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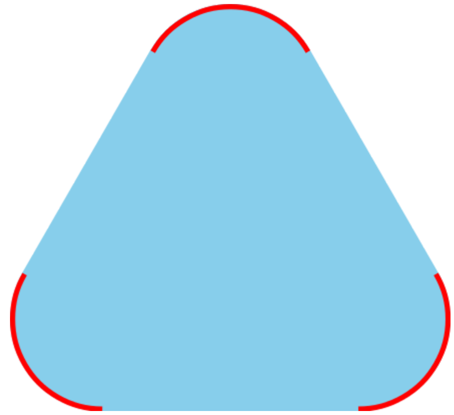
Extreme point

In mathematics, an **extreme point** of a convex set S in a real or complex vector space is a point in S which does not lie in any open line segment joining two points of S. In linear programming problems, an extreme point is also called vertex or corner point of S.[1]



A convex set in light blue, and its extreme points in red.

Contents

Definition

Throughout, it is assumed that X is a real or complex vector space.

For any $p, x, y \in X$, say that p **lies between**[2] x and y if $x \neq y$ and there exists a $0 < t < 1$ such that $p = tx + (1-t)y$.

If K is a subset of X and $p \in K$, then p is called an **extreme point**[2] of K if it does not lie between any two *distinct* points of K. That is, if there does *not* exist $x, y \in K$ and $0 < t < 1$ such that $x \neq y$ and $p = tx + (1-t)y$. The set of all extreme points of K is denoted by $\operatorname{extreme}(K)$.

Characterizations

The **midpoint**[2] of two elements x and y in a vector space is the vector $\frac{1}{2}(x+y)$.

For any elements x and y in a vector space, the set $[x, y] = \{tx + (1-t)y : 0 \leq t \leq 1\}$ is called the **closed line segment** or **closed interval** between x and y. The **open line segment** or **open interval** between x and y is $(x, x) = \varnothing$ when $x = y$ while it is $(x, y) = \{tx + (1-t)y : 0 < t < 1\}$ when $x \neq y$.[2] The points x and y are called the **endpoints** of these interval. An interval is said to be a **non−degenerate interval** or a **proper interval** if its endpoints are distinct. The **midpoint of an interval** is the midpoint of its endpoints.

The closed interval $[x, y]$ is equal to the convex hull of (x, y) if (and only if) $x \neq y$. So if K is convex and $x, y \in K$, then $[x, y] \subseteq K$.

If K is a nonempty subset of X and F is a nonempty subset of K, then F is called a **face**[2] of K if whenever a point $p \in F$ lies between two points of K, then those two points necessarily belong to F.

Theorem[2] — Let K be a non-empty convex subset of a vector space X and let $p \in K$. Then the following statements are equivalent:

1. p is an extreme point of K.
2. $K \setminus \{p\}$ is convex.
3. p is not the midpoint of a non-degenerate line segment contained in K.
4. for any $x, y \in K$, if $p \in [x, y]$ then $x = p$ or $y = p$.
5. if $x \in X$ is such that both $p + x$ and $p - x$ belong to K, then $x = 0$.
6. $\{p\}$ is a face of K.

Examples

If $a < b$ are two real numbers then a and b are extreme points of the interval $[a, b]$. However, the open interval (a, b) has no extreme points.[2] Any open interval in $\mathbb{R}$ has no extreme points while any non-degenerate closed interval not equal to $\mathbb{R}$ does have extreme points (that is, the closed interval's endpoint(s)). More generally, any open subset of finite-dimensional Euclidean space $\mathbb{R}^n$ has no extreme points.

The extreme points of the closed unit disk in $\mathbb{R}^2$ is the unit circle.

The perimeter of any convex polygon in the plane is a face of that polygon.[2] The vertices of any convex polygon in the plane $\mathbb{R}^2$ are the extreme points of that polygon.

An injective linear map $F : X \to Y$ sends the extreme points of a convex set $C \subseteq X$ to the extreme points of the convex set $F(X)$.[2] This is also true for injective affine maps.

Properties

The extreme points of a compact convex form a Baire space (with the subspace topology) but this set may *fail* to be closed in X.[2]

Theorems

Krein–Milman theorem

The Krein–Milman theorem is arguably one of the most well-known theorems about extreme points.

> **Krein–Milman theorem** — If S is convex and compact in a locally convex topological vector space, then S is the closed convex hull of its extreme points: In particular, such a set has extreme points.

For Banach spaces

These theorems are for Banach spaces with the Radon–Nikodym property.

A theorem of Joram Lindenstrauss states that, in a Banach space with the Radon–Nikodym property, a nonempty closed and bounded set has an extreme point. (In infinite-dimensional spaces, the property of compactness is stronger than the joint properties of being closed and being bounded).[3]

> **Theorem** (Gerald Edgar) — Let E be a Banach space with the Radon-Nikodym property, let C be a separable, closed, bounded, convex subset of E, and let a be a point in C. Then there is a probability measure p on the universally measurable sets in C such that a is the barycenter of p, and the set of extreme points of C has p-measure 1.[4]

Edgar's theorem implies Lindenstrauss's theorem.

Related notions

A closed convex subset of a topological vector space is called *strictly convex* if every one of its (topological) boundary points is an extreme point.[5] The unit ball of any Hilbert space is a strictly convex set.[5]

k-extreme points

More generally, a point in a convex set S is **k-extreme** if it lies in the interior of a k-dimensional convex set within S, but not a $k+1$-dimensional convex set within S. Thus, an extreme point is also a 0-extreme point. If S is a polytope, then the k-extreme points are exactly the interior points of the k-dimensional faces of S. More generally, for any convex set S, the k-extreme points are partitioned into k-dimensional open faces.

The finite-dimensional Krein-Milman theorem, which is due to Minkowski, can be quickly proved using the concept of k-extreme points. If S is closed, bounded, and n-dimensional, and if p is a point in S, then p is k-extreme for some $k \leq n$. The theorem asserts that p is a convex combination of extreme points. If $k = 0$ then it is immediate. Otherwise p lies on a line segment in S which can be maximally extended (because S is closed and bounded). If the endpoints of the segment are q and r, then their extreme rank must be less than that of p, and the theorem follows by induction.

See also

- Choquet theory

Citations

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5. Halmos 1982, p. 5.

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