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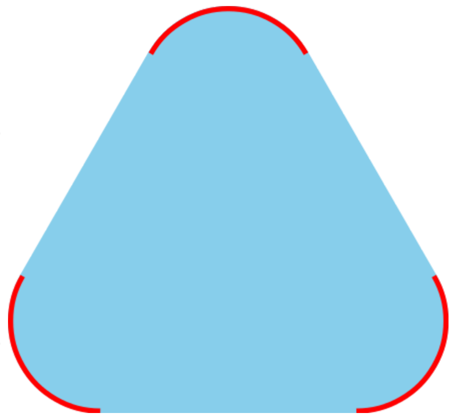
Krein–Milman theorem

In the mathematical theory of functional analysis, the **Krein–Milman theorem** is a proposition about compact convex sets in locally convex topological vector spaces (TVSs).



Given a convex shape K (light blue) and its set of extreme points B (red), the convex hull of B is K.

> **Krein–Milman theorem** — A compact convex subset of a Hausdorff locally convex topological vector space is equal to the closed convex hull of its extreme points.

This theorem generalizes to infinite-dimensional spaces and to arbitrary compact convex sets the following basic observation: a convex (i.e. "filled") triangle, including its perimeter and the area "inside of it", is equal to the convex hull of its three vertices, where these vertices are exactly the extreme points of this shape. This observation also holds for any other convex polygon in the plane $\mathbb{R}^2$.

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Preliminaries and definitions

Throughout, X will be a real or complex vector space.

For any elements x and y in a vector space, the set $[x, y] := \{tx + (1-t)y : 0 \leq t \leq 1\}$ is called the **closed line segment** or **closed interval** between x and y. The **open line segment** or **open interval** between x and y is $(x, x) := \varnothing$ when $x = y$ while it is $(x, y) := \{tx + (1-t)y : 0 < t < 1\}$ when $x \neq y$;[1] it satisfies $(x, y) = [x, y] \setminus \{x, y\}$ and $[x, y] = (x, y) \cup \{x, y\}$. The points x and y are called the **endpoints** of these interval. An interval is said to be **non-degenerate** or **proper** if its endpoints are distinct.

The intervals $[x, x] = \{x\}$ and $[x, y]$ always contain their endpoints while $(x, x) = \varnothing$ and (x, y) never contain either of their endpoints. If x and y are points in the real line $\mathbb{R}$ then the above definition of $[x, y]$ is the same as its usual definition as a closed interval.

For any $p, x, y \in X$, the point p is said to (strictly) **lie between** x and y if p belongs to the open line segment (x, y).[1]

If K is a subset of X and $p \in K$, then p is called an **extreme point** of K if it does not lie between any two *distinct* points of K. That is, if there does *not* exist $x, y \in K$ and $0 < t < 1$ such that $x \neq y$ and $p = tx + (1-t)y$. In this article, the set of all extreme points of K will be denoted by $\operatorname{extreme}(K)$.[1]

For example, the vertices of any convex polygon in the plane $\mathbb{R}^2$ are the extreme points of that polygon. The extreme points of the closed unit disk in $\mathbb{R}^2$ is the unit circle. Every open interval and degenerate closed interval in $\mathbb{R}$ has no extreme points while the extreme points of a non-degenerate closed interval $[x, y]$ are x and y.

A set S is called **convex** if for any two points $x, y \in S$, S contains the line segment $[x, y]$. The smallest convex set containing S is called the **convex hull** of S and it is denoted by $\operatorname{co} S$. The **closed convex hull** of a set S, denoted by $\overline{\operatorname{co}}(S)$, is the smallest closed and convex set containing S. It is also equal to the intersection of all closed convex subsets that contain S and to the closure of the convex hull of S; that is,

$$\overline{\operatorname{co}}(S) = \overline{\operatorname{co}(S)},$$

where the right hand side denotes the closure of $\operatorname{co}(S)$ while the left hand side is notation. For example, the convex hull of any set of three distinct points forms either a closed line segment (if they are collinear) or else a solid (that is, "filled") triangle, including its perimeter. And in the plane $\mathbb{R}^2$, the unit circle is *not* convex but the closed unit disk is convex and furthermore, this disk is equal to the convex hull of the circle.

Statement

Krein–Milman theorem[1] — Suppose X is a Hausdorff locally convex topological vector space (for example, a normed space) and K is a compact and convex subset of X. Then K is equal to the closed convex hull of its extreme points:

$$K = \overline{\operatorname{co}}(\operatorname{extreme}(K)).$$

Moreover, if $B \subseteq K$ then K is equal to the closed convex hull of B if and only if $\operatorname{extreme} K \subseteq \operatorname{cl} B$, where $\operatorname{cl} B$ is closure of B.

The convex hull of the extreme points of K forms a convex subset of K so the main burden of the proof is to show that there are enough extreme points so that their convex hull covers all of K. For this reason, the following corollary to the above theorem is also often called the Krein–Milman theorem.

(KM) Krein–Milman theorem (Existence)[1] — Every non-empty compact convex subset of a Hausdorff locally convex topological vector space has an extreme point; that is, the set of its extreme points is not empty.

A particular case of this theorem, which can be easily visualized, states that given a convex polygon, the corners of the polygon are all that is needed to recover the polygon shape. The statement of the theorem is false if the polygon is not convex, as then there can be many ways of drawing a polygon having given points as corners.

The requirement that the convex set be compact can be weakened to give the following strengthened version of the theorem.

(SKM) Generalization of Krein–Milman theorem (Existence)[2] — Suppose X is a Hausdorff locally convex topological vector space and K is a non-empty convex subset of X with the property that whenever $\mathcal{C}$ is a cover of K by *convex* closed subsets of X such that $\{K \cap C : C \in \mathcal{C}\}$ has the finite intersection property, then $K \cap \bigcap_{C \in \mathcal{C}} C$ is not empty. Then $\operatorname{extreme}(K)$ is not empty.

More general settings

The assumption of local convexity for the ambient space is necessary, because James Roberts (1977) constructed a counter-example for the non-locally convex space $L^p[0,1]$ where $0 < p < 1$.[3]

Linearity is also needed, because the statement fails for weakly compact convex sets in CAT(0) spaces, as proved by Nicolas Monod (2016).[4] However, Theo Buehler (2006) proved that the Krein–Milman theorem does hold for *metrically* compact CAT(0) spaces.[5]

Related results

Under the previous assumptions on K, if T is a subset of K and the closed convex hull of T is all of K, then every extreme point of K belongs to the closure of T. This result is known as *Milman's* (partial) *converse* to the Krein–Milman theorem.[6]

The Choquet–Bishop–de Leeuw theorem states that every point in K is the barycenter of a probability measure supported on the set of extreme points of K.

Relation to the axiom of choice

Under the Zermelo–Fraenkel set theory (**ZF**) axiomatic framework, the axiom of choice (**AC**) suffices to prove all version of the Krein–Milman theorem given above, including statement **KM** and its generalization **SKM**. The axiom of choice also implies, but is not equivalent to, the Boolean prime ideal theorem (**BPI**), which is equivalent to the Banach–Alaoglu theorem. Conversely, the Krein–Milman theorem **KM** together with the Boolean prime ideal theorem (**BPI**) imply the axiom of choice.[7] In summary, **AC** holds if and only if both **KM** and **BPI** hold.[2] It follows that under **ZF**, the axiom of choice is equivalent to the following statement:

> The closed unit ball of the continuous dual space of any real normed space has an extreme point.[2]

Furthermore, **SKM** together with the Hahn–Banach theorem for real vector spaces (**HB**) are also equivalent to the axiom of choice.[2] It is known that **BPI** implies **HB**, but that it is not equivalent to it (said differently, **BPI** is strictly stronger than **HB**).

History

The original statement proved by Mark Krein and David Milman (1940) was somewhat less general than the form stated here.[8]

Earlier, Hermann Minkowski (1911) proved that if X is 3-dimensional then K equals the convex hull of the set of its extreme points.[9] This assertion was expanded to the case of any finite dimension by Ernst Steinitz (1916).[10] The Krein–Milman theorem generalizes this to arbitrary locally convex X; however, to generalize from finite to infinite dimensional spaces, it is necessary to use the closure.

See also

- Banach–Alaoglu theorem – The closed unit ball in the dual of a normed vector space is compact in the weak* topology
- Carathéodory's theorem (convex hull) – Point in the convex hull of a set P in Rd, is the convex combination of d+1 points in P
- Choquet theory
- Helly's theorem – Theorem about the intersections of d-dimensional convex sets
- Radon's theorem – Says d+2 points in d dimensions can be partitioned into two subsets whose convex hulls intersect
- Shapley–Folkman lemma – Sums of sets of vectors are nearly convex
- Topological vector space – Vector space with a notion of nearness

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