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Pseudoconvexity

In mathematics, more precisely in the theory of functions of several complex variables, a **pseudoconvex set** is a special type of open set in the n-dimensional complex space $\mathbf{C}^n$. Pseudoconvex sets are important, as they allow for classification of domains of holomorphy.

Let

$$G \subset \mathbb{C}^n$$

be a domain, that is, an open connected subset. One says that G is *pseudoconvex* (or *Hartogs pseudoconvex*) if there exists a continuous plurisubharmonic function φ on G such that the set

$$\{z \in G \mid \varphi(z) < x\}$$

is a relatively compact subset of G for all real numbers x. In other words, a domain is pseudoconvex if G has a continuous plurisubharmonic exhaustion function. Every (geometrically) convex set is pseudoconvex. However, there are pseudoconvex domains which are not geometrically convex.

When G has a C^2 (twice continuously differentiable) boundary, this notion is the same as Levi pseudoconvexity, which is easier to work with. More specifically, with a C^2 boundary, it can be shown that G has a defining function; i.e., that there exists $\rho : \mathbb{C}^n \to \mathbb{R}$ which is C^2 so that $G = \{\rho < 0\}$, and $\partial G = \{\rho = 0\}$. Now, G is pseudoconvex iff for every $p \in \partial G$ and w in the complex tangent space at p, that is,

$$\nabla \rho(p) w = \sum_{i=1}^{n} \frac{\partial \rho(p)}{\partial z_j} w_j = 0, \text{ we have}$$

$$\sum_{i,j=1}^{n} \frac{\partial^2 \rho(p)}{\partial z_i \partial \bar{z}_j} w_i \bar{w}_j \geq 0.$$

If G does not have a C^2 boundary, the following approximation result can be useful.

Proposition 1 *If G is pseudoconvex, then there exist bounded, strongly Levi pseudoconvex domains $G_k \subset G$ with C^∞ (smooth) boundary which are relatively compact in G, such that*

$$G = \bigcup_{k=1}^{\infty} G_k.$$

This is because once we have a φ as in the definition we can actually find a C^∞ exhaustion function.

Contents

The case n = 1

In one complex dimension, every open domain is pseudoconvex. The concept of pseudoconvexity is thus more useful in dimensions higher than 1.

See also

- Holomorphically convex hull
- Stein manifold
- Analytic polyhedron
- Eugenio Elia Levi

References

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- Lars Hörmander, *An Introduction to Complex Analysis in Several Variables*, North-Holland, 1990. (ISBN 0-444-88446-7).
- Steven G. Krantz. *Function Theory of Several Complex Variables*, AMS Chelsea Publishing, Providence, Rhode Island, 1992.

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External links

- Range, R. Michael (February 2012), "WHAT IS...a Pseudoconvex Domain?" (https://www.ams.org/notices/201202/rtx120200301p.pdf) (PDF), *Notices of the American Mathematical Society*, **59** (2): 301–303, doi:10.1090/noti798 (https://doi.org/10.1090%2Fnoti798)
- "Pseudo-convex and pseudo-concave" (https://www.encyclopediaofmath.org/index.php?title=Pseudo-convex_and_pseudo-concave), *Encyclopedia of Mathematics*, EMS Press, 2001 [1994]