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K-convex function

***K*-convex functions**, first introduced by Scarf,[1] are a special weakening of the concept of convex function which is crucial in the proof of the optimality of the (s, S) policy in inventory control theory. The policy is characterized by two numbers s and S, $S \geq s$, such that when the inventory level falls below level s, an order is issued for a quantity that brings the inventory up to level S, and nothing is ordered otherwise. Gallego and Sethi [2] have generalized the concept of K-convexity to higher dimensional Euclidean spaces.

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Definition

Two equivalent definitions are as follows:

Definition 1 (The original definition)

Let K be a non-negative real number. A function $g : \mathbb{R} \to \mathbb{R}$ is K-convex if

$$g(u) + z\left[\frac{g(u) - g(u-b)}{b}\right] \leq g(u+z) + K$$

for any $u, z \geq 0$, and $b > 0$.

Definition 2 (Definition with geometric interpretation)

A function $g : \mathbb{R} \to \mathbb{R}$ is K-convex if

$$g(\lambda x + \bar{\lambda} y) \leq \lambda g(x) + \bar{\lambda}[g(y) + K]$$

for all $x \leq y, \lambda \in [0,1]$, where $\bar{\lambda} = 1 - \lambda$.

This definition admits a simple geometric interpretation related to the concept of visibility.[3] Let $a \geq 0$. A point $(x, f(x))$ is said to be visible from $(y, f(y) + a)$ if all intermediate points $(\lambda x + \bar{\lambda} y, f(\lambda x + \bar{\lambda} y)), 0 \leq \lambda \leq 1$ lie below the line segment joining these two points. Then the geometric characterization of *K*-convexity can be obtain as:

> A function g is *K*-convex if and only if $(x, g(x))$ is visible from $(y, g(y) + K)$ for all $y \geq x$.

Proof of Equivalence

It is sufficient to prove that the above definitions can be transformed to each other. This can be seen by using the transformation

$$\lambda = z/(b+z), \quad x = u - b, \quad y = u + z.$$

Properties

[4]

Property 1

If $g : \mathbb{R} \to \mathbb{R}$ is *K*-convex, then it is *L*-convex for any $L \geq K$. In particular, if g is convex, then it is also *K*-convex for any $K \geq 0$.

Property 2

If g_1 is *K*-convex and g_2 is *L*-convex, then for $\alpha \geq 0, \beta \geq 0,\ g = \alpha g_1 + \beta g_2$ is $(\alpha K + \beta L)$-convex.

Property 3

If g is *K*-convex and ξ is a random variable such that $E|g(x - \xi)| < \infty$ for all x, then $Eg(x - \xi)$ is also *K*-convex.

Property 4

If $g : \mathbb{R} \to \mathbb{R}$ is *K*-convex, restriction of g on any convex set $\mathbb{D} \subset \mathbb{R}$ is *K*-convex.

Property 5

If $g : \mathbb{R} \to \mathbb{R}$ is a continuous *K*-convex function and $g(y) \to \infty$ as $|y| \to \infty$, then there exit scalars s and S with $s \leq S$ such that

- $g(S) \leq g(y)$, for all $y \in \mathbb{R}$;
- $g(S) + K = g(s) < g(y)$, for all $y < s$;
- $g(y)$ is a decreasing function on $(-\infty, s)$;
- $g(y) \leq g(z) + K$ for all y, z with $s \leq y \leq z$.

References

1. Scarf, H. (1960). *The Optimality of (S, s) Policies in the Dynamic Inventory Problem*. Stanford, CA: Stanford University Press. p. Chapter 13.
2. Gallego, G. and Sethi, S. P. (2005). *K*-convexity in $\Re^n$. *Journal of Optimization Theory & Applications,* 127(1):71-88.
3. Kolmogorov, A. N.; Fomin, S. V. (1970). *Introduction to Real Analysis*. New York: Dover Publications Inc.
4. Sethi S P, Cheng F. Optimality of (s, S) Policies in Inventory Models with Markovian Demand. INFORMS, 1997.

External links

- Gallego, Guillermo; Sethi, Suresh (16 September 2004). "K-CONVEXITY IN $\Re^n$" (https://www.utdallas.edu/~sethi/Postscript/Kconvexity091504.pdf) (PDF): 21. Retrieved January 21, 2016.