CONTINUOUS QUIVERS OF TYPE A (III) EMBEDDINGS OF CLUSTER THEORIES

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Abstract. We continue the work started in parts (I) and (II) ([13] and [21], respectively). In this part we classify which continuous type A quivers are derived equivalent and introduce the new continuous cluster category with $\mathbf{E}$-clusters, which are a generalization of clusters. In the middle we provide a rigorous connection between the previous construction of the continuous cluster category and the new construction. We conclude with the introduction of a cluster theory, generalizing the notion of a cluster structure. Using this new notion, we demonstrate how one embeds known type A cluster theories into the new $\mathbf{E}$-cluster theory in a way compatible with mutation. This is part (III) in a series of work that will conclude with a continuous generalization of mutation for cluster theories.

Contents

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Introduction

History. Cluster algebras were introduced by Fomin and Zelevinksy in [5, 6, 7, 8]. One application is in particle physics to study scattering diagrams (see work of Golden, Goncharov, Spradlin, Vergud, and Volovicha in [10]). Two different categorifications of the cluster structure in a cluster algebra were introduced independently in [3] and [4]. The first and third authors introduced a continuous version of a cluster category in [15].

In Part (I) [13] the authors defined continuous quivers of type A, generalizing quivers of type A, and proved results about decomposition of pointwise finite-dimensional representations and the category of finitely generated representations. In Part (II) [21] the second author introduced the Auslander-Reiten space, a continuous analog to the Auslander-Reiten quiver, and proved results relating the AR-space to extensions in the representation category and distinguished triangles in the derived category. This is Part (III) of this series; the goal is to form a continuous generalization of clusters and mutations.

Contributions. We begin this work with classifying the derived categories of representations of continuous quivers of type A. In the finite case, the derived categories for any orientation of an A_n quiver are all equivalent [11]. This is not true for the continuum.

Theorem A (Theorem 2.3.5)**.** *Let $A_{\mathbb{R}}$ and $A'_{\mathbb{R}}$ be possibly different orientations of continuous type A quivers. Then $\mathcal{D}^b(A_{\mathbb{R}})$ and $\mathcal{D}^b(A'_{\mathbb{R}})$ are equivalent as triangulated categories if and only if one of the following holds:*

1. *both $A_{\mathbb{R}}$ and $A'_{\mathbb{R}}$ have finitely many sinks and sources,*
2. *the sinks and sources of $A_{\mathbb{R}}$ and $A'_{\mathbb{R}}$ are each bounded on exactly one side, or*
3. *the sinks and sources of $A_{\mathbb{R}}$ and $A'_{\mathbb{R}}$ are unbounded on both sides.*

We then define the new continuous cluster category $\mathcal{C}(A_{\mathbb{R}})$ as the orbit category of the doubling of $\mathcal{D}^b(A_{\mathbb{R}})$ via almost-shift (same method as in [15]). We define **E**-clusters (Definition 3.2.1) in $\mathcal{C}(A_{\mathbb{R}})$ and prove that if an element is **E**-mutable then the choice is unique and yields another **E**-cluster. In an **E**-cluster T there may exist an indecomposable X such that X is not **E**-mutable in T but is **E**-mutable in some other **E**-cluster T'. Thus we will not call an indecomposable frozen if it is not **E**-mutable in some particular **E**-cluster.

Theorem B (Theorem 3.2.8)**.** *Let T be a **E**-cluster and $V \in T$ be **E**-mutable with choice W. Then $(T \setminus \{V\}) \cup \{W\}$ is a **E**-cluster and any other choice W' for V is isomorphic to W.*

The new construction is not unrelated to the previous construction in [15]. In fact, the new continuous cluster category and the previous continuous cluster category are related by a localization of the derived category $\mathcal{D}^b(A_{\mathbb{R}})$ when $A_{\mathbb{R}}$ has finitely-many sinks and sources. In the previous construction, the category playing the role of the derived category was denoted $\mathcal{D}_\pi$.

Theorem C (Theorems 4.2.4 and 4.2.5)**.** *Assume $A_{\mathbb{R}}$ has finitely-many sinks and sources. Then there exist triangulated localizations $\mathcal{D}^b(A_{\mathbb{R}}) \to \mathcal{D}^b(A_{\mathbb{R}})[\mathcal{M}^{-1}]$ and $\mathcal{C}(A_{\mathbb{R}}) \to \mathcal{C}(A_{\mathbb{R}})[\mathcal{N}^{-1}]$ and triangulated equivalences of categories $G : \mathcal{D}^b(A_{\mathbb{R}})[\mathcal{M}^{-1}] \to \mathcal{D}_\pi$ and $H : \mathcal{C}(A_{\mathbb{R}})[\mathcal{N}^{-1}] \to \mathcal{C}_\pi$.*

The key difference between these new **E**-clusters and existing cluster structures (including the previous construction) is that weakening of the requirement for mutable elements. Instead of *there exists a unique choice* we require *there exists none or one*. This leads to our final contribution: that of cluster theories (Definition 5.1.1) which generalize cluster structures. In this definition a cluster theory on a skeletally-small Krull-Schmidt additive category $\mathcal{C}$ is a groupoid $\mathscr{T}_{\mathbf{P}}(\mathcal{C})$ induced by a pairwise compatibility condition $\mathbf{P}$ on $\mathrm{Ind}(\mathcal{C})$. In this language **E**-clusters form the **E**-cluster theory of $\mathcal{C}(A_{\mathbb{R}})$ (Example 5.1.8). With this new definition we define a rigorous description of an embedding of cluster theories (Definition 5.1.10).

Theorem D (Theorems 5.2.7, 5.3.10, 5.4.9)**.**

- *For any A_n quiver there is an embedding of cluster theories $\mathscr{T}_{\mathbf{N}_n}(\mathcal{C}(A_n)) \to \mathscr{T}_{\mathbf{E}}(\mathcal{C}(A_{\mathbb{R}}))$.*
- *For the straight A_∞ quiver there is an embedding of cluster theories $\mathscr{T}_{\mathbf{N}_\infty}(\mathcal{C}(A_\infty)) \to \mathscr{T}_{\mathbf{E}}(\mathcal{C}(A_{\mathbb{R}}))$.*
- *There is an embedding of cluster theories from the previous construction's cluster theory in* [15] *to our new construction: $\mathscr{T}_{\mathbf{N}_{\mathbb{R}}}(\mathcal{C}_\pi) \to \mathscr{T}_{\mathbf{E}}(\mathcal{C}(A_{\mathbb{R}}))$.*

Theorem D uses the cluster theories from [4], [12], and [15], respectively.

Future Work. The next (and final) part of this series will address the continuous generalization of the mutation and the embedding of cluster theories from the groupoid of A_∞ clusters using the completed infinity-gon (introduced in [1]). In particular, continuous mutation will allow for the encoding of the transfinite mutation introduced in the same paper along side new mutations of **E**-clusters. These embeddings and continuous structure should be useful as intuition, if not also as machinery, for a continuous cluster algebra that can handle the embeddings of all existing type A cluster algebras. The final part of this series will also address generalizations of the geometric models of type A cluster theories.

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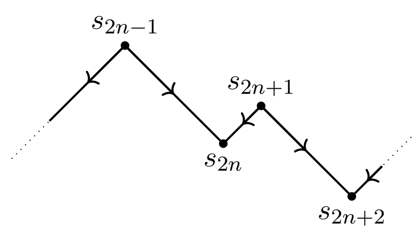
1. Parts (I) and (II)

In this section we recall the requisite information from parts (I) and (II) of this series.

1.1. Continuous Quivers of Type A and Their Representations.

In this subsection we recall relevant definitions and theorems about continuous quivers of type A and their representations from Part (I) of this series [13].

Fix a field k. We begin with definitions of a continuous quivers of type A and its representations. However, we first use a picture to illustrate the concept.



Definition 1.1.1. A quiver of continuous type A, denoted by $A_{\mathbb{R}}$, is a triple $(\mathbb{R}, S, \preceq)$, where:

(1) (a) $S \subset \mathbb{R}$ is a discrete subset, possibly empty, with no accumulation points.
  (b) Order on $S \cup \{\pm\infty\}$ is induced by the order of $\mathbb{R}$, and $-\infty < s < +\infty$ for $\forall s \in S$.
  (c) Elements of $S \cup \{\pm\infty\}$ are indexed by a subset of $\mathbb{Z} \cup \{\pm\infty\}$ so that s_n denotes the element of $S \cup \{\pm\infty\}$ with index n. The indexing must adhere to the following two conditions:
    i1 There exists $s_0 \in S \cup \{\pm\infty\}$.
    i2 If $m \leq n \in \mathbb{Z} \cup \{\pm\infty\}$ and $s_m, s_n \in S \cup \{\pm\infty\}$ then for all $p \in \mathbb{Z} \cup \{\pm\infty\}$ such that $m \leq p \leq n$ the element s_p is in $S \cup \{\pm\infty\}$.

(2) New partial order $\preceq$ on $\mathbb{R}$, which we call the orientation of $A_{\mathbb{R}}$, is defined as:
  p1 The $\preceq$ order between consecutive elements of $S \cup \{\pm\infty\}$ does not change.
  p2 Order reverses at each element of S.
  p3 If n is even s_n is a sink.
  p3' If n is odd s_n is a source.

Definition 1.1.2. Let $A_\mathbb{R} = (\mathbb{R}, S \preceq)$ be a continuous quiver of type A. A representation V of $A_\mathbb{R}$ is the following data:

- A vector space $V(x)$ for each $x \in \mathbb{R}$.
- For every pair $y \preceq x$ in $A_\mathbb{R}$ a linear map $V(x, y) : V(x) \to V(y)$ such that if $z \preceq y \preceq x$ then $V(x, z) = V(y, z) \circ V(x, y)$.

We say V is pointwise finite-dimensional if $\dim V(x) < \infty$ for all $x \in \mathbb{R}$.

Definition 1.1.3. Let $A_\mathbb{R}$ be a continuous quiver of type A and $I \subset \mathbb{R}$ be an interval. We denote by M_I the representation of $A_\mathbb{R}$ where

$$M_I(x) = \begin{cases} k & x \in I \\ 0 & \text{otherwise} \end{cases} \qquad\qquad M_I(x, y) = \begin{cases} 1_k & y \preceq x \in I \\ 0 & \text{otherwise.} \end{cases}$$

We call M_I an interval indecomposable.

Notation 1.1.4. Let $a < b \in \mathbb{R} \cup \{\pm\infty\}$. By the notation $|a, b|$ we mean an interval subset of $\mathbb{R}$ whose endpoints are a and b. The $|$'s indicate that a and b may or may not be in the interval. In practice this is (i) clear from context, (ii) does not matter in its context, or (iii) intentionally left as an unknown. For example, we may write $M_{|a,b|}$ to refer to one of four possible interval indecomposables. There is one exception: if a or b is $-\infty$ or $+\infty$, respectively, then the $|$ always means (or), respectively.

There are essentially three important results that we will use from [13]. The first is about the structure of a pointwise finite-dimensional representation of a continuous quiver of type A. It is important to note the last statement of the theorem recovers a result by Botnan and Crawley-Boevey in [2].

Theorem 1.1.5 (Theorems 2.3.2 and 2.4.13 in [13])**.** *Let $A_\mathbb{R}$ be a continuous quiver of type A. For any interval $I \subset \mathbb{R}$, the representation M_I of $A_\mathbb{R}$ is indecomposable. Any indecomposable pointwise finite-dimensional representation of $A_\mathbb{R}$ is isomorphic to M_I for some interval I. Furthermore, for any indecomposable representations V and W of $A_\mathbb{R}$, $V \cong W$ if and only if $\operatorname{supp} V = \operatorname{supp} W$. Finally, any pointwise finite-dimensional representation V of $A_\mathbb{R}$ is the direct sum of interval indecomposables.*

Definition 1.1.6. Let $A_\mathbb{R}$ be a continuous quiver of type A. By $\mathrm{Rep}_k^{\mathrm{pwf}}(A_\mathbb{R})$ we denote the category of pointwise finite-dimensional representations of $A_\mathbb{R}$. That is, for any representation V in $\mathrm{Rep}_k^{\mathrm{pwf}}(A_\mathbb{R})$ and $x \in \mathbb{R}$, the k-vector space $V(x)$ is finite-dimensional.

In Section 3 we will need the following classification of indecomposable pointwise finite-dimensional projective representations.

Theorem 1.1.7 (Theorem 2.1.6 and Remark 2.4.16 in [13])**.** *Let P be a projective indecomposable in $\mathrm{Rep}_k^{\mathrm{pwf}}(A_\mathbb{R})$. Then there exists $a \in \mathbb{R} \cup \{\pm\infty\}$ such that P is isomorphic to one of the following indecomposables: P_a, $P_{(a}$, or $P_{a)}$:*

$$P_a(x) = \begin{cases} k & x \preceq a \\ 0 & \textit{otherwise} \end{cases} \qquad\qquad P_a(x, y) = \begin{cases} 1_k & y \preceq x \preceq a \\ 0 & \textit{otherwise} \end{cases}$$

$$P_{(a}(x) = \begin{cases} k & x \preceq a \textit{ and } x > a \textit{ in } \mathbb{R} \\ 0 & \textit{otherwise} \end{cases} \qquad P_{(a}(x, y) = \begin{cases} 1_k & y \preceq x \preceq a \textit{ and } x, y > a \\ 0 & \textit{otherwise} \end{cases}$$

$$P_{a)}(x) = \begin{cases} k & x \preceq a \textit{ and } x < a \textit{ in } \mathbb{R} \\ 0 & \textit{otherwise} \end{cases} \qquad P_{a)}(x, y) = \begin{cases} 1_k & y \preceq x \preceq a \textit{ and } x, y < a \\ 0 & \textit{otherwise} \end{cases}$$

Definition 1.1.8. Let $A_\mathbb{R}$ be a continuous quiver of type A. By $\mathrm{rep}_k(A_\mathbb{R})$ we denote the full subcategory of $\mathrm{Rep}_k^{\mathrm{pwf}}(A_\mathbb{R})$ whose objects are finitely generated by the indecomposable projectives in Theorem 1.1.7. In particular, the indecomposable projectives in both categories are the same.

Theorem 1.1.9 (Theorem 3.0.1 in [13]). *Let $A_{\mathbb{R}}$ be a continuous quiver of type A. The following hold.*

1. *For any pair of indecomposable representations M_I and M_J in $\mathrm{rep}_k(A_{\mathbb{R}})$, either $\mathrm{Hom}(M_I, M_J) \cong k$ or $\mathrm{Hom}(M_I, M_J) = 0$.*
2. *The category $\mathrm{rep}_k(A_{\mathbb{R}})$ is abelian.*
3. *The category $\mathrm{rep}_k(A_{\mathbb{R}})$ is Krull-Schmidt but not artinian.*
4. *The global dimension of $\mathrm{rep}_k(A_{\mathbb{R}})$ is 1.*
5. *For any indecomposables M_I and M_J in $\mathrm{rep}_k(A_{\mathbb{R}})$, either $Ext^1(M_I, M_J) \cong k$ or $Ext^1(M_I, M_J) = 0$.*
6. *The category $\mathrm{rep}_k(A_{\mathbb{R}})$ has some, but not all, Auslander-Reiten sequences (fully classified in [21, Table 3.1.3]).*

1.2. **The AR-space of $\mathrm{rep}_k(A_{\mathbb{R}})$ and $\mathcal{D}^b(A_{\mathbb{R}})$.** In this subsection we recall the necessary definitions and theorems from Part (II) of this series, [21]. We start with λ functions, which are used to construct the Auslander-Reiten space, or AR-space, of $\mathrm{rep}_k(A_{\mathbb{R}})$ and $\mathcal{D}^b(A_{\mathbb{R}})$, the latter of which is constructed in the usual way out of an abelian category.

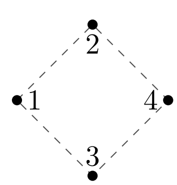
Definition 1.2.1. Let $z \in \mathbb{R}$. Then $z = 2n\pi + w$ for $n \in \mathbb{Z}$ and $0 \leq w \leq 2\pi$. So let the function $\lambda : \mathbb{R} \to \mathbb{R}$ be given by

$$\lambda(z) = \lambda(2n\pi + w) = \begin{cases} w - \frac{\pi}{2} & 0 \leq w \leq \pi \\ -w + \frac{3\pi}{2} & \pi \leq w \leq 2\pi. \end{cases}$$

A <u>λ function</u> is a function $\mathbb{R} \to [-\frac{\pi}{2}, \frac{\pi}{2}]$ defined by $x \mapsto \lambda(x - \kappa)$ for a fixed $\kappa \in [-\pi, \pi]$.

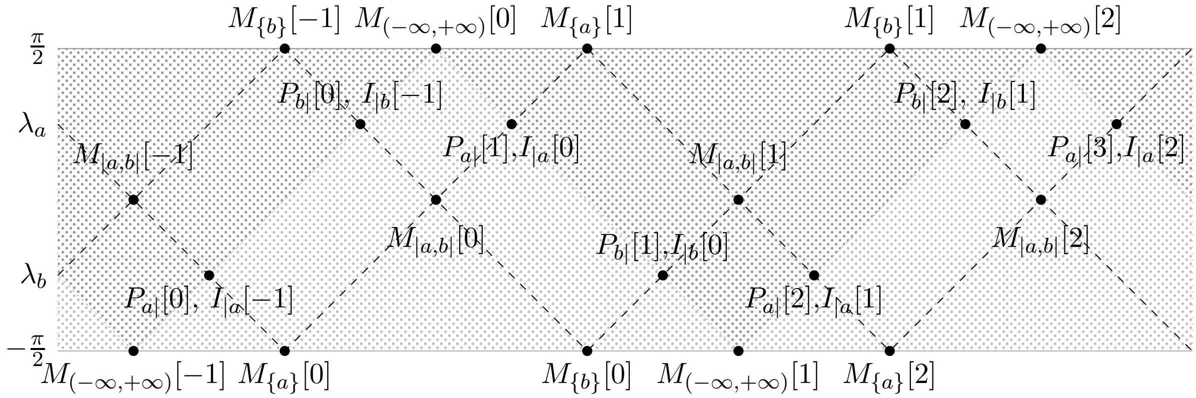
Recall $\mathbf{\Gamma} : \mathrm{Ind}(\mathrm{rep}_k(A_{\mathbb{R}})) \to \mathbb{R} \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ is a function from the isomorphism classes of indecomposables in $\mathrm{rep}_k(A_{\mathbb{R}})$ to the real plane. The map $\mathbf{\Gamma}^b$ is the natural extension of $\mathbf{\Gamma}$ to all the indecomposables of $\mathcal{D}^b(A_{\mathbb{R}})$ where $\mathbf{\Gamma}^b\, V[1] = (x + \pi, -y)$ for each indecomposable V with $\mathbf{\Gamma}^b\, V = (x, y)$. These functions are used to define the continuous generalization of the Auslander-Reiten quiver, called the Auslander-Reiten space, of $\mathrm{rep}_k(A_{\mathbb{R}})$ and $\mathcal{D}^b(A_{\mathbb{R}})$ ([21, Definitions 4.1.9 and 5.2.5]).

Recall each (isomorphism class of) indecomposables in $\mathrm{rep}_k(A_{\mathbb{R}})$ or $\mathcal{D}^b(A_{\mathbb{R}})$ have a position ([21, Definition 4.1.2]) 1, 2, 3, or 4 in the AR-space of $\mathrm{rep}_k(A_{\mathbb{R}})$ or $\mathcal{D}^b(A_{\mathbb{R}})$, respectively. The positions are to be thought of as occupying the four corners of a diamond:



In particular, two indecomposables V and W in $\mathrm{rep}_k(A_{\mathbb{R}})$ or $\mathcal{D}^b(A_{\mathbb{R}})$ are isomorphic if and only if their position and image under $\mathbf{\Gamma}$ or $\mathbf{\Gamma}^b$, respectively, are the same.

Example 1.2.2. Recall from [21] that when $A_{\mathbb{R}}$ has the straight descending orientation the following is part of the AR-space:



In [21], an extra generalized metric is defined; this allows for the description of lines and their slopes in the AR-space of $\text{rep}_k(A_{\mathbb{R}})$ and $\mathcal{D}^b(A_{\mathbb{R}})$. In particular this also allows for rectangles and almost-complete rectangles in the AR-space of these categories as well. This results in the following theorems that will be used in Section 2.3. The "good slopes" can be thought of as analogous to a 45° angle and by "scaling" we mean "scaling of morphisms".

Theorem 1.2.3 (Theorem 4.3.11 in [21])**.** *Let $V = M_{|a,b|}$ and $W = M_{|c,d|}$ be indecomposables in $\text{rep}_k(A_{\mathbb{R}})$ such that $V \not\cong W$. Then there is a nontrivial extension $V \hookrightarrow E \twoheadrightarrow W$ if and only if there exists a rectangle or almost complete rectangle whose corners are the indecomposables in the sequence with V as the left-most corner and W as the right-most corner.*

- *If the rectangle is complete E is a direct sum of two indecomposables.*
- *If the rectangle is almost complete E is indecomposable.*

Furthermore, there is a bijection

$$\begin{gathered}\{\textit{rectangles and almost complete rectangles with "good" slopes of sides in AR-space of } \text{rep}_k(A_{\mathbb{R}})\} \\ \updownarrow \cong \\ \{\textit{nontrivial extensions of indecomposables by indecomposables up to scaling and isomorphisms}\}\end{gathered}$$

In the following theorem, we consider a triangle and any of its rotations to be distinct for the purposes of the statement of the theorem. We also say "nontrivial triangle" to mean a distinguished triangle that is not of the form $(A \to A \to 0 \to)$, $(A \to 0 \to A[1] \to)$, or $(0 \to A \to A \to)$.

Theorem 1.2.4 (Theorem 5.2.10 in [21])**.** *Let $V = M_{|a,b|}[m]$ and $W = M_{|c,d|}[n]$ be indecomposables in $\mathcal{D}^b(A_{\mathbb{R}})$ such that $V \not\cong W$. Then there is a nontrivial distinguished triangle $V \to U \to W \to$ if and only if there exists a rectangle or almost complete rectangle in the AR-space of $\mathcal{D}^b(A_{\mathbb{R}})$ whose corners are the indecomposables in the triangle with V as the left-most corner and W as the right-most corner.*

- *If the rectangle is complete E is a direct sum of two indecomposables.*
- *If the rectangle is almost complete E is indecomposable.*

Furthermore, there is a bijection

$$\begin{gathered}\{\textit{rectangles and almost complete rectangles with "good" slopes of sides in AR-space of } \mathcal{D}^b(A_{\mathbb{R}})\} \\ \updownarrow \cong \\ \{\textit{nontrivial triangles with first and third term indecomposable up to scaling and isomorphisms}\}\end{gathered}$$

2. Derived Equivalence

In this section we classify the derived categories of finitely generated representations of quivers of continuous type A.

2.1. **The Octahedral Axiom.** We recall the octahedral axiom of a triangulated category, which we use explicitly in Lemma 2.2.6. In particular we use [19, Proposition 1.4.6] in Neeman's book, which the author proves is equivalent to the octahedral axiom. Suppose the following are distinguished triangles:

$$U \xrightarrow{f} V \xrightarrow{g'} W' \xrightarrow{h'} U[1]$$

$$V \xrightarrow{g} W \xrightarrow{h} U' \xrightarrow{f'} V[1].$$

Then the following distinguished triangles also exist:

$$U \xrightarrow{g\circ f} W \xrightarrow{i} V' \xrightarrow{j} U[1]$$

$$W' \xrightarrow{i'} V' \xrightarrow{j'} U' \xrightarrow{g'[1]\circ f'} W'[1],$$

such that $h = j' \circ i$ and $h' = j \circ i'$ (note the mismatched primes). This is often drawn as the lower and upper part of an octahedron. Another way of stating the axiom is that given the "lower cap" on the left below the "upper cap" on the right exists as well:

$$\begin{tikzcd}
U' \arrow[dd, "{[1]}"'] \arrow[dr, "{[1]}"] & & W \arrow[ll] \\
& V \arrow[ur] \arrow[dl] & \\
W' \arrow[rr, "{[1]}"'] & & U \arrow[uu] \arrow[ul]
\end{tikzcd}
\qquad
\begin{tikzcd}
U' \arrow[dd, "{[1]}"'] & & W \arrow[ll] \arrow[dl] \\
& V' \arrow[ul] \arrow[dr, "{[1]}"'] & \\
W' \arrow[ur] \arrow[rr, "{[1]}"'] & & U. \arrow[uu]
\end{tikzcd}$$

By an immediate result in [19], we also have the following distinguished triangles:

$$V \xrightarrow{\begin{bmatrix} g \\ g' \end{bmatrix}} W \oplus W' \xrightarrow{\begin{bmatrix} i & -i' \end{bmatrix}} V' \xrightarrow{f[1]\circ j = f'\circ j'} V[1].$$

$$V' \xrightarrow[\begin{bmatrix} j \\ j' \end{bmatrix}]{} U[1] \oplus U' \xrightarrow[\begin{bmatrix} f[1] & -f' \end{bmatrix}]{} V[1] \xrightarrow[(i\circ g)[1]=(i'\circ g')[1]]{} V'[1].$$

We will need the following well-known facts, also from [19], albeit in a slightly different form. Thus, we will state it as a proposition and provide a concise proof.

Proposition 2.1.1. *Let $\mathcal{D}$ be a triangulated category and let*

$$V \xrightarrow{f} W_1 \xrightarrow{p_1} U_1 \xrightarrow{q_1} V[1]$$

be a distinguished triangle where f is nontrivial. Let $h : W_1 \to W_2$ such that h and $h \circ f$ are nontrivial. Then

$$V \xrightarrow{\begin{bmatrix} f \\ h \circ f \end{bmatrix}} W_1 \oplus W_2 \xrightarrow{\begin{bmatrix} p_1 & 0 \\ h & -1 \end{bmatrix}} U_1 \oplus W_2 \xrightarrow{\begin{bmatrix} q_1 & 0 \end{bmatrix}} V[1] \tag{1}$$

exists as a distinguished triangle in $\mathcal{D}$. Dually, given the following distinguished triangle in $\mathcal{D}$:

$$V_2 \xrightarrow{g} W \xrightarrow{p_2} U_2 \xrightarrow{q_2} V_2[2],$$

and morphism $h : V_1 \to V_2$ where g, h, and $g \circ h$ are nontrivial there exists

$$V_1 \oplus V_2 \xrightarrow{\begin{bmatrix} g\circ h & g \end{bmatrix}} W \xrightarrow{\begin{bmatrix} 0 \\ p_2 \end{bmatrix}} V_1[1] \oplus U_2 \xrightarrow{\begin{bmatrix} 1 & 0 \\ h & q_2 \end{bmatrix}} (V_1 \oplus V_2)[1] \tag{2}$$

as a distinguished triangle in $\mathcal{D}$.

Proof. We prove that (1) is a distinguished triangle. The proof of (2) is similar. Set $g = h \circ f$. We start with the distinguished triangle from the statement and the following distinguished triangles:

$$W_1 \xrightarrow{h} W_2 \longrightarrow E \longrightarrow W_1[1]$$

$$V \xrightarrow{g} W_2 \xrightarrow{p_2} U_2 \xrightarrow{q_2} V[1].$$

By the octahedral axiom this yields the distinguished triangles

$$U_1 \xrightarrow{r} U_2 \xrightarrow{s} E \xrightarrow{t} U_1[1]$$

$$W_1 \xrightarrow{[p_1\, h]^t} U_1 \oplus W_2 \xrightarrow{[r\, -p_2]} U_2 \xrightarrow{q_3} W_1[1].$$

We start again with the triangles

$$U_2[-1] \xrightarrow{q_2[-1]} V \xrightarrow{g} W_2 \xrightarrow{p_2} U_2$$

$$V \xrightarrow{f} W_1 \xrightarrow{p_1} U_1 \xrightarrow{q_1} V[1]$$

$$U_2[-1] \xrightarrow{q_3[-1]} W_1 \xrightarrow{[p_1\, h]^t} U_1 \oplus W_2 \xrightarrow{[r\, -p_2]} U_2.$$

Noting that $[p_1\, h]^t \circ f = [0\, 1]^t \circ g$ we see $q_3 = f[1] \circ q_2$. So, we may apply the octahedral axiom again to obtain the desired triangle (1). □

We specifically desire to use Proposition 2.1.1 as the following corollary.

Corollary 2.1.2. *Let $\mathcal{D}$ be a triangulated category. Consider the distinguished triangles from Proposition 2.1.1:*

$$V \xrightarrow{f} W_1 \xrightarrow{p_1} U_1 \xrightarrow{q_1} V[1]$$

$$V \xrightarrow{h\circ f} W_2 \xrightarrow{p_2} U_2 \xrightarrow{q_2} V[1].$$

where $h : W_1 \to W_2$, f, and $h \circ f$ are nontrivial. If U_2 is not isomorphic to summands of V, W_1, W_2, or U_1 then it does not appear as a summand in any distinguished triangle of the form

$$V \xrightarrow{\begin{bmatrix} f \\ h\circ f \end{bmatrix}} W_1 \oplus W_2 \longrightarrow U \longrightarrow V[1]. \tag{*}$$

Proof. The distinguished triangle $*$ will be isomorphic to distinguished triangle (1) in Proposition 2.1.1. Since U_2 does not appear in (1) as a summand it will not appear in $*$ as a summand. □

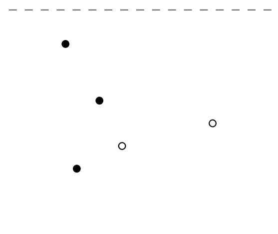
Proposition 2.1.3. *Let V and W be indecomposables in $\mathcal{D}^b(A_{\mathbb{R}})$ such that $V \not\cong W$. Then* $\mathrm{Hom}(V, W) \cong k$ *implies* $\mathrm{Hom}(W, V) = 0$.

Proof. This follows directly from [21, Proposition 4.4.2, Lemma 5.2.9]. □

Construction 2.1.4. Consider a finite direct sum $W = \bigoplus W_i$ of indecomposables in $\mathcal{D}^b(A_{\mathbb{R}})$. Using Proposition 2.1.3 we can consider a subset X_W of summands of W determined by

- If $W_i, W_j \in X_W$ and $i \neq j$ then $\mathrm{Hom}(W_i, W_j) = 0$.
- If $W_i \in X_W$ and there exists W_j such that $\mathrm{Hom}(W_j, W_i) \cong k$ then $W_i \cong W_j$.

(Notice X_W may not be unique.) Let Y_W be the multisubset of summands of W such that $\bigoplus_{X_W \amalg Y_W} W_i \cong W$. Below is an example depicted in the AR-space of $\mathcal{D}^b(A_{\mathbb{R}})$; members of X_W are filled in and members of Y_W are not.



X_W and Y_W

Now consider a finite sum $V = \bigoplus V_i$ of indecomposables in $\mathcal{D}^b(A_{\mathbb{R}})$. Consider the subset ${}_VX$ of summands of V determined by

- If $V_i, V_j \in_V X$ and $i \neq j$ then $\mathrm{Hom}(V_i, V_j) = 0$.
- If $V_i \in {}_VX$ and there exists W_j such that $\mathrm{Hom}(V_i, V_j) \cong k$ then $V_i \cong V_j$.

(Notice ${}_VX$ may not be unique.) Let ${}_VY$ be the multisubset of summands of V such that $\bigoplus_{{}_VX \amalg {}_VY} V_i \cong V$.

Lemma 2.1.5. *Let $f : V \to W$ be a morphism in $\mathcal{D}^b(A_{\mathbb{R}})$. Suppose V is indecomposable and for each summand of W the composition $f_i : V \xrightarrow{f} W \xrightarrow{\pi} W_i$ is nonzero. Choose a set X_W and Y_W as in Construction 2.1.4. Denoting by π the projection $W \twoheadrightarrow \bigoplus_{Y_W} W_i$, if*

$$V \xrightarrow{[f_i]_{W_i \in X_W}} \bigoplus_{X_W} W_i \xrightarrow{\quad g \quad} E \xrightarrow{\quad h \quad} V[1]$$

is a distinguished triangle in $\mathcal{D}^b(A_{\mathbb{R}})$ so is

$$V \xrightarrow{\quad f \quad} W \xrightarrow{\begin{bmatrix} g \\ \pi \end{bmatrix}} E \oplus \left(\bigoplus_{Y_W} W_i\right) \xrightarrow{\begin{bmatrix} h & 0 \end{bmatrix}} V[1]. \tag{1}$$

Dually, suppose instead W is indecomposable, each $f_i : V_i \to W$ is nonzero, and we've chosen ${}_VX$ and ${}_VY$ from Construction 2.1.4. Denoting by ι the inclusion $\bigoplus_{{}_VY} V_i \hookrightarrow V$, if

$$\bigoplus_{{}_VX} V_i \xrightarrow{[f_i]_{V_i \in {}_VX}} W \xrightarrow{\quad g \quad} E \xrightarrow{\quad h \quad} (\bigoplus_{{}_VX} V_i)[1]$$

is a distinguished triangle in $\mathcal{D}^b(A_{\mathbb{R}})$ so is

$$V \xrightarrow{\quad f \quad} W \xrightarrow{\begin{bmatrix} 0 \\ g \end{bmatrix}} \left(\left(\bigoplus_{{}_VY} V_i\right)[1]\right) \oplus E \xrightarrow{\begin{bmatrix} \iota & h \end{bmatrix}} V[1]. \tag{2}$$

Proof. We'll prove (1) since the proof of (2) is similar. Since $\mathrm{Hom}(A, B) \cong k$ or $=0$ for all indecomposables A and B in $\mathcal{D}^b(A_{\mathbb{R}})$, if, for a third indecomposable C as well,

$$\mathrm{Hom}(A, B) \cong \mathrm{Hom}(A, C) \cong \mathrm{Hom}(B, C) \cong k$$

then given any pair of morphisms $f : A \to B$ and $g : A \to C$ there exists $h : B \to C$ such that $g = hf$. We may then apply Corollary 2.1.2 and obtain (1). $\square$

2.2. **Triangles and the Geometry of the AR-space of $\mathcal{D}^b(A_\mathbb{R})$.** In this subsection we show how the geometry of the AR-space of $\mathcal{D}^b(A_\mathbb{R})$ is closely tied to the distinguished triangles in $\mathcal{D}^b(A_\mathbb{R})$. We will use these connections in Section 2.3.

Definition 2.2.1. For each object V in $\mathcal{D}^b(A_\mathbb{R})$, $V \cong \bigoplus_{i=1}^{\ell} M_{|a_i,b_i|}[n_i]$ for intervals $|a_i, b_i|$ and $n_i \in \mathbb{Z}$. Reindex the $|a_i, b_i|$'s such that the following hold.

- if $n_i < n_j$ then $i < j$,
- if $n_i = n_j$ and $a_i < a_j$ then $i < j$,
- if $n_i = n_j$, $a_i = a_j$, and $b_i < b_j$ then $i < j$,
- if $n_i = n_j$, $a_i = a_j$, $b_i = b_j$, $a_i \in |a_i, b_i|$, and $a_j \notin |a_j, b_j|$ then $i < j$, and
- if $n_i = n_j$, $a_i = a_j$, $b_i = b_j$, $a_i, a_j \in |a_i, b_i|$ or $a_i, a_j \notin |a_i, b_i|$, $b_i \notin |a_i, b_i|$, and $b_j \in |a_j, b_j|$ then $i < j$.

This ordering determines a unique object ιV, **not just up to isomorphism**, such that

$$V \cong \underbrace{(((\cdots(M_{|a_1,b_1|}[n_1] \oplus M_{|a_2,b_2|}[n_2]) \oplus \cdots) \oplus M_{|a_{\ell-1},b_{\ell-1}|}[n_{\ell-1}]) \oplus M_{|a_\ell,b_\ell|}[n_\ell])}_{\iota V}.$$

Fix an isomorphism $\iota_V : V \to \iota V$ (where $\iota V = \bigoplus_{i=1}^{\ell} M_{|a_i,b_i|}[n_i]$ as we've described). Note that in some cases ι_V will be the identity.

Let $f : V \to W$ be a morphism in $\mathcal{D}^b(A_\mathbb{R})$. Then there exists a unique morphism $\iota(f) : \iota(V) \to \iota(W)$ that makes the following diagram commute:

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\iota_V"'] & W \arrow[d, "\iota_W"] \\ \iota(V) \arrow[r, dashed, "\exists!\iota(f)"'] & \iota(W). \end{tikzcd}$$

.

Remark 2.2.2. It is straightforward to check that $\iota : \mathcal{D}^b(A_\mathbb{R}) \to \mathcal{D}^b(A_\mathbb{R})$ is a triangulated equivalence of categories as the image of ι is a skeleton of $\mathcal{D}^b(A_\mathbb{R})$.

Definition 2.2.3. Let $V \cong \bigoplus V_i$ be an object in $\mathcal{D}^b(A_\mathbb{R})$ where each V_i is indecomposable and let $\{(x_i, y_i)\} = \{\mathbf{\Gamma}^b\, V_i\}$. Let $r \in \mathbb{R}$ and set $z_i = x_i - r$ for each V_i. If each (z_i, y_i) is in the image of $\mathbf{\Gamma}^b$ then there exists $M_{|a_i,b_i|}[n_i]$ for each V_i such that $\mathbf{\Gamma}^b\, M_{|a_i,b_i|}[n_i] = (z_i, y_i)$ and the position of $M_{|a_i,b_i|}[n_i]$ is the same as the position of V_i. Then we write $T_r V$ to mean $\bigoplus M_{|a_i,b_i|}[n_i]$, indexed and parenthesized in the same way as in Definition 2.2.1. (If $r = 0$ then $T_r V = \iota(V)$.)

Let $f : M_{|a,b|}[m] \to M_{|c,d|}[n]$ be a morphism in $\mathcal{D}^b(A_\mathbb{R})$ such that $T_r M_{|a,b|}[m]$ and $T_r M_{|c,d|}[n]$ are defined. By [21, Proposition 5.2.8 and Lemma 5.2.9],

$$\mathrm{Hom}_{\mathcal{D}^b(A_\mathbb{R})}(M_{|a,b|}[m], M_{|c,d|}[n]) \cong \mathrm{Hom}_{\mathcal{D}^b(A_\mathbb{R})}(T_r M_{|a,b|}[m], T_r M_{|c,d|}[n]).$$

If f is nonzero then both Hom sets are k. Then f is a scalar in $\mathrm{Hom}_{\mathcal{D}^b(A_\mathbb{R})}(M_{|a,b|}[m], M_{|c,d|}[n])$. We define $T_r f$ to be the same scalar in $\mathrm{Hom}_{\mathcal{D}^b(A_\mathbb{R})}(T_r M_{|a,b|}[m], T_r M_{|c,d|}[n])$.

For arbitrary map $f : V \to W$ in $\mathcal{D}^b(A_\mathbb{R})$, $\iota(f)$ is a direct sum of morphisms $\hat{f}_{i,j} : M_{|a_i,b_i|}[m_i] \to M_{|c_j,d_j|}[n_j]$. Let $f : V \to W$ be a morphism in $\mathcal{D}^b(A_\mathbb{R})$ and $r \in \mathbb{R}$ such that both $T_r V$ and $T_r W$ are defined. Then we define $T_r f$ to be $\bigoplus_{\hat{f}_{i,j}} T_r \hat{f}_{i,j}$.

Remark 2.2.4. Note that aside from our choice of r, T_r does not depend on any additional choices beyond those for ι in Definition 2.2.1.

Definition 2.2.5. Let $V \cong \bigoplus V_i$ and $W \cong \bigoplus W_j$ be objects in $\mathcal{D}^b(A_{\mathbb{R}})$ where each V_i and W_j are indecomposable. Let $f : V \to W$ be a morphism. Consider the distinguished triangle $V \xrightarrow{f} W \xrightarrow{g} U \xrightarrow{h}$. We say U is determined by geometry and f if for any $r \in \mathbb{R}$ such that $T_r V$ and $T_r W$ are defined, there exists a distinguished triangle $T_r V \xrightarrow{T_r f} T_r W \to T_r U \to$ in $\mathcal{D}^b(A_{\mathbb{R}})$.

A visual example of the proof technique to the following lemma is exhibited in Example 2.2.7, stated afterwards.

Lemma 2.2.6. *Let V be indecomposable and W an object in $\mathcal{D}^b(A_{\mathbb{R}})$. Suppose $f : V \to W$ is a nonzero morphism and consider the distinguished triangle $V \xrightarrow{f} W \to U \to$. Then U is determined by geometry and f. The conclusion holds if instead W is indecomposable and V is some object.*

Furthermore:

1. *If V and W together have n indecomposable summands then U has at most n indecomposable summands.*
2. *If V is indecomposable, for each indecomposable summand U_j of U there is an indecomposable summand W_i of W such that the line segment with endpoints W_i and U_j has slope $\pm(1,1)$.*
3. *If W is indecomposable, for each indecomposable summand U_i of U there is an indecomposable summand V_j of V such that the line segment with endpoints $U_i[-1]$ and V_j has slope $\pm(1,1)$.*

Proof. **Setup and Base Case.** We prove the statement where V is indecomposable as the proof when W is indecomposable is similar. In particular we'll prove (1) and (2) in the enumerated list (as (3) is part of the case when W is the indecomposable). The proof will be by induction on the number of summands of W. Our base case is 1; i.e. W is also indecomposable. The base case follows from Theorem 1.2.4.

Induction Setup and Trivial Case. For induction assume statements (1) and (2) hold when V is indecomposable and W has n or fewer indecomposable summands. Suppose $W \cong \bigoplus_{i=1}^n W_i$ where each W_i is indecomposable. For each summand W_i, let $f_i : V \to W_i$ be such that $f = [f_1 \ \cdots \ f_n]^t$.

Let W_{n+1} be an additional indecomposable and $f_{n+1} : V \to W_{n+1}$ a morphism. If $f_{n+1} = 0$ we are done as we obtain the distinguished triangle

$$V \xrightarrow{\begin{bmatrix} f \\ 0 \end{bmatrix}} W \oplus W_{n+1} \xrightarrow{\begin{bmatrix} g & 0 \\ 0 & 1 \end{bmatrix}} U \oplus W_{n+1} \xrightarrow{\begin{bmatrix} h & 0 \end{bmatrix}} V[1].$$

$\mathbf{Hom(W,W_{n+1}) \neq 0}$ **or** $\mathbf{Hom(W_{n+1},W) \neq 0.}$ Now assume $f_{n+1} \neq 0$. If $\operatorname{Hom}(W, W_{n+1})$ is not 0 such that f_{n+1} factors through f then again the lemma follows using Lemma 2.1.5. If $\operatorname{Hom}(W_{n+1}, W) \neq 0$ then there is at least one summand W_j of W such that $\operatorname{Hom}(W_{n+1}, W_i) \neq 0$. In this case, if $f_i = 0$ then we can reverse the roles of W_{n+1} and W_i and use the induction hypothesis where $f_{n+1} = 0$. If $f_i \neq 0$ then our induction hypothesis holds for $V \to (\bigoplus_1^{j-1} W_i) \oplus (\bigoplus_{j+1}^{n+1} W_i)$ and we apply Lemma 2.1.5 again.

If $\operatorname{Hom}(W, W_{n+1}) \neq 0$ but f_{n+1} does not factor through W via f then for each summand W_i of W such that $\operatorname{Hom}(W_i, W_{n+1}) \neq 0$ we know $f_i = 0$. Then we can reverse the roles of one such W_i and W_{n+1} and use the induction hypothesis where $f_{n+1} = 0$.

$\mathbf{Hom(W,W_{n+1}) = 0 = Hom(W_{n+1},W).}$ Now suppose $f_{n+1} \neq 0$ and $\operatorname{Hom}(W, W_{n+1}) = 0 = \operatorname{Hom}(W_{n+1}, W)$. Choose an X_W (Construction 2.1.4). If X_W has fewer than n elements then we are done since $X_W \cup \{W_{n+1}\}$ has n or fewer elements and we may then apply the induction hypothesis. So suppose X_W contains each summand of W.

Note $\operatorname{Hom}(W, W_{n+1}) = 0$ but $f_i \neq 0$ for all $1 \leq i \leq n+1$. Thus the W_i, for all $1 \leq i \leq n+1$, are totally ordered by y-coordinate and position (2 is greater than 1 and 4 which are greater than 3).

Reindex the W_is such that $W_i < W_{i+1}$ in the total order and replace W with $\bigoplus_{i=1}^{n} W_i$ in the new index. We have the following distinguished triangles:

$$U'[-1] \xrightarrow{h'[-1]} V \xrightarrow{f_{n+1}} W_{n+1} \xrightarrow{g'} U'$$

$$V \xrightarrow{f} W \xrightarrow{g} U \xrightarrow{h} V[1]$$

$$U'[-1] \xrightarrow{f\circ h'[-1]} W \xrightarrow{p} E \xrightarrow{q} U'.$$

By our base case U' has at most two indecomposable summands and by the rest of Theorem 1.2.4 the slopes of the line segments from W_{n+1} to each summand is $\pm(1,1)$. By [21, Lemma 5.2.9] W_n can map to at most one of the indecomposable summands of U' and that is the only possible summand that can map to $W_n[1]$.

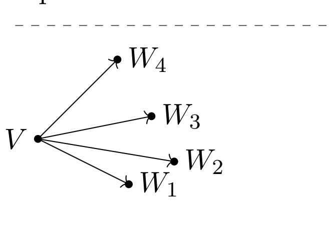
Thus W can map to at most one indecomposable summand of U'. Then by induction E is determined by geometry and $f\circ h'[-1]$. By the formulation of the octahedral axiom we've stated, we obtain the distinguished triangles

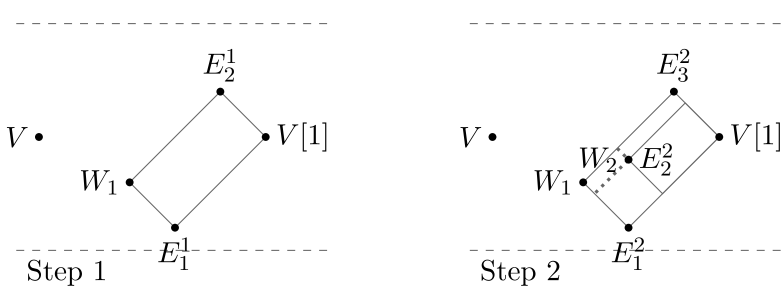
$$W_{n+1} \xrightarrow{r} E \xrightarrow{s} U \xrightarrow{t} W_{n+1}[1]$$

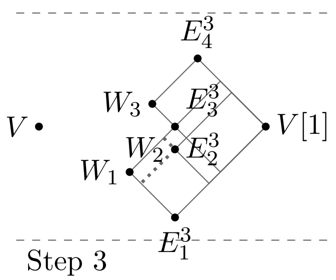
$$V \xrightarrow[\begin{bmatrix} f \\ f_{n+1} \end{bmatrix}]{} W \oplus W_{n+1} \xrightarrow[\begin{bmatrix} p & -r \end{bmatrix}]{} E \xrightarrow[h\circ s = h'\circ q]{} V[1].$$

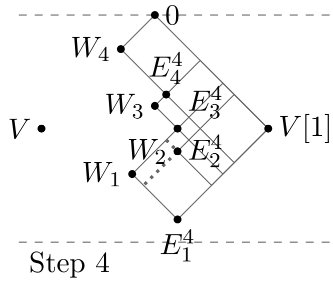
Thus E is determined by geometry and $[f\, f_{n+1}]^t$. Furthermore, the cone has the desired number of indecomposable summands. $\square$

Example 2.2.7. The practical technique for doing this may be somewhat opaque to the reader at first. As an example, one might have $f : V \to \bigoplus_{i=1}^{4} W_i$. According to the proposition the cone E should be the direct sum of no more than 5 indecomposables, the slopes between select indecomposables should be as described, and E should be determined by geometry and f. We start with $V \to W_i$ for each i, then using the technique in the proof, splice together the triangles one by one. At each step $i > 1$, the new cone E^i is given by using most of the old cone E^{i-1} with a slight change to account for the new indecomposable.

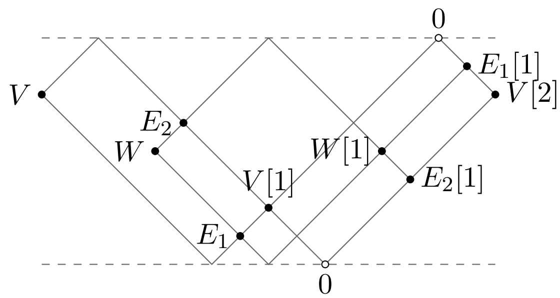








Example 2.2.8. Another notable example is what happens when we check the cone $E_1 \oplus E_2 \to V[1]$ from a map of indecomposable $V \to W$. Since E_1 and E_2 both create line segments with other endpoint V and slope $\pm(1,1)$, any triangle $E_i \to V \to F_i \to$ has F_i indecomposable. Furthermore, each F_i will also share an endpoint with V. Of course, algebraically we *must* have that $W[1]$ is the cone of $E_1 \oplus E_2 \to V[1]$ but this is also correct *geometrically*. We end up with the following picture:



Theorem 2.2.9. *Let $f : V \to W$ be a morphism in $\mathcal{D}^b(A_{\mathbb{R}})$ and consider the distinguished triangle*

$$V \xrightarrow{f} W \xrightarrow{g} U \xrightarrow{h} V[1].$$

Then U is determined by geometry and f.

Proof. Since V and W are each finite direct sums of indecomposables we'll prove the statement by induction on the number of indecomposable summands of V, using Lemma 2.2.6 as the base case.

Assume the lemma holds for all positive integers $\leq n$ and let V_i for $1 \leq i \leq n+1$ be indecomposables and let $V = \bigoplus_{i=1}^n V_i$. Let $f : V \to W$ and $f_{n+1} : V_{n+1} \to W$ be a nontrivial morphisms. We begin with the following distinguished triangles:

$$V \xrightarrow{f} W \xrightarrow{g} U \xrightarrow{h} V[1]$$

$$W \xrightarrow{g_{n+1}} U' \xrightarrow{h_{n+1}} V_{n+1}[1] \xrightarrow{f_{n+1}[1]} W[1]$$

$$V \xrightarrow{g_{n+1}\circ f} U' \xrightarrow{p} E \xrightarrow{q} V[1].$$

Applying our octahedral axiom we obtain the following distinguished triangles:

$$U \xrightarrow{p'} E \xrightarrow{q'} V_{n+1}[1] \xrightarrow{(g\circ f_{n+1})[1]} U[1]$$

$$E \xrightarrow[\begin{bmatrix} q \\ q' \end{bmatrix}]{} V[1] \oplus V_{n+1}[1] \xrightarrow[\begin{bmatrix} f[1] & -f_{n+1}[1] \end{bmatrix}]{} W[1] \xrightarrow[(p'\circ g)[1]=(p\circ g_{n+1})[1]]{} E[1].$$

By the induction hypothesis, E is determined by geometry and $g_{n+1}\circ f$. Therefore, E is determined by geometry and $[f\ -f_{n+1}]$. $\square$

2.3. **Derived Equivalence.** In this subsection we prove the first main result of this paper: the bounded derived categories of two continuous type A quivers are triangulated-equivalent if and only if they belong to the same of the following three classes:

- finitely many sinks and sources,
- the existence of a minimal or maximal sink/source, but not both, or
- no minimal or maximal sink/source.

Note that these are all disjoint.

Lemma 2.3.1. *Let $A_{\mathbb{R}}$ and $A'_{\mathbb{R}}$ be possibly different continuous quivers of type A and let $\mathcal{G}:\mathcal{D}^b(A_{\mathbb{R}})\to\mathcal{D}^b(A'_{\mathbb{R}})$ be an equivalence of categories. Let V and W be indecomposables in $\mathcal{D}^b(A_{\mathbb{R}})$ and suppose the slope of the line segment from V to W in the AR-space of $\mathcal{D}^b(A_{\mathbb{R}})$ is $\pm(1,1)$. Then the slope of the line segment from $\mathcal{G}V$ to $\mathcal{G}W$ in the AR-space of $\mathcal{D}^b(A'_{\mathbb{R}})$ is $\pm(1,1)$.*

Proof. For contradiction, suppose the conclusion is false. We know by [21, Lemma 5.2.9] that if the slope of the line segment from $\mathcal{G}V$ to $\mathcal{G}W$ in the AR-space of $\mathcal{D}^b(A'_{\mathbb{R}})$ is greater than $(1,1)$ or less than $-(1,1)$ then $\mathrm{Hom}_{\mathcal{D}^b(A'_{\mathbb{R}})}(\mathcal{G}V,\mathcal{G}W)=0$. This contradicts that $\mathcal{G}$ is an equivalence of categories.

Now suppose the slope is less than $(1,1)$ and greater than $-(1,1)$. We know $\mathcal{G}W$ must be close enough to $\mathcal{G}V$ by [21, Lemma 5.2.9]. Then we may use Theorem 1.2.4 and construct a distinguished triangle $\mathcal{G}V\to U\to\mathcal{G}W\to$. We may also choose a U' in $\mathcal{D}^b(A'_{\mathbb{R}})$ such that $\mathrm{Hom}(U,U')=\mathrm{Hom}(U',U)=0$ on the line segement from $\mathcal{G}V$ to $\mathcal{G}W$. Then any morphism $f:\mathcal{G}V\to\mathcal{G}W$ factors through either U or U' but not both. This contradicts the fact that any morphism from V to W in $\mathcal{D}^b(A_{\mathbb{R}})$ that factors through some X and X' factors through both and $\mathrm{Hom}(X,X')\cong k$ or $\mathrm{Hom}(X',X)\cong k$. $\square$

Definition 2.3.2. For each isomorphism class of indecomposable objects in $\mathcal{D}^b(A_{\mathbb{R}})$ we call the representation V such that $\iota V=V$ the representative.

Definition 2.3.3. Let $A_{\mathbb{R}}$ and $A'_{\mathbb{R}}$ be possibly different continuous type A quivers. Suppose also that both $A_{\mathbb{R}}$ and $A'_{\mathbb{R}}$ have either (a) finitely many sinks and sources or (b) infinitely many sinks and sources but a minimal or maximal sink/source. We construct a functor $\mathcal{F}$ from $\mathcal{D}^b(A_{\mathbb{R}})$ to $\mathcal{D}^b(A'_{\mathbb{R}})$.

If (b) is true then by the proof of [21, Proposition 2.4.4] there is a λ function $\lambda_{*\infty}$ (Definition 1.2.1) whose graph in $\mathbb{R}^2$ is disjoint from the image of $\mathbf{\Gamma}^b:\mathrm{Ind}(\mathcal{D}^b(A_{\mathbb{R}}))\to\mathbb{R}^2$. All other points in $\mathbb{R}\times[-\frac{\pi}{2},\frac{\pi}{2}]$ are in the image of $\mathbf{\Gamma}^b$. For $A'_{\mathbb{R}}$ there is a similar $\lambda'_{*\infty}$ disjoint from the image of $(\mathbf{\Gamma}')^b$.

If (b) is true let T be the translation of $\mathbb{R}\times[-\frac{\pi}{2},\frac{\pi}{2}]$ such that the graph of $\lambda_{*\infty}$ is taken to the graph of $\lambda'_{*\infty}$. If (a) is true instead let T be the identity.

For each indecomposable V in $\mathcal{D}^b(A_{\mathbb{R}})$ with position i define $\mathcal{F}V$ be the representative indecomposable in $\mathcal{D}^b(A'_{\mathbb{R}})$ such that $(\mathbf{\Gamma}')^b\mathcal{F}V=(T\circ\mathbf{\Gamma}^b)V$ and the position of $\mathcal{F}$ is i. Since $\mathcal{D}^b(A_{\mathbb{R}})$ is Krull-Schmidt ([21, Proposition 5.1.2]) this induces a mapping on all objects given by

$$\mathcal{F}\left(\bigoplus V_i\right):=\bigoplus\mathcal{F}V_i.$$

with the indexing and parenthesizing from Definition 2.2.1. The Hom space between any two indecomposables in both $\mathcal{D}^b(A_{\mathbb{R}})$ and $\mathcal{D}^b(A'_{\mathbb{R}})$ is k or 0. For any pair of representatives V' and W' in $\mathcal{D}^b(A'_{\mathbb{R}})$ we have can take each nontrivial morphism $f:V'\to W'$ to be a nonzero value of k. Define $\mathcal{F}f:=f$ for all morphisms of representative indecomposables $V\xrightarrow{f}W$ in $\mathcal{D}^b(A_{\mathbb{R}})$. For indecomposables V and W in $\mathcal{D}^b(A_{\mathbb{R}})$ with representatives ιV and ιW, any nontrivial morphism

$f : V \to W$ factors as in Definition 2.2.1:

$$V \xrightarrow{\iota_V} \iota V \xrightarrow{\iota f} \iota W \xrightarrow{\iota_W^{-1}} W.$$

We set $\mathcal{F}f := \mathcal{F}\iota f$. For any two objects $\bigoplus^m V_i$ and $\bigoplus^n W_j$ in $\mathcal{D}^b(A_{\mathbb{R}})$ we can extend bilinearly with a basis is given by the ordered summands of $\iota(\bigoplus^m V_i)$ and $\iota(\bigoplus^n W_j)$.

Lemma 2.3.4. *Given $A_{\mathbb{R}}$ and $A'_{\mathbb{R}}$ in Definition 2.3.3, $\mathcal{F}$ is a well-defined additive functor. Furthermore, $\mathcal{F}$ is an equivalence of categories and $\mathcal{F}(V[1]) \cong (\mathcal{F}V)[1]$.*

Proof. By using the translation T in Definition 2.3.3 for any indecomposable V in $\mathcal{D}^b(A_{\mathbb{R}})$, $\mathcal{F}V$ is well-defined and indeed $\mathcal{F}$ induces a bijection on indecomposable objects. Furthermore, since $\mathcal{D}^b(A_{\mathbb{R}})$ and $\mathcal{D}^b(A'_{\mathbb{R}})$ are both Krull-Schmidt, we have a bijection on all objects.

Recall the Hom support of an indecomposable V in $\mathcal{D}^b(A_{\mathbb{R}})$ is completely determined by the coordinates of $\mathbf{\Gamma}^b V$ and the position of V. Thus for any pair of indecomposables V and W in $\mathcal{D}^b(A_{\mathbb{R}})$ we have an isomorphism of vector spaces:

$$\mathrm{Hom}_{\mathcal{D}^b(A_{\mathbb{R}})}(V, W) \cong \mathrm{Hom}_{\mathcal{D}^b(A'_{\mathbb{R}})}(\mathcal{F}V, \mathcal{F}W).$$

Extending bilinearly gives us

$$\mathrm{Hom}_{\mathcal{D}^b(A_{\mathbb{R}})}\left(\bigoplus^m V_j, \bigoplus^n W_j\right) \cong \mathrm{Hom}_{\mathcal{D}^b(A'_{\mathbb{R}})}\left(\bigoplus^m \mathcal{F}V_i, \bigoplus^n \mathcal{F}W_j\right).$$

It is clear composition is preserved and that the functor is additive.

Finally let $(x, y) = \mathbf{\Gamma}^b V$ for an indecomposable V in $\mathcal{D}^b(A_{\mathbb{R}})$. We know $\mathbf{\Gamma}^b V[1] = (x + \pi, -y)$ by definition. Then if $(T \circ \mathbf{\Gamma}^b)V = (\mathbf{\Gamma}')^b \mathcal{F}V$ we know

$$(T \circ \mathbf{\Gamma}^b)(V[1]) = (\mathbf{\Gamma}')^b \mathcal{F}(V[1]) = (\mathbf{\Gamma}')^b((\mathcal{F}V)[1]). \qquad \square$$

Theorem 2.3.5. *Let $A_{\mathbb{R}}$ and $A'_{\mathbb{R}}$ be possibly different orientations of continuous type A quivers. One of the following is true if and only if $\mathcal{D}^b(A_{\mathbb{R}})$ and $\mathcal{D}^b(A'_{\mathbb{R}})$ are equivalent as triangulated categories:*

(1) *both $A_{\mathbb{R}}$ and $A'_{\mathbb{R}}$ have finitely many sinks and sources,*
(2) *the sinks and sources of $A_{\mathbb{R}}$ and $A'_{\mathbb{R}}$ are each bounded on exactly one side, or*
(3) *the sinks and sources of $A_{\mathbb{R}}$ and $A'_{\mathbb{R}}$ are unbounded on both sides.*

Proof. If (3) holds then by Proposition [13, Proposition 3.2.1] the representation categories are equivalent as abelian categories and so the derived categories will also be equivalent as triangulated categories. If (1) or (2) holds then the categories are already equivalent as additive categories, by Lemma 2.3.4. By the same Lemma, $\mathcal{F}$ commutes with the shift in each derived category. Thus it remains to show that $\mathcal{F}$ takes cones in distinguished triangles to cones in distinguished triangles.

Let $U \overset{f}{\to} V \overset{g}{\to} W \overset{h}{\to}$ be distinguished in $\mathcal{D}^b(A_{\mathbb{R}})$. By Theorem 2.2.9 we see W is determined by geometry and f. Then the same geometry and image of the maps under $\mathcal{F}$ determine a distinguished triangle $\mathcal{F}U \overset{\mathcal{F}f}{\to} \mathcal{F}V \overset{g'}{\to} W' \overset{h'}{\to}$ in $\mathcal{D}^b(A'_{\mathbb{R}})$. However, by Definition 2.3.3, we see that $W' \cong \mathcal{F}W$. Thus we have

$$\begin{tikzcd}
\mathcal{F}V \arrow[r, "\mathcal{F}f"] & \mathcal{F}U \arrow[r, "g'"] & W' \arrow[r, "h'"] \arrow[d, leftrightarrow, "\cong"] & \mathcal{F}V[1] \\
\mathcal{F}V \arrow[r, "\mathcal{F}f"'] & \mathcal{F}U \arrow[r, "\mathcal{F}g"'] & \mathcal{F}W \arrow[r, "\mathcal{F}h"'] & \mathcal{F}V[1].
\end{tikzcd}$$

Therefore up to changing signs on the morphisms between indecomposables that make up h' the bottom row is also a distinguished triangle and so $\mathcal{F}$ is a triangulated equivalence.

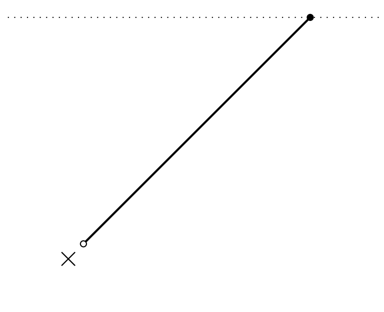
Since every continuous quiver of type A with finitely-many sinks and sources is derived equivalent we will show that classes (2) and (3) are disjoint from (1) using the straight descending orientation. Afterwards we will show (2) and (3) are also disjoint.

Let $A_{\mathbb{R}}$ be a continuous quiver of type A with infinitely-many sinks and sources. Let $A'_{\mathbb{R}}$ be the straight descending continuous quiver of type A. Let $\mathcal{P}$ be the set of indecomposable projectives of $\mathrm{rep}_k(A'_{\mathbb{R}})$ included as indecomposables in degree 0 in $\mathcal{D}^b(A'_{\mathbb{R}})$.

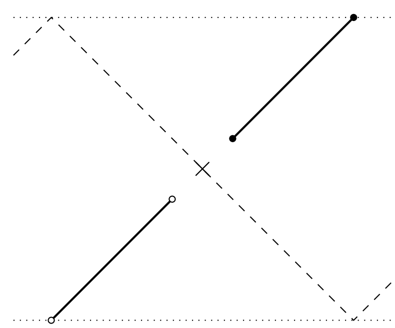
For contradiction, suppose there is an equivalence of categories $\mathcal{G} : \mathcal{D}^b(A'_{\mathbb{R}}) \to \mathcal{D}^b(A_{\mathbb{R}})$. By Lemma 2.3.1 we know that for any pair $P, P' \in \mathcal{P}$, the slope of any line segment from $\mathcal{G}P$ to $\mathcal{G}P'$ (switching roles if necessary) is $\pm(1,1)$. Since $\mathcal{G}$ is an equivalence we know that if there exists V in $\mathcal{D}^b(A_{\mathbb{R}})$ such that $\mathrm{Hom}_{\mathcal{D}^b(A_{\mathbb{R}})}(\mathcal{G}P, V) \cong \mathrm{Hom}_{\mathcal{D}^b(A_{\mathbb{R}})}(V, \mathcal{G}P') \cong k$ for $P, P' \in \mathcal{P}$ then $V \cong \mathcal{G}P''$ for some $P'' \in \mathcal{P}$.

Note that $\mathcal{P}$ actually induces an almost-complete line segment in the AR-space of $\mathcal{D}^b(A'_{\mathbb{R}})$. Then there does not exist an indecomposable object V in $\mathcal{D}^b(A'_{\mathbb{R}})$ such that (i) V is not isomorphic to an element of $\mathcal{P}$ and (ii) $\mathrm{Hom}_{\mathcal{D}^b(A'_{\mathbb{R}})}(V, P) \cong k$ for all $P \in \mathcal{P}$. We also know that if $\mathrm{Hom}_{\mathcal{D}^b(A'_{\mathbb{R}})}(P_{+\infty}, U) \cong \mathrm{Hom}_{\mathcal{D}^b(A'_{\mathbb{R}})}(P_{+\infty}, U') \cong k$ then $\mathrm{Hom}_{\mathcal{D}^b(A'_{\mathbb{R}})}(U, U') \cong k$ or $\mathrm{Hom}_{\mathcal{D}^b(A'_{\mathbb{R}})}(U', U) \cong k$. Since $\mathcal{G}$ is an equivalence of categories these properties must be preserved.

This forces the image $\mathcal{G}\mathcal{P}$ to induce an almost complete line segment whose existing endpoint $\mathcal{G}P_{+\infty}$ is on the top or bottom boundary of the AR-space of $\mathcal{D}^b(A_{\mathbb{R}})$. We describe how we will arrive at a contradiction and include pictures after this paragraph. Since $A_{\mathbb{R}}$ has infinitely many sinks and sources we must split the set $\mathcal{G}\mathcal{P}$ into (at least) two pieces since there is a missing graph of a λ function from the image of $\mathbf{\Gamma}^b$. The set $\mathbf{\Gamma}^b\,\mathcal{G}\mathcal{P}$ is homeomorphic to the disjoint union of (at least) two intervals. The set $(\mathbf{\Gamma}')^b\mathcal{P}$ is homeomorphic to one interval. We can't reorder the elements in the almost-complete line segment and we can't allow a strict inclusion of $\mathcal{P}$ into the almost-complete line segment in the AR-space of $\mathcal{D}^b(A_{\mathbb{R}})$. This finally leads us to the contradiction. Below we depict two of the issues with mapping the projective line:



$\times$ prevents an equivalence



$\times$ prevents an equivalence

Now we show that (2) and (3) are also disjoint. Suppose $A_{\mathbb{R}}$ has half bounded sinks and sources and $A'_{\mathbb{R}}$ has unbounded sinks and sources on both sides. The image in $\mathbb{R}^2$ under $\mathbf{\Gamma}^b$ of an almost complete line segment in the AR-space of $\mathcal{D}^b(A_{\mathbb{R}})$ has at most two path components. However, one may construct an almost complete line segment in the AR-space of $\mathcal{D}^b(A'_{\mathbb{R}})$ that whose image in $\mathbb{R}^2$ under $(\mathbf{\Gamma}')^b$ has three path components. Knowing this we apply a similar argument as before and see that an equivalence of categories cannot exist between $\mathcal{D}^b(A_{\mathbb{R}})$ and $\mathcal{D}^b(A'_{\mathbb{R}})$. $\square$

Remark 2.3.6. It follows from Theorem 1.2.4 that the only Auslander-Reiten triangles in $\mathcal{D}^b(A_{\mathbb{R}})$, for some continuous quiver of type A, are those of the form

$$V_1 \longrightarrow \begin{matrix} V_2 \\ \oplus \\ V_3 \end{matrix} \longrightarrow V_4$$

where each V_i has position i and $\mathbf{\Gamma}^b V_i = \mathbf{\Gamma}^b V_j \in \mathbb{R} \times (-\frac{\pi}{2}, \frac{\pi}{2})$ for all i, j.

2.4. **Naïve "Fixes".** For type A_n quivers, the path algebra of every orientation is derived equivalent to the rest. However, we see in Theorem 2.3.5 that different orientations of continuous type A quivers may yield different derived categories.

One naïve approach to "fixing" this is to first note that when $A_{\mathbb{R}}$ has infinitely many sinks and sources that some indecomposables are not finitely-generated. One may think to include these anyway and then obtains the category of *locally* finitely-generated representations. These are representations such that the restriction of a module to any finite interval is finitely-generated.

This leads to a problem with the AR-space of the representation category and then the derived category. The missing λ functions in the image of $\mathbf{\Gamma}$ (and $\mathbf{\Gamma}^b$) do get filled but instead of 4 modules at every point the modules that are new modules have only 2 modules per point and thus no Auslander-Reiten sequence. While this may not appear to be particularly problematic it still fails to unify all the continuous type A quivers' derived categories.

The other possibility is to include $\pm\infty$ in a continuous quiver of type A. However, if a continuous quiver of type A has infinitely-many sinks and sources then the vertex at $\pm\infty$ becomes disjoint from the rest of the quiver. There can be no path from $\pm\infty$ to a finite value if there are infinitely-many sinks and sources in the way. So, the finitely-generated representation category just ends up with two additional simples and the locally finitely-generated representation category has the same fate.

In essence, the added complexity of either locally finitely-generated representations or adding $\pm\infty$ to the continuous quivers does not yield any new useful structure. Futhermore, the trinary classification of the derived categories appears to be inherent to the continuous quivers themselves regardless of these algebraic tricks.

3. New Continuous Cluster Category

3.1. **Definition and g-vectors.** In this section we will define a new model for the continuous cluster category based on the previous construction by Igusa and Todorov in [15]. In Remark 2.3.6 we noted what all the Auslander-Reiten triangles look like. This also means we don't have left or right Auslander-Reiten triangles for every object. In fact, those indecomposable objects V such that $\mathbf{\Gamma}^b V$ has y-coordinate $\pm\frac{\pi}{2}$ do not belong to *any* Auslander-Reiten triangles.

In [3] the cluster category associated to a quiver Q is constructed using the orbit category of the bounded derived category of representations via the composition of the shift functor followed by the inverse Auslander-Reiten translation. The result is a 2-Calabi-Yau orbit category. We do not have enough Auslander-Reiten triangles to form the (inverse) Auslander-Reiten translation. However, one may go looking for another possible functor instead; this won't work.

Without all Auslander-Reiten triangles we cannot have a Serre functor. Without a Serre functor we cannot construct a 2-Calabi-Yau orbit category. (See work by Reiten and Van den Bergh [20].) Thus, as in [15] we will use a functor which is almost-shift.

Definition 3.1.1. Let $A_{\mathbb{R}}$ be a continuous quiver of type A and $\mathcal{D}^b(A_{\mathbb{R}})$ its bounded derived category. The continuous $\mathbf{E}$-cluster category, denoted $\mathcal{C}(A_{\mathbb{R}})$, is an orbit category of the doubling of $\mathcal{D}^b(A_{\mathbb{R}})$. We take the doubling of $\mathcal{D}^b(A_{\mathbb{R}})$ as in [15], which is equivalent to $\mathcal{D}^b(A_{\mathbb{R}})$, and call the functor with respect to which we take the orbit the almost-shift. We denote this orbit category by $\mathcal{C}(A_{\mathbb{R}})$. As in [15], $\mathcal{C}(A_{\mathbb{R}})$ is also a triangulated category.

Proposition 3.1.2. *Let V and W be indecomposable objects in $\mathcal{C}(A_{\mathbb{R}})$. Then* $\operatorname{Hom}_{\mathcal{C}(A_{\mathbb{R}})}(V, W) = 0$ *or* $\operatorname{Hom}_{\mathcal{C}(A_{\mathbb{R}})}(V, W) \cong k$.

Proof. Suppose $\operatorname{Hom}_{\mathcal{C}(A_{\mathbb{R}})}(V, W) \neq 0$. We know $\operatorname{Hom}_{\mathcal{C}(A_{\mathbb{R}})}(V, W) \cong \bigoplus_{n\in\mathbb{Z}} \operatorname{Hom}_{\mathcal{D}^b(A_{\mathbb{R}})}(V, W[n])$ and so there exists at least one n such that $\operatorname{Hom}_{\mathcal{D}^b(A_{\mathbb{R}})}(V, W[n]) \cong k$. By [21, Propositino 4.4.2 and Lemma 5.2.9] if $V \not\cong W$ we know that $\operatorname{Hom}_{\mathcal{D}^b(A_{\mathbb{R}})}(V, W[m]) = 0$ for $m > n$ and $m < n$. If $V \cong W$

we note that $\mathrm{Ext}^1(U,U) = 0$ for all indecomposables U in $\mathrm{rep}_k(A_{\mathbb{R}})$. Thus $\mathrm{Hom}_{\mathcal{C}(A_{\mathbb{R}})}(V,W) \cong \mathrm{Hom}_{\mathcal{D}^b(A_{\mathbb{R}})}(V,W[n]) \cong k$. □

Recall that by taking the orbit category, the class of objects in $\mathcal{C}(A_{\mathbb{R}})$ is the same as that in the doubling of $\mathcal{D}^b(A_{\mathbb{R}})$ even though the isomorphism classes have changed. Then the following proposition is straight-forward.

Proposition 3.1.3. *The orbit category $\mathcal{C}(A_{\mathbb{R}})$ is Krull-Schmidt.*

We follow Jørgensen and Yakimov in [16] with the following definition.

Definition 3.1.4. Denote by $\mathcal{P}$ the collection of indecomposable objects isomorphic to $P[n]$ in $\mathcal{C}(A_{\mathbb{R}})$ where P was a projective indecomposable in $\mathrm{rep}_k(A_{\mathbb{R}})$. Let V be an indecomposable in $\mathcal{C}(A_{\mathbb{R}})$. We define the g-vector or index of V to be the element $[P_V] - [Q_V]$ in $K_0^{\text{split}}(Add\mathcal{P})$ such that $Q_V \to P_V \to V \to$ is a distinguished triangle in $\mathcal{C}(A_{\mathbb{R}})$ that comes from the projective resolution of V.

Definition 3.1.5. Let $[A] = \Sigma m_i[A_i]$ and $[B] = \Sigma n_j[B_j]$ be elements of $K_0^{\text{split}}(\mathcal{C}(A_{\mathbb{R}}))$ where the A_is and B_js are indecomposable. The Euler bilinear form $\langle [A]\,,\,[B]\rangle$ is given in the following way. First, for a pair of indecomposables A_i and B_j in $\mathcal{C}(A_{\mathbb{R}})$ we define

$$\langle m_i[A_i]\,,\,n_j[B_j]\rangle := (m_i \cdot n_j)(\dim \mathrm{Hom}_{\mathcal{C}(A_{\mathbb{R}})}(A_i, B_j)).$$

Thus the form is defined to be

$$\langle [A]\,,\,[B]\rangle := \sum_i \sum_j \langle [A_i]\,,\,[B_j]\rangle.$$

Since $\mathcal{C}(A_{\mathbb{R}})$ is Krull-Schmidt this is always a finite sum and thus well-defined.

Definition 3.1.6. We say two g-vectors $[P_V] - [Q_V]$ and $[P_W] - [Q_W]$ are **E**-compatible if

$$\langle [P_V] - [Q_V]\,,\,[P_W] - [Q_W]\rangle \geq 0 \qquad \text{and} \qquad \langle [P_W] - [Q_W]\,,\,[P_V] - [Q_V]\rangle \geq 0.$$

We call this compatibility **E**-compatibility to align better with Section 5, where we introduce the general definition of a cluster theory (Definition 5.1.1).

Proposition 3.1.7. *Let $[P_V]-[Q_V]$ and $[P_W]-[Q_W]$ be two g-vectors of indecomposables V and W in $\mathcal{C}(A_{\mathbb{R}})$, both in degree 0. Consider V and W as images of the composite $\mathrm{rep}_k(A_{\mathbb{R}}) \hookrightarrow \mathcal{D}^b(A_{\mathbb{R}}) \to \mathcal{C}(A_{\mathbb{R}})$. Then $[P_V]-[Q_V]$ and $[P_W]-[Q_W]$ are not **E**-compatible if and only if there is an extension $V \hookrightarrow U \twoheadrightarrow W$ or $W \hookrightarrow U \twoheadrightarrow V$ in $\mathrm{rep}_k(A_{\mathbb{R}})$.*

Proof. Take P_V, P_W, Q_V, and Q_W to be the projectives in $\mathrm{rep}_k(A_{\mathbb{R}})$ whose image is in the isomorphism classes indicated by the g-vectors. If there is an extension $V \hookrightarrow U \twoheadrightarrow W$ in $\mathrm{rep}_k(A_{\mathbb{R}})$ then there is a nontrivial morphism $W \to V[1]$ in $\mathcal{D}^b(A_{\mathbb{R}})$. By Theorem 1.1.9 there is, up to scaling and isomorphism, a unique extension. This extension exists because there is a morphism $Q_W \to P_V$ that does not factor through $Q_V \oplus P_W$.

By the proof of [13, Proposition 3.2.4] we see this means there must be some indecomposable summand of Q_W that maps to at least one indecomposable summand of P_V but does not factor through Q_V or P_W. By Theorem 1.1.7, $\mathrm{rep}_k(A_{\mathbb{R}})$ is hereditary and so Q_W is a subrepresentation of P_W and Q_V is a subrepresentation of P_V. Thus

$$\langle [P_V] - [Q_V]\,,\,[P_W] - [Q_W]\rangle < 0. \qquad \square$$

If we start with incompatible g-vectors then we reverse the argument and see that, up to symmetry, there is a morphism $Q_W \to P_V$ that does not factor through $Q_V \oplus P_W$. Thus there is an extension $V \hookrightarrow U \twoheadrightarrow W$ in $\mathrm{rep}_k(A_{\mathbb{R}})$.

Proposition 3.1.8. *Let $V[m]$ and $W[n]$ be indecomposable objects in $\mathcal{C}(A_{\mathbb{R}})$ where V and W are indecomposables in the 0th degree. Then the g-vectors $[P_V]-[Q_V]$ and $[P_W]-[Q_W]$ are not* **E***-compatible if and only if there is a rectangle or almost complete rectangle in the AR-space of $\mathrm{rep}_k(A_{\mathbb{R}})$, as a subspace of the AR-space of $\mathcal{D}^b(A_{\mathbb{R}})$, whose sides have slopes $\pm(1,1)$ and whose left and right corners are V and W (not necessarily respectively).*

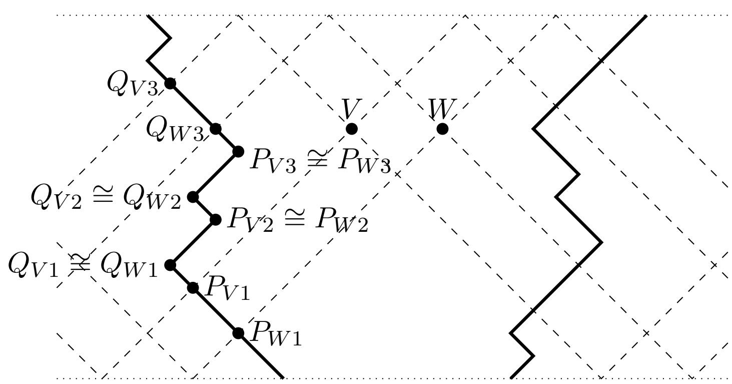
Proof. Without loss of generality suppose V is the left corner and W is the right corner of a rectangle or almost complete rectangle in the AR-space of $\mathrm{rep}_k(A_{\mathbb{R}})$. Then by Theorem 1.2.4 there is a distinguished triangle $V \to U \to W \to$. This corresponds to an extension $V \hookrightarrow U \twoheadrightarrow W$ in $\mathrm{rep}_k(A_{\mathbb{R}})$ and so by Proposition 3.1.7 we know $[P_V]-[Q_V]$ and $[P_W]-[Q_W]$ are not **E**-compatible.

Now suppose $[P_V]-[Q_V]$ and $[P_W]-[Q_W]$ are not **E**-compatible. Without loss of generality suppose

$$\langle [P_W]-[Q_W]\,,\,[P_V]-[Q_V]\rangle < 0.$$

Then in $\mathrm{rep}_k(A_{\mathbb{R}})$ there is an extension $V \hookrightarrow U \twoheadrightarrow W$. Thus by Theorem 1.2.3 there is a rectangle or almost complete rectangle in the AR-space of $\mathrm{rep}_k(A_{\mathbb{R}})$ whose left corner is V and right corner is W. □

Example 3.1.9. Below, the bold lines are the isomorphism classes of objects $P[0]$ and $P[1]$ where P is projective $\mathrm{rep}_k(A_{\mathbb{R}})$. The objects V and W are two indecomposables whose g-vectors are not **E**-compatible and are clearly the left and right corners of a rectangle in the AR-space of $\mathcal{D}^b(A_{\mathbb{R}})$ whose sides have slope $\pm(1,1)$. The points labeled P and Q with subscripts Vi or Wj are indecomposables of the objects P_V, P_W, Q_V, and Q_W.



We then perform the following computations:

$$\begin{aligned} &\langle [P_W]-[Q_W]\,,\,[P_V]\rangle \\ &= \langle [P_{W1}]+[P_{W2}]+[P_{W3}]-[Q_{W1}]-[Q_{W2}]-[Q_{W3}],\ [P_{V1}]+[P_{V2}]+[P_{V3}]\rangle \\ &= -1-1-1 = -3 \end{aligned}$$

$$\begin{aligned} &\langle [P_W]-[Q_W]\,,\,-[Q_V]\rangle \\ &= \langle [P_{W1}]+[P_{W2}]+[P_{W3}]-[Q_{W1}]-[Q_{W2}]-[Q_{W3}],\ -[Q_{V1}]-[Q_{V2}]-[Q_{V3}]\rangle \\ &= 1+1+0 = 2 \end{aligned}$$

And so we have

$$\langle [P_W]-[Q_W]\,,\,[P_V]-[Q_V]\rangle = -1.$$

3.2. **E-Clusters.** In this section we define **E**-clusters and show how to mutate elements in a way similar to the usual notion of mutation.

Definition 3.2.1. Let T be a collection of (isomorphism classes of) indecomposable objects in $\mathcal{C}(A_{\mathbb{R}})$. We say T is <u>**E**-compatible</u> if for any pair $V, W \in T$ the g-vectors $[P_V]-[Q_V]$ and $[P_W]-[Q_W]$ are **E**-compatible.

If, for any $U \notin T$, there exists a $V \in T$ such that $[P_U]-[Q_U]$ and $[P_V]-[Q_V]$ are not **E**-compatible are call T an **E**-cluster.

Example 3.2.2. Let $\mathcal{P}$ be the set of (isomorphism classes of) indecomposables P whose g-vector is of the form $[P]$. Then $\mathcal{P}$ is **E**-compatible.

Suppose V is ay other indecomposable with g-vector $[P_V]-[Q_V]$. Then $[P_V]-[Q_V]$ is not **E**-compatible with $[Q_V]$. Therefore $\mathcal{P}$ is an **E**-cluster.

Lemma 3.2.3. *Let $V \to U_1 \oplus U_2 \to W \to$ be a distinguished triangle in $\mathcal{C}(A_{\mathbb{R}})$ where V, U_1, U_2, and W are indecomposable. Suppose further that one may take representatives of each isomorphism class in degree 0 and obtain a (almost complete) rectangle entirely in the AR-space of $\mathrm{rep}_k(A_{\mathbb{R}})$ as a subspace of the AR-space of $\mathcal{D}^b(A_{\mathbb{R}})$, where V is the left corner and W is the right corner. Then for any indecomposable X in $\mathcal{C}(A_{\mathbb{R}})$ if $\{X, V, W\}$ is **E**-compatible so is $\{X, U_1, U_2\}$. Furthermore, if $\{X, V, U_1, U_2\}$ or $\{X, W, U_1, U_2\}$ is **E**-compatible so is $\{X, W\}$ or $\{X, V\}$, respectively.*

Proof. We will instead prove that if $\{X, U_1, U_2\}$ is not **E**-compatible then $\{X, V, W\}$ is not **E**-compatible, which is equivalent. Thus we may assume, without loss of generality, that $\{X, U_1\}$ is not **E**-compatible. We need only to prove that $\{X, V\}$ or $\{X, W\}$ is not **E**-compatible.

Suppose one of $\{X, V\}$ or $\{X, W\}$ is **E**-compatible. We shall assume $\{X, V\}$ is **E**-compatible as the other assumption is symmetric. By Proposition 3.1.8, since $\{X, U_1\}$ is not **E**-compatible there is a (almost complete) rectangle in the AR-space of $\mathrm{rep}_k(A_{\mathbb{R}})$ whose left and right corners are X and U_1 (possibly not respectively).

Since $\{X, V\}$ is **E**-compatible X must be to the left of U_1 or else there would be a (almost complete) rectangle with left corner V and right corner X in the AR-space of $\mathrm{rep}_k(A_{\mathbb{R}})$. But then there is a (almost complete) rectangle with left corner X and right corner W and so $\{X, W\}$ is not **E**-compatible.

Thus, if $\{X, U_1\}$ or $\{X, U_2\}$ is not **E**-compatible then at least one of $\{X, V\}$ or $\{X, W\}$ is not **E**-compatible. Therefore, if $\{X, U_1, U_2\}$ is not **E**-compatible then $\{X, V, W\}$ is not **E**-compatible. The furthermore in the lemma follows from a similar argument using AR-space geometry. $\square$

Definition 3.2.4. Let T be an **E**-cluster and $V \in T$. If there exists W such that $\{V, W\}$ is not **E**-compatible but $(T \setminus \{V\}) \cup \{W\}$ is **E**-compatible we say V is **E**-mutable.

Remark 3.2.5. Note that we have not required that $(T \setminus \{V\}) \cup \{W\}$ be an **E**-cluster. We only require that if V is replaced with W then the new set is **E**-compatible. We will prove later that this means $(T \setminus \{V\}) \cup \{W\}$ is indeed an **E**-cluster.

Proposition 3.2.6. *Let T be an **E**-cluster and $V \in T$ be **E**-mutable with choice W. Then one of the following is the distinguished triangle associated to the (almost complete) rectangle in Proposition 3.1.8 and whichever of U_1 and U_2 are nonzero are in T.*

$$\text{(1)} \qquad V \longrightarrow U_1 \oplus U_2 \longrightarrow W \longrightarrow V$$

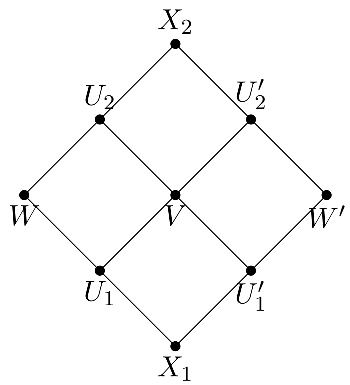
$$\text{(2)} \qquad W \longrightarrow U_1 \oplus U_2 \longrightarrow V \longrightarrow W.$$

Proof. We will prove the statement for (1) as (2) is similar. We know $\{V, U_1, U_2\}$ and $\{W, U_1, U_2\}$ are **E**-compatible by Proposition 3.1.8. We also know that for all $X \in T$ both $\{X, V\}$ and $\{X, W\}$ are **E**-compatible. Then, by Lemma 3.2.3, for all $X \in T$ we know $\{X, U_1, U_2\}$ is **E**-compatible. Since T is an **E**-cluster, this means $U_1, U_2 \in T$. $\square$

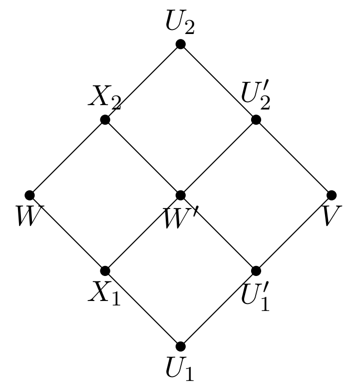
Lemma 3.2.7. *Let V, W, and W' be indecomposables in $\mathcal{C}(A_{\mathbb{R}})$ such that $\{V, W\}$ and $\{V, W'\}$ are not **E**-compatible. Let U_1, U_2, U_1', and U_2' be the indecomposables from the distinguished triangles in Proposition 3.1.8. If $W \not\cong W'$ then at least one of $\{W, U_1', U_2'\}$, $\{W', U_1, U_2\}$, or $\{U_1, U_2, U_1', U_2'\}$ is not **E**-compatible.*

Proof. There are two cases: (1) when $\{W, W'\}$ is not **E**-compatible and (2) when $\{W, W'\}$ is **E**-compatible. By Proposition 3.1.8 this is equivalent to: (1) when there is a rectangle or almost complete rectangle in the AR-space of $\operatorname{rep}_k(A_{\mathbb{R}})$ whose left and right corners are W and W' and (2) when there is no such (almost complete) rectangle. By symmetry we will assume that, in the AR-space of $\operatorname{rep}_k(A_{\mathbb{R}})$, W is never to the right of W'.

We assume (1) first. Then by our symmetry assumption W is the left corner and W' is the right corner. We already know there is a rectangle or almost complete rectangle with left and right corners W and V and similarly for W' and V. There are then three possible places for V horizontally in the AR-space of $\operatorname{rep}_k(A_{\mathbb{R}})$: (i) to the left of W, (ii) between W and W', and (iii) to the right of W'. We see that (i) and (iii) are similar so we'll just focus on (ii) and (iii). We have the following schematics in the AR-space of $\operatorname{rep}_k(A_{\mathbb{R}})$:



Case (1)(ii)



Case (1)(iii)

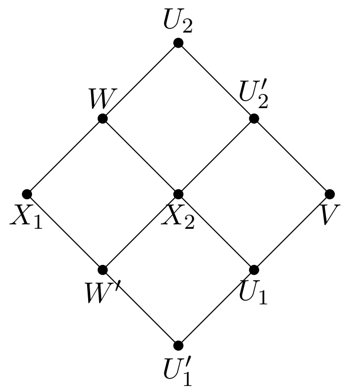
In the two schematics, at least one of X_1, X_2, least one of U_1, U_2, and both of U_1', U_2' must be nonzero. In particular, if one of X_1, X_2 is 0 we must be in case (1)(ii) so both U_1 and U_2 are nonzero and similarly for case (1)(iii).

In case (1)(ii), choose X_i to be one of X_1 or X_2 and nonzero. Let $j \in \{1, 2\}$ such that $\{i, j\} = \{1, 2\}$. Then X_i and U_j' are the top and bottom corners of a rectangle in the AR-space of $\operatorname{rep}_k(A_{\mathbb{R}})$ whose left corner is U_i and right corner is W'. By Proposition 3.1.8 we see thus means $\{W', U_i\}$ is not **E**-compatible and so $\{W', U_1, U_2\}$ is not **E**-compatible.

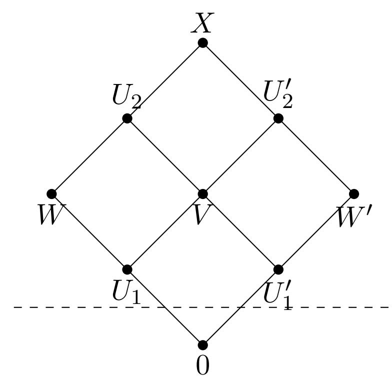
In case (1)(iii) choose U_i to be one of U_1 or U_2 and nonzero and $j \in \{1, 2\}$ such that $\{i, j\} = \{1, 2\}$. We have a rectangle in the AR-space of $\operatorname{rep}_k(A_{\mathbb{R}})$ with left corner W, top and bottom corners U_i and X_j, and right corner U_i' By Proposition 3.1.8 again we have $\{W, U_1', U_2'\}$ is not **E**-compatible.

Now we assume (2). This also comes with subcases. Either (i) W and W' are on the 'same side' of V in the AR-space of $\operatorname{rep}_k(A_{\mathbb{R}})$ or (ii) W and W' are on 'opposite sides' of V. In case (2)(i) this means $\mathbf{\Gamma}^b W$ and $\mathbf{\Gamma}^b W'$ both lie in the H_V as described in [21, Lemma 2.5.2]. By the same lemma, case (2)(ii) means one of $\mathbf{\Gamma}^b W$ and $\mathbf{\Gamma}^n W'$ lies in H_V and the other in $H_{V[-1]}$. Since $\{W, W'\}$ is **E**-compatible these are equivalent to either (i) one of W or W' being 'above' the other in the AR-space of $\operatorname{rep}_k(A_{\mathbb{R}})$ or (ii) W and W' are 'too far apart' to be the left and right corners of a rectangle or almost complete rectangle. In case (2)(ii), if we draw a rectangle with left and right corners $\mathbf{\Gamma}^b W$ and $\mathbf{\Gamma}^b W'$ in $\mathbb{R}^2$ one of the top or bottom corners will be in the image of the AR-space of $\operatorname{rep}_k(A_{\mathbb{R}})$ under $\mathbf{\Gamma}$. (Otherwise, W and W' would not both lie in the AR-space of $\operatorname{rep}_k(A_{\mathbb{R}})$ as a subspace of the AR-space of $\mathcal{D}^b(A_{\mathbb{R}})$.)

We have the following schematics (where the horizontal dashed line is the lower boundary of the AR-space of $\mathcal{D}^b(A_{\mathbb{R}})$) for cases (2)(i) and (2)(ii):



Case (2)(i)



Case (2)(ii)

At least one of U_2 and U_1' must be nonzero in case (2)(i). Up to reversing roles, we see that there is a rectangle with left corner W, top and bottom corners U_2 and X_2, and right corner U_2'. Thus $\{W, U_1', U_2'\}$ is not $\mathbf{E}$-compatible. In case (2)(ii) we have argued that X must be nonzero so we have a rectangle with left corner U_2, top and bottom corners X and V, and right corner U_2'. This means $\{U_1, U_2, U_1', U_2'\}$ is not $\mathbf{E}$-compatible because $\{U_2, U_2'\}$ is not $\mathbf{E}$-compatible. Therefore the proposition holds. $\square$

Theorem 3.2.8. *Let T be an $\mathbf{E}$-cluster and $V \in T$ $\mathbf{E}$-mutable with choice W. Then $(T\setminus\{V\})\cup\{W\}$ is an $\mathbf{E}$-cluster and any other choice W' for V is isomorphic to W.*

Proof. First we prove the choice of W is unique up to isomorphism. By Proposition 3.2.6 we know there are two dinstinguished triangles with indecomposables U_1, U_2, U_1', and U_2', all of which are in T. By Lemma 3.2.7 we know that if $W \not\cong W'$ then one of $\{W, U_1', U_2'\}$, $\{W', U_1, U_2\}$, or $\{U_1, U_2, U_1', U_2'\}$ is not $\mathbf{E}$-compatible. Therefore since both W and W' are choices for V we must have $W \cong W'$.

Now let X be an indecomposable in $\mathcal{C}(A_{\mathbb{R}})$ such that $((T \setminus \{V\}) \cup \{W\}) \cup \{X\}$ is $\mathbf{E}$-compatible. Then X is $\mathbf{E}$-compatible with U_1 and U_2 from the distinguished triangle. By Lemma 3.2.3, since $\{X, W, U_1, U_2\}$ is $\mathbf{E}$-compatible so is $\{X, V\}$. Therefore $X \in T$. $\square$

We may now use the theorem and state the following definition.

Definition 3.2.9. Let T be an $\mathbf{E}$-cluster and $V \in T$ be $\mathbf{E}$-mutable. We call the indecomposable W such that $(T \setminus \{V\}) \cup \{W\}$ is an $\mathbf{E}$-cluster (Theorem 3.2.8) the $\underline{\mathbf{E}\text{-replacement}}$ for V.

Let $\mu : T \to (T \setminus \{V\}) \cup \{W\}$ be the bijection that sends X to X if $X \not\cong V$ and sends V to W. We call μ an $\underline{\mathbf{E}\text{-mutation}}$ and say we have $\underline{\mathbf{E}\text{-mutated}}$ V to W.

4. Relation to Previous Construction

This section is dedicated to providing a rigorous connection to the previous construction of the continuous cluster category in [15].

4.1. **Localizations.**

In this section we create a calculus of fractions in order to construct a triangulated localization of $\mathcal{D}^b(A_{\mathbb{R}})$ and $\mathcal{C}(A_{\mathbb{R}})$. We do this using a null system.

Definition 4.1.1. Let V be an indecomposable in $\mathcal{D}^b(A_{\mathbb{R}})$. If $\mathbf{\Gamma}\, V = (x, \pm\frac{\pi}{2})$ or $\mathbf{\Gamma}^b\, V = (x, \pm\frac{\pi}{2})$, respectively, for some x we say V is $\underline{\text{degenerate}}$. Let V now be an indecomposable in $\mathcal{C}(A_{\mathbb{R}})$, which comes from an indecomposable object V' in $\mathcal{D}^b(A_{\mathbb{R}})$. If V' is degenerate we say V is also degenerate.

For simplicity we also say the 0 object in each of these categories is degenerate. Let $V \cong \bigoplus V_i$ be an object in $\mathrm{rep}_k(A_{\mathbb{R}})$, $\mathcal{D}^b(A_{\mathbb{R}})$, or $\mathcal{C}(A_{\mathbb{R}})$ where each V_i is indecomposable. If each V_i is degenerate we say V is degenerate.

Proposition 4.1.2. *Let V and W be degenerate indecomposable objects in $\mathrm{rep}_k(A_{\mathbb{R}})$, $\mathcal{D}^b(A_{\mathbb{R}})$, or $\mathcal{C}(A_{\mathbb{R}})$, and $f : V \to W$ a morphism. Then f is either 0 or an isomorphism.*

Proof. When V and W are in $\mathrm{rep}_k(A_{\mathbb{R}})$ or $\mathcal{D}^b(A_{\mathbb{R}})$ this follows from [21, Section 4.2 and Lemma 5.2.9], respectively. Now suppose V and W are in $\mathcal{C}(A_{\mathbb{R}})$ and f is not 0.

Since $\mathrm{Hom}_{\mathcal{C}(A_{\mathbb{R}})}(V, W) \cong \bigoplus_{\mathbb{Z}} \mathrm{Hom}_{\mathcal{D}^b(A_{\mathbb{R}})}(V, W[n])$ some shift of W must be isomorphic to V in $\mathcal{D}^b(A_{\mathbb{R}})$. Then in $\mathcal{C}(A_{\mathbb{R}})$ we see $V \cong W$. Since $\mathrm{Hom}(V, V) \cong k$ and f is not 0 f must be an isomorphism. □

Definition 4.1.3. Let $\mathcal{D}$ be a triangulated category and $\mathcal{N}$ a full subcategory of $\mathcal{D}$ such that

(1) the 0 object is in $\mathcal{N}$,
(2) if X is an object in $\mathcal{N}$ and $Y \cong X$ in $\mathcal{D}$ then Y is an object in $\mathcal{N}$,
(3) an object X is in $\mathcal{N}$ if and only if $X[1]$ is in $\mathcal{N}$, and
(4) if $X \to Y \to Z \to X[1]$ is a distinguished triangle in $\mathcal{D}$ and X and Z are both objects in $\mathcal{N}$ then so is Y.

We call $\mathcal{N}$ a null system in $\mathcal{D}$.

Remark 4.1.4. We note that Definition 4.1.3 is the traditional definition used. However, we also note that (3) and (4) imply (1) and (2).

We recall the following fact from [17] as a proposition.

Proposition 4.1.5. *Let $\mathcal{D}$ be a triangulated category and $\mathcal{N}$ a null system in $\mathcal{D}$. Then the class of morphisms*

$$\mathcal{N}Q := \{X \xrightarrow{f} Y : \exists \text{ distinguished triangle } X \xrightarrow{f} Y \to Z \to, Z \text{ in } \mathcal{N}\}$$

admits a left and right calculus of fractions in $\mathcal{D}$.

Proposition 4.1.6. *Let $\mathcal{D}$ be $\mathcal{D}^b(A_{\mathbb{R}})$ or $\mathcal{C}(A_{\mathbb{R}})$. Let $\mathcal{N}$ be the full, wide subcategory of $\mathcal{D}$ whose objects are the degenerate objects in $\mathcal{D}$. Then $\mathcal{N}$ is a null system and so $\mathcal{N}Q$ admits a left and right calculus of fractions.*

Proof. Items (1)–(3) in Definition 4.1.3 are clear by Definition 4.1.1. Item (4) follows from Proposition 4.1.2. The second half of the conclusion follows from Proposition 4.1.5. □

We now recall the following well known categorical fact.

Proposition 4.1.7. *Let $\mathcal{D}$ be a triangulated category and $\mathcal{M}$ a class of morphisms that admits a left and right calculus of fractions. Then there exists a localization $\mathcal{D} \to \mathcal{D}[\mathcal{M}^{-1}]$ such that $\mathcal{D}[\mathcal{M}^{-1}]$ is a triangulated category whose distinguished triangles are exactly the images of distinguished triangles in $\mathcal{D}$.*

Proposition 4.1.8. *Let $\mathcal{D}$ be $\mathcal{D}^b(A_{\mathbb{R}})$ or $\mathcal{C}(A_{\mathbb{R}})$ and $\mathcal{N}$ the full, wide subcategory of degenerate objects in $\mathcal{D}$. Then $\mathcal{D}[\mathcal{N}Q^{-1}]$ is a triangulated category whose distinguished triangles are images of distinguished triangles in $\mathcal{D}$.*

Proof. This follows from Propositions 4.1.6 and 4.1.7. □

Lemma 4.1.9. *Let $\mathcal{D}$ be either $\mathcal{D}^b(A_{\mathbb{R}})$ or $\mathcal{C}(A_{\mathbb{R}})$. Let V and W be not degenerate indecomposables in $\mathcal{D}$.*

(1) *If $\mathcal{D} = \mathcal{D}^b(A_{\mathbb{R}})$ then $V \cong W$ in $\mathcal{D}[\mathcal{N}Q^{-1}]$ if and only if $\mathbf{\Gamma}^b V = \mathbf{\Gamma}^b W$.*

(2) *If $\mathcal{D} = \mathcal{C}(A_{\mathbb{R}})$ then $V \cong W$ in $\mathcal{D}[\mathcal{N}Q^{-1}]$ if and only if there exists $n \in \mathbb{Z}$ such that $\mathbf{\Gamma}^b V = \mathbf{\Gamma}^b(W[n])$.*

Proof. In $\mathcal{C}(A_{\mathbb{R}})$, $W \cong W[n]$ for all $n \in \mathbb{Z}$. Thus, by replacing W with $W[n]$, and choosing the appropriate n, we can prove (2) for $\mathbf{\Gamma}^b V = \mathbf{\Gamma}^b W$.

First suppose $\mathbf{\Gamma}^b V = \mathbf{\Gamma}^b W$ in $\mathcal{D}^b(A_{\mathbb{R}})$. Without loss of generality suppose $\mathrm{Hom}_{\mathcal{D}^b(A_{\mathbb{R}})}(V,W) \cong k$. Then by Theorem 1.2.4 any nonzero indecomposable summands of U in the distinguished triangle $W \to U \to V[1] \to$ in $\mathcal{D}^b(A_{\mathbb{R}})$ are degenerate. Then U is degenerate and so $V \cong W$ in $\mathcal{D}^b(A_{\mathbb{R}})[\mathcal{N}Q^{-1}]$. Furthermore if V and W are instead in $\mathcal{C}(A_{\mathbb{R}})$ then U is still degenerate and so $V \cong W$ in $\mathcal{C}(A_{\mathbb{R}})[\mathcal{N}Q^{-1}]$.

Now suppose $V \cong W$ in $\mathcal{D}^b(A_{\mathbb{R}})[\mathcal{N}Q^{-1}]$. Then U in the distinguished triangle $W \to U \to V[1] \to$ in $\mathcal{D}^b(A_{\mathbb{R}})$ is degenerate. We use Theorem 1.2.4 again and see $\mathbf{\Gamma}^b V = \mathbf{\Gamma}^b W$. Now suppose $V \cong W$ in $\mathcal{C}(A_{\mathbb{R}})[\mathcal{N}Q^{-1}]$; so U is degenerate in the distinguished triangle $W \to U \to V \to$ in $\mathcal{C}(A_{\mathbb{R}})$. As an orbit category by almost-shift there are choices of lifts W, U, and $V[1]$ in $\mathcal{D}^b(A_{\mathbb{R}})$ that yield the triangle in $\mathcal{C}(A_{\mathbb{R}})$. Again we apply Theorem 1.2.4 and see $\mathbf{\Gamma}^b V = \mathbf{\Gamma}^b W$. □

Lemma 4.1.10. *Let $\mathcal{D}$ be either $\mathcal{D}^b(A_{\mathbb{R}})$ or $\mathcal{C}(A_{\mathbb{R}})$ and V and W be indecomposables in $\mathcal{D}$. Let V_1 be an indecomposable in $\mathcal{D}$ such that $\mathbf{\Gamma}^b V = \mathbf{\Gamma}^b V_1$ and V_1 has position 1. Then*

$$\mathrm{Hom}_{\mathcal{D}[\mathcal{N}Q^{-1}]}(V,W) = 0 \textit{ if and ony if } \mathrm{Hom}_{\mathcal{D}}(V_1,W) = 0.$$

Proof. Suppose $\mathrm{Hom}_{\mathcal{D}}(V_1,W) = 0$. Consider a roof

$$\underline{f} = V \xleftarrow{f} U \xrightarrow{g} W$$

in $\mathcal{D}[\mathcal{N}Q^{-1}]$. Since $f \in \mathcal{N}Q$ we know (shifting if necessary in $\mathcal{C}(A_{\mathbb{R}})$) that $\mathbf{\Gamma}^b V = \mathbf{\Gamma}^b U$. Then $\mathbf{\Gamma}^b V_1 = \mathbf{\Gamma}^b U = \mathbf{\Gamma}^b V$ and so $\mathrm{Hom}_{\mathcal{D}}(V_1,U) \cong k \cong \mathrm{Hom}_{\mathcal{D}}(V_1,V)$. Let $s : V_1 \to U$ be a nontrivial morphism. We then have the following commutative diagram in $\mathcal{D}$:

$$\begin{tikzcd}
V \arrow[dd, equal] & U \arrow[l, "f"'] \arrow[r, "g"] & W \arrow[dd, equal] \\
& V_1 \arrow[u, "s"] \arrow[d, equal] & \\
V & V_1 \arrow[l, "f\circ s"] \arrow[r, "g\circ s"'] & W.
\end{tikzcd}$$

Since $f \circ s \in \mathcal{N}Q$ we see that these two roofs are equivalent in $\mathcal{D}[\mathcal{N}Q^{-1}]$. Denote the bottom roof by $\underline{f}'$. If $g = 0$ then $\underline{f}$ was 0 all along. If $g \neq 0$ we have $g \circ s = 0$ but $f \circ s \in \mathcal{N}Q$ and so $\underline{f}' = 0$. Thus $\underline{f}$ must be 0 and so $\mathrm{Hom}_{\mathcal{D}[\mathcal{N}Q^{-1}]}(V,W) = 0$.

Now suppose $\mathrm{Hom}_{\mathcal{D}}(V_1,W) \neq 0$. Choose nonzero morphisms $f : V_1 \to V$ and $g : V_1 \to W$. Then we have the following roof

$$\underline{f} = V \xleftarrow{f} V_1 \xrightarrow{g} W.$$

For contradiction, suppose $\underline{f} = 0$. Then there is a roof

$$\underline{h} = V \xleftarrow{h} U \xrightarrow{g'} W$$

where $g' = 0$ and the following commutative diagram in $\mathcal{D}$:

$$\begin{tikzcd} V \arrow[dd, equal] & V_1 \arrow[l, "f"'] \arrow[r, "g"] & W \arrow[dd, equal] \\ & \tilde{U} \arrow[u, "s"] \arrow[d, "t"'] & \\ V & U \arrow[l, "h"'] \arrow[r, "g'"] & W. \end{tikzcd}$$

However, this means (up to shifting in $\mathcal{C}(A_{\mathbb{R}})$) $\mathbf{\Gamma}^b\,\tilde{U} = \mathbf{\Gamma}^b\,V_1$ and so $\tilde{U} \cong V_1$, a contradiction as then the right side of the diagram would not commute. Therefore there exists a nonzero $\underline{f} \in \mathrm{Hom}_{\mathcal{D}[\mathcal{N}Q^{-1}]}(V, W)$. $\square$

Proposition 4.1.11. *Let $\mathcal{D}$ be either $\mathcal{D}^b(A_{\mathbb{R}})$ or $\mathcal{C}(A_{\mathbb{R}})$. Let V and W be indecomposable objects in $\mathcal{D}[\mathcal{N}Q^{-1}]$. Then* $\mathrm{Hom}_{\mathcal{D}[\mathcal{N}Q^{-1}]}(V, W) \cong k$ *or* $\mathrm{Hom}_{\mathcal{D}[\mathcal{N}Q^{-1}]}(V, W) = 0$.

Proof. Choose $\mathcal{D} = \mathcal{D}^b(A_{\mathbb{R}})$ or $\mathcal{D} = \mathcal{C}(A_{\mathbb{R}})$. Let V and W be indecomposables in $\mathcal{D}[\mathcal{N}Q^{-1}]$ such that $\mathrm{Hom}_{\mathcal{D}[\mathcal{N}Q^{-1}]}(V, W) \neq 0$.

Let V_1 be an indecomposable $\mathcal{D}$ such that $\mathbf{\Gamma}^b\,V_1 = \mathbf{\Gamma}^b\,V$ and the position of V_1 is 1. By Lemmas 4.1.9 and 4.1.10 we know $V \cong_{\mathcal{D}[\mathcal{N}Q^{-1}]} V_1$ and $\mathrm{Hom}_{\mathcal{D}}(V_1, W) \neq 0$. We will show

$$\mathrm{Hom}_{\mathcal{D}[\mathcal{N}Q^{-1}]}(V, W) \cong \mathrm{Hom}_{\mathcal{D}}(V_1, W)$$

by defining two maps

$$\Phi : \mathrm{Hom}_{\mathcal{D}[\mathcal{N}Q^{-1}]}(V, W) \to \mathrm{Hom}_{\mathcal{D}}(V_1, W)$$
$$\Psi : \mathrm{Hom}_{\mathcal{D}}(V_1, W) \to \mathrm{Hom}_{\mathcal{D}[\mathcal{N}Q^{-1}]}(V, W).$$

Fix a nonzero morphism $f' : V_1 \to V$ in $\mathcal{D}$. As we saw in the proof of Lemma 4.1.10 ever nonzero morphism $V \to W$ in $\mathcal{D}[\mathcal{N}Q^{-1}]$ is equivalent to a roof whose middle term is V_1. Let

$$\underline{f} = \; V \xleftarrow{\;f\;} V_1 \xrightarrow{\;g\;} W$$

in $\mathcal{D}[\mathcal{N}Q^{-1}]$. Since $\mathrm{Hom}_{\mathcal{D}}(V_1, V) \cong k$ there is a unique $s : V_1 \to V_1$ in $\mathcal{D}$ such that $f \circ s = f'$. Similarly, there is a unique $g' : V_1 \to W$ in $\mathcal{D}$ such that $g' = g \circ s$. So we set $\Phi(\underline{f}) = g'$.

Let $g' : V_1 \to W$ be a morphism in $\mathcal{D}$. Then there is a roof

$$\underline{f'} = \; V \xleftarrow{\;f'\;} V_1 \xrightarrow{\;g'\;} W$$

in $\mathcal{D}[\mathcal{N}Q^{-1}]$. So we set $\Psi(g') = \underline{f'}$.

Note that $\Psi\Phi(\underline{f}) = \underline{f'}$ but the following diagram commutes in $\mathcal{D}$:

$$\begin{tikzcd} V \arrow[dd, equal] & V_1 \arrow[l, "f"'] \arrow[r, "g"] & W \arrow[dd, equal] \\ & V_1 \arrow[u, "s"] \arrow[d, equal] & \\ V & V_1 \arrow[l, "f'"'] \arrow[r, "g'"] & W \end{tikzcd}$$

Thus $\underline{f'} = \underline{f}$ in $\mathcal{D}[\mathcal{N}Q^{-1}]$ and so $\Psi\Phi(\underline{f}) = \underline{f}$. We see $\Phi\Psi(g') = g'$ since in this case the s from our definition of $\Phi(\Psi(g))$ is the identity. Therefore $\Phi = \Psi^{-1}$ and $\Psi = \Phi^{-1}$. Finally, it is straightforward to check that Φ and Ψ preserve addition and send 0 to 0. Thus we have the desired isomorphism and the proposition holds. $\square$

4.2. **Triangulated Equivalences.** Here we show the localization of the derived and new continuous cluster categories are triangulated equivalent to the previous derived and continuous cluster categories, respectively. The localization of the new continuous cluster category is *not* equivalent to the previous continuous cluster category in a way that is **E**-compatible with mutation. See Section 4.3.

We recall the construction of the continuous derived category $\mathcal{D}_r$ from [15] as modified in [9]. The idea is that $\mathcal{D}_\pi$ is the limit as $n \to \infty$ of the bounded derived category of $\text{rep}_k(Q_n)$ where Q_n is the quiver of type A_n with straight orientation. This will be a topological category with a continuous triangulation. The original definition constructed $\mathcal{D}_r$ for any positive real number r as the stable category of a Frobenius category. Here we take the approach given in [9] which does not involve construction of an auxiliary category.

Definition 4.2.1. The object space of $\mathcal{D}_r$ is a subset of the plane:

$$\mathcal{O}b(\mathcal{D}_r) = \{(x, y) \in \mathbb{R}^2 \mid |y - x| < r\}.$$

Equivalently, $x - r < y < x + r$. We take the discrete topology on the field k and let

$$\text{Hom}_{\mathcal{D}_r}(X, Y) = \begin{cases} k & \text{if } x_1 \le x_2 < y_1 + r \text{ and } y_1 \le y_2 < x_1 + r \\ 0 & \text{otherwise} \end{cases}$$

Thus, for $X = (x, y)$, the support of $\text{Hom}_{\mathcal{D}_r}(X, -)$ is the half-open rectangle

$$[x, y + r) \times [y, x + r)$$

and nonzero morphisms are specified by scalars in k. When Y converges to a limit point of this half-open interval, morphisms converge to zero. Then $\mathcal{D}_r$ is a topological k-category in the sense that the object and morphism sets are topological spaces and the structure maps of the category (source, target, composition, identity, scalar multiplication) are continuous.

Following [9, Section 4.3], the distinguished triangles in $\mathcal{D}_r$ are constructed out of a family of distinguished triangles called <u>universal virtual triangles</u>. For each object $X = (x, y)$ these is a family of distinguished triangles:

$$X \xrightarrow{\binom{1}{1}} I_1^\varepsilon X \oplus I_2^\varepsilon X \xrightarrow{(-1,1)} T^\varepsilon X \xrightarrow{1} TX \tag{1}$$

for sufficiently small $\varepsilon > 0$ where $I_1^\varepsilon X = (x, x + r - \varepsilon)$, $I_2^\varepsilon X = (y + r - \varepsilon, y)$ and $T^\varepsilon X = (y + r - \varepsilon, x + r - \varepsilon)$. Morphisms are given by the indicated scalars. If X has several components, we take the direct sum of the virtual triangles (1) over all components of X. Note that, as $\varepsilon \to 0$, $T^\varepsilon X$ converges to TX and $T^\varepsilon X \xrightarrow{1} TX$ converges to the identity morphism on TX. The objects $I_1^\varepsilon X$, $I_2^\varepsilon X$ converge to 0 as $\varepsilon \to 0$.

The distinguished triangles in $\mathcal{D}_r$ are given as follows. For any morphism $f : X \to Y$ in $\mathcal{D}_r$, we defined the distinguished triangle $X \to_f Y \to_g Z \to_h TX$ to be the limit as $\varepsilon \to 0$ of the following pushout diagram.

$$\begin{tikzcd}
X \arrow[r, "\binom{1}{1}"] \arrow[d, "f"'] & I_1^\varepsilon X \oplus I_2^\varepsilon X \arrow[r, "(-1,1)"] \arrow[d] & T^\varepsilon X \arrow[d] \\
Y \arrow[r, "g"] & Z^\varepsilon \arrow[r, "h"] & T^\varepsilon X
\end{tikzcd}$$

As $\varepsilon \to 0$, the object Z^ε converges to an object Z. The morphisms g, h also stabilize and the limit is well-defined. (See [9] for details.)

Definition 4.2.3. We define a functor $G : \mathcal{D}^b(A_\mathbb{R})[\mathcal{N}Q^{-1}] \to \mathcal{D}_\pi$. We will use the representative objects as in Definition 2.3.2. Noting Lemma 4.1.9 we will assume our representative object in each isomorphism class has position 1 in the AR-space of $\mathcal{D}^b(A_\mathbb{R})$.

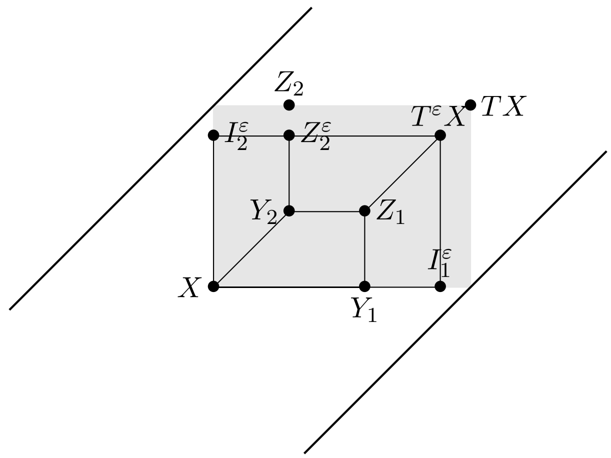


FIGURE 4.2.2. In this example, $Y = Y_1 \oplus Y_2$ where Y_1 has the same y-coordinate as X. Then $Z^\varepsilon = I_1^\varepsilon \oplus Z_1 \oplus Z_2^\varepsilon$. As $\varepsilon \to 0$, I_1^ε moves to the right until it becomes 0, Z_2^ε moves up to Z_2, Z_1 stays where it is. Also, $T^\varepsilon X$ goes to TX. Thus the distinguished triangle is $X \to Y_1 \oplus Y_2 \to Z_1 \oplus Z_2 \to TX$. The support of $\mathrm{Hom}_{\mathcal{D}_r}(X, -)$ is shaded.

Let V be an indecomposable representative object in $\mathcal{D}^b(A_{\mathbb{R}})[\mathcal{N}Q^{-1}]$. Let $(x, y) = \mathbf{\Gamma}^b V$. We define GV to be $M(x - y, x + y)$ in $\mathcal{D}_\pi$. It is easy to check that $|x - y - x - y| < \pi$. Since $\mathrm{Hom}_{\mathcal{D}^b(A_{\mathbb{R}})[\mathcal{N}Q^{-1}]}(V, W) = k$ (Proposition 4.1.11) for two indecomposable representatives V and W, we send a morphism $f \in k$ to $f \in k = \mathrm{Hom}_{\mathcal{D}_\pi}(GV, GW)$. Since both categories are Krull-Schmidt the rest of G is defined by extending bilinearly.

Theorem 4.2.4. *Assume $A_{\mathbb{R}}$ has finitely-many sinks and sources. Then G in Definition 4.2.3 is a triangulated equivalence.*

Proof. One quickly verifies that G induces a bijection on isomorphism classes of objects and bijections on Hom spaces. It remains to show that cones in distinguished triangles are taken to cones in distinguished triangles.

Let $V \to U \to W \to$ be a distinguished triangle in $\mathcal{D}^b(A_{\mathbb{R}})[\mathcal{N}Q^{-1}]$ such that V, U, and W are all nonzero and distinct and V and W are indecomposable. Then this comes from a triangle $\tilde{V} \to \tilde{U} \to \tilde{W} \to$ in $\mathcal{D}^b(A_{\mathbb{R}})$ and so $U = U_1 \oplus U_2$ where U_1 and U_2 are indecomposable and at most one is 0. Furthermore, $\tilde{V}$ and $\tilde{W}$ are indecomposable. Then by Theorem 1.2.4 there is a rectangle or almost complete rectangle in the AR-space of $\mathcal{D}^b(A_{\mathbb{R}})$ whose sides have slopes $\pm(1,1)$ and whose left and right corner are $\tilde{V}$ and $\tilde{W}$, respectively. Thus $\tilde{U}$ has at most two indecomposable summands.

Since U is not 0, $\mathbf{\Gamma}^b \tilde{V}, \mathbf{\Gamma}^b \tilde{W}$, and $\mathbf{\Gamma}^b \tilde{U}$ form the corners of a rectangle in $\mathbb{R}^2$ whose left and right corners are $\mathbf{\Gamma}^b \tilde{V}$ and $\mathbf{\Gamma}^b \tilde{W}$, respectively. Since $U \neq 0$ no more than one indecomposable summand of $\tilde{U}$ may be sent to $\mathbb{R} \times (-\frac{\pi}{2}, \frac{\pi}{2})$ by $\mathbf{\Gamma}^b$.

We give the coordinates of the image of $\mathbf{\Gamma}^b$ of the lifts of each indecomposable and compute G on each of our indecomposables below. First, a few notes. If one of $\tilde{U}_1$ or $\tilde{U}_2$ is 0 then there is no $\mathbf{\Gamma}^b$ of that indecomposable. If one of U_1 or U_2 is 0 then G of that indecomposable will be 0. However, we will compute $\mathbf{\Gamma}^b$ and G for both possibilities in each case for when those situations arise.

$$\begin{aligned} \mathbf{\Gamma}^b\tilde{V} &= (x,y) & GV &= M(x-y,x+y) \\ \mathbf{\Gamma}^b\tilde{U}_1 &= (x+\alpha,y-\alpha) & GU_1 &= M(x-y+2\alpha,x+y) \\ \mathbf{\Gamma}^b\tilde{U}_2 &= (x+\beta,y+\beta) & GU_2 &= M(x-y,x+y+2\beta) \\ \mathbf{\Gamma}^b\tilde{W} &= (x+\alpha+\beta,y-\alpha+\beta) & GW &= M(x-y+2\alpha,x+y+2\beta). \end{aligned}$$

By the description in [15] the four indecomposables in the image of G also form a distinguished triangle. In particular, if α or β are 0 then the distinguished triangles in $\mathcal{D}^b(A_{\mathbb{R}})[\mathcal{N}Q^{-1}]$ and $\mathcal{D}_\pi$ are split/trivial. With the same techniques used to prove Theorems 2.2.9 and 2.3.5 we see that G takes cones in distinguished triangles to cones in distinguished triangles. $\square$

Theorem 4.2.5. *Assume $A_{\mathbb{R}}$ has finitely-many sinks and sources. Then there is a triangulated equivalence $H : \mathcal{C}(A_{\mathbb{R}})[\mathcal{N}Q^{-1}] \to \mathcal{C}_\pi$.*

Proof. Let $\mathcal{C}$ be the orbit category of $\mathcal{D}^b(A_{\mathbb{R}})[\mathcal{N}Q^{-1}]$ via doubling and almost-shift as in [15]. Since $\mathcal{D}^b(A_{\mathbb{R}})[\mathcal{N}Q^{-1}]$ is triangulated equivalent to $\mathcal{D}_\pi$, by Theorem 4.2.4, we see there must be a triangulated equivalence $H_2 : \mathcal{C} \to \mathcal{C}_\pi$. We will define a triangulated equivalence $H_1 : \mathcal{C}(A_{\mathbb{R}})[\mathcal{N}Q^{-1}] \to \mathcal{C}$. Afterwards we let $H = H_2 \circ H_1$, completing the proof.

Since $\mathcal{C}$ is an orbit category we choose our fundamental domain. We choose those indecomposables V that come from an indecomposable $\tilde{V}$ in $\mathcal{D}^b(A_{\mathbb{R}})$ such that $(\alpha,\beta) = \mathbf{\Gamma}^b\, V$ satisfy

$$\begin{aligned} -\frac{\pi}{2} <&\beta < \frac{\pi}{2} \\ \beta \leq&\alpha < \pi - \beta. \end{aligned}$$

This is precisely the image of the the 0th degree indecomposables in $\mathcal{D}^b(A_{\mathbb{R}})$, excluding the injective indecomposables from $\operatorname{rep}_k(A_{\mathbb{R}})$.

Recall $\mathcal{C}(A_{\mathbb{R}})$ has the same objects as $\mathcal{D}^b(A_{\mathbb{R}})$ but different isomorphism classes and similarly for $\mathcal{C}$ and $\mathcal{D}^b(A_{\mathbb{R}})[\mathcal{N}Q^{-1}]$, respectively. For each indecomposable V in $\mathcal{C}(A_{\mathbb{R}})[\mathcal{N}Q^{-1}]$ there exists a $\tilde{V}$ in $\mathcal{D}^b(A_{\mathbb{R}})$ in degree 0 such that after taking the orbit, $\tilde{V}$ is sent to V in the localization of $\mathcal{C}(A_{\mathbb{R}})$. We define H_1V to be the indecomposable in $\mathcal{C}$ that comes from an indecomposable $\hat{V}$ in $\mathcal{D}^b(A_{\mathbb{R}})[\mathcal{N}Q^{-1}]$ that also comes from $\tilde{V}$ in $\mathcal{D}^b(A_{\mathbb{R}})$.

Let V and W be indecomposables in $\mathcal{C}(A_{\mathbb{R}})[\mathcal{N}Q^{-1}]$. We will show $\operatorname{Hom}_{\mathcal{C}(A_{\mathbb{R}})[\mathcal{N}Q^{-1}]}(V,W) \cong k$ if and only if $\operatorname{Hom}_{\mathcal{C}}(H_1V,H_1W) \cong k$ and similarly for 0 hom spaces. First, there are $\tilde{V}$ and $\tilde{W}$ in degree 0 in $\mathcal{D}^b(A_{\mathbb{R}})$ that are sent to V and W after taking the orbit and localization. Then either $\operatorname{Hom}_{\mathcal{D}^b(A_{\mathbb{R}})}(\tilde{V},\tilde{W}) \cong k$ or $\operatorname{Hom}_{\mathcal{D}^b(A_{\mathbb{R}})}(\tilde{V},\tilde{W}[1]) \cong k$. Let $\tilde{V}_1$ be an indecomposable in $\mathcal{D}^b(A_{\mathbb{R}})$ such that $\mathbf{\Gamma}^b\,\tilde{V}_1 = \mathbf{\Gamma}^b\,\tilde{V}$ and $\tilde{V}_1$ has position 1. If $\operatorname{Hom}_{\mathcal{D}^b(A_{\mathbb{R}})}(\tilde{V}_1,\tilde{W})$ and $\operatorname{Hom}_{\mathcal{D}^b(A_{\mathbb{R}})}(\tilde{V}_1,\tilde{W}[1])$ were both 0 then $\operatorname{Hom}_{\mathcal{C}(A_{\mathbb{R}})[\mathcal{N}Q^{-1}]}(V,W)$ would be 0. Since this is not the case, $\operatorname{Hom}_{\mathcal{D}^b(A_{\mathbb{R}})}(\tilde{V}_1,\tilde{W}) \cong k$ or $\operatorname{Hom}_{\mathcal{D}^b(A_{\mathbb{R}})}(\tilde{V}_1,\tilde{W}[1]) \cong k$. Then $\operatorname{Hom}_{\mathcal{D}^b(A_{\mathbb{R}})[\mathcal{N}Q^{-1}]}(\hat{V},\hat{W}) \cong k$ or $\operatorname{Hom}_{\mathcal{D}^b(A_{\mathbb{R}})[\mathcal{N}Q^{-1}]}(\hat{V},\hat{W}[1]) \cong k$. In either case $\operatorname{Hom}_{\mathcal{C}}(H_1V,H_1W) \cong k$. In the case that $\operatorname{Hom}_{\mathcal{C}(A_{\mathbb{R}})[\mathcal{N}Q^{-1}]}(V,W) = 0$ we have $\operatorname{Hom}_{\mathcal{D}^b(A_{\mathbb{R}})}(\tilde{V}_1,\tilde{W}) = 0$ and $\operatorname{Hom}_{\mathcal{D}^b(A_{\mathbb{R}})}(\tilde{V}_1,\tilde{W}[1]) = 0$ and so $\operatorname{Hom}_{\mathcal{C}}(H_1V,H_1W) = 0$ as well.

Now, as in Definition 2.3.3 for our triangulated equivalence of derived categories for different continuous quivers of type A, we can choose representatives from each isomorphism class of indecomposables and fix isomorphisms between indecomposables and their respective representatives. With such a construction we set $H_1(\operatorname{Hom}_{\mathcal{C}(A_{\mathbb{R}})[\mathcal{N}Q^{-1}]}(V,W)) := \operatorname{Hom}_{\mathcal{C}}(H_1V,H_1W)$ for each pair of representatives V and W in $\mathcal{C}(A_{\mathbb{R}})[\mathcal{N}Q^{-1}]$. This gives us an equivalence of categories but must still check triangles.

However, each distinguished triangle $U \to V \to W \to$ in $\mathcal{C}(A_{\mathbb{R}})[\mathcal{N}Q^{-1}]$ comes from a triangle $\tilde{U} \to \tilde{V} \to \tilde{W} \to$ in the doubling of $\mathcal{D}^b(A_{\mathbb{R}})$. But after taking localization and then the orbit, the image of $\tilde{U} \to \tilde{V} \to \tilde{W} \to$ is a distinguished triangle in $\mathcal{C}$. This is, by definition, precisely the image of $U \to V \to W \to$ under H_1. Thus, H_1 is a triangulated equivalence and so is $H = H_2 \circ H_1$. □

4.3. **Comparing the Constructions.** In this section we highlight the differences and similarities in between the previous and new constructions.

The cluster structure introduced by Igusa and Todorov in [15] requires that the clusters be discrete. This was required so that every object in the cluster be mutable. The new theory given in Section 3 does not come with this restriction. Accordingly, not all objects in a cluster are **E**-mutable in the new theory. We refer back to Example 3.2.2 for the description of the cluster in the following example.

Example 4.3.1. Choose a particular continuous quiver $A_{\mathbb{R}}$ of type A. Consider again the **E**-cluster $\mathcal{P}$ in Example 3.2.2. Let $a \in \mathbb{R}$ such that a is neither a source nor a sink. By Theorem 1.1.7 there are exactly two projectives at a in $\text{rep}_k(A_{\mathbb{R}})$.

Let P be the object in $\mathcal{C}(A_{\mathbb{R}})$ that comes from P_a and Q the object that comes from $P_{(a}$ or $P_{a)}$, whichever exists in $\text{rep}_k(A_{\mathbb{R}})$. Then we have a distinguished triangle $Q \to P \to V \to$ in $\mathcal{C}(A_{\mathbb{R}})$ where V is degenerate. In particular, there is no distinguished triangle $P' \to W \to V \to$ in $\mathcal{C}(A_{\mathbb{R}})$ where P' is in $\mathcal{P}$ and not isomorphic to Q. Thus $(\mathcal{P} \setminus \{Q\}) \cup \{V\}$ is **E**-compatible and so Q is **E**-mutable.

However, for any object V such that $\{V, P\}$ is not **E**-compatible we have $\{V, P_{a+\varepsilon}\}$ is not **E**-compatible for $0 < \varepsilon << 1$. Thus P is not **E**-mutable.

We would like to mutate all the projectives from $\text{rep}_k(A_{\mathbb{R}})$ into all the injectives from $\text{rep}_k(A_{\mathbb{R}})$. However, this example appears to present a problem. The next paper in this series will address this with a continuous generalization of mutation.

5. Embeddings of Cluster Theories

In this section we demonstrate how to embed existing cluster theories (Definition 5.1.1) in the literature into the new continuous cluster theory in a way that is compatible with mutation. It should be noted that we do not make an attempt at embedding the cluster categories themselves, as this can lead to unanticipated problems (see Section 5.5). Thus we create new machinery in Section 5.1 to rigorously describe what we mean to embed one cluster theory within another. Other relationships between between cluster theories is outside the scope of this thesis but may be of interest in the future.

The goal of embedding cluster theories is the following. We hope, and anticipate, a continuous cluster *algebra* at some point in the future. If we can embed the existing type A cluster theories into the new **E**-cluster theory then this will provide the intuition, and perhaps some machinery, useful for the embedding of the relevant algebras. The A_n type cluster algebras already exist; the cluster category constructions in [3, 4] came after. In the reverse order, an A_∞ type cluster structure was introduced by Holm and Jørgensen in [12] and the cluster algebra type came later, introduced by Ndouné in [18]. A continuous version of a cluster algebra related to this work or the work in [15] is not known to the authors.

5.1. **Cluster Theories: $\mathscr{T}_{\mathbf{P}}(\mathcal{C})$.** This subsection is dedicated to providing a framework in which to talk about embedding cluster theories without requiring a functor between cluster categories. It should be noted that cluster theories are not cluster structures as they are often defined (for examples, [4, 3, 12]). While cluster structures require each indecomposable in a cluster be mutable and mutation be given by approximations, cluster theories do not make such a requirement. Instead, we require that *if* an indecomposable is mutable then there is a unique choice for a replacement. In practice this should be related to some homological or other algebraic property but we do not explicitly make this requirement. The reader should recall that a pairwise compability condition on indecomposable objects takes unordered pairs of indecomposables and determines their compatibility, thus allowing larger sets to be called compatible if every subset of 2 elements is pairwise compatible.

Definition 5.1.1. Let $\mathcal{C}$ be a skeletally small Krull-Schmidt additive category in which there exists a pairwise compatibility condition $\mathbf{P}$ on (isomorphism classes of) indecomposable objects. Suppose also that for each (isomorphism class of) indecomposable X in a maximally $\mathbf{P}$-compatible set T there exists none or one (isomorphism class of) indecomposable Y such that $\{X, Y\}$ is not $\mathbf{P}$-compatible but $(T \setminus \{X\}) \cup \{Y\}$ is maximally $\mathbf{P}$-compatible. Then

- We call the maximally $\mathbf{P}$-compatible sets $\underline{\mathbf{P}\text{-clusters}}$.
- We call a function of the form $\mu : T \to (T \setminus \{X\}) \cup \{Y\}$ such that $\mu Z = Z$ when $Z \neq X$ and $\mu X = Y$ a $\underline{\mathbf{P}\text{-mutation}}$ or $\underline{\mathbf{P}\text{-mutation at } X}$.
- If there exists a $\mathbf{P}$-mutation $\mu : T \to (T \setminus \{X\}) \cup \{Y\}$ we say $X \in T$ is $\underline{\mathbf{P}\text{-mutable}}$.
- The subcategory $\mathscr{T}_{\mathbf{P}}(\mathcal{C})$ of $\mathcal{S}$et whose objects are $\mathbf{P}$-clusters and whose morphisms are generated by $\mathbf{P}$-mutations (and identity functions) is called $\underline{\text{the } \mathbf{P}\text{-cluster theory of } \mathcal{C}}$.
- The functor $I_{\mathbf{P},\mathcal{C}} : \mathscr{T}_{\mathbf{P}}(\mathcal{C}) \to \mathcal{S}$et is the inclusion of the subcategory.

From now on, when we say "a Krull-Schmidt category" we mean "a skeletally small Krull-Schmidt additive category."

Remark 5.1.2. Let $\mathcal{C}$ be a Krull-Schmidt category and $\mathbf{P}$ a pairwise compatibility condition on the indecomposable objects in $\mathcal{C}$. If the $\mathbf{P}$-cluster theory of $\mathcal{C}$ exists then it is completely determined by $\mathbf{P}$. Thus we say that $\mathbf{P}$ $\underline{\text{induces}}$ the $\mathbf{P}$-cluster theory of $\mathcal{C}$.

Remark 5.1.3. Let $\mathcal{C}$ be a Krull-Schmidt category and $\mathbf{P}$ a pairwise compatibility condition on $\text{Ind}(\mathcal{C})$. Using Zorn's lemma we note that there exist maximally $\mathbf{P}$-compatible sets of indecomposables of $\mathcal{C}$.

Definition 5.1.4. Let $\mathcal{C}$ be a Krull-Schmidt category and $\mathbf{P}$ a pairwise compatibility condition such that $\mathbf{P}$ induces the $\mathbf{P}$-cluster theory of $\mathcal{C}$. If for every $\mathbf{P}$-cluster T and $X \in T$ there is a $\mathbf{P}$-mutation at X then we call $\mathscr{T}_{\mathbf{P}}(\mathcal{C})$ the $\underline{\text{tilting } \mathbf{P}\text{-cluster theory}}$.

Remark 5.1.5. The reader may be familiar with frozen elements of a cluster and notice that sometimes an indecomposable X in a $\mathbf{P}$-cluster T may not be $\mathbf{P}$-mutable. However, we do not call X frozen. This is because in the next paper of this series we will introduce a continuous generalization of mutation that allows some X which are not $\mathbf{P}$-mutable to *become* $\mathbf{P}$-mutable. The word frozen, then, should be reserved for indecomposables we have intentionally frozen or those that may *never* be made $\mathbf{P}$-mutable.

Proposition 5.1.6. *Let $\mathcal{C}$ be a Krull-Schmidt category and $\mathbf{P}$ a pairwise compatibility condition such that $\mathbf{P}$ induces the $\mathbf{P}$-cluster theory of $\mathcal{C}$. Let T be a $\mathbf{P}$-cluster and $X \in T$ such that there exists a $\mathbf{P}$-mutation $T \to (T \setminus \{X\}) \cup \{Y\}$. Then there exists a $\mathbf{P}$-mutation $T' = (T \setminus \{X\}) \cup \{Y\} \to (T' \setminus \{Y\}) \cup \{X\}$.*

Proof. We know $\{Y, X\}$ is not $\mathbf{P}$-compatible but $(T' \setminus \{Y\}) \cup \{X\}$ is maximally $\mathbf{P}$-compatible. □

Example 5.1.7. Our first example of a cluster theory is the cluster structure defined in [3] for cluster categories $\mathcal{C}(Q)$ of dynkin quivers Q. Consider clusters in such cluster categories like those described in [3] as maximally rigid sets of indecomposables instead of the subcategories generated by those indecomposables.

We define our pairwise compatibility condition $\mathbf{R}$ to be rigidity. Then the $\mathbf{R}$-clusters are maximally rigid sets of indecomposables in $\mathcal{C}(Q)$ and $\mathbf{R}$-mutations are traditional cluster tilting in $\mathcal{C}(Q)$. This yields the tilting $\mathbf{R}$-cluster theory of $\mathcal{C}(Q)$.

Example 5.1.8. The Euler form on $K_0^{\text{split}}(\mathcal{C}(A_{\mathbb{R}}))$ in Definition 3.1.5 is by definition a pairwise compatibility condition that we have already called $\mathbf{E}$-compatibility. We have shown in Theorem 3.2.8 that $\mathbf{E}$ induces the $\mathbf{E}$-cluster theory of $\mathcal{C}(A_{\mathbb{R}})$. In the $\mathbf{E}$-cluster T in Example 4.3.1 we see that there exist indecomposables P which are not $\mathbf{E}$-mutable. Thus the $\mathbf{E}$-cluster theory is not tilting.

Example 5.1.9. Our final example for now is the triangulations of the $(n+3)$-gon model introduced in [4]. In [4], the authors describe the A_n cluster structure as triangulations of the $(n+3)$-gon. This arises in a category $\mathcal{C}(A_n)$ whose indecomposable objects are diagonals of the $(n+3)$-gon and rigidity is given by two diagonals not crossing.

We let $\mathbf{N}_n$ be the pairwise compatibility condition of not crossing. We let $\mathbf{N}_n$-clusters be maximal sets of noncrossing diagonals and let $\mathbf{N}_n$-mutations be the exchanging of one diagonal for another to produce a different triangulation. This is the tilting $\mathbf{N}_n$-cluster theory of $\mathcal{C}(A_n)$.

Definition 5.1.10. Let $\mathcal{C}$ and $\mathcal{D}$ be two Krull-Schmidt categories with respective pairwise compatibility conditions $\mathbf{P}$ and $\mathbf{Q}$. Suppose these compatibility conditions induce the $\mathbf{P}$-cluster theory and $\mathbf{Q}$-cluster theory of $\mathcal{C}$ and $\mathcal{D}$, respectively.

Suppose there exists a functor $F : \mathscr{T}_{\mathbf{P}}(\mathcal{C}) \to \mathscr{T}_{\mathbf{Q}}(\mathcal{D})$ such that F is an injection on objects and an injection from $\mathbf{P}$-mutations to $\mathbf{Q}$-mutations. Suppose also there is a natural transformation $\eta : I_{\mathbf{P},\mathcal{C}} \to I_{\mathbf{Q},\mathcal{D}} \circ F$ whose morphisms $\eta_T : I_{\mathbf{P},\mathcal{C}}(T) \to I_{\mathbf{Q},\mathcal{D}} \circ F(T)$ are all injections. Then we call $(F,\eta) : \mathscr{T}_{\mathbf{P}}(\mathcal{C}) \to \mathscr{T}_{\mathbf{Q}}(\mathcal{D})$ an embedding of cluster theories.

Remark 5.1.11. Observing Definition 5.1.10 we see that if we can produce the functions involved in the natural transformation then we obtain the embedding of cluster theories

We will not immediately provide an example of Definition 5.1.10 as Sections 5.2 and 5.3 are devoted to such examples.

We conclude this subsection with the definition of an $\mathbf{E}$-compatible set T_∞ for straight descending A_R we will use in Sections 5.2 and 5.3.

Definition 5.1.12. Let $\{a_i\}_{i\in\mathbb{Z}}$ be a collection of real numbers such that

- $a_i < a_{i+1}$ for all $i \in \mathbb{Z}$ and
- $\lim_{i\to-\infty} a_i, \lim_{i\to+\infty} a_i \in \mathbb{Z}$.

Let $a_{-\infty} = \lim_{i\to-\infty} a_i$ and $a_{+\infty} = \lim_{i\to+\infty} a_i$. For each $i, j, \ell \in \mathbb{Z}$ such that $l \geq 0$ and $0 \leq j \leq 2^\ell$ define

$$a_{i,j,\ell} := a_i + \left(\frac{j}{2^\ell}\right)(a_{i+1} - a_i).$$

For each a_i, we define the following $\mathbf{E}$-compatible set:

$$\begin{aligned} T_{a_i} := &\left\{ M_{(a_{i,j,\ell},\, a_{i,j+1,\ell})} : j,\ell \in \mathbb{Z},\, \ell \geq 0,\, 0 \leq j < 2^\ell \right\} \\ &\cup \left\{ M_{\{x\}} : x \in (a_i, a_i+1),\, x \neq a_{i,j,l},\ j,\ell \in \mathbb{Z},\, \ell \geq 0,\, 0 \leq j < 2^\ell \right\}, \end{aligned}$$

Note that for any a_i and a_j then $T_{a_i} \cup T_{a_j}$ is $\mathbf{E}$-compatible. Now, for each $i \in \mathbb{Z}$ such that $i < a_{-\infty}$ or $i \geq a_{+\infty}$ define a similar type of $\mathbf{E}$-compatible set:

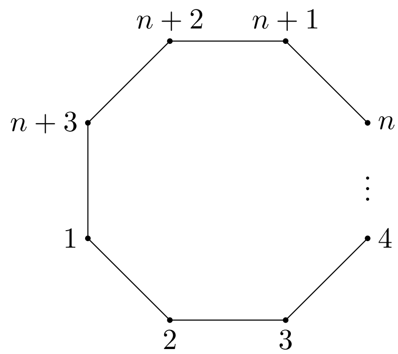
$$\begin{aligned} T_i := &\left\{ M_{(i+j/2^\ell, i+(j+1)/2^\ell)} : j,\ell \in \mathbb{Z},\, \ell \geq 0,\, 0 \leq j < 2^\ell \right\} \cup \{P_{i+1}\} \\ &\cup \left\{ M_{\{x\}} : x \in (i, i+1),\, x \neq i + j/2^\ell,\, j,\ell \in \mathbb{Z},\, \ell \geq 0,\, 0 \leq j < 2^\ell \right\} \end{aligned}$$

The $\mathbf{E}$-compatible set we want is

$$\begin{aligned} T_\infty := &\left(\bigcup_{i\in\mathbb{Z}} T_{a_i} \right) \cup \left(\bigcup_{i<a_{-\infty} \text{ or } i\geq a_{+\infty}} T_i \right) \\ &\cup \{M_{(a_{-\infty},a_{+\infty})}, P_{+\infty}\} \cup \{P_i : i \leq a_{-\infty} \text{ or } i \geq a_{+\infty}\}. \end{aligned}$$

5.2. **Embeddings $\mathscr{T}_{\mathbf{N}_n}(\mathcal{C}(A_n)) \to \mathscr{T}_{\mathbf{E}}(\mathcal{C}(A_{\mathbb{R}}))$.** In this section we demonstrate how to embed type A_n cluster theories into the $\mathbf{E}$-cluster theory of $\mathcal{C}(A_{\mathbb{R}})$. We will assume that $A_{\mathbb{R}}$ has the straight descending orientation.

We will use the $\mathbf{N}_n$-cluster theory in Example 5.1.9. Choose n and label the vertices of the $(n+3)$-gon counterclockwise.



We will label a diagonal in the $(n+3)$-gon by $i{\bullet\!\!-\!\!\bullet}j$, where $i<j$. An $\mathbf{N}_n$-cluster is a maximal collection of noncrossing diagonals; this is also called a triangulation of the $(n+3)$-gon. A pair of diagonals $i{\bullet\!\!-\!\!\bullet}j$ and $i'{\bullet\!\!-\!\!\bullet}j'$ cross if and only if $i<i'<j<j'$ or $i'<i<j'<j$.

Recall the notation $M_{|a,b|}$ from Defintion 1.1.3 and Notation 1.1.4.

Definition 5.2.1. Assume $A_{\mathbb{R}}$ has the straight orientation. Again let a_i (for all $i \in \mathbb{Z}$), $a_{-\infty}$, and $a_{+\infty}$ be as in Definition 5.1.12. Let

$$T_n := T_\infty \cup \{M_{(a_i,a_1)} : i<0\} \cup \{M_{(a_1,a_j)} : j \geq n+3\} \cup \{M_{(a_{-\infty},a_1)}, M_{(a_1,a_{+\infty})}\}$$

It is straightforward to check that T_n is $\mathbf{E}$-compatible.

Definition 5.2.2. Let $T_{\mathbf{N}_n}$ be an $\mathbf{N}_n$-cluster in $\mathscr{T}_{\mathbf{N}_n}(\mathcal{C}(A_n))$ as described. We will construct an $\mathbf{E}$-cluster $T_{\mathbf{E}}$ in $\mathscr{T}_{\mathbf{E}}(\mathcal{C}(A_{\mathbb{R}}))$ based on $T_{\mathbf{N}_n}$. Thus, define $M_{i{\bullet\!\!-\!\!\bullet}j} := M_{(a_i,a_j)}$.

Proposition 5.2.3. *A set of diagonals $\{i{\bullet\!\!-\!\!\bullet}j, i'{\bullet\!\!-\!\!\bullet}j'\}$ is $\mathbf{N}_n$-compatible if and only if the set $\{M_{i{\bullet\!\!-\!\!\bullet}j}, M_{i'{\bullet\!\!-\!\!\bullet}j'}\}$ is $\mathbf{E}$-compatible.*

Proof. Suppose $i{\bullet\!\!-\!\!\bullet}j$ is not $\mathbf{N}_n$-compatible with $i'{\bullet\!\!-\!\!\bullet}j'$. Then, up to symmetry, $i<i'<j<j'$. We then know there exists a rectangle in the AR-space of $\text{rep}_k(A_{\mathbb{R}})$ whose left corner is $M_{i{\bullet\!\!-\!\!\bullet}j}$, top corner is $M_{(a_i,a_{j'})}$, bottom corner is $M_{(a_{i'},a_j)}$, and right corner is $M_{i'{\bullet\!\!-\!\!\bullet}j'}$. Thus $M_{i{\bullet\!\!-\!\!\bullet}j}$ and $M_{i'{\bullet\!\!-\!\!\bullet}j'}$ are not $\mathbf{E}$-compatible.

If we start with $M_{i{\bullet\!\!-\!\!\bullet}j}$ and $M_{i'{\bullet\!\!-\!\!\bullet}j'}$ are not $\mathbf{E}$-compatible we get the rectangle in the AR-space of $\text{rep}_k(A_{\mathbb{R}})$ again which implies (up to symmetry) that $a_i<a_{i'}<a_j<a_{j'}$ and so $i<i'<j<j'$. Therefore $i{\bullet\!\!-\!\!\bullet}j$ and $i'{\bullet\!\!-\!\!\bullet}j'$ are not $\mathbf{N}_n$-compatible so the proposition holds. □

Definition 5.2.4. Given an $\mathbf{N}_n$-cluster $T_{\mathbf{N}_n}$, let

$$T_{\mathbf{E}_n} = T_n \cup \{M_{i{\bullet\!\!-\!\!\bullet}j} : i{\bullet\!\!-\!\!\bullet}j \in T_{\mathbf{N}_n}\}.$$

With Proposition 5.2.3 it is straightforward to check $T_{\mathbf{E}_n}$ is $\mathbf{E}$-compatible.

Proposition 5.2.5. *The $\mathbf{E}$-compatible set $T_{\mathbf{E}_n}$ is an $\mathbf{E}$-cluster.*

Proof. Choose some indecomposable $M_{|c,d|}$ in $\mathcal{C}(A_{\mathbb{R}})$ such that $\{M_{|c,d|}\} \cup T_{\mathbf{E}}$ is $\mathbf{E}$-compatible. We will show that $M_{|c,d|} \in T_{\mathbf{E}}$. Recall the |s in our notation mean we not assuming whether or not c or d is in the interval $|c,d|$. We will check the various possible values of c to complete the proof; note that $c<+\infty$.

Suppose $c=-\infty$. Then either $d \leq a_{-\infty}$ or $d \geq a_{+\infty}$. If $d=a_{\pm\infty}$ then $M_{|c,d|}$ must be the corresponding open projective at $a_{\pm\infty}$. If $d<a_{-\infty}$ or $d>a_{+\infty}$ then $d \in \mathbb{Z}$ or $d=+\infty$ as

$M_{(i,i+1)} \in T_i$ from Definition 5.1.12. Thus in all these cases, $M_{|c,d|}$ must be P_i for some $i \in \mathbb{Z}$ outside $(a_{-\infty}, a_{+\infty})$ or $i = +\infty$. If $-\infty < c < a_{-\infty}$ or $a_{+\infty} \le c < +\infty$ then either $c = i + j/2^\ell$ and $d = i + (j+1)/2^\ell$ for some i, $\ell \ge 0$, $0 \le j < 2^\ell$ or $c = d$ and $M_{|c,d|} = M_{\{c\}}$.

Suppose $c = a_{-\infty}$ Then $d = a_1$ or $d = a_{+\infty}$. Thus, $M_{|c,d|}$ must be $M_{(a_{-\infty},a_1)}$ or $M_{(a_{-\infty},a_{+\infty})}$. Suppose $a_{-\infty} < c < a_{+\infty}$ and $c \neq a_i$ for any i. If $c = a_{i,j,\ell}$ for some $i \in \mathbb{Z}$, $\ell \ge 0$, and $0 < j < (2^\ell)$ then, up to adjusting ℓ, $d = a_{i,j+1,\ell}$. If c is not of this form then $d = c$ and $M_{|c,d|} = M_{\{c\}}$.

If $c = a_i$ for some $i \notin \{1, \dots, n+2\}$ then either $d = a_{i+1}$, $d = a_1$, or $d = +\infty$. Thus $M_{|c,d|}$ is one of $M_{(a_i,a_{i+1})|}$, $M_{(a_i,a_1)}$, or $I_{(a_1}$.

So now we check $c \in \{a_i\}_{i=1}^{n+2}$. First assume $M_{|c,d|} \neq M_{i \bullet\!\!-\!\!\bullet j}$ for any $i \bullet\!\!-\!\!\bullet j$. If $c = a_1$ then either $d = a_{n+3}$, $d = a_{+\infty}$, or $d \in (a_1, a_2]$. If $c = a_i$ for $1 < i < n+3$ then $d \in (a_i, a_{i+1}]$. In any of these cases, $M_{|c,d|} = M_{(a_1,a_{n+3})}$, $M_{|c,d|} = M_{(a_i,a_{i,j,l})}$, $M_{|c,d|} = M_{(a_1,a_{+\infty})}$, or $M_{|c,d|} = M_{(a_i,a_{i+1})}$.

Now the only possibility left to check is $c \in \{a_i\}_{i=1}^{n+2}$ and $M_{|c,d|} = M_{i \bullet\!\!-\!\!\bullet j}$ for some $i \bullet\!\!-\!\!\bullet j$, not necessarily in $T_{\mathbf{N}_n}$. For contradiction, assume $i \bullet\!\!-\!\!\bullet j \notin T_{\mathbf{N}_n}$. Then there is a $i' \bullet\!\!-\!\!\bullet j' \in T_{\mathbf{N}_n}$ such that $\{i \bullet\!\!-\!\!\bullet j, i' \bullet\!\!-\!\!\bullet j'\}$ is not $\mathbf{N}_n$-compatible. By symmetry suppose $i < i' < j < j'$. However, $M_{i \bullet\!\!-\!\!\bullet j} \to M_{(a_{i'},a_j)} \oplus M_{(a_i,a_{j'})} \to M_{i' \bullet\!\!-\!\!\bullet j'}$ is an extension in $\operatorname{rep}_k(A_{\mathbb{R}})$ and so there is a rectangle in the AR-space of $\operatorname{rep}_k(A_{\mathbb{R}})$ whose left and right corners are $M_{i \bullet\!\!-\!\!\bullet j}$ and $M_{i' \bullet\!\!-\!\!\bullet j'}$, respectively (Theorem 1.2.3). Then $M_{i \bullet\!\!-\!\!\bullet j}$ and $M_{i' \bullet\!\!-\!\!\bullet j'}$ are not $\mathbf{E}$-compatible by Proposition 3.1.8. Since $M_{i' \bullet\!\!-\!\!\bullet j'} \in T_{\mathbf{E}}$, we have a contradiction. □

Lemma 5.2.6. *Consider an $\mathbf{N}_n$-cluster $T_{\mathbf{N}_n}$ as above and the induced $\mathbf{E}$-cluster $T_{\mathbf{E}_n}$. Suppose $T_{\mathbf{N}_n} \to (T_{\mathbf{N}_n} \setminus \{i \bullet\!\!-\!\!\bullet j\}) \cup \{i' \bullet\!\!-\!\!\bullet j'\}$ is an $\mathbf{N}_n$-mutation. Then $T_{\mathbf{E}_n} \to (T_{\mathbf{E}_n} \setminus \{M_{i \bullet\!\!-\!\!\bullet j}\}) \cup \{M_{i' \bullet\!\!-\!\!\bullet j'}\}$ is an $\mathbf{E}$-mutation.*

Proof. It suffices to show that $\{M_{i \bullet\!\!-\!\!\bullet j}, M_{i' \bullet\!\!-\!\!\bullet j'}\}$ is not $\mathbf{E}$-compatible and $(T_{\mathbf{E}_n} \setminus \{M_{i \bullet\!\!-\!\!\bullet j}\}) \cup \{M_{i' \bullet\!\!-\!\!\bullet j'}\}$ is $\mathbf{E}$-compatible. By the end of the proof of Proposition 5.2.5, we see that $\{M_{i \bullet\!\!-\!\!\bullet j}, M_{i' \bullet\!\!-\!\!\bullet j'}\}$ is not $\mathbf{E}$-compatible. Let $T'_{\mathbf{N}_n} = (T_{\mathbf{N}_n} \setminus \{i \bullet\!\!-\!\!\bullet j\}) \cup \{i' \bullet\!\!-\!\!\bullet j'\}$. Note that $(T_{\mathbf{E}_n} \setminus \{M_{i \bullet\!\!-\!\!\bullet j}\}) \cup \{M_{i' \bullet\!\!-\!\!\bullet j'}\} = T'_{\mathbf{E}_n}$. By Proposition 5.2.5, $T'_{\mathbf{E}_n}$ is an $\mathbf{E}$-cluster. Therefore $T_{\mathbf{E}_n} \to (T_{\mathbf{E}_n} \setminus \{M_{i \bullet\!\!-\!\!\bullet j}\}) \cup \{M_{i' \bullet\!\!-\!\!\bullet j'}\}$ is an $\mathbf{E}$-mutation. □

Theorem 5.2.7. *There exists an embedding of cluster theories $(F, \eta) : \mathscr{T}_{\mathbf{N}_n}(\mathcal{C}(A_n)) \to \mathscr{T}_{\mathbf{E}_n}(\mathcal{C}(A_{\mathbb{R}}))$.*

Proof. By Lemma 5.2.6 we see that defining $F(T_{\mathbf{N}_n}) := T_{\mathbf{E}_n}$ and sending $\mathbf{N}_n$-mutations to the corresponding $\mathbf{E}$-mutations described in the lemma yields a functor. Since $M_{i \bullet\!\!-\!\!\bullet j} \not\cong M_{i' \bullet\!\!-\!\!\bullet j'}$ if $i \neq i'$ or $j \neq j'$ we see that if $T_{\mathbf{N}_n} \neq T'_{\mathbf{N}_n}$ then $T_{\mathbf{E}_n} \neq T'_{\mathbf{E}_n}$. Furthermore, F is injective on clusters and generating morphisms between clusters.

Let $\eta_{T_{\mathbf{N}_n}} : T_{\mathbf{N}_n} \to T_{\mathbf{E}_n}$ be given by $\eta_{T_{\mathbf{N}_n}}(i \bullet\!\!-\!\!\bullet j) = M_{i \bullet\!\!-\!\!\bullet j}$. We see this is an injection and as described in Lemma 5.2.6 these injections commute with mutations. Therefore (F, η) is an embedding of cluster theories. □

5.3. **Embedding $\mathscr{T}_{\mathbf{N}_\infty}(\mathcal{C}(A_\infty)) \to \mathscr{T}_{\mathbf{E}}(\mathcal{C}(A_{\mathbb{R}}))$.** In this section we demonstrate how to embed type A_∞ cluster theory into the $\mathbf{E}$-cluster theory of $\mathcal{C}(A_{\mathbb{R}})$. Consider again $A_{\mathbb{R}}$ with the straight descending orientation. In particular, we address the structure introduced by Holm and Jørgensen in [12] using triangulations of the infinity-gon.

We will discuss how to embed the closely related structure on the completed infinity-gon (introduced by Baur and Graz in [1]) in the next paper of this series.

Definition 5.3.1 (From [12])**.** The infinity-gon is has its vertices indexed by $\mathbb{Z}$ and no vertex at infinity. An arc is a pair of integers (i, j) such that $i < j$ and $i - j \ge 2$. Two arcs (i, j) and (i', j') are defined to cross if and only if $i < i' < j < j'$ or $i' < i < j' < j$. We will call this pairwise compatibility condition $\mathbf{N}_\infty$. Note the similarity to the definitions in Section 5.2. Thus we will write our arcs as $i \bullet\!\!-\!\!\bullet j$.

The authors show there is a triangulated category whose indecomposables are the diagonals that are compatible if and only if they do not cross. We will denote the cluster category in which the arcs exist by $\mathcal{C}(A_\infty)$.

We continue to let $\{a_i\}$ be the sequence from Definition 5.1.12 and define $M_{i{\bullet\!\!-\!\!\bullet}j} := M_{(a_i,a_j)}$. The following proposition is proved in precisely the same fashion as Proposition 5.2.3.

Proposition 5.3.2. *Two arcs $i{\bullet\!\!-\!\!\bullet}j$ and $i'{\bullet\!\!-\!\!\bullet}j'$ are $\mathbf{N}_\infty$-compatible if and only if $M_{i{\bullet\!\!-\!\!\bullet}j}$ and $M_{i'{\bullet\!\!-\!\!\bullet}j'}$ are $\mathbf{E}$-compatible.*

Definition 5.3.3. Let $T_{\mathbf{N}_\infty}$ be an $\mathbf{N}_\infty$-cluster. Define $T^\circ_{\mathbf{E}_\infty}$ to be

$$T^\circ_{\mathbf{E}_\infty} = T_\infty \cup \{M_{i{\bullet\!\!-\!\!\bullet}j} : i{\bullet\!\!-\!\!\bullet}j \in T_{\mathbf{N}_\infty}\}.$$

Similar to Section 5.2 we see that with Proposition 5.3.2 it is straightforward to check that $T_{\mathbf{E}_\infty}$ is $\mathbf{E}$-compatible.

It is not true that $T^\circ_{\mathbf{E}_\infty}$ is always an $\mathbf{E}$-cluster. That is, the clusters considered in [12] form a *proper* subcategory of $\mathscr{T}_{\mathbf{N}_\infty}(\mathcal{C}(A_\infty))$. The $\mathbf{N}_\infty$-cluster theory includes what the authors in [12] called "weak clusters." Put another way: the $\mathbf{N}_\infty$-cluster theory of $\mathcal{C}(A_\infty)$ is not tilting.

Example 5.3.4. Consider the $\mathbf{N}_\infty$-cluster

$$T_{\mathbf{N}_\infty} = \{i{\bullet\!\!-\!\!\bullet}0 : i < -1\} \cup \{1{\bullet\!\!-\!\!\bullet}j : j > 2\}$$

as in [12, Sketch 3, p.279]. This is maximally $\mathbf{N}_\infty$-compatible but $T^\circ_{\mathbf{E}_\infty}$ is not maximally $\mathbf{E}$-compatible. However, one may check that

$$T_{\mathbf{E}_\infty} := T_{\mathbf{E}_\infty} \cup \{M_{(a_{-\infty},a_0)}, M_{(a_{-\infty},a_1)}, M_{(a_1,a_{+\infty})}\}$$

is maximally $\mathbf{E}$-compatible.

One issue with $T_{\mathbf{N}_\infty}$ is addressed by the authors in [12]: add $T_{\mathbf{N}_\infty}$ is not functorially finite in $\mathcal{C}(A_\infty)$. There is not truly a problem with embedding *too many* cluster-like objects. Once the embedding has been established, one may take the subgroupoid of $\mathscr{T}_{\mathbf{N}_\infty}(\mathcal{C}(A_\infty))$ consisting of only those $\mathbf{N}_\infty$-clusters that are part of the cluster structure in [12]. Thus there is still an embedding into the $\mathbf{E}$-cluster theory of $\mathcal{C}(A_\mathbb{R})$.

To create the embedding of cluster theories $\mathscr{T}_{\mathbf{N}_\infty}(\mathcal{C}(A_\infty)) \to \mathscr{T}_{\mathbf{E}}(\mathcal{C}(A_\mathbb{R}))$ we need the following definitions adapted from [12, Definition 3.2].

Definition 5.3.5. Let T be an $\mathbf{N}_\infty$-compatible set of arcs.

- If, for all $n \in \mathbb{Z}$, there are only finitely many arcs in the set $\{i{\bullet\!\!-\!\!\bullet}j \in T : i = n \text{ or } j = n\}$ we say T is <u>locally finite</u>.
- If there exists $n \in \mathbb{Z}$ such that $\{i{\bullet\!\!-\!\!\bullet}j \in T : j = n\}$ is infinite we call this set of arcs a left-fountain.
- If there exists $n \in \mathbb{Z}$ such that $\{i{\bullet\!\!-\!\!\bullet}j \in T : i = n\}$ is infinite we call this set of arcs a right-fountain.
- If there exists n that has both a left- and right-fountain we say $\{i{\bullet\!\!-\!\!\bullet}j \in T : i = n \text{ or } j = n\}$ is a fountain.

The authors note in [12, Lemma 3.3] that if a left- or right-fountain exists in a $\mathbf{N}_\infty$-cluster then it must be unique. Just before the lemma the authors note that a left-fountain exists if and only if a right-fountain exists, crediting Collin Bleak. This will become quite important. It is also prudent to note that if there is a left-fountain at m and a right-fountain at n then $m \leq n$.

Definition 5.3.6. We now define $T_{\mathbf{E}_\infty}$ given $T_{\mathbf{N}_\infty}$.

- If $T_{\mathbf{N}_\infty}$ is locally finite then
$$T_{\mathbf{E}_\infty} = T^\circ_{\mathbf{E}_\infty}.$$

- If $T_{\mathbf{N}_\infty}$ has a left- or right-fountain it has the other. Let m be the vertex with the left-fountain and n the vertex with the right-fountain; note that it is possible $m = n$. Set
$$T_{\mathbf{E}_\infty} = T^\circ_{\mathbf{E}_\infty} \cup \{M_{(a_{-\infty}, a_m)}, M_{(a_{-\infty}, a_n)}, M_{(a_n, a_{+\infty})}\}.$$

Proposition 5.3.7. *Let $T_{\mathbf{N}_\infty}$ be an $\mathbf{N}_\infty$ cluster. Suppose there exists $\ell \in \mathbb{Z}$ such that for all $i{\bullet\!\!-\!\!\bullet}j \in T_{\mathbf{N}_\infty}$, $\ell \leq i$ or $j \leq \ell$. Then there exists a left- and right-fountain in $T_{\mathbf{N}_\infty}$.*

Proof. For contradiction, suppose $T_{\mathbf{N}_\infty}$ is locally finite. Let
$$i_\ell = \min_i \{i \geq \ell \in \mathbb{Z} : \exists\, i{\bullet\!\!-\!\!\bullet}j \in T_{\mathbf{N}_\infty}\}$$
$$j_\ell = \max_j \{j \leq \ell \in \mathbb{Z} : \exists\, i{\bullet\!\!-\!\!\bullet}j \in T_{\mathbf{N}_\infty}\}.$$
By the maximality of $T_{\mathbf{N}_\infty}$ we see that $0 \leq i_\ell - j_\ell \leq 1$ and $\ell \in \{i_\ell, j_\ell\}$. Since we have assumed $T_{\mathbf{N}_\infty}$ is locally finite, let
$$j_0 = \max_j \{j \in \mathbb{Z} : i_\ell{\bullet\!\!-\!\!\bullet}j \in T\}$$
$$i_0 = \min_i \{i \in \mathbb{Z} : i{\bullet\!\!-\!\!\bullet}j_\ell \in T\}.$$
We will show $i_0{\bullet\!\!-\!\!\bullet}j_0 \in T_{\mathbf{N}_\infty}$, contradicting our assumption about ℓ.

For contradiction, suppose there exists $i{\bullet\!\!-\!\!\bullet}j \in T_{\mathbf{N}_\infty}$ such that $i_0 < i < j_0 < j$. Since $i_0 < i < j_0$, we must have $j_\ell \leq i \leq i_\ell$. But then $i = i_\ell$ by our definition of i_ℓ. However $j_0 < j$, contradiction our definition of j_0. Thus there cannot be such a $i{\bullet\!\!-\!\!\bullet}j \in T_{\mathbf{N}_\infty}$. Similarly, there can be no $i'{\bullet\!\!-\!\!\bullet}j' \in T_{\mathbf{N}_\infty}$ such that $i' < i_0 < j' < j_0$.

This means $\{i_0{\bullet\!\!-\!\!\bullet}j_0\} \cup T_{\mathbf{N}_\infty}$ is $\mathbf{N}_\infty$-compatible. Since $T_{\mathbf{N}_\infty}$ is an $\mathbf{N}_\infty$-cluster we have $i_0{\bullet\!\!-\!\!\bullet}j_0 \in T_{\mathbf{N}_\infty}$. This contradicts our assumption about ℓ since $i_0 < \ell < j_0$. Therefore $T_{\mathbf{N}_\infty}$ is not locally finite; i.e. there exists a left- and right-fountain in $T_{\mathbf{N}_\infty}$. □

Proposition 5.3.8. *Let $T_{\mathbf{N}_\infty}$ be an $\mathbf{N}_\infty$-cluster. Then $T_{\mathbf{E}_\infty}$ is an $\mathbf{E}$-cluster.*

Proof. Recall $T_{\mathbf{E}_\infty}$ is an $\mathbf{E}$-compatible set. Now suppose $M_{|c,d|}$ is an indecomposable in $\mathcal{C}(A_\mathbb{R})$ and $T_{\mathbf{E}_\infty} \cup \{M_{|c,d|}\}$ is $\mathbf{E}$-compatible. We will first assume $M_{|c,d|}$ is not of the form $M_{i{\bullet\!\!-\!\!\bullet}j}$ for any pair of integers $i < j$ where $j - i \geq 2$. Similar to Proposition 5.2.5 we will check various possibilities for c. The argument when $c < a_{-\infty}$ or $c \geq a_{+\infty}$ is identical to Proposition 5.2.5.

Suppose $c = a_{-\infty}$. Suppose $T_{\mathbf{N}_\infty}$ has a left- and right-fountain at m and n, respectively. Then $d = a_m$, $d = a_n$, or $d = a_{+\infty}$ since there is a left-fountain at m and a right-fountain at n. Note that if $d = a_{+\infty}$ then either $c = a_{-\infty}$ or $c = a_n$. If $T_{\mathbf{N}_\infty}$ is locally finite then $c = a_{-\infty}$ if and only if $d = a_{+\infty}$.

We now suppose neither c nor d is in $\{a_{-\infty}, a_{+\infty}\}$. If $a_i < c < a_{i+1}$ for some $i \in \mathbb{Z}$ then $M_{|c,d|} = M_{\{c\}}$ or $M_{|c,d|} = M_{(a_{i,j,\ell}, a_{i,j+1,\ell})}$ for $0 \leq \ell$ and $0 \leq j < 2^\ell$ similar to the proof of Proposition 5.2.5. If $c = a_i$ for some $i \in \mathbb{Z}$ then, since we are still assuming $M_{|c,d|} \neq M_{i'{\bullet\!\!-\!\!\bullet}j'}$ for any $i' < i' + 1 < j'$, $M_{|c,d|} = M_{(a_i, a_{i,j,\ell})}$ for some $0 \leq \ell$ and $0 \leq j \leq 2^\ell$.

Now we finally suppose $M_{|c,d|} = M_{i{\bullet\!\!-\!\!\bullet}j}$ for some $i < i + 1 < j$. For contradiction suppose $i{\bullet\!\!-\!\!\bullet}j \notin T_{\mathbf{N}_\infty}$. Since $T_{\mathbf{N}_\infty}$ is an $\mathbf{N}_\infty$-cluster we know there exists $i'{\bullet\!\!-\!\!\bullet}j' \in T_{\mathbf{N}_\infty}$ such that $i < i' < j < j'$ or $i' < i < j' < j$. But then by Proposition 5.3.2 $\{M_{i{\bullet\!\!-\!\!\bullet}j}, M_{i'{\bullet\!\!-\!\!\bullet}j'}\}$ is not $\mathbf{E}$-compatible, a contradiction since $M_{i'{\bullet\!\!-\!\!\bullet}j'} \in T_{\mathbf{E}_\infty}$. Therefore, in all possibilities, $M_{|c,d|} \in T_{\mathbf{E}_\infty}$ already and so $T_{\mathbf{E}_\infty}$ is an $\mathbf{E}$-cluster. □

Lemma 5.3.9. *Consider an $\mathbf{N}_\infty$-cluster $T_\mathbf{N}$ and the induced $\mathbf{E}$-cluster $T_{\mathbf{E}_\infty}$. Suppose $T_{\mathbf{N}_\infty} \to (T_{\mathbf{N}_\infty} \setminus \{i{\bullet\!\!-\!\!\bullet}j\}) \cup \{i'{\bullet\!\!-\!\!\bullet}j'\}$ is an $\mathbf{N}_\infty$-mutation. Then $T_{\mathbf{E}_\infty} \to (T_{\mathbf{E}_\infty} \setminus \{M_{i{\bullet\!\!-\!\!\bullet}j}\}) \cup \{M_{i'{\bullet\!\!-\!\!\bullet}j'}\}$ is an $\mathbf{E}$-mutation.*

Proof. As with Lemma 5.2.6 it suffices to show $\{M_{i{\bullet\!\!-\!\!\bullet}j}, M_{i'{\bullet\!\!-\!\!\bullet}j'}\}$ is not $\mathbf{E}$-compatible but $(T_{\mathbf{E}_\infty} \setminus \{M_{i{\bullet\!\!-\!\!\bullet}j}\}) \cup \{M_{i'{\bullet\!\!-\!\!\bullet}j'}\}$ is $\mathbf{E}$-compatible. Since $\{i{\bullet\!\!-\!\!\bullet}j, i'{\bullet\!\!-\!\!\bullet}j'\}$ is not $\mathbf{N}_\infty$-compatible we know $\{M_{i{\bullet\!\!-\!\!\bullet}j}, M_{i'{\bullet\!\!-\!\!\bullet}j'}\}$

is not **E**-compatible by Proposition 5.3.2. Since $T'_{\mathbf{N}_\infty} = (T_{\mathbf{N}_\infty} \setminus \{i \bullet\!\!\!\longrightarrow\!\!\bullet j\}) \cup \{i' \bullet\!\!\!\longrightarrow\!\!\bullet j'\}$ is an $\mathbf{N}_\infty$-cluster and one mutation cannot introduce or remove a left- or right-fountain we see $T'_{\mathbf{E}_\infty} = (T_{\mathbf{E}_\infty} \setminus \{M_{i \bullet\!\!-\!\!\bullet j}\}) \cup \{M_{i' \bullet\!\!-\!\!\bullet j'}\}$ and $T'_{\mathbf{E}_\infty}$ is an **E**-cluster by Proposition 5.3.8. □

Theorem 5.3.10. *There exists an embedding of cluster theories* $(F,\eta) : \mathscr{T}_{\mathbf{N}_\infty}(\mathcal{C}(A_\infty)) \to \mathscr{T}_{\mathbf{E}}(\mathcal{C}(A_\mathbb{R}))$.

Proof. By Lemma 5.3.9 we see that defining $F(T_{\mathbf{N}_\infty}) := T_{\mathbf{E}_\infty}$ and sending $\mathbf{N}_\infty$-mutations to the corresponding **E**-mutations yields a functor $\mathscr{T}_{\mathbf{N}_\infty}(\mathcal{C}(A_\infty)) \to \mathscr{T}_{\mathbf{E}}(\mathcal{C}(A_\mathbb{R}))$. As with Theorem 5.2.7 F is an injection on clusters and mutations.

Let $\eta_{T_{\mathbf{N}_\infty}} : T_{\mathbf{N}_\infty} \to T_{\mathbf{E}_\infty}$ be defined by $\eta_{T_{\mathbf{N}_\infty}}(i \bullet\!\!\!\longrightarrow\!\!\bullet j) := M_{i \bullet\!\!-\!\!\bullet j}$. This is an injection by definition and by Lemma 5.3.9 the η's commute with mutation. Therefore (F,η) is an embedding of cluster theories. □

5.4. **Embedding $\mathscr{T}_{\mathbf{N}_\mathbb{R}}(\mathcal{C}_\pi) \to \mathscr{T}_{\mathbf{E}}(\mathcal{C}(A_\mathbb{R}))$.** In this section we demonstrate how to embed the previous continuous cluster theory into the new continuous cluster theory. Let $A_\mathbb{R}$ again have the straight orientation.

In [15] the continuous cluster category $\mathcal{C}_\pi$ is the orbit category of the doubling of $\mathcal{D}_\pi$ category (Definition 4.2.1) via almost-shift. Two indecomposables V and W in $\mathcal{C}_\pi$ are defined to be compatible if and only if

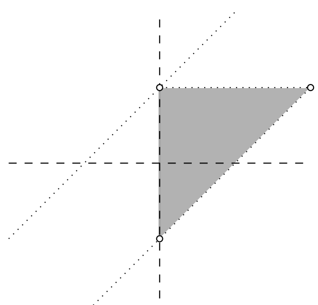
$$\dim(\mathrm{Ext}(V,W) \oplus \mathrm{Ext}(W,V)) \leq 1.$$

We will denote this compatibility condition by $\mathbf{N}_\mathbb{R}$.

Equivalently, V and W are not compatible in $\mathcal{C}_\pi$ if there exists $n \in \mathbb{Z}$ such that there is a rectangle contained in $\mathcal{D}_\pi$ with lower left corner and upper right corner equal to V and $W[n]$ and at most one of the other two corners on the boundary. I.e., there may be up to one point missing from the rectangle and it must be one of the corners not equal to V of W.

Let V and W be indecomposables in $\mathcal{C}_\pi$, which come from indecomposables in $\mathcal{D}_\pi$. Using the functor G (Section 4.2, Definition 4.2.3), we have a guide for where indecomposables from $\mathcal{C}_\pi$ should approximately be sent. Let P_a be a projective indecomposable from $\mathrm{rep}_k(A_\mathbb{R})$, where $a \neq +\infty$, as an indecomposable in degree 0 in $\mathcal{D}^b(A_\mathbb{R})$. Then $\mathbf{\Gamma}^b\, P_a = (\tan^{-1} a, \tan^{-1} a)$.

We will take our fundamental domain of $\mathcal{C}_\pi$ to be those indecomposables between the lines given by $M(0,y)$ and $M(x,\pi)$, including the $M(0,y)$ indecomposables and excluding the $M(x,\pi)$ indecomposables.



Will send send each of the indecomposables in this fundamental domain to an indecomposable in $\mathcal{C}(A_\mathbb{R})$. Recall that for an interval $|a,b|$ we denote by $M_{|a,b|}$ the indecomposable in $\mathrm{rep}_k(A_\mathbb{R})$ (and its image in $\mathcal{D}^b(A_\mathbb{R})$ and $\mathcal{C}(A_\mathbb{R})$) corresponding to the interval. To avoid confusion in notation, we let ${}_{(x,y)}M$ denote the indecompoable in $\mathcal{C}(A_\mathbb{R})$ that we obtain from $M(x,y)$ in $\mathcal{C}_\pi$. Each of our ${}_{(x,y)}M$ indecomposables will be representatives chosen from the 0th degree in $\mathcal{D}^b(A_\mathbb{R})$.

The line segment (without its endpoints) from $(-\frac{\pi}{2}, -\frac{\pi}{2})$ to $(\frac{\pi}{2}, \frac{\pi}{2})$ in $\mathbb{R} \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ will be the image of the indecomposables in $\mathcal{D}_\pi$ of the form $M(0,y)$. It is also the image of the projective indecomposables from $\mathrm{rep}_k(A_\mathbb{R})$ in the 0th degree, with the exception of $P_{+\infty}$, in $\mathcal{D}^b(A_\mathbb{R})$. The dotted line bordering the fundamental domain in the picture are the indecomposables in $\mathcal{D}_\pi$ of the form $M(x,\pi)$. These *would* be sent to the line segment from $(\frac{\pi}{2},\frac{\pi}{2})$ to $(\pi,-\frac{\pi}{2})$. This is precisely the image of the injectives from $\text{rep}_k(A_\mathbb{R})$ in the 0th degree in $\mathcal{D}^b(A_\mathbb{R})$ under $\mathbf{\Gamma}^b$. As we've shown the rest of the the shaded triangle will then correspond to indecomposables in degree 0 in $\mathcal{D}^b(A_\mathbb{R})$ that (i) are not degenerate and (ii) are from neither projectives nor injectives in $\text{rep}_k(A_\mathbb{R})$.

Definition 5.4.1. For each $M(x,y)$ in the fundamental domain, set

$$\alpha_{x,y} = \frac{y+x}{2} \qquad\qquad \beta_{x,y} = \frac{y-x}{2}.$$

It is straightforward to see that $-\frac{\pi}{2} < \beta_{x,y} < \frac{\pi}{2}$ and $\beta_{x,y} \le \alpha_{x,y} < \pi - \beta_{x,y}$.

We define ${}_{(x,y)}M$ to be the indecomposable $M_{(a,b)}$ in $\mathcal{C}(A_\mathbb{R})$ whose image under $\mathbf{\Gamma}^b$ is $(\alpha_{x,y},\beta_{x,y})$. We set

$$a_{x,y} = \tan\left(\frac{\alpha_{x,y}-\beta_{x,y}-\pi}{2}\right) \qquad\qquad b_{x,y} = \tan\left(\frac{\alpha_{x,y}+\beta_{x,y}}{2}\right).$$

Note it is possible that $a_{x,y} = -\infty$, in which case we have $\mathbf{\Gamma}^b\, P_{b_{x,y}}$. Then we set

$${}_{(x,y)}M = M_{(a_{x,y},b_{x,y})}.$$

We will use the following definition a few times.

Definition 5.4.2. Let $\mathcal{C}A$, $\mathcal{C}B$, and $\mathcal{C}C$ be the sets below:

$$\begin{aligned}
\mathcal{C}A &= \{(x,y)\in\mathbb{R}^2 : |x-y|<\pi, x\ge 0, y<\pi\} \\
\mathcal{C}B &= \left\{(\alpha,\beta)\in\mathbb{R}^2 : -\frac{\pi}{2}<\beta<\frac{\pi}{2} \text{ and } \beta\le\alpha<\pi-\beta\right\} \\
\mathcal{C}C &= \{(a,b)\in(\mathbb{R}\cup\{-\infty\})\times\mathbb{R} : -\infty\le a<b<+\infty\}.
\end{aligned}$$

Using Definition 5.4.1, let

$$\begin{aligned}
\mathfrak{g} : \mathcal{C}A &\to \mathcal{C}B \\
(x,y) &\mapsto (\alpha_{x,y},\beta_{x,y}) \\
\mathfrak{h} : \mathcal{C}B &\to \mathcal{C}C \\
(\alpha_{x,y},\beta_{x,y}) &\mapsto (a_{x,y},b_{x,y}).
\end{aligned}$$

Let $\mathfrak{f} : \mathcal{C}A \to \mathcal{C}C$ be the composite $\mathfrak{h}\circ\mathfrak{g}$. For $i\in\{1,2\}$ define $\mathfrak{g}_i$, $\mathfrak{h}_i$, and $\mathfrak{f}_i$ to be the projection onto the ith coordinate.

Proposition 5.4.3. *The function $\mathfrak{f}$ in Definition 5.4.2 is a bijection.*

Proof. We first show $\mathfrak{f}$ is well-defined. The set $\mathcal{C}A$ is precisely the set corresponding to the fundamental domain of $\mathcal{C}_\pi$ we have chosen. We know $\alpha_{x,y}$ and $\beta_{x,y}$ are defined in terms of x and y. Since $-\frac{\pi}{2} < \beta_{x,y} < \frac{\pi}{2}$ and $\beta_{x,y} \le \alpha_{x,y} < \pi - \beta_{x,y}$ we see that

$$-\pi \le \alpha_{x,y} - \beta_{x,y} - \pi < \pi.$$

so $-\infty \le a < +\infty$. We also see that

$$\alpha_{x,y} - \beta_{x,y} - \pi < \alpha_{x,y} + \beta_{x,y} < \pi.$$

Thus, $-\infty \le a_{x,y} < b_{x,y} < +\infty$ and so $f(x,y)\in B$.

Now suppose $(x,y)\ne(x',y')$. Then $(\alpha_{x,y},\beta_{x,y}) \ne (\alpha'_{x,y},\beta'_{x,y})$ and so $(a_{x,y},b_{x,y}) \ne (a'_{x,y},b'_{x,y})$. Thus, $\mathfrak{f}$ is injective.

Let $(a,b)\in\mathcal{C}C$. Set

$$\begin{aligned}
\alpha &:= \tan^{-1} b + \tan^{-1} a + \frac{\pi}{2} \\
\beta &:= \tan^{-1} b - \tan^{-1} a - \frac{\pi}{2}.
\end{aligned}$$

We immediately see that $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$. It is straightforward to see that $\beta \leq \alpha < \pi - \beta$. Then we define

$$x := \alpha - \beta$$
$$y := \alpha + \beta.$$

We see $x \geq 0$, $y < \pi$, and $|x - y| < \pi$. Thus $\mathfrak{f}(x, y) = (a, b)$ and so $\mathfrak{f}$ is surjective. □

Lemma 5.4.4. *Let $M(x, y)$ and $M(x', y')$ be indecomposables in the fundamental domain of $\mathcal{C}_\pi$. The set $\{M(x, y), M(x', y')\}$ is $\mathbf{N}_\mathbb{R}$-compatible if and only if $\{_{(x,y)}M, _{(x',y')}M\}$ is $\mathbf{E}$-compatible.*

Proof. First suppose $\{M(x, y), M(x', y')\}$ is not $\mathbf{N}_\mathbb{R}$-compatible. Then $M(x', y)$ and $M(x, y')$ are indecomposables in the fundamental domain of $\mathcal{C}_\pi$. This means $_{(x',y)}M$ and $_{(x,y')}M$ are well-defined indecomposables in $\mathcal{C}(A_\mathbb{R})$ that, with $_{(x,y)}M$ and $_{(x',y')}M$, form a rectangle in the AR-space of $\mathcal{D}^b(A_\mathbb{R})$ which is entirely contained in the AR-space of $\text{rep}_k(A_\mathbb{R})$. Thus, $\{_{(x,y)}M, _{(x',y')}M\}$ is not $\mathbf{E}$-compatible.

Now suppose $\{_{(x,y)}M, _{(x',y')}M\}$ is not $\mathbf{E}$-compatible. We reverse the argument and see that $M(x', y)$ and $M(x, y')$ are indecomposables in the fundamental domain of $\mathcal{C}_\pi$. Therefore, $\{M(x, y), M(x', y')\}$ is not $\mathbf{N}_\mathbb{R}$-compatible. □

Definition 5.4.5. Let $T_{\mathbf{N}_\mathbb{R}}$ be an $\mathbf{N}_\mathbb{R}$-cluster. We define

$$\begin{aligned} T^\circ_{\mathbf{E}_\mathbb{R}} =& \{P_{+\infty}\} \cup \left\{_{(x,y)}M = M_{\mathfrak{f}(x,y)} : M(x, y) \in T_{\mathbf{N}_\mathbb{R}}\right\} \\ & \cup \{M_{\{z\}} : z \in \mathbb{R},\ \nexists M(x, y) \in T_{\mathbf{N}_\mathbb{R}} \text{ such that } (\mathfrak{f}_1(x, y) = z \text{ or } \mathfrak{f}_2(x, y) = z)\} \end{aligned}$$

For each $_{(x,y)}M = M_{(a,b)} \in T^\circ_{\mathbf{E}_\mathbb{R}}$ we will define the set $\tau(a, b)$ that we will use to construct $T_{\mathbf{E}_\mathbb{R}}$.

Definition 5.4.6. At a we can check the following conditions.

(1) There exist $M_{(c,a)} \in T^\circ_{\mathbf{E}_\mathbb{R}}$.
(2) There exist $M_{(a,b')} \in T^\circ_{\mathbf{E}_\mathbb{R}}$ where $b' > b$.
(3) $a = -\infty$.

If a satisfies any of (1), (2), or (3) we say it is satisfactory. We check similar conditions to (1) and (2) for b and use the same definition of satisfactory. Now we define $\tau(a, b)$.

- If both a and b are satisfactory let $\tau(a, b) = \emptyset$.
- If a is satisfactory but b is not let $\tau(a, b) = \{M_{(a,b]}\}$
- If b is satisfactory but a is not let $\tau(a, b) = \{M_{[a,b)}\}$.
- If neither a nor b are satisfactory let $\tau(a, b) = \{M_{[a,b]}, M_{[a,b)}\}$.

We now define $T_{\mathbf{E}_\mathbb{R}}$ in one of two ways. Let $\mathcal{P} = \{P_{b)} : P_{b)} \in T^\circ_{\mathbf{E}_\mathbb{R}}, b < +\infty\}$ with total order given by $P_{b)} \leq P_{b')}$ if and only if $b \leq b'$.

- If $\mathcal{P}$ is empty or has no maximal element then define

$$T_{\mathbf{E}_\mathbb{R}} := \left(\bigcup_{M_{(a,b)} \in T^\circ_{\mathbf{E}_\mathbb{R}}} \tau(a, b) \right) \cup T^\circ_{\mathbf{E}_\mathbb{R}}.$$

- If $\mathcal{P}$ is nonempty with a maximal element $P_{b)}$ then define

$$T_{\mathbf{E}_\mathbb{R}} := \left(\bigcup_{P_{b)} \neq M_{(a,b)} \in T^\circ_{\mathbf{E}_\mathbb{R}}} \tau(a, b) \right) \cup \{I_{(b}\} \cup T^\circ_{\mathbf{E}_\mathbb{R}}.$$

Proposition 5.4.7. *Let $T_{\mathbf{N}_\mathbb{R}}$ be an $\mathbf{N}_\mathbb{R}$-cluster. Then $T_{\mathbf{E}_\mathbb{R}}$ is an $\mathbf{E}$-cluster*

Proof. Using Lemma 5.4.4 it is straightforward to check that $T_{\mathbf{E}_\mathbb{R}}$ is $\mathbf{E}$-compatible. Let $M_{|c,d|}$ be an indecomposable in $\mathcal{C}(A_\mathbb{R})$ such that $T_{\mathbf{E}_\mathbb{R}} \cup \{M_{|c,d|}\}$ is $\mathbf{E}$-compatible. We will show $M_{|c,d|} \in T_{\mathbf{E}_\mathbb{R}}$.

As before we can check various values for c but in a different fashion than before. First suppose there exists $M(x,y) \in T_{\mathbf{N}_\mathbb{R}}$ such that $\mathfrak{f}_1(x,y) = c$. Then there is no $M(x',y') \in T_{\mathbf{N}_\mathbb{R}}$ such that $\mathfrak{f}(x,y) = (a',b')$ and $a' < d < b'$. Thus, there exist $M(x',y') \in T_{\mathbf{N}_\mathbb{R}}$ such that $d = \mathfrak{f}_1(x',y')$ or $d = \mathfrak{f}_2(x',y')$. Then $M_{\{c\}}$ and $M_{\{d\}}$ are not in $T_{\mathbf{E}_\mathbb{R}}$.

Now we have $\{M_{(c,d)}\} \cup T_{\mathbf{E}_\mathbb{R}}$ is $\mathbf{E}$-compatible. By Proposition 5.4.3 there is a $M(x'',y'')$ that is compatible with $T_{\mathbf{N}_\mathbb{R}}$ such that $\mathfrak{f}(x'',y'') = (c,d)$. However, since $T_{\mathbf{N}_\mathbb{R}}$ is an $\mathbf{N}_\mathbb{R}$-cluster $M(x'',y'') \in T_{\mathbf{N}_\mathbb{R}}$ and so $M_{(c,d)} \in T_{\mathbf{E}_\mathbb{R}}$. If $c \in |c,d|$ or $c = -\infty$ then there is no $M_{(a,c)} \in T_{\mathbf{E}_\mathbb{R}}$ and if $d \in |c,d|$ there is no $M_{(d,b)} \in T_{\mathbf{E}_\mathbb{R}}$. Thus, either $|c,d| = (c,d)$ or $M_{|c,d|} \in \tau(c,d)$. In either case $M_{|c,d|} \in T_{\mathbf{E}_\mathbb{R}}$.

Now suppose there is no $M(x,y) \in T_{\mathbf{N}_\mathbb{R}}$ such that $\mathfrak{f}_1(x,y) = c$. For contradiction suppose $d > c$. Then $\{M(\mathfrak{f}^{-1}(c,d))\} \cup T_{\mathbf{N}_\mathbb{R}}$ is not $\mathbf{N}_\mathbb{R}$-compatible and so by Lemma 5.4.4 there is $M(x',y') \in T_{\mathbf{N}_\mathbb{R}}$ such tha (i) $\mathfrak{f}(x',y') = (a,b)$ and (ii) $a < c < b < d$ or $c < a < d < b$. Thus, $\{M_{|c,d|}\} \cup T_{\mathbf{E}_\mathbb{R}}$ is not $\mathbf{E}$-compatible, a contradiction. Thus $d = c$ and $M_{|c,d|} = M_{\{c\}}$. Then we know $d = c$ cannot be $\mathfrak{f}_2(x',y')$ for some $M(x',y') \in T_{\mathbf{N}_\mathbb{R}}$ or else $\{M_{\{c\}}\} \cup T_{\mathbf{E}_\mathbb{R}}$ would not be $\mathbf{E}$-compatible. Therefore $M_{\{c\}}$ is already in $T^\circ_{\mathbf{E}_\mathbb{R}} \subset T_{\mathbf{E}_\mathbb{R}}$. □

Lemma 5.4.8. *Let $T_{\mathbf{N}_\mathbb{R}}$ be an $\mathbf{N}_\mathbb{R}$-cluster and $T_{\mathbf{E}_\mathbb{R}}$ the induced $\mathbf{E}$-cluster. If $T_{\mathbf{N}_\mathbb{R}} \to (T_{\mathbf{N}_\mathbb{R}} \setminus \{M(x,y)\}) \cup \{M(x',y')\}$ is an $\mathbf{N}_\mathbb{R}$-mutation then $T_{\mathbf{E}_\mathbb{R}} \to (T_{\mathbf{E}_\mathbb{R}} \setminus \{{}_{(x,y)}M\}) \cup \{{}_{(x',y')}M\}$ is an $\mathbf{E}$-mutation.*

Proof. By Lemma 5.4.4 we know $\{{}_{(x,y)}M, {}_{(x',y')}M\}$ is not $\mathbf{E}$-compatible. Thus it remains to show $(T_{\mathbf{E}_\mathbb{R}} \setminus \{{}_{(x,y)}M\}) \cup \{{}_{(x',y')}M\}$ is an $\mathbf{E}$-cluster. Without loss of generality we will assume $y' > y$ which implies $x' > x$, since $\{M(x,y), M(x',y')\}$ is not $\mathbf{N}_\mathbb{R}$-compatible. This also means $x' - \pi \leq y$, $x - \pi < x' - \pi$, $x' \leq \pi + y$, and $\pi + y < \pi + y'$.

Let $M(w,z) \in T_{\mathbf{N}_\mathbb{R}}$ such that $(w,z) \neq (x,y)$. Then $\{M(w,z), M(x,y)\}$ and $\{M(w,z), M(x',y')\}$ are both $\mathbf{N}_\mathbb{R}$-compatible. We will check various possibilities for z starting with the highest possible values and and working down.

- If $y' < z < \pi$ then $w \leq x$ or $\pi + y' < w$.
- If $z = y'$ then $w \leq x$ or $\pi + y \leq w < \pi + y'$.
- If $y < z < y'$ then $\pi + y \leq w$.
- If $z = y$ then $x' \leq w < \pi + y$.
- If $x' - \pi < z < y$ then $x' \leq w$.
- If $z = x' - \pi$ then $x \leq w < x'$.
- If $x - \pi < z < x' - \pi$ then $x \leq w$.
- If $z = x - \pi$ then $w < x$.
- If $z < x - \pi$ then $w \leq x$.

In each of these cases $\{M(w,z), M(x',y), M(x,y')\}$ is $\mathbf{N}_\mathbb{R}$-compatible. Thus, $M(x',y), M(x,y') \in T_{\mathbf{N}_\mathbb{R}}$. We can further work with $M(w,z)$.

- We have $x \leq w \leq x'$ if and only if $x - \pi < z \leq x' - \pi$.
- We have $\pi + y \leq w \leq \pi + y'$ if and only if $y < z \leq y'$.

Thus $\{M(w,z), M(x,x'), M(y,y')\}$ is also $\mathbf{N}_\mathbb{R}$-compatible and so $M(x,x'), M(y,y') \in T_{\mathbf{N}_\mathbb{R}}$.

Let $(a,b) = \mathfrak{f}(x,y)$ and $(a',b') = \mathfrak{f}(x',y')$. Then

$$(a',b) = \mathfrak{f}(x',y) \qquad (a,a') = \mathfrak{f}(x,x')$$
$$(a,b') = \mathfrak{f}(x,y') \qquad (b,b') = \mathfrak{f}(y,y').$$

We see that for $\tau(a,b)$ both a and b are satisfactory. Furthermore, for $\tau(a,a')$ and $\tau(b,b')$ all of a, a', b, and b' are satisfactory. Thus $(T_{\mathbf{E}_\mathbb{R}} \setminus \{{}_{(x,y)}M\}) \cup \{{}_{(x',y')}M\}$ is $\mathbf{E}$-compatible. Furthermore, let

$T'_{\mathbf{N}_\mathbb{R}} = (T_{\mathbf{N}_\mathbb{R}} \setminus \{M(x,y)\}) \cup \{M(x',y')\}$. Then

$$T'_{\mathbf{E}_\mathbb{R}} = (T_{\mathbf{E}_\mathbb{R}} \setminus \{{}_{(x,y)}M\}) \cup \{{}_{(x',y')}M\}.$$

Therefore $T'_{\mathbf{E}_\mathbb{R}}$ is an $\mathbf{E}$-cluster by Proposition 5.4.7 and the lemma holds. □

Theorem 5.4.9. *There exists an embedding of cluster theories* $(F,\eta) : \mathscr{T}_{\mathbf{N}_\mathbb{R}}(\mathcal{C}_\pi) \to \mathscr{T}_{\mathbf{E}}(\mathcal{C}(A_\mathbb{R}))$.

Proof. By Proposition 5.4.7 and Lemma 5.4.8 we see that defining $F(T_{\mathbf{N}_\mathbb{R}}) := T_{\mathbf{E}_\mathbb{R}}$ and sending $\mathbf{N}_\mathbb{R}$-mutations to the corresponding $\mathbf{E}$-mutations yields a functor $\mathscr{T}_{\mathbf{N}_\mathbb{R}}(\mathcal{C}_\pi) \to \mathscr{T}_{\mathbf{E}}(\mathcal{C}(A_\mathbb{R}))$. It is straightforward to check F is an injection on clusters and mutations.

Let $\eta_{T_{\mathbf{N}_\mathbb{R}}} : T_{\mathbf{N}_\mathbb{R}} \to T_{\mathbf{E}_\mathbb{R}}$ be defined by $\eta_{T_{\mathbf{N}_\mathbb{R}}}(M(x,y)) := {}_{(x,y)}M$. This is an injection by definition and by Lemma 5.4.8 the η's commute with mutation. Therefore (F,η) is an embedding of cluster theories. □

Remark 5.4.10. We remark here that all clusters in [15] are $\mathbf{N}_\mathbb{R}$-clusters. However, there are $\mathbf{N}_\mathbb{R}$-clusters that are not part of the cluster structure in [15]. For example, the vertical line $\{M(0,y) : -\pi < y < \pi\}$ is an $\mathbf{N}_\mathbb{R}$-cluster but not part of the cluster structure in $\mathcal{C}_\pi$.

However, as with A_∞, this is not truly an issue. We have an injection on objects and so taking a subgroupoid of $\mathscr{T}_{\mathbf{N}_\mathbb{R}}(\mathcal{C}_\pi)$ that only contains the clusters in the cluster structure in $\mathcal{C}_\pi$ still embeds into $\mathscr{T}_{\mathbf{E}}(\mathcal{C}_\pi)$ while preserving mutation. This is done in the next paper in this series.

5.5. **Issues with Functors Between Cluster Categories.** In this short subsection we describe issues to any simple or straightforward embedding $\mathcal{C}_\pi \to \mathcal{C}(A_\mathbb{R})$ that is somehow compatible with mutation. The first issue is the following: any embedding of $\mathcal{C}_\pi \to \mathcal{C}(A_\mathbb{R})$ cannot originate as a functor $\mathcal{D}_\pi \to \mathcal{D}^b(A_\mathbb{R})$.

Since they are triangulated equivalent, we will consider $\mathcal{D}^b(A_\mathbb{R})[\mathcal{N}Q^{-1}]$ instead of $\mathcal{D}_\pi$. As an example, consider an embedding that sends all the indcemposables in $\mathcal{D}_\pi$ to the indecomposable in position 1 in the inverse image of the localization $\mathcal{D}^b(A_\mathbb{R}) \to \mathcal{D}^b(A_\mathbb{R})[\mathcal{N}Q^{-1}]$. This type of embedding will preserve the triangulated structure but cause problems with compatibility after taking the orbit category. In $\mathcal{C}_\pi$, $M(x,y)$ and $M(\pi+y,y')$ are compatible if $y<y'$. However, using an embedding $\mathcal{D}^b(A_\mathbb{R})[\mathcal{N}Q^{-1}] \to \mathcal{D}^b(A_\mathbb{R})$ and then taking the orbit sends $M(x,y)$ and $M(\pi+y,y')$ to $M_{[a,b)}$ and $M_{[b,b')}$. These are not $\mathbf{E}$-compatible. Thus any such embedding would have to embed to positions 2 and 3, using position 3 in even degrees and position 2 in odd degrees to preserve the triangulated structure.

This creates a new problem. Consider $M(x,y)$ and $M(x,y')$ where $y<y'$. Suppose $M(x,y)$ is sent to degree 0 and $M(x,y')$ to degree 1 in $\mathcal{D}^b(A_\mathbb{R})$. But now the slope from $M_{(a,b)}$ to $M_{(c,a)}[1]$ is greater than $(1,1)$ in the AR-space of $\mathcal{D}^b(A_\mathbb{R})$ and so $\mathrm{Hom}(M_{(a,b)}, M_{(c,a)}[1]) = 0$. While this doesn't necessarily prevent such a functor from being triangulated, or even preserving mutation in some way after taking orbits, this is no longer an embedding.

So we are left with a functor $\mathcal{C}_\pi \to \mathcal{C}(A_\mathbb{R})$. As we've seen, we need to send the fundamental domain to position 3 as we did in Section 5.4. But then we have the same problem as we've just described with Hom support, and so we don't have an embedding. Since $\mathrm{Ext}(V,W) = \mathrm{Hom}(W,V)$ in both $\mathcal{C}_\pi$ and $\mathcal{C}(A_\mathbb{R})$ and compatibility in $\mathcal{C}_\pi$ is determined by Ext this is fundamentally a problem. Therefore, the authors resorted to cluster theories in order to create an embedding that makes sense.

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