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Ekeland's variational principle

In mathematical analysis, **Ekeland's variational principle**, discovered by Ivar Ekeland,[1][2][3] is a theorem that asserts that there exist nearly optimal solutions to some optimization problems.

Ekeland's variational principle can be used when the lower level set of a minimization problems is not compact, so that the Bolzano–Weierstrass theorem cannot be applied. Ekeland's principle relies on the completeness of the metric space.[4]

Ekeland's principle leads to a quick proof of the Caristi fixed point theorem.[4][5]

Ekeland's principle has been shown to be equivalent to completeness of metric spaces.[6]

Ekeland was associated with the Paris Dauphine University when he proposed this theorem.[1]

Contents

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Preliminary definitions

Let $f : X \to \mathbb{R} \cup \{-\infty, +\infty\}$ be a function valued in the extended real numbers $\mathbb{R} \cup \{-\infty, +\infty\} = [-\infty, +\infty]$. Then

- $\operatorname{dom} f := \{x \in X : f(x) \neq +\infty\}$ denotes the **effective domain** of f.
- f is **proper** if $\operatorname{dom} f \neq \varnothing$ (that is, if f is not identically $+\infty$).
- f is **bounded below** if $\inf_{x \in X} f(x) > -\infty$.
- given $x_0 \in X$, say that f is *lower semicontinuous* at x_0 if for every real $y < f(x_0)$ there exists a neighborhood U of x_0 such that $f(x) > y$ for all x in U.
- f is lower semicontinuous if it is lower semicontinuous at every point of X.
  - A function is lower semi-continuous if and only if $\{x \in X : \ f(x) > y\}$ is an open set for every $y \in \mathbb{R}$; alternatively, a function is lower semicontinuous if and only if all of its lower level sets $\{x \in X : \ f(x) \leq y\}$ are closed.

Statement of the theorem

Ekeland's variational principle[7] — Let (X, d) be a complete metric space and $f : X \to \mathbb{R} \cup \{+\infty\}$ a proper (that is, not identically $+\infty$) lower semicontinuous function that is bounded below. Pick $\epsilon > 0$ and $x_0 \in X$ such that $f(x_0) \neq +\infty$ (or equivalently, $f(x_0) \in \mathbb{R}$). There exists some $v \in X$ such that

$$f(v) \ \leq \ f(x_0) - \epsilon \, d(x_0, v)$$

and for all $x \neq v$,

$$f(v) \ < \ f(x) + \epsilon \, d(v, x).$$

Proof

Define a function $G : X \times X \to \mathbb{R} \cup \{+\infty\}$ by

$$G(x, y) := f(x) + \epsilon \, d(x, y)$$

which is lower semicontinuous because it is the sum of the lower semicontinuous function f and the continuous function $(x, y) \mapsto \epsilon \, d(x, y)$. Given $z \in X$, define the functions

$$G_z := G(z, \cdot) : X \to \mathbb{R} \cup \{+\infty\} \qquad \text{and} \qquad G^z := G(\cdot, z) : X \to \mathbb{R} \cup \{+\infty\}$$

and define the set

$$F(z) := \{y \in Y : G_z(y) \leq f(z)\} = \{y \in Y : f(z) + \epsilon \, d(z, y) \leq f(z)\}.$$

It may be verified that for all $x \in X$,

1. $F(x)$ is closed (because $G_x := G(x, \cdot) : X \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous);
2. if $x \notin \operatorname{dom} f$ then $F(x) = X$;
3. if $x \in \operatorname{dom} f$ then $x \in F(x) \subseteq \operatorname{dom} f$; in particular, $x_0 \in F(x_0) \subseteq \operatorname{dom} f$;
4. if $y \in F(x)$ then $F(y) \subseteq F(x)$.

Let $s_0 = \inf_{x \in F(x_0)} f(x)$, which is a real number because f was assumed to be bounded below. Pick $x_1 \in F(x_0)$ such that $f(x_1) < s_0 + 2^{-1}$. Having defined s_{n-1} and x_n, let

$$s_n := \inf_{x \in F(x_n)} f(x)$$

and pick $x_{n+1} \in F(x_n)$ such that $f(x_{n+1}) < s_n + 2^{-(n+1)}$.

These sequences have the following properties:

- for all $n \geq 0$, $F(x_{n+1}) \subseteq F(x_n)$ because $x_{n+1} \in F(x_n)$, where this now implies that $s_{n+1} \geq s_n$;
- for all $n \geq 1$, because $x_{n+1} \in F(x_n)$

$$\epsilon\, d(x_{n+1}, x_n) \;\leq\; f(x_n) - f(x_{n+1}) \;\leq\; f(x_n) - s_n \;\leq\; f(x_n) - s_{n-1} \;<\; 2^{-n}.$$

It follows that for all $n, p \geq 1$,

$$d(x_{n+1}, x_n) \;\leq\; \epsilon\, 2^{1-n},$$

which proves that $x_\bullet := (x_n)_{n=0}^{\infty}$ is a Cauchy sequence. Because X is a complete metric space, there exists some $v \in X$ such that $x_\bullet$ converges to v. The fact that $x_m \in F(x_n)$ for all $m \geq n$ implies that $v \in \operatorname{cl}_Y F(x_n) = F(x_n)$ for all $n \geq 0$, where in particular, $v \in F(x_0)$.

The conclusion of the theorem will follow once it is shown that $F(v) = \{v\}$. So let $x \in F(v)$. Because $x \in F(x_n)$ for all $n \geq 0$, it follows as above that $\epsilon\, d(x, x_n) \leq 2^{-n}$, which implies that $x_\bullet$ converges to x. Since the limit of $x_\bullet$ is unique, it follows that $x = v$. Thus $F(v) = \{v\}$, as desired. ■

Corollaries

Corollary[8] — **Corollary**: Let (X, d) be a complete metric space, and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous functional on X that is bounded below and not identically equal to $+\infty$. Fix $\epsilon > 0$ and a point $x_0 \in X$ such that

$$f(x_0) \;\leq\; \varepsilon + \inf_{x \in X} f(x).$$

Then, for every $\lambda > 0$, there exists a point $v \in X$ such that

$$f(v) \;\leq\; f(x_0),$$

$$d(x_0, v) \;\leq\; \lambda,$$

and, for all $x \neq v$,

$$f(x) \;>\; f(v) - \frac{\varepsilon}{\lambda} d(v, x).$$

A good compromise is to take $\lambda := \sqrt{\epsilon}$ in the preceding result.[8]

References

1. Ekeland, Ivar (1974). "On the variational principle" (https://doi.org/10.1016%2F0022-247X%2874%2990025-0). *J. Math. Anal. Appl.* **47** (2): 324–353. doi:10.1016/0022-247X(74)90025-0 (https://doi.org/10.1016%2F0022-247X%2874%2990025-0). ISSN 0022-247X (https://www.worldcat.org/issn/0022-247X).
2. Ekeland, Ivar (1979). "Nonconvex minimization problems" (https://doi.org/10.1090%2FS0273-0979-1979-14595-6). *Bulletin of the American Mathematical Society*. New Series. **1** (3): 443–474. doi:10.1090/S0273-0979-1979-14595-6 (https://doi.org/10.1090%2FS0273-0979-1979-14595-6). MR 0526967 (https://www.ams.org/mathscinet-getitem?mr=0526967).
3. Ekeland, Ivar; Temam, Roger (1999). *Convex analysis and variational problems*. Classics in applied mathematics. Vol. 28 (Corrected reprinting of the (1976) North-Holland ed.). Philadelphia, PA: Society for Industrial and Applied Mathematics (SIAM). pp. 357–373. ISBN 0-89871-450-8. MR 1727362 (https://www.ams.org/mathscinet-getitem?mr=1727362).
4. Kirk, William A.; Goebel, Kazimierz (1990). *Topics in Metric Fixed Point Theory*. Cambridge University Press. ISBN 0-521-38289-0.
5. Ok, Efe (2007). "D: Continuity I". *Real Analysis with Economic Applications* (http://homepages.nyu.edu/~eo1/Book-PDF/Ekeland.pdf) (PDF). Princeton University Press. p. 664. ISBN 978-0-691-11768-3. Retrieved January 31, 2009.
6. Sullivan, Francis (October 1981). "A characterization of complete metric spaces" (https://doi.org/10.1090%2FS0002-9939-1981-0624927-9). *Proceedings of the American Mathematical Society*. **83** (2): 345–346. doi:10.1090/S0002-9939-1981-0624927-9 (https://doi.org/10.1090%2FS0002-9939-1981-0624927-9). MR 0624927 (https://www.ams.org/mathscinet-getitem?mr=0624927).
7. Zalinescu 2002, p. 29.
8. Zalinescu 2002, p. 30.

Bibliography

- Ekeland, Ivar (1979). "Nonconvex minimization problems" (https://doi.org/10.1090%2FS0273-0979-1979-14595-6). *Bulletin of the American Mathematical Society*. New Series. **1** (3): 443–474. doi:10.1090/S0273-0979-1979-14595-6 (https://doi.org/10.1090%2FS0273-0979-1979-14595-6). MR 0526967 (https://www.ams.org/mathscinet-getitem?mr=0526967).
- Kirk, William A.; Goebel, Kazimierz (1990). *Topics in Metric Fixed Point Theory*. Cambridge University Press. ISBN 0-521-38289-0.
- Zalinescu, C (2002). *Convex analysis in general vector spaces*. River Edge, N.J. London: World Scientific. ISBN 981-238-067-1. OCLC 285163112 (https://www.worldcat.org/oclc/285163112).
- Zălinescu, Constantin (30 July 2002). *Convex Analysis in General Vector Spaces* (https://archive.org/details/convexanalysisge00zali_934). River Edge, N.J. London: World Scientific Publishing. ISBN 978-981-4488-15-0. MR 1921556 (https://www.ams.org/mathscinet-getitem?mr=1921556). OCLC 285163112 (https://www.worldcat.org/oclc/285163112) – via Internet Archive.