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M. Riesz extension theorem

The **M. Riesz extension theorem** is a theorem in mathematics, proved by Marcel Riesz[1] during his study of the problem of moments.[2]

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Formulation

Let E be a real vector space, $F \subset E$ be a vector subspace, and $K \subset E$ be a convex cone.

A linear functional $\phi : F \to \mathbb{R}$ is called K-*positive*, if it takes only non-negative values on the cone K:

$$\phi(x) \geq 0 \quad \text{for} \quad x \in F \cap K.$$

A linear functional $\psi : E \to \mathbb{R}$ is called a K-positive *extension* of ϕ, if it is identical to ϕ in the domain of ϕ, and also returns a value of at least 0 for all points in the cone K:

$$\psi|_F = \phi \quad \text{and} \quad \psi(x) \geq 0 \quad \text{for} \quad x \in K.$$

In general, a K-positive linear functional on F cannot be extended to a K-positive linear functional on E. Already in two dimensions one obtains a counterexample. Let $E = \mathbb{R}^2,\ K = \{(x,y) : y > 0\} \cup \{(x,0) : x > 0\}$, and F be the x-axis. The positive functional $\phi(x,0) = x$ can not be extended to a positive functional on E.

However, the extension exists under the additional assumption that $E \subset K + F$, namely for every $y \in E$, there exists an $x \in F$ such that $y - x \in K$.

Proof

The proof is similar to the proof of the Hahn–Banach theorem (see also below).

By transfinite induction or Zorn's lemma it is sufficient to consider the case dim $E/F = 1$.

Choose any $y \in E \setminus F$. Set

$$a = \sup\{\, \phi(x) \mid x \in F,\ y - x \in K \,\},\ b = \inf\{\, \phi(x) \mid x \in F, x - y \in K \,\}.$$

We will prove below that $-\infty < a \leq b$. For now, choose any c satisfying $a \leq c \leq b$, and set $\psi(y) = c$, $\psi|_F = \phi$, and then extend ψ to all of E by linearity. We need to show that ψ is K-positive. Suppose $z \in K$. Then either $z = 0$, or $z = p(x + y)$ or $z = p(x - y)$ for some $p > 0$ and $x \in F$. If $z = 0$, then $\psi(z) > 0$. In the first remaining case $x + y = y - (-x) \in K$, and so

$$\psi(y) = c \geq a \geq \phi(-x) = \psi(-x)$$

by definition. Thus

$$\psi(z) = p\psi(x + y) = p(\psi(x) + \psi(y)) \geq 0.$$

In the second case, $x - y \in K$, and so similarly

$$\psi(y) = c \leq b \leq \phi(x) = \psi(x)$$

by definition and so

$$\psi(z) = p\psi(x - y) = p(\psi(x) - \psi(y)) \geq 0.$$

In all cases, $\psi(z) > 0$, and so ψ is K-positive.

We now prove that $-\infty < a \leq b$. Notice by assumption there exists at least one $x \in F$ for which $y - x \in K$, and so $-\infty < a$. However, it may be the case that there are no $x \in F$ for which $x - y \in K$, in which case $b = \infty$ and the inequality is trivial (in this case notice that the third case above cannot happen). Therefore, we may assume that $b < \infty$ and there is at least one $x \in F$ for which $x - y \in K$. To prove the inequality, it suffices to show that whenever $x \in F$ and $y - x \in K$, and $x' \in F$ and $x' - y \in K$, then $\phi(x) \leq \phi(x')$. Indeed,

$$x' - x = (x' - y) + (y - x) \in K$$

since K is a convex cone, and so

$$0 \leq \phi(x' - x) = \phi(x') - \phi(x)$$

since ϕ is K-positive.

Corollary: Krein's extension theorem

Let E be a real linear space, and let $K \subset E$ be a convex cone. Let $x \in E \backslash (-K)$ be such that $\mathbf{R}\, x + K = E$. Then there exists a K-positive linear functional $\varphi: E \to \mathbf{R}$ such that $\varphi(x) > 0$.

Connection to the Hahn–Banach theorem

The Hahn–Banach theorem can be deduced from the M. Riesz extension theorem.

Let V be a linear space, and let N be a sublinear function on V. Let φ be a functional on a subspace $U \subset V$ that is dominated by N:

$$\phi(x) \leq N(x), \quad x \in U.$$

The Hahn–Banach theorem asserts that φ can be extended to a linear functional on V that is dominated by N.

To derive this from the M. Riesz extension theorem, define a convex cone $K \subset \mathbf{R} \times V$ by

$$K = \{(a, x) \mid N(x) \leq a\}.$$

Define a functional φ_1 on $\mathbf{R} \times U$ by

$$\phi_1(a, x) = a - \phi(x).$$

One can see that φ_1 is K-positive, and that $K + (\mathbf{R} \times U) = \mathbf{R} \times V$. Therefore φ_1 can be extended to a K-positive functional ψ_1 on $\mathbf{R} \times V$. Then

$$\psi(x) = -\psi_1(0, x)$$

is the desired extension of φ. Indeed, if $\psi(x) > N(x)$, we have: $(N(x), x) \in K$, whereas

$$\psi_1(N(x), x) = N(x) - \psi(x) < 0,$$

leading to a contradiction.

Notes

1. Riesz (1923)
2. Akhiezer (1965)

References

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