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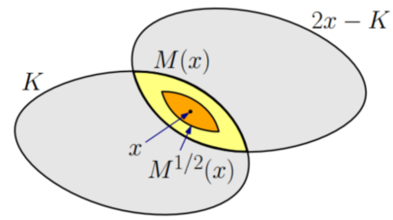
Macbeath region

In mathematics, a **Macbeath Region** is an explicitly defined region in convex analysis on a bounded convex subset of d-dimensional Euclidean space $\mathbb{R}^d$. The idea was introduced by Alexander Macbeath (1952)[1] and dubbed by G. Ewald, D. G. Larman and C. A. Rogers in 1970.[2] Macbeath regions have been used to solve certain complex problems in the study of the boundaries of convex bodies.[3] Recently they have been used in the study of convex approximations and other aspects of computational geometry.[4][5]



The Macbeath around a point x in a convex body K and the scaled Macbeath region around a point x in a convex body K

Contents

Definition

Let K be a bounded convex set in a Euclidean space. Given a point x and a scaler λ the λ-scaled the Macbeath region around a point x is:

$$M_K(x) = K \cap (2x - K) = x + ((K - x) \cap (x - K)) = \{k' \in K | \exists k \in K \text{ and } k' - x = x - k\}$$

The scaled Macbeath region at x is defined as:

$$M_K^\lambda(x) = x + \lambda((K - x) \cap (x - K)) = \{(1 - \lambda)x + \lambda k' | k' \in K, \exists k \in K \text{ and } k' - x = x - k\}$$

This can be seen to be the intersection of K with the reflection of K around x scaled by λ.

Example uses

- Macbeath regions can be used to create ϵ approximations, with respect to the Hausdorff distance, of convex shapes within a factor of $O(\log^{\frac{d+1}{2}}(\frac{1}{\epsilon}))$ combinatorial complexity of the lower bound.[5]
- Macbeath regions can be used to approximate balls in the Hilbert metric, e.g. given any convex K, containing an x and a $0 \leq \lambda < 1$ then:[4][6]

$$B_H\left(x, \frac{1}{2}\ln(1+\lambda)\right) \subset M^\lambda(x) \subset B_H\left(x, \frac{1}{2}\ln\frac{1+\lambda}{1-\lambda}\right)$$

Properties

- The $M_K^\lambda(x)$ is centrally symmetric around x.
- Macbeath regions are convex sets.
- If $x, y \in K$ and $M^{\frac{1}{2}}(x) \cap M^{\frac{1}{2}}(y) \neq \emptyset$ then $M^1(y) \subset M^5(x)$.[3][4] Essentially if two Macbeath regions intersect, you can scale one of them up to contain the other.
- If some convex *K* in R^d containing both a ball of radius r and a half-space *H*, with the half-space disjoint from the ball, and the cap $K \cap H$ of our convex set has a width less than or equal to $\frac{r}{2}$, we get $K \cap H \subset M^{3d(x)}$ for *x*, the center of gravity of K in the bounding hyper-plane of *H*.[3]
- Given a convex body $K \subset R^d$ in canonical form, then any cap of *K* with width at most $\frac{1}{6d}$ then $C \subset M^{3d}(x)$, where x is the centroid of the base of the cap.[5]
- Given a convex *K* and some constant $\lambda > 0$, then for any point *x* in a cap *C* of *K* we know $M^\lambda(x) \cap K \subset C^{1+\lambda}$. In particular when $\lambda \leq 1$, we get $M^\lambda(x) \subset C^{1+\lambda}$.[5]
- Given a convex body *K*, and a cap *C* of *K*, if x is in *K* and $C \cap M'(x) \neq \emptyset$ we get $M'(x) \subset C^2$.[5]
- Given a small $\epsilon > 0$ and a convex $K \subset R^d$ in canonical form, there exists some collection of $O\left(\frac{1}{\epsilon^{\frac{d-1}{2}}}\right)$ centrally symmetric disjoint convex bodies $R_1, \ldots, R_k$ and caps $C_1, \ldots, C_k$ such that for some constant β and λ depending on d we have:[5]
  - Each C_i has width $\beta\epsilon$, and $R_i \subset C_i \subset R_i^\lambda$
  - If C is any cap of width ϵ there must exist an *i* so that $R_i \subset C$ and $C_i^{\frac{1}{\beta^2}} \subset C \subset C_i$

References

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Further reading

- Dutta, Kunal; Ghosh, Arijit; Jartoux, Bruno; Mustafa, Nabil (2019). "Shallow Packings, Semialgebraic Set Systems, Macbeath Regions, and Polynomial Partitioning" (https://drops.dagstuhl.de/opus/volltexte/2017/7199/). *Discrete & Computational Geometry*. **61** (4): 756–777. doi:10.1007/s00454-019-00075-0 (https://doi.org/10.1007%2Fs00454-019-00075-0). S2CID 127559205 (https://api.semanticscholar.org/CorpusID:127559205).