Typed Linear Algebra for Efficient Analytical Querying

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ABSTRACT

This paper uses typed linear algebra (LA) to represent data and perform analytical querying in a single, unified framework. The typed approach offers strong type checking (as in modern programming languages) and a diagrammatic way of expressing queries (paths in LA diagrams). A kernel of LA operators has been implemented so that paths extracted from LA diagrams can be executed. The approach is validated and evaluated taking TPC-H benchmark queries as reference. The performance of the LA-based approach is compared with popular database competitors (PostgreSQL and MySQL).

KEYWORDS

Analytical querying, typed linear algebra, OLAP

1 INTRODUCTION

"Only by taking infinitesimally small units for observation (the differential of history, that is, the individual tendencies of men) and attaining to the art of integrating them (that is, finding the sum of these infinitesimals) can we hope to arrive at the laws of history."

Leo Tolstoy, "War and Peace" - Book XI, Chap.II (1869)

J. Gantz *et al* predict that the amount of digital bits created and consumed each year in the USA will grow to 6.6 zettabytes by 2020 [12]. In this age of the information society, *data records* are the *differentials* of Tolstoy's quote given above. Through analytical querying, strategy planners are mining such infinitesimals and *integrating* them so as to infer the "laws of our society". A century and a half in advance, Tolstoy has given us perhaps the oldest definition of data mining that we have.

Since the early days of psychometrics in the social sciences (1970s), *linear algebra* (LA) has been central to data analysis, namely through tensor decomposition, incremental tensor analysis [26]

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and so on. Abadir and Magnus [2] stress the need for a *standardized* notation for linear algebra in the field of econometrics and statistics.

More recently, it has been shown that data consolidation can be elegantly expressed in *typed* linear algebra [17, 18], a categorial approach to linear algebra [16] in which matrices are represented

by arrows $m \xrightarrow{M} n$ where M is a LA-expression denoting a matrix, m is the number of columns of M and n is the corresponding number of rows. The approach is *strongly typed* in the sense that it is free from matrix dimension errors, by construction [16]. Moreover, it is *type-polymorphic*, making room for proving properties of data constructions relying on types alone. For instance, the "free theorem" [27] of the *data cube* operator given in [18] is proved in that way.

The main aim of [17, 18] was to show how analytical querying theory benefits from the typed LA approach, essentially from a foundation point of view. In the current paper, we propose, validate and evaluate such a typed LA approach as a means of data analysis programming.

We implemented a minimal kernel of LA operators needed when scripting analytic queries. Such scripts essentially describe paths of diagrams whose arrows record data encoded as typed matrices. We also show how to infer such LA scripts from standard SQL code. Finally, we evaluate the approach by running the scripts of some queries of the TPC-H benchmark suite. This gives evidence of such scripts being faster and more efficient than standard SQL-based solutions.

Our main contributions are:

- A typed linear algebra approach to complex data querying based on a minimal LA kernel.
- A new way to build query plans as *paths* of typed LA diagrams, ensuring *type-correctness* by construction.
- An evaluation of the approach using queries of the TPC-H benchmark suite, which includes a comparison with two widely used and industry proven databases, PostgreSQL[13]

and MySQL [8]. Although our system is still a prototype, its measured performance is better in the majority of the benchmarks.

2 ALGEBRAIC ENCODING OF DATA

As starting point for describing our approach, consider the following samples of two relational data tables randomly generated by the TPC-H benchmark suite [9] — table *orders*

and table lineitem:

```
# | l_orderkey | l_quantity | l_linestatus | l_extendedprice
    2723 | 4.00 | 0 | 2124.32
          1.00 I
                 0 |
                      16994 56
           2.00 | F
    5699 I
                      32735.70
           9.00
                į o
    4354
                      1902.10
    3392
                       42846.80
    4354
           5.00
                İΕ
                      35707.22
                      44064.48
    5699
           5.00
           3.00 | O |
                      7168.84
```

Further consider the following, much simplified version of query number 3 of the same benchmark:

```
select o_{o}rderpriority, o_{o}rderpriority, o_{o}rderdate, sum (l_{q}uantity * l_{e}xtendedprice) from lineitem, orders where l_{o}rderkey = o_{o}rderkey group by o_{o}rderdate, o_{o}rderpriority
```

In general, given a table t, one of its attributes a and the number i of one of its records (rows), we denote the corresponding data value by the expression t[i].a. For instance, $orders[3].o_orderkey = 551$ above.

Our typed LA encoding of data is inherently *columnar* [11]. As is usual, we shall split columns (attributes) into two groups: the so-called *dimensions* and the so-called *measures*.

Every *dimension* attribute d of a given table t is represented by a so-called *projection* function $t_d: \#t \to |d|$, where |d| is the type containing the range of values of attribute d, and $\#t = \{1, ..., \#t\}$ is the set of row indices of t. Clearly,

$$t_d(i) = t[i].d (1)$$

e.g. $orders_{orderkey}$ (3) = $orders[3].orderkey = 551.^2$

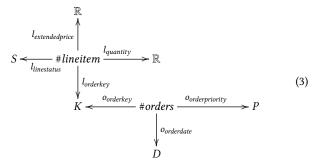
Projection functions represent relational data sets with no loss of information insofar as the original tuples are concerned, as these can be recovered from them by a function combinator called *pairing*. Let $f:A\to B$ and $g:A\to C$ be functions with the same source type. We denote by $f\triangledown g:A\to B\times C$ the *pairing* of f and g, that is, the function defined by

$$(f \lor g) \ a = (f \ a, g \ a) \tag{2}$$

For instance, abbreviating *orders* to *o* for economy of notation:

and so on.

Since columns (dimension attributes) "are" functions, we can represent the schema of the (simplified) TPC-H data base by a diagram of function types, where we associate type variables to the ranges of the corresponding attributes:



For instance, $|l_orderkey| = K$ and so $l_{orderkey}$: $\#l \to K$. That the target type of $l_{orderkey}$ has to be the same as that of $o_{orderkey}$, i.e. K, is entailed by the clause $l_orderkey = o_orderkey$ in the query, otherwise this clause wouldn't type.

Besides K (for Key), the other type variables associated to each of the SQL concrete types present in the model [9] have the following meaning: P stands for Priority, D for Date, and S for Status. \mathbb{R} denotes the set of all real numbers, where prices are valuated.

One striking observation about the diagram above is that all types are ranges of values (and therefore finite) with the exception of \mathbb{R} , the target type of $l_{extendedprice}$ and $l_{quantity}$, corresponding to DECIMAL types in the SQL code. Attributes of this kind should not be regarded as *dimensions* but rather as *measures* because one can operate over them to produce consolidated data. The minimum algebraic structure for consolidation to take place is that of a *semiring*, offering a multiplicative (×) and an additive operator (+) with the expected properties (as in \mathbb{R}), notably $a \times (b+c) = a \times b + a \times c$ ensuring *linearity*.

Treating measures in the same way as dimensions would have the disadvantage that data consolidation has to be carried and reasoned out explicitly, involving lots of (nested) summations and quantifiers alike. To circumvent this problem, in the following section we generalize projection functions to matrices and move from the (functional) λ -calculus to the realm of matrix calculi, i.e., linear algebra.

3 LINEAR ALGEBRAIC ENCODING OF DATA

The typed LA approach to data representation [18] consists in representing projection functions by (Boolean) matrices, as follows:

 $^{^1}$ Note the slight abuse of notation, which also enables us to define the singleton type by 1 = {1}, useful in the sequel. As expected, 0 = { }.

²Since notation t_d makes it clear which table an attribute belongs to, we henceforth drop TPC-H prefixes "o_'" and "1_" f from attribute names.

and $[o_{orderdate}]$ is the matrix:

	1	2	3	4	5
1992-07-30	1	0	0	0	0
1994-09-30	0	1	0	0	0
1995-05-30	0	0	1	0	0
1995-10-06	0	0	0	1	0
1995-10-28	0	0	0	0	1

Likewise, it can be easily shown that

$$(b,c) \llbracket f \triangledown g \rrbracket \ a = (b \llbracket f \rrbracket \ a) \times (c \llbracket g \rrbracket \ a) \tag{5}$$

holds, since multiplication within $\{0,1\}$ implements logic conjunction

As in [18] we shall abuse of notation and (very conveniently, as we shall see) drop the parentheses from $[\![f]\!]$. This is consistent with writing $f \lor g$ to denote the operation above, which in fact corresponds to a well-known matrix operator. It is called the *Khatri-Rao product* [22] $M \lor N$ of two arbitrary matrices M and N and is defined index-wise by:

$$(b,c) (M \triangledown N) a = (b M a) \times (c N a)$$
 (6)

Thus $[o_{orderdate}] \lor [o_{orderpriority}]$ is the typed matrix:

1	2	3	4	5
1	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	1	0	0	0
0	0	1	0	0
0	0	0	0	0
0	0	0	1	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	1
	0 0 0 0 0 0 0	1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0	1 0 0 0 0 0 0 0 0 0 0 1 0 0 0 1 0 0 0 0 0 0 0 0 0	1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0

Matrices representing projection functions can be chained with each other (thus yielding *queries*, as we shall see) thanks to two main LA operations: *composition* and *converse*. Given two functions $g:A\to B$ and $f:B\to C$, their *composition* $f\cdot g$ is defined by

$$(f \cdot g) \ a = f \ (g \ a).$$

Matrix-wise, one can define $\llbracket g \rrbracket \cdot \llbracket f \rrbracket$ too, where we overload the dot to also mean matrix *multiplication*. In general, given two matrices $N:A \to B$ and $M:B \to C$, their *composition* (or matrix-matrix multiplication, MMM) is the matrix $M \cdot N$ defined by:

$$c(M \cdot N) a = \langle \Sigma b :: c M b \times b N a \rangle \tag{7}$$

Note how we extend the arrow notation used to type functions to also type arbitrary matrices, $M:A\to B$ meaning that matrix

A has *A*-many columns and *B*-many rows. Writing $B \stackrel{M}{\longleftarrow} A$ means the same as $A \stackrel{M}{\longrightarrow} B$ or as $M: A \rightarrow B$.

Wherever a matrix has one sole row it is said to be a *row vector* and we write e.g. $v:A \to 1$ to say this. Type $1 = \{1\}$ is the singleton type whose unique element is number 1, as we have already said. Clearly, type 1 is monomorphic.

Given a type A, there is a unique row vector wholly filled with 1s. This is termed "bang" [18] and denoted by $!: A \rightarrow 1$. Clearly,

$$M \circ ! = M = ! \circ M \tag{8}$$

cf. e.g.

$$\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 \times 1 & 3 \times 1 \\ 4 \times 1 & 5 \times 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$$

There is also a unique square matrix of type $A \to A$ whose diagonal is wholly filled with 1s and otherwise filled with 0s — it is termed the *identity* matrix and is denoted by $id: A \to A$. This is the unit of matrix composition:

$$M \cdot id = M = id \cdot M$$
.

The other capital LA operator for building analytical queries is *transposition*, or *converse* $M^{\circ}: B \to A$, given matrix $M: A \to B$. It is defined by swapping rows with columns: $a \ M^{\circ} \ b = b \ M \ a$. The following laws hold: $(M^{\circ})^{\circ} = M$ (idempotence) and $(M \cdot N)^{\circ} = N^{\circ} \cdot M^{\circ}$ (contravariance). Clearly, the converse of a *row* vector $v: A \to 1$ is a *column* vector $v^{\circ}: 1 \to A$. A row vector which is also a column vector is a *scalar*. For instance, given $!: A \to 1$, the scalar

$$1 \stackrel{!\cdot !^{\circ}}{-} 1$$
 counts the number of elements of the finite type A .

Vectors provide a very convenient LA representation of measure attributes, as we shall see shortly. Before this, we introduce another matrix product of capital importance — the so-called Hadamard product:

$$b(M \times N) a = (b M a) \times (b N a)$$
(9)

Typewise, M, N and $M \times N$ are all of the same type, for instance

$$\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 \times 1 & 3 \times 0 \\ 4 \times 0 & 5 \times 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$$

all of type $2 \to 2$. Finally, for vectors $v, u : A \to 1$,

$$v \lor u = v \times u \tag{10}$$

3.1 Measures

As already said, the meaning of *projection matrix* t_d , for d a dimension of table t, is given by

$$v \ t_d \ i = 1 \quad \Leftrightarrow \quad t[i].d = v \tag{11}$$

However, it does not seem a good idea to represent measure attribute $l_extendedprice$ by projection matrix $[\![l_{extendedprice}]\!]: \#l \to \mathbb{R}$, an infinite dimensional Boolean matrix.

Instead, LA offers the alternative of "internalizing" the real values of the dimension into a row vector, of type $\#l \to 1$ in the case of $l_extendedprice$:

³Following the infix notation usually adopted for relations (which are Boolean matrices), for instance $y \le x$, we write $y \ M \ x$ to denote the contents of the cell in matrix M addressed by row y and column x.

 $^{^4}$ Again we abbreviate *lineitem* by l, taking the converse so that the vector fits into one line of text.

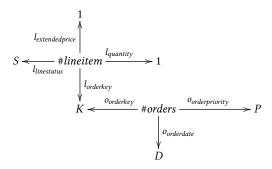


Figure 1: Typed LA diagram representing the (simplified) schema of the TPC-H database used as running example in the paper.

#
$$l \stackrel{(lextendedprice)^{\circ}}{<} 1 = \begin{bmatrix} 2124.32 \\ 16994.56 \\ 32735.70 \\ 1902.10 \\ 42846.80 \\ 35707.22 \\ 44064.48 \end{bmatrix}$$

Also note that, to stress the difference between *matrices* representing *dimension* attributes and *vectors* representing *measure* attributes, we adopt a superscripted notation for the latter.

In general, a measure column (attribute) m of a source tabe t (e.g. $m = l_extendedprice$) is represented by a row vector of type $\#t \to 1$ whose cells contain the corresponding numeric data. Superscripted notation $t^m: \#t \to 1$ distinguishes dimension matrices (e.g. t_d) from measure vectors (e.g. t^m). When needing to address the specific values of cells in measure vectors, instead of (say) 1 t^m i, for $i \in \#t$, we shall write t[i].m.

The typed LA diagram of Figure 1 depicts our definite model of the schema of the TPC-H database used as running example in this paper. Summarizing:

- measures are (typed) vectors
- dimensions are (typed) matrices.

Diagrams of this kind offer two main facilities. One is the ability to represent data in a fully typed way. For instance, the information contained in table *lineitem* with respect to quantities is captured by the column vector

$$v = (l_{orderkev} \lor l_{linestatus}) \cdot (l^{quantity})^{\circ}$$
(12)

of type $K \times S \longleftarrow 1$. To see what this means, we shall use the following rules interfacing index-free and index-wise matrix notation, where M is an arbitrary matrix and f and g functional

	$K \times S$	1								
	(2723,F)	0.0								
	(2723,O)	4.0				0	0	0	9	∞
	(3392,F)	1.0				1992-07-30	1994-09-30	1995-05-30	1995-10-06	1995-10-28
	(3392,O)	3.0				2-0	4-0	2-0	5-1	5-1
v =	(4354,F)	5.0	9	Q =		199	199	199	199	199
	(4354, O)	9.0		-	F	2	1	0	0	1
	(551,F)	0.0			o	0	1	1	1	1
	(551,O)	1.0			0	·	•	•	•	-
	(5699,F)	7.0								
	(5699,O)	0.0								
	(a)						(b)			

Figure 2: Vector $v: 1 \to (K \times S)$ (12) and query $Q: D \to S$ (16).

matrices:6

$$y(g^{\circ} \cdot M \cdot f)x = (g y)M(f x)$$

$$y(f \cdot M)x = \langle \sum z : y = f z : zMx \rangle$$

$$y(M \cdot f^{\circ})x = \langle \sum z : x = f z : yMz \rangle$$
(13)

Then (abbreviating $l_{orderkey}$ to lky and $l_{linestatus}$ to lst, for saving space):

$$v[k, s]$$
= { expanding shorthand notation $v[k, s]$ }
$$(k, s) v 1$$
= { (12) and (14) }
$$\langle \sum i : (k, s) = (lky \lor lst) i : i(l^{quantity})^{\circ} 1 \rangle$$
= { (2) and vector pointwise notation }
$$\langle \sum i : k = lky \ i \land s = lst \ i : l_{q}uantity[i] \rangle$$

Vector *v* is depicted in Figure 2a. The extra zeros correspond to combinations of values in the attributes that could not be found in the original data set. Strictly speaking, in a typed setting *every pair* of a cartesian product of types has to be taken into account.

The other feature is that data querying is performed simply by evaluating paths of LA diagrams, just by composing the arrows, which are matrices. *Converse* is of capital importance to build these queries because it enables paths that would otherwise not be available. For instance, the path

$$Q = l_{linestatus} \cdot l_{orderkey}^{\circ} \cdot o_{orderkey} \cdot o_{orderdate}^{\circ}$$
 (16)

is the matrix of type $S \leftarrow D$ depicted in Figure 2b, where $s \ Q \ d$ answers the query:

how many items are there with status s of orders issued on date d?

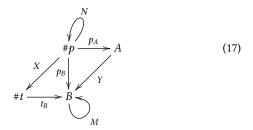
This query is illustrative of two patterns that invariably turn up in querying diagrams, as presented next.

 $[\]frac{5}{5}$ Measure columns are easy to spot in SQL data models by searching for DECIMAL SQL types.

⁶These rules, expressed in the style of the Eindhoven quantifier calculus, are convenient shorthands for the corresponding instances of matrix composition (7). See Appendix A for an explanation of this style and notation.

3.2 Joins and tabulations

Let us consider two data sources p and t with attributes A and B as shown in the following diagram:



Let $N: \#p \to \#p$ and $M: B \to B$ be two arbitraty matrices of their type. We refer to matrix $X: \#p \to \#t$ defined by

$$X = t_B^{\circ} \cdot M \cdot p_B \tag{18}$$

as a (generic) *join*, and to matrix $Y: A \rightarrow B$ defined by

$$Y = p_B \cdot N \cdot p_A^{\circ} \tag{19}$$

as a (generic) *tabulation*. Note (from their types) that joins are matrices typed by *indices* of two *different* tables, while tabulations are matrices typed by *attributes* of the *same* table.

Query Q (16) can be regarded as the composition of two tabulations in which N=id. These are called *counting* tabulations since the cells of the resulting matrices are all natural numbers. The corresponding situation in a *join*, M=id in (18), is known as an *equi-join*, e.g.

Equi-joins are always Boolean matrices that represent difunctional relations [24]. Their pointwise meaning is the expected (make $X=l_{orderkey}^{\circ} \cdot o_{orderkey}$)

$$j X i = l[j].orderkey = o[i].orderkey$$

recall rules (13) and (1). That is, equi-join matrices correspond to the *columnar-joins* of [1], compare the example

excerpted from [1] with the same device encoded as a difunctional matrix:

	1	2 0 0 0 1	3	4	5
1	0	0	0	0	1
2	1	0	1	0	0
3	0	0	0	0	0
4	0	1	0	0	0

Alternatively, (16) can be regarded as tabulation

$$Q = l_{linestatus} \cdot M \cdot o_{orderdate}^{\circ}$$

where M is the equi-join

$$M = l_{orderkey}^{\circ} \cdot o_{orderkey}.$$

Other forms of joins and tabulations are obtained from (18,19) by suitably instantiating matrices M and N. For instance, let $M = \langle \leqslant \rangle$ in (18), the matrix encoding some ordering on type B. Then $X = t_B^{\circ} \cdot \langle \leqslant \rangle \cdot p_B$ is the relational join

$$j X i = t[j].B \leq p[i].B$$

and so on and so forth.

Some explanation about the notation $\langle _ \rangle$ encoding predicates into Boolean matrices is needed. Given a *binary* predicate $p:B\times A\to \mathbb{B}$, denote by $\langle p\rangle:B\leftarrow A$ the Boolean matrix which encodes p, that is

$$b \langle p \rangle a = \text{if } p(b, a) \text{ then } 1 \text{ else } 0$$
 (20)

In case of a *unary* predicate $q:A\to\mathbb{B}, \langle q\rangle:1\leftarrow A$ is the Boolean vector such that:

$$1 \langle q \rangle a = \text{if } q \text{ a then } 1 \text{ else } 0$$
 (21)

We very often abbreviate the scalar 1 $\langle q \rangle$ *a* by q[a]. By (10), the conjunction of two predicates is represented to the Hadamard or Khatri-Rao product of the corresponding vectors: $\langle p \wedge q \rangle = \langle p \rangle \, \forall \langle q \rangle = \langle p \rangle \times \langle q \rangle$.

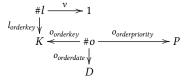
3.3 Group-by's

Group-by's are captured by tabulations (19) where matrix N "absorbs" the measure data involved.

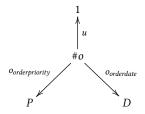
Let us see an example first, taken from the SQL query already given. The aggregation is performed over the product l_q uantity * l_e xtendedprice. In LA this corresponds to the pointwise product of the two measure vectors, a particular case of the matrix Hadamard product (9). So we can define

$$\#l \xrightarrow{v} 1 = l^{extendedprice} \times l^{quantity}$$
 (22)

and incorportate this vector in the part of the diagram involved in this query:



By defining vector $\#o \xrightarrow{u} 1 = v \cdot l_{orderkey}^{\circ} \cdot o_{orderkey}$ we get the "pairing wheel" (33):



As seen in appendix B, this can be expressed in several (isomorphic) ways. One is the immediate

$$D \times P \stackrel{(o_{orderdate} \nabla o_{orderpriority}) \cdot u^{\circ}}{\longleftarrow} 1 \tag{23}$$

- cf. (34) - yielding the tensor presentation⁷

$D \times P$	1
(1992-07-30,2-HIGH)	285793.8
(1994-09-30,2-HIGH)	16994.56
(1994-09-30,3-MEDIUM)	8497.28
(1995-10-06,2-HIGH)	195655.0
(1995-10-28,3-MEDIUM)	64353.3

for the sample data of our running example, removing zero entries for space economy. Now, by (35) in appendix B we can also express the same information by

$$D \stackrel{o_{orderdate} \nabla u}{\leftarrow} \#_{O} \stackrel{o_{orderpriority}}{\leftarrow} P$$

or by

$$P < o_{orderpriority}^{\circ} \vee u + o < o_{orderdate}^{\circ} D$$
 (24)

in this case yielding the same data in tabulated format:

Path (24) can be further re-written, by (38), into the tabulation format (19)

$$P \stackrel{o_{orderpriority}}{\longleftarrow} \#o \stackrel{id \nabla u}{\longleftarrow} \#o \stackrel{o_{orderdate}^{\circ}}{\longleftarrow} D$$

for N=id \forall u, the diagonal matrix representing vector u. That is, non-zeros can only be found in the diagonal of this square matrix, cf:

$$j (id \lor u) \ i = (j \ id \ i) \times u[i]$$

$$\equiv \qquad \{ \text{ Khatri-Rao (6)} \ ; id \ x = x \ \}$$

$$j (id \lor u) \ i = (j = i) \times u[i]$$

$$\equiv \qquad \{ \text{ pointwise LA rule (20)} \}$$

$$j (id \lor u) \ i = \text{if } j = i \text{ then } u[i] \text{ else } 0$$

Altogther, query

$$P \stackrel{Q}{\longleftarrow} D = o_{orderpriority} \cdot \underbrace{id \vee u}_{N} \cdot o_{orderdate}^{\circ}$$

evaluates in the following way: for each $p \in P$ and $d \in D$, it finds those records $i \in \#o$ with p and d as attributes and adds up prices

$$u[i], \text{ cf.}$$

$$p \ Q \ d$$

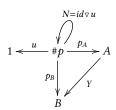
$$= \left\{ \text{ rule (14) etc } \right\}$$

$$\left\langle \sum j, i : p = o_{orderpriority} \ j \land d = o_{orderdate} \ i : j \ N \ i \right\}$$

$$= \left\{ \text{ since } j \ N \ i = \text{ if } j = i \text{ then } u[i] \text{ else } 0 \right\}$$

$$\left\langle \sum i : p = o_{orderpriority} \ i \land d = o_{orderdate} \ i : u[i] \right\rangle$$

In summary: given



there are three equivalent ways of expressing *tabulation Y*:

$$Y = p_B \cdot (id \nabla u) \cdot p_A^{\circ}$$

= $(p_B \nabla u) \cdot p_A^{\circ}$
= $p_B \cdot (p_A \nabla u)^{\circ}$. (25)

That is, a *measure* or *condition* captured by a vector u can be incorporated in a tabulation by multiplying (in the Khatri-Rao sense) any of its "legs" p_B or p_A by u. The second and third alternatives are handy in the sense that they allow for simplifying LA paths, as we shall soon see.

In the following sections we shall see how to evaluate typed LA paths on top of an LA kernel implementing the operations involved (Khatri-Rao, etc), making it possible to benchmark our approach. Relying on law (37) of the appendix,

$$P \cdot v^{\circ} = (v \nabla P) \cdot !^{\circ}$$

tabulations of type $B \leftarrow A$ are rendered in the type $B \times A \leftarrow 1$, that is, in the same columnar format as (23). In the LA kernel, $(\cdot!^{\circ})$ is implemented by a convenient operation that adds matrices rowwise. For instance, in the case of (23), the implemented LA script is equivalent to the expression:

$$(u \triangledown (o_{orderdate} \triangledown o_{orderpriority})) \cdot !^{\circ}$$

We detail the query conversion into a LA script in section 5, while implementation details are in section 6. Before that, we address an important feature of the LA approach proposed in this paper: incrementality (i.e. differential querying).

4 INCREMENTAL QUERYING

One advantage of the typed LA approach to data querying is *incrementality*. Suppose that, having evaluated a query Q(M) involving some given data encoded into matrix M, data envolves so that M becomes M'.

Because M and M' are matrices, we can calculate their difference, $\delta M = M' - M$. Then, instead of calculating Q(M') we can add to the already calculated Q(M) the corresponding *differential* query $Q'(\delta M)$, cf:

$$Q(M + \delta M) = Q(M) + Q'(\delta M)$$
(26)

 $^{^7\}mathrm{Recall}$ that matrices or vectors of higher-rank such as this are normally known as tensors [26].

The derivation of Q' can always be performed thanks to the *linearity* of the LA operators, notably of composition,

$$M \cdot (N + P) = M \cdot N + M \cdot P$$

 $(N + P) \cdot M = N \cdot M + P \cdot M$

of converse,

$$(M+N)^{\circ} = M^{\circ} + N^{\circ}$$

and of the Khatri-Rao product:

$$(M+N) \triangledown P = M \triangledown P + N \triangledown P$$

 $P \triangledown (M+N) = P \triangledown M + P \triangledown N$

In the case of queries involving several parameters, for instance three parameters such as in

 $(o_{orderdate} \triangledown o_{orderpriority}) \cdot u^{\circ}$

one performs accordingly for all the parameters.

The main advantage is that, assuming Q(M) already calculated in (26), $Q'(\delta M)$ costs far less than $Q(M + \delta M)$ due to the reduced size of δ *M* compared to *M*.

5 FROM SQL TO TYPED LA SCRIPTS

We are ready to explain the process of converting a realistic SQL query into its equivalent typed LA script. Query 3 in the TPC-H benchmark suite [9] is taken as example.

select

 $l_orderkey$, $o_orderdate$, $o_shippriority$; $sum (l_extendedprice * (1 - l_discount))$ as revenue

orders, customer, lineitem

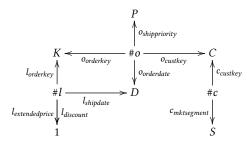
where

 $c_m ktsegment = 'MACHINERY'$ and c_c ustkey = o_c ustkey and l_0 rderkey = o_0 rderkey and $o_orderdate < date$ '1995-03-10' and lshipdate > date '1995-03-10' group by l_0 rderkey, o_0 rderdate, o_s hippriority

order by

revenue desc, oorderdate;

The type diagram given before needs to be extended with a new table - *customer*, abbreviated to c — with the attributes (projection functions) as shown in the diagram:



Looking at the group-by clause,

group by $l_orderkey$, $o_orderdate$, $o_shippriority$

we aim at building a path of type $K \times (D \times P) \leftarrow 1$ (column vector) or of any equivalent (isomorphic) type. We start by encoding the predicates specified in the where-clause of the query by Boolean vectors

$$u: \#l \to 1$$
$$v: \#o \to 1$$

defined by:

$$u[i] = l[i].shipdate > '1995-03-10'$$

 $v[j] = o[j].orderdate < '1995-03-10'$

recall (21). Data value MACHINERY is captured by the constant func-

tion $1 \xrightarrow{\text{MACHINERY}} S$ such that $\underline{\text{MACHINERY}} 1 = \text{MACHINERY}.$ Moreover, clauses

 $c_m kt segment = 'MACHINERY'$ and c_c ustkey = o_c ustkey

amount to Boolean path (vector)

$$z = 1 \frac{\text{MACHINERY}^{\circ}}{\leftarrow} S \stackrel{c_{mktsegment}}{\leftarrow} \#c \stackrel{c_{custkey}^{\circ}}{\leftarrow} C \stackrel{o_{custkey}}{\leftarrow} \#o$$

which counts how many customers exhibit the specified market segment:

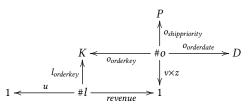
$$z[k] = \langle \sum i : sel i : 1 \rangle$$

where $sel i =$
 $c[i].custkey = o[k].custkey \land$
 $c[i].mktsegment = MACHINERY$

Finally, we also define *measure* vector $\#l \xrightarrow{revenue} 1$ in the expected way:

$$revenue = l_{extendedprice} \times (! - l_{discount})$$
 (27)

Altogether, we obtain:

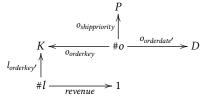


By (25) we define

$$o_{orderdate'} = o_{orderdate} \triangledown (v \times z)$$

 $l_{orderkey'} = l_{orderkey} \triangledown u$

we merge predicates with projections, which thus become partial functions:



Note how vector

$$r = revenue \cdot l_{orderkey'}^{\circ}$$

⁸Constant functions of this kind, also called *points*, are a standard algebraic way of describing specific data values.

computes measure *revenue* (filtered by u), for each key k:

$$r[k] = \langle \sum i : k = l_{orderkey} \land u[i] : revenue[i] \rangle$$

We can merge r with $o_{orderkey}$,

$$o_{orderkey'} = (r \triangledown o_{orderkey}^{\circ})^{\circ}$$

to obtain as outcome the expected 'pairing wheel':

$$P$$
 $O_{orderpriority}$
 $\#O$
 $O_{orderdate'}$
 V
 V
 V
 V
 V

Finally, putting everything together:

$$P \times D \xleftarrow{Q} K = H \cdot X$$

where

 $H = o_{orderpriority} \triangledown ((o_{orderdate} \triangledown v) \triangledown z)$
 $X = ((revenue \cdot l_{orderkey'}) \triangledown o_{orderkey}^{\circ})$

The actual implementation of Q over the developed LA kernel offers the output in isomorphic type $K \times (P \times D) \leftarrow 1$ using the rotating isomorphism α of appendix B. Written in a simple, single-assignment language defined for guiding the implementation process, Q reads like this:

```
v = filter( o_orderdate < '1995-03-10' )
B = krao( v, o_orderdate )
C = filter( c_mktsegment = 'MACHINERY' )
u = filter( l_shipdate > '1995-03-10' )
z = dot( C, o_custkey )
F = krao( l_orderkey, u )
G = krao( B, z )
H = krao( G, o_shippriority )
I = dot( H, l_orderkey )
J = krao( F, I )
K = lift( l_extendedprice * (1-l_discount) )
L = krao( J, K )
Q = sum( L )
```

Each equality corresponds to a step in the path extracted from the typed LA diagram. Type-correctness is ensured by the diagram itself. The simplified (abstract) BNF syntax of this kernel language is

```
\begin{split} dsl &:= (v = m)^* \\ m &:= v \mid \text{krao} (m, m') \mid \\ \text{dot} (m, m') \mid \text{filter} (p) \mid \\ \text{tr} (m) \mid \text{lift} (e) \mid \\ \text{sum} (m) \end{split}
```

where dsl is the axiom and v, m, p and e are non-terminals for variables, matrices, attribute-level predicates and attribute level expressions, respectively. The mapping between the combinators of this language and operations of the LA kernel is fairly obvious. Matrix composition $M \cdot N$ and Khatri-Rao product $M \cdot N$ are encoded by dot (M,N) and krao (M,N), respectively. filter (p) represents predicate p in the form of a vector, recall (21). sum (m) implements $m \cdot !^\circ$, cf. (37). Every cell-level operation is lifted to the corresponding matrix operation through lift. Finally, although converse is avoided in our scripts, it is available with syntax tr (m) (cf. transpose).

6 IMPLEMENTATION

Database environments place a strong demand for real-time query processing of always increasing volumes of data. The typed LA approach can take advantage of the powerful capabilities of current processor architectures, namely to operate on vectors and matrices. The data representation and the LA operation encoding are key factors to speedup query processing.

This section focuses on the key issues required to efficiently process and execute a query: the data representation and the key LA operations, grouped into a LA kernel. This LA kernel has been used to validate and evaluate the performance of the proposed LA approach using the TPC-H benchmark suite.

6.1 Data representation

As dimension projection matrices are very sparse, each column bearing a single non-zero value — recall e.g. (4) — the chosen representation format must be memory efficient and support efficient implementations of operations on such matrices. On the other hand, the row/column orientation is dictated by the LA operations themselves. Consider, for instance, matrix composition:

$$M \cdot N = \begin{bmatrix} A & B \end{bmatrix} \cdot \begin{bmatrix} C \\ D \end{bmatrix} = A \cdot C + B \cdot D$$
 (28)

This calls for a columnar representation of matrix M, and a rowwise representation of N. This contrasts with the Hadamard and Kronecker products, which require the same row and column-wise formats. Concerning the Khatri-Rao product, instead of

$$M \vee N = \left[\frac{A}{B} \right] \vee N = \left[\frac{A \vee N}{B \vee N} \right]$$
 (29)

one may rely on property

$$[A \mid B] \vee [C \mid D] = [A \vee C \mid B \vee D] \tag{30}$$

and stay with a columnar representation of both argument matrices. This matches with the LA approach proposed in this paper, which is inherently columnar. Thus, the adopted format for matrices is CSC, the *compressed sparse column* format [10]. It represents a sparse matrix by a set of three 1-D arrays:

- values, with the values of each non-zero element to be represented;
- row_index, with the row indices of each non-zero value;
- column_pointer, with size n_columns+1, that specifies the position of the non-zero elements of a given column in the other arrays, together with the number of non-zero values of each column; the 1st value is 0 and the column_pointer[column] is given by adding the number of non-zero values in [column-1] to the column_pointer[column-1].

Figure 3 illustrates how a matrix is compressed in CSC.

Sample matrix				Same matrix in CSC format					
	0	3	0]	1	ſ.	0	-	- 1	
)	0	0	7	values	[1	3	5	7]	
)	0	5	0	row_indices	ĺΟ	0	2	1]	,
)	0	0	0	column_pointers	[0	1	1	3	4]
	~	9	~]						

Figure 3: A sparse matrix in CSC format

Recall that each row of the projection matrix of an attribute corresponds to a data label. The number of rows is the number of distinct labels that the attribute has. One of the challenges of the LA approach to data representation is how to represent such *data labels*, that is, how to map attribute values to unique matrix indices.

To ensure that every label maps to a distinct matrix row, they are dynamically inserted in the hash table 9 , incrementally taking the first available integer value, in an AUTO INCREMENT fashion, as in other DBMSs. A double hashing approach is followed to improve performance, since it leads to a complexity of O(1) on both directions of the association. This strategy is applied independently for each attribute. However, for referential integrity to hold among all database tables, such structures are shared by both primary and foreign key attributes.

Primary key attribute values are always mapped first. Since these keys must be DISTINCT and NOT NULL, there is a *bijection* between row numbers (say #o) and key values (say #o), of type $\#o \to \#K$; for instance, consider the *orders* table of our running example:

	1	2	3	4	5
2723	0	0	0	1	0
3392	0	0	0	0	1
4354	0	1	0	0	0
551	0	0	1	0	0
5699	1	0	0	0	0

By respecting the order in which key attributes arise in tables, this bijection can actually be represented by the identity matrix¹⁰:

	1	2	3	4	5
5699	1	0	0	0	0
4354	0	1	0	0	0
551	0	0	1	0	0
2723	0	0	0	1	0
3392	0	0	0	0	1

Therefore, by mapping primary key attribute values first, in this fashion, one can *cancel out* their projection functions from LA scripts altogether, resulting in great performance gains wherever joins are evaluated. That is, the matrix representing an equi-join is the same as the one representing the foreign key.

By contrast, measures are represented by dense vectors.

6.2 Operations

A LA kernel of the main six algebraic operators was implemented containing three matrix products — composition, Hadamard and Khatri-Rao — plus attribute filtering, matrix aggregation and the *lift* operator.

Three key features lead to simplified and more efficient versions of the matrix product algorithms:

- matrices are sparse: only the non-zero values need to be accounted for;
- if measures are introduced at a later state, such sparse matrices are Boolean (no need to access cell-values, checking for non-zeros suffices);
- there is at most one element per column: this avoids iterating each row index array in the CSC structure, significantly reducing memory accesses.

These modified versions take advantage of the performance capabilities of the CSC format, leading to operations faster than those in high performance BLAS implementations. The following paragraphs address each of these key operations.

The *Hadamard product* is mainly used to combine filters, which are sparse Boolean vectors. It only needs to iterate through nonzeros of one vector and check if there is a non-zero in the same position in the other vector.

When applied to vectors, the Hadamard and *Khatri-Rao* products coincide, recall (10). For matrices $n \xrightarrow{M} m$ and $p \xrightarrow{N} q$, the algorithm used in $M \times N$ (Hadamard) is the foundation of $M \triangledown N$ (Khatri-Rao). For each column index i, both algorithms iterate through M (using index j, for instance), checking if there is an element in N at row (say) k. Then the new element to be inserted in the output matrix obeys the following rule: (jq+k) R i = j M i*k N i, where $R = M \triangledown N$.

Attribute filtering has two distinct implementations for dimensions and measures. In dimensions, this operation has two steps: (i) apply the filter predicate to the labels of the matrix, resulting in a sparse vector that will be used to filter the attribute, and (ii) multiply this sparse vector by the Boolean attribute, using matrix composition. The result of the operation is also a sparse vector, with as many columns as the original matrix.

Since measures are stored in dense vectors, attribute filtering here only requires one step: to iterate over the dense vector, filtering the values in a similar way as the first step for dimensions.

Matrix aggregation "folds" each row (resp. column) of a given matrix over a monoidal operator — e.g. a sum, a count, an average, a min or a max operation — by replacing the row (resp. column) by the aggregated value. Thus it converts a matrix $n \stackrel{M}{\longleftarrow} m$ into a vector $n \stackrel{v}{\longleftarrow} 1$ (resp. $1 \stackrel{v}{\longleftarrow} m$). That is, the result is a single sparse vector with as many rows (resp. columns) as the original number of matrix rows (resp. columns).

Finally, the *lift* operator promotes a matrix-cell-level operation to a matrix-operation. For instance, the Hadamard product $M \times N$ is the lifting of cell multiplication, M+N is that of cell addition, and so on. It is implemented in a "polymorphic" way, accepting operations of different arities.

7 VALIDATION AND EVALUATION

To guide and validate the LA implementation and to compare its performance with competitor solutions, several queries of the TPC-H benchmark suite were selected:

- query 6: useful to validate the implementation of the most basic filter operations, namely equality, relational, and between;
- query 3, richer than 6: also explores joins and group-by clauses;
- query 12 and 14: rich in filter and logical operations, such as CASE, LIKE, IN and NOT statements;
- query 11: contains sub-queries and filters after group-by (HAVING);
- query 4: explores semi-joins (EXISTS).

⁹The Glib Hash Table is used for this purpose [23].

 $^{^{10}\}mathrm{Compare}$ with a change of basis in vector space terminology.

Although TPC-H Composite Query-per-Hour is the standard TPC-H performance metric, this is not adequate for a fair comparative evaluation of the LA solution. The goal of this evaluation being to assess the efficiency of the LA approach to process SQL queries on several dataset sizes and only after the datasets are loaded into RAM, execution of multiple concurrent queries is not a suitable metric here. Measuring the efficiency of disk access was deferred to a later stage.

Instead, we opted to measure, for each query, the execution times for different dataset sizes (from 1 GiB to 64 GiB) and compare the measured figures of the LA approach with those of two open-source competitor database management systems: PostgreSQL and MySQL.

Before presenting benchmark figures, the next subsection details the testbed environment: the measurement methodology, the server specifications and how the DBMSs were configured.

7.1 Testbed environment

Trustworthy experimental results must be reproducible. When these results are code execution times, several runs give a clue on the execution stability. To minimize external unwanted interference, we always look for the faster times, but more than just one value. Our measurements are based on the 3-best runs out of 10, within a 5% max error interval; the best of these 3 is the recorded time.

Table 1 shows the key features of the testbed environment.

#PU-chips	2
Model	Intel Xeon E5-2683v4
Base clock freq	2.10 GHz (up to 3.00 GHz)
#PU cores	2 x 16 (2-way SMT support)
L1 cache	2 x 16 x 32 KiB
L2 cache	2 x 16 x 256 KiB
L3 cache	2 x 40 MiB
RAM	2 x 128 GiB (NUMA)
OS	CentOS 6.3
PostgreSQL	V. 10.2
MySQL	V. 5.7

Table 1: Testbed environment

To ensure fairness, each competitor DBMS was properly configured with support from their technical staff.

MySQL has three available storage engines: MyISAM and InnoDB are the most common ones, and MEMORY (HEAP). Since the goal in this comparative evaluation is to measure in-memory performance, the MEMORY (HEAP) alternative was chosen.

PostgreSQL has no storage engine equivalent to MEMORY (HEAP) in MySQL; a fair measure of execution times requires two runs to warm up the cache (DB-cache in RAM) before the 10 runs. The recommended size for the shared buffers is 1/4 of the RAM size, but this value was set to 3/4 to ensure that all queries data could fit in these buffers.¹¹

7.2 Result analysis

This section plots and discusses measured performance: execution times and memory usage for the sequential version (single-threaded) of six selected queries from the TPC-H benchmark suite, in each of the three competing environments, PostgreSQL, MySQL and the LA approach. It also compares execution times of the sequential and parallel versions of a single query (query 6), using up to all available cores in the server. MySQL was excluded from this comparison since it has no parallel version of a single query.

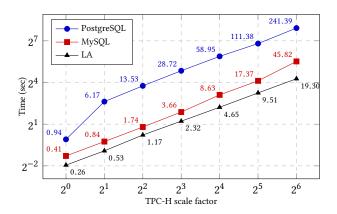


Figure 4: TPC-H Query 3

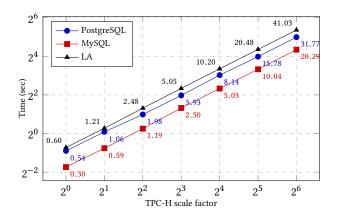


Figure 5: TPC-H Query 4

The plots of Figures 4 and 5 show the measured times for two queries, where the scale factors have a close relationship with the dataset size (in GiB). The standard deviation is not displayed in each data-point since it is so small that stay hidden in most cases.

The LA approach is clearly the fastest in query 3 for all dataset sizes, while the redundant filtering operations in query 4 take over 60% of the overall execution time, considerably degrading its performance (the same happens in query 12). The removal of this bottleneck is the subject of ongoing work [3].

Since the 6 tested queries display almost the same linear behaviour across all scale factor values, there is no need to display

 $^{^{11}\}mbox{Details}$ of the overall implementation including these configurations can be found in the repository of the project: https://github.com/Hubble83/LA_benchmarks.

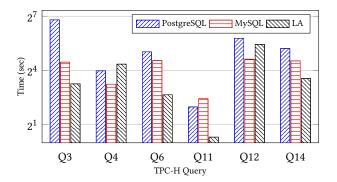


Figure 6: Execution times (scale factor: 2⁵)

their plots. Instead, Figure 6 compares the performance of all systems for each of the sequential versions of the six queries, for a scale factor 2^5 . The LA approach has proved to be faster than its competitors, in its current prototype version; it only has a lower performance in queries 4 and 12 due to redundant filtering (already explained).

The use of column-oriented tables in the LA approach (attribute oriented) gives advantages over the competitors, namely because:

- the columnar approach loads less data;
- it avoids operations that implicitly require a row orientation (namely, converse and matrix composition); thus the overall emphasis on the Khatri-Rao product;
- measures are incorporated as late as possible, taking advantage of using Boolean matrices as long as possible;
- it saves one matrix composition by equi-join, due to the smart encoding of primary keys.

Performance can be further improved if the available cores in the PU-chips are adequately used: while MySQL explores parallelism by concurrently processing multiple queries, PostgreSQL can use in parallel up to all available cores to process any query, and each kernel operation in LA approach can also use up to all available cores.

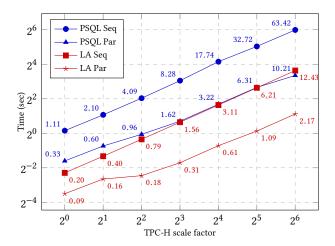


Figure 7: Sequential and Parallel Query 6

Figure 7 compares the single and multi-threaded versions of query 6 in the LA approach with the PostgreSQL versions. As expected, both parallel versions run consistently faster than the corresponding sequential ones, and the gain is lower when the dataset size is small.

However, the parallel efficiency in both systems is quite low: using 32 cores, only the larger scale factor managed to reach 6x speedup. PostgreSQL is configured by default to use only 2 workers per query (equivalent to 3 threads) and the scalability analysis of the LA approach has not been performed yet. Performance tuning will follow soon.

In some queries the parallel version of PostgreSQL has unstable behaviour for larger dataset sizes; for instance, in query 3 its query planner fails in such a way that the parallel execution times in datasets larger than 2^4 are longer than the sequential version.

Overall, the LA approach shows again its parallel superiority against PostgreSQL and both versions (sequential and parallel) have a consistent behaviour through all queries.

Code efficiency is also related to the required memory to run each query. The three systems follow different approaches: while PostgreSQL loads blocks of data in RAM, keeping those that it may need (disk cache), all data in MySQL is directly inserted in RAM (in these tests). The LA approach only places in RAM the attributes it will need.

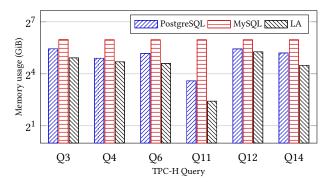


Figure 8: Memory usage (scale factor: 2⁵)

Figure 8 shows the maximum RAM space required for each of the sequential versions of the six queries in each DBMS (using the 3-worst case out of 10), for the scale factor 2^5 , and measured using detat.

This plot clearly shows that the LA approach is very efficient in managing the used RAM and these figures can be further improved.

8 CONCLUSIONS

This paper presents and validates a typed linear algebra (LA) approach to analytical data querying. The main novelty consists in representing data by matrices which are aggregated in a strongly typed manner, in contrast with the modest typing facilities of languages such as e.g. SQL, and with other LA techniques used in data analysis [2, 14], including incremental tensor analysis [25, 26], which are by and large untyped.

The theory behind the overall strategy, categorial linear algebra [16], enables a diagrammatic representation of both data (matrices,

the arrows of diagrams) and queries (paths in diagrams involving the required dimensions and measures) in a type-safe way. The laws of LA enable a number of path transformations which preserve correctness while improving efficiency. In particular, and because the overall approach is inherently columnar, the so-called Khatri-Rao matrix product [22] gains prominence among the other LA operations used in queries.

A prototype of these LA kernel operations was implemented, showing our strategy performing better than other standard technologies in the majority of the tests carried out. However, there is still much room for improvement and consolidation before we can say we have a winning approach, as indicated below in the ongoing and future work section.

8.1 Related work

This work is a follow up of previous research in adopting typed linear algebra and its diagram-orientation to data analysis [17, 18]. Although it may seem at first sight that this approach bears some resemblance to graph databases [28], nodes in our diagrams represent data types and not individual data items.

There is much work on scalable execution models for data bases, namely in the *columnar* trend. Kernert *et al* [14, 15] present an approach to integrate sparse matrices in column-oriented in-memory database systems. This includes API-level support for performing elementary linear algebra operations.

Qin and Rusu [21] study linear algebra scalability for big model analytics. The emphasis is on gaining efficiency in dot-product joins, a primary operation for training linear models. This contrasts with the current paper, where joins are not the main problem, the focus being on the Khatr-Rao product, an useful operation often absent from the LA kernels described in the literature.

Finally note that the projection matrices used in this research also intersect with the bitmaps used to represent table indexes. Wu *et al* [29] propose a way to overtake the large memory demands of the IBM's Model 204 bitmap representation [19]. However, our sparse representation inherently compresses the data, and so such efforts are not required.

8.2 Ongoing and future work

This paper validates and evaluates a proof of concept framework for analytical querying on a strong formal basis, with efficiency gains. However, to convert this proof of concept into an usable software toolset, additional effort is required. The implementation of the framework architecture shown in Figure 9 is on-going work [3, 4].

Moreover, there is still room to improve the parallel efficiency of the overall design and implementation:

- to evaluate the scalability of the LA operations to define the max number of sustainable concurrent cores per query without degrading performance;
- to explore matrix splitting into blocks of columns to allow operations in micro batching or even streaming [5], if the blocks are short enough; this not only will speedup most LA operations with vectors and matrices, but also require less memory usage;

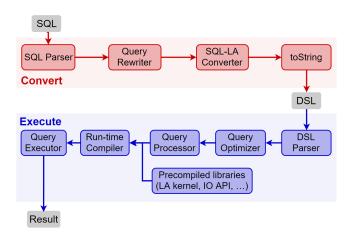


Figure 9: The framework architecture

- to dynamically explore parallelism in the execution of independent pipeline stages for any heterogeneous server architecture, with or without computing accelerators (e.g., GPU, FPGA) [7, 20];
- to implement the LA kernels in distributed memory; by splitting or replicating matrices across servers, attributes larger than single server memories can be queried.

In another direction, we would like to study the practical impact of incrementality (section 4) on efficient data cube updating, combining the results of this paper with those of reference [18].

ACKNOWLEDGMENTS

This work was supported by the EU Framework Programme for Research and Innovation 2014-2020, Horizon 2020, under grant agreement No. 732051 (CloudDB Appliance project) and by Project Search-ON2 (NORTE-07-0162- FEDER-000086), co-funded by the North Portugal Regional Operational Programme (ON.2 - O Novo Norte), under the National Strategic Reference Framework, through the European Regional Development Fund.

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APPENDIX

Two appendices are included: one describing the typed linear algebra pontwise notation adopted in the paper and the other about the so-called "pairing wheel" rule, which can be used to improve the performance of queries involving the Khatri-Rao product.

A TYPED LINEAR ALGEBRA POINTWISE NOTATION

The notation adopted for expressing index-wise matrix expressions is known as Eindhoven quantifier calculus, see e.g. [6]. In this notation, a quantified expression is always of the form

$$\langle \sum x : R : T \rangle$$

where *R* is a predicate specifying the *range* of the quantification and *T* is a numeric term. In case $T = B \times M$ where Boolean $B = \langle P \rangle$ encodes predicate P — recall (20) — we have the *trading rule*:

$$\langle \sum x : R : \langle P \rangle \times M \rangle = \langle \sum x : R \wedge P : M \rangle$$
 (31)

Another (very) useful law of this calculus is known as the $\emph{one-point}$ rule

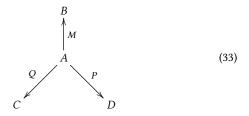
$$\langle \sum k : k = e : T \rangle = T[k := e] \tag{32}$$

where expression T[k := e] denotes T with every occurrence of k replaced by e.

The LA (index-free) properties given in this paper, namely (13, 14, 15) can be derived from (31) and (32).

B THE "PAIRING WHEEL" RULE

In typed LA, the information captured by the three matrices M, P and Q in



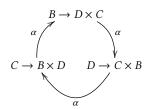
can be aggregated in several ways, namely

$$B \xrightarrow{(P \vee Q) \cdot M^{\circ}} D \times C$$

$$D \xrightarrow{(Q \vee M) \cdot P^{\circ}} C \times E$$

$$C \xrightarrow{(M \vee P) \cdot Q^{\circ}} B \times C$$

all isomorphic to each other:



The rotation among matrices and types justifies the name "pairing wheel" given to (33). Isomorphism α holds in the sense that every cell of one of the aggregates is uniquely represented by another

cell in any other aggregate, for instance:

$$(d, c) ((P \lor Q) \cdot M^{\circ}) b$$

$$= \{ \text{ composition ; Khatri-Rao } \}$$

$$\langle \sum a :: (d P a \times c Q a) \times a M^{\circ} b \rangle$$

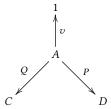
$$= \{ \text{ converse; } \times \text{ is associative and commutative } \}$$

$$\langle \sum a :: (c Q a \times b M a) \times a P^{\circ} d \rangle$$

$$= \{ \text{ composition ; Khatri-Rao } \}$$

$$(c, b) ((Q \lor M) \cdot P^{\circ}) d$$

Thus: α $((P \lor Q) \cdot M^{\circ}) = (Q \lor M) \cdot P^{\circ}$. In the special case of one of the matrices being a vector, say M = v in



we get that

$$1 \xrightarrow{(P \vee Q) \cdot \nu^{\circ}} D \times C \tag{34}$$

bears the same information as

$$C \leftarrow \frac{(Q \nabla v) \cdot P^{\circ}}{D}$$
 (35)

which is the converse of

$$C \xrightarrow{(v \nabla P) \cdot Q^{\circ}} D$$

Thus the rule:

$$P \cdot (Q \triangledown \nu)^{\circ} = (\nu \triangledown P) \cdot Q^{\circ}$$
(36)

For Q = ! in (36) we have, via (8),

$$P \cdot v^{\circ} = (v \, \forall \, P) \cdot !^{\circ} \tag{37}$$

Moreover, for Q = id in (36), we get

$$P \cdot (id \nabla v)^{\circ} = v \nabla P$$

Finally, since $id \nabla v$ is diagonal and therefore symmetric,

$$P \cdot (id \nabla v) = v \nabla P \tag{38}$$

and, of course, for P = id,

$$id \triangledown v = v \triangledown id \tag{39}$$