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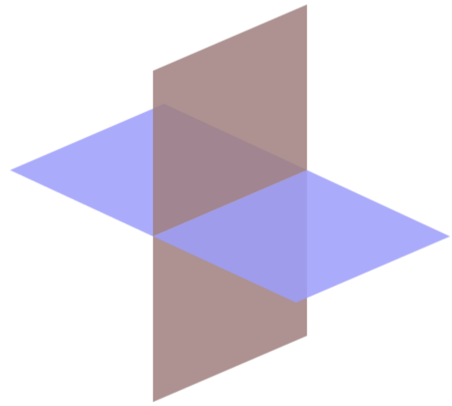
Hyperplane

In geometry, a **hyperplane** is a subspace whose dimension is one less than that of its *ambient space*. For example, if a space is 3-dimensional then its hyperplanes are the 2-dimensional planes, while if the space is 2-dimensional, its hyperplanes are the 1-dimensional lines. This notion can be used in any general space in which the concept of the dimension of a subspace is defined.



Two intersecting planes in three-dimensional space. A plane is a hyperplane of dimension 2, when embedded in a space of dimension 3.

In different settings, hyperplanes may have different properties. For instance, a hyperplane of an n-dimensional affine space is a flat subset with dimension $n - 1$[1] and it separates the space into two half spaces. While a hyperplane of an n-dimensional projective space does not have this property.

The difference in dimension between a subspace S and its ambient space X is known as the codimension of S with respect to X. Therefore, a necessary condition for S to be a hyperplane in X is for S to have codimension one in X.

Contents

Technical description

In geometry, a **hyperplane** of an n-dimensional space V is a subspace of dimension $n - 1$, or equivalently, of codimension 1 in V. The space V may be a Euclidean space or more generally an affine space, or a vector space or a projective space, and the notion of hyperplane varies correspondingly since the definition of subspace differs in these settings; in all cases however, any hyperplane can be given in coordinates as the solution of a single (due to the "codimension 1" constraint) algebraic equation of degree 1.

If V is a vector space, one distinguishes "vector hyperplanes" (which are linear subspaces, and therefore must pass through the origin) and "affine hyperplanes" (which need not pass through the origin; they can be obtained by translation of a vector hyperplane). A hyperplane in a Euclidean space separates that space into two half spaces, and defines a reflection that fixes the hyperplane and interchanges those two half spaces.

Special types of hyperplanes

Several specific types of hyperplanes are defined with properties that are well suited for particular purposes. Some of these specializations are described here.

Affine hyperplanes

An **affine hyperplane** is an affine subspace of codimension 1 in an affine space. In Cartesian coordinates, such a hyperplane can be described with a single linear equation of the following form (where at least one of the a_i's is non-zero and b is an arbitrary constant):

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b.$$

In the case of a real affine space, in other words when the coordinates are real numbers, this affine space separates the space into two half-spaces, which are the connected components of the complement of the hyperplane, and are given by the inequalities

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n < b$$

and

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n > b.$$

As an example, a point is a hyperplane in 1-dimensional space, a line is a hyperplane in 2-dimensional space, and a plane is a hyperplane in 3-dimensional space. A line in 3-dimensional space is not a hyperplane, and does not separate the space into two parts (the complement of such a line is connected).

Any hyperplane of a Euclidean space has exactly two unit normal vectors.

Affine hyperplanes are used to define decision boundaries in many machine learning algorithms such as linear-combination (oblique) decision trees, and perceptrons.

Vector hyperplanes

In a vector space, a vector hyperplane is a subspace of codimension 1, only possibly shifted from the origin by a vector, in which case it is referred to as a flat. Such a hyperplane is the solution of a single linear equation.

Projective hyperplanes

Projective hyperplanes, are used in projective geometry. A projective subspace is a set of points with the property that for any two points of the set, all the points on the line determined by the two points are contained in the set.[2] Projective geometry can be viewed as affine geometry with vanishing points (points at infinity) added. An affine hyperplane together with the associated points at infinity forms a projective hyperplane. One special case of a projective hyperplane is the **infinite** or **ideal hyperplane**, which is defined with the set of all points at infinity.

In projective space, a hyperplane does not divide the space into two parts; rather, it takes two hyperplanes to separate points and divide up the space. The reason for this is that the space essentially "wraps around" so that both sides of a lone hyperplane are connected to each other.

Applications

In convex geometry, two disjoint convex sets in n-dimensional Euclidean space are separated by a hyperplane, a result called the hyperplane separation theorem.

In machine learning, hyperplanes are a key tool to create support vector machines for such tasks as computer vision and natural language processing.

Dihedral angles

The dihedral angle between two non-parallel hyperplanes of a Euclidean space is the angle between the corresponding normal vectors. The product of the transformations in the two hyperplanes is a rotation whose axis is the subspace of codimension 2 obtained by intersecting the hyperplanes, and whose angle is twice the angle between the hyperplanes.

Support hyperplanes

A hyperplane H is called a "support" hyperplane of the polyhedron P if P is contained in one of the two closed half-spaces bounded by H and $H \cap P \neq \varnothing$.[3] The intersection of P and H is defined to be a "face" of the polyhedron. The theory of polyhedra and the dimension of the faces are analyzed by the looking at these intersections involving hyperplanes.

See also

- Hypersurface
- Decision boundary
- Ham sandwich theorem
- Arrangement of hyperplanes
- Supporting hyperplane theorem

References

1. "Excerpt from Convex Analysis, by R.T. Rockafellar" (http://www.u.arizona.edu/~mwalker/econ519/RockafellarExcerpt.pdf) (PDF). *u.arizona.edu*.

2. Beutelspacher, Albrecht; Rosenbaum, Ute (1998), *Projective Geometry: From Foundations to Applications*, Cambridge University Press, p. 10, ISBN 9780521483643
3. Polytopes, Rings and K-Theory by Bruns-Gubeladze

- Binmore, Ken G. (1980). *The Foundations of Topological Analysis: A Straightforward Introduction: Book 2 Topological Ideas* (https://books.google.com/books?id=o485AAAAIAAJ&pg=PA13). Cambridge University Press. p. 13. ISBN 0-521-29930-6.
- Charles W. Curtis (1968) *Linear Algebra*, page 62, Allyn & Bacon, Boston.
- Heinrich Guggenheimer (1977) *Applicable Geometry*, page 7, Krieger, Huntington ISBN 0-88275-368-1 .
- Victor V. Prasolov & VM Tikhomirov (1997,2001) *Geometry*, page 22, volume 200 in *Translations of Mathematical Monographs*, American Mathematical Society, Providence ISBN 0-8218-2038-9 .

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- Weisstein, Eric W. "Flat" (https://mathworld.wolfram.com/Flat.html). *MathWorld*.