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Weak duality

In applied mathematics, **weak duality** is a concept in optimization which states that the duality gap is always greater than or equal to 0. That means the solution to the dual (minimization) problem is *always* greater than or equal to the solution to an associated primal problem. This is opposed to strong duality which only holds in certain cases.[1]

Contents

Uses

Many primal-dual approximation algorithms are based on the principle of weak duality.[2]

Weak duality theorem

The *primal* problem:

Maximize $\mathbf{c}^T\mathbf{x}$ subject to $A\,\mathbf{x} \le \mathbf{b}$, $\mathbf{x} \ge 0$;

The *dual* problem,

Minimize $\mathbf{b}^T\mathbf{y}$ subject to $A^T\mathbf{y} \ge \mathbf{c}$, $\mathbf{y} \ge 0$.

The weak duality theorem states $\mathbf{c}^T\mathbf{x} \le \mathbf{b}^T\mathbf{y}$.

Namely, if $(x_1, x_2, \ldots, x_n)$ is a feasible solution for the primal maximization linear program and $(y_1, y_2, \ldots, y_m)$ is a feasible solution for the dual minimization linear program, then the weak duality theorem can be stated as $\sum_{j=1}^{n} c_j x_j \le \sum_{i=1}^{m} b_i y_i$, where c_j and b_i are the coefficients of the respective objective functions.

Proof: $\mathbf{c}^T\mathbf{x} = \mathbf{x}^T\mathbf{c} \le \mathbf{x}^T A^T\mathbf{y} \le \mathbf{b}^T\mathbf{y}$

Generalizations

More generally, if x is a feasible solution for the primal maximization problem and y is a feasible solution for the dual minimization problem, then weak duality implies $f(x) \leq g(y)$ where f and g are the objective functions for the primal and dual problems respectively.

See also

- Convex optimization
- Max–min inequality

References

1. Boţ, Radu Ioan; Grad, Sorin-Mihai; Wanka, Gert (2009), *Duality in Vector Optimization* (https://books.google.com/books?id=nwB0qExrF00C&pg=PA1), Berlin: Springer-Verlag, p. 1, doi:10.1007/978-3-642-02886-1 (https://doi.org/10.1007%2F978-3-642-02886-1), ISBN 978-3-642-02885-4, MR 2542013 (https://www.ams.org/mathscinet-getitem?mr=2542013).
2. Gonzalez, Teofilo F. (2007), *Handbook of Approximation Algorithms and Metaheuristics* (https://books.google.com/books?id=QK3_VU8ngK8C&pg=SA2-PA12), CRC Press, p. 2-12, ISBN 9781420010749.