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Proper convex function

In mathematical analysis, in particular the subfields of convex analysis and optimization, a **proper convex function** is an extended real-valued convex function with a non-empty domain, that never takes on the value $-\infty$ and also is not identically equal to $+\infty$.

In convex analysis and variational analysis, a point (in the domain) at which some given function f is minimized is typically sought, where f is valued in the extended real number line $[-\infty, \infty] = \mathbb{R} \cup \{\pm\infty\}$.[1] Such a point, if it exists, is called a *global minimum point* of the function and its value at this point is called the *global minimum* (*value*) of the function. If the function takes $-\infty$ as a value then $-\infty$ is necessarily the global minimum value and the minimization problem can be answered; this is ultimately the reason why the definition of "*proper*" requires that the function never take $-\infty$ as a value. Assuming this, if the function's domain is empty or if the function is identically equal to $+\infty$ then the minimization problem once again has an immediate answer. Extended real-valued function for which the minimization problem is not solved by any one of these three trivial cases are exactly those that are called *proper*. Many (although not all) results whose hypotheses require that the function be proper add this requirement specifically to exclude these trivial cases.

If the problem is instead a maximization problem (which would be clearly indicated, such as by the function being concave rather than convex) then the definition of "*proper*" is defined in an analogous (albeit technically different) manner but with the same goal: to exclude cases where the maximization problem can be answered immediately. Specifically, a concave function g is called *proper* if its negation $-g$, which is a convex function, is proper in the sense defined above.

Contents

Definitions

Suppose that $f : X \to [-\infty, \infty]$ is a function taking values in the extended real number line $[-\infty, \infty] = \mathbb{R} \cup \{\pm\infty\}$. If f is a convex function or if a minimum point of f is being sought, then f is called ***proper*** if

$$f(x) > -\infty \qquad \text{for } \textit{every } x \in \operatorname{domain} f$$

and if there also exists *some* point x_0 in its domain such that

$$f(x_0) < +\infty.$$

That is, a function is *proper* if its effective domain is nonempty and it never attains the value $-\infty$.[2] This means that there exists some $x \in \operatorname{domain} f$ at which $f(x) \in \mathbb{R}$ and f is also *never* equal to $-\infty$. Convex functions that are not proper are called ***improper*** convex functions.[3]

A *proper* *concave function* is by definition, any function $g : X \to [-\infty, \infty]$ such that $f := -g$ is a proper convex function. Explicitly, if $g : X \to [-\infty, \infty]$ is a concave function or if a maximum point of g is being sought, then g is called ***proper*** if its domain is not empty, it *never* takes on the value $+\infty$, and it is not identically equal to $-\infty$.

Properties

For every proper convex function $f : \mathbb{R}^n \to [-\infty, \infty]$, there exist some $b \in \mathbb{R}^n$ and $r \in \mathbb{R}$ such that

$$f(x) \geq x \cdot b - r$$

for every $x \in X$.

The sum of two proper convex functions is convex, but not necessarily proper.[4] For instance if the sets $A \subset X$ and $B \subset X$ are non-empty convex sets in the vector space X, then the characteristic functions I_A and I_B are proper convex functions, but if $A \cap B = \varnothing$ then $I_A + I_B$ is identically equal to $+\infty$.

The infimal convolution of two proper convex functions is convex but not necessarily proper convex.[5]

See also

- Effective domain

Citations

1. Rockafellar & Wets 2009, pp. 1–28.
2. Aliprantis, C.D.; Border, K.C. (2007). *Infinite Dimensional Analysis: A Hitchhiker's Guide* (3 ed.). Springer. p. 254. doi:10.1007/3-540-29587-9 (https://doi.org/10.1007%2F3-540-29587-9). ISBN 978-3-540-32696-0.
3. Rockafellar, R. Tyrrell (1997) [1970]. *Convex Analysis*. Princeton, NJ: Princeton University Press. p. 24. ISBN 978-0-691-01586-6.
4. Boyd, Stephen (2004). *Convex Optimization*. Cambridge, UK: Cambridge University Press. p. 79. ISBN 978-0-521-83378-3.
5. Ioffe, Aleksandr Davidovich; Tikhomirov, Vladimir Mikhaĭlovich (2009), *Theory of extremal problems* (https://books.google.com/books?id=iDRVxznSxUsC&pg=PA168), Studies in Mathematics and its Applications, vol. 6, North-Holland, p. 168, ISBN 9780080875279.

References

- Rockafellar, R. Tyrrell; Wets, Roger J.-B. (26 June 2009). *Variational Analysis*. Grundlehren der mathematischen Wissenschaften. Vol. 317. Berlin New York: Springer Science & Business Media.

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