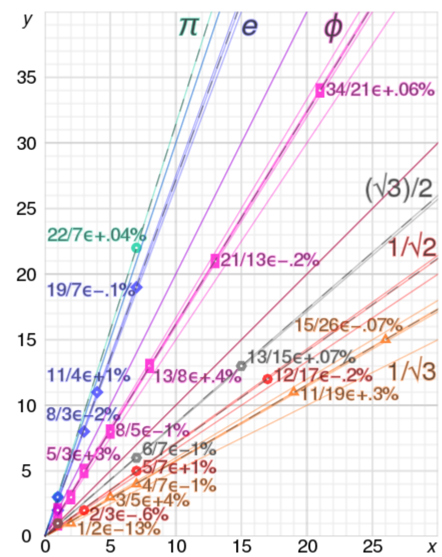
WIKIPEDIA

Geometry of numbers

Geometry of numbers is the part of number theory which uses geometry for the study of algebraic numbers. Typically, a ring of algebraic integers is viewed as a lattice in $\mathbb{R}^n$, and the study of these lattices provides fundamental information on algebraic numbers.[1] The geometry of numbers was initiated by Hermann Minkowski (1910).

The geometry of numbers has a close relationship with other fields of mathematics, especially functional analysis and Diophantine approximation, the problem of finding rational numbers that approximate an irrational quantity.[2]



Best rational approximants for π (green circle), e (blue diamond), ϕ (pink oblong), (√3)/2 (grey hexagon), 1/√2 (red octagon) and 1/√3 (orange triangle) calculated from their continued fraction expansions, plotted as slopes y/x with errors from their true values (black dashes)

Contents

Minkowski's results

Suppose that Γ is a lattice in n-dimensional Euclidean space $\mathbb{R}^n$ and K is a convex centrally symmetric body. Minkowski's theorem, sometimes called Minkowski's first theorem, states that if $\operatorname{vol}(K) > 2^n \operatorname{vol}(\mathbb{R}^n/\Gamma)$, then K contains a nonzero vector in Γ.

The successive minimum λ_k is defined to be the inf of the numbers λ such that λK contains k linearly independent vectors of Γ. Minkowski's theorem on successive minima, sometimes called Minkowski's second theorem, is a strengthening of his first theorem and states that[3]

$$\lambda_1 \lambda_2 \cdots \lambda_n \operatorname{vol}(K) \leq 2^n \operatorname{vol}(\mathbb{R}^n/\Gamma).$$

Later research in the geometry of numbers

In 1930-1960 research on the geometry of numbers was conducted by many number theorists (including Louis Mordell, Harold Davenport and Carl Ludwig Siegel). In recent years, Lenstra, Brion, and Barvinok have developed combinatorial theories that enumerate the lattice points in some convex bodies.[4]

Subspace theorem of W. M. Schmidt

In the geometry of numbers, the subspace theorem was obtained by Wolfgang M. Schmidt in 1972.[5] It states that if n is a positive integer, and $L_1,\ldots,L_n$ are linearly independent linear forms in n variables with algebraic coefficients and if ε>0 is any given real number, then the non-zero integer points x in n coordinates with

$$|L_1(x)\cdots L_n(x)| < |x|^{-\varepsilon}$$

lie in a finite number of proper subspaces of $\mathbf{Q}^n$.

Influence on functional analysis

Minkowski's geometry of numbers had a profound influence on functional analysis. Minkowski proved that symmetric convex bodies induce norms in finite-dimensional vector spaces. Minkowski's theorem was generalized to topological vector spaces by Kolmogorov, whose theorem states that the symmetric convex sets that are closed and bounded generate the topology of a Banach space.[6]

Researchers continue to study generalizations to star-shaped sets and other non-convex sets.[7]

References

1. MSC classification, 2010, available at http://www.ams.org/msc/msc2010.html, Classification 11HXX.
2. Schmidt's books. Grötschel et alii, Lovász et alii, Lovász.
3. Cassels (1971) p. 203
4. Grötschel et alii, Lovász et alii, Lovász, and Beck and Robins.
5. Schmidt, Wolfgang M. *Norm form equations.* Ann. Math. (2) **96** (1972), pp. 526-551. See also Schmidt's books; compare Bombieri and Vaaler and also Bombieri and Gubler.
6. For Kolmogorov's normability theorem, see Walter Rudin's *Functional Analysis*. For more results, see Schneider, and Thompson and see Kalton et alii.
7. Kalton et alii. Gardner

Bibliography

- Matthias Beck, Sinai Robins. *Computing the continuous discretely: Integer-point enumeration in polyhedra*, Undergraduate Texts in Mathematics, Springer, 2007.
- Enrico Bombieri; Vaaler, J. (Feb 1983). "On Siegel's lemma". *Inventiones Mathematicae*. **73** (1): 11–32. Bibcode:1983InMat..73...11B (https://ui.adsabs.harvard.edu/abs/1983InMat..73...11B). doi:10.1007/BF01393823 (https://doi.org/10.1007%2FBF01393823). S2CID 121274024 (https://api.semanticscholar.org/CorpusID:121274024).
- Enrico Bombieri & Walter Gubler (2006). *Heights in Diophantine Geometry*. Cambridge U. P.
- J. W. S. Cassels. *An Introduction to the Geometry of Numbers*. Springer Classics in Mathematics, Springer-Verlag 1997 (reprint of 1959 and 1971 Springer-Verlag editions).
- John Horton Conway and N. J. A. Sloane, *Sphere Packings, Lattices and Groups*, Springer-Verlag, NY, 3rd ed., 1998.
- R. J. Gardner, *Geometric tomography,* Cambridge University Press, New York, 1995. Second edition: 2006.

- P. M. Gruber, *Convex and discrete geometry,* Springer-Verlag, New York, 2007.
- P. M. Gruber, J. M. Wills (editors), *Handbook of convex geometry. Vol. A. B,* North-Holland, Amsterdam, 1993.
- M. Grötschel, Lovász, L., A. Schrijver: *Geometric Algorithms and Combinatorial Optimization*, Springer, 1988
- Hancock, Harris (1939). *Development of the Minkowski Geometry of Numbers*. Macmillan. (Republished in 1964 by Dover.)
- Edmund Hlawka, Johannes Schoißengeier, Rudolf Taschner. *Geometric and Analytic Number Theory*. Universitext. Springer-Verlag, 1991.
- Kalton, Nigel J.; Peck, N. Tenney; Roberts, James W. (1984), *An F-space sampler*, London Mathematical Society Lecture Note Series, 89, Cambridge: Cambridge University Press, pp. xii+240, ISBN 0-521-27585-7, MR 0808777 (https://www.ams.org/mathscinet-getitem?mr=0808777)
- C. G. Lekkerkererker. *Geometry of Numbers*. Wolters-Noordhoff, North Holland, Wiley. 1969.
- Lenstra, A. K.; Lenstra, H. W. Jr.; Lovász, L. (1982). "Factoring polynomials with rational coefficients" (http://infoscience.epfl.ch/record/164484/files/nscan4.PDF) (PDF). *Mathematische Annalen*. **261** (4): 515–534. doi:10.1007/BF01457454 (https://doi.org/10.1007%2FBF01457454). hdl:1887/3810 (https://hdl.handle.net/1887%2F3810). MR 0682664 (https://www.ams.org/mathscinet-getitem?mr=0682664). S2CID 5701340 (https://api.semanticscholar.org/CorpusID:5701340).
- Lovász, L.: *An Algorithmic Theory of Numbers, Graphs, and Convexity*, CBMS-NSF Regional Conference Series in Applied Mathematics 50, SIAM, Philadelphia, Pennsylvania, 1986
- Malyshev, A.V. (2001) [1994], "Geometry of numbers" (https://www.encyclopediaofmath.org/index.php?title=Geometry_of_numbers), *Encyclopedia of Mathematics*, EMS Press
- Minkowski, Hermann (1910), *Geometrie der Zahlen* (https://archive.org/details/geometriederzahl00minkrich), Leipzig and Berlin: R. G. Teubner, JFM 41.0239.03 (https://zbmath.org/?format=complete&q=an:41.0239.03), MR 0249269 (https://www.ams.org/mathscinet-getitem?mr=0249269), retrieved 2016-02-28
- Wolfgang M. Schmidt. *Diophantine approximation*. Lecture Notes in Mathematics 785. Springer. (1980 [1996 with minor corrections])
- Schmidt, Wolfgang M. (1996). *Diophantine approximations and Diophantine equations*. Lecture Notes in Mathematics. Vol. 1467 (2nd ed.). Springer-Verlag. ISBN 3-540-54058-X. Zbl 0754.11020 (https://zbmath.org/?format=complete&q=an:0754.11020).
- Siegel, Carl Ludwig (1989). *Lectures on the Geometry of Numbers* (https://archive.org/details/lecturesongeomet0000sieg). Springer-Verlag.
- Rolf Schneider, *Convex bodies: the Brunn-Minkowski theory,* Cambridge University Press, Cambridge, 1993.
- Anthony C. Thompson, *Minkowski geometry,* Cambridge University Press, Cambridge, 1996.
- Hermann Weyl. Theory of reduction for arithmetical equivalence . Trans. Amer. Math. Soc. 48 (1940) 126–164. doi:10.1090/S0002-9947-1940-0002345-2 (https://doi.org/10.1090%2FS0002-9947-1940-0002345-2)
- Hermann Weyl. Theory of reduction for arithmetical equivalence. II . Trans. Amer. Math. Soc. 51 (1942) 203–231. doi:10.2307/1989946 (https://doi.org/10.2307%2F1989946)