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Hypograph (mathematics)

In mathematics, the **hypograph** or **subgraph** of a function $f : \mathbb{R}^n \to \mathbb{R}$ is the set of points lying on or below its graph. A related definition is that of such a function's epigraph, which is the set of points on or above the function's graph.

The domain (rather than the codomain) of the function is not particularly important for this definition; it can be an arbitrary set[1] instead of $\mathbb{R}^n$.

Contents

Definition

The definition of the hypograph was inspired by that of the graph of a function, where the ***graph*** of $f : X \to Y$ is defined to be the set

$$\operatorname{graph} f := \{(x, y) \in X \times Y ~:~ y = f(x)\}.$$

The ***hypograph*** or ***subgraph*** of a function $f : X \to [-\infty, \infty]$ valued in the extended real numbers $[-\infty, \infty] = \mathbb{R} \cup \{\pm\infty\}$ is the set[2]

$$\begin{aligned} \operatorname{hyp} f &= \{(x, r) \in X \times \mathbb{R} ~:~ r \leq f(x)\} \\ &= \left[f^{-1}(\infty) \times \mathbb{R}\right] \cup \bigcup_{x \in f^{-1}(\mathbb{R})} \{x\} \times (-\infty, f(x)]. \end{aligned}$$

Similarly, the set of points on or above the function is its epigraph. The ***strict hypograph*** is the hypograph with the graph removed:

$$\begin{aligned} \operatorname{hyp}_S f &= \{(x, r) \in X \times \mathbb{R} ~:~ r < f(x)\} \\ &= \operatorname{hyp} f \setminus \operatorname{graph} f \\ &= \bigcup_{x \in X} \{x\} \times (-\infty, f(x)). \end{aligned}$$

Despite the fact that f might take one (or both) of $\pm\infty$ as a value (in which case its graph would *not* be a subset of $X \times \mathbb{R}$), the hypograph of f is nevertheless defined to be a subset of $X \times \mathbb{R}$ rather than of $X \times [-\infty, \infty]$.

Properties

The hypograph of a function f is empty if and only if f is identically equal to negative infinity.

A function is concave if and only if its hypograph is a convex set. The hypograph of a real affine function $g : \mathbb{R}^n \to \mathbb{R}$ is a halfspace in $\mathbb{R}^{n+1}$.

A function is upper semicontinuous if and only if its hypograph is closed.

See also

- Effective domain
- Epigraph (mathematics)
- Proper convex function

Citations

1. Charalambos D. Aliprantis; Kim C. Border (2007). *Infinite Dimensional Analysis: A Hitchhiker's Guide* (https://books.google.com/books?id=4hIq6ExH7NoC&pg=PA8) (3rd ed.). Springer Science & Business Media. pp. 8–9. ISBN 978-3-540-32696-0.
2. Rockafellar & Wets 2009, pp. 1–37.

References

- Rockafellar, R. Tyrrell; Wets, Roger J.-B. (26 June 2009). *Variational Analysis*. Grundlehren der mathematischen Wissenschaften. Vol. 317. Berlin New York: Springer Science & Business Media. ISBN 9783642024313. OCLC 883392544 (https://www.worldcat.org/oclc/883392544).