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Closed convex function

In mathematics, a function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be **closed** if for each $\alpha \in \mathbb{R}$, the sublevel set $\{x \in \mathbf{dom} f | f(x) \leq \alpha\}$ is a closed set.

Equivalently, if the epigraph defined by $\mathbf{epi} f = \{(x,t) \in \mathbb{R}^{n+1} | x \in \mathbf{dom} f,\ f(x) \leq t\}$ is closed, then the function f is closed.

This definition is valid for any function, but most used for convex functions. A proper convex function is closed if and only if it is lower semi-continuous.[1] For a convex function which is not proper there is disagreement as to the definition of the *closure* of the function.

Properties

- If $f : \mathbb{R}^n \to \mathbb{R}$ is a continuous function and $\mathbf{dom} f$ is closed, then f is closed.
- If $f : \mathbb{R}^n \to \mathbb{R}$ is a continuous function and $\mathbf{dom} f$ is open, then f is closed if and only if it converges to ∞ along every sequence converging to a boundary point of $\mathbf{dom} f$.[2]
- A closed proper convex function *f* is the pointwise supremum of the collection of all affine functions *h* such that *h* ≤ *f* (called the affine minorants of *f*).

References

1. *Convex Optimization Theory* (https://www.amazon.com/Convex-Optimization-Theory-Dimitri-Bertsekas/dp/1886529310/). Athena Scientific. 2009. pp. 10, 11. ISBN 978-1886529311.
2. Boyd, Stephen; Vandenberghe, Lieven (2004). *Convex optimization* (https://www.stanford.edu/~boyd/cvxbook/bv_cvxbook.pdf) (PDF). New York: Cambridge. pp. 639–640. ISBN 978-0521833783.

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