Category Theory

—

a silly walk

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Abstract: Being a syntax-oriented formalist I always had problems with the usual informal and graphical presentation of categories. This is my trial to overcome these problems.

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Contents

1 Basic Notions and Notations

'$\mathbb{N}$' denotes the set of and '$\prec$' the ordering on natural numbers. We define $\mathbb{N}_+ := \{\, n \in \mathbb{N} \mid 0 \neq n \,\}$.

We use '$\uplus$' for the union of disjoint classes, 'id' for the identity function, and '$\mathcal{V}$' for the class of all sets. For a class R we define *domain*, *range*, and *restriction to* and *image* and *reverse-image of a class* A by

$$\begin{array}{lcll} \mathrm{dom}(R) & := & \{\, a \mid \exists b\colon\ (a,b) \in R \,\} & ; \\ \mathrm{ran}(R) & := & \{\, b \mid \exists a\colon\ (a,b) \in R \,\} & ; \\ {}_A|R & := & \{\, (a,b) \in R \mid\ a \in A \,\} & ; \\ \langle A \rangle R & := & \{\, b \mid \exists a \in A\colon\ (a,b) \in R \,\} & ; \\ R\langle B \rangle & := & \{\, a \mid \exists b \in B\colon\ (a,b) \in R \,\} & . \end{array}$$

For a class A we define its *power class* and the *class of its non-empty finite subsets* by

$$\begin{array}{lcll} \mathfrak{P}(A) & := & \{\ B \mid\ \ B \subseteq A\ \ \} & ; \\ \mathfrak{P}_{\mathbb{N}_+}(A) & := & \{\ B \mid\ \ B \subseteq A\ \wedge\ |B| \in \mathbb{N}_+ \,\} & . \end{array}$$

For a (possibly) partial function $B\colon\ \mathcal{V} \rightsquigarrow \mathfrak{P}(\mathcal{V})$ and class[1] A with $A \subseteq \mathrm{dom}(B)$ we define their *product* and *co-product* (or *free sum*) by

$$\prod_{a \in A} B_a \ := \ \{\ f\colon\ A \to \mathcal{V} \mid\ \ \forall a \in A\colon\ f(a) \in B_a\ \ \} \quad ;$$

$$\coprod_{a \in A} B_a \ := \ \{\ (a,b) \in A \times \mathcal{V} \mid\ \ b \in B_a\ \ \} \quad .$$

[1]Note that the product is meaningless unless A is a set: If A is a proper class, then any function f with $\mathrm{dom}(f) = A$ is a proper class, so that the product will be the empty set.

2 Categories and Functors

The notion of an *(abstract) category* can reasonably be defined in several different ways. The following two are particularly useful. As prerequisites, define the *morphism*, *domain*, and *codomain* functions $\mathrm{mor}, \mathrm{Dom}, \mathrm{Cod}\colon\ \mathcal{V}\times\mathcal{V}\times\mathcal{V} \to \mathcal{V}$ by $((h,s,t) \in \mathcal{V}\times\mathcal{V}\times\mathcal{V})$ $\mathrm{mor}(h,s,t) := h$, $\mathrm{Dom}(h,s,t) := s$, and $\mathrm{Cod}(h,s,t) := t$.

Definition 2.1 (Category)

A *category* C consists of (left indices deliberately omitted):

- a class ${}_{\mathrm{C}}\mathrm{Obj}$ of *objects*,
- a class ${}_{\mathrm{C}}\mathrm{Arr}$ of *arrows* with ${}_{\mathrm{C}}\mathrm{Arr} \subseteq \mathcal{V}\times{}_{\mathrm{C}}\mathrm{Obj}\times{}_{\mathrm{C}}\mathrm{Obj}$,
- for each $s \in \mathrm{Obj}$ an *identity* ${}_{\mathrm{C}}\mathrm{Id}_s$ with $(\mathrm{Id}_s, s, s) \in {}_{\mathrm{C}}\mathrm{Arr}$,
- a binary partial *composition* function ${}_{\mathrm{C}}\diamond\ :\ ({}_{\mathrm{C}}\mathrm{Arr}\times{}_{\mathrm{C}}\mathrm{Arr}) \rightsquigarrow {}_{\mathrm{C}}\mathrm{Arr}$ with infix notation.

Now the following must hold:

1. $\mathrm{dom}(\,\diamond\,) \;=\; \{\ (a,b) \in \mathrm{Arr}\times\mathrm{Arr} \mid \mathrm{Cod}(a) = \mathrm{Dom}(b)\ \}$
2. $\forall a,b \in \mathrm{Arr}\colon \left(\mathrm{Cod}(a) = \mathrm{Dom}(b) \ \Rightarrow\ \left(\begin{array}{rl} & (a \diamond b) \in \mathrm{Arr} \\ \wedge & \mathrm{Dom}(a \diamond b) = \mathrm{Dom}(a) \\ \wedge & \mathrm{Cod}(a \diamond b) = \mathrm{Cod}(b) \end{array} \right) \right)$
3. $\forall s \in \mathrm{Obj}\colon\ \forall a \in \mathrm{Arr}\colon \left(\begin{array}{rl} & \left(\mathrm{Dom}(a) = s \ \Rightarrow (\mathrm{Id}_s, s, s) \diamond a = a \right) \\ \wedge & \left(\mathrm{Cod}(a) = s \ \Rightarrow\ a \diamond (\mathrm{Id}_s, s, s) = a \right) \end{array} \right)$
4. $\forall a,b,c \in \mathrm{Arr}\colon \left(\left(\begin{array}{rl} & \mathrm{Cod}(a) = \mathrm{Dom}(b) \\ \wedge & \mathrm{Cod}(b) = \mathrm{Dom}(c) \end{array} \right) \Rightarrow (a \diamond b) \diamond c = a \diamond (b \diamond c) \right)$

Definition 2.2 (Category *alternate***)**

A *category* C consists of (left indices deliberately omitted):

- a class ${}_{\mathrm{C}}\mathrm{Obj}$ of *objects*,
- for each $(s,t) \in \mathrm{Obj}\times\mathrm{Obj}$ a class ${}_{\mathrm{C}}\mathrm{Mor}_{s,t}$ of *morphisms*,
- for each $s \in \mathrm{Obj}$ an *identity* ${}_{\mathrm{C}}\mathrm{Id}_s$ with ${}_{\mathrm{C}}\mathrm{Id}_s \in {}_{\mathrm{C}}\mathrm{Mor}_{s,s}$,
- a binary partial *composition* function ${}_{\mathrm{C}}\diamond\ :\ (\mathcal{V}\times\mathcal{V}) \rightsquigarrow \mathcal{V}$ with infix notation.

Now the following must hold:

1. $\mathrm{dom}(\diamond) \;=\; \bigcup_{s,t,u \in \mathrm{Obj}} \mathrm{Mor}_{s,t} \times \mathrm{Mor}_{t,u}$
2. $\forall s,t,u \in \mathrm{Obj}\colon\ \forall f \in \mathrm{Mor}_{s,t}\colon\ \forall g \in \mathrm{Mor}_{t,u}\colon\ (f \diamond g) \in \mathrm{Mor}_{s,u}$
3. $\forall s \in \mathrm{Obj}\colon \left(\begin{array}{rl} & \forall t \in \mathrm{Obj}\colon\ \forall f \in \mathrm{Mor}_{s,t}\colon\ \mathrm{Id}_s \diamond f = f \\ \wedge & \forall r \in \mathrm{Obj}\colon\ \forall g \in \mathrm{Mor}_{r,s}\colon\ g \diamond \mathrm{Id}_s = g \end{array} \right)$
4. $\forall s,t,u,v \in \mathrm{Obj}\colon\ \forall f \in \mathrm{Mor}_{s,t}\colon\ \forall g \in \mathrm{Mor}_{t,u}\colon\ \forall h \in \mathrm{Mor}_{u,v}\colon\ (f \diamond g) \diamond h = f \diamond (g \diamond h)$

First consider C to be given according to Def. 2.1: We can define for $s,t\in\mathrm{Obj}$:

$$\mathrm{Mor}_{s,t} \;:=\; \{\; f \mid (f,s,t)\in{}_{\mathrm{C}}\mathrm{Arr} \;\}.$$

Now assume[2] that the composition function depends only on the first component of the arrows but not on their domain and codomain, formally:

$$\forall s,t,u,s',t',u'\in\mathrm{Obj}\colon\ \forall f,g,h\colon \left(\begin{array}{l}\left(\begin{array}{ll} & (f,s,t),(f,s',t'),(g,t,u),(g,t',u')\in\mathrm{Arr}\\ \wedge & (f,s,t)\diamond(g,t,u)=(h,s,u)\end{array}\right)\\ \Rightarrow\ (f,s',t')\diamond(g,t',u')=(h,s',u')\end{array}\right).$$

Note that this is always the case if

$$\forall s,t,u,v\in\mathrm{Obj}\colon \left(\begin{array}{ll} & \mathrm{Mor}_{s,t}\cap\mathrm{Mor}_{u,v}=\emptyset\\ \vee & (s,t)=(u,v)\end{array}\right). \tag{:§}$$

Under this assumption, the composition function on morphisms f,g can be defined just by adding any appropriate s,t,u such that $(f,s,t),(g,t,u)\in\mathrm{Arr}$ and taking the first component of the triple resulting from their composition.

Now the enumerated items of Def. 2.2 follow from the respective items of Def. 2.1.

Now, however, consider C to be given according to Def. 2.2: Then we can define

$$\mathrm{Arr} := \{\; (f,s,t) \mid\; f\in\mathrm{Mor}_{s,t} \;\wedge\; s,t\in\mathrm{Obj} \;\}.$$

Again, the enumerated items of Def. 2.1 follow from the respective items of Def. 2.2. Moreover, if (§) holds, we can treat Dom and Cod as if they were directly defined on

$$\mathrm{Mor} \;:=\; \textstyle\bigcup_{s,t\in\mathrm{Obj}}\ \mathrm{Mor}_{s,t}$$

via[3]

$$\mathrm{Dom}(f) := \bigcap_{\exists t\colon f\in\mathrm{Mor}_{s,t}} s\ , \qquad\qquad \text{and} \qquad\qquad \mathrm{Cod}(f) := \bigcap_{\exists s\colon f\in\mathrm{Mor}_{s,t}} t\ .$$

All in all, if we have to define some concrete category, we can always choose the more convenient way and check the category property without translation. Moreover, by use of the translation given above, we can always use '$\mathrm{Mor}_{s,t}$', 'Arr', and 'Mor', no matter which way the category has been defined. In case of (§), we can even always use Dom and Cod and write the composition function on morphisms instead of arrows.

We will abbreviate "$f\diamond g$" by "fg" and "$f(gh)$" (or "$(fg)h$") by "fgh". Moreover, when "fg" appears in any atomic formula A, this formula is to be replaced with the formula $(A\wedge B)$, where B is a formula saying that the composition is defined, i.e. B is either $\mathrm{Cod}(f)=\mathrm{Dom}(g)$ or $\exists s,t,u\in\mathrm{Obj}\colon\ (f\in\mathrm{Mor}_{s,t}\ \wedge\ g\in\mathrm{Mor}_{t,u})$.

Finally we write

$$f{::}s{\to}t$$

and intend it to mean any of either $f\in\mathrm{Mor}_{s,t}$ or $(f,s,t)\in\mathrm{Arr}$.

[2]Even if this is not the case we could get a category in the sense of Def. 2.2 by setting $\mathrm{Mor}_{s,t} := \{\; (f,s,t) \mid (f,s,t)\in{}_{\mathrm{C}}\mathrm{Arr} \;\}$, thereby (together with the transformation in the other direction) establishing the equivalence of Def. 2.1 and Def. 2.2. We use the less general transformation because it is more useful in practice.

[3]Note that in a set theory with ur-elements, we need Hilbert's ε-operator to define $\mathrm{Dom}(f) \;:=\; \varepsilon s\colon \exists t\colon f\in\mathrm{Mor}_{s,t}$ etc.. Anyway, the simple meaning of this is: "Let $\mathrm{Dom}(f)$ be such that $f\in\mathrm{Mor}_{\mathrm{Dom}(f),t}$."

Corollary 2.3
Let C *be a category.*
Now Id_s *is unique in* $\mathrm{Mor}_{s,s}$ *satisfying item (3) in Def. 2.1 or Def. 2.2. More precisely:*
For $s \in \mathrm{Obj}$ *and* $i \in \mathrm{Mor}_{s,s}$ *with* $\forall f \in \mathrm{Mor}_{s,s}\colon\ (i,s,s)(f,s,s) = (f,s,s)$ *or* $\forall g \in \mathrm{Mor}_{s,s}\colon$ $(g,s,s)(i,s,s) = (g,s,s)$ *we have* $i = \mathrm{Id}_s$.

Definition 2.4 (Functor) Let C, D be categories.

A (*co-variant*) *functor* F *from* C *to* D (written "F: C $\twoheadrightarrow$ D") is a triple

$$\mathrm{F} \ = \ ((\mathrm{F}_{\mathrm{Obj}}, \mathrm{F}_{\mathrm{Arr}}), \mathrm{C}, \mathrm{D})$$

or else

$$\mathrm{F} \ = \ ((\mathrm{F}_{\mathrm{Obj}}, \mathrm{F}_{\mathrm{Mor}}), \mathrm{C}, \mathrm{D})$$

with:

- An *object function* $\mathrm{F}_{\mathrm{Obj}}\colon {}_{\mathrm{C}}\mathrm{Obj} \to {}_{\mathrm{D}}\mathrm{Obj}$. We abbreviate "$\mathrm{F}_{\mathrm{Obj}}(s)$" by "$s\mathrm{F}$".
- An *arrow function* $\mathrm{F}_{\mathrm{Arr}}\colon {}_{\mathrm{C}}\mathrm{Arr} \to {}_{\mathrm{D}}\mathrm{Mor}$ or else a *morphism function* $\mathrm{F}_{\mathrm{Mor}}\colon$ ${}_{\mathrm{C}}\mathrm{Mor} \to {}_{\mathrm{D}}\mathrm{Mor}$. We abbreviate "$\mathrm{F}_{\mathrm{Arr}}(a)$" by "${}_a\mathrm{F}$" and "$\mathrm{F}_{\mathrm{Mor}}(f)$" by "${}_f\mathrm{F}$". Note that any morphism function $\mathrm{F}_{\mathrm{Mor}}$ defines the corresponding arrow function $\mathrm{F}_{\mathrm{Arr}}$ via $\mathrm{F}_{\mathrm{Arr}}(f,s,t) := \mathrm{F}_{\mathrm{Mor}}(f)$, but the converse definition is not always possible.

Now for the array function (morphism function analogously) the following must hold:

1. $\forall s,t \in {}_{\mathrm{C}}\mathrm{Obj}\colon \forall f \in {}_{\mathrm{C}}\mathrm{Mor}_{s,t}\colon \ ({}_{(f,s,t)}\mathrm{F}, s\mathrm{F}, t\mathrm{F}) \ \in \ {}_{\mathrm{D}}\mathrm{Arr}$.
2. $\forall s \in {}_{\mathrm{C}}\mathrm{Obj}\colon \ {}_{({}_{\mathrm{C}}\mathrm{Id}_s,s,s)}\mathrm{F} \ = \ {}_{\mathrm{D}}\mathrm{Id}_{s\mathrm{F}}$.
3. $\forall s,t,u \in {}_{\mathrm{C}}\mathrm{Obj}\colon \forall f \in {}_{\mathrm{C}}\mathrm{Mor}_{s,t}\colon \forall g \in {}_{\mathrm{C}}\mathrm{Mor}_{t,u}\colon$
$$\big({}_{((f,s,t)\ {}_{\mathrm{C}}\diamond\ (g,t,u))}\mathrm{F}, s\mathrm{F}, u\mathrm{F}\big) = \big({}_{(f,s,t)}\mathrm{F}, s\mathrm{F}, t\mathrm{F}\big)\ {}_{\mathrm{D}}\diamond\ \big({}_{(g,t,u)}\mathrm{F}, t\mathrm{F}, u\mathrm{F}\big) \qquad .$$

When C and D are given according to Def. 2.2 this reads:

1. $\forall s,t \in {}_{\mathrm{C}}\mathrm{Obj}\colon \ \forall f \in {}_{\mathrm{C}}\mathrm{Mor}_{s,t}\colon \ {}_f\mathrm{F} \ \in \ {}_{\mathrm{D}}\mathrm{Mor}_{s\mathrm{F},t\mathrm{F}}$.
2. $\forall s \in {}_{\mathrm{C}}\mathrm{Obj}\colon \ {}_{{}_{\mathrm{C}}\mathrm{Id}_s}\mathrm{F} \ = \ {}_{\mathrm{D}}\mathrm{Id}_{s\mathrm{F}}$.
3. $\forall s,t,u \in {}_{\mathrm{C}}\mathrm{Obj}\colon \forall f \in {}_{\mathrm{C}}\mathrm{Mor}_{s,t}\colon \forall g \in {}_{\mathrm{C}}\mathrm{Mor}_{t,u}\colon \ {}_{(f\ {}_{\mathrm{C}}\diamond\ g)}\mathrm{F} = {}_f\mathrm{F}\ {}_{\mathrm{D}}\diamond\ {}_g\mathrm{F}$.

F is *full* if $\forall s,t \in {}_{\mathrm{C}}\mathrm{Obj}\colon \forall f' \in {}_{\mathrm{D}}\mathrm{Mor}_{s\mathrm{F},t\mathrm{F}}\colon \exists f \in {}_{\mathrm{C}}\mathrm{Mor}_{s,t}\colon \ {}_{(f,s,t)}\mathrm{F} = f'$.
F is *faithful* if $\forall s,t \in {}_{\mathrm{C}}\mathrm{Obj}\colon \forall f,g \in {}_{\mathrm{C}}\mathrm{Mor}_{s,t}\colon \Big({}_{(f,s,t)}\mathrm{F} = {}_{(g,s,t)}\mathrm{F} \ \Rightarrow \ f = g \Big)$.

The *identity functor* $\mathbb{I}_{\mathrm{C}}$ *for* C is the functor $\mathbb{I}_{\mathrm{C}} \ := \ (({}_{{}_{\mathrm{C}}\mathrm{Obj}}|\mathrm{id}, {}_{{}_{\mathrm{C}}\mathrm{Mor}}|\mathrm{id}), \mathrm{C}, \mathrm{C})$.

If $(({}_{{}_{\mathrm{C}}\mathrm{Obj}}|\mathrm{id}, {}_{{}_{\mathrm{C}}\mathrm{Mor}}|\mathrm{id}), \mathrm{C}, \mathrm{D})$ is a functor, then it is called the *inclusion functor from* C *to D* and C is called a *sub-category of* D. If this functor is full, C is a *full sub-category* of D.

A *contra-variant functor* (*or contra-functor*) F *from* C *to* D (written "F: C $\rightleftharpoons$ D") differs from a co-variant functor only in the following (for $\mathrm{F}_{\mathrm{Arr}}$ analogously):

1. $\forall s,t \in {}_{\mathrm{C}}\mathrm{Obj}\colon \ \forall f \in {}_{\mathrm{C}}\mathrm{Mor}_{s,t}\colon \ {}_f\mathrm{F} \ \in \ {}_{\mathrm{D}}\mathrm{Mor}_{t\mathrm{F},s\mathrm{F}}$.

3. $\forall s,t,u \in {}_{\mathrm{C}}\mathrm{Obj}\colon \forall f \in {}_{\mathrm{C}}\mathrm{Mor}_{s,t}\colon \forall g \in {}_{\mathrm{C}}\mathrm{Mor}_{t,u}\colon \ {}_{(f\ {}_{\mathrm{C}}\diamond\ g)}\mathrm{F} = {}_g\mathrm{F}\ {}_{\mathrm{D}}\diamond\ {}_f\mathrm{F}$

Note that, due to our special notation for the application of $\mathrm{F}_{\mathrm{Arr}}$ and $\mathrm{F}_{\mathrm{Mor}}$, we can get the references to C and D, resp., from the level and thus may omit them. E.g., we will write "${}_{\mathrm{Id}_s}\mathrm{F} = \mathrm{Id}_{s\mathrm{F}}$" for "${}_{{}_{\mathrm{C}}\mathrm{Id}_s}\mathrm{F} = {}_{\mathrm{D}}\mathrm{Id}_{s\mathrm{F}}$" as well as "${}_{(f \diamond g)}\mathrm{F} = {}_f\mathrm{F} \diamond {}_g\mathrm{F}$" (or even "${}_{fg}\mathrm{F} = {}_f\mathrm{F}_g\mathrm{F}$") for "${}_{(f\ {}_{\mathrm{C}}\!\diamond\ g)}\mathrm{F} = {}_f\mathrm{F}\ {}_{\mathrm{D}}\!\diamond\ {}_g\mathrm{F}$".

Furthermore, note that $\forall s, t \in {}_{\mathrm{C}}\mathrm{Obj}\colon\ \forall f' \in {}_{\mathrm{D}}\mathrm{Mor}_{s\mathrm{F},t\mathrm{F}}\colon\ \exists a \in {}_{\mathrm{C}}\mathrm{Arr}\colon\ {}_a\mathrm{F} = f'$ does in general not imply that F is full, and that a faithful F does not have to satisfy $\forall a, b \in {}_{\mathrm{C}}\mathrm{Arr}\colon$ $({}_a\mathrm{F} = {}_b\mathrm{F} \Rightarrow a = b)$. This is one of the unusual cases in category theory where the domain and codomain information becomes crucial.

Definition 2.5 (Small, Large, and Superlarge)
A category C is *small* if ${}_{\mathrm{C}}\mathrm{Arr}$ is a set.[4] It is *large* if it is not small.
A *superlarge category* C is like a category, but where ${}_{\mathrm{C}}\mathrm{Obj}$ and ${}_{\mathrm{C}}\mathrm{Arr}$ are collections of classes.

Corollary 2.6
Let C *and* D *be categories.*
If C *and* D *are a small, then* ${}_{\mathrm{C}}\mathrm{Obj}$, $({}_{\mathrm{C}}\mathrm{Id}_s)_{s \in {}_{\mathrm{C}}\mathrm{Obj}}$, ${}_{\mathrm{C}}\diamond$, $({}_{\mathrm{C}}\mathrm{Mor}_{s,t})_{(s,t) \in {}_{\mathrm{C}}\mathrm{Obj} \times {}_{\mathrm{C}}\mathrm{Obj}}$, ${}_{\mathrm{C}}\mathrm{Arr}$, C, *and all functors* F: C $\twoheadrightarrow$ D *are sets.*
If C *is large, then* ${}_{\mathrm{C}}\diamond$, ${}_{\mathrm{C}}\mathrm{Arr}$, C, *and all functors* F: C $\twoheadrightarrow$ D *are proper classes.*

Example 2.7 (Categories of Sets)
The *relation category of sets* is the large category REL given as follows:

$$
\begin{aligned}
&{}_{\mathrm{REL}}\mathrm{Obj} := \mathcal{V} \\
&{}_{\mathrm{REL}}\mathrm{Mor}_{A,B} \ := \ \{\ R \mid R \subseteq A \times B\ \} \ \text{ for } A, B \in \mathcal{V}. \\
&R\ {}_{\mathrm{REL}}\!\diamond\ S := R \circ S \ \text{ for } R, S \in {}_{\mathrm{REL}}\mathrm{Mor}. \\
&{}_{\mathrm{REL}}\mathrm{Id}_A \ := \ {}_A|\mathrm{id} \ \text{ for } A \in \mathcal{V}.
\end{aligned}
$$

The *partial function category of sets* is the large sub-category FUN of the category REL given by ${}_{\mathrm{FUN}}\mathrm{Obj} := \mathcal{V}$ and for $A, B \in \mathcal{V}$:

$$ {}_{\mathrm{FUN}}\mathrm{Mor}_{A,B} \ := \ A \rightsquigarrow B \ := \ \{\ f \mid f\colon\ A \rightsquigarrow B\ \} $$

The (*total function*) *category of sets* is the large sub-category SET of the category FUN given by ${}_{\mathrm{SET}}\mathrm{Obj} := \mathcal{V}$ and for $A, B \in \mathcal{V}$:

$$ {}_{\mathrm{SET}}\mathrm{Mor}_{A,B} \ := \ A \to B \ := \ \{\ f \mid f\colon\ A \to B\ \} $$

[4]This is, of course, equivalent to ${}_{\mathrm{C}}\mathrm{Obj}$ and ${}_{\mathrm{C}}\mathrm{Mor}_{s,t}$ being sets, for any $s, t \in \mathrm{Obj}$.

Example 2.8 (Categories of Classes)

The *relation category of classes* is the superlarge category RELCLASS given as follows:

${}_{\mathrm{RELCLASS}}\mathrm{Obj}$ is the collection of all classes $A \subseteq \mathcal{V}$.

${}_{\mathrm{RELCLASS}}\mathrm{Mor}_{A,B}$ is the collection all all classes $R \subseteq A \times B$ for $A, B \subseteq \mathcal{V}$.

$R \;{}_{\mathrm{RELCLASS}}\diamond\; S \;:=\; R \circ S$ for R, S in ${}_{\mathrm{RELCLASS}}\mathrm{Mor}$.

${}_{\mathrm{RELCLASS}}\mathrm{Id}_A \;:=\; {}_A|\mathrm{id}$ for $A \subseteq \mathcal{V}$.

The (*total function*) *category of classes* is the superlarge sub-category CLASS of the superlarge category RELCLASS given by ${}_{\mathrm{CLASS}}\mathrm{Obj} := {}_{\mathrm{RELCLASS}}\mathrm{Obj}$ and for A, B in ${}_{\mathrm{RELCLASS}}\mathrm{Obj}$:

${}_{\mathrm{CLASS}}\mathrm{Mor}_{A,B}$ is the collection of all $f\colon\ A \to B$.

Example 2.9 (Category of $\mathbb{S}$-Sorted Sets)

Let $\mathbb{S}$ be some set of *sorts*. The *function category of* $\mathbb{S}$*-sorted sets* is the category SORT($\mathbb{S}$) given as follows:

${}_{\mathrm{SORT}(\mathbb{S})}\mathrm{Obj} := \mathbb{S} \to \mathfrak{P}(\mathcal{V})$ is the class of all $\mathbb{S}$-sorted families of sets.

${}_{\mathrm{SORT}(\mathbb{S})}\mathrm{Mor}_{A,B} := \{\ f\colon\ \mathbb{S} \to (\mathcal{V} \rightsquigarrow \mathcal{V}) \mid \forall s \in \mathbb{S}\colon\ f_s\colon\ A_s \to B_s\ \}$
is the set of $\mathbb{S}$-sorted functions from A to B for $A, B \in {}_{\mathrm{SORT}(\mathbb{S})}\mathrm{Obj}$.

$f \;{}_{\mathrm{SORT}(\mathbb{S})}\diamond\; g \;:=\; (f_s \circ g_s)_{s \in \mathbb{S}}$ for $f, g \in {}_{\mathrm{SORT}(\mathbb{S})}\mathrm{Mor}$.

${}_{\mathrm{SORT}(\mathbb{S})}\mathrm{Id}_A := ({}_{A_s}|\mathrm{id})_{s \in \mathbb{S}}$ for $A \in {}_{\mathrm{SORT}(\mathbb{S})}\mathrm{Obj}$.

Note that SORT($\mathbb{S}$) is large iff $\mathbb{S} \neq \emptyset$. Even in this case — although ${}_{\mathrm{SORT}(\mathbb{S})}\mathrm{Obj}$ and ${}_{\mathrm{SORT}(\mathbb{S})}\mathrm{Arr}$ are proper classes — ${}_{\mathrm{SORT}(\mathbb{S})}\mathrm{Mor}_{A,B}$ is always a set because it is contained in the set $\mathfrak{P}(\bigcup_{s \in \mathbb{S}} A_s \times \bigcup_{s \in \mathbb{S}} B_s)$.

Example 2.10 (Functor Categories of Categories)

The *functor category of of small categories* is the large category SMALL given as follows:

${}_{\mathrm{SMALL}}\mathrm{Obj} := \{\ \mathrm{C} \mid \mathrm{C} \text{ is a category}\ \}$ is the proper class of all small categories.

${}_{\mathrm{SMALL}}\mathrm{Mor}_{\mathrm{C},\mathrm{D}} := \{\ \mathrm{F} \mid \mathrm{F}\colon\ \mathrm{C} \twoheadrightarrow \mathrm{D}\ \}$ is the set of all functors from
the small category C to the small category D.

$(\mathrm{F}_{\mathrm{Obj}}, \mathrm{F}_{\mathrm{Arr}}) \;{}_{\mathrm{SMALL}}\diamond\; (\mathrm{G}_{\mathrm{Obj}}, \mathrm{G}_{\mathrm{Arr}}) \;:=\; (\mathrm{F}_{\mathrm{Obj}} \circ \mathrm{G}_{\mathrm{Obj}}, \mathrm{H}_{\mathrm{Arr}})$ where $\mathrm{H}_{\mathrm{Arr}}$ is given by
${}_{(f,s,t)}\mathrm{H} \;:=\; {}_{({}_{(f,s,t)}\mathrm{F}, s\mathrm{F}, t\mathrm{F})}\mathrm{G}$ for F: $\mathrm{B} \twoheadrightarrow \mathrm{C}$ and F: $\mathrm{C} \twoheadrightarrow \mathrm{D}$.

${}_{\mathrm{SMALL}}\mathrm{Id}_{\mathrm{C}} := \mathbb{I}_{\mathrm{C}}$.

The *functor category of of all categories* is the superlarge category ALL given as follows:

${}_{\mathrm{ALL}}\mathrm{Obj}$ is the collection of all categories
— excluding the superlarge categories.

${}_{\mathrm{ALL}}\mathrm{Mor}_{\mathrm{C},\mathrm{D}}$ is the collection of all functors F: $\mathrm{C} \twoheadrightarrow \mathrm{D}$ from
the category C to the category D.

The rest is analogous to SMALL.

3 Duality

The *duality principle* is a very important tool for doing universal category theory because it halves the proof work. The principle is based on the cognition that things do not change when all arrows and their composition are reversed.

Definition 3.1 (Dual Category and Functor)

Let C be a category. The *dual*[5] *category of* C is the category Dual(C) given by ${}_{\mathrm{Dual(C)}}\mathrm{Obj} := {}_{\mathrm{C}}\mathrm{Obj}$; ${}_{\mathrm{Dual(C)}}\mathrm{Arr} := \{\ (f,t,s) \mid (f,s,t) \in {}_{\mathrm{C}}\mathrm{Arr}\ \}$; ${}_{\mathrm{Dual(C)}}\mathrm{Id}_s := {}_{\mathrm{C}}\mathrm{Id}_s$; ${}_{\mathrm{Dual(C)}}\mathrm{Mor}_{s,t} := {}_{\mathrm{C}}\mathrm{Mor}_{t,s}$; $(g,u,t)\ {}_{\mathrm{Dual(C)}}\!\diamond\ (f,t,s) \ := \ (f,s,t)\ {}_{\mathrm{C}}\!\diamond\ (g,t,u)$.

Let F: C $\twoheadrightarrow$ D. The *dual functor of* F

$$\mathrm{Dual(F)}\colon\ \mathrm{Dual(C)} \twoheadrightarrow \mathrm{Dual(D)}$$

is given by $\mathrm{Dual(F)}_{\mathrm{Obj}} := \mathrm{F}_{\mathrm{Obj}}$ and ${}_{(f,t,s)}\mathrm{Dual(F)} := {}_{(f,s,t)}\mathrm{F}$.

The *dual contra-functor of* F: C $\twoheadrightarrow$ D is

$$\mathrm{F}\colon\ \mathrm{C} \rightleftharpoons \mathrm{Dual(D)}.$$

Of special interest is $\mathbb{I}_{\mathrm{C}}$: C $\rightleftharpoons$ Dual(C), which in general is no co-variant functor because item (1) of Def. 2.4 is satisfied iff $|{}_{\mathrm{C}}\mathrm{Obj}| \preceq 1$.

The *dual expression* of some expression ϕ is obtained by replacing each occurrence of a category or functor symbol X by Dual(X), reversing the arguments of all concatenations, replacing ${}_{\mathrm{C}}\mathrm{Mor}_{s,t}$ with ${}_{\mathrm{C}}\mathrm{Mor}_{t,s}$, Dom with Cod, and Cod with Dom.

Simply speaking of the *dual* we mean what is obtained by substituting everything by its dual.

Note that the dual expression has the same truth value in the dual as the original expression in the original. Since Dual(Dual(C)) = C and Dual(Dual(F)) = F for F: C $\twoheadrightarrow$ D, the dual of the dual is the original. Moreover, the dual of all categories and functors are all categories and functors: Indeed, the dual of the expression "C is a category" is logically equivalent to the expression "C is a category" itself. The same holds for the expression "F is a functor". Since the dual of a logic inference is a logic inference, if a set of axioms A has a consequence ϕ, then the dual of ϕ is a consequence of the dual of A. Thus, the universal theory of categories and functors does not differ from its dual. Stated differently: If a general theorem about categories and functors is valid, then the dual theorem is valid, too. This "*duality principle*" is a very important tool for doing universal category theory because it halves the proof work.

[5]called *opposite* by some authors

4 On the Inside of Categories

In the whole section we assume C to be some category and omit the left indices to C.

Definition 4.1

$t \in \mathrm{Obj}$ is called …

… *initial* if $\forall u \in \mathrm{Obj}\colon\ |\mathrm{Mor}_{t,u}| = 1$.

… *final* (or *terminal*) if $\forall s \in \mathrm{Obj}\colon\ |\mathrm{Mor}_{s,t}| = 1$.

Thus, “final” is the dual of “initial”.

… a *zero object* if t is initial and final.

Thus, “being a zero object” is the dual of itself.

E.g. in SET, $\emptyset$ is the only initial object, the terminal objects are exactly the sets with a single element, and zero objects do not exist.

Definition 4.2

$a \in \mathrm{Arr}$ is called …

… *monic* if $\forall b, c \in \mathrm{Arr}\colon \left(ba = ca \ \Rightarrow\ b = c \right)$.

… *epi* if $\forall b, c \in \mathrm{Arr}\colon \left(ab = ac \ \Rightarrow\ b = c \right)$.

Thus, “epi” is the dual of “monic”.

… *split-monic* if $\exists r \in \mathrm{Arr}\colon\ ar = (\mathrm{Id}_{\mathrm{Dom}(a)}, \mathrm{Dom}(a), \mathrm{Dom}(a))$.
Such an r is called a *retraction of* a.

… *split-epi* if $\exists l \in \mathrm{Arr}\colon\ la = (\mathrm{Id}_{\mathrm{Cod}(a)}, \mathrm{Cod}(a), \mathrm{Cod}(a))$.
Such an l is called a *section of* a.

Thus, “split-epi” and “section” are the dual of “split-monic” and “retraction”.

… *iso* if $\exists a' \in \mathrm{Arr}\colon \left(\begin{array}{rl} & aa' = (\mathrm{Id}_{\mathrm{Dom}(a)}, \mathrm{Dom}(a), \mathrm{Dom}(a)) \\ \wedge & a'a = (\mathrm{Id}_{\mathrm{Cod}(a)}, \mathrm{Cod}(a), \mathrm{Cod}(a)) \end{array} \right)$.
Such an a' is called the *inversion of* a.

… *endo* if $\mathrm{Dom}(a) = \mathrm{Cod}(a)$.

… *auto* if a is iso and endo.

… *idempotent* if $aa = a$.

Thus, “iso”, “endo”, “auto”, and “idempotent” are the duals of themselves.

… an *equalizer for* p, q if $ap = aq$ and $\forall a' \in \mathrm{Arr}\colon \left(a'p = a'q \Rightarrow \exists! c \in \mathrm{Arr}\colon\ a' = ca \right)$.

… a *co-equalizer for* p, q if $pa = qa$ and $\forall a' \in \mathrm{Arr}\colon \left(pa' = qa' \Rightarrow \exists! c \in \mathrm{Arr}\colon\ a' = ac \right)$.

Thus, “co-equalizer” is the dual of “equalizer”.

Definition 4.3

$s, t \in \mathrm{Obj}$ are called ...

... *isomorphic* if $\exists f \in \mathrm{Mor}_{s,t}$: ((f, s, t) is iso).

... *uniquely isomorphic* if $\exists! f \in \mathrm{Mor}_{s,t}$: ((f, s, t) is iso).

Corollary 4.4 *"isomorphic" and "uniquely isomorphic" are the duals of themselves.*

Corollary 4.5 *Let $a \in \mathrm{Arr}$. a is iso iff a is split-monic and split-epi.*

Lemma 4.6

(1) *Two initial objects are uniquely isomorphic.*

(2) *An object isomorphic to an initial object is initial.*

Dual(1) *Two final objects are uniquely isomorphic.*

Dual(2) *An object isomorphic to an final object is final.*

Proof of Lemma 4.6
(1): Suppose that s, t are both initial. Define $\{f\} := \mathrm{Mor}_{s,t}$ and $\{g\} := \mathrm{Mor}_{t,s}$. Now $\mathrm{mor}((f, s, t)(g, t, s)) \in \mathrm{Mor}_{s,s} = \{\mathrm{Id}_s\}$ and $\mathrm{mor}((g, t, s)(f, s, t)) \in \mathrm{Mor}_{t,t} = \{\mathrm{Id}_t\}$.

Lemma 4.7

(1) *If a is split-monic, then a is monic.*

(2) *If a is an equalizer for p, q, then a is monic.*

Dual(1) *If a is split-epi, then a is epi.*

Dual(2) *If a is a co-equalizer for p, q, then a is epi.*

Proof of Lemma 4.7
(1): For $ba = ca$ we get $b = b\,(\mathrm{Id}_{\mathrm{Dom}(a)}, \mathrm{Dom}(a), \mathrm{Dom}(a)) = bar = car = c\,(\mathrm{Id}_{\mathrm{Dom}(a)}, \mathrm{Dom}(a), \mathrm{Dom}(a)) = c$.
(2): By the first property of an equalizer we get $(ba)p = (ba)q$ for each $b \in \mathrm{Arr}$ with $\mathrm{Cod}(b) = \mathrm{Dom}(a)$. Thus, (by the uniqueness part of the second property of an equalizer) $ba = ca$ implies $b = c$.

Lemma 4.8
In the relation category[6] REL *of sets the following items are logically equivalent for* $R{::}A{\to}B$ *and* $C \ := \ \{\ y \mid \quad |R\langle\{y\}\rangle| = 1 \ \}$ *:*

1. $R{::}A{\to}B$ *is split-monic.*
2. $R{::}A{\to}B$ *is monic.*
3. $\forall X, Y \subseteq A$: $(\ \langle X\rangle R = \langle Y\rangle R \ \Rightarrow\ X{=}Y\ )$.
4. $\forall x \in A$: $\langle A\rangle R \neq \langle A\backslash\{x\}\rangle R$.
5. $\mathrm{dom}(R \circ {}_C|\mathrm{id}) = A$.
6. $R \circ {}_C|\mathrm{id} \circ R^{-1} \ = \ {}_A|\mathrm{id}$.

Proof of Lemma 4.8
$\underline{(1) \Rightarrow (2)}$: By Lemma 4.7.
$\underline{(2) \Rightarrow (3)}$: Suppose $\langle X\rangle R = \langle Y\rangle R$. Define $S := \{\emptyset\}{\times}X$ and $T := \{\emptyset\}{\times}Y$. Then $S{\circ}R = \{\emptyset\}{\times}\langle X\rangle R = \{\emptyset\}{\times}\langle Y\rangle R = T{\circ}R$. Since $R{::}A{\to}B$ is monic, we get $S{=}T$, i.e. $X{=}Y$.
$\underline{(3) \Rightarrow (4)}$: If (4) were not true, (3) would imply $A = A\backslash\{x\}$.
$\underline{(4) \Rightarrow (5)}$: "$\subseteq$" is trivial. For showing "$\supseteq$" assume $x \in A$ but $y \notin C$ for all y with $(x, y) \in \mathrm{R}$. Then we get $\langle A\rangle R = \langle A\backslash\{x\}\rangle R$, which contradicts (4).
$\underline{(5) \Rightarrow (6)}$: "$\subseteq$" is true by definition of C. "$\supseteq$" follows from (5).
$\underline{(6) \Rightarrow (1)}$: We get ${}_C|\mathrm{id} \circ R^{-1}{::}B{\to}A$ and $R \circ {}_C|\mathrm{id} \circ R^{-1} \ = \ {}_A|\mathrm{id} \ = \ {}_{\mathrm{REL}}\mathrm{Id}_A$.

Lemma 4.9
In the relation category[6] REL *of sets the following items are logically equivalent for* $R{::}A{\to}B$ *and* $C \ := \ \{\ y \mid \quad |\langle\{y\}\rangle R| = 1 \ \}$ *:*

1. $R{::}A{\to}B$ *is split-epi.*
2. $R{::}A{\to}B$ *is epi.*
3. $\forall X, Y \subseteq B$: $\big(\ R\langle X\rangle = R\langle Y\rangle \ \Rightarrow\ X{=}Y\ \big)$.
4. $\forall x \in B$: $R\langle B\rangle \neq R\langle B\backslash\{x\}\rangle$.
5. $\mathrm{ran}({}_C|\mathrm{id} \circ R) = B$.
6. $R^{-1} \circ {}_C|\mathrm{id} \circ R \ = \ {}_B|\mathrm{id}$.

Since Dual(REL) is isomorphic to REL via F: Dual(REL) $\twoheadrightarrow$ REL given by $A\mathrm{F} := A$ and ${}_R\mathrm{F} := R^{-1}$ and since Lemma 4.9 is the F-isomorphic of the dual of Lemma 4.8, we do not give a proof of Lemma 4.9 — even if the duality principle[7] is actually only sufficient to infer the logical equivalence of (1) and (2).

[6] Cf. Ex. 2.7.
[7] Cf. section 3.

Lemma 4.10
In the (total function) category[8] SET *of sets the following items are logically equivalent for* $f{::}A{\to}B$:

(1) $f{::}A{\to}B$ *is an equalizer*
[for each $p, q{::}B{\to}C$ *with* $\forall b \in B$: $(\ p(b) = q(b) \ \Leftrightarrow \ b \in \mathrm{ran}(f)\ )$*]*.

(2) $f{::}A{\to}B$ *is monic.*

(3) $f\colon\ A \to B$ *is injective.*

Furthermore, (1) - (3) are implied by

(4) $f{::}A{\to}B$ *is split-monic.*

In general, however, (4) is not implied by (1) - (3), but it is equivalent to

(5) $f\colon\ A \to B$ *is injective and* $(\ A = \emptyset \ \Rightarrow \ \mathrm{B} = \emptyset\ )$.

Proof of Lemma 4.10
(1) $\Rightarrow$ (2): By Lemma 4.7.
(2) $\Rightarrow$ (3): If f were not injective, there would be $a, b \in \mathrm{dom}(f)$ with $a \neq b$ and $f(a) = f(b)$. Define $g, h{::}\{\emptyset\}{\to}A$ by $g(\emptyset) := a$ and $h(\emptyset) := b$. Thus, $g \circ f = h \circ f$ and $g \neq h$, which contradicts (2).
(3) $\Rightarrow$ (1): For p, q as indicated, we get $f \circ p = f \circ q$. Furthermore, for $f'{::}D{\to}B$ with $f' \circ p = f' \circ q$ we get $\mathrm{ran}(f') \subseteq \mathrm{ran}(f)$. Thus, since f is injective, we then can define $h{::}D{\to}A$ by $h := f' {\circ} f^{-1}$, and get $f' = f' \circ {}_{\mathrm{ran}(f)}|\mathrm{id} = f' \circ f^{-1} \circ f = h \circ f$. Finally, for any other $h'{::}D{\to}A$ with $f' = h' \circ f$ we get $h' = h' \circ {}_A|\mathrm{id} = h' \circ f \circ f^{-1} = f' \circ f^{-1} = h$.
(4) $\Rightarrow$ (2): By Lemma 4.7.
(4) $\Rightarrow$ (5): If f has a retraction $r{::}B{\to}\emptyset$, we must have $B = \emptyset$.
(5) $\Rightarrow$ (4): If $A = \emptyset = B$, f has the retraction $\emptyset{::}\emptyset{\to}\emptyset$. If there is some $a \in A$, f has the retraction $r{::}B{\to}A$ given by $r(x) := f^{-1}(x)$ for $x \in \mathrm{ran}(f)$ and $r(x) := a$ for $x \in B \backslash \mathrm{ran}(f)$.
In general not "(3) $\Rightarrow$ (4)": Take $f := A := \emptyset$ and $B := \{\emptyset\}$. Now f is injective but cannot have a retraction since ${}_{\mathrm{SET}}\mathrm{Mor}_{B,\emptyset} = \emptyset$.

[8] Cf. Ex. 2.7.

Lemma 4.11
In the (total function) category[8] SET *of sets the following items are logically equivalent for* $f{::}A{\to}B$*:*

(1) $f{::}A{\to}B$ *is a co-equalizer*
[for $p,q{::}C{\to}A$*;* $C := \ker(f) := f \circ f^{-1}$*; and*
p,q *given by* $((x,y) \in C)$ $p(x,y) := x$ *and* $q(x,y) := y$ *].*

(2) $f{::}A{\to}B$ *is epi.*

(3) $f\colon\ A \to B$ *is surjective (i.e.* $\mathrm{ran}(f) = B$*).*

Furthermore, (1) - (3) are implied by

(4) $f{::}A{\to}B$ *is split-epi.*

"(3) ⇒ (4)", however, is equivalent to the set-version of (the function choice function version of) the Axiom of Choice.

Proof of Lemma 4.11
(1) ⇒ (2): By Lemma 4.7.
(2) ⇒ (3): If $\mathrm{ran}(f) \neq B$, there would be some $b \in B\backslash\mathrm{ran}(f)$. Define $g,h{::}B{\to}\{0,1\}$ by $g(b) := 1$, $g(x) := 0$ for $x \in B\backslash\{b\}$, and $h(x) := 0$ for $x \in B$. Thus, $fg = fh$ and $g \neq h$, which contradicts the assumption that $f{::}A{\to}B$ is epi.
(3) ⇒ (1): For p, q as indicated, we get $p \circ f = q \circ f$. Furthermore, for $f'{::}A{\to}D$ with $p \circ f' = q \circ f'$ we get $\ker(f) \subseteq \ker(f')$ because f' is total on A. Thus, since $f\colon\ A \to B$ is surjective, we then can define $h{::}B{\to}D$ by $h := f^{-1} \circ f'$ and get $f' = f \circ f^{-1} \circ f'$ because of ${}_A|\mathrm{id} \subseteq f \circ f^{-1} \subseteq \ker(f')$. Thus $f' = f \circ h$. Finally, for any other $h'{::}B{\to}D$ with $f' = f \circ h'$, since f is surjective we get $h' = {}_B|\mathrm{id} \circ h' = {}_{\mathrm{ran}(f)}|\mathrm{id} \circ h' = f^{-1} \circ f \circ h' = f^{-1} \circ f' = h$.
(4) ⇒ (2): By Lemma 4.7.
Axiom of Choice, (3) ⇒ (4): Since $\mathrm{ran}(f) = B$, let $f''\colon\ B \to A$ be a function choice function of $f'\colon\ B \to \mathfrak{P}(A)$ given by $(x \in B)$ $f'(x) := f\langle\{x\}\rangle$. Since f is a function and then by $\mathrm{ran}(f) = B$ we get $f'' \circ f = {}_{\mathrm{ran}(f)}|\mathrm{id} = {}_B|\mathrm{id}$. Thus $f''{::}B{\to}A$ is a section of $f{::}A{\to}B$.
((3) ⇒ (4)) ⇒ Axiom of Choice: Assume $g\colon\ B \to \mathfrak{P}(\mathcal{V})$ with $\forall x \in B\colon\ g(x) \neq \emptyset$. Define $A := \bigcup_{x \in B}(\{x\} \times g(x))$ and the surjective $f\colon\ A \to B$ by $f(x,y) := x$. If this is supposed to imply that $f{::}A{\to}B$ is split-epi, then there must be a section $f'{::}B{\to}A$ of $f{::}A{\to}B$. Define $h\colon\ A \to \mathcal{V}$ by $h(x,y) := y$. Now $f' \circ h$ is a function choice function of g. Indeed, for $x \in B$ we get $f(f'(x)) = x$, $f'(x) \in \{x\} \times g(x)$, and $h(f'(x)) \in g(x)$.

Lemma 4.12
In the partial function category[9] FUN *of sets the following items are logically equivalent for* $f{::}A{\to}B$:

(1) $f{::}A{\to}B$ *is split-monic.*

(2) $f{::}A{\to}B$ *is monic.*

(3) $f\colon A \rightsquigarrow B$ *is injective and* $\mathrm{dom}(f) = A$.

Proof of Lemma 4.12
(1) $\Rightarrow$ (2): By Lemma 4.7.
(2) $\Rightarrow$ (3): If f were not total on A, there would be an $a \in A\backslash\mathrm{dom}(f)$. Thus, ${}_{\{a\}}|\mathrm{id}{\circ}f = \emptyset{\circ}f$ and ${}_{\{a\}}|\mathrm{id} \neq \emptyset$, which contradicts (2) due to ${}_{\{a\}}|\mathrm{id}{::}A{\to}A$ and $\emptyset{::}A{\to}A$. Thus, $f{::}A{\to}B$ is also in SET. Since SET is a sub-category of FUN and $f{::}A{\to}B$ is monic in FUN, $f{::}A{\to}B$ is also monic in SET. By Lemma 4.10, we conclude that f is injective.
(3) $\Rightarrow$ (1): For injective $f\colon A \rightsquigarrow B$ we get $f^{-1}{::}B{\to}A$ and $f \circ f^{-1} = {}_{\mathrm{dom}(f)}|\mathrm{id} = {}_{A}|\mathrm{id} = {}_{\mathrm{FUN}}\mathrm{Id}_A$.

Lemma 4.13
In the partial function category[9] FUN *of sets the following items are logically equivalent for* $f{::}A{\to}B$:

(1) $f{::}A{\to}B$ *is epi.*

(2) $f\colon A \rightsquigarrow B$ *is surjective (i.e.* $\mathrm{ran}(f) = B$*).*

Furthermore, (1) and (2) are implied by

(3) $f{::}A{\to}B$ *is split-epi.*

"(1) $\Rightarrow$ (3)*", however, is equivalent to the set-version of (the function choice function version of) the Axiom of Choice.*

Proof of Lemma 4.13
(1) $\Rightarrow$ (2): If $\mathrm{ran}(f) \neq B$, there would be some $b \in B\backslash\mathrm{ran}(f)$. Define $g, h{::}B{\to}\{0,1\}$ by $g(b) := 1$, $g(x) := 0$ for $x \in B\backslash\{b\}$, and $h(x) := 0$ for $x \in B$. Thus, $f \circ g = f \circ h$ and $g \neq h$, which contradicts the assumption that $f{::}A{\to}B$ is epi.
(2) $\Rightarrow$ (1): Assume $g, h{::}B{\to}C$ with $f \circ g = f \circ h$. Since f is a function and then by (2) we get $f^{-1} \circ f = {}_{\mathrm{ran}(f)}|\mathrm{id} = {}_{B}|\mathrm{id}$. Thus $g = {}_{B}|\mathrm{id} \circ g = f^{-1} \circ f \circ g = f^{-1} \circ f \circ h = {}_{B}|\mathrm{id} \circ h = h$.
(3) $\Rightarrow$ (1): By Lemma 4.7.
Logical equivalence with the Axiom of Choice: The proof for this does not differ from the respective part of the proof of Lemma 4.11.

[9] Cf. Ex. 2.7.

5 Algebras

Definition 5.1 (Signature)

A *signature* $(\mathbb{S}, \mathbb{F}, \mathrm{rt})$ consists of

- a set of sorts $\mathbb{S}$ and
- an $\mathbb{S}$-sorted family $\mathbb{F} = (\mathbb{F}_s)_{s \in \mathbb{S}^*}$ of sets $\mathbb{F}_s$ of function symbols with argument sort s, and
- a result-type function $\mathrm{rt}\colon \coprod_{s \in \mathbb{S}^*} \mathbb{F}_s \to \mathbb{S}$.

Example 5.2 (Signature)

Set $\mathbb{S} := \{1, \mathsf{zero}, \mathsf{posnat}, \mathsf{emptylist}, \mathsf{nonemptylist}, \mathsf{zero{+}posnat}, \mathsf{zero{+}posnat{+}1}, \mathsf{emptylist{+}nonemptylist}\}$.

Note that 1 is included in $\mathbb{S}$ in order to model the range of "undefined" function applications.

Set $\mathbb{F}_{\emptyset} := \{\mathsf{0}, \mathsf{nil}\}$; $\mathbb{F}_{\mathsf{zero}} := \{\mathsf{s}\}$; $\mathbb{F}_{\mathsf{posnat}} := \{\mathsf{s}, \mathsf{p}\}$;
$\mathbb{F}_{\mathsf{emptylist}} := \{\mathsf{length}\}$; $\mathbb{F}_{\mathsf{nonemptylist}} := \{\mathsf{length}, \mathsf{car}, \mathsf{cdr}\}$;
$\mathbb{F}_{\mathsf{zero\,zero}} := \mathbb{F}_{\mathsf{zero\,posnat}} := \mathbb{F}_{\mathsf{posnat\,posnat}} := \{+, -\}$;
$\mathbb{F}_{\mathsf{posnat\,zero}} :=$
$\mathbb{F}_{\mathsf{emptylist\,emptylist}} := \mathbb{F}_{\mathsf{emptylist\,nonemptylist}} := \mathbb{F}_{\mathsf{nonemptylist\,emptylist}} := \mathbb{F}_{\mathsf{nonemptylist\,nonemptylist}} := \{+\}$;
$\mathbb{F}_{\mathsf{zero\,emptylist}} := \mathbb{F}_{\mathsf{zero\,nonemptylist}} := \mathbb{F}_{\mathsf{posnat\,emptylist}} := \mathbb{F}_{\mathsf{posnat\,nonemptylist}} := \{\mathsf{cons}\}$; and
$\mathbb{F}_s := \emptyset$ for any other $s \in \mathbb{S}^*$.

Set $\mathrm{rt}(\emptyset, \mathsf{0}) := \mathrm{rt}(\mathsf{emptylist}, \mathsf{length}) := \mathrm{rt}(\mathsf{zero\,zero}, +) := \mathrm{rt}(\mathsf{zero\,zero}, -) := \mathsf{zero}$;
$\mathrm{rt}(\mathsf{zero}, \mathsf{s}) := \mathrm{rt}(\mathsf{posnat}, \mathsf{s}) := \mathrm{rt}(\mathsf{nonemptylist}, \mathsf{length}) := \mathrm{rt}(\mathsf{zero\,posnat}, +) :=$
$\mathrm{rt}(\mathsf{posnat\,zero}, +) := \mathrm{rt}(\mathsf{posnat\,posnat}, +) := \mathrm{rt}(\mathsf{posnat\,zero}, -) := \mathsf{posnat}$;
$\mathrm{rt}(\mathsf{posnat}, \mathsf{p}) := \mathrm{rt}(\mathsf{nonemptylist}, \mathsf{car}) := \mathsf{zero{+}posnat}$;
$\mathrm{rt}(\mathsf{posnat\,posnat}, -) := \mathsf{zero{+}posnat{+}1}$;
$\mathrm{rt}(\emptyset, \mathsf{nil}) := \mathrm{rt}(\mathsf{emptylist\,emptylist}, +) := \mathsf{emptylist}$;
$\mathrm{rt}(\mathsf{emptylist\,nonemptylist}, +) := \mathrm{rt}(\mathsf{nonemptylist\,emptylist}, +) :=$
$\mathrm{rt}(\mathsf{nonemptylist\,nonemptylist}, +) := \mathrm{rt}(\mathsf{zero\,emptylist}, \mathsf{cons}) :=$
$\mathrm{rt}(\mathsf{zero\,nonemptylist}, \mathsf{cons}) := \mathrm{rt}(\mathsf{posnat\,emptylist}, \mathsf{cons}) :=$
$\mathrm{rt}(\mathsf{posnat\,nonemptylist}, \mathsf{cons}) := \mathsf{nonemptylist}$; and
$\mathrm{rt}(\mathsf{nonemptylist}, \mathsf{cdr}) := \mathsf{emptylist{+}nonemptylist}$.

The following definition is classical.

Definition 5.3 (**($\mathbb{S}, \mathbb{F}, \mathrm{rt}$)-Algebra**)

Let $(\mathbb{S}, \mathbb{F}, \mathrm{rt})$ be a signature.

A (total) $(\mathbb{S}, \mathbb{F}, \mathrm{rt})$-*algebra* consists of

- a family $A = (A_{s'})_{s' \in \mathbb{S}}$ of non-empty sets ("universes") and,
- for each $(s, f) \in \coprod_{s \in \mathbb{S}^*} \mathbb{F}_s$, a function

$$(s, f)^{\mathcal{A}}\colon \quad \prod_{i \prec |s|} A_{s_i} \quad \rightarrow \quad A_{\mathrm{rt}(s,f)}.$$

Example 5.4

Let $(\mathbb{S}, \mathbb{F}, \mathrm{rt})$ be the signature of Ex. 5.2.

Set $A_1 := \{\omega\}$; $A_{\mathsf{zero}} := \{0\}$; $A_{\mathsf{posnat}} := \mathbb{N}_+$; $A_{\mathsf{emptylist}} := \{\emptyset\}$; $A_{\mathsf{nonemptylist}} := \mathbb{N}^+$; and $A_{\mathsf{zero+posnat}} := A_{\mathsf{zero}} \uplus A_{\mathsf{posnat}}$; $A_{\mathsf{zero+posnat+1}} := A_{\mathsf{zero+posnat}} \uplus A_1$; $A_{\mathsf{emptylist+nonemptylist}} := A_{\mathsf{emptylist}} \uplus A_{\mathsf{nonemptylist}}$.

For $x, y \in \mathbb{N}$ and $k, l \in \mathbb{N}^*$ define: $(\emptyset, \mathsf{0})^{\mathcal{A}}(\emptyset) := 0$; $(\emptyset, \mathsf{nil})^{\mathcal{A}}(\emptyset) := \emptyset$;
$(\mathsf{zero}, \mathsf{s})^{\mathcal{A}}(x) := (\mathsf{posnat}, \mathsf{s})^{\mathcal{A}}(x) := x{+}1$;
$(\mathsf{emptylist}, \mathsf{length})^{\mathcal{A}}(l) := (\mathsf{nonemptylist}, \mathsf{length})^{\mathcal{A}}(l) := |l|$;
$(\mathsf{zero\ zero}, +)^{\mathcal{A}}(x, y) := (\mathsf{zero\ posnat}, +)^{\mathcal{A}}(x, y) :=$
$(\mathsf{posnat\ zero}, +)^{\mathcal{A}}(x, y) := (\mathsf{posnat\ posnat}, +)^{\mathcal{A}}(x, y) := x{+}y$;
$(\mathsf{posnat}, \mathsf{p})^{\mathcal{A}}(x{+}1) := (\mathsf{posnat\ zero}, -)^{\mathcal{A}}(x, 0) := (\mathsf{nonemptylist}, \mathsf{car})^{\mathcal{A}}(xl) := x$;
$(\mathsf{posnat\ posnat}, -)^{\mathcal{A}}(x, y) := x{-}y$ for $x \succeq y$;
$(\mathsf{posnat\ posnat}, -)^{\mathcal{A}}(x, y) := \omega$ for $x \prec y$;
$(\mathsf{emptylist\ emptylist}, +)^{\mathcal{A}}(k, l) := (\mathsf{emptylist\ nonemptylist}, +)^{\mathcal{A}}(k, l) :=$
$(\mathsf{nonemptylist\ emptylist}, +)^{\mathcal{A}}(k, l) := (\mathsf{nonemptylist\ nonemptylist}, +)^{\mathcal{A}}(k, l) := kl$;
$(\mathsf{zero\ emptylist}, \mathsf{cons})^{\mathcal{A}}(x, l) := (\mathsf{zero\ nonemptylist}, \mathsf{cons})^{\mathcal{A}}(x, l) :=$
$(\mathsf{posnat\ emptylist}, \mathsf{cons})^{\mathcal{A}}(x, l) := (\mathsf{posnat\ nonemptylist}, \mathsf{cons})^{\mathcal{A}}(x, l) := xl$; and
$(\mathsf{nonemptylist}, \mathsf{cdr})^{\mathcal{A}}(xl) := l$.

Note that the signature of Ex. 5.2 is rather dull because $\mathsf{s}(\mathsf{p}(\mathsf{s}(\mathsf{s}(\mathsf{0}))))$ is not a term although we can evaluate it: $\mathsf{p}(\mathsf{s}(\mathsf{s}(\mathsf{0})))$ has the sort $\mathsf{zero{+}posnat}$ and we have $\mathbb{F}_{\mathsf{zero+posnat}} = \emptyset$. Note that a signature cannot properly model functions with different range sorts for a single argument sort. This will be different with co-signatures in the next section.

While the notion of an algebra given in Def. 5.3 is nice, it is not very well suited for category theory. The first step for coming to the categorical definition of an algebra is to encode $(\mathbb{S}, \mathbb{F}, \mathrm{rt})$ into the following functor $((\Sigma_{\mathrm{Obj}}, \Sigma_{\mathrm{Mor}}), \mathrm{SORT}(\mathbb{S}), \mathrm{SORT}(\mathbb{S}))$ from the category $\mathrm{SORT}(\mathbb{S})$ of $\mathbb{S}$-sorted sets[10] into itself:

$$(A\Sigma)_{s'} \quad := \quad \coprod_{(s,f)\in\mathrm{rt}\langle\{s'\}\rangle} \quad \prod_{i\prec|s|} A_{s_i}$$

for $A \in {}_{\mathrm{SORT}(\mathbb{S})}\mathrm{Obj}$ and $s' \in \mathbb{S}$, and

$$({}_h\Sigma)_{s'}\Big(\ (s,f),\ \ (a_i)_{i\prec|s|}\ \Big) \quad := \quad \Big(\ (s,f),\ \ (h_{s_i}(a_i))_{i\prec|s|}\ \Big)$$

for $h{::}A{\to}B$ in $\mathrm{SORT}(\mathbb{S})$; $s' \in \mathbb{S}$; $(s,f) \in \mathrm{rt}\langle\{s'\}\rangle$; $\forall i \prec |s|:\ a_i \in A_{s_i}$.

Now we can categorically represent the above algebra over $(\mathbb{F}, \mathbb{S}, \mathrm{rt})$ as the arrow $(\alpha, A\Sigma, A) \in {}_{\mathrm{SORT}(\mathbb{S})}\mathrm{Arr}$ given by $\alpha_{s'}((s,f),a) := (s,f)^{\mathcal{A}}(a)$ for $((s,f),a) \in (A\Sigma)_{s'}$.

Definition 5.5 (Σ-Algebras and Σ-Homomorphisms)
Let Σ: $\mathrm{SORT}(\mathbb{S}) \twoheadrightarrow \mathrm{SORT}(\mathbb{S})$.
A Σ-*algebra* is an arrow $(\alpha, A\Sigma, A) \in {}_{\mathrm{SORT}(\mathbb{S})}\mathrm{Arr}$ with $\forall s \in \mathbb{S}:\ A_s \neq \emptyset$.
Let $(\alpha, A\Sigma, A)$ and $(\beta, B\Sigma, B)$ be Σ-algebras.
A Σ-*homomorphism from* $(\alpha, A\Sigma, A)$ *to* $(\beta, B\Sigma, B)$ is a morphism $h{::}A{\to}B$ in $\mathrm{SORT}(\mathbb{S})$ such that h is compatible with Σ in the sense that $\alpha\ {}_{\mathrm{SORT}(\mathbb{S})}\!\diamond\ h = {}_h\Sigma\ {}_{\mathrm{SORT}(\mathbb{S})}\!\diamond\ \beta$.

$$\begin{tikzcd}
A\Sigma \arrow[r, "{}_h\Sigma"'] \arrow[d, "\alpha"] \arrow[dr, phantom, "="] & B\Sigma \arrow[d, "\beta"] \\
A \arrow[r, "h"'] & B
\end{tikzcd}$$

Note that for Σ defined from $(\mathbb{S}, \mathbb{F}, \mathrm{rt})$ as above, the latter condition means that for $s' \in \mathbb{S}$; $(s,f) \in \mathrm{rt}\langle\{s'\}\rangle$; $\forall i \prec |s|:\ a_i \in A_{s_i}$:

$$h_{s'}\Big(\ \alpha_{s'}\Big(\ (s,f),\ \ (a_i)_{i\prec|s|}\ \Big)\ \Big) \quad = \quad \beta_{s'}\Big(\ (s,f),\ \ (h_{s_i}(a_i))_{i\prec|s|}\ \Big),$$

i.e.

$$h_{s'}\Big(\ (s,f)^{\mathcal{A}}\Big(\ (a_i)_{i\prec|s|}\ \Big)\ \Big) \quad = \quad (s,f)^{\mathcal{B}}\Big(\ (h_{s_i}(a_i))_{i\prec|s|}\ \Big).$$

Definition 5.6 (ALG)

The Σ-*homomorphism category* ALG *of* Σ-*algebras* is given as follows:

${}_{\mathrm{ALG}}\mathrm{Obj}$ is the class of all Σ-algebras.

${}_{\mathrm{ALG}}\mathrm{Mor}_{(\alpha,A\Sigma,A),(\beta,B\Sigma,B)}$ is the class of all Σ-homomorphisms $h{::}A{\to}B$.

$h\ {}_{\mathrm{ALG}}\!\diamond\ k := h\ {}_{\mathrm{SORT}(\mathbb{S})}\!\diamond\ k$
for $h \in {}_{\mathrm{ALG}}\mathrm{Mor}_{(\alpha,A\Sigma,A),(\beta,B\Sigma,B)}$ and $h \in {}_{\mathrm{ALG}}\mathrm{Mor}_{(\beta,B\Sigma,B),(\gamma,C\Sigma,C)}$.

${}_{\mathrm{ALG}}\mathrm{Id}_{(\alpha,A\Sigma,A)} := {}_{\mathrm{SORT}(\mathbb{S})}\mathrm{Id}_A$.

Note that, while the morphisms of ALG are also morphisms of $\mathrm{SORT}(\mathbb{S})$ and their composition is the same, ALG is no sub-category of $\mathrm{SORT}(\mathbb{S})$ because the objects of ALG are no *objects* of $\mathrm{SORT}(\mathbb{S})$.

[10] Cf. Ex. 2.9.

6 Co-Algebras

Definition 6.1 (Co-Signature)

A *co-signature* $(\mathbb{S}, \mathbb{F}, \mathrm{rt})$ consists of

- a set of sorts $\mathbb{S}$ and
- an $\mathbb{S}$-sorted family $\mathbb{F} = (\mathbb{F}_s)_{s \in \mathbb{S}}$ of sets $\mathbb{F}_s$ of function symbols with argument sort s, and
- a result-type function $\mathrm{rt}\colon \coprod_{s \in \mathbb{S}} \mathbb{F}_s \to \mathfrak{P}_{\mathbb{N}_+}(\mathbb{S})$.

Example 6.2 (Co-Signature)

Set $\mathbb{S} := \{1, \mathsf{zero}, \mathsf{posnat}, \mathsf{emptylist}, \mathsf{nonemptylist}\} \cup P_{\mathsf{nat}} \cup P_{\mathsf{list}} \cup Q$;
$P_{\mathsf{nat}} := \{\mathsf{zero}{\times}\mathsf{zero}, \mathsf{zero}{\times}\mathsf{posnat}, \mathsf{posnat}{\times}\mathsf{zero}, \mathsf{posnat}{\times}\mathsf{posnat}\}$;
$P_{\mathsf{list}} := \{\mathsf{emptylist}{\times}\mathsf{emptylist}, \mathsf{emptylist}{\times}\mathsf{nonemptylist}, \mathsf{nonemptylist}{\times}\mathsf{emptylist}, \mathsf{nonemptylist}{\times}\mathsf{nonemptylist}\}$; and
$Q := \{\mathsf{zero}{\times}\mathsf{emptylist}, \mathsf{zero}{\times}\mathsf{nonemptylist}, \mathsf{posnat}{\times}\mathsf{emptylist}, \mathsf{posnat}{\times}\mathsf{nonemptylist}\}$.

Note that 1 is included in $\mathbb{S}$ in order to model the domain of nullary functions and the range of "undefined" function applications.

Set $\mathbb{F}_1 := \{\mathsf{0}, \mathsf{nil}\}$; $\mathbb{F}_{\mathsf{zero}} := \{\mathsf{s}\}$; $\mathbb{F}_{\mathsf{posnat}} := \{\mathsf{s}, \mathsf{p}\}$;
$\mathbb{F}_{\mathsf{emptylist}} := \{\mathsf{length}\}$; $\mathbb{F}_{\mathsf{nonemptylist}} := \{\mathsf{length}, \mathsf{car}, \mathsf{cdr}\}$;
$\mathbb{F}_{\mathsf{zero}\times\mathsf{zero}} := \mathbb{F}_{\mathsf{posnat}\times\mathsf{zero}} := \mathbb{F}_{\mathsf{posnat}\times\mathsf{posnat}} := \{+, -\}$;
$\mathbb{F}_{\mathsf{zero}\times\mathsf{posnat}} := \mathbb{F}_s := \{+\}$ for $s \in P_{\mathsf{list}}$; and $\mathbb{F}_s := \{\mathsf{cons}\}$ for $s \in Q$.

Set $\mathrm{rt}(1, \mathsf{0}) := \mathrm{rt}(\mathsf{emptylist}, \mathsf{length}) := \mathrm{rt}(\mathsf{zero}{\times}\mathsf{zero}, +) := \mathrm{rt}(\mathsf{zero}{\times}\mathsf{zero}, -) := \{\mathsf{zero}\}$;
$\mathrm{rt}(\mathsf{zero}, \mathsf{s}) := \mathrm{rt}(\mathsf{posnat}, \mathsf{s}) := \mathrm{rt}(\mathsf{nonemptylist}, \mathsf{length}) := \mathrm{rt}(\mathsf{zero}{\times}\mathsf{posnat}, +) :=$
$\mathrm{rt}(\mathsf{posnat}{\times}\mathsf{zero}, +) := \mathrm{rt}(\mathsf{posnat}{\times}\mathsf{posnat}, +) := \mathrm{rt}(\mathsf{posnat}{\times}\mathsf{zero}, -) := \{\mathsf{posnat}\}$;
$\mathrm{rt}(\mathsf{posnat}, \mathsf{p}) := \mathrm{rt}(\mathsf{nonemptylist}, \mathsf{car}) := \{\mathsf{zero}, \mathsf{posnat}\}$;
$\mathrm{rt}(\mathsf{posnat}{\times}\mathsf{posnat}, -) := \{1, \mathsf{zero}, \mathsf{posnat}\}$;
$\mathrm{rt}(1, \mathsf{nil}) := \mathrm{rt}(\mathsf{emptylist}{\times}\mathsf{emptylist}, +) := \{\mathsf{emptylist}\}$;
$\mathrm{rt}(\mathsf{emptylist}{\times}\mathsf{nonemptylist}, +) := \mathrm{rt}(\mathsf{nonemptylist}{\times}\mathsf{emptylist}, +) :=$
$\mathrm{rt}(\mathsf{nonemptylist}{\times}\mathsf{nonemptylist}, +) := \mathrm{rt}(s, \mathsf{cons}) := \{\mathsf{nonemptylist}\}$ for $s \in Q$; and
$\mathrm{rt}(\mathsf{nonemptylist}, \mathsf{cdr}) := \{\mathsf{emptylist}, \mathsf{nonemptylist}\}$.

Definition 6.3 ($(\mathbb{S}, \mathbb{F}, \mathrm{rt})$-Co-Algebra)

Let $(\mathbb{S}, \mathbb{F}, \mathrm{rt})$ be a co-signature.

A $(\mathbb{S}, \mathbb{F}, \mathrm{rt})$-*co-algebra* consists of

- a family $A = (A_s)_{s\in\mathbb{S}}$ of non-empty sets ("universes") and,
- for each $(s, f) \in \coprod_{s\in\mathbb{S}} \mathbb{F}_s$, a function
$$(s, f)^{\mathcal{A}}\colon \quad A_s \quad \to \quad \coprod_{s'\in\mathrm{rt}(s,f)} A_{s'}.$$

Example 6.4

Let $(\mathbb{S}, \mathbb{F}, \mathrm{rt})$ be the co-signature of Ex. 6.2.

Set $A_{\mathsf{1}} := \{\omega\}$; $A_{\mathsf{zero}} := \{0\}$; $A_{\mathsf{posnat}} := \mathbb{N}_+$; $A_{\mathsf{emptylist}} := \{\emptyset\}$; $A_{\mathsf{nonemptylist}} := \mathbb{N}^+$; and $A_{s\times t} := A_s \times A_t$ for $s{\times}t \in P_{\mathsf{nat}} \cup P_{\mathsf{list}} \cup Q$.

For $x, y \in \mathbb{N}$ and $k, l \in \mathbb{N}^*$ define:
$(\mathsf{1}, \mathsf{0})^{\mathcal{A}}(\omega) := (\mathsf{posnat}, \mathsf{p})^{\mathcal{A}}(1) := (\mathsf{emptylist}, \mathsf{length})^{\mathcal{A}}(\emptyset) := (\mathsf{zero}{\times}\mathsf{zero}, +)^{\mathcal{A}}(0, 0) :=$
$(\mathsf{zero}{\times}\mathsf{zero}, -)^{\mathcal{A}}(0, 0) := (\mathsf{posnat}{\times}\mathsf{posnat}, -)^{\mathcal{A}}(x, x) := (\mathsf{nonemptylist}, \mathsf{car})^{\mathcal{A}}(0l) := (\mathsf{zero}, 0)$;
$(\mathsf{1}, \mathsf{nil})^{\mathcal{A}}(\omega) := (\mathsf{emptylist}{\times}\mathsf{emptylist}, +)^{\mathcal{A}}(\emptyset, \emptyset) := (\mathsf{nonemptylist}, \mathsf{cdr})^{\mathcal{A}}(x) := (\mathsf{emptylist}, \emptyset)$;
$(\mathsf{zero}, \mathsf{s})^{\mathcal{A}}(x) := (\mathsf{posnat}, \mathsf{s})^{\mathcal{A}}(x) := (\mathsf{posnat}, x{+}1)$;
$(\mathsf{nonemptylist}, \mathsf{length})^{\mathcal{A}}(l) := (\mathsf{posnat}, |l|)$;
$(\mathsf{zero}{\times}\mathsf{posnat}, +)^{\mathcal{A}}(x, y) := (\mathsf{posnat}{\times}\mathsf{zero}, +)^{\mathcal{A}}(x, y) := (\mathsf{posnat}{\times}\mathsf{posnat}, +)^{\mathcal{A}}(x, y) :=$
$(\mathsf{posnat}, x{+}y)$;
$(\mathsf{posnat}, \mathsf{p})^{\mathcal{A}}(x{+}1) := (\mathsf{posnat}{\times}\mathsf{zero}, -)^{\mathcal{A}}(x, 0) := (\mathsf{nonemptylist}, \mathsf{car})^{\mathcal{A}}(xl) :=$
(posnat, x) for $x \in \mathbb{N}_+$;
$(\mathsf{posnat}{\times}\mathsf{posnat}, -)^{\mathcal{A}}(x, y) := (\mathsf{posnat}, x{-}y)$ for $x \succ y$;
$(\mathsf{posnat}{\times}\mathsf{posnat}, -)^{\mathcal{A}}(x, y) := (\mathsf{1}, \omega)$ for $x \prec y$;
$(\mathsf{emptylist}{\times}\mathsf{nonemptylist}, +)^{\mathcal{A}}(k, l) := (\mathsf{nonemptylist}{\times}\mathsf{emptylist}, +)^{\mathcal{A}}(k, l) :=$
$(\mathsf{nonemptylist}{\times}\mathsf{nonemptylist}, +)^{\mathcal{A}}(k, l) := (\mathsf{nonemptylist}, kl)$;
$(s, \mathsf{cons})^{\mathcal{A}}(x, l) := (\mathsf{nonemptylist}, xl)$; and
$(\mathsf{nonemptylist}, \mathsf{cdr})^{\mathcal{A}}(xl) := (\mathsf{nonemptylist}, l)$ for $l \in \mathbb{N}^+$.

Note that the co-signature of Ex. 6.2 is rather dull because we cannot add three natural numbers: The co-signature does not know that the result of the first addition can be paired with a natural number in order to add it. Actually, we cannot even add two natural numbers, but the sorts in P_{nat} have hidden this. Note that a co-signature cannot contain pairing functions because they always need more than one argument.

While the notion of a co-algebra given in Def. 6.3 is nice, it is not very well suited for category theory. The first step for coming to the categorical definition of a co-algebra is to encode $(\mathbb{S}, \mathbb{F}, \mathrm{rt})$ into the following functor $((\Delta_{\mathrm{Obj}}, \Delta_{\mathrm{Mor}}), \mathrm{SORT}(\mathbb{S}), \mathrm{SORT}(\mathbb{S}))$ from the category $\mathrm{SORT}(\mathbb{S})$ of $\mathbb{S}$-sorted sets[11] into itself:

$$(A\Delta)_s \quad := \quad \prod_{f \in \mathbb{F}_s} \quad \coprod_{s' \in \mathrm{rt}(s,f)} \quad A_{s'}$$

for $A \in {}_{\mathrm{SORT}(\mathbb{S})}\mathrm{Obj}$ and $s \in \mathbb{S}$, and

$$({}_h\Delta)_s\Big(\ (s'_f, a'_f)_{f \in \mathbb{F}_s}\ \Big) \quad := \quad (s'_f, h_{s'_f}(a'_f))_{f \in \mathbb{F}_s}$$

for $h{::}A{\to}B$ in $\mathrm{SORT}(\mathbb{S})$; $s \in \mathbb{S}$; $\forall f \in \mathbb{F}_s$: $(s'_f \in \mathrm{rt}(s,f) \wedge a'_f \in A_{s'_f})$.

Now we can categorically represent the above co-algebra over $(\mathbb{F}, \mathbb{S}, \mathrm{rt})$ as the arrow $(\alpha, A, A\Delta) \in {}_{\mathrm{SORT}(\mathbb{S})}\mathrm{Arr}$ given by $\alpha_s(a)_f := (s,f)^{\mathcal{A}}(a)$ for $a \in A_s$ and $f \in \mathbb{F}_s$.

Definition 6.5 (Δ-Co-Algebras and Δ-Co-Homomorphisms)
Let Δ: $\mathrm{SORT}(\mathbb{S}) \twoheadrightarrow \mathrm{SORT}(\mathbb{S})$.
A Δ-*co-algebra* is an arrow $(\alpha, A, A\Delta) \in {}_{\mathrm{SORT}(\mathbb{S})}\mathrm{Arr}$ with $\forall s \in \mathbb{S}$: $A_s \neq \emptyset$.
Let $(\alpha, A, A\Delta)$ and $(\beta, B, B\Delta)$ be Δ-co-algebras.
A Δ-*co-homomorphism from* $(\alpha, A, A\Delta)$ *to* $(\beta, B, B\Delta)$ is a morphism $h{::}A{\to}B$ in $\mathrm{SORT}(\mathbb{S})$ such that h is compatible with Δ in the sense that $\alpha\ {}_{\mathrm{SORT}(\mathbb{S})}\!\diamond\ {}_h\Delta = h\ {}_{\mathrm{SORT}(\mathbb{S})}\!\diamond\ \beta$.

$$\begin{tikzcd} A \arrow[r, "h"'] \arrow[d, "\alpha"] \arrow[dr, phantom, "="] & B \arrow[d, "\beta"] \\ A\Delta \arrow[r, "{}_h\Delta"'] & B\Delta \end{tikzcd}$$

Note that for Δ defined from $(\mathbb{S}, \mathbb{F}, \mathrm{rt})$ as above, the latter condition means that for $s \in \mathbb{S}$; $f \in \mathbb{F}_s$; $a \in A_s$; and

$$\text{for } (s', a') := \alpha_s(a)_f\text{:} \qquad \Big(\ s',\ \ h_{s'}(a')\ \Big) \quad = \quad \beta_s\Big(\ h_s(a)\ \Big)_f,$$

i.e.

$$\text{for } (s', a') := (s,f)^{\mathcal{A}}(a)\text{:} \qquad \Big(\ s',\ \ h_{s'}(a')\ \Big) \quad = \quad (s,f)^{\mathcal{B}}\Big(\ h_s(a)\ \Big).$$

Definition 6.6 (COALG)

The Δ-*co-homomorphism category* COALG *of* Δ-*co-algebras* is given as follows:

${}_{\mathrm{COALG}}\mathrm{Obj}$ is the class of all Δ-co-algebras.

${}_{\mathrm{COALG}}\mathrm{Mor}_{(\alpha,A,A\Delta),(\beta,B,B\Delta)}$ is the class of all Δ-co-homomorphisms $h{::}A{\to}B$.

$h\ {}_{\mathrm{COALG}}\!\diamond\ k := h\ {}_{\mathrm{SORT}(\mathbb{S})}\!\diamond\ k$
for $h \in {}_{\mathrm{COALG}}\mathrm{Mor}_{(\alpha,A,A\Delta),(\beta,B,B\Delta)}$ and $h \in {}_{\mathrm{COALG}}\mathrm{Mor}_{(\beta,B,B\Delta),(\gamma,C,C\Delta)}$.

${}_{\mathrm{COALG}}\mathrm{Id}_{(\alpha,A,A\Delta)} := {}_{\mathrm{SORT}(\mathbb{S})}\mathrm{Id}_A$.

Note that, while the morphisms of COALG are also morphisms of $\mathrm{SORT}(\mathbb{S})$ and their composition is the same, COALG is no sub-category of $\mathrm{SORT}(\mathbb{S})$ because the objects of COALG are no *objects* of $\mathrm{SORT}(\mathbb{S})$.

[11] Cf. Ex. 2.9.

7 Di-Algebras

Definition 7.1 (Di-Signature)

A *di-signature* $(\mathbb{S}, \mathbb{F}, \mathrm{rt})$ consists of

- a set of sorts $\mathbb{S}$ and
- an $\mathbb{S}$-sorted family $\mathbb{F} = (\mathbb{F}_s)_{s \in \mathbb{S}^*}$ of sets $\mathbb{F}_s$ of function symbols with argument sort s, and
- a result-type function $\mathrm{rt}\colon \coprod_{s \in \mathbb{S}^*} \mathbb{F}_s \to \mathfrak{P}_{\mathbb{N}_+}(\mathbb{S})$.

Example 7.2 (Di-Signature)

Set $\mathbb{S} := \{1, \mathsf{zero}, \mathsf{posnat}, \mathsf{emptylist}, \mathsf{nonemptylist}\}$.

Note that 1 is included in $\mathbb{S}$ in order to model the range of "undefined" function applications.

Set $\mathbb{F}_{\emptyset} := \{\mathsf{0}, \mathsf{nil}\}$; $\mathbb{F}_{\mathsf{zero}} := \{\mathsf{s}\}$; $\mathbb{F}_{\mathsf{posnat}} := \{\mathsf{s}, \mathsf{p}\}$;
$\mathbb{F}_{\mathsf{emptylist}} := \{\mathsf{length}\}$; $\mathbb{F}_{\mathsf{nonemptylist}} := \{\mathsf{length}, \mathsf{car}, \mathsf{cdr}\}$;
$\mathbb{F}_{\mathsf{zero\,zero}} := \mathbb{F}_{\mathsf{zero\,posnat}} := \mathbb{F}_{\mathsf{posnat\,posnat}} := \{+, -\}$;
$\mathbb{F}_{\mathsf{posnat\,zero}} :=$
$\mathbb{F}_{\mathsf{emptylist\,emptylist}} := \mathbb{F}_{\mathsf{emptylist\,nonemptylist}} := \mathbb{F}_{\mathsf{nonemptylist\,emptylist}} := \mathbb{F}_{\mathsf{nonemptylist\,nonemptylist}} := \{+\}$;
$\mathbb{F}_{\mathsf{zero\,emptylist}} := \mathbb{F}_{\mathsf{zero\,nonemptylist}} := \mathbb{F}_{\mathsf{posnat\,emptylist}} := \mathbb{F}_{\mathsf{posnat\,nonemptylist}} := \{\mathsf{cons}\}$; and
$\mathbb{F}_s := \emptyset$ for any other $s \in \mathbb{S}^*$.

Set $\mathrm{rt}(\emptyset, \mathsf{0}) := \mathrm{rt}(\mathsf{emptylist}, \mathsf{length}) := \mathrm{rt}(\mathsf{zero\,zero}, +) := \mathrm{rt}(\mathsf{zero\,zero}, -) := \{\mathsf{zero}\}$;
$\mathrm{rt}(\mathsf{zero}, \mathsf{s}) := \mathrm{rt}(\mathsf{posnat}, \mathsf{s}) := \mathrm{rt}(\mathsf{nonemptylist}, \mathsf{length}) := \mathrm{rt}(\mathsf{zero\,posnat}, +) :=$
$\mathrm{rt}(\mathsf{posnat\,zero}, +) := \mathrm{rt}(\mathsf{posnat\,posnat}, +) := \mathrm{rt}(\mathsf{posnat\,zero}, -) := \{\mathsf{posnat}\}$;
$\mathrm{rt}(\mathsf{posnat}, \mathsf{p}) := \mathrm{rt}(\mathsf{nonemptylist}, \mathsf{car}) := \{\mathsf{zero}, \mathsf{posnat}\}$;
$\mathrm{rt}(\mathsf{posnat\,posnat}, -) := \{\mathsf{zero}, \mathsf{posnat}, 1\}$;
$\mathrm{rt}(\emptyset, \mathsf{nil}) := \mathrm{rt}(\mathsf{emptylist\,emptylist}, +) := \{\mathsf{emptylist}\}$;
$\mathrm{rt}(\mathsf{emptylist\,nonemptylist}, +) := \mathrm{rt}(\mathsf{nonemptylist\,emptylist}, +) :=$
$\mathrm{rt}(\mathsf{nonemptylist\,nonemptylist}, +) := \mathrm{rt}(\mathsf{zero\,emptylist}, \mathsf{cons}) :=$
$\mathrm{rt}(\mathsf{zero\,nonemptylist}, \mathsf{cons}) := \mathrm{rt}(\mathsf{posnat\,emptylist}, \mathsf{cons}) :=$
$\mathrm{rt}(\mathsf{posnat\,nonemptylist}, \mathsf{cons}) := \{\mathsf{nonemptylist}\}$; and
$\mathrm{rt}(\mathsf{nonemptylist}, \mathsf{cdr}) := \{\mathsf{emptylist}, \mathsf{nonemptylist}\}$.

Definition 7.3 (($\mathbb{S}, \mathbb{F}, \mathrm{rt}$)**-Di-Algebra**)

Let $(\mathbb{S}, \mathbb{F}, \mathrm{rt})$ be a di-signature.

A $(\mathbb{S}, \mathbb{F}, \mathrm{rt})$-*di-algebra* consists of

- a family $A = (A_s)_{s \in \mathbb{S}}$ of non-empty sets ("universes") and,
- for each $(s, f) \in \coprod_{s \in \mathbb{S}} \mathbb{F}_s$, a function

$$(s, f)^{\mathcal{A}}\colon \quad \prod_{i \prec |s|} A_{s_i} \quad \to \quad \coprod_{s' \in \mathrm{rt}(s,f)} A_{s'}.$$

Example 7.4

Let $(\mathbb{S}, \mathbb{F}, \mathrm{rt})$ be the di-signature of Ex. 7.2.

Set $A_1 := \{\omega\}$; $A_{\mathsf{zero}} := \{0\}$; $A_{\mathsf{posnat}} := \mathbb{N}_+$; $A_{\mathsf{emptylist}} := \{\emptyset\}$; $A_{\mathsf{nonemptylist}} := \mathbb{N}^+$.

For $x, y \in \mathbb{N}$ and $k, l \in \mathbb{N}^*$ define:

$(\emptyset, \mathsf{0})^{\mathcal{A}}(\omega) := (\mathsf{posnat}, \mathsf{p})^{\mathcal{A}}(1) := (\mathsf{emptylist}, \mathsf{length})^{\mathcal{A}}(\emptyset) := (\mathsf{zero\,zero}, +)^{\mathcal{A}}(0, 0) :=$
$(\mathsf{zero\,zero}, -)^{\mathcal{A}}(0, 0) := (\mathsf{posnat\,posnat}, -)^{\mathcal{A}}(x, x) := (\mathsf{nonemptylist}, \mathsf{car})^{\mathcal{A}}(0l) := (\mathsf{zero}, 0)$;
$(\emptyset, \mathsf{nil})^{\mathcal{A}}(\omega) := (\mathsf{emptylist\,emptylist}, +)^{\mathcal{A}}(\emptyset, \emptyset) := (\mathsf{nonemptylist}, \mathsf{cdr})^{\mathcal{A}}(x) := (\mathsf{emptylist}, \emptyset)$;
$(\mathsf{zero}, \mathsf{s})^{\mathcal{A}}(x) := (\mathsf{posnat}, \mathsf{s})^{\mathcal{A}}(x) := (\mathsf{posnat}, x{+}1)$;
$(\mathsf{nonemptylist}, \mathsf{length})^{\mathcal{A}}(l) := (\mathsf{posnat}, |l|)$;
$(\mathsf{zero\,posnat}, +)^{\mathcal{A}}(x, y) := (\mathsf{posnat\,zero}, +)^{\mathcal{A}}(x, y) := (\mathsf{posnat\,posnat}, +)^{\mathcal{A}}(x, y) :=$
$(\mathsf{posnat}, x{+}y)$;
$(\mathsf{posnat}, \mathsf{p})^{\mathcal{A}}(x{+}1) := (\mathsf{posnat\,zero}, -)^{\mathcal{A}}(x, 0) := (\mathsf{nonemptylist}, \mathsf{car})^{\mathcal{A}}(xl) :=$
(posnat, x) for $x \in \mathbb{N}_+$;
$(\mathsf{posnat\,posnat}, -)^{\mathcal{A}}(x, y) := (\mathsf{posnat}, x{-}y)$ for $x \succ y$;
$(\mathsf{posnat\,posnat}, -)^{\mathcal{A}}(x, y) := (\mathsf{1}, \omega)$ for $x \prec y$;
$(\mathsf{emptylist\,nonemptylist}, +)^{\mathcal{A}}(k, l) := (\mathsf{nonemptylist\,emptylist}, +)^{\mathcal{A}}(k, l) :=$
$(\mathsf{nonemptylist\,nonemptylist}, +)^{\mathcal{A}}(k, l) := (\mathsf{nonemptylist}, kl)$;
$(s, \mathsf{cons})^{\mathcal{A}}(x, l) := (\mathsf{nonemptylist}, xl)$; and
$(\mathsf{nonemptylist}, \mathsf{cdr})^{\mathcal{A}}(xl) := (\mathsf{nonemptylist}, l)$ for $l \in \mathbb{N}^+$.

Note that the di-signature of Ex. 7.2 is rather fine because the problems with the co-signature of Ex. 6.2 and the problems with the signature of Ex. 5.2 are both overcome. On the one hand, we can now add three natural numbers: The di-signature does know that the result of the first addition can be paired with a natural number in order to add it. On the other hand, $\mathsf{s}(\mathsf{p}(\mathsf{s}(\mathsf{s}(\mathsf{0}))))$ is a term w.r.t. the di-algebra of Ex. 7.4 because $\mathsf{p}(\mathsf{s}(\mathsf{s}(\mathsf{0})))$ has the sort posnat. Note, however, that in order to tell whether $\mathsf{s}(\mathsf{p}(\mathsf{s}(\mathsf{s}(\mathsf{0}))))$ really denotes something we have to know to which specific di-algebra the question refers. The di-signature must accept $\mathsf{s}(\mathsf{p}(\mathsf{s}(\mathsf{s}(\mathsf{0}))))$ as a term without knowing whether it denotes something. This problem of di-signatures already exists for co-signatures.

While the notion of a di-algebra given in Def. 7.3 is nice, it is not very well suited for category theory. The first step for coming to the categorical definition of a di-algebra is to encode $(\mathbb{S}, \mathbb{F}, \mathrm{rt})$ into the following two functors $((\Gamma_{\mathrm{Obj}}, \Gamma_{\mathrm{Mor}}), \mathrm{SORT}(\mathbb{S}), \mathrm{SORT}(\mathbb{S}^*))$ and $((\Delta_{\mathrm{Obj}}, \Delta_{\mathrm{Mor}}), \mathrm{SORT}(\mathbb{S}), \mathrm{SORT}(\mathbb{S}^*))$ from the category $\mathrm{SORT}(\mathbb{S})$ of $\mathbb{S}$-sorted sets into the category $\mathrm{SORT}(\mathbb{S}^*)$ of $\mathbb{S}^*$-sorted sets:

$$
\begin{aligned}
(A\Gamma)_s &:= \prod_{i \prec |s|} A_{s_i} \\
(A\Delta)_s &:= \coprod_{f \in \mathbb{F}_s} \prod_{s' \in \mathrm{rt}(s,f)} A_{s'} \\
({}_h\Gamma)_s\Big(\ (a_i)_{i \prec |s|}\ \Big) &:= (h_{s_i}(a_i))_{i \prec |s|} \\
({}_h\Delta)_s\Big(\ (s'_f, a'_f)_{f \in \mathbb{F}_s}\ \Big) &:= (s'_f, h_{s'_f}(a'_f))_{f \in \mathbb{F}_s}
\end{aligned}
$$

for $A \in {}_{\mathrm{SORT}(\mathbb{S})}\mathrm{Obj}$; $h{::}A \rightarrow B$ in $\mathrm{SORT}(\mathbb{S})$; $s \in \mathbb{S}^*$, $\forall i \prec |s|\colon\ a_i \in A_{s_i}$; and $\forall f \in \mathbb{F}_s$: $(s'_f \in \mathrm{rt}(s,f) \wedge a'_f \in A_{s'_f})$.

Now we can categorically represent the above di-algebra over $(\mathbb{F}, \mathbb{S}, \mathrm{rt})$ using the morphism $\alpha{::}A\Gamma \rightarrow A\Delta$ in $\mathrm{SORT}(\mathbb{S}^*)$ given by $\alpha_s(a)_f := (s,f)^{\mathcal{A}}(a)$ for $a \in A_s$ and $f \in \mathbb{F}_s$.

While the definition of a Σ-algebra is standard and the definition of a Γ-co-algebra is its dual, the definition of a (Γ,Δ)-di-algebra below is our invention.

Definition 7.5 (**(Γ,Δ)-Di-Algebras and (Γ,Δ)-Di-Homomorphisms**)
(Cf. Fokkinga (1996).)
Let Γ, Δ: $\mathrm{SORT}(\mathbb{S}) \twoheadrightarrow \mathrm{SORT}(\mathbb{S}^*)$.
A (Γ,Δ)-*di-algebra* is a pair (A, α) with $A \in {}_{\mathrm{SORT}(\mathbb{S})}\mathrm{Obj}$; $\forall s \in \mathbb{S}$: $A_s \neq \emptyset$; and $\alpha{::}A\Gamma \rightarrow A\Delta$ in $\mathrm{SORT}(\mathbb{S}^*)$.
Let (A, α) and (B, β) be (Γ,Δ)-di-algebras.
A (Γ,Δ)-*di-homomorphism from* (A, α) *to* (B, β) is a morphism $h{::}A \rightarrow B$ in $\mathrm{SORT}(\mathbb{S})$ such that h is compatible with (Γ,Δ) in the sense that $\alpha \ {}_{\mathrm{SORT}(\mathbb{S}^*)}\!\diamond\ {}_h\Delta = {}_h\Gamma \ {}_{\mathrm{SORT}(\mathbb{S}^*)}\!\diamond\ \beta$.

$$
\begin{tikzcd}
A\Gamma \arrow[r, "{}_h\Gamma"'] \arrow[d, "\alpha"] \arrow[dr, phantom, "="] & B\Gamma \arrow[d, "\beta"] \\
A\Delta \arrow[r, "{}_h\Delta"'] & B\Delta
\end{tikzcd}
$$

Note that for (Γ,Δ) defined from $(\mathbb{S}, \mathbb{F}, \mathrm{rt})$ as above, the latter condition means that for $s \in \mathbb{S}^*$; $f \in \mathbb{F}_s$; $\forall i \prec |s|\colon\ a_i \in A_{s_i}$; and

$$\text{for } (s', a') := \alpha_s\Big(\ (a_i)_{i \prec |s|}\ \Big)_f\colon \qquad \Big(\ s',\ \ h_{s'}(a')\ \Big) \quad = \quad \beta_s\Big(\ (h_{s_i}(a)_i)_{i \prec |s|}\ \Big)_f,$$

i.e.

$$\text{for } (s', a') := (s,f)^{\mathcal{A}}\Big(\ (a_i)_{i \prec |s|}\ \Big)\colon \qquad \Big(\ s',\ \ h_{s'}(a')\ \Big) \quad = \quad (s,f)^{\mathcal{B}}\Big(\ (h_{s_i}(a_i))_{i \prec |s|}\ \Big).$$

Definition 7.6 (SUALG)
The (Γ,Δ)-*di-homomorphism category* SUALG *of* (Γ,Δ)-*di-algebras* is given as follows:

> ${}_{\mathrm{SUALG}}\mathrm{Obj}$ is the class of all (Γ,Δ)-di-algebras.
>
> ${}_{\mathrm{SUALG}}\mathrm{Mor}_{(A,\alpha),(B,\beta)}$ is the class of all (Γ,Δ)-di-homomorphisms $h{::}A \rightarrow B$.
>
> $h \ {}_{\mathrm{SUALG}}\!\diamond\ k := h \ {}_{\mathrm{SORT}(\mathbb{S})}\!\diamond\ k$ for $h \in {}_{\mathrm{SUALG}}\mathrm{Mor}_{(A,\alpha),(B,\beta)}$ and $h \in {}_{\mathrm{SUALG}}\mathrm{Mor}_{(B,\beta),(C,\gamma)}$.
>
> ${}_{\mathrm{SUALG}}\mathrm{Id}_{(A,\alpha)} := {}_{\mathrm{SORT}(\mathbb{S})}\mathrm{Id}_A$.

Note that, while the morphisms of SUALG are also morphisms of $\mathrm{SORT}(\mathbb{S})$ and their composition is the same, SUALG is no sub-category of $\mathrm{SORT}(\mathbb{S})$ because the objects of SUALG are no objects of $\mathrm{SORT}(\mathbb{S})$.

References

Maarten M. Fokkinga (1996). *Datatype Lawes without Signatures.* Math. Struct. in Comp. Sci. **6**, pp. 1-32, Cambridge Univ. Press.