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Hadwiger's theorem

In integral geometry (otherwise called geometric probability theory), **Hadwiger's theorem** characterises the valuations on convex bodies in $\mathbb{R}^n$. It was proved by Hugo Hadwiger.

Contents

Introduction

Valuations

Let $\mathbb{K}^n$ be the collection of all compact convex sets in $\mathbb{R}^n$. A **valuation** is a function $v : \mathbb{K}^n \to \mathbb{R}$ such that $v(\varnothing) = 0$ and for every $S, T \in \mathbb{K}^n$ that satisfy $S \cup T \in \mathbb{K}^n$,

$$v(S) + v(T) = v(S \cap T) + v(S \cup T) \ .$$

A valuation is called continuous if it is continuous with respect to the Hausdorff metric. A valuation is called invariant under rigid motions if $v(\varphi(S)) = v(S)$ whenever $S \in \mathbb{K}^n$ and φ is either a translation or a rotation of $\mathbb{R}^n$.

Quermassintegrals

The quermassintegrals $W_j : \mathbb{K}^n \to \mathbb{R}$ are defined via Steiner's formula

$$\mathrm{Vol}_n(K + tB) = \sum_{j=0}^{n} \binom{n}{j} W_j(K) t^j \ ,$$

where B is the Euclidean ball. For example, W_o is the volume, W_1 is proportional to the surface measure, W_{n-1} is proportional to the mean width, and W_n is the constant $\mathrm{Vol}_n(B)$.

W_j is a valuation which is homogeneous of degree $n - j$, that is,

$$W_j(tK) = t^{n-j} W_j(K) \ , \quad t \geq 0 \ .$$

Statement

Any continuous valuation v on $\mathbb{K}^n$ that is invariant under rigid motions can be represented as

$$v(S) = \sum_{j=0}^{n} c_j W_j(S) \ .$$

Corollary

Any continuous valuation v on $\mathbb{K}^n$ that is invariant under rigid motions and homogeneous of degree j is a multiple of W_{n-j}.

See also

- Minkowski functional
- Set function – Function from sets to numbers

References

An account and a proof of Hadwiger's theorem may be found in

- Klain, D.A.; Rota, G.-C. (1997). *Introduction to geometric probability* (https://archive.org/details/introductiontoge0000klai). Cambridge: Cambridge University Press. ISBN 0-521-59362-X. MR 1608265 (https://www.ams.org/mathscinet-getitem?mr=1608265).

An elementary and self-contained proof was given by Beifang Chen in

- Chen, B. (2004). "A simplified elementary proof of Hadwiger's volume theorem". *Geom. Dedicata*. **105**: 107–120. doi:10.1023/b:geom.0000024665.02286.46 (https://doi.org/10.1023%2Fb%3Ageom.0000024665.02286.46). MR 2057247 (https://www.ams.org/mathscinet-getitem?mr=2057247).