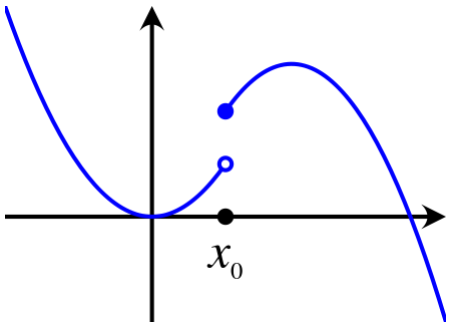
WIKIPEDIA

Fenchel–Moreau theorem

In convex analysis, the **Fenchel–Moreau theorem** (named after Werner Fenchel and Jean Jacques Moreau) or **Fenchel biconjugation theorem** (or just **biconjugation theorem**) is a theorem which gives necessary and sufficient conditions for a function to be equal to its biconjugate. This is in contrast to the general property that for any function $f^{**} \leq f$.[1][2] This can be seen as a generalization of the bipolar theorem.[1] It is used in duality theory to prove strong duality (via the perturbation function).



A function that is not lower semi-continuous. By the Fenchel-Moreau theorem, this function is not equal to its biconjugate.

Statement

Let (X, τ) be a Hausdorff locally convex space, for any extended real valued function $f : X \to \mathbb{R} \cup \{\pm\infty\}$ it follows that $f = f^{**}$ if and only if one of the following is true

1. f is a proper, lower semi-continuous, and convex function,
2. $f \equiv +\infty$, or
3. $f \equiv -\infty$.[1][3][4]

References

1. Borwein, Jonathan; Lewis, Adrian (2006). *Convex Analysis and Nonlinear Optimization: Theory and Examples* (2 ed.). Springer. pp. 76–77. ISBN 9780387295701.
2. Zălinescu, Constantin (2002). *Convex analysis in general vector spaces*. River Edge, NJ: World Scientific Publishing Co., Inc. pp. 75–79. ISBN 981-238-067-1. MR 1921556 (https://www.ams.org/mathscinet-getitem?mr=1921556).
3. Hang-Chin Lai; Lai-Jui Lin (May 1988). "The Fenchel-Moreau Theorem for Set Functions" (https://doi.org/10.2307%2F2047532). *Proceedings of the American Mathematical Society*. American Mathematical Society. **103** (1): 85–90. doi:10.2307/2047532 (https://doi.org/10.2307%2F2047532).
4. Shozo Koshi; Naoto Komuro (1983). "A generalization of the Fenchel–Moreau theorem". *Proc. Japan Acad. Ser. A Math. Sci.*. **59** (5): 178–181.