

Unitary transformations of fibre functors

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Abstract

We study unitary pseudonatural transformations (UPTs) between fibre functors $\text{Rep}(G) \rightarrow \text{Hilb}$, where G is a compact quantum group. For fibre functors F_1, F_2 we show that the category of UPTs $F_1 \rightarrow F_2$ and modifications is isomorphic to the category of finite-dimensional $*$ -representations of the corresponding bi-Hopf-Galois object. We give a constructive classification of fibre functors accessible by a UPT from the canonical fibre functor, as well as UPTs themselves, in terms of Frobenius algebras in the category $\text{Rep}(A_G)$, where A_G is the Hopf $*$ -algebra dual to the compact quantum group. As an example, we show that finite-dimensional quantum isomorphisms from a quantum graph X are UPTs between fibre functors on $\text{Rep}(G_X)$, where G_X is the quantum automorphism group of X .

1 Introduction

1.1 Overview

Compact symmetry groups play a crucial role in quantum physics. By Tannaka duality, a compact group G is interchangeable with its category of finite dimensional representations $\text{Rep}(G)$, with canonical unitary monoidal *fibre functor* $F : \text{Rep}(G) \rightarrow \text{Hilb}$, where Hilb is the category of finite-dimensional Hilbert spaces and linear maps. One can therefore equivalently say that categories of representations of compact groups with a fibre functor play a crucial role in quantum physics. One way to interpret such categories is as encoding a consistent structure of system types, fusion rules, permissible transitions, etc. The fibre functor then assigns state spaces to all of the systems in the theory, and can be seen as a representation of this compositional structure as part of finite-dimensional quantum theory. Schematically:

$$\begin{array}{ccc} \text{Compositional category} & \xrightarrow{\quad} & \text{Representation} \\ \text{Rep}(G) & \xrightarrow{\quad} & \text{Hilb} \end{array} \quad \begin{array}{c} (1) \\ (2) \end{array}$$

One can generalise this notion of representation to theories more general than those described by the representation category of a compact group. In particular, we can consider representations of *C^* -tensor categories with conjugates*. Such categories and their fibre functors are described by the representation theory of compact quantum groups, and their associated Hopf-Galois objects.

In [18] we introduced a notion of *unitary pseudonatural transformation* relating two monoidal functors, or more generally two pseudofunctors. This paper is a study of unitary pseudonatural transformations between fibre functors on C^* -tensor categories with conjugates, or equivalently (provided a fibre functor exists) representation categories of compact quantum groups. The physical significance of these transformations will be explained in forthcoming work.

Unitary pseudonatural transformations. Unitary pseudonatural transformations are a generalisation of unitary monoidal natural isomorphisms, defined as follows. Let \mathcal{C} be a C^* -tensor category with conjugates, and let $F, F' : \mathcal{C} \rightarrow \text{Hilb}$ be fibre functors. Then a unitary pseudonatural transformation specifies:

- A Hilbert space H .
- For every object X of \mathcal{C} , a unitary linear map $F(X) \otimes H \rightarrow H \otimes F'(X)$.

These unitaries must obey equations generalising the monoidality and naturality conditions for unitary monoidal natural isomorphisms $F \rightarrow F'$, which are recovered when $H = \mathbb{C}$.

Hopf-Galois theory. Let \mathcal{C} be a C^* -tensor category with conjugates. Whenever a fibre functor $F : \mathcal{C} \rightarrow \text{Hilb}$ exists, we can construct a monoidal equivalence $\mathcal{C} \cong \text{Rep}(G)$ for a *compact quantum group* G . The category \mathcal{C} can therefore be understood in terms of the compact quantum group G , or rather its dual Hopf $*$ -algebra A_G .

Let $F_1, F_2 : \mathcal{C} \rightarrow \text{Hilb}$ be fibre functors corresponding to compact quantum groups G_1, G_2 . Then one can construct an A_{G_1} - A_{G_2} -bi-Hopf-Galois object Z linking the two fibre functors. This is a $*$ -algebra with a compatible left and right coactions of the algebras A_{G_1}, A_{G_2} .

Here we show (Theorem 3.13) that there is an isomorphism of categories between:

- The category $\text{Rep}(Z)$ of finite-dimensional $*$ -representations of Z and intertwining linear maps.
- The category $\text{Hom}(F_1, F_2)$ of unitary pseudonatural transformations $F_1 \rightarrow F_2$ and modifications.

This generalises the known fact [4, Thm 4.4.1] that the 1-dimensional $*$ -representations of an A_{G_1} - A_{G_2} -bi-Hopf-Galois object correspond to unitary monoidal natural transformations $F_1 \rightarrow F_2$.

Morita theory. In [18] we showed that the 2-category $\text{Fun}(\mathcal{C}, \text{Hilb})$ of unitary fibre functors, unitary pseudonatural transformations and modifications has certain nice properties; in particular, it is a dagger 2-category with duals and split dagger idempotents. This makes it an appropriate setting for Morita theory [15, Appendix].

Let us fix some fibre functor $F : \mathcal{C} \rightarrow \text{Hilb}$. By the results just discussed, the endomorphism category $\text{End}(F)$ of UPTs $F \rightarrow F$ and modifications is isomorphic to the category $\text{Rep}(A_G)$ of f.d. $*$ -representations of the associated compact quantum group algebra.

We use Morita theory to classify fibre functors F' such that there exists a UPT $F \rightarrow F'$, as well as UPTs $F \rightarrow F'$, as well as UPTs themselves, in terms of certain algebraic structures called *simple Frobenius monoids* in the category $\text{Rep}(A_G)$. In particular, we give constructions setting up a correspondence between the following structures (Theorem 4.16):

- Unitary monoidal isomorphism classes of unitary fibre functors accessible from F by a UPT; and Morita equivalence classes of simple Frobenius monoids in $\text{Rep}(A_G)$.
- Equivalence classes of UPTs $\alpha : F \rightarrow F'$ for some accessible fibre functor F' ; and $*$ -isomorphism classes of simple Frobenius monoids in $\text{Rep}(A_G)$.

As a consequence, we obtain a concrete construction of fibre functors accessible from F by a UPT in terms of idempotent splitting (Theorem 4.11).

Quantum graph isomorphisms. As an example of UPTs between fibre functors, we show that, for finite quantum graphs X, Y , the finite-dimensional quantum graph isomorphisms $X \rightarrow Y$ (Definition 5.11) considered in quantum information theory [1, 6] are UPTs between accessible fibre functors on the category $\text{Rep}(G_X)$ of representations of the quantum automorphism group of X . This sets up an equivalence between the following 2-categories (Theorem 5.19):

- QGraph_X : Objects — quantum graphs f.d. quantum isomorphic to X . 1-morphisms — f.d. quantum isomorphisms. 2-morphisms — intertwiners.
- $\text{Fun}(\text{Rep}(G_X), \text{Hilb})_X$: Objects — Fibre functors accessible by a UPT from the canonical fibre functor on $\text{Rep}(G_X)$. 1-morphisms — UPTs. 2-morphisms — modifications.

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1.3 Structure

In Section 2 we introduce necessary mathematical background material for this paper. In Section 3 we discuss the relationship between UPTs and Hopf-Galois theory. In Section 4 we discuss the Morita classification/construction of accessible UPTs and fibre functors. In Section 5 we show that finite-dimensional quantum graph isomorphisms are UPTs.

2 Background

2.1 Pivotal dagger categories and their diagrammatic calculus

2.1.1 Monoidal categories

We assume the reader is familiar with the definition of a monoidal category [11]. We use the standard coherence theorem [12] to assume that all our monoidal categories are strict, allowing the use of the following well-known diagrammatic calculus [17].

We read diagrams from bottom to top. Objects are drawn as wires, while morphisms are drawn as boxes whose type corresponds to their input and output wires. Composition of morphisms is represented by vertical juxtaposition, while monoidal product is represented by horizontal juxtaposition. For example, two morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ can be composed as follows:

$$\begin{array}{c} |Z \\ \boxed{g} \\ |Y \\ \boxed{f} \\ |X \end{array} \quad \begin{array}{c} |Y \\ \boxed{f} \quad \boxed{g} \\ |Z \\ |X \quad |Y \end{array} \quad (3)$$

$$g \circ f : X \rightarrow Z \quad f \otimes g : X \otimes Y \rightarrow Y \otimes Z \quad (4)$$

The wire for the monoidal unit $\mathbb{1}$, and the identity morphism id_X for any object X , are invisible in the diagram. Two diagrams which are planar isotopic represent the same morphism [17].

2.1.2 Pivotal categories

We recall the notion of duality in a monoidal category.

Definition 2.1. Let X be an object in a monoidal category. A *right dual* $[X^*, \eta, \epsilon]$ for X is:

- An object X^* .
- Morphisms $\eta : \mathbb{1} \rightarrow X^* \otimes X$ and $\epsilon : X \otimes X^* \rightarrow \mathbb{1}$ satisfying the following *snake equations*:

$$\begin{array}{ccc} \begin{array}{c} X^* \\ s \quad \boxed{\epsilon} \quad r \\ X \quad X^* \end{array} & = & \begin{array}{c} s \quad r \\ |X^* \end{array} \\ \begin{array}{c} r \quad \boxed{\epsilon} \quad s \\ X \quad X^* \end{array} & = & \begin{array}{c} r \quad s \\ |X \end{array} \end{array} \quad (5)$$

A *left dual* $[{}^*X, \eta, \epsilon]$ is defined similarly, with morphisms $\eta : \mathbb{1} \rightarrow X \otimes {}^*X$ and $\epsilon : {}^*X \otimes X \rightarrow \mathbb{1}$ satisfying the analogues of (5).

We say that a monoidal category \mathcal{C} *has right duals* (resp. *has left duals*) if every object X in \mathcal{C} has a chosen right dual $[X^*, \eta, \epsilon]$ (resp. a chosen left dual).

To represent duals in the graphical calculus, we draw an upward-facing arrow on the X -wire and a downward-facing arrow on the X^* - or *X -wire, and write η and ϵ as a cup and a cap, respectively. Then the equations (5) become purely topological:

$$\begin{array}{ccc} s \downarrow \cup r & = & s \downarrow r \\ \text{right dual} & & \end{array} \quad \begin{array}{ccc} r \uparrow \cup s & = & r \uparrow s \\ \text{right dual} & & \end{array} \quad \begin{array}{ccc} r \uparrow \cup s & = & r \uparrow s \\ \text{left dual} & & \end{array} \quad \begin{array}{ccc} s \downarrow \cup r & = & s \downarrow r \\ \text{left dual} & & \end{array}$$

Proposition 2.2 ([8, Lemmas 3.6, 3.7]). *If $[X^*, \eta_X, \epsilon_X]$ and $[Y^*, \eta_Y, \epsilon_Y]$ are right duals for X and Y respectively, then $[Y^* \otimes X^*, \eta_{X \otimes Y}, \epsilon_{X \otimes Y}]$ is right dual to $X \otimes Y$, where $\eta_{X \otimes Y}$ and $\epsilon_{X \otimes Y}$ are defined by:*

$$\begin{array}{ccc} \text{Diagram for } \eta_{X \otimes Y}: & & \text{Diagram for } \epsilon_{X \otimes Y}: \\ \text{A cup with wires labeled } Y^* X^* \text{ and } X Y. & & \text{A cap with wires labeled } X Y \text{ and } Y^* X^*. \\ \eta_{X \otimes Y} & & \epsilon_{X \otimes Y} \end{array} \quad (6)$$

Moreover, $[\mathbb{1}, \text{id}_{\mathbb{1}}, \text{id}_{\mathbb{1}}]$ is right dual to $\mathbb{1}$. Analogous statements hold for left duals.

Duals are unique up to isomorphism.

Proposition 2.3 ([8, Lemma 3.4]). *Let X be an object of a monoidal category, and let $[X^*, \eta, \epsilon], [X^{*\prime}, \eta', \epsilon']$ be right duals. Then there is a unique isomorphism $\alpha : X^* \rightarrow X^{*\prime}$ such that*

$$\begin{array}{ccc} \text{Diagram for } \alpha: & & \text{Diagram for } \epsilon': \\ \text{A box labeled } \eta' \text{ with an upward arrow on the left wire and a downward arrow on the right wire.} & = & \text{A box labeled } \eta \text{ with an upward arrow on the left wire and a downward arrow on the right wire.} \\ \text{A box labeled } \alpha \text{ with an upward arrow on the left wire and a downward arrow on the right wire.} & & \text{A box labeled } \epsilon' \text{ with an upward arrow on the left wire and a downward arrow on the right wire.} \\ \text{A box labeled } \alpha^{-1} \text{ with an upward arrow on the left wire and a downward arrow on the right wire.} & & \text{A box labeled } \epsilon \text{ with an upward arrow on the left wire and a downward arrow on the right wire.} \end{array} \quad (7)$$

An analogous statement holds for left duals.

In a category with duals, we can define a notion of transposition for morphisms.

Definition 2.4. Let X, Y be objects with chosen right duals $[X^*, \eta_X, \epsilon_X]$ and $[Y^*, \eta_Y, \epsilon_Y]$. For any morphism $f : X \rightarrow Y$, we define its *right transpose* $f^* : Y^* \rightarrow X^*$ as follows:

$$\begin{array}{ccc} \text{Diagram for } f^*: & & \text{Diagram for } f^*: \\ \text{A box labeled } f^* \text{ with an upward arrow on the left wire and a downward arrow on the right wire.} & = & \text{A box labeled } f \text{ with an upward arrow on the left wire and a downward arrow on the right wire.} \\ \text{A box labeled } f \text{ with an upward arrow on the left wire and a downward arrow on the right wire.} & & \text{A box labeled } f \text{ with an upward arrow on the left wire and a downward arrow on the right wire.} \end{array} \quad (8)$$

For left duals ${}^*X, {}^*Y$, a *left transpose* may be defined analogously.

In this work we are mostly interested in categories with compatible left and right duals. Such categories are called *pivotal*.

Let \mathcal{C} be a monoidal category with right duals. It is straightforward to check that the following defines an identity-on-objects monoidal functor $\mathcal{C} \rightarrow \mathcal{C}$, which we call the *double duals* functor:

- Objects X are taken to the double dual $X^{**} := (X^*)^*$.

- Morphisms $f : X \rightarrow Y$ are taken to the double transpose $f^{**} := (f^*)^*$.
- The multiplicators $m_{X,Y}$ and unitors u_r are defined using the isomorphisms of Proposition 2.3.

Definition 2.5. We say that a monoidal category \mathcal{C} with right duals is *pivotal* if the double duals functor is monoidally naturally isomorphic to the identity functor.

Roughly, the existence of a monoidal natural isomorphism in Definition 2.5 comes down to the following statement:

- For every object $X : r \rightarrow s$, there is an isomorphism $\iota_X : X^{**} \rightarrow X$.
- These $\{\iota_X\}$ can be chosen compatibly with composition in \mathcal{C} .

In a pivotal category, for any object X the right dual X^* is also a left dual for X by the following cup and cap (we have drawn a double upwards arrow on the double dual):

$$\begin{array}{c} \cup \\ \downarrow \end{array} := \begin{array}{c} \uparrow \\ \square_X \\ \uparrow \end{array} \quad \begin{array}{c} \cap \\ \uparrow \end{array} := \begin{array}{c} \uparrow \\ \square_{X^*} \\ \uparrow \end{array} \quad (9)$$

With these left duals, the left transpose of a morphism is equal to the right transpose. Whenever we refer to a pivotal category from now on, we suppose that the left duals are chosen in this way.

There is a very useful graphical calculus for these compatible left and right dualities in a pivotal category. To represent the transpose, we make our morphism boxes asymmetric by tilting the right vertical edge. We now write the transpose by rotating the boxes, as though we had ‘yanked’ both ends of the wire in the RHS of (8):

$$\begin{array}{c} \downarrow \\ f \\ \downarrow \end{array} \quad \begin{array}{c} \downarrow \\ f^* \\ \downarrow \end{array}$$

Using this notation, morphisms now freely slide around cups and caps.

Proposition 2.6 ([8, Lemma 3.12, Lemma 3.26]). *Let \mathcal{C} be a pivotal category and $f : X \rightarrow Y$ a modification. Then:*

$$\begin{array}{ccccccc} \begin{array}{c} \uparrow \\ f \\ \downarrow \end{array} & = & \begin{array}{c} \uparrow \\ f \\ \downarrow \end{array} & \begin{array}{c} \uparrow \\ f \\ \downarrow \end{array} & = & \begin{array}{c} \uparrow \\ f \\ \downarrow \end{array} & \begin{array}{c} \uparrow \\ f \\ \downarrow \end{array} = \begin{array}{c} \uparrow \\ f \\ \downarrow \end{array} \quad (10) \\ \cap & & \cup & \cap & & \cap & \cap \end{array}$$

The diagrammatic calculus is summarised by the following theorem.

Theorem 2.7 ([17, Theorem 4.14]). *Two diagrams for a morphism in a pivotal category represent the same morphism if there is a planar isotopy between them, which may include sliding of morphisms as in Proposition 2.6.*

In a pivotal category we can define notions of dimension for objects and trace for morphisms.

Definition 2.8. Let X be an object and let $f : X \rightarrow X$ be a morphism in a pivotal category \mathcal{C} . We define the *right trace* of f to be the following morphism $\text{Tr}_R(f) : \mathbb{1} \rightarrow \mathbb{1}$:

$$\begin{array}{c} \uparrow \\ r \\ \square_f \\ \uparrow \\ \downarrow \end{array}$$

We define the *right dimension* $\dim_R(r)$ of an object X of \mathcal{C} to be $\text{Tr}_R(\text{id}_X) : \mathbb{1} \rightarrow \mathbb{1}$. The *left traces* Tr_L and *left dimensions* \dim_L are defined analogously using the right cup and left cap.

Definition 2.9. We call a pivotal category \mathcal{C} *spherical* if, for object X , and any morphism $f : X \rightarrow X$, $\text{Tr}_L(f) = \text{Tr}_R(f) =: \text{Tr}(f)$. In this case we call $\text{Tr}(f)$ and $\dim(f)$ simply the *trace* and the *dimension*.

2.1.3 Pivotal dagger categories

Definition 2.10. A monoidal category \mathcal{C} is *dagger* if:

- There is a contravariant identity-on-objects functor $\dagger : \mathcal{C} \rightarrow \mathcal{C}$, which is *involutive*: for any morphism $f : X \rightarrow Y$, $\dagger(\dagger(f)) = f$. We write the dagger as a power on morphisms, i.e. $\dagger(f) =: f^\dagger$.
- The dagger is compatible with monoidal product: for any objects X, X', Y, Y' and morphisms $\alpha : X \rightarrow X'$ and $\beta : Y \rightarrow Y'$ we have $(\alpha \otimes \beta)^\dagger = \alpha^\dagger \otimes \beta^\dagger$.

We call the image of a morphism $f : X \rightarrow Y$ under $\dagger_{r,s}$ its *dagger*, and write it as $f^{\dagger_{r,s}}$.

In the graphical calculus, we represent the dagger of a morphism by reflection in a horizontal axis, preserving the direction of any arrows:

$$\begin{array}{ccc} \begin{array}{c} \text{X} \\ \uparrow \\ \boxed{f} \\ \downarrow \\ \text{Y} \end{array} & := & \begin{array}{c} \text{X} \\ \uparrow \\ \boxed{f^\dagger} \\ \downarrow \\ \text{Y} \end{array} \end{array} \quad (11)$$

Definition 2.11. Let \mathcal{C} be a dagger category. We say that a morphism $\alpha : X \rightarrow Y$ is an *isometry* if $\alpha^\dagger \circ \alpha = \text{id}_X$. We say that it is *unitary* if it is an isometry and additionally $\alpha \circ \alpha^\dagger = \text{id}_Y$.

We now give the condition for compatibility of dagger and pivotal structure.

Definition 2.12. Let \mathcal{C} be a pivotal category which is also a monoidal dagger category. We say that \mathcal{C} is *pivotal dagger* when, for all objects X :

$$\begin{array}{ccc} \cup \downarrow & = & \left(\cap \uparrow \right)^\dagger & \cap \uparrow & = & \left(\cup \downarrow \right)^\dagger \end{array}$$

Remark 2.13. Definition 2.12 implies compatibility between the graphical calculi of the duality and the dagger.

For any morphism $f : X \rightarrow Y$, a pivotal dagger structure implies the following *conjugate* morphism f_* is graphically well-defined:

$$\begin{array}{ccc} \begin{array}{c} \text{Y} \\ \downarrow \\ \boxed{f} \\ \uparrow \\ \text{X} \end{array} & := & \begin{array}{c} \downarrow \\ \cup \cap \\ \boxed{f} \\ \cap \cup \\ \downarrow \\ \text{Y} \end{array} & = & \left[\begin{array}{c} \text{X} \\ \downarrow \\ \boxed{f} \\ \uparrow \\ \text{Y} \end{array} \right]^\dagger \end{array} \quad (12)$$

Remark 2.14. Following the bra-ket convention, we draw morphisms $f : \mathbb{1} \rightarrow X$ and $f^\dagger : X \rightarrow \mathbb{1}$ — called *states* and *effects* of X respectively — as triangles rather than as boxes. The morphisms f and f^\dagger can be distinguished from f^T and f_* by the direction of the arrows:

$$\begin{array}{cccc} \begin{array}{c} \uparrow \\ \triangle \\ \downarrow \\ f \end{array} & \begin{array}{c} \triangle \\ \uparrow \\ f \\ \downarrow \end{array} & \begin{array}{c} \downarrow \\ \triangle \\ \uparrow \\ f \end{array} & \begin{array}{c} \triangle \\ \uparrow \\ f \\ \downarrow \end{array} \end{array} \quad (13)$$

$$f : \mathbb{1} \rightarrow X \qquad f^\dagger : X \rightarrow \mathbb{1} \qquad f_* : \mathbb{1} \rightarrow X^* \qquad f^T : X^* \rightarrow \mathbb{1} \quad (14)$$

2.1.4 Example: the category Hilb

A basic example of a pivotal dagger category is the category Hilb . The objects of the monoidal category Hilb are finite-dimensional Hilbert spaces, and the morphisms are linear maps between them; composition of morphisms is composition of linear maps. The monoidal product is given on objects by the tensor product of Hilbert spaces, and on morphisms by the tensor product of linear maps; the unit object is the 1-dimensional Hilbert space \mathbb{C} .

For any object H , its right dual is defined to be the dual Hilbert space H^* . Any basis $\{|v_i\rangle\}$ for H defines a cup and cap:

$$\begin{array}{c} \text{cup: } V \nearrow \curvearrowright V^* \\ |v\rangle \otimes \langle w| \mapsto \langle w|v\rangle \end{array} \quad \begin{array}{c} \text{cap: } V^* \downarrow \curvearrowleft V \\ 1 \mapsto \sum_i \langle v_i| \otimes |v_i\rangle \end{array} \quad (15)$$

It may easily be checked that this cup and cap fulfil the snake equations (5). This duality is pivotal; the monoidal natural isomorphism from the identity functor to the double duals functor is given by the standard isomorphism from a Hilbert space to its double dual.

The dagger structure is given by the Hermitian adjoint of a linear map. As long as the basis $\{|v_i\rangle\}$ is orthonormal Hilb is pivotal dagger. The transpose (8) and conjugate (12) are simply the usual transpose and complex conjugate of a linear map with respect to the orthonormal basis defining the duality.

In fact, Hilb is a *compact closed* category — it is symmetric monoidal in a way which is compatible with its pivotal dagger structure. Because it is symmetric monoidal, diagrams in Hilb should be considered as embedded in four-dimensional space. In particular, for any two Hilbert spaces V, W there is a *swap map* $\sigma_{V,W} : V \otimes W \rightarrow W \otimes V$. In four dimensions there is no difference between overcrossings and undercrossings, so we simply draw this as an intersection:

$$\begin{array}{c} \text{swap map: } \sigma_{V,W} \\ \text{two intersecting arcs} \end{array} \quad (16)$$

$$|v\rangle \otimes |w\rangle \mapsto |w\rangle \otimes |v\rangle \quad (17)$$

The four-dimensional calculus allows us to untangle arbitrary diagrams and remove any twists, as exemplified by the following equations, which hold regardless of the direction of the arrows on the wires:

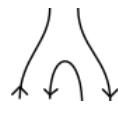
$$\begin{array}{ccccccccc} \text{complex crossing} & = & \text{three parallel lines} & = & \text{single vertical line} & = & \text{U-shaped wire} & = & \text{inverted U-shaped wire} \end{array} \quad (18)$$

It immediately follows that Hilb is spherical. The trace and dimension of Definition 2.8 reduce to the usual notion of trace and dimension of linear maps and Hilbert spaces.

The endomorphism algebra. The diagrammatic calculus in Hilb also allows us to conveniently express the endomorphism algebra $B(H)$ of a Hilbert space H using the pivotal dagger structure.

Definition 2.15. Let H be a Hilbert space. We define the following *endomorphism $*$ -algebra* on

$H \otimes H^*$:



$$m : (H \otimes H^*) \otimes (H \otimes H^*) \rightarrow H \otimes H^* \quad (20)$$



$$(19)$$

$$*: H \otimes H^* \rightarrow H \otimes H^* \quad (20)$$

It is straightforward to check that the endomorphism algebra is indeed a $*$ -algebra using the diagrammatic calculus of the pivotal dagger category Hilb . In fact, it is a Frobenius monoid (Definition 4.1).

Proposition 2.16. *There is an $*$ -isomorphism between the endomorphism algebra $H \otimes H^*$ and the $*$ -algebra $B(H)$.*

Proof. Consider the linear bijection $H \otimes H^* \rightarrow B(H)$ defined on orthonormal basis elements by $|i\rangle \otimes \langle j| \mapsto |i\rangle \langle j|$. It is multiplicative:

$$(21)$$

It is also unital, since $\sum_i |v_i\rangle \langle v_i| = \mathbb{1}$. Finally, the involution is preserved:

$$(22)$$

□

2.2 Monoidal functors

2.2.1 Diagrammatic calculus for monoidal functors

While our monoidal categories are strict, allowing us to use the diagrammatic calculus, we will consider functors between them which are not strict. For this, we use a graphical calculus of *functorial boxes* [13].

Definition 2.17. Let \mathcal{C}, \mathcal{D} , be monoidal categories. A *monoidal functor* $F : \mathcal{C} \rightarrow \mathcal{D}$ consists of the following data.

- A functor $F : \mathcal{C} \rightarrow \mathcal{D}$.

In the graphical calculus, we represent the effect of the functor F by drawing a shaded box around objects and morphisms in \mathcal{C} . For example, let X, Y be objects and $f : X \rightarrow Y$ a morphism in \mathcal{C} .

Then the morphism $F(f) : F(X) \rightarrow F(Y)$ in \mathcal{D} is represented as:

$$\begin{array}{c} F(Y) \\ \downarrow \\ \boxed{F(f)} \\ \downarrow \\ F(X) \end{array} = \quad \begin{array}{c} Y \\ | \\ f \\ | \\ X \end{array}$$

- For every pair of objects X, Y of \mathcal{C} , an invertible *multiplicator* morphism $m_{X,Y} : F(X) \otimes_D F(Y) \rightarrow F(X \otimes_C Y)$. In the graphical calculus, these morphisms and their inverses are represented as follows:

$$m_{X,Y} : F(X) \otimes_D F(Y) \rightarrow F(X \otimes_C Y) \quad m_{X,Y}^{-1} : F(X \otimes_C Y) \rightarrow F(X) \otimes_D F(Y) \quad (23)$$

- An invertible ‘unit’ morphism $u : \mathbb{1}_D \rightarrow F(\mathbb{1}_C)$. In the diagrammatic calculus, this morphism and its inverse are represented as follows (recall that the monoidal unit is invisible in the diagrammatic calculus):

The multiplicators and unit obey the following coherence equations:

- Naturality.* For any objects X, X', Y, Y' and morphisms $f : X \rightarrow X', g : Y \rightarrow Y'$ in \mathcal{C} :

$$\begin{array}{c} X' \\ | \\ f \\ | \\ X \end{array} \quad \begin{array}{c} Y' \\ | \\ g \\ | \\ Y \end{array} = \quad \begin{array}{c} X' \\ | \\ f \\ | \\ X \end{array} \quad \begin{array}{c} Y' \\ | \\ g \\ | \\ Y \end{array} \quad (25)$$

- Associativity.* For any objects X, Y, Z of \mathcal{C} :

$$\begin{array}{c} X \\ | \\ \text{---} \\ | \\ Y \\ | \\ \text{---} \\ | \\ Z \end{array} = \quad \begin{array}{c} X \\ | \\ \text{---} \\ | \\ Y \\ | \\ \text{---} \\ | \\ Z \end{array} \quad (26)$$

- *Unitality.* For any object X of \mathcal{C} :

$$\text{Diagram (27)} \quad \begin{array}{c} \text{Box with blue U-cutout} \\ | \\ \text{X} \end{array} = \begin{array}{c} \text{Solid blue box} \\ | \\ \text{X} \end{array} = \begin{array}{c} \text{Box with blue U-cutout} \\ | \\ \text{X} \end{array}$$

We observe that the analogous *conaturality*, *coassociativity* and *counitality* equations for the inverses $\{m_{X,Y}^{-1}\}, u^{-1}$, obtained by reflecting (25-27) in a horizontal axis, are already implied by (25-27). To give some idea of the calculus of functorial boxes, we explicitly prove the following lemma and proposition. From now on we will unclutter the diagrams by omitting object labels, unless adding the labels seems to significantly aid comprehension.

Lemma 2.18. *For any objects X, Y, Z of \mathcal{C} , the following equations are satisfied:*

$$\text{Diagram (25-27)} \quad \begin{array}{c} \text{Box with blue U-cutout} \\ | \\ \text{Box with blue U-cutout} \end{array} = \begin{array}{c} \text{Box with blue U-cutout} \\ | \\ \text{Box with blue U-cutout} \end{array} \quad \begin{array}{c} \text{Box with blue U-cutout} \\ | \\ \text{Box with blue U-cutout} \end{array} = \begin{array}{c} \text{Box with blue U-cutout} \\ | \\ \text{Box with blue U-cutout} \end{array}$$

Proof. We prove the left equation; the right equation is proved similarly.

$$\text{Diagram (25-27)} \quad \begin{array}{c} \text{Box with blue U-cutout} \\ | \\ \text{Box with blue U-cutout} \end{array} = \begin{array}{c} \text{Box with blue U-cutout} \\ | \\ \text{Box with blue circle} \end{array} = \begin{array}{c} \text{Box with blue circle} \\ | \\ \text{Box with blue U-cutout} \end{array} = \begin{array}{c} \text{Box with blue U-cutout} \\ | \\ \text{Box with blue U-cutout} \end{array}$$

Here the first and third equalities are by invertibility of $m_{X,Y}$, and the second is by coassociativity. \square

With Lemma 2.18, the equations (25-27) are sufficient to deform functorial boxes topologically as required. From now on we will do this mostly without comment.

2.2.2 Induced duals

We first observe that the duals in \mathcal{C} induce duals in \mathcal{D} under a monoidal functor $F : \mathcal{C} \rightarrow \mathcal{D}$.

Proposition 2.19 (Induced duals). *Let X be an object in \mathcal{C} and $[X^*, \eta, \epsilon]$ a right dual. Then $F(X^*)$ is a right dual of $F(X)$ in \mathcal{D} with the following cup and cap:*

$$\text{Diagram (28)} \quad \begin{array}{c} \text{Blue U-shape} \\ \downarrow \\ F(\eta) \end{array} \quad \begin{array}{c} \text{Blue rainbow shape} \\ \uparrow \\ F(\epsilon) \end{array}$$

The analogous statement holds for left duals.

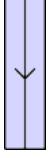
Proof. We show one of the snake equations (5) in the case of right duals; the others are all proved similarly.

$$\text{Diagram (5)} \quad \begin{array}{c} \text{Box with blue U-cutout, blue arrow up} \\ | \\ \text{Box with blue U-cutout, blue arrow up} \end{array} = \begin{array}{c} \text{Box with blue U-cutout, blue arrow up} \\ | \\ \text{Box with blue U-cutout, blue arrow up} \end{array} = \begin{array}{c} \text{Box with blue U-cutout, blue arrow up} \\ | \\ \text{Box with blue U-cutout, blue arrow up} \end{array} = \begin{array}{c} \text{Vertical blue bar} \end{array}$$

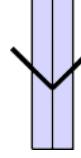
Here the first equality is by Lemma 2.18, the second by (25) and the third by (27) and the right snake equation in \mathcal{C} . \square

For any 1-morphism X of \mathcal{C} , then, we have two sets of left and right duals on $F(X)$; the first from the pivotal structure in \mathcal{C} by Proposition 2.19, and the second from the pivotal structure in \mathcal{D} .

In the diagrammatic calculus we distinguish between these two duals by drawing a large downwards arrowhead on the dual in \mathcal{D} , like so:



Induced duals $F(X^*)$ from \mathcal{C}



Duals $F(X)^*$ in \mathcal{D}

(28)

2.3 Compact quantum groups

In this work we will restrict our attention to the specific case of UPTs between linear monoidal functors from C^* -tensor categories with conjugates \mathcal{C} into the category Hilb of finite-dimensional Hilbert spaces and linear maps. Such functors are of physical interest, since Hilb is the category in which quantum mechanics is formulated. Provided that such a functor exists, there is a duality theory which identifies \mathcal{C} as the category of corepresentations of a certain algebraic object.

2.3.1 C^* -tensor categories

We first recall the definition of a C^* -tensor category with conjugates.

Definition 2.20. A dagger category is \mathbb{C} -linear if:

- For every pair of objects X, Y , $\text{Hom}(X, Y)$ is a complex vector space.
- For every triple of objects X, Y, Z , composition $\circ : \text{Hom}(X, Y) \times \text{Hom}(Y, Z) \rightarrow \text{Hom}(X, Z)$ is a bilinear map.
- For every pair of objects X, Y , the dagger $\dagger : \text{Hom}(X, Y) \rightarrow \text{Hom}(Y, X)$ is an antilinear and positive map, i.e. $\alpha \circ \alpha^\dagger = 0$ iff $\alpha = 0$.

A monoidal dagger category is \mathbb{C} -linear if, additionally:

- For every quadruple of objects X, X', Y, Y' , the tensor product $\otimes : \text{Hom}(X, X') \times \text{Hom}(Y, Y') \rightarrow \text{Hom}(X \otimes X', Y \otimes Y')$ is a bilinear map.

A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ between \mathbb{C} -linear categories is called \mathbb{C} -linear if the induced maps on Hom-spaces $F_{X,Y} : \text{Hom}_{\mathcal{C}}(X, Y) \rightarrow \text{Hom}_{\mathcal{D}}(F(X), F(Y))$ are \mathbb{C} -linear.

Definition 2.21. A \mathbb{C} -linear monoidal dagger category is called a C^* -tensor category if

- $\text{Hom}(X, Y)$ is a Banach space for all objects X, Y , and $\|fg\| \leq \|f\| \|g\|$.
- The dagger satisfies the following properties for any morphism $f : X \rightarrow Y$:
 - $\|f^\dagger \circ f\| = \|f\|^2$.
 - $f^\dagger \circ f$ is positive.

Following [16] we also assume that our C^* -tensor categories have the following completeness properties:

- There is an object $\mathbf{0}$ such that $\dim(\text{Hom}(\mathbf{0}, X)) = 0$ for every object X .
- There are binary direct sums: for every pair of objects X_1, X_2 , there is an object $X_1 \oplus X_2$ and morphisms $s_i : X_i \rightarrow X_1 \oplus X_2$ (for $i \in \{1, 2\}$) such that

$$s_i^\dagger s_i = \text{id}_{X_i} \quad s_1 s_1^\dagger + s_2 s_2^\dagger = \text{id}_{X_1 \oplus X_2} \quad (29)$$

- Dagger idempotents split: for any morphism $f : X \rightarrow X$ such that $f \circ f^\dagger = f$, there exists an object Y and an isometry $\iota : Y \rightarrow X$ such that $\iota \circ \iota^\dagger = f$.
- The unit object $\mathbb{1}$ is *irreducible*, i.e. $\text{End}(\mathbb{1}) = \mathbb{C}\text{id}_{\mathbb{1}}$.

In the setting of C^* -tensor categories, one normally speaks of *conjugates*¹ rather than duals.

Definition 2.22. Let X be an object of a C^* -tensor category \mathcal{C} . We say that another object X^* is *conjugate* to X if there exist morphisms $R : \mathbb{1} \rightarrow X^* \otimes X$ and $\bar{R} : \mathbb{1} \rightarrow X \otimes X^*$ such that $[X^*, R, \bar{R}]$ is right dual to X .

In any C^* -tensor category with conjugates each object possesses a distinguished conjugate $[X^*, R, \bar{R}]$, unique up to unitary isomorphism, called a *standard solution* [16, Def 2.2.14].

Theorem 2.23 ([16, Thm. 2.2.21]). *Let \mathcal{C} be a C^* -tensor category with conjugates, and let $\{[X^*, R_X, \bar{R}_X]\}$ be a choice of standard solutions for all objects X in \mathcal{C} so that \mathcal{C} has right duals. Then there is a unitary monoidal natural isomorphism ι from the double duals functor to the identity functor, such that $\bar{R}_X = (\iota_X \otimes \mathbb{1})R_{X^*}$.*

A C^* -tensor category equipped with a right duality arising from standard solutions is therefore a pivotal dagger category (in fact, a spherical dagger category [16, Thm 2.2.16]) and can be treated using the graphical calculus just discussed.

Definition 2.24. Let \mathcal{C} be a C^* -tensor category with conjugates. We call a unitary \mathbb{C} -linear monoidal functor $F : \mathcal{C} \rightarrow \text{Hilb}$ a *fibre functor*.

2.3.2 Compact quantum groups

We now introduce the algebraic objects dual to C^* -tensor categories with conjugates and a chosen fibre functor. All algebras are taken over \mathbb{C} .

Definition 2.25 ([16, Definition 1.6.1]). A unital $*$ -algebra A equipped with a unital $*$ -homomorphism $\Delta : A \rightarrow A \otimes A$ (the *comultiplication*) is called a *Hopf- $*$ -algebra* if $(\Delta \otimes \text{id}_A) \circ \Delta = (\text{id}_A \otimes \Delta) \circ \Delta$ and there exist linear maps $\epsilon : A \rightarrow \mathbb{C}$ (the *counit*) and $S : A \rightarrow A$ (the *antipode*) such that

$$(\epsilon \otimes \text{id}_A) \circ \Delta = \text{id}_A = (\text{id}_A \otimes \epsilon) \circ \Delta \quad m \circ (S \otimes \text{id}_A) \circ \Delta = u \circ \epsilon = m \circ (\text{id}_A \otimes S) \circ \Delta \quad (30)$$

where $m : A \otimes A \rightarrow A$ is the multiplication and $u : \mathbb{C} \rightarrow A$ the unit of the algebra A .

Definition 2.26 ([16, Definition 1.6.5]). A *corepresentation* of a Hopf $*$ -algebra A on a vector space H is a linear map $\delta : H \rightarrow H \otimes A$ such that

$$(\delta \otimes \text{id}_A) \circ \delta = (\text{id}_H \otimes \Delta) \circ \delta \quad (\text{id}_H \otimes \epsilon) \circ \delta = \text{id}_H \quad (31)$$

The corepresentation is called *unitary* if H is a Hilbert space and

$$\langle \delta(\xi), \delta(\zeta) \rangle = (\xi, \zeta) \mathbb{1}_A \quad \text{for all } \xi, \zeta \in H$$

where the A -valued inner product $\langle \cdot, \cdot \rangle$ on $H \otimes A$ is defined by linear extension of $\langle \xi \otimes a, \zeta \otimes b \rangle = (\xi, \zeta) a^* b$.

For $(H_1, \delta_1), (H_2, \delta_2)$ corepresentations of a Hopf $*$ -algebra A , we say that a linear map $f : H_1 \rightarrow H_2$ is an *intertwiner* $f : (H_1, \delta_1) \rightarrow (H_2, \delta_2)$ if $\delta_2 \circ f = (f \otimes \text{id}_A) \circ \delta_1$.

Definition 2.27. Let (H, δ) be a finite-dimensional unitary corepresentation of A , and let $\{|v_i\rangle\}$ be an orthonormal basis of H . Then $(\langle v_j | \otimes \text{id}_A) \circ \delta(|v_i\rangle)$ defines an A -valued matrix U_{ij} ; we say that the entries of this matrix are the *matrix coefficients* of the representation in the basis $\{|v_i\rangle\}$.

¹Note that these are not the conjugates of (12); they are rather dual objects.

A monoidal product of corepresentations can be defined [16, Definition 1.3.2], as can a notion of conjugate corepresentation [16, Def. 1.4.5]. This yields a C^* -tensor category with conjugates $\text{Corep}(A)$ whose objects are finite-dimensional unitary corepresentations of the algebra (A, Δ) and whose morphisms are intertwiners, with an obvious canonical fibre functor $F : \text{Corep}(A) \rightarrow \text{Hilb}$ which forgets the representation. Taking standard solutions to the conjugate equations, $\text{Corep}(A)$ has the structure of a pivotal dagger category.

Definition 2.28 (c.f. [16, Theorem 1.6.7]). We say that a Hopf-*algebra is a *compact quantum group algebra* if it is generated as an algebra by matrix coefficients of its finite-dimensional unitary corepresentations.

Such an algebra is considered as the algebra of matrix coefficients of representations of some ‘compact quantum group’ G , such that $\text{Rep}(G) = \text{Corep}(A)$. We will refer to compact quantum groups G , and write $\text{Rep}(G)$, in order to emphasise the similarity with representation theory of compact groups. However, the algebra A_G is the concrete object in general.

We now recall the theorem relating C^* -tensor categories with conjugates to compact quantum groups.

Theorem 2.29 ([16, Theorem 2.3.2]). *Let \mathcal{C} be a C^* -tensor category with conjugates, and let $U : \mathcal{C} \rightarrow \text{Hilb}$ be a fibre functor. Then there exists a compact quantum group algebra A (uniquely determined up to isomorphism) and a unitary monoidal equivalence $E_U : \mathcal{C} \rightarrow \text{Rep}(G_A)$, such that U is unitarily monoidally naturally isomorphic to $F \circ E_U$.*

2.4 Unitary pseudonatural transformations

A pseudonatural transformation between pseudofunctors is the 2-categorical generalisation of a natural transformation between functors [10]. Monoidal category theory is a special instance of 2-category theory, since a monoidal category is precisely a 2-category with a single object. However, pseudonatural transformations between monoidal functors are rarely discussed in generality; it is more usual to consider only monoidal natural transformations, which are a special case. In this paper we consider these transformations in general.

2.4.1 Definition

If the categories \mathcal{C} and \mathcal{D} are pivotal dagger categories, there is a notion of unitarity for pseudonatural transformations between monoidal functors $\mathcal{C} \rightarrow \mathcal{D}$ which generalises unitary monoidal natural transformations [18]. We here deal only with unitary pseudonatural transformations.

Definition 2.30. Let \mathcal{C}, \mathcal{D} be pivotal dagger categories, and let $F_1, F_2 : \mathcal{C} \rightarrow \mathcal{D}$ be unitary monoidal functors. (We colour the functorial boxes blue and red respectively.) A *unitary pseudonatural transformation* $(\alpha, H) : F_1 \rightarrow F_2$ is defined by the following data:

- An object H of \mathcal{D} (drawn as a green wire).
- For every object X of \mathcal{C} , a unitary morphism $\alpha_X : F_1(X) \otimes H \rightarrow H \otimes F_2(X)$ (drawn as a white vertex):

(32)

The unitary morphisms α_X must satisfy the following conditions:

- *Naturality.* For every morphism $f : X \rightarrow Y$ in \mathcal{C} :

(33)

- *Monoidality.*

- For every pair of objects X, Y of \mathcal{C} :

(34)

- α_1 is defined as follows:

(35)

Remark 2.31. Unitary pseudonatural transformations generalise the notion of *unitary monoidal natural isomorphism*, which we recover when $H \cong \mathbb{1}$.

Remark 2.32. The diagrammatic calculus shows that pseudonatural transformation is a planar notion. The H -wire forms a boundary between two regions of the \mathcal{D} -plane, one in the image of F and the other in the image of G . By pulling through the H -wire, morphisms from \mathcal{C} can move between the two regions (33).

UPTs $(\alpha, H) : F_1 \rightarrow F_2$ and $(\beta, H') : F_2 \rightarrow F_3$ can be composed associatively to obtain a UPT $(\alpha \circ \beta, H \otimes H') : F_1 \rightarrow F_3$ whose components $(\alpha \circ \beta)_X$ are as follows (we colour the H' -wire orange, and the F_3 -box brown):

(36)

There are also morphisms between pseudonatural transformations, known as *modifications* [10].

Definition 2.33. Let $(\alpha, H), (\beta, H') : F_1 \Rightarrow F_2$ be UPTs. (We colour the H -wire green and the H' -wire orange.) A *modification* $f : \alpha \rightarrow \beta$ is a morphism $f : H \rightarrow H'$ satisfying the following equation for all components $\{\alpha_X, \beta_X\}$:

(37)

Modifications can themselves be composed horizontally and vertically — vertical composition is composition of morphisms in \mathcal{D} , while horizontal composition is monoidal product of morphisms in \mathcal{D} . The dagger of a modification is also a modification. Altogether, the compositional structure is that of a dagger 2-category.²

Definition 2.34. Let \mathcal{C}, \mathcal{D} be pivotal dagger categories. The dagger 2-category $\text{Fun}(\mathcal{C}, \mathcal{D})$ is defined as follows:

- Objects: unitary monoidal functors $F_1, F_2, \dots, \cdot : \mathcal{C} \rightarrow \mathcal{D}$.
- 1-morphisms: unitary pseudonatural transformations $\alpha, \beta, \dots : F_1 \rightarrow F_2$.
- 2-morphisms: modifications $f, g, \dots : \alpha \rightarrow \beta$.

Because we are able to assume that \mathcal{C} and \mathcal{D} are strict, $\text{Fun}(\mathcal{C}, \mathcal{D})$ is a strict 2-category.

Remark 2.35. Because $\text{Fun}(\mathcal{C}, \mathcal{D})$ is a dagger 2-category, the endomorphism categories $\text{End}(F)$ of UPTs $F \rightarrow F$ and modifications for a given functor F are monoidal dagger categories.

2.4.2 Duals

Definition 2.36. Let $(\alpha, H) : F_1 \rightarrow F_2$ be a UPT. Then the *dual* of α is a UPT $(\alpha^*, H^*) : F_2 \rightarrow F_1$ whose components α_X^* are defined as follows:

(38)

Here the second equality is by unitarity of α [18, Prop. 5.2]. We sometimes put a $*$ next to the vertex for the dual UPT α^* to distinguish it from α , although the orientation of the H -wire is sufficient for this.

For the composition of a UPT with its dual, the cups and caps of the dagger duality in \mathcal{D} are modifications:

(39)

(40)

3 UPTs and Hopf-Galois theory

The Tannaka duality relating C^* -tensor categories to compact quantum group algebras extends to the characterisation of their fibre functors. For a compact quantum group G , the algebraic objects corresponding to fibre functors on $\text{Rep}(G)$ are *Hopf-Galois objects* for the CQG algebra A_G , also known as *noncommutative torsors*. In this section we will show that UPTs between fibre functors can be classified in terms of the finite-dimensional representation theory of these Hopf-Galois objects.

²For a definition of dagger 2-category, see [18, Def. 2.13].

3.1 Background on Hopf-Galois theory

We now review the basics of Hopf-Galois theory for compact quantum groups, based on [4].

Definition 3.1. Let A be a CQG algebra, and Z a $*$ -algebra. We say that a left corepresentation $\delta : Z \rightarrow A \otimes Z$ is a *left coaction* if δ is a $*$ -homomorphism.

Definition 3.2. [4, Def. 4.1.1] Let A be a CQG algebra, and let Z be an $*$ -algebra with a left A -coaction. We say that Z is a *left A -Hopf-Galois object* if the following linear map is a bijection:

$$(\text{id}_A \otimes m_Z) \circ (\delta \otimes \text{id}_Z) : Z \otimes Z \rightarrow A \otimes Z$$

Right A -Hopf-Galois objects may be defined similarly. For two CQG algebras A_1, A_2 , if Z is a left A_1 - and a right A_2 -Hopf Galois object, we say that it is an *A_1 - A_2 -bi-Hopf-Galois object*.

From any left Hopf-Galois object for A_G it is possible to construct a unitary fibre functor on $\text{Rep}(G)$.

Construction 3.3 ([4, Prop. 4.3.1]). Let G be a compact quantum group, and (Z, δ_Z) a left Hopf-Galois object for A_G . Then a unitary fibre functor $F_Z : \text{Rep}(G) \rightarrow \text{Hilb}$ may be constructed, part of whose definition is as follows:

- *On objects.* For any corepresentation (V, δ_V) of A_G , as a vector space $F(V) = V \wedge Z$, where $V \wedge Z$ is the equaliser of the double arrow:

$$\delta_V \otimes \text{id}_Z, \text{id}_V \otimes \delta_Z : V \otimes Z \rightrightarrows V \otimes A_G \otimes Z$$

- *On morphisms.* For any intertwiner $f : V \rightarrow W$, $F(f) = f \otimes \text{id}_Z : V \wedge Z \rightarrow W \wedge Z$.

Remark 3.4. The Hopf-Galois object corresponding to the canonical fibre functor is A_G itself.

Likewise, from any unitary fibre functor on $\text{Rep}(G)$ one can construct an left A_G -Hopf-Galois object.

Construction 3.5 ([4, Prop 4.3.3]). Let \mathcal{C} be a semisimple pivotal dagger category, let $F_1, F_2 : \text{Rep}(G) \rightarrow \text{Hilb}$ be two unitary fibre functors, and let G_1, G_2 be the two compact quantum groups obtained by Tannaka reconstruction (Theorem 2.29). The predual $\text{Hom}^\vee(F_1, F_2)$ of the vector space $\text{Hom}(F_1, F_2)$ of natural transformations $F_1 \rightarrow F_2$ has the structure of an A_{G_1} - A_{G_2} -bi-Hopf-Galois object.

These constructions lead to a classification of unitary fibre functors on $\text{Rep}(G)$. Let $\text{Gal}(A)$ be the category whose objects are A -Hopf-Galois objects and whose morphisms are algebra $*$ -homomorphisms intertwining the A -coactions. Let $\text{Fib}(G)$ be the category whose objects are unitary fibre functors on $\text{Rep}(G)$ and whose morphisms are unitary monoidal natural isomorphisms.

Theorem 3.6 ([4, Thm 4.3.4]). *Constructions 3.3 and 3.5 yield an equivalence of categories $\text{Gal}(A) \cong \text{Fib}(G)$.*

We also note the following fact characterising the spectrum of a bi-Hopf-Galois object, which Theorem 3.13 will generalise to all finite-dimensional $*$ -representations.

Proposition 3.7 ([4, Thm 4.4.1]). *Let \mathcal{C} be a C^* -tensor category with conjugates, let $F_1, F_2 : \mathcal{C} \rightarrow \text{Hilb}$ be fibre functors, and let Z be the corresponding A_{G_1} - A_{G_2} -bi-Hopf-Galois object. Then there is a linear isomorphism between the vector space of unitary monoidal natural isomorphisms $F_1 \rightarrow F_2$ and the vector space of 1-dimensional $*$ -representations of Z .*

A generators-and-relations description of Z . We will need the following generators-and-relations description of the bi-Hopf-Galois object $Z = \text{Hom}^\vee(F_1, F_2)$ linking two fibre functors $F_1, F_2 : \mathcal{C} \rightarrow \text{Hilb}$, taken from [4, 9].

Consider the vector space $\bigoplus_{V \in \text{Obj}(\mathcal{C})} \text{Hom}(F_2(V), F_1(V))$, where the sum is taken over all objects of \mathcal{C} . Let \mathcal{N} be the following subspace:

$$\langle F_1(f) \circ v - v \circ F_2(f) \mid \forall W \in \text{Obj}(\mathcal{C}), \forall f \in \text{Hom}_{\mathcal{C}}(V, W), \forall v \in \text{Hom}(F_2(W), F_1(V)) \rangle \quad (41)$$

Then, as a vector space:

$$\text{Hom}^\vee(F_1, F_2) := \bigoplus_{V \in \text{Obj}(\mathcal{C})} \text{Hom}(F_2(V), F_1(V)) / \mathcal{N} \quad (42)$$

We denote $v \in \text{Hom}(F_2(V), F_1(V))$ by $[V, v]$ as an element of $\text{Hom}^\vee(F_1, F_2)$, which is clearly generated as a vector space by all the $[V, v]$ up to the relations (41). The algebra structure is defined as follows on the generators, where we draw F_1 with a blue box and F_2 with a red box:

$$[V, v] \cdot [W, w] = [V \otimes W, m_{F_1} \circ (v \otimes w) \circ m_{F_2}^\dagger] \quad (44)$$

The unit of the algebra is $[\mathbb{1}, u_1 \circ u_2^\dagger]$, where u_i is the unitor for F_i .

The involution is defined as follows on the generators:

$$[V, v]^* = [V^*, (v^*)^T] \quad (46)$$

3.2 UPTs as $*$ -representations of Hopf-Galois objects

We have just recalled that, for a compact quantum group G , the category $\text{Fib}(G)$ of fibre functors on the pivotal dagger category $\text{Rep}(G)$ and unitary monoidal natural isomorphisms is characterised by Hopf-Galois theory. In [18] we showed that $\text{Fib}(G)$ can be generalised to a dagger 2-category $\text{Fun}(\text{Rep}(G), \text{Hilb})$ of fibre functors, unitary pseudonatural transformations and modifications. We will now see that this dagger 2-category is also characterised by Hopf-Galois theory: UPTs correspond to finite-dimensional $*$ -representations of bi-Hopf-Galois objects, just as unitary monoidal natural transformations correspond to their one-dimensional $*$ -representations.

Let \mathcal{C} be a C^* -tensor category with conjugates, let $F_1, F_2 : \mathcal{C} \rightarrow \text{Hilb}$ be fibre functors, and let Z be the corresponding G_1 - G_2 -bi-Hopf-Galois object. We will first show that for every unitary pseudonatural transformation $(\alpha, H) : F_1 \rightarrow F_2$, one can construct a $*$ -representation $\pi_\alpha : Z \rightarrow B(H)$.

Construction 3.8. Recall the generators-and-relations description of Z from Section 3.1. We define the following map $\pi_\alpha : \bigoplus_{V \in \text{Obj}(\mathcal{C})} \text{Hom}(F_2(V), F_1(V)) \rightarrow H \otimes H^* \cong B(H)$ by its action on generators

$[V, v]$:

(47)

Note that here we are taking the trace with respect to the dual $F_2(V)^*$ in Hilb (see (28)).

Proposition 3.9. *Construction 3.8 defines a $*$ -representation $\pi_\alpha : Z \rightarrow B(H)$. Moreover, any modification $f : (\alpha, H) \rightarrow (\beta, H')$ induces an intertwiner $H \rightarrow H'$.*

Proof. We first show that π_α induces a well-defined map on $Z = \bigoplus_{V \in \text{Obj}(\mathcal{C})} \text{Hom}(F_2(V), F_1(V))/\mathcal{N}$, where \mathcal{N} is the subspace defined in (41). For this it is sufficient to show that, for any $f : V \rightarrow W$ in \mathcal{C} and $x : F_2(W) \rightarrow F_1(V)$ in \mathcal{D} , we have $\pi_\alpha([V, x \circ F_2(f)]) = \pi_\alpha([W, F_1(f) \circ x])$:

(48)

Here for the first equality we slide the morphism $F_2(f) : F_2(V) \rightarrow F_2(W)$ in the dotted box around the loop using the graphical calculus of Hilb (Theorem 2.7); the second equality is by naturality of α . We therefore indeed have a map $Z \rightarrow B(H)$, which we now show is a $*$ -homomorphism.

- *Multiplicative.* Recalling the definition of the multiplication (43) of Z , we show $\pi_\alpha([V, v] \cdot [W, w]) = \pi_\alpha(v)\pi_\alpha(w)$:

(49)

Here the first equality is by monoidality of α , and for the second we slide the multiplicator $F_2(V \otimes W) \rightarrow F_2(V) \otimes F_2(W)$ around the loop and use unitarity of F_2 to cancel it and its inverse. The third equality is by the graphical calculus of the symmetric monoidal category Hilb . In the last diagram we recognise the multiplication of $B(H) \cong H \otimes H^*$ (19).

- *Unital.* Recalling that the unit of Z is $[\mathbb{1}, u_1 \circ u_2^\dagger]$, we observe:

(50)

Here the first equality is by monoidality of α ; for the second we slide the unitor around the loop and use unitarity of F_2 to cancel it with its inverse. In the final diagram we recognise the unit of $B(H) \cong H \otimes H^*$ (19).

- *Involution-preserving.* Recalling the definition of the involution (45) on generators $[V, v]$, we have the following equations. We use a large double upwards arrow for the dual $F_2(V^*)^*$ in Hilb, and a large downwards arrow for the dual $F_2(V)^*$ in Hilb:

(51)

Here the first equality is by unitarity of α (38), and the second equality is by a snake equation for the induced duality on $F_1(V)$. We now observe that, since $F_2(V^*)$ is dual to $F_2(V)$ in Hilb by the induced cup and cap, by Proposition 2.3 there is an invertible morphism $f : F_2(V^*)^* \rightarrow F_2(V)$ relating the two caps:

(52)

We now continue (51):

(53)

(54)

For the first equality we used (52); for the second we used a snake equation for the induced dual on $F_2(V)$; for the third we slid f^{-1} around the loop and cancelled it with f ; for the fourth we used the graphical calculus of Hilb to pull the loop around; the fifth equality is by the ‘horizontal reflection’ calculus of the dagger in Hilb; and for the sixth equality we used the graphical calculus of Hilb to untangle the H -wires. In the final diagram we observe the involution of $B(H)$ (19).

Finally, every modification clearly induces an intertwiner:

(55)

□

We will now produce a construction in the other direction, which gives a UPT $F_1 \rightarrow F_2$ for any $*$ -representation of Z .

Construction 3.10. Let $\pi : Z \rightarrow B(H)$ be a $*$ -representation. Recall that $Z \cong \bigoplus_{V \in \text{Obj}(\mathcal{C})} F_1(V) \otimes F_2(V)^*/\mathcal{N}$. For every object V of \mathcal{C} we define a map $U_V : F_1(V) \rightarrow F_2(V) \otimes Z$ as follows (c.f. [16, Thm 2.3.11]):

(56)

We then define a map $(\alpha_\pi)_V : F_1(V) \otimes H \rightarrow H \otimes F_2(V)$ (here $\pi : Z \rightarrow B(H) \cong H \otimes H^*$ is represented

by a white vertex):



Remark 3.11. Here we are abusing notation by including a Z -wire in the Hilb-diagrams, since it is not generally finite-dimensional and we have not even defined an inner product on it. The diagrams could easily be modified to remove the Z -wire at the cost of some lack of clarity.

Proposition 3.12. *Construction 3.10 defines a UPT $(\alpha_\pi, H) : F_1 \rightarrow F_2$. Moreover, any intertwiner $f : (\pi, H) \rightarrow (\pi', H')$ induces a modification $f : (\alpha_\pi, H) \rightarrow (\alpha_{\pi'}, H')$.*

Proof. We show that the map U_V satisfies certain properties which will imply that α_π is a UPT.

- *Naturality.* For any $f : V \rightarrow W$ in \mathcal{C} :

(58)

(59)

Here the first and fifth equalities are by definition, and the second and fourth equalities are by naturality of the symmetry in Hilb. The third equality can be seen by picking orthonormal bases

$\{|i\rangle\}$ for $F(V)$ and $\{|j\rangle\}$ for $G(W)$ — for every pair $|i\rangle, |j\rangle$ we then have:

(60)

Here the second equality is by (41) and the others by the graphical calculus of Hilb .

- **Monoidality.** For any objects V, W in \mathcal{C} we have the following two equations:

(61)

(62)

Here the first and third equalities are by definition (note we have written the multiplication of Z as a black vertex). For the second equality we use the graphical calculus of Hilb to untangle the wires and then slide the transpose of the comultiplicator of F_2 around the loop to the left, cancelling it with the multiplicator.

(63)

—

Here the first equality is by unitarity of the unitors of F, F' ; the second equality is by definition of U_1 ; the third equality is by the graphical calculus of Hilb; and the fourth equality is by definition of the unit of Z (which we have drawn as a black vertex).

- *Unitarity.* Choose orthonormal bases of $F(V), F'(V)$. We show that with respect to such a basis U_V is a unitary matrix. We prove the first unitarity equation $\sum_k [U_V]_{ki}^* [U_V]_{kj} = \delta_{ij} 1_Z$. Let $\{|i\rangle\}, \{|j\rangle\}$ be elements of the orthonormal basis of $F(V)$ and let $\{|k\rangle\}$ be the orthonormal basis of $F'(V)$. Then:

$$\sum_k \left[\begin{array}{c} \text{red vertical line with } v \uparrow, i \downarrow \\ \text{blue vertical line with } v \uparrow, j \downarrow \\ \text{white parallelogram } U_V \\ \text{black dot } Z \end{array} \right] = \sum_k \left[\begin{array}{c} \text{red U-shaped curves with arrows} \\ \text{blue vertical line with } v \uparrow, i \downarrow \\ \text{white parallelogram } U_V \\ \text{black dot } Z \end{array} \right]^*$$
(64)

$$= \sum_k \left[\begin{array}{c} \text{red U-shaped curve with arrows} \\ \text{blue vertical line with } v \uparrow, i \downarrow \\ \text{white parallelogram } U_V \\ \text{black dot } Z \end{array} \right]^* = \sum_k \left[\begin{array}{c} \text{two blue vertical lines with } v \uparrow, j \downarrow \\ \text{red U-shaped curve with arrows} \\ \text{white parallelogram } U_V \\ \text{black dot } Z \end{array} \right]$$
(65)

$$= \sum_k \left[\begin{array}{c} \text{two blue vertical lines with } i \downarrow, j \downarrow \\ \text{red U-shaped curve with arrows} \\ \text{white parallelogram } U_V \end{array} \right]^* = \text{red U-shaped curve with arrows} = \text{red U-shaped curve with arrows}$$
(66)

$$= \delta_{ij} \quad (67)$$

Here for the first equality we used the definition of U_V ; for the second equality we used the graphical calculus of Hilb ; for the third equality we used the definition of multiplication and involution in Z ; for the fourth equality we used the induced duality in \mathcal{C} ; for the fifth equality we removed the resolution of the identity and used unitarity of the functor F' ; for the sixth equality we used the definition of the quotient space defining Z (41); for the seventh equality we used unitarity of the functor F ; and for the eighth equality we used the induced duality from \mathcal{C} and orthogonality of the basis. In the final diagram we recognise the unit of Z .

The second unitarity equation, $\sum_k [U_V]_{ik} [U_V]^*_{jk} = \delta_{ij} 1_Z$ is shown similarly.

We now show that α_π is a UPT.

- *Naturality.* Follows immediately from naturality of $\{U_V\}$ and of the swap.
- *Monoidality.*

$$= \quad = \quad (68)$$

In the first equation we recognised the multiplication of $B(H) \cong H \otimes H^*$ (19) and used the fact that π is a homomorphism. In the second equation we used monoidality of U .

- *Unitarity.* We must show that α_V^\dagger is a 2-sided inverse for α_V . Consider the first of the 2

unitarity equations:

$$(69)$$

To see this, we choose orthonormal bases for $F_1(V)$ and $F_2(V)$. Let $|i\rangle, |j\rangle$ be any two elements of the orthonormal basis for $F_1(V)$ and let $\{|k\rangle\}$ be the orthonormal basis elements of $F_2(V)$. Then we have the following equation:

$$(70)$$

Here we simply inserted a resolution of the identity in $F_2(V)$. Now we define the following elements of $B(H)$:

$$(71)$$

With these elements (70) becomes:

$$\sum_k \text{[complex structure involving } j, v, U_V, k\text{]} = \sum_k [U_V(j,k)] = \sum_k [U_V(j,k)^*] = \delta_{ij} \quad (72)$$

Here in the second equality we noticed the involution and multiplication of $B(H) \cong H \otimes H^*$ (19). For the third equality we used involutivity and multiplicativity of π , and then unitarity of U_V . Therefore (69) is proved. The proof of the other unitarity equation is similar.

Finally, an intertwiner clearly induces a modification:

$$\text{[Diagram showing the effect of an intertwiner } f\text{ on a UPT structure]} = \text{[Diagram showing the modified UPT structure with } f\text{]} \quad (73)$$

□

Theorem 3.13. Let \mathcal{C} be a C^* -tensor category with conjugates, and let $F_1, F_2 : \mathcal{C} \rightarrow \text{Hilb}$ be fibre functors. Let Z be the corresponding G_1-G_2 -Hopf-Galois-object. There is an isomorphism of categories between:

- The category $\text{Rep}(Z)$ of f.d. $*$ -representations of Z and intertwiners.
- The category $\text{Fun}(F_1, F_2)$ of UPTs $F_1 \rightarrow F_2$ and modifications.

Proof. We show that Constructions 3.8 and 3.10 are strictly inverse. First we observe that, for a UPT $\alpha : F_1 \rightarrow F_2$, the $*$ -homomorphism $\pi_\alpha : Z \rightarrow B(H)$ is defined on the subspace in the image of $\iota_V : F_1(V) \otimes F_2(V)^* \rightarrow Z$ as follows:

$$\text{[Diagram showing a red curved ribbon with a self-intersection, representing a morphism in the category Rep(Z)]} \quad (74)$$

We also observe that, for a $*$ -homomorphism $\pi : Z \rightarrow B(H)$, the UPT $\alpha_\pi : F_1 \rightarrow F_2$ is defined on V as follows:

(75)

For one direction, let $\alpha : F_1 \rightarrow F_2$ be a UPT and use Construction (3.8) to obtain a $*$ -representation $\pi_\alpha : Z \rightarrow B(H)$. Now use Construction 3.10 to obtain a UPT $F_1 \rightarrow F_2$. That the resulting UPT is α follows immediately from the graphical calculus of Hilb:

(76)

In the other direction, let π be a $*$ -representation $\pi_\alpha : Z \rightarrow B(H)$ and use Construction (3.10) to obtain a UPT $\alpha_\pi : F_1 \rightarrow F_2$. Now use Construction 3.8 to obtain a $*$ -representation $Z \rightarrow B(H)$. That the resulting UPT is α again follows immediately from the graphical calculus of Hilb:

(77)

That the maps on modifications and intertwiners are inverse is clear. \square

Corollary 3.14. *Two fibre functors on $\text{Rep}(G)$ are related by a UPT precisely when the corresponding bi-Hopf-Galois object has a finite-dimensional $*$ -representation.*

Definition 3.15. Let \mathcal{C} be a semisimple pivotal dagger category and let $F : \mathcal{C} \rightarrow \text{Hilb}$ be a fibre functor. We say that a fibre functor $F' : \mathcal{C} \rightarrow \text{Hilb}$ is *accessible from F* if there exists a UPT $\alpha : F \rightarrow F'$.

Let G be a compact quantum group and $F : \text{Rep}(G) \rightarrow \text{Hilb}$ the canonical fibre functor. We observe that a UPT — since its components are unitary — must preserve dimensions of Hilbert spaces in the

sense that $\dim(F(V)) = \dim(F'(V))$, so an A_G -Hopf-Galois object corresponding to a fibre functor accessible from F must be *cleft* [5, Thm 1.17]. It is unknown to the author whether all cleft Hopf-Galois objects admit a finite-dimensional $*$ -representation.

At least in one case the situation is clear, since cleft Hopf-Galois objects for a compact quantum group algebra A are all obtained as cocycle twists of A [5, Thm 1.8], so for finite-dimensional A they are also finite-dimensional with a faithful state [5, Prop. 4.2.5, Cor. 4.3.5].

Corollary 3.16. *When G is a finite CQG (i.e the algebra A_G is finite-dimensional) all fibre functors on $\text{Rep}(G)$ are accessible from the canonical fibre functor by a UPT.*

4 Morita theory of accessible fibre functors and UPTs

In Section 3.2 we showed that a fibre functor on the category $\text{Rep}(G)$ is accessible from the canonical fibre functor F by a UPT precisely when the corresponding A_G -Hopf-Galois object admits a finite-dimensional $*$ -representation. In this Section we will use Morita theory to show that accessible fibre functors F' and UPTs $F \rightarrow F'$ can be constructively classified in terms of the finite-dimensional representation theory of the compact quantum group algebra A_G .

4.1 Background on Frobenius monoids and Morita equivalence

Morita theory relates 1-morphisms $X : r \rightarrow s$ out of an object r of a 2-category \mathcal{C} to Frobenius monoids in its category of endomorphisms $\mathcal{C}(r, r)$. In our case, the 2-category in question is $\text{Fun}(\text{Rep}(G), \text{Hilb})$, and we consider the category $\text{End}(F)$ of endomorphisms of the canonical fibre functor $F : \text{Rep}(G) \rightarrow \text{Hilb}$, which we have just shown (Theorem 3.13) is isomorphic to $\text{Rep}(A_G)$. We now recall the definition of a Frobenius monoid, and two notions of equivalence which will be important in our classification.

Frobenius monoids.

Definition 4.1. Let \mathcal{C} be a monoidal dagger category. A *monoid* in \mathcal{C} is an object A with multiplication and a unit morphisms, depicted as follows:

$$\begin{array}{ccc} \text{Diagram: } & & \text{Diagram: } \\ \text{A vertical line with a circle at the top and a curved line connecting it to a point below.} & & \text{A vertical line with a circle at the top.} \\ m : A \otimes A \rightarrow A & & u : \mathbb{1} \rightarrow A \end{array} \quad (78)$$

These morphisms satisfy the following associativity and unitality equations:

$$\begin{array}{ccccc} \text{Diagram: } & = & \text{Diagram: } & = & \text{Diagram: } \\ \text{Two vertical lines meeting at a point with a circle above them.} & & \text{Two vertical lines meeting at a point with a circle above them.} & & \text{A vertical line with a circle at the top and a vertical line below it.} \\ & & & & \text{A vertical line with a circle at the top and a vertical line below it.} \end{array} \quad (79)$$

Analogously, a *comonoid* is an object A with a coassociative comultiplication $\delta : A \rightarrow A \otimes A$ and a counit $\epsilon : A \rightarrow \mathbb{C}$. The dagger of a monoid (A, m, u) is a comonoid $(A, m^\dagger, u^\dagger)$. A monoid (A, m, u) in \mathcal{C} is called *Frobenius* if the monoid and adjoint comonoid structures are related by the following *Frobenius equation*:

$$\begin{array}{ccc} \text{Diagram: } & = & \text{Diagram: } \\ \text{Two vertical lines meeting at a point with a circle above them, connected by a curved line.} & & \text{Two vertical lines meeting at a point with a circle above them, connected by a curved line.} \\ & & \text{Two vertical lines meeting at a point with a circle above them, connected by a curved line.} \end{array} \quad (80)$$

Frobenius monoids are canonically self-dual. Indeed, it is easy to see that for any Frobenius monoid the following cup and cap fulfil the snake equations (5):

$$\begin{array}{c} \text{cup} \\ \text{cap} \end{array} := \begin{array}{c} \text{cup} \\ \text{cap} \end{array} \quad \begin{array}{c} \text{cup} \\ \text{cap} \end{array} := \begin{array}{c} \text{cup} \\ \text{cap} \end{array} \quad (81)$$

We now consider two equivalence relations on Frobenius monoids in a dagger category with split idempotents. The stricter of the relations is **-isomorphism*.

*-isomorphism.

Definition 4.2. Let A, B be Frobenius monoids in a monoidal dagger category. We say that a morphism $f : A \rightarrow B$ is a **-homomorphism* precisely when it satisfies the following equations:

$$\begin{array}{c} \text{circle with } f \\ \text{cup} \end{array} = \begin{array}{c} \text{circle with } f \\ \text{cap} \end{array} \quad \begin{array}{c} \text{circle with } f \\ \text{cap} \end{array} = \begin{array}{c} \text{circle with } f \\ \text{cup} \end{array} \quad \begin{array}{c} \text{circle with } f \\ \text{cap} \end{array} = \begin{array}{c} \text{circle with } f \\ \text{cup} \end{array} \quad (82)$$

We say that a morphism $f : A \rightarrow B$ is a **-cohomomorphism* precisely when it satisfies the following equations:

$$\begin{array}{c} \text{cup} \\ \text{circle with } f \end{array} = \begin{array}{c} \text{cap} \\ \text{circle with } f \end{array} \quad \begin{array}{c} \text{circle with } f \\ \text{cap} \end{array} = \begin{array}{c} \text{circle with } f \\ \text{cup} \end{array} \quad \begin{array}{c} \text{circle with } f \\ \text{cap} \end{array} = \begin{array}{c} \text{circle with } f \\ \text{cup} \end{array} \quad (83)$$

A **-isomorphism* $A \rightarrow B$ is a morphism $f : A \rightarrow B$ which is both a **-homomorphism* and a **-cohomomorphism*. This yields an equivalence relation which we call **-isomorphism*.

Morita equivalence. To define the weaker equivalence relation, we introduce the notion of a dagger bimodule.

Definition 4.3. Let A and B be special dagger Frobenius monoids in a monoidal dagger category. An $A-B$ -dagger bimodule is an object M together with a morphism $\rho : A \otimes M \otimes B \rightarrow M$ fulfilling the following equations:

$$\begin{array}{c} \text{cup} \\ \text{square with } \rho \end{array} = \begin{array}{c} \text{cap} \\ \text{square with } \rho \end{array} \quad \begin{array}{c} \text{circle with } \rho \\ \text{cap} \end{array} = \begin{array}{c} \text{circle with } \rho \\ \text{cup} \end{array} \quad \begin{array}{c} \text{cup} \\ \text{square with } \rho \end{array} = \begin{array}{c} \text{cap} \\ \text{square with } \rho \end{array} \quad (84)$$

We usually denote an $A-B$ -dagger bimodule M by ${}_A M_B$.

Definition 4.4. A *morphism of dagger bimodules* ${}_A M_B \rightarrow {}_A N_B$ is a morphism $f : M \rightarrow N$ that commutes with the action of the Frobenius monoids:

$$\begin{array}{c} \text{square with } \rho \\ \text{cup} \end{array} = \begin{array}{c} \text{square with } \rho \\ \text{cap} \end{array} \quad \begin{array}{c} \text{square with } \rho \\ \text{cap} \end{array} = \begin{array}{c} \text{square with } \rho \\ \text{cup} \end{array} \quad (85)$$

Two dagger bimodules are *isomorphic*, here written ${}_A M_B \cong {}_A N_B$, if there is a unitary morphism of dagger bimodules ${}_A M_B \rightarrow {}_A N_B$.

In a monoidal dagger category in which dagger idempotents split (Definition 2.21), we can compose dagger bimodules ${}_A M_B$ and ${}_B N_C$ to obtain an $A-C$ -dagger bimodule ${}_A M \otimes_B N_C$ as follows. First note that the following endomorphism is a dagger idempotent:

(86)

The *relative tensor product* ${}_A M \otimes_B N_C$ is defined as the image of the splitting of this idempotent. We depict the isometry $i : M \otimes_B N \rightarrow M \otimes N$ as a downwards pointing triangle:

(87)

For dagger bimodules ${}_A M_B$ and ${}_B N_C$, the relative tensor product $M \otimes_B N$ is itself an $A-C$ -dagger bimodule with the following action $A \otimes (M \otimes_B N) \otimes C \rightarrow M \otimes_B N$:

(88)

Definition 4.5. Two dagger Frobenius monoids A and B are *Morita equivalent* if there are dagger bimodules ${}_A M_B$ and ${}_B N_A$ such that ${}_A M \otimes_B N_A \cong {}_A A_A$ and ${}_B N \otimes_A M_B \cong {}_B B_B$.

It may straightforwardly be verified that $*$ -isomorphic Frobenius monoids are Morita equivalent.

4.2 Morita classification of accessible fibre functors and UPTs

We first observe that every UPT out of the canonical fibre functor F gives rise to a Frobenius monoid in $\text{End}(F)$.

Proposition 4.6. *Let G be a compact quantum group, let $F : \text{Rep}(G) \rightarrow \text{Hilb}$ be the canonical fibre functor, and let F' be another fibre functor and $\alpha : F \rightarrow F'$ a UPT. Then the object $\alpha^* \circ \alpha$ of $\text{End}(F)$ has the structure of a Frobenius monoid in $\text{End}(F)$ with the following multiplication and unit modifications:*

(89)

$$m : (\alpha^* \circ \alpha) \circ (\alpha^* \circ \alpha) \rightarrow (\alpha^* \circ \alpha) \quad (90)$$

Proof. That these are modifications as claimed follows from the pull-through equations for the cup and cap for the dual UPT (39). It is a Frobenius monoid because the underlying algebra of the modifications is the endomorphism algebra on $H \otimes H^*$ in Hilb (Definition 2.15). \square

Definition 4.7. The monoidal functor $\text{For} : \text{End}(F) \rightarrow \text{Hilb}$ is defined as follows:

- *On objects:* Every UPT $(\alpha, H) : F \rightarrow F$ is taken to its underlying Hilbert space H .
- *On morphisms:* Every modification $f : (\alpha_1, H_1) \rightarrow (\alpha_2, H_2)$ is taken to its underlying linear map $f : H_1 \rightarrow H_2$.

Remark 4.8. By the isomorphism $\text{End}(F) \cong \text{Rep}(A_G)$ of Theorem 3.13, For is precisely the canonical fibre functor on the category $\text{Rep}(A_G)$.

Definition 4.9. A Frobenius monoid $A = ((\alpha, H), m, u)$ in $\text{End}(F)$ is *simple* if $\text{For}(A)$ is $*$ -isomorphic to an endomorphism algebra in Hilb .

Every fibre functor F' and UPT $F \rightarrow F'$ gives rise to a simple Frobenius monoid in $\text{End}(F)$ by the construction of Proposition 4.6. We now give a construction in the other direction — any simple Frobenius monoid in $\text{End}(F) \cong \text{Rep}(A_G)$ can be ‘split’ to obtain a fibre functor F' and a UPT $F \rightarrow F'$. First observe that we may conjugate any simple symmetric Frobenius monoid $((\tilde{\alpha}, \tilde{H}), \tilde{m}, \tilde{u})$ in $\text{End}(F)$ by the $*$ -isomorphism of Definition 4.9 to obtain a simple Frobenius monoid $((\alpha, H \otimes H^*), m, u)$ where the modifications m, u have the standard form:

$$m : \alpha \circ \alpha \rightarrow \alpha \quad (91)$$

$$u : \text{id}_F \rightarrow \alpha \quad (92)$$

Lemma 4.10. For a simple symmetric Frobenius monoid $((\alpha, H \otimes H^*), m, u)$ in $\text{End}(F)$ and any object V of $\text{Rep}(G)$, the following defines a dagger idempotent on $H^* \otimes F(V) \otimes H$:

$$\frac{1}{d} \quad (93)$$

Proof. Idempotency follows from the fact that the algebra multiplication is a modification:

$$\frac{1}{d^2} = \frac{1}{d^2} = \frac{1}{d} \quad (94)$$

We must now show that the idempotent is dagger. Recall that a Frobenius algebra has a self-duality with the cup and cap (81). By Proposition 2.3 applied in $\text{End}(F)$, there is therefore a canonical unitary modification $P : (\alpha, H \otimes H^*) \rightarrow (\alpha^*, (H \otimes H^*)^*)$ satisfying the following equations in Hilb :

(95)

Here we drew the chosen right dual of $(H \otimes H^*)$ using a thick wire and a thick downward arrow in the spirit of (28). Now we show that the idempotent is dagger:

(96)

Here for the first equality we used the graphical calculus of the dagger; for the second equality we used unitarity of α (38); for the third equality we used (95); and for the fourth equality we used the fact that P is a modification $\alpha \rightarrow \alpha^*$ to cancel P with its inverse. \square

This dagger idempotent splits to give a new Hilbert space, which, foreshadowing Theorem 4.11, we call $F_\alpha(V)$ and draw as V surrounded by a red box, and an isometry $\iota_V : F_\alpha(V) \rightarrow H^* \otimes V \otimes H$ satisfying the following equations:

(97)

Theorem 4.11. Let $((\alpha, H \otimes H^*), m, u)$ be a simple dagger Frobenius monoid in $\text{End}(F) \cong \text{Rep}(A_G)$. For every representation V of G , let $F_\alpha(V)$ and $\iota_V : F_\alpha(V) \rightarrow H^* \otimes V \otimes H$ be the Hilbert space and isometry defined in the foregoing discussion.

Then the following defines a fibre functor $F_\alpha : \text{Rep}(G) \rightarrow \text{Hilb}$:

- On objects: $V \mapsto F_\alpha(V)$.
- On morphisms:

(98)

- Monoidal structure:

$m_{V,W} : F_\alpha(V) \otimes F_\alpha(W) \rightarrow F_\alpha(V \otimes W)$ (99)
 $u : \mathbb{C} \rightarrow F_\alpha(\mathbb{1})$ (100)

Moreover, there is a UPT $(\sqrt{\alpha}, H) : F \rightarrow F_\alpha$ with the following components $\sqrt{\alpha}_V$:

(101)

This UPT ‘splits’ A in the sense that $\sqrt{\alpha}^* \circ \sqrt{\alpha} = \alpha$.

Proof. We first show that F_α is a fibre functor. Compositionality is seen as follows:

(102)

Here the first equality is by the second equation of (97); the second equality is naturality of the UPT $(\alpha, H \otimes H^*) : F \rightarrow F$; the third equality is by the second equation of (97); and the fourth equality is by the first equation of (97).

G clearly takes identity morphisms to identity morphisms since ι is an isometry, and the functor preserves the dagger by symmetry of (98) in a horizontal axis. We therefore already have a unitary functor. For monoidality, we must check that $\{m_{V,W}\}$ and u are unitary and that they obey the associativity and unitality equations (26) and (27).

- *Unitarity of $\{m_{V,W}\}$.* One equation of unitarity is straightforward:

(103)

Here for the first and third equalities we used the first equation of (97), and for the second equality we evaluated the loop and used unitarity of the monoidal functor F .

For the other unitarity equation:

(104)

(105)

Here for the first equality we used the first equation of (97); for the second equality we used the second equation of (97) and monoidality of α ; for the third equality we used the fact that the multiplication of the Frobenius algebra is a modification; and for the fourth equality we evaluated the loops and used the first equation of (97).

- *Unitarity of u .* For the first equation:

(106)

Here for the first equality we used the second equation of (97); for the second equality we used monoidality of α (35); and for the third equality we evaluated the loops and used unitarity of F .

For the second equation:

(107)

For the first equality we used monoidality of α (35); for the second equality we used the second equation of (97); and for the third equality we used the first equation of (97).

- *Associativity* (26). We have the following sequence of equations:

(108)

(109)

Here for the first equality we used the first equation of (97) to insert isometries and their inverses on all three legs, and the second equation of (97) and monoidality of α . For the second equality we used the second equation of (97). For the third equality we used the fact that the multiplication of the Frobenius algebra is a modification and evaluated the resulting loop.

We leave the rest of the proof to the reader: use monoidality of α on the two rightmost legs, use the second equation of (97) to eliminate all occurrences of α , then cancel isometries using the first equation of (97).

- *Unitality* (27). The left unitality equation is shown as follows:

(110)

The right unitality equation is similar.

We have therefore shown that F_α is a fibre functor on $\text{Rep}(G)$. We must now show that $\sqrt{\alpha}$ is a UPT $F \rightarrow F_\alpha$. For this we must show naturality and monoidality (33-35).

- *Naturality.* For any $f : V \rightarrow W$:

(111)

Here the first and fourth equalities are by definition; the second equality is by the second equation of (97) and naturality of α ; and the third equality is by the first equation of (97).

- *Monoidality.*

– We show (34):

(112)

(113)

Here the first equality is by definition of $\sqrt{\alpha}$ and the multiplicator of F_α ; the second equality is by the second equation of (97); the third equality is by monoidality of α ; the fourth equality is by the second equation of (97); and the fifth equality is by the first equation of (97) and the definition of $\sqrt{\alpha}$.

– We show (35):

Here the first equality is by definition of $\sqrt{\alpha}$ and by the first equation of (97); the second equality is by the second equation of (97); the third equality is by (97); and the fourth equality is by definition of the unit of F_α .

- *Unitarity.* We show the first equation of unitarity:

Here for the first equality we used the first equation of (97); for the second equality we used the second equation of (97); for the third equality we used the fact that the comultiplication of $((\alpha, H \otimes H^*), m, u)$ is a modification; for the fourth equality we used the second equation of (97); and for the final equality we used the first equation of (97).

We show the second equation of unitarity:

For the first equality we used the second equation of (97), and for the second equality we used that the counit of $((\alpha, H \otimes H^*), m, u)$ is a modification.

We have therefore shown that $\alpha : G \rightarrow F$ is a UPT. Finally, we show that $\sqrt{\alpha}$ splits α , i.e. $\sqrt{\alpha} \circ \sqrt{\alpha}^* = \alpha$. By unitarity of $\sqrt{\alpha}$ it is equivalent to show that $\sqrt{\alpha} \circ \sqrt{\alpha}^\dagger = A$, which follows immediately from

the second equation of (97):

The diagram consists of two parts separated by an equals sign (=). On the left, there are two horizontal boxes labeled t_V . Between them is a vertical red rectangle. A green curved arrow enters from the top-left, passes over the top box, then under the red rectangle, then over the bottom box, and exits to the bottom-right. Another green curved arrow enters from the bottom-left, passes under the bottom box, then over the red rectangle, then under the top box, and exits to the top-right. On the right, there is a single vertical purple rectangle. A green curved arrow enters from the top-left, passes over the top of the purple rectangle, then under its side, and exits to the bottom-right. Another green curved arrow enters from the bottom-left, passes under the side of the purple rectangle, then over its top, and exits to the top-right.

(117)

□

Classification of UPTs and accessible fibre functors. We have seen that every UPT $\alpha : F \rightarrow F'$, where F' is some fibre functor accessible from F , yields a simple Frobenius algebra in $\text{End}(F)$, and that every isomorphism class of simple Frobenius algebras is obtained in this way. By means of this correspondence we can classify UPTs and accessible fibre functors. The proof of this classification requires some 2-categorical material; see [18] for clarification.

We first observe a couple of technical facts.

Definition 4.12. We say that a dagger 2-category has *split dagger idempotents* if dagger idempotents split in all of its Hom-categories.

Proposition 4.13. *The category $\text{Fun}(\text{Rep}(G), \text{Hilb})$ has split dagger idempotents.*

Proof. Let $(\alpha, H) : F_1 \rightarrow F_2$ be a UPT and let $f : \alpha \rightarrow \alpha$ be a dagger idempotent modification. Since f is also a dagger idempotent in Hilb , there exists a Hilbert space I and an isometry $\iota_f : I \rightarrow H$ such that:

$$\iota_f \circ \iota_f^\dagger = \text{id}_I \quad \iota_f^\dagger \circ \iota_f = f \quad (118)$$

Now we define a new UPT α'^f whose components α'^f_V are given as in [18, Eq. 32]. It is clear that this is a UPT and that ι_f is a modification $\alpha \rightarrow \alpha'^f$ satisfying (118) as a modification. □

Proposition 4.14. *Let G be a compact quantum group. In $\text{Fun}(\text{Rep}(G), \text{Hilb})$ there exists a dagger equivalence [18, Def. 2.15] between two objects F_1, F_2 iff these functors are unitarily monoidally naturally isomorphic.*

Proof. For a pseudonatural transformation $(\alpha, H) : F_1 \rightarrow F_2$ to be a dagger equivalence in $\text{Fun}(\text{Rep}(G), \text{Hilb})$, there must exist a pseudonatural transformation $(\alpha^{-1}, H^*) : G \rightarrow F$ and an unitary isomorphism $f : \mathbb{C} \rightarrow H \otimes H^*$. But then H must be 1-dimensional, and therefore unitarily isomorphic to the unit object \mathbb{C} . Conjugating (α, H) by this isomorphism as in [18, Eq. 32], we obtain a unitary monoidal natural isomorphism $F_1 \rightarrow F_2$. In the other direction, a unitary monoidal natural isomorphism is clearly a dagger equivalence. □

We now state the classification result.

Definition 4.15. We say that two UPTs $\alpha_1 : F \rightarrow F_1$ and $\alpha_2 : F \rightarrow F_2$ are *equivalent* if there exist a unitary monoidal natural isomorphism $X : F_2 \rightarrow F_1$ and a unitary modification $u : \alpha_1 \rightarrow X \circ \alpha_2$.

The following theorem is an immediate application of the results of [15, Appendix].

Theorem 4.16. *Let F be the canonical fibre functor $\text{Rep}(G) \rightarrow \text{Hilb}$. Then the constructions of Proposition 4.6 and Theorem 4.11 give a bijective, constructive correspondence between:*

- Unitary monoidal isomorphism classes of unitary fibre functors accessible from F by a UPT; and Morita equivalence classes of simple Frobenius monoids in $\text{Rep}(A_G)$.
- Equivalence classes of UPTs $\alpha : F \rightarrow F'$ for some accessible fibre functor F' ; and $*$ -isomorphism classes of simple Frobenius monoids in $\text{Rep}(A_G)$.

5 Quantum graphs and their isomorphisms

In this Section we give an example of UPTs arising in the study of finite quantum graph theory [14]. Specifically, we will show that finite-dimensional quantum graph isomorphisms from a quantum graph X are UPTs from the canonical fibre functor on the category of representations of its quantum automorphism group.

5.1 UPTs for compact matrix quantum groups

Definition 5.1. We say that a C^* -tensor category \mathcal{C} is *generated* by a family of objects Q if, for any object V of \mathcal{C} , there exists a family $\{b_k\}$ of *reduction morphisms* $b_k \in \text{Hom}(r_k, V)$, where each r_k is a monoidal product of objects in Q , such that $\sum_k b_k b_k^\dagger = \text{id}_X$.

Definition 5.2. We say that a pair (G, u) of a compact quantum group G and some representation u is a *compact matrix quantum group* when $\text{Rep}(G)$ is generated by the objects $\{u, u^*\}$.

For (G, u) a compact matrix quantum group, we will now show that a UPT between fibre functors on $\text{Rep}(G)$ is completely determined by its component on the fundamental representation u .

First we define some notation: for a vector $\vec{x} \in \{\pm 1\}^n$, $n \in \mathbb{N}$, we write $u^{\vec{x}}$ for the object $u^{x_1} \otimes \dots \otimes u^{x_n}$, where we take $u^{-1} := u^*$. We additionally define $u^0 := \mathbb{1}$.

Definition 5.3. Let (G, u) be a compact matrix quantum group, and let $F, F' : \text{Rep}(G) \rightarrow \text{Hilb}$ be fibre functors. We define a *reduced unitary pseudonatural transformation* $(\tilde{\alpha}, H) : F \rightarrow F'$ to be:

- A Hilbert space H (drawn as a green wire).
- A unitary morphism $\tilde{\alpha} : F(u) \otimes H \rightarrow H \otimes F'(u)$ (drawn as a white vertex) which is:
 - *Natural*. For any 2-morphism $f : u^{\vec{x}} \rightarrow u^{\vec{y}}$ in \mathcal{C} :

(119)

Here the empty horizontal blue and red rectangles represent manipulation of functorial boxes. For the purpose of drawing the diagram (119) we have supposed that \vec{x}, \vec{y} are both of the form $(1, -1, \dots, 1, -1)$; it should be clear how to generalise to other \vec{x}, \vec{y} or to u^0 (e.g. if $f : u^0 \rightarrow u^{\vec{y}}$, on the RHS of (119) the blue rectangle will be the counitor of F , the red will be the unitor of F' and there will be no white vertices). We also used the following definition:

(120)

It is immediate from Definition 2.30 and unitarity of α (38) that a UPT $\alpha : F \rightarrow F'$ restricts to a reduced UPT α_u . We now show that this correspondence is bijective: every reduced UPT induces a unique UPT.

Lemma 5.4. If $\tilde{\alpha}$ is a reduced UPT, then the morphism defined in (120) is unitary.

Proof. We show one of the two unitarity equations; the other is proved similarly.

$$(121)$$

Here the first equality is by definition (120), the second is by naturality (119) of the reduced UPT $\tilde{\alpha}$ for the morphism $\epsilon : u^{(-1,1)} \rightarrow u^0$, and the third is by unitarity of the monoidal functor F' . \square

Proposition 5.5. Let (G, u) be a compact matrix quantum group, let F, F' be fibre functors, and let $(\tilde{\alpha}, H) : F \rightarrow F'$ be a reduced UPT. There is a unique UPT $(\alpha, H) : F \rightarrow F'$ which restricts to $\tilde{\alpha}$ on α_u , whose components α_V are defined as follows for any representation V of G :

$$(122)$$

Here $\{b_k : u^{\vec{x}_k} \rightarrow V\}$ is any family of reduction morphisms.

Proof. First we show that α is well-defined, i.e. it does not depend on the choice of reduction morphisms. Let V be some representation of G and let $\{b_k : u^{\vec{x}_k} \rightarrow V\}$ and $\{c_l : u^{\vec{y}_l} \rightarrow V\}$ be two families of reduction morphisms. Then:

$$(123)$$

$$= \sum_l \quad (124)$$

Here the first equality is by $\sum_l c_l c_l^\dagger = \text{id}_V$, the second is by naturality of the reduced UPT $\tilde{\alpha}$, and the third is by $\sum_k b_k b_k^\dagger = \text{id}_V$.

We now show that α is indeed a UPT.

- *Naturality.* Let $\{b_k : u^{\vec{x}_k} \rightarrow V\}$ and $\{c_l : u^{\vec{y}_l} \rightarrow W\}$ be reduction morphisms for representations V, W of \mathcal{C} . We show (33) for any morphism $f : V \rightarrow W$:

$$= \sum_{k,l} \quad = \sum_k \quad (125)$$

Here the first equality is by $\sum_k b_k b_k^\dagger = \text{id}_V$, and the second is by naturality of the reduced UPT $\tilde{\alpha}$ and $\sum_l c_l c_l^\dagger = \text{id}_W$.

- *Monoidality.*

- Let V, W be some representations of G , and pick some reduction morphisms $\{b_k : u^{\vec{x}_k} \rightarrow V\}, \{c_l : u^{\vec{y}_l} \rightarrow W\}$. It is clear that $\{b_k \otimes c_l : u^{\vec{x}_k} \otimes u^{\vec{y}_l} \rightarrow V \otimes W\}$ are reduction morphisms

for $V \otimes W$. Now (34) is immediate by manipulation of functorial boxes:

$$\sum_{k,l} \begin{array}{c} \text{Diagram showing two stacked red layers with } b_k \text{ and } c_l \text{ boxes, and two stacked blue layers with } b_k \text{ and } c_l \text{ boxes. Green arrows indicate vertical connections between corresponding layers.} \\ \text{Diagram showing the same structure with a central white oval connecting the middle layers.} \end{array} = \sum_{k,l} \begin{array}{c} \text{Diagram showing the same structure with a central white oval connecting the middle layers.} \\ \text{Diagram showing the same structure with a central white oval connecting the middle layers.} \end{array} \quad (126)$$

- The equation (35) is precisely (122) for the object $\mathbb{1}$ where the reduction morphism is $\text{id}_{\mathbb{1}} : \mathbb{1} \rightarrow u^0$.

- *Unitarity.* We show one of the unitarity equations; the other is proved similarly. For any 1-morphism X of \mathcal{C} :

$$\sum_{k,l} \begin{array}{c} \text{Diagram showing three stacked red layers with } b_l, b_l, b_k, b_k, b_k \text{ boxes, and two stacked blue layers with } b_k, b_k \text{ boxes. Green arrows indicate vertical connections between corresponding layers.} \\ \text{Diagram showing the same structure with a central white oval connecting the middle layers.} \end{array} = \sum_{k,l} \begin{array}{c} \text{Diagram showing the same structure with a central white oval connecting the middle layers.} \\ \text{Diagram showing the same structure with a central white oval connecting the middle layers.} \end{array} = \sum_l \begin{array}{c} \text{Diagram showing the same structure with a central white oval connecting the middle layers.} \\ \text{Diagram showing the same structure with a central white oval connecting the middle layers.} \end{array} \quad (127)$$

$$= \sum_l \begin{array}{c} \text{Diagram showing a single red layer with } b_l \text{ box, and a single blue layer with } b_l \text{ box. Green arrows indicate vertical connections.} \\ \text{Diagram showing a single red layer with } b_l \text{ box, and a single blue layer with } b_l \text{ box. Green arrows indicate vertical connections.} \end{array} = \begin{array}{c} \text{Diagram showing a single red layer with } b_l \text{ box, and a single blue layer with } b_l \text{ box. Green arrows indicate vertical connections.} \\ \text{Diagram showing a single red layer with } b_l \text{ box, and a single blue layer with } b_l \text{ box. Green arrows indicate vertical connections.} \end{array} \quad (128)$$

Here the first equation is by naturality for $\tilde{\alpha}$, the second is by $\sum_k b_k b_k^\dagger = \text{id}_V$, the third is by unitarity of α_u and Lemma 5.4, and the fourth is by manipulation of functorial boxes and $\sum_l b_l b_l^\dagger = \text{id}_V$.

Finally, uniqueness of the induced morphism α follows from the fact that, for a UPT $\alpha : F \rightarrow F'$, the component α_V for any V is defined by α_U and α_{U^*} by (122). \square

We can also introduce a notion of modification for reduced UPTs.

Definition 5.6. Let (G, u) be a compact matrix quantum group, let F, F' be fibre functors, and let $(\tilde{\alpha}, H_\alpha), (\tilde{\beta}, H_\beta)$ be reduced UPTs (the first drawn with a green wire, the second with an orange wire). Then a *modification* $f : \tilde{\alpha} \rightarrow \tilde{\beta}$ is a linear map $f : H_\alpha \rightarrow H_\beta$ satisfying the following equations:

$$(129)$$

Proposition 5.7. Let $(\tilde{\alpha}, H_\alpha), (\tilde{\beta}, H_\beta) : F \rightarrow F'$ be reduced UPTs and let $(\alpha, H_\alpha), (\beta, H_\beta) : F \rightarrow F'$ be the unique induced UPTs. Then a modification $f : \tilde{\alpha} \rightarrow \tilde{\beta}$ is precisely a modification $f : \alpha \rightarrow \beta$.

Proof. It is clear that every modification $\alpha \rightarrow \beta$ is in particular a modification $\tilde{\alpha} \rightarrow \tilde{\beta}$. That a modification $f : \tilde{\alpha} \rightarrow \tilde{\beta}$ is also a modification $f : \alpha \rightarrow \beta$ is clear from the definition (122) of the induced UPTs; f pulls through all the vertices, thereby satisfying the equations (37). \square

The results of this section are summarised by the following theorem.

Theorem 5.8. Let (G, u) be a compact matrix quantum group and let F, F' be fibre functors. Then restriction of UPTs $\alpha \mapsto \alpha_u$ defines an isomorphism of categories between:

- UPTs $F \rightarrow F'$ and modifications.
- Reduced UPTs $F \rightarrow F'$ and modifications.

5.2 Quantum graphs and their isomorphisms

We have seen that for a compact matrix quantum group (G, u) , a UPT $\alpha : F_1 \rightarrow F_2$ between fibre functors $F_1, F_2 : \text{Rep}(G) \rightarrow \text{Hilb}$ is determined by a single unitary $(\alpha_U, H) : F_1(u) \otimes H \rightarrow H \otimes F_2(u)$ obeying the naturality condition (119) for the intertwiner spaces $\text{Hom}(u^{\otimes m}, u^{\otimes n})$.

We now recall the notions of quantum graph and finite-dimensional quantum graph isomorphism. For more information about these structures, their significance in quantum information theory, and how they generalise their classical counterparts, see [1, 6, 7, 14].

Definition 5.9 ([14, Def. 5.1]). A *quantum graph* $X = (A, \Gamma)$ is a pair of:

- A Frobenius monoid (A, m, u) in the category Hilb (Definition 4.1) satisfying the following *symmetry* equation:

$$(130)$$

- A self-adjoint linear map $\Gamma : A \rightarrow A$ satisfying the following equations:

$$\begin{array}{c} \text{Diagram showing } \Gamma \circ \Gamma = \Gamma \\ \text{Diagram showing } \Gamma \circ \Gamma^\dagger = \Gamma^\dagger \\ \text{Diagram showing } \Gamma^\dagger \circ \Gamma = \Gamma \end{array} = \boxed{\quad} \quad (131)$$

Remark 5.10. A symmetric Frobenius monoid in Hilb is precisely a f.d. C^* -algebra equipped with a faithful trace (the inner product is obtained as $\langle a|b \rangle = \text{Tr}(a^*b)$, and the trace is the counit of the Frobenius monoid). There are various ways to normalise this trace. One approach is to require that the Frobenius monoid be *special* — this approach was taken in the definition of quantum graphs given in [14]:

$$\begin{array}{c} \text{Diagram showing } \Gamma = \Gamma^\dagger \\ \text{Diagram showing } \Gamma = \delta^2 \end{array} = \boxed{\quad} \quad (132)$$

Another normalisation was chosen in [6], where the trace was required to be a δ -form for some $\delta > 0$:

$$\begin{array}{c} \text{Diagram showing } \Gamma = 1 \\ \text{Diagram showing } \Gamma = \delta^2 \end{array} = \boxed{\quad} \quad (133)$$

These normalisations produce slightly different definitions of quantum graph. Definition 5.9 includes them both by not stipulating any normalisation for the trace. In view of Lemma 5.15 we remark that, if either of these normalisations is chosen, it will be preserved under an accessible fibre functor.

Definition 5.11 ([14, Def. 5.11]). Let $X = (A, \Gamma)$ and $Y = (A', \Gamma')$ be quantum graphs. A *finite-dimensional quantum graph isomorphism* $(P, H) : X \rightarrow Y$ is a pair of a Hilbert space H and a unitary linear map $P : A \otimes H \rightarrow H \otimes A'$ satisfying the following equations, where the monoids A and A' are depicted as white and grey nodes respectively:

$$\begin{array}{c} \text{Diagram showing } P \circ P = P \\ \text{Diagram showing } P \circ P^\dagger = P^\dagger \\ \text{Diagram showing } P^\dagger \circ P = P \end{array} = \boxed{\quad} \quad (134)$$

$$\begin{array}{c} \text{Diagram showing } P^\dagger \circ P = P \\ \text{Diagram showing } P \circ P^\dagger = P^\dagger \end{array} = \boxed{\quad} \quad (135)$$

An *intertwiner* of quantum isomorphisms $(H, P) \rightarrow (H', P')$ is a linear map $f : H \rightarrow H'$ such that the following holds:

$$\begin{array}{c} \text{Diagram showing two configurations of nodes } P' \text{ and } f \text{ connected by arrows.} \\ = \\ \text{Diagram showing node } f \text{ connected to node } P \text{ via arrows.} \end{array} \quad (136)$$

We now recall the definition of the quantum automorphism group of a quantum graph.

Definition 5.12 ([6]). Let $X = (A, \Gamma)$ be a quantum graph with $\dim(A) = n$, and let $\{|i\rangle\}$ be a basis for A orthonormal with respect to the cup and cap (81). Then the *quantum automorphism group algebra* $O(G_X)$ is the universal unital $*$ -algebra generated by the coefficients of a unitary matrix $[u_{ij}]_{i,j=1}^n$ subject to the relations making the map

$$\rho : A \rightarrow A \otimes G_X \quad \rho(|i\rangle) = \sum_j |j\rangle \otimes u_{ij}$$

a unital $*$ -homomorphism satisfying $\rho \circ \Gamma = (\Gamma \otimes \text{id}_{O(G_X)}) \circ \rho$. Its Hopf- $*$ -algebra structure is defined in [6].

Remark 5.13. $O(G_X)$ can also be defined as the $*$ -algebra of matrix coefficients of corepresentations of the C^* -algebra obtained by Woronowicz's Tannaka-Krein construction [19] for a suitable concrete W^* -category (c.f. [2, Prop. 1.1]). In particular, we have the following facts:

1. G_X is a compact matrix quantum group with fundamental representation A .
2. The intertwiner spaces $\text{Hom}_{\text{Rep}(G_X)}(A^{\otimes m}, A^{\otimes n})$ are generated by three morphisms $m : A \otimes A \rightarrow A$, $u : \mathbb{C} \rightarrow A$ and $\Gamma : A \rightarrow A$, satisfying the equations of a Frobenius monoid and of a quantum graph, under composition, monoidal product, dagger and linear combination.
3. The fundamental representation A is self-dual in $\text{Rep}(G_X)$ with cup and cap (81).
4. The image of $((A, m, u), \Gamma)$ under the canonical fibre functor $F : \text{Rep}(G_X) \rightarrow \text{Hilb}$ is the quantum graph X .

Definition 5.14 ([6]). For any two quantum graphs $X = (A_X, \Gamma_X)$, $Y = (A_Y, \Gamma_Y)$ we define $O(G_Y, G_X)$ to be the universal $*$ -algebra generated by the coefficients of a unitary matrix $p = [p_{ij}]_{ij} \in O(G_Y, G_X) \otimes B(A_X, A_Y)$ with relations ensuring that

$$\rho_{Y,X} : A_X \rightarrow A_Y \otimes O(G_Y, G_X) \quad |i\rangle \mapsto \sum_j |j\rangle \otimes p_{ij}$$

is a unital $*$ -homomorphism satisfying $\rho \circ \Gamma_X = (\Gamma_Y \otimes \text{id}_{O(G_Y, G_X)}) \circ \rho$.

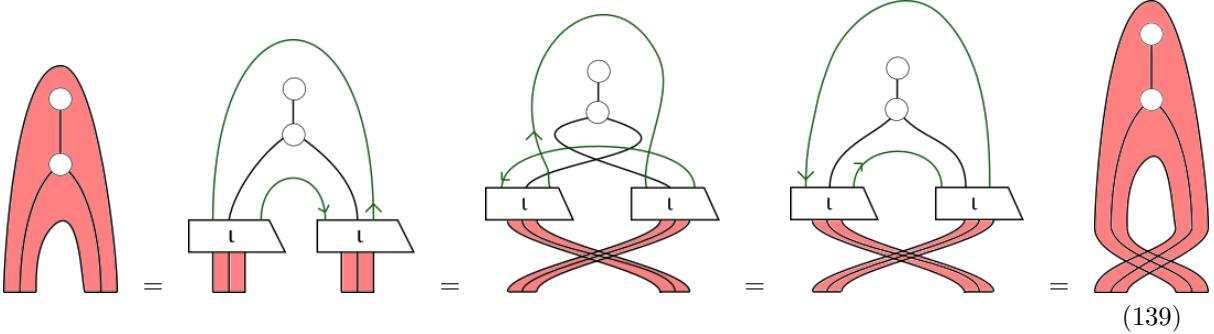
Lemma 5.15. Let G_X be the automorphism group of a quantum graph $F(X) = ((F(A), F(m), F(u)), F(\Gamma))$, where $F : \text{Rep}(G_X) \rightarrow \text{Hilb}$ is the canonical fibre functor, A is the generating object of $\text{Rep}(G_X)$ and m, u, Γ are the generating morphisms. Let $F' : \text{Rep}(G) \rightarrow \text{Hilb}$ be any other fibre functor accessible from F by a UPT. Then $F'(X) := ((F'(A), F'(m), F'(u)), F'(\Gamma))$ is a quantum graph f.d. quantum isomorphic to X :

$$(137)$$

$$F'(m) : F'(A) \otimes F'(A) \rightarrow F'(A) \quad F'(u) : \mathbb{C} \rightarrow F'(A) \quad F'(\Gamma) : F'(A) \rightarrow F'(A) \quad (138)$$

All quantum graphs f.d. quantum isomorphic to $F(X)$ are obtained in this way.

Proof. All axioms of a quantum graph except symmetry follow straightforwardly from unitarity of the functor F' (we need Lemma 2.18 for the Frobenius axiom). For the symmetry condition (130), we recall that any accessible fibre functor has the form given in Theorem (4.11). We can then show symmetry of $F'(X)$ as follows:



Here for the first equality we used the definition of the accessible fibre functor F' (Theorem 4.11); for the second equality pulled one leg over the other and undid twists in the green wires; for the third equality we used symmetry of $F(X)$; and for the fourth equality we again used the definition of F' .

It follows that $F'(X)$ is a quantum graph. To see that $F'(X)$ is f.d. quantum isomorphic to $F(X)$, consider any UPT $\alpha : F \rightarrow F'$. Then its restriction α_A is a quantum isomorphism: the equations (134) follow from monoidality and naturality of α , and the equation (135) follows from unitarity of α (38) and self-duality of A .

To see that all quantum graphs f.d. quantum isomorphic to $F(X)$ are thus obtained, we observe that a f.d. quantum isomorphism $P : Y \rightarrow X$ is an f.d. $*$ -representation of $O(G_X, G_Y)$, which is therefore nonzero. By [6, Thm. 4.5] $O(G_X, G_Y)$ is an $O(G_X)$ -Hopf-Galois object and the associated fibre functor $\text{Rep}(G_X) \rightarrow \text{Hilb}$ takes $(A, m, u, \Gamma) \mapsto Y$. This fibre functor F' is accessible from F precisely because $O(G_X, G_Y)$ possesses a finite dimensional $*$ -representation (Theorem 3.13). \square

Lemma 5.16. *Let $X = (A, \Gamma)$ be a quantum graph, let G_X be its quantum automorphism group and let $F_1, F_2 : \text{Rep}(G_X) \rightarrow \text{Hilb}$ be two fibre functors accessible from the canonical fibre functor. Then there is an isomorphism of categories between:*

- UPTs $F_1 \rightarrow F_2$, and modifications.
- Quantum isomorphisms $F_1(X) \rightarrow F_2(X)$, and intertwiners.

Proof. We observe that a quantum isomorphism $F_1(X) \rightarrow F_2(X)$ is precisely a reduced UPT $F_1 \rightarrow F_2$. Indeed, A is the generating object for $\text{Rep}(G_X)$ and the quantum isomorphism is a unitary linear map of the right type. We must therefore show the naturality equations (119). But these are given on generators precisely by (134), which is sufficient. The equation (135) follows from self-duality of A .

We also observe that an intertwiner of quantum graph isomorphisms is precisely a modification of reduced UPTs. This follows from self-duality of A , which reduces the equations (129) to (136).

The result then follows immediately from Theorem 5.8. \square

We now state the main result of this section.

Definition 5.17. Let X be a quantum graph. The 2-category QGraph_X is defined as follows:

- *Objects:* Quantum graphs f.d. quantum isomorphic to X .
- *1-morphisms:* Finite-dimensional quantum isomorphisms.
- *2-morphisms:* Intertwiners.

Definition 5.18. Let X be a quantum graph. The 2-category $\text{Fun}(\text{Rep}(G_X, \text{Hilb}))_F$ is defined as follows:

- *Objects*: Fibre functors accessible from the canonical fibre functor $F : \text{Rep}(G_X) \rightarrow \text{Hilb}$.
- *1-morphisms*: UPTs.
- *2-morphisms*: Modifications.

Theorem 5.19. Let X be a quantum graph. Then there is an equivalence of 2-categories $\text{Fun}(\text{Rep}(G_X), \text{Hilb})_F \cong \text{QGraph}_X$. Moreover, this equivalence is an isomorphism on Hom-categories.

Proof. We define a strict pseudofunctor $\text{Fun}(\text{Rep}(G_X, \text{Hilb})_F \rightarrow \text{QGraph}_X$ witnessing the equivalence as follows.

- *On objects*: An accessible fibre functor $F' : \text{Rep}(G) \rightarrow \text{Hilb}$ is taken to the quantum graph (137).
- *On 1-morphisms*: A UPT $\alpha : F' \rightarrow F''$ is taken to its component α_A .
- *On 2-morphisms*: A modification $\alpha \rightarrow \beta$ is taken to an intertwiner $\alpha_A \rightarrow \beta_A$.

We first show that this is a well-defined strict pseudofunctor. That the quantum graph (137) is f.d. quantum isomorphic to X was shown in Lemma 5.15, so the pseudofunctor is well-defined on objects. Well-definition on 1-morphisms and 2-morphisms follows from Lemma 5.16. Compositionality is clear by comparing the composition of quantum graph isomorphisms and intertwiners [14, Def. 3.18] to that of UPTs and modifications. Essential surjectivity on objects follows immediately from the last statement of Lemma 5.15. That the equivalence is in fact an isomorphism on Hom-categories follows from Lemma 5.16. \square

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