Parameter estimation in the logistic equation

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Inference with differential equations

Ideas

- Study parameter estimation with ordinary differential equations
- Minimal model: scalar equation with two parameters
- Logistic equations is integrable.
- We can analyze the inference process in terms of what we know of the analytical solution

Logistic equation

Ordinary differential equation

$$\frac{dx(t)}{dt} = rx(t)\left(1 - \frac{x(t)}{K}\right)$$
$$x(0) = x_0$$

Parameters

- Initial condition x_0
- Growth rate r
- Carrying capacity K

Logistic equation

Ordinary differential equation

$$
\begin{aligned}
\frac{dx(t)}{dt} &= rx(t)\left(1 - \frac{x(t)}{K}\right) \\
x(0) &= x_0
\end{aligned}
$$

Analytic solution

$$
x(t) = \frac{x_0 K}{x_0 + (K - x_0)\exp(-rt)}
$$

Logistic equation

Direct problem

Solve the initial value problem for the logistic equation given $\theta = (x_0, r, K)$

Inverse problem

Estimate x_0, r, K given noisy measurements z_i, $i = 1, ..., k$ of the state variable $x(t, \theta)$, assuming $x(t, \theta)$ is solution of the initial value poblem for the logistic equation

Plan

Parameter estimation

- We have (real) yeast growth data z_i at times t_1,t_2,...,t_k
- Because data consists of counts of cells , a natural and simple statistical model is the Poisson distribution

$$z_i \sim \mathrm{Poisson(x(t_i, \theta))}$$

  where $\theta = (r, K)$, $z = (z_1, ..., z_k)$, we assume x_0 known
- Assuming that the observations are independent between them, the joint distribution of the observed cell abundances is a good approximation to the conditional probability (likelihood) $\pi(z \mid \theta)$, which would simply be defined by the product of the individual probability density functions of the observations:

$$\pi(z \mid \theta) = \prod_{i=1}^{k} \mathrm{Poisson(x(t_i, \theta))}$$

Plan

Bayesian Inference

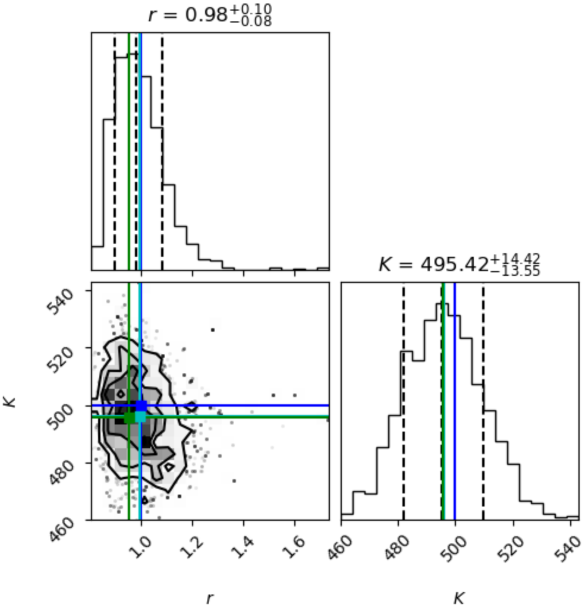
- We need an apriori probability density function $\pi(\theta)$
- Suppose $r \sim \text{Gamma}(a_r, b_r)$ and $K \sim \text{Gamma}(a_K, b_K)$ and are independent, i.e. $\pi(\theta) = \pi(r) \times \pi(K)$
- Bayes theorem implies

$$\begin{aligned}\pi(\theta \mid z) &\propto \pi(z \mid \theta) \times \pi(\theta) \\ &= \prod_{i=1}^{k} \mathrm{Poisson}(\mathrm{x}(\mathrm{t_i}, \theta)) \times \text{Gamma}(a_r, b_r) \times \text{Gamma}(a_K, b_k)\end{aligned}$$

- We carry out Markov Chain Monte Carlo (MCMC) to construct a statistically meaningful sample of the posterior conditional distribution

$$\pi(\theta \mid z)$$

Posterior distribution



Predictive posterior distribution