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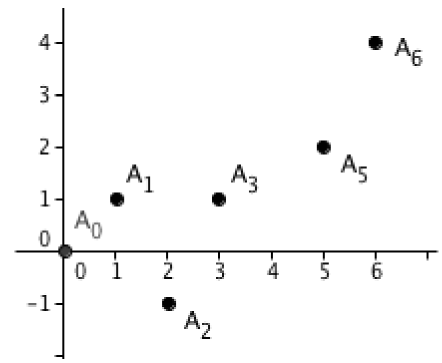
# Newton polygon

In mathematics, the **Newton polygon** is a tool for understanding the behaviour of polynomials over local fields.



Construction of the Newton polygon of the polynomial $P(X) = 1 + 5X + 1/5X^2 + 35X^3$ with respect to the 5-adic valuation.

In the original case, the local field of interest was the field of formal Laurent series in the indeterminate $X$, i.e. the field of fractions of the formal power series ring

$$K[[X]],$$

over $K$, where $K$ was the real number or complex number field. This is still of considerable utility with respect to Puiseux expansions. The Newton polygon is an effective device for understanding the leading terms

$$aX^r$$

of the power series expansion solutions to equations

$$P(F(X)) = 0$$

where $P$ is a polynomial with coefficients in $K[X]$, the polynomial ring; that is, implicitly defined algebraic functions. The exponents $r$ here are certain rational numbers, depending on the branch chosen; and the solutions themselves are power series in

$$K[[Y]]$$

with $Y = X^{1/d}$ for a denominator $d$ corresponding to the branch. The Newton polygon gives an effective, algorithmic approach to calculating $d$.

After the introduction of the p-adic numbers, it was shown that the Newton polygon is just as useful in questions of ramification for local fields, and hence in algebraic number theory. Newton polygons have also been useful in the study of elliptic curves.

## Contents

# Definition

A priori, given a polynomial over a field, the behaviour of the roots (assuming it has roots) will be unknown. Newton polygons provide one technique for the study of the behaviour of the roots.

Let $K$ be a local field with discrete valuation $v_K$ and let

$$f(x) = a_n x^n + \cdots + a_1 x + a_0 \in K[x]$$

with $a_0 a_n \neq 0$. Then the Newton polygon of $f$ is defined to be the lower convex hull of the set of points

$$P_i = (i, v_K(a_i)),$$

ignoring the points with $a_i = 0$. Restated geometrically, plot all of these points $P_i$ on the $xy$-plane. Let's assume that the points indices increase from left to right ($P_0$ is the leftmost point, $P_n$ is the rightmost point). Then, starting at $P_0$, draw a ray straight down parallel with the $y$-axis, and rotate this ray counter-clockwise until it hits the point $P_{k_1}$ (not necessarily $P_1$). Break the ray here. Now draw a second ray from $P_{k_1}$ straight down parallel with the $y$-axis, and rotate this ray counter-clockwise until it hits the point $P_{k_2}$. Continue until the process reaches the point $P_n$; the resulting polygon (containing the points $P_0$, $P_{k_1}$, $P_{k_2}$, ..., $P_{k_m}$, $P_n$) is the Newton polygon.

Another, perhaps more intuitive way to view this process is this : consider a rubber band surrounding all the points $P_0$, ..., $P_n$. Stretch the band upwards, such that the band is stuck on its lower side by some of the points (the points act like nails, partially hammered into the xy plane). The vertices of the Newton polygon are exactly those points.

For a neat diagram of this see Ch6 §3 of "Local Fields" by JWS Cassels, LMS Student Texts 3, CUP 1986. It is on p99 of the 1986 paperback edition.

# History

Newton polygons are named after Isaac Newton, who first described them and some of their uses in correspondence from the year 1676 addressed to Henry Oldenburg.[1]

# Applications

A Newton Polygon is sometimes a special case of a Newton polytope, and can be used to construct asymptotic solutions of two-variable polynomial equations like $3x^2y^3 - xy^2 + 2x^2y^2 - x^3y = 0$

Another application of the Newton polygon comes from the following result:

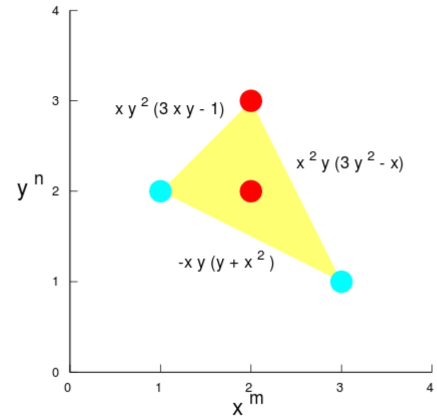
Let

$$\mu_1, \mu_2, \ldots, \mu_r$$

be the slopes of the line segments of the Newton polygon of $f(x)$ (as defined above) arranged in increasing order, and let

$$\lambda_1, \lambda_2, \ldots, \lambda_r$$

be the corresponding lengths of the line segments projected onto the x-axis (i.e. if we have a line segment stretching between the points $P_i$ and $P_j$ then the length is $j - i$). Then for each integer $1 \leq \kappa \leq r$, $f(x)$ has exactly $\lambda_\kappa$ roots with valuation $-\mu_\kappa$.




This diagram shows the Newton polygon for $P(x,y) = 3x^2 y^3 - xy^2 + 2x^2y^2 - x^3y$, with positive monomials in red and negative monomials in cyan. Faces are labelled with the limiting terms they correspond to.

## Symmetric function explanation

In the context of a valuation, we are given certain information in the form of the valuations of elementary symmetric functions of the roots of a polynomial, and require information on the valuations of the actual roots, in an algebraic closure. This has aspects both of ramification theory and singularity theory. The valid inferences possible are to the valuations of power sums, by means of Newton's identities.

## See also

- F-crystal
- Eisenstein's criterion
- Newton–Okounkov body
- Newton polytope

## References

1. Egbert Brieskorn, Horst Knörrer (1986). *Plane Algebraic Curves*, pp. 370–383.

- Goss, David (1996), *Basic structures of function field arithmetic*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 35, Berlin, New York: Springer-Verlag, doi:10.1007/978-3-642-61480-4 (https://doi.org/10.1007%2F978-3-642-61480-4), ISBN 978-3-540-61087-8, MR 1423131 (https://www.ams.org/mathscinet-getitem?mr=1423131)
- Gouvêa, Fernando: p-adic numbers: An introduction. Springer Verlag 1993. p. 199.

## External links

- Applet drawing a Newton Polygon (http://www.math.sc.edu/~filaseta/newton/newton.html)