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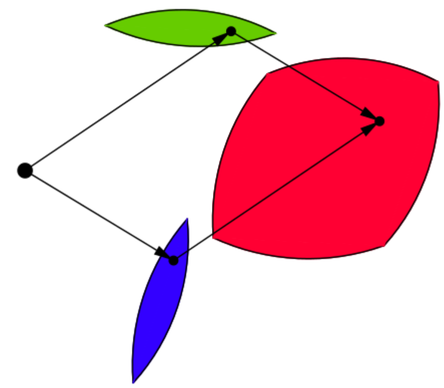
Minkowski addition

In geometry, the **Minkowski sum** (also known as dilation) of two sets of position vectors A and B in Euclidean space is formed by adding each vector in A to each vector in B, i.e., the set

$$A + B = \{\mathbf{a} + \mathbf{b} \,|\, \mathbf{a} \in A,\ \mathbf{b} \in B\}.$$



The red figure is the Minkowski sum of blue and green figures.

Analogously, the **Minkowski difference** (or geometric difference)[1] is defined using the complement operation as

$$A - B = (A^c + (-B))^c$$

In general $A - B \neq A + (-B)$. For instance, in a one-dimensional case $A = [-2, 2]$ and $B = [-1, 1]$ the Minkowski difference $A - B = [-1, 1]$, whereas $A + (-B) = A + B = [-3, 3]$.

In a two-dimensional case, Minkowski difference is closely related to erosion (morphology) in image processing.

The concept is named for Hermann Minkowski.

Contents

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Example

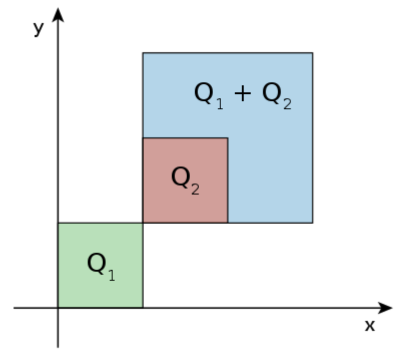
For example, if we have two sets A and B, each consisting of three position vectors (informally, three points), representing the vertices of two triangles in $\mathbb{R}^2$, with coordinates

$$A = \{(1,0), (0,1), (0,-1)\}$$

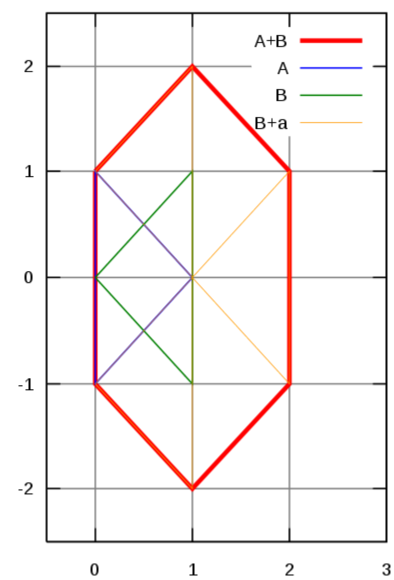
and

$$B = \{(0,0), (1,1), (1,-1)\}$$

then their Minkowski sum is



Minkowski addition of sets. The sum of the squares $Q_1 = [0,1]^2$ and $Q_2 = [1,2]^2$ is the square $Q_1 + Q_2 = [1,3]^2$.



Minkowski sum $A + B$

$$A + B = \{(1,0), (2,1), (2,-1), (0,1), (1,2), (1,0), (0,-1), (1,0), (1,-2)\}$$

which comprises the vertices of a hexagon.

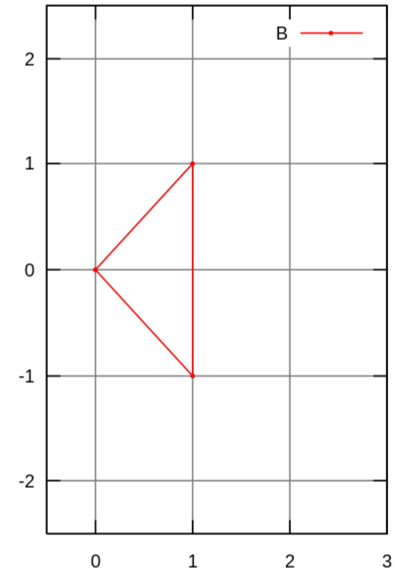
For Minkowski addition, the *zero set*, $\{0\}$, containing only the zero vector, 0, is an identity element: for every subset S of a vector space,

$$S + \{0\} = S.$$

The empty set is important in Minkowski addition, because the empty set annihilates every other subset: for every subset S of a vector space, its sum with the empty set is empty:

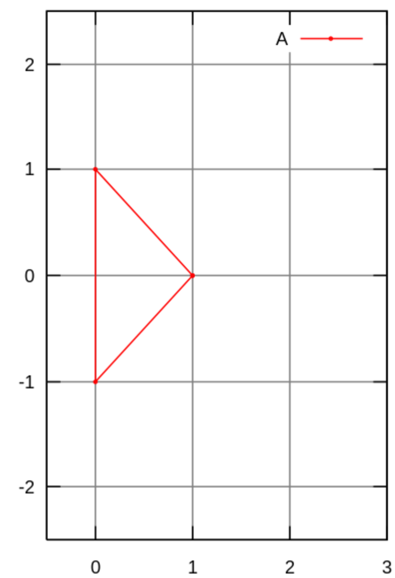
$$S + \emptyset = \emptyset.$$

For another example, consider the Minkowski sums of open or closed balls in the field $\mathbb{K}$, which is either the real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$. If $B_r := \{s \in \mathbb{K} : |s| \leq r\}$ is the closed ball of radius $r \in [0, \infty]$ centered at 0 in $\mathbb{K}$ then for any $r, s \in [0, \infty]$, $B_r + B_s = B_{r+s}$ and also $cB_r = B_{|c|r}$ will hold for any scalar $c \in \mathbb{K}$ such that the product $|c|r$ is defined (which happens when $c \neq 0$ or $r \neq \infty$). If $r, s,$ and c are all non-zero then the same equalities would still hold had B_r been defined to be the open ball, rather than the closed ball, centered at 0 (the non-zero assumption is needed because the open ball of radius 0 is the empty set). The Minkowski sum of a closed ball and an open ball is an open ball. More generally, the Minkowski sum of an open subset with *any* other set will be an open subset.



B

If $G = \{(x, 1/x) : 0 \neq x \in \mathbb{R}\}$ is the graph of $f(x) = \frac{1}{x}$ and if and $Y = \{0\} \times \mathbb{R}$ is the y-axis in $X = \mathbb{R}^2$ then the Minkowski sum of these two closed subsets of the plane is the open set $G + Y = \{(x, y) \in \mathbb{R}^2 : x \neq 0\} = \mathbb{R}^2 \setminus Y$ consisting of everything other than the y-axis. This shows that the Minkowski sum of two closed sets is not necessarily a closed set. However, the Minkowski sum of two closed subsets will be a closed subset if at least one of these sets is also a compact subset.



A

Convex hulls of Minkowski sums

Minkowski addition behaves well with respect to the operation of taking convex hulls, as shown by the following proposition:

> For all non-empty subsets S_1 and S_2 of a real vector space, the convex hull of their Minkowski sum is the Minkowski sum of their convex hulls:
>
> $$\operatorname{Conv}(S_1 + S_2) = \operatorname{Conv}(S_1) + \operatorname{Conv}(S_2).$$

This result holds more generally for any finite collection of non-empty sets:

$$\operatorname{Conv}\left(\sum S_n\right) = \sum \operatorname{Conv}(S_n).$$

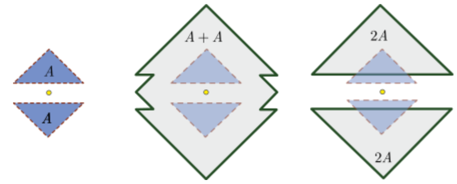
In mathematical terminology, the operations of Minkowski summation and of forming convex hulls are commuting operations.[2][3]

If S is a convex set then $\mu S + \lambda S$ is also a convex set; furthermore

$$\mu S + \lambda S = (\mu + \lambda) S$$

for every $\mu, \lambda \geq 0$. Conversely, if this "distributive property" holds for all non-negative real numbers, μ, λ, then the set is convex.[4]

The figure to the right shows an example of a non-convex set for which $A + A \supsetneq 2A$.



An example of a non-convex set such that $A + A \neq 2A$.

An example in 1 dimension is: $B = [1, 2] \cup [4, 5]$. It can be easily calculated that $2B = [2, 4] \cup [8, 10]$ but $B + B = [2, 4] \cup [5, 7] \cup [8, 10]$, hence again $B + B \supsetneq 2B$.

Minkowski sums act linearly on the perimeter of two-dimensional convex bodies: the perimeter of the sum equals the sum of perimeters. Additionally, if K is (the interior of) a curve of constant width, then the Minkowski sum of K and of its 180° rotation is a disk. These two facts can be combined to give a short proof of Barbier's theorem on the perimeter of curves of constant width.[5]

Applications

Minkowski addition plays a central role in mathematical morphology. It arises in the brush-and-stroke paradigm of 2D computer graphics (with various uses, notably by Donald E. Knuth in Metafont), and as the solid sweep operation of 3D computer graphics. It has also been shown to be closely connected to the Earth mover's distance, and by extension, optimal transport.[6]

Motion planning

Minkowski sums are used in motion planning of an object among obstacles. They are used for the computation of the configuration space, which is the set of all admissible positions of the object. In the simple model of translational motion of an object in the plane, where the position of an object may be uniquely specified by the position of a fixed point of this object, the configuration space are the Minkowski sum of the set of obstacles and the movable object placed at the origin and rotated 180 degrees.

Numerical control (NC) machining

In numerical control machining, the programming of the NC tool exploits the fact that the Minkowski sum of the cutting piece with its trajectory gives the shape of the cut in the material.

3D solid modeling

In OpenSCAD Minkowski sums are used to outline a shape with another shape creating a composite of both shapes.

Aggregation theory

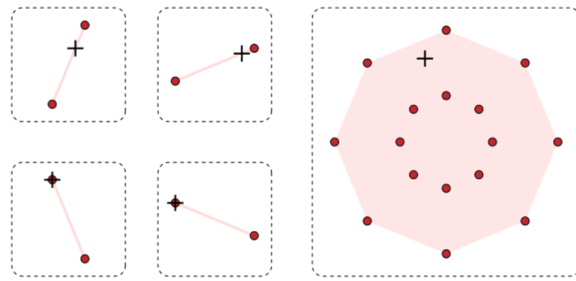
Minkowski sums are also frequently used in aggregation theory when individual objects to be aggregated are characterized via sets.[7][8]

Collision detection

Minkowski sums, specifically Minkowski differences, are often used alongside GJK algorithms to compute collision detection for convex hulls in physics engines.

Algorithms for computing Minkowski sums

Planar case



Minkowski addition and convex hulls. The sixteen dark-red points (on the right) form the Minkowski sum of the four non-convex sets (on the left), each of which consists of a pair of red points. Their convex hulls (shaded pink) contain plus-signs (+): The right plus-sign is the sum of the left plus-signs.

Two convex polygons in the plane

For two convex polygons P and Q in the plane with m and n vertices, their Minkowski sum is a convex polygon with at most $m + n$ vertices and may be computed in time O($m + n$) by a very simple procedure, which may be informally described as follows. Assume that the edges of a polygon are given and the direction, say, counterclockwise, along the polygon boundary. Then it is easily seen that these edges of the convex polygon are ordered by polar angle. Let us merge the ordered sequences of the directed edges from P and Q into a single ordered sequence S. Imagine that these edges are solid arrows which can be moved freely while keeping them parallel to their original direction. Assemble these arrows in the order of the sequence S by attaching the tail of the next arrow to the head of the previous arrow. It turns out that the resulting polygonal chain will in fact be a convex polygon which is the Minkowski sum of P and Q.

Other

If one polygon is convex and another one is not, the complexity of their Minkowski sum is O(nm). If both of them are nonconvex, their Minkowski sum complexity is O($(mn)^2$).

Essential Minkowski sum

There is also a notion of the **essential Minkowski sum** $+_e$ of two subsets of Euclidean space. The usual Minkowski sum can be written as

$$A + B = \{z \in \mathbb{R}^n \mid A \cap (z - B) \neq \emptyset\}.$$

Thus, the **essential Minkowski sum** is defined by

$$A +_e B = \{z \in \mathbb{R}^n \mid \mu\left[A \cap (z - B)\right] > 0\},$$

where μ denotes the n-dimensional Lebesgue measure. The reason for the term "essential" is the following property of indicator functions: while

$$1_{A+B}(z) = \sup_{x \in \mathbb{R}^n} 1_A(x) 1_B(z-x),$$

it can be seen that

$$1_{A+_e B}(z) = \operatorname*{ess\,sup}_{x \in \mathbb{R}^n} 1_A(x) 1_B(z-x),$$

where "ess sup" denotes the essential supremum.

L^p Minkowski sum

For K and L compact convex subsets in $\mathbb{R}^n$, the Minkowski sum can be described by the support function of the convex sets:

$$h_{K+L} = h_K + h_L.$$

For $p \geq 1$, Firey[9] defined the **$\mathbf{L^p}$ Minkowski sum** $K +_p L$ of compact convex sets K and L in $\mathbb{R}^n$ containing the origin as

$$h^p_{K+_pL} = h^p_K + h^p_L.$$

By the Minkowski inequality, the function h_{K+_pL} is again positive homogeneous and convex and hence the support function of a compact convex set. This definition is fundamental in the L^p Brunn-Minkowski theory.

See also

- Blaschke sum
- Brunn–Minkowski theorem, an inequality on the volumes of Minkowksi sums
- Convolution
- Dilation
- Erosion
- Interval arithmetic
- Mixed volume (a.k.a. Quermassintegral or intrinsic volume)
- Parallel curve
- Shapley–Folkman lemma
- Sumset
- Topological vector space#Properties
- Zonotope

Notes

1. Hadwiger, Hugo (1950), "Minkowskische Addition und Subtraktion beliebiger Punktmengen und die Theoreme von Erhard Schmidt", *Math. Z.*, **53** (3): 210–218, doi:10.1007/BF01175656 (https://doi.org/10.1007%2FBF01175656)
2. Theorem 3 (pages 562–563): Krein, M.; Šmulian, V. (1940). "On regularly convex sets in the space conjugate to a Banach space". *Annals of Mathematics*. Second Series. Vol. 41. pp. 556–

583. doi:10.2307/1968735 (https://doi.org/10.2307%2F1968735). JSTOR 1968735 (https://www.jstor.org/stable/1968735). MR 0002009 (https://www.ams.org/mathscinet-getitem?mr=0002009).

3. For the commutativity of Minkowski addition and convexification, see Theorem 1.1.2 (pages 2–3) in Schneider; this reference discusses much of the literature on the convex hulls of Minkowski sumsets in its "Chapter 3 Minkowski addition" (pages 126–196): Schneider, Rolf (1993). *Convex bodies: The Brunn–Minkowski theory* (https://archive.org/details/convexbodiesbrun0000schn). Encyclopedia of mathematics and its applications. Vol. 44. Cambridge: Cambridge University Press. pp. xiv+490. ISBN 978-0-521-35220-8. MR 1216521 (https://www.ams.org/mathscinet-getitem?mr=1216521).
4. Chapter 1: Schneider, Rolf (1993). *Convex bodies: The Brunn–Minkowski theory* (https://archive.org/details/convexbodiesbrun0000schn). Encyclopedia of mathematics and its applications. Vol. 44. Cambridge: Cambridge University Press. pp. xiv+490. ISBN 978-0-521-35220-8. MR 1216521 (https://www.ams.org/mathscinet-getitem?mr=1216521).
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8. Mayer, A.; Zelenyuk, V. (2014). "Aggregation of Malmquist productivity indexes allowing for reallocation of resources" (https://ideas.repec.org/a/eee/ejores/v238y2014i3p774-785.html). *European Journal of Operational Research*. **238** (3): 774–785. doi:10.1016/j.ejor.2014.04.003 (https://doi.org/10.1016%2Fj.ejor.2014.04.003).
9. Firey, William J. (1962), "*p*-means of convex bodies", *Math. Scand.*, **10**: 17–24, doi:10.7146/math.scand.a-10510 (https://doi.org/10.7146%2Fmath.scand.a-10510)

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External links

- "Minkowski addition" (https://www.encyclopediaofmath.org/index.php?title=Minkowski_addition), *Encyclopedia of Mathematics*, EMS Press, 2001 [1994]
- Howe, Roger (1979), *On the tendency toward convexity of the vector sum of sets* (http://econpapers.repec.org/RePEc:cwl:cwldpp:538), Cowles Foundation discussion papers, vol. 538, Cowles Foundation for Research in Economics, Yale University
- Minkowski Sums (https://www.cgal.org/Pkg/MinkowskiSum2), in Computational Geometry Algorithms Library
- The Minkowski Sum of Two Triangles (http://demonstrations.wolfram.com/TheMinkowskiSumOfTwoTriangles/) and The Minkowski Sum of a Disk and a Polygon (http://demonstrations.wolfram.com/TheMinkowskiSumOfADiskAndAPolygon/) by George Beck, The Wolfram Demonstrations Project.
- Minkowski's addition of convex shapes (http://www.cut-the-knot.org/Curriculum/Geometry/PolyAddition.shtml) by Alexander Bogomolny: an applet
- Wikibooks:OpenSCAD User Manual/Transformations#minkowski by Marius Kintel: Application
- Application of Minkowski Addition to robotics (https://minkowski-sum.herokuapp.com/minkownski-sum.html) by Joan Gerard