Image Metrics on Noisy Images

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Definition of an Image

An 8-bit, gray scale image is represented as

$$A_{m\times n} = [a_{i,j}], \qquad a_{i,j} \in \{0, 1, \dots, 255\}$$

where 0 corresponds to black, and 255 to white.
A color image (rgb) may be represented similarly as an $m \times n \times 3$ matrix, where entries correspond to red, green, and blue intensity values.

Motivation

- Acquired images in medical contexts are often corrupted by noise or distortion.
- In the presence of noise, using metrics to mathematically compare two images is often ineffective.
- We seek methods to mathematically quantify the behavior of image metrics in the presence of noise.

Metrics

Let A, B be images of resolution $m \times n$.

- The mean squares metric is defined as the sum

$$MS(A,B) := \frac{1}{mn}\sum_{i=0}^{m}\sum_{j=0}^{n}(A_{i,j} - B_{i,j})^2.$$

- The normalized correlation metric is the sum

$$NC(A,B) := -\frac{\sum A_{ij}B_{ij}}{\sqrt{\sum A_{i,j}^2 \sum B_{i,j}^2}}$$

- The mutual information metric is defined as the sum

$$MI(A,B) := \sum_{i=0}^{255}\sum_{j=0}^{255}\Pr(i,j)\log_2\left(\frac{\Pr(i,j)}{\Pr_A(i)\Pr_B(j)}\right)$$

where Pr is the PDF for the distribution of pixels.

Noise Model

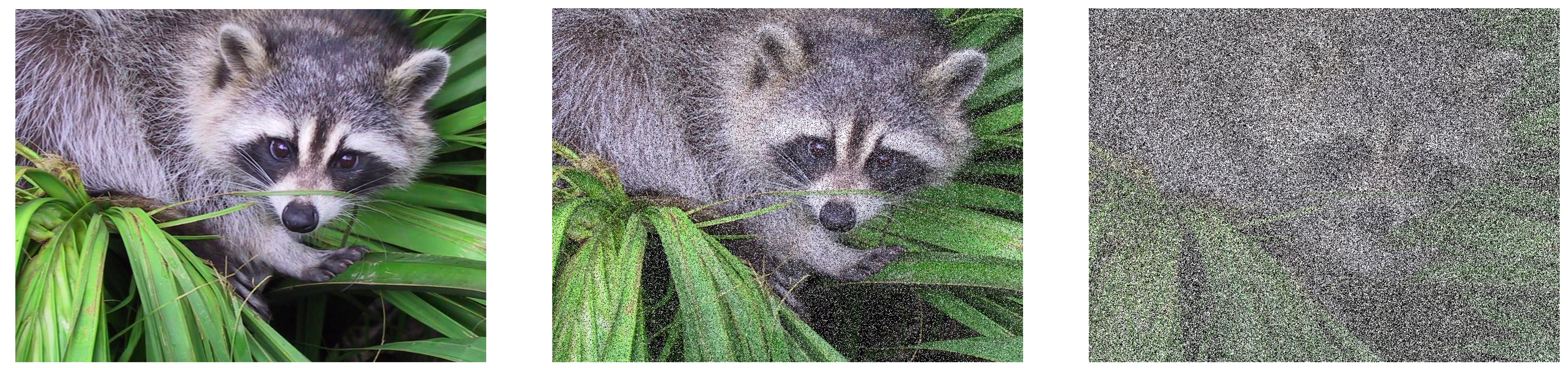
Impulse noise, or salt and pepper noise, is estimated by the following model.

An image f with applied impulse noise can be estimated by

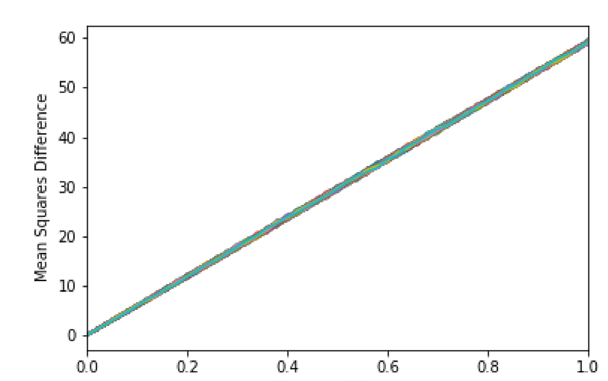
$$f_{i,j}(\delta) = \begin{cases} f_{i,j}, & \text{with prob. } (1-\delta) \\ \alpha, & \text{with prob. } p\delta \\ \beta, & \text{with prob. } (1-p)\delta \end{cases}$$

where $\delta \in [0,1]$ is the *density* of the noise, and $\alpha, \beta \in I$, if I is the set of possible intensity values, and p defines the *stability* of our noise.

Impulse Noise as a Function of δ

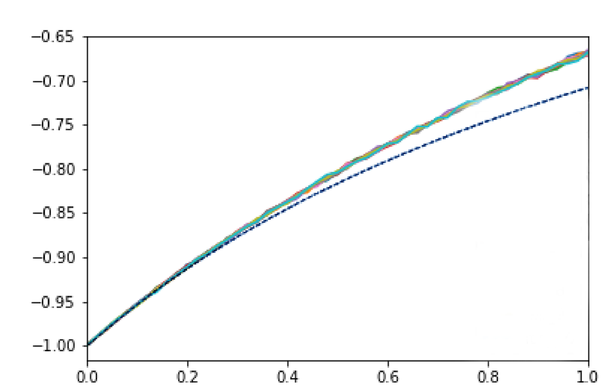


Mean Squares Metric



1000 repetitions for long term patterns

Normalized Correlation Metric

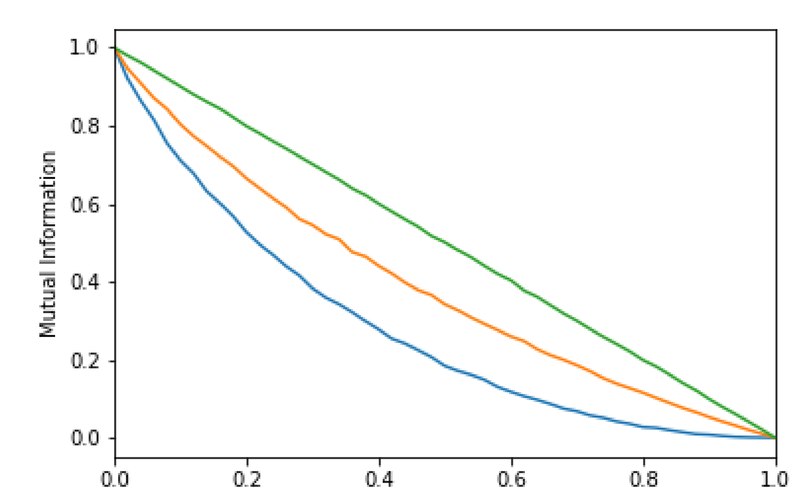


Predicted output vs actual

Mutual Information Metric

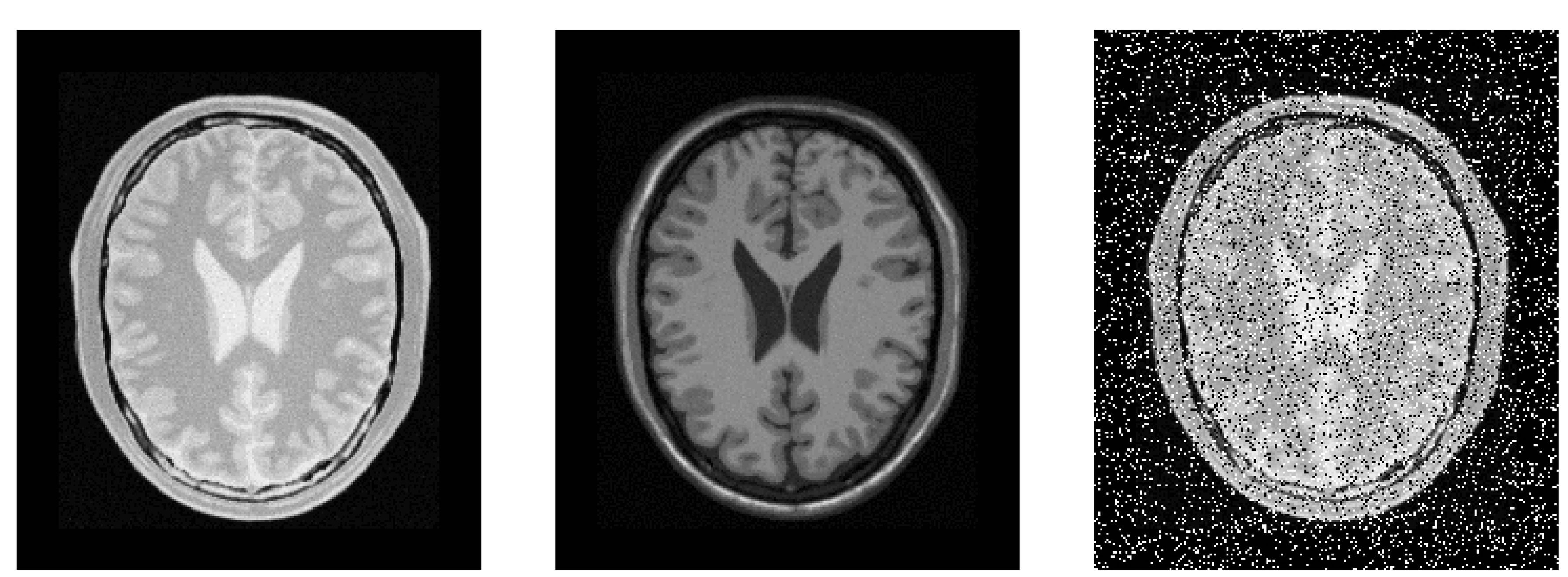
By constructing a probability distribution function from an image's histogram, we can decompose the mutual information metric into a linear and nonlinear portion.

$$MI(A,B) \approx \sum_{a\neq 0,k}\left(\log_2\left(\frac{1}{\Pr_A(a)}\right)(\delta(1+\Pr_A(a)) + \Pr_A(a))\right) + \sum_{a,b\in\{0,k\}}\Pr(a,b)\log_2\left(\frac{\Pr(a,b)}{\Pr_A(a)\Pr_B(b)}\right)$$



Checkerboard clipped at $\{0,255\}_B$, $\{1,255\}_O$, and $\{1,254\}_G$

Multimodality Images



Medical images of a brain from multiple modalities. From left to right: proton density, T1-weighted MRI, proton density slice with impulse noise.

First Result

Theorem

Given a $m \times n$ image f, we have

$\mathbb{E}[MS(f(0), f(\delta))]$ is approximately linear in δ.

Second Result

Theorem

Given a constant $m \times n$ image f, we have

$$\mathbb{E}[NC(f(0), f(\delta))] = -\frac{a+(\beta(1-p)-a)\delta}{\sqrt{a^2+(\beta^2(1-p)-a^2)\delta}}$$

where $f_{i,j} = a$ for all i, j.

Third Result

Theorem

Given an $m \times n$ image f, we have

$\mathbb{E}[MI(f(0), f(\delta))]$ is linear in δ
if $f_{i,j} \notin \{\alpha, \beta\}$ for all i, j.

Further Investigation

How can we model noise for different types of noise in our image such as Gaussian additive or multiplicative? How does our model for noise using MI change if we have a different PDF estimation? Can we define a metric based off of a multiscale image decomposition that better handles noisy images?

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