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Effective domain

In convex analysis, a branch of mathematics, the **effective domain** is an extension of the domain of a function defined for functions that take values in the extended real number line $[-\infty, \infty] = \mathbb{R} \cup \{\pm\infty\}$.

In convex analysis and variational analysis, a point at which some given extended real-valued function is minimized is typically sought, where such a point is called a global minimum point. The effective domain of this function is defined to be the set of all points in this function's domain at which its value is not equal to $+\infty$,[1] where the effective domain is defined this way because it is only these points that have even a remote chance of being a global minimum point. Indeed, it is common practice in these fields to set a function equal to $+\infty$ at a point specifically to *exluded* that point from even being considered as a potential solution (to the minimization problem).[1] Points at which the function takes the value $-\infty$ (if any) belong to the effective domain because such points are considered acceptable solutions to the minimization problem,[1] with the reasoning being that if such a point was not acceptable as a solution then the function would have already been set to $+\infty$ at that point instead.

When a minimum point (in X) of a function $f : X \to [-\infty, \infty]$ is to be found but f's domain X is a proper subset of some vector space V, then it often technically useful to extend f to all of V by setting $f(x) := +\infty$ at every $x \in V \setminus X$.[1] By definition, no point of $V \setminus X$ belongs to the effective domain of f, which is consistent with the desire to find a minimum point of the original function $f : X \to [-\infty, \infty]$ rather than of the newly defined extension to all of V.

If the problem is instead a maximization problem (which would be clearly indicated) then the effective domain instead consists of all points in the function's domain at which it is not equal to $-\infty$.

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Definition

Suppose $f : X \to [-\infty, \infty]$ is a map valued in the extended real number line $[-\infty, \infty] = \mathbb{R} \cup \{\pm\infty\}$ whose domain, which is denoted by $\operatorname{domain} f$, is X (where X will be assumed to be a subset of some vector space whenever this assumption is necessary). Then the ***effective domain*** of f is denoted by $\operatorname{dom} f$ and typically defined to be the set[1][2][3]

$$\operatorname{dom} f = \{x \in X \ : \ f(x) < +\infty\}$$

unless f is a concave function or the maximum (rather than the minimum) of f is being sought, in which case the ***effective domain*** of f is instead the set[2]

$$\operatorname{dom} f = \{x \in X \ : \ f(x) > -\infty\}.$$

In convex analysis and variational analysis, $\operatorname{dom} f$ is usually assumed to be $\operatorname{dom} f = \{x \in X \ : \ f(x) < +\infty\}$ unless clearly indicated otherwise.

Characterizations

Let $\pi_X : X \times \mathbb{R} \to X$ denote the canonical projection onto X, which is defined by $(x, r) \mapsto x$. The effective domain of $f : X \to [-\infty, \infty]$ is equal to the image of f's epigraph $\operatorname{epi} f$ under the canonical projection π_X. That is

$$\operatorname{dom} f = \pi_X(\operatorname{epi} f) = \{x \in X \ : \ \text{there exists } y \in \mathbb{R} \text{ such that } (x, y) \in \operatorname{epi} f\}.$$ [4]

For a maximization problem (such as if the f is concave rather than convex), the effective domain is instead equal to the image under π_X of f's hypograph.

Properties

If a function *never* takes the value $+\infty$, such as if the function is real-valued, then its domain and effective domain are equal.

A function $f : X \to [-\infty, \infty]$ is a proper convex function if and only if f is convex, the effective domain of f is nonempty, and $f(x) > -\infty$ for every $x \in X$.[4]

See also

- Proper convex function
- Epigraph (mathematics)
- Hypograph (mathematics)

References

1. Rockafellar & Wets 2009, pp. 1–28.
2. Aliprantis, C.D.; Border, K.C. (2007). *Infinite Dimensional Analysis: A Hitchhiker's Guide* (3 ed.). Springer. p. 254. doi:10.1007/3-540-29587-9 (https://doi.org/10.1007%2F3-540-29587-9). ISBN 978-3-540-32696-0.
3. Föllmer, Hans; Schied, Alexander (2004). *Stochastic finance: an introduction in discrete time* (2 ed.). Walter de Gruyter. p. 400. ISBN 978-3-11-018346-7.
4. Rockafellar, R. Tyrrell (1997) [1970]. *Convex Analysis*. Princeton, NJ: Princeton University Press. p. 23. ISBN 978-0-691-01586-6.

- Rockafellar, R. Tyrrell; Wets, Roger J.-B. (26 June 2009). *Variational Analysis*. Grundlehren der mathematischen Wissenschaften. Vol. 317. Berlin New York: Springer Science & Business Media.

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