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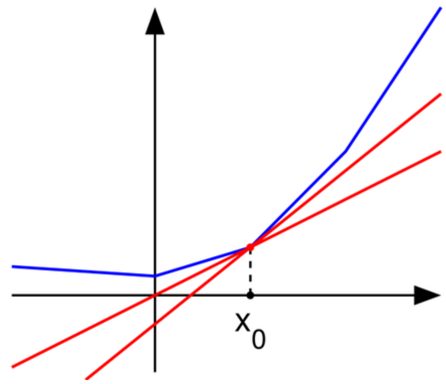
Subderivative

In mathematics, the **subderivative**, **subgradient**, and **subdifferential** generalize the derivative to convex functions which are not necessarily differentiable. Subderivatives arise in convex analysis, the study of convex functions, often in connection to convex optimization.



A convex function (blue) and "subtangent lines" at x_0 (red).

Let $f: I \to \mathbb{R}$ be a real-valued convex function defined on an open interval of the real line. Such a function need not be differentiable at all points: For example, the absolute value function $f(x)=|x|$ is nondifferentiable when $x=0$. However, as seen in the graph on the right (where $f(x)$ in blue has non-differentiable kinks similar to the absolute value function), for any x_0 in the domain of the function one can draw a line which goes through the point $(x_0, f(x_0))$ and which is everywhere either touching or below the graph of f. The slope of such a line is called a *subderivative* (because the line is under the graph of f).

Contents

Definition

Rigorously, a *subderivative* of a convex function $f: I \to \mathbb{R}$ at a point x_0 in the open interval I is a real number c such that

$$f(x) - f(x_0) \geq c(x - x_0)$$

for all x in I. One may show that the set of subderivatives at x_0 for a convex function is a nonempty closed interval $[a, b]$, where a and b are the one-sided limits

$$a = \lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0}$$

$$b = \lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0}$$

which are guaranteed to exist and satisfy $a \leq b$.

The set $[a, b]$ of all subderivatives is called the **subdifferential** of the function f at x_0. If f is convex and its subdifferential at x_0 contains exactly one subderivative, then f is differentiable at x_0.[1]

Example

Consider the function $f(x)=|x|$ which is convex. Then, the subdifferential at the origin is the interval $[-1, 1]$. The subdifferential at any point $x_0<0$ is the singleton set $\{-1\}$, while the subdifferential at any point $x_0>0$ is the singleton set $\{1\}$. This is similar to the sign function, but is not a single-valued function at 0, instead including all possible subderivatives.

Properties

- A convex function $f:I\to\mathbf{R}$ is differentiable at x_0 if and only if the subdifferential is made up of only one point, which is the derivative at x_0.
- A point x_0 is a global minimum of a convex function f if and only if zero is contained in the subdifferential, that is, in the figure above, one may draw a horizontal "subtangent line" to the graph of f at $(x_0, f(x_0))$. This last property is a generalization of the fact that the derivative of a function differentiable at a local minimum is zero.
- If f and g are convex functions with subdifferentials $\partial f(x)$ and $\partial g(x)$ with x being the interior point of one of the functions, then the subdifferential of $f+g$ is $\partial(f+g)(x) = \partial f(x) + \partial g(x)$ (where the addition operator denotes the Minkowski sum). This reads as "the subdifferential of a sum is the sum of the subdifferentials."[2]

The subgradient

The concepts of subderivative and subdifferential can be generalized to functions of several variables. If $f:U\to\mathbf{R}$ is a real-valued convex function defined on a convex open set in the Euclidean space $\mathbf{R}^n$, a vector v in that space is called a **subgradient** at a point x_0 in U if for any x in U one has

$$f(x) - f(x_0) \geq v \cdot (x - x_0)$$

where the dot denotes the dot product. The set of all subgradients at x_0 is called the **subdifferential** at x_0 and is denoted $\partial f(x_0)$. The subdifferential is always a nonempty convex compact set.

These concepts generalize further to convex functions $f:U\to\mathbf{R}$ on a convex set in a locally convex space V. A functional v^* in the dual space V^* is called *subgradient* at x_0 in U if for all x in U

$$f(x) - f(x_0) \geq v^*(x - x_0).$$

The set of all subgradients at x_0 is called the subdifferential at x_0 and is again denoted $\partial f(x_0)$. The subdifferential is always a convex closed set. It can be an empty set; consider for example an unbounded operator, which is convex, but has no subgradient. If f is continuous, the subdifferential is nonempty.

History

The subdifferential on convex functions was introduced by Jean Jacques Moreau and R. Tyrrell Rockafellar in the early 1960s. The *generalized subdifferential* for nonconvex functions was introduced by F.H. Clarke and R.T. Rockafellar in the early 1980s.[3]

See also

- Weak derivative
- Subgradient method

References

1. Rockafellar, R. T. (1970). *Convex Analysis*. Princeton University Press. p. 242 [Theorem 25.1]. ISBN 0-691-08069-0.
2. Lemaréchal, Claude; Hiriart-Urruty, Jean-Baptiste (2001). *Fundamentals of Convex Analysis* (https://archive.org/details/fundamentalsconv00hiri). Springer-Verlag Berlin Heidelberg. p. 183 (https://archive.org/details/fundamentalsconv00hiri/page/n193). ISBN 978-3-642-56468-0.
3. Clarke, Frank H. (1983). *Optimization and nonsmooth analysis* (https://archive.org/details/optimizationnons0000clar). New York: John Wiley & Sons. pp. xiii+308. ISBN 0-471-87504-X. MR 0709590 (https://www.ams.org/mathscinet-getitem?mr=0709590).

- Borwein, Jonathan; Lewis, Adrian S. (2010). *Convex Analysis and Nonlinear Optimization : Theory and Examples* (2nd ed.). New York: Springer. ISBN 978-0-387-31256-9.
- Hiriart-Urruty, Jean-Baptiste; Lemaréchal, Claude (2001). *Fundamentals of Convex Analysis*. Springer. ISBN 3-540-42205-6.
- Zălinescu, C. (2002). *Convex analysis in general vector spaces*. World Scientific Publishing Co., Inc. pp. xx+367. ISBN 981-238-067-1. MR 1921556 (https://www.ams.org/mathscinet-getitem?mr=1921556).

External links

- "Uses of $\lim_{h\to 0} \frac{f(x+h)-f(x-h)}{2h}$" (https://math.stackexchange.com/q/65569). *Stack Exchange*. July 15, 2002.