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Convex series

In mathematics, particularly in functional analysis and convex analysis, a *convex series* is a series of the form $\sum_{i=1}^{\infty} r_i x_i$ where $x_1, x_2, \ldots$ are all elements of a topological vector space X, and all $r_1, r_2, \ldots$ are non-negative real numbers that sum to 1 (that is, such that $\sum_{i=1}^{\infty} r_i = 1$).

Contents

Types of Convex series

Suppose that S is a subset of X and $\sum_{i=1}^{\infty} r_i x_i$ is a convex series in X.

- If all $x_1, x_2, \ldots$ belong to S then the convex series $\sum_{i=1}^{\infty} r_i x_i$ is called a **convex series with elements of** S.
- If the set $\{x_1, x_2, \ldots\}$ is a (von Neumann) bounded set then the series called a **b-convex series**.
- The convex series $\sum_{i=1}^{\infty} r_i x_i$ is said to be a **convergent series** if the sequence of partial sums $\left(\sum_{i=1}^{n} r_i x_i\right)_{n=1}^{\infty}$ converges in X to some element of X, which is called the **sum of the convex series**.

- The convex series is called **Cauchy** if $\sum_{i=1}^{\infty} r_i x_i$ is a Cauchy series, which by definition means that the sequence of partial sums $\left(\sum_{i=1}^{n} r_i x_i\right)_{n=1}^{\infty}$ is a Cauchy sequence.

Types of subsets

Convex series allow for the definition of special types of subsets that are well-behaved and useful with very good stability properties.

If S is a subset of a topological vector space X then S is said to be a:

- **cs-closed set** if any convergent convex series with elements of S has its (each) sum in S.
  - In this definition, X is *not* required to be Hausdorff, in which case the sum may not be unique. In any such case we require that every sum belong to S.
- **lower cs-closed set** or a **lcs-closed set** if there exists a Fréchet space Y such that S is equal to the projection onto X (via the canonical projection) of some cs-closed subset B of $X \times Y$ Every cs-closed set is lower cs-closed and every lower cs-closed set is lower ideally convex and convex (the converses are not true in general).
- **ideally convex set** if any convergent b-series with elements of S has its sum in S.
- **lower ideally convex set** or a **li-convex set** if there exists a Fréchet space Y such that S is equal to the projection onto X (via the canonical projection) of some ideally convex subset B of $X \times Y$. Every ideally convex set is lower ideally convex. Every lower ideally convex set is convex but the converse is in general not true.
- **cs-complete set** if any Cauchy convex series with elements of S is convergent and its sum is in S.
- **bcs-complete set** if any Cauchy b-convex series with elements of S is convergent and its sum is in S.

The empty set is convex, ideally convex, bcs-complete, cs-complete, and cs-closed.

Conditions (Hx) and (Hwx)

If X and Y are topological vector spaces, A is a subset of $X \times Y$, and $x \in X$ then A is said to satisfy:[1]

- **Condition (H*x*)**: Whenever $\sum_{i=1}^{\infty} r_i (x_i, y_i)$ is a *convex series* with elements of A such that $\sum_{i=1}^{\infty} r_i y_i$ is convergent in Y with sum y and $\sum_{i=1}^{\infty} r_i x_i$ is Cauchy, then $\sum_{i=1}^{\infty} r_i x_i$ is convergent in X and its sum x is such that $(x, y) \in A$.

- **Condition (Hw*x*)**: Whenever $\sum_{i=1}^{\infty} r_i(x_i, y_i)$ is a *b-convex series* with elements of A such that $\sum_{i=1}^{\infty} r_i y_i$ is convergent in Y with sum y and $\sum_{i=1}^{\infty} r_i x_i$ is Cauchy, then $\sum_{i=1}^{\infty} r_i x_i$ is convergent in X and its sum x is such that $(x, y) \in A$.
  - If X is locally convex then the statement "and $\sum_{i=1}^{\infty} r_i x_i$ is Cauchy" may be removed from the definition of condition (Hw*x*).

Multifunctions

The following notation and notions are used, where $\mathcal{R} : X \rightrightarrows Y$ and $\mathcal{S} : Y \rightrightarrows Z$ are multifunctions and $S \subseteq X$ is a non-empty subset of a topological vector space X :

- The **graph of a multifunction of** $\mathcal{R}$ is the set $\operatorname{gr} \mathcal{R} := \{(x, y) \in X \times Y : y \in \mathcal{R}(x)\}$.
- $\mathcal{R}$ is **closed** (respectively, **cs-closed**, **lower cs-closed**, **convex**, **ideally convex**, **lower ideally convex**, **cs-complete**, **bcs-complete**) if the same is true of the graph of $\mathcal{R}$ in $X \times Y$.
  - The mulifunction $\mathcal{R}$ is convex if and only if for all $x_0, x_1 \in X$ and all $r \in [0, 1]$, $r\mathcal{R}(x_0) + (1 - r)\mathcal{R}(x_1) \subseteq \mathcal{R}(rx_0 + (1 - r)x_1)$.
- The **inverse of a multifunction** $\mathcal{R}$ is the multifunction $\mathcal{R}^{-1} : Y \rightrightarrows X$ defined by $\mathcal{R}^{-1}(y) := \{x \in X : y \in \mathcal{R}(x)\}$. For any subset $B \subseteq Y$, $\mathcal{R}^{-1}(B) := \cup_{y \in B} \mathcal{R}^{-1}(y)$.
- The **domain of a multifunction** $\mathcal{R}$ is $\operatorname{Dom} \mathcal{R} := \{x \in X : \mathcal{R}(x) \neq \emptyset\}$.
- The **image of a multifunction** $\mathcal{R}$ is $\operatorname{Im} \mathcal{R} := \cup_{x \in X} \mathcal{R}(x)$. For any subset $A \subseteq X$, $\mathcal{R}(A) := \cup_{x \in A} \mathcal{R}(x)$.
- The composition $\mathcal{S} \circ \mathcal{R} : X \rightrightarrows Z$ is defined by $(\mathcal{S} \circ \mathcal{R})(x) := \cup_{y \in \mathcal{R}(x)} \mathcal{S}(y)$ for each $x \in X$.

Relationships

Let X, Y, and Z be topological vector spaces, $S \subseteq X, T \subseteq Y$, and $A \subseteq X \times Y$. The following implications hold:

> complete $\implies$ cs-complete $\implies$ cs-closed $\implies$ lower cs-closed (lcs-closed) *and* ideally convex.
> lower cs-closed (lcs-closed) *or* ideally convex $\implies$ lower ideally convex (li-convex) $\implies$ convex.
> (H*x*) $\implies$ (Hw*x*) $\implies$ convex.

The converse implications do not hold in general.

If X is complete then,

1. S is cs-complete (respectively, bcs-complete) if and only if S is cs-closed (respectively, ideally convex).
2. A satisfies (H*x*) if and only if A is cs-closed.

3. A satisfies (Hw*x*) if and only if A is ideally convex.

If Y is complete then,

1. A satisfies (H*x*) if and only if A is cs-complete.
2. A satisfies (Hw*x*) if and only if A is bcs-complete.
3. If $B \subseteq X \times Y \times Z$ and $y \in Y$ then:
    1. B satisfies (H*(x, y)*) if and only if B satisfies (H*x*).
    2. B satisfies (Hw*(x, y)*) if and only if B satisfies (Hw*x*).

If X is locally convex and $\operatorname{Pr}_X(A)$ is bounded then,

1. If A satisfies (H*x*) then $\operatorname{Pr}_X(A)$ is cs-closed.
2. If A satisfies (Hw*x*) then $\operatorname{Pr}_X(A)$ is ideally convex.

Preserved properties

Let X_0 be a linear subspace of X. Let $\mathcal{R} : X \rightrightarrows Y$ and $\mathcal{S} : Y \rightrightarrows Z$ be multifunctions.

- If S is a cs-closed (resp. ideally convex) subset of X then $X_0 \cap S$ is also a cs-closed (resp. ideally convex) subset of X_0.
- If X is first countable then X_0 is cs-closed (resp. cs-complete) if and only if X_0 is closed (resp. complete); moreover, if X is locally convex then X_0 is closed if and only if X_0 is ideally convex.
- $S \times T$ is cs-closed (resp. cs-complete, ideally convex, bcs-complete) in $X \times Y$ if and only if the same is true of both S in X and of T in Y.
- The properties of being cs-closed, lower cs-closed, ideally convex, lower ideally convex, cs-complete, and bcs-complete are all preserved under isomorphisms of topological vector spaces.
- The intersection of arbitrarily many cs-closed (resp. ideally convex) subsets of X has the same property.
- The Cartesian product of cs-closed (resp. ideally convex) subsets of arbitrarily many topological vector spaces has that same property (in the product space endowed with the product topology).
- The intersection of countably many lower ideally convex (resp. lower cs-closed) subsets of X has the same property.
- The Cartesian product of lower ideally convex (resp. lower cs-closed) subsets of countably many topological vector spaces has that same property (in the product space endowed with the product topology).
- Suppose X is a Fréchet space and the A and B are subsets. If A and B are lower ideally convex (resp. lower cs-closed) then so is $A + B$.
- Suppose X is a Fréchet space and A is a subset of X. If A and $\mathcal{R} : X \rightrightarrows Y$ are lower ideally convex (resp. lower cs-closed) then so is $\mathcal{R}(A)$.
- Suppose Y is a Fréchet space and $\mathcal{R}_2 : X \rightrightarrows Y$ is a multifunction. If $\mathcal{R}, \mathcal{R}_2, \mathcal{S}$ are all lower ideally convex (resp. lower cs-closed) then so are $\mathcal{R} + \mathcal{R}_2 : X \rightrightarrows Y$ and $\mathcal{S} \circ \mathcal{R} : X \rightrightarrows Z$.

Properties

If S be a non-empty convex subset of a topological vector space X then,

1. If S is closed or open then S is cs-closed.
2. If X is Hausdorff and finite dimensional then S is cs-closed.
3. If X is first countable and S is ideally convex then $\operatorname{int} S = \operatorname{int}(\operatorname{cl} S)$.

Let X be a Fréchet space, Y be a topological vector spaces, $A \subseteq X \times Y$, and $\operatorname{Pr}_Y : X \times Y \to Y$ be the canonical projection. If A is lower ideally convex (resp. lower cs-closed) then the same is true of $\operatorname{Pr}_Y(A)$.

If X is a barreled first countable space and if $C \subseteq X$ then:

1. If C is lower ideally convex then $C^i = \operatorname{int} C$, where $C^i := \operatorname{aint}_X C$ denotes the algebraic interior of C in X.
2. If C is ideally convex then $C^i = \operatorname{int} C = \operatorname{int}(\operatorname{cl} C) = (\operatorname{cl} C)^i$.

See also

- Ursescu theorem – Generalization of closed graph, open mapping, and uniform boundedness theorem

Notes

1. Zălinescu 2002, pp. 1–23.

References

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