A 2CAT-INSPIRED MODEL STRUCTURE FOR DOUBLE CATEGORIES

LYNE MOSER, MARU SARAZOLA, AND PAULA VERDUGO

Abstract. We construct a model structure on the category DblCat of double categories and double functors. Unlike previous model structures for double categories, it recovers the homotopy theory of 2-categories through the horizontal embedding functor $\mathbb{H}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$, which is both left and right Quillen, and homotopically fully faithful. Furthermore, we show that Lack's model structure on 2Cat is right-induced along $\mathbb{H}$ from our model structure on DblCat; thus, a 2-functor F is a biequivalence (resp. fibration) if and only if $\mathbb{H}F$ is a weak equivalence (resp. fibration) in DblCat. In addition, we obtain a 2Cat-enrichment of our model structure on DblCat, by using a variant of the Gray tensor product.

Finally, under certain conditions, a characterization of our weak equivalences allows us to prove a Whitehead theorem for double categories, which retrieves the Whitehead theorem for 2-categories as a special case.

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1. Introduction

In category theory as well as homotopy theory, we strive to find the correct notion of "sameness", often with a specific context or perspective in mind. When working with categories themselves, it is commonly agreed that having an isomorphism between categories is much too strong a requirement, and we instead concur that the right condition to demand is the existence of an *equivalence* of categories.

There are many ways one can justify this in practice, but, at heart, it is due to the fact that the category Cat of categories and functors actually forms a 2-category, with 2-cells given by the natural transformations. Therefore, instead of asking that a functor $F\colon \mathcal{A} \to \mathcal{B}$ has an inverse $G\colon \mathcal{B} \to \mathcal{A}$ such that their composites are *equal* to the identities, it is more natural to ask for the existence of *natural isomorphisms* $\mathrm{id} \cong FG$ and $GF \cong \mathrm{id}$. In particular, this characterizes F as a functor that is essentially surjective on objects (i.e., surjective up to an isomorphism) and fully faithful on morphisms.

Ever since Quillen's seminal work [17], and even more so in the last two decades, we have come to expect that any reasonable notion of equivalence in a category should lend itself to defining the class of weak equivalences of a model structure. This is in fact the case of the categorical equivalences: the category Cat can be endowed with a model structure, called the *canonical model structure*, in which the weak equivalences are precisely the equivalences of categories.

Going one dimension up and focusing on 2-categories, the 2-functors themselves now form a 2-category, with higher cells given by the pseudo natural transformations, and the so-called modifications between them. We can then define a 2-functor $F\colon \mathcal{A} \to \mathcal{B}$ to be a *biequivalence* if it has an inverse $G\colon \mathcal{B} \to \mathcal{A}$ together with pseudo natural equivalences $\mathrm{id} \simeq FG$ and $GF \simeq \mathrm{id}$, i.e., equivalences in the corresponding 2-categories of 2-dimensional functors. Note that this inverse G is in general a pseudo functor rather than a 2-functor. Furthermore, a Whitehead theorem for 2-categories [10, Theorem 7.4.1] is available, and characterizes the biequivalences as the 2-functors that are bi-essentially surjective on objects (i.e., surjective up to an equivalence in the target 2-category), essentially full on morphisms (i.e., full up to an invertible 2-cell), and fully faithful on 2-cells.

As in the case of the equivalences of categories, the biequivalences of 2-categories are part of the data of a model structure. Indeed, in [11, 12], Lack defines a model structure on the category 2Cat of 2-categories and 2-functors in which the weak equivalences are precisely the biequivalences; we henceforth refer to it as the *Lack model structure*. In particular, the canonical homotopy theory of categories embeds reflectively in this homotopy theory of 2-categories.

In this paper, we consider another type of 2-dimensional objects, called *double categories*, which have both horizontal and vertical morphisms between pairs of objects, related by 2-dimensional cells called *squares*. These are more structured than 2-categories, in the sense that a 2-category $\mathcal{A}$ can be seen as a horizontal double category $\mathbb{H}\mathcal{A}$ with only trivial vertical morphisms. As a consequence, the study of various notions of 2-category theory benefits from a passage to double categories. For example, a 2-limit of a 2-functor F does not coincide with a 2-terminal object in the slice 2-category of cones, as shown in [2, Counterexample 2.12]. However, by considering the 2-functor F as a horizontal double functor $\mathbb{H}F$, we can see that a 2-limit of F is precisely a double terminal object in the slice double category of cones over $\mathbb{H}F$; see [6, §4.2] and [5, Theorem 5.6.5].

This horizontal embedding of 2-categories into double categories is fully faithful, and we expect to have a homotopy theory of double categories that contains that of 2-categories; constructing such a homotopy theory is the aim of this paper.

The idea of defining a model structure on the category of double categories is scarcely a new one. In [3], Fiore, Paoli, and Pronk construct several model structures on the category DblCat of double categories and double functors, but the horizontal embedding of 2-categories does not induce a Quillen pair between Lack's model structure and any of these model structures; this follows from Lemma 6.7. Some intuition is provided by the fact that their categorical model structures on DblCat are constructed from the canonical model structure on Cat. As a result, the weak equivalences in each of these model structures induce two equivalences of categories: one between the categories of objects and horizontal morphisms, and one between the categories of vertical morphisms and squares. However, a biequivalence between 2-categories does not generally induce an equivalence between the underlying categories. Therefore, the horizontal embedding of 2Cat into DblCat will not preserve weak equivalences.

In order to remedy this loss of higher data, we can instead extract from a double category $\mathbb{A}$ two 2-categories whose underlying categories are precisely the ones above. One is given by the underlying horizontal 2-category $\mathbf{H}\mathbb{A}$ of objects, horizontal morphisms, and squares with only trivial boundaries, and the other is given by the 2-category $\mathcal{V}\mathbb{A}$ of vertical morphisms, squares, and 2-cells as described in Definition 2.10. We then get a notion of *double biequivalence*: a double functor $F\colon \mathbb{A} \to \mathbb{B}$ such that the two induced 2-functors $\mathbf{H}F$ and $\mathcal{V}F$ are biequivalences in 2Cat.

A double biequivalence can alternatively be characterized as a double functor that is horizontally bi-essentially surjective on objects, essentially full on horizontal morphisms, bi-essentially surjective on vertical morphisms, and fully faithful on squares. From this characterization and the fact that $\mathbf{H}\mathbb{H} = \mathrm{id}$, we directly get that a 2-functor $F\colon \mathcal{A} \to \mathcal{B}$ is a biequivalence if and only if $\mathbb{H}F\colon \mathbb{H}\mathcal{A} \to \mathbb{H}\mathcal{B}$ is a double biequivalence. This can be seen as a first step towards showing that the homotopy theory of 2-categories sits inside that of double categories.

More surprisingly, double biequivalences are similarly well-behaved with respect to other double categories typically constructed from a 2-category $\mathcal{A}$, and which have $\mathcal{A}$ itself as their underlying horizontal 2-category; namely, the double category of quintets $\mathbb{Q}\mathcal{A}$ (see [5, §3.1.4]) and the double category of adjunctions $\mathbb{A}\mathrm{dj}\mathcal{A}$ (see [5, §3.1.5]). Indeed, one can prove that a 2-functor $F\colon \mathcal{A} \to \mathcal{B}$ is a biequivalence if and only if the induced double functor $\mathbb{Q}F\colon \mathbb{Q}\mathcal{A} \to \mathbb{Q}\mathcal{B}$ is a double biequivalence, and similarly for $\mathbb{A}\mathrm{dj}F$.

As further evidence supporting this notion of weak equivalence for double functors, we obtain a Whitehead theorem for double categories under an additional hypothesis.

Theorem 10.14 (Whitehead Theorem for double categories)**.** *Let $\mathbb{A}$ and $\mathbb{B}$ be double categories such that $\mathbb{A}$ is weakly horizontally invariant or $\mathbb{B}$ has only trivial vertical morphisms. Then a double functor $F\colon \mathbb{A} \to \mathbb{B}$ is a double biequivalence if and only if there exists a pseudo double functor $G\colon \mathbb{A} \to \mathbb{B}$ together with horizontal pseudo natural equivalences* $\mathrm{id} \simeq GF$ *and* $FG \simeq \mathrm{id}$*.*

Our first main result, Theorem 3.16, provides a model structure on the category of double categories in which the weak equivalences are precisely the double biequivalences, and it is obtained as a right-induced model structure along $(\mathbf{H}, \mathcal{V})$ from two copies of the Lack model structure on 2Cat × 2Cat.

Theorem 3.16. *Consider the adjunction*

$$\begin{tikzcd} 2\mathrm{Cat} \times 2\mathrm{Cat} \arrow[r, bend left, "\mathbb{H} \sqcup \mathbb{L}"] \arrow[r, phantom, "\perp"] & \mathrm{DblCat} \arrow[l, bend left, "(\mathbf{H}, \mathcal{V})"] \end{tikzcd},$$

where each copy of 2Cat *is endowed with the Lack model structure. Then the right-induced model structure on* DblCat *exists. In particular, a double functor is a weak equivalence (resp. fibration) in this model structure if and only if it is a double biequivalence (resp. double fibration).*

Since the Lack model structure on 2Cat is cofibrantly generated, we also get a cofibrantly generated model structure on DblCat from this construction. Moreover, every double category is fibrant, since all objects are fibrant in 2Cat.

This model structure on DblCat is defined in such a way that it is compatible with the model structure on 2Cat. More precisely, the horizontal embedding $\mathbb{H}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$ is both a left and a right Quillen functor, and it is homotopically fully faithful. This implies that the functor $\mathbb{H}$ embeds the homotopy theory of 2-categories in that of double categories in a reflective and coreflective way. Furthermore, the Lack model structure can be shown to be right-induced along $\mathbb{H}$ from our model structure on DblCat.

Theorem 4.7. *The Lack model structure on* 2Cat *is right-induced from the adjunction*

$$\begin{tikzcd} \mathrm{DblCat} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & 2\mathrm{Cat} \arrow[l, bend left, "\mathbb{H}"] \end{tikzcd},$$

where DblCat *is endowed with the model structure of Theorem 3.16.*

As a consequence, the functor $\mathbb{H}$ preserves cofibrations, and creates weak equivalences and fibrations.

Having established a first compatibility of our model structure on DblCat with the Lack model structure on 2Cat, we want to further investigate their relation. Lack shows in [11] that the model structure on 2Cat is monoidal with respect to the Gray tensor product. In the double categorical setting, there is an analogous monoidal structure on DblCat given by the Gray tensor product constructed by Böhm in [1]. However, this monoidal structure is not compatible with our model structure on DblCat (see Remark 5.8), since it treats the vertical and horizontal directions symmetrically, while our model structure does not. Nevertheless, restricting this Gray tensor product for double categories in one of the variables to 2Cat via $\mathbb{H}$ removes this symmetry and provides an enrichment of DblCat over 2Cat that is compatible with our model structure. More precisely, this enrichment is given by the hom-2-categories of double functors, pseudo horizontal natural transformations, and modifications between them.

Theorem 5.11. *The model structure on* DblCat *of Theorem 3.16 is a* 2Cat*-enriched model structure, where the enrichment is given by* $[-,-]_{\mathrm{ps}}$*.*

The fact that pseudo horizontal natural transformations play a key role was to be expected, since they are the type of transformations that detect our weak equivalences, as established in our version of the Whitehead theorem above.

Just as the composition in 2-categories can be weakened to obtain the notion of *bicategories*, double categories also admit a weaker version, called *weak double categories*, where horizontal composition is associative and unital up to vertically invertible squares. A bicategory can then be seen as a horizontal weak double category. In [12], Lack shows that the category of bicategories and strict functors admits a model structure, in which the weak equivalences are again the biequivalences. Moreover, the full embedding of 2-categories into bicategories induces a Quillen equivalence. In a forthcoming paper [16], we endow the category of weak double categories and strict double functors with a model structure, whose weak equivalences are the double biequivalences. We prove that the main results of this paper also hold in the weaker setting; notably, the homotopy theory of bicategories is embedded in that of weak double categories in a reflective and coreflective way, and the Lack model structure for bicategories is right-induced from the model structure for weak double categories. Moreover, the full inclusion of double categories into that of weak double categories gives a Quillen equivalence.

1.1. **Outline.** This paper is divided in three parts. The first part, comprising Sections 2 to 6, contains all of the key information about our model structure. This includes its construction, together with characterizations of the relevant classes of maps, as well as its enrichment over 2Cat. To facilitate the reading of this paper, we postpone lengthy technical proofs to the second part, containing Sections 7 to 9. The reader can be assured that no important results are introduced in the second part, and thus it can be omitted, if they are willing to trust the claims made in the first part. The third part is self-contained (except for the use of Definition 3.5, which introduces double biequivalences), and addresses the statement and proof of a Whitehead theorem for double categories.

Let us provide a more detailed outline of the paper. In Section 2, we introduce the double categorical notions that will be used. In particular, we present the different adjunctions of interest between 2Cat and DblCat. After recalling the important features of the Lack model structure on 2Cat at the beginning of Section 3, we use it to construct a model structure on DblCat which is right-induced from $2\mathrm{Cat} \times 2\mathrm{Cat}$. A proof of the existence of this model structure is provided here, with the exception of the construction of a path object for double categories, which is postponed to the second part. We also state a description of the weak equivalences and (trivial) fibrations, as well as of the fibrant and cofibrant objects. In Section 4, we establish the different Quillen pairs between 2Cat and DblCat, and show that the Lack model structure is in fact right-induced from our model structure. We also give a Quillen pair between Cat and DblCat. In analogy to the Lack model structure being monoidal, we examine in Section 5 the enrichment of our model structure on DblCat over itself and over 2Cat. We show that it is not monoidal with respect to either the cartesian or Gray tensor product on DblCat. However, we provide a 2Cat-enrichment on DblCat that makes our model structure into a 2Cat-enriched model structure.

Now that the model structure on DblCat and its relation with 2Cat are established, in Section 6 we turn to a comparison with previously existing model structures on DblCat, defined by Fiore, Paoli, and Pronk in [3]. We show that the adjunction given by the identity functors on DblCat is not a Quillen pair between our model structure and any of the model structures of [3].

In the second part, we first complete the proof of the main result, Theorem 3.16, by constructing a path object in double categories in Section 7. We then prove in Section 8 the characterizations of weak equivalences and fibrations stated in Section 3, which are given by the notions of *double biequivalences* and *double fibrations*. Finally, in Section 9, we provide a better description of the generating (trivial) cofibrations, and give a characterization of the cofibrations in our model structure. As a corollary, we get the characterization of cofibrant objects stated in Section 3.

As mentioned above, the last part addresses the Whitehead theorem for double categories, Theorem 10.14.

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Part I. The model structure

2. Double categorical preliminaries

In this section, we recall the basic notions about double categories, and also introduce non-standard definitions and terminology that will be used throughout the paper. The reader familiar with double categories may wish to jump directly to Definition 2.10.

Definition 2.1. A **double category** $\mathbb{A}$ consists of

(i) objects A, B, C, ...,
(ii) horizontal morphisms $a\colon A \to B$ with composition denoted by $b \circ a$ or ba,
(iii) vertical morphisms $u\colon A \to\!\!\!\!\bullet\; A'$ with composition denoted by $v \bullet u$ or vu,
(iv) squares (or cells) $\alpha\colon (u \,{}^{a}_{b}\, v)$ of the form

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "u"', "\bullet" marking] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "v", "\bullet" marking] \\ A' \arrow[r, "b"'] & B' \end{tikzcd}$$

with both horizontal composition along their vertical boundaries and vertical composition along their horizontal boundaries, and

(v) horizontal identities $\mathrm{id}_A\colon A \to A$ and vertical identities $e_A\colon A \to\!\!\!\!\bullet\; A$ for each object A, vertical identity squares $e_a\colon (\mathrm{id}_A \,{}^{a}_{a}\, \mathrm{id}_B)$ for each horizontal morphism $a\colon A \to B$, horizontal identity squares $\mathrm{id}_u\colon (u \,{}^{\mathrm{id}_A}_{\mathrm{id}_{A'}}\, u)$ for each vertical morphism $u\colon A \to\!\!\!\!\bullet\; A'$, and identity squares $\square_A = \mathrm{id}_{e_A} = e_{\mathrm{id}_A}$ for each object A,

such that all compositions are unital and associative, and such that the horizontal and vertical compositions of squares satisfy the interchange law.

Definition 2.2. Let $\mathbb{A}$ and $\mathbb{B}$ be double categories. A **double functor** $F\colon \mathbb{A} \to \mathbb{B}$ consists of maps on objects, horizontal morphisms, vertical morphisms, and squares, which are compatible with domains and codomains and preserve all double categorical compositions and identities strictly.

The category of double categories is cartesian closed, and therefore there is a double category whose objects are the double functors. We describe the horizontal morphisms, vertical morphisms, and squares of this double category.

Definition 2.3. Let $F, G, F', G'\colon \mathbb{A} \to \mathbb{B}$ be four double functors.

A **horizontal natural transformation** $h\colon F \Rightarrow G$ consists of

(i) a horizontal morphism $h_A\colon FA \to GA$ in $\mathbb{B}$, for each object $A \in \mathbb{A}$, and
(ii) a square $h_u\colon (Fu \,{}^{h_A}_{h_{A'}}\, Gu)$ in $\mathbb{B}$, for each vertical morphism $u\colon A \to\!\!\!\!\bullet\; A'$ in $\mathbb{A}$,

such that the assignment of squares is functorial with respect to the composition of vertical morphisms, and these data satisfy a naturality condition with respect to horizontal morphisms and squares.

Similarly, a **vertical natural transformation** $r\colon F \Rightarrow F'$ consists of

(i) a vertical morphism $r_A\colon FA \,\mathrlap{\hspace{0.3em}\bullet}{\to}\, F'A$ in $\mathbb{B}$, for each object $A \in \mathbb{A}$, and
(ii) a square $r_a\colon (r_A \,{}^{Fa}_{F'a}\, r_B)$ in $\mathbb{B}$, for each horizontal morphism $a\colon A \to B$ in $\mathbb{A}$,

satisfying transposed conditions.

Given another horizontal natural transformation $k\colon F' \Rightarrow G'$ and another vertical natural transformation $s\colon G \Rightarrow G'$, a **modification** $\mu\colon (r \,{}^{h}_{k}\, s)$ consists of

(i) a square $\mu_A\colon (r_A \,{}^{h_A}_{k_A}\, s_A)$ in $\mathbb{B}$, for each object $A \in \mathbb{A}$,

satisfying horizontal and vertical coherence conditions with respect to the squares of the transformations h, k, r, and s.

See [5, §3.2.7] for more explicit definitions.

Definition 2.4. Let $\mathbb{A}$ and $\mathbb{B}$ be double categories. We define the double category $[\mathbb{A}, \mathbb{B}]$ whose

(i) objects are the double functors $\mathbb{A} \to \mathbb{B}$,
(ii) horizontal morphisms are the horizontal natural transformations,
(iii) vertical morphisms are the vertical natural transformations, and
(iv) squares are the modifications.

Proposition 2.5 ([3, Proposition 2.11])**.** *For every double category $\mathbb{A}$, there is an adjunction*

$$\begin{tikzcd} \mathrm{DblCat} \arrow[r, bend left, "- \times \mathbb{A}"] \arrow[r, phantom, "\perp"] & \mathrm{DblCat} \arrow[l, bend left, "{[\mathbb{A}, -]}"] \end{tikzcd}.$$

As mentioned in the introduction, there is a full horizontal embedding of the category of 2-categories into that of double categories. This is given by the following functor.

Definition 2.6. We define the functor $\mathbb{H}\colon \mathrm{2Cat} \to \mathrm{DblCat}$. It takes a 2-category $\mathcal{A}$ to the double category $\mathbb{H}\mathcal{A}$ having the same objects as $\mathcal{A}$, the morphisms of $\mathcal{A}$ as horizontal morphisms, only identities as vertical morphisms, and squares

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, equal] \arrow[dr, phantom, "\alpha"] & B \arrow[d, equal] \\ A \arrow[r, "b"'] & B \end{tikzcd}$$

given by the 2-cells $\alpha\colon a \Rightarrow b$ in $\mathcal{A}$. It sends a 2-functor $F\colon \mathcal{A} \to \mathcal{B}$ to the double functor $\mathbb{H}F\colon \mathbb{H}\mathcal{A} \to \mathbb{H}\mathcal{B}$ that acts as F does on the corresponding data.

The functor $\mathbb{H}$ admits a right adjoint given by the following.

Definition 2.7. We define the functor $\mathbf{H}\colon \mathrm{DblCat} \to \mathrm{2Cat}$. It takes a double category $\mathbb{A}$ to its underlying horizontal 2-category $\mathbf{H}\mathbb{A}$, i.e., the 2-category whose objects are the objects of $\mathbb{A}$, whose morphisms are the horizontal morphisms of $\mathbb{A}$, and whose 2-cells $\alpha\colon a \Rightarrow b$ are given by the squares in $\mathbb{A}$ of the form

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\alpha"] & B \arrow[d, equal, "\bullet" marking] \\ A \arrow[r, "b"'] & B \end{tikzcd} .$$

It sends a double functor $F\colon \mathbb{A} \to \mathbb{B}$ to the 2-functor $\mathbf{H}F\colon \mathbf{H}\mathbb{A} \to \mathbf{H}\mathbb{B}$ that acts as F does on the corresponding data.

Proposition 2.8 ([3, Proposition 2.5])**.** *The functors* $\mathbf{H}$ *and* $\mathbb{H}$ *form an adjunction*

$$\begin{tikzcd} 2\mathrm{Cat} \arrow[r, bend left, "\mathbb{H}"] \arrow[r, phantom, "\perp"] & \mathrm{DblCat} \arrow[l, bend left, "\mathbf{H}"] \end{tikzcd} .$$

Moreover, the unit $\eta\colon \mathrm{id} \Rightarrow \mathbf{H}\mathbb{H}$ *is the identity.*

Remark 2.9. Similarly, we can define a functor $\mathbb{V}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$, sending a 2-category to its associated vertical double category with only trivial horizontal morphisms, and a functor $\mathbf{V}\colon \mathrm{DblCat} \to 2\mathrm{Cat}$, sending a double category to its underlying vertical 2-category. They also form an adjunction $\mathbb{V} \dashv \mathbf{V}$.

We now introduce a functor that extracts, from a double category, a 2-category whose objects and morphisms are the vertical morphisms and squares; this is the functor $\mathcal{V}$ mentioned in the introduction. In order to do this, we need the category $\mathbb{V}\mathbb{2}$, where $\mathbb{2}$ is the (2-)category $\{0 \to 1\}$. This double category $\mathbb{V}\mathbb{2}$ contains exactly one vertical morphism.

Definition 2.10. We define the functor $\mathcal{V}\colon \mathrm{DblCat} \to 2\mathrm{Cat}$ to be the functor $\mathbf{H}[\mathbb{V}\mathbb{2}, -]$. More explicitly, it sends a double category $\mathbb{A}$ to the 2-category $\mathcal{V}\mathbb{A} = \mathbf{H}[\mathbb{V}\mathbb{2}, \mathbb{A}]$ given by the following data.

(i) An object in $\mathcal{V}\mathbb{A}$ is a vertical morphism $u\colon A \to A'$ in $\mathbb{A}$.
(ii) A morphism $(a, b, \alpha)\colon u \to v$ is a square in $\mathbb{A}$

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "u"', "\bullet" marking] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "v", "\bullet" marking] \\ A' \arrow[r, "b"'] & B' \end{tikzcd} .$$

(iii) Composition of morphisms is given by the horizontal composition of squares in $\mathbb{A}$.
(iv) A 2-cell $(\sigma_0, \sigma_1)\colon (a, b, \alpha) \Rightarrow (c, d, \beta)$ consists of two squares in $\mathbb{A}$

$$
\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\sigma_0"] & B \arrow[d, equal, "\bullet" marking] \\
A \arrow[r, "c"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
A' \arrow[r, "b"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\sigma_1"] & B' \arrow[d, equal, "\bullet" marking] \\
A' \arrow[r, "d"'] & B'
\end{tikzcd}
$$

such that the following pasting equality holds.

$$
\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\sigma_0"] & B \arrow[d, equal, "\bullet" marking] \\
A \arrow[r, "c"] \arrow[d, "u"', "\bullet" marking] \arrow[dr, phantom, "\beta"] & B \arrow[d, "v", "\bullet" marking] \\
A' \arrow[r, "d"'] & B'
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "u"', "\bullet" marking] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "v", "\bullet" marking] \\
A' \arrow[r, "b"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\sigma_1"] & B' \arrow[d, equal, "\bullet" marking] \\
A' \arrow[r, "d"'] & B'
\end{tikzcd}
$$

(v) Horizontal and vertical compositions of 2-cells are given by the componentwise horizontal and vertical compositions of squares in $\mathbb{A}$.

Proposition 2.11. *The functor $\mathcal{V}$ has a left adjoint $\mathbb{L}$*

$$
\begin{tikzcd}
2\mathrm{Cat} \arrow[r, bend left, "\mathbb{L}"] \arrow[r, phantom, "\perp"] & \mathrm{DblCat} \arrow[l, bend left, "\mathcal{V}"]
\end{tikzcd}
$$

given by $\mathbb{L} = \mathbb{H}(-) \times \mathbb{V}\mathbb{2}$.

Proof. By definition, the functor $\mathcal{V}\colon \mathrm{DblCat} \to 2\mathrm{Cat}$ is given by the composite

$$\mathrm{DblCat} \xrightarrow{[\mathbb{V}\mathbb{2}, -]} \mathrm{DblCat} \xrightarrow{\mathbf{H}} 2\mathrm{Cat}.$$

Since $[\mathbb{V}\mathbb{2}, -]$ has a left adjoint $- \times \mathbb{V}\mathbb{2}$ given by Proposition 2.5, and $\mathbf{H}$ has a left adjoint $\mathbb{H}$ given by Proposition 2.8, it follows that $\mathcal{V}$ has a left adjoint given by the composite of the two left adjoints, namely, $\mathbb{L} = \mathbb{H}(-) \times \mathbb{V}\mathbb{2}$. $\square$

We conclude this section by introducing notions of weak invertibility for horizontal morphisms and squares, together with some technical results that will be of use later in the paper. We do not prove these results here, but proofs will be provided in forthcoming work by the first author [14].

Definition 2.12. A horizontal morphism $a\colon A \to B$ in a double category $\mathbb{A}$ is a **horizontal equivalence** if it is an equivalence in the 2-category $\mathbf{H}\mathbb{A}$.

Definition 2.13. A square $\alpha\colon (u \; {}^{a}_{b} \; v)$ in a double category $\mathbb{A}$ is **weakly horizontally invertible** if it is an equivalence in the 2-category $\mathcal{V}\mathbb{A}$. In other words, the square α is weakly horizontally invertible if there exists a cell $\beta\colon (v \; {}^{a'}_{b'} \; u)$ in $\mathbb{A}$ and four vertically invertible cells η_a, η_b, ϵ_a, and ϵ_b as in the pasting diagrams below.

$$
\begin{tikzcd}
A \arrow[rr, equal] \arrow[d, equal, "\bullet" marking] & \arrow[d, phantom, "\eta_a \, \cong"] & A \arrow[d, equal, "\bullet" marking] \\
A \arrow[r, "a"] \arrow[d, "u"', "\bullet" marking] \arrow[dr, phantom, "\alpha"] & B \arrow[r, "a'"] \arrow[d, "v", "\bullet" marking] \arrow[dr, phantom, "\beta"] & A \arrow[d, "u", "\bullet" marking] \\
A' \arrow[r, "b"'] & B' \arrow[r, "b'"'] & A'
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[rr, equal] \arrow[d, "u"', "\bullet" marking] & \arrow[d, phantom, "\mathrm{id}_u"] & A \arrow[d, "u", "\bullet" marking] \\
A' \arrow[rr, equal] \arrow[d, equal, "\bullet" marking] & \arrow[d, phantom, "\eta_b \, \cong"] & A' \arrow[d, equal, "\bullet" marking] \\
A' \arrow[rr, equal] & & A'
\end{tikzcd}
$$

$$
\begin{tikzcd}
B \arrow[r, "a'"] \arrow[d, equal, "\bullet" marking] & A \arrow[r, "a"] \arrow[d, phantom, "\epsilon_a \, \cong"] & B \arrow[d, equal, "\bullet" marking] \\
B \arrow[rr, equal] \arrow[d, "v"', "\bullet" marking] & \arrow[d, phantom, "\mathrm{id}_v"] & B \arrow[d, "v", "\bullet" marking] \\
B' \arrow[rr, equal] & & B'
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
B \arrow[r, "a'"] \arrow[d, "v"', "\bullet" marking] \arrow[dr, phantom, "\beta"] & A \arrow[r, "a"] \arrow[d, "u", "\bullet" marking] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "v", "\bullet" marking] \\
B' \arrow[r, "b'"'] \arrow[d, equal, "\bullet" marking] & A' \arrow[r, "b"'] \arrow[d, phantom, "\epsilon_b \, \cong"] & B' \arrow[d, equal, "\bullet" marking] \\
A' \arrow[rr, equal] & & A'
\end{tikzcd}
$$

We call β a **weak inverse** of α.

Remark 2.14. In particular, the horizontal boundaries a and b of a weakly horizontally invertible square α as above are horizontal equivalences witnessed by the data $(a, a', \eta_a, \epsilon_a)$ and $(b, b', \eta_b, \epsilon_b)$. We call these tuples the *horizontal equivalence data* of (α, β).

Since an equivalence in a 2-category can always be promoted to an adjoint equivalence, we get the following result.

Lemma 2.15. *Every horizontal equivalence can be promoted to a horizontal* adjoint *equivalence. Similarly, every weakly horizontally invertible square can be promoted to one with horizontal* adjoint *equivalence data.*

Finally, we conclude with two results concerning weakly horizontally invertible cells.

Lemma 2.16. [14] *Given a weakly horizontally invertible square $\alpha\colon (u \,{}^{a}_{b}\, v)$ and two horizontal adjoint equivalences $(a, a', \eta_a, \epsilon_a)$ and $(b, b', \eta_b, \epsilon_b)$, there exists a unique weak inverse $\beta\colon (v \,{}^{a'}_{b'}\, u)$ of α with respect to these horizontal adjoint equivalences.*

Lemma 2.17. [14] *A square whose horizontal boundaries are horizontal equivalences, and whose vertical boundaries are identities, is weakly horizontally invertible if and only if it is vertically invertible.*

Remark 2.18. It follows that, for a 2-category $\mathcal{A}$, a weakly horizontally invertible square in the double category $\mathbb{H}\mathcal{A}$ corresponds to an invertible 2-cell in $\mathcal{A}$.

3. Model structure for double categories

This section contains our first main result, which proves the existence of a model structure on DblCat that is right-induced along the functor $(\mathbf{H}, \mathcal{V})\colon \mathrm{DblCat} \to 2\mathrm{Cat} \times 2\mathrm{Cat}$, where both copies of 2Cat are endowed with the Lack model structure. After recalling the main features of the Lack model structure on 2Cat in Section 3.1, we define in Section 3.2 notions of *double biequivalences* and *double fibrations* in DblCat, which extend the notions of biequivalences and fibrations in 2Cat. These appear to be exactly the weak equivalences and fibrations of the right-induced model structure mentioned above. We then prove, using a result inspired by the Quillen Path Object Argument, that this right-induced model structure on DblCat exists. Moreover, as the model structure on 2Cat × 2Cat is cofibrantly generated, so is our model structure on DblCat. We describe in Section 3.3 sets of generating cofibrations and generating trivial cofibrations, as well as the cofibrant objects.

The technical proofs of the results stated in this section are postponed to Part II.

3.1. **Lack model structure on** 2Cat**.** We start by recalling the relevant classes of maps in Lack's model structure on 2Cat; see [11, 12]. The weak equivalences are given by the *biequivalences*, and we refer to the fibrations in this model structure as *Lack fibrations*.

Definition 3.1. Given 2-categories $\mathcal{A}$ and $\mathcal{B}$, a 2-functor $F\colon \mathcal{A} \to \mathcal{B}$ is a **biequivalence** if

- (b1) for every object $B \in \mathcal{B}$, there exist an object $A \in \mathcal{A}$ and an equivalence $B \xrightarrow{\simeq} FA$,
- (b2) for every morphism $b\colon FA \to FC$ in $\mathcal{B}$, there exist a morphism $a\colon A \to C$ in $\mathcal{A}$ and an invertible 2-cell $b \cong Fa$, and
- (b3) for every 2-cell $\beta\colon Fa \Rightarrow Fc$ in $\mathcal{B}$, there exists a unique 2-cell $\alpha\colon a \Rightarrow c$ in $\mathcal{A}$ such that $F\alpha = \beta$.

Definition 3.2. Given 2-categories $\mathcal{A}$ and $\mathcal{B}$, a 2-functor $F\colon \mathcal{A} \to \mathcal{B}$ is a **Lack fibration** if

- (f1) for every equivalence $b\colon B \xrightarrow{\simeq} FC$ in $\mathcal{B}$, there exists an equivalence $a\colon A \xrightarrow{\simeq} C$ in $\mathcal{A}$ such that $Fa = b$, and
- (f2) for every morphism $c\colon A \to C$ in $\mathcal{A}$ and every invertible 2-cell $\beta\colon b \cong Fc$, there exists an invertible 2-cell $\alpha\colon a \cong c$ in $\mathcal{A}$ such that $F\alpha = \beta$.

There exists a model structure on 2Cat determined by the classes above.

Theorem 3.3 ([12, Theorem 4])**.** *There is a cofibrantly generated model structure on* 2Cat*, called the* Lack model structure*, in which the weak equivalences are the biequivalences and the fibrations are the Lack fibrations.*

Remark 3.4. Note that every 2-category is fibrant in the Lack model structure.

3.2. **Constructing the model structure for** DblCat**.** We define double biequivalences in DblCat inspired by the characterization of biequivalences in 2Cat in terms of 2-functors that are bi-essentially surjective on objects, essentially full on morphisms, and fully faithful on 2-cells. Our convention of regarding 2-categories as horizontal double categories justifies the choice of directions when emulating this characterization of biequivalences in the context of double categories. Thus, a double biequivalence will be required to be bi-essentially surjective on objects up to a *horizontal equivalence* (see Definition 2.12), essentially full on horizontal morphisms, and fully faithful on squares. However, this does not take into account the vertical structure of double categories, and so we need to add a condition of bi-essential surjectivity on vertical morphisms given up to a *weakly horizontally invertible square* (see Definition 2.13).

Definition 3.5. Given double categories $\mathbb{A}$ and $\mathbb{B}$, a double functor $F\colon \mathbb{A} \to \mathbb{B}$ is a **double biequivalence** if

(db1) for every object $B \in \mathbb{B}$, there exist an object $A \in \mathbb{A}$ and a horizontal equivalence $B \xrightarrow{\simeq} FA$,

(db2) for every horizontal morphism $b\colon FA \to FC$ in $\mathbb{B}$, there exist a horizontal morphism $a\colon A \to C$ in $\mathbb{A}$ and a vertically invertible square in $\mathbb{B}$

$$\begin{tikzcd} FA \arrow[r, "b"] \arrow[d, equal] \arrow[dr, phantom, "\cong"] & FC \arrow[d, equal] \\ FA \arrow[r, "Fa"'] & FC \end{tikzcd},$$

(db3) for every vertical morphism $v\colon B \to B'$ in $\mathbb{B}$, there exist a vertical morphism $u\colon A \to A'$ in $\mathbb{A}$ and a weakly horizontally invertible square in $\mathbb{B}$

$$\begin{tikzcd} B \arrow[r, "\simeq"] \arrow[d, "v"'] \arrow[dr, phantom, "\simeq"] & FA \arrow[d, "Fu"] \\ B' \arrow[r, "\simeq"'] & FA' \end{tikzcd},$$

(db4) for every square in $\mathbb{B}$ of the form

$$\begin{tikzcd} FA \arrow[r, "Fa"] \arrow[d, "Fu"'] \arrow[dr, phantom, "\beta"] & FC \arrow[d, "Fu'"] \\ FA' \arrow[r, "Fc"'] & FC' \end{tikzcd},$$

there exists a unique square $\alpha\colon (u \, {}^{a}_{c} \, u')$ in $\mathbb{A}$ such that $F\alpha = \beta$.

Remark 3.6. In 2Cat, one can prove that a 2-functor $F\colon \mathcal{A} \to \mathcal{B}$ is a biequivalence if and only if there exists a pseudo functor $G\colon \mathcal{B} \to \mathcal{A}$ together with two pseudo natural equivalences $\mathrm{id} \simeq GF$ and $FG \simeq \mathrm{id}$. Under certain hypotheses, we can show a similar characterization of double biequivalences using pseudo *horizontal* natural equivalences. This is done in Section 10.

Similarly to the definition of double biequivalence, we take inspiration from the Lack fibrations to define a notion of *double fibrations.*

Definition 3.7. Given double categories $\mathbb{A}$ and $\mathbb{B}$, a double functor $F\colon \mathbb{A} \to \mathbb{B}$ is a **double fibration** if (df1) for every horizontal equivalence $b\colon B \xrightarrow{\simeq} FC$ in $\mathbb{B}$, there exists a horizontal equivalence $a\colon A \xrightarrow{\simeq} C$ in $\mathbb{A}$ such that $Fa = b$,

(df2) for every horizontal morphism $c\colon A \to C$ in $\mathbb{A}$ and for every vertically invertible square $\beta\colon (e_{FA}\, {}^{b}_{Fc}\, e_{FC})$ in $\mathbb{B}$ as depicted below left, there exists a vertically invertible square $\alpha\colon (e_A\, {}^{a}_{c}\, e_C)$ in $\mathbb{A}$ as depicted below right such that $F\alpha = \beta$,

$$\begin{tikzcd} FA \arrow[r, "b"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\beta\ \cong"] & FC \arrow[d, equal, "\bullet" marking] \\ FA \arrow[r, "Fc"'] & FC \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "a"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\alpha\ \cong"] & C \arrow[d, equal, "\bullet" marking] \\ A \arrow[r, "c"'] & C \end{tikzcd}$$

(df3) for every vertical morphism $u'\colon C \mathrel{\mathrlap{\rightarrow}\mkern-1mu\bullet} C'$ in $\mathbb{A}$ and every weakly horizontally invertible square $\beta\colon (v \underset{\simeq}{\overset{\simeq}{}} Fu')$ in $\mathbb{B}$ as depicted below left, there exists a weakly horizontally invertible square $\alpha\colon (u \underset{\simeq}{\overset{\simeq}{}} u')$ in $\mathbb{A}$ as depicted below right such that $F\alpha = \beta$.

$$\begin{tikzcd} B \arrow[r, "\simeq"] \arrow[d, "v"', "\bullet" marking] \arrow[dr, phantom, "\beta\ \simeq"] & FC \arrow[d, "Fu'", "\bullet" marking] \\ B' \arrow[r, "\simeq"'] & FC' \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\simeq"] \arrow[d, "u"', "\bullet" marking] \arrow[dr, phantom, "\alpha\ \simeq"] & C \arrow[d, "u'", "\bullet" marking] \\ A \arrow[r, "\simeq"'] & C \end{tikzcd}$$

By requiring that a double functor is both a double biequivalence and a double fibration, we get a notion of *double trivial fibration.*

Definition 3.8. Given double categories $\mathbb{A}$ and $\mathbb{B}$, a double functor $F\colon \mathbb{A} \to \mathbb{B}$ is a **double trivial fibration** if

(dt1) for every object $B \in \mathbb{B}$, there exists an object $A \in \mathbb{A}$ such that $B = FA$,

(dt2) for every horizontal morphism $b\colon FA \to FC$ in $\mathbb{B}$, there exists a horizontal morphism $a\colon A \to C$ such that $b = Fa$,

(dt3) for every vertical morphism $v\colon B \mathrel{\mathrlap{\rightarrow}\mkern-1mu\bullet} B'$ in $\mathbb{B}$, there exists a vertical morphism $u\colon A \mathrel{\mathrlap{\rightarrow}\mkern-1mu\bullet} A'$ in $\mathbb{A}$ such that $v = Fu$, and

(dt4) for every square in $\mathbb{B}$ of the form

$$\begin{tikzcd} FA \arrow[r, "Fa"] \arrow[d, "Fu"', "\bullet" marking] \arrow[dr, phantom, "\beta"] & FC \arrow[d, "Fu'", "\bullet" marking] \\ FA' \arrow[r, "Fc"'] & FC' \end{tikzcd},$$

there exists a unique square $\alpha\colon (u\, {}^{a}_{c}\, u')$ in $\mathbb{A}$ such that $F\alpha = \beta$.

Remark 3.9. Note that (dt2) says that a double trivial fibration is *full* on horizontal morphisms, while (dt3) says that a double trivial fibration is only *surjective* on vertical morphisms.

We can characterize double biequivalences and double fibrations through biequivalences and Lack fibrations in 2Cat. Recall the functors $\mathbf{H}, \mathcal{V} \colon \mathrm{DblCat} \to 2\mathrm{Cat}$ defined in Section 2, which respectively extract from a double category its underlying horizontal 2-category and a 2-category whose objects and morphisms are given by its vertical morphisms and squares. Then double biequivalences and double fibrations can be characterized as the double functors whose images under both $\mathbf{H}$ and $\mathcal{V}$ are biequivalences or Lack fibrations. This is intuitively sound, since horizontal equivalences and weakly horizontally invertible squares were defined to be the equivalences in the 2-categories induced by $\mathbf{H}$ and $\mathcal{V}$, respectively. We state these characterizations here, and defer their proofs to Section 8.

Proposition 3.10. *Given double categories $\mathbb{A}$ and $\mathbb{B}$, a double functor $F \colon \mathbb{A} \to \mathbb{B}$ is a double biequivalence in* DblCat *if and only if $\mathbf{H}F \colon \mathbf{H}\mathbb{A} \to \mathbf{H}\mathbb{B}$ and $\mathcal{V}F \colon \mathcal{V}\mathbb{A} \to \mathcal{V}\mathbb{B}$ are biequivalences in* 2Cat*.*

Proposition 3.11. *Given double categories $\mathbb{A}$ and $\mathbb{B}$, a double functor $F \colon \mathbb{A} \to \mathbb{B}$ is a double fibration in* DblCat *if and only if $\mathbf{H}F \colon \mathbf{H}\mathbb{A} \to \mathbf{H}\mathbb{B}$ and $\mathcal{V}F \colon \mathcal{V}\mathbb{A} \to \mathcal{V}\mathbb{B}$ are Lack fibrations in* 2Cat*.*

As a corollary, we get a similar characterization for double trivial fibrations.

Corollary 3.12. *Given double categories $\mathbb{A}$ and $\mathbb{B}$, a double functor $F \colon \mathbb{A} \to \mathbb{B}$ is a double trivial fibration in* DblCat *if and only if $\mathbf{H}F \colon \mathbf{H}\mathbb{A} \to \mathbf{H}\mathbb{B}$ and $\mathcal{V}F \colon \mathcal{V}\mathbb{A} \to \mathcal{V}\mathbb{B}$ are trivial fibrations in the Lack model structure on* 2Cat*.*

Proof. It is a routine exercise to check that a double functor that is both a double biequivalence and a double fibration is precisely a double trivial fibration as defined in Definition 3.8. Therefore, this follows directly from Propositions 3.10 and 3.11. $\square$

In order to build a model structure on DblCat with these classes of morphisms as its weak equivalences and (trivial) fibrations, we make use of the notion of *right-induced model structure*. Given a model category $\mathcal{M}$ and an adjunction

$$\begin{tikzcd} \mathcal{M} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathcal{N} \arrow[l, bend left, "R"] \end{tikzcd} \tag{3.13}$$

we can, under certain conditions, induce a model structure on $\mathcal{N}$ along the right adjoint R, in which a weak equivalence (resp. fibration) is a morphism F in $\mathcal{N}$ such that RF is a weak equivalence (resp. fibration) in $\mathcal{M}$.

Propositions 3.10 and 3.11 suggest that the model structure on DblCat we desire, with double biequivalences as the weak equivalences and double fibrations as the fibrations, corresponds to the right-induced model structure, if it exists, along the adjunction

$$\begin{tikzcd} 2\mathrm{Cat} \times 2\mathrm{Cat} \arrow[r, bend left, "\mathbb{H} \sqcup \mathbb{L}"] \arrow[r, phantom, "\perp"] & \mathrm{DblCat} \arrow[l, bend left, "(\mathbf{H}, \mathcal{V})"] \end{tikzcd},$$

where each copy of 2Cat is endowed with the Lack model structure. To prove the existence of this model structure, we use results by Garner, Hess, Kędziorek, Riehl, and Shipley in [4, 8]. In particular, we use the following theorem, inspired by the original Quillen Path Object Argument [17].

Theorem 3.14. *Let $\mathcal{M}$ be an accessible model category, and let $\mathcal{N}$ be a locally presentable category. Suppose we have an adjunction $L \dashv R$ between them as in (3.13). Suppose moreover that every object is fibrant in $\mathcal{M}$ and that, for every $X \in \mathcal{N}$, there exists a factorization*

$$X \xrightarrow{W} \mathrm{Path}(X) \xrightarrow{P} X \times X$$

of the diagonal morphism in $\mathcal{N}$ such that RP is a fibration in $\mathcal{M}$ and RW is a weak equivalence in $\mathcal{M}$. Then the right-induced model structure on $\mathcal{N}$ exists.

Proof. This follows directly from [15, Theorem 6.2], which is the dual of [8, Theorem 2.2.1]. Indeed, if every object in $\mathcal{M}$ is fibrant, then the underlying fibrant replacement of conditions (i) and (ii) of [15, Theorem 6.2] are trivially given by the identity. □

Our strategy is then to construct a path object $\mathcal{P}\mathbb{A}$ for a double category $\mathbb{A}$ together with double functors W and P factorizing the diagonal morphism $\mathbb{A} \to \mathbb{A} \times \mathbb{A}$, such that their images under $(\mathbf{H}, \mathcal{V})$ give a weak equivalence and a fibration in $2\mathrm{Cat} \times 2\mathrm{Cat}$ respectively; or equivalently, by Propositions 3.10 and 3.11, such that W is a double biequivalence and P is a double fibration. This construction is inspired by the path object construction in 2Cat of [11, Proof of Theorem 5.1], and its statement is summarized in the following technical result, whose proof is the content of Section 7.

Proposition 3.15. *For every double category $\mathbb{A}$, there exists a double category $\mathcal{P}\mathbb{A}$ together with a factorization of the diagonal double functor*

$$\mathbb{A} \xrightarrow{W} \mathcal{P}\mathbb{A} \xrightarrow{P} \mathbb{A} \times \mathbb{A}$$

such that W is a double biequivalence and P is a double fibration.

We are finally ready to prove the existence of the right-induced model structure on DblCat along the adjunction $\mathbb{H} \sqcup \mathbb{L} \dashv (\mathbf{H}, \mathcal{V})$.

Theorem 3.16. *Consider the adjunction*

$$\begin{tikzcd} 2\mathrm{Cat} \times 2\mathrm{Cat} \arrow[r, bend left, "\mathbb{H} \sqcup \mathbb{L}"] \arrow[r, phantom, "\perp"] & \mathrm{DblCat} \arrow[l, bend left, "(\mathbf{H}, \mathcal{V})"] \end{tikzcd},$$

where each copy of 2Cat *is endowed with the Lack model structure. Then the right-induced model structure on* DblCat *exists. In particular, a double functor is a weak equivalence (resp. fibration) in this model structure if and only if it is a double biequivalence (resp. double fibration).*

Proof. We first describe the weak equivalences and fibrations in this model structure on DblCat. The weak equivalences (resp. fibrations) in the right-induced model structure on DblCat are given by the double functors F such that $(\mathbf{H}, \mathcal{V})F$ is a weak equivalence (resp. fibration) in $2\mathrm{Cat} \times 2\mathrm{Cat}$, or equivalently, such that both $\mathbf{H}F$ and $\mathcal{V}F$ are biequivalences (resp. Lack fibrations) in 2Cat. Then it follows from Propositions 3.10 and 3.11 that the weak equivalences (resp. fibrations) in DblCat are precisely the double biequivalences (resp. double fibrations).

We now prove the existence of the model structure. For this purpose, we want to apply Theorem 3.14 to our setting. First note that 2Cat and DblCat are locally presentable, and that the Lack model structure on 2Cat is cofibrantly generated. In particular, this implies that the product $2\mathrm{Cat} \times 2\mathrm{Cat}$ endowed with two copies of the Lack model structure is combinatorial, hence accessible. Moreover, every pair of 2-categories is fibrant in $2\mathrm{Cat} \times 2\mathrm{Cat}$, since every object is fibrant in the Lack model structure. Finally, for every double category $\mathbb{A}$, Proposition 3.15 gives a factorization

$$\mathbb{A} \xrightarrow{W} \mathcal{P}\mathbb{A} \xrightarrow{P} \mathbb{A} \times \mathbb{A}$$

such that W is a double biequivalence and P is a double fibration. By Theorem 3.14, this proves that the right-induced model structure along $(\mathbf{H}, \mathcal{V})$ on DblCat exists. □

Remark 3.17. Note that every double category is fibrant in this model structure. Indeed, this follows directly from the fact that it is right-induced from a model structure in which every object is fibrant.

3.3. **Generating (trivial) cofibrations and cofibrant objects.** We now turn our attention to the cofibrations. As stated in Theorem 3.3, the Lack model structure on 2Cat is cofibrantly generated. Since our model structure on DblCat is right-induced from copies of it, it is also cofibrantly generated. The next result provides sets of generating (trivial) cofibrations for our model structure on DblCat.

Proposition 3.18. *Let $\mathcal{I}_2$ and $\mathcal{J}_2$ denote sets of generating cofibrations and generating trivial cofibrations, respectively, for the Lack model structure on* 2Cat*. Then, the sets of morphisms in* DblCat

$$\mathcal{I} = \{\mathbb{H}i,\ \mathbb{H}i \times \mathbb{V}\mathbb{2} \mid i \in \mathcal{I}_2\}, \quad \text{and} \quad \mathcal{J} = \{\mathbb{H}j,\ \mathbb{H}j \times \mathbb{V}\mathbb{2} \mid j \in \mathcal{J}_2\}$$

give sets of generating cofibrations and generating trivial cofibrations, respectively, for the model structure on DblCat *of Theorem 3.16.*

Proof. Since the model structure on DblCat is right-induced from two copies of the Lack model structure on 2Cat along the adjunction $\mathbb{H} \sqcup \mathbb{L} \dashv (\mathbf{H}, \mathcal{V})$, the sets of generating cofibrations and of generating trivial cofibrations are given by the image under the left adjoint $\mathbb{H} \sqcup \mathbb{L}$ of the sets of generating cofibrations and generating trivial cofibrations in $2\mathrm{Cat} \times 2\mathrm{Cat}$.

Let i and i' be generating cofibrations of $\mathcal{I}_2$ in 2Cat. Then $\mathbb{H}i$ and $\mathbb{L}i = \mathbb{H}i \times \mathbb{V}\mathbb{2}$ are cofibrations in DblCat. To see this apply $\mathbb{H} \sqcup \mathbb{L}$ to the cofibrations $(i, \mathrm{id}_\emptyset)$ and $(\mathrm{id}_\emptyset, i)$, respectively. Similarly, $\mathbb{H}i'$ and $\mathbb{L}i' = \mathbb{H}i' \times \mathbb{V}\mathbb{2}$ are cofibrations in DblCat. Since coproducts of cofibrations are cofibrations, $(\mathbb{H} \sqcup \mathbb{L})(i, i') = \mathbb{H}i \sqcup \mathbb{L}i'$ can be obtained from $\mathbb{H}i$ and $\mathbb{L}i' = \mathbb{H}i' \times \mathbb{V}\mathbb{2}$. This shows that $\mathcal{I}$ is a set of generating cofibrations of DblCat.

Similarly, we can show that $\mathcal{J}$ is a set of generating trivial cofibrations of DblCat. $\square$

In the Lack model structure on 2Cat, the cofibrant objects are precisely the 2-categories whose underlying categories are free, by [11, Theorem 4.8]. We get a similar characterization of the cofibrant objects in our model structure. Since double trivial fibrations are full on horizontal morphisms and fully faithful on squares, the underlying horizontal category of a cofibrant double category is also free. To characterize these cofibrant double categories completely, we also need a condition for vertical morphisms. This is expressed in terms of the underlying vertical category, which is not only required to be free, but, in addition, it cannot contain any composition of morphisms. This is intuitively coming from the fact that double trivial fibrations are only surjective on vertical morphisms instead of full.

Notation 3.19. We write $U\colon \mathrm{2Cat} \to \mathrm{Cat}$ for the functor that sends a 2-category to its underlying category.

Proposition 3.20. *A double category $\mathbb{A}$ is cofibrant if and only if its underlying horizontal category $U\mathbf{H}\mathbb{A}$ is free and its underlying vertical category $U\mathbf{V}\mathbb{A}$ is a disjoint union of copies of $\mathbb{1}$ and $\mathbb{2}$.*

We prove this proposition in Section 9, using a description of the cofibrations presented therein. In that same section, we also provide smaller sets of generating cofibrations and generating trivial cofibrations.

4. Quillen pairs between DblCat, 2Cat, and Cat

In this paper, the model structure on DblCat was constructed in such a way as to be compatible with the Lack model structure on 2Cat. This section investigates the interactions between our model structure and Lack's model structure by looking at different Quillen pairs between them. Since our model structure on DblCat was right-induced along the adjunction $\mathbb{H} \sqcup \mathbb{L} \dashv (\mathbf{H}, \mathcal{V})$, this adjunction is itself a Quillen pair. From it, we induce two Quillen pairs between DblCat and 2Cat: $\mathbb{H} \dashv \mathbf{H}$, presented in Section 4.1, and $\mathbb{L} \dashv \mathcal{V}$, which is the content of Section 4.3.

Since the derived unit of the adjunction $\mathbb{H} \dashv \mathbf{H}$ is an identity, this further implies that the functor $\mathbb{H}\colon \mathrm{2Cat} \to \mathrm{DblCat}$ is homotopically fully faithful. In Section 4.1, we also prove that the functor $\mathbb{H}$ and its left adjoint form a Quillen pair. These results imply that the functor $\mathbb{H}$ embeds the homotopy theory of 2Cat into that of DblCat in a reflective and coreflective way, in the sense that the $(\infty, 1)$-category of 2-categories is fully embedded in the $(\infty, 1)$-category of double categories, and this embedding has both adjoints at the level of $(\infty, 1)$-categories. Among other things, this means that the functor $\mathbb{H}$ creates all homotopy limits and colimits. Finally, our last result in Section 4.1 shows that the Lack model structure on 2Cat is right-induced from our model structure along $\mathbb{H}$.

In Section 4.2, we give a Quillen pair between Cat and DblCat, which horizontally embeds the canonical homotopy theory of Cat into that of DblCat in a reflective way. We also show that the canonical model structure on Cat is right-induced from ours.

We first give the following lemma, which allows us to directly induce the Quillen pairs $\mathbb{H} \dashv \mathbf{H}$ and $\mathbb{L} \dashv \mathcal{V}$ from the one given by $\mathbb{H} \sqcup \mathbb{L} \dashv (\mathbf{H}, \mathcal{V})$.

Lemma 4.1. *The following adjunctions*

$$\begin{tikzcd} 2\mathrm{Cat} \arrow[r, bend left, "(\mathrm{id}, \emptyset)"] \arrow[r, phantom, "\perp"] & 2\mathrm{Cat} \times 2\mathrm{Cat} \arrow[l, bend left, "\mathrm{pr}_1"] \end{tikzcd} \qquad \begin{tikzcd} 2\mathrm{Cat} \arrow[r, bend left, "(\emptyset, \mathrm{id})"] \arrow[r, phantom, "\perp"] & 2\mathrm{Cat} \times 2\mathrm{Cat} \arrow[l, bend left, "\mathrm{pr}_2"] \end{tikzcd}$$

induced from the adjunction given by the initial object $\emptyset$ *of* 2Cat

$$\begin{tikzcd} \mathbb{1} \arrow[r, bend left, "\emptyset"] \arrow[r, phantom, "\perp"] & 2\mathrm{Cat} \arrow[l, bend left, "!"] \end{tikzcd}$$

are Quillen pairs, where all copies of 2Cat *are endowed with the Lack model structure.*

Proof. It is clear that the projection functors $\mathrm{pr_i}$ for $i = 1, 2$ preserve fibrations and trivial fibrations. Therefore they are right Quillen functors and the adjunctions are Quillen pairs. □

4.1. **Quillen pairs involving** $\mathbb{H}$**.** We present here the two Quillen pairs involving the functor $\mathbb{H}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$ and its right and left adjoints.

Proposition 4.2. *The adjunction*

$$\begin{tikzcd} 2\mathrm{Cat} \arrow[r, bend left, "\mathbb{H}"] \arrow[r, phantom, "\perp"] & \mathrm{DblCat} \arrow[l, bend left, "\mathbf{H}"] \end{tikzcd}$$

is a Quillen pair, where 2Cat *is endowed with the Lack model structure and* DblCat *is endowed with the model structure of Theorem 3.16. Moreover, the unit* $\eta_{\mathcal{A}}\colon \mathcal{A} \to \mathbf{H}\mathbb{H}\mathcal{A}$ *is an identity for all* $\mathcal{A} \in 2\mathrm{Cat}$*.*

Proof. By composing the Quillen pairs $(\mathrm{id}, \emptyset) \dashv \mathrm{pr}_1$ of Lemma 4.1 and $\mathbb{H} \sqcup \mathbb{L} \dashv (\mathbf{H}, \mathcal{V})$ of Theorem 3.16, we directly get the Quillen pair above. Moreover, we have $\mathbf{H}\mathbb{H} = \mathrm{id}_{2\mathrm{Cat}}$. □

Remark 4.3. In particular, since every object is fibrant, the derived unit of the adjunction $\mathbb{H} \dashv \mathbf{H}$ is given by the components of the unit at cofibrant objects, and is therefore an identity. This implies that the functor $\mathbb{H}$ is homotopically fully faithful.

The functor $\mathbb{H}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$ also admits a left adjoint L. Indeed, this is given by the Adjoint Functor Theorem, since $\mathbb{H}$ preserves all limits and the categories involved are locally presentable. The next theorem shows that $\mathbb{H}$ is also a right Quillen functor.

Theorem 4.4. *The adjunction*

$$\begin{tikzcd} \mathrm{DblCat} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & 2\mathrm{Cat} \arrow[l, bend left, "\mathbb{H}"] \end{tikzcd}$$

is a Quillen pair, where 2Cat *is endowed with the Lack model structure and* DblCat *is endowed with the model structure of Theorem 3.16. Moreover, the counit* $\epsilon_{\mathcal{A}} \colon L\mathbb{H}\mathcal{A} \to \mathcal{A}$ *is an isomorphism of* 2*-categories for all* $\mathcal{A} \in 2\mathrm{Cat}$*.*

Proof. We show that $\mathbb{H}$ is right Quillen, i.e., it preserves fibrations and trivial fibrations.

Let $F \colon \mathcal{A} \to \mathcal{B}$ be a fibration in 2Cat; we prove that $\mathbb{H}F \colon \mathbb{H}\mathcal{A} \to \mathbb{H}\mathcal{B}$ is a double fibration in DblCat. Since $\mathbf{H}\mathbb{H}F = F$ and F is a fibration, (df1-2) of Definition 3.7 are satisfied. It remains to show (df3) of Definition 3.7. Let us consider a weakly horizontally invertible square in $\mathbb{H}\mathcal{B}$

$$\begin{tikzcd} B \arrow[r, "\simeq", "b"'] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\beta \ \rotatebox{90}{$\cong$}"] & FC \arrow[d, equal, "\bullet" marking] \\ B \arrow[r, "\simeq", "d"'] & FC \end{tikzcd}.$$

Note that its vertical boundaries must be trivial, since all vertical morphisms in $\mathbb{H}\mathcal{B}$ are identities. The square β is, in particular, vertically invertible by Lemma 2.17. Since F is a fibration in 2Cat, there exists an equivalence $c \colon A \xrightarrow{\simeq} C$ such that $Fc = d$, by (f1) of Definition 3.2. Now β can be rewritten as

$$\begin{tikzcd} FA \arrow[r, "\simeq", "b"'] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\beta \ \rotatebox{90}{$\cong$}"] & FC \arrow[d, equal, "\bullet" marking] \\ FA \arrow[r, "\simeq", "Fc"'] & FC \end{tikzcd}.$$

Then β is equivalently an invertible 2-cell $\beta \colon b \Rightarrow Fc$ in $\mathcal{B}$. Since F is a fibration in 2Cat, there exist a morphism $a \colon A \to C$ in $\mathcal{A}$ and an invertible 2-cell $\alpha \colon a \Rightarrow c$ in $\mathcal{A}$ such that $F\alpha = \beta$, by (f2) of Definition 3.2. In particular, since c is an equivalence, then so is a. This gives a vertically invertible square in $\mathbb{H}\mathcal{A}$

$$\begin{tikzcd} A \arrow[r, "\simeq", "a"'] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\alpha \ \rotatebox{90}{$\cong$}"] & C \arrow[d, equal, "\bullet" marking] \\ A \arrow[r, "\simeq", "c"'] & C \end{tikzcd}$$

such that $F\alpha = \beta$; furthermore, by Lemma 2.17, the square α is weakly horizontally invertible. This shows that $\mathbb{H}F$ is a double fibration.

Now let $F\colon \mathcal{A} \to \mathcal{B}$ be a trivial fibration in 2Cat. We show that $\mathbb{H}F\colon \mathbb{H}\mathcal{A} \to \mathbb{H}\mathcal{B}$ is a double trivial fibration in DblCat. Since $\mathbf{H}\mathbb{H}F = F$ and F is a trivial fibration, (dt1-2) of Definition 3.8 are satisfied. Then (dt3) of Definition 3.8 follows from the fact that F is surjective on objects, since all vertical morphisms are identities. Finally, (dt4) of Definition 3.8 is a direct consequence of F being fully faithful on 2-cells, since all squares in $\mathbb{H}\mathcal{A}$ and $\mathbb{H}\mathcal{B}$ are equivalently 2-cells in $\mathcal{A}$ and $\mathcal{B}$, respectively. This shows that $\mathbb{H}F$ is a double trivial fibration, and concludes the proof of $L \dashv \mathbb{H}$ being a Quillen pair.

Let $\mathcal{A} \in$ 2Cat; it remains to show that the counit $\epsilon_{\mathcal{A}}\colon L\mathbb{H}\mathcal{A} \to \mathcal{A}$ is an isomorphism of 2-categories. Since $\mathbb{H}$ is fully faithful, this follows directly from evaluating at $\mathrm{id}_{\mathcal{A}}$ the following isomorphism

$$\mathrm{2Cat}(\mathcal{A},\mathcal{A}) \cong \mathrm{DblCat}(\mathbb{H}\mathcal{A},\mathbb{H}\mathcal{A}) \cong \mathrm{2Cat}(L\mathbb{H}\mathcal{A},\mathcal{A}),$$

where the first isomorphism is induced by $\mathbb{H}$ and the second comes from the adjunction $L \dashv \mathbb{H}$. □

Remark 4.5. As we have seen in Remark 4.3, the functor $\mathbb{H}$ is homotopically fully faithful, and therefore the derived counit of the adjunction $L \dashv \mathbb{H}$ is levelwise a biequivalence.

Remark 4.6. By Proposition 4.2 and Theorem 4.4, we can see that $\mathbb{H}\colon \mathrm{2Cat} \to \mathrm{DblCat}$ is both left and right Quillen, and so it preserves all cofibrations, fibrations, and weak equivalences. In particular, since $\mathbb{H}$ is homotopically fully faithful by Remark 4.3, this says that the homotopy theory of 2Cat is reflectively and coreflectively embedded in that of DblCat via the functor $\mathbb{H}$.

In fact, more is true: the Lack model structure on 2Cat is right-induced from our model structure on DblCat along the adjunction $L \dashv \mathbb{H}$, which implies that the functor $\mathbb{H}$ also reflects fibrations and weak equivalences.

Theorem 4.7. *The Lack model structure on* 2Cat *is right-induced from the adjunction*

$$\begin{tikzcd} \mathrm{DblCat} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathrm{2Cat} \arrow[l, bend left, "\mathbb{H}"] \end{tikzcd},$$

where DblCat *is endowed with the model structure of Theorem 3.16.*

Proof. We show that a 2-functor $F\colon \mathcal{A} \to \mathcal{B}$ is a biequivalence (resp. Lack fibration) if and only if $\mathbb{H}F\colon \mathbb{H}\mathcal{A} \to \mathbb{H}\mathcal{B}$ is a double biequivalence (resp. double fibration).

Since $\mathbb{H}$ is right Quillen by Theorem 4.4, it preserves fibrations. Moreover, since all objects in 2Cat are fibrant, by Ken Brown's Lemma (see [9, Lemma 1.1.12]), we have that $\mathbb{H}$ also preserves weak equivalences. This shows that if F is a biequivalence (resp. Lack fibration), then $\mathbb{H}F$ is a double biequivalence (resp. double fibration).

Conversely, if $\mathbb{H}F$ is a double biequivalence (resp. double fibration), then $\mathbf{H}\mathbb{H}F = F$ is a biequivalence (resp. Lack fibration) by definition of the model structure on DblCat.

Since model structures are uniquely determined by their classes of weak equivalences and fibrations, this shows that the Lack model structure on 2Cat is right-induced along $\mathbb{H}$ from the model structure on DblCat of Theorem 3.16. □

We saw that the derived unit (resp. counit) of the adjunction $\mathbb{H} \dashv \mathbf{H}$ (resp. $L \dashv \mathbb{H}$) is levelwise a biequivalence. However, these adjunctions are not expected to be Quillen equivalences, since the homotopy theory of double categories should be richer than that of 2-categories. This is indeed the case, as shown in the following remarks.

Remark 4.8. The components of the (derived) counit of the adjunction $\mathbb{H} \dashv \mathbf{H}$ are not double biequivalences. To see this, consider the double category $\mathbb{V}\mathbb{2}$. The component of the counit at $\mathbb{V}\mathbb{2}$ is given by the inclusion

$$\epsilon_{\mathbb{V}\mathbb{2}} \colon \mathbb{H}\mathbf{H}(\mathbb{V}\mathbb{2}) \cong \mathbb{1} \sqcup \mathbb{1} \to \mathbb{V}\mathbb{2}$$

which is not a double biequivalence, as it does not satisfy (db3) of Definition 3.5. Since $\mathbf{H}\mathbb{V}\mathbb{2} \cong \mathbb{1} \sqcup \mathbb{1}$ is cofibrant in 2Cat, this is also the component of the derived counit at $\mathbb{V}\mathbb{2}$.

Remark 4.9. The components of the (derived) unit of the adjunction $L \dashv \mathbb{H}$ are not double biequivalences. Since the unique map $\emptyset \to \mathbb{1}$ is a generating cofibration in 2Cat [11, §3], then by Proposition 3.18 the unique map $\emptyset \to \mathbb{V}\mathbb{2}$ is a generating cofibration in DblCat, so that $\mathbb{V}\mathbb{2}$ is cofibrant in DblCat. But we have that

$$\eta_{\mathbb{V}\mathbb{2}} \colon \mathbb{V}\mathbb{2} \to \mathbb{H}L(\mathbb{V}\mathbb{2}) \cong \mathbb{1}$$

is not a double biequivalence, where the isomorphism comes from the fact that the left adjoint L collapses the vertical structure and thus $L\mathbb{V}\mathbb{2} \cong \mathbb{1}$.

4.2. **Quillen pairs to** Cat**.** The category Cat of categories and functors also admits a model structure, called the *canonical model structure*, in which the weak equivalences are the equivalences of categories and the fibrations are the isofibrations. As shown by Lack in [11], with this model structure, the homotopy theory of Cat is reflectively embedded in the homotopy theory of 2Cat. Combining this result with the one of Theorem 4.4, we get that the homotopy theory of Cat is also reflectively embedded in that of DblCat.

Notation 4.10. We write $D \colon \mathrm{Cat} \to 2\mathrm{Cat}$ for the functor that sends a category to the 2-category with the same objects and morphisms, and with only identity 2-cells. We write $P \colon 2\mathrm{Cat} \to \mathrm{Cat}$ for its left adjoint. In particular, the functor only P sends a 2-category $\mathcal{A}$ to the category $P\mathcal{A}$ with the same objects as $\mathcal{A}$ and with hom-sets $P\mathcal{A}(A,B) = \pi_0\mathcal{A}(A,B)$, where $\pi_0 \colon \mathrm{Cat} \to \mathrm{Set}$ is the functor sending a category to its set of connected components.

Remark 4.11. In [11, Theorem 8.2], Lack shows that the adjunction $P \dashv D$ is a Quillen pair between the Lack model structure on 2Cat and the canonical model structure on Cat, whose derived counit is levelwise a weak equivalence, but the unit is not. Composing this Quillen pair with the one of Theorem 4.4, we get a Quillen adjunction $PL \dashv \mathbb{H}D$ between the model structure on DblCat of Theorem 3.16 and the canonical model structure on Cat whose derived counit is levelwise an equivalence of categories. In particular, $\mathbb{H}D \colon \mathrm{Cat} \to \mathrm{DblCat}$ is homotopically fully faithful.

The above remark guarantees that the functors $D \colon \mathrm{Cat} \to 2\mathrm{Cat}$ and $\mathbb{H}D \colon \mathrm{Cat} \to \mathrm{DblCat}$ preserve fibrations and weak equivalences, since all objects are fibrant. Furthermore, the following results imply that these two functors create fibrations and weak equivalences, since the canonical model structure on Cat is right-induced from the ones on 2Cat and DblCat.

Proposition 4.12. *The canonical model structure on* Cat *is right-induced from the adjunction*

$$\begin{tikzcd} \mathrm{2Cat} \arrow[r, bend left, "P"] \arrow[r, phantom, "\perp"] & \mathrm{Cat} \arrow[l, bend left, "D"] \end{tikzcd},$$

where 2Cat *is endowed with the Lack model structure.*

Proof. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor in Cat. It suffices to show that F is an equivalence (resp. isofibration) if and only if DF is a biequivalence (resp. Lack fibration). Indeed, both statements can easily seen to be true, due to the fact that for any category $\mathcal{A}$, the 2-category $D\mathcal{A}$ only has trivial 2-cells, and thus a morphism in $D\mathcal{A}$ is an equivalence precisely if it is an isomorphism in $\mathcal{A}$. $\square$

Corollary 4.13. *The canonical model structure on* Cat *is right-induced from the adjunction*

$$\begin{tikzcd} \mathrm{DblCat} \arrow[r, bend left, "PL"] \arrow[r, phantom, "\perp"] & \mathrm{Cat} \arrow[l, bend left, "\mathbb{H}D"] \end{tikzcd},$$

where DblCat *is endowed with the model structure of Theorem 3.16.*

Proof. This follows directly from Proposition 4.12 and Theorem 4.7. $\square$

4.3. **The Quillen pair $\mathbb{L} \dashv \mathcal{V}$.** Since we induced the model structure on DblCat from the adjunction $\mathbb{H} \sqcup \mathbb{L} \dashv (\mathbf{H}, \mathcal{V})$, we directly get that the adjunction $\mathbb{L} \dashv \mathcal{V}$ forms a Quillen pair. The (derived) unit and counit of this adjunction, however, are not levelwise weak equivalences.

Proposition 4.14. *The adjunction*

$$\begin{tikzcd} \mathrm{2Cat} \arrow[r, bend left, "\mathbb{L}"] \arrow[r, phantom, "\perp"] & \mathrm{DblCat} \arrow[l, bend left, "\mathcal{V}"] \end{tikzcd}$$

is a Quillen pair, where 2Cat *is endowed with the Lack model structure and* DblCat *is endowed with the model structure of Theorem 3.16.*

Proof. By composing the Quillen pairs $(\emptyset, \mathrm{id}) \dashv \mathrm{pr}_2$ of Lemma 4.1 and $\mathbb{H} \sqcup \mathbb{L} \dashv (\mathbf{H}, \mathcal{V})$ of Theorem 3.16, we directly get the Quillen pair above. $\square$

Remark 4.15. The components of the (derived) unit of the adjunction $\mathbb{L} \dashv \mathcal{V}$ are not biequivalences. From [11, §3], the unique map $\emptyset \to \mathbb{1}$ is a cofibration, so that $\mathbb{1}$ is cofibrant in 2Cat. But we have that

$$\eta_{\mathbb{1}} \colon \mathbb{1} \to \mathcal{V}\mathbb{L}(\mathbb{1}) \cong \mathbf{H}[\mathbb{V}\mathbb{2}, \mathbb{V}\mathbb{2}] \cong \mathbb{1} \sqcup \mathbb{1} \sqcup \mathbb{1}$$

is not a biequivalence.

Remark 4.16. The components of the (derived) counit of the adjunction $\mathbb{L} \dashv \mathcal{V}$ are not double biequivalences. For example, if we consider the double category $\mathbb{A}$ generated by

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, "\bullet" marking] \arrow[r, phantom, shift right=4ex, "\alpha\quad\beta"] & 1 \arrow[d, "\bullet" marking] \\ 0' \arrow[r] & 1' , \end{tikzcd}$$

then one can check that the double functor

$$\epsilon_{\mathbb{A}} \colon \mathbb{L}\mathcal{V}\mathbb{A} = \mathbb{H}\mathbf{H}[\mathbb{V}\mathbb{2}, \mathbb{A}] \times \mathbb{V}\mathbb{2} \to \mathbb{A}$$

given by evaluation is not full on squares, hence, it is not a biequivalence. By [11, Theorem 4.8], the 2-category $\mathcal{V}\mathbb{A}$ is cofibrant in 2Cat since its underlying category is free. Therefore, the double functor $\epsilon_{\mathbb{A}}$ is also the component of the derived counit at $\mathbb{A}$.

5. 2Cat-enrichment of the model structure on DblCat

The aim of this section is to provide a 2Cat-enrichment on DblCat which is compatible with the model structure introduced in Theorem 3.16. Recall that a model category $\mathcal{M}$ is said to be *enriched* over a closed monoidal category $\mathcal{N}$ that is also a model category, if it is a tensored and cotensored $\mathcal{N}$-category and satisfies the pushout-product axiom (see [15, §5] for more details). In particular, $\mathcal{N}$ is said to be a *monoidal model category*, if the model structure is enriched over itself.

In [11], it is shown that the Lack model structure is not monoidal with respect to the cartesian product. However, it is established that such a compatibility exists when considering instead the closed symmetric monoidal structure on 2Cat given by the Gray tensor product. We first recall this result in Section 5.1.

Similarly, the category of double categories admits two closed symmetric monoidal structures, given by the cartesian product, and by an analogue of the Gray tensor product defined by Böhm in [1]. We show in Section 5.2 that the category DblCat is not a monoidal model category with respect to either of these monoidal structures.

Nevertheless, the Gray tensor product on DblCat is not entirely unrelated to our model structure. By restricting this tensor product in one of the variables to 2Cat along $\mathbb{H}$, we obtain in Section 5.3 a 2Cat-enrichment on DblCat compatible with our model structure.

5.1. **The Lack model structure is monoidal.** Let us recall how the Gray tensor product on 2Cat is defined, and state its compatibility with the Lack model structure.

Definition 5.1. The **Gray tensor product** $\otimes_2 \colon \mathrm{2Cat} \times \mathrm{2Cat} \to \mathrm{2Cat}$ is defined by the following universal property: for all 2-categories $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, there is an isomorphism

$$\mathrm{2Cat}(\mathcal{A} \otimes_2 \mathcal{B}, \mathcal{C}) \cong \mathrm{2Cat}(\mathcal{A}, \mathrm{Ps}[\mathcal{B}, \mathcal{C}])$$

natural in $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, where $\mathrm{Ps}[\mathcal{B}, \mathcal{C}]$ denotes the 2-category of 2-functors $\mathcal{B} \to \mathcal{C}$, pseudo natural transformations, and modifications.

Theorem 5.2 ([11, Theorem 7.5]). *The category* 2Cat *endowed with the Lack model structure is a monoidal model category with respect to the closed monoidal structure given by the Gray tensor product.*

5.2. **The model structure on** DblCat **is not monoidal.** We now turn to double categories. As shown in the remark below, a similar argument to Lack's [11, Example 7.2] also applies in the case of DblCat, to see that the model structure on DblCat is not monoidal with respect to the cartesian product.

Remark 5.3. Since the inclusion 2-functor $i \colon \mathbb{1} \sqcup \mathbb{1} \to \mathbb{2}$ is a generating cofibration in 2Cat by [11, §3], the double functor $\mathbb{H}i \colon \mathbb{1} \sqcup \mathbb{1} \to \mathbb{H}\mathbb{2}$ is a generating cofibration in DblCat, due to Proposition 3.18. However, the pushout-product $\mathbb{H}i \,\square\, \mathbb{H}i$ with respect to the cartesian product is the double functor from the non-commutative square of horizontal morphisms to the commutative square of horizontal morphisms, as in [11, Example 7.2]. As we will see in Corollary 9.10, cofibrations in DblCat are in particular faithful on horizontal morphisms, and therefore the pushout-product $\mathbb{H}i \,\square\, \mathbb{H}i$ cannot be a cofibration in DblCat.

As we mentioned before, a Gray tensor product for DblCat is introduced by Böhm in [1]. In the same vein as in the 2-categorical case, the corresponding hom-double categories make use of the notions of pseudo horizontal and vertical transformations, and modifications between them.

Definition 5.4. Let $F, G \colon \mathbb{A} \to \mathbb{B}$ be double functors. A **pseudo horizontal natural transformation** $h \colon F \Rightarrow G$ consists of

(i) a horizontal morphism $h_A \colon FA \to GA$ in $\mathbb{B}$, for each object $A \in \mathbb{A}$,
(ii) a square $h_u \colon (Fu \,{}^{h_A}_{h_{A'}}\, Gu)$ in $\mathbb{B}$, for each vertical morphism $u \colon A \mathrel{\mkern3mu\vcenter{\hbox{$\bullet$}}\mkern-14mu{\to}} A'$ in $\mathbb{A}$, and
(iii) a vertically invertible square $h_a \colon (e_{FA} \,{}^{Ga \circ h_A}_{h_B \circ Fa}\, e_{GB})$ in $\mathbb{B}$, for each horizontal morphism $a \colon A \to B$ in $\mathbb{A}$, expressing a pseudo naturality condition for horizontal morphisms.

These assignments of squares are functorial with respect to compositions of horizontal and vertical morphisms, and these data satisfy a naturality condition with respect to squares.

Similarly, one can define a transposed notion of **pseudo vertical natural transformation** between double functors.

A **modification** in a square of pseudo horizontal and vertical transformations is defined similarly to Definition 2.3, with the horizontal and vertical coherence conditions taking the pseudo data of the transformations into account.

See [5, §3.8] or [1, §2.2] for precise definitions.

Definition 5.5. Let $\mathbb{A}$ and $\mathbb{B}$ be double categories. We define the double category $[\mathbb{A},\mathbb{B}]_{\mathrm{ps}}$ whose

(i) objects are the double functors $\mathbb{A}\to\mathbb{B}$,
(ii) horizontal morphisms are the pseudo horizontal transformations,
(iii) vertical morphisms are the pseudo vertical transformations, and
(iv) squares are the modifications.

Proposition 5.6 ([1, §3]). *There is a symmetric monoidal structure on* DblCat *given by the Gray tensor product*

$$\otimes_{\mathrm{Gr}}\colon \mathrm{DblCat}\times\mathrm{DblCat}\to\mathrm{DblCat}.$$

Moreover, this monoidal structure is closed: for all double categories $\mathbb{A}$, $\mathbb{B}$, and $\mathbb{C}$, there is an isomorphism

$$\mathrm{DblCat}(\mathbb{A}\otimes_{\mathrm{Gr}}\mathbb{B},\mathbb{C})\cong\mathrm{DblCat}(\mathbb{A},[\mathbb{B},\mathbb{C}]_{\mathrm{ps}}),$$

natural in $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$.

Since this Gray tensor product deals with the horizontal and vertical directions in an equal manner, while our model structure does not, it is not surprising that these two structures are not compatible.

Notation 5.7. Let $I\colon\mathbb{A}\to\mathbb{B}$ and $J\colon\mathbb{A}'\to\mathbb{B}'$ be double functors in DblCat. We write $I\,\square_{\mathrm{Gr}}\,J$ for their pushout-product

$$I\,\square_{\mathrm{Gr}}\,J\colon \mathbb{A}\otimes_{\mathrm{Gr}}\mathbb{B}'\coprod_{\mathbb{A}\otimes_{\mathrm{Gr}}\mathbb{A}'}\mathbb{B}\otimes_{\mathrm{Gr}}\mathbb{A}'\to\mathbb{B}\otimes_{\mathrm{Gr}}\mathbb{B}'$$

with respect to the Gray tensor product $\otimes_{\mathrm{Gr}}$ on DblCat.

Remark 5.8. The model structure defined in Theorem 3.16 is not compatible with the Gray tensor product $\otimes_{\mathrm{Gr}}$. To see this, recall that $i\colon\emptyset\to\mathbb{1}$ is a generating cofibration in 2Cat and therefore $\mathbb{L}i\colon\emptyset\to\mathbb{V}\mathbb{2}$ is a generating cofibration in DblCat, by Proposition 3.18. However the pushout-product

$$\mathbb{L}i\,\square_{\mathrm{Gr}}\,\mathbb{L}i\colon\delta(\mathbb{V}\mathbb{2}\otimes_{\mathrm{Gr}}\mathbb{V}\mathbb{2})\to\mathbb{V}\mathbb{2}\otimes_{\mathrm{Gr}}\mathbb{V}\mathbb{2}$$

is not a cofibration, where $\mathbb{V}\mathbb{2}\otimes_{\mathrm{Gr}}\mathbb{V}\mathbb{2}$ is

$$\begin{tikzcd} 0 \arrow[r, equal] \arrow[d, "\bullet" marking] & 1 \arrow[d, "\bullet" marking] \\ 0' \arrow[d, "\bullet" marking] \arrow[r, phantom, "\cong"] & 1' \arrow[d, "\bullet" marking] \\ 0'' \arrow[r, equal] & 1'' \end{tikzcd}$$

and $\delta(\mathbb{V}\mathbb{2}\otimes_{\mathrm{Gr}}\mathbb{V}\mathbb{2})$ is its sub-double category without the horizontally invertible square (just four vertical morphisms sharing some boundaries and no other relation). The fact that $\mathbb{L}i\,\square_{\mathrm{Gr}}\,\mathbb{L}i$ is not a cofibration is a consequence of (dt3) of Definition 3.8 requiring only surjectivity on vertical morphisms, instead of fullness.

5.3. 2Cat**-enrichment of the model structure on** DblCat**.** By restricting the Gray tensor product on DblCat along $\mathbb{H}$ in one of the variables, we get rid of the issue concerning the vertical structure that obstructs the compatibility with the model structure of Theorem 3.16. With this variation, we show that DblCat is a tensored and cotensored 2Cat-category, and that the corresponding enrichment is now compatible with our model structure.

Definition 5.9. We define the tensoring functor $\otimes\colon 2\mathrm{Cat}\times\mathrm{DblCat}\to\mathrm{DblCat}$ to be the composite

$$2\mathrm{Cat}\times\mathrm{DblCat}\xrightarrow{\mathbb{H}\times\mathrm{id}}\mathrm{DblCat}\times\mathrm{DblCat}\xrightarrow{\otimes_{\mathrm{Gr}}}\mathrm{DblCat}.$$

Proposition 5.10. *The category* DblCat *is enriched, tensored, and cotensored over* 2Cat*, with*

*(i) hom-*2*-categories given by* $\mathbf{H}[\mathbb{A},\mathbb{B}]_{\mathrm{ps}}$*, for all* $\mathbb{A},\mathbb{B}\in\mathrm{DblCat}$*,*
(ii) tensors given by $\mathcal{C}\otimes\mathbb{A}$*, where* $\otimes$ *is the tensoring functor of Definition 5.9, for all* $\mathbb{A}\in\mathrm{DblCat}$ *and* $\mathcal{C}\in 2\mathrm{Cat}$*, and*
(iii) cotensors given by $[\mathbb{H}\mathcal{C},\mathbb{B}]_{\mathrm{ps}}$*, for all* $\mathbb{B}\in\mathrm{DblCat}$ *and* $\mathcal{C}\in 2\mathrm{Cat}$*.*

Proof. This follows directly from the definition of $\otimes$, and the universal properties of the tensor $\otimes_{\mathrm{Gr}}$ and of the adjunction $\mathbb{H}\dashv\mathbf{H}$. □

We now present the main result of this section.

Theorem 5.11. *The model structure on* DblCat *of Theorem 3.16 is a* 2Cat*-enriched model structure, where the enrichment is given by* $[-,-]_{\mathrm{ps}}$*.*

The rest of this section is devoted to the proof of this theorem. With that goal, we first prove several auxiliary lemmas.

Notation 5.12. Let $i\colon\mathcal{A}\to\mathcal{B}$ and $j\colon\mathcal{A}'\to\mathcal{B}'$ be 2-functors in 2Cat, and let $I\colon\mathbb{A}\to\mathbb{B}$ be a double functor in DblCat. We denote by $i\square_2 j$ the pushout-product

$$i\,\square_2\,j\colon\mathcal{A}\otimes_2\mathcal{B}'\coprod_{\mathcal{A}\otimes_2\mathcal{A}'}\mathcal{B}\otimes_2\mathcal{A}'\to\mathcal{B}\otimes_2\mathcal{B}'$$

with respect to the Gray tensor product $\otimes_2$ on 2Cat, and we denote by $i\,\square\,I$ the pushout-product

$$i\,\square\,I\colon\mathcal{A}\otimes\mathbb{B}\coprod_{\mathcal{A}\otimes\mathbb{A}}\mathcal{B}\otimes\mathbb{A}\to\mathcal{B}\otimes\mathbb{B}$$

with respect to the tensoring functor $\otimes\colon 2\mathrm{Cat}\times\mathrm{DblCat}\to\mathrm{DblCat}$. In particular, we have that $i\,\square\,I=\mathbb{H}i\,\square_{\mathrm{Gr}}\,I$.

Lemma 5.13. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be* 2*-categories. There is an isomorphism of double categories*

$$\mathcal{A}\otimes\mathbb{H}\mathcal{B}\cong\mathbb{H}(\mathcal{A}\otimes_2\mathcal{B}),$$

natural in $\mathcal{A}$ *and* $\mathcal{B}$*.*

Proof. This follows from the universal properties of $\otimes$ and $\otimes_2$, and the isomorphism $\mathbf{H}[\mathbb{H}\mathcal{B}, \mathbb{C}]_{\mathrm{ps}} \cong \mathrm{Ps}[\mathcal{B}, \mathbf{H}\mathbb{C}]$, natural in $\mathcal{B} \in 2\mathrm{Cat}$ and $\mathbb{C} \in \mathrm{DblCat}$. This isomorphism holds, since there are no non trivial vertical morphisms in $\mathbb{H}\mathcal{B}$, and therefore pseudo horizontal natural transformations out of $\mathbb{H}\mathcal{B}$ are canonically the same as pseudo natural transformations out of $\mathcal{B}$. □

Remark 5.14. In particular, the natural isomorphism $\mathbf{H}[\mathbb{H}(-), -]_{\mathrm{ps}} \cong \mathrm{Ps}[-, \mathbf{H}(-)]$ implies that the adjunction $\mathbb{H} \dashv \mathbf{H}$ is enriched with respect to the 2Cat-enrichments $\mathbf{H}[-,-]_{\mathrm{ps}}$ and $\mathrm{Ps}[-,-]$ of DblCat and 2Cat, respectively.

Lemma 5.15. *Let $\mathcal{A}$ be a 2-category. There is an isomorphism of double categories*

$$\mathcal{A} \otimes \mathbb{V}\mathbb{2} \cong \mathbb{H}\mathcal{A} \times \mathbb{V}\mathbb{2},$$

natural in $\mathcal{A}$.

Proof. This follows from the universal property of $\otimes$ and of $\times$, and the fact that we have $\mathbf{H}[\mathbb{V}\mathbb{2}, \mathbb{B}]_{\mathrm{ps}} = \mathbf{H}[\mathbb{V}\mathbb{2}, \mathbb{B}]$, for all $\mathbb{B} \in \mathrm{DblCat}$. This equality holds, since there are no non trivial horizontal morphisms in $\mathbb{V}\mathbb{2}$, and therefore pseudo horizontal natural transformations out of $\mathbb{V}\mathbb{2}$ correspond to (strict) horizontal natural transformations out of $\mathbb{V}\mathbb{2}$. □

Lemma 5.16. *Let $i \colon \mathcal{A} \to \mathcal{B}$ and $j \colon \mathcal{A}' \to \mathcal{B}'$ be 2-functors in* 2Cat*. Then the following statements hold.*

- *(i) There exists an isomorphism $i \square \mathbb{H}j \cong \mathbb{H}(i \square_2 j)$ in the arrow category.*
- *(ii) There exists an isomorphism $i \square (\mathbb{H}j \times \mathbb{V}\mathbb{2}) \cong \mathbb{H}(i \square_2 j) \times \mathbb{V}\mathbb{2}$ in the arrow category.*

Proof. Since $\mathbb{H}$ is a left adjoint, it preserves pushouts. Moreover, by Lemma 5.13, we have that it is compatible with the tensors $\otimes$ and $\otimes_2$. Therefore, $i \square \mathbb{H}j \cong \mathbb{H}(i \square_2 j)$. By Lemma 5.15, by associativity of $\otimes_{\mathrm{Gr}}$, and by (i), we get that

$$i \square (\mathbb{H}j \times \mathbb{V}\mathbb{2}) \cong i \square (j \otimes \mathbb{V}\mathbb{2}) \cong (i \square \mathbb{H}j) \otimes_{\mathrm{Gr}} \mathbb{V}\mathbb{2} \cong (i \square_2 j) \otimes \mathbb{V}\mathbb{2} \cong \mathbb{H}(i \square_2 j) \times \mathbb{V}\mathbb{2}.$$

□

We are now ready to prove Theorem 5.11.

Proof of Theorem 5.11. Recall from Proposition 3.18 that a set $\mathcal{I}$ of generating cofibrations and a set $\mathcal{J}$ of generating trivial cofibrations for the model structure on DblCat are given by morphisms of the form $\mathbb{H}j$ and $\mathbb{H}j \times \mathbb{V}\mathbb{2}$, where j is a generating cofibration or a generating trivial cofibration in 2Cat, respectively.

We show that the pushout-product of a generating cofibration in $\mathcal{I}$ with any (trivial) cofibration in 2Cat is a (trivial) cofibration in DblCat, and that the pushout-product of a generating trivial cofibration in $\mathcal{J}$ with any cofibration in 2Cat is a trivial cofibration in DblCat.

Given cofibrations i and j in 2Cat, we know by Lemma 5.16 that

$$i \square \mathbb{H}j \cong \mathbb{H}(i \square_2 j) \quad \text{and} \quad i \square (\mathbb{H}j \times \mathbb{V}\mathbb{2}) \cong \mathbb{H}(i \square_2 j) \times \mathbb{V}\mathbb{2} = \mathbb{L}(i \square_2 j),$$

and by Theorem 5.2 that $i \square_2 j$ is also a cofibration in 2Cat, which is trivial when either i or j is. Since $\mathbb{H}$ and $\mathbb{L}$ preserve (trivial) cofibrations, then $\mathbb{H}(i \square_2 j)$ and $\mathbb{L}(i \square_2 j)$ are cofibrations in DblCat, which are trivial if either i or j is.

Taking j to be a generating cofibration or generating trivial cofibration in 2Cat, we get the desired results. More precisely, for all cofibrations i in 2Cat and all generating cofibrations $I \in \mathcal{I}$, we have that $i \square I$ is a cofibration in DblCat, which is trivial if i is trivial. Similarly, for all cofibrations i of 2Cat and all generating trivial cofibrations $J \in \mathcal{J}$, we have that $i \square J$ is a trivial cofibration in DblCat. $\square$

6. Comparison with other model structures on DblCat

In [3], Fiore, Paoli, and Pronk construct several model structures on the category DblCat of double categories. We show in this section that our model structure on DblCat is not related to these model structures in the following sense: the identity adjunction on DblCat is not a Quillen pair between the model structure of Theorem 3.16 and any of the model structures of [3]. This is not surprising, since our model structure was constructed in such a way that the functor $\mathbb{H}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$ embeds the homotopy theory of 2Cat into that of DblCat, while there seems to be no such relation between their model structures on DblCat and the Lack model structure on 2Cat, e.g. see end of Section 9 in [3].

We start by recalling the categorical model structures on DblCat constructed in [3]. Since our primary interest is to compare them to our model structure, we only describe the weak equivalences; the curious reader is encouraged to visit their paper for further details.

The first model structure we recall is induced from the canonical model structure on Cat by means of the *vertical nerve.*

Definition 6.1 ([3, Definition 5.1])**.** The **vertical nerve** of double categories is the functor

$$N_v\colon \mathrm{DblCat} \to \mathrm{Cat}^{\Delta^{\mathrm{op}}}$$

sending a double category $\mathbb{A}$ to the simplicial category $N_v\mathbb{A}$ such that $(N_v\mathbb{A})_0$ is the category of objects and horizontal morphisms of $\mathbb{A}$, $(N_v\mathbb{A})_1$ is the category of vertical morphisms and squares of $\mathbb{A}$, and $(N_v\mathbb{A})_n = (N_v\mathbb{A})_1 \times_{(N_v\mathbb{A})_0} \dots \times_{(N_v\mathbb{A})_0} (N_v\mathbb{A})_1$, for $n \geq 2$.

Proposition 6.2 ([3, Theorem 7.17])**.** *There is a model structure on* DblCat *in which a double functor F is a weak equivalence if and only if N_vF is levelwise an equivalence of categories.*

The next model structure on DblCat requires a different perspective. For a 2-category $\mathcal{A}$ whose underlying 1-category $U\mathcal{A}$ admits limits and colimits, there exists a model structure on $U\mathcal{A}$ in which the weak equivalences are precisely the equivalences of the 2-category $\mathcal{A}$; see [13]. When applying this construction to the 2-category DblCat_h of double categories, double functors, and horizontal natural transformations, we obtain the following model structure on DblCat.

Proposition 6.3. *There is a model structure on* DblCat*, called the* trivial model structure*, in which a double functor $F\colon \mathbb{A} \to \mathbb{B}$ is a weak equivalence if and only if it is an equivalence in the* 2*-category* DblCat_h*, i.e., there exist a double functor $G\colon \mathbb{B} \to \mathbb{A}$ and two horizontal natural isomorphisms* $\mathrm{id} \cong GF$ *and* $FG \cong \mathrm{id}$*.*

The last model structure is of a more algebraic flavor. Let T be a 2-monad on a 2-category $\mathcal{A}$. In [13], Lack gives a construction of a model structure on the category of T-algebras, in which the weak equivalences are the morphisms of T-algebras whose underlying morphism in $\mathcal{A}$ is an equivalence. In particular, double categories can be seen as the algebras over a 2-monad on the 2-category Cat(Graph) whose objects are the category objects in graphs; see [3, §9]. This gives the following model structure.

Proposition 6.4. *There is a model structure on* DblCat*, called the* algebra model structure*, in which a double functor F is a weak equivalence if and only if its underlying morphism in the* 2*-category* Cat(Graph) *is an equivalence.*

Remark 6.5. In [3, Corollary 8.29 and Theorems 8.52 and 9.1], Fiore, Paoli, and Pronk show that the model structures on DblCat of Propositions 6.2 to 6.4 coincide with model structures given by Grothendieck topologies, when double categories are seen as internal categories to Cat. Then, it follows from [3, Propositions 8.24 and 8.38] that a weak equivalence in the algebra model structure is in particular a weak equivalence in the model structure induced by the vertical nerve N_v.

Remark 6.6. At this point, we must mention that [3] defines one other model structure on DblCat, which is not equivalent to any of the above. However, this is constructed from the Thomason model structure on Cat, and is therefore not expected to have any relation to our model structure, which is closely related to the canonical model structure on Cat, as shown in Corollary 4.13.

We now proceed to compare these three model structures on DblCat to the one defined in Theorem 3.16. Our strategy will be to find a trivial cofibration in our model structure that is not a weak equivalence in any of the other model structures. Let E_{adj} be the 2-category containing two objects 0 and 1 and an adjoint equivalence between them. By [12, §6], the inclusion 2-functor $j_0\colon \mathbb{1} \to E_{\mathrm{adj}}$ at 0 is a generating trivial cofibration in the Lack model structure on 2Cat. By Proposition 3.18, the double functor $\mathbb{H}j_0\colon \mathbb{1} \to \mathbb{H}E_{\mathrm{adj}}$ is a generating trivial cofibration in the model structure of Theorem 3.16.

Lemma 6.7. *The double functor $\mathbb{H}j_0\colon \mathbb{1} \to \mathbb{H}E_{\mathrm{adj}}$ is not a weak equivalence in any of the model structures of Propositions 6.2 to 6.4.*

Proof. We first prove that $\mathbb{H}j_0$ is not a weak equivalence in the model structure on DblCat of Proposition 6.2 induced by the vertical nerve. For this, we need to show that

$$N_v(\mathbb{H}j_0)\colon N_v(\mathbb{1}) = \Delta\mathbb{1} \to N_v(\mathbb{H}E_{\mathrm{adj}})$$

is not a levelwise equivalence of categories. Indeed, the category $N_v(\mathbb{H}E_{\mathrm{adj}})_0$ is the free category generated by $\{0 \rightleftarrows 1\}$ which is not equivalent to $\mathbb{1}$.

By Remark 6.5, a weak equivalence in the algebra model structure on DblCat of Proposition 6.4 is in particular a weak equivalence in the model structure induced by the vertical nerve. Therefore, $\mathbb{H}j_0$ is not a weak equivalence in the algebra model structure either.

Finally, we show $\mathbb{H}j_0$ is not a weak equivalence in the trivial model structure on DblCat of Proposition 6.3. If $\mathbb{H}j_0$ was an equivalence in the 2-category DblCat_h, then its weak inverse would be given by the unique double functor $!\colon \mathbb{H}E_{\mathrm{adj}} \to \mathbb{1}$ and we would have a horizontal natural isomorphism $\mathrm{id}_{\mathbb{H}E_{\mathrm{adj}}} \cong \mathbb{H}j_0\,!$, where $\mathbb{H}j_0\,!$ is constant at 0. But such a horizontal natural isomorphism does not exist since 1 is not isomorphic to 0 in $\mathbb{H}E_{\mathrm{adj}}$. Therefore $\mathbb{H}j_0$ is not an equivalence. □

Proposition 6.8. *The identity adjunction on* DblCat *is not a Quillen pair between the model structure of Theorem 3.16 and any of the model structures of Propositions 6.2 to 6.4.*

Proof. We consider the identity functor $\mathrm{id}\colon \mathrm{DblCat} \to \mathrm{DblCat}$ from the model structure of Theorem 3.16 to any of the other model structures of Propositions 6.2 to 6.4, and we show that it is neither left nor right Quillen.

Since $\mathbb{H}j_0$ is a trivial cofibration in the model structure of Theorem 3.16, but is not a weak equivalence in any of the other model structures by Lemma 6.7, we see that id does not preserve trivial cofibrations; therefore, it is not left Quillen. Moreover, every object is fibrant in the model structure of Theorem 3.16, so that if id was right Quillen, it would preserve all weak equivalences by Ken Brown's Lemma (see [9, Lemma 1.1.12]). However, it does not preserve the weak equivalence $\mathbb{H}j_0$, and thus it is not right Quillen. □

Part II. Technical results

7. Path objects in double categories

This section is devoted to the proof of Proposition 3.15, regarding the existence of path objects for double categories. For the reader's convenience, we recall the precise statement of Proposition 3.15 here.

Proposition 3.15. *For every double category $\mathbb{A}$, there exists a double category $\mathcal{P}\mathbb{A}$ together with a factorization of the diagonal double functor*

$$\mathbb{A} \xrightarrow{W} \mathcal{P}\mathbb{A} \xrightarrow{P} \mathbb{A} \times \mathbb{A}$$

such that W is a double biequivalence and P is a double fibration.

We start by defining the path object $\mathcal{P}\mathbb{A}$ for a double category $\mathbb{A}$.

Definition 7.1. Let $\mathbb{A}$ be a double category. The **path object** $\mathcal{P}\mathbb{A}$ is the double category defined by the following data.

(i) An object (A, a_0, a_1) in $\mathcal{P}\mathbb{A}$ consists of a triple (A, A_0, A_1) of objects in $\mathbb{A}$ together with horizontal equivalences $a_0\colon A_0 \xrightarrow{\simeq} A$ and $a_1\colon A_1 \xrightarrow{\simeq} A$ in $\mathbb{A}$.

(ii) A horizontal morphism $(f, \varphi_0, \varphi_1)\colon (A, a_0, a_1) \to (B, b_0, b_1)$ in $\mathcal{P}\mathbb{A}$ consists of three horizontal morphisms in $\mathbb{A}$

$$f\colon A \to B, \quad f_0\colon A_0 \to B_0, \quad \text{and} \quad f_1\colon A_1 \to B_1,$$

together with two vertically invertible squares φ_0 and φ_1 in $\mathbb{A}$ as depicted below.

$$
\begin{tikzcd}
A_0 \arrow[r, "f_0"] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "\varphi_0\ \cong"] & B_0 \arrow[r, "\simeq", "b_0"'] & B \arrow[d, equal, "\bullet" marking] \\
A_0 \arrow[r, "\simeq", "a_0"'] & A \arrow[r, "f"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
A_1 \arrow[r, "f_1"] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "\varphi_1\ \cong"] & B_1 \arrow[r, "\simeq", "b_1"'] & B \arrow[d, equal, "\bullet" marking] \\
A_1 \arrow[r, "\simeq", "a_1"'] & A \arrow[r, "f"'] & B
\end{tikzcd}
$$

(iii) Composition of horizontal morphisms is defined as follows: given horizontal morphisms $(f, \varphi_0, \varphi_1)\colon (A, a_0, a_1) \to (B, b_0, b_1)$ and $(g, \psi_0, \psi_1)\colon (B, b_0, b_1) \to (C, c_0, c_1)$, their composite is given by the composites

$$gf\colon A \to C, \quad g_0 f_0\colon A_0 \to C_0, \quad \text{and} \quad g_1 f_1\colon A_1 \to C_1,$$

together with the vertically invertible squares in $\mathbb{A}$ obtained by the pastings

$$
\begin{tikzcd}
A_0 \arrow[r, "f_0"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "e_{f_0}"] & B_0 \arrow[r, "g_0"] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "\psi_0\ \cong"] & C_0 \arrow[r, "\simeq", "c_0"'] & C \arrow[d, equal, "\bullet" marking] \\
A_0 \arrow[r, "f_0"'] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "\varphi_0\ \cong"] & B_0 \arrow[r, "\simeq", "b_0"'] & B \arrow[r, "g"'] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "e_g"] & C \arrow[d, equal, "\bullet" marking] \\
A_0 \arrow[r, "\simeq", "a_0"'] & A \arrow[r, "f"'] & B \arrow[r, "g"'] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
A_1 \arrow[r, "f_1"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "e_{f_1}"] & B_1 \arrow[r, "g_1"] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "\psi_1\ \cong"] & C_1 \arrow[r, "\simeq", "c_1"'] & C \arrow[d, equal, "\bullet" marking] \\
A_1 \arrow[r, "f_1"'] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "\varphi_1\ \cong"] & B_1 \arrow[r, "\simeq", "b_1"'] & B \arrow[r, "g"'] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "e_g"] & C \arrow[d, equal, "\bullet" marking] \\
A_1 \arrow[r, "\simeq", "a_1"'] & A \arrow[r, "f"'] & B \arrow[r, "g"'] & C\,.
\end{tikzcd}
$$

(iv) A vertical morphism $(u, \mu_0, \mu_1)\colon (A, a_0, a_1) \mathrel{\to\!\!\!\!\!\bullet\;} (A', a'_0, a'_1)$ consists of three vertical morphisms in $\mathbb{A}$

$$u\colon A \mathrel{\to\!\!\!\!\!\bullet\;} A', \quad u_0\colon A_0 \mathrel{\to\!\!\!\!\!\bullet\;} A'_0, \quad \text{and} \quad u_1\colon A_1 \mathrel{\to\!\!\!\!\!\bullet\;} A'_1,$$

together with weakly horizontally invertible squares μ_0 and μ_1 in $\mathbb{A}$ as depicted below.

$$
\begin{tikzcd}
A_0 \arrow[r, "\simeq", "a_0"'] \arrow[d, "u_0"', "\bullet" marking] \arrow[dr, phantom, "\mu_0\ \simeq"] & A \arrow[d, "u", "\bullet" marking] \\
A'_0 \arrow[r, "\simeq", "a'_0"'] & A'
\end{tikzcd}
\qquad
\begin{tikzcd}
A_1 \arrow[r, "\simeq", "a_1"'] \arrow[d, "u_1"', "\bullet" marking] \arrow[dr, phantom, "\mu_1\ \simeq"] & A \arrow[d, "u", "\bullet" marking] \\
A'_1 \arrow[r, "\simeq", "a'_1"'] & A'
\end{tikzcd}
$$

(v) Composition of vertical morphisms is given by the vertical composition in each component of vertical morphisms and squares in $\mathbb{A}$.

(vi) A square in $\mathcal{P}\mathbb{A}$

$$
\begin{tikzcd}
(A, a_0, a_1) \arrow[r, "{(f, \varphi_0, \varphi_1)}"] \arrow[d, "{(u, \mu_0, \mu_1)}"', "\bullet" marking] \arrow[dr, phantom, "{(\alpha, \alpha_0, \alpha_1)}"] & (B, b_0, b_1) \arrow[d, "{(v, \nu_0, \nu_1)}", "\bullet" marking] \\
(A', a'_0, a'_1) \arrow[r, "{(g, \psi_0, \psi_1)}"'] & (B', b'_0, b'_1)
\end{tikzcd}
$$

consists of three squares in $\mathbb{A}$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"', "\bullet" marking] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "v", "\bullet" marking] \\ A' \arrow[r, "g"'] & B' \end{tikzcd}, \qquad \begin{tikzcd} A_0 \arrow[r, "f_0"] \arrow[d, "u_0"', "\bullet" marking] \arrow[dr, phantom, "\alpha_0"] & B_0 \arrow[d, "v_0", "\bullet" marking] \\ A'_0 \arrow[r, "g_0"'] & B'_0 \end{tikzcd}, \qquad \begin{tikzcd} A_1 \arrow[r, "f_1"] \arrow[d, "u_1"', "\bullet" marking] \arrow[dr, phantom, "\alpha_1"] & B_1 \arrow[d, "v_1", "\bullet" marking] \\ A'_1 \arrow[r, "g_1"'] & B'_1 \end{tikzcd}$$

satisfying the following pasting equality, for each $i = 0, 1$.

$$\begin{tikzcd} A_i \arrow[r, "f_i"] \arrow[d, equal, "\bullet" marking] & B_i \arrow[r, "\simeq", "b_i"'] \arrow[d, phantom, "\varphi_i\ \cong"] & B \arrow[d, equal, "\bullet" marking] \\ A_i \arrow[r, "\simeq", "a_i"'] \arrow[d, "u_i"', "\bullet" marking] \arrow[dr, phantom, "\mu_i\ \cong"] & A \arrow[r, "f"] \arrow[d, "u", "\bullet" marking] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "v", "\bullet" marking] \\ A'_0 \arrow[r, "\simeq", "a'_i"'] & A' \arrow[r, "g"'] & B' \end{tikzcd} = \begin{tikzcd} A_i \arrow[r, "f_i"] \arrow[d, "u"', "\bullet" marking] \arrow[dr, phantom, "\alpha_i"] & B_i \arrow[r, "\simeq", "b_i"'] \arrow[d, "u_i"', "\bullet" marking] \arrow[dr, phantom, "\nu_i\ \cong"] & B \arrow[d, "v_i", "\bullet" marking] \\ A'_i \arrow[r, "g_i"'] \arrow[d, equal, "\bullet" marking] & B'_i \arrow[r, "\simeq", "b'_i"'] \arrow[d, phantom, "\psi_i\ \cong"] & B' \arrow[d, equal, "\bullet" marking] \\ A'_i \arrow[r, "\simeq", "a'_i"'] & A' \arrow[r, "g"'] & B' \end{tikzcd}$$

(vii) Horizontal and vertical compositions of squares are given by componentwise horizontal and vertical compositions of squares in $\mathbb{A}$.

This path object comes with two double functors

$$\mathbb{A} \xrightarrow{W} \mathcal{P}\mathbb{A} \xrightarrow{P} \mathbb{A} \times \mathbb{A}.$$

Definition 7.2. The double functor $W\colon \mathbb{A} \to \mathcal{P}\mathbb{A}$ sends

(i) an object $A \in \mathbb{A}$ to the object $(A, \mathrm{id}_A, \mathrm{id}_A) \in \mathcal{P}\mathbb{A}$,
(ii) a horizontal morphism f in $\mathbb{A}$ to the horizontal morphism (f, e_f, e_f) in $\mathcal{P}\mathbb{A}$,
(iii) a vertical morphism u in $\mathbb{A}$ to the vertical morphism $(u, \mathrm{id}_u, \mathrm{id}_u)$ in $\mathcal{P}\mathbb{A}$, and
(iv) a square $\alpha\colon (u \, {}^{f}_{g} \, v)$ to the square (α, α, α) in $\mathcal{P}\mathbb{A}$.

Definition 7.3. The double functor $P\colon \mathcal{P}\mathbb{A} \to \mathbb{A} \times \mathbb{A}$ sends

(i) an object $(A, a_0, a_1) \in \mathcal{P}\mathbb{A}$ to the object $(A_0, A_1) \in \mathbb{A} \times \mathbb{A}$,
(ii) a horizontal morphism $(f, \varphi_0, \varphi_1)$ in $\mathcal{P}\mathbb{A}$ to the horizontal morphism (f_0, f_1) in $\mathbb{A} \times \mathbb{A}$,
(iii) a vertical morphism (u, μ_0, μ_1) in $\mathcal{P}\mathbb{A}$ to the vertical morphism (u_0, u_1) in $\mathbb{A} \times \mathbb{A}$, and
(iv) a square $(\alpha, \alpha_0, \alpha_1)$ in $\mathcal{P}\mathbb{A}$ to the square (α_0, α_1) in $\mathbb{A} \times \mathbb{A}$.

It follows directly from the definitions that the composite PW is the diagonal functor $\mathbb{A} \to \mathbb{A} \times \mathbb{A}$. Furthermore, the functors W and P have the desired properties, as we show in the following lemmas.

Lemma 7.4. *The double functor $W\colon \mathbb{A} \to \mathcal{P}\mathbb{A}$ is a double biequivalence.*

Proof. We prove (db1-4) of Definition 3.5.

We first prove (db1). Let (A, a_0, a_1) be an object in $\mathcal{P}\mathbb{A}$. Then $WA = (A, \mathrm{id}_A, \mathrm{id}_A)$ comes with a horizontal equivalence $(A, a_0, a_1) \xrightarrow{\simeq} WA$ in $\mathcal{P}\mathbb{A}$ given by

$$\mathrm{id}_A \colon A \xrightarrow{\simeq} A, \quad a_0 \colon A_0 \xrightarrow{\simeq} A, \quad \text{and} \quad a_1 \colon A_1 \xrightarrow{\simeq} A,$$

and the weakly invertible squares e_{a_0} and e_{a_1}.

We now prove (db2). Let $(f, \varphi_0, \varphi_1) \colon WA \to WB$ be a morphism in $\mathcal{P}\mathbb{A}$, i.e., φ_0 and φ_1 are vertically invertible squares in $\mathbb{A}$ of the form

$$\begin{tikzcd} A \arrow[r, "f_0"] \arrow[d, equal] \arrow[dr, phantom, "\varphi_0 \cong"] & B \arrow[d, equal] \\ A \arrow[r, "f"'] & B \end{tikzcd}, \qquad \begin{tikzcd} A \arrow[r, "f_1"] \arrow[d, equal] \arrow[dr, phantom, "\varphi_1 \cong"] & B \arrow[d, equal] \\ A \arrow[r, "f"'] & B \end{tikzcd}.$$

Then $Wf = (f, e_f, e_f) \colon WA \to WB$ comes with a vertically invertible square in $\mathcal{P}\mathbb{A}$

$$\begin{tikzcd} WA \arrow[r, "{(f,\varphi_0,\varphi_1)}"] \arrow[d, equal] \arrow[dr, phantom, "\cong"] & WB \arrow[d, equal] \\ WA \arrow[r, "Wf"'] & WB \end{tikzcd}$$

given by the triple of vertically invertible squares $(e_f, \varphi_0, \varphi_1)$.

We now prove (db3). Let $(u, \mu_0, \mu_1) \colon (A, a_0, a_1) \to (A', a'_0, a'_1)$ be a vertical morphism in $\mathcal{P}\mathbb{A}$. Then $Wu = (u, \mathrm{id}_u, \mathrm{id}_u) \colon WA \to WA'$ comes with a weakly horizontally invertible square in $\mathcal{P}\mathbb{A}$

$$\begin{tikzcd} (A, a_0, a_1) \arrow[r, "\simeq", "{(\mathrm{id}_A, e_{a_0}, e_{a_1})}"'] \arrow[d, "{(u,\mu_0,\mu_1)}"'] \arrow[dr, phantom, "\simeq"] & WA \arrow[d, "Wu"] \\ (A', a'_0, a'_1) \arrow[r, "\simeq", "{(\mathrm{id}_{A'}, e_{a'_0}, e_{a'_1})}"'] & WA' \end{tikzcd}$$

given by the triple of weakly horizontally invertible squares $(\mathrm{id}_u, \mu_0, \mu_1)$.

Finally, we prove (db4). Let $(\alpha, \alpha_0, \alpha_1) \colon (Wu \; {}^{Wf}_{Wg} \; Wv)$ be a square in $\mathcal{P}\mathbb{A}$. By the relations in Definition 7.1 (vi), we must have $\alpha = \alpha_0 = \alpha_1$. Then $W\alpha = (\alpha, \alpha, \alpha) = (\alpha, \alpha_0, \alpha_1)$ and it is the unique square in $\mathbb{A}$ satisfying this equality. □

Lemma 7.5. *The double functor $P \colon \mathcal{P}\mathbb{A} \to \mathbb{A} \times \mathbb{A}$ is a double fibration.*

Proof. We prove (df1-3) of Definition 3.7.

We first prove (df1). Let (C, c_0, c_1) be an object in $\mathcal{P}\mathbb{A}$ with $P(C, c_0, c_1) = (C_0, C_1)$, and let $(f_0, f_1)\colon (B_0, B_1) \xrightarrow{\simeq} (C_0, C_1)$ be a horizontal equivalence in $\mathbb{A} \times \mathbb{A}$. We define the object $(C, c_0 f_0, c_1 f_1)$ in $\mathcal{P}\mathbb{A}$ through the horizontal equivalences

$$B_0 \xrightarrow[f_0]{\simeq} C_0 \xrightarrow[c_0]{\simeq} C\,, \qquad B_1 \xrightarrow[f_1]{\simeq} C_1 \xrightarrow[c_1]{\simeq} C\,,$$

and then the horizontal equivalence $(\mathrm{id}_C, e_{c_0 f_0}, e_{c_1 f_1})\colon (C, c_0 f_0, c_1 f_1) \xrightarrow{\simeq} (C, c_0, c_1)$ in $\mathcal{P}\mathbb{A}$ given by $(\mathrm{id}_C, f_0, f_1)$ is sent via P to (f_0, f_1).

We now prove (df2). Let $(g, \psi_0, \psi_1)\colon (A, a_0, a_1) \to (C, c_0, c_1)$ be a horizontal morphism in $\mathcal{P}\mathbb{A}$, and let (β_0, β_1) be a vertically invertible square in $\mathbb{A} \times \mathbb{A}$ as below,

$$\begin{tikzcd} A_0 \arrow[r, "f_0"] \arrow[d, equal] \arrow[dr, phantom, "\beta_0 \cong"] & C_0 \arrow[d, equal] \\ A_0 \arrow[r, "g_0"'] & C_0 \end{tikzcd} \qquad \begin{tikzcd} A_1 \arrow[r, "f_1"] \arrow[d, equal] \arrow[dr, phantom, "\beta_1 \cong"] & C_1 \arrow[d, equal] \\ A_1 \arrow[r, "g_1"'] & C_1 \end{tikzcd}$$

where $(g_0, g_1) = P(g, \psi_0, \psi_1)$. We define the morphism $(g, \varphi_0, \varphi_1)\colon (A, a_0, a_1) \to (C, c_0, c_1)$ through the horizontal morphisms

$$g\colon A \to C, \quad f_0\colon A_0 \to C_0, \quad \text{and} \quad f_1\colon A_1 \to C_1,$$

and the vertically invertible squares φ_0 and φ_1 given by the following pastings.

$$\begin{tikzcd} A_0 \arrow[r, "f_0"] \arrow[d, equal] \arrow[dr, phantom, "\beta_0 \cong"] & C_0 \arrow[r, "\simeq", "c_0"'] \arrow[d, equal] \arrow[dr, phantom, "e_{c_0}"] & C \arrow[d, equal] \\ A_0 \arrow[r, "g_0"'] \arrow[d, equal] \arrow[drr, phantom, "\psi_0 \cong"] & C_0 \arrow[r, "\simeq", "c_0"'] & C \arrow[d, equal] \\ A_0 \arrow[r, "\simeq", "a_0"'] & A \arrow[r, "g"'] & C \end{tikzcd} \qquad \begin{tikzcd} A_1 \arrow[r, "f_1"] \arrow[d, equal] \arrow[dr, phantom, "\beta_1 \cong"] & C_1 \arrow[r, "\simeq", "c_1"'] \arrow[d, equal] \arrow[dr, phantom, "e_{c_1}"] & C \arrow[d, equal] \\ A_1 \arrow[r, "g_1"'] \arrow[d, equal] \arrow[drr, phantom, "\psi_1 \cong"] & C_1 \arrow[r, "\simeq", "c_1"'] & C \arrow[d, equal] \\ A_1 \arrow[r, "\simeq", "a_1"'] & A \arrow[r, "g"'] & C \end{tikzcd}$$

Then the vertically invertible square in $\mathcal{P}\mathbb{A}$

$$\begin{tikzcd} (A, a_0, a_1) \arrow[r, "{(g, \varphi_0, \varphi_1)}"] \arrow[d, equal] \arrow[dr, phantom, "\cong"] & (C, c_0, c_1) \arrow[d, equal] \\ (A, \varphi_0, \varphi_1) \arrow[r, "{(g, \psi_0, \psi_1)}"'] & (C, \psi_0, \psi_1) \end{tikzcd}$$

given by (e_g, β_0, β_1) is sent via P to (β_0, β_1).

Finally, we prove (df3). Let $(u, \nu_0, \nu_1)\colon (C, c_0, c_1) \twoheadrightarrow (C', c_0', c_1')$ be a vertical morphism in $\mathcal{P}\mathbb{A}$, and let (β_0, β_1) be a weakly horizontally invertible square in $\mathbb{A} \times \mathbb{A}$ as below,

$$\begin{tikzcd} B_0 \arrow[r, "\simeq", "f_0"'] \arrow[d, "v_0"', "\bullet" marking] \arrow[dr, phantom, "\beta_0\ \simeq"] & C_0 \arrow[d, "u_0", "\bullet" marking] \\ B_0' \arrow[r, "\simeq", "g_0"'] & C_0' \end{tikzcd} \qquad \begin{tikzcd} B_1 \arrow[r, "\simeq", "f_1"'] \arrow[d, "v_1"', "\bullet" marking] \arrow[dr, phantom, "\beta_1\ \simeq"] & C_1 \arrow[d, "u_1", "\bullet" marking] \\ B_1' \arrow[r, "\simeq", "g_1"'] & C_1' \end{tikzcd}$$

where $(u_0, u_1) = P(u, \nu_0, \nu_1)$. We set $a_0 = c_0 f_0$, $a_1 = c_1 f_1$, $a_0' = c_0' g_0$, and $a_1' = c_1' g_1$. We define the morphism $(u, \mu_0, \mu_1)\colon (C, a_0, a_1) \to (C', a_0', a_1')$ in $\mathcal{P}\mathbb{A}$ through the vertical morphisms

$$u\colon C \twoheadrightarrow C', \quad v_0\colon B_0 \twoheadrightarrow B_0', \quad \text{and} \quad v_1\colon B_1 \twoheadrightarrow B_1'$$

and the weakly invertible squares μ_0 and μ_1 given by the following pastings.

$$\begin{tikzcd} B_0 \arrow[r, "\simeq", "f_0"'] \arrow[d, "v_0"', "\bullet" marking] \arrow[dr, phantom, "\beta_0\ \simeq"] & C_0 \arrow[r, "\simeq", "c_0"'] \arrow[d, "u_0", "\bullet" marking] \arrow[dr, phantom, "\nu_0 \simeq"] & C \arrow[d, "u", "\bullet" marking] \\ B_0' \arrow[r, "\simeq", "g_0"'] & C_0' \arrow[r, "\simeq", "c_0'"'] & C' \end{tikzcd} \qquad \begin{tikzcd} B_1 \arrow[r, "\simeq", "f_1"'] \arrow[d, "v_1"', "\bullet" marking] \arrow[dr, phantom, "\beta_1\ \simeq"] & C_1 \arrow[r, "\simeq", "c_1"'] \arrow[d, "u_1", "\bullet" marking] \arrow[dr, phantom, "\nu_1 \simeq"] & C \arrow[d, "u", "\bullet" marking] \\ B_1' \arrow[r, "\simeq", "g_1"'] & C_1' \arrow[r, "\simeq", "c_1'"'] & C' \end{tikzcd}$$

Then the weakly horizontally invertible square in $\mathcal{P}\mathbb{A}$

$$\begin{tikzcd} (C, a_0, a_1) \arrow[r, "\simeq", "(\mathrm{id}_C, e_{a_0}, e_{a_1})"'] \arrow[d, "(u, \mu_0, \mu_1)"', "\bullet" marking] \arrow[dr, phantom, "\simeq"] & (C, c_0, c_1) \arrow[d, "(u, \nu_0, \nu_1)", "\bullet" marking] \\ (C', a_0', a_1') \arrow[r, "\simeq", "(\mathrm{id}_{C'}, e_{a_0'}, e_{a_1'})"'] & (C', c_0', c_1') \end{tikzcd}$$

given by $(\mathrm{id}_u, \beta_0, \beta_1)$ is sent via P to (β_0, β_1). □

The above constructions provide a proof of Proposition 3.15, as we now summarize.

Proof of Proposition 3.15. Let $\mathbb{A}$ be a double category, and let $\mathcal{P}\mathbb{A}$ be the path object constructed in Definition 7.1. Consider the functors

$$\mathbb{A} \xrightarrow{W} \mathcal{P}\mathbb{A} \xrightarrow{P} \mathbb{A} \times \mathbb{A}$$

as in Definitions 7.2 and 7.3; these provide a factorization of the diagonal functor. Moreover, Lemmas 7.4 and 7.5 show that W is a double biequivalence and P is a double fibration. □

8. Characterization of weak equivalences and fibrations

This section provides proofs of Propositions 3.10 and 3.11, which claim that the weak equivalences and fibrations of the right-induced model structure on DblCat of Theorem 3.16 are precisely the double biequivalences of Definition 3.5 and the double fibrations of Definition 3.7.

We first focus on Proposition 3.10, dealing with weak equivalences. In order to characterize the functors F such that $(\mathbf{H},\mathcal{V})F$ is a weak equivalence, we express what it means for $\mathbf{H}F$ and $\mathcal{V}F$ to be biequivalences in 2Cat; this is done by translating (b1-3) of Definition 3.1 for these 2-functors.

Remark 8.1. Let $F\colon \mathbb{A}\to\mathbb{B}$ be a double functor. Then $\mathbf{H}F\colon \mathbf{H}\mathbb{A}\to\mathbf{H}\mathbb{B}$ is a biequivalence in 2Cat if and only if it satisfies (db1-2) of Definition 3.5, and the following condition:

(hb3) for every square in $\mathbb{B}$ of the form

$$\begin{tikzcd} FA \arrow[r, "Fa"] \arrow[d, equal] \arrow[dr, phantom, "\beta"] & FC \arrow[d, equal] \\ FA \arrow[r, "Fc"'] & FC \end{tikzcd},$$

there exists a unique square $\alpha\colon (e_A \; {}^{a}_{c} \; e_C)$ in $\mathbb{A}$ such that $F\alpha=\beta$.

Remark 8.2. Let $F\colon \mathbb{A}\to\mathbb{B}$ be a double functor. Then $\mathcal{V}F\colon \mathcal{V}\mathbb{A}\to\mathcal{V}\mathbb{B}$ is a biequivalence in 2Cat if and only if it satisfies (db3) of Definition 3.5, and the following conditions:

(vb2) for every square $\beta\colon (Fu \; {}^{b}_{d} \; Fu')$ in $\mathbb{B}$, there exist a square $\alpha\colon (u \; {}^{a}_{c} \; u')$ in $\mathbb{A}$ and two vertically invertible squares in $\mathbb{B}$ such that the following pasting equality holds,

$$\begin{tikzcd} FA \arrow[r, "b"] \arrow[d, equal] \arrow[dr, phantom, "\cong"] & FC \arrow[d, equal] \\ FA \arrow[r, "Fa"] \arrow[d, "Fu"'] \arrow[dr, phantom, "F\alpha"] & FC \arrow[d, "Fu'"] \\ FA' \arrow[r, "Fc"'] & FC' \end{tikzcd} = \begin{tikzcd} FA \arrow[r, "b"] \arrow[d, "Fu"'] \arrow[dr, phantom, "\beta"] & FC \arrow[d, "Fu'"] \\ FA' \arrow[r, "d"'] \arrow[d, equal] \arrow[dr, phantom, "\cong"] & FC' \arrow[d, equal] \\ FA' \arrow[r, "Fc"'] & FC' \end{tikzcd}$$

(vb3) for all squares τ_0 and τ_1 as in the pasting equality below left, there exist unique squares $\sigma_0\colon (e_A \; {}^{a}_{a'} \; e_C)$ and $\sigma_1\colon (e_{A'} \; {}^{c}_{c'} \; e_{C'})$ in $\mathbb{A}$ satisfying the pasting equality below right, and such that $F\sigma_0=\tau_0$ and $F\sigma_1=\tau_1$.

$$\begin{tikzcd}
FA \arrow[r, "Fa"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\tau_0"] & FC \arrow[d, equal, "\bullet" marking] \\
FA \arrow[r, "Fa'"] \arrow[d, "Fu"', "\bullet" marking] \arrow[dr, phantom, "F\alpha'"] & FC \arrow[d, "Fu'", "\bullet" marking] \\
FA' \arrow[r, "Fc'"'] & FC'
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "Fa"] \arrow[d, "Fu"', "\bullet" marking] \arrow[dr, phantom, "F\alpha"] & FC \arrow[d, "Fu'", "\bullet" marking] \\
FA' \arrow[r, "Fc"'] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\tau_1"] & FC' \arrow[d, equal, "\bullet" marking] \\
FA' \arrow[r, "Fc'"'] & FC'
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\sigma_0"] & C \arrow[d, equal, "\bullet" marking] \\
A \arrow[r, "a'"] \arrow[d, "u"', "\bullet" marking] \arrow[dr, phantom, "\alpha'"] & C \arrow[d, "u'", "\bullet" marking] \\
A' \arrow[r, "c'"'] & C'
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "u"', "\bullet" marking] \arrow[dr, phantom, "\alpha"] & C \arrow[d, "u'", "\bullet" marking] \\
A' \arrow[r, "c"'] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\sigma_1"] & C' \arrow[d, equal, "\bullet" marking] \\
A' \arrow[r, "c'"'] & C'
\end{tikzcd}$$

The reader may have noticed that condition (db4) in Definition 3.7 has not been mentioned so far. The following lemma explains how the additional conditions (hb3) and (vb2-3) introduced in Remarks 8.1 and 8.2 relate to (db4).

Lemma 8.3. *Suppose $F\colon \mathbb{A} \to \mathbb{B}$ is a double functor satisfying (hb3) of Remark 8.1, and (vb2-3) of Remark 8.2. Then, for every square in $\mathbb{B}$ of the form*

$$\begin{tikzcd}
FA \arrow[r, "Fa"] \arrow[d, "Fu"', "\bullet" marking] \arrow[dr, phantom, "\beta"] & FC \arrow[d, "Fu'", "\bullet" marking] \\
FA' \arrow[r, "Fc"'] & FC' ,
\end{tikzcd}$$

there exists a unique square $\alpha\colon (u \,{}^{a}_{c}\, u')$ in $\mathbb{A}$ such that $F\alpha = \beta$.

Proof. Suppose $\beta\colon (Fu \,{}^{Fa}_{Fc}\, Fu')$ is a square in $\mathbb{B}$ as above. By (vb2) of Remark 8.2, there exists a square $\overline{\alpha}\colon (u \,{}^{\overline{a}}_{\overline{c}}\, u')$ in $\mathbb{A}$ and two vertically invertible squares ψ_0, ψ_1 in $\mathbb{B}$ such that the following pasting equality holds.

$$\begin{tikzcd}
FA \arrow[r, "Fa"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\psi_0 \, \cong"] & FC \arrow[d, equal, "\bullet" marking] \\
FA \arrow[r, "F\overline{a}"] \arrow[d, "Fu"', "\bullet" marking] \arrow[dr, phantom, "F\overline{\alpha}"] & FC \arrow[d, "Fu'", "\bullet" marking] \\
FA' \arrow[r, "F\overline{c}"'] & FC'
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "Fa"] \arrow[d, "Fu"', "\bullet" marking] \arrow[dr, phantom, "\beta"] & FC \arrow[d, "Fu'", "\bullet" marking] \\
FA' \arrow[r, "Fc"'] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\psi_1 \, \cong"] & FC' \arrow[d, equal, "\bullet" marking] \\
FA' \arrow[r, "F\overline{c}"'] & FC'
\end{tikzcd}$$

By (hb3) of Remark 8.1, there exist unique squares φ_0 and φ_1 in $\mathbb{A}$

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\varphi_0"] & C \arrow[d, equal, "\bullet" marking] \\
A \arrow[r, "\overline{a}"'] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
A' \arrow[r, "c"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\varphi_1"] & C' \arrow[d, equal, "\bullet" marking] \\
A' \arrow[r, "\overline{c}"'] & C'
\end{tikzcd}$$

such that $F\varphi_0 = \psi_0$ and $F\varphi_1 = \psi_1$. Moreover, the squares φ_0 and φ_1 are vertically invertible by the unicity condition in (hb3) of Remark 8.1. Therefore, the square α given by the following vertical pasting

$$
\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "u"', "\bullet" marking] \arrow[dr, phantom, "\alpha"] & C \arrow[d, "u'", "\bullet" marking] \\
A' \arrow[r, "c"'] & C'
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\varphi_0 \, \cong"] & C \arrow[d, equal, "\bullet" marking] \\
A \arrow[r, "\overline{a}"] \arrow[d, "u"', "\bullet" marking] \arrow[dr, phantom, "\overline{\alpha}"] & C \arrow[d, "u'", "\bullet" marking] \\
A' \arrow[r, "\overline{c}"'] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\varphi_1^{-1} \, \cong"] & C' \arrow[d, equal, "\bullet" marking] \\
A' \arrow[r, "c"'] & C'
\end{tikzcd}
$$

is such that $F\alpha = \beta$. This settles the matter of the existence of the square α. Now suppose there are two squares α and α' in $\mathbb{A}$

$$
\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "u"', "\bullet" marking] \arrow[dr, phantom, "\alpha"] & C \arrow[d, "u'", "\bullet" marking] \\
A \arrow[r, "c"'] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
A' \arrow[r, "a"] \arrow[d, "u"', "\bullet" marking] \arrow[dr, phantom, "\alpha'"] & C' \arrow[d, "u'", "\bullet" marking] \\
A' \arrow[r, "c"'] & C'
\end{tikzcd}
$$

such that $F\alpha = \beta = F\alpha'$. Take $\tau_0 = e_{Fa}$ and $\tau_1 = e_{Fc}$ in (vb3) of Remark 8.2. This gives unique squares σ_0 and σ_1 in $\mathbb{A}$ such that the following pasting equality holds

$$
\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\sigma_0"] & C \arrow[d, equal, "\bullet" marking] \\
A \arrow[r, "a"] \arrow[d, "u"', "\bullet" marking] \arrow[dr, phantom, "\alpha'"] & C \arrow[d, "u'", "\bullet" marking] \\
A' \arrow[r, "c"'] & C'
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "u"', "\bullet" marking] \arrow[dr, phantom, "\alpha"] & C \arrow[d, "u'", "\bullet" marking] \\
A' \arrow[r, "c"'] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\sigma_1"] & C' \arrow[d, equal, "\bullet" marking] \\
A' \arrow[r, "c"'] & C'
\end{tikzcd}
$$

and $F\sigma_0 = e_{Fa}$ and $F\sigma_1 = e_{Fc}$. By unicity in (hb3) of Remark 8.1, we must have $\sigma_0 = e_a$ and $\sigma_1 = e_c$. Replacing σ_0 and σ_1 by e_a and e_c in the pasting diagram above implies that $\alpha = \alpha'$. This proves unicity. □

We can now use the above results to prove Proposition 3.10, giving the characterization of the weak equivalences.

Proof of Proposition 3.10. Suppose that $F \colon \mathbb{A} \to \mathbb{B}$ is a double functor such that both $\mathbf{H}F$ and $\mathcal{V}F$ are biequivalences in 2Cat. By Remarks 8.1 and 8.2, we directly have (db1-3) of

Definition 3.5. Moreover, by Lemma 8.3, we also have (db4) of Definition 3.5. This shows that F is a double biequivalence.

Now suppose that $F\colon \mathbb{A} \to \mathbb{B}$ is a double biequivalence. We want to show that both $\mathbf{H}F$ and $\mathcal{V}F$ are biequivalences in 2Cat. To show that $\mathbf{H}F$ is a biequivalence, it suffices to show that (hb3) of Remark 8.1 is satisfied; this follows directly from taking u and u' to be vertical identities in (db4) of Definition 3.5.

It remains to show that $\mathcal{V}F$ is a biequivalence; we do so by proving (vb2-3) of Remark 8.2. To prove (vb2), let β be a square in $\mathbb{B}$ of the form

$$\begin{tikzcd}
FA \arrow[r, "b"] \arrow[d, "Fu"', "\bullet" marking] \arrow[dr, phantom, "\beta"] & FC \arrow[d, "Fu'", "\bullet" marking] \\
FA' \arrow[r, "d"'] & FC' .
\end{tikzcd}$$

By (db2) of Definition 3.5, there exist horizontal morphisms $a\colon A \to C$ and $c\colon A' \to C'$ in $\mathbb{A}$ and vertically invertible squares φ_0 and φ_1 in $\mathbb{B}$ as depicted below.

$$\begin{tikzcd}
FA \arrow[r, "b"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\varphi_0\ \cong"] & FC \arrow[d, equal, "\bullet" marking] \\
FA \arrow[r, "Fa"'] & FC
\end{tikzcd}
\qquad
\begin{tikzcd}
FA' \arrow[r, "d"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\varphi_1\ \cong"] & FC' \arrow[d, equal, "\bullet" marking] \\
FA' \arrow[r, "Fc"'] & FC'
\end{tikzcd}$$

By (db4) of Definition 3.5, there exists a unique square $\alpha\colon (u \,{}^{a}_{c}\, u')$ in $\mathbb{A}$ such that

$$\begin{tikzcd}
& & & FA \arrow[r, "Fa"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\varphi_0^{-1}\ \cong"] & FC \arrow[d, equal, "\bullet" marking] \\
FA \arrow[r, "Fa"] \arrow[d, "Fu"', "\bullet" marking] \arrow[dr, phantom, "F\alpha"] & FC \arrow[d, "Fu'", "\bullet" marking] & = & FA \arrow[r, "b"] \arrow[d, "Fu"', "\bullet" marking] \arrow[dr, phantom, "\beta"] & FC \arrow[d, "Fu'", "\bullet" marking] \\
FA' \arrow[r, "Fc"'] & FC' & & FA' \arrow[r, "d"'] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\varphi_1\ \cong"] & FC' \arrow[d, equal, "\bullet" marking] \\
& & & FA' \arrow[r, "Fc"'] & FC' ,
\end{tikzcd}$$

which gives (vb2). Finally, we prove (vb3). Suppose we have the following pasting equality in $\mathbb{B}$.

$$
\begin{tikzcd}
FA \arrow[r, "Fa"] \arrow[d, equal] \arrow[dr, phantom, "\tau_0"] & FC \arrow[d, equal] \\
FA \arrow[r, "Fa'"] \arrow[d, "Fu"'] \arrow[dr, phantom, "F\alpha'"] & FC \arrow[d, "Fu'"] \\
FA' \arrow[r, "Fc'"'] & FC'
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
FA \arrow[r, "Fa"] \arrow[d, "Fu"'] \arrow[dr, phantom, "F\alpha"] & FC \arrow[d, "Fu'"] \\
FA' \arrow[r, "Fc"'] \arrow[d, equal] \arrow[dr, phantom, "\tau_1"] & FC' \arrow[d, equal] \\
FA' \arrow[r, "Fc'"'] & FC'
\end{tikzcd}
$$

Applying (db4) of Definition 3.5 to τ_0 and τ_1 gives unique squares $\sigma_0 \colon (e_A \; {}^{a}_{a'} \; e_C)$ and $\sigma_1 \colon (e_{A'} \; {}^{c}_{c'} \; e_{C'})$ in $\mathbb{A}$ such that $F\sigma_0 = \tau_0$ and $F\sigma_1 = \tau_1$. Moreover, by unicity in (db4) of Definition 3.5, we have that

$$
\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, equal] \arrow[dr, phantom, "\sigma_0"] & C \arrow[d, equal] \\
A \arrow[r, "a'"] \arrow[d, "u"'] \arrow[dr, phantom, "\alpha'"] & C \arrow[d, "u'"] \\
A' \arrow[r, "c'"'] & C'
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "u"'] \arrow[dr, phantom, "\alpha"] & C \arrow[d, "u'"] \\
A' \arrow[r, "c"'] \arrow[d, equal] \arrow[dr, phantom, "\sigma_1"] & C' \arrow[d, equal] \\
A' \arrow[r, "c'"'] & C'
\end{tikzcd},
$$

since applying F to each vertical composite yields the same squares in $\mathbb{B}$. This proves (vb3), and thus concludes our proof. $\square$

Now we turn our attention to Proposition 3.11, dealing with fibrations. Our treatment is analogous to that of weak equivalences: in order to characterize the functors F such that $(\mathbf{H}, \mathcal{V})F$ is a fibration, we express what it means for $\mathbf{H}F$ and $\mathcal{V}F$ to be Lack fibrations in 2Cat; this is done by translating (f1-2) of Definition 3.2 for these 2-functors.

Remark 8.4. Let $F \colon \mathbb{A} \to \mathbb{B}$ be a double functor. Then $\mathbf{H}F \colon \mathbf{H}\mathbb{A} \to \mathbf{H}\mathbb{B}$ is a fibration in 2Cat if and only if it satisfies (df1-2) of Definition 3.7.

Remark 8.5. Let $F \colon \mathbb{A} \to \mathbb{B}$ be a double functor. Then $\mathcal{V}F \colon \mathcal{V}\mathbb{A} \to \mathcal{V}\mathbb{B}$ is a fibration in 2Cat if and only if it satisfies (df3) of Definition 3.7, and the following condition:

(vf2) for every square $\alpha' \colon (u \; {}^{a'}_{c'} \; u')$ in $\mathbb{A}$ and every square $\beta \colon (Fu \; {}^{b}_{d} \; Fu')$ in $\mathbb{B}$, together with vertically invertible squares τ_0 and τ_1 in $\mathbb{B}$ as in the pasting equality below left, there exists a square $\alpha \colon (u \; {}^{a}_{c} \; u')$, together with vertically invertible squares σ_0 and σ_1 in $\mathbb{A}$ as in the pasting equality below right, such that $F\alpha = \beta$, $F\sigma_0 = \tau_0$, and $F\sigma_1 = \tau_1$.

$$
\begin{tikzcd}
FA \arrow[r, "b"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\tau_0 \cong"] & FC \arrow[d, equal, "\bullet" marking] \\
FA \arrow[r, "Fa'"] \arrow[d, "Fu"', "\bullet" marking] \arrow[dr, phantom, "F\alpha'"] & FC \arrow[d, "Fu'", "\bullet" marking] \\
FA' \arrow[r, "Fc'"'] & FC'
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "b"] \arrow[d, "Fu"', "\bullet" marking] \arrow[dr, phantom, "\beta"] & FC \arrow[d, "Fu'", "\bullet" marking] \\
FA' \arrow[r, "d"'] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\tau_1 \cong"] & FC' \arrow[d, equal, "\bullet" marking] \\
FA' \arrow[r, "Fc'"'] & FC'
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\sigma_0 \cong"] & C \arrow[d, equal, "\bullet" marking] \\
A \arrow[r, "a'"] \arrow[d, "u"', "\bullet" marking] \arrow[dr, phantom, "\alpha'"] & C \arrow[d, "u'", "\bullet" marking] \\
A' \arrow[r, "c'"'] & C'
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "u"', "\bullet" marking] \arrow[dr, phantom, "\alpha"] & C \arrow[d, "u'", "\bullet" marking] \\
A' \arrow[r, "c"'] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\sigma_1 \cong"] & C' \arrow[d, equal, "\bullet" marking] \\
A' \arrow[r, "c'"'] & C'
\end{tikzcd}
$$

We can now use the above remarks to provide a proof of Proposition 3.11, giving the characterization of the fibrations.

Proof of Proposition 3.11. It is clear that if a double functor $F\colon \mathbb{A} \to \mathbb{B}$ is such that both $\mathbf{H}F$ and $\mathcal{V}F$ are Lack fibrations in 2Cat, then it is a double fibration, by Remarks 8.4 and 8.5.

Suppose now that $F\colon \mathbb{A} \to \mathbb{B}$ is a double fibration. By Remark 8.4, we directly get that $\mathbf{H}F$ is a Lack fibration in 2Cat. To show that $\mathcal{V}F$ is is also a Lack fibration, it suffices to show that (vf3) of Remark 8.5 is satisfied. Let $\alpha'\colon (u \,{}^{a'}_{c'}\, u')$ be a square in $\mathbb{A}$ and $\beta\colon (Fu \,{}^{b}_{d}\, Fu')$ be a square in $\mathbb{B}$, together with vertically invertible squares τ_0 and τ_1 in $\mathbb{B}$ such that the following pasting equality holds.

$$
\begin{tikzcd}
FA \arrow[r, "b"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\tau_0 \cong"] & FC \arrow[d, equal, "\bullet" marking] \\
FA \arrow[r, "Fa'"] \arrow[d, "Fu"', "\bullet" marking] \arrow[dr, phantom, "F\alpha'"] & FC \arrow[d, "Fu'", "\bullet" marking] \\
FA' \arrow[r, "Fc'"'] & FC'
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "b"] \arrow[d, "Fu"', "\bullet" marking] \arrow[dr, phantom, "\beta"] & FC \arrow[d, "Fu'", "\bullet" marking] \\
FA' \arrow[r, "d"'] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\tau_1 \cong"] & FC' \arrow[d, equal, "\bullet" marking] \\
FA' \arrow[r, "Fc'"'] & FC'
\end{tikzcd}
$$

By (df2) of Definition 3.7, there exist vertically invertible squares σ_0 and σ_1 in $\mathbb{A}$

$$
\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\sigma_0 \cong"] & C \arrow[d, equal, "\bullet" marking] \\
A \arrow[r, "a'"'] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
A' \arrow[r, "c"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\sigma_1 \cong"] & C' \arrow[d, equal, "\bullet" marking] \\
A' \arrow[r, "c'"'] & C'
\end{tikzcd}
$$

such that $F\sigma_0 = \tau_0$ and $F\sigma_1 = \tau_1$. Then the square α given by the vertical composite

$$
\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "u"'] \arrow[dr, phantom, "\alpha"] & C \arrow[d, "u'"] \\
A' \arrow[r, "c"'] & C'
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, equal] \arrow[dr, phantom, "\sigma_0 \, \cong"] & C \arrow[d, equal] \\
A \arrow[r, "a'"] \arrow[d, "u"'] \arrow[dr, phantom, "\alpha'"] & C \arrow[d, "u'"] \\
A' \arrow[r, "c'"'] \arrow[d, equal] \arrow[dr, phantom, "\sigma_1^{-1} \, \cong"] & C' \arrow[d, equal] \\
A' \arrow[r, "c"'] & C'
\end{tikzcd}
$$

is such that $F\alpha = \beta$, which proves (vf3). □

9. Generating (trivial) cofibrations and cofibrant objects

In this section, we take a closer look at the (trivial) cofibrations and cofibrant objects in our model structure on DblCat. In Proposition 3.18, we specified sets of generating cofibrations and generating trivial cofibrations for this model structure. In fact, there exist smaller sets of generating cofibrations and generating trivial cofibrations, determined directly by their left lifting properties, as we show in Proposition 9.2.

A closer study of the lifting properties further reveals that cofibrations can be characterized through their underlying horizontal and vertical (1-)functors; this is done in Proposition 9.5. Finally, in Proposition 9.11, we use this characterization to describe the cofibrant double categories in our model structure.

Let us first describe new sets of generating cofibrations and generating trivial cofibrations.

Notation 9.1. Let $\mathbb{S}$ be the double category containing a square, $\delta\mathbb{S}$ be its boundary, and $\mathbb{S}_2$ be the double category containing two squares with same boundaries.

$$
\mathbb{S} = \begin{tikzcd}
0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\alpha"] & 1 \arrow[d] \\
0' \arrow[r] & 1'
\end{tikzcd} ; \quad
\delta\mathbb{S} = \begin{tikzcd}
0 \arrow[r] \arrow[d] & 1 \arrow[d] \\
0' \arrow[r] & 1'
\end{tikzcd} ; \quad
\mathbb{S}_2 = \begin{tikzcd}
0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\alpha_0 \ \alpha_1"] & 1 \arrow[d] \\
0' \arrow[r] & 1'
\end{tikzcd}
$$

We fix notation for the following double functors, which form a set of generating cofibrations for our model structure on DblCat:

- the unique map $I_1 \colon \emptyset \to \mathbb{1}$,
- the inclusion $I_2 \colon \mathbb{1} \sqcup \mathbb{1} \to \mathbb{H}\mathbb{2}$,
- the unique map $I_3 \colon \emptyset \to \mathbb{V}\mathbb{2}$,
- the inclusion $I_4 \colon \delta\mathbb{S} \to \mathbb{S}$, and
- the double functor $I_5 \colon \mathbb{S}_2 \to \mathbb{S}$ sending both squares in $\mathbb{S}_2$ to the non-trivial square of $\mathbb{S}$.

We also fix notation for the following double functors, which form a set of generating trivial cofibrations for our model structure on DblCat:

- the inclusion $J_1 \colon \mathbb{1} \to \mathbb{H}E_{\mathrm{adj}}$, where E_{adj} is the 2-category containing an adjoint equivalence,
- the inclusion $J_2 \colon \mathbb{H}\mathbb{2} \to \mathbb{H}C_{\mathrm{inv}}$, and
- the inclusion $J_3 \colon \mathbb{V}\mathbb{2} \to \mathbb{H}E_{\mathrm{adj}} \times \mathbb{V}\mathbb{2}$; note that $\mathbb{H}E_{\mathrm{adj}} \times \mathbb{V}\mathbb{2}$ is the double category containing a weakly horizontally invertible square.

Proposition 9.2. *In the model structure on* DblCat *of Theorem 3.16, a set of generating cofibrations is given by*

$$\mathcal{I}' = \{I_1 \colon \emptyset \to \mathbb{1},\ I_2 \colon \mathbb{1} \sqcup \mathbb{1} \to \mathbb{H}\mathbb{2},\ I_3 \colon \emptyset \to \mathbb{V}\mathbb{2},\ I_4 \colon \delta\mathbb{S} \to \mathbb{S},\ I_5 \colon \mathbb{S}_2 \to \mathbb{S}\}$$

and a set of generating trivial cofibrations is given by

$$\mathcal{J}' = \{J_1 \colon \mathbb{1} \to \mathbb{H}E_{\mathrm{adj}},\ J_2 \colon \mathbb{H}\mathbb{2} \to \mathbb{H}C_{\mathrm{inv}},\ J_3 \colon \mathbb{V}\mathbb{2} \to \mathbb{H}E_{\mathrm{adj}} \times \mathbb{V}\mathbb{2}\}.$$

Proof. It is a routine exercise to check that a double functor is a double trivial fibration as defined in Definition 3.8 if and only if it has the right-lifting property with respect to the cofibrations in $\mathcal{I}'$, and that a double functor is a double fibration as defined in Definition 3.7 if and only if it has the right-lifting property with respect to the trivial cofibrations of $\mathcal{J}'$. This shows that $\mathcal{I}'$ and $\mathcal{J}'$ are sets of generating cofibrations and generating trivial cofibration for DblCat, respectively. □

Our next goal is to provide a characterization of the cofibrations in DblCat. In [11, Lemma 4.1], Lack shows that a 2-functor is a cofibration in 2Cat if and only if its underlying functor has the left lifting property with respect to all surjective on objects and full functors. A similar result applies to our model structure; indeed, we show that a double functor is a cofibration in DblCat if and only if its underlying horizontal functor and its underlying vertical functor satisfy respective lifting properties.

Remark 9.3. The functor $U\mathbf{H} \colon \mathrm{DblCat} \to \mathrm{Cat}$, which sends a double category to its underlying category of objects and horizontal morphisms, has a right adjoint. It is given by the functor $R_h \colon \mathrm{Cat} \to \mathrm{DblCat}$ that sends a category $\mathcal{C}$ to the double category with the same objects as $\mathcal{C}$, horizontal morphisms given by the morphisms of $\mathcal{C}$, a unique vertical morphism between every pair of objects and a unique square $! \colon (! \; {}^{f}_{g} \; !)$ for every pair of morphisms f, g in $\mathcal{C}$.

Remark 9.4. The functor $U\mathbf{V} \colon \mathrm{DblCat} \to \mathrm{Cat}$, which sends a double category to its underlying category of objects and vertical morphisms, has a right adjoint. It is given by the functor $R_v \colon \mathrm{Cat} \to \mathrm{DblCat}$ that sends a category $\mathcal{C}$ to the double category with the same objects of $\mathcal{C}$, a unique horizontal morphism between every pair of objects, vertical morphisms given by the morphisms of $\mathcal{C}$, and a unique square $! \colon (u \,\substack{!\\!}\, v)$ for every pair of morphisms u, v in $\mathcal{C}$.

Proposition 9.5. *A double functor $F \colon \mathbb{A} \to \mathbb{B}$ is a cofibration in* DblCat *if and only if*

(i) the functor $U\mathbf{H}F \colon U\mathbf{H}\mathbb{A} \to U\mathbf{H}\mathbb{B}$ has the left lifting property with respect to all surjective on objects and full functors, and

(ii) *the functor* $U\mathbf{V}F\colon U\mathbf{V}\mathbb{A} \to U\mathbf{V}\mathbb{B}$ *has the left lifting property with respect to all surjective on objects and surjective on morphisms functors.*

Proof. Suppose first that $F\colon \mathbb{A} \to \mathbb{B}$ is a cofibration in DblCat, i.e., it has the left lifting property with respect to all double trivial fibrations. In order to show (i), let $P\colon \mathcal{X} \to \mathcal{Y}$ be a surjective on objects and full functor. By the adjunction $U\mathbf{H} \dashv R_h$, saying that $U\mathbf{H}F$ has the left lifting property with respect to P is equivalent to saying that F has the left lifting property with respect to R_hP. We now prove this latter statement.

Note that the double functor $R_hP\colon R_h\mathcal{X} \to R_h\mathcal{Y}$ is surjective on morphisms and full on horizontal morphisms, since P is. Moreover, by construction of R_h, there is exactly one vertical morphism and one square for each boundary in both its source and target; therefore R_hP is surjective on vertical morphisms and fully faithful on squares. Hence R_hP is a double trivial fibration, and F has the left lifting property with respect to R_hP since it is a cofibration in DblCat.

Similarly, one can show that (ii) holds, by considering the adjunction $U\mathbf{V} \dashv R_v$ and replacing fullness by surjectivity on morphisms.

Now suppose that $F\colon \mathbb{A} \to \mathbb{B}$ satisfies (i) and (ii). Given a double trivial fibration $P\colon \mathbb{X} \to \mathbb{Y}$ and a commutative square as below, we want to find a lift $L\colon \mathbb{B} \to \mathbb{X}$.

$$\begin{tikzcd}
\mathbb{A} \arrow[r, "G"] \arrow[d, "F"'] & \mathbb{X} \arrow[d, "P"] \\
\mathbb{B} \arrow[r, "H"'] \arrow[ur, dotted, "L"] & \mathbb{Y}
\end{tikzcd}$$

Using (ii), since $U\mathbf{V}P$ is surjective on objects and surjective on morphisms, we have a lift L_v in the following diagram.

$$\begin{tikzcd}
U\mathbf{V}\mathbb{A} \arrow[r, "U\mathbf{V}G"] \arrow[d, "U\mathbf{V}F"'] & U\mathbf{V}\mathbb{X} \arrow[d, "U\mathbf{V}P"] \\
U\mathbf{V}\mathbb{B} \arrow[r, "U\mathbf{V}H"'] \arrow[ur, dotted, "L_v"] & U\mathbf{V}\mathbb{Y}
\end{tikzcd}$$

Now, using (i), since $U\mathbf{H}P$ is surjective on objects and full, we can choose a lift L_h in the following diagram

$$\begin{tikzcd}
U\mathbf{H}\mathbb{A} \arrow[r, "U\mathbf{H}G"] \arrow[d, "U\mathbf{H}F"'] & U\mathbf{H}\mathbb{X} \arrow[d, "U\mathbf{H}P"] \\
U\mathbf{H}\mathbb{B} \arrow[r, "U\mathbf{H}H"'] \arrow[ur, dotted, "L_h"] & U\mathbf{H}\mathbb{Y}
\end{tikzcd}$$

such that L_h coincides with L_v on objects. This comes from the fact that, by fullness of $U\mathbf{H}P$, we can first choose an assignment on objects and then choose a compatible assignment on morphisms. Then, since $P\colon \mathbb{X} \to \mathbb{Y}$ is fully faithful on squares, the assignment on objects, horizontal morphisms, and vertical morphisms given by L_h and L_v uniquely extend to a double functor $L\colon \mathbb{B} \to \mathbb{Y}$, which gives the desired lift. □

To understand what it means for a double functor to satisfy (i) and (ii) of Proposition 9.5, we state a characterization of the functors in Cat that have the left lifting property with respect to all surjective on objects and full (resp. surjective on morphisms) functors.

Proposition 9.6. *There is a cofibrantly generated weak factorization system $(\mathcal{G}, \mathcal{R})$ on* Cat*, where $\mathcal{R}$ is the set of surjective on objects and full functors. A set of generating morphisms for $\mathcal{G}$ can be chosen to be $\{i_1 : \emptyset \to \mathbb{1},\, i_2 \colon \mathbb{1} \sqcup \mathbb{1} \to \mathbb{2}\}$.*

Corollary 9.7. *A functor $F\colon \mathcal{A} \to \mathcal{B}$ is in $\mathcal{G}$ if and only if*

(i) the functor F is injective on objects and faithful, and
(ii) there exist functors $I\colon \mathcal{B} \to \mathcal{C}$ and $R\colon \mathcal{C} \to \mathcal{B}$ such that $RI = \mathrm{id}_{\mathcal{B}}$, where the category $\mathcal{C}$ is obtained from the image of F by freely adjoining objects and then freely adjoining morphisms between specified objects.

Moreover, a functor $\emptyset \to \mathcal{A}$ is in $\mathcal{G}$ if and only if $\mathcal{A}$ is a retract of a free category $\mathcal{C}$. In particular, the category $\mathcal{A}$ is itself free.

Proposition 9.8. *There is a cofibrantly generated weak factorization system $(\mathcal{H}, \mathcal{S})$ on* Cat*, where $\mathcal{S}$ is the set of surjective on objects and surjective on morphisms functors. A set of generating morphisms for $\mathcal{H}$ can be chosen to be $\{i_1 : \emptyset \to \mathbb{1},\, i_3 : \emptyset \to \mathbb{2}\}$.*

Corollary 9.9. *A functor $F\colon \mathcal{A} \to \mathcal{B}$ is in $\mathcal{H}$ if and only if*

(i) the functor F is injective on objects and faithful, and
(ii) there exist functors $I\colon \mathcal{B} \to \mathcal{C}$ and $R\colon \mathcal{C} \to \mathcal{B}$ such that $RI = \mathrm{id}_{\mathcal{B}}$, where the category $\mathcal{C}$ is obtained from the image of F by freely adjoining objects and morphisms.

Moreover, a functor $\emptyset \to \mathcal{A}$ is in $\mathcal{H}$ if and only if $\mathcal{A}$ is a retract of a category $\mathcal{C}$ that is a disjoint union of copies of $\mathbb{1}$ and $\mathbb{2}$. In particular, the category $\mathcal{A}$ is itself a disjoint union of copies of $\mathbb{1}$ and $\mathbb{2}$.

From these characterizations, we get the following result.

Corollary 9.10. *If a double functor is a cofibration in* DblCat*, then it is injective on objects, faithful on horizontal morphisms, and faithful on vertical morphisms.*

Proof. This follows directly from Proposition 9.5 and Corollaries 9.7 and 9.9. □

Finally, we use the above results to obtain a characterization of the cofibrant double categories in terms of their underlying horizontal and vertical categories.

Proposition 9.11. *A double category $\mathbb{A}$ is cofibrant if and only if its underlying horizontal category $U\mathbf{H}\mathbb{A}$ is free and its underlying vertical category $U\mathbf{V}\mathbb{A}$ is a disjoint union of copies of $\mathbb{1}$ and $\mathbb{2}$.*

Proof. By Proposition 9.5, a double category $\mathbb{A}$ is cofibrant if and only if

(i) the functor $\emptyset \to U\mathbf{H}\mathbb{A}$ has the left lifting property with respect to all surjective on objects and full functors, and
(ii) the functor $\emptyset \to U\mathbf{V}\mathbb{A}$ has the left lifting property with respect to all surjective on objects and surjective on morphisms functors.

By Corollary 9.7, (i) is equivalent to the category $U\mathbf{H}\mathbb{A}$ being free, and, by Corollary 9.9, (ii) is equivalent to the category $U\mathbf{V}\mathbb{A}$ being a disjoint union of copies of $\mathbb{1}$ and $\mathbb{2}$. $\square$

Remark 9.12. Note that the functor $\mathbf{H}\colon \mathrm{DblCat} \to \mathrm{Cat}$ preserves cofibrant objects. Indeed, if $\mathbb{A}$ is a cofibrant double category, then its underlying horizontal category $U\mathbf{H}\mathbb{A}$ is free, by Proposition 9.11. Since a 2-category is cofibrant if and only if its underlying category is free by [11, Theorem 4.8], it follows that $\mathbf{H}\mathbb{A}$ is cofibrant in 2Cat.

Part III. The Whitehead Theorem

10. A Whitehead Theorem for double categories

In this section, we show that a Whitehead Theorem for double biequivalences is available in some cases. As we will see, this continues to highlight the close connection between our model structure and Lack's model structure on 2Cat.

Recall the statement of the Whitehead Theorem for biequivalences between 2-categories: a 2-functor $F\colon \mathcal{A} \to \mathcal{B}$ is a biequivalence if and only if there exists a pseudo functor $G\colon \mathcal{B} \to \mathcal{A}$ together with two pseudo natural equivalences $\mathrm{id} \simeq GF$ and $FG \simeq \mathrm{id}$. This is a long-established result in the literature; a proof can be found, for example, in [10, Theorem 7.4.1].

Under certain conditions, we can show an analogous characterization of double biequivalences using pseudo *horizontal* natural equivalences; this is done in Theorem 10.14, which we interpret as a Whitehead Theorem for double categories. In particular, this holds for double biequivalences of the form $\mathbb{H}F\colon \mathbb{H}\mathcal{A} \to \mathbb{H}\mathcal{B}$, for any 2-functor $F\colon \mathcal{A} \to \mathcal{B}$, and so this result recovers the 2-categorical version.

Let us first introduce the notions of pseudo double functors and pseudo horizontal natural equivalences, which are needed to state the theorem.

Definition 10.1. A **pseudo double functor** $G\colon \mathbb{B} \to \mathbb{A}$ consists of maps on objects, horizontal morphisms, vertical morphisms, and squares, compatible with sources and targets, which preserve

(i) horizontal compositions and identities up to coherent vertically invertible squares

$$\begin{tikzcd} GB \arrow[r, "Gb"] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "\Phi_{b,d}\ \wr\wr"] & GC \arrow[r, "Gd"] & GD \arrow[d, equal, "\bullet" marking] \\ GB \arrow[rr, "G(db)"'] & & GD \end{tikzcd} \qquad \begin{tikzcd} GB \arrow[r, equal] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\Phi_B\ \wr\wr"] & GB \arrow[d, equal, "\bullet" marking] \\ GB \arrow[r, "G\mathrm{id}_B"'] & GB \end{tikzcd}$$

for all $B \in \mathbb{B}$, and all composable horizontal morphisms $b\colon B \to C$ and $d\colon C \to D$ in $\mathbb{B}$,

(ii) vertical compositions and identities up to coherent horizontally invertible squares

$$
\begin{tikzcd}
GB \arrow[r, equal] \arrow[d, "Gv"'] & GB \arrow[dd, "G(v'v)"] \\
GB' \arrow[d, "Gv'"'] \arrow[r, phantom, "\overset{\cong}{\Psi_{v,v'}}"] & \\
GB'' \arrow[r, equal] & GB''
\end{tikzcd}
\qquad
\begin{tikzcd}
GB \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\overset{\cong}{\Psi_B}"] & GB \arrow[d, "G\mathrm{id}_B"] \\
GB \arrow[r, equal] & GB
\end{tikzcd}
$$

for all $B \in \mathbb{B}$, and all composable vertical morphisms $v\colon B \to B'$ and $v'\colon B' \to B''$ in $\mathbb{B}$.

For a detailed description of the coherences, the reader can see [5, Definition 3.5.1].

The pseudo double functor G is said to be **normal** if the squares Φ_B and Ψ_B are identities for all $B \in \mathbb{B}$.

Remark 10.2. There are also notions of pseudo horizontal natural transformations between (normal) pseudo double functors, and modifications between them (with trivial vertical boundaries). These are defined analogously to Definition 5.4 and satisfy similar coherence conditions to the ones in [5, §3.8].

Definition 10.3. Let $F, G\colon \mathbb{A} \to \mathbb{B}$ be (normal) pseudo double functors. A **pseudo horizontal natural equivalence** $\varphi\colon F \Rightarrow G$ is an equivalence in the 2-category of (normal) pseudo double functors $\mathbb{A} \to \mathbb{B}$, pseudo natural horizontal transformations, and modifications with trivial vertical boundaries.

Remark 10.4. Equivalently, a pseudo horizontal natural equivalence is a pseudo horizontal natural transformation $h\colon F \Rightarrow G$ such that the horizontal morphisms $h_A\colon FA \xrightarrow{\simeq} GA$ are horizontal equivalences, for all $A \in \mathbb{A}$, and the squares $h_u\colon (Fu\ {}^{h_A}_{h_{A'}}\ Gu)$ are weakly horizontally invertible, for all vertical morphisms $u\colon A \to A'$ in $\mathbb{A}$; see [14].

We now introduce a notion of *horizontal biequivalence* for a double functor which admits a pseudo weak inverse.

Definition 10.5. A double functor $F\colon \mathbb{A} \to \mathbb{B}$ is a **horizontal biequivalence** if there exist a pseudo double functor $G\colon \mathbb{B} \to \mathbb{A}$ and pseudo horizontal natural equivalences $\eta\colon \mathrm{id} \Rightarrow GF$ and $\epsilon\colon FG \Rightarrow \mathrm{id}$.

Remark 10.6. Let $F\colon \mathbb{A} \to \mathbb{B}$ be a double functor. If F is a horizontal biequivalence, its data (G, η, ϵ) can always be promoted to the following data:

(i) a *normal* pseudo double functor $G\colon \mathbb{B} \to \mathbb{A}$,

(ii) a pseudo horizontal natural *adjoint* equivalence

$$(\eta\colon \mathrm{id} \Rightarrow GF, \eta'\colon GF \Rightarrow \mathrm{id}, \lambda\colon \mathrm{id} \cong \eta'\eta, \kappa\colon \eta\eta' \cong \mathrm{id}),$$

where λ and κ satisfy the triangle identities,

(iii) a pseudo horizontal natural *adjoint* equivalence

$$(\epsilon\colon FG \Rightarrow \mathrm{id}, \epsilon'\colon \mathrm{id} \Rightarrow FG, \mu\colon \mathrm{id} \cong \epsilon'\epsilon, \nu\colon \epsilon\epsilon' \cong \mathrm{id}),$$

where μ and ν satisfy the triangle identities,

(iv) two invertible modifications $\Theta\colon \mathrm{id}_F \cong \epsilon_F \circ F\eta$ and $\Sigma\colon \mathrm{id}_G \cong G\epsilon \circ \eta_G$, expressing the triangle (pseudo-)identities for η and ϵ.

This follows from the fact that a pseudo double functor can always be promoted to a normal one, and from a result by Gurski [7, Theorem 3.2], saying that a biequivalence can always be promoted to a biadjoint biequivalence.

Our next goal is to show one direction of the characterization provided in the Whitehead Theorem; namely, that a horizontal biequivalence is in particular a double biequivalence. In order to prove this result, we need the following lemma.

Lemma 10.7. *The data of Remark 10.6 induces an invertible modification $\theta\colon F\eta' \cong \epsilon_F$.*

Proof. Given an object $A \in \mathbb{A}$, we define the data of θ at A to be the vertically invertible square

$$\begin{tikzcd}
FGFA \arrow[r, "F\eta'_A"] \arrow[d, equal] \arrow[dr, phantom, "\theta_A \cong"] & FA \arrow[d, equal] \\
FGFA \arrow[r, "\epsilon_{FA}"'] & FA
\end{tikzcd}
=
\begin{tikzcd}
FGFA \arrow[r, "F\eta'_A"] \arrow[d, equal] \arrow[dr, phantom, "e_{F\eta'_A}"] & FA \arrow[rr, equal] \arrow[d, equal] \arrow[drr, phantom, "\Theta_A \cong"] & & FA \arrow[d, equal] \\
FGFA \arrow[r, "F\eta'_A"'] \arrow[d, equal] \arrow[drr, phantom, "F\kappa_A \cong"] & FA \arrow[r, "F\eta_A"] & FGFA \arrow[r, "\epsilon_{FA}"] \arrow[d, equal] \arrow[dr, phantom, "e_{\epsilon_{FA}}"] & FA \arrow[d, equal] \\
FGFA \arrow[rr, equal] & & FGFA \arrow[r, "\epsilon_{FA}"'] & FA
\end{tikzcd}.$$

The proof of horizontal and vertical coherences for θ is a standard check that stems from the constructions of the squares θ_A and from the horizontal and vertical coherences of the modifications $F\kappa\colon (F\eta)(F\eta') \cong \mathrm{id}$ and $\Theta\colon \mathrm{id} \cong \epsilon_F \circ F\eta$. □

Proposition 10.8. *If $F\colon \mathbb{A} \to \mathbb{B}$ is a horizontal biequivalence, then F is a double biequivalence.*

Proof. We proceed to check that F satisfies (db1-4) of Definition 3.5. Let (F, G, η, ϵ) be the data of a horizontal adjoint biequivalence as in Remark 10.6.

We first show (db1). For every object $B \in \mathbb{B}$, we want to find an object $A \in \mathbb{A}$ and a horizontal equivalence $B \xrightarrow{\simeq} FA$ in $\mathbb{B}$. Setting $A = GB$, we have that $\epsilon'_B\colon B \xrightarrow{\simeq} FGB = FA$ gives such a horizontal equivalence.

We now show (db2). Let $b\colon FA \to FC$ be a horizontal morphism in $\mathbb{B}$. We want to find a horizontal morphism $a\colon A \to C$ in $\mathbb{A}$ and a vertically invertible square in $\mathbb{B}$

$$\begin{tikzcd}
FA \arrow[r, "b"] \arrow[d, equal, "\bullet" description] \arrow[dr, phantom, "\cong"] & FC \arrow[d, equal, "\bullet" description] \\
FA \arrow[r, "Fa"'] & FC .
\end{tikzcd}$$

Let $a\colon A \to C$ be the composite

$$A \xrightarrow{\eta_A} GFA \xrightarrow{Gb} GFC \xrightarrow{\eta'_C} C\,;$$

we then have a vertically invertible square as desired,

$$\begin{tikzcd}
FA \arrow[rr, equal] \arrow[d, equal, "\bullet" description] \arrow[drr, phantom, "\Theta_A \cong"] & & FA \arrow[r, "b"] \arrow[d, equal, "\bullet" description] \arrow[dr, phantom, "e_b"] & FC \arrow[d, equal, "\bullet" description] \\
FA \arrow[r, "F\eta_A"] \arrow[d, equal, "\bullet" description] \arrow[dr, phantom, "e_{F\eta_A}"] & FGFA \arrow[r, "\epsilon_{FA}"] \arrow[d, equal, "\bullet" description] \arrow[drr, phantom, "\epsilon_b \cong"] & FA \arrow[r, "b"] & FC \arrow[d, equal, "\bullet" description] \\
FA \arrow[r, "F\eta_A"] \arrow[d, equal, "\bullet" description] \arrow[drr, phantom, "e_{(FGb)(F\eta_A)}"] & FGFA \arrow[r, "FGb"] & FGFC \arrow[r, "\epsilon_{FC}"] \arrow[d, equal, "\bullet" description] \arrow[dr, phantom, "\theta_C^{-1} \cong"] & FC \arrow[d, equal, "\bullet" description] \\
FA \arrow[r, "F\eta_A"'] & FGFA \arrow[r, "FGb"'] & FGFC \arrow[r, "F\eta'_C"'] & FC
\end{tikzcd}$$

where θ_C is the component at C of the invertible modification θ of Lemma 10.7.

We now show (db3). Let $v\colon B \twoheadrightarrow B'$ be a vertical morphism in $\mathbb{B}$. We want to find a vertical morphism $u\colon A \twoheadrightarrow A'$ in $\mathbb{A}$ and a weakly horizontally invertible square in $\mathbb{B}$

$$\begin{tikzcd}
B \arrow[r, "\simeq"] \arrow[d, "v"', "\bullet" description] \arrow[dr, phantom, "\simeq"] & FA \arrow[d, "Fu", "\bullet" description] \\
B' \arrow[r, "\simeq"'] & FA' .
\end{tikzcd}$$

Let $u\colon A \twoheadrightarrow A'$ be the vertical morphism $Gv\colon GB \twoheadrightarrow GB'$. Then ϵ'_v gives the desired weakly horizontally invertible square.

$$\begin{tikzcd}
B \arrow[r, "\epsilon'_B", "\simeq"'] \arrow[d, "v"', "\bullet" description] \arrow[dr, phantom, "\epsilon'_v \simeq"] & FGB \arrow[d, "FGv", "\bullet" description] \\
B' \arrow[r, "\simeq", "\epsilon'_{B'}"'] & FGB'
\end{tikzcd}$$

We finally show (db4). Let β be a square in $\mathbb{B}$ of the form

$$\begin{tikzcd}
FA \arrow[r, "Fa"] \arrow[d, "Fu"', "\bullet" marking] \arrow[dr, phantom, "\beta"] & FC \arrow[d, "Fu'", "\bullet" marking] \\
FA' \arrow[r, "Fc"'] & FC' .
\end{tikzcd}$$

We want to show that there exists a unique square $\alpha\colon (u \,{}^{a}_{c}\, u')$ in $\mathbb{A}$ such that $F\alpha = \beta$. Define α to be the square given by the following pasting.

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "u"', "\bullet" marking] \arrow[dr, phantom, "\alpha"] & C \arrow[d, "u'", "\bullet" marking] \\
A' \arrow[r, "c"'] & C'
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[rr, equal] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "\mu_A \cong"] & & A \arrow[r, "a"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "e_a"] & C \arrow[d, equal, "\bullet" marking] \\
A \arrow[r, "\eta_A"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "e_{\eta_A}"] & GFA \arrow[r, "\eta'_A"] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "\eta'_a \cong"] & A \arrow[r, "a"] & C \arrow[d, equal, "\bullet" marking] \\
A \arrow[r, "\eta_A"] \arrow[d, "u"', "\bullet" marking] \arrow[dr, phantom, "\eta_u"] & GFA \arrow[r, "GFa"] \arrow[d, "GFu", "\bullet" marking] \arrow[dr, phantom, "G\beta"] & GFC \arrow[r, "\eta'_C"] \arrow[d, "GFu'", "\bullet" marking] \arrow[dr, phantom, "\eta'_{u'}"] & C \arrow[d, "u'", "\bullet" marking] \\
A' \arrow[r, "\eta_{A'}"'] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "e_{\eta_{A'}}"] & GFA' \arrow[r, "GFc"'] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "\eta'^{-1}_c \cong"] & GFC' \arrow[r, "\eta'_{C'}"'] & C' \arrow[d, equal, "\bullet" marking] \\
A' \arrow[r, "\eta_{A'}"'] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "\mu^{-1}_{A'} \cong"] & GFA' \arrow[r, "\eta'_{A'}"'] & A' \arrow[r, "c"'] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "e_c"] & C' \arrow[d, equal, "\bullet" marking] \\
A' \arrow[rr, equal] & & A' \arrow[r, "c"'] & C'
\end{tikzcd}$$

The thorough reader might check that $F\alpha = \beta$ by completing the following steps. First transform $F\eta'_{u'}$ by using the invertible modification $\theta\colon F\eta' \cong \epsilon_F$ of Lemma 10.7; then apply, in the given order: the horizontal coherence of the modification $F\nu\colon (F\eta')(F\eta) \cong \mathrm{id}$, the horizontal coherence of the modification $\Theta\colon \mathrm{id} \cong \epsilon_F \circ F\eta$, the triangle identity for (μ,ν), the compatibility of $\epsilon_F\colon FGF \Rightarrow F$ with $FG\beta$ and β, and finally the horizontal coherence of the modification $\Theta\colon \mathrm{id} \cong \epsilon_F \circ F\eta$.

Suppose now that $\alpha'\colon (u \,{}^{a}_{c}\, u')$ is another square in $\mathbb{A}$ such that $F\alpha' = \beta$. If we replace $G\beta$ with $GF\alpha'$ in the pasting diagram above, then it follows from the compatibility of $\eta'\colon GF \Rightarrow \mathrm{id}$ with $GF\alpha'$ and α', and the vertical coherence of the modification $\mu\colon \mathrm{id} \cong \eta'\eta$, that this pasting is also equal to α'. Therefore, we must have $\alpha = \alpha'$. This proves both the existence and unicity required in (db4). $\square$

The second part of this section is largely devoted to proving a converse statement for Proposition 10.8. Such a result will not hold in the same generality, and we will have to require either a condition saying that all vertical morphisms are trivial, or another condition, which we now introduce. The formulation of this second condition is inspired by the definition of *horizontal invariance* for a double category given by Grandis in [5, Theorem and Definition 4.1.7]. This latter definition is needed to prove a version of the Whitehead theorem for a stricter notion of equivalence of double categories; see [5, Theorem 4.4.5].

Definition 10.9. A double category $\mathbb{A}$ is **weakly horizontally invariant** if, for all horizontal equivalences $a\colon A \xrightarrow{\simeq} C$ and $c\colon A' \xrightarrow{\simeq} C'$ in $\mathbb{A}$ and every vertical morphism $u'\colon C \to C'$ in $\mathbb{A}$, there exist a vertical morphism $u\colon A \to A'$ and a weakly horizontally invertible square in $\mathbb{A}$ as depicted below.

$$\begin{tikzcd} A \arrow[r, "a", "\simeq"'] \arrow[d, "u"'] \arrow[dr, phantom, "\simeq"] & C \arrow[d, "u'"] \\ A' \arrow[r, "\simeq", "c"'] & C' \end{tikzcd}$$

Example 10.10. The class of weakly horizontally invertible double categories contains many examples of interest. For instance, one can easily check that the (flat) double category $\mathbb{R}\mathrm{elSet}$ of relations of sets satisfies this condition. More relevantly, this class also contains the double categories of quintets $\mathbb{Q}\mathcal{A}$ and of adjunctions $\mathbb{A}\mathrm{dj}\mathcal{A}$ built from any 2-category $\mathcal{A}$. A precise description of these double categories can be found in [5, §3.1]; in fact, the reader may check that all examples presented in that section are weakly horizontally invertible.

The following is a technical lemma, needed for the proof of the next proposition. It shows that the lift along a double biequivalence of a horizontal equivalence is again a horizontal equivalence.

Lemma 10.11. *Let $F\colon \mathbb{A} \to \mathbb{B}$ be a double biequivalence, and let $b\colon FA \xrightarrow{\simeq} FC$ be a horizontal equivalence in $\mathbb{B}$. Then any horizontal morphism $a\colon A \to C$ in $\mathbb{A}$ such that there exists a vertically invertible square β in $\mathbb{B}$ as in the diagram below*

$$\begin{tikzcd} FA \arrow[r, "b", "\simeq"'] \arrow[d, equal] \arrow[dr, phantom, "\beta\ \cong"] & FC \arrow[d, equal] \\ FA \arrow[r, "Fa"'] & FC \end{tikzcd}$$

is a horizontal equivalence in $\mathbb{A}$.

Proof. Let (b, b', η, ϵ) be the data of a horizontal equivalence. By (db2) of Definition 3.5, there exists a horizontal morphism $a'\colon C \to A$ in $\mathbb{A}$ together with a vertically invertible square β' in $\mathbb{B}$ as depicted below left. Let us denote the two right-hand side pastings shown below by Λ_η and Λ_ϵ, respectively.

$$
\begin{tikzcd}
FC \arrow[r, "b'", "\simeq"'] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\beta' \cong"] & FA \arrow[d, equal, "\bullet" marking] \\
FC \arrow[r, "Fa'"'] & FA
\end{tikzcd}
\qquad
\begin{tikzcd}
FA \arrow[rr, equal] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "\eta \cong"] & & FA \arrow[d, equal, "\bullet" marking] \\
FA \arrow[r, "b"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\beta \cong"] & FC \arrow[r, "b'"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\beta' \cong"] & FA \arrow[d, equal, "\bullet" marking] \\
FA \arrow[r, "Fa"'] & FC \arrow[r, "Fa'"'] & FA
\end{tikzcd}
\qquad
\begin{tikzcd}
FC \arrow[r, "Fa"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\beta^{-1} \cong"] & FA \arrow[r, "Fa'"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\beta'^{-1} \cong"] & FC \arrow[d, equal, "\bullet" marking] \\
FC \arrow[r, "b'"'] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "\epsilon \cong"] & FA \arrow[r, "b"'] & FC \arrow[d, equal, "\bullet" marking] \\
FC \arrow[rr, equal] & & FC
\end{tikzcd}
$$

By (db4) of Definition 3.5, there exist unique vertically invertible squares $\overline{\eta}$ and $\overline{\epsilon}$ in $\mathbb{A}$

$$
\begin{tikzcd}
A \arrow[rr, equal] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "\overline{\eta} \cong"] & & A \arrow[d, equal, "\bullet" marking] \\
A \arrow[r, "a"'] & C \arrow[r, "a'"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[r, "a'"] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "\overline{\epsilon} \cong"] & A \arrow[r, "a"] & C \arrow[d, equal, "\bullet" marking] \\
C \arrow[rr, equal] & & C
\end{tikzcd}
$$

such that $F\overline{\eta} = \Lambda_\eta$ and $F\overline{\epsilon} = \Lambda_\epsilon$. This provides the data of a horizontal equivalence $(a, a', \overline{\eta}, \overline{\epsilon})$. □

We now prove a converse of Proposition 10.8, under the additional assumption that our domain double category is weakly horizontally invariant.

Proposition 10.12. *Let $F\colon \mathbb{A} \to \mathbb{B}$ be a double biequivalence, where the double category $\mathbb{A}$ is weakly horizontally invariant. Then F is a horizontal biequivalence.*

Proof. We simultaneously define the pseudo double functor $G\colon \mathbb{B} \to \mathbb{A}$ and the pseudo horizontal natural transformation $\epsilon\colon FG \Rightarrow \mathrm{id}$.

G **and** ϵ **on objects.** Let $B \in \mathbb{B}$ be an object. By (db1) of Definition 3.5, there exist an object $A \in \mathbb{A}$ and a horizontal equivalence $b\colon FA \xrightarrow{\simeq} B$ in $\mathbb{B}$. We set $GB \coloneqq A$ and $\epsilon_B \coloneqq b\colon FGB \xrightarrow{\simeq} B$, and also fix a horizontal equivalence data $(\epsilon_B, \epsilon'_B, \mu_B, \nu_B)$.

G **and** ϵ **on horizontal morphisms.** Now let $b\colon B \to C$ be a horizontal morphism in $\mathbb{B}$. By (db2) of Definition 3.5, there exist a horizontal morphism $a\colon GB \to GC$ in $\mathbb{A}$ and a vertically invertible square $\overline{\epsilon}_b$ as in

$$
\begin{tikzcd}
FGB \arrow[r, "\epsilon_B"] \arrow[d, equal, "\bullet" marking] \arrow[drrr, phantom, "\overline{\epsilon}_b \cong"] & B \arrow[r, "b"] & C \arrow[r, "\epsilon'_C"] & FGC \arrow[d, equal, "\bullet" marking] \\
FGB \arrow[rrr, "Fa"'] & & & FGC\,.
\end{tikzcd}
$$

We set $Gb \coloneqq a\colon GB \to GC$ and ϵ_b to be the square given by the following pasting.

$$
\begin{tikzcd}
FGB \arrow[r, "\epsilon_B"] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "\epsilon_b \cong"] & B \arrow[r, "b"] & C \arrow[d, equal, "\bullet" marking] \\
FGB \arrow[r, "FGb"'] & FGC \arrow[r, "\epsilon_C"'] & C
\end{tikzcd}
=
\begin{tikzcd}
FGB \arrow[r, "\epsilon_B"] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "e_{b\epsilon_B}"] & B \arrow[r, "b"] & C \arrow[rr, equal] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "\nu_C^{-1} \cong"] & & C \arrow[d, equal, "\bullet" marking] \\
FGB \arrow[r, "\epsilon_B"] \arrow[d, equal, "\bullet" marking] \arrow[drrr, phantom, "\overline{\epsilon}_b \cong"] & B \arrow[r, "b"] & C \arrow[r, "\epsilon'_C"] & FGC \arrow[r, "\epsilon_C"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "e_{\epsilon_C}"] & C \arrow[d, equal, "\bullet" marking] \\
FGB \arrow[rrr, "FGb"'] & & & FGC \arrow[r, "\epsilon_C"'] & C
\end{tikzcd}
$$

If $b = \mathrm{id}_B$, we can choose $G\mathrm{id}_B \coloneqq \mathrm{id}_{GB}$ and $\overline{\epsilon}_{\mathrm{id}_B} \coloneqq \mu_B^{-1}$. Then $\epsilon_{\mathrm{id}_B} = e_{\epsilon_B}$ by the triangle identities for (μ_B, ν_B).

Horizontal coherence. Given horizontal morphisms $b\colon B \to C$ and $d\colon C \to D$ in $\mathbb{B}$, we define the vertically invertible comparison square between $Gd \circ Gb$ and $G(db)$ as follows. Let us denote by $\Theta_{b,d}$ the following pasting.

$$
\begin{tikzcd}
FGB \arrow[rrr, "FGb"] \arrow[d, equal, "\bullet" marking] \arrow[drrr, phantom, "\overline{\epsilon}_b^{-1} \cong"] & & & FGC \arrow[rrr, "FGd"] \arrow[d, equal, "\bullet" marking] \arrow[drrr, phantom, "\overline{\epsilon}_d^{-1} \cong"] & & & FGD \arrow[d, equal, "\bullet" marking] \\
FGB \arrow[r, "\epsilon_B"] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "e_{b\epsilon_B}"] & B \arrow[r, "b"] & C \arrow[r, "\epsilon'_C"] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "\nu_C \cong"] & FGC \arrow[r, "\epsilon_C"] & C \arrow[r, "d"] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "e_{\epsilon'_D d}"] & D \arrow[r, "\epsilon'_D"] & FGD \arrow[d, equal, "\bullet" marking] \\
FGB \arrow[r, "\epsilon_B"'] \arrow[d, equal, "\bullet" marking] \arrow[drrrrrr, phantom, "\overline{\epsilon}_{db} \cong"] & B \arrow[r, "b"'] & C \arrow[rr, equal] & & C \arrow[r, "d"'] & D \arrow[r, "\epsilon'_D"'] & FGD \arrow[d, equal, "\bullet" marking] \\
FGB \arrow[rrrrrr, "FG(db)"'] & & & & & & FGD
\end{tikzcd}
$$

Then, by (db4) of Definition 3.5, there exists a unique vertically invertible square

$$
\begin{tikzcd}
GB \arrow[r, "Gb"] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "\Phi_{b,d} \cong"] & GC \arrow[r, "Gd"] & GD \arrow[d, equal, "\bullet" marking] \\
GB \arrow[rr, "G(db)"'] & & GD
\end{tikzcd}
$$

such that $F\Phi_{b,d} = \Theta_{b,d}$. In particular, one can check that, with this definition of $\Phi_{b,d}$, the squares ϵ_b, ϵ_d, and ϵ_{db} satisfy the following pasting equality.

$$
\begin{tikzcd}
B \arrow[r, "\epsilon_B"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "e_{\epsilon_B}"] & FGB \arrow[r, "FGb"] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "F\Phi_{b,d}\ \rotatebox{90}{$\simeq$}"] & FGC \arrow[r, "FGd"] & FGD \arrow[d, equal, "\bullet" marking] & & B \arrow[r, "\epsilon_B"] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "\epsilon_b\ \rotatebox{90}{$\simeq$}"] & FGB \arrow[r, "FGb"] & FGC \arrow[r, "FGd"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "e_{FGd}"] & FGD \arrow[d, equal, "\bullet" marking] \\
B \arrow[r, "\epsilon_B"'] \arrow[d, equal, "\bullet" marking] \arrow[drrr, phantom, "\epsilon_{db}\ \rotatebox{90}{$\simeq$}"] & FGB \arrow[rr, "FG(db)"'] & & FGD \arrow[d, equal, "\bullet" marking] & = & B \arrow[r, "b"'] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "e_b"] & C \arrow[r, "\epsilon_C"'] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "\epsilon_d\ \rotatebox{90}{$\simeq$}"] & FGC \arrow[r, "FGd"'] & FGD \arrow[d, equal, "\bullet" marking] \\
B \arrow[r, "b"'] & C \arrow[r, "d"'] & D \arrow[r, "\epsilon_D"'] & FGD & & B \arrow[r, "b"'] & C \arrow[r, "d"'] & D \arrow[r, "\epsilon_D"'] & FGD
\end{tikzcd}
$$

G and ϵ on vertical morphisms. Now let $v\colon B \to\!\!\!\bullet\; B'$ be a vertical morphism in $\mathbb{B}$. By (db3) of Definition 3.5, there exist a vertical morphism $u'\colon A \to\!\!\!\bullet\; A'$ and a weakly horizontally invertible square γ_v as in

$$
\begin{tikzcd}
B \arrow[r, "b", "\simeq"'] \arrow[d, "v"', "\bullet" marking] \arrow[dr, phantom, "\gamma_v\ \simeq"] & FA \arrow[d, "Fu'", "\bullet" marking] \\
B' \arrow[r, "\simeq", "d"'] & FA' .
\end{tikzcd}
$$

If we consider the horizontal equivalences $b\epsilon_B\colon FGB \xrightarrow{\simeq} FA$ and $d\epsilon_{B'}\colon FGB' \xrightarrow{\simeq} FA'$, there exist horizontal morphisms $a\colon GB \to A$ and $c\colon GB' \to A'$ in $\mathbb{A}$ and vertically invertible squares γ_b and γ_d as depicted below.

$$
\begin{tikzcd}
FGB \arrow[r, "\epsilon_B", "\simeq"'] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "\gamma_b\ \rotatebox{90}{$\simeq$}"] & B \arrow[r, "b", "\simeq"'] & FA \arrow[d, equal, "\bullet" marking] & FGB' \arrow[r, "\epsilon_{B'}", "\simeq"'] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "\gamma_d\ \rotatebox{90}{$\simeq$}"] & B' \arrow[r, "d", "\simeq"'] & FA' \arrow[d, equal, "\bullet" marking] \\
FGB \arrow[rr, "Fa"'] & & FA & FGB' \arrow[rr, "Fc"'] & & FA'
\end{tikzcd}
$$

By Lemma 10.11, we have that $a\colon GB \xrightarrow{\simeq} A$ and $c\colon GB' \xrightarrow{\simeq} A'$ are horizontal equivalences in $\mathbb{A}$; thus, since $\mathbb{A}$ is weakly horizontally invariant, there exist a vertical morphism $u\colon GB \to\!\!\!\bullet\; GB'$ and a weakly horizontally invertible square

$$
\begin{tikzcd}
GB \arrow[r, "a", "\simeq"'] \arrow[d, "u"', "\bullet" marking] \arrow[dr, phantom, "\alpha_v\ \simeq"] & A \arrow[d, "u'", "\bullet" marking] \\
GB' \arrow[r, "\simeq", "c"'] & A' .
\end{tikzcd}
$$

We set $Gv \coloneqq u\colon GB \to\!\!\!\bullet\; GB'$. To define the weakly horizontally invertible square ϵ_v, let us first fix a weak inverse γ'_v of γ_v with respect to some horizontal equivalences (b, b', λ, κ) and $(d, d', \lambda', \kappa')$. We set ϵ_v to be the square given by the following pasting.

$$
\begin{tikzcd}
FGB \arrow[r, "\epsilon_B"] \arrow[d, "FGv"', "\bullet" marking] \arrow[dr, phantom, "\epsilon_v"] & B \arrow[d, "v", "\bullet" marking] \\
FGB' \arrow[r, "\epsilon_{B'}"'] & B'
\end{tikzcd}
=
\begin{tikzcd}
FGB \arrow[r, "\epsilon_B"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "e_{\epsilon_B}"] & B \arrow[rr, equal] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "\lambda \cong"] & & B \arrow[d, equal, "\bullet" marking] \\
FGB \arrow[r, "\epsilon_B"] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "\gamma_b \cong"] & B \arrow[r, "b"] & FA \arrow[r, "b'"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "e_{b'}"] & B \arrow[d, equal, "\bullet" marking] \\
FGB \arrow[rr, "Fa"] \arrow[d, "FGv"', "\bullet" marking] \arrow[drr, phantom, "F\alpha_v \simeq"] & & FA \arrow[r, "b'"] \arrow[d, "Fu'"', "\bullet" marking] \arrow[dr, phantom, "\gamma'_v \simeq"] & B \arrow[d, "v", "\bullet" marking] \\
FGB' \arrow[rr, "Fc"'] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "\gamma_d^{-1} \cong"] & & FA' \arrow[r, "d'"'] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "e_{d'}"] & B' \arrow[d, equal, "\bullet" marking] \\
FGB' \arrow[r, "\epsilon_{B'}"'] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "e_{\epsilon_{B'}}"] & B' \arrow[r, "d"'] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "\lambda'^{-1} \cong"] & FA' \arrow[r, "d'"'] & B' \arrow[d, equal, "\bullet" marking] \\
FGB' \arrow[r, "\epsilon_{B'}"'] & B' \arrow[rr, equal] & & B'
\end{tikzcd}
$$

Note that all the squares in the pasting are weakly horizontally invertible by Lemma 2.17, and thus so is ϵ_v. We write ϵ'_v for its unique weak inverse with respect to the horizontal adjoint equivalences $(\epsilon_B, \epsilon'_B, \mu_B, \nu_B)$ and $(\epsilon_{B'}, \epsilon'_{B'}, \mu_{B'}, \nu_{B'})$, as given by Lemma 2.16.

If $v = e_B$, we can choose $Ge_B \coloneqq e_{GB}$ and $\gamma_{e_B} \coloneqq e_{\epsilon_B}$. Then α_{e_B} can be chosen to be the identity square at the object GB and we get $\epsilon_{e_B} = e_{\epsilon_B}$.

Vertical coherence. Given vertical morphisms $v\colon B \twoheadrightarrow B'$ and $v'\colon B' \twoheadrightarrow B''$ in $\mathbb{B}$, we define the horizontally invertible comparison square between $Gv' \bullet Gv$ and $G(v'v)$ as follows. Let us denote by $\Omega_{v,v'}$ the following pasting.

$$
\begin{tikzcd}
FGB \arrow[rr, equal] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "\mu_B \cong"] & & FGB \arrow[d, equal, "\bullet" marking] \\
FGB \arrow[r, "\epsilon_B"] \arrow[d, "FGv"', "\bullet" marking] \arrow[dr, phantom, "\epsilon_v"] & B \arrow[r, "\epsilon'_B"] \arrow[d, "v", "\bullet" marking] & FGB \arrow[dd, "FG(v'v)", "\bullet" marking] \\
FGB' \arrow[r, "\epsilon_{B'}"] \arrow[d, "FGv'"', "\bullet" marking] \arrow[dr, phantom, "\epsilon_{v'}"] & B' \arrow[d, "v'", "\bullet" marking] \arrow[r, phantom, "\epsilon'_{v'v}"] & {} \\
FGB'' \arrow[r, "\epsilon_{B''}"'] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "\mu_{B''}^{-1} \cong"] & B'' \arrow[r, "\epsilon'_{B''}"'] & FGB'' \arrow[d, equal, "\bullet" marking] \\
FGB'' \arrow[rr, equal] & & FGB''
\end{tikzcd}
$$

Note that this square is horizontally invertible, since it is weakly horizontally invertible and its horizontal boundaries are identities. By (db4) of Definition 3.5, there exists a unique horizontally invertible square $\Psi_{v,v'}$ as depicted below left such that $F\Psi_{v,v'} = \Omega_{v,v'}$. In particular, one can check that, with this definition of $\Psi_{v,v'}$, the squares ϵ_v, $\epsilon_{v'}$ and $\epsilon_{v'v}$ satisfy the pasting equality below right.

$$
\begin{tikzcd}
GB \arrow[r, equal] \arrow[d, "Gv"'] & GB \arrow[dd, "G(v'v)"] \\
GB' \arrow[r, phantom, "\underset{\Psi_{v,v'}}{\cong}"] \arrow[d, "Gv'"'] & \\
GB'' \arrow[r, equal] & GB''
\end{tikzcd}
\qquad
\begin{tikzcd}
FGB \arrow[r, equal] \arrow[d, "FGv"'] & FGB \arrow[r, "\epsilon_B"] \arrow[dd, "FG(v'v)"] & B \arrow[dd, "v'v"] \\
FGB' \arrow[r, phantom, "\underset{F\Psi_{v,v'}}{\cong}"] \arrow[d, "FGv'"'] & \arrow[r, phantom, "\epsilon_{v'v}"] & \\
FGB'' \arrow[r, equal] & FGB'' \arrow[r, "\epsilon_{B''}"'] & B''
\end{tikzcd}
=
\begin{tikzcd}
FGB \arrow[r, "\epsilon_B"] \arrow[d, "FGv"'] \arrow[dr, phantom, "\epsilon_v"] & B \arrow[d, "v"] \\
FGB' \arrow[r, "\epsilon_{B'}"] \arrow[d, "FGv'"'] \arrow[dr, phantom, "\epsilon_{v'}"] & B' \arrow[d, "v'"] \\
FGB'' \arrow[r, "\epsilon_{B''}"'] & B''
\end{tikzcd}
$$

G **on squares.** Let $\beta\colon (v \, {}^{b}_{d} \, v')$ be a square in $\mathbb{B}$. Let us denote by δ the following pasting.

$$
\begin{tikzcd}
FGB \arrow[rrr, "FGb"] \arrow[d, equal] & & & FGC \arrow[d, equal] \\
FGB \arrow[r, "\epsilon_B"] \arrow[d, "FGv"'] \arrow[rrr, phantom, bend left=40, "\overline{\epsilon}_b^{-1} \cong"] & B \arrow[r, "b"] \arrow[d, "v"] & C \arrow[r, "\epsilon'_C"] \arrow[d, "v'"] & FGC \arrow[d, "FGv'"] \\
FGB' \arrow[r, "\epsilon_{B'}"'] \arrow[d, equal] \arrow[ur, phantom, "\epsilon_v"] & B' \arrow[r, "d"'] \arrow[ur, phantom, "\beta"] & C' \arrow[r, "\epsilon'_{C'}"'] \arrow[ur, phantom, "\epsilon'_{v'}"] & FGC' \arrow[d, equal] \\
FGB' \arrow[rrr, "FGd"'] \arrow[rrr, phantom, bend left=40, "\overline{\epsilon}_d \cong"] & & & FGC'
\end{tikzcd}
$$

Then, by (db4) of Definition 3.5, there exists a unique square

$$
\begin{tikzcd}
GB \arrow[r, "Gb"] \arrow[d, "Gv"'] \arrow[dr, phantom, "\alpha"] & GC \arrow[d, "Gv'"] \\
GB' \arrow[r, "Gd"'] & GC'
\end{tikzcd}
$$

such that $F\alpha = \delta$. We set $G\beta \coloneqq \alpha\colon (Gv \, {}^{Gb}_{Gd} \, Gv')$.

Let $b\colon B \to C$ be a horizontal morphism in $\mathbb{B}$, and $\beta = e_b\colon (e_B \, {}^{b}_{b} \, e_C)$. Then we have that $\delta = e_{FGb}$, since $\epsilon_{e_B} = e_{\epsilon_B}$ and $\epsilon'_{e_C} = e_{\epsilon_C}$, and the unique square $\alpha\colon (e_{GB} \, {}^{Gb}_{Gb} \, e_{GC})$ such that $F\alpha = e_{FGb}$ is given by e_{Gb}. Therefore, $Ge_b = e_{Gb}$.

Now let $v\colon B \mathrel{\bullet\!\!\!\to} B'$ be a vertical morphism in $\mathbb{B}$, and $\beta = \mathrm{id}_v \colon (v \,{}^{\mathrm{id}_B}_{\mathrm{id}_{B'}}\, v)$. Then we have that $\delta = \mathrm{id}_{FGv}$, since $\overline{\epsilon}^{-1}_{\mathrm{id}_B} = \mu_B$ and $\overline{\epsilon}_{\mathrm{id}_{B'}} = \mu^{-1}_{B'}$ and ϵ'_v is the weak inverse of ϵ_B with respect to the horizontal adjoint equivalence data $(\epsilon_B, \epsilon'_B, \mu_B, \nu_B)$ and $(\epsilon_{B'}, \epsilon'_{B'}, \mu_{B'}, \nu_{B'})$. The unique square $\alpha\colon (Gv \,{}^{\mathrm{id}_{GB}}_{\mathrm{id}_{GB'}}\, Gv)$ such that $F\alpha = \mathrm{id}_{FGv}$ is given by id_{Gv}. Therefore, $G\mathrm{id}_v = \mathrm{id}_{Gv}$.

Naturality and adjointness of ϵ and ϵ'. The assignment of G on squares is natural with the data of ϵ_B, ϵ_b and ϵ_v, and therefore the latter assemble into a pseudo horizontal natural equivalence $\epsilon\colon FG \Rightarrow \mathrm{id}$. Moreover, since $(\epsilon_B, \epsilon'_B, \mu_B, \nu_B)$ are horizontal adjoint equivalences, the data of ϵ'_B, ϵ'_b and ϵ'_v also assemble into a pseudo horizontal natural equivalence $\epsilon'\colon \mathrm{id} \Rightarrow FG$, where ϵ'_b is defined in a similar manner as ϵ_b was. In particular, $\epsilon\colon FG \Rightarrow \mathrm{id}$ and $\epsilon'\colon \mathrm{id} \Rightarrow FG$ are adjoint equivalences, where the invertible modifications are given by $\mu\colon \mathrm{id} \cong \epsilon'\epsilon$ and $\nu\colon \epsilon\epsilon' \cong \mathrm{id}$.

It remains to define the pseudo horizontal natural equivalence $\eta\colon \mathrm{id} \Rightarrow GF$. For this purpose, we use the pseudo horizontal natural equivalence $\epsilon'\colon \mathrm{id} \Rightarrow FG$.

η on objects. Let $A \in \mathbb{A}$, and consider the horizontal equivalence $\epsilon'_{FA}\colon FA \xrightarrow{\simeq} FGFA$. By (db2) of Definition 3.5, there exist a horizontal morphism $a\colon A \to GFA$ and a vertically invertible square

$$\begin{tikzcd}
FA \arrow[r, "\epsilon'_{FA}"] \arrow[d, equal] \arrow[dr, phantom, "\rho_A\ \cong"] & FGFA \arrow[d, equal] \\
FA \arrow[r, "Fa"'] & FGFA
\end{tikzcd}.$$

We set $\eta_A \coloneqq a\colon A \to GFA$. Note that $\eta_A\colon A \xrightarrow{\simeq} GFA$ is a horizontal equivalence by Lemma 10.11.

η on horizontal morphisms. Let $a\colon A \to C$ be a horizontal morphism in $\mathbb{A}$. We denote by ψ_a the following pasting.

$$\begin{tikzcd}
FA \arrow[r, "F\eta_A"] \arrow[d, equal] \arrow[dr, phantom, "\rho_A^{-1}\ \cong"] & FGFA \arrow[r, "FGFa"] \arrow[d, equal] \arrow[dr, phantom, "e_{FGFa}"] & FGFC \arrow[d, equal] \\
FA \arrow[r, "\epsilon'_{FA}"'] \arrow[d, equal] \arrow[drr, phantom, "\epsilon'_{Fa}\ \cong"] & FGFA \arrow[r, "FGFa"'] & FGFC \arrow[d, equal] \\
FA \arrow[r, "Fa"'] \arrow[d, equal] \arrow[dr, phantom, "e_{Fa}"] & FC \arrow[r, "\epsilon'_{FC}"'] \arrow[d, equal] \arrow[dr, phantom, "\rho_C\ \cong"] & FGFC \arrow[d, equal] \\
FA \arrow[r, "Fa"'] & FC \arrow[r, "F\eta_C"'] & FGFC
\end{tikzcd}$$

By (db4) of Definition 3.5, there exists a unique vertically invertible square

$$\begin{tikzcd}
A \arrow[r, "\eta_A"] \arrow[d, equal, "\bullet" marking] & GFA \arrow[r, "GFa"] \arrow[d, phantom, "\alpha \cong"] & GFC \arrow[d, equal, "\bullet" marking] \\
A \arrow[r, "a"'] & C \arrow[r, "\eta_C"'] & GFC
\end{tikzcd}$$

such that $F\alpha = \psi_a$; let $\eta_a \coloneqq \alpha$.

η **on vertical morphisms.** Let $u\colon A \to A'$ be a vertical morphism in $\mathbb{A}$. We denote by ψ_u the following pasting.

$$\begin{tikzcd}
FA \arrow[r, "F\eta_A"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\rho_A^{-1} \cong"] & FGFA \arrow[d, equal, "\bullet" marking] \\
FA \arrow[r, "\epsilon'_{FA}"] \arrow[d, "Fu"', "\bullet" marking] \arrow[dr, phantom, "\epsilon'_{Fu} \simeq"] & FGFA \arrow[d, "FGFu", "\bullet" marking] \\
FA' \arrow[r, "\epsilon'_{FA'}"'] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\rho_{A'} \cong"] & FGFA' \arrow[d, equal, "\bullet" marking] \\
FA' \arrow[r, "F\eta_{A'}"'] & FGFA'
\end{tikzcd}$$

Note that all the squares in the pasting are weakly horizontally invertible by Lemma 2.17, and thus so is ψ_u. By (db4) of Definition 3.5, there exists a unique weakly horizontally invertible square

$$\begin{tikzcd}
A \arrow[r, "\eta_A"] \arrow[d, "u"', "\bullet" marking] \arrow[dr, phantom, "\gamma"] & GFA \arrow[d, "GFu", "\bullet" marking] \\
A' \arrow[r, "\eta_{A'}"'] & GFA'
\end{tikzcd}$$

such that $F\gamma = \psi_u$; let $\eta_u \coloneqq \gamma$.

Naturality of η**.** Since $\epsilon'\colon \mathrm{id} \Rightarrow FG$ is a pseudo horizontal natural transformation, then η_A, η_a, and η_u assemble into a pseudo horizontal natural transformation $\eta\colon \mathrm{id} \Rightarrow GF$. Note that η is a pseudo horizontal natural equivalence, because η_A are horizontal equivalences and η_u are weakly horizontally invertible squares. Moreover, $\rho\colon \epsilon'_F \cong F\eta$ gives the data of an invertible modification. □

Remark 10.13. A careful study of the proof of Proposition 10.12 reveals that, if all the vertical morphisms in the double category $\mathbb{B}$ are identities, then the result holds without requiring that $\mathbb{A}$ be weakly horizontally invariant, since this condition is only needed for defining the pseudo functor $G\colon \mathbb{B} \to \mathbb{A}$ on vertical morphisms.

Put together, Propositions 10.8 and 10.12 and Remark 10.13 give the following characterization of the double biequivalences, when the source double category is weakly horizontally invariant or the target double category is horizontal.

Theorem 10.14 (Whitehead Theorem for double categories)**.** *Let $\mathbb{A}$ and $\mathbb{B}$ be double categories such that $\mathbb{A}$ is weakly horizontally invariant or $\mathbb{B}$ has only trivial vertical morphisms. Then a double functor $F\colon \mathbb{A} \to \mathbb{B}$ is a double biequivalence if and only if there exists a pseudo double functor $G\colon \mathbb{A} \to \mathbb{B}$ together with horizontal pseudo natural equivalences $\mathrm{id} \simeq GF$ and $FG \simeq \mathrm{id}$.*

In particular, we can restrict our results to double functors arising from 2-functors, and recover the well-known statement of the aforementioned Whitehead Theorem for 2-categories.

Corollary 10.15 (Whitehead Theorem for 2-categories)**.** *Let $F\colon \mathcal{A} \to \mathcal{B}$ be a 2-functor. Then F is a biequivalence if and only if there exists a pseudo functor $G\colon \mathcal{B} \to \mathcal{A}$ together with two pseudo natural equivalences $\mathrm{id} \simeq GF$ and $FG \simeq \mathrm{id}$.*

Proof. This can be obtained as a direct application of Theorem 10.14 to the double functor $\mathbb{H}F\colon \mathbb{H}\mathcal{A} \to \mathbb{H}\mathcal{B}$. It follows from the fact that pseudo double functors and pseudo horizontal natural equivalences between double categories in the image of $\mathbb{H}$ are equivalently pseudo functors and pseudo natural transformations between their preimages. □

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UPHESS BMI FSV, Ecole Polytechnique Fédérale de Lausanne, Station 8, 1015 Lausanne, Switzerland

E-mail address: lyne.moser@epfl.ch

Department of Mathematics, Cornell University, Ithaca NY, 14853, USA

E-mail address: mes462@cornell.edu

Department of Mathematics, Macquarie University, NSW 2109, Australia

E-mail address: paula.verdugo@hdr.mq.edu.au