

THE EXISTENCE AND UTILITY OF GIRY ALGEBRAS IN PROBABILITY THEORY

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ABSTRACT. Giry algebras are barycenters maps, which are coequalizers of contractible coequalizer pairs (like any algebras), and their existence, in general, requires the measurable space be coseparated by the discrete two point space, and the hypothesis that no measurable cardinals exist. Under that hypothesis, every measurable space which is coseparated has an algebra, and the category of Giry algebras provides a convenient setting for probability theory because it is a symmetric monoidal closed category with all limits and colimits, as well as having a separator and coseparator. This is in stark contrast to the Kleisli category of the Giry monad, which is often used to model conditional probability, which has a separator but not much else.

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1. INTRODUCTION

For \mathbf{T} a monad on any category \mathcal{X} , the category of \mathbf{T} -algebras, $\mathcal{X}^{\mathbf{T}}$, is the largest category which the monad \mathbf{T} functors through, while the Kleisli category of that monad, $\mathcal{X}_{\mathbf{T}}$, is the smallest category through which the monad factors. The category $\mathcal{X}_{\mathbf{T}}$ embeds into $\mathcal{X}^{\mathbf{T}}$, and the objects of $\mathcal{X}^{\mathbf{T}}$ are in effect all possible quotients of the free objects which are the codomain objects of maps in $\mathcal{X}_{\mathbf{T}}$, which explains why

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it has much better categorical properties than the Kleisi category. Nowhere is this more evident than in the case of the Giry monad (\mathcal{G}, η, μ) defined on the category of measurable spaces, \mathbf{Meas} . $\mathbf{Meas}^{\mathcal{G}}$ is a symmetric monoidal closed category with all limits, colimits, and a separator and coseparator, whereas $\mathbf{Meas}_{\mathcal{G}}$ has neither limits, colimits, or a coseparator. Because of the wide chasm between the categories $\mathbf{Meas}^{\mathcal{G}}$ and $\mathbf{Meas}_{\mathcal{G}}$ it is imperative that we do not identify probability theory with the Kleisi category.

The lack of understanding about $\mathbf{Meas}^{\mathcal{G}}$ arises from the fact $\mathbf{Meas}^{\mathcal{G}}$, like any category of algebras, is defined descriptively by defining the objects and arrows in terms of commutative diagrams, and says nothing about their existence. Moreover, the characterization of the objects and arrows of $\mathbf{Meas}^{\mathcal{G}}$ do not naturally lend themselves to how we represent models in probability. But that shortcoming simply suggests that an equivalent representation of $\mathbf{Meas}^{\mathcal{G}}$ is necessary to bridge the gap in our lack of understanding of $\mathbf{Meas}^{\mathcal{G}}$.

1.1. Equivalent representations using enrichment. In seeking to determine an equivalent representation for any Eilenberg-Moore category $\mathcal{X}^{\mathbf{T}}$ we can use the fact that the objects in $\mathcal{X}^{\mathbf{T}}$ are in effect all possible quotients of the free objects. Of course, to take a quotient of those free objects we need to know which category those objects should be viewed in to obtain an equivalent representation, and hence it is essential to determine the *appropriate* category in which the arrows of $\mathbf{Meas}_{\mathcal{G}}$ are enriched in. Knowing the enriching category, we can proceed to construct a category which is equivalent to $\mathbf{Meas}^{\mathcal{G}}$. The adjective *appropriate* makes reference to the fact that the arrows of a Kleisi category $\mathcal{X}_{\mathbf{T}}$ are usually enriched in several different categories, and the *appropriate* one is the one which is equivalent to $\mathcal{X}^{\mathbf{T}}$. For that reason we often need to know some basic facts about $\mathcal{X}^{\mathbf{T}}$ so we can compare it against the enriching category.

So just exactly what categories are the arrows of $\mathbf{Meas}_{\mathcal{G}}$ enriched in? It is well known that $\mathbf{Meas}_{\mathcal{G}}$ is enriched over the category of convex spaces, \mathbf{Cvx} . The proof that $\mathbf{Meas}_{\mathcal{G}}$ is enriched over \mathbf{Cvx} , which can be found in Meng[9], shows upon extending finite affine sums to countable affine sums, that $\mathbf{Meas}_{\mathcal{G}}$ is enriched over the category of super convex spaces, \mathbf{SCvx} . As the name suggests, the category \mathbf{SCvx} extends the concept of \mathbf{Cvx} , so an object in \mathbf{SCvx} is a set of points A such that for any countable number of points $a_1, a_2, \dots \in A$, every countable affine sum of elements also lies in A , $\sum_{i=1}^{\infty} \alpha_i a_i \in A$, where the family of elements α_i are elements $\alpha_i \in [0, 1]$ and satisfy $\lim_{n \rightarrow \infty} \sum_{i=1}^n \alpha_i = 1$. The morphisms in \mathbf{SCvx} are required to preserve those countable affine sums.

The category \mathbf{SCvx} does not have a coseparator, whereas $\mathbf{Meas}^{\mathcal{G}}$ does have a coseparator $\mathcal{G}\mathbf{2} \xrightarrow{\epsilon_2} \mathbf{2}$, where $\mathbf{2}$ is the discrete space, $\mathcal{G}\mathbf{2} \cong [0, 1]$, and $\epsilon_2(u) = 0$ for all $u \in [0, 1)$, and $\epsilon(1) = 1$. Thus to find the appropriate category for which $\mathbf{Meas}_{\mathcal{G}}$ is enriched over, we need to descend to a subcategory of \mathbf{SCvx} which does have a coseparator. Taking the full subcategory of \mathbf{SCvx} given by those objects which are coseparated by the object \mathbb{R}_{∞} , which is the one point extension of the real line, we obtain the category of \mathbb{R}_{∞} -coseparated super convex spaces, denoted $\mathbb{R}_{\infty}\text{-}\mathbf{SCvx}$. A more detailed definition of \mathbf{SCvx} and $\mathbb{R}_{\infty}\text{-}\mathbf{SCvx}$ are given in §2.1. A simple verification shows that $\mathbf{Meas}_{\mathcal{G}}$ is enriched over $\mathbb{R}_{\infty}\text{-}\mathbf{SCvx}$. The close relationship between the coseparator ϵ_2 in $\mathbf{Meas}^{\mathcal{G}}$ and the coseparator \mathbb{R}_{∞} in $\mathbb{R}_{\infty}\text{-}\mathbf{SCvx}$ is discussed in §2.2.

Using the above argument as a road map, in this article we address two questions: (1) When do \mathcal{G} -algebras exist?, and (2) Why is $\mathbf{Meas}^{\mathcal{G}}$ important for probability theory? These two questions could be restated as (1') Does there exist a category which is equivalent to $\mathbf{Meas}^{\mathcal{G}}$, and (2') Is $\mathbf{Meas}^{\mathcal{G}}$ or an equivalent category useful for probability theory?

Answering the first question consumes the bulk of our effort, and to answer it we analyze \mathcal{G} -algebras, recognizing them as countably affine measurable maps between \mathbb{R}_{∞} -coseperated super convex spaces, and apply Beck's precise tripleability theorem to show equivalence between $\mathbf{Meas}^{\mathcal{G}}$ and the category $\mathbb{R}_{\infty}\text{-}\mathbf{SCvx}$. In this introduction, we also address the second question which speaks to the significance of using $\mathbf{Meas}^{\mathcal{G}}$, or an equivalent category, in probability theory and applications.

1.2. When do \mathcal{G} -algebras exist? Given any measurable space X , a necessary conditions for a map $\mathcal{G}X \xrightarrow{h} X$ to be a \mathcal{G} -algebra is that X be coseperated by the discrete two point space $\mathbf{2}$. Otherwise the composite map $h \circ \eta_X$, where η_X is the unit of the Giry monad at component X mapping $x \mapsto \delta_x$, will not satisfy $h \circ \eta_X = id_X$. This necessary condition provides no insight on how to construct a \mathcal{G} -algebra on X , and hence the question of existence of a \mathcal{G} -algebra on a coseperated space X still remains.

However if we recognize that a \mathcal{G} -algebra $\mathcal{G}X \xrightarrow{h} X$ is in fact a countably affine map between \mathbb{R}_{∞} -coseperable super convex spaces then we can say quite a bit. The measurable space of all probability measures on X , $\mathcal{G}(X)$, has a super convex space structure defined on it pointwise. That is, for $\{\alpha_i\}_{i=1}^{\infty}$ a sequence of elements with $\alpha_i \in [0, 1]$ such that $\lim_{n \rightarrow \infty} \sum_{i=1}^n \alpha_i = 1$, it follows that the countable affine sum $\sum_{i=1}^{\infty} \alpha_i P_i$, defined pointwise on the measurable sets, defines a unique probability measure on X . $\mathcal{G}X$ is coseperated by \mathbb{R}_{∞} because it is coseperated by the set of evaluation maps, $\{ev_U\}_{U \in \Sigma_X}$, where each map $\mathcal{G}X \xrightarrow{ev_U} [0, 1]$ sends $P \mapsto P(U)$.

If h is a \mathcal{G} -algebra then due to the super convex space structure on $\mathcal{G}(X)$, we obtain an induced super convex space structure on X defined by $\sum_{i=1}^{\infty} \alpha_i x_i := h(\sum_{i=1}^{\infty} \alpha_i \delta_{x_i})$. Let us refer to the space X , when viewed as a super convex space, as X_h . Using the induced super convex space structure, X_h , it follows that the measurable map h itself is a countably affine map, $h(\sum_{i=1}^{\infty} \alpha_i P_i) = \sum_{i=1}^{\infty} \alpha_i h(P_i)$. Furthermore, if $\mathcal{G}(Y) \xrightarrow{k} Y$ is a \mathcal{G} -algebra and $f : h \rightarrow k$ is a morphism of \mathcal{G} -algebras, then the computation

$$\begin{aligned} f(\sum_{i=1}^{\infty} \alpha_i x_i) &= f(h(\sum_{i=1}^{\infty} \alpha_i \delta_{x_i})) \\ &= k\mathcal{G}(f)(\sum_{i=1}^{\infty} \alpha_i \delta_{x_i}) \\ &= k(\sum_{i=1}^{\infty} \alpha_i \delta_{f(x_i)}) \\ &= \sum_{i=1}^{\infty} \alpha_i f(x_i) \end{aligned}$$

proves that $f : X_h \rightarrow Y_k$ is a countably affine map between super convex spaces. Thus we have

Lemma 1.1. *Every \mathcal{G} -algebra $h : \mathcal{G}X \rightarrow X$ is a countably affine map under the induced super convex space structure on X , and every morphism of \mathcal{G} -algebras is a countably affine (measurable) map.*

This lemma by itself shows the existence of a functor $\mathbf{Meas}^{\mathcal{G}} \rightarrow \mathbb{R}_{\infty}\text{-}\mathbf{SCvx}$. In trying to construct a functor $\mathbb{R}_{\infty}\text{-}\mathbf{SCvx} \rightarrow \mathbf{Meas}^{\mathcal{G}}$ we are led to the idea that given any \mathbb{R}_{∞} -coseperated super convex space A , that there should exist a functor $\mathbb{R}_{\infty}\text{-}\mathbf{SCvx} \xrightarrow{\Sigma} \mathbf{Meas}$ such that $\mathcal{G}(\Sigma A) \xrightarrow{\epsilon_A} \Sigma A$ specifies a \mathcal{G} -algebra. More thought

suggest the following two functors, (1) $\mathbf{Meas} \xrightarrow{\mathcal{P}} \mathbb{R}_\infty\text{-}\mathbf{SCvx}$ is the functor \mathcal{G} viewed as a functor into $\mathbb{R}_\infty\text{-}\mathbf{SCvx}$, and (2) $\mathbb{R}_\infty\text{-}\mathbf{SCvx} \xrightarrow{\Sigma} \mathbf{Meas}$ is the functor endowing each \mathbb{R}_∞ -coseperable super convex space A with the smallest σ -algebra such that all the countably affine functions $A \xrightarrow{m} (\mathbb{R}_\infty, \mathcal{B})$ are measurable, where \mathcal{B} is the Borel σ -algebra on the extended real-line.

In attempting to prove the existence of an adjunction, $\langle \mathcal{P}, \Sigma, \eta, \epsilon \rangle : \mathbf{Meas} \rightarrow \mathbb{R}_\infty\text{-}\mathbf{SCvx}$, the only difficult aspect resides in the construction of the counit ϵ , which amounts to showing there exists a barycenter map $(\mathcal{P} \circ \Sigma)A \xrightarrow{\epsilon_A} A$ which satisfies the property that for all $P \in \mathcal{P}(\Sigma A)$ and for all countably affine maps $A \xrightarrow{m} \mathbb{R}_\infty$ that $\int_A m dP = m(\epsilon_A(P))$. Under the hypothesis that *no measurable cardinals exist*, which is consistent with the ZFC axioms, all such barycenter maps exist. The proof of their existence is given in Theorem 3.2. From the category theoretical point of view, the hypothesis that no measurable cardinals exist is equivalent to the hypothesis that the only exact endofunctor on \mathbf{Set} is, up to a natural isomorphism, the identity functor.[1, 2].

The connection between the existence of barycenter maps and the hypothesis that no measurable cardinals arises in the problem of showing that the full subcategory of $\mathbb{R}_\infty\text{-}\mathbf{SCvx}$ consisting of the single object \mathbb{R}_∞ is right adequate (codense) in $\mathbb{R}_\infty\text{-}\mathbf{SCvx}$. That problem is closely analogous to the observation that Isbell made in his article *Adequate subcategories* concerning the right-adequacy of \mathbb{N} in the category \mathbf{Set} . [7]

For subsequent use, let us state the defining properties of a measurable cardinal. Let X be any set and $\mathcal{P}X$ the set of all subsets of X . If there exists a function $\mu : \mathcal{P}X \rightarrow \{0, 1\}$ such that (1) $\mu(\emptyset) = 0$ and $\mu(X) = 1$, (2) $\mu(\{x\}) = 0$ for all $x \in X$, and (3) μ is countably additive, then we say μ is a Ulam measure on X , and say $|X|$ is a measurable cardinal. We reiterate that the hypothesis that no measurable cardinals exist is consistent with the ZFC axioms.

1.3. Why use Giry-algebras? The hypothesis that no measurable cardinals exist leads to an elegant and more complete theory for probability since models can be constructed using $\mathbf{Meas}^{\mathcal{G}}$, or any of its equivalent representations, and those categories have nice categorical properties compared to $\mathbf{Meas}_{\mathcal{G}}$. Hence various constructions are possible in the category $\mathbf{Meas}^{\mathcal{G}}$ which are simply not possible within the framework of modeling probability using $\mathbf{Meas}_{\mathcal{G}}$.

For example in modeling stochastic processes, the existence of a limit of diagram of finite tensor product spaces and the projection maps between those spaces are required. Because $\mathbf{Meas}^{\mathcal{G}}$ has all limits, those limits exist for all families of spaces $\{X_i\}_{i \in I}$, where I is an indexing set. Such is not the case for $\mathbf{Meas}_{\mathcal{G}}$, and one is forced to restrict to Borel measurable spaces over a countable indexing set. We can express this fact by saying all infinite tensor products[5] exist in $\mathbf{Meas}^{\mathcal{G}}$, but those infinite tensor products do not exist in $\mathbf{Meas}_{\mathcal{G}}$.¹

¹The category $\mathbf{Meas}^{\mathcal{G}}$ has two symmetric monoidal structures, one is a cartesian symmetric monoidal structure arising from \mathbf{Meas} using the product $X \times Y$, with the product σ -algebra. The other monoidal structure arises from using the tensor σ -algebra, which is determined by the constant graph maps into the cartesian product $X \times Y$ of sets, and taking the final σ -algebra on the product set so that the constant graph maps are measurable. The product σ -algebra is a sub σ -algebra of the tensor σ -algebra, and the tensor σ -algebra construction makes $(\mathbf{Meas}, \otimes, \mathbf{1})$ a monoidal closed category. Both monoidal structures contain all infinite tensor products. The tensor monoidal structure is only a semicartesian symmetric monoidal structure; it is not a Markov category. For example,

We have already noted that in using the Kleisli category $\mathbf{Meas}_{\mathcal{G}}$ we are restricted to modeling conditional probabilities using maps into free objects. Hence any modeling problem requiring a quotient cannot be modeled within $\mathbf{Meas}_{\mathcal{G}}$. The simplest example of this is

Example 1.2. Consider the simplex $\Delta_n := \mathcal{G}(\Sigma \mathbf{n})$, consisting of all probability measures on a discrete space with n points, and a kernel map $X \rightarrow \Delta_n$. Given any permutation of the labels, $\phi : \mathbf{n} \rightarrow \mathbf{n}$, we desire to identify a probability measure $P \in \mathcal{G}(\Sigma \mathbf{n})$ with the probability measure $P\phi^{-1}$. That is, we want to define a congruence relation \sim on the super convex space Δ_n by $P \sim Q$ if and only if there exists a permutation $\phi : \mathbf{n} \rightarrow \mathbf{n}$ so that we can identify when one probability measure is the same as another probability measure up to a relabeling. The congruence relation yields a quotient space Δ_n / \sim which is not a free object. In particular, taking $n = 2$ we have $\Delta_2 = [0, 1]$ and the quotient space is $\mathbb{R}_{\infty}\text{-SCvx}$ -isomorphic to $[0, \frac{1}{2}]$.

If we view a quotient space simply as a “parameter space” we immediately recognize that it can be usefully employed for modeling purposes, and the restriction of parameter spaces to be free objects is an unnecessary restriction on our modeling toolbox. While the above example is trivial in the sense that the problem can be addressed using other elementary methods without invoking $\mathbf{Meas}_{\mathcal{G}}$, but it is clear that constructing arbitrary quotient spaces of free objects do require $\mathbf{Meas}_{\mathcal{G}}$.

For understanding various properties of $\mathbf{Meas}_{\mathcal{G}}$ it is easier to analyze $\mathbb{R}_{\infty}\text{-SCvx}$. For example, in the article *Commutative monads as a theory of distributions* various biaffine maps are discussed.[8] Taking the monad \mathbf{T} as \mathcal{G} , and viewed within the framework of $\mathbb{R}_{\infty}\text{-SCvx}$, which has both a cartesian symmetric monoidal and a tensor symmetric monoidal structure, those biaffine maps can be viewed as countably affine maps. For example, the integration pairing map $\langle P, f \rangle \mapsto \int f dP$ can be viewed as a countably affine mapping $P \otimes f \mapsto \int f dP$. More generally, we have the countably affine map $\sum_{i=1}^{\infty} \alpha_i (P_i \otimes f_i) \mapsto \sum_{i=1}^{\infty} \alpha_i \int f_i dP_i$ where $\{\alpha_i\}_{i=1}^{\infty}$ is a sequence of elements $\alpha_i \in [0, 1]$ such that $\lim_{n \rightarrow \infty} \sum_{i=1}^n \alpha_i = 1$, and each $f_i : X \rightarrow \mathbb{R}_{\infty}$ is a measurable function and $P_i \in \mathcal{G}(X)$. Moreover, by applying the functor Σ to the appropriate diagram, the integration pairing map is a countably affine measurable function.

2. \mathbb{R}_{∞} -COSEPERATED SUPER CONVEX SPACES

2.1. Definition and basic properties. Let \mathbb{N} denotes the set of natural numbers, and let Ω denote the set of all countable partitions of one,

$$\Omega := \{\alpha = \{\alpha_i\}_{i=1}^{\infty} \mid \sum_{i=1}^{\infty} \alpha_i = 1, \alpha_i \in [0, 1]\},$$

take the real line \mathbb{R} with the countable/cocountable σ -algebra. The copy map $x \mapsto (x, x)$ is not a measurable function. The fact all infinite tensor products exists is due to the fact that $\mathbf{Meas}_{\mathcal{G}}$ has all limits (because \mathbf{Meas} has all limits). Whether the Markov category assumption is necessary to characterize *categorical probability* or whether a weaker hypothesis suffices I do not know. But it is not necessary for infinite tensor products where the semicartesian symmetric monoidal condition is sufficient. At this juncture, the term *categorical probability* itself is still somewhat vague, though like all new ideas, the concept will take time to arrive at a precise meaning.

where “ $\sum_{i=1}^{\infty} \alpha_i = 1$ ” is shorthand notation for the limit condition, $\lim_{N \rightarrow \infty} \{\sum_{i=1}^N \alpha_i\} = 1$. A super convex space A is a set A together with a “structural” map

$$\begin{array}{ccc} \Omega & \xrightarrow{st_A} & \mathbf{Set}(A^{\mathbb{N}}, A) \\ \alpha \vdash & \longrightarrow & A^{\mathbb{N}} \xrightarrow{\alpha_A} A \\ & & \{a_i\}_{i=1}^{\infty} \vdash \longrightarrow \sum_{i=1}^{\infty} \alpha_i a_i \end{array}$$

which satisfies the following two properties:

- (i) $\sum_{i=1}^{\infty} \delta_i^j a_i = a_j$ for all $j \in \mathbb{N}$, and all $\{a_i\}_{i=1}^{\infty} \in A^{\mathbb{N}}$, and
- (ii) $\sum_{i=1}^{\infty} \alpha_i (\sum_{j=1}^{\infty} \beta_j a_j) = \sum_{j=1}^{\infty} (\sum_{i=1}^{\infty} \alpha_i \beta_j^i) a_j$ for all $\alpha, \beta^i \in \mathbb{R}$.

The definition implies that the convex space $\mathbb{R} = (-\infty, \infty)$ is not a super convex space because we can choose the countable partition of one given by $\{\frac{1}{2^i}\}_{i=1}^{\infty}$ and choose a sequence of points $\{i2^i\}_{i=1}^{\infty}$ in \mathbb{R} such that countable affine sum $\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{i2^i}{2^i} \notin \mathbb{R}$.

A morphism from a super convex space A to a super convex space B is a set map $A \xrightarrow{m} B$ making the following **Set**-diagram

$$\begin{array}{ccc} A^{\mathbb{N}} & \xrightarrow{\alpha_A} & A \\ m^{\mathbb{N}} \downarrow & & \downarrow m \\ B^{\mathbb{N}} & \xrightarrow{\alpha_B} & B \end{array} \quad m(\sum_{i=1}^{\infty} \alpha_i a_i) = \sum_{i=1}^{\infty} \alpha_i m(a_i)$$

commute, where $A^{\mathbb{N}} \xrightarrow{m^{\mathbb{N}}} B^{\mathbb{N}}$ is defined componentwise. Thus, a set map between super convex spaces, $A \xrightarrow{m} B$, is a morphism in **SCvx** if and only if it preserves countable affine sums. Super convex spaces form a category, with composition of morphisms being the set-theoretical one.

The convex space $\mathbb{R} = (-\infty, \infty)$, with the natural convex space structure, has a one point extension yielding the convex space $\mathbb{R}_{\infty} = (-\infty, \infty]$, specified on (finite) affine sums by the property, for all $u \in (-\infty, \infty)$ and all $r \in (0, 1]$, that $(1-r)u + r\infty = \infty$.

The object \mathbb{R}_{∞} can also be viewed as a super convex space specified, for all $u_i \in \mathbb{R}_{\infty}$ and all countable partitions of one, $\{\alpha_i\}_{i=1}^{\infty}$, by

$$\sum_{i=1}^{\infty} \alpha_i u_i = \begin{cases} \lim_{N \rightarrow \infty} \{\sum_{i=1}^N \alpha_i u_i\} & \text{provided the limit exist} \\ \infty & \text{otherwise} \end{cases}.$$

It is convenient to view the space \mathbb{R}_{∞} as the doubly extended real line, $[-\infty, \infty]$, modulo the relation $-\infty = \infty$. The object, \mathbb{R}_{∞} , is a coseparator in **Cvx**. [3] But there are no coseparators for **SCvx**. [4]

To be able to construct an equivalence with the category \mathcal{G} -algebras it is necessary to restrict our consideration to the subcategory of **SCvx** consisting of those super convex spaces A which are coseparable by \mathbb{R}_{∞} , meaning that given any two points $a_1, a_2 \in A$ there exists a countably affine map $A \xrightarrow{m} \mathbb{R}_{\infty}$ such that $m(a_1) \neq m(a_2)$. We denote this subcategory by \mathbb{R}_{∞} -**SCvx**.

Note that applying the functor Σ to the super convex space A gives the measurable space ΣA which is coseparated by the discrete measurable space **2**. The proof follows

from the fact A is coseparated by the super convex space \mathbb{R}_∞ , so that we have a measurable map $\Sigma A \xrightarrow{m} \Sigma \mathbb{R}_\infty \rightarrow \Sigma \mathbf{2}$. (Note that $\Sigma \mathbb{R}_\infty = (\mathbb{R}_\infty, \mathcal{B})$ because $\mathbb{R}_\infty \xrightarrow{id} (\mathbb{R}_\infty, \mathcal{B})$ is a countably affine map, and hence the induced σ -algebra on the super convex space \mathbb{R}_∞ is the Borel σ -algebra on the extended real line with the two points $-\infty$ and ∞ identified. We denote the measurable space $(\mathbb{R}_\infty, \mathcal{B})$ by \mathbb{R}_∞ also.)

Hereafter, we make no further use of the category \mathbf{SCvx} , and for brevity we often drop the qualifying adjective “ \mathbb{R}_∞ -coseparated”, and use the term “super convex space” to mean “ \mathbb{R}_∞ -coseparated super convex space”. When we do use the qualifying adjective, it is used for emphasis.

2.2. Epi-mono factorization. In \mathbb{R}_∞ - \mathbf{SCvx} the monomorphisms are precisely the countably affine one-to-one maps, while the epimorphisms are the countably affine surjective maps. The proof of both of these statements are elementary.

Lemma 2.1. *Every morphism $A \xrightarrow{m} \mathbb{R}_\infty$ in \mathbb{R}_∞ - \mathbf{SCvx} factorizes as an epi-mono pair,*

$$\begin{array}{ccc} A & \xrightarrow{m} & \mathbb{R}_\infty \\ & \searrow q & \nearrow \iota \\ & Im(m) & \end{array}$$

where the image of m , denoted $Im(m)$, is itself is an object in \mathbb{R}_∞ - \mathbf{SCvx} , and ι is the inclusion mapping which is monic.

Proof. Define $Im(m) = \{r \in \mathbb{R}_\infty \mid \exists a \in A \text{ s.t. } r = m(a)\}$. To prove $Im(m)$ is a coseparated super convex space, let $\{r_i\}_{i=1}^\infty$ be a countable family of elements in $Im(m)$, say $r_i = m_i(a_i)$. For $\{\alpha_i\}_{i=1}^\infty$ a countable partition of one, we have $\sum_{i=1}^\infty \alpha_i r_i = \sum_{i=1}^\infty \alpha_i m(a_i) = m(\sum_{i=1}^\infty \alpha_i a_i)$ which lies in $Im(m)$. Thus $Im(m)$ is a super convex space, and the map q defined by $q(a) = m(a)$ is an epimorphism. It is clear that $Im(m)$ is coseparated because it is a subobject of \mathbb{R}_∞ .

The inclusion mapping of $Im(m)$ into \mathbb{R}_∞ is injective, and hence monic. \square

The super convex space $\mathbf{2}$, consisting of two points, $\{0, 1\}$, plays a prominent role in the category \mathbb{R}_∞ - \mathbf{SCvx} . The super convex space structure on it is defined by

$$\sum_{i=1}^\infty \alpha_i u_i = \begin{cases} 0 & \text{if there exists an index } k \text{ such that } u_k = 0 \text{ and } \alpha_k > 0 \\ 1 & \text{otherwise} \end{cases}.$$

We have, for every $\omega \in \mathbb{R}$, a countably affine map

$$\begin{array}{ccc} \mathbf{2} & \xrightarrow{\hat{\omega}} & \mathbb{R}_\infty \\ u & \mapsto & \begin{cases} \infty & \text{for } u = 0 \\ \omega & \text{for } u = 1 \end{cases} \end{array}$$

and the image space $Im(\hat{\omega})$ is isomorphic to $\mathbf{2}$. Hence $\mathbf{2}$ is a subobject of \mathbb{R}_∞ . The space $\mathbf{2}$ is a coseparator for discrete (combinatorial) super convex spaces, but not general spaces. On the other hand, the unit interval $[0, 1]$ is adequate for coseparating geometric spaces. The space \mathbb{R}_∞ effectively combines these two “component type coseparators” so that \mathbb{R}_∞ coseparates any super convex space, regardless of whether it is geometric, discrete, or of mixed type.

2.3. The expected value of a map into \mathbb{R}_∞ . We have already noted that the real line \mathbb{R} is not a super convex space. On the other hand, by Lemma 2.1 the image of any countably affine map is a subobject of \mathbb{R}_∞ . Such subobjects can be characterized as one of two general forms:

- (1) A discrete subspace of \mathbb{R}_∞ isomorphic to either **1** or **2**.
- (2) The inclusion of a nondegenerate interval into \mathbb{R}_∞ , or a nondegenerate interval adjoined with the singleton set $\{\infty\}$. By an interval we mean those of the form (u, v) , $[u, v)$, $[u, v]$, $(u, v]$ as well as $[\infty, v)$, $(u, \infty]$, $[u, \infty]$, $[\infty, v]$. In cases where we do not wish to distinguish whether the boundary is open or closed, we use the notation $\{u, v\}$, $\{u, \infty\}$, etc..

Of course, dropping the adjective “nondegenerate” in the second case, yields the subobject $[\omega, \omega] \cup \{\infty\} \cong \mathbf{2}$ which is a discrete subspace mentioned in (1).

For A any super convex space, $P \in \mathcal{G}(\Sigma A)$, and any countably affine map $A \xrightarrow{m} \mathbb{R}_\infty$, define

$$\mathbb{E}_P(m) := \int_A m \, dP.$$

Lemma 2.2. *Given a countably affine map $\hat{\omega} : \mathbf{2} \rightarrow \mathbb{R}_\infty$ and any $P \in \mathcal{G}(\Sigma \mathbf{2})$, it follows that $\mathbb{E}_P(\hat{\omega}) \in \text{Im}(\hat{\omega})$.*

Proof. Since every probability measure on **2** is of the form $(1 - \alpha)\delta_0 + \alpha\delta_1$ the integral can be computed as

$$\int_{\mathbf{2}} \hat{\omega} \, d((1 - \alpha)\delta_0 + \alpha\delta_1) = \infty \cdot (1 - \alpha) + \omega \cdot \alpha = \begin{cases} \infty = \hat{\omega}(0) & \text{for } \alpha \in [0, 1) \\ \omega = \hat{\omega}(1) & \text{for } \alpha = 1 \end{cases}$$

□

We now proceed to show that same property, $\mathbb{E}_P(\iota) \in \text{Im}(\iota)$, holds for the subobjects of \mathbb{R}_∞ which include a nondegenerate interval. If $S = [u, v]$ then, letting \bar{u} denote the constant function on S with value u , since $\bar{u} \leq \iota \leq \bar{v}$ it follows that $\mathbb{E}_P(\iota) \in \text{Im}(\iota)$.

Lemma 2.3. *Suppose $S = (u, v)$ is a subobject of \mathbb{R}_∞ , where u and v are finite. Then every probability measure $P \in \mathcal{G}(\Sigma S)$ satisfies the property $\mathbb{E}_P(\iota) \in \text{Im}(\iota)$.*

Proof. By the remark preceding the lemma it follows that $\mathbb{E}_P(\iota) \in [u, v]$.

For every affine transformation on \mathbb{R}_∞ specified by a scale factor λ and translation t , $\mathbb{E}_P(\lambda \cdot \text{id} + t) = \lambda \mathbb{E}_P(\text{id}) + t$. Therefore, without loss of generality we can assume that (u, v) is the interval $(0, 1)$. Let us show that $\mathbb{E}_P(\iota) = 0$ is impossible, which therefore implies that $\mathbb{E}_P(\iota) = u$ is not possible.

Since the insertion map ι is the identity function it is a measurable function. We can therefore write it as a pointwise limit of a sequence of simple functions, $\{\psi_n\}_{n=1}^\infty$, which are (pointwise) monotonically increasing, $\psi_n(x) \leq \psi_{n+1}(x)$, where

$$\psi_n = \sum_{k=0}^{2^n-1} \frac{k}{2^n} \chi_{E_{n,k}}$$

and

$$E_{n,k} = \{x \in (0, 1) \mid \frac{k}{2^n} \leq x < \frac{k+1}{2^n}\} \quad \text{for } k = 0, 1, \dots, 2^n - 1.$$

We have

$$\begin{aligned}\mathbb{E}_P(\iota) &= \lim_{n \rightarrow \infty} \int_S \psi_n dP \\ &= \lim_{n \rightarrow \infty} \left\{ \sum_{k=0}^{2^n-1} \frac{k}{2^n} P(E_{n,k}) \right\}.\end{aligned}$$

Suppose, to obtain a contradiction, that $\mathbb{E}_P(\iota) = 0$. Note that

$$\mathbb{E}_P(\iota) = \lim_n \mathbb{E}_P(\psi_n) = \mathbb{E}_P(\lim_n \psi_n),$$

where the second equality follows from the monotone convergence theorem. The hypothesis $\mathbb{E}_P(\iota) = 0$ implies that $\lim_n P(E_{n,0}) = 1$. But $\lim_n E_{n,0} = \bigcap_{n=1}^{\infty} E_{n,0} = \emptyset$, and since P is a probability measure it follows that P is continuous at \emptyset , i.e., $P(\lim_n E_{n,0}) = P(\emptyset) = 0$. Hence $\mathbb{E}_P(\iota) = 0$ is impossible.

A similiar argument as that given above, using the fact $\lim_n P(E_{n,2^n-1}) = P(\emptyset) = 0$, shows that $\mathbb{E}_P(\iota) = 1$ is also impossible. Thus $\mathbb{E}_P(\iota) \in (0, 1)$. Translation and scaling by the inverse transformation of $\lambda x + t$ then yields the result for the general interval (u, v) . \square

Using the proof of this lemma, it is evident that for $S = \{u, \infty\}$ or $S = [-\infty, v\}$, where we identify $-\infty$ and ∞ , that $\mathbb{E}_P(\iota) \in Im(\iota)$.

Lemma 2.4. *Let S a subobject of \mathbb{R}_{∞} which is a bounded interval plus the singleton set $\{\infty\}$. If $P \in \mathcal{G}(\Sigma S)$ then $\mathbb{E}_P(\iota) \in Im(\iota)$.*

Proof. If $P(\{\infty\}) > 0$ then $\mathbb{E}_P(\iota) = \infty$, and the lemma holds. So let us suppose that $P(\{\infty\}) = 0$. Then P can be viewed as a probability measure on a bounded interval and the previous results show $\mathbb{E}_P(\iota)$ lies in the bounded interval. \square

Theorem 2.5. *Let A be a super convex space A , and $A \xrightarrow{m} \mathbb{R}_{\infty}$ a countably affine map. Then for every probability measure $P \in \mathcal{G}(\Sigma A)$ it follows that $\mathbb{E}_P(m) \in Im(m)$. In other words, there exists an element $a \in A$ such that $\int_A m dP = m(a)$.*

Proof. Given any countably affine map $m : A \rightarrow \mathbb{R}_{\infty}$ and any $P \in \mathcal{G}(\Sigma A)$ it follows, using the factorization of $m = \iota \circ q$, that

$$\mathbb{E}_P(m) = \int_A m dP = \int_A (\iota \circ q) dP = \int_{Im(m)} \iota d(Pq^{-1}) = \mathbb{E}_{Pq^{-1}}(\iota).$$

Since $Im(m)$ is a subobject of \mathbb{R}_{∞} , it follows by the previous three lemmas that $\mathbb{E}_{Pq^{-1}}(\iota) \in Im(\iota)$. Viewed as sets, $Im(\iota) = Im(m)$, from whence the result follows. \square

3. EXISTENCE OF THE BARYCENTER MAP

Let A be a super convex space and suppose that $P \in \mathcal{P}(\Sigma A)$. Define a two-valued function on the set of all subsets of A by

$$\begin{aligned}\mu_P &: \mathcal{P}(A) \rightarrow \{0, 1\} \\ &: V \mapsto \begin{cases} 1 & \text{iff } \mathbb{E}_P(m) \in m(V) \quad \forall m \in \mathbb{R}_{\infty}\text{-}\mathbf{SCvx}(A, \mathbb{R}_{\infty}) \\ 0 & \text{otherwise} \end{cases}\end{aligned}$$

Lemma 3.1. *The function μ_P satisfies the following properties:*

- (1) $\mu_P(\emptyset) = 0$ and $\mu_P(A) = 1$,

(2) μ_P is countably additive.

Proof. (1) The property $\mu(\emptyset) = 0$ is obvious. The property $\mu_P(A) = 1$ is equivalent to saying that $\mathbb{E}_P(m) \in \text{Im}(m)$, which follows from Theorem 2.5.

(2) Let $\{S_i\}_{i=1}^\infty$ be a countable partition of S . Note that μ_P is monotone: if $S_i \subset S$ and $\mu_P(S_i) = 1$ then $\mu_P(S) \geq 1$. Hence if $\mu_P(S) = 0$ then $\mu_P(S_i) = 0$ for all indices i .

On the other hand, if $\mu_P(S) = 1$ then we must show there exists exactly one index k such that $\mu_P(S_k) = 1$. Since $\mu_P(S) = \mu_P(\bigcup_{i=1}^\infty S_i) \leq \sum_{i=1}^\infty \mu_P(S_i)$ it follows there exists at least one such index k such that $\mu_P(S_k) = 1$. To show that there exists at most one such index k such that $\mu_P(S_k) = 1$, suppose, to obtain a contradiction, that $\mu_P(S_{k_1}) = 1 = \mu_P(S_{k_2})$. For every pair of points $(a_1, a_2) \in S_{k_1} \times S_{k_2}$ there exists an $m \in \mathbb{R}_\infty\text{-SCvx}(A, \mathbb{R}_\infty)$ such that $m(a_1) \neq m(a_2)$, hence it cannot be the case that $\mathbb{E}_P(m) = m(a_1) = m(a_2)$ for all $m \in \mathbb{R}_\infty\text{-SCvx}(A, \mathbb{R}_\infty)$. \square

Theorem 3.2. *Assume there are no measurable cardinals. For every coseparable super convex space A and P any probability measure on ΣA , there exists a unique point $a_\star \in A$ such that, for all countably affine maps $A \xrightarrow{m} \mathbb{R}_\infty$, $\mathbb{E}_P(m) = m(a_\star)$. That is, P is a Dirac measure at a point $a_\star \in A$ when viewed as a functional on the set of all countably affine measurable functions on A into \mathbb{R}_∞ .*

Proof. By Lemma 3.1 the function μ_P , defined on the power set of A , is a countably additive two-valued measure. By hypothesis, no measurable cardinals exist, which implies that the measure μ_P must be a Dirac measure $\mu_P = \delta_a$ for some element $a \in A$.

The condition $\mu_P = \delta_a$ implies, for every $m \in \mathbb{R}_\infty\text{-SCvx}(A, \mathbb{R}_\infty)$ and every subset $U \subseteq \mathbb{R}_\infty$, that

$$\mu_P(m^{-1}(U)) = \delta_{m(a)}(U) = \begin{cases} 1 & \text{iff } \mathbb{E}_P(m) \in m(m^{-1}(U)) \\ 0 & \text{otherwise} \end{cases}.$$

Let $U = \{\mathbb{E}_P(m)\}$. By Theorem 2.5 it follows that $m^{-1}(\{\mathbb{E}_P(m)\}) \neq \emptyset$, and hence $\mathbb{E}_P(m) \in m(m^{-1}(\{\mathbb{E}_P(m)\}))$. Thus it follows that $\delta_{m(a)}(\{\mathbb{E}_P(m)\}) = 1$ which implies $m(a) = \mathbb{E}_P(m)$ for every $m \in \mathbb{R}_\infty\text{-SCvx}(A, \mathbb{R}_\infty)$.

The uniqueness of the element a follows from the condition that A is a \mathbb{R}_∞ -coseparated super convex space. \square

A simple diagram chase then proves

Corollary 3.3. *The set of maps, defined for each object A in $\mathbb{R}_\infty\text{-SCvx}$ by the map $\mathcal{P}(\Sigma A) \xrightarrow{\epsilon_A} A$ sending a probability measure on ΣA to the unique element $a \in A$ such that $m(a) = \mathbb{E}_P(m)$ for all $m \in \mathbb{R}_\infty\text{-SCvx}(A, \mathbb{R}_\infty)$ specify a natural transformation $\mathcal{P} \circ \Sigma \xrightarrow{\epsilon} \text{id}_{\mathbb{R}_\infty\text{-SCvx}}$.*

The natural transformation ϵ specifies all the barycenter maps. It is now straight forward, taking the unit $\eta : \text{id}_{\mathbf{Meas}} \Rightarrow \Sigma \circ \mathcal{P}$ specified by $x \mapsto \delta_x$, and the counit as the natural transformation ϵ , to check $\mathcal{P} \dashv \Sigma$ by verifying the two triangular identities, $\epsilon_{\mathcal{P}} \circ \mathcal{P}\eta = 1_{\mathcal{P}}$ and $\Sigma\epsilon \circ \eta_\Sigma = 1_\Sigma$.

4. APPLYING BECK'S PRECISE TRIPLEABILITY THEOREM

In applying Becks theorem to prove the equivalence, we use the following form.

Let

$$\langle \mathcal{F}, \mathcal{G}, \eta, \epsilon \rangle : X \rightarrow A$$

be an adjunction, $\langle T, \eta, \mu \rangle$ the monad it defines in X , X^T the category of T -algebras for this monad, and

$$\langle \mathcal{F}^T, \mathcal{G}^T, \eta^T, \epsilon^T \rangle : X \rightarrow X^T$$

the corresponding adjunction. Then the following conditions are equivalent:

- (1) The comparison functor $K : A \rightarrow X^T$ is an equivalence of categories.
- (2) If f and g are any parallel pair in A for which the pair $\mathcal{G}f$ and $\mathcal{G}g$ has a split coequalizer, then A has a coequalizer for the pair f and g , and \mathcal{G} preserves and reflects coequalizers for these pairs.

Let $f : A \rightarrow B$ and $g : A \rightarrow B$ be two parallel arrows in $\mathbb{R}_\infty\text{-}\mathbf{SCvx}$, and let the measurable space X , along with the measurable function $\Sigma B \xrightarrow{e} X$ be the coequalizer of the parallel pair Σf and Σg in \mathbf{Meas} . Suppose s and t are a split coequalizer of Σf and Σg in \mathbf{Meas} ,

$$\Sigma A \begin{array}{c} \xrightarrow{\Sigma f} \\ \xleftarrow[\Sigma g]{t} \end{array} \Sigma B \begin{array}{c} \xrightarrow{e} \\ \xleftarrow[s]{} \end{array} X$$

hence the following equations hold,

$$\begin{array}{rcl} e \circ \Sigma f & = & e \circ \Sigma g \\ e \circ s & = & 1_X \\ \Sigma f \circ t & = & 1_B \\ \Sigma g \circ t & = & s \circ e \end{array} \quad .$$

Define a function $h : \mathcal{G}X \rightarrow X$ using the composite

$$\begin{array}{ccc} \mathcal{G}(\Sigma B) & \xleftarrow{\mathcal{G}s} & \mathcal{G}(X) \\ \Sigma \epsilon_B \downarrow & & \downarrow h \\ \Sigma B & \xrightarrow{e} & X \end{array}$$

4.1. Proving the induced map h is a \mathcal{G} -algebra. The composite map h is a \mathcal{G} -algebra because it satisfies the two necessary properties,

- (1) The unit condition: $h \circ \eta_X = id_X$ follows from

$$\begin{aligned} \left(e \circ \Sigma \epsilon_B \circ \mathcal{G}s \circ \eta_X \right)(x) &= e(\epsilon_B(\mathcal{G}(s)(\delta_x))) \\ &= e(\epsilon_B(\delta_{s(x)})) \\ &= e(s(x)) \\ &= x \end{aligned}$$

- (2) The associativity property $h \circ \mathcal{G}(h) = h \circ \mu_X$ follows from analysis of the following \mathbf{Meas} -diagram

$$\begin{array}{ccccc}
\mathcal{G}^2(\Sigma A) & \xrightleftharpoons[\mathcal{G}^2(\Sigma g)]{\mathcal{G}^2(\Sigma f)} & \mathcal{G}^2(\Sigma B) & \xrightarrow{\mathcal{G}^2 e} & \mathcal{G}^2(X) \\
\downarrow \mathcal{G}(\Sigma \epsilon_A) \quad \mu_{\Sigma A} & & \downarrow \mathcal{G}(\Sigma \epsilon_B) \quad \mu_{\Sigma B} & & \downarrow \mathcal{G}h \quad \mu_X \\
\mathcal{G}(\Sigma A) & \xrightleftharpoons[\mathcal{G}(\Sigma g)]{\mathcal{G}(\Sigma f)} & \mathcal{G}(\Sigma B) & \xrightarrow{\mathcal{G}e} & \mathcal{G}(X) \\
\downarrow \Sigma \epsilon_A & & \downarrow \Sigma \epsilon_B & & \downarrow h \\
\Sigma A & \xrightleftharpoons[\Sigma g]{\Sigma f} & \Sigma B & \xrightarrow{e} & X
\end{array}$$

DIAGRAM 1. The basic construction used to show the associativity property of the \mathcal{G} -algebra h .

Because f and g are countably affine maps, and the fact ϵ is a natural transformation, the left hand square commutes serially, i.e., $\Sigma \epsilon_B \circ \mathcal{G}(\Sigma f) = \Sigma f \circ \Sigma \epsilon_A$ and $\Sigma \epsilon_B \circ \mathcal{G}(\Sigma g) = \Sigma g \circ \Sigma \epsilon_A$. Using the fact the middle row is a coequalizer - $\mathcal{G}e$ is a coequalizer of the pair $\mathcal{G}(\Sigma f), \mathcal{G}(\Sigma g)$ - and $e \circ \Sigma \epsilon_B$ also coequalizes the pair, it follows that h is the unique map such that $h \circ \mathcal{G}e = d \circ c$. Thus the lower right hand square also commutes.

The upper right hand square commutes serially, the square with the μ 's because μ is a natural transformation, and the square with $\mathcal{G}(\Sigma \epsilon_B)$ and $\mathcal{G}h$ because that square is just the bottom right square with the functor \mathcal{G} applies to it. Thus we have the three equations,

$$\begin{aligned}
\mathcal{G}h \circ \mathcal{G}^2 e &= \mathcal{G}e \circ \mathcal{G}(\Sigma \epsilon_B) \\
\mu_X \circ \mathcal{G}^2 e &= \mathcal{G}e \circ \mu_{\Sigma B} \\
h \circ \mathcal{G}e &= e \circ \Sigma \epsilon_B
\end{aligned}$$

Taking the first two equations and composing on the left by h yields

$$\begin{aligned}
h \circ \mathcal{G}h \circ \mathcal{G}^2 e &= h \circ (\mathcal{G}e \circ \mathcal{G}(\Sigma \epsilon_B)) = (h \circ \mathcal{G}e) \circ \mathcal{G}(\Sigma \epsilon_B) = (e \circ \Sigma \epsilon_B) \circ \mathcal{G}(\Sigma \epsilon_B) \\
h \circ \mu_X \circ \mathcal{G}^2 e &= h \circ (\mathcal{G}e \circ \mu_{\Sigma B}) = (h \circ \mathcal{G}e) \circ \mu_{\Sigma B} = (e \circ \Sigma \epsilon_B) \circ \mu_{\Sigma B}
\end{aligned}$$

Now since $\Sigma \epsilon_B \circ \mathcal{G}(\Sigma \epsilon_B) = \Sigma \epsilon_B \circ \mu_{\Sigma B}$ both rows of the above set of equations are equal,

$$h \circ \mathcal{G}h \circ \mathcal{G}^2 e = h \circ \mu_X \circ \mathcal{G}^2 e.$$

Since e is split by s it follows that $h \circ \mathcal{G}(h) = h \circ \mu_X$.

Thus $h : \mathcal{G}X \rightarrow X$ is a \mathcal{G} -algebra.

Using Lemma 1.1, it follows that there exists a super convex space structure on X defined by $\sum_{i=1}^{\infty} \alpha_i x_i = h(\sum_{i=1}^{\infty} \alpha_i \delta_{x_i})$ for every countable partition of one, which makes h a countably affine map. Moreover, by the same lemma, since e is a morphism of \mathcal{G} -algebras, it follows that e is a countably affine function.

4.2. Σ reflects coequalizers. To see that the countably affine map e (which is also measurable) is the coequalizer of the pair $\{f, g\}$ in $\mathbb{R}_{\infty}\text{-}\mathbf{SCvx}$ note that Σe is the coequalizer of $\{\Sigma f, \Sigma g\}$ in \mathbf{Meas} , and hence if q also factorizes the pair f, g

in $\mathbb{R}_\infty\text{-}\mathbf{SCvx}$, then Σq also coequalizes the pair Σf and Σg , and hence factorizes uniquely through X , say $\theta : X \rightarrow \Sigma C$,

$$\begin{array}{ccc}
 \mathcal{G}(\Sigma B) & \xrightarrow{\mathcal{G}e} & \mathcal{G}(X) \\
 \Sigma\epsilon_B \downarrow & & \downarrow h \\
 \Sigma B & \xrightarrow{e} & X \\
 & \searrow \Sigma q & \searrow \theta \\
 & & \Sigma C
 \end{array}
 \quad \begin{array}{l}
 \theta \circ h \\
 \theta
 \end{array}$$

Since $e \circ s = id_X$ it follows that

$$\begin{aligned}
 \theta &= \theta \circ (e \circ s) \\
 &= (\theta \circ e) \circ s \\
 &= \Sigma q \circ s
 \end{aligned}$$

Moreover, because s is a section of e , θ is the unique such arrow satisfying $\Sigma q = \theta \circ e$.

Now we proceed to show that θ is a morphism of \mathcal{G} -algebras, and hence countably affine. We need to show that $\theta \circ h = \Sigma\epsilon_C \circ \mathcal{G}(\theta)$. Substituting $h = e \circ \Sigma\epsilon_B \circ \mathcal{G}s$ into the left hand side of that equation yields

$$\begin{aligned}
 \theta \circ h &= \theta \circ e \circ \Sigma\epsilon_B \circ \mathcal{G}s \\
 &= \Sigma q \circ \Sigma\epsilon_B \circ \mathcal{G}s
 \end{aligned}$$

On the other hand, we have

$$\begin{aligned}
 \Sigma\epsilon_C \circ \mathcal{G}\theta &= \Sigma\epsilon_C \circ \mathcal{G}(\Sigma q \circ s) \\
 &= \Sigma\epsilon_C \circ \mathcal{G}(\Sigma q) \circ \mathcal{G}s
 \end{aligned}
 \tag{1}$$

Since q is a countably affine map, the naturality of ϵ gives the commutative square

$$\begin{array}{ccc}
 \mathcal{P}(\Sigma B) & \xrightarrow{\mathcal{P}\Sigma q} & \mathcal{P}(\Sigma C) \\
 \epsilon_B \downarrow & & \downarrow \epsilon_C \\
 B & \xrightarrow{q} & C
 \end{array}
 \quad \text{in } \mathbb{R}_\infty\text{-}\mathbf{SCvx}$$

and application of Σ then yields a morphism of the \mathcal{G} -algebras, $\Sigma q : \Sigma\epsilon_B \rightarrow \Sigma\epsilon_C$. Substituting that result, $\Sigma\epsilon_C \circ \mathcal{G}q = \Sigma q \circ \Sigma\epsilon_B$, into the right hand side of equation (1) then shows the desired result, $\theta \circ h = \Sigma\epsilon_C \circ \mathcal{G}\theta$, thereby proving θ is a \mathcal{G} -algebra morphism, and hence countably affine. That completes the proof that Σ reflects coequalizers.

4.3. Σ preserves coequalizers. Let $\{f, g\}$ be a parallel pair in $\mathbb{R}_\infty\text{-}\mathbf{SCvx}$ as above, and let e be the coequalizer of the parallel pair in $\mathbb{R}_\infty\text{-}\mathbf{SCvx}$. Applying the functor Σ gives the parallel pair $\{\Sigma f, \Sigma g\}$, and, up to an isomorphism, the coequalizer of that pair is given by the projection $q : \Sigma B \rightarrow B/E$ on the quotient set of B by the least equivalence relation $E \subseteq B \times B$ which contain all pairs $\langle f(a), g(a) \rangle$ for all $a \in A$. The σ -algebra of the quotient space is given by the largest σ -algebra on that set so that the canonical projection map is measurable. We denote that quotient set B/E

by X . The map q is an epimorphism (=surjection) in **Meas**, and using the axiom of choice, we can choose a section $s : X \rightarrow \Sigma B$ of q .

We have the following **Meas**-diagram

$$\begin{array}{ccccc} & & & & X \\ & & & \nearrow s & \downarrow \theta \\ \Sigma A & \xrightarrow[\Sigma g]{\Sigma f} & \Sigma B & \xrightarrow[\Sigma e]{q} & \Sigma C \end{array}$$

Since (X, q) is the coequalizer of $\{\Sigma f, \Sigma g\}$ and $(\Sigma C, \Sigma e)$ coequalizes that pair, there exists a unique map θ making the above diagram commute. Since q has a section s , that map θ is necessarily given by $\theta = \Sigma e \circ s$.

Now we can use Diagram 1 with e replaced by q , and the identical argument used there to conclude that $h = q \circ \Sigma e_B \circ \mathcal{G}s$ is a \mathcal{G} -algebra, and that the measurable function q is a morphisms of \mathcal{G} -algebras, $q : \Sigma e_B \rightarrow h$, and hence countably affine. Thus, X with the induced super convex space structure, is a super convex space, and q is a countably affine map. Therefore, viewed back in $\mathbb{R}_\infty\text{-}\mathbf{SCvx}$ where (C, e) is the coequalizer, there exists a unique map ψ such that $\psi \circ e = q$. Note that both ψ and θ are countably affine; θ is affine because $e = \theta \circ q$ where e and q are both countably affine, and q is surjective.

Using the four equations, $\theta = e \circ s$, $q = \psi \circ e$, $e = \theta \circ q$, and $q \circ s = 1_X$, we obtain

$$\psi \circ \theta = \psi \circ e \circ s = q \circ s = 1_X,$$

and

$$(\theta \circ \psi) \circ e = \theta \circ (\psi \circ e) = \theta \circ q = e = 1_C \circ e,$$

which implies, because e is an epimorphism in $\mathbb{R}_\infty\text{-}\mathbf{SCvx}$, that $\theta \circ \psi = 1_C$. Hence, for the induces super convex space structure on X , X is isomorphic to C and it follows that Σ preserves coequalizers.

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