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Duality gap

In optimization problems in applied mathematics, the **duality gap** is the difference between the primal and dual solutions. If d^* is the optimal dual value and p^* is the optimal primal value then the duality gap is equal to $p^* - d^*$. This value is always greater than or equal to 0 (for minimization problems). The duality gap is zero if and only if strong duality holds. Otherwise the gap is strictly positive and weak duality holds.[1]

In general given two dual pairs separated locally convex spaces (X, X^*) and (Y, Y^*). Then given the function $f : X \to \mathbb{R} \cup \{+\infty\}$, we can define the primal problem by

$$\inf_{x \in X} f(x).$$

If there are constraint conditions, these can be built into the function f by letting $f = f + I_{\text{constraints}}$ where I is the indicator function. Then let $F : X \times Y \to \mathbb{R} \cup \{+\infty\}$ be a perturbation function such that $F(x, 0) = f(x)$. The *duality gap* is the difference given by

$$\inf_{x \in X}[F(x,0)] - \sup_{y^* \in Y^*}[-F^*(0, y^*)]$$

where F^* is the convex conjugate in both variables.[2][3][4]

In computational optimization, another "duality gap" is often reported, which is the difference in value between any dual solution and the value of a feasible but suboptimal iterate for the primal problem. This alternative "duality gap" quantifies the discrepancy between the value of a current feasible but suboptimal iterate for the primal problem and the value of the dual problem; the value of the dual problem is, under regularity conditions, equal to the value of the *convex relaxation* of the primal problem: The convex relaxation is the problem arising replacing a non-convex feasible set with its closed convex hull and with replacing a non-convex function with its convex closure, that is the function that has the epigraph that is the closed convex hull of the original primal objective function.[5][6][7][8][9][10][11][12][13]

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