2)}R^{(1)})\cdot\mu^{-1}(m)\otimes\alpha^{-1}(r^{(1)}R^{(2)})\cdot\nu^{-1}(n)\\
&= 1_H\cdot\mu^{-1}(m)\otimes 1_H\cdot\nu^{-1}(n) = m\otimes n.
\end{aligned}$$

It follows that the braiding $C_{M,N}$ is symmetric. The proof is completed.

Proposition 4.2. Let $(B,\langle|\rangle,\beta)$ be a cotriangular Hom-Hopf algebra and (H,α) a Hom-Hopf algebra. Then the category ${}^B\mathbb{M}$ of left (B,β)-Hom-comodules is a symmetric subcategory of ${}_H^B\mathbb{L}$ under the left (H,α)-module action $h\cdot m=\epsilon(h)\mu(m)$, where $h\in H, m\in(M,\mu)\in{}^B\mathbb{M}$, and the braiding is given by

$$C_{M,N}: M\otimes N\to N\otimes M, m\otimes n\to\langle m_{(-1)}|n_{(-1)}\rangle\nu^{-2}(n_{(0)})\otimes\mu^{-2}(m_{(0)}),$$

for all $m\in(M,\mu)\in{}^B\mathbb{M}, n\in(N,\nu)\in{}^B\mathbb{M}$.

Proof. We first show that the left (H,α)-module action defined above forces (M,μ) to be a left (H,α)-module, but this is easy to check. For the compatible condition Eq. (2.1), we take $h\in H, m\in(M,\mu)\in{}^B\mathbb{M}$ and calculate as follows:

$$\rho(h\cdot m) = 1_B\otimes\mu(h\cdot m) = 1_B\otimes\epsilon(h)\mu(m) = \beta(m_{(-1)})\otimes\alpha(h)\cdot m_{(0)}.$$

So, Eq. (2.1) holds, as required. Therefore, (M,ρ,μ) is an (H,B)-Hom-Long dimodule.

Now we verify that any morphism in ${}^B\mathbb{M}$ is left (H,α)-linear, too. Indeed, for any $m\in(M,\mu)\in{}^B\mathbb{M}$ and $n\in(N,\nu)\in{}^B\mathbb{M}$. Assume that $f:(M,\mu)\to(N,\nu)$ is a morphism in ${}^B\mathbb{M}$, then

$$f(h\cdot m) = f(\epsilon(h)\mu(m)) = \epsilon(h)\mu(f(m)) = h\cdot f(m).$$

So f is left (H,α)-linear, as desired. Therefore, ${}^B\mathbb{M}$ is a subcategory of ${}_H^B\mathbb{L}$.

Finally, we show that ${}^B\mathbb{M}$ is a symmetric subcategory of ${}_H^B\mathbb{L}$. Since $C_{M,N}(m\otimes n) = \langle m_{(-1)}|n_{(-1)}\rangle\nu^{-1}(n_{(0)})\otimes\mu^{-1}(m_{(0)})$, for all $m\in(M,\mu)\in{}^B\mathbb{M}$ and $n\in(N,\nu)\in{}^B\mathbb{M}$, then

$$\begin{aligned}
&C_{N,M}\circ C_{M,N}(m\otimes n)\\
=\ &\langle m_{(-1)}|n_{(-1)}\rangle C_{N,M}(\nu^{-1}(n_{(0)})\otimes\mu^{-1}(m_{(0)}))\\
=\ &\langle m_{(-1)}|n_{(-1)}\rangle\langle\beta^{-1}(n_{(0)(-1)})|\beta^{-1}(m_{0(-1)})\rangle(\mu^{-2}(m_{(0)(0)})\otimes\nu^{-2}(n_{(0)(0)})\\
=\ &\langle\beta^{-1}(m_{(-1)1})|\beta^{-1}(n_{(-1)1})\rangle\langle\beta^{-1}(n_{(-1)2})|\beta^{-1}(m_{(-1)2})\rangle\mu^{-1}(m_{(0)})\otimes\nu^{-1}(n_{(0)})\\
=\ &\epsilon(m_{(-1)})\epsilon(n_{(-1)})\mu^{-1}(m_{(0)})\otimes\nu^{-1}(n_{(0)}) = m\otimes n,
\end{aligned}$$

where the fourth equality holds since $\langle|\rangle$ is β-invariant. It follows that the braiding $C_{M,N}$ is symmetric. The proof is completed.

Theorem 4.3. Let (H,α) be a triangular Hom-Hopf algebra and $(B,\langle|\rangle,\beta)$ a cotriangular Hom-Hopf algebra. Then the category ${}^B_H\mathbb{L}$ is symmetric.

Proof. For any $m\in(M,\mu)\in{}^B_H\mathbb{L}$ and $n\in(N,\nu)\in{}^B_H\mathbb{L}$, we have

$$\begin{aligned}
& C_{N,M}\circ C_{M,N}(m\otimes n)\\
= & \langle m_{(-1)}|n_{(-1)}\rangle C_{N,M}(R^{(2)}\cdot\nu^{-2}(n_{(0)})\otimes R^{(1)}\cdot\mu^{-2}(m_{(0)}))\\
= & \langle m_{(-1)}|n_{(-1)}\rangle\langle\beta(n_{(0)(-1)})|\beta(m_{(0)(-1)})\rangle\\
& \qquad r^{(2)}\cdot\mu^{-2}(\alpha(R^{(1)})\cdot\mu^{-2}(m_{(0)(0)}))\otimes r^{(1)}\cdot\nu^{-2}(\alpha(R^{(2)})\cdot\nu^{-2}(n_{(0)(0)}))\\
= & \langle\beta^{-1}(m_{(-1)1})|\beta^{-1}(n_{(-1)1})\rangle\langle\beta^{-1}(n_{(-1)2})|\beta^{-1}(m_{(-1)2})\rangle\\
& \qquad \alpha^{-1}(r^{(2)}R^{(1)})\cdot\mu^{-2}(m_{(0)})\otimes\alpha^{-1}(r^{(1)}R^{(2)})\cdot\nu^{-2}(n_{(0)})\\
= & \epsilon(m_{(-1)})\epsilon(n_{(-1)})1_H\cdot\mu^{-2}(m_{(0)})\otimes 1_H\cdot\nu^{-2}(n_{(0)})\\
= & \epsilon(m_{(-1)})\epsilon(n_{(-1)})\mu^{-1}(m_{(0)})\otimes\nu^{-1}(n_{(0)})\\
= & m\otimes n,
\end{aligned}$$

as desired. This finishes the proof.

5 New solutions of the Hom-Long Equation

In this section, we will present a kind of new solutions of the Hom-Long equation.

Definition 5.1. Let (H,α) be a Hom-bialgebra and (M,μ) a Hom-module over (H,α). Then $R\in End(M\otimes M)$ is called the solution of the Hom-Long equation if it satisfies the nonlinear equation:

$$R^{12}\circ R^{23}=R^{23}\circ R^{12}, \tag{5.1}$$

where $R^{12}=R\otimes\mu, R^{23}=\mu\otimes R$.

Example 5.2. If $R\in End(M\otimes M)$ is invertible, then it is easy to see that R is a solution of the Hom-Long equation if and only if R^{-1} is too.

Example 5.3. Let (M,μ) an (H,α)-Hom-module with a basis $\{m_1,m_2,\cdots,m_n\}$. Assume that μ is given by $\mu(m_i)=a_im_i$, where $a_i\in k,\ i=1,2,\cdots,n$. Define a map

$$R:\ M\otimes M\to M\otimes M,\quad R(m_i\otimes m_j)=b_{ij}m_i\otimes m_j,$$

where $b_{ij}\in k,\ i,j=1,2,,\cdots,n$. Then R is a solution of the Hom-Long equation (5.1). Furthermore, if $a_i=1$, for all $i=1,2,\cdots,n$, then R is a solution of the classical Long equation.

Proposition 5.4. Let (M,μ) an (H,α)-Hom-module with a basis $\{m_1, m_2, \cdots, m_n\}$. Assume that $R, S \in End(M\otimes M, \mu\otimes\mu^{-1})$ given by the matrix formula

$$R(m_k\otimes m_l) = x_{kl}^{ij} m_i \otimes \mu^{-1}(m_j), \quad S(m_k\otimes m_l) = y_{kl}^{ij} m_i \otimes \mu^{-1}(m_j),$$

and $\mu(m_l) = z_l^i m_i$, where $x_{kl}^{ij}, y_{kl}^{ij}, z_l^i \in k$. Then $S^{12}\circ R^{23} = R^{23}\circ S^{12}$ if and only if

$$z_u^i x_{vw}^{jk} y_{ij}^{pq} = z_i^p x_{jw}^{qk} y_{uv}^{ij},$$

for all $k, p, q, u, v, w = 1, 2, \cdots, n$. In particular, R is a solution of Hom-Long equation if and only if

$$z_u^i x_{vw}^{jk} x_{ij}^{pq} = z_i^p x_{jw}^{qk} x_{uv}^{ij}.$$

Proof. According to the definition of R, S, μ, we have

$$\begin{aligned} S^{12}\circ R^{23}(m_u\otimes m_v\otimes m_w) &= S^{12}(z_u^i m_i \otimes x_{vw}^{jk} m_j \otimes \mu^{-1}(m_k)) \\ &= z_u^i x_{vw}^{jk} y_{ij}^{pq} (m_p \otimes \mu^{-1}(m_q)\otimes m_k), \\ R^{23}\circ S^{12}(m_u\otimes m_v\otimes m_w) &= R^{23}(y_{uv}^{ij} m_i \otimes \mu^{-1}(m_j)\otimes m_w) \\ &= y_{uv}^{ij} z_i^p x_{jw}^{qk} (m_p\otimes \mu^{-1}(m_q)\otimes m_k). \end{aligned}$$

It follows that $S^{12}\circ R^{23} = R^{23}\circ S^{12}$ if and only if $z_u^i x_{vw}^{jk} y_{ij}^{pq} = z_i^p x_{jw}^{qk} y_{uv}^{ij}$. Furthermore, $R^{12}\circ R^{23} = R^{23}\circ R^{12}$ if and only if $z_u^i x_{vw}^{jk} x_{ij}^{pq} = z_i^p x_{jw}^{qk} x_{uv}^{ij}$. The proof is completed.

In the following proposition, we use the notation: for any $F\in End(M\otimes M)$, we denote $F^{12} = F\otimes\mu, F^{23} = \mu\otimes F, F^{13} = (id\otimes\tau)\circ(F\otimes\mu)\circ(id\otimes\tau)$, and $\tau^{(123)}(x\otimes y\otimes z) = (z, x, y)$.

Proposition 5.5. Let (M,μ) an (H,α)-Hom-module and $R \in End(M\otimes M)$. The following statements are equivalent:

(1) R is a solution of the Hom-Long equation.

(2) $U = \tau\circ R$ is a solution of the equation:

$$U^{13}\circ U^{23} = \tau^{(123)}\circ U^{13}\circ U^{12}.$$

(3) $T = R\circ\tau$ is a solution of the equation:

$$T^{12}\circ T^{13} = T^{23}\circ T^{13}\circ\tau^{(123)}.$$

(4) $W = \tau\circ R\circ\tau$ is a solution of the equation:

$$\tau^{(123)}\circ W^{23}\circ W^{13} = W^{12}\circ W^{13}\circ\tau^{(123)}.$$

Proof. We just prove(1) $\Leftrightarrow$ (2), and similar for (1) $\Leftrightarrow$ (3) and (1) $\Leftrightarrow$ (4). Since $R = \tau\circ U$, R is a solution of the Hom-Long equation if and only if $R^{12}\circ R^{23} = R^{23}\circ R^{12}$, that is,

$$\tau^{12}\circ U^{12}\circ\tau^{23}\circ U^{23} = \tau^{23}\circ U^{23}\circ\tau^{12}\circ U^{12}. \tag{5.2}$$

While $\tau^{12} \circ U^{12} \circ \tau^{23} = \tau^{23} \circ \tau^{13} \circ U^{13}$ and $\tau^{23} \circ U^{23} \circ \tau^{12} = \tau^{23} \circ \tau^{12} \circ U^{13}$, (5.2) is equivalent to

$$\tau^{23} \circ \tau^{13} \circ U^{13} \circ U^{23} = \tau^{23} \circ \tau^{12} \circ U^{13} \circ U^{12},$$

which is equivalent to $U^{13} \circ U^{23} = \tau^{(123)} \circ U^{13} \circ U^{12}$ from the fact $\tau^{23} \circ \tau^{12} = \tau^{(123)}$.

Next we will present a new solution for Hom-Long equation by the Hom-Long dimodule structures. For this, we give the notion of (H, α)-Hom-Long dimodules.

Definition 5.6. Let (H, α) a Hom-bialgebra. A left-left (H, α)-Hom-Long dimodule is a quadrupl $(M, \cdot, \rho, \mu)$, where $(M, \cdot, \mu)$ is a left (H, α)-Hom-module and (M, ρ, μ) is a left (H, α)-Hom-comodule such that

$$\rho(h \cdot m) = \alpha(m_{(-1)}) \otimes \alpha(h) \cdot m_0, \tag{5.3}$$

for all $h \in H$ and $m \in M$.

Remark 5.7. Clearly, left-left (H, α)-Hom-Long dimodules is a special case of (H, B)-Hom-Long dimodules in Definition 2.1 by setting $(H, \alpha) = (B, \beta)$.

Example 5.8. Let (H, α) be a Hom-bialgebra and $(M, \cdot, \mu)$ be a left (H, α)-Hom-module. Define a left (H, α)-Hom-module structure and a left (H, α)-Hom-comodule structure on $(H \otimes M, \alpha \otimes \mu)$ as follows:

$$h \cdot (g \otimes m) = \alpha(g) \otimes h \cdot \mu(m), \quad \rho(g \otimes m) = g_1 \otimes g_2 \otimes \mu(m),$$

for all $h, g \in H$ and $m \in M$. Then $(H \otimes M, \alpha \otimes \mu)$ is an (H, α)-Hom-Long dimodule.

Example 5.9. Let (H, α) be a Hom-bialgebra and (M, ρ, μ) be a left (H, α)-Hom-comodule. Define a left (H, α)-Hom-module structure and be a left (H, α)-Hom-comodule structure on $(H \otimes M, \alpha \otimes \mu)$ as follows:

$$h \cdot (g \otimes m) = hg \otimes \mu(m), \quad \rho(g \otimes m) = m_{-1} \otimes \alpha(g) \otimes m_0,$$

for all $h, g \in H$ and $m \in M$. Then $(H \otimes M, \alpha \otimes \mu)$ is an (H, α)-Hom-Long dimodule.

Theorem 5.10. Let (H, α) be a Hom-bialgebra and $(M, \cdot, \rho, \mu)$ be a (H, α)-Hom-Long dimodule. Then the map

$$R_M : M \otimes M \to M \otimes M, \qquad m \otimes n \mapsto n_{-1} \cdot m \otimes n_0, \tag{5.4}$$

is a solution of the Hom-Long equation, for any $m, n \in M$.

Proof. For any $l, m, n \in M$, we calculate

$$\begin{aligned} R_M^{12} \circ R_M^{23}(l \otimes m \otimes n) &= R_M^{12}(\mu(l) \otimes n_{(-1)} \cdot m \otimes n_0) \\ &= (n_{(-1)} \cdot m)_{(-1)} \cdot \mu(l) \otimes (n_{(-1)} \cdot m)_0 \otimes \mu(n_0) \end{aligned}$$

$$
\begin{aligned}
&= \alpha(m_{(-1)}) \cdot \mu(l) \otimes \alpha(n_{(-1)}) \cdot m_0 \otimes \mu(n_0), \\
R_M^{23} \circ R_M^{12}(l \otimes m \otimes n) &= R_M^{23}(m_{(-1)} \cdot l \otimes m_0 \otimes \mu(n)) \\
&= \mu(m_{(-1)} \cdot l) \otimes \alpha(n_{(-1)})) \cdot m_0 \otimes \mu(n_0) \\
&= \alpha(m_{(-1)}) \cdot \mu(l) \otimes \alpha(n_{(-1)}) \cdot m_0 \otimes \mu(n_0).
\end{aligned}
$$

So we have $R_M^{12} \circ R_M^{23} = R_M^{23} \circ R_M^{12}$, as desired. And this finishes the proof.

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