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Categorifying cardinal arithmetic

MAA MathFest, August 4, 2018



Goal: prove $a \times (b+c) = (a \times b) + (a \times c)$ for any natural numbers a, b, and c.



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- Step I: categorification
- Step 2: the Yoneda lemma
- Step 3: representability
- Step 4: the proof



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- Step 1: categorification
- Step 2: the Yoneda lemma
- Step 3: representability
- Step 4: the proof
- Epilogue: what was the point of that?





Step I: categorification

The idea of categorification



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as expressing some deeper truth about mathematical structures.

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Natural numbers a, b, and c encode the sizes of finite sets A, B, and C.

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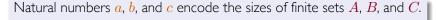
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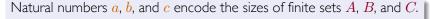
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Categorification: the truth behind a = b is $A \cong B$.





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Q: What is the deeper meaning of the symbols "+" and " \times "?

Categorifying +



Q: If b := |B| and c := |C| what set has b + c elements?

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A: The disjoint union B+C is a set with b+c elements.

$$B = \left\{ \begin{array}{c} \sharp \\ \flat \\ \natural \end{array} \right\} , \qquad C = \left\{ \begin{array}{c} \spadesuit & \heartsuit \\ \diamondsuit & \clubsuit \end{array} \right\} , \qquad B + C = \left\{ \begin{array}{c} \sharp & \flat & \spadesuit & \heartsuit \\ \natural & \diamondsuit & \clubsuit \end{array} \right\}$$

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$$b + c \coloneqq |B + C|$$

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A: The cartesian product $A \times B$ is a set with $a \times b$ elements.

$$A = \left\{ \begin{array}{cc} * & \star \end{array} \right\} , \qquad B = \left\{ \begin{array}{c} \sharp \\ \flat \\ \natural \end{array} \right\} , \qquad A \times B = \left\{ \begin{array}{cc} (*, \sharp) & (\star, \sharp) \\ (*, \flat) & (\star, \flat) \\ (*, \natural) & (\star, \natural) \end{array} \right\}$$

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In summary:

• Natural numbers define cardinalities: there are sets A, B, and C so that a := |A|, b := |B|, and c := |C|.



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A: It means that the sets $A \times (B+C)$ and $(A \times B) + (A \times C)$ are isomorphic!

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(*,,) & (\star,,) \\
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2

Step 2: the Yoneda lemma



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• for all sets X, the sets of functions

$$\operatorname{Fun}(A,X) := \{\phi \colon A \to X\} \quad \text{and} \quad \operatorname{Fun}(B,X) := \{\psi \colon B \to X\}$$
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Proof (\Leftarrow) : Suppose $\operatorname{Fun}(A,X)\cong\operatorname{Fun}(B,X)$ for all X. Taking X=A and X=B, we use the bijections:

$$\operatorname{Fun}(A,A) \quad \cong \quad \operatorname{Fun}(B,A) \qquad \quad \operatorname{Fun}(A,B) \quad \cong \quad \operatorname{Fun}(B,B)$$



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to define functions $\psi \colon B \to A$ and $\phi \colon A \to B$.



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Exercise: Use the "naturality" of $\operatorname{Fun}(A,X) \cong \operatorname{Fun}(B,X)$ to prove that $\psi \circ \phi = \operatorname{id}_A$ and $\phi \circ \psi = \operatorname{id}_B$.

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Step 3: representability



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A: For each $b \in B$, we need to specify $f(b) \in X$, and for each $c \in C$, we need to specify $f(c) \in X$. So the function $f \colon B + C \to X$ is determined by two functions $f_B \colon B \to X$ and $f_C \colon C \to X$.



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By "pairing"
$$\begin{array}{ccc} \operatorname{Fun}(B+C,X) &\cong & \operatorname{Fun}(B,X) \times \operatorname{Fun}(C,X) \\ & & & & \\ \downarrow & & & \\ f & & & \\ \end{array}$$



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A: For each $b \in B$ and $a \in A$, we need to specify an element $f(a,b) \in X$. Thus, for each $b \in B$, we need to specify a function $f(-,b) \colon A \to X$ sending a to f(a,b).



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By "currying"
$$\begin{array}{cccc} \operatorname{Fun}(A\times B,X) &\cong & \operatorname{Fun}(B,\operatorname{Fun}(A,X)) \\ & & & & & \\ & & & & \\ f\colon A\times B\to X & \leftrightsquigarrow & f\colon B\to \operatorname{Fun}(A,X) \end{array}$$

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By the Yoneda lemma:

$$\begin{array}{l} \text{Step 2 summary: To prove } A \times (B+C) \cong (A \times B) + (A \times C) \\ \Rightarrow \text{ we'll instead define a "natural" isomorphism} \\ \text{Fun}(A \times (B+C), X) \cong \text{Fun}((A \times B) + (A \times C), X). \end{array}$$

Summary of Steps 1, 2, and 3



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By representability:

Step 3 summary:

• $\operatorname{Fun}(B+C,X) \cong \operatorname{Fun}(B,X) \times \operatorname{Fun}(C,X)$ by "pairing"

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By representability:

Step 3 summary:

- $\operatorname{Fun}(B+C,X) \cong \operatorname{Fun}(B,X) \times \operatorname{Fun}(C,X)$ by "pairing" and
- $\operatorname{Fun}(A \times B, X) \cong \operatorname{Fun}(B, \operatorname{Fun}(A, X))$ by "currying."





Step 4: the proof



Theorem. For any natural numbers a, b, and c,

$$a \times (b+c) = (a \times b) + (a \times c).$$

Proof:



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5

Epilogue: what was the point of that?



Note we didn't actually need the sets A, B, and C to be finite.



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Exercise: Find a similar proof for other identities of cardinal arithmetic:

$$\alpha^{\beta+\gamma}=\alpha^{\beta}\times\alpha^{\gamma}\quad \text{ and }\quad (\alpha^{\beta})^{\gamma}=\alpha^{\beta\times\gamma}=(\alpha^{\gamma})^{\beta}.$$

Generalization to other mathematical contexts



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• For nice topological spaces X, Y, Z,

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• For abelian groups A, B, C,

$$A \otimes_{\mathbb{Z}} (B \oplus C) \cong (A \otimes_{\mathbb{Z}} B) \oplus (A \otimes_{\mathbb{Z}} C).$$

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Thank you!