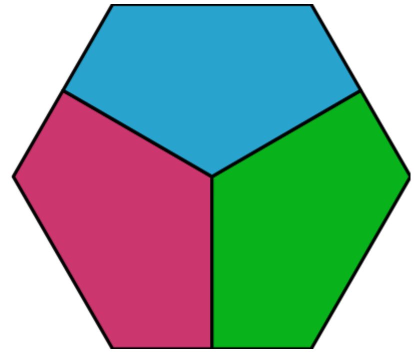
Borsuk's conjecture

The **Borsuk problem in geometry**, for historical reasons[note 1] incorrectly called **Borsuk's conjecture**, is a question in discrete geometry. It is named after Karol Borsuk.



An example of a hexagon cut into three pieces of smaller diameter.

Contents

Problem

In 1932, Karol Borsuk showed[2] that an ordinary 3-dimensional ball in Euclidean space can be easily dissected into 4 solids, each of which has a smaller diameter than the ball, and generally n-dimensional ball can be covered with $n+1$ compact sets of diameters smaller than the ball. At the same time he proved that n subsets are not enough in general. The proof is based on the Borsuk–Ulam theorem. That led Borsuk to a general question:

> *Die folgende Frage bleibt offen: Lässt sich jede beschränkte Teilmenge E des Raumes* $\mathbb{R}^n$ *in* $(n+1)$ *Mengen zerlegen, von denen jede einen kleineren Durchmesser als E hat?*[2]

This can be translated as:

> *The following question remains open: Can every bounded subset E of the space* $\mathbb{R}^n$ *be partitioned into* $(n+1)$ *sets, each of which has a smaller diameter than E?*

The question was answered in the positive in the following cases:

- $n = 2$ — which is the original result by Karol Borsuk (1932).
- $n = 3$ — shown by Julian Perkal (1947),[3] and independently, 8 years later, by H. G. Eggleston (1955).[4] A simple proof was found later by Branko Grünbaum and Aladár Heppes.
- For all n for smooth convex bodies — shown by Hugo Hadwiger (1946).[5][6]
- For all n for centrally-symmetric bodies — shown by A.S. Riesling (1971).[7]
- For all n for bodies of revolution — shown by Boris Dekster (1995).[8]

The problem was finally solved in 1993 by Jeff Kahn and Gil Kalai, who showed that the general answer to Borsuk's question is *no*.[9] They claim that their construction shows that $n + 1$ pieces do not suffice for $n = 1325$ and for each $n > 2014$. However, as pointed out by Bernulf Weißbach,[10] the first part of this claim is in fact false. But after improving a suboptimal conclusion within the corresponding derivation, one can indeed verify one of the constructed point sets as a counterexample for $n = 1325$ (as well as all higher dimensions up to 1560).[11]

Their result was improved in 2003 by Hinrichs and Richter, who constructed finite sets for $n \geq 298$, which cannot be partitioned into $n + 11$ parts of smaller diameter.[1]

In 2013, Andriy V. Bondarenko had shown that Borsuk's conjecture is false for all $n \geq 65$.[12][13] Shortly after, Thomas Jenrich derived a 64-dimensional counterexample from Bondarenko's construction, giving the best bound up to now.[14][15]

Apart from finding the minimum number n of dimensions such that the number of pieces $\alpha(n) > n + 1$, mathematicians are interested in finding the general behavior of the function $\alpha(n)$. Kahn and Kalai show that in general (that is, for n sufficiently large), one needs $\alpha(n) \geq (1.2)^{\sqrt{n}}$ many pieces. They also quote the upper bound by Oded Schramm, who showed that for every ε, if n is sufficiently large, $\alpha(n) \leq \left(\sqrt{3/2} + \varepsilon\right)^n$.[16] The correct order of magnitude of $\alpha(n)$ is still unknown.[17] However, it is conjectured that there is a constant $c > 1$ such that $\alpha(n) > c^n$ for all $n \geq 1$.

See also

- Hadwiger's conjecture on covering convex bodies with smaller copies of themselves

Note

1. As Hinrichs and Richter say in the introduction to their work,[1] the *“Borsuk's conjecture [was] believed by many to be true for some decades”* (hence commonly called 'a conjecture') so *“it came as a surprise when Kahn and Kalai constructed finite sets showing the contrary”*. It's worth noting, however, that Karol Borsuk has formulated the problem just as a question, not suggesting the expected answer would be positive.

References

1. Hinrichs, Aicke; Richter, Christian (28 August 2003). "New sets with large Borsuk numbers" (https://doi.org/10.1016%2FS0012-365X%2802%2900833-6). *Discrete Mathematics*. Elsevier. **270** (1–3): 137–147. doi:10.1016/S0012-365X(02)00833-6 (https://doi.org/10.1016%2FS0012-365X%2802%2900833-6).
2. Borsuk, Karol (1933), "Drei Sätze über die n-dimensionale euklidische Sphäre" (http://matwbn.icm.edu.pl/ksiazki/fm/fm20/fm20117.pdf) (PDF), *Fundamenta Mathematicae* (in German), **20**: 177–190, doi:10.4064/fm-20-1-177-190 (https://doi.org/10.4064%2Ffm-20-1-177-190)
3. Perkal, Julian (1947), "Sur la subdivision des ensembles en parties de diamètre inférieur", *Colloquium Mathematicum*, **2**: 45
4. Eggleston, H. G. (1955), "Covering a three-dimensional set with sets of smaller diameter", *Journal of the London Mathematical Society*, **30**: 11–24, doi:10.1112/jlms/s1-30.1.11 (https://doi.org/10.1112%2Fjlms%2Fs1-30.1.11), MR 0067473 (https://www.ams.org/mathscinet-getitem?mr=0067473)

5. Hadwiger, Hugo (1945), "Überdeckung einer Menge durch Mengen kleineren Durchmessers", *Commentarii Mathematici Helvetici*, **18** (1): 73–75, doi:10.1007/BF02568103 (https://doi.org/10.1007%2FBF02568103), MR 0013901 (https://www.ams.org/mathscinet-getitem?mr=0013901), S2CID 122199549 (https://api.semanticscholar.org/CorpusID:122199549)
6. Hadwiger, Hugo (1946), "Mitteilung betreffend meine Note: Überdeckung einer Menge durch Mengen kleineren Durchmessers", *Commentarii Mathematici Helvetici*, **19** (1): 72–73, doi:10.1007/BF02565947 (https://doi.org/10.1007%2FBF02565947), MR 0017515 (https://www.ams.org/mathscinet-getitem?mr=0017515), S2CID 121053805 (https://api.semanticscholar.org/CorpusID:121053805)
7. Riesling, A. S. (1971), "Проблема Борсука в трехмерных пространствах постоянной кривизны" (http://geometry.karazin.ua/resources/articles/594b744b34d8b035cdea7128bbae7d64.pdf) [Borsuk's problem in three-dimensional spaces of constant curvature] (PDF), *Ukr. Geom. Sbornik* (in Russian), Kharkov State University (now National University of Kharkiv), **11**: 78–83
8. Dekster, Boris (1995), "The Borsuk conjecture holds for bodies of revolution", *Journal of Geometry*, **52** (1–2): 64–73, doi:10.1007/BF01406827 (https://doi.org/10.1007%2FBF01406827), MR 1317256 (https://www.ams.org/mathscinet-getitem?mr=1317256), S2CID 121586146 (https://api.semanticscholar.org/CorpusID:121586146)
9. Kahn, Jeff; Kalai, Gil (1993), "A counterexample to Borsuk's conjecture", *Bulletin of the American Mathematical Society*, **29** (1): 60–62, arXiv:math/9307229 (https://arxiv.org/abs/math/9307229), doi:10.1090/S0273-0979-1993-00398-7 (https://doi.org/10.1090%2FS0273-0979-1993-00398-7), MR 1193538 (https://www.ams.org/mathscinet-getitem?mr=1193538), S2CID 119647518 (https://api.semanticscholar.org/CorpusID:119647518)
10. Weißbach, Bernulf (2000), "Sets with Large Borsuk Number" (https://www.emis.de/journals/BAG/vol.41/no.2/b41h2wb1.pdf) (PDF), *Beiträge zur Algebra und Geometrie*, **41** (2): 417–423
11. Jenrich, Thomas (2018), *On the counterexamples to Borsuk's conjecture by Kahn and Kalai*, arXiv:1809.09612v4 (https://arxiv.org/abs/1809.09612v4)
12. Bondarenko, Andriy V. (2013), *On Borsuk's conjecture for two-distance sets*, arXiv:1305.2584 (https://arxiv.org/abs/1305.2584), Bibcode:2013arXiv1305.2584B (https://ui.adsabs.harvard.edu/abs/2013arXiv1305.2584B)
13. Bondarenko, Andriy (2014), "On Borsuk's Conjecture for Two-Distance Sets", *Discrete & Computational Geometry*, **51** (3): 509–515, doi:10.1007/s00454-014-9579-4 (https://doi.org/10.1007%2Fs00454-014-9579-4), MR 3201240 (https://www.ams.org/mathscinet-getitem?mr=3201240)
14. Jenrich, Thomas (2013), *A 64-dimensional two-distance counterexample to Borsuk's conjecture*, arXiv:1308.0206 (https://arxiv.org/abs/1308.0206), Bibcode:2013arXiv1308.0206J (https://ui.adsabs.harvard.edu/abs/2013arXiv1308.0206J)
15. Jenrich, Thomas; Brouwer, Andries E. (2014), "A 64-Dimensional Counterexample to Borsuk's Conjecture", *Electronic Journal of Combinatorics*, **21** (4): #P4.29, doi:10.37236/4069 (https://doi.org/10.37236%2F4069), MR 3292266 (https://www.ams.org/mathscinet-getitem?mr=3292266)
16. Schramm, Oded (1988), "Illuminating sets of constant width", *Mathematika*, **35** (2): 180–189, doi:10.1112/S0025579300015175 (https://doi.org/10.1112%2FS0025579300015175), MR 0986627 (https://www.ams.org/mathscinet-getitem?mr=0986627)
17. Alon, Noga (2002), "Discrete mathematics: methods and challenges", *Proceedings of the International Congress of Mathematicians, Beijing*, **1**: 119–135, arXiv:math/0212390 (https://arxiv.org/abs/math/0212390), Bibcode:2002math.....12390A (https://ui.adsabs.harvard.edu/abs/2002math.....12390A)

Further reading

- Oleg Pikhurko, *Algebraic Methods in Combinatorics (https://web.archive.org/web/20160304000819/http://www.math.cmu.edu/~pikhurko/AlgMet.ps)*, course notes.
- Andrei M. Raigorodskii, The Borsuk partition problem: the seventieth anniversary, *Mathematical Intelligencer* **26** (2004), no. 3, 4–12.
- Raigorodskii, Andreii M. (2008). "Three lectures on the Borsuk partition problem". In Young, Nicholas; Choi, Yemon (eds.). *Surveys in contemporary mathematics*. London Mathematical Society Lecture Note Series. Vol. 347. Cambridge University Press. pp. 202–247. ISBN 978-0-521-70564-6. Zbl 1144.52005 (https://zbmath.org/?format=complete&q=an:1144.52005).

External links

- Weisstein, Eric W. "Borsuk's Conjecture" (https://mathworld.wolfram.com/BorsuksConjecture.html). *MathWorld*.