$*$-AUTONOMOUS ENVELOPES

MICHAEL SHULMAN

Abstract. We show that every closed symmetric monoidal category can be fully embedded in a $*$-autonomous category, preserving both tensor products and internal-homs, as well as any desired nonempty limits and colimits. Thus, linear logics and graphical calculi for $*$-autonomous categories can also be applied to closed symmetric monoidal categories. The embedding uses an enhanced "Hyland envelope" that can be regarded as a polycategorical form of Day convolution. It also applies to other fragments of $*$-autonomous structure, such as linear distributivity, and can be combined with a form of double gluing to prove full completeness results.

1. Introduction

Duality in monoidal categories is useful for many purposes. For instance, if A has a dual A^*, then the internal-hom $A \multimap B$ is isomorphic to $A^* \otimes B$. Thus, graphical calculi for compact closed categories (where all objects have duals) allow strings to be simply bent around to represent duals, whereas it appears that additional "clasps" and "bubbles" (as in [BS11]) are needed for general internal-homs.

Unfortunately, a symmetric monoidal category cannot be fully embedded in a compact closed one unless it is *traced* [JSV96]. But a weaker notion of duality is that of a $*$-*autonomous* category [Bar79], where instead $A \multimap B$ is isomorphic to $(A \otimes B^*)^*$. Here no traces are induced, and indeed every closed symmetric monoidal category with pullbacks can be embedded in a $*$-autonomous category by the Chu construction [Chu79]; but this embedding does not preserve the internal-homs. Thus, it is natural to ask whether any closed symmetric monoidal category $\mathcal{C}$ can be fully embedded in a $*$-autonomous category preserving the closed structure.

We will show the answer is yes, using a more refined Chu construction. The problem with the ordinary Chu construction on $\mathcal{C}$ is that there is no *canonical* dualizing object $\perp$, and in particular none for which all objects of $\mathcal{C}$ are reflexive (i.e. $A \cong (A \multimap \perp) \multimap \perp$). Now in Isbell duality [Isb66], the hom-functor $\mathcal{C}(-,-)$ is a canonical "dualizing object" between presheaf and co-presheaf categories, for which all representables are reflexive. But to apply the Chu construction, we need the dualizing object to live in the same category as the representables themselves.

The *modules* of [Hyl02] achieve this for polycategories, with a resulting Chu construction called the *envelope*. We modify Hyland's envelope so that the embedding

Date: April 21, 2020.

This material is based on research sponsored by The United States Air Force Research Laboratory under agreement number FA9550-15-1-0053. The U.S. Government is authorized to reproduce and distribute reprints for Governmental purposes notwithstanding any copyright notation thereon. The views and conclusions contained herein are those of the author and should not be interpreted as necessarily representing the official policies or endorsements, either expressed or implied, of the United States Air Force Research Laboratory, the U.S. Government, or Carnegie Mellon University.

preserves any desired tensor and cotensor products, by adapting the standard trick for making Yoneda embeddings preserve colimits. We then deduce the result by representing closed symmetric monoidal categories by certain polycategories.

It follows that we can use circuit diagrams for $*$-autonomous categories [BCST96], or term calculi for linear logic (e.g. [Red91, Red93]), to reason about closed symmetric monoidal categories. We also obtain a semantic proof of the conservativity of multiplicative linear logic over its intuitionistic variant. And combining it with with a "double gluing" construction [Tan98, HS03, Has99] and the technique of [Laf88], we can prove "full completeness":[1] the *universal* functor from a closed symmetric monoidal category to a $*$-autonomous category is already a full embedding. This gives a new perspective on why the same "Kelly-MacLane graphs" (and enhanced versions incorporating units) appear in coherence for closed symmetric monoidal categories and for $*$-autonomous categories [KM71, Tri94, Blu91, Blu93, Hug12].

Throughout the paper, we use multicategories and polycategories systematically, obtaining cleaner and more general proofs and constructions. For instance, polycategorical Chu constructions and double gluing, and their $*$-autonomous operations, are characterized by universal properties. Our embedding theorem also applies to more general polycategories, yielding full completeness of $*$-autonomous categories over other structures such as linearly distributive categories (this was proven by cut-elimination in [BCST96]). We can also make the embedding preserve any desired polycategorical limits and colimits, although only *nonempty* limits and colimits in a closed symmetric monoidal categories are polycategorical; this gives a semantic explanation for why only the inclusion of an initial object $\mathbf{0}$ breaks the conservativity of classical multiplicative-additive linear logic over its intuitionistic version [Sch91].

Thanks to Sam Staton, Kirk Sturtz, and Todd Trimble for useful conversations.

2. Adding duals to polycategories

First, recall that a closed symmetric monoidal category $(\mathcal{C}, \otimes, \mathbb{1}, \multimap)$ can be regarded as a symmetric multicategory that is representable and closed:

$$\mathcal{C}(\Gamma; C) \cong \mathcal{C}(\Gamma, \mathbb{1}; C) \qquad \mathcal{C}(\Gamma, A, B; C) \cong \mathcal{C}(\Gamma, A \otimes B; C)$$
$$\mathcal{C}(\Gamma, A; B) \cong \mathcal{C}(\Gamma; A \multimap B).$$

Moreover, a symmetric multicategory is just a symmetric polycategory [Sza75] that is **co-unary**, i.e. all codomains have one object. (All our multi- and polycategories will be symmetric, so we henceforth drop the adjective.) Similarly, any $*$-autonomous category $\mathcal{E}$ can be regarded as a representable[2] polycategory with duals:

$$\mathcal{E}(\Gamma, A, B; \Delta) \cong \mathcal{E}(\Gamma, A \otimes B; \Delta) \qquad \mathcal{E}(\Gamma; \Delta, A, B) \cong \mathcal{E}(\Gamma; \Delta, A \parr B)$$
$$\mathcal{E}(\Gamma; \Delta) \cong \mathcal{E}(\Gamma, \mathbb{1}; \Delta) \qquad \mathcal{E}(\Gamma; \Delta) \cong \mathcal{E}(\Gamma; \Delta, \bot) \qquad \mathcal{E}(\Gamma, A; \Delta) \cong \mathcal{E}(\Gamma; \Delta, A^*)$$

A **$*$-polycategory** [Hyl02] is a polycategory with specified strictly involutive duals ($A^{**} = A$). In particular, a representable $*$-polycategory is $*$-autonomous.

The forgetful functor from $*$-polycategories to polycategories has a left adjoint $\mathcal{P} \mapsto \circledast\mathcal{P}$. The objects of $\circledast\mathcal{P}$ consist of two copies of the objects of $\mathcal{P}$, denoted A

[1]It is claimed in [HS03, Has99] that this can be proven using double gluing with an ordinary Yoneda embedding, but this doesn't appear to be true; see Remark 10.7.

[2]Note that a representable multicategory is not representable as a co-unary polycategory.

and A^* respectively. The morphisms are determined by saying that

$$\circledast\mathcal{P}(\Gamma, \Pi^*; \Delta, \Sigma^*) = \mathcal{P}(\Gamma, \Sigma; \Delta, \Pi).$$

Composition is inherited from $\mathcal{P}$, perhaps in the other order. For instance, if $f \in \circledast\mathcal{P}(A, B^*; C, D^*) = \mathcal{P}(A, D; C, B)$ and $g \in \circledast\mathcal{P}(C, X^*; B^*, Y) = \mathcal{P}(C, B; Y, X)$ then $g \circ_C^{\circledast\mathcal{P}} f = g \circ_C^{\mathcal{P}} f$ and $f \circ_{B^*}^{\circledast\mathcal{P}} g = g \circ_B^{\mathcal{P}} f$. Thus $\mathcal{P}$ embeds fully-faithfully in $\circledast\mathcal{P}$.

Any polycategory functor preserves duals, so if $\mathcal{P}$ has duals the map $\mathcal{P} \to \circledast\mathcal{P}$ is essentially surjective, hence an equivalence. Thus any $*$-autonomous category is equivalent to a representable $*$-polycategory.[3]

Any tensor product $A \otimes B$ in $\mathcal{P}$ is also a tensor product in $\circledast\mathcal{P}$, while $(A \otimes B)^*$ is a cotensor product $A^* \parr B^*$ in $\circledast\mathcal{P}$. The situation for cotensor products $A \parr B$ is dual, while that for units and counits is similar. However, even if $\mathcal{P}$ has all tensor and/or cotensor products, $\circledast\mathcal{P}$ will not in general have $A^* \otimes B$ or $A^* \parr B$.

If $\mathcal{C}$ is a multicategory regarded as a co-unary polycategory, then $\circledast\mathcal{C}(\Gamma, \Pi^*; \Delta, \Sigma^*)$ is nonempty just when $|\Delta \cup \Pi| = 1$. (This left adjoint to the forgetful functor from $*$-polycategories to multicategories appears in [DCH18].) In this case, if $\mathcal{C}$ is closed, the hom-object $A \multimap B$ is a cotensor product $A^* \parr B$ in $\circledast\mathcal{C}$. For by definition:

$$\circledast\mathcal{C}(\Gamma, \Pi^*; \Delta, \Sigma^*, A \multimap B) \cong \mathcal{C}(\Gamma, \Sigma; \Pi, \Delta, A \multimap B)$$
$$\circledast\mathcal{C}(\Gamma, \Pi^*; \Delta, \Sigma^*, A^*, B) \cong \mathcal{C}(\Gamma, \Sigma, A; \Pi, \Delta, B)$$

and both right-hand sides are nonempty only if $\Pi = \Delta = \emptyset$, in which case they are naturally isomorphic by the universal property of $A \multimap B$ in $\mathcal{C}$. Let U_{multi} denote the forgetful functor from polycategories to multicategories; then we have shown:

Theorem 2.1. *If $\mathcal{C}$ is closed symmetric monoidal, there is a $*$-polycategory $\circledast\mathcal{C}$ and a fully faithful functor $\mathcal{C} \to U_{\mathrm{multi}}(\circledast\mathcal{C})$ that preserves tensor products (including the unit) and takes internal-homs $A \multimap B$ to cotensor products $A^* \parr B$.* □

Therefore, to embed $\mathcal{C}$ in a $*$-autonomous category preserving both tensor products and internal-homs, it will suffice to embed the $*$-polycategory $\circledast\mathcal{C}$ in a $*$-autonomous category preserving *those tensor and cotensor products that exist.*

3. Modules

Let $\mathcal{P}$ be a polycategory. Following [Hyl02, §5], a $\mathcal{P}$**-module** is a family of sets $\mathcal{U}(\Gamma; \Delta)$ with symmetric group actions and left and right actions by $\mathcal{P}$

$$\mathcal{P}(\Pi; \Sigma, A) \times \mathcal{U}(A, \Gamma; \Delta) \to \mathcal{U}(\Pi, \Gamma; \Sigma, \Delta)$$
$$\mathcal{U}(\Gamma; \Delta, A) \times \mathcal{P}(A, \Pi; \Sigma) \to \mathcal{U}(\Gamma, \Pi; \Delta, \Sigma)$$

satisfying the same associativity and unit laws as the composition in a polycategory.

Examples 3.1.

(i) The hom-sets $\mathcal{P}(\Gamma; \Delta)$ form a $\mathcal{P}$-module, which we denote simply $\mathcal{P}$.
(ii) For any Π, Σ and module $\mathcal{U}$, we have a **shifted** module $\mathcal{U}[\Pi; \Sigma]$ defined by $\mathcal{U}[\Pi; \Sigma](\Gamma; \Delta) = \mathcal{U}(\Gamma, \Pi; \Delta, \Sigma)$.
(iii) For an object $A \in \mathcal{P}$, we have the **representable** modules at A:

$$\text{よ}_A = \mathcal{P}[\,; A] \qquad \text{and} \qquad {}_A\text{よ} = \mathcal{P}[A;\,].$$

[3] This was shown by [JCS06] using a *right* adjoint instead of our left adjoint. In fact $*$-polycategories are both 2-monadic and 2-comonadic over the 2-category of polycategories, functors, and natural isomorphisms, and the 2-monad and 2-comonad are pseudo-idempotent [KL97].

A **module (multi-)morphism** $(\mathcal{U}_1, \dots, \mathcal{U}_n) \to \mathcal{V}$ consists of functions

$$\mathcal{U}_1(\Gamma_1; \Delta_1) \times \cdots \times \mathcal{U}_n(\Gamma_n; \Delta_n) \to \mathcal{V}(\Gamma_1, \dots, \Gamma_n; \Delta_1, \dots, \Delta_n)$$

that commute with the symmetric group actions and the actions of $\mathcal{P}$. (A nullary morphism $() \to \mathcal{V}$ is just an element of $\mathcal{V}(\,;\,)$.) This defines a multicategory $\mathsf{Mod}_{\mathcal{P}}$.

Theorem 3.2. *The multicategory $\mathsf{Mod}_{\mathcal{P}}$ is closed and representable.*

Proof. Let $(\mathcal{U} \multimap \mathcal{V})(\Gamma; \Delta)$ be the set of (unary) module morphisms $\mathcal{U} \to \mathcal{V}[\Gamma; \Delta]$. The unit $\mathcal{I}$ is defined by $\mathcal{I}(\,;\,) = 1$ and all other sets empty. The tensor products can be obtained formally by an adjoint functor theorem (since $\mathsf{Mod}_{\mathcal{P}}$ is locally presentable), or constructed more explicitly as in [Hyl02, §5.2]. $\square$

Remark 3.3. For an alternative proof of Theorem 3.2, let $F_\otimes U_{\mathrm{multi}} \circledast \mathcal{P}$ be the free symmetric strict monoidal category on the underlying multicategory of $\circledast\mathcal{P}$. Its objects are finite lists of objects and $\mathcal{P}$ and their formal duals, but by symmetry each is isomorphic to one of the form (Γ, Δ^*) where Γ and Δ consist of objects of $\mathcal{P}$. A $\mathcal{P}$-module is then equivalent to an ordinary presheaf on $F_\otimes U_{\mathrm{multi}} \circledast \mathcal{P}$ defined by $(\Gamma, \Delta^*) \mapsto \mathcal{U}(\Gamma; \Delta)$. Now since $F_\otimes U_{\mathrm{multi}} \circledast \mathcal{P}$ is symmetric monoidal, its presheaf category has a closed symmetric monoidal Day convolution [Day70] monoidal structure, which coincides with that described above.

Theorem 3.4 (Polycategorical Yoneda lemmas)**.** *We have natural isomorphisms*

$$\mathsf{Mod}_{\mathcal{P}}(\text{よ}_A; \mathcal{V}) \cong \mathcal{V}(A;\,) \qquad \mathsf{Mod}_{\mathcal{P}}({}_A\reflectbox{よ}; \mathcal{V}) \cong \mathcal{V}(\,;A) \tag{3.5}$$

$$(\text{よ}_A \multimap \mathcal{V}) \cong \mathcal{V}[A;\,] \qquad ({}_A\reflectbox{よ} \multimap \mathcal{V}) \cong \mathcal{V}[\,;A] \tag{3.6}$$

$$\mathsf{Mod}_{\mathcal{P}}(\Gamma, \text{よ}_A; \mathcal{V}) \cong \mathsf{Mod}_{\mathcal{P}}(\Gamma; \mathcal{V}[A;\,]) \qquad \mathsf{Mod}_{\mathcal{P}}(\Gamma, {}_A\reflectbox{よ}; \mathcal{V}) \cong \mathsf{Mod}_{\mathcal{P}}(\Gamma; \mathcal{V}[\,;A]) \tag{3.7}$$

Proof. This follows formally from Remark 3.3 and properties of Day convolution, but we can also give an explicit proof. Since $1_A \in \mathcal{P}(A;A) = \text{よ}_A(A;\,)$, any $\phi : \text{よ}_A \to \mathcal{V}$ induces $\phi(1_A) \in \mathcal{V}(A;\,)$. Conversely, from $x \in \mathcal{V}(A;\,)$ we define $\psi_x : \text{よ}_A \to \mathcal{V}$ by:

$$\Big(f \in \text{よ}_A(\Gamma;\Delta) = \mathcal{P}(\Gamma; \Delta, A)\Big) \mapsto \Big(x \circ_A f \in \mathcal{V}(\Gamma;\Delta)\Big).$$

The associativity of the $\mathcal{P}$-action on $\mathcal{V}$ ensures that this is a $\mathcal{P}$-module morphism. Clearly $\psi_x(1_A) = x \circ_A 1_A = x$, while on the other side we have

$$\psi_{\phi(1_A)}(f) = \phi(1_A) \circ_A f = \phi(1_A \circ_A f) = \phi(f).$$

This, and a dual calculation, proves (3.5). For (3.6), we have

$$(\text{よ}_A \multimap \mathcal{V})(\Gamma;\Delta) = \mathsf{Mod}_{\mathcal{P}}(\text{よ}_A, \mathcal{V}[\Gamma;\Delta]) \cong \mathcal{V}[\Gamma;\Delta](A;\,) = \mathcal{V}(A, \Gamma; \Delta) \cong \mathcal{V}[A;\,](\Gamma;\Delta)$$

and dually. Finally, (3.7) follows from (3.6) and the universal property of $\multimap$. $\square$

Corollary 3.8. $(\text{よ}_A \multimap \mathcal{P}) \cong {}_A\reflectbox{よ}$ *and* $({}_A\reflectbox{よ} \multimap \mathcal{P}) \cong \text{よ}_A$. $\square$

In particular, both kinds of representable are reflexive: $\text{よ}_A \cong ((\text{よ}_A \multimap \mathcal{P}) \multimap \mathcal{P})$ and ${}_A\reflectbox{よ} \cong (({}_A\reflectbox{よ} \multimap \mathcal{P}) \multimap \mathcal{P})$.

Corollary 3.9. *There are full embeddings of multicategories*

$$\text{よ} : U_{\mathrm{multi}}(\mathcal{P}) \to \mathsf{Mod}_{\mathcal{P}} \qquad \textit{and} \qquad \reflectbox{よ} : U_{\mathrm{multi}}(\mathcal{P}^{\mathrm{op}}) \to \mathsf{Mod}_{\mathcal{P}}.$$

Proof. From Theorem 3.4, we have

$$\begin{aligned}
\mathsf{Mod}_{\mathcal{P}}(\text{よ}_{A_1}, \dots, \text{よ}_{A_n}; \text{よ}_B) &\cong \text{よ}_B(A_1, \dots, A_n;\ ) = \mathcal{P}(A_1, \dots, A_n; B) \\
\mathsf{Mod}_{\mathcal{P}}({}_{A_1}\reflectbox{よ}, \dots, {}_{A_n}\reflectbox{よ}; {}_B\reflectbox{よ}) &\cong {}_B\reflectbox{よ}(\,; A_1, \dots, A_n) = \mathcal{P}(B; A_1, \dots, A_n).
\end{aligned}$$

Functoriality is easy to check. □

4. The polycategorical Chu construction

Let $\mathcal{E}$ be a multicategory; a **presheaf** on $\mathcal{E}$ can be defined equivalently as either:

(i) A module (as in §3) over $\mathcal{E}$ *qua* co-unary polycategory, whose only nonempty values have the form $\mathcal{U}(\Gamma;\ )$.
(ii) An ordinary presheaf on the free symmetric strict monoidal category $F_\otimes \mathcal{E}$ generated by $\mathcal{E}$.
(iii) An extension of $\mathcal{E}$ to a *co-subunary polycategory*, i.e. one in which all morphisms have codomain arity 0 or 1.
(iv) Explicitly as in [Shu20, §2]; here we consider only set-valued presheaves.

When $\perp\!\!\!\perp$ is a presheaf on $\mathcal{E}$, we write $\mathcal{E}(\Gamma; \perp\!\!\!\perp)$ for the set $\perp\!\!\!\perp(\Gamma;\ )$ as in (i). This is not very abusive, since by Corollary 3.9 it is isomorphic to $\mathsf{Mod}_{\mathcal{E}}(\text{よ}_\Gamma; \perp\!\!\!\perp)$.

The following multicategorical Chu construction first appeared, to my knowledge, in [Shu20], although [CKS03, Example 1.8(2)] contains a similar construction for bicategories. It explains the Chu tensor product [Chu79] by a universal property.

Definition 4.1. The **Chu construction** $\mathsf{Chu}(\mathcal{E}, \perp\!\!\!\perp)$ is the following $*$-polycategory:

- Its objects are triples $A = (A^+, A^-, \underline{A})$ where A^+ and A^- are objects of $\mathcal{E}$, and $\underline{A} \in \mathcal{E}(A^+, A^-; \perp\!\!\!\perp)$. We have $(A^+, A^-, \underline{A})^* = (A^-, A^+, \underline{A}\sigma)$.
- Its morphisms $f : (A_1, \dots, A_m) \to (B_1, \dots, B_n)$ are families of morphisms in $\mathcal{E}$:

$$\begin{aligned}
f_j^+ &: (A_1^+, \dots, A_m^+, B_1^-, \dots, \widehat{B_j^-}, \dots B_n^-) \longrightarrow B_j^+ \qquad (1 \le j \le n) \\
f_i^- &: (A_1^+, \dots, \widehat{A_i^+}, \dots A_m^+, B_1^-, \dots, B_n^-) \longrightarrow A_i^- \qquad (1 \le i \le m) \\
\underline{f} &: (A_1^+, \dots, A_m^+, B_1^-, \dots, B_n^-) \longrightarrow \perp\!\!\!\perp
\end{aligned}$$

  (where hats indicate omitted entries) such that $\underline{B_j} \circ_{B_j^+} f_j^+ = \underline{f}$ and $\underline{A_i} \circ_{A_i^-} f_i^- = \underline{f}$ (modulo permutations). If $m = n = 0$, the only datum is $\underline{f} : () \to \perp\!\!\!\perp$.
- The identity of $(A^+, A^-, \underline{A})$ is $(1_{A^+}, 1_{A^-}, \underline{A})$; composition is induced from $\mathcal{E}$.

Theorem 4.2. *If $\mathcal{E}$ is representable and closed with pullbacks, and $\perp\!\!\!\perp = \text{よ}_\perp$ is representable, then $\mathsf{Chu}(\mathcal{E}, \perp)$ is representable, hence $*$-autonomous.*

Proof. The usual formulas $\mathbb{1} = (\mathbb{1}, \perp, \ell)$ and $A \otimes B = (A^+ \otimes B^+, P, \rho)$ can be verified to have the correct universal properties, where P is the pullback

$$\begin{tikzcd}
P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & A^+ \multimap B^- \arrow[d] \\
B^+ \multimap A^- \arrow[r] & (A^+ \otimes B^+) \multimap \perp.
\end{tikzcd}$$

□

In addition, the multicategorical Chu construction itself has a simple universal property, which generalizes and strictifies that of [Pav93]. Let $*\mathsf{Poly}$ denote the category of $*$-polycategories, and $\mathsf{Poly}_{\le 1}$ that of co-subunary polycategories.

Theorem 4.3. Chu *is right adjoint to the forgetful functor* $U^*_{\le 1} : *\mathsf{Poly} \to \mathsf{Poly}_{\le 1}$, *and the adjunction is comonadic.*

Proof. The counit $U^*_{\le 1}\mathsf{Chu}(\mathcal{E}) \to \mathcal{E}$ extracts A^+ from A, f_1^+ from a morphism $f : (A_1, \dots, A_m) \to B_1$, and $\underline{f}$ from a morphism $f : (A_1, \dots, A_n) \to ()$. The unit $\mathcal{P} \to \mathsf{Chu}(U^*_{\le 1}(\mathcal{P}))$ sends an object A to (A, A^*, ε_A), where $\varepsilon_A \in \mathcal{P}(A, A^*;\ )$ is the duality counit, and a morphism $f \in \mathcal{P}(\Gamma; \Delta)$ to the family of all its co-subunary duality images. The coalgebras for the induced comonad are *co-subunary $*$-polycategories*, which by [Shu20, §7] are equivalent to ordinary $*$-polycategories. □

5. Envelopes

Definition 5.1 ([Hyl02, §5])**.** The **envelope** of $\mathcal{P}$ is the Chu construction

$$\mathsf{Env}_{\mathcal{P}} = \mathsf{Chu}(\mathsf{Mod}_{\mathcal{P}}, \mathcal{P}).$$

Thus, an object of $\mathsf{Env}_{\mathcal{P}}$ is two modules $\mathcal{U}$, $\mathcal{V}$ with a module map $(\mathcal{U}, \mathcal{V}) \to \mathcal{P}$. Since $\mathsf{Mod}_{\mathcal{P}}$ is representable and closed with pullbacks, $\mathsf{Env}_{\mathcal{P}}$ is $*$-autonomous, and contains $\mathsf{Mod}_{\mathcal{P}}$ as a symmetric monoidal full subcategory via $\mathcal{U} \mapsto (\mathcal{U}, \mathcal{U} \multimap \mathcal{P}, \mathsf{ev})$. Hence it also contains $U_{\mathrm{multi}}(\mathcal{P})$ as a full sub-multicategory via $A \mapsto (\text{よ}_A, {}_A\reflectbox{よ}, \gamma_A)$, where $\gamma_A : (\text{よ}_A, {}_A\reflectbox{よ}) \to \mathcal{P}$ is composition in $\mathcal{P}$. In fact, Hyland showed:

Theorem 5.2 ([Hyl02, §5])**.** *The assignment $A \mapsto (\text{よ}_A, {}_A\reflectbox{よ}, \gamma_A)$ extends to a full embedding of* polycategories $\mathcal{P} \hookrightarrow \mathsf{Env}_{\mathcal{P}}$.

Proof. A morphism $g : (\gamma_{A_1}, \dots, \gamma_{A_m}) \to (\gamma_{B_1}, \dots, \gamma_{B_n})$ in $\mathsf{Env}_{\mathcal{P}}$ is a compatible family of morphisms in $\mathsf{Mod}_{\mathcal{P}}$:

$$\begin{aligned}
g_j^+ &: (\text{よ}_{A_1}, \dots, \text{よ}_{A_m}, {}_{B_1}\reflectbox{よ}, \dots, \widehat{{}_{B_j}\reflectbox{よ}}, \dots, {}_{B_n}\reflectbox{よ}) \to \text{よ}_{B_j} \\
g_i^- &: (\text{よ}_{A_1}, \dots, \widehat{\text{よ}_{A_i}}, \dots, \text{よ}_{A_m}, {}_{B_1}\reflectbox{よ}, \dots, {}_{B_n}\reflectbox{よ}) \to {}_{A_i}\reflectbox{よ} \\
\underline{g} &: \qquad (\text{よ}_{A_1}, \dots, \text{よ}_{A_m}, {}_{B_1}\reflectbox{よ}, \dots, {}_{B_n}\reflectbox{よ}) \to \mathcal{P}.
\end{aligned}$$

By (3.7), each of these is equivalent to a morphism $(A_1, \dots, A_m) \to (B_1, \dots, B_n)$ in $\mathcal{P}$, and the compatibility conditions say exactly that they correspond to the same such morphism. Functoriality is straightforward to verify. □

Thus, any polycategory $\mathcal{P}$ embeds fully-faithfully in a $*$-autonomous category, and indeed in a Chu construction. An explicit way to extract a morphism in $\mathcal{P}$ from a morphism $g : (\gamma_{A_1}, \dots, \gamma_{A_m}) \to (\gamma_{B_1}, \dots, \gamma_{B_n})$ is to evaluate the component

$$\text{よ}_{A_1}(A_1;) \times \cdots \times \text{よ}_{A_m}(A_m;) \times {}_{B_1}\reflectbox{よ}(;B_1) \times \cdots \times {}_{B_n}\reflectbox{よ}(;B_n) \to \mathcal{P}(A_1, \dots, A_m; B_1, \dots, B_n)$$

of $\underline{g}$ at the identities $(1_{A_1}, \dots, 1_{A_m}, 1_{B_1}, \dots, 1_{B_n})$.

Remark 5.3. If $\mathcal{P}$ is an ordinary category regarded as a unary co-unary polycategory, then the categories $[\mathcal{P}^{\mathrm{op}}, \mathsf{Set}]$ and $[\mathcal{P}, \mathsf{Set}]$ are equivalent to the categories of modules whose only nonempty values have the form $\mathcal{U}(A;\ )$ and $\mathcal{U}(\ ;A)$, respectively. The functor $(-) \multimap \mathcal{P}$ restricts to Isbell conjugation interchanging these two subcategories, and the Hyland envelope contains the Isbell envelope [Isb66].

6. Preserving tensors and cotensors

Hyland's envelope does not preserve any tensor or cotensor products that exist in $\mathcal{P}$, but we can modify it to do so along the lines of [Kel82, §3.12].

Suppose $\mathcal{P}$ is equipped with a set $\mathcal{J}$ of tensor and cotensor products that exist, which we call **distinguished**. Our intended example is $\mathcal{P} = \circledast\mathcal{C}$ from §2, with tensor products coming from $\mathcal{C}$ and cotensor products $(A^* \parr B) = (A \multimap B)$.

Definition 6.1. A $(\mathcal{P}, \mathcal{J})$-**module** is a module that respects the distinguished tensor and cotensor products. E.g. if $(A \otimes B) \in \mathcal{J}$, the induced maps such as $\mathcal{U}(\Gamma, A \otimes B; \Delta) \to \mathcal{U}(\Gamma, A, B; \Delta)$ are isomorphisms.

Let $\mathsf{Mod}_{(\mathcal{P},\mathcal{J})} \subseteq \mathsf{Mod}_{\mathcal{P}}$ consist of the $(\mathcal{P}, \mathcal{J})$-modules. Of course, $\mathcal{P}$ is a $(\mathcal{P}, \mathcal{J})$-module, as is any shift of a $(\mathcal{P}, \mathcal{J})$-module; thus $よ_A, {}_Aよ \in \mathsf{Mod}_{(\mathcal{P},\mathcal{J})}$.

Theorem 6.2. *The embedding $よ : U_{\mathrm{multi}}(\mathcal{P}) \to \mathsf{Mod}_{(\mathcal{P},\mathcal{J})}$ preserves distinguished tensor products. Dually, the embedding $よ : U_{\mathrm{multi}}(\mathcal{P}^{\mathrm{op}}) \to \mathsf{Mod}_{(\mathcal{P},\mathcal{J})}$ takes distinguished cotensor products (which are tensor products in $\mathcal{P}^{\mathrm{op}}$) to tensor products.*

Proof. Let $\mathcal{U}$ be a $(\mathcal{P}, \mathcal{J})$-module and Γ a list of $(\mathcal{P}, \mathcal{J})$-modules. Then by (3.7) and the assumption on $\mathcal{U}$ we have natural bijections:

$$\begin{aligned}
\mathsf{Mod}_{\mathcal{P}}(\Gamma, よ_A, よ_B\,; \mathcal{U}) &\cong \mathsf{Mod}_{\mathcal{P}}(\Gamma\,; \mathcal{U}[A, B;\ ]) \\
&\cong \mathsf{Mod}_{\mathcal{P}}(\Gamma\,; \mathcal{U}[A \otimes B;\ ]) \\
&\cong \mathsf{Mod}_{\mathcal{P}}(\Gamma, よ_{A\otimes B}\,; \mathcal{U}).
\end{aligned}$$

Thus, $よ_{A\otimes B}$ is a tensor product $よ_A \otimes よ_B$. The dual statement is similar. ☐

Theorem 6.3.

(i) $\mathsf{Mod}_{(\mathcal{P},\mathcal{J})}$ *is a reflective subcategory of* $\mathsf{Mod}_{\mathcal{P}}$.
(ii) If $\mathcal{U} \in \mathsf{Mod}_{\mathcal{P}}$ *and* $\mathcal{V} \in \mathsf{Mod}_{(\mathcal{P},\mathcal{J})}$ *then* $(\mathcal{U} \multimap \mathcal{V}) \in \mathsf{Mod}_{(\mathcal{P},\mathcal{J})}$.
(iii) $\mathsf{Mod}_{(\mathcal{P},\mathcal{J})}$ *has a closed symmetric monoidal structure such that the reflector is strong monoidal and the inclusion preserves internal-homs.*

Proof. $\mathsf{Mod}_{(\mathcal{P},\mathcal{J})}$ is locally presentable, closed under limits, and accessibly embedded, so (i) follows from the adjoint functor theorem. Now by definition

$$(\mathcal{U} \multimap \mathcal{V})(\Gamma; \Delta) = \mathsf{Mod}_{\mathcal{P}}(\mathcal{U}, \mathcal{V}[\Gamma; \Delta]),$$

so (ii) follows since $\mathcal{V}$ is a $(\mathcal{P}, \mathcal{J})$-module. Finally, (iii) follows formally [Day72]; the tensor product $\otimes_{\mathcal{J}}$ of $\mathsf{Mod}_{(\mathcal{P},\mathcal{J})}$ is the reflection of that of $\mathsf{Mod}_{\mathcal{P}}$. ☐

Example 6.4. By the formula for $\mathcal{U} \otimes \mathcal{V}$ on p28 of [Hyl02], any elements $u \in \mathcal{U}(A;\ )$ and $v \in \mathcal{V}(B;\ )$ induce an element of $(\mathcal{U} \otimes \mathcal{V})(A, B;\ )$, hence of $(\mathcal{U} \otimes_{\mathcal{J}} \mathcal{V})(A \otimes B;\ )$ and thence $(\mathcal{U} \otimes_{\mathcal{J}} \mathcal{V})(C;\ )$ for any $C \to A \otimes B$ in $\mathcal{P}$. Thus we have a map

$$\mathcal{P}(C, A \otimes B) \times \mathcal{U}(A;\ ) \times \mathcal{V}(B;\ ) \longrightarrow (\mathcal{U} \otimes_{\mathcal{J}} \mathcal{V})(C;\ )$$

showing that $\otimes_{\mathcal{J}}$ is similar to Day convolution [Day70].

Definition 6.5. Given $(\mathcal{P}, \mathcal{J})$, its **envelope** is the Chu construction

$$\mathsf{Env}_{(\mathcal{P},\mathcal{J})} = \mathsf{Chu}(\mathsf{Mod}_{(\mathcal{P},\mathcal{J})}, \mathcal{P}).$$

Theorem 6.6. $\mathsf{Env}_{(\mathcal{P},\mathcal{J})}$ *is $*$-autonomous, contains $\mathcal{P}$ as a full sub-polycategory, and the inclusion preserves the distinguished tensor and cotensor products.*

Proof. Since $\mathsf{Mod}_{(\mathcal{P},\mathcal{J})}$ is a full sub-multicategory of $\mathsf{Mod}_{\mathcal{P}}$, $\mathsf{Env}_{(\mathcal{P},\mathcal{J})}$ is a full sub-polycategory of $\mathsf{Env}_{\mathcal{P}}$. Since it contains the image of $\mathcal{P}$, which is a full sub-polycategory of $\mathsf{Env}_{\mathcal{P}}$ by Theorem 5.2, the inclusion $\mathcal{P} \hookrightarrow \mathsf{Env}_{(\mathcal{P},\mathcal{J})}$ is also full.

Now the embedding of $\mathsf{Mod}_{(\mathcal{P},\mathcal{J})}$ in $\mathsf{Env}_{(\mathcal{P},\mathcal{J})}$, like that of any monoidal category in its Chu construction, preserves tensor products. Since the embedding of $U_{\mathrm{multi}}(\mathcal{P})$ in $\mathsf{Mod}_{(\mathcal{P},\mathcal{J})}$ preserves distinguished tensor products by Theorem 6.2, so does the composite $U_{\mathrm{multi}}(\mathcal{P}) \hookrightarrow \mathsf{Mod}_{(\mathcal{P},\mathcal{J})} \hookrightarrow \mathsf{Env}_{(\mathcal{P},\mathcal{J})}$. Dually, the composite embedding $U_{\mathrm{multi}}(\mathcal{P}) \hookrightarrow \mathsf{Mod}^{\mathrm{op}}_{(\mathcal{P},\mathcal{J})} \hookrightarrow \mathsf{Env}_{(\mathcal{P},\mathcal{J})}$ preserves distinguished cotensor products, and by Corollary 3.8 the two embeddings coincide. □

Combining Theorems 2.1 and 6.6, we obtain our main theorem.

Theorem 6.7. *Any closed symmetric monoidal category $\mathcal{C}$ can be fully embedded in a $*$-autonomous category by a strong symmetric monoidal closed functor.*

Proof. By Theorem 2.1, $\mathcal{C}$ embeds fully in the $*$-polycategory $\circledast\mathcal{C}$ preserving tensor products and taking homs $A \multimap B$ to cotensor products $A^* \parr B$. Let $\mathcal{J}$ consist of these tensor and cotensor products, and embed $\circledast\mathcal{C}$ in the $*$-autonomous category $\mathsf{Env}_{(\circledast\mathcal{C},\mathcal{J})}$, which by Theorem 6.6 preserves these tensor and cotensor products. □

7. Preserving limits and colimits

By a **limit** in a polycategory $\mathcal{P}$ we mean a cone of unary morphisms such that

$$\mathcal{P}(\Gamma;\Delta,\lim_i A_i) \to \lim_i \mathcal{P}(\Gamma;\Delta,A_i)$$

is always an isomorphism. A **colimit** is a limit in $\mathcal{P}^{\mathrm{op}}$. Note that since limits of modules are pointwise, we have

$$\text{よ}_{(\lim_i A_i)} \cong \lim_i(\text{よ}_{A_i}) \qquad \text{and} \qquad {}_{(\mathrm{colim}_i A_i)}\reflectbox{よ} \cong \lim_i({}_{A_i}\reflectbox{よ}),$$

but in $\mathsf{Mod}_{\mathcal{P}}$ or $\mathsf{Mod}_{(\mathcal{P},\mathcal{J})}$ we can say nothing about $\text{よ}_{(\mathrm{colim}_i A_i)}$ or ${}_{(\lim_i A_i)}\reflectbox{よ}$. Now let $(\mathcal{P},\mathcal{J})$ be as in §6, and $\mathcal{K}$ a set of distinguished limit and colimit cones in $\mathcal{P}$.

Remark 7.1. We have been ignoring size, but $\mathcal{P}$ should everywhere be a *small* polycategory, and $\mathcal{K}$ should be a small set. ($\mathcal{J}$ is automatically small once $\mathcal{P}$ is.)

Definition 7.2. A $(\mathcal{P},\mathcal{J},\mathcal{K})$**-module** is a $(\mathcal{P},\mathcal{J})$-module that in addition respects the distinguished limits and colimits in $\mathcal{K}$, i.e. the map

$$\mathcal{U}(\Gamma;\Delta,\lim_i D_i) \to \lim_i \mathcal{U}(\Gamma;\Delta,D_i)$$

is an isomorphism for all distinguished limit cones, and dually for colimits.

$\mathcal{P}$ is a $(\mathcal{P},\mathcal{J},\mathcal{K})$-module, as is any shift of a $(\mathcal{P},\mathcal{J},\mathcal{K})$-module; hence so are よ_A and ${}_A\reflectbox{よ}$. Let $\mathsf{Mod}_{(\mathcal{P},\mathcal{J},\mathcal{K})} \subseteq \mathsf{Mod}_{\mathcal{P}}$ consist of the $(\mathcal{P},\mathcal{J},\mathcal{K})$-modules.

Theorem 7.3.

(i) $\mathsf{Mod}_{(\mathcal{P},\mathcal{J},\mathcal{K})}$ *is a reflective subcategory of* $\mathsf{Mod}_{\mathcal{P}}$.
(ii) If $\mathcal{U} \in \mathsf{Mod}_{\mathcal{P}}$ *and* $\mathcal{V} \in \mathsf{Mod}_{(\mathcal{P},\mathcal{J},\mathcal{K})}$ *then* $(\mathcal{U} \multimap \mathcal{V}) \in \mathsf{Mod}_{(\mathcal{P},\mathcal{J},\mathcal{K})}$.
(iii) $\mathsf{Mod}_{(\mathcal{P},\mathcal{J},\mathcal{K})}$ *has a closed symmetric monoidal structure such that the reflector is strong monoidal and the inclusion preserves internal-homs.*
(iv) The embedding $\text{よ} : U_{\mathrm{multi}}(\mathcal{P}) \to \mathsf{Mod}_{(\mathcal{P},\mathcal{J},\mathcal{K})}$ *preserves distinguished tensor products. Dually, the embedding* $\reflectbox{よ} : U_{\mathrm{multi}}(\mathcal{P}^{\mathrm{op}}) \to \mathsf{Mod}_{(\mathcal{P},\mathcal{J},\mathcal{K})}$ *takes distinguished cotensor products to tensor products.*
(v) The embeddings よ *and* $\reflectbox{よ}$ *preserve distinguished limits and colimits.*

Proof. Parts (i)–(iv) are just like Theorems 5.2 and 6.3. For (v), it remains to show that よ preserves distinguished colimits and $\rotatebox[origin=c]{180}{よ}$ preserves distinguished limits (i.e. takes them to colimits). To see this, let $\mathcal{U}$ be a $(\mathcal{P},\mathcal{J},\mathcal{K})$-module; then we have

$$\begin{aligned}
\mathsf{Mod}_{(\mathcal{P},\mathcal{J},\mathcal{K})}(\Gamma, \text{よ}_{(\mathrm{colim}_i\, A_i)}; \mathcal{U}) &\cong \mathsf{Mod}_{(\mathcal{P},\mathcal{J},\mathcal{K})}(\Gamma; \mathcal{U}[\mathrm{colim}_i\, A_i\,;\,])\\
&\cong \mathsf{Mod}_{(\mathcal{P},\mathcal{J},\mathcal{K})}(\Gamma; \lim_i \mathcal{U}[A_i\,;\,])\\
&\cong \lim_i \mathsf{Mod}_{(\mathcal{P},\mathcal{J},\mathcal{K})}(\Gamma; \mathcal{U}[A_i\,;\,])\\
&\cong \lim_i \mathsf{Mod}_{(\mathcal{P},\mathcal{J},\mathcal{K})}(\Gamma, \text{よ}_{A_i}; \mathcal{U})\\
&\cong \mathsf{Mod}_{(\mathcal{P},\mathcal{J},\mathcal{K})}(\Gamma, \mathrm{colim}_i(\text{よ}_{A_i}); \mathcal{U}),
\end{aligned}$$

using the assumption on $\mathcal{U}$ in the second step. The claim about $\rotatebox[origin=c]{180}{よ}$ is dual. □

Definition 7.4. Given $(\mathcal{P},\mathcal{J},\mathcal{K})$, its **envelope** is the Chu construction

$$\mathsf{Env}_{(\mathcal{P},\mathcal{J},\mathcal{K})} = \mathsf{Chu}(\mathsf{Mod}_{(\mathcal{P},\mathcal{J},\mathcal{K})}, \mathcal{P}).$$

Theorem 7.5. $\mathsf{Env}_{(\mathcal{P},\mathcal{J},\mathcal{K})}$ *is a complete and cocomplete $*$-autonomous category, contains $\mathcal{P}$ as a full sub-polycategory, and the inclusion preserves the distinguished tensor and cotensor products and the distinguished limits and colimits.*

Proof. Just like Theorem 6.6 except for the final claim, which follows from the formulas for limits and colimits in a Chu construction and Theorem 7.3(v):

$$\begin{aligned}
\lim_i(A_i^+, A_i^-, e_i) &= (\lim_i A_i^+, \mathrm{colim}_i\, A_i^-, _)\\
\mathrm{colim}_i(A_i^+, A_i^-, e_i) &= (\mathrm{colim}_i\, A_i^+, \lim_i A_i^-, _)
\end{aligned}$$

□

We want to combine this with Theorem 2.1 as before, but there is a complication. For any limit cone in a closed symmetric monoidal category $\mathcal{C}$, we might expect

$$\begin{aligned}
¿\qquad \circledast\mathcal{C}(\Gamma, \Pi^*; \Delta, \Sigma^*, \lim_i A_i) &= \mathcal{C}(\Gamma, \Sigma; \Delta, \Pi, \lim_i A_i)\\
&\cong \lim_i \mathcal{C}(\Gamma, \Sigma; \Delta, \Pi, A_i)\\
&= \lim_i \circledast\mathcal{C}(\Gamma, \Pi^*; \Delta, \Sigma^*, A_i) \qquad ?
\end{aligned}$$

yielding a limit in $\circledast\mathcal{C}$. However, the isomorphism in the second line is only valid if $\Delta = \Pi = \emptyset$, whereas to have a limit in $\circledast\mathcal{C}$ the composite isomorphism must hold for all Δ and Π. Of course, if $\Delta \cup \Pi$ is nonempty, then $\circledast\mathcal{C}(\Gamma, \Pi^*; \Delta, \Sigma^*, \lim_i A_i)$ is empty, as is each $\circledast\mathcal{C}(\Gamma, \Pi^*; \Delta, \Sigma^*, A_i)$ — but the latter only implies that their limit is empty *if the diagram is nonempty!* And indeed, a terminal object in $\mathcal{C}$ need not be terminal in $\circledast\mathcal{C}$: the latter requires $\circledast\mathcal{C}(\Gamma, \Pi^*; \Delta, \Sigma^*, 1) = 1$ always, but the former only ensures that this holds when $\Pi = \Delta = \emptyset$. Similar considerations apply for colimits, so the best enhancement of Theorem 6.7 we can manage is:

Theorem 7.6. *Any closed symmetric monoidal category $\mathcal{C}$ can be fully embedded in a $*$-autonomous category by a strong symmetric monoidal closed functor, which preserves any chosen family of* nonempty *limits and colimits that exist in $\mathcal{C}$.* □

With other choices of $(\mathcal{P},\mathcal{J},\mathcal{K})$, we can embed other kinds of structures into $*$-autonomous categories as well. For instance:

- Any linearly distributive category can be fully embedded in a $*$-autonomous category, preserving tensors, cotensors, and any set of colimits that are preserved in each variable by $\otimes$ and limits that are preserved in each variable by $\invamp$.

- Any ordinary category can be fully embedded in a $*$-autonomous category preserving any set of nonempty limits and colimits. (Recall from Remark 5.3 that this embedding is closely related to the Isbell envelope.)

8. Conservativity

Since closed symmetric monoidal and $*$-autonomous categories model intuitionistic and classical linear logic, respectively, Theorem 7.6 implies a conservativity result.

Theorem 8.1. *Classical linear logic with* $(-)^\perp, \otimes, \parr, \mathbb{1}, \perp, \multimap, \&, \oplus$ *is conservative over intuitionistic linear logic with* $\otimes, \mathbb{1}, \multimap, \&, \oplus$ *(but not* $\mathbf{0}, \top$*).*

Proof. Let $\mathrm{MA}^-\mathrm{LL}$ and $\mathrm{IMA}^-\mathrm{LL}$ denote the given fragments, and let T be a theory in $\mathrm{IMA}^-\mathrm{LL}$. By imposing an appropriate equivalence relation on proofs from T in $\mathrm{IMA}^-\mathrm{LL}$ and $\mathrm{MA}^-\mathrm{LL}$ respectively, we obtain a closed symmetric monoidal category $\mathcal{C}_T$ and a $*$-autonomous category $\mathcal{D}_T$, both with binary products and coproducts, and each freely generated by a model of T among categories with the same structure.

By Theorem 7.6, $\mathcal{C}_T$ embeds fully in $\mathsf{Env}_{\mathcal{C}_T}$ preserving all the structure. Since $\mathsf{Env}_{\mathcal{C}_T}$ is a $*$-autonomous model of T, this embedding factors up to isomorphism through $\mathcal{D}_T$. Now any sequent in the language of $\mathrm{IMA}^-\mathrm{LL}$ over T that is provable in $\mathrm{MA}^-\mathrm{LL}$ yields a morphism in $\mathcal{D}_T$ between objects in the image of $\mathcal{C}_T$. Hence such a morphism also exists in $\mathsf{Env}_{\mathcal{C}_T}$, and thus also in $\mathcal{C}_T$. $\square$

Similarly, we can show that $\mathrm{MA}^-\mathrm{LL}$ is conservative over any smaller fragment of $\mathrm{IMA}^-\mathrm{LL}$, and that full MALL is conservative over any of its fragments. (IMALL is not a fragment of MALL in this sense, since its judgmental structure is different.)

Perhaps surprisingly, Theorem 8.1 is almost best possible. It is shown in [Sch91] that classical linear logic with $\mathbf{0}$ (an initial object) and $\multimap$ is *not* conservative over intuitionistic linear logic with the same connectives: the sequent

$$C \multimap ((\mathbf{0} \multimap X) \multimap A), (C \multimap B) \multimap \mathbf{0} \vdash A$$

is provable in the former but not the latter. Thus, not every closed symmetric monoidal category with an initial object admits a strong monoidal closed embedding into a $*$-autonomous category preserving the initial object.

Theorem 8.1 says equivalently that for the universal functor $\Phi : \mathcal{C} \to \mathcal{D}$ from a closed symmetric monoidal category to a $*$-autonomous one (both with binary products and coproducts), if $\mathcal{D}(\Phi(\Gamma); \Phi(\Delta))$ is nonempty, so is $\mathcal{C}(\Gamma; \Delta)$. Similarly:

Theorem 8.2. *Let* $\Phi : \mathcal{C} \to \mathcal{D}$ *be the universal functor from a closed symmetric monoidal category* $\mathcal{C}$ *to a* $*$*-autonomous category (perhaps with specified nonempty limits and/or colimits). Then* Φ *is faithful and conservative.*

Proof. As before, $\mathcal{C}$ embeds in the $*$-autonomous $\mathsf{Env}_{\mathcal{C}}$ preserving all its structure. This embedding is fully faithful, hence faithful and conservative. This embedding factors through Φ, so Φ is also faithful and conservative. $\square$

Obviously a more categorically desirable result would be that Φ is fully faithful; this is sometimes called a *full completeness* theorem. We cannot show this in the same way, since fully faithful functors are not cancelable. But we can use the general technique of Lafont [Laf88] for proving full completeness with Artin gluing along a restricted Yoneda embedding (a.k.a. a "Kripke logical relation"), as generalized to the $*$-autonomous case by [Tan98, HS03, Has99] using *double gluing*.

9. Double gluing

The name "double gluing" presumably refers to the appearance of *two* "logical relation" families, but fortuitously it can also be expressed using *double categories*. Recall that a double category is a category internal to Cat; by a **poly double category** we mean a category internal to Poly. For example, any polycategory $\mathcal{P}$ induces a poly double category $\mathbb{Q}\mathcal{P}$ consisting of the following structure:

- The objects and the horizontal poly-arrows are those of $\mathcal{P}$.
- The vertical arrows are the unary and co-unary morphisms of $\mathcal{P}$. (Note that the vertical arrows in any poly double category are only an ordinary category.)
- The 2-cells are "commutative squares" in $\mathcal{P}$ of the form

$$\begin{array}{ccc} (A_1, \dots, A_m) & \xrightarrow{f} & (B_1, \dots, B_n) \\ {\scriptstyle u_1}\big\downarrow \quad \cdots \quad \big\downarrow{\scriptstyle u_m} & & {\scriptstyle v_1}\big\downarrow \quad \cdots \quad \big\downarrow{\scriptstyle v_n} \\ (C_1, \dots, C_m) & \xrightarrow[g]{} & (D_1, \dots, D_n) \end{array}$$

  i.e. the assertion that $g \circ (u_1, \dots, u_m) = (v_1, \dots, v_n) \circ f$.

Now since the functor Chu is a right adjoint, it preserves internal categories. Thus any multicategory $\mathcal{E}$ with a presheaf $\perp\!\!\!\perp$ has a **double Chu construction** [Shu20]

$$\mathbb{C}\mathsf{hu}(\mathcal{E}, \perp\!\!\!\perp) \coloneqq \mathsf{Chu}(\mathbb{Q}(\mathcal{E}, \perp\!\!\!\perp)).$$

This is a poly double category described as follows.

- Its objects and horizontal poly-arrows are those of $\mathsf{Chu}(\mathcal{E}, \perp\!\!\!\perp)$.
- A vertical arrow $u : A \to B$ is a pair $(u^+ : A^+ \to B^+, u^- : A^- \to B^-)$ such that $\underline{B} \circ (u^+, u^-) = \underline{A}$.
- If $m + n > 0$, a 2-cell

$$\begin{array}{ccc} (A_1, \dots, A_m) & \xrightarrow{f} & (B_1, \dots, B_n) \\ {\scriptstyle u_1}\big\downarrow \quad \cdots \quad \big\downarrow{\scriptstyle u_m} & \Downarrow\mu & {\scriptstyle v_1}\big\downarrow \quad \cdots \quad \big\downarrow{\scriptstyle v_n} \\ (C_1, \dots, C_m) & \xrightarrow[g]{} & (D_1, \dots, D_n) \end{array}$$

  consists of a family of commuting squares in $\mathscr{E}$:

$$\begin{array}{ccc} (A_1^+, \dots, A_m^+, B_1^-, \dots, \widehat{B_j^-}, \dots B_n^-) & \xrightarrow{f_j^+} & B_j^+ \\ \big\downarrow \; {\scriptstyle \cdots(u_1^+, \dots, u_m^+, v_1^-, \dots, \widehat{v_j^-}, \dots, v_n^-)\cdots} \; \big\downarrow & & \big\downarrow{\scriptstyle v_j^+} \\ (C_1^+, \dots, C_m^+, D_1^-, \dots, \widehat{D_j^-}, \dots D_n^-) & \xrightarrow[g_j^+]{} & D_j^+ \end{array} \qquad \begin{array}{ccc} (A_1^+, \dots, \widehat{A_i^+}, \dots A_m^+, B_1^-, \dots, B_n^-) & \xrightarrow{f_i^-} & A_i^- \\ \big\downarrow \; {\scriptstyle \cdots(u_1^+, \dots, \widehat{u_i^+}, \dots u_m^+, v_1^-, \dots, v_n^-)\cdots} \; \big\downarrow & & \big\downarrow{\scriptstyle u_i^-} \\ (C_1^+, \dots, \widehat{C_i^+}, \dots C_m^+, D_1^-, \dots, D_n^-) & \xrightarrow[g_i^-]{} & C_i^- \end{array}$$

$$\begin{array}{ccc} (A_1^+, \dots, A_m^+, B_1^-, , \dots B_n^-) & \xrightarrow{\underline{f}} & \perp\!\!\!\perp \\ \big\downarrow {\scriptstyle \cdots(u_1^+, \dots, u_m^+, v_1^-, \dots, v_n^-)\cdots} \big\downarrow & & \big\| \\ (C_1^+, \dots, C_m^+, D_1^-, , \dots D_n^-) & \xrightarrow[\underline{g}]{} & \perp\!\!\!\perp \end{array}$$

  (the last follows from the others unless $m = n = 0$, when it is the only condition).

Now let $\mathcal{D}$ be a polycategory, and $\Lambda : \mathcal{D} \to \mathsf{Chu}(\mathcal{E}, \perp\!\!\!\perp)$ a functor, whose action on objects we write as $\Lambda(A) = (L(A), K(A), \lambda_A)$. On underlying 1-categories, L and K are functors $\mathcal{D} \to \mathcal{E}$ and $\mathcal{D} \to \mathcal{E}^{\mathrm{op}}$ respectively.

Example 9.1. If $\mathcal{D}$ and $\mathcal{E}$ are representable multicategories and $\perp\!\!\!\perp = 1$ is the terminal presheaf, then such a Λ reduces to the input data of [HS03, §4.2.1]. For applying Λ to the universal morphism $(A, B) \to A \otimes B$ yields three morphisms in $\mathcal{E}$:

$$\begin{aligned} m &: L(A) \otimes L(B) \to L(A \otimes B) \\ k &: L(A) \otimes K(A \otimes B) \to K(B) \\ k' &: L(B) \otimes K(A \otimes B) \to K(A) \end{aligned}$$

of which k and k' determine each other by the symmetries. Applying Λ to units and triple tensors makes m a lax symmetric monoidal structure on L and implies that k is a "contraction" as in [HS03, §4.2.1], and these data determine Λ.

Example 9.2. By Theorem 4.3, if $\mathcal{D}$ is a $*$-polycategory and $\perp\!\!\!\perp = 1$, a $*$-polycategory functor $\Lambda : \mathcal{D} \to \mathsf{Chu}(\mathcal{E}, 1)$ is uniquely determined by a functor $L : U_{\mathrm{multi}}(\mathcal{D}) \to \mathcal{E}$, with $K(A)$ defined as $L(A^*)$. If $\mathcal{D}$ and $\mathcal{E}$ are representable (hence $\mathcal{D}$ is $*$-autonomous), such an L is equivalently a lax symmetric monoidal functor; thus in this case our Λ reduces to the input data of [HS03, §4.3.1]. Up to isomorphism, the same is true when $\mathcal{D}$ is $*$-autonomous without strict duals.

Definition 9.3. Let $\psi : \perp\!\!\!\perp_1 \to \perp\!\!\!\perp_2$ be a morphism of presheaves on $\mathcal{E}$, and $\Lambda : \mathcal{D} \to \mathsf{Chu}(\mathcal{E}, \perp\!\!\!\perp_2)$. The **double gluing** $\mathsf{Gl}(\Lambda, \psi)$ is the following comma object in the 2-category of poly double categories (whose 2-cells are *vertical* transformations):

$$\begin{tikzcd} \mathsf{Gl}(\Lambda,\psi) \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow"] & \mathsf{Chu}(\mathcal{E}, \perp\!\!\!\perp_1) \arrow[d, "\psi"] \\ \mathcal{D} \arrow[r, "\Lambda"'] & \mathbb{C}\mathsf{hu}(\mathcal{E}, \perp\!\!\!\perp_2). \end{tikzcd}$$

Here $\mathcal{D}$ and $\mathsf{Chu}(\mathcal{E}, \perp\!\!\!\perp_1)$ are regarded as vertically discrete poly double categories. Hence so is $\mathsf{Gl}(\Lambda, \psi)$, i.e. it is a plain polycategory. Its objects consist of

- An object $A^1 \in \mathcal{D}$.
- An object $(A^+, A^-, \underline{A})$ of $\mathsf{Chu}(\mathcal{E}, \perp\!\!\!\perp_1)$.
- A vertical morphism $(A^+, A^-, \psi \circ \underline{A}) \to (L(A^1), K(A^1), \lambda_A)$ in $\mathbb{C}\mathsf{hu}(\mathcal{E}, \perp\!\!\!\perp_2)$, consisting of $A^+ \to L(A^1)$ and $A^- \to K(A^1)$ in $\mathcal{E}$ such that the composites $(A^+, A^-) \to \perp\!\!\!\perp_1 \xrightarrow{\psi} \perp\!\!\!\perp_2$ and $(A^+, A^-) \to (L(A^1), K(A^1)) \xrightarrow{\lambda} \perp\!\!\!\perp_2$ agree.

Similarly, a morphism $(A_1, \dots, A_m) \to (B_1, \dots, B_n)$ in $\mathsf{Gl}(\Lambda, \psi)$ consists of

- A morphism $f : (A_1^1, \dots, A_m^1) \to (B_1^1, \dots, B_n^1)$ in $\mathcal{D}$.
- A morphism $(f_j^+, f_i^-, \underline{f})$ in $\mathsf{Chu}(\mathcal{E}, \perp\!\!\!\perp_1)$.
- The following squares commute:

$$\begin{tikzcd} (A_1^+, \dots, A_m^+, B_1^-, \dots, \widehat{B_j^-}, \dots B_n^-) \arrow[r, "f_j^+"] \arrow[d] & B_j^+ \arrow[d] \\ (L(A_1^1), \dots, L(A_m^1), K(B_1^1), \dots, \widehat{K(B_j^1)}, \dots K(B_n^1)) \arrow[r, "\Lambda(f)_j^+"'] & L(B_j^1) \end{tikzcd} \tag{9.4}$$

$$\begin{tikzcd}[column sep=small]
(& A_1^+ \arrow[d] & ,\dots, & \widehat{A_i^+} & ,\dots & A_m^+ \arrow[d] & , & B_1^- \arrow[d] & ,\dots, & B_n^- \arrow[d] &) \arrow[r, "f_i^-"] & A_i^- \arrow[d] \\
 & (L(A_1^1) & ,\dots, & \widehat{L(A_i^1)} & ,\dots & L(A_m^1), & & K(B_1^1) & ,\dots, & K(B_n^1)) & \arrow[r, "\Lambda(f)_i^-"'] & K(A_i^1)
\end{tikzcd} \tag{9.5}$$

$$\begin{tikzcd}[column sep=small]
(& A_1^+ \arrow[d] & ,\dots, & A_m^+ \arrow[d] & , & B_1^- \arrow[d] & ,\dots, & B_n^- \arrow[d] &) \arrow[r, "f"] & \Perp_1 \arrow[d, "\psi"] \\
 & (L(A_1^1) & ,\dots, & L(A_m^1), & & K(B_1^1) & ,\dots, & K(B_n^1)) & \arrow[r, "\underline{\Lambda(f)}"'] & \Perp_2
\end{tikzcd} \tag{9.6}$$

(the last follows from the others unless $m = n = 0$, when it is the only condition).

Theorem 9.7. *Suppose $\mathcal{E}$ is representable and closed with pullbacks, and $\Perp_1$ and $\Perp_2$ are either both terminal ($\Perp_1 = \Perp_2 = 1$) or both representable ($\Perp_1 = \text{よ}_{\perp_1}$ and $\Perp_2 = \text{よ}_{\perp_2}$). Then tensor products, cotensor products, duals, and internal-homs exist in $\mathsf{Gl}(\Lambda, \psi)$ insofar as they do for the relevant underlying objects in $\mathcal{D}$. In particular, if $\mathcal{D}$ is ∗-autonomous, so is $\mathsf{Gl}(\Lambda, \psi)$.*

Proof. When $\Perp_1$ and $\Perp_2$ are terminal, we use the formulas from [HS03, §4.2]. When they are representable, we modify the formulas slightly; for tensor products we have

$$(A \otimes B)^+ = A^+ \otimes B^+ \to L(A^1) \otimes L(B^1) \to L(A^1 \otimes B^1)$$

and the limit of the following diagram (drawn in the middle):

$$\begin{tikzcd}[column sep=small]
 & & (A^+ \otimes B^+) \multimap \perp_1 & & \\
 & A^+ \multimap B^- \arrow[ur] \arrow[dl] & (A \otimes B)^- \arrow[l, dashed] \arrow[r, dashed] \arrow[d, dashed] & B^+ \multimap A^- \arrow[ul] \arrow[dr] & \\
A^+ \multimap K(B^1) & L(A^1) \multimap K(B^1) \arrow[l] & K(A^1 \otimes B^1) \arrow[l] \arrow[r] & L(B^1) \multimap K(A^1) \arrow[r] & B^+ \multimap K(A^1).
\end{tikzcd}$$

The unit consists of $\mathbb{1}^+ = \mathbb{1} \to L(\mathbb{1})$ and the pullback

$$\begin{tikzcd}
\mathbb{1}^- \arrow[rrr] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & & & \perp_1 \arrow[d] \\
K(\mathbb{1}) \arrow[r, "\cong"'] & \mathbb{1} \otimes K(\mathbb{1}) \arrow[r] & L(\mathbb{1}) \otimes K(\mathbb{1}) \arrow[r, "\lambda"'] & \perp_2.
\end{tikzcd}$$

The dual of $(A^1, A^+, A^-, \underline{A})$ is $((A^1)^*, A^-, A^+, \underline{A}\sigma)$, and for the internal-hom we have

$$\begin{tikzcd}[column sep=small]
 & & (A^+ \otimes B^-) \multimap \perp_1 & & \\
 & A^+ \multimap B^+ \arrow[ur] \arrow[dl] & (A \multimap B)^+ \arrow[l, dashed] \arrow[r, dashed] \arrow[d, dashed] & B^- \multimap A^- \arrow[ul] \arrow[dr] & \\
A^+ \multimap L(B^1) & L(A^1) \multimap L(B^1) \arrow[l] & L(A^1 \multimap B^1) \arrow[l] \arrow[r] & K(B^1) \multimap K(A^1) \arrow[r] & B^- \multimap K(A^1).
\end{tikzcd}$$

and

$$(A \multimap B)^- = A^+ \otimes B^- \to L(A^1) \otimes K(B^1) \to K(A^1 \multimap B^1).$$

We leave cotensor products to the reader. □

Example 9.8. Double gluing is usually described only when $\perp\!\!\!\perp_1 = \perp\!\!\!\perp_2 = 1$, but our more general version appears implicitly in at least one place. Specifically, consider Example 9.1 with $L = \mathcal{C}(\mathbb{1}, -)$ and $K = \mathcal{C}(-, J)$ for some J, as in [HS03, §4.2.2]. Then every object $\Lambda(A) = (LA, KA)$ comes with a pairing $LA \times KA = \mathcal{C}(\mathbb{1}, A) \times \mathcal{C}(A, J) \to \mathcal{C}(\mathbb{1}, J)$, which is respected by every morphism in the image of Λ; thus the codomain of Λ lifts to $\mathsf{Chu}(\mathsf{Set}, \perp_2)$ where $\perp_2 = \mathcal{C}(\mathbb{1}, J)$. Now for any $F \subseteq \mathcal{C}(\mathbb{1}, J)$ we can take $\psi : \perp_1 = F \hookrightarrow \mathcal{C}(\mathbb{1}, J) = \perp_2$, and the resulting $\mathsf{Gl}(\Lambda, \psi)$ coincides (modulo a restriction to monomorphisms) with the *slack orthogonality category* of the *focused orthogonality* on $\mathsf{Gl}(\Lambda, 1)$ determined by F as in [HS03, §5].

10. Full completeness

Let $\mathcal{P}$ be a polycategory with $\mathcal{J}$ as in §6, and let $\Phi : \mathcal{P} \to \mathcal{D}$ be its universal functor to a $*$-autonomous category. That is, Φ is a polycategory functor preserving the tensor and cotensor products in $\mathcal{J}$ (up to isomorphism), and such that any polycategory functor $\mathcal{P} \to \mathcal{Q}$ that preserves $\mathcal{J}$ and where $\mathcal{Q}$ is $*$-autonomous factors essentially uniquely through Φ. This "up to isomorphism" version is categorically "correct", and seems to be necessary since the functor Ξ below does not preserve $\mathcal{J}$ strictly. The existence of such Φ follows from 2-categorical algebra [BKP89].

Theorem 10.1. *The universal functor $\Phi : \mathcal{P} \to \mathcal{D}$ from any $(\mathcal{P}, \mathcal{J})$ to a $*$-autonomous category is fully faithful.*

Proof. For any $\mathcal{D}$-module $\mathcal{U}$, we write $\mathcal{U}[\Phi]$ for the $\mathcal{P}$-module with $\mathcal{U}[\Phi](\Gamma; \Delta) = \mathcal{U}(\Phi(\Gamma); \Phi(\Delta))$. This defines a functor $(-)[\Phi] : \mathsf{Mod}_{\mathcal{D}} \to \mathsf{Mod}_{\mathcal{P}}$; and if $\mathcal{U}$ respects the tensor and cotensor products of $\mathcal{D}$, then $\mathcal{U}[\Phi]$ is a $(\mathcal{P}, \mathcal{J})$-module since Φ preserves $\mathcal{J}$. Now we let Λ be the composite polycategory functor

$$\mathcal{D} \xrightarrow{(\text{よ},\reflectbox{よ},\gamma)} \mathsf{Env}_{\mathcal{D}} = \mathsf{Chu}(\mathsf{Mod}_{\mathcal{D}}, \mathcal{D}) \xrightarrow{(-)[\Phi]} \mathsf{Chu}(\mathsf{Mod}_{\mathcal{P}}, \mathcal{D}[\Phi]),$$

where $(\mathsf{Mod}_{\mathcal{D}}, \mathcal{D}) \to (\mathsf{Mod}_{\mathcal{P}}, \mathcal{D}[\Phi])$ is induced by Φ. Thus on objects we have

$$\Lambda(R) = (L(R), K(R), \lambda_R) = (\mathcal{D}[\,;R][\Phi], \mathcal{D}[R;\,][\Phi], \gamma_{\Phi R}). \tag{10.2}$$

Note that Λ lands in $\mathsf{Chu}(\mathsf{Mod}_{(\mathcal{P},\mathcal{J})}, \mathcal{D}[\Phi])$. Now Φ also induces a map $\phi : \mathcal{P} \to \mathcal{D}[\Phi]$ in $\mathsf{Mod}_{(\mathcal{P},\mathcal{J})}$. Since $\mathcal{D}$ is $*$-autonomous and $\mathsf{Mod}_{(\mathcal{P},\mathcal{J})}$ is representable and closed with pullbacks, the double gluing category $\mathsf{Gl}(\Lambda, \phi)$ is $*$-autonomous.

We now define a functor $\Xi : \mathcal{P} \to \mathsf{Gl}(\Lambda, \phi)$. By the definition of $\mathsf{Gl}(\Lambda, \phi)$ and the universal property of comma objects, to define Ξ it suffices to give the following diagram in poly double categories (most of which are vertically discrete):

$$\begin{tikzcd}
\mathcal{P} \arrow[r, "(\text{よ},\reflectbox{よ},\gamma)"] \arrow[d, "\Phi"'] & \mathsf{Env}_{(\mathcal{P},\mathcal{J})} \arrow[r, equal] \arrow[d, phantom, "\Downarrow"] & \mathsf{Chu}(\mathsf{Mod}_{(\mathcal{P},\mathcal{J})}, \mathcal{P}) \arrow[d, "\phi"] \\
\mathcal{D} \arrow[rr, "\Lambda"'] & & \mathbb{C}\mathsf{hu}(\mathsf{Mod}_{(\mathcal{P},\mathcal{J})}, \mathcal{D}[\Phi])
\end{tikzcd}$$

The necessary 2-cell has components $\text{よ}_A \to \text{よ}_{\Phi A}[\Phi]$ and ${}_A\reflectbox{よ} \to {}_{\Phi A}\reflectbox{よ}[\Phi]$, which we take to be the action of Φ on hom-sets. Thus $\Xi(A)$ consists of $\Phi A \in \mathcal{D}$,

$(よ_A, {}_A\reflectbox{よ}, \gamma_A) \in \mathsf{Chu}(\mathsf{Mod}_{(\mathcal{P},\mathcal{J})}, \mathcal{P})$, and the maps

$$よ_A = \mathcal{P}[\,;A] \to \mathcal{D}[\Phi][\,;A] = \mathcal{D}[\,;\Phi A][\Phi] = よ_{\Phi A}[\Phi] \tag{10.3}$$

$${}_A\reflectbox{よ} = \mathcal{P}[A;\,] \to \mathcal{D}[\Phi][A;\,] = \mathcal{D}[\Phi A;\,][\Phi] = {}_{\Phi A}\reflectbox{よ}[\Phi] \tag{10.4}$$

We claim that Ξ preserves the tensor and cotensor products in $\mathcal{J}$. For the tensors, we use that Φ and よ preserve them, and also need to calculate the limit

$$\begin{tikzcd}
 & & (よ_A \otimes よ_B) \multimap \mathcal{P} & & \\
 & よ_A \multimap {}_B\reflectbox{よ} \arrow[ur] \arrow[dl] & \bullet \arrow[l, dashed] \arrow[r, dashed] \arrow[d, dashed] & よ_B \multimap {}_A\reflectbox{よ} \arrow[ul] \arrow[dr] & \\
よ_A \multimap K(\Phi B) & & K(\Phi A \otimes \Phi B) \arrow[ll] \arrow[rr] & & よ_B \multimap K(\Phi A).
\end{tikzcd}$$

Using (10.2) and (3.6), this becomes

$$\begin{tikzcd}
 & & \mathcal{P}[A,B;\,] & & \\
 & \mathcal{P}[A,B;\,] \arrow[ur] \arrow[dl] & \bullet \arrow[l, dashed] \arrow[r, dashed] \arrow[d, dashed] & \mathcal{P}[A,B;\,] \arrow[ul] \arrow[dr] & \\
\mathcal{D}[\Phi B;\,][\Phi][A;\,] \arrow[d, "\cong"] & & \mathcal{D}[\Phi A \otimes \Phi B;\,][\Phi] \arrow[ll] \arrow[rr] \arrow[d, "\cong"] & & \mathcal{D}[\Phi A;\,][\Phi][B;\,] \arrow[d, "\cong"] \\
\mathcal{D}[\Phi][A,B;\,] & & \mathcal{D}[\Phi][A,B;\,] \arrow[ll] \arrow[rr] & & \mathcal{D}[\Phi][A,B;\,]
\end{tikzcd}$$

whose limit is $\mathcal{P}[A,B;\,] \cong {}_{A\otimes B}\reflectbox{よ}$. For a unit, we instead consider the pullback

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{P} \arrow[d] \\
\mathcal{D}[\Phi\mathbb{1};\,][\Phi] \arrow[r] & \mathcal{D}[\Phi].
\end{tikzcd}$$

But the bottom map is an isomorphism, hence the pullback is isomorphic to $\mathcal{P}$, which is isomorphic to $\mathcal{P}[\mathbb{1};\,] = {}_{\mathbb{1}}\reflectbox{よ}$. Cotensors are dual.

Now, since the composite $\mathcal{P} \xrightarrow{\Xi} \mathsf{Gl}(\Lambda, \mathcal{P}) \to \mathcal{D}$ is equal to Φ, we can extend Ξ by the universal property of $\mathcal{D}$ to a functor $\Xi' : \mathcal{D} \to \mathsf{Gl}(\Lambda, \mathcal{P})$. By the universal property of $\mathsf{Gl}(\Lambda, \mathcal{P})$, this means we have a diagram

$$\begin{tikzcd}
\mathcal{P} \arrow[r, "(よ{,}\reflectbox{よ}{,}\gamma)"] \arrow[d, "\Phi"'] & \mathsf{Env}_{(\mathcal{P},\mathcal{J})} \arrow[r, equal] & \mathsf{Chu}(\mathsf{Mod}_{(\mathcal{P},\mathcal{J})}, \mathcal{P}) \arrow[d, "\phi"] \\
\mathcal{D} \arrow[urr, dashed, "\Xi'"] \arrow[rr, "\Lambda"'] & \arrow[r, phantom, "\Downarrow\mu"] & \mathbb{C}\mathsf{hu}(\mathsf{Mod}_{(\mathcal{P},\mathcal{J})}, \mathcal{D}[\Phi])
\end{tikzcd}$$

where $\mu\Phi$ equals (10.3)–(10.4). This determines the components of μ on objects ΦA; its naturality on all morphisms in $\mathcal{D}$ between such objects then entails multiple commuting squares like (9.4)–(9.6). Considering only those like (9.6), we conclude that the quadrilateral involving Ξ' in the following diagram commutes:

$$\begin{tikzcd}
\mathcal{P}(A_1,\dots,A_m;B_1,\dots,B_n) \arrow[rr, "\cong"] \arrow[d, "\Phi"'] & & \mathsf{Mod}_{(\mathcal{P},\mathcal{J})}(よ_{A_1},\dots,よ_{A_m},{}_{B_1}\reflectbox{よ},\dots,{}_{B_n}\reflectbox{よ},\dots;\mathcal{P}) \arrow[d] \\
\mathcal{D}(\Phi A_1,\dots,\Phi A_m;\Phi B_1,\dots,\Phi B_n) \arrow[rru, dashed, "\Xi'"] \arrow[rr, dotted, "\cong"'] \arrow[dr] & & \mathsf{Mod}_{(\mathcal{P},\mathcal{J})}(よ_{A_1},\dots,よ_{A_m},{}_{B_1}\reflectbox{よ},\dots,{}_{B_n}\reflectbox{よ};\mathcal{D}[\Phi]) \\
& \mathsf{Mod}_{(\mathcal{P},\mathcal{J})}(\mathcal{D}[\Phi][;A_1],\dots,\mathcal{D}[\Phi][;A_m],\mathcal{D}[\Phi][B_1;],\dots,\mathcal{D}[\Phi][B_n;];\mathcal{D}[\Phi]) \arrow[ur] &
\end{tikzcd}$$

Now the two horizontal arrows are isomorphisms by the Yoneda lemma, Theorem 3.4. Thus, by the 2-out-of-6 property for isomorphisms, the left-hand vertical map

$$\mathcal{P}(A_1,\dots,A_m;B_1,\dots,B_n) \xrightarrow{\Phi} \mathcal{D}(\Phi A_1,\dots,\Phi A_m;\Phi B_1,\dots,\Phi B_n)$$

is also an isomorphism; i.e. Φ is fully faithful. □

Remark 10.5. Most of the proof would work using the traditional $\mathsf{Gl}(\Lambda,1)$ instead of our $\mathsf{Gl}(\Lambda,\phi)$. The first difference is that when showing that Ξ preserves $\mathcal{J}$, omitting $(よ_A \otimes よ_B) \multimap \mathcal{P}$ from the diagram changes the limit to the kernel-pair of $\mathcal{P}[A,B;] \to \mathcal{D}[\Phi][A,B;]$. But we already know Φ is faithful by Theorem 8.2, so this morphism is monic, hence its kernel pair is just its domain.

The second difference is that the squares like (9.6) would carry no information. We could use the squares like (9.4)–(9.5) instead — unless $m=n=0$, in which case there are no such squares! We can ignore this case if $\mathcal{J}$ includes a unit or counit, as it usually does; but it seems preferable to have a proof that works in general.

Corollary 10.6. *Let $\mathcal{C}$ be a closed symmetric monoidal category and $\Phi:\mathcal{C}\to\mathcal{D}$ its universal functor to a $*$-autonomous one. Then Φ is fully faithful.*

Proof. A closed symmetric monoidal functor from $\mathcal{C}$ to a $*$-autonomous category factors essentially uniquely through $(\circledast\mathcal{C},\mathcal{J})$. Thus $\Phi':\circledast\mathcal{C}\to\mathcal{D}$ is the universal functor from $(\circledast\mathcal{C},\mathcal{J})$ to a $*$-autonomous category; now apply Theorem 10.1. □

To emphasize how this proof deals with the internal-homs, we check explicitly that the induced $\Xi:\mathcal{C}\to\mathsf{Gl}(\Lambda,\phi)$ preserves them. We must inspect the limit

$$\begin{tikzcd}
& & (よ_A \otimes {}_B\reflectbox{よ}) \multimap \circledast\mathcal{C} & & \\
& よ_A \multimap よ_B \arrow[ur] \arrow[dl] & \bullet \arrow[l, dashed] \arrow[r, dashed] \arrow[d, dashed] & {}_B\reflectbox{よ} \multimap {}_A\reflectbox{よ} \arrow[ul] \arrow[dr] & \\
よ_A \multimap \mathcal{D}[;B][\Phi] & & \mathcal{D}[;A\multimap B][\Phi] \arrow[ll] \arrow[rr] & & {}_B\reflectbox{よ} \multimap \mathcal{D}[A;][\Phi].
\end{tikzcd}$$

which by (10.2) and (3.6) becomes

$$\begin{tikzcd}[column sep=small]
 & & \ast\mathcal{C}[A;B] & & \\
 & \ast\mathcal{C}[A;B] \arrow[ur] \arrow[dl] & \bullet \arrow[l, dashed] \arrow[r, dashed] \arrow[d, dashed] & \ast\mathcal{C}[A;B] \arrow[ul] \arrow[dr] & \\
\mathcal{D}[;\Phi B][\Phi][A;] \arrow[d, "\cong"] & & \mathcal{D}[;\Phi(A \multimap B)][\Phi] \arrow[ll] \arrow[rr] \arrow[d, "\cong"] & & \mathcal{D}[\Phi A;][\Phi][;B] \arrow[d, "\cong"] \\
\mathcal{D}[\Phi][A;B] & & \mathcal{D}[\Phi][A;B] \arrow[ll] \arrow[rr] & & \mathcal{D}[\Phi][A;B],
\end{tikzcd}$$

whose limit is $\ast\mathcal{C}[A;B] \cong \ast\mathcal{C}[;A \multimap B] = \text{よ}_{A \multimap B}$. We also need to check that

$$\text{よ}_A \otimes {}_B\reflectbox{よ} \to \mathcal{D}[;\Phi A \multimap \Phi B][\Phi] \quad \text{is isomorphic to} \quad {}_{A \multimap B}\reflectbox{よ} \to \mathcal{D}[;\Phi(A \multimap B)][\Phi].$$

But by Theorem 6.2, we have ${}_{A \multimap B}\reflectbox{よ} = {}_{A^* \parr B}\reflectbox{よ} \cong {}_{A^*}\reflectbox{よ} \otimes {}_B\reflectbox{よ} \cong \text{よ}_A \otimes {}_B\reflectbox{よ}$. This is the crucial point: the polycategorical framework supplies a contravariant Yoneda embedding $\reflectbox{よ}$ that maps internal-homs to tensor products.

Remark 10.7. It is claimed in [HS03, Has99] that full completeness for $\ast$-autonomous extensions of closed symmetric monoidal categories can be proven using double gluing into an ordinary presheaf category. However, it seems that this does not work. An ordinary Yoneda embedding has no contravariant version like $\reflectbox{よ}$, so the "dual parts" of the functor Ξ have to be chosen "tautologically"; but then Ξ fails to preserve the internal-homs. In the notation of [Has99, §4.5], $(\mathbb{P}_A \multimap \mathbb{P}_B)_t(X)$ is the set of morphisms $\mathbb{I}X \to \mathbb{I}A \otimes (\mathbb{I}B)^\perp$ in $\mathbb{C}_1$ that factor as $(\mathbb{I}f \otimes g) \circ \mathbb{I}h$ for some $h \in \mathbb{C}_0(X, Y \otimes Z)$, $f \in \mathbb{C}_0(Y, A)$, and $g \in \mathbb{C}_1(\mathbb{I}Z, (\mathbb{I}B)^\perp)$, whereas $(\mathbb{P}_{A \multimap B})_t(X)$ is the set of *all* morphisms in $\mathbb{C}_1(\mathbb{I}X, \mathbb{I}A \otimes (\mathbb{I}B)^\perp)$. There seems no reason why every such morphism should factor in that way.

Theorem 10.1 also specializes to other polycategorical structures. For instance, we have the following result (shown in [BCST96] by cut-elimination).

Corollary 10.8. *The universal functor $\Phi : \mathcal{P} \to \mathcal{D}$ from a linearly distributive category $\mathcal{P}$ to a $\ast$-autonomous one is fully faithful.*

Proof. By [CS97], $\mathcal{P}$ can be regarded as a representable polycategory. Now apply Theorem 10.1 with all tensors and cotensors in $\mathcal{J}$. □

We can also include a family $\mathcal{K}$ of limits and colimits by double gluing with $\mathsf{Mod}_{(\mathcal{P},\mathcal{J},\mathcal{K})}$ instead. The formulas in [HS03, Proposition 31] for products and coproducts in double gluing categories still work, as do similar ones for other limits and colimits, and the functor Ξ preserves them. Of course, as in §7, only nonempty limits and colimits in a multicategory $\mathcal{C}$ induce polycategorical ones in $\ast\mathcal{C}$.

References

[Bar79] Michael Barr. *$\ast$-autonomous categories*, volume 752 of *Lecture Notes in Mathematics*. Springer, 1979. (On p. 1)

[BCST96] R. Blute, R. Cockett, R. Seely, and T. Trimble. Natural deduction and coherence for weakly distributive categories. *J. Pure and Appl. Algebra*, 113:229–296, 1996. (On pp. 2 and 17)

[BKP89] R. Blackwell, G. M. Kelly, and A. J. Power. Two-dimensional monad theory. *J. Pure Appl. Algebra*, 59(1):1–41, 1989. (On p. 14)

[Blu91] Richard Blute. Proof nets and coherence theorems. In *Category Theory and Computer Science*, pages 121–137, Berlin, Heidelberg, 1991. Springer Berlin Heidelberg. (On p. 2)

[Blu93] Richard Blute. Linear logic, coherence and dinaturality. *Theoretical Computer Science*, 115(1):3–41, 1993. (On p. 2)

[BS11] John Baez and Mike Stay. Physics, topology, logic and computation: A Rosetta stone. In *New Structures for Physics*, volume 813 of *Lecture Notes in Physics*, pages 95–174. Springer, 2011. (On p. 1)

[Chu79] Po-Hsaing Chu. Constructing $*$-autonomous categories. In *$*$-autonomous categories*, volume 752 of *Lecture Notes in Mathematics*, chapter Appendix. Springer-Verlag, 1979. (On pp. 1 and 5)

[CKS03] J.R.B. Cockett, J. Koslowski, and R.A.G. Seely. Morphisms and modules for poly-bicategories. *Theory Appl. Categ.*, 11(2):15–74, 2003. (On p. 5)

[CS97] Robin Cockett and Robert Seely. Weakly distributive categories. *Journal of Pure and Applied Algebra*, 114(2):133–173, 1997. (On p. 17)

[Day70] Brian Day. On closed categories of functors. In *Reports of the Midwest Category Seminar, IV*, Lecture Notes in Mathematics, Vol. 137, pages 1–38. Springer, Berlin, 1970. (On pp. 4 and 7)

[Day72] Brian Day. A reflection theorem for closed categories. *J. Pure Appl. Algebra*, 2(1):1–11, 1972. (On p. 7)

[DCH18] Gabriel C. Drummond-Cole and Philip Hackney. Dwyer–Kan homotopy theory for cyclic operads. arXiv:1809.06322, 2018. (On p. 3)

[Has99] M. Hasegawa. Categorical glueing and logical predicates for models of linear logic. Preprint RIMS-1223, Kyoto University, 1999. http://www.kurims.kyoto-u.ac.jp/~hassei/papers/full.pdf. (On pp. 2, 10, and 17)

[HS03] Martin Hyland and Andrea Schalk. Glueing and orthogonality for models of linear logic. *Theoretical Computer Science*, 294(1):183 – 231, 2003. Category Theory and Computer Science. (On pp. 2, 10, 12, 13, 14, and 17)

[Hug12] Dominic J.D. Hughes. Simple free star-autonomous categories and full coherence. *Journal of Pure and Applied Algebra*, 216(11):2386 – 2410, 2012. arXiv:math/0506521. (On p. 2)

[Hyl02] J.M.E. Hyland. Proof theory in the abstract. *Annals of Pure and Applied Logic*, 114:43–78, 2002. (On pp. 1, 2, 3, 4, 6, and 7)

[Isb66] John R. Isbell. Structure of categories. *Bull. Amer. Math. Soc.*, 72:619–655, 1966. (On pp. 1 and 6)

[JCS06] M. Hasegawa J.R.B. Cockett and R.A.G. Seely. Coherence of the double involution on $*$-autonomous categories. *Theory and Applications of Categories*, 17(2):17–29, 2006. (On p. 3)

[JSV96] André Joyal, Ross Street, and Dominic Verity. Traced monoidal categories. *Math. Proc. Cambridge Philos. Soc.*, 119(3):447–468, 1996. (On p. 1)

[Kel82] G. M. Kelly. *Basic concepts of enriched category theory*, volume 64 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, 1982. Also available online in Reprints in Theory and Applications of Categories, No. 10 (2005) pp. 1-136. (On p. 7)

[KL97] G. M. Kelly and Stephen Lack. On property-like structures. *Theory Appl. Categ.*, 3(9):213–250, 1997. (On p. 3)

[KM71] G.M. Kelly and S. Maclane. Coherence in closed categories. *Journal of Pure and Applied Algebra*, 1(1):97 – 140, 1971. (On p. 2)

[Laf88] Yves Lafont. *Logiques, catégories & machines : implantation de langages de programmation guidée par la logique catégorique.* PhD thesis, Paris 7, 1988. (On pp. 2 and 10)

[Pav93] Duško Pavlović. Chu I: cofree equivalences, dualities, and $*$-autonomous categories. *Math. Struct. in Comp. Science*, 11, 1993. (On p. 5)

[Red91] Uday S. Reddy. Acceptors as values. http://www.cs.bham.ac.uk/~udr/, 1991. (On p. 2)

[Red93] Uday S. Reddy. A typed foundation for directional logic programming. In E. Lamma and P. Mello, editors, *Extensions of Logic Programming*, pages 282–318, Berlin, Heidelberg, 1993. Springer Berlin Heidelberg. (On p. 2)

[Sch91] Harold Schellinx. Some syntactical observations on linear logic. *Journal of Logic and Computation*, 1(4):537–559, 1991. (On pp. 2 and 10)

[Shu20] Michael Shulman. The 2-Chu-Dialectica construction and the polycategory of multivariable adjunctions. *Theory Appl. Categ.*, 35(4):89–136, 2020. arXiv:1806.06082. (On pp. 5, 6, and 11)

[Sza75] M.E. Szabo. Polycategories. *Communications in Algebra*, 3(8):663–689, 1975. (On p. 2)

[Tan98] Audrey Tan. *Full completeness for models of Linear Logic*. PhD thesis, Cambridge University, 1998. (On pp. 2 and 10)

[Tri94] Todd Trimble. *Linear Logic, Bimodules, and Full Coherence for Autonomous Categories*. PhD thesis, Rutgers University, 1994. (On p. 2)