WIKIPEDIA

Hermite–Hadamard inequality

In mathematics, the **Hermite–Hadamard inequality**, named after Charles Hermite and Jacques Hadamard and sometimes also called **Hadamard's inequality**, states that if a function $f : [a, b] \to \mathbf{R}$ is convex, then the following chain of inequalities hold:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}.$$

The inequality has been generalized to higher dimensions: if $\Omega \subset \mathbb{R}^n$ is a bounded, convex domain and $f : \Omega \to \mathbb{R}$ is a positive convex function, then

$$\frac{1}{|\Omega|}\int_\Omega f(x)\,dx \leq \frac{c_n}{|\partial\Omega|}\int_{\partial\Omega} f(y)\,d\sigma(y)$$

where c_n is a constant depending only on the dimension.

A corollary on Vandermonde-type integrals

Suppose that $-\infty < a < b < \infty$, and choose n distinct values $\{x_j\}_{j=1}^{n}$ from (a, b). Let f:$[a, b] \to \mathbb{R}$ be convex, and let I denote the "integral starting at a" operator; that is,

$$(If)(x) = \int_a^x f(t)\,dt.$$

Then

$$\sum_{i=1}^n \frac{(I^{n-1}F)(x_i)}{\prod_{j\neq i}(x_i - x_j)} \leq \frac{1}{n!}\sum_{i=1}^n f(x_i)$$

Equality holds for all $\{x_j\}_{j=1}^{n}$ iff f is linear, and for all f iff $\{x_j\}_{j=1}^{n}$ is constant, in the sense that

$$\lim_{\{x_j\}_j \to \alpha} \sum_{i=1}^n \frac{(I^{n-1}F)(x_i)}{\prod_{j\neq i}(x_i - x_j)} = \frac{f(\alpha)}{(n-1)!}$$

The result follows from induction on n.

References

- Jacques Hadamard, "Étude sur les propriétés des fonctions entières et en particulier d'une fonction considérée par Riemann", *Journal de Mathématiques Pures et Appliquées*, volume 58, 1893, pages 171–215.

- Zoltán Retkes, "An extension of the Hermite–Hadamard Inequality", *Acta Sci. Math. (Szeged)*, 74 (2008), pages 95–106.
- Mihály Bessenyei, "The Hermite–Hadamard Inequality on Simplices", *American Mathematical Monthly*, volume 115, April 2008, pages 339–345.
- Flavia-Corina Mitroi, Eleutherius Symeonidis, "The converse of the Hermite-Hadamard inequality on simplices", Expo. Math. 30 (2012), pp. 389–396. doi:10.1016/j.exmath.2012.08.011 (https://doi.org/10.1016%2Fj.exmath.2012.08.011); ISSN 0723-0869 (https://www.worldcat.org/search?fq=x0:jrnl&q=n2:0723-0869)
- Stefan Steinerberger, The Hermite-Hadamard Inequality in Higher Dimensions, The Journal of Geometric Analysis, 2019.

Retrieved from "https://en.wikipedia.org/w/index.php?title=Hermite–Hadamard_inequality&oldid=1007941195"