Convex Optimization and Applications to Image Registration

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In everyday life, many of the problems we seek to solve are, at heart, optimization problems. *We seek to maximize or minimize some quantity* in the presence of constraints.

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For the most part, these problems are *easy to state,* and *hard to solve.*

For example, most people are attempting to solve some sort of optimization problem related to happiness or joy or satisfaction in life. We want to maximize this quantity that is subject to the many different constraints faced day to day.

$$\begin{aligned} &\text{maximize} \quad \text{Joy} \\ &\text{subject to} \quad \text{Responsibilities.} \end{aligned}$$

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Definition (*Mathematical Optimization Problem*)

A *mathematical optimization problem* is a problem of the form

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & f_0(x) \quad (\textit{objective function}) \\ \text{subject to} \quad & f_i(x) \leq b_i \text{ for } i = 1, 2, \ldots, m, \\ & h_i(x) = c_i \text{ for } i = 1, 2, \ldots, p, \\ & \qquad\qquad (\textit{constraints}) \end{aligned}$$

where $b_i, \in \mathbb{R}$ for $i = 1, 2, \ldots, m$, $c_i, \in \mathbb{R}$ for $i = 1, 2, \ldots, p$.

Familiar Optimization Examples

Example (Minimizing a Quadratic)

$$\text{minimize}_{x \in \mathbb{R}^n} \, f(x) = x^T P x + q^T x + c$$

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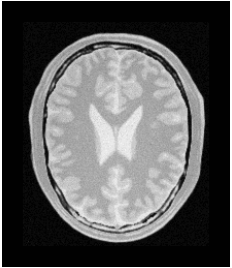
Example (Portfolio Optimization)

$$\begin{aligned} \text{maximize}_{x \in \mathbb{R}^n} \quad & c^T x \\ \text{subject to} \quad & x \succcurlyeq 0 \\ & \mathbf{1}^T x = 1 \end{aligned}$$

The Image Registration Problem

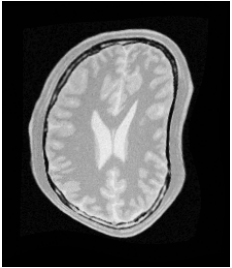
We will be particularly interested in the *image registration problem*, which is formulated mathematically as follows

> *Given two images A and B, a distance function D, and a transformation space τ, find the transformation Φ such that*
>
> $$\Phi = \mathrm{argmin}_{\varphi \in \tau} D(A, \varphi(B)).$$



Base, B

$$\xrightarrow{\varphi(B)}$$



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Our goal is to state the image registration problem as an optimization problem with a unique solution.

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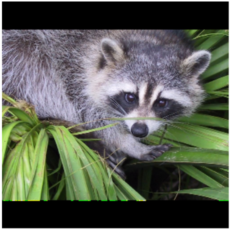
- Color images (RGB) will be functions into $\mathbb{R}^3$ with intensity values as triples in $\{0, \ldots, 255\}^3$.

The Image Registration Problem

For this talk we consider each $\varphi \in \tau$ to be a spacial transformation on coordinates in the physical space of the image:



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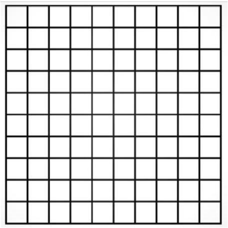

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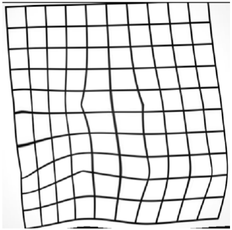

Image Grid

Transformed Image Grid

It is important to notice that our minimization problem

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is highly dependent upon this distance function D.

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Definition (*Mean Squares Metric*)

Given two images A and B, the *Mean Squares Metric* is the function

$$MS(A, B) = \frac{1}{mn} \sum_{x=1}^{n} \sum_{y=1}^{m} (A_{xy} - B_{xy})^2,$$

where A_{xy} and B_{xy} represent the intensity values at pixel (x, y) of the images A and B respectively.

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- Normalized Correlation Metric:

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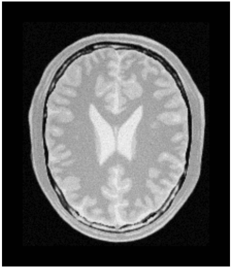
- Mutual Information Metric (allows for multi-modal registration):

$$MI(A,B) = \sum_{x \in X} \sum_{y \in Y} p(x,y) \log \left(\frac{p(x,y)}{p(x)p(y)} \right),$$

  where $p(x,y)$ is the joint probability of intensities between images, and $p(x), p(y)$ are the probabilities of intensities in image A and B respectively.

Consider a more challenging registration problem of registering two medical images of different modality (*e.g.* CT vs. MRI):



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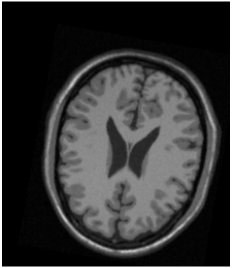

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  does that mean that B is 'closer' to A than C is?
- If $\varphi, \sigma \in \tau$ are two transformations, and

  $$MS(A, \varphi(B)) \leq MS(A, \sigma(B)),$$

  is φ a 'better' transformation than σ?

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The issues that arise in solving the Image Registration problem are not unique to it. In all optimization problems we are constantly running into the issue that

$$\text{local minima} \;\not\Leftrightarrow\; \text{global minima} ☹.$$

Again the mathematician is faced with an ambiguity that must be reconciled.

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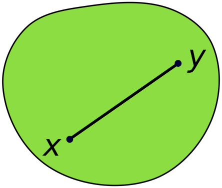
The central idea in all of these conditions is that of *convexity*.

Definition (*convex set*)

A set $C \subseteq \mathbb{R}^n$ is *convex* if for any two points $x_1, x_2 \in C$, the line segment between them lies in C. i.e.

$$\forall\, \theta \in [0,1], \quad \theta x_1 + (1-\theta) x_2 \in C$$



Cartoon of a Convex Set in $\mathbb{R}^2$

Example

Give any norm $\|\cdot\|$ on $\mathbb{R}^n$, the closed norm ball centered at $x_c \in \mathbb{R}^n$ of radius $r > 0$ defined by

$$B(x_c, r) = \{y \in \mathbb{R}^n \mid \|x_c - y\| \leq r\},$$

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Example

The hyperplane parallel to $a \in \mathbb{R}^n$ at b is the set

$$\{x \in \mathbb{R}^n \mid a^T x = b\}.$$

A hyperplane is a convex set, and each hyperplane splits $\mathbb{R}^n$ into two *halfspaces*

$$\{x \in \mathbb{R}^n \mid a^T x \leq b\} \text{ and } \{x \in \mathbb{R}^n \mid a^T x \geq b\},$$

both of which are convex sets.

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- A *polyhedron* in $\mathbb{R}^n$ can be expressed as the the intersection of finitely many hyperplanes and half-spaces, and thus is convex.
- If we denote $\mathbf{S}^n$ as the set of symmetric matrices in $\mathbb{R}^{n\times n}$, i.e.

$$\mathbf{S}^n = \{X \in \mathbb{R}^{n\times n} | X = X^T\},$$

then the set of positive semidefinite matrices

$$\mathbf{S}^n_+ = \{X \in \mathbf{S}^n | X \succcurlyeq 0\} = \{X \in \mathbb{R}^{n\times n} | x^T X x \geq 0 \, \forall x \in \mathbb{R}^n\},$$

is the intersection of halfspaces in $\mathbf{S}^n$,

$$\bigcap_{x\in\mathbb{R}^n} \left\{X \in \mathbf{S}^n \mid x^T X x \geq 0\right\},$$

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Definition (*convex hull*)

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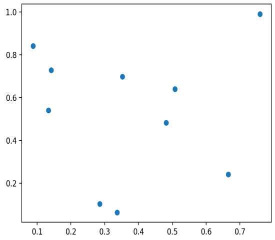
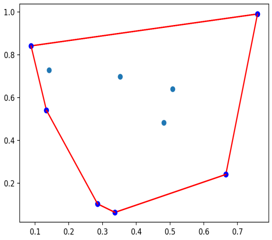


C



conv C

By considering the *epigraph* of a function $f : \mathbb{R} \to \mathbb{R}$, defined as

$$\textbf{epi}\, f = \{(x, t) \in \mathbb{R}^2 \mid f(x) \leq t\}$$

we can immediately relate our definition of a convex set to the codomain of a function.

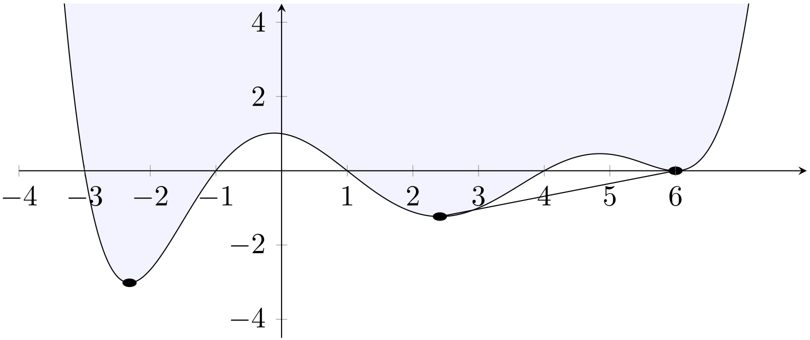


Figure: A non-convex epigraph, local minima $\not\Leftrightarrow$ global minimum

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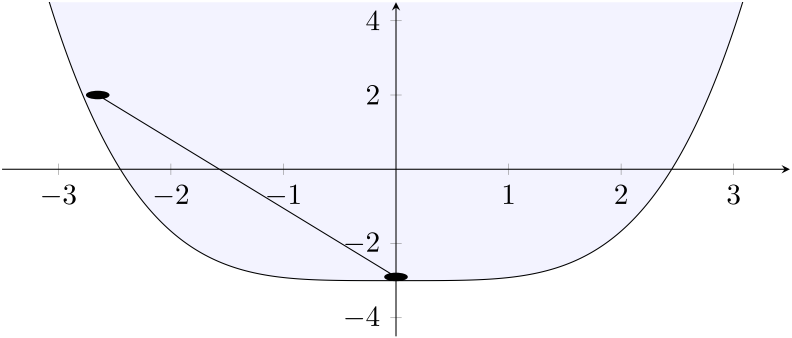

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A function $f : C \subseteq \mathbb{R}^n \to \mathbb{R}$ is *convex* if $C = \mathbf{dom}\, f$ is a convex set and if for all $x, y \in \mathbf{dom}\, f$ and $\theta \in [0, 1]$, the inequality

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  $$(x_1,t_1),(x_2,t_2) \in \mathbf{epi}\, f \implies f(x_1) \leq t_1 \text{ and } f(x_2) \leq t_2.$$

  So, for $\theta \in [0,1]$, $f(\theta x_1 + (1-\theta)x_2) \leq \theta t_1 + (1-\theta)t_2$.

Let $f : C \subseteq \mathbb{R}^n \to \mathbb{R}$ be a convex function. The following sets associated with f are all convex:

- The *α-sublevel set* of f defined as $C_\alpha = \{x \mid f(x) \leq \alpha\}$.
  Proof:
  $$f(\theta x+(1-\theta)y) \leq \theta f(x)+(1-\theta)f(y) \leq \theta\alpha+(1-\theta)\alpha = \alpha. \quad \square$$

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  So, for $\theta \in [0,1]$, $f(\theta x_1 + (1-\theta)x_2) \leq \theta t_1 + (1-\theta)t_2$. Thus
  $\theta(x_1,t_1) + (1-\theta)(x_2,t_2) \in \mathbf{epi}\, f$. $\square$

Example (Non-negative sums)

Given two convex functions f and g, the sum $f + g$ is convex on $\mathbf{dom}\, f \cap \mathbf{dom}\, g$.

Proof: The domain $\mathbf{dom}\,(f+g) = \mathbf{dom}\, f \cap \mathbf{dom}\, g$ is convex. ✓
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This can be extended to arbitrary finite sums, non-negatively weighted sums, so that

$$\mathcal{F} = \{ f : \mathbb{R}^n \to \mathbb{R} \mid f \text{ is convex } \}$$

forms a convex cone.

With the notion of convex sets and convex functions understood, we can now define a *convex optimization problem*.

Convex Optimization Problems

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A standard *convex optimization problem* is an optimization problem of the form:

$$\begin{aligned} \underset{x \in \mathcal{D}}{\text{minimize}} \quad & f_0(x) \\ \text{subject to} \quad & f_i(x) \leq 0, \quad i = 1, \ldots, m, \\ & a_i^T x = b_i, \quad i = 1, \ldots, p, \end{aligned}$$

where f_i are all convex functions, and $a_i \in \mathbb{R}^n$ (equivalently, $Ax = b$ for $A \in \mathbb{R}^{p \times n}$).

Given our standard convex optimization problem, we have some immediate consequences that inform us our definition is pretty good:

- The domain of the problem, $\mathcal{D}$, is a convex set, as

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- The set of all elements in $\mathcal{D}$ that satisfy all constraints, called the *feasible set*,

$$\mathcal{D} \cap \left(\bigcap_{i=1}^{m} \{x \mid f_i(x) \leq 0\} \right) \cap \left(\bigcap_{i=1}^{p} \{x \mid a_i^T x = b_i\} \right)$$

  is convex, as it is the intersection of $\mathcal{D}$, 0-sublevel sets of convex functions, and hyperplanes.

Convex Optimization Problems

Theorem (Reconciliation of the Minima)

Give a convex optimization problem, any local minima $\tilde{x}$ of f_0 in the feasible set is a global minima of f_0 over the feasible set.

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Since f_0 is convex we have that

$$\begin{aligned} f_0(\theta\tilde{x} + (1-\theta)x^*) &\leq \theta f_0(\tilde{x}) + (1-\theta)f_0(x^*) \\ &\leq f_0(\tilde{x}) \text{ for any } \theta \in [0,1]. \end{aligned}$$

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- We have just shown that convex optimization problems have the structure that we desired our optimization problems to have; now we set out to solve them.

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Let's start by considering an unconstrained convex optimization problem with f sufficiently differentiable on an open domain:

$$\underset{x \in \mathbf{dom}\, f}{\text{minimize}} \quad f(x).$$

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- This must occur at a point such that $\nabla f(x^*) = 0$.
- Since local minima are global minima, a solution to the above equation solves the minimization problem.
- Sometimes this may be done analytically, but most of the time we use numerical approximations given by various descent methods.

A More Complicated Case

Now we consider the convex optimization problem

$$\begin{array}{ll} \underset{x \in \mathbf{dom}\, f}{\text{minimize}} & f_0(x) \\ \text{subject to} & f_i(x) \le b_i, \quad i = 1, \ldots, m, \end{array}$$

or equivalently

$$\begin{array}{ll} \underset{x \in \mathbf{dom}\, f}{\text{minimize}} & f_0(x) \\ \text{subject to} & f_i(x) - b_i \le 0, \quad i = 1, \ldots, m, \end{array}$$

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$$I_-(u) = \begin{cases} \infty & u < 0 \\ 0 & \text{otherwise.} \end{cases}$$

- If $f_i(x) > b_i$ for any i, $f(x) = \infty$, and so if this problem has a solution it must be feasible.
- $f(x)$ is still convex (sum of two convex functions), but is clearly no longer differentiable, so descent methods won't work.

To solve this problem, we introduce the *logarithmic barrier function* to approximate a solution to the previous problem:

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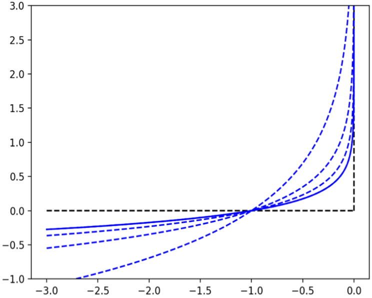

Figure: Plots of the $\hat{I}(u,t)$ for increasing values of t from $t = 1$ to $t = 4$ in blue. Solid blue is $t = 4$. Black dashed line is the function $I_{-}(u)$.

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- As $t \to \infty$, the solution approaches the solution to the original inequality constrained problem.
- We solve iteratively in t to ensure a feasible starting position x at as t increases in some predefined step size.

Now recall the image registration problem over some transformation space τ with $D = MS$. Given by

$$\underset{\varphi \in \tau}{\text{minimize}} \quad MS(A, \varphi(B)) = \frac{1}{mn} \sum_{x,y} e_{xy}(\varphi(B)_{xy})$$

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1. τ to be a convex set.
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We construct a convex transformation space $\tau \subseteq \mathbb{R}^{2k}$ by parameterizing transformations in the following way:

- Define a collection functions in x and y, that form a set of basis vectors for a given image when evaluated at each pixel. *e.g.*

$$(1, x, y), \quad (1, x, y, xy, x^2, y^2), \quad (\gamma(\|(x, y) - (x_i, y_i)\|))_{i=1}^{k}$$

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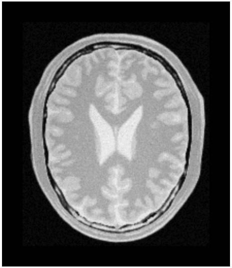
- Introduce variables $\mathbf{p_X}$ and $\mathbf{p_Y}$ in $\subseteq \mathbb{R}^k$, where k is the number of basis functions chosen, and define the displacement from (x, y) as a linear function of $\mathbf{p_X}$ and $\mathbf{p_Y}$. *e.g.*

$$(x, y) \mapsto \left(x + \mathbf{p_X}^T(1, x, y),\ \ y + \mathbf{p_Y}^T(1, x, y)\right)$$

$$(x, y) \mapsto \left(x + \sum_{i=1}^{k} p_{X_i}\gamma(r_i),\ \ y + \sum_{i=1}^{k} p_{Y_i}\gamma(r_i)\right)$$

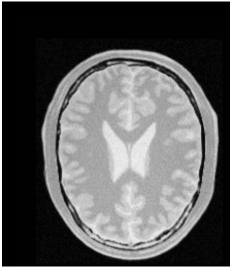
A Convex Image Registration Problem

To exemplify this, we model the translation below:



Base, B

$$\xrightarrow{\varphi(B)}$$



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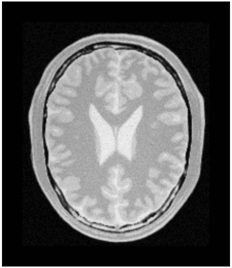

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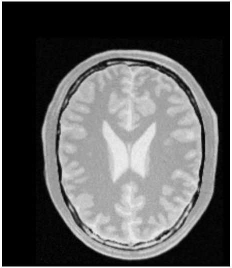

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Using basis functions $(1, x, y)$ with

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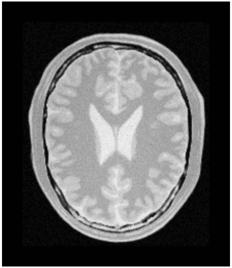

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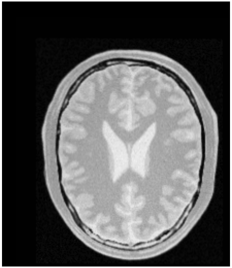

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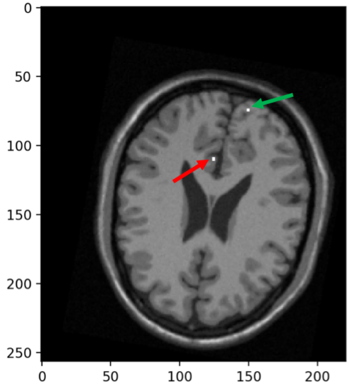

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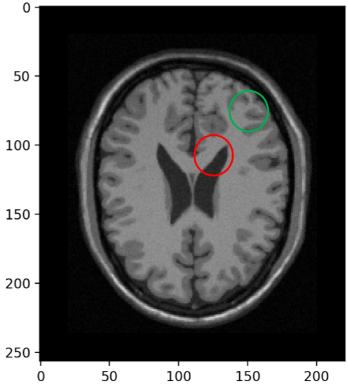
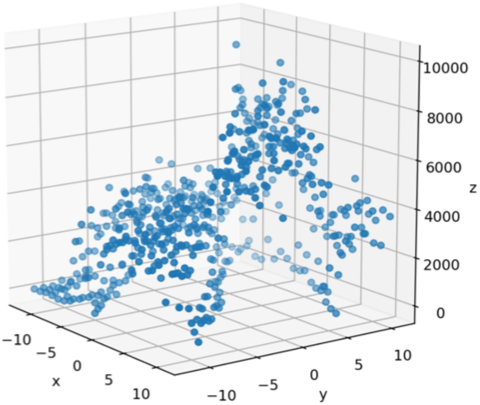


Base, B

But clearly this is not the circumstance we find ourselves in.



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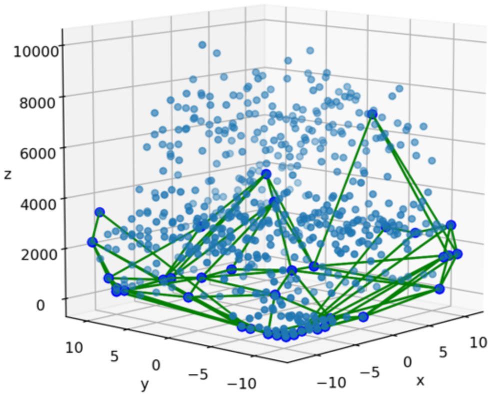

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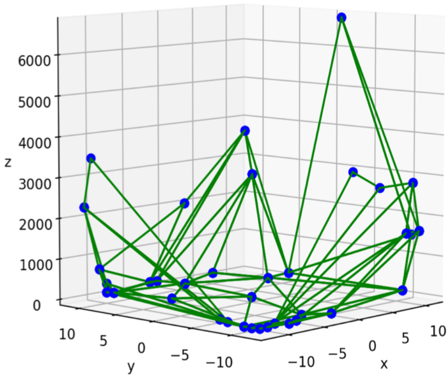

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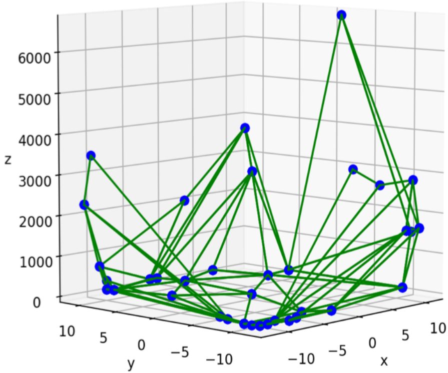

$MS' = \frac{1}{mn} \sum_{x,y} e'_{xy}(\varphi(B)_{xy})$ gives a convex approximation to MS.

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- For each pixel (x, y) we have a some number, F_{xy}, of planes that define it, each with these four coefficients.

$$a_x^i(x,y)\tilde{x} + a_y^i(x,y)\tilde{y} + a_z^i(x,y)\tilde{z} = b_i(x,y), \quad i = 1, \ldots, F_{xy}.$$

The lower hull of a set of points in $\mathbb{R}^3$, as we have, can be described by the planar facets that define it.

- A plane in $\mathbb{R}^3$ is defined by 4 coefficients a_1, a_2, a_3, b, so that a point (x, y, z) is in the plane if

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  Here $\tilde{x}$ and $\tilde{y}$ represent displacements from x and y respectively, and $\tilde{z}$ is the point on the corresponding facet.

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lies on or above the convex error surface. To reflect this we introduce an auxiliary variable $z(x, y)$ for each pixel and require that, for $i = 1, \ldots, F_{xy}$,

$$a_x^i(x, y)D_x(x, y, \mathbf{p_X}) + a_y^i(x, y)D_y(x, y, \mathbf{p_Y}) - z(x, y) \geq b_i(x, y).$$

i.e. $z(x, y)$ must be on or above each planar facet of the new error surface, and then we have an associated cost $e'_{xy} = z(x, y)$.

A Convex Image Registration Problem

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$$\text{minimize}_{z,\mathbf{p_X},\mathbf{p_Y}} \sum_{x,y} z(x,y) = \mathbf{1}^T z$$

subject to

$$z(x,y) \geq a_x^i(x,y) D_x(x,y,\mathbf{p_X}) + a_y^i(x,y) D_y(x,y,\mathbf{p_Y}) - b_i(x,y)$$

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for all i, for all x, y.

- This is a linear function of z with linear inequality constraints, of which there are $F = \sum_{x,y} F_{xy}$, one for each planar facet.

This is precisely the form of the problem we used the barrier method to solve.

We claim that, with blood, sweat and tears, this problem may be reformulated as

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$$\begin{aligned}
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&\text{subject to} \\
&A_X C\mathbf{p_X} + A_Y C\mathbf{p_Y} - I_z z \preccurlyeq b.
\end{aligned}$$

It may be shown that, in solving each problem given by the barrier method, computing the descent direction $\delta_{\mathbf{p}}$, can be boiled down to solving the following equation:

$$(H_{\mathbf{p}} - H_z D^{-1} H_z^T)\delta_{\mathbf{p}} = g_{\mathbf{p}} - H_z D^{-1} g_z.$$

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- *e.g.*, an affine transformation described by the basis functions $(1, x, y)$ requires $\mathbf{p_X}, \mathbf{p_Y} \in \mathbb{R}^3$, so that the above is as system of 6 equations.

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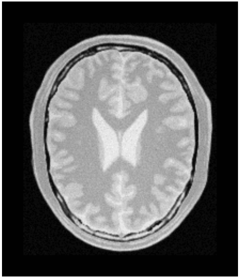
Results

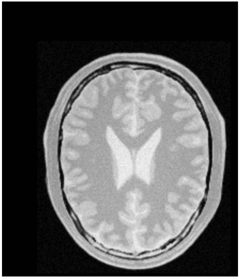
So we have successfully constructed a convex problem related to the image registration problem, but is it at all meaningful or useful?

- We saw that the cost associated with a particular pixel may be severely underestimated via our construction. Isn't that bad?
- These are images, after all, how feasible is it to construct this approximation for each pixel?

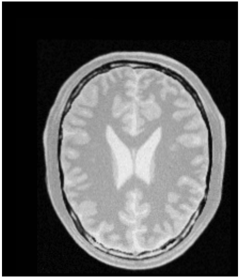
Results (Rigid)



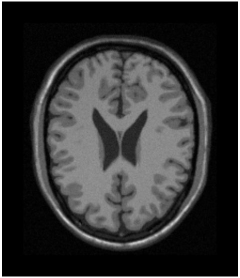
Base, B



Target, A



Registered, $\Phi(B)$



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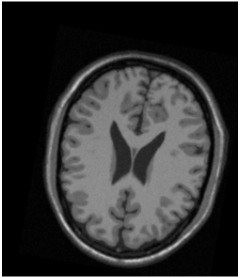

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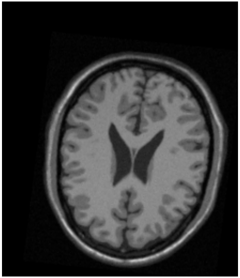

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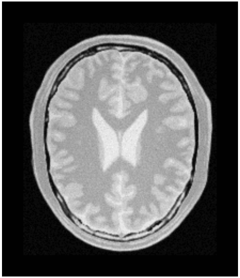
Results (Non-Rigid)



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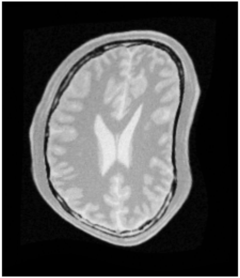

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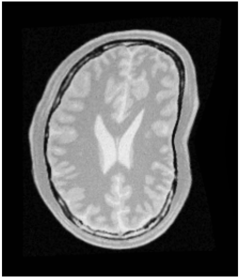

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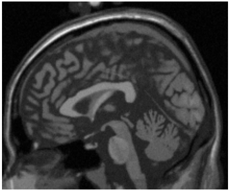
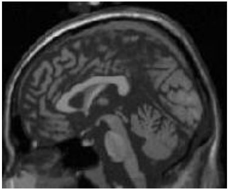


Base, B



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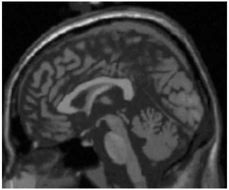

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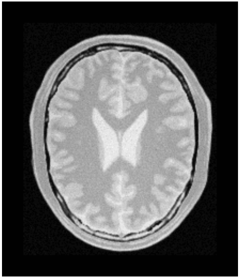

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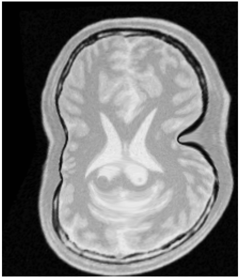

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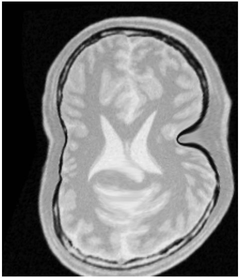

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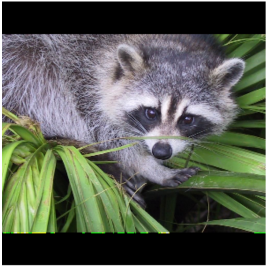
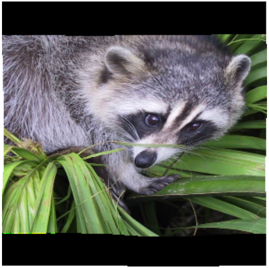


Base, B



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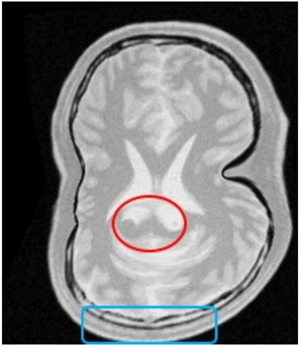

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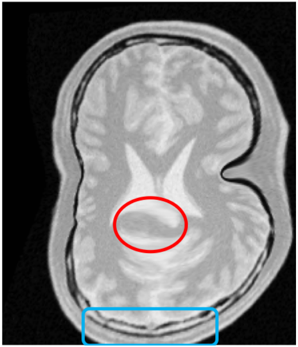

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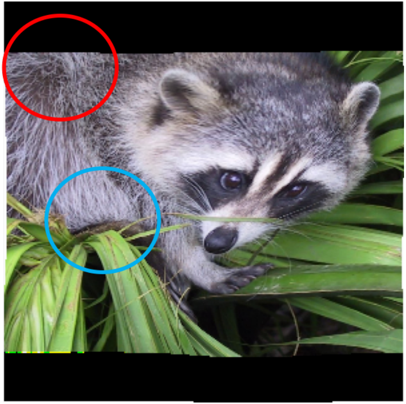

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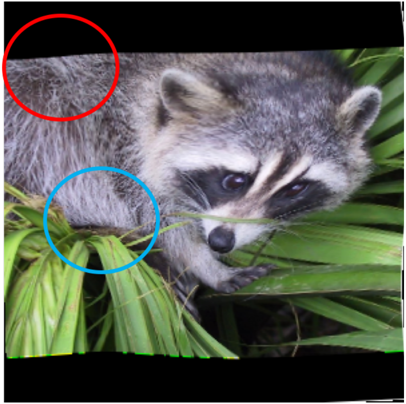

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Limitations and Future Work

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- A multi-modal convex registration algorithm

Thank You

Thank you to my advisor, the entire math department, my family and my friends.

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Questions?