

THE DUSKIN NERVE OF 2-CATEGORIES IN JOYAL'S DISK CATEGORY Θ_2

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ABSTRACT. We give an explicit and purely combinatorial description of the Duskin nerve of any $(r + 1)$ -point suspension 2-category, and in particular of any 2-category belonging to Joyal's disk category Θ_2 .

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OVERVIEW OF RESULTS

The 2-categorical analog of the nerve of ordinary categories goes by the name of *Duskin nerve*. It was implicitly defined by Street [Str87] and further studied by Duskin [Dus02]. Roughly speaking, the objects, 1-morphisms and 2-morphisms of the given 2-category are incorporated suitably in the 0-, 1- and 2-simplices of the Duskin nerve, which is 3-coskeletal.

The Duskin nerve is a classical construction, and many of its homotopical properties have been established. For instance, Duskin [Dus02] showed that the Duskin nerve of a $(2, 0)$ -category is always a Kan complex and that the Duskin nerve of a $(2, 1)$ -category is always a quasi-category. Bullejos, Carasco, Cegarra, and Garzón showed in different combinations that analogs of Quillen's Theorems A and B hold for the Duskin nerve of 2-categories [BC03, Ceg11], and that the Duskin nerve is homotopically equivalent to other nerve constructions for 2-categories [CCG10]. To mention one application, Nanda [Nan19] then built on their work showing that the Duskin nerve of the discrete flow 2-category associated to a simplicial complex (with extra structure) has the same homotopy type as that of the simplicial complex.

The machinery developed by Steiner [Ste07] implicitly provides methods to study the Duskin nerve of 2-categories. However, we are not aware of many explicit computations and descriptions of the Duskin nerve, even for small and rather simple 2-categories.

As a first analysis in this direction, one could observe that in the nerve of finite 1-categories there are only finitely many non-degenerate simplices, and imagine that the Duskin nerve of finite 2-categories would enjoy the same property. Somewhat surprisingly, we discovered that the Duskin nerve of 2-categories is much more complex than expected. For instance, we show in Section 3 that the Duskin nerve of the free 2-cell

$$\begin{array}{ccc} & f_0 & \\ x & \begin{array}{c} \curvearrowright \\ \Downarrow \alpha \\ \curvearrowleft \end{array} & y, \\ & f_1 & \end{array}$$

which is a very simple 2-category that does not contain any non-trivial composition, has non-degenerate simplices in each dimension.

Proposition. *The Duskin nerve of the free 2-cell has precisely two non-degenerate simplices in each positive dimension.*

This result was unexpected to us, and we were able to conjecture it in the first place only after having a computer produce all n -simplices of the Duskin nerve of the free 2-cell for $n \leq 6$. In order to prove the proposition, we developed a more general study of the Duskin nerve of 2-categories of the form $\Sigma\mathcal{D}$, sometimes referred to as a *suspension 2-categories*, of which the free 2-cell is an example for $\mathcal{D} = [1]$.

The suspension 2-category $\Sigma\mathcal{D}$ of a category \mathcal{D} , which can be pictured as

$$\Sigma\mathcal{D} := \quad [0] \curvearrowright x \begin{array}{c} \xleftarrow{\quad \emptyset \quad} \\ \xrightarrow{\quad \mathcal{D} \quad} \end{array} y \curvearrowleft [0],$$

appears often in the literature as a special case of a simplicial suspension. For instance, the homwise nerve $N_*(\Sigma\mathcal{D})$ of the suspension $\Sigma\mathcal{D}$ is a simplicial category that agrees with what would be denoted as $U(N\mathcal{D})$ in [Ber07], as $S(N\mathcal{D})$ in [Joy07], as $[1]_{N\mathcal{D}}$ in [Lur09], and as $2[N\mathcal{D}]$ in [RV18].

In Section 1 we prove the following description for the Duskin nerve of suspension 2-categories. Our methods were inspired by those used in [BGLS15], where Buckley, Garner, Lack and Street face a similar situation, showing that the monoidal nerve of a rather simple monoidal category is the highly non-trivial ‘‘Catalan simplicial set’’.

Theorem A. *An n -simplex of the Duskin nerve of the suspension $\Sigma\mathcal{D}$ can be uniquely described as a grid valued in \mathcal{D} of the form*

$$\begin{array}{ccccccc}
d_{0l} & \longrightarrow & d_{0(l-1)} & \longrightarrow & \cdots & \longrightarrow & d_{00} \\
\downarrow & & \downarrow & & & & \downarrow \\
d_{1l} & \longrightarrow & d_{1(l-1)} & \longrightarrow & \cdots & \longrightarrow & d_{10} \\
\downarrow & & \downarrow & & & & \downarrow \\
\vdots & & \vdots & & \ddots & & \vdots \\
\downarrow & & \downarrow & & & & \downarrow \\
d_{kl} & \longrightarrow & d_{k(l-1)} & \longrightarrow & \cdots & \longrightarrow & d_{k0},
\end{array}$$

for $k, l \geq -1$ and $k + l = n - 1$, and the simplicial structure is understood as suitably removing or doubling rows or columns.

The proof of the theorem relies on a coskeletality argument, which hides the meaning of this correspondence for simplices in dimension higher than 4. To address this, we devote Section 3 to explaining how to convert a simplex of $N(\Sigma\mathcal{D})$ to a \mathcal{D} -valued grid and vice versa.

After having understood the Duskin nerve of suspension 2-categories, we then study the Duskin nerve of $(r + 1)$ -point suspension 2-categories $\Sigma[\mathcal{D}_1, \dots, \mathcal{D}_r]$, which are 2-categories obtained by pasting together suspension 2-categories of categories $\mathcal{D}_1, \dots, \mathcal{D}_r$ along objects as in the following picture:

$$\Sigma[\mathcal{D}_1, \dots, \mathcal{D}_r] := x_0 \begin{array}{c} \xleftarrow{\emptyset} \\ \xrightarrow{\mathcal{D}_1} \end{array} x_1 \begin{array}{c} \xleftarrow{\emptyset} \\ \xrightarrow{\mathcal{D}_2} \end{array} x_2 \begin{array}{c} \xleftarrow{\emptyset} \\ \xrightarrow{\mathcal{D}_3} \end{array} \cdots \begin{array}{c} \xleftarrow{\emptyset} \\ \xrightarrow{\mathcal{D}_r} \end{array} x_r.$$

This type of horizontal gluing of suspension 2-categories appears e.g. in [Ver08a, §11.5]. Motivating examples of $(r + 1)$ -point suspension 2-categories are the 2-categories that belong to Joyal's disk category Θ_2 from [Joy97], which are all $(r + 1)$ -point suspension 2-categories of the form $\Sigma[[n_1], \dots, [n_r]]$ for $n_1, \dots, n_r \geq 0$. An example would be the 2-category $\Sigma[[2], [0], [1]]$, which is generated by the following data:

$$\begin{array}{ccccc}
& & f & & m \\
& \searrow & \Downarrow \alpha & \searrow & \searrow \\
x & \xrightarrow{\quad g \quad} & y & \xrightarrow{\quad l \quad} & z & \xrightarrow{\quad k \quad} & w \\
& \swarrow & \Downarrow \beta & \swarrow & \swarrow & \swarrow & \\
& & h & & & &
\end{array}$$

We are able to describe the Duskin nerve of $(r + 1)$ -point suspension 2-categories in Section 4.

Theorem B. *Let $\mathcal{D}_1, \dots, \mathcal{D}_r$ be given 1-categories. An n -simplex of the Duskin nerve of the $(r+1)$ -point suspension $\Sigma[\mathcal{D}_1, \dots, \mathcal{D}_r]$ can be uniquely described as a list of grids valued in \mathcal{D}_i whose numbers of rows are suitably increasing.*

The explicit description of the Duskin nerve of 2-categories from this paper can then also be used to prove finer homotopical properties. For instance, in ongoing work, we use these results to show that the canonical inclusion

$$N(\Sigma[1]) \coprod_{N(\Sigma[0])} N(\Sigma[1]) \hookrightarrow N(\Sigma[2])$$

is a categorical equivalence.

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1. THE DUSKIN NERVE OF SUSPENSION 2-CATEGORIES - THE THEORY

We start by recalling the definition of the Duskin nerve of 2-categories¹.

Definition 1.1. The *Duskin nerve* $N(\mathcal{C})$ of a 2-category \mathcal{C} is a 3-coskeletal simplicial set in which

(0) a 0-simplex consists of an object of \mathcal{C} :

$$x;$$

(1) a 1-simplex consists of a 1-morphism of \mathcal{C} :

$$x \xrightarrow{a} y;$$

(2) a 2-simplex consists of a 2-cell of \mathcal{C} of the form $c \Rightarrow b \circ a$:

$$\begin{array}{ccc} & y & \\ a \nearrow & \Uparrow & \searrow b \\ x & \xrightarrow{c} & z \end{array};$$

(3) a 3-simplex consists of four 2-cells of \mathcal{C} that satisfy the following relation.

$$\begin{array}{ccc} \begin{array}{ccccc} w & \xleftarrow{e} & z & & \\ \uparrow a & \nearrow & \uparrow c & & \\ & b & & & \\ x & \xrightarrow{d} & y & & \end{array} & = & \begin{array}{ccccc} w & \xleftarrow{e} & z & & \\ \uparrow a & \nwarrow & \uparrow c & & \\ & f & & & \\ x & \xrightarrow{d} & y & & \end{array} \end{array}$$

The following type of 2-category is of interest in this paper.

Definition 1.2. The *suspension* of a 1-category \mathcal{D} is the 2-category $\Sigma\mathcal{D}$ with two objects x, y and hom categories given by

$$\underline{\text{Map}}_{\Sigma\mathcal{D}}(x, y) = \mathcal{D}, \quad \text{Map}_{\Sigma\mathcal{D}}(y, x) = [-1], \quad \text{Map}_{\Sigma\mathcal{D}}(x, x) = \text{Map}_{\Sigma\mathcal{D}}(y, y) = [0].$$

¹In this paper we are only concerned with *strict* 2-categories.

We want to identify each n -simplex of the Duskin nerve of the suspension 2-category $\Sigma\mathcal{D}$ with a functor $\sigma: [k] \times [l]^{\text{op}} \rightarrow \mathcal{D}$, which is in turn completely described by $(k+1) \times (l+1)$ objects of \mathcal{D} connected horizontally and vertically by morphisms of \mathcal{D}

$$\begin{array}{ccccccc}
d_{0l} & \longrightarrow & d_{0(l-1)} & \longrightarrow & \cdots & \longrightarrow & d_{00} \\
\downarrow & & \downarrow & & & & \downarrow \\
d_{1l} & \longrightarrow & d_{1(l-1)} & \longrightarrow & \cdots & \longrightarrow & d_{10} \\
\downarrow & & \downarrow & & & & \downarrow \\
\vdots & & \vdots & & \ddots & & \vdots \\
\downarrow & & \downarrow & & & & \downarrow \\
d_{kl} & \longrightarrow & d_{k(l-1)} & \longrightarrow & \cdots & \longrightarrow & d_{k0},
\end{array}$$

such that all the resulting squares are all commutative. We call such a diagram a *matrix*² valued in \mathcal{D} . We follow the convention that $[-1] = \emptyset$ is the empty category.

To this end, we first discuss how the collections of such morphisms assemble into a simplicial set.

Lemma 1.3. *Let \mathcal{D} be a category. The collection of \mathcal{D} -matrices*

$$\text{Mat}_n(\mathcal{D}) := \{\sigma: [k] \times [l]^{\text{op}} \rightarrow \mathcal{D} \mid k, l \geq -1, k + l = n - 1\}$$

for $n \geq 0$ defines a 3-coskeletal simplicial set $\text{Mat}(\mathcal{D})$ with respect to the following simplicial structure. The faces and degeneracies of a \mathcal{D} -valued matrix $\sigma: [k] \times [l]^{\text{op}} \rightarrow \mathcal{D}$ are given by

$$\begin{aligned}
d_i \sigma &= \begin{cases} \sigma(d^i \times \text{id}_{[l]^{\text{op}}}) : [k-1] \times [l]^{\text{op}} \rightarrow \mathcal{D} & \text{for } 0 \leq i \leq k, \\ \sigma(\text{id}_{[k]} \times d^{i-(k+1)}) : [k] \times [l-1]^{\text{op}} \rightarrow \mathcal{D} & \text{for } k+1 \leq i \leq n; \end{cases} \\
s_i \sigma &= \begin{cases} \sigma(s^i \times \text{id}_{[l]^{\text{op}}}) : [k+1] \times [l]^{\text{op}} \rightarrow \mathcal{D} & \text{for } 0 \leq i \leq k, \\ \sigma(\text{id}_{[k]} \times s^{i-(k+1)}) : [k] \times [l+1]^{\text{op}} \rightarrow \mathcal{D} & \text{for } k+1 \leq i \leq n. \end{cases}
\end{aligned}$$

Roughly speaking, in the simplicial set $\text{Mat}(\mathcal{D})$:

- (1) faces are given by removing precisely one row or one column;
- (2) degeneracies are given by doubling precisely one row or one column;
- (3) the non-degenerate simplices are the ones where no two rows and no two columns coincide.

Proof. The fact that $\text{Mat}(\mathcal{D})$ is indeed a simplicial set can be verified by means of a straightforward computation. For simplicity of exposition, we show 3-coskeletality of $\text{Mat}(\mathcal{D})$ for the case of \mathcal{D} being a poset; the general case only requires the treatment of a slightly larger distinction of cases.

Suppose we are given a collection of \mathcal{D} -valued matrices $\tau_i: [k-1] \times [n-1-k]^{\text{op}} \rightarrow \mathcal{D}$ for $0 \leq i \leq k$ and $\tau_i: [k] \times [(n-1-k)-1]^{\text{op}} \rightarrow \mathcal{D}$ for

²We warn the reader that the use of matrices from this paper is not directly related with the matrices used by Duskin in [Dus02].

$k + 1 \leq i \leq n$ satisfying the relation $d_i \tau_j = d_{j-1} \tau_i$ for all $0 \leq i < j \leq n$ with $n \geq 4$; we then need to define a functor $\tau: [k] \times [n - 1 - k]^{\text{op}} \rightarrow \mathcal{D}$ so that $d_i \tau = \tau_i$, and show its uniqueness. If $k = -1$ or $k = n$, then the uniqueness and existence are immediate; assume thus $0 \leq k \leq n - 1$. Since $k + (n - 1 - k) = n - 1 \geq 3$, either $k \geq 2$ or $(n - 1 - k) \geq 2$ or both. If $k \geq 2$, a map $[k] \times [n - 1 - k]^{\text{op}} \rightarrow \mathcal{D}$ is the same data as a k -simplex in $N(\mathcal{D}^{[n-1-k]^{\text{op}}})$, which is a 1-coskeletal simplicial set being the nerve of a poset, so it is completely determined by its boundary. A dual argument applies to the case $n - 1 - k \geq 2$. \square

As announced in Theorem A, we now identify the Duskin nerve of the suspension 2-category $\Sigma\mathcal{D}$ with the simplicial set of \mathcal{D} -valued matrices.

Theorem 1.4. *Let \mathcal{D} be a 1-category. There is an isomorphism of simplicial sets*

$$N(\Sigma\mathcal{D}) \cong \text{Mat}(\mathcal{D}).$$

In particular, an n -simplex of the Duskin nerve of the suspension $\Sigma\mathcal{D}$ can be described uniquely as a functor $[k] \times [l]^{\text{op}} \rightarrow \mathcal{D}$, with $k + l = n - 1$ and $k, l \geq -1$.

Proof. We recall that the Duskin nerve of $\Sigma\mathcal{D}$ is 3-coskeletal, and we showed in Lemma 1.3 that the set of \mathcal{D} -valued matrices also assembles into a 3-coskeletal simplicial set. Therefore, to prove the theorem it is enough to identify the simplices of these two simplicial sets up to dimension 3 compatibly with the simplicial structure.

We identify all simplices in dimension up to 3 with \mathcal{D} -valued matrices as follows.

(0) Any of the two objects

$$x \quad \text{and} \quad y$$

of $\Sigma\mathcal{D}$ defines a 0-simplex of the Duskin nerve of $\Sigma\mathcal{D}$; we identify them with the unique functor $[0] \times [-1]^{\text{op}} \rightarrow \mathcal{D}$ and the unique functor $[-1] \times [0]^{\text{op}} \rightarrow \mathcal{D}$, respectively. Similarly, for all $n = 1, 2, 3$ any of the two objects x and y of $\Sigma\mathcal{D}$ defines a unique degenerate n -simplex of the Duskin nerve; we identify them with the unique functor $[n] \times [-1]^{\text{op}} \rightarrow \mathcal{D}$ and the unique functor $[-1] \times [n]^{\text{op}} \rightarrow \mathcal{D}$, respectively.

(1) Any object a in \mathcal{D} gives rise to a 1-simplex in the Duskin nerve:

$$x \xrightarrow{a} y$$

and all 1-simplices of the Duskin nerve of $\Sigma\mathcal{D}$ that are not degeneracies of a 0-simplex can uniquely be written in this form for some a in \mathcal{D} . These 1-simplices can be identified with the functor $[0] \times [0]^{\text{op}} \rightarrow \mathcal{D}$ with image

$$a.$$

- (2) Any morphism $\varphi: a \rightarrow b$ in \mathcal{D} gives rise to two 2-simplices in the Duskin nerve of $\Sigma\mathcal{D}$:

$$\begin{array}{ccc} & x & \\ s_0x \nearrow & \Uparrow \varphi & \nwarrow b \\ x & \xrightarrow{a} & y \end{array} \quad \text{and} \quad \begin{array}{ccc} & y & \\ b \nearrow & \Uparrow \varphi & \nwarrow s_0y \\ x & \xrightarrow{a} & y. \end{array}$$

Moreover, all 2-simplices of the Duskin nerve of $\Sigma\mathcal{D}$ that are not degeneracies of a 0-simplex can uniquely be written in one of these two forms for some $\varphi: a \rightarrow b$ in \mathcal{D} . These 2-simplices can be identified with the functors $[1] \times [0]^{\text{op}} \rightarrow \mathcal{D}$ and $[0] \times [1]^{\text{op}} \rightarrow \mathcal{D}$ with image

$$\begin{array}{c} a \\ \downarrow \varphi \\ b \end{array} \quad \text{and} \quad a \xrightarrow{\varphi} b.$$

- (3) Any commutative square

$$\begin{array}{ccc} a & \xrightarrow{\varphi} & b \\ \psi \downarrow & & \downarrow \gamma \\ f & \xrightarrow{\theta} & c \end{array}$$

in \mathcal{D} gives rise to three 3-simplices in the Duskin nerve of $\Sigma\mathcal{D}$:

$$\begin{array}{ccc} \begin{array}{ccc} y & \xleftarrow{c} & x \\ \uparrow \theta\psi & \nearrow & \uparrow \text{id}_x \\ a & \xrightarrow{\text{id}_x} & x \\ \uparrow \text{id}_x & & \uparrow \text{id}_x \end{array} & = & \begin{array}{ccc} y & \xleftarrow{c} & x \\ \uparrow \theta & \nearrow & \uparrow \text{id}_x \\ a & \xrightarrow{\psi} & x \\ \uparrow \psi & & \uparrow \text{id}_x \end{array} \\ \\ \begin{array}{ccc} y & \xleftarrow{\text{id}_y} & y \\ \uparrow \varphi & \nearrow & \uparrow c \\ a & \xrightarrow{\text{id}_x} & x \\ \uparrow \text{id}_x & & \uparrow \text{id}_x \end{array} & = & \begin{array}{ccc} y & \xleftarrow{\text{id}_y} & y \\ \uparrow \theta & \nearrow & \uparrow c \\ a & \xrightarrow{\psi} & x \\ \uparrow \psi & & \uparrow \text{id}_x \end{array} \\ \\ \begin{array}{ccc} y & \xleftarrow{\text{id}_y} & y \\ \uparrow \varphi & \nearrow & \uparrow \text{id}_y \\ a & \xrightarrow{\text{id}_x} & y \\ \uparrow \text{id}_x & & \uparrow \text{id}_x \end{array} & = & \begin{array}{ccc} y & \xleftarrow{\text{id}_y} & y \\ \uparrow \text{id}_y & \nearrow & \uparrow \text{id}_y \\ a & \xrightarrow{\gamma\varphi} & y \\ \uparrow \gamma\varphi & & \uparrow \text{id}_x \end{array} \end{array}$$

Moreover, it can be seen by direct inspection that all 3-simplices of the Duskin nerve that are not degeneracies of a 0-simplex can uniquely be written in one of these three forms for some commutative square in \mathcal{D} as

above. These 3-simplices can be identified with the functors $[2] \times [0]^{\text{op}} \rightarrow \mathcal{D}$, $[1] \times [1]^{\text{op}} \rightarrow \mathcal{D}$ and $[0] \times [2]^{\text{op}} \rightarrow \mathcal{D}$ displayed as

$$\begin{array}{c} a \\ \downarrow \psi \\ f \\ \downarrow \theta \\ c \end{array} \quad \text{and} \quad \begin{array}{ccc} a & \xrightarrow{\varphi} & b \\ \psi \downarrow & & \downarrow \gamma \\ f & \xrightarrow{\theta} & c \end{array} \quad \text{and} \quad a \xrightarrow{\varphi} b \xrightarrow{\gamma} c.$$

The given identification between simplices of the Duskin nerve in dimension up to 3 and \mathcal{D} -valued matrices can be checked to be compatible with the simplicial structure, using the explicit formulas from Lemma 1.3. \square

2. THE DUSKIN NERVE OF SUSPENSION 2-CATEGORIES - THE PRACTICE

The proof of Theorem 1.4 relies on the coskeletality of the simplicial sets $N(\Sigma\mathcal{D})$ and $\text{Mat}(\mathcal{D})$, and does not enlighten how the correspondence between \mathcal{D} -valued matrices $[k] \times [n-1-k]^{\text{op}} \rightarrow \mathcal{D}$ and n -simplices in the Duskin nerve of $\Sigma\mathcal{D}$ really works for $n \geq 4$. In this section we collect a few useful observations in this direction, and illustrate with an example how one can reconstruct a matrix from a simplex and vice versa.

Remark 2.1. Given the fact that $\Sigma\mathcal{D}$ has only two objects, for $n \geq 2$ any n -simplex σ in $N(\Sigma\mathcal{D})$ has exactly zero or one non-degenerate edges of the form $(k, k+1)$. More precisely,

- (a) if the simplex σ is the degeneracy of one of the 0-simplices x or y , each edge of σ is degenerate at the same vertex x or y . In this case, we read from the proof of Theorem 1.4 that the matrix corresponding to σ is the unique functor $[n] \times [-1]^{\text{op}} \rightarrow \mathcal{D}$ or the unique functor $[-1] \times [n]^{\text{op}} \rightarrow \mathcal{D}$.
- (b) if the simplex is not the degeneracy of a 0-simplex, it has precisely one non-degenerate edge of the form $(k, k+1)$ for some $0 \leq k \leq n-1$.

The fact that these two very different behaviours partition the simplices of $N(\Sigma\mathcal{D})$ is fundamental, and we therefore make the following definition.

Definition 2.2. For $n \geq 1$ we say that an n -simplex σ in $N(\Sigma\mathcal{D})$ is

- (a) *maximally degenerate* if it is the degeneracy of one of the 0-simplices x or y .
- (b) *of type k* for $0 \leq k \leq n-1$ if it has one non-degenerate edge of the form $(k, k+1)$.

Example 2.3. The 3-simplex of $N(\Sigma\mathcal{D})$

$$\begin{array}{ccc}
 y & \xleftarrow{\text{id}_y} & y \\
 \uparrow a & \nearrow \varphi & \uparrow c \\
 x & \xrightarrow{\text{id}_x} & x \\
 & \nwarrow \gamma & \\
 & &
 \end{array}
 =
 \begin{array}{ccc}
 y & \xleftarrow{\text{id}_y} & y \\
 \uparrow a & \nearrow \theta & \uparrow c \\
 x & \xrightarrow{\text{id}_x} & x \\
 & \nwarrow \psi & \\
 & &
 \end{array}$$

is of type 1, whereas the 3-simplex

$$\begin{array}{ccc}
 y & \xleftarrow{c} & x \\
 \uparrow a & \nearrow \theta\psi & \uparrow \text{id}_x \\
 x & \xrightarrow{\text{id}_x} & x \\
 & \nwarrow \text{id}_x & \\
 & &
 \end{array}
 =
 \begin{array}{ccc}
 y & \xleftarrow{c} & x \\
 \uparrow a & \nearrow \theta & \uparrow \text{id}_x \\
 x & \xrightarrow{\text{id}_x} & x \\
 & \nwarrow \psi & \\
 & &
 \end{array}$$

is of type 2.

In particular, it is consistent to think of the maximally degenerate n -simplex of the 0-simplex y as the (unique) n -simplex of *type* -1 and to the maximally degenerate n -simplex of the 0-simplex x as the (unique) n -simplex of *type* n . The following corollary relates the type k of an n -simplex σ to the size of the matrix corresponding to σ .

Corollary 2.4. *Let \mathcal{D} be a 1-category, $n \geq 1$ and $-1 \leq k \leq n$. There is a bijective correspondence between the n -simplices of $N(\Sigma\mathcal{D})$ of type k , and the functors $[k] \times [n-1-k]^{\text{op}} \rightarrow \mathcal{D}$.*

Proof. The corollary is proven by induction on $n \geq 1$. For $n = 1, 2$, the list of simplices of n -simplices of $N(\Sigma\mathcal{D})$ has been matched explicitly to a \mathcal{D} -valued matrix in the proof of Theorem 1.4, and the reader can see by direct inspection that the statement holds. Suppose now that we are given an n -simplex σ of $N(\Sigma\mathcal{D})$ for $n > 2$ and that σ is of type k for $-1 \leq k \leq n$. We know that the i -th $(n-1)$ -face of the n -simplex σ is of type $k-1$ if $0 \leq i \leq k$ and of type k if $k+1 \leq i \leq n$. By induction hypothesis, the face $d_i\sigma$ therefore corresponds to a functor of the form $\tau_i: [k-1] \times [n-1-k]^{\text{op}} \rightarrow \mathcal{D}$ for $0 \leq i \leq k$ and $\tau_i: [k] \times [(n-1-k)-1]^{\text{op}} \rightarrow \mathcal{D}$ for $k+1 \leq i \leq n$. Recall that, for $n > 2$, any n -simplex in $N(\Sigma\mathcal{D})$ or $\text{Mat}(\mathcal{D})$ is determined by its boundary. The only elements in $\text{Mat}(\mathcal{D})$ having such boundary are functors of the form $[k] \times [n-1-k]^{\text{op}} \rightarrow \mathcal{D}$, as desired. A similar argument shows that the n -simplex corresponding to a functor $M: [k] \times [n-1-k]^{\text{op}} \rightarrow \mathcal{D}$ must be of type k . \square

The next corollary describes a correspondence between *triangulations* labeled in the 2-faces of a given simplex of the Duskin nerve of $\Sigma\mathcal{D}$, and *monotone paths* inside the corresponding \mathcal{D} -valued matrix.

For triangulations, we make use of the formalism from [DK12, Ex. 2.2.15]. A *triangulation* \mathcal{T} of a convex $(n+1)$ -gon with cyclically numbered vertices

only contains triangles with vertices being vertices of the original polygon. To any such triangulation \mathcal{T} , we can associate a simplicial subset $\Delta[\mathcal{T}] \subset \text{sk}_2 \Delta[n] \subset \Delta[n]$ by choosing the 2-faces corresponding to the triangles in the triangulation.

Definition 2.5. Let $n \geq 2$. Given an n -simplex σ of $N(\Sigma\mathcal{D})$ of type k for $0 \leq k \leq n-1$, a σ -labeled triangulation consists of a triangulation \mathcal{T} of an $(n+1)$ -gon that does not have any triangle completely contained neither in $\{0, \dots, k\}$ nor in $\{k+1, \dots, n\}$, together with the composite

$$\Delta[\mathcal{T}] \hookrightarrow \Delta[n] \xrightarrow{\sigma} N(\Sigma\mathcal{D}).$$

In particular, the definition requires a compatibility between the triangulation \mathcal{T} and the simplex σ , namely that the 2-simplices in the image of the composite $\Delta[\mathcal{T}] \rightarrow N(\Sigma\mathcal{D})$ above are not degeneracies of a 0-simplex.

Example 2.6. Given σ a 3-simplex of $N(\Sigma\mathcal{D})$ of type 1 given by

$$\begin{array}{ccc} y & \xleftarrow{\text{id}_y} & y \\ \uparrow a & \nearrow \varphi & \uparrow c \\ x & \xrightarrow{\text{id}_x} & x \end{array} \quad = \quad \begin{array}{ccc} y & \xleftarrow{\text{id}_y} & y \\ \uparrow a & \nearrow \theta & \uparrow c \\ x & \xrightarrow{\text{id}_x} & x \end{array}$$

(Note: The original image shows a more complex diagram with multiple arrows labeled a, b, c, x, y, id_x, id_y, phi, gamma, theta, psi. The diagram above is a simplified representation of the structure shown in the image.)

an example of a σ -labeled triangulation is

$$\begin{array}{ccc} y & \xleftarrow{\text{id}_y} & y \\ \uparrow a & \nearrow \varphi & \uparrow c \\ x & \xrightarrow{\text{id}_x} & x \end{array}$$

Recall e.g. from [Ver08b, Def. 65] that a *shuffle* of $\Delta[k] \times \Delta[l]$ for $k, l \geq 0$ is a non-degenerate $(k+l)$ -simplex of $\Delta[k] \times \Delta[l]$. An easy and useful characterization of these is that they are precisely the functors³

$$S := (\alpha, \beta): [k+l] \rightarrow [k] \times [l]^{\text{op}}$$

that satisfy the *ordinate summation property*: for all $0 \leq i \leq k+l$

$$\alpha(i) + l - \beta(i) = i.$$

Definition 2.7. Given a \mathcal{D} -valued matrix $M: [k] \times [n-1-k]^{\text{op}} \rightarrow \mathcal{D}$ with $0 \leq k \leq n-1$, a *monotone path inside the matrix* M consists of a shuffle $S: [n-1] \rightarrow [k] \times [n-1-k]^{\text{op}}$, together with the restriction

$$[n-1] \xrightarrow{S} [k] \times [n-1-k]^{\text{op}} \xrightarrow{M} \mathcal{D}$$

of M along S .

³We choose to have $[l]^{\text{op}}$ rather than $[l]$ in the second factor. This convention is more convenient to the setup of the paper and not restrictive, modulo consistent adaptation of the relation between the indices $\alpha(i)$ and $\beta(i)$ in the next formula.

Example 2.8. If $M: [1] \times [1]^{\text{op}} \rightarrow \mathcal{D}$ is a functor given by

$$\begin{array}{ccc} a & \xrightarrow{\varphi} & b \\ \psi \downarrow & & \downarrow \gamma \\ f & \xrightarrow{\theta} & c \end{array}$$

a monotone path inside M is for instance

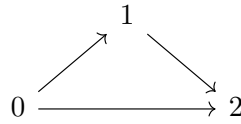
$$\begin{array}{ccc} a & \xrightarrow{\varphi} & b \\ & & \downarrow \gamma \\ & & c. \end{array}$$

Corollary 2.9. *Let \mathcal{D} be a 1-category, $n \geq 2$, σ an n -simplex of $N(\Sigma\mathcal{D})$ of type k for $0 \leq k \leq n-1$, and $M_\sigma: [k] \times [n-1-k]^{\text{op}} \rightarrow \mathcal{D}$ the corresponding \mathcal{D} -valued matrix according to Theorem 1.4. There is a bijective correspondence between σ -labeled triangulations $\Delta[\mathcal{T}] \rightarrow N(\Sigma\mathcal{D})$ and monotone paths $P: [n-1] \rightarrow \mathcal{D}$ inside $M_\sigma: [k] \times [n-1-k]^{\text{op}} \rightarrow \mathcal{D}$.*

The corollary is a direct consequence of Theorem 1.4 along with the following combinatorial fact⁴.

Lemma 2.10. *Let $n \geq 2$ and let $0 \leq k \leq n-1$. Then there is a bijective correspondence between triangulations $\Delta[\mathcal{T}] \rightarrow \Delta[n]$ of an $(n+1)$ -gon which do not have a triangle completely contained neither in $\{0, \dots, k\}$ nor in $\{k+1, \dots, n\}$ and shuffles $S: [n-1] \rightarrow [k] \times [n-1-k]^{\text{op}}$.*

Proof of the lemma. The lemma is proven by induction on $n \geq 2$. If $n = 2$ there is a unique triangulation of the $(2+1)$ -gon given by



both when $k = 0$ or $k = 1$. On the other side, there is a unique shuffle given by $[1] \rightarrow [0] \times [1]^{\text{op}}$ for $k = 0$ and a unique shuffle $[1] \rightarrow [1] \times [0]^{\text{op}}$ for $k = 1$.

If $n > 2$, we first show that for a given triangulation as above the edge $(0, n)$ is contained exactly in one triangle that is of the form $(0, n-1, n)$ or $(0, 1, n)$. To see this, assume otherwise that $(0, n)$ is contained in the triangle $(0, p, n)$ for some $1 < p < n-1$. We only consider $p \leq k$, the other case being symmetric. Then the triangulation of the $(n+1)$ -gon we started with

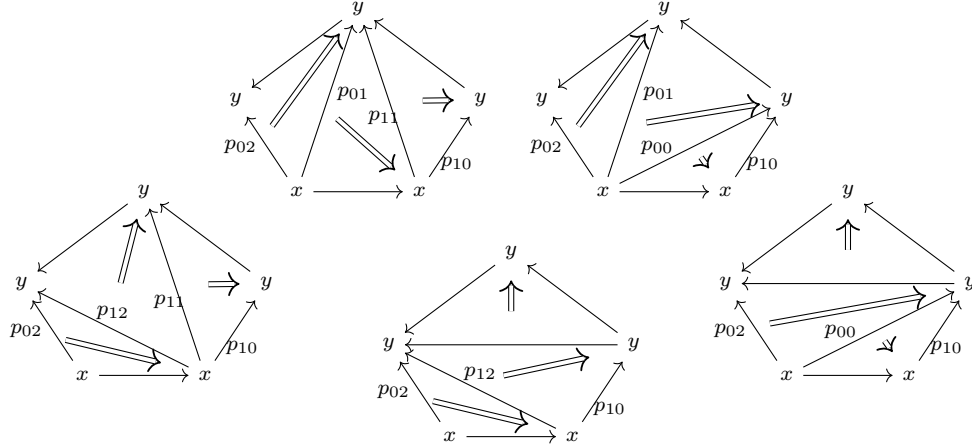
⁴ The lemma appears to be a variant of the classical fact that the Catalan number C_{n-1} can be expressed in two equivalent ways: as the number of triangulations of an $(n+1)$ -gon, or as the number of monotone lattice paths along the edges of a grid with $(n-1) \times (n-1)$ square cells which do not pass above the diagonal. However, we are not aware of a direct comparison with the statement of our lemma.

induces a triangulation of the $(p+1)$ -gon with vertices $0, 1, \dots, p$, since $(0, p)$ is one of the edges in the triangulation. But we assumed that no triangles include only vertices in $0, 1, \dots, k$, leading to a contradiction.

It then follows that the given triangulation of the $(n+1)$ -gon includes exactly one of the triangles $(0, n-1, n)$ and $(0, 1, n)$, and is completely and uniquely described by such a triangle and the triangulation of the remaining n -gon. By induction hypothesis, this corresponds to a shuffle of the form $[n-1-1] \rightarrow [k] \times [n-2-k]^{\text{op}}$ in the first case and of the form $[n-1-1] \rightarrow [k-1] \times [n-1-k]^{\text{op}}$ in the second case, together with an extra arrow that can be connected to this shuffle (horizontally in the first case and vertically in the second case). By connecting these two pieces together we obtain a shuffle of the form $[n-1] \rightarrow [k] \times [n-1-k]^{\text{op}}$, and all such shuffles arise in this way. \square

We illustrate with an example how the proposition can be used to write down the matrix associated to a simplex and vice versa. The idea is that, given a triangulation labeled in a simplex, each simplex with a degenerate 2-nd face contributes as a vertical step in the corresponding path and each 2-simplex with a degenerate 0-th face contributes as a horizontal step.

Example 2.11. Let \mathcal{P} be a poset. Consider the 4-simplex σ of type 1 of the Duskin nerve of $\Sigma\mathcal{P}$ determined by the following 2-skeleton

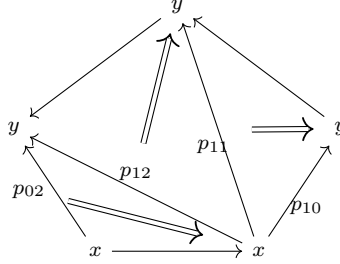


where p_{ij} belongs to \mathcal{P} and let's determine the \mathcal{P} -valued matrix M_σ that corresponds to it according to Theorem 1.4.

Given that the edge $(1, 2)$ of σ is non-degenerate, the given 4-simplex is of type 1 and we can use Corollary 2.4 to assert that the matrix M_σ has to be of the form

$$M_\sigma: [1] \times [4-1-1]^{\text{op}} = [1] \times [2]^{\text{op}} \rightarrow \mathcal{P}.$$

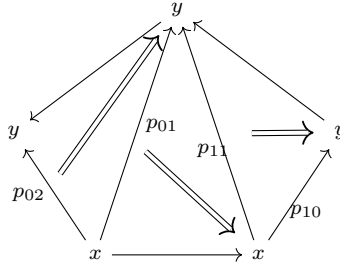
The σ -triangulation



corresponds to the monotone path in M_σ that covers fully the left column and the bottom row and the 1-st row, and is as follows:

$$\begin{array}{c} p_{02} \\ \downarrow \\ p_{12} \longrightarrow p_{11} \longrightarrow p_{10}. \end{array}$$

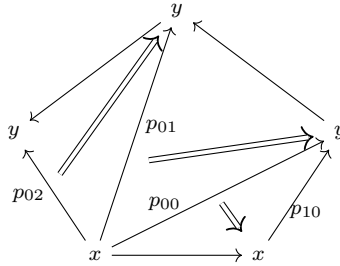
The σ -triangulation



corresponds to the monotone path in M_σ that goes through the 1-st column, and is as follows:

$$\begin{array}{c} p_{02} \longrightarrow p_{01} \\ \downarrow \\ p_{11} \longrightarrow p_{10}. \end{array}$$

The σ -triangulation



corresponds to the monotone path in M_σ that covers fully the 0-th row and the last column, and is as follows:

$$\begin{array}{ccccc} p_{02} & \longrightarrow & p_{01} & \longrightarrow & p_{00} \\ & & & & \downarrow \\ & & & & p_{10}. \end{array}$$

We conclude that M_σ is the functor $[1] \times [2]^{\text{op}} \rightarrow \mathcal{P}$ given by

$$\begin{array}{ccccc} p_{02} & \longrightarrow & p_{01} & \longrightarrow & p_{00} \\ \downarrow & & \downarrow & & \downarrow \\ p_{12} & \longrightarrow & p_{11} & \longrightarrow & p_{10}. \end{array}$$

3. THE DUSKIN NERVE OF THE FREE 2-CELL

As an instance of Theorem A, we obtain a full description of the non-degenerate simplices of the Duskin nerve of the free 2-cell

$$\begin{array}{ccc} & f_0 & \\ x & \begin{array}{c} \curvearrowright \\ \Downarrow \alpha \\ \curvearrowleft \end{array} & y, \\ & f_1 & \end{array}$$

it being the suspension of the 1-category [1].

Proposition 3.1. *In dimension n , the Duskin nerve of the free 2-cell has precisely two non-degenerate simplices σ_n and σ'_n . More precisely, $\sigma_0 := y$ and $\sigma'_0 := x$ are the two 0-simplices of the Duskin nerve, $\sigma_1 := 1: x \rightarrow y$ and $\sigma'_1 := 0: x \rightarrow y$ are the two 1-simplices of the Duskin nerve, and for $n > 1$ the n -simplices σ_n and σ'_n are described as follows.*

- if $n = 2m$, the non-degenerate $2m$ -simplices σ_{2m} and σ'_{2m} are uniquely determined by the relations

$$d_i \sigma_{2m} = \begin{cases} s_{m-1+i} \sigma_{2m-2} & \text{for } 0 \leq i \leq m-1 \\ \sigma'_{2m-1} & \text{for } i = m \\ s_{i-m-1} \sigma_{2m-2} & \text{for } m+1 \leq i \leq 2m-1 \\ \sigma_{2m-1} & \text{for } i = 2m \end{cases}$$

$$d_i \sigma'_{2m} = \begin{cases} \sigma_{2m-1} & \text{for } i = 0 \\ s_{m-1+i} \sigma'_{2m-2} & \text{for } 1 \leq i \leq m-1 \\ \sigma'_{2m-1} & \text{for } i = m \\ s_{i-m-1} \sigma'_{2m-2} & \text{for } m+1 \leq i \leq 2m; \end{cases}$$

- if $n = 2m+1$, the non-degenerate $(2m+1)$ -simplices σ_{2m+1} and σ'_{2m+1} are uniquely determined by the relations

$$d_i \sigma_{2m+1} = \begin{cases} s_{m+i} \sigma_{2m-1}, & \text{for } 0 \leq i \leq m-1 \\ \sigma_{2m}, & \text{for } i = m \\ \sigma'_{2m}, & \text{for } i = m+1 \\ s_{i-m-2} \sigma_{2m-1}, & \text{for } m+2 \leq i \leq 2m+1 \end{cases}$$

$$d_i \sigma'_{2m+1} = \begin{cases} \sigma_{2m}, & \text{for } i = 0 \\ s_{m-1+i} \sigma'_{2m-1}, & \text{for } 1 \leq i \leq m \\ s_{i-m-1} \sigma'_{2m-1}, & \text{for } m+1 \leq i \leq 2m \\ \sigma'_{2m}, & \text{for } i = 2m+1. \end{cases}$$

Proof. By Theorem 1.4 we know that simplices of the Duskin nerve of the free 2-cell can be enumerated by means of functors $[k] \times [l]^{\text{op}} \rightarrow [1]$, with $k, l \geq -1$ and $k + l = n - 1$. Moreover, an n -simplex is non-degenerate if and only if all rows are different and all columns are different, meaning that $k + 1 \leq l + 2$ and $l + 1 \leq k + 2$. Since we have $k + l = n - 1$, we obtain that $n - 2 \leq 2k \leq n$.

According to this analysis, in dimension n the Duskin nerve of the free 2-cells has precisely two non-degenerate simplices σ_n and σ'_n .

- If $n = 2m$, the non-degenerate simplices σ_{2m} and σ'_{2m} correspond to the functors

$$M_{2m}: [m-1] \times [m]^{\text{op}} \rightarrow [1] \text{ and } M'_{2m}: [m] \times [m-1]^{\text{op}} \rightarrow [1]$$

given on objects by

$$M_{2m}(i, j) = \begin{cases} 0, & \text{if } i < j, \\ 1, & \text{else.} \end{cases} \quad \text{and} \quad M'_{2m}(i, j) = \begin{cases} 0, & \text{if } i \leq j, \\ 1, & \text{else.} \end{cases}$$

- If $n = 2m + 1$, the non-degenerate simplices σ_{2m+1} and σ'_{2m+1} correspond to the functors

$$M_{2m+1}: [m] \times [m]^{\text{op}} \rightarrow [1] \text{ and } M'_{2m+1}: [m] \times [m]^{\text{op}} \rightarrow [1]$$

given on objects by

$$M_{2m+1}(i, j) = \begin{cases} 0, & \text{if } i < j, \\ 1, & \text{else.} \end{cases} \quad \text{and} \quad M'_{2m+1}(i, j) = \begin{cases} 0, & \text{if } i \leq j, \\ 1, & \text{else.} \end{cases}$$

In particular, in both M and M' no two rows or columns are equal, and each row and column is increasing. They can be depicted as follows.

$$\begin{array}{ccccccc} 0 & \rightarrow & 0 & \rightarrow & \cdots & \rightarrow & 0 \rightarrow 0 \rightarrow 1 \\ \downarrow & & \downarrow & & & & \downarrow & \downarrow & \downarrow \\ 0 & \rightarrow & 0 & \rightarrow & \cdots & \rightarrow & 0 \rightarrow 1 \rightarrow 1 \\ \downarrow & & \downarrow & & & & \downarrow & \downarrow & \downarrow \\ 0 & \rightarrow & 0 & \rightarrow & \cdots & \rightarrow & 1 \rightarrow 1 \rightarrow 1 \\ \downarrow & & \downarrow & & & & \downarrow & \downarrow & \downarrow \\ \vdots & & \vdots & & \ddots & & \vdots & \vdots & \vdots \\ \downarrow & & \downarrow & & & & \downarrow & \downarrow & \downarrow \\ 0 & \rightarrow & 1 & \rightarrow & \cdots & \rightarrow & 1 \rightarrow 1 \rightarrow 1 \\ \downarrow & & \downarrow & & & & \downarrow & \downarrow & \downarrow \\ 1 & \rightarrow & 1 & \rightarrow & \cdots & \rightarrow & 1 \rightarrow 1 \rightarrow 1 \end{array}$$

An induction argument shows that these simplices satisfy the desired relations. Uniqueness can be checked directly for simplices in dimension 1, 2, 3

and follows from 3-coskeletality of $N(\Sigma[1])$ for simplices in dimension at least 4. \square

4. THE DUSKIN NERVE OF $(r + 1)$ -POINT SUSPENSION 2-CATEGORIES

As announced informally in Theorem B, we now describe the Duskin nerve of $(r + 1)$ -point suspension 2-categories $\Sigma[\mathcal{D}_1, \dots, \mathcal{D}_r]$, of which a large class of examples is given by all elements of Joyal's disk category Θ_2 . This description was inspired by the argument used in [Rez10, Prop.4.9].

Definition 4.1. The $(r + 1)$ -point suspension of given 1-categories $\mathcal{D}_1, \dots, \mathcal{D}_r$ is the 2-category $\Sigma[\mathcal{D}_1, \dots, \mathcal{D}_r]$ constructed inductively as the pushout of 2-categories

$$\begin{array}{ccc} [0] & \xrightarrow{\text{first object}} & \Sigma\mathcal{D}_r \\ \text{last object} \downarrow & & \downarrow \\ \Sigma[\mathcal{D}_1, \dots, \mathcal{D}_{r-1}] & \longrightarrow & \Sigma[\mathcal{D}_1, \dots, \mathcal{D}_{r-1}, \mathcal{D}_r]. \end{array}$$

Alternatively, it can be seen using [AM14, §7.2] that the $(r + 1)$ -point suspension $\Sigma[\mathcal{D}_1, \dots, \mathcal{D}_r]$ can be described as the 2-category with $r + 1$ objects x_0, \dots, x_r and hom categories given by

$$\text{Map}_{\Sigma[\mathcal{D}_1, \dots, \mathcal{D}_r]}(x_i, x_j) \cong \begin{cases} \mathcal{D}_{i+1} \times \dots \times \mathcal{D}_j & \text{if } i < j \\ [0] & \text{if } i = j \\ [-1] & \text{if } i > j. \end{cases}$$

Remark 4.2. Given $r \geq 1$, there is a map of categories

$$s_r: [r] \hookrightarrow [1]^r$$

given on objects by

$$s_r: i \mapsto (\underbrace{1, \dots, 1}_{i \text{ times}}, \underbrace{0, \dots, 0}_{(r-i) \text{ times}}).$$

This map is injective on objects and fully faithful. When taking nerves, the induced simplicial map

$$N(s_r): \Delta[r] \hookrightarrow \Delta[1]^r$$

is a monomorphism, and the image is described as follows.

Any n -simplex of $\Delta[1]^r$ is of the form $f_k: [n] \rightarrow [1]^r$ for $k = -1, \dots, n$, with f_k defined on objects by

$$f_k: i \mapsto \begin{cases} 0 & i \leq k \\ 1 & i > k. \end{cases}$$

In particular, $s_r = (f_0, \dots, f_{r-1})$. According to this notation an n -simplex $(f_{k_1}, \dots, f_{k_r}): \Delta[n] \rightarrow \Delta[1]^r$ of $\Delta[1]^r$ is in the image of $N(s_r)$ if and only if $k_1 \leq \dots \leq k_r$.

Remark 4.3. Given categories $\mathcal{D}_1, \dots, \mathcal{D}_r$, there are canonical maps of 2-categories defined on $\Sigma[\mathcal{D}_1, \dots, \mathcal{D}_r]$.

- (1) For any $1 \leq i \leq r$ there are canonical maps of 2-categories

$$\Sigma[\mathcal{D}_1, \dots, \mathcal{D}_r] \rightarrow \Sigma[\mathcal{D}_i],$$

which are induced by collapsing all 2-categories $\Sigma\mathcal{D}_j$ for $j < i$ to the point x_{i-1} and all 2-categories $\Sigma\mathcal{D}_j$ for $j > i$ to the point x_i ,

- (2) There is a canonical map of 2-categories

$$\Sigma[\mathcal{D}_1, \dots, \mathcal{D}_r] \rightarrow \Sigma[[0], \dots, [0]] \cong [r],$$

which is induced by collapsing each generating 2-cell to a 1-cell.

- (3) As a special case of the previous one, for any $1 \leq i \leq r$ there are canonical maps of 2-categories

$$\Sigma[\mathcal{D}_i] \rightarrow \Sigma[0].$$

When taking nerves, the induced map

$$N(\Sigma[\mathcal{D}_i]) \rightarrow N(\Sigma[0]) = N([1]) = \Delta[1],$$

sends an n -simplex σ of $N(\Sigma[\mathcal{D}_i])$

- to the n -simplex f_{-1} of $\Delta[1]$ if σ is maximally degenerate at y ;
- to the n -simplex f_k of $\Delta[1]$ if σ is of type k ;
- to the n -simplex f_n of $\Delta[1]$ if σ is maximally degenerate at x .

Similarly, using the identification from Theorem 1.4 the induced map

$$\text{Mat}(\mathcal{D}_i) \rightarrow \Delta[1],$$

sends a matrix $M: [k] \times [n-1-k]^{\text{op}} \rightarrow \mathcal{D}_i$ to f_k .

Theorem 4.4. *Given categories $\mathcal{D}_1, \dots, \mathcal{D}_r$, there is a pullback square of simplicial sets*

$$\begin{array}{ccc} N(\Sigma[\mathcal{D}_1, \dots, \mathcal{D}_r]) & \longrightarrow & N(\Sigma[\mathcal{D}_1]) \times \dots \times N(\Sigma[\mathcal{D}_r]) \\ \downarrow & & \downarrow \\ \Delta[r] & \longrightarrow & \Delta[1] \times \dots \times \Delta[1], \end{array}$$

where

- the top horizontal map is induced by the canonical maps $\Sigma[\mathcal{D}_1, \dots, \mathcal{D}_r] \rightarrow \Sigma[\mathcal{D}_i]$;
- the left vertical map is induced by the canonical map $\Sigma[\mathcal{D}_1, \dots, \mathcal{D}_r] \rightarrow [r]$;
- the right vertical arrow is induced by the canonical maps $\Sigma[\mathcal{D}_i] \rightarrow [1]$;
- the bottom horizontal arrow is induced by the canonical map $s_r: [r] \hookrightarrow [1]^r$.

Before proving the theorem, we observe that combining the result with Theorem 1.4, we obtain that there is also a pullback of simplicial sets

$$\begin{array}{ccc} N(\Sigma[\mathcal{D}_1, \dots, \mathcal{D}_r]) & \longrightarrow & \text{Mat}(\mathcal{D}_1) \times \dots \times \text{Mat}(\mathcal{D}_r) \\ \downarrow & & \downarrow \\ \Delta[r] & \longrightarrow & \Delta[1] \times \dots \times \Delta[1]. \end{array}$$

In particular, it follows from Remarks 4.2 and 4.3 that an n -simplex of the Duskin nerve of the $(r+1)$ -point suspension $\Sigma[\mathcal{D}_1, \dots, \mathcal{D}_r]$ can be uniquely described as a r -uple of functors $[k_i] \times [l_i]^{\text{op}} \rightarrow \mathcal{D}_i$ for $i = 1, \dots, r$, with $k_i, l_i \geq -1$, $k_i + l_i = n - 1$ and subject to the condition that $k_i \leq k_j$ for $0 \leq i \leq j \leq r$.

Proof. We argue that there is a pullback square of 2-categories

$$\begin{array}{ccc} \Sigma[\mathcal{D}_1, \dots, \mathcal{D}_r] & \longrightarrow & \Sigma[\mathcal{D}_1] \times \dots \times \Sigma[\mathcal{D}_r] \\ \downarrow & & \downarrow \\ [r] & \xrightarrow{s_r} & [1] \times \dots \times [1], \end{array}$$

From there we can then conclude, given that the Duskin nerve respects pullbacks and products, being a right adjoint.

The square of 2-categories above commutes by direct inspection. In order to prove that the square is a pullback of 2-categories, we check that it is a pullback at the level of objects, and that it is a locally a pullback at the level of hom-categories of any pair of objects in $\Sigma[\mathcal{D}_1, \dots, \mathcal{D}_r]$.

At the level of objects, we ought to look at the commutative square of sets

$$\begin{array}{ccc} \text{Ob}(\Sigma[\mathcal{D}_1, \dots, \mathcal{D}_r]) & \longrightarrow & \text{Ob}(\Sigma[\mathcal{D}_1] \times \dots \times \Sigma[\mathcal{D}_r]) \\ \downarrow & & \downarrow \\ \text{Ob}([r]) & \xrightarrow{s_r} & \text{Ob}([1] \times \dots \times [1]). \end{array}$$

This square is expressed as the following square, where both vertical maps are bijections,

$$\begin{array}{ccc} \{x_0, \dots, x_r\} & \longrightarrow & \{x, y\} \times \dots \times \{x, y\} \\ \downarrow \cong & & \downarrow \cong \\ \{0, \dots, r\} & \xrightarrow{s_r} & \{0, 1\} \times \dots \times \{0, 1\}. \end{array}$$

The square is therefore a pullback of sets.

At the level of hom-categories, given any two objects x_i and x_j of $\Sigma[\mathcal{D}_1, \dots, \mathcal{D}_r]$, we ought to look at the commutative square of categories

$$\begin{array}{ccc} \mathrm{Map}_{\Sigma[\mathcal{D}_1, \dots, \mathcal{D}_r]}(x_i, x_j) & \longrightarrow & \mathrm{Map}_{\Sigma[\mathcal{D}_1] \times \dots \times \Sigma[\mathcal{D}_r]}(\vec{x}_{s_r(i)}, \vec{x}_{s_r(j)}) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{[r]}(i, j) & \longrightarrow & \mathrm{Map}_{[1] \times \dots \times [1]}(s_r(i), s_r(j)), \end{array}$$

where $\vec{x}_{s_r(i)}$ and $\vec{x}_{s_r(j)}$ denote the images of x_i and x_j in $\Sigma[\mathcal{D}_1] \times \dots \times \Sigma[\mathcal{D}_r]$. If $i > j$, this square is easily expressed as the following square, where the left vertical map is an isomorphism of empty categories $[-1]$,

$$\begin{array}{ccc} [-1] & \longrightarrow & \mathrm{Map}_{\Sigma[\mathcal{D}_1] \times \dots \times \Sigma[\mathcal{D}_r]}(\vec{x}_{s_r(i)}, \vec{x}_{s_r(j)}) \\ \downarrow \cong & & \downarrow \\ [-1] & \longrightarrow & \mathrm{Map}_{[1] \times \dots \times [1]}(s_r(i), s_r(j)). \end{array}$$

If instead $i \leq j$, this square is easily expressed as the following square, where both horizontal arrows are isomorphisms of categories,

$$\begin{array}{ccc} \mathrm{Map}_{\Sigma[\mathcal{D}_1, \dots, \mathcal{D}_r]}(x_i, x_j) & \xrightarrow{\cong} & \mathcal{D}_{i+1} \times \dots \times \mathcal{D}_j \\ \downarrow & & \downarrow \\ [0] & \xrightarrow{\cong} & [0] \times \dots \times [0]. \end{array}$$

The square is therefore a pullback of categories in both cases. \square

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