APPROXIMATE INJECTIVITY AND SMALLNESS IN METRIC-ENRICHED CATEGORIES

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Abstract. Properties of categories enriched over the category of metric spaces are investigated and applied to a study of constructions known from that category and the category of Banach spaces. We prove e.g. that weighted limits and colimits exist in a metric-enriched category iff ordinary limits and colimits exist and ε-(co)equalizers are given by ε-(co)isometries for all ε.

An object is called approximately injective w.r.t. a morphism $h : A \to A'$ iff morphisms from A into it are arbitrarily close to those morphisms that factorize through h. We investigate classes of objects specified by their approximate injectivity w.r.t. given morphisms. They are called approximate-injectivity classes. And we also study, conversely, classes of morphisms specified by the property that certain objects are approximately injective w.r.t. to them.

For every class of morphisms satisfying a mild smallness condition we prove the corresponding approximate-injectivity class is weakly reflective, and we study the properties of the reflection morphisms. As an application we present a new categorical proof of the essential uniqueness of the Gurarii space.

1. Introduction

Categories enriched over **Met**, the category of metric spaces and nonexpanding maps, play an important role in various realms, e.g., in the study of quantitative algebras [22], in continuous logic [14] and in a related theory of approximate Fraïssé limits [8], [17] and [21]. These papers have led to a general theory of approximate injectivity developed in [24] and applied e.g. to a categorical proof of the existence of the Gurarii space, see [12].

Recall that injectivity of objects w.r.t a morphism $h : A \to A'$ is one of the fundamental categorical concepts in algebra and topology: an object K is called injective if every morphism $f : A \to K$ is equal to $f' \cdot h$ for some $f' : A' \to K$. In **Met**-enriched categories, K is called *approximately injective* if every morphism $f : A \to K$ has arbitrarily small distances from the morphisms of the form $f' \cdot h$, see [24]. We present applications of this concept for which we need ε-pushouts introduced in [24] but previously considered in special cases in [21], [11], or [12]. An ε-pushout of a span of morphisms is a square universal among squares commuting up to ε, see 3.7. Similarly, we work with ε-coequalizers and, in general, ε-colimits introduced in [24].

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We show that they are weighted colimits in the sense of enriched category theory. In fact, we prove that a metric enriched category has weighted limits and colimits iff it has ordinary ones and ε-equalizers exist and are formed by isometries (introduced in [24]) and dually. We also show that isometries lead to an important factorization system on **Met**-enriched categories. Another useful concept is that of an ε-isometry introduced in [17] which form the ε-cancellable closure of isometries. Their role for approximate Fraïssé limits was established in [17] and we transfer it to approximate injectivity.

Our main examples of **Met**-enriched categories are **Met** itself, its full subcategory **CMet** on complete metric spaces, and the category **Ban** of (real or complex) Banach spaces and linear maps of norm ≤ 1.

Approximate injectivity classes, i.e. classes of objects approximately injective to a class $\mathcal{H}$ of morphisms, were studied in [24]. In the present paper, we develop this theory further, e.g. we investigate classes of morphisms consisting, for a given class $\mathcal{X}$ of objects, of precisely those morphisms for which all objects of $\mathcal{X}$ are approximately injective. In case of injectivity, these classes are closed under transfinite composites, and they are stable under pushout and are (left) cancellable. In the approximate injectivity case, pushouts are replaced by ε-pushouts and cancellability by approximate cancellability. We also develop an approximate small-object argument corresponding to the small-object argument in case of injectivity. The latter constructs weak reflections in the full subcategory of injective objects, which is important in homotopy theory where it yields e.g. fibrant replacements. Our approximate small-object argument provides weak reflections for the full subcategory of approximately injective objects. For this purpose, we introduce approximately small objects ensuring the convergence of the small-object argument. We prove that finite-dimensional Banach spaces are approximately small w.r.t. isometries and apply it for of a a new categorical proof of the essential uniqueness of the Gurarii space.

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2. Metric spaces

We denote by **Met** the category of (generalized) metric spaces and nonexpanding maps. A metric is a function from $X \times X$ to $[0, \infty]$ (distance ∞ is allowed, therefore the proper name would be 'generalized metric') satisfying the usual axioms. Given metric spaces C and C', a function $f\colon C \to C'$ is *nonexpanding* iff for all x, $y \in C$ we have

$$d(x, y) \geq d\big(f(x), f(y)\big) .$$

In case that for all x, $y \in C$ we have

$$d(x, y) = d\big(f(x), f(y)\big) ,$$

f is called an *isometry*.

Notation 2.1. For every real number $\varepsilon \geq 0$ we write

$$x \sim_\varepsilon y \quad \text{instead of} \quad d(x,y) \leq \varepsilon\,.$$

We use the letter ε to denote a (variable) real number ≥ 0.

Remark 2.2. (1) The category **Met** is symmetric monoidal closed w.r.t. $A \otimes B$ having the underlying set $A \times B$ and the metric

$$d\big((a,b),(a',b')\big) = d_A(a,a') + d_B(b,b')\,.$$

Here $[A,B]$ is the set of all nonexpanding maps with the metric

$$d(f,g) = \inf_{\alpha \in A} d_B\big(f(a), g(a)\big)\,.$$

(2) Let λ be a regular cardinal. Recall that an object A of a category $\mathcal{K}$ is λ-*presentable* if its hom-functor $\mathcal{K}(A,-) : \mathcal{K} \to \mathbf{Set}$ preserves λ-directed colimits (see [6]). That is, given a λ-directed diagram D with colimit cocone $k_i : K_i \to K$ $(i \in I)$, for every morphism $f : A \to K$

(a) a factorization through k_i exists for some $i \in I$, and
(b) given two factorizations $f', f'' : A \to K_i$ of f, we can find a connecting morphism $k_{i,j} : K_i \to K_j$ of D merging them, i.e., $k_{i,j} f' = k_{i,j} f''$.

A category $\mathcal{K}$ is *locally λ-presentable* if it is cocomplete and has a set of λ-presentable objects whose closure under λ-directed colimits is all of $\mathcal{K}$. Every such category is complete, wellpowered and cowellpowered, see [6].

Examples 2.3. (1) **Met** is locally $\aleph_1$-presentable. Moreover, the forgetful functor $U : \mathbf{Met} \to \mathbf{Set}$ preserves $\aleph_1$-directed colimits (see [23] 4.5(3)). In particular, **Met** is complete and cocomplete. U has a left adjoint sending a set X to the metric space on X where all distinct points have distance ∞. These metric spaces are called *discrete*.

(2) The subcategory **CMet** of all complete spaces is a full reflective subcategory of **Met** closed under $\aleph_1$-directed colimits. Thus it also is locally $\aleph_1$-presentable, see [6], 1.46. **CMet** is also a symmetric monoidal closed category w.r.t. the structure inherited from **Met**.

(3) The category **Ban** of real (or complex) Banach spaces and linear maps of norm at most 1 is locally $\aleph_1$-presentable (see [6] 1.48).

(4) Let **PMet** be the category of (generalized) pseudometric spaces and nonexpansive maps. (The difference is just that for a pseudometric we do not require $d(x,y) > 0$ if $x \neq y$.) Following [23] 4.5(3), **PMet** is locally $\aleph_1$-presentable. Moreover, the forgetful functor from **PMet** to **Set** is topological, see [1], 21.8(1). Indeed, given pseudometric spaces K_i, $i \in I$, and a cocone in **Set**, $f_i \colon K_i \to K$ $(i \in I)$, the following pseudometric d on the set K is the final one making all f_i nonexpanding:

$$(*) \qquad d(x,y) = \inf \sum_{k=1}^{n} r_k\,.$$

The infimum ranges over n-tuples of pairs u_k, $v_k \in K_{i_k}$ (for $i_1, \dots, i_n \in I$) of distance r_k such that $x = f_{i_1}(u_1)$, $y = f_{i_n}(v_n)$ and for all $k = 1, \dots, n-1$ we have $f_{i_k}(v_k) = f_{i_{k+1}}(u_{k+1})$.

The full subcategory **Met** of **PMet** is reflective: the reflection of a pseudometric space C is its metric quotient

$$q_K : K \to K/\cong$$

where $x \cong y$ iff $d(x, y) = 0$.

Lemma 2.4. *Let $(k_{ij} : K_i \to K_j)_{i \le j \in I}$ be a directed diagram in* **Met** *with a colimit $(k_i : K_i \to K)_{i \in I}$. Then*

(1) *k_i are jointly surjective, $K = \bigcup_{i \in I} k_i[K_i]$, and*

(2) *for every pair x, $y \in K_i$*

$$d(k_i(x), k_i(y)) = \inf_{j \ge i} d(k_{ij}(x), k_{ij}(y)).$$

Proof. Let $\bar{k}_i \colon UK_i \to \bar{K}$ $(i \in I)$ be the colimit of $U(k_{ij})$ in **Set**. Denote by $\bar{d}$ the final pseudometric on $\bar{K}$, see 2.3(4). For this pseudometric space the cocone $\bar{k}_i \colon K_i \to \bar{K}$ $(i \in I)$ is a colimit in **PMet**. If $q \colon \bar{K} \to K$ denotes the metric quotient of $\bar{K}$, then the cocone

$$k_i = q \cdot \bar{k}_i \colon K_i \to K$$

is a colimit in **Met**, see 2.3(4).

Property (1) is clear. To verify (2) since our diagram is directed, $(*)$ from 2.3(4) reduces to

$$\bar{d}(\bar{k}_i(x), \bar{k}_i(y)) = \inf_{j \ge i} d(k_{ij}(x), k_{ij}(y)).$$

Thus (2) follows from

$$d(k_i(x), k_i(y)) = \bar{d}(\bar{k}_i(x), \bar{k}_i(y)).$$

□

Remark 2.5. (1) For a λ-directed diagram, condition (2) can be strengthened to

(2') for every subset M of K_i of power less than λ there exists a connecting map $k_{ij} \colon K_i \to K_j$ of D such that for all x, y in Mwe have

$$d\big(k_{ij}(x), k_{ij}(y)\big) = d\big(k_i(x), k_i(y)\big).$$

(2) For every directed diagram of isometries k_{ij}, all k_i are isometries.

(3) Directed colimits in **CMet** are completions of those in **Met**.

(4) Analogously, directed colimits in **Ban** are completions of those in the category **Norm** of normed vector spaces and linear maps of norm ≤ 1. Those colimits are described analogously to 2.4 with (2) replaced by

$$\| k_i(x) \| = \inf_{j \ge i} \| k_{ij}(x) \| .$$

The verification is analogous to the above Lemma: use the category **PNorm** of pseudonormed spaces and linear maps of norm ≤ 1. (The difference is that nonzero vectors can have norm 0.) This category is topological over the category of vector spaces and linear maps, and **Norm** is reflective in **PNorm** with reflections given by cokernel modulo the subspace of all vectors of norm 0.

Lemma 2.6. *For an uncountable regular cardinal λ, a metric space is λ-presentable in* **Met** *iff it has a cardinality less than λ.*

Proof. Every metric space is a λ-directed colimit of its subspaces of cardinality less than λ. If A is λ-presentable, the identity $\mathrm{id}_A : A \to A$ factorizes through one of these subspaces and thus A has cardinality less than λ. Conversely, let A has cardinality less than λ and let $k_i : K_i \to K$ be a λ-directed colimit of metric spaces K_i, $i \in I$, with connecting mappings $k_{ij} : K_i \to K_j$ for $i < j \in I$. Let $f : A \to K$ be a morphism. Since U preserves λ-directed colimits, there is a mapping $f' : A \to K_i$ such that $k_i f' = f$. Given $a, b \in A$ then, following 2.5(1), $d(fa, fb) = d(k_{ij}fa, k_{ij}fb)$ for some $i < j \in I$. Since A has less than λ elements, there is $i < j \in I$ such that $k_{ij}f'$ is nonexpanding. Hence A is λ-presentable.

□

Remark 2.7. (1) The only $\aleph_0$-presentable object in **Met** or **CMet** is the empty space. Indeed, let 2_ε be the two-element space with distance ε between the elements. The chain

$$2_1 \xrightarrow{\mathrm{id}} 2_{\frac{1}{2}} \xrightarrow{\mathrm{id}} 2_{\frac{1}{3}} \cdots$$

has the one-point space 1 as a colimit. Given a nonempty space A, the two distinct constant morphisms $f_1, f_2 : A \to 2_1$ are not identified by any $\mathrm{id} : 2_1 \to 2_{\frac{1}{n}}$. Hence A is not $\aleph_0$-presentable.

Analogously, the only $\aleph_0$-presentable object in **Ban** is the trivial space 0: for every space A use the sequence of spaces A_n obtained from A by dividing the norm by $1/n$, whose colimit is 0.

(2) For λ uncountable, λ-presentable objects in **CMet** are precisely complete metric spaces of density character less than λ, i.e., those having a dense subset of cardinality less that λ (see [23]). Similarly, λ-presentable objects in **Ban** are precisely the Banach spaces of density character less than λ.

3. **Met**-enriched categories

We consider categories enriched over the symmetric monoidal closed category **Met**. Every such category $\mathcal{K}$ has its underlying category $\mathcal{K}_0$ and a metric is given on every hom-set $\mathcal{K}_0(A, B)$ such that composition is nonexpanding.

Remark 3.1. We have to distinguish limits in $\mathcal{K}_0$ from conical limits in $\mathcal{K}$. The latter are those limits in $\mathcal{K}_0$ which have a *collectively isometric* limit cone. This means a limit cone $p_i\colon P \to P_i \ \ (i \in I)$ such that given a parallel pair u, v

$$\begin{tikzcd} K \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & P \arrow[d, "p_i"] \\ & P_i \end{tikzcd}$$

we have $d(u,v) = \sup\limits_{i\in I} d(p_i u, p_i v)$. As we see in Section 4 below these are precisely the weighted limits with the trivial weight. Observe that every object K of $\mathcal{K}$ yields an obvious functor $\mathcal{K}(K,-)\colon \mathcal{K}_0 \to \mathbf{Met}$ and that conical limits are precisely those that each such hom-funktor preserves.

Analogously for conical colimits: this means that the colimit cocone $c_i : C_i \to C$ $(i \in I)$ fulfils for parallel pairs $u, v : C \to K$ that $d(u.v) = \inf\limits_{i\in I} d(uc_i, vc_i)$.

Example 3.2. The categories **Met**, **CMet** and **Ban** have conical limits and colimits, see 4.5.

Definition 3.3 ([24])**.** A morphism $f\colon K \to L$ is called an *isometry* if for every parallel pair $u, v\colon Q \to K$ we have

$$d(u,v) = d(fu, fv)\,.$$

Dually, f is called a *coisometry* if $d(uf, vf) = d(u,v)$ for all $u, v : L \to Q$.

Example 3.4. In **Met** isometries precisely represent inclusions of subspaces.

Coisometries are precisely the morphisms with dense image. Indeed, every such morphism is clearly a coisometry. Conversely, let $f : A \to B$ be a coisometry and decompose it as $A \to \overline{f[A]} \to B$. Form a pushout

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \overline{f[A]} \arrow[d, "v"] \\ \overline{f[A]} \arrow[r, "u"'] & C \end{tikzcd}$$

This pushout replaces every element in $B \setminus \overline{f[A]}$ by a pair of points having distance ∞ (for this, one needs that $\overline{f[A]}$ is closed in B). Then $uf = vf$ but $d(u,v) = \infty$, which is not possible.

Remark 3.5. (1) Every isometry is a monomorphism.

(2) Let $\mathcal{K}_0$ be an ordinary category and enich it trivially over **Met** by putting $d(f,g) = \infty$ iff $f \neq g$. Then every monomorphism in $\mathcal{K}_0$ is an isometry in $\mathcal{K}$. Thus, isometries do not need to be regular monomorphisms.

(3) A composition of two isometries is an isometry.

(4) Isometries are *left cancellable*, i.e., if gf is an isometry, then so is f. In fact, for a parallel pair u, v we have $d(fu, fv) \leq d(u, v)$. On the other hand, $d(u, v) = d(gfu, gfv) \leq d(fu, fv)$.

(5) $f : K \to L$ is an isometry iff $\mathcal{K}(Q, f)$ is an isometry in **Met** for every Q in $\mathcal{K}$.

(6) Since f is a coisometry in $\mathcal{K}$ iff it is an isometry in $\mathcal{K}^{\mathrm{op}}$, we have dual statements for coisometries.

Notation 3.6. (1) For parallel morphisms $f,\ g\colon X \to Y$ we write $f \sim_\varepsilon g$ if their distance in $\mathcal{K}(X, Y)$ is at most ε. (This corresponds well with 2.1)

Analogously, we denote situations with $f \sim_\varepsilon g_2 \cdot g_1$ by triangles as follows

$$\begin{tikzcd} \cdot \arrow[rr, "f"] \arrow[dr, "g_1"'] & \arrow[d, phantom, "\sim_\varepsilon"] & \cdot \\ & \cdot \arrow[ur, "g_2"'] & \end{tikzcd}$$

(2) Note that $f \sim_\varepsilon g$ in **Ban** iff $\| f - g \| \leq \varepsilon$, which means that $\| fx - gx \| \leq \varepsilon$ for all $x \in X$, $\| x \| \leq 1$.

Definition 3.7 (see [24])**.** (1) By an ε*-pushout* of morphisms $f_i\colon A \to B_i$ $(i = 1, 2)$ is meant a universal pair of morphisms $g_i\colon B_i \to C$ with $g_1 f_1 \sim_\varepsilon g_2 f_2$.

$$\begin{tikzcd} & A \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\ B_1 \arrow[dr, "g_1"'] & \sim_\varepsilon & B_2 \arrow[dl, "g_2"] \\ & C & \end{tikzcd}$$

Universality is in the usual (strict) sense: for every other such square $g_1' f_1 \sim_\varepsilon g_2' f_2$ (where $g_i'\colon B_i \to C'$) there exists a unique $h\colon C \to C'$ with $g_1' = h \cdot g_1$ and $g_2' = h \cdot g_2$.

(2) Analogously, an ε*-coequalizer* of a parallel pair $f_1,\ f_2$ is a (strictly) universal morphism c w.r.t. $cf_1 \sim_\varepsilon f_2$.

(3) The dual concepts are ε-pullback of a cospan and ε-equalizer of a parallel pair.

Remark 3.8. Every ε-coequalizer is an epimorphism. This follows from the universal property.

It need not be a regular epimorphism. Denote by 2_ε the two-point metric space with points of distance ε. The $\varepsilon/2$-coequalizer of the two points $p_1, p_2 : 1 \to 2_\varepsilon$ is the identity map from 2_ε to $2_\varepsilon/2$.

Example 3.9. ε-pushouts in **Met** were constructed in [24]. ε-pullbacks in **Met** are easy to construct: given morphisms u, v, form the following square

$$\begin{tikzcd} D \arrow[r, "\bar{u}"] \arrow[d, "\bar{v}"'] & C \arrow[d, "v"] \\ B \arrow[r, "u"'] & A \end{tikzcd}$$

where D is the subspace of $B \times C$ consisting of all pairs (b, c) such that $d(b, c) \leq \varepsilon$ and $\bar{u}$, $\bar{v}$ are the projections.

Definition 3.10. We say that $\mathcal{K}$ *has isometric ε-equalizers* if for every parallel pair an ε-equalizer exists and is formed by an isometry.

Dually: $\mathcal{K}$ has *coisometric ε-coequalizers.*

Example 3.11. **Met** has isometric ε-equalizers and coisometric ε-coequalizers. Also **CMet** and **Ban** have this property. This follows from 4.5 and 4.1 below.

Definition 3.12. A morphism $f\colon K \to L$ is called an *isometry-extremal epimorphism* if every isometry $M \to L$ through which f factorizes is an isomorphism.

Dually, $f\colon K \to L$ is called an *coisometry-extremal monomorphism* if every coisometry $K \to M$ through which f factorizes is an isomorphism.

Remark 3.13. (1) If $\mathcal{K}$ has conical equalizers, every isometry-extremal epimorphism $f : K \to L$ is an epimorphism. Indeed, assume that $uf = vf$ and let e be an equalizer of u and v. Since e is an isometry, it is an isomorphism and thus $u = v$.

Dually, coisometry-extremal monomorphisms are monomorphisms if coequalizers are conical.

(2) If $\mathcal{K}$ has finite conical products and isometric ε-equalizers, then it has ε-pullbacks which, moreover, are jointly isometric. This follows from the classical construction of a pullback of morphisms $f_i : A_i \to B$ for $i = 1, 2$ as an equalizer of the pair $f_1\pi_1, f_2\pi_2 : A_1 \times A_2 \to B$. This applies to ε-pulbacks, too, whenever the product $A_1 \times A_2$ with projections π_i is conical.

Theorem 3.14. *If $\mathcal{K}_0$ is wellpowered and $\mathcal{K}$ has conical limits, then it has the factorization system $(\mathcal{E}, \mathcal{M})$ with $\mathcal{M}$ = isometries and $\mathcal{E}$ = isometry-extremal epimorphisms.*

Proof. (1) We first verify that an intersection of isometries $f_i\colon X_i \to Y \quad (i \in I)$ is an isometry. Indeed, since our category is wellpowered and isometries are monic, an intersection f exists:

$$\begin{tikzcd} X \arrow[r, "\pi_i"] \arrow[d, "f"'] & X_i \arrow[dl, "f_i"] \\ Y & \end{tikzcd} \qquad (i \in I)$$

Given u_1, $u_2 \colon Z \to X$, we have, since limits are conical,

$$d(u_1, u_2) = \sup_{i \in I} d(\pi_i u_1, \pi_i u_2)$$

and since f_i is an isometry,

$$d(\pi_i u_1, \pi_i u_2) = d(f_i \pi_i u_1, f_i \pi_i u_2) = d(f u_1, f u_2)\,.$$

This proves $d(u_1, u_2) = d(fu_1, fu_2)$.

(2) For every morphism $g \colon X \to Y$ let $m : Z \to Y$ be the intersection of all isometries through which g factorizes. Morevover m is an isometry and $g = mh$ for a unique $h \colon X \to Z$. Then h is an isometry-extremal epimorphism. Indeed, suppose $h = m_0 k$ for some isometry $m_0 \colon Z_0 \to Z$. Then mm_0 is also an isometry, and since g factorizes through it ($g = mm_0 k$), we see that m is a subobject of mm_0, hence, m_0 is invertible as required.

(3) The diagonal fill-in property holds. Indeed, given a commutative square as follows

$$\begin{tikzcd}
A \arrow[r, "e"] \arrow[d, "f"'] & B \arrow[d, "g"] \\
X \arrow[r, "m"'] & Y
\end{tikzcd}$$

with m an isometry and e an isometry-extremal epimorphism, form the pullback P of m along g:

$$\begin{tikzcd}
A \arrow[rr, "e"] \arrow[dd, "f"'] \arrow[dr, dotted] & & B \arrow[dd, "g"] \\
& P \arrow[ur, "\bar{m}"] \arrow[dl, "\bar{g}"] & \\
X \arrow[rr, "m"'] & & Y
\end{tikzcd}$$

Then $\bar{m}$ is an isometry: given a parallel pair u_1, $u_2 \colon Z \to P$, then since the pullback is conical, we have

$$d(u_1, u_2) = \sup\{d(\bar{g}u_1, \bar{g}u_2), d(\bar{m}u_1, \bar{m}u_2)\}\,.$$

Since m is an isometry, we conclude

$$d(\bar{g}u_1, \bar{g}u_2) = d(m\bar{g}u_1, m\bar{g}u_2) = d(g\bar{m}u_1, g\bar{m}u_2) \leq d(\bar{m}u_1, \bar{m}u_2)\,.$$

This proves

$$d(u_1, u_2) = d(\bar{m}u_1, \bar{m}u_2)$$

as required.

Since e is an isometry-extremal epimorphism which factorizes through $\bar{m}$ (using the universal property of P), we conclude that $\bar{m}$ is invertible. The desired diagonal is

$$\bar{g} \cdot \bar{m}^{-1} \colon B \to X\,.$$

□

Remark 3.15. (1) Dually, we have the (coisometry, isometry-extremal monomorphism) factorization system, whenever $\mathcal{K}_0$ is cowellpowered and $\mathcal{K}$ has conical colimits.

(2) We have shown that isometry-extremal epimorphisms coincide with isometry-strong epimorphisms, i.e., with those having the diagonal fill-in property w.r.t. isometries. Dually, isometry-extremal monomorphisms coincide with isometry-strong monomorphisms.

Examples 3.16. (1) Isometry-extremal epimorphisms in **Met** are precisely the surjective morphisms. Thus we have (surjective, isometry) factorizations. Coisometry-exremal momomorphisms are precisely the closed isometries, i.e., isometries with a closed image. Since coisometries are precisely the dense morphisms, the second factorization system is (dense, closed isometry).

(2) In **CMet** and **Ban** the two factorization systems coincide: we get the (dense, isometry) factorization system,

Lemma 3.17. *If $\mathcal{K}$ has ε-coequalizers for every ε, then every limit is conical.*

Proof. Let D be a diagram with a limit cone $p_i : L \to D_i$ $(i \in I)$. Given a parallel pair $u, u' : Q \to P$, we are to prove $d(u, u') = \varepsilon$, where ε is the supremum of $d(p_i u, p_i u')$ for $i \in I$. Let $e : P \to R$ be an ε-coequalizer of u and u', which is an epimorphism, see 3.8. Since for every i we have $d(p_i u, p_i u') \le \varepsilon$, there is a factorization $p_i = f_i e$. These factorizations form a cone of the diagram D since e is an epimorphism. Therefore, there exists $f : R \to L$ with $f_i = p_i f$ for all $i \in I$. We conclude $fe = \mathrm{id}$, since for all i we have $p_i f e = p_i$. Therefore e is an isomorphism, proving $d(u, u') = \varepsilon$. □

Theorem 3.18. *Let $\mathcal{K}_0$ be complete, wellpowered and cowellpowered. Then $\mathcal{K}$ has conical limits iff it has ε-coequalizers for every ε. If, moreover $\mathcal{K}_0$ has finite coproducts, then conical limits imply that ε-pushouts exist for all ε.*

Proof. (1) We first prove the last statement.

Given morphisms $f_i : A \to B_i$, consider an arbitrary ε-commutative square

$$\begin{tikzcd} & A \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\ B_1 \arrow[dr, "c_1"'] & \sim_\varepsilon & B_2 \arrow[dl, "c_2"] \\ & C & \end{tikzcd}$$

Factorize $[c_1, c_2]\colon B_1 + B_2 \to C$ as an isometry-extremal epimorphism $\bar{c}\colon B_1 + B_2 \to \bar{C}$ followed by an isometry $m_c\colon \bar{C} \to C$ (see 3.14). Since $\bar{c}$ is an epimorphism (see 3.13(1)), these quotients have a set $\bar{c}^t\colon B_1 + B_2 \to \bar{C}^t$ $(t \in T)$ of representatives.

Denote by $\bar{P}$ the product of all $\bar{C}^t$ with projections $\pi^t\colon \bar{P} \to \bar{C}^t$. The morphism

$$\bar{c} = \langle \bar{c}^t \rangle_{t \in T}\colon B_1 + B_2 \to \bar{P}$$

has a factorization as a isometry-extremal epimorphism $p\colon B_1 + B_2 \to P$ followed by an isometry $m_p\colon P \to \bar{P}$. Let $p_i\colon B_i \to P$ be the components of p, then we prove that the following square

$$\begin{tikzcd}
 & A \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\
B_1 \arrow[dr, "p_1"'] & & B_2 \arrow[dl, "p_2"] \\
 & P &
\end{tikzcd}$$

is an ε-pushout.

(a) We prove $p_1 f_1 \sim_\varepsilon p_2 f_2$, i.e. $d(pv_1 f_1, pv_2 f_2) \le \varepsilon$, where v_i denotes the injections of $B_1 + B_2$. For every $t \in T$ we know that

$$m_c \cdot \bar{c}^t \cdot v_1 \cdot f_1 \sim_\varepsilon m_c \cdot \bar{c}^t \cdot v_2 \cdot f_2$$

which, since m_c is an isometry, implies

$$\bar{c}^t \cdot v_1 \cdot f_1 \sim_\varepsilon \bar{c}^t \cdot v_2 \cdot f_2 \,.$$

The product $\bar{P} = \prod \bar{C}^t$ is conical, thus

$$\begin{tikzcd}
 & & P \arrow[d, "\pi^t"] \\
A \arrow[r, shift left, "v_1 f_1"] \arrow[r, shift right, "v_2 f_2"'] & B_1 + B_2 \arrow[ur, "\bar{c}"] \arrow[r, "\bar{c}^t"] & \bar{C}^t
\end{tikzcd}$$

$d(\bar{c} v_1 f_1, \bar{c} v_2 f_2)$ is the supremum of all $d(\bar{c}^t v_1 f_1, \bar{c}^t v_2 f_2)$, which proves

$$d(\bar{c} v_1 f_1, \bar{c} v_2 f_2) \le \varepsilon \,.$$

Since $\bar{c} = m_p p$ and m_p is an isometry, this proves the desired inequality

$$d(pv_1 f_1, pv_2 f_2) \le \varepsilon \,.$$

(b) The universal property needs only be verified for every c_1^t, c_2^t $(t \in T)$ since $\bar{c}^t$ represent all the above quotients $\bar{c}$ (and the isometries m_c play no role). For every $t \in T$ the pair c_1^t, c_2^t factorizes through p_1, p_2 since

$$\bar{c}^t = \pi^t \cdot \bar{c} = \pi^t \cdot m_p \cdot p$$

which precomposed with v_i yields $c_i^t = (\pi^t \cdot m_p) \cdot p_i$. The factorization is clearly unique.

(2) Conical limits imply ε-coequalizers. This is completely analogous: given morphisms $f_1, f_2 : A \to B$ consider the collection of all quotients $\bar{c}^t \colon B \to \bar{C}^t$ $(t \in T)$ with $\bar{c}^t f_1 \sim_\varepsilon \bar{c}^t f_2$. Factorize $\bar{c} = \langle \bar{c}^t \rangle_{t \in T}$ as $m_P p$ as above, then p is the ε-coequalizer of f_1, f_2.

(3) ε-coequalizers imply conical limits by the previous lemma.

□

Corollary 3.19. *If isometries are stable under pushouts, then they are stable under ε-pushout for every $\varepsilon > 0$.*

Proof. The result follows from 3.5(4) and the fact that pushouts factorize through ε-pushouts. □

Lemma 3.20. *Let $\mathcal{K}_0$ be complete, wellpowered and cowellpowered, and let ε-equalizers and ε-coequalizers exist for all $\varepsilon > 0$. Then the following conditions are equivalent:*

1. *Every isometry-extremal epimorphism is a coisometry, and*
2. *Every ε-equalizer is an isometry.*

Proof. (1)⇒(2). Let $e : E \to K$ be an ε-equalizer of $u, v : K \to L$. Factorize it as $e = gf$ where $f : E \to A$ is an isometry-extremal epimorphism and $g : A \to K$ is an isometry, see 3.14. Then $ug \sim_\varepsilon vg$ and thus there exist $t : A \to E$ such that $g = et = gft$, hence $tf = \mathrm{id}_E$. Since limits are conical by Lemma 3.17,f is an epimorphism (see 3.13(1)), thus, it is an isomorphism. Hence e is an isometry.

(2)⇒(1). Let $f : K \to L$ be isometry-extremal epimorphism and $u, v : L \to A$ such that $fu \sim_\varepsilon fv$. Let $e : E \to K$ be an ε-equalizer of u and v. There exist $g : K \to E$ such that $eg = f$. Since e is an isometry and f is isometry-extremal, e is an isomorphism. Hence $u \sim_\varepsilon v$. □

Definition 3.21 ([17])**.** A morphism $f : A \to B$ is called an *ε-isometry* provided that there are isometries $g : B \to C$ and $h : A \to C$ such that $gf \sim_\varepsilon h$.

Lemma 3.22. *A morphism $f : A \to B$ is an ε-isometry iff in the following ε-pushout*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\mathrm{id}_A"'] & B \arrow[d, "g"] \\ A \arrow[r, "\overline{f}"'] & P \end{tikzcd}$$

$\overline{f}$ is an isometry.

Proof. If $\overline{f}$ is an isometry then (since g is also an isometry) f is an ε-isometry. Conversely, assume that f is an ε-isometry. Then there are isometries $g : B \to C$ and $h : A \to C$ such that $gf \sim_\varepsilon h$. Let $p : P \to C$ be the induced morphism. Since $p\overline{f} = h$, $\overline{f}$ is an isometry (see 3.5 (4)). □

4. Weighted limits and colimits

The appropriate concept of a (co)limit in a **Met**-enriched category $\mathcal{K}$ is that of a weighted (co)limit. We are going to prove that ordinary (co)limits and (co)isometric ε-(co)equalizers imply that weighted limits and colimits exist.

Let us recall the concept of a limit of a diagram D in $\mathcal{K}$ weighted by a weight W. The diagram scheme is a small **Met**-enriched category $\mathcal{D}$, we denote by $[\mathcal{D}, \mathbf{Met}]$ the enriched category of all enriched functors from it to **Met**. That is, objects are all (ordinary) functors $W : \mathcal{D} \to \mathbf{Met}$ such that the induced map from $\mathcal{D}(X, X')$ to $\mathbf{Met}(WX, WX')$ is nonexpanding for all pairs X, X' in $\mathcal{D}$. Morphisms are ordinary natural transformations. (They are automaticly enriched due to the fact that for the unit 1 of the monoidal category **Met** the hom-functor is faithful.)

A *weight* is an enriched functor $W : \mathcal{D} \to \mathbf{Met}$. Given a diagram in $\mathcal{K}$, i.e. an enriched functor $D : \mathcal{D} \to \mathcal{K}$, its *limit weighted by* W is an object R, usually written $R = \lim_{\mathcal{W}} D$, such that for all objects K of $\mathcal{K}$ we have isomorphisms

$$\mathcal{K}(K, R) \cong [\mathcal{D}, \mathbf{Met}], (W, \mathcal{K}(K, D-)).$$

natural in K.

Examples 4.1. (1) Conical limits (see Remark 3.1) are precisely the weighted limits with the trivial weight constant to the one-point space. Here $\mathcal{D}$ is enriched by putting all distances ∞ or 0.

(2) For the one-morphism category $\mathcal{D}$ a diagram is a choice of an object L of $\mathcal{K}$, and a weight is a choice of a metric space M. The weighted limit is called the *cotensor of* L *and* M and is denoted by $[M, L]$. It is characterized by the natural isomorphisms

$$\mathcal{K}(K, [M, L]) \cong \mathbf{Met}(M, \mathcal{K}(K, L)).$$

(3) Let $\mathcal{D}$ consist of a parallel pair with distance ∞. Then a diagram is precisely a parallel pair of morphisms $u_1, u_2 : K \to L$ in $\mathcal{K}$. To express their ε-equalizer, choose the weight W given by the two points $p_1, p_2 : 1 \to 2_\varepsilon$, where 2_ε is the space of two points with distance ε. A weighted limit is then precisely an ε-equalizer of u_1, u_2 which, moreover, is an isometry.

(4) Analogously for ε-pullbacks. Let $\mathcal{D}$ be a cospan and let D correspond to a cospan $u_i : K_i \to L$ $(i = 1, 2)$ in $\mathcal{K}$. Here W assigns to the cospan the parallel pair $p_1, p_2 : 1 \to 2_\varepsilon$ above. Then a weighted limit is precisely an ε-pullback $v_i : P \to K_i$ which, moreover, is collectively isometric, see Remark 3.1.

Consequently, this weighted limit is nothing else than an ε-equalizer of

$$\langle u_1\pi_1, u_2\pi_2 \rangle : K_1 \times K_2 \to L$$

formed by an isometry.

Remark 4.2. The dual concept is that of a *weighted colimit*. Here the weight is an enriched functor $W : \mathcal{D}^{op} \to \mathbf{Met}$. A weighted colimit of a diagram D is an object

R, usually written $R = \mathrm{colim}_{\mathcal{W}} D$, with isomorphisms

$$\mathcal{K}(R, K) \cong [\mathcal{D}^{op}, \mathbf{Met}]((W, \mathcal{K}(D-, K))$$

natural in K. The dual construct to cotensor is *tensor* of an object L and a metric space M. It is denoted by $M \otimes L$ and is an object with natural isomorphisms

$$\mathcal{K}(M \otimes L, K) \cong \mathbf{Met}(M, \mathcal{K}(L, K)).$$

Theorem 4.3 ([9], 6.6.16). *A* **Met***-enriched category* $\mathcal{K}$ *has weighted limits and colimits iff it has tensors and cotensors and* $\mathcal{K}_0$ *is complete and cocomplete.*

Assumption 4.4. *Throughout the rest of the paper we assume that a* **Met***-enriched category* $\mathcal{K}$ *is given with the underlying category* $\mathcal{K}_0$ *complete and cocomplete. And that* $\mathcal{K}$ *has for all* $\varepsilon > 0$ *isometric* ε*-equalizers and coisometric* ε*-coequalizers.*

Examples 4.5. All of our running examples are categories with weighted limits and colimits (and thus satisfy Assumptions). Inded, they are all complete and cocomplete. Moreover:

(1) **Met** has tensors $M \otimes L$ and cotensors $[M, L]$ as described in Remark 2.2(1).

(2) **CMet** has tensors $M \otimes L = M^* \otimes L$ where M^* is the completion of M because

$$\mathbf{CMet}(M^* \otimes L, K) \cong \mathbf{CMet}(M^*, \mathbf{CMet}(L, K)) \cong \mathbf{Met}(M, \mathbf{CMet}(L, K)).$$

Cotensors are $[M, L] = \mathbf{Met}(M^*, L)$, i.e., cotensors $[M^*, L]$ in **Met**. Indeed,

$$\mathbf{CMet}(K, [M^*, L]) \cong \mathbf{Met}(M^*, \mathbf{Met}(K, L)) \cong \mathbf{Met}(M, \mathbf{Met}(K, L)).$$

(3) In order to describe tensors and cotensors in **Ban**, denote by U the unit-ball functor to **Met**. Since U preserves limits and $\aleph_1$-directed colimits, and **Ban** and **Met** are locally presentable, U has a left adjoint $F : \mathbf{Met} \to \mathbf{Ban}$ (see [6] 1.66). The Banach spaces FM are also called Lipschitz-free (see [10]). Moreover, this adjunction is **Met**-enriched.

The category **Ban** is symmetric monoidal closed where $\otimes$ is the projective tensor product, and internal hom is the space $\{K, L\}$ consisting of *all* bounded linear mappings (not necessarily of norm at most 1) from K to L (see [9] 6.1.9h). Observe that $U\{K, L\} = \mathbf{Ban}(K, L)$.

Ban has tensors $M \otimes L = FM \otimes L$ and cotensors $[M, L] = \{FM, L\}$. Indeed,

$$\mathbf{Ban}(FM \otimes L, K) \cong \mathbf{Ban}(FM, \{L, K\}) \cong \mathbf{Met}(M, U\{FM, L\})$$

which is $\mathbf{Met}(M, \mathbf{Ban}(L, K))$. Similarly,

$$\mathbf{Ban}(K, \{FM, L\}) \cong \mathbf{Ban}(K \otimes FM, L) \cong \mathbf{Ban}(FM, \{K, L\})$$

which is $\mathbf{Met}(M, \mathbf{Ban}(K, L))$.

Theorem 4.6. *All weighted limits and colimits exist in* $\mathcal{K}$.

Proof. We only need to prove that cotensors exist. The dual result then yields tensors and we can apply the above theorem.

Consider a metric space M. We present cotensors for all objects L.

(1) If M is the one-element space, then cotensors are trivial: $[M, L] = L$.

(2) If M has just two elements of distance ε, then $\mathbf{Met}(M, \mathcal{K}(K, L))$ is given by a parallel pair $u_1, u_2 : K \to L$ in $\mathcal{K}$ of distance ε. Form the following ε-pullback

$$\begin{tikzcd} P_\varepsilon \arrow[r, "v_\varepsilon"] \arrow[d, "u_\varepsilon"'] & L \arrow[d, "\mathrm{id}_L"] \\ L \arrow[r, "\mathrm{id}_L"'] & L \end{tikzcd}$$

Then $u_\varepsilon, v_\varepsilon$ is a universal parallel pair of distance ε. Indeed, these morphisms are collectively isometric by 3.17 and 3.13(2). Hence P is the desired cotensor $[M, L]$ due to the natural isomorphism

$$\mathcal{K}(K, P) \cong \mathbf{Met}(M, \mathcal{K}(K, L)).$$

(3) Let M be an arbitrary space. Form all subspaces $M_{x,y}$ on at most two elements $x, y \in M$. Then M is a canonical colimit of the diagram of all these subspaces and all inclusions $M_{x,x} \hookrightarrow M_{x,y}$, where the colimit maps are also the inclusions. By the above items we get a diagram of all cotensors $[M_{x,y}, L]$ and all the derived morphisms $[M_{x,y}, L] \to [M_{x,x}, L]$. The desired cotensor is then a limit of this diagram:

$$[M, L] = \lim_{(x,y) \in M \times M} [M_{x,y}, L].$$

Indeed, for every object K the desired natural isomorphism is obtained as the following composite

$$\mathcal{K}(K, [M, L]) = \mathcal{K}(K, \lim[M_{x,y}, L]) \cong \lim \mathcal{K}(K, [M_{x,y}, L]) \cong \lim \mathbf{Met}(M_{x,y}, \mathcal{K}(K, L)),$$

where we used the fact that limits are conical (3.18), thus preserved by $\mathcal{K}(K, -)$, and then we applied the universal property of cotensors. From this we get

$$\mathcal{K}(K, [M, L]) \cong \mathbf{Met}(\operatorname{colim} M_{x,y}, \mathcal{K}(K, L)) = \mathbf{Met}(M, \mathcal{K}(K, L))).$$

□

5. Approximate Injectivity and Approximate Smallness

Throughout this section we assume that a class $\mathcal{H}$ of morphisms in a category satisfying 4.6 (and thus having weighted limits and colimits) is given. We now come to the central cocepts of our paper: objects that are approximately injective or approximately small with respect to $\mathcal{H}$.

In an ordinary category an object X is called injective w.r.t. $\mathcal{H}$ if for every member $h\colon A \to A'$ of $\mathcal{H}$ all morphisms $f\colon A \to X$ factorize through h (i.e. $f = f'h$ for some $f' : A' \to X$). We denote by

$$\operatorname{Inj}\mathcal{H}$$

the class of all such objects. Classes of objects of this form are called *injectivity classes*.

Definition 5.1 (See [24]). (1) An object X is *approximately injective* w.r.t. $\mathcal{H}$ if for every member $h\colon A \to A'$ of $\mathcal{H}$ and every morphism $f\colon A \to X$ there exist ε-factorizations $f'\colon A' \to X$ through h for all $\varepsilon > 0$:

$$\begin{tikzcd} A \arrow[rr, "h"] \arrow[dr, "f"'] & \arrow[d, phantom, "\sim_\varepsilon"] & A' \arrow[dl, "f'"] \\ & X & \end{tikzcd}$$

(2) The class of all these objects is denoted by

$$\operatorname{Inj}_{\mathrm{ap}}\mathcal{H}\,.$$

An *approximate injectivity class* is a class of objects of the form $\operatorname{Inj}_{\mathrm{ap}}\mathcal{H}$.

Remark 5.2. An object X is approximately injective w.r.t. $h : A \to A'$ iff the induced morphism $\mathcal{K}(h, X) : \mathcal{K}(A', X) \to \mathcal{K}(A, X)$ is a coisometry. This means that X is $\mathcal{E}$-injective in the sense of [20] where $\mathcal{E}$ is the class of coisometries in **Met**.

Example 5.3. If $\mathcal{K}$ is locally λ-presentable in the enriched sense (see Section 6), every approximate injectivity class is an injectivity class, as proved in [24], but not conversely: see Example 5.25.

Remark 5.4. Using the terminology from [2], we say that a morphism h is an *injectivity consequence* of $\mathcal{H}$ if $\operatorname{Inj}\mathcal{H} \subseteq \operatorname{Inj}\{h\}$. That is, objects injective w.r.t. $\mathcal{H}$ are also injective w.r.t. h. Here are two 'approximate' versions:

Definition 5.5. (1) A morphism h is an *approximate-injectivity consequence* of $\mathcal{H}$ if $\operatorname{Inj}_{\mathrm{ap}}\mathcal{H} \subseteq \operatorname{Inj}_{\mathrm{ap}}\{h\}$, i.e., objects approximately injective w.r.t. $\mathcal{H}$ are also approximately injective w.r.t. h.

(2) h is called a *strict approximate-injectivity consequence* of $\mathcal{H}$ if $\operatorname{Inj}_{\mathrm{ap}}\mathcal{H} \subseteq \operatorname{Inj}\{h\}$, i.e., objects approximately injective w.r.t. $\mathcal{H}$ are injective w.r.t h.

Example 5.6. Given an ε-pushout

$$\begin{tikzcd} \cdot \arrow[r, "h"] \arrow[d, "k"'] & \cdot \arrow[d, "k'"] \\ \cdot \arrow[r, "h'"'] & \cdot \end{tikzcd}$$

then h' is a strict approximate-injectivity consequence of h.

In fact, if X is approximately injective w.r.t. h, then every morphism u as in the following diagram

$$\begin{tikzcd} {} \arrow[r, "h"] \arrow[d, "p"'] & {} \arrow[d, "p'"] \arrow[dd, bend left=60, "v"] \\ {} \arrow[r, "h'"] \arrow[dr, "u"'] & {} \arrow[d, dotted, "w"] \\ & X \end{tikzcd}$$

factorizes through h'. To see this, use the approximate injectivity of X to choose a morphism v with $up \sim_\varepsilon vh$. Then the universal property yields w with $u = wh'$.

Lemma 5.7. *Let $h : A \to B$ and $h' : A \to B'$ be morphisms. Given triangles*

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "h'"'] \arrow[dr, phantom, "\sim_{1/n}"] & B \\ B' \arrow[ur, "f_n"'] & \end{tikzcd}$$

$(n = 1, 2, 3, \dots)$ *then h' is an approximate-injectivity consequence of h.*

Proof. Let X be approximately injective w.r.t. h and let $u\colon A \to B$ be given:

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "h'"] \arrow[dd, bend right=60, "u"'] & B \arrow[ddl, "v"] \\ B' \arrow[ur, "f_n"] \arrow[d, dotted] & \\ X & \end{tikzcd}$$

For every $\varepsilon > 0$ choose n with $\frac{2}{n} < \varepsilon$. We have a morphism v with $u \sim_{1/n} v \cdot h$ which implies $u \sim_\varepsilon (v \cdot f_n) \cdot h'$, as desired. This follows from $u \sim_{1/n} v \cdot h$ and $v \cdot h \sim_{1/n} v \cdot f_n \cdot h'$, thus, $u \sim_{2/n} (v \cdot f_n) \cdot h'$. $\square$

Remark 5.8. (1) Recall the concept of a *transfinite composite* of morphisms:

(a) Given an ordinal α, an α-chain (of objects K_i, $i < \alpha$, and morphisms $k_{ji}\colon K_j \to K_i$ for $j \leq i$) is called *smooth* if for every limit ordinal $i < \alpha$ we have

$$K_i = \operatorname*{colim}_{j<i} K_j$$

with the colimit cocone $(k_{ji})_{j<i}$.

(b) Given a smooth $(\alpha+1)$-chain, the morphism $k_{0\alpha}\colon K_0 \to K_\alpha$ is called the *α-composite* of the morphisms $(k_{i,i+1})_{i<\alpha}$.

Thus $\alpha = 2$ yields the usual concept of a composite of two morphisms $K_0 \to K_1 \to K_2$ up to isomorphism of the codomain K_2.

The case $\alpha = 0$ means that every isomorphism is a 0-composite (of the empty set of morphisms).

(c) Transfinite composites are α-composites where α is an arbitrary ordinal. If α has cofinality at least λ, then these chains is λ-directed. We then speak about λ-directed transfinite composites.

(2) The closure of $\mathcal{H}$ under pushout and transfinite composite is denoted by $\mathrm{cell}(\mathcal{H})$ (the *cellular morphisms for $\mathcal{H}$*). A well-known fact is that every object injective w.r.t. $\mathcal{H}$ is also injective w.r.t. cellular morphisms for $\mathcal{H}$ (see, e.g., [4]).

In ordinary categories an object is called λ-small if its hom-functor preserves λ-directed transfinite composites of cellular morphisms. We are using the appropriate enriched variant:

Definition 5.9. An object A is called *λ-small* w.r.t. $\mathcal{H}$ if $\mathcal{K}(A,-) : \mathcal{K} \to \mathbf{Met}$ preserves λ-directed transfinite composites of cellular morphisms for $\mathcal{H}$.

Explicitly: for every λ-directed smooth chain $k_{ij}\colon K_i \to K_j$ where $k_{i,i+1} \in \mathrm{cell}(\mathcal{H})$ $(i < \mu)$ with a colimit $k_i\colon K_i \to K_\mu$ $(i < \mu)$

(a) given a morphism $f\colon A \to K_\mu$, there is $i < \mu$ such that f factorizes through k_i: we have $f = k_i f'$,

(b) if $k_{i\mu} f' \sim_\varepsilon k_{i\mu} f''$ for $f', f'' : A \to K_i$ and $i < \mu$, then $k_{ij} f' \sim_\varepsilon k_{ij} f''$ for some $i \le j < \mu$.

Let us introduce the corresponding approximate concepts:

Definition 5.10. (1) The closure of $\mathcal{H}$ under transfinite composites and ε-pushouts (for all $\varepsilon > 0$) is denoted by $\mathrm{cell}_{\mathrm{op}}(\mathcal{H})$. Its members are called *approximately cellular* morphisms for $\mathcal{H}$.

(2) Let λ be a regular cardinal. An object A is called *approximately λ-small* w.r.t. $\mathcal{H}$ if for every λ-directed transfinite composite $(k_{ij} : K_i \to K_j)_{i \le j \le \mu}$ of morphisms approximately cellular for $\mathcal{H}$ and every $\varepsilon > 0$ we have that

(a) every morphism $f : A \to K_\mu$ has an ε-factorization $f' : A \to K_i$ through $k_{i\mu}$ for some $i < \mu$, i.e. $f \sim_\varepsilon k_{i\mu} f'$,

(b) if $k_{i\mu} f' \sim_\varepsilon k_{i\mu} f''$ for $f', f'' : A \to K_i$ and $i < \mu$, then $k_{ij} f' \sim_\varepsilon k_{ij} f''$ for some $i \le j < \mu$.

Remark 5.11. (1) Every λ-small object w.r.t. $\mathcal{H}$ is approximately λ-small w.r.t. $\mathcal{H}$.

(2) If $\mathrm{cell}_{\mathrm{ap}}(\mathcal{H})$ consists of isometries, then condition (b) can be omitted in 5.10.

(3) If $\mathcal{K}$ is enriched over **CMet** then K is approximately λ-small w.r.t. $\mathcal{H}$ iff $\mathcal{K}(K,-) : \mathcal{K} \to \mathbf{CMet}$ preserves λ-directed transfinite composites of approximately cellular morphisms for $\mathcal{H}$.

From 5.8(2) and 5.6 we conclude the following fact.

Lemma 5.12. *All approximately cellular morphisms are strict injectivity consequences of the given class $\mathcal{H}$, i.e.,*

$$h \in \mathrm{cell}_{\mathrm{ap}}(\mathcal{H}) \quad \textit{implies} \quad \mathcal{H} \models h.$$

Lemma 5.13. *A coproduct of less than λ approximately λ-small objects w.r.t. $\mathcal{H}$ is approximately λ-small w.r.t. $\mathcal{H}$.*

Proof. Let $u_t : A_t \to \coprod_{t\in I} A_t$ with A_t approximately λ-small where $|I| < \lambda$, and let a morphism $f : \coprod A_t \to K_\mu$ be given. Since $|I| < \lambda$ and $(k_{ij} : K_i \to K_j)_{i\le j\le \mu}$ is λ-directed, there exist $i < \mu$ such that for every $t \in I$ we have $f'_t : A_t \to K_i$ with $k_{i\mu} f'_t \sim_\varepsilon f u_t$. Let $f' : \coprod A_t \to K_i$ be the induced morphism, i.e., $f' u_t = f'_t$ for every $t \in I$. Since coproducts are conical, we have $k_{i\mu} f' \sim_\varepsilon f$. Given morphisms $f', f'' \colon \coprod A_t \to K_i$ with $k_{i\mu} f'' \sim_\varepsilon k_{i\mu} f'$, we conclude $k_{i\mu} f' u_t \sim_\varepsilon k_{i\mu} f'' u_t$ for each $t \in I$. There is $j \ge i$ such that $k_{ij} f' u_t \sim_\varepsilon k_{ij} f'' u_t$ for each $t \in I$. Hence $k_{ik} f' \sim_\varepsilon k_{ij} f''$. $\square$

Theorem 5.14. *For every uncountable regular cardinal λ and every $\varepsilon > 0$ all approximately λ-small objects w.r.t. $\mathcal{H}$ are stable under ε-pushouts.*

Proof. Let

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "v"'] & B \arrow[d, "\bar{v}"] \\ C \arrow[r, "\bar{u}"'] & D \arrow[ul, phantom, "\ulcorner \varepsilon_0", very near start] \end{tikzcd}$$

be an ε_0-pushout with A, B and C approximately λ-small w.r.t. $\mathcal{H}$. Given a morphism $f\colon D \to K_\mu$ where $(k_{ij} : K_i \to K_j)_{i\le j\le\mu}$ is given as in 5.10(2) and given $\varepsilon > 0$ we verify Conditions (a) and (b) for f.

Condition (a). Since C satisfies (a) we can find for $f \cdot \bar{u}\colon C \to K_\mu$ a $\frac{1}{n}$-factorization g_1 through some $k_{i\mu}$ ($i < \mu$). Analogously for $f \cdot \bar{v}$ a $\frac{1}{n}$-factorization g_2 (through the same $k_{i\mu}$):

(1)
$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "v"'] & B \arrow[d, "\bar{v}"] \arrow[dddrr, "g_2"] & & \\ C \arrow[r, "\bar{u}"'] \arrow[ddrrr, "g_1"'] & D \arrow[ul, phantom, "\ulcorner \varepsilon_0", very near start] \arrow[dr, "f"] & & \\ & & K_\mu \arrow[ul, phantom, "\sim_{1/n}"] \arrow[ur, phantom, "\sim_{1/n}"] & \\ & & & K_i \arrow[ul, "k_{i\mu}"'] \end{tikzcd}$$

We conclude for $\bar{\varepsilon} = \varepsilon_0 + \frac{2}{n}$ that

$$k_{i\mu} \cdot g_1 \cdot v \sim_{\bar{\varepsilon}} k_{i\mu} \cdot g_2 \cdot u$$

due to

$$k_{i\mu} \cdot g_1 \cdot v \sim_{1/n} k_{i\mu} \cdot \bar{u} \cdot v \sim_{\varepsilon_0} k_{i\mu} \cdot \bar{v} \cdot u \sim_{1/n} k_{i\mu} \cdot g_2 \cdot u\,.$$

Since A is approximately λ-small, Condition (b) in 5.10(2) implies that there exists $i \leq j < \mu$ such that k_{ij} $\bar{\varepsilon}$-factorizes $g_1 \cdot u$ and $g_2 \cdot v$:

(2)
$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "v"'] & B \arrow[r, "g_1"] & K_i \arrow[dd, "k_{ij}"] \\
C \arrow[d, "g_2"'] & \sim_{\varepsilon_0 + \frac{2}{n}} & \\
K_i \arrow[rr, "k_{ij}"'] & & K_j
\end{tikzcd}$$

This ordinal j can be chosen independent of n ($\in \mathbb{N}$) since λ is uncountable and regular. Let us form an $\bar{\varepsilon}$-pushout D_n of u and v:

(3)
$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "v"'] & B \arrow[d, "\bar{v}_n"] \arrow[ddr, dashed, "k_{ij} \cdot g_1"] & \\
C \arrow[r, "\bar{u}_n"] \arrow[drr, dashed, "k_{ij} \cdot g_2"'] & {}^{\ulcorner \bar{\varepsilon}} D_n \arrow[dr, "h_n"] & \\
 & & K_j
\end{tikzcd}$$

We obtain the unique factorization $h_n \colon D_n \to K_j$ as indicated above.

The objects D_n, $n \geq 1$ form an ω-chain, where $d_{nm} \colon D_n \to D_m$ is the obvious morphism we get from $\varepsilon_0 + \frac{1}{n} \geq \varepsilon_0 + \frac{1}{m}$ (for all $n \geq m$). The object D is a colimit of this chain with colimit morphisms $d_n \colon D_n \to D$ uniquely determined by

(4) $$\bar{u} = d_n \cdot \bar{u}_n \quad \text{and} \quad \bar{v} = d_n \cdot \bar{v}_n \quad \text{for all} \quad n < \omega\,.$$

And the morphisms h_n form a cocone: if $n \leq m$ then $h_n = h_m \cdot d_{nm}$ because

$$h_n \cdot \bar{u}_n = k_{ij} \cdot g_2 = h_m \cdot \bar{u}_m = (h_m \cdot d_{nm}) \cdot \bar{u}_n$$

and analogously for v_n. Thus we get a unique

(5) $$f' \colon D \to K_j \quad \text{with} \quad f' \cdot d_n = h_n \quad (n \geq 1)\,.$$

This is the desired ε-factorization of f: choose n with $\frac{1}{n} < \varepsilon$, then $f \sim_\varepsilon k_{j\mu} \cdot f'$ because $f \sim_{1/n} k_{j\mu} \cdot f'$. Indeed, to verify this last statement, we only need proving that $f \cdot \bar{u} \sim_{1/n} k_{j\mu} \cdot f' \cdot \bar{u}$ (and analogously for $\bar{v}$):

$$\begin{aligned} f \cdot \bar{u} &\sim_{1/n} k_{i\mu} \cdot g_1 && \text{see (1)} \\ &= k_{j\mu} \cdot k_{ij} \cdot g_1 \\ &= k_{j\mu} \cdot h_n \cdot \bar{v}_n && \text{see(3)} \\ &= k_{j\mu} \cdot f' \cdot d_n \cdot \bar{v}_n && \text{see (5)} \\ &= k_{j\mu} \cdot f' \cdot \bar{u} && \text{see (4).} \end{aligned}$$

Condition (b). Assume that $k_{i\mu} \cdot f' \sim_\varepsilon k_{i\mu} \cdot f''$ for $f', f'' : A \to K_i$ and $i < \mu$. Since C is approximately λ-small and

$$f \cdot \bar{u} \sim_{1/n} k_{j\mu} \cdot f' \cdot \bar{u}\,, \quad f \cdot \bar{u} \sim_{1/n} c_j \cdot f'' \cdot \bar{u}$$

implies $k_{j\mu} \cdot f' \cdot \bar{u} \sim_{\frac{2}{n}} k_{j\mu} \cdot f'' \cdot \bar{u}$ (and analogously for $\bar{v}$) there exists $\bar{j} < \lambda$ such that $k_{j\bar{j}} \cdot f' \cdot \bar{u} \sim_{\frac{2}{n}} k_{j\bar{j}} \cdot f'' \cdot \bar{u}$ (and analogously for $\bar{v}$). Choose n with $\varepsilon \geq \frac{2}{n}$ to get $k_{j\bar{j}} \cdot f' \cdot \bar{u} \sim_\varepsilon k_{j\bar{j}} \cdot f'' \cdot \bar{u}$ as well as $k_{j\bar{j}} \cdot f' \cdot \bar{v} \sim_\varepsilon k_{j\bar{j}} \cdot f'' \cdot \bar{v}$. By the dual of Example 4.1(4), our ε_0-pushout is a weighted colimit, thus,$k_{j\bar{j}}$ ε-merges f' and f'', as desired. □

Remark 5.15. For $\lambda = \aleph_0$ 5.14 does not hold, see 5.20.

Definition 5.16. For a regular cardinal λ, an object A in $\mathcal{K}$ will be called *approximately λ-generated* if it is approximately λ-small w.r.t. the class of all isometries.

Remark 5.17. This concept was inroduced in [24] 6.4 where, however, instead of λ-directed composites, general λ-directed colimits were used. For $\lambda = \aleph_0$ this makes no difference provided that isometries are closed under directed colimits in the category $\mathcal{K}^{\to}$ of morphisms of $\mathcal{K}$ (see [6] 1.7). This is true in **Met**, **CMet** and **Ban**.

Examples 5.18. (1) Finite metric spaces are approximately $\aleph_0$-generated in **Met** (use 2.5(2)).

(2) Finite discrete spaces are approximately $\aleph_0$-generated in **CMet**. Just use the fact that a directed colimit in **CMet** is a completion of that in **Met**. Conversely:

Proposition 5.19. *Every finite space which is approximately $\aleph_0$-generated in* **CMet** *is discrete.*

Proof. If a finite space A is not discrete, we prove that it is not approximately $\aleph_0$-generated. Let d be the minimum distance of distinct elements of A, then $d < \infty$. Choose x, $y \in A$ of distance d. Consider the following subspace K of the real line

$$K = \{0, d\} \cup \left\{ d + \frac{1}{n}; n = 1, 2, 3, \dots \right\}.$$

This is a colimit of the ω-chain of its subspaces $K_r = \{0\} \cup \left\{ d + \frac{1}{n}; n = 1, 2, \dots, r \right\}$ for $r = 1, 2, 3 \dots$ in **CMet**. The function $f \colon A \to K$ mapping x to 0 and all other points to d is clearly nonexpanding. But for no r is there a nonexpanding factorization g of f through the colimit map $K_r \hookrightarrow K$ (since $d(x, y) = d$, and given $g \colon A \to K_r$

with $g(x) = 0$, there is no point of $K_r \backslash \{d\}$ of distance at most d from 0, thus, no possibility for $g(y)$). Therefore, A is not approximately $\aleph_0$-generated. $\square$

Remark 5.20. For $\lambda = \aleph_0$ 5.14 does not hold. Consider the ε-pushout below in **CMet**:

$$\begin{tikzcd} 1 \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & 1 \arrow[d] \\ 1 \arrow[r] & \ulcorner\varepsilon D \end{tikzcd}$$

Then D is a space of two points of distance ε. Although 1 is approximately $\aleph_0$-generated, D is not.

Remark 5.21. Every finite space A is 'almost' approximately $\aleph_0$-generated in **CMet** in the following sense: let $(k_{ij} : K_i \to K_j)_{i \leq j \leq \mu}$ a transfinite composite of isometries in **CMet** and $f : A \to K_\mu$ an isometry. Then for every $\varepsilon > 0$ there is an ε-isometry in the sense of 7.4 (1) (not necessarily nonexpanding) $f' : A \to K_i$ with $i < \mu$ such that $k_i f' \sim_\varepsilon f$. In fact, there is $i < \mu$ such that

$$\varepsilon \geq |d(k_i f' x, k_i f' y) - d(fx, fy)| = |d(f' x, f' y) - d(x, y)|.$$

Lemma 5.22. *Let λ be an uncountable regular cardinal and $\mathcal{K}$ be* **CMet***-enriched. Then every approximately λ-small object A is λ-small.*

Proof. Following **??**, we have to check condition (a) of 5.8(3). So, let $(k_{ij} : K_i \to K_j)_{i \leq j \leq \mu}$ a λ-directed transfinite composite of morphisms from $\mathcal{H}$ and let $f : A \to K_\mu$ be a morphism. For every $n > 0$, there is $f'_n : A \to K_{i_n}$ such that $k_{i_n \mu} f'_n \sim_{\frac{1}{n}} f$. Hence $k_{i_n \mu} f'_n \sim_{\frac{1}{n} + \frac{1}{m}} k_{i_m \mu} f'_m$. Following 2.5(1), there is $i > i_n$ for every n such that $k_{i_n i} f'_n \sim_{\frac{1}{n} + \frac{1}{m}} k_{i_m i} f'_m$ (for all n, m). Thus $k_{i_n i} f'_n$ form a Cauchy sequence in $\mathcal{K}(A, K_i)$. Let $f' : A \to K_i$ be its limit. Then $k_{i\mu} f'$ is the limit of $k_{i\mu} k_{i_n i} f'_n$ and thus $k_{i\mu} f' = f$. $\square$

Definition 5.23. (1) A morphism $m \colon K \to L$ is called *approximately split* if for every $\varepsilon > 0$ there is a morphism $e \colon L \to K$ with $e \cdot m \sim_\varepsilon \mathrm{id}_K$.

(2) A full subcategory $\mathcal{L}$ of $\mathcal{K}$ is *closed under approximately split morphisms* if for every approximately split morphism $K \to L$ with L in $\mathcal{L}$ one also has K in $\mathcal{L}$.

Remark 5.24. Every approximate injectivity class is closed under approximately split morphisms (see [24] 5.5 and 3.4(3)).

Example 5.25. The class **CMet** of complete metric spaces is an injectivity class in **Met** which fails to be an approximate injectivity class.

(1) To verify that **CMet** is not an approximate injectivity class in **Met**, observe that in the real line the inclusion $m : (0, 1] \to [0, 1]$ is approximately split. Indeed, $g_n \colon [0, 1] \to (0, 1]$ defined by $g_n(x) = x + \frac{1-x}{n}$ is nonexpanding and fulfils $g_n \cdot m \sim_{1/n} \mathrm{id}$.

Indeed $d\big(x, g_n(x)\big) \leq \frac{1}{n}$ because $d^2\big(x, g_n(x)\big) = \frac{(x-1)^2}{n^2} < \frac{1}{n^2}$. Thus, **CMet** is not closed under approximately split morphisms .

(2) **CMet** is an injectivity class: For every countable metric space A denote by $i_A \colon A \to A^*$ its Cauchy completion. Then **CMet** is the injectivity class of the (essentially small) set of all i_A's. Indeed, a Cauchy sequence in a space X has a limit in X iff for the embedding $f \colon A \hookrightarrow X$ of the corresponding (countable) subspace a non-expanding extension to A^* exists.

Example 5.26. (1) Finite spaces are closed in **Met** under approximately split morphisms. In fact, assume that an infinite metric space B has an approximately split morphism $u : B \to A$ to a finite space A. There are morphisms $s_n : A \to B$ such that $s_n u \sim_{\frac{1}{n}} \mathrm{id}_B$. Choose pairwise distinct elements $b_i \in B$, $i < \omega$ with $u(b_i) = u(b_j)$ for every $i, j < \omega$. Then given $i \neq j$ for every n we conclude $b_i \sim_{2/n} b_j$:

$$b_i \sim_{\frac{1}{n}} s_n u(b_i) = s_n u(b_j) \sim_{\frac{1}{n}} b_j.$$

Hence $b_i = b_j$, a contradiction.

(2) Separable metric spaces are also closed in **Met** under approximately split morphisms. Let $u : B \to A$ be approximately split and B separable. Let X be a countable dense subset of A. For s_n as in (1) we have a countable dense subset $\bigcup\limits_{n<\omega} s_n(X)$ of A.

Proposition 5.27. *Let λ be an uncountable regular cardinal. Then approximately λ-small objects w.r.t. $\mathcal{H}$ are closed under approximately split morphisms.*

Proof. Let $u : A \to B$ be an approximately split morphism with B approximately λ-small w.r.t. $\mathcal{H}$. Consider $(k_{ij} : K_i \to K_j)_{i \leq j \leq \mu}$ from 5.10. Choose a morphism $f : A \to K_\mu$. For every $\varepsilon > 0$ and arbitrary morphisms $t : B \to A$ and $f' : B \to K_i$ there is a morphism u such that $tu \sim_{\frac{\varepsilon}{2}} \mathrm{id}_A$ and $k_{i\mu} f' \sim_{\frac{\varepsilon}{2}} ft$. Hence $k_{i\mu} f' u \sim_{\frac{\varepsilon}{2}} ftu \sim_{\frac{\varepsilon}{2}} f$ and thus $k_{i\mu} f' u \sim_\varepsilon f$. This verifies (a) in Definition 5.10. To verify (b), assume that $k_{i\mu} f' \sim_\varepsilon k_{i\mu} f''$ for $f', f'' : A \to K_i$. For $n > 0$, there are morphisms $t_n : B \to A$ such that $t_n u \sim_{\frac{1}{n}} \mathrm{id}_A$ and we choose $j_n \geq i$ such that $k_{ij_n} f' t_n \sim_\varepsilon k_{ij_n} f'' t_n$. Hence

$$k_{ij_n} f' \sim_{\frac{1}{n}} k_{ij_n} f' t_n u \sim_\varepsilon k_{ij_n} f'' t_n u \sim_{\frac{1}{n}} k_{ij_n} f''$$

and thus $k_{ij_n} f' \sim_{\varepsilon + \frac{2}{n}} k_{ij_n} f''$. Since λ is uncountable, we can choose $j > j_n$ for every n, and we get $k_{ij} f' \sim_\varepsilon k_{ij} f''$. $\square$

Theorem 5.28. *Assume that the domains of morphisms from $\mathcal{H}$ are approximately small w.r.t. $\mathcal{H}$. Then* $\mathrm{Inj}_{\mathrm{ap}}\, \mathcal{H}$ *is weakly reflective in $\mathcal{K}$ with approximately cellular weak reflections.*

Proof. For every object K we construct a weak reflection in $\mathrm{Inj}_{\mathrm{ap}}\, \mathcal{H}$ in two transfinite steps: we first define a morphism $t_K \colon K \to K^*$ as a composite of a certain transfinite chain formed by ε-pushouts of morphisms of $\mathcal{H}$. Then we iterate this step from K to K^* transfinitely in order to obtain the desired weak reflection $\widehat{K}$.

Construction (1): Consider the set $\mathcal{X}_K$ of all triples (h, u, n) where u and h form a span

$$\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "u"'] & B \\
K &
\end{tikzcd}$$

with $h \in \mathcal{H}$ and $n > 0$ is a natural number. Put $\mu_K = \operatorname{card} \mathcal{X}_K$. We will index members of $\mathcal{X}_{\mathcal{K}}$ by ordinals $i < \mu_{\mathcal{K}}$. That is, $\mathcal{X}_K = \{(h_i, u_i, n_i); i < n_K\}$. We define a chain $k_{ij} : K_i \to K_j$, $i \leq j \leq \mu_K$ and morphisms $b_i \colon B_i \to K_{i+1}$ by the following transfinite recursion:

First step: $K_0 = K$.

Isolated step: K_{i+1}, r_i and $k_{i,i+1}$ are given by an $\frac{1}{n_i}$-pushout as follows

$$\begin{tikzcd}
A_i \arrow[r, "h_i"] \arrow[d, "u_i"'] & B_i \arrow[dd, "r_i"] \\
K \arrow[d, "k_{0i}"'] & \\
K_i \arrow[r, "k_{i,i+1}"'] \arrow[ur, phantom, "\ulcorner 1/n_i" near end] & K_{i+1}
\end{tikzcd}$$

We put $k_{j,i+1} = k_{i,i+1} \cdot k_{j,i}$. for all $j < i$.

Limit step: K_i is the colimit of the chain $(K_j)_{j<i}$ and $k_{ji} : K_j \to K_i$ are given as the colimit cocone for all $j < i$.

The object K_{μ_K} will be denoted by K^* and the morphism $k_{0\mu_k} : K \to K^*$ by t_K. For every i we obtain the following $\frac{1}{n_i}$-commutative square

$$\begin{tikzcd}
A_i \arrow[r, "h_i"] \arrow[dd, "u_i"'] & B_i \arrow[d, "r_i"] \\
 & K_{i+1} \arrow[d, "k_{i+1,\mu_K}"] \\
K \arrow[r, "t_K"'] & K^*
\end{tikzcd}$$

Construction (2): We are ready to construct a weak reflection $\widehat{K}$ of K. By assumption on $\mathcal{H}$ there is a regular cardinal λ such that the domains of all morphisms in $\mathcal{H}$ are λ-small. We define a λ-chain by iterating Construction (1) λ-times. That is, we define $m_{ij} : M_i \to M_j$, $i \leq j \leq \lambda$ by the following transfinite recursion:

First step: $M_0 = K$.

Isolated step: $M_{i+1} = M_i^*$ and $m_{i,i+1} = t_{M_i}$.

Limit step: M_i is the colimit of the chain $(M_j)_{j<i}$ with the colimit cocone $(m_{ji})_{j<i}$. In particular, M_λ is a colimit of $(M_j)_{j<\lambda}$. We put

$$\widehat{K} = M_\lambda \quad \text{and} \quad r_K = m_{0,\lambda} \colon K \to \widehat{K}\,.$$

We will show that this is a desired weak reflection of K.

(2a) $\widehat{K}$ is approximately injective. Indeed, given $h : A \to B$ in $\mathcal{H}$, a morphism $u : A \to \widehat{K}$ and $n > 0$, we provide an $\frac{1}{n}$-factorization of u through h as follows. Since the object A is λ-approximately small and $\widehat{K}$ is a directed colimit of M_i, $i < \lambda$, there is a $\frac{1}{2n}$-factorization

$$\begin{tikzcd} & M_i \arrow[d, "m_{i\lambda}"] \\ A \arrow[ur, "u'"] \arrow[r, "u"'] \arrow[ur, phantom, shift right=3ex, "\sim_{1/2n}"] & \widehat{K} \end{tikzcd}$$

of u through $m_{i\lambda}$ for some $i < \lambda$ (i.e. the triangle is $\frac{1}{2n}$-commuting). Since the triple $(h, u', 2n)$ lies in the set $\mathcal{X}_{M_i}$, we obtain a $\frac{1}{2n}$-commutative square as follows

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "u'"'] \arrow[dr, phantom, "\sim_{1/2n}"] & B \arrow[d, "v"] \\ M_i \arrow[r, "m_{i,i+1}"'] & M_{i+1} \end{tikzcd}$$

We have

$$u \sim_{\frac{1}{2n}} m_{i\lambda} u' = m_{i+1,\lambda} m_{i,i+1} u' \sim_{\frac{1}{2n}} m_{i+1,\lambda} v g\,.$$

Hence $u \sim_{\frac{1}{n}} m_{i+1,\lambda} v g$, which proves that $\widehat{K}$ is approximately injective w.r.t. $\mathcal{H}$.

(2b) Since $m_{0\lambda} : K \to \widehat{K}$ is approximately cellular, it is the desired weak reflection (following Lemma 5.12). □

Remark 5.29. (1) In the proof of 5.28 we only need 5.10(a).

(2) The proof mimics the construction of a weak reflection to $\operatorname{Inj} \mathcal{H}$ (see, e.g., [4]) – just pushouts are replaced by ε-pushouts. By using pushouts, we would get $\bar{r}_K = \bar{m}_{0\lambda} : K \to \bar{K}$ with $\bar{K}$ approximately injective w.r.t. $\mathcal{H}$ but not a weak reflection to $\operatorname{Inj}_{\mathrm{ap}} \mathcal{H}$. There is a morphism $t : \widehat{K} \to \bar{K}$ such that $t r_K = \bar{r}_K$ given by comparison morphisms from ε-pushouts to pushouts.

(3) Observe that if $\mathcal{H}'$ is the class of all the weak reflections $r_K : K \to \widehat{K}$, $K \in \mathcal{K}$, then $\operatorname{Inj}_{\mathrm{ap}} \mathcal{H}$ is the corresponding injectivity class:

$$\operatorname{Inj}_{\mathrm{ap}} \mathcal{H} = \operatorname{Inj} \mathcal{H}'.$$

Definition 5.30. The *approximately cancellable closure* of $\mathcal{H}$ consists of all morphisms $g : A \to B'$ such that for every $n > 0$ there are morphisms $h : A \to B$ in $\mathcal{H}$ and $f_n : B' \to B$ such that $f_n g \sim_{\frac{1}{n}} h$.

Proposition 5.31. *Suppose that domains of all morphisms of $\mathcal{H}$ are approximately small w.r.t. $\mathcal{H}$. Then a morphism in $\mathcal{K}$ is an approximate injectivity consequence of $\mathcal{H}$ iff it belongs to the approximately cancellable closure of* $\mathrm{cell}_{\mathrm{ap}}(\mathcal{H})$.

Proof. Following 5.12 and 5.7, every morphism from the approximately cancellative closure of $\mathrm{cell}_{\mathrm{ap}}(\mathcal{H})$ is an approximate injectivity consequence of $\mathcal{H}$. Conversely, assume that a morphism $h \colon K \to L$ is an approximate injectivity consequence of $\mathcal{H}$. Let $r_K : K \to \widehat{K}$ be the approximately cellular weak reflection of 5.28. Since $\widehat{K}$ is approximately injective, for each $n > 0$ there is $f_n : L \to \widehat{K}$ such that $f_n \cdot g \sim_{\frac{1}{n}} r_K$. Hence g belongs to the approximately cancellative closure of $\mathcal{H}$. $\square$

Remark 5.32. (1) Recall $\mathrm{cell}(\mathcal{H})$, the class of cellular morphisms from 5.8 (2). Let $\overline{\mathrm{cell}}(\mathcal{H})$ be its cancellative closure given by morphisms f with $hf \in \mathrm{cell}\,\mathcal{H}$ for some h. Then

$$\mathrm{cell}_{\mathrm{ap}}(\mathcal{H}) \subseteq \overline{\mathrm{cell}}(\mathcal{H}).$$

Indeed, given an approximately cellular morphism $f : K \to L$, we take the corresponding cellular morphism $g : K \to M$ where we replace every ε-pushout by a pushout. Then $g = hf$ for some $h : L \to M$.

(2) If the domains of morphisms from $\mathcal{H}$ are λ-small for some λ, then $\overline{\mathrm{cell}}(\mathcal{H})$ is the class of injectivity consequences of $\mathcal{H}$. Thus every approximately cellular morphism w.r.t. $\mathcal{H}$ is an injectivity consequence of $\mathcal{H}$.

(3) Under the assumption of 5.31, every strict approximate injectivity consequence of $\mathcal{H}$ (see Definition 5.5) belongs to $\overline{\mathrm{cell}}_{\mathrm{ap}}(\mathcal{H})$.

Indeed, if $h : A \to B$ is a strict approximate injectivity consequence of $\mathcal{H}$ and $r : A \to A^*$ is the approximately cellular weak reflection from 5.28, then $r = gf$ for some g.

On the other hand, one cannot expect that all morphisms from $\mathcal{H}$ are strict approximate injectivity consequences of $\mathcal{H}$.

6. Approximate injectivity in locally presentable categories

We have recalled the concept of a locally presentable (ordinary) category in Remark 2.2(2). Since we work with **Met**-enriched categories, we need the enriched concept. Let λ be a regular cardinal. An object A is called λ*-presentable in the enriched sense* if its hom-functor $\mathcal{K}(A, -) : \mathcal{K} \to \mathbf{Met}$ preserves λ-directed colimits. This means that (b) in Remark 2.2(2)is strengthened to

(b') given $f', f'' : A \to K_i$ then

$$d(k_i f', k_i f'') = \inf_{j \geq i} d(k_{i,j} f', k_{i,j} f'').$$

Following [16], a **Met**-enriched $\mathcal{K}$ is called *localy λ-presentable in the enriched sense* if it has weighted colimits and a set of λ-presentable objects in the enriched sense whose closure under λ-directed colimits is all of $\mathcal{K}$. Every such category has weighted limits.

For λ uncountable, a **Met**-enriched category $\mathcal{K}$ with weighted colimits is locally λ-presentable in the enriched sense iff $\mathcal{K}_0$ is locally λ-presentable and every λ-presentable object in $\mathcal{K}_0$ is λ-presentable in the enriched sense.

Proposition 6.1. *Let $\mathcal{K}_0$ be locally λ-presentable. Then $\mathcal{K}$ is locally λ-presentable in the enriched sense iff λ-presentable objects in $\mathcal{K}_0$ are closed under ε-coequalizers for every $\varepsilon > 0$.*

Proof. (1) Assume that λ-presentable objects in $\mathcal{K}_0$ are closed under ε-coequalizers for every $\varepsilon > 0$. Let A be λ-presentable in $\mathcal{K}_0$. Given a λ-directed colimit $k_i : K_i \to K$ $(i \in I)$, it is our task to prove that for every parallel pair of morphisms $f', f'' : A \to K_i$ with $d(k_i f', k_i f'') \leq \varepsilon$ there exists a connecting morphism $k_{i,j} : K_i \to K_j$ with $d(k_{ij} f', k_{i,j} f'') \leq \varepsilon$. Since $\mathcal{K}_0$ is locally λ-presentable, K_i is a λ-directed colimit $l_\alpha : L_\alpha \to K_i$, $\alpha \in J$ of λ-presentable objects L_α. Since A is λ-presentable in $\mathcal{K}_0$, there are $\alpha \in J$ and $g', g'' : A \to L_\alpha$ with $l_\alpha g' = f'$ and $l_\alpha g'' = f''$. Let $c : L_\alpha \to C$ be an ε- coequalizer of g' and g''. We have, due to

$$d(k_i l_\alpha g,' k_i l_\alpha g'') = d(k_i f', k_i f'') \leq \varepsilon,$$

a factorization $k_i l_\alpha = hc$ for some $h : C \to K$. Since C is an ε-coequalizer of λ-presentable objects in $\mathcal{K}_0$, it is λ-presentable in $\mathcal{K}_0$. Hence there is $i \leq j' \in I$ and $h' : C \to K_{j'}$ such that $h = k_{j'} h'$.

We have

$$k_{j'} k_{ij'} l_\alpha = k_i l_\alpha = hc = k_{j'} h' c$$

Since C is λ-presentable in $\mathcal{K}_0$, there is $j' \leq j \in I$ such that

$$k_{j'j} k_{ij'} l_\alpha = k_{j'j} h' c.$$

Hence $k_{ij} l_\alpha = k_{j'j} k_{ij'} l_\alpha$ factorizes through c. Thus $d(k_{ij} l_\alpha g', k_{ij} l_\alpha g'') \leq \varepsilon$. Therefore $d(k_{ij} f', k_{ij} f'') \leq \varepsilon$.

(2) Conversely, assume that λ-presentable objects in the enriched sense. Let $c : B \to C$ be an ε-coequalizer of $u, v : A \to B$ with A and B λ-presentable, then we are to prove that C is λ-presentable. Let $k_i : K_i \to K$ $(i \in I)$ be a λ-directed colimit and $f : C \to K$. There are $i \in I$ and $f' : B \to K_i$ such that $fc = k_i f'$. Hence $k_i f' u \sim_\varepsilon k_i f' v$ and, since B is λ-presentable in $\mathcal{K}$, there is $i \leq j \in I$ such that $k_{ij} f' u \sim_\varepsilon k_{ij} f' v$. Thus $k_i f'$ factorizes through c, $k_{ij} f' = f'' c$. Hence f factorizes through k_j: we have $f = k_j f$" because c is epic by 3.8 and

$$cf = k_i f' = k_j f\text{"} c.$$

The essential uniqueness of this factorization is evident since c is a coisometry by Assumptions 4.4.

∎

Corollary 6.2. *Let $\mathcal{K}_0$ be locally λ-presentable with λ-presentable objects closed under coisometric quotients. Then $\mathcal{K}$ is locally λ-presentable in the enriched sense.*

Proof. This follows from 6.1 and 4.4. □

Example 6.3. The categories **Met**, **CMet** and **Ban** are locally $\aleph_1$-presentable in the enriched sense.

Indeed, we have seen in Section 2 that the underlying ordinary categories are locally $\aleph_1$-presentable, thus, we only need to observe that for every regular cardinal λ the two concepts of λ-presentable object coincide. This follows from the above corollary: recall from Section 2 that in **Met** if λ is uncountable, then presentability is just the cardinality smaller than λ, such spaces are clearly closed under coisometric quotients. And the only $\aleph_0$-presentable space is the empty one. The situation with the other two categories is analogous.

The following theorem improves [24] 5.8. Recall the set-theory axiom

Weak Vopěnka's Principle

stating that no full embedding $\mathbf{Ord}^{\mathrm{op}} \hookrightarrow \mathbf{Gra}$ exists. Here **Ord** is the ordered category of all ordinals and **Gra** the category of graphs. This principle implies that measurable cardinals exist, thus, its negation is consistent with set theory by [6], A.7. Weak Vopěnka's Principle is also consistent with set theory by [6], 6.21 and A.12. Moreover, Weak Vopěnka's Principle follows from *Vopěnka's Principle* which states that a large discrete category cannot be fully embedded into **Gra**.

Theorem 6.4. *Assume Weak Vopěnka's Principle. If $\mathcal{K}$ is locally λ-presentable in the enriched sense, then a class of objects is an approximate injectivity clas iff it is closed under products and approximately split morphisms.*

For the proof see [24], Theorem 5.8. The stronger assumption that Vopěnka's Principle holds made there was not fully used: by inspecting the proof one sees that all that was needed was the result that a full subcategory of a locally presentable category closed under products and split subobjects in weakly reflective. However, this was proved in [5] under the following assumption:

Semi-Weak Vopěnka's Principle

stating that no class L_i ($i \in \mathbf{Ord}$) of graphs fulfils $\mathbf{Gra}(L_i, L_j) = \emptyset$ iff $i < j$. Recently Wilson proved that this principle is equivalent to Weak Vopěnka's Principle [26].

7. Banach spaces

We now turn to the category **Ban** and apply our results to prove that the Gurarii space is essentially unique.

Remark 7.1. (1) We will show that, in **Ban**, ε-isometries of Definition 3.21 coincide with the usual concept of an ε-isometry of Banach spaces of norm ≤ 1. Recall that a linear mapping $f : A \to B$ between Banach spaces is called an *ε-isometry* if it satisfies

$$(\sharp) \qquad (1-\varepsilon) \parallel x \parallel \leq \parallel fx \parallel \leq (1+\varepsilon) \parallel x \parallel$$

for every $x \in A$ (see [7]). It is evident that it suffices to take x with $\parallel x \parallel = 1$ only.

(2) It should be a folklore that a linear mapping $f : A \to B$ between Banach spaces satisfies $(\sharp)$ iff it satisfies

$$(*) \qquad | \parallel fx \parallel - \parallel x \parallel | \leq \varepsilon$$

for every $x \in A$, $\parallel x \parallel \leq 1$. We provide a short proof for the convenience of the reader.

Let f satisfy $(\sharp)$ and $\parallel x \parallel \leq 1$ such that $\parallel fx \parallel > \parallel x \parallel$. Then

$$\parallel fx \parallel - \parallel x \parallel \leq (1+\varepsilon) \parallel x \parallel - \parallel x \parallel = \varepsilon \parallel x \parallel \leq \varepsilon.$$

If $\parallel fx \parallel \leq \parallel x \parallel$ then $(1-\varepsilon) \parallel x \parallel \leq \parallel fx \parallel$ and thus

$$\parallel x \parallel - \parallel fx \parallel \leq \varepsilon \parallel x \parallel \leq \varepsilon.$$

Therefore f satisfies $(*)$.

Conversely, let f satisfy $(*)$ and $\parallel x \parallel = 1$. The inequality $\parallel fx \parallel \leq (1+\varepsilon) \parallel x \parallel$ is evident for $\parallel fx \parallel \leq \parallel x \parallel$. Assume that $\parallel x \parallel \leq \parallel fx \parallel$. Following $(*)$,

$$\frac{| \parallel fx \parallel - \parallel x \parallel |}{\parallel x \parallel |} = |\frac{\parallel fx \parallel}{\parallel x \parallel} - 1| = | \parallel f(\frac{x}{\parallel x \parallel}) \parallel - \parallel \frac{x}{\parallel x \parallel} \parallel | \leq \varepsilon$$

and thus

$$\parallel fx \parallel \leq (1+\varepsilon) \parallel x \parallel .$$

The other inequality $(1-\varepsilon) \parallel x \parallel \leq \parallel fx \parallel$ is evident for $\parallel x \parallel \leq \parallel fx \parallel$. Assume that $\parallel x \parallel \geq \parallel fx \parallel$. Following $(*)$,

$$\parallel x \parallel - \parallel fx \parallel \leq \varepsilon \parallel x \parallel$$

and thus

$$(1-\varepsilon) \parallel x \parallel \leq \parallel fx \parallel .$$

Hence f satisfies $(\sharp)$.

Lemma 7.2. *A morphism $f : A \to B$ in* **Ban** *is an ε-isometry iff it satisfies $(\sharp)$.*

Proof. Following [11] 2.1, if a morphism $f : A \to B$ in **Ban** satisfies $(\sharp)$ then f has the property from 3.22 and thus it is an ε-isometry. Conversely, let f be an ε-isometry. Then

$$\parallel x \parallel - \parallel fx \parallel = | \parallel x \parallel - \parallel fx \parallel | = | \parallel hx \parallel - \parallel gfx \parallel | \leq \parallel hx - gfx \parallel \leq \varepsilon$$

Hence f satisfies $(\sharp)$. □

Remark 7.3. If a morphism $f : A \to B$ in **Ban** is an ε-isometry between finite-dimensional Banach spaces, then the Banach space C from 3.21 can be taken finite-dimensional. The reason is that the ε-pushout from 3.22 is finite-dimensional (see [11] 2.1).

Remark 7.4. (1) A mapping $f : A \to B$ between metric spaces (not necessarily non-expanding) is called an ε-*isometry* in [15]) if

$$(\sharp\sharp) \qquad |d(x,y) - d(fx,fy)| \leq \varepsilon$$

for every $x, y \in A$. Let us show that a morphism $f : A \to B$ in **Met** satisfies ($\sharp\sharp$) iff it is a $\frac{\varepsilon}{2}$-isometry in our sense.

Let $f : A \to B$ be a $\frac{\varepsilon}{2}$-isometry, i.e., there are isometries $g : B \to C$ and $h : A \to C$ such that $gf \sim_{\frac{\varepsilon}{2}} h$. Then, for $x, y \in A$, we have

$$d(x,y) = d(hx,hy) \leq d(hx,gfx) + d(gfx,gfy) + d(gfy,hy) = d(fx,fy) + \varepsilon.$$

Hence f satisfies ($\sharp\sharp$).

Conversely, assume that f satisfies ($\sharp\sharp$). Consider the $\frac{\varepsilon}{2}$-pushout used in 3.22. Since u is an isometry (see 3.19), we have

$$d(\overline{f}x,\overline{f}y) \geq d(\overline{f}x,ufx)+d(ufx,ufy)+d(ufy,\overline{f}y) = \varepsilon+d(fx,fy)+\varepsilon = d(fx,fy)+2\varepsilon.$$

Since f satisfies ($\sharp\sharp$), $d(fx,fy) + 2\varepsilon \geq d(x,y)$. Following the construction of ε-pushouts in **Met** from [24] 2.3, we have $d(\overline{f}x,\overline{f}y) \geq d(x,y)$. Hence $\overline{f}$ is an isometry.

Remark 7.5. The class of all isometries in **Ban** is stable under pushouts (see [7] A.19) and hence under ε-pushouts (see 3.19). Since isometries in **Ban** are closed under transfinite composites, they are both cellularly closed and approximately cellularly closed.

In [24] 6.5(1) it is claimed that finite-dimensional spaces are approximately $\aleph_0$-generated. We now prove this claim using Definition 5.16. This is, as remarked in 5.17, weaker than that of [24] but the proof works for the stronger variant as well.

Proposition 7.6. *Finite-dimensional Banach spaces are approximately $\aleph_0$-generated.*

Proof. Let A be a finite-dimensional Banach space and $(k_{ij} \colon K_i \to K_j)_{i \leq j \leq \mu}$ be a transfinite composite of isometries.

I. Let $f \colon A \to K_\mu$ be an isometry.

(a) For every $\varepsilon > 0$ we prove that there exists an ε-isometry $f' \colon A \to K_i$ (not necessarily of norm ≤ 1, i.e. satisfying $\sharp$ in 7.1) for some $i < \mu$ with $f \sim_\varepsilon k_{i\mu} f'$.

Let $e_1, \dots, e_n$ be a basis of A. Since any two norms on a finite-dimensional Banach space are equivalent, there is a number r such that

$$\sum_{0<j\leq n} |a_j| \leq r \parallel \sum_{0<j\leq n} a_j e_j \parallel .$$

Let $\delta = \frac{\varepsilon}{r}$. There are elements $u_1, \dots, u_n \in K_i$, $i < \mu$, such that

$$| \parallel u_j \parallel - \parallel e_j \parallel | = | \parallel k_i u_j \parallel - \parallel f e_j \parallel | \leq \delta$$

for $j = 1, \dots, n$. Let $f' : A \to K_i$ be the linear mapping such that $f'e_j = u_j$ for $j = 1, \dots, n$. We have

$$\| (k_{i,\mu}f'-f)(\sum_{0<j\leq n} a_je_j) \|\leq \sum_{0<j\leq n} |a_j| \| (k_{i,\mu}f'-f)(e_j) \|\leq \sum_{0<j\leq n} |a_j|\delta \leqslant \varepsilon \| \sum_{0<j\leq n} a_je_j \| .$$

Hence $k_{i,\mu}f' \sim_\varepsilon f$.

Following 7.1, f' is an ε-isometry because

$$| \| f' \sum_{0<j\leq n} a_je_j \| - \| \sum_{0<j\leq n} a_je_j \| | = | \| k_if' \sum_{0<j\leq b} a_je_j| \| - \| f \sum_{0<j\leq n} a_je_j \| |$$

and this value is at most

$$\| k_if' \sum_{0<j\leq n} a_je_j - f \sum_{0<j\leq n} a_je_j \|=\| (k_if' - f) \sum_{0<j\leq n} a_je_j \|\leq \varepsilon$$

provided that $\| \sum_{0<j\leq n} a_je_j \|\leq 1$.

(b) For every $\varepsilon > 0$ we next prove that there is $i < \mu$ and a morphism $f'' : A \to K_i$ such that $\| k_if'' - f \|\leq \varepsilon$.

Take f' from the proof of (a) for $\varepsilon' = \frac{\varepsilon}{2}$. Let $\| \sum_{0<j\leq n} a_je_j \|= 1$. Then

$$| \| f'(\sum_{0<j\leq n} a_je_j) \| -1| \leq \varepsilon'.$$

If $\| f'(\sum_{0<j\leq n} a_je_j) \|\geq 1$ then $\| f'(\sum_{0<j\leq n} a_je_j) \|\leq 1+\varepsilon'$. If $\| f'(\sum_{0<j\leq n} a_je_j) \|\leq 1$ then, again, $\| f'(\sum_{0<j\leq n} a_je_j) \|\leq 1 + \varepsilon'$. We have proved that $\| f' \|\leq 1 + \varepsilon'$. Hence $f'' = \frac{1}{1+\varepsilon'}f'$ is a morphism in **Ban**.

For $a = \sum_{0<j\leq n} a_je_j$ we have

$$\| f'a-f''a \|= \frac{1}{1+\varepsilon'} \| (1+\varepsilon')f'a-f'a) \|= \frac{\varepsilon'}{1+\varepsilon'} \| f'a \|\leq \frac{\varepsilon'}{1+\varepsilon'}(1+\varepsilon') \| a \|= \varepsilon' \| a \| .$$

Hence $\| f' - f'' \|\leq \varepsilon'$ and thus $\| k_if' - k_if'' \|\leq \varepsilon'$. Since $\| k_if' - f \|\leq \varepsilon'$, we have $\| k_if'' - f \|\leq \varepsilon$.

Since f'' is an ε'-isometry, we have $(1 - \varepsilon') \| a \|\leq\| f''a \|$. Hence

$$\| f''a \|= \frac{1}{1+\varepsilon'} \| f'a \|\geq \frac{1-\varepsilon'}{1+\varepsilon'} \| a \| .$$

Since

$$\frac{1-\varepsilon'}{1+\varepsilon'} = 1 - \frac{2\varepsilon'}{1+\varepsilon'} = \frac{\varepsilon}{1+\frac{\varepsilon}{2}} = \frac{2\varepsilon}{2+\varepsilon},$$

f'' is an $\frac{2\varepsilon}{2+\varepsilon}$-isometry. Since $\frac{2\varepsilon}{2+\varepsilon} \leq \varepsilon$, f'' is an ε-isometry.

II. Every morphism $f : A \to K_\mu$ has a decomposition

$$A \to f[A] \to K_\mu$$

where $f_1 : A \to f[A]$ is an epimorphism and $f_2 : f[A] \to K_\mu$ is an isometry. Since $f[A]$ is a finite-dimensional Banach space and f_2 is an isometry, following I. we have a morphism $f_2' : f(A) \to K_i$ such that $\| k_i f_2' - f_2 \| \leq \varepsilon$. Now, for $f' = f_2' f_1$ we have $\| k_i f' - f \| \leq \varepsilon$. $\square$

Remark 7.7. (1) Following [24] 6.5(2), every approximately $\aleph_0$-generated Banach space A admits for every $\varepsilon > 0$ an ε-split morphism $u : A \to B$ to a finite-dimensional Banach space B. This means that there exists $r : B \to A$ such that $ru \sim_\varepsilon \mathrm{id}_A$. Moreover, r can be taken as an isometry.

Conversely, every A with morphisms r and u as above is approximately $\aleph_0$-generated. Indeed, let $(k_{ij} : K_i \to K_j)_{i \leq j \leq \mu}$ a transfinite composite of isometries and let $f : A \to K_\mu$ and $\varepsilon > 0$ be given. There is an $\frac{\varepsilon}{2}$-split morphism $u : A \to B$ with B finite-dimensional. Since B is approximately $\aleph_0$-generated, there is $f' : B \to K_i$ for some $i < \mu$ such that $k_i f' \sim_{\frac{\varepsilon}{2}} fr$ (where r ε-splits u). Hence $k_i f' u \sim_{\frac{\varepsilon}{2}} fru$ and $fru \sim_{\frac{\varepsilon}{2}} f'$. Therefore $k_i f' \sim_\varepsilon f$.

(2) Every approximately $\aleph_0$-generated Banach space A is separable. Indeed, for every $n > 0$ there is a $\frac{1}{n}$-split morphism $u_n : A \to B_n$ to a finite-dimensional Banach space B_n. Let X_n be a countable dense set in B_n. Then for $r_n : B_n \to A$ which $\frac{1}{n}$-splits u_n we conclude that $\bigcup\limits_{n>0} r_n(X_n)$ is a countable dense set in A.

We do not know whether every approximately $\aleph_0$-generated Banach space is finite-dimensional.

Remark 7.8. (1) *Approximately* $\aleph_0$*-saturated* objects were defined in [24] 6.6. as objects approximately injective w.r.t. isometries between approximately $\aleph_0$-generated objects in the category $\mathcal{K}_{\text{iso}}$ of $\mathcal{K}$-objects and isometries. This means that in 5.1 $\mathcal{H}$ consists of isometries between approximately $\aleph_0$-generated objects and f, f' are isometries.

(2) In **Ban**, approximately $\aleph_0$-saturated objects coincide with Banach spaces of almost universal disposition for finite dimensional Banach spaces in [7]. The definition is formulated differently in op. cit., but its equivalence with approximate $\aleph_0$-saturation is shown in [18].

(3) An approximately $\aleph_0$-saturated object in **Ban** is approximately injective w.r.t. isometries between approximately $\aleph_0$-generated Banach spaces. It suffices to take an arbitrary $f : A \to X$ from 5.1, factorize it as $f = f_2 f_1$ where $f_2 : A_0 \to X$ is an isometry and A_0 is finite-dimensional and take the following pushout

$$\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "f_1"'] & A' \arrow[d, "\overline{f}_1"] \\
A_0 \arrow[r, "\overline{h}"'] & A_0'
\end{tikzcd}$$

Then for $f_2 : A_0 \to X$, we take $f_2' : A_0' \to X$ such that $f_2' \overline{h} \sim_\varepsilon f_2$. We conclude $f_2' \overline{f}_1 h \sim_\varepsilon f$.

(4) There exists no separable Banach space injective w.r.t. isometries between finite-dimensional Banach spaces in **Ban** (see [7], 3.10). On the other hand, the Gurarii space is a (unique) separable approximately $\aleph_0$-saturated Banach space, see [19].

(5) In [24] 6.7, a

arii space was obtained as $\bar{r}_{K_0} = \bar{m}_{0\omega} : K_0 \to \bar{K}_0$ (from 5.29) where $K_0 = 0$ is the null space. It could also be obtained as $r_{K_0} = m_{0\omega} : K_0 \to \widehat{K}_0$ from 5.28.

We are going to prove the uniqueness of a Gurarii space based on the results above, but using the approximate back-and-forth method of [19].

Lemma 7.9. *Let K be an approximately $\aleph_0$-saturated Banach space. Given an ε-isometry $h : A \to A'$ in* **Ban** *between finite-dimensional Banach spaces and an isometry $f : A \to K$, then for every $\delta > 0$ there is an isometry $f' : A' \to K$ in* **Ban** *such that $f'h \sim_{\varepsilon+\delta} f$.*

Proof. Following 7.3, there are isometries $u : A' \to B$ and $v : A \to B$ where B is finite-dimensional such that $uh \sim_\varepsilon v$. Given $\delta > 0$, there is an isometry $f'' : C \to K$ such that $f''v \sim_\delta f$. Let $f' = f''u$. Then $f'h \sim_{\varepsilon+\delta} f$. □

Proposition 7.10. *Any two separable approximately $\aleph_0$-saturated Banach spaces are isomorphic.*

Proof. Let K and L be separable approximately $\aleph_0$-saturated Banach spaces. Both K and L are colimits of an ω-chain of finite-dimensional Banach spaces and linear isometries: $K = \operatorname{colim}_{i<\omega} K_i$ and $L = \operatorname{colim}_{j<\omega} L_j$ where $K_0 = L_0 = 0$ and with the connecting morphisms $k_{i_1 i_2} : K_{i_1} \to K_{i_2}$ and $l_{j_1 j_2} : L_{j_1} \to L_{j_2}$. We will produce increasing sequences $i_1 < \dots i_n < \dots$ and $j_0 = 0 < j_1 < \dots j_n < \dots$, together with $\frac{1}{n}$-isometries $f_n : K_{i_n} \to L_{j_n}$, $n > 0$ in **Ban** and $\frac{1}{n+1}$-isometries $g_n : L_{j_n} \to K_{i_n+1}$, $n \geq 0$ in **Ban** such that for every $n > 0$ we have

$$(*) \qquad \| g_n f_n - k_{i_n, i_{n+1}} \| \leq \frac{2}{n+1}$$

and

$$(**) \qquad \| f_{n+1} g_n - l_{j_n, j_{n+1}} \| \leq \frac{2}{n+1}.$$

We proceed by induction, $g_0 = \mathrm{id}_0 : 0 \to 0$. Assume that we have g_n. Following 7.9, there is an isometry $t : K_{i_n} \to L$ such that

$$\| t g_n - l_{j_n} \| \leq \frac{1}{n+1}.$$

There is an index $j_{n+1} > j_n$ for which we have a morphism $f_{n+1} : K_{i_{n+1}} \to L_{j_{n+1}}$ such that $\| l_{j_{n+1}} f_{n+1} - t \| \leq \frac{1}{n+1}$. Following 3.21, f_{n+1} is an $\frac{1}{n+1}$-isometry. Moreover,

$$\| l_{j_{n+1}} f_{n+1} g_n - l_{j_{n+1}} l_{j_n j_{n_1}} \| = \| l_{j_{n+1}} f_{n+1} g_n - t g_n + t g_n - l_{j_{n+1}} l_{j_n j_{n_1}} \| \leq \frac{1}{n+1} + \frac{1}{n+1}.$$

Hence we have $(**)$.

Next, assume that we have f_n. Following 7.9, there is an $\frac{1}{n+1}$-isometry $t : L_{j_n} \to K$ such that

$$\| t f_n - k_{i_n} \| \leq \frac{1}{n+1}.$$

There is an index $i_{n+1} > i_n$ for which we have a morphism $g_n : L_{j_n} \to K_{i_{n+1}}$ such that $\| k_{i_{n+1}} g_n - t \| \leq \frac{1}{n+1}$. Following 3.21, g_n is an $\frac{1}{n+1}$-isometry. Moreover,

$$\| k_{i_{n+1}} g_n f_n - k_{i_{n+1}} k_{i_n i_{n_1}} \| = \| k_{i_{n+1}} g_n f_n - t f_n + t f_n - k_{i_{n+1}} k_{i_n i_{n_1}} \| \leq \frac{1}{n+1} + \frac{1}{n+1}.$$

Hence we have $(*)$.

Then the morphisms $f = \lim_n f_n : K \to L$ and $g = \lim_n g_n : L \to K$ are mutually inverse. In more detail, every $x \in K$ is a limit $x = \lim_n x_n$ where $x_n \in K_{i_n}$. Then $fx = \lim_n f_n x_n$. Similarly, we define g.

Hence $K \cong L$. □

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