Exploring Binary

Converting a Bicimal to a Fraction (Subtraction Method)

By Rick Regan June 30th, 2012

In my article “Binary Division” I showed how binary long division converts a fraction to a repeating bicimal. In this article, I’ll show you a well-known procedure — what I call the *subtraction method* — to do the reverse: convert a repeating bicimal to a fraction.

$$\frac{101111}{1100} \rightleftarrows 11.11\overline{10}$$

Equivalent Representations of 47/12, in Binary

Bicimals

A bicimal is the base-two analog of a decimal; it has a bicimal point and bicimal places, and can be terminating or repeating.

A **terminating** bicimal has a finite number of bicimal places; a **repeating** bicimal (also known as a recurring bicimal or periodic bicimal) has an infinite number of bicimal places, due to a sequence of digits that repeat forever. There are two types of repeating bicimals: **pure repeating** (also known as immediate repeating or strictly repeating) and **mixed repeating** (also known as delayed repeating). In a pure repeating bicimal, the repeating part (also known as the *repetend*) starts immediately after the bicimal point. In a mixed repeating bicimal, a non-repeating part starts immediately after the bicimal point; it is then followed by a repeating part.

Here are some examples:

- 0.1101 (a terminating bicimal).
- $101.\overline{01}$ (a pure repeating bicimal).
- $11.11\overline{10}$ (a mixed repeating bicimal).

Like decimals, bicimals are created from fractions through long division. Also like decimals, bicimals can be converted *back* to fractions. You convert a bicimal to a fraction the same way you convert a decimal to a fraction — you just work in binary instead of decimal, and use powers of two instead of powers of ten.

A terminating bicimal is easy to convert to a fraction: the numerator of the resulting fraction is the bicimal itself, treated as an integer; the denominator is 2^n, where *n* is the number of bicimal places. For example, 0.1101 = $1101/2^4$ = 1101/10000 (which in decimal equals 13/16 = 0.8125). (Notice how I mixed decimal and binary numerals. Although potentially confusing, it is common practice, and actually makes things easier. I will be doing this throughout the article.)

Repeating bicimals take more work to convert. There are several methods; I will describe the *subtraction method.*

Subtraction Method

In the subtraction method, you take a bicimal *b* and create an expression that subtracts out its repeating part, and then rewrite that expression so that *b* is expressed as a fraction. Specifically, you create two nonnegative power of two multiples of *b*, *mb* and *nb*, such that *mb* − *nb* = *i*, where *i* is an integer. Using simple algebra, you rewrite this expression as a fraction: *b(m-n)* = *i*, or *b* = *i/(m-n)*.

Here is a straightforward way to pick *m* and *n*. Let *p* be the length of the non-repeating part, and let *r* be the length of one cycle of the repeating part. Choose *m* = 2^{p+r}, and choose *n* = 2^p. These values of *m* and *n* shift *b* left by differing numbers of places, creating new bicimals with identical fractional parts but different integer parts. (For pure repeating bicimals, *p* will be 0 and thus *n* will be 1, so *nb* = *b*; that is, it's not shifted.)

Examples

Example 1: $101.\overline{01}$ = 10000/11

Consider the pure repeating bicimal *b* = $101.\overline{01}$. Picking *m* = 2^2 = 4 shifts *b* left by two places, giving $10101.\overline{01}$. Now if we subtract *b* from that, we will remove the fractional part (leaving *b* as is means we've picked $n = 2^0 = 1$). Let's show the subtraction algebraically, in mixed decimal and binary numerals:

$mb - nb$

$4b - b = 10101.\overline{01} - 101.\overline{01}$

$3b = 10000$

$11b = 10000$

$b = 10000/11$

which in decimal equals $16/3 = 5.\overline{3}$.

Example 2: $11.11\overline{10} = 101111/1100$

Consider the mixed repeating bicimal $b = 11.11\overline{10}$. Picking $m = 2^4 = 16$ will shift b left by four places — pulling out the non-repeating part and one cycle of the repeating part — giving $111110.\overline{10}$. Picking $n = 2^2 = 4$ will shift a copy of b left by two places — pulling out just the non-repeating part — giving $1111.\overline{10}$. Now let's do the algebra:

$mb - nb$

$16b - 4b = 111110.\overline{10} - 1111.\overline{10}$

$12b = 101111$

$1100b = 101111$

$b = 101111/1100$

which in decimal equals $47/12 = 3.91\overline{6}$.

Examples Redone Using Only Binary Numerals

You can perform the subtraction method using only binary numerals (thanks James for the comment); I will redo the two examples in this way.

Example 1: $101.\overline{01}$ = 10000/11

$mb - nb$

$100b - b = 10101.\overline{01} - 101.\overline{01}$

$11b = 10000$

$b = 10000/11$

Example 2: $11.11\overline{10}$ = 101111/1100

$mb - nb$

$10000b - 100b = 111110.\overline{10} - 1111.\overline{10}$

$1100b = 101111$

$b = 101111/1100$

A Way to Avoid Binary Arithmetic on the Left Hand Side

The binary subtraction on the left hand side of the equation can be avoided; let's manipulate the left hand side algebraically (remember that $m = 2^{p+r}$ and $n = 2^p$):

$mb - nb$

$2^{p+r}b - 2^pb$

$2^p(2^rb - b)$

$2^pb(2^r - 1)$

$(2^r - 1)2^p b$

You don't need to do any arithmetic to compute $(2^r - 1)2^p$: it is a string of r 1s followed by p 0s. For *example 1*, $r = 2$ and $p = 0$ gives 11; for *example 2*, $r = 2$ and $p = 2$ gives 1100.

On Going Directly to a Fraction Written in Binary

If you look around the Web (for example, Wikipedia), you'll see this procedure is typically used to convert a bicimal straight to a fraction written in *decimal* numerals. For *example 2*, the steps would be:

$mb - nb$

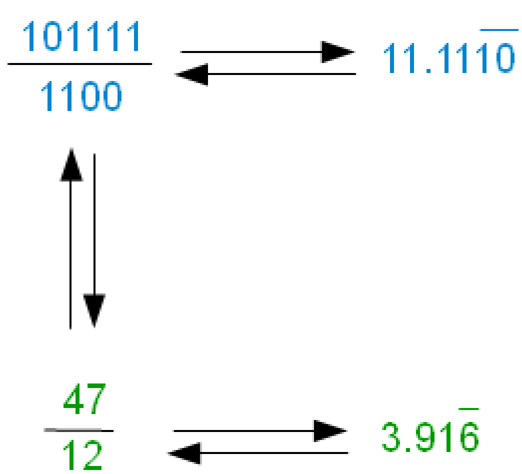
$16b - 4b = 111110.\overline{10} - 1111.\overline{10}$

$12b = 47$

$b = 47/12$

which you'd then say is 101111/1100 in binary.

Conceptually, I see the process as bicimal -> fraction written in binary -> fraction written in decimal -> decimal, which is why I prefer my presentation.



Equivalent Representations of 47/12, in Binary and Decimal

Other Methods

Read my articles about the direct method and the series method — two other ways to convert a bicimal to a fraction.

EB

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2 comments

1. **James**

   July 1, 2012 at 6:59 am

   Good method! Am I right in saying that this should work in all bases?

   Decimal:
   3.4'83 (apostrophe signifies start of repetend)
   md – nd
   1000d – 10d = 3483.'83 – 34.'83
   990d = 3449
   d = 3449/990
   Correct!

   Dozenal/uncial (0123456789XƐ):
   3.4'83
   mu – nu
   1000u – 10u = 3483.'83 – 34.'83
   ƐƐ0u = 344Ɛ
   u = 344Ɛ/ƐƐ0
   Correct!

   Hexadecimal (0123456789ABCDEF):
   3.4'83
   mx – nx
   1000x – 10x = 3483.'83 – 34.'83
   FF0x = 344F
   x = 344F/FF0
   Correct!

   (Checked on W|A).

2. **Rick Regan**

July 1, 2012 at 10:01 am

@James,

Yes, it's good for any base (of course you must have the means — like Wolfram Alpha — to do arithmetic and base conversion in that base).

You do the method slightly differently than I do. You do the left hand side arithmetic in the non-decimal base. For example, here is your base 12 example (copied here in my formatting):

mu – *nu*
$1000u - 10u = 3483.\overline{83} - 34.\overline{83}$
ƐƐ0*u* = 344Ɛ
u = 344Ɛ/ƐƐ0

I would have done it this way (I'm using $12^3 = 1728$ and $12^1 = 12$):

mu – *nu*
$1728u - 12u = 3483.\overline{83} - 34.\overline{83}$
1716*u* = 344Ɛ
ƐƐ0*u* = 344Ɛ
u = 344Ɛ/ƐƐ0

I think I like your way better: all numerals are in one base, and the required left shifting is more apparent.

Your examples also highlight the form the denominators take: decimal, 9s followed by 0s; dozenal, Ɛs followed by 0s; hexadecimal, Fs followed by 0s. I will talk about this more in my next article, which will discuss what I call the ~~denominator method~~ direct method.

As usual, thanks for the comment.

Comments are closed.