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Concave function

In mathematics, a **concave function** is the negative of a convex function. A concave function is also synonymously called **concave downwards**, **concave down**, **convex upwards**, **convex cap**, or **upper convex**.

Contents

Definition

A real-valued function f on an interval (or, more generally, a convex set in vector space) is said to be *concave* if, for any x and y in the interval and for any $\alpha \in [0, 1]$,[1]

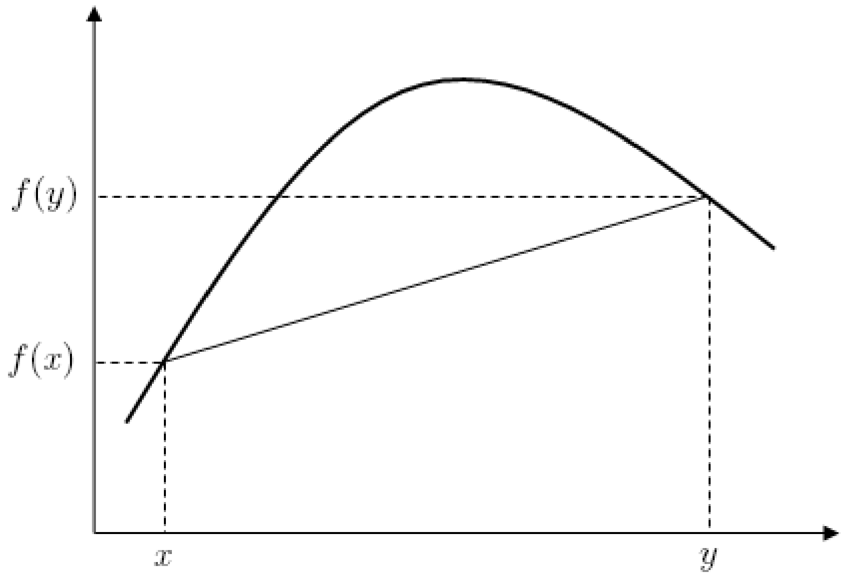
$$f((1-\alpha)x + \alpha y) \geq (1-\alpha)f(x) + \alpha f(y)$$

A function is called *strictly concave* if

$$f((1-\alpha)x + \alpha y) > (1-\alpha)f(x) + \alpha f(y)$$

for any $\alpha \in (0, 1)$ and $x \neq y$.

For a function $f : \mathbb{R} \to \mathbb{R}$, this second definition merely states that for every z strictly between x and y, the point $(z, f(z))$ on the graph of f is above the straight line joining the points $(x, f(x))$ and $(y, f(y))$.



A function f is quasiconcave if the upper contour sets of the function $S(a) = \{x : f(x) \geq a\}$ are convex sets.[2]

Properties

Functions of a single variable

1. A differentiable function f is (strictly) concave on an interval if and only if its derivative function f' is (strictly) monotonically decreasing on that interval, that is, a concave function has a non-increasing (decreasing) slope.[3][4]
2. Points where concavity changes (between concave and convex) are inflection points.[5]
3. If f is twice-differentiable, then f is concave if and only if f'' is non-positive (or, informally, if the "acceleration" is non-positive). If its second derivative is negative then it is strictly concave, but the converse is not true, as shown by $f(x) = -x^4$.
4. If f is concave and differentiable, then it is bounded above by its first-order Taylor approximation:[2]

$$f(y) \leq f(x) + f'(x)[y - x]$$

5. A Lebesgue measurable function on an interval $\mathbf{C}$ is concave if and only if it is midpoint concave, that is, for any x and y in $\mathbf{C}$

$$f\left(\frac{x+y}{2}\right) \geq \frac{f(x) + f(y)}{2}$$

6. If a function f is concave, and $f(0) \geq 0$, then f is subadditive on $[0, \infty)$. Proof:
   - Since f is concave and $1 \geq \mathrm{t} \geq 0$, letting $y = 0$ we have

   $$f(tx) = f(tx + (1-t) \cdot 0) \geq tf(x) + (1-t)f(0) \geq tf(x).$$

   - For $a, b \in [0, \infty)$:

   $$f(a) + f(b) = f\left((a+b)\frac{a}{a+b}\right) + f\left((a+b)\frac{b}{a+b}\right) \geq \frac{a}{a+b}f(a+b) + \frac{b}{a+b}f(a +$$

Functions of *n* variables

1. A function f is concave over a convex set if and only if the function $-f$ is a convex function over the set.
2. The sum of two concave functions is itself concave and so is the pointwise minimum of two concave functions, i.e. the set of concave functions on a given domain form a semifield.
3. Near a local maximum in the interior of the domain of a function, the function must be concave; as a partial converse, if the derivative of a strictly concave function is zero at some point, then that point is a local maximum.
4. Any local maximum of a concave function is also a global maximum. A *strictly* concave function will have at most one global maximum.

Examples

- The functions $f(x) = -x^2$ and $g(x) = \sqrt{x}$ are concave on their domains, as their second derivatives $f''(x) = -2$ and $g''(x) = -\frac{1}{4x^{3/2}}$ are always negative.
- The logarithm function $f(x) = \log x$ is concave on its domain $(0, \infty)$, as its derivative $\frac{1}{x}$ is a strictly decreasing function.
- Any affine function $f(x) = ax + b$ is both concave and convex, but neither strictly-concave nor strictly-convex.
- The sine function is concave on the interval $[0, \pi]$.
- The function $f(B) = \log |B|$, where $|B|$ is the determinant of a nonnegative-definite matrix *B*, is concave.[6]

Applications

- Rays bending in the computation of radiowave attenuation in the atmosphere involve concave functions.
- In expected utility theory for choice under uncertainty, cardinal utility functions of risk averse decision makers are concave.
- In microeconomic theory, production functions are usually assumed to be concave over some or all of their domains, resulting in diminishing returns to input factors.[7]

See also

- Concave polygon
- Jensen's inequality
- Logarithmically concave function
- Quasiconcave function
- Concavification

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