Team Paqasaurus Rex

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Motivation

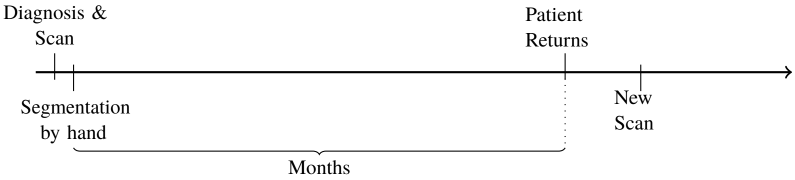

Figure: Current Timeline for Cancer Diagnosis and Treatment

Mathematics of Images

We can define an image as $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ for gray-scale images, where $f \in L^2(\mathbb{R}^2)$ and outputs intensity values

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

and

$$L^2(\mathbb{R}^2) = \left\{ f : \left(\int_{\mathbb{R}^2} |f(\mathbf{x})|^2 \, d\mathbf{x} \right)^{1/2} < \infty \right\}$$

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For color images, we can view the image as a function

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

where

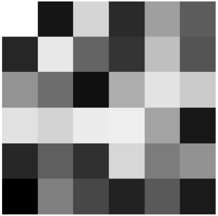
$$(i, j) \longmapsto (R, G, B)$$

where each coordinate has intensity values for red, green, and blue, similar to gray-scale

Discrete Images

- Gray-scale: $m \times n$ matrix with entries corresponding to sampled pixel intensities
- Color: $m \times n \times 3$ matrix with entries corresponding to sampled rgb intensities
- Discrete $\longrightarrow$ Continuous by defining an interpolation

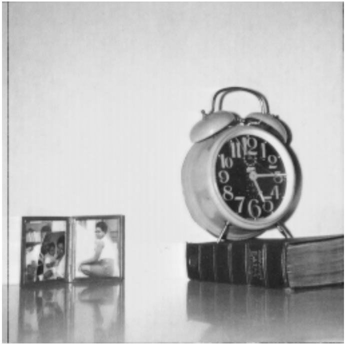
Example



$$\rightarrow \begin{bmatrix} 240 & 30 & 202 & 49 & 154 & 95 \\ 45 & 219 & 101 & 58 & 182 & 87 \\ 143 & 109 & 26 & 166 & 215 & 193 \\ 213 & 201 & 223 & 224 & 157 & 33 \\ 46 & 96 & 54 & 204 & 122 & 142 \\ 11 & 126 & 75 & 42 & 91 & 36 \end{bmatrix}$$

Noise

Noise may be thought of as a corruption to the image.



Noise

Salt and Pepper Noise Model. Let A be an image with entries $\{a_{i,j}\}$

$$N_\delta(a_{i,j}) = \begin{cases} a_{i,j} & \text{with prob } 1-\delta \\ 0 & \text{with prob } \delta/2 \\ 255 & \text{with prob } \delta/2 \end{cases}$$

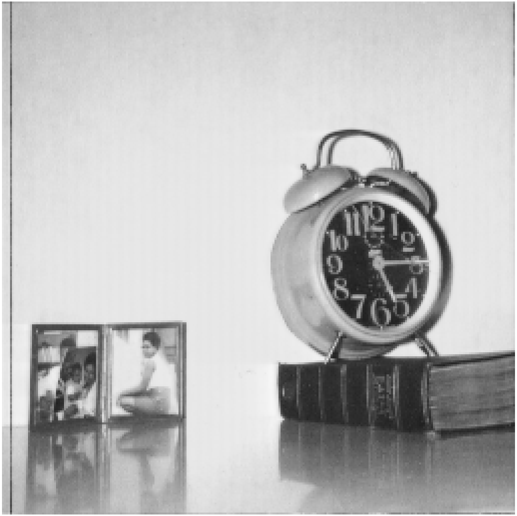
Denoising

For this talk, we have two examples of denoising algorithms

- Median Filter
- Nonlinear Total Variation Noise Removal (Rudin, Osher, Fatemi)

Median Filter

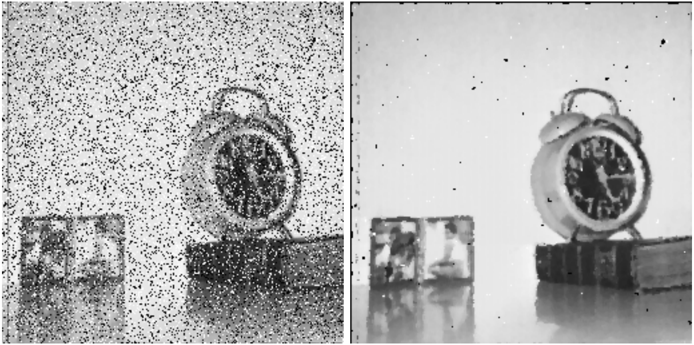
The median filter compares each pixel to its 8 neighbors and assigns to that pixel the median value of the 9 pixels.



Introducing .5 salt and pepper noise

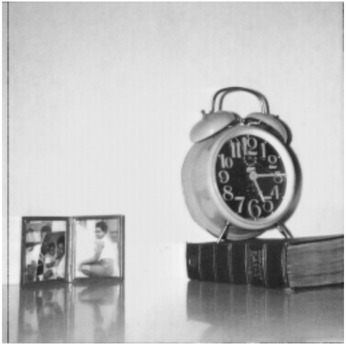
Running the algorithm

Median Filter



Most noise is removed, image is only slightly corrupted.

Median Filter




However, it’s blurry and the border is corrupted. This is why we are interested in stronger algorithms.

Basic denoising techniques, such as the median filter, fail to preserve detail of the image. In medical imaging, it is vital to retain as much detail as possible in order to optimize patient treatment.

i.e. Blurring edges of tumours/bones/ etc. can be detrimental to further image analysis and thus to treatment.

Original Image

ROF TV Denoising

Adding Salt & Pepper Noise

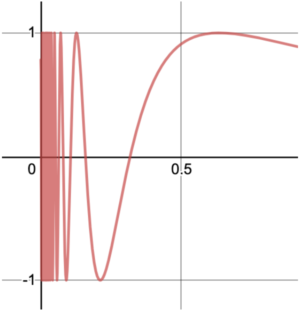
ROF TV Denoising

Denoising

ROF TV Denoising

Denoising

$$BV = \left\{ f : \|f\|_{BV} := \sup_{h\neq 0} |h|^{-1} \|f(\cdot + h) - f(\cdot)\|_{L^1} < \infty \right\}$$



1d example: The curve $\sin(\frac{1}{x})$ is not of bounded variation

(BV, L^2) Decomposition

Given an image $f \in L^2(\mathbb{R}^2)$, the J-functional below allows us to decompose our image into a BV image and an L^2 image:

$$J(f,\lambda) := \inf_{u+v=f} \left\{ \lambda \|v\|_{L^2}^2 + \|u\|_{BV} \right\}$$

$$(u_\lambda, v_\lambda) = \operatorname*{arginf}_{u+v=f} \left\{ \lambda \|v\|_{L^2}^2 + \|u\|_{BV} \right\}$$

for $\lambda > 0$. This decomposition will depend on the parameter λ chosen.

- If λ is too small, 'too many details' will be removed.
- If λ is too large, u_λ will be 'too similar' to the original image.

(BV, L^2) Decomposition

Original Image, f

(BV, L^2) Decomposition

BV Component, u_λ

(BV, L^2) Decomposition

L^2 Component, v_λ

Multiscale Decomposition

With the (BV, L^2) decomposition in place, we can iteratively construct a multiscale decomposition for an image f in the following way:

- Set $\lambda = \lambda_0 > 0$. Decompose f.

$$f = u_0 + v_0, \qquad [u_0, v_0] = \underset{u+v=f}{\operatorname{arginf}} J(f, \lambda_0)$$

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- Set $\lambda_1 = 2\lambda_0$. Decompose v_0,

$$v_0 = u_1 + v_1, \qquad [u_1, v_1] = \operatorname*{arginf}_{u+v=v_0} J(v_0, 2\lambda_0)$$

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Continuing in this fashion, letting

$$v_j = u_{j+1} + v_{j+1}, \qquad [u_{j+1}, v_{j+1}] = \underset{u+v=v_j}{\operatorname{arginf}} J(v_j, 2^{j+1}\lambda_0),$$

we get the k-scale decomposition

$$f = u_0 + u_1 + \cdots + u_k + v_k = C_k(f) + v_k.$$

and, for sufficiently large k, $f \sim u_0 + u_1 + \cdots + u_k = C_k(f)$.

Multiscale Decomposition

f

u_0

v_0

$$f = u_0 + v_0, \qquad [u_0, v_0] = \operatorname{arginf}_{u+v=f} J(f, \lambda_0)$$

v_0

u_1

v_1

$$v_0 = u_1 + v_1, \qquad [u_1, v_1] = \operatorname{arginf}_{u+v=f} J(f, 2\lambda_0)$$

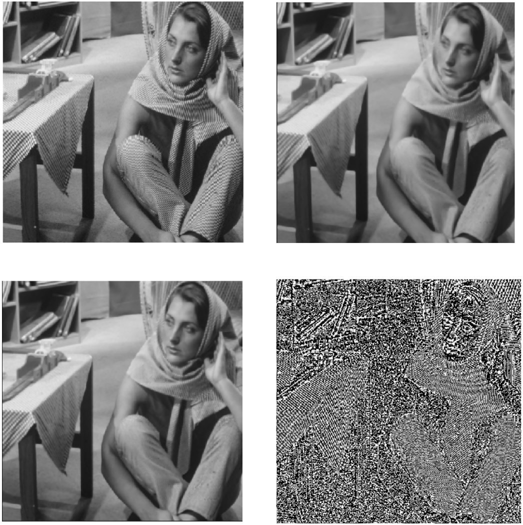
v_1

u_2

v_2

$$v_1 = u_2 + v_2, \qquad [u_2, v_2] = \operatorname{arginf}_{u+v=f} J(f, 2^2\lambda_0)$$

Multiscale Decomposition



Top: f, $C_0(f)$

Bottom: $C_1(f)$, v_1

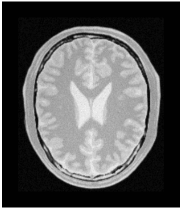
Registration and Segmentation

Definition (Registration of Images)

Let A and B be images and d some metric on the space of images. The registration of B to A is

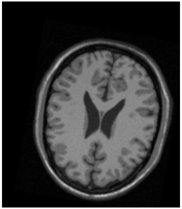
$$\varphi = \operatorname*{argmin}_{\psi} \left[d(A, \psi(B)) \right].$$

Where ψ is a function from some transformation space.



A

$\xrightarrow{\varphi(A)}$



B

Registration and Segmentation

We see 2 types of mappings in registration algorithms:

- Rigid
- Non-rigid

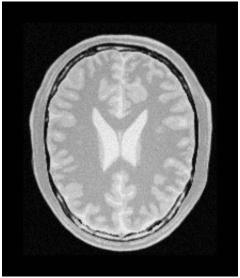
Original

A

B

Definition (Segmentation of Images)

The process of partitioning an image with the goal of locating objects and boundaries.



A

$$\xrightarrow{S(A)}$$

Summer Objective

Our goal was to integrate multiscale decomposition with segmentation and registration for a better way to analyze medical images.

Image Metrics

The 3 metrics we used to quantify the registrations were:

- Mean-Squares Metric
- Normalized Correlation Metric
- Mutual Information Metric

Note: These are not true mathematical metrics but share many important properties with metrics.

Definition (Mean Squares Metric)

Let A, B be discrete grayscale images of the same size. The mean squares metric is the function

$$MS(A,B) := \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}(a_{i,j} - b_{i,j})^2$$

Note: $MS(A,B) \geq 0$ and $MS(A,B) = 0$ if and only if $A = B$

Image Metrics

Definition (Normalized Correlation Metric)

Let A, B be discrete grayscale images of the same size. The normalized correlation metric is the function

$$NC(A,B) := \frac{-\sum_{i=1}^{m}\sum_{j=1}^{n} a_{i,j}b_{i,j}}{\sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} a_{i,j}^2 \sum_{i=1}^{m}\sum_{j=1}^{n} b_{i,j}^2}} = -\left\langle \frac{A}{||A||}, \frac{B}{||B||} \right\rangle.$$

Where $||\cdot||$ is the usual vector norm.

Note: If $A = \lambda B$ for nonzero λ then $NC(A,B) = -1$, this is our optimal value.

Histogram

Definition (Histogram)

Let A be an image. The histogram of A, h_A, is a vector of size 256×1 where the i^{th} entry is the number of pixels of intensity i in A.

Definition (Normalized Histogram)

Let A be an image of size $m \times n$. The normalized histogram of A is $\frac{h_A}{mn}$.

The normalized histogram gives an estimate of the probability distribution function of the pixel intensities of the image A.

Mutual Information

Mutual information is an information theoretic metric. It compares the probability distributions and the entropy within images.

Definition (Mutual Information)

$$MI(A,B) := \sum_{i=0}^{255} \sum_{j=0}^{255} \Pr(i,j) \log_2 \left(\frac{\Pr(i,j)}{\Pr_A(i) \Pr_B(j)} \right)$$

where i, j are intensity values.

Metric Analysis

- Is there a concavity change in the metrics as we introduce more noise?
- Are any of the metrics better suited to deal with different types of noise?

Metric Analysis

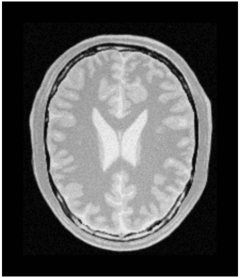
- Is there a concavity change in the metrics as we introduce more noise?
- Are any of the metrics better suited to deal with different types of noise?
- When are Mutual Information and the Mean Squares metrics linear?

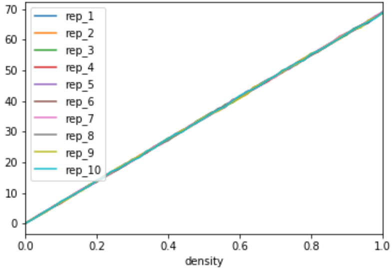
Metric Analysis on Salt and Pepper Noise

Given a $m \times n$ image A, and $0 \leq \delta \leq 1$

Theorem (Linearity of Expected value of Mean Squares)

$\mathbb{E}[MS(A, N_\delta(A))]$ is linear in δ.





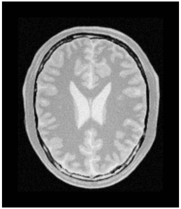
Metric Analysis on Salt and Pepper Noise

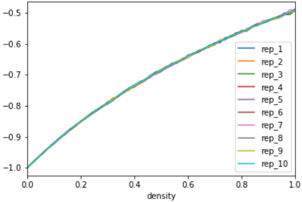
Given a $m \times n$ image A, and $0 \leq \delta \leq 1$

Theorem (Modeling Expected Value of Normalized Correlation)

Let a be the average pixel value of our image, then,

$$\mathbb{E}[NC(A, N_\delta(A))] = -\frac{a + \left(\frac{255}{2} - a\right)\delta}{\sqrt{a^2 + \left(\frac{255^2}{2} - a^2\right)\delta}}$$



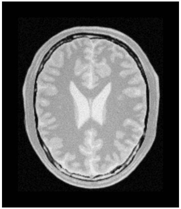


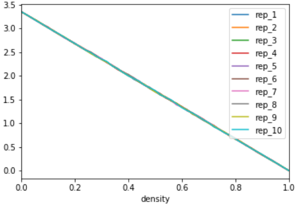
Metric Analysis on Salt and Pepper Noise

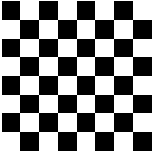
Given a $m \times n$ image A, and $0 \leq \delta \leq 1$

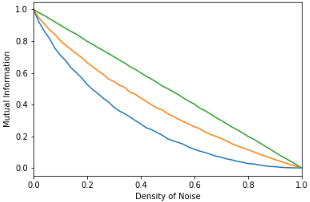
Theorem (Conditions on Linearity of Mutual Information)

$\mathbb{E}[MI(A, N_\delta(A))]$ is linear in δ if $f_{i,j} \notin \{0, 255\}$ for all i, j.





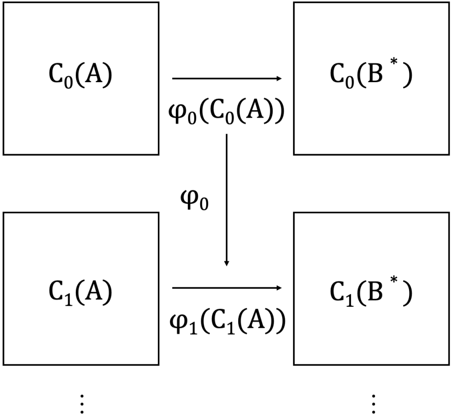




Iterative Multinodal Multiscale Registration Algorithm

Moving image A, and fixed image B^*, we seek to register A to B by some φ.

Idea: utilize multi-scale decomposition

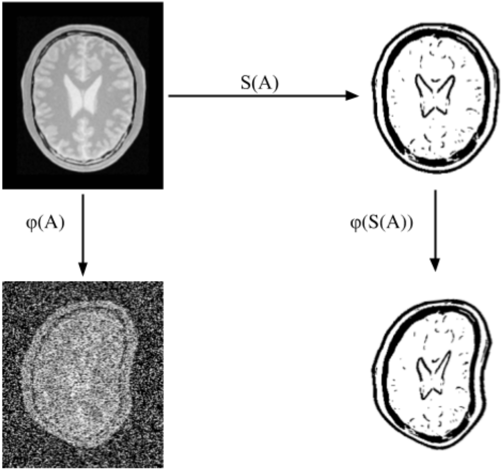


IMMRSA - Iterative Multinode Multiscale Registration Segmentation Algorithm

Given moving image A, a fixed image B^*, and a segmentation algorithm, S,

1. Segment A, to find $S(A)$.
2. Find $\varphi = \text{IMMRA}(A, B^*)$.
3. Map $S(A)$ through φ to get a segmentation of B^*.

IMMRSA Schematic



Moving Forward

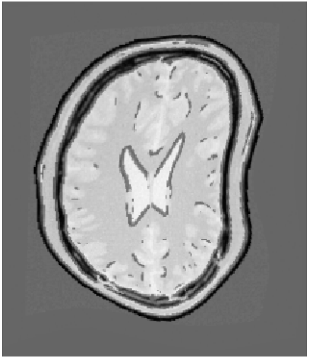
Additional topics within our scope:

- Can we create models for the other types of noise?
- Can we get a better PDF estimation and adjust our Mutual Information model?
- With improved registration algorithms, how much more effective can IMMRSA be with respect to standard segmentation algorithms?
- Is IMMRSA a computationally feasible algorithm?

We'd like to thank Bill and Linda Frost for making this project possible,
and open the floor to any questions.

IMMRSA Results

$\varphi(S(A))$



Overlayed

IMMRSA Result for $\delta = 0.5$

IMMRSA Results

δ	$S(M(B^*))$	$\Phi(S(A))$	$\varphi(S(A))$
0	0.01648	0.02473	0.01909
5	0.01942	0.02452	0.01909
10	0.02069	0.02460	0.01908
20	0.02556	0.02677	0.02185
30	0.02768	0.02710	0.02039
40	0.03463	0.03293	0.02376
50	0.04375	0.03481	0.01946
60	0.10087	0.04249	0.06214
70	0.25651	0.04666	0.04486
80	0.51770	0.04760	0.07296
90	0.78675	0.05826	0.09339
95	0.84934	0.10406	0.11041

Figure: Distance to benchmark with mean squared metric.