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Mixed volume

In mathematics, more specifically, in convex geometry, the **mixed volume** is a way to associate a non-negative number to an r-tuple of convex bodies in n-dimensional space. This number depends on the size and shape of the bodies and on their relative orientation to each other.

Contents

Definition

Let $K_1, K_2, \dots, K_r$ be convex bodies in $\mathbb{R}^n$ and consider the function

$$f(\lambda_1, \dots, \lambda_r) = \mathrm{Vol}_n(\lambda_1 K_1 + \cdots + \lambda_r K_r), \qquad \lambda_i \geq 0,$$

where Vol_n stands for the n-dimensional volume and its argument is the Minkowski sum of the scaled convex bodies K_i. One can show that f is a homogeneous polynomial of degree n, therefore it can be written as

$$f(\lambda_1, \dots, \lambda_r) = \sum_{j_1, \dots, j_n = 1}^{r} V(K_{j_1}, \dots, K_{j_n}) \lambda_{j_1} \cdots \lambda_{j_n},$$

where the functions V are symmetric. For a particular index function $j \in \{1, \dots, r\}^n$, the coefficient $V(K_{j_1}, \dots, K_{j_n})$ is called the mixed volume of $K_{j_1}, \dots, K_{j_n}$.

Properties

- The mixed volume is uniquely determined by the following three properties:

1. $V(K, \dots, K) = \mathrm{Vol}_n(K)$;
2. V is symmetric in its arguments;
3. V is multilinear: $V(\lambda K + \lambda' K', K_2, \dots, K_n) = \lambda V(K, K_2, \dots, K_n) + \lambda' V(K', K_2, \dots, K_n)$ for $\lambda, \lambda' \geq 0$.

- The mixed volume is non-negative and monotonically increasing in each variable: $V(K_1, K_2, \ldots, K_n) \leq V(K_1', K_2, \ldots, K_n)$ for $K_1 \subseteq K_1'$.
- The Alexandrov–Fenchel inequality, discovered by Aleksandr Danilovich Aleksandrov and Werner Fenchel:

$$V(K_1, K_2, K_3, \ldots, K_n) \geq \sqrt{V(K_1, K_1, K_3, \ldots, K_n) V(K_2, K_2, K_3, \ldots, K_n)}.$$

  Numerous geometric inequalities, such as the Brunn–Minkowski inequality for convex bodies and Minkowski's first inequality, are special cases of the Alexandrov–Fenchel inequality.

Quermassintegrals

Let $K \subset \mathbb{R}^n$ be a convex body and let $B = B_n \subset \mathbb{R}^n$ be the Euclidean ball of unit radius. The mixed volume

$$W_j(K) = V(\overbrace{K, K, \ldots, K}^{n-j \text{ times}}, \overbrace{B, B, \ldots, B}^{j \text{ times}})$$

is called the j-th **quermassintegral** of K.[1]

The definition of mixed volume yields the **Steiner formula** (named after Jakob Steiner):

$$\mathrm{Vol}_n(K + tB) = \sum_{j=0}^{n} \binom{n}{j} W_j(K) t^j.$$

Intrinsic volumes

The j-th **intrinsic volume** of K is a different normalization of the quermassintegral, defined by

$$V_j(K) = \binom{n}{j} \frac{W_{n-j}(K)}{\kappa_{n-j}}, \text{ or in other words } \mathrm{Vol}_n(K + tB) = \sum_{j=0}^{n} V_j(K)\, \mathrm{Vol}_{n-j}(tB_{n-j}).$$

where $\kappa_{n-j} = \mathrm{Vol}_{n-j}(B_{n-j})$ is the volume of the $(n-j)$-dimensional unit ball.

Hadwiger's characterization theorem

Hadwiger's theorem asserts that every valuation on convex bodies in $\mathbb{R}^n$ that is continuous and invariant under rigid motions of $\mathbb{R}^n$ is a linear combination of the quermassintegrals (or, equivalently, of the intrinsic volumes).[2]

Notes

1. McMullen, P. (1991). "Inequalities between intrinsic volumes" (https://doi.org/10.1007%2Fbf01299276). *Monatsh. Math.* **111** (1): 47–53. doi:10.1007/bf01299276 (https://doi.org/10.1007%2Fbf01299276). MR 1089383 (https://www.ams.org/mathscinet-getitem?mr=1089383).

2. Klain, D.A. (1995). "A short proof of Hadwiger's characterization theorem". *Mathematika*. **42** (2): 329–339. doi:10.1112/s0025579300014625 (https://doi.org/10.1112%2Fs0025579300014625). MR 1376731 (https://www.ams.org/mathscinet-getitem?mr=1376731).

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Burago, Yu.D. (2001) [1994], "Mixed volume theory" (https://www.encyclopediaofmath.org/index.php?title=Mixed_volume_theory), *Encyclopedia of Mathematics*, EMS Press