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Ursescu theorem

In mathematics, particularly in functional analysis and convex analysis, the **Ursescu theorem** is a theorem that generalizes the closed graph theorem, the open mapping theorem, and the uniform boundedness principle.

Contents

Ursescu Theorem

The following notation and notions are used, where $\mathcal{R} : X \rightrightarrows Y$ is a multivalued function and S is a non-empty subset of a topological vector space X:

- the affine span of S is denoted by $\operatorname{aff} S$ and the linear span is denoted by $\operatorname{span} S$.
- $S^i := \operatorname{aint}_X S$ denotes the algebraic interior of S in X.
- ${}^i S := \operatorname{aint}_{\operatorname{aff}(S-S)} S$ denotes the relative algebraic interior of S (i.e. the algebraic interior of S in $\operatorname{aff}(S-S)$).
- ${}^{ib} S := {}^i S$ if $\operatorname{span}(S - s_0)$ is barreled for some/every $s_0 \in S$ while ${}^{ib} S := \varnothing$ otherwise.
  - If S is convex then it can be shown that for any $x \in X$, $x \in {}^{ib} S$ if and only if the cone generated by $S - x$ is a barreled linear subspace of X or equivalently, if and only if $\cup_{n \in \mathbb{N}} n(S - x)$ is a barreled linear subspace of X
- The **domain of** $\mathcal{R}$ is $\operatorname{Dom} \mathcal{R} := \{x \in X : \mathcal{R}(x) \neq \varnothing\}$.
- The **image of** $\mathcal{R}$ is $\operatorname{Im} \mathcal{R} := \cup_{x \in X} \mathcal{R}(x)$. For any subset $A \subseteq X$, $\mathcal{R}(A) := \cup_{x \in A} \mathcal{R}(x)$.
- The **graph of** $\mathcal{R}$ is $\operatorname{gr} \mathcal{R} := \{(x, y) \in X \times Y : y \in \mathcal{R}(x)\}$.

- $\mathcal{R}$ is **closed** (respectively, **convex**) if the graph of $\mathcal{R}$ is closed (resp. convex) in $X \times Y$.
  - Note that $\mathcal{R}$ is convex if and only if for all $x_0, x_1 \in X$ and all $r \in [0,1]$, $r\mathcal{R}(x_0) + (1-r)\mathcal{R}(x_1) \subseteq \mathcal{R}(rx_0 + (1-r)x_1)$.
- The **inverse of** $\mathcal{R}$ is the multifunction $\mathcal{R}^{-1} : Y \rightrightarrows X$ defined by $\mathcal{R}^{-1}(y) := \{x \in X : y \in \mathcal{R}(x)\}$. For any subset $B \subseteq Y$, $\mathcal{R}^{-1}(B) := \cup_{y \in B} \mathcal{R}^{-1}(y)$.
  - If $f : X \to Y$ is a function, then its inverse is the multifunction $f^{-1} : Y \rightrightarrows X$ obtained from canonically identifying f with the multifunction $f : X \rightrightarrows Y$ defined by $x \mapsto \{f(x)\}$.
- $\operatorname{int}_T S$ is the topological interior of S with respect to T, where $S \subseteq T$.
- $\operatorname{rint} S := \operatorname{int}_{\operatorname{aff} S} S$ is the interior of S with respect to $\operatorname{aff} S$.

Statement

Theorem[1] **(Ursescu)** — Let X be a complete semi-metrizable locally convex topological vector space and $\mathcal{R} : X \rightrightarrows Y$ be a closed convex multifunction with non-empty domain. Assume that $\operatorname{span}(\operatorname{Im}\mathcal{R} - y)$ is a barrelled space for some/every $y \in \operatorname{Im}\mathcal{R}$. Assume that $y_0 \in {}^{i}(\operatorname{Im}\mathcal{R})$ and let $x_0 \in \mathcal{R}^{-1}(y_0)$ (so that $y_0 \in \mathcal{R}(x_0)$). Then for every neighborhood U of x_0 in X, y_0 belongs to the relative interior of $\mathcal{R}(U)$ in $\operatorname{aff}(\operatorname{Im}\mathcal{R})$ (that is, $y_0 \in \operatorname{int}_{\operatorname{aff}(\operatorname{Im}\mathcal{R})} \mathcal{R}(U)$). In particular, if ${}^{ib}(\operatorname{Im}\mathcal{R}) \neq \varnothing$ then ${}^{ib}(\operatorname{Im}\mathcal{R}) = {}^{i}(\operatorname{Im}\mathcal{R}) = \operatorname{rint}(\operatorname{Im}\mathcal{R})$.

Corollaries

Closed graph theorem

Closed graph theorem — Let X and Y be Fréchet spaces and $T : X \to Y$ be a linear map. Then T is continuous if and only if the graph of T is closed in $X \times Y$.

Proof

For the non-trivial direction, assume that the graph of T is closed and let $\mathcal{R} := T^{-1} : Y \rightrightarrows X$. It is easy to see that $\operatorname{gr}\mathcal{R}$ is closed and convex and that its image is X. Given $x \in X$, (Tx, x) belongs to $Y \times X$ so that for every open neighborhood V of Tx in Y, $\mathcal{R}(V) = T^{-1}(V)$ is a neighborhood of x in X. Thus T is continuous at x. ∎

Uniform boundedness principle

Uniform boundedness principle — Let X and Y be Fréchet spaces and $T : X \to Y$ be a bijective linear map. Then T is continuous if and only if $T^{-1} : Y \to X$ is continuous. Furthermore, if T is continuous then T is an isomorphism of Fréchet spaces.

Proof

Apply the closed graph theorem to T and T^{-1}. ■

Open mapping theorem

Open mapping theorem — Let X and Y be Fréchet spaces and $T : X \to Y$ be a continuous surjective linear map. Then T is an open map.

Proof

Clearly, T is a closed and convex relation whose image is Y. Let U be a non-empty open subset of X, let y be in $T(U)$, and let x in U be such that $y = Tx$. From the Ursescu theorem it follows that $T(U)$ is a neighborhood of y. ■

Additional corollaries

The following notation and notions are used for these corollaries, where $\mathcal{R} : X \rightrightarrows Y$ is a multifunction, S is a non-empty subset of a topological vector space X:

- a **convex series with elements of** S is a series of the form $\sum_{i=1}^{\infty} r_i s_i$ where all $s_i \in S$ and $\sum_{i=1}^{\infty} r_i = 1$ is a series of non-negative numbers. If $\sum_{i=1}^{\infty} r_i s_i$ converges then the series is called **convergent** while if $(s_i)_{i=1}^{\infty}$ is bounded then the series is called **bounded** and **b-convex**.
- S is **ideally convex** if any convergent b-convex series of elements of S has its sum in S.
- S is **lower ideally convex** if there exists a Fréchet space Y such that S is equal to the projection onto X of some ideally convex subset *B* of $X \times Y$. Every ideally convex set is lower ideally convex.

Corollary — Let X be a barreled first countable space and let C be a subset of X. Then:

1. If C is lower ideally convex then $C^i = \operatorname{int} C$.
2. If C is ideally convex then $C^i = \operatorname{int} C = \operatorname{int}(\operatorname{cl} C) = (\operatorname{cl} C)^i$.

Related theorems

Simons' theorem

Simons' theorem[2] — Let X and Y be first countable with X locally convex. Suppose that $\mathcal{R} : X \rightrightarrows Y$ is a multimap with non-empty domain that satisfies condition (Hw*x*) or else assume that X is a Fréchet space and that $\mathcal{R}$ is lower ideally convex. Assume that $\operatorname{span}(\operatorname{Im} \mathcal{R} - y)$ is barreled for some/every $y \in \operatorname{Im} \mathcal{R}$. Assume that $y_0 \in {}^i(\operatorname{Im} \mathcal{R})$ and let $x_0 \in \mathcal{R}^{-1}(y_0)$. Then for every neighborhood U of x_0 in X, y_0 belongs to the relative interior of $\mathcal{R}(U)$ in $\operatorname{aff}(\operatorname{Im} \mathcal{R})$ (i.e. $y_0 \in \operatorname{int}_{\operatorname{aff}(\operatorname{Im} \mathcal{R})} \mathcal{R}(U)$). In particular, if ${}^{ib}(\operatorname{Im} \mathcal{R}) \neq \varnothing$ then ${}^{ib}(\operatorname{Im} \mathcal{R}) = {}^i(\operatorname{Im} \mathcal{R}) = \operatorname{rint}(\operatorname{Im} \mathcal{R})$.

Robinson–Ursescu theorem

The implication (1) $\implies$ (2) in the following theorem is known as the Robinson–Ursescu theorem.[3]

Robinson–Ursescu theorem[3] — Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be normed spaces and $\mathcal{R} : X \rightrightarrows Y$ be a multimap with non-empty domain. Suppose that Y is a barreled space, the graph of $\mathcal{R}$ verifies condition condition (Hw*x*), and that $(x_0, y_0) \in \operatorname{gr} \mathcal{R}$. Let C_X (resp. C_Y) denote the closed unit ball in X (resp. Y) (so $C_X = \{x \in X : \|x\| \leq 1\}$). Then the following are equivalent:

1. y_0 belongs to the algebraic interior of $\operatorname{Im} \mathcal{R}$.
2. $y_0 \in \operatorname{int} \mathcal{R}(x_0 + C_X)$.
3. There exists $B > 0$ such that for all $0 \leq r \leq 1$, $y_0 + BrC_Y \subseteq \mathcal{R}(x_0 + rC_X)$.
4. There exist $A > 0$ and $B > 0$ such that for all $x \in x_0 + AC_X$ and all $y \in y_0 + AC_Y$, $d(x, \mathcal{R}^{-1}(y)) \leq B \cdot d(y, \mathcal{R}(x))$.
5. There exists $B > 0$ such that for all $x \in X$ and all $y \in y_0 + BC_Y$, $d(x, \mathcal{R}^{-1}(y)) \leq \dfrac{1 + \|x - x_0\|}{B - \|y - y_0\|} \cdot d(y, \mathcal{R}(x))$.

See also

- Closed graph theorem – Theorem relating continuity to graphs
- Closed graph theorem (functional analysis) – Theorems for deducing continuity
- Open mapping theorem (functional analysis) – Condition for a linear operator to be open
- Surjection of Fréchet spaces – Characterization of surjectivity
- Uniform boundedness principle – A theorem stating that pointwise boundedness implies uniform boundedness
- Webbed space – Spaces where open mapping and closed graphs theorems hold

Notes

1. Zălinescu 2002, p. 23.
2. Zălinescu 2002, p. 22-23.
3. Zălinescu 2002, p. 24.

References

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