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Mazur's lemma

In mathematics, **Mazur's lemma** is a result in the theory of Banach spaces. It shows that any weakly convergent sequence in a Banach space has a sequence of convex combinations of its members that converges strongly to the same limit, and is used in the proof of Tonelli's theorem.

Statement of the lemma

Let $(X, \|\cdot\|)$ be a Banach space and let $(u_n)_{n\in\mathbb{N}}$ be a sequence in X that converges weakly to some u_0 in X:

$$u_n \rightharpoonup u_0 \text{ as } n \to \infty.$$

That is, for every continuous linear functional $f \in X'$, the continuous dual space of X,

$$f(u_n) \to f(u_0) \text{ as } n \to \infty.$$

Then there exists a function $N : \mathbb{N} \to \mathbb{N}$ and a sequence of sets of real numbers

$$\{\alpha(n)_k : k = n, \ldots, N(n)\}$$

such that $\alpha(n)_k \geq 0$ and

$$\sum_{k=n}^{N(n)} \alpha(n)_k = 1$$

such that the sequence $(v_n)_{n\in\mathbb{N}}$ defined by the convex combination

$$v_n = \sum_{k=n}^{N(n)} \alpha(n)_k u_k$$

converges strongly in X to u_0; that is

$$\|v_n - u_0\| \to 0 \text{ as } n \to \infty.$$

See also

- Banach–Alaoglu theorem – The closed unit ball in the dual of a normed vector space is compact in the weak* topology
- Bishop–Phelps theorem

- Eberlein–Šmulian theorem – Relates three different kinds of weak compactness in a Banach space
- James's theorem
- Goldstine theorem

References

- Renardy, Michael & Rogers, Robert C. (2004). *An introduction to partial differential equations*. Texts in Applied Mathematics 13 (Second ed.). New York: Springer-Verlag. p. 350. ISBN 0-387-00444-0.