

Symmetric monoidal category

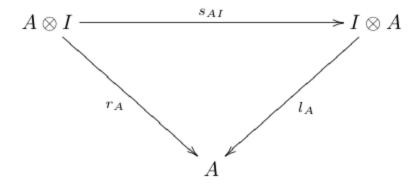
(Redirected from Symmetric monoidal categories)

In category theory, a branch of mathematics, a **symmetric monoidal category** is a monoidal category (i.e. a category in which a "tensor product" \otimes is defined) such that the tensor product is symmetric (i.e. $A \otimes B$ is, in a certain strict sense, naturally isomorphic to $B \otimes A$ for all objects A and B of the category). One of the prototypical examples of a symmetric monoidal category is the category of vector spaces over some fixed field k, using the ordinary tensor product of vector spaces.

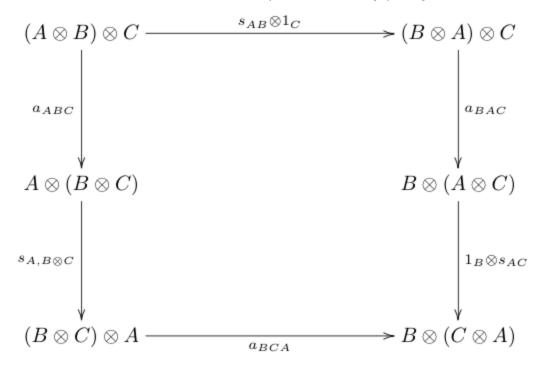
Definition

A symmetric monoidal category is a monoidal category (C, \otimes, I) such that, for every pair A, B of objects in C, there is an isomorphism $s_{AB}: A \otimes B \to B \otimes A$ called the *swap map*^[1] that is <u>natural</u> in both A and B and such that the following diagrams commute:

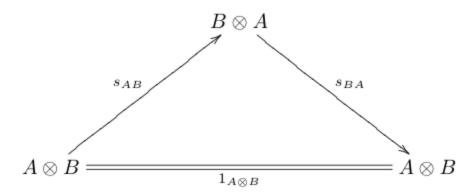
The unit coherence:



• The associativity coherence:



The inverse law:



In the diagrams above, a, l, and r are the associativity isomorphism, the left unit isomorphism, and the right unit isomorphism respectively.

Examples

Some examples and non-examples of symmetric monoidal categories:

- The category of sets. The tensor product is the set theoretic cartesian product, and any singleton can be fixed as the unit object.
- The category of groups. Like before, the tensor product is just the cartesian product of groups, and the trivial group is the unit object.
- More generally, any category with finite products, that is, a cartesian monoidal category, is symmetric monoidal. The tensor product is the direct product of objects, and any terminal object (empty product) is the unit object.
- The category of bimodules over a ring R is monoidal (using the ordinary tensor product of modules), but not necessarily symmetric. If R is commutative, the category of left R-modules is

- symmetric monoidal. The latter example class includes the category of all vector spaces over a given field.
- Given a field k and a group (or a Lie algebra over k), the category of all k-linear representations of the group (or of the Lie algebra) is a symmetric monoidal category. Here the standard tensor product of representations is used.
- The categories (Ste,*) and (Ste,⊙) of stereotype spaces over C are symmetric monoidal, and moreover, (Ste,*) is a closed symmetric monoidal category with the internal hom-functor ⊘.

Properties

The classifying space (geometric realization of the nerve) of a symmetric monoidal category is an E_{∞} space, so its group completion is an infinite loop space. [2]

Specializations

A $\underline{\text{dagger symmetric monoidal category}}$ is a symmetric monoidal category with a compatible $\underline{\text{dagger}}$ structure.

A cosmos is a complete cocomplete closed symmetric monoidal category.

Generalizations

In a symmetric monoidal category, the natural isomorphisms $s_{AB}: A \otimes B \to B \otimes A$ are their *own* inverses in the sense that $s_{BA} \circ s_{AB} = 1_{A \otimes B}$. If we abandon this requirement (but still require that $A \otimes B$ be naturally isomorphic to $B \otimes A$), we obtain the more general notion of a <u>braided monoidal</u> category.

References

- Fong, Brendan; Spivak, David I. (2018-10-12). "Seven Sketches in Compositionality: An Invitation to Applied Category Theory". arXiv:1803.05316 (https://arxiv.org/abs/1803.05316) [math.CT (https://arxiv.org/archive/math.CT)].
- 2. Thomason, R.W. (1995). "Symmetric Monoidal Categories Model all Connective Spectra" (http://www.tac.mta.ca/tac/volumes/1995/n5/v1n5.pdf) (PDF). Theory and Applications of Categories. 1 (5): 78–118. CiteSeerX 10.1.1.501.2534 (https://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.5 01.2534).
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