Symmetric monoidal category

(Redirected from Symmetric monoidal categories)

In category theory, a branch of mathematics, a **symmetric monoidal category** is a monoidal category (i.e. a category in which a "tensor product" $\otimes$ is defined) such that the tensor product is symmetric (i.e. $A \otimes B$ is, in a certain strict sense, naturally isomorphic to $B \otimes A$ for all objects A and B of the category). One of the prototypical examples of a symmetric monoidal category is the category of vector spaces over some fixed field *k*, using the ordinary tensor product of vector spaces.

Definition

A symmetric monoidal category is a monoidal category (C, $\otimes$, I) such that, for every pair A, B of objects in C, there is an isomorphism $s_{AB} : A \otimes B \to B \otimes A$ called the *swap map*[1] that is natural in both A and B and such that the following diagrams commute:

- The unit coherence:

$$\begin{tikzcd} A \otimes I \arrow[rr, "s_{AI}"] \arrow[dr, "r_A"'] & & I \otimes A \arrow[dl, "l_A"] \\ & A & \end{tikzcd}$$

- The associativity coherence:

$$\begin{tikzcd}
(A \otimes B) \otimes C \arrow[rr, "s_{AB} \otimes 1_C"] \arrow[d, "a_{ABC}"'] & & (B \otimes A) \otimes C \arrow[d, "a_{BAC}"] \\
A \otimes (B \otimes C) \arrow[d, "s_{A, B \otimes C}"'] & & B \otimes (A \otimes C) \arrow[d, "1_B \otimes s_{AC}"] \\
(B \otimes C) \otimes A \arrow[rr, "a_{BCA}"'] & & B \otimes (C \otimes A)
\end{tikzcd}$$

- The inverse law:

$$\begin{tikzcd}
& B \otimes A \arrow[dr, "s_{BA}"] & \\
A \otimes B \arrow[ur, "s_{AB}"] \arrow[rr, equal, "1_{A \otimes B}"'] & & A \otimes B
\end{tikzcd}$$

In the diagrams above, a, l, and r are the associativity isomorphism, the left unit isomorphism, and the right unit isomorphism respectively.

Examples

Some examples and non-examples of symmetric monoidal categories:

- The category of sets. The tensor product is the set theoretic cartesian product, and any singleton can be fixed as the unit object.
- The category of groups. Like before, the tensor product is just the cartesian product of groups, and the trivial group is the unit object.
- More generally, any category with finite products, that is, a cartesian monoidal category, is symmetric monoidal. The tensor product is the direct product of objects, and any terminal object (empty product) is the unit object.
- The category of bimodules over a ring *R* is monoidal (using the ordinary tensor product of modules), but not necessarily symmetric. If *R* is commutative, the category of left *R*-modules is

symmetric monoidal. The latter example class includes the category of all vector spaces over a given field.

- Given a field *k* and a group (or a Lie algebra over *k*), the category of all *k*-linear representations of the group (or of the Lie algebra) is a symmetric monoidal category. Here the standard tensor product of representations is used.
- The categories (**Ste**,⊛) and (**Ste**,⊙) of stereotype spaces over ℂ are symmetric monoidal, and moreover, (**Ste**,⊛) is a closed symmetric monoidal category with the internal hom-functor ⊘.

Properties

The classifying space (geometric realization of the nerve) of a symmetric monoidal category is an E_∞ space, so its group completion is an infinite loop space.[2]

Specializations

A dagger symmetric monoidal category is a symmetric monoidal category with a compatible dagger structure.

A cosmos is a complete cocomplete closed symmetric monoidal category.

Generalizations

In a symmetric monoidal category, the natural isomorphisms $s_{AB} : A \otimes B \to B \otimes A$ are their *own* inverses in the sense that $s_{BA} \circ s_{AB} = 1_{A \otimes B}$. If we abandon this requirement (but still require that $A \otimes B$ be naturally isomorphic to $B \otimes A$), we obtain the more general notion of a braided monoidal category.

References

1. Fong, Brendan; Spivak, David I. (2018-10-12). "Seven Sketches in Compositionality: An Invitation to Applied Category Theory". arXiv:1803.05316 (https://arxiv.org/abs/1803.05316) [math.CT (https://arxiv.org/archive/math.CT)].
2. Thomason, R.W. (1995). "Symmetric Monoidal Categories Model all Connective Spectra" (http://www.tac.mta.ca/tac/volumes/1995/n5/v1n5.pdf) (PDF). *Theory and Applications of Categories*. **1** (5): 78–118. CiteSeerX 10.1.1.501.2534 (https://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.501.2534).

- Symmetric monoidal category (https://ncatlab.org/nlab/show/symmetric+monoidal+category) at the *n*Lab
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