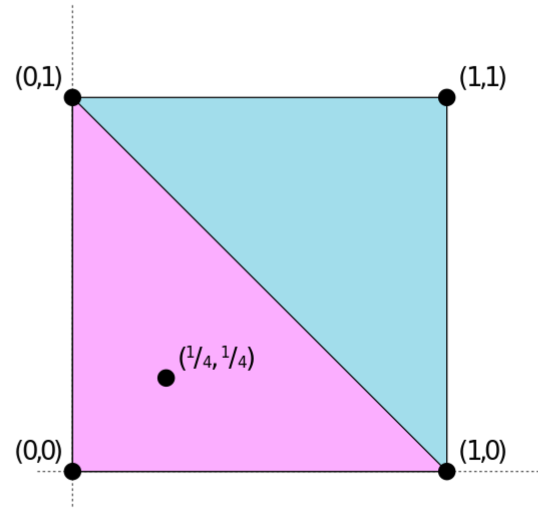
WIKIPEDIA

# Carathéodory's theorem (convex hull)

**Carathéodory's theorem** is a theorem in convex geometry. It states that if a point $x$ of $\mathbf{R}^d$ lies in the convex hull of a set $P$, then $x$ can be written as the convex combination of at most $d + 1$ points in P. Namely, there is a subset $P'$ of $P$ consisting of $d + 1$ or fewer points such that $x$ lies in the convex hull of $P'$. Equivalently, $x$ lies in an $r$-simplex with vertices in $P$, where $r \leq d$. The smallest $r$ that makes the last statement valid for each $x$ in the convex hull of $P$ is defined as the *Carathéodory's number* of $P$. Depending on the properties of $P$, upper bounds lower than the one provided by Carathéodory's theorem can be obtained.[1] Note that $P$ need not be itself convex. A consequence of this is that $P'$ can always be extremal in $P$, as non-extremal points can be removed from $P$ without changing the membership of $x$ in the convex hull.




An illustration of Carathéodory's theorem for a square in $\mathbf{R}^2$

The similar theorems of Helly and Radon are closely related to Carathéodory's theorem: the latter theorem can be used to prove the former theorems and vice versa.[2]

The result is named for Constantin Carathéodory, who proved the theorem in 1911 for the case when $P$ is compact.[3] In 1914 Ernst Steinitz expanded Carathéodory's theorem for any sets $P$ in $\mathbf{R}^d$.[4]

## Contents



## Example

Consider a set $P = \{(0,0), (0,1), (1,0), (1,1)\}$ which is a subset of $\mathbf{R}^2$. The convex hull of this set is a square. Consider now a point $x = (1/4, 1/4)$, which is in the convex hull of $P$. We can then construct a set $\{(0,0),(0,1),(1,0)\} = P'$, the convex hull of which is a triangle and encloses $x$, and thus the theorem works for this instance, since $|P'| = 3$. It may help to visualise Carathéodory's theorem in 2 dimensions, as saying that we can construct a triangle consisting of points from $P$ that encloses any point in $P$.

# Proof

Let $\boldsymbol{x}$ be a point in the convex hull of $P$. Then, $\boldsymbol{x}$ is a convex combination of a finite number of points in $P$:

$$\mathbf{x} = \sum_{j=1}^{k} \lambda_j \mathbf{x}_j$$

where every $\boldsymbol{x}_j$ is in $P$, every $\lambda_j$ is (w.l.o.g.) positive, and $\sum_{j=1}^{k} \lambda_j = 1$.

Suppose $k > d + 1$ (otherwise, there is nothing to prove). Then, the vectors $\boldsymbol{x}_2 - \boldsymbol{x}_1, \ldots, \boldsymbol{x}_k - \boldsymbol{x}_1$ are linearly dependent,

so there are real scalars $\mu_2, \ldots, \mu_k$, not all zero, such that

$$\sum_{j=2}^{k} \mu_j (\mathbf{x}_j - \mathbf{x}_1) = \mathbf{0}.$$

If $\mu_1$ is defined as

$$\mu_1 := -\sum_{j=2}^{k} \mu_j$$

then

$$\sum_{j=1}^{k} \mu_j \mathbf{x}_j = \mathbf{0}$$
$$\sum_{j=1}^{k} \mu_j = 0$$

and not all of the $\mu_j$ are equal to zero. Therefore, at least one $\mu_j > 0$. Then,

$$\mathbf{x} = \sum_{j=1}^{k} \lambda_j \mathbf{x}_j - \alpha \sum_{j=1}^{k} \mu_j \mathbf{x}_j = \sum_{j=1}^{k} (\lambda_j - \alpha \mu_j) \mathbf{x}_j$$

for any real $\alpha$. In particular, the equality will hold if $\alpha$ is defined as

$$\alpha := \min_{1 \leq j \leq k} \left\{ \frac{\lambda_j}{\mu_j} : \mu_j > 0 \right\} = \frac{\lambda_i}{\mu_i}.$$

Note that $\alpha > 0$, and for every $j$ between 1 and $k$,

$$\lambda_j - \alpha\mu_j \geq 0.$$

In particular, $\lambda_i - \alpha\mu_i = 0$ by definition of $\alpha$. Therefore,

$$\mathbf{x} = \sum_{j=1}^{k} (\lambda_j - \alpha\mu_j)\mathbf{x}_j$$

where every $\lambda_j - \alpha\mu_j$ is nonnegative, their sum is one , and furthermore, $\lambda_i - \alpha\mu_i = 0$. In other words, $\boldsymbol{x}$ is represented as a convex combination of at most $k$-1 points of $P$. This process can be repeated until $\boldsymbol{x}$ is represented as a convex combination of at most $d + 1$ points in $P$.

Alternative proofs uses Helly's theorem or the Perron–Frobenius theorem.[5][6]

# Variants

## Carathéodory's theorem for the conical hull

If a point $x$ of $\mathbf{R}^d$ lies in the **conical hull** of a set $P$, then $x$ can be written as the **conical combination** of at most $\boldsymbol{d}$ points in P. Namely, there is a subset $P'$ of $P$ consisting of $\boldsymbol{d}$ or fewer points, such that $x$ lies in the conical hull of $P'$.[7]:257 The proof is similar to the original theorem; the difference is that, in a $d$-dimensional space, the maximum size of a linearly-independent set is $d$, while the maximum size of an affinely-independent set is $d$+1.[8]

## Dimensionless variant

Recently, Adiprasito, Barany, Mustafa and Terpai proved a variant of Caratheodory's theorem that does not depend on the dimension of the space.[9]

## Colorful Carathéodory theorem

Let $X_1, ..., X_{d+1}$ be sets in $\mathbf{R}^d$ and let $x$ be a point contained in the intersection of the convex hulls of all these $d$+1 sets.

Then there is a set $T = \{x_1, ..., x_{d+1}\}$, where $x_1 \in X_1, ..., x_{d+1} \in X_{d+1}$, such that the convex hull of $T$ contains the point $x$.[10]

By viewing the sets $X_1, ..., X_{d+1}$ as different colors, the set $T$ is made by points of all colors, hence the "colorful" in the theorem's name.[11] The set $T$ is also called a *rainbow simplex*, since it is a $d$-dimensional simplex in which each corner has a different color.[12]

This theorem has a variant in which the convex hull is replaced by the conical hull.[10]:Thm.2.2 Let $X_1$, ..., $X_d$ be sets in $\mathbf{R}^d$ and let $x$ be a point contained in the intersection of the *conical hulls* of all these $d$ sets. Then there is a set $T = \{x_1, ..., x_d\}$, where $x_1 \in X_1$, ..., $x_d \in X_d$, such that the *conical hull* of $T$ contains the point $x$.[10]

Mustafa and Ray extended this colorful theorem from points to convex bodies.[12]

## See also

- Shapley–Folkman lemma
- Helly's theorem
- Kirchberger's theorem
- Radon's theorem, and its generalization Tverberg's theorem
- Krein–Milman theorem
- Choquet theory

## Notes

1. Bárány, Imre; Karasev, Roman (2012-07-20). "Notes About the Carathéodory Number". *Discrete & Computational Geometry*. **48** (3): 783–792. arXiv:1112.5942 (https://arxiv.org/abs/1112.5942). doi:10.1007/s00454-012-9439-z (https://doi.org/10.1007%2Fs00454-012-9439-z). ISSN 0179-5376 (https://www.worldcat.org/issn/0179-5376). S2CID 9090617 (https://api.semanticscholar.org/CorpusID:9090617).
2. Danzer, L.; Grünbaum, B.; Klee, V. (1963). "Helly's theorem and its relatives". *Convexity*. Proc. Symp. Pure Math. Vol. 7. American Mathematical Society. pp. 101–179. See in particular p.109
3. Carathéodory, C. (1911). "Über den Variabilitätsbereich der Fourier'schen Konstanten von positiven harmonischen Funktionen" (https://link.springer.com/article/10.1007%2FBF03014795). *Rendiconti del Circolo Matematico di Palermo (1884–1940)* (in German). **32** (1): 193–217[see p.200 bottom]. doi:10.1007/BF03014795 (https://doi.org/10.1007%2FBF03014795). S2CID 120032616 (https://api.semanticscholar.org/CorpusID:120032616).
4. Steinitz, Ernst (1913). "Bedingt konvergente Reihen und konvexe Systeme". *J. Reine Angew. Math*. **1913** (143): 128–175. doi:10.1515/crll.1913.143.128 (https://doi.org/10.1515%2Fcrll.1913.143.128). S2CID 120411668 (https://api.semanticscholar.org/CorpusID:120411668).
5. Eggleston, H. G. (1958). *Convexity* (http://ebooks.cambridge.org/ref/id/CBO9780511566172). Cambridge University Press. doi:10.1017/cbo9780511566172 (https://doi.org/10.1017%2Fcbo9780511566172). ISBN 9780511566172. See pages 40–41.
6. Naszódi, Márton; Polyanskii, Alexandr (2021). "Perron and Frobenius Meet Carathéodory". *The Electronic Journal of Combinatorics*. **28** (3). arXiv:1901.00540 (https://arxiv.org/abs/1901.00540). doi:10.37236/9996 (https://doi.org/10.37236%2F9996). S2CID 119656227 (https://api.semanticscholar.org/CorpusID:119656227).
7. Lovász, László; Plummer, M. D. (1986). *Matching Theory*. Annals of Discrete Mathematics. Vol. 29. North-Holland. ISBN 0-444-87916-1. MR 0859549 (https://www.ams.org/mathscinet-getitem?mr=0859549).
8. "convex geometry - Caratheodory's theorem for vectors in a cone" (https://math.stackexchange.com/questions/3759861/caratheodorys-theorem-for-vectors-in-a-cone). *Mathematics Stack Exchange*. Retrieved 2020-07-22.

9. Adiprasito, Karim; Bárány, Imre; Mustafa, Nabil H.; Terpai, Tamás (2019-08-28). "Theorems of Carathéodory, Helly, and Tverberg without dimension". arXiv:1806.08725 (https://arxiv.org/abs/1806.08725) [math.MG (https://arxiv.org/archive/math.MG)].
10. Bárány, Imre (1982-01-01). "A generalization of carathéodory's theorem" (https://doi.org/10.1016%2F0012-365X%2882%2990115-7). *Discrete Mathematics*. **40** (2–3): 141–152. doi:10.1016/0012-365X(82)90115-7 (https://doi.org/10.1016%2F0012-365X%2882%2990115-7). ISSN 0012-365X (https://www.worldcat.org/issn/0012-365X).
11. Montejano, Luis; Fabila, Ruy; Bracho, Javier; Bárány, Imre; Arocha, Jorge L. (2009-09-01). "Very Colorful Theorems" (https://doi.org/10.1007%2Fs00454-009-9180-4). *Discrete & Computational Geometry*. **42** (2): 142–154. doi:10.1007/s00454-009-9180-4 (https://doi.org/10.1007%2Fs00454-009-9180-4). ISSN 1432-0444 (https://www.worldcat.org/issn/1432-0444).
12. Mustafa, Nabil H.; Ray, Saurabh (2016-04-06). "An optimal generalization of the Colorful Carathéodory theorem" (https://doi.org/10.1016%2Fj.disc.2015.11.019). *Discrete Mathematics*. **339** (4): 1300–1305. doi:10.1016/j.disc.2015.11.019 (https://doi.org/10.1016%2Fj.disc.2015.11.019). ISSN 0012-365X (https://www.worldcat.org/issn/0012-365X).


## Further reading


- Eckhoff, J. (1993). "Helly, Radon, and Carathéodory type theorems". *Handbook of Convex Geometry*. Vol. A, B. Amsterdam: North-Holland. pp. 389–448.
- Mustafa, Nabil; Meunier, Frédéric; Goaoc, Xavier; De Loera, Jesús (2019). "The discrete yet ubiquitous theorems of Carathéodory, Helly, Sperner, Tucker, and Tverberg". *Bulletin of the American Mathematical Society*. **56** (3): 415–511. arXiv:1706.05975 (https://arxiv.org/abs/1706.05975). doi:10.1090/bull/1653 (https://doi.org/10.1090%2Fbull%2F1653). ISSN 0273-0979 (https://www.worldcat.org/issn/0273-0979). S2CID 32071768 (https://api.semanticscholar.org/CorpusID:32071768).


## External links

- Concise statement of theorem (https://planetmath.org/caratheodorystheorem) in terms of convex hulls (at PlanetMath)