# Some Price Theory of Buyer-Optimal Learning
# Roesler and Szentes (2017)

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## 1 Introduction

Roesler and Szentes (2017) study the following problem: suppose that a buyer is facing a monopolist and the buyer is uncertain about her valuation. However, she can learn more about her valuation through an experiment. They ask, what is the optimal experiment from the *ex ante* buyer's perspective?

Their main result derives the buyer-optimal distribution and corresponding demand curve. Their derivation uses mean-preserving spreads and stochastic dominance. I found this formulation, while based on standard results, somewhat unintuitive. This note finds the optimal distribution for a simpler problem, where the derivation relies more heavily on the properties of the inverse demand and marginal revenue curves. I aim for simple exposition and leave issues like the existence of a derivate or integral for granted. This note is not a re-derivation of their result—I solve a simpler problem—but is better seen as an exploration of their problem using price theory.

## 2 Model

A seller has a single unit to sell to a buyer. The seller has a marginal cost of zero. The buyer's valuation can be either high with probability $\mu$ or low with probability $1-\mu$. To keep the calculations simple, let the high valuation be one and the low valuation be zero, which implies that the expected value of the good is $\mu$. Before buying, the buyer can observe an unbiased signal about her valuation and updates her beliefs. Before observing a particular signal, the buyer has a distribution of signals. The problem is to find as the modeler (or let the buyer choose) the optimal signal distribution.

The timing is as follows: 1) the buyer observes an unbiased signal about her valuation from a signal distribution, 2) the seller sees the distribution but not the signal and chooses a take-it-or-leave-it offer, and 3) the buyer decides whether to buy or not.

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## 2.1 Constraint Set: Feasible Inverse Demand Curves

Instead of thinking about signal distributions, the problem can be thought of as choosing an inverse demand curve. This section discusses that transformation.

To translate the signal distribution into inverse demand, notice that any general distribution of signals generates a corresponding demand curve for seller which can be turned into an inverse demand curve. For example, take the CDF of an arbitrary signal distribution, given on the left of Figure 1. The CDF gives the probability that the buyer's valuation will be less than or equal to $v$.

From the CDF of valuations, we can directly find the demand curve. The buyer will demand the good if his valuation is greater than or equal to the price $P$. That leads to a demand curve given in the middle of Figure 1. The only part that takes some care is that valuations that have positive mass translate into vertical parts of the demand curve. The inverse demand curve comes directly from the demand curve, given on the right of Figure 1.

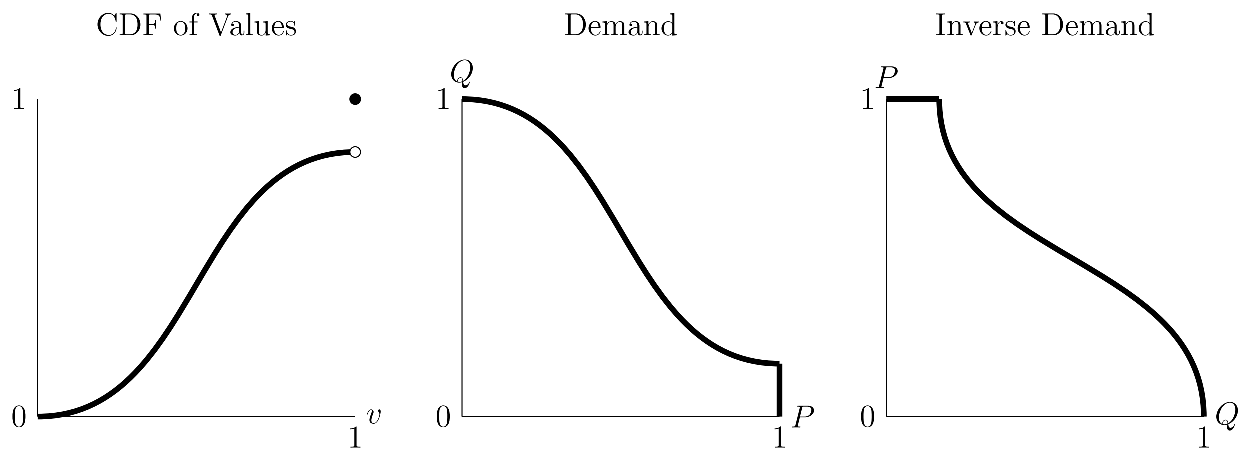


Figure 1: CDF to Inverse Demand

Bayes' Theorem restricts the feasible posterior CDFs (and therefore the feasible inverse demand curves), given a prior distribution. Roesler and Szentes use the observation that the set of feasible CDFs is the set of CDFs such that the prior CDF is a mean-preserving spread of the posterior CDF (see Definition 6.D.2 and Proposition 6.D.2 in Mas-Colell, Winston, and Green (1995)).

However, I made a simplification; the original valuation is either zero or one. This means that the prior CDF is flat at a height of $1-\mu$. Therefore, loosely, the "mean-preserving" is the restriction, and not the "spread," because the prior distribution is a spread of all distributions with the same mean. Therefore, the only restriction on the posterior distribution from Bayes' Theorem (besides that it is a CDF) is that the integral of the posterior equals the prior. This restriction has an even easier graphical interpretation. It requires that the area under the CDF equals $1 - \mu$.[1]

1. This is the same simple transformation I use in Albrecht (2017), although I have not seen it used

Translating that restriction into demand terms, the set of feasible demand curves are those with an area underneath of $\mu$. Similarly, the constraint on the choice of inverse demand curve is that the integral of the inverse demand curve equals $\mu$. The following is the formal constrained set of *Bayes-plausible* inverse demand curves:

$$\mathcal{D}(Q) = \left\{ D(Q) \;\middle|\; \int_0^1 D(Q)dQ = \mu \wedge D(Q) \geq D(Q') \; \forall Q \leq Q' \wedge D(Q) \in [0,1] \; \forall Q \right\}$$

In words, the choice set for the buyer is the set of inverse demand curves that 1) have an area under the curve that is equal to the prior, 2) are downward sloping—I repeat myself—and 3) are between zero and one. While still a large set, it is a very simple set to think about.

## 2.2 Objective: Consumer Surplus

From the constrained set, the problem now becomes to choose the inverse demand curve that maximizes buyer welfare, or, as it often label when discussing the inverse demand curve, consumer surplus.

To calculate consumer surplus for a given inverse demand curve, I will use the observation, shown in Figure 2, that consumer surplus is the area between the inverse demand curve and the marginal revenue curve up to the equilibrium quantity.[2] The observation that consumer surplus is the area between inverse demand and marginal revenue, while obvious in retrospect, completely eluded me until reading Bulow and Klemperer (2012). I suspect that I am not alone and encourage everyone to read their paper.

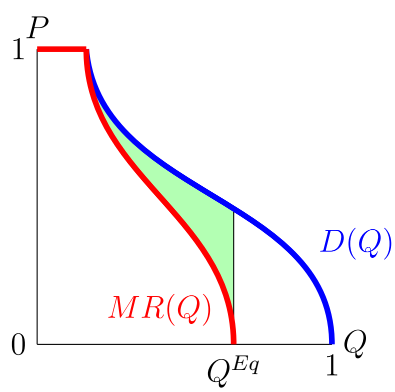


Figure 2: Demand, Marginal Revenue, and Consumer Surplus

elsewhere. It comes from integration by parts of the Riemann-Stieltjes integral:

$$\int_a^b x dF(x) = bF(b) - aF(b) - \int_a^b F(x)dx = 1 - \int_0^1 F(x)dx.$$

2. To see this, note that revenue is the integral of the marginal revenue curve **and** revenue is price multiplied by quantity: $D(Q)Q$.

To pin down equilibrium consumer surplus, we need to now consider the supply side. The monopolist will only choose a quantity where marginal revenue equals marginal cost, which is zero. The optimal quantity does not need to be unique but can be a range. Let $\underline{Q}_D$ be the minimum $Q$ and $\overline{Q}_D$ be the maximum $Q$ where the marginal revenue is zero. The subscript is to denote that the quantity bounds depend on the inverse demand curve. In general, $\overline{Q}_D$ will quantity we are interested in, since it will give the consumer higher surplus, but $\underline{Q}_D$ will be used in calculations.

## 2.3 Optimization Problem

We now have all of the parts in place to find the buyer-optimal inverse demand curve. The problem is to choose an inverse demand curve (which pins down a maximum quantity that the monopolist will supply) to maximize the area between the inverse demand curve and the marginal revenue curve, subject to demand curve being feasible (in $\mathcal{D}$) and the monopolist is best-responding by choosing a quantity $\overline{Q}$ where MR=MC. Formally, the maximization is:

$$\begin{aligned} \max_{D(Q)} \quad & \int_0^{\overline{Q}_D} D(Q)dQ - \int_0^{\overline{Q}_D} \left[D'(Q)Q + D(Q)\right] dQ \\ \text{subject to} \quad & \int_0^1 D(Q)dQ = \mu \\ & D(Q) \geq D(Q') \quad \forall Q \leq Q' \\ & D(Q) \in [0,1] \quad \forall Q \\ & D'(\overline{Q}_D)\overline{Q}_D + D(\overline{Q}_D) = MC = 0. \end{aligned} \tag{1}$$

This maximization looks cumbersome (choosing a function with constraints on the integrals and derivatives), but we can use the economics to further simplify the problem. First, we can split the quantities into those where the below the minimum price the monopolist will see, $\underline{Q}$, and those above. The objective function can be rewritten

$$\int_0^{\underline{Q}_D} D(Q)dQ + \int_{\underline{Q}_D}^{\overline{Q}_D} D(Q)dQ - \int_0^{\underline{Q}_D} \left[D'(Q)Q + D(Q)\right] dQ - \int_{\underline{Q}_D}^{\overline{Q}_D} \left[D'(Q)Q + D(Q)\right] dQ. \tag{2}$$

The first integral and the second part of the third integral cancel. We also know that the marginal revenue is zero for each $Q$ between $\underline{Q}$ and $\overline{Q}$, so the last term is zero. The objective then simplifies to

$$\int_{\underline{Q}_D}^{\overline{Q}_D} D(Q)dQ - \int_0^{\underline{Q}_D} D'(Q)QdQ. \tag{3}$$

The second term says that the objective is decreasing in the derivative of the inverse demand curve between 0 and $\underline{Q}$. Set $D'(Q) = 0$ between 0 and $\underline{Q}$. This gives us the first result about the shape of the optimal inverse demand curve:

**Observation 1.** The optimal inverse demand curve is flat below $\underline{Q}$.

Now let's again use that the marginal revenue must be zero for all quantities between $\underline{Q}$ and $\overline{Q}$. This holds if and only if the inverse demand curve is of the form $C/Q$ between $\underline{Q}$ and $\overline{Q}$, where $C$ is a constant. Note that $D(\underline{Q}) \leq 1$ from the third constraint and $C = D(\underline{Q})$. Therefore, $C \leq 1$. In fact, it is easy to verify that it is binding[3]

**Observation 2.** The optimal inverse demand curve is $1/Q$ between $\underline{Q}$ and $\overline{Q}$.

Notice now that $\overline{Q}$ only shows up as a constraint in the final constraint. However, from the shape of $1/Q$, the final constraints hold for any $Q$. Therefore, since the objective is increasing in $\overline{Q}$ and the constraint is always satisfied. It is optimal to raise $\overline{Q}$ to 1.

**Observation 3.** The optimal inverse demand curve has $\overline{Q} = 1$.

The three observations give the shape of the optimal inverse demand curve, given in Figure 3, which is the same shape that Roesler and Szentes show is optimal for their more complicated problem. The shaded green area between the inverse demand curve and the marginal revenue curve is consumer surplus. The producer surplus is the area under the marginal revenue curve, which is equal to $\underline{Q}$. The equilibrium quantity is one and price is $\underline{Q}$.

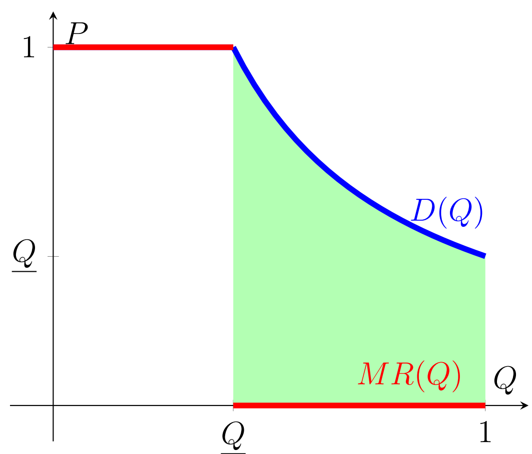


Figure 3: Buyer-Optimal Inverse Demand

We can use Bayes-plausibility to pin down $\underline{Q}$, the only remaining unknown. The Bayes-plausible constraint is

$$\int_0^1 D(Q)dQ = \mu = 1\underline{Q} + \int_{\underline{Q}}^1 \frac{1}{Q}dQ. \tag{4}$$

Therefore, $\underline{Q}$ is pinned down by

$$\underline{Q}[1 - \log \underline{Q}] = \mu. \tag{5}$$

3. Solve the maximization without the constraint and you will find that the optimal $C$ is unbounded.

## 2.4 Continuous Valuations

It is worth mentioning how my simplification, that the buyer can have high or low valuation, plays in. If I did not make this simplification, then finding the height of the demand curve below $\underline{Q}$ would be more complicated. It is no longer true that any demand curve with an area equal to the prior is feasible. We are back to the world of mean-preserving spreads. However, otherwise the logic goes through. The shape is the same; there is a flat part and a $C/Q$ part of the inverse demand curve. However, finding the optimal level that is feasible is more complicated.

# 3 Interpretation

Left to the reader.

# References


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