Exploring Binary

Seeing Powers of Five in Powers of Two and Vice Versa

By Rick Regan November 16th, 2009

The decimal representations of oppositely signed powers of two and powers of five look alike, as seen in these examples: $2^{-3} = 0.125$ and $5^3 = 125$; $5^{-5} = 0.00032$ and $2^5 = 32$. The significant digits in each pair of powers is the same, even though one is a fraction and one is an integer. In other words, a negative power of one base looks like a positive power of the other.

$$2^{-3} = 0.125 \qquad 125 = 5^3$$

$$5^{-5} = 0.00032 \qquad 32 = 2^5$$

Powers of Two and Powers of Five that Look Alike

This relationship is not coincidence; it's a by-product of how fractions are represented as decimals. I'll show you simple algebra that proves it, as well as algebra that proves similar properties — in products involving negative powers.

Positive Powers of Five in Negative Powers of Two

A negative power of two in decimal form looks like a positive power of five, except that it has a decimal point and possibly some leading zeros. Here are some examples, the first eight pairs of positive powers of five and negative powers of two:

n	5^n	2^{-n}
1	5	0.5
2	25	0.25
3	125	0.125
4	625	0.0625
5	3125	0.03125
6	15625	0.015625
7	78125	0.0078125

8	390625	0.00390625

The powers that are paired have exponents that are *additive inverses* of each other. It's easy to show this relationship holds for all such pairs of powers. A negative power of two is a number of the form 2^n, n and integer less than 0, or equivalently, a number of the form 2^{-n}, n > 0. By the laws of exponents, $2^{-n} = 1/2^n = (1/2)^n = (5/10)^n = \mathbf{5^n/10^n}$. Written as a decimal, $5^n/10^n$ looks like a positive power of five, except that it's a fraction with n decimal places.

A different transformation gives another way to look at this: $2^{-n} = 1/2^n = (1/2)^n = (0.5)^n$. If you were to multiply this out by hand, you would ignore the decimal point, multiply all the factors of five, and then place the decimal point when you were done. You'd end up with a positive power of five, but preceded with a decimal point and zero or more leading 0s — to pad it out to n decimal places.

Positive Powers of Two in Negative Powers of Five

Not surprisingly, a similar relationship exists in reverse; that is, between the positive powers of two and the negative powers of five. This makes sense, given the role of the numbers two and five in the decimal system.

A negative power of five in decimal form looks like a positive power of two, except that it has a decimal point and possibly some leading zeros. Here are some examples, the first eight pairs of positive powers of two and negative powers of five:

n	2^n	5^{-n}
1	2	0.2
2	4	0.04
3	8	0.008
4	16	0.0016
5	32	0.00032
6	64	0.000064
7	128	0.0000128
8	256	0.00000256

Again, the powers that are paired have exponents that are additive inverses of each other. Using a similar transformation as above, we can show this relationship holds for all such pairs of powers: $5^{-n} = 1/5^n = (1/5)^n = (2/10)^n = \mathbf{2^n/10^n}$. Written as a decimal, $2^n/10^n$ looks like a positive power of two, except that it's a fraction with n decimal places.

There's also an alternate way to view this: $5^{-n} = 1/5^n = (1/5)^n = (0.2)^n$. As above, think of it as multiplying the factors to get a power of two and then prefixing a decimal point and leading zeros.

Products of Negative Powers of Two and Positive Powers of Five

Because a negative power of two looks like a positive power of five, multiplying it by a positive power of five makes it look like yet another positive power of five. The resulting power of five has an exponent that is the sum of the absolute values of the exponents of the multiplied powers. For example: $2^{-3} \cdot 5^4 = 78.125$, which looks like $5^7 = 78125$; $2^{-7} \cdot 5^3 = 0.9765625$, which looks like $5^{10} = 9765625$.

Algebraically, the multiplication is expressed as $2^{-t} \cdot 5^f$, with t, f > 0. The result looks like $5^{(t+f)}$; here's why: $2^{-t} = 5^t/10^t$, so $2^{-t} \cdot 5^f = \mathbf{5^{(t+f)}/10^t}$. This expression describes a positive power of five, only shifted right t decimal places.

You can think of these as looking like negative powers of two as well. The product $2^{-t} \cdot 5^f$ can be written equivalently as $2^{-t} \cdot 5^f \cdot (2^{-f} \cdot 2^f) = \mathbf{2^{-(t+f)} \cdot 10^f}$, which is $2^{-(t+f)}$ shifted left f decimal places.

Products of Negative Powers of Five and Positive Powers of Two

Similarly, the product of a negative power of five and a positive power of two looks like a positive power of two. For example, $5^{-3} \cdot 2^3 = 0.064$, which looks like $2^6 = 64$, and $5^{-4} \cdot 2^{12} = 6.5536$, which looks like $2^{16} = 65536$.

Algebraically, the multiplication is expressed as $5^{-f} \cdot 2^t$, with f, t > 0. The result looks like $2^{(f+t)}$; here's why: $5^{-f} = 2^f/10^f$, so $5^{-f} \cdot 2^t = \mathbf{2^{(f+t)}/10^f}$. This expression describes a positive power of two, only shifted right f decimal places.

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Products of Negative Powers of Two and Negative Powers of Five

A negative power of two times a negative power of five results in a string of digits that looks like *either* a positive power of five or a positive power of two, depending on which exponent is smaller. For example, $2^{-4}\cdot5^{-2}$ = 0.0025, which looks like a positive power of five, and $2^{-2}\cdot5^{-4}$ = 0.0004, which looks like a positive power of two.

Algebraically, the multiplication is expressed as $2^{-t}\cdot5^{-f}$, with t, f > 0, and f ≠ t. There are two cases:

- **f < t**: the result looks like $5^{(t-f)}$, with t decimal places. Here's why:
  Recall from above that $2^{-t} = 5^{t}/10^{t}$ and that $5^{-f} = 2^{f}/10^{f}$. This gives $2^{-t}\cdot5^{-f} = (5^{t}/10^{t})(2^{f}/10^{f})$ $= (5^{t}\cdot2^{f})/10^{(t+f)}$. Now factor out 10^{f} from the numerator, which we can do since f < t: $(5^{t}\cdot2^{f})/10^{(t+f)} = (10^{f}\cdot5^{(t-f)})/10^{(t+f)} = \mathbf{5^{(t-f)}/10^{t}}$.

- **t < f**: the result looks like $2^{(f-t)}$, with f decimal places. Here's why:
  Starting from $(5^{t}\cdot2^{f})/10^{(t+f)}$, factor out 10^{t} from the numerator, which we can do since t < f: $(5^{t}\cdot2^{f})/10^{(t+f)} = (10^{t}\cdot2^{(f-t)})/10^{(t+f)} = \mathbf{2^{(f-t)}/10^{f}}$.

(If you allow f = t, the answer would be $2^{0} = 5^{0} = 1$, a nonnegative power, with f = t decimal places.)

Discussion

I came up with the algebra for the multiplication cases *after* I played around with some numbers in [PARI/GP](). I noticed a pattern, and then sought to state it mathematically.

The patterns hold for quotients of powers as well, since divisions can be converted to multiplications by inverting exponents.

Some Practical Uses

If you know the positive powers of five, or if you learn them, you will recognize negative powers of two more readily — similarly for the positive powers of two and negative powers of five.

If you wanted to print a table of negative powers of two or negative powers of five using a computer program, you could do so easily, just by using integers: convert the positive powers to strings, and then prefix each with a decimal point and the appropriate number of leading zeros.

References

- learner.org article "Fractions to Decimals".

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2 comments

1. **Niranjan**

   July 4, 2018 at 10:59 pm

   Very nice!! Does this logic hold good in binary representation also?

2. **Rick Regan**

   July 7, 2018 at 9:29 pm

@Niranjan,

No. One way to show it is that $2^{(+/-)i}$ always has only one 1 bit in binary; $5^{(+/-)i}$ always has more than one 1 bit.

There would be similar patterns in other bases though; for example, base 6, with factors 2 and 3: $(2^{15})_6$ = 411412, and $(3^{-15})_6$ = 0.000000000411412.

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