Bounded gaps between products of special primes

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- Maynard-Tao: $C \leq 600$,
- Polymath: $C \leq 270$.

Generalizations

Generalizations:

- Bounded gaps between E_r numbers (product of r distinct primes) for $r \geq 2$?
- Restrict to a "nice" subset of primes $\mathcal{P}$?

Work of Thorne

Let $\mathcal{P}$ be a "well-distributed" subset of primes,
q_n be the n^{th} E_r number with all prime factors in $\mathcal{P}$.

Theorem (Thorne)

For any such $\mathcal{P}$ and $r \geq 2$, there exists an explicit constant $C(r, \mathcal{P})$ such that

$$\liminf_{n\to\infty}(q_{n+1} - q_n) \leq C(r, \mathcal{P}).$$

Applications

Thorne's main result leads to several corollaries:

Bounded gaps between integers n with special properties:

1. The class group $\mathrm{Cl}(\mathbb{Q}(\sqrt{-n}))$ contains order 4 elements.
2. Given an elliptic curve $E/\mathbb{Q}$, the quadratic twist $E(n)$ has rank 0 and $L(E(n), 1) \neq 0$.

Our observation

We observe that in Thorne's applications, without a restriction on the number of prime factors, much improved bounds are possible.

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We will make the statement precise. Then we revisit Thorne's examples and give better bounds on gaps.

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Goal: Prove bounded gaps between $E_{\mathcal{P}}$ numbers.

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Density δ_i = the proportion of $E_{\mathcal{P}}$ numbers represented by $a_i n + b_i$.

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Minimum density δ = minimum of δ_i.

Strategy: If at least two $Mn + b_i$ represent $E_{\mathcal{P}}$ numbers infinitely often, our k-tuple gives bounded gaps (bounded by $b_k - b_1$).

Precise Statement

Let $\mathcal{P}$ be a set of prime with density $\alpha > 0$ that satisfies $SW(M)$, $\{a_i n + b_i\}$ be an M-admissible k-tuple with minimum density δ.

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Theorem (C., Li)

There are at least $\nu + 1$ forms among them which infinitely often simultaneously represent $E_{\mathcal{P}}$ numbers, provided that

$$k > \nu \frac{4^{1-\alpha}}{\delta\phi(M)} \frac{\mathfrak{b}(0,\alpha)\mathfrak{b}(1,\alpha)}{\mathfrak{b}(k,\alpha)} (1-\alpha)^2 \Gamma(\alpha)\Gamma(1-\alpha),$$

where

$$\mathfrak{b}(k,\alpha) := \frac{\Gamma(1-\alpha)\Gamma(k(1-\alpha)+1)}{\Gamma((k+1)(1-\alpha)+1)}.$$

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As a corollary (for $\nu = 1$, $k = 2$), we have a twin-prime type result.

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Let $\mathcal{P}$ be a set of prime with density $\alpha > 0$ that satisfies $SW(M)$, δ be the minimum density of the pair $(Mn, Mn + d)$.

Corollary

For any even number d, assume that $\delta\phi(M) > F(\alpha)$, where

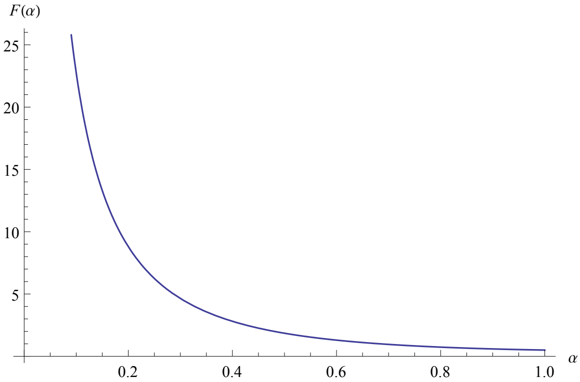
$$F(\alpha) = 2^{1-2\alpha} \frac{\mathfrak{b}(0,\alpha)\mathfrak{b}(1,\alpha)}{\mathfrak{b}(2,\alpha)} (1-\alpha)^2 \Gamma(\alpha)\Gamma(1-\alpha).$$

Then there are infinitely many n for which n and $n + d$ are simultaneously $E_{\mathcal{P}}$ numbers.

Plot

Corollary

For any even number d, assume that $\delta\phi(M) > F(\alpha)$, then there are infinitely many n for which n and $n+d$ are simultaneously $E_{\mathcal{P}}$ numbers.



Sketch of the proof

Following the same idea as GPY/GGPY and Thorne, consider

$$S = \sum_{n=N}^{2N} \left(\sum_{i=1}^{k} \chi_{\mathcal{P}}(a_i n + b_i) - \nu \right) \left(\sum_{d \mid \prod_i (a_i n + b_i)} \lambda_d \right)^2 ,$$

where

- $\chi_{\mathcal{P}}$ is the characteristic function of $E_{\mathcal{P}}$ numbers,
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If $S > 0$ for all large N, the theorem is true.

Remark: The additional requirement that $p \nmid d$ for all $p \in \mathcal{P}$ gives a bigger sieve weight to those n for which $\prod_i (a_i n + b_i)$ is divisible by many primes in $\mathcal{P}$.

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Break S up into

$$S_0 := \sum_{n=N}^{2N} \left(\sum_{\substack{d | \prod_i (a_i n + b_i) \\ p \nmid d \forall p \in \mathcal{P}}} \lambda_d \right)^2,$$

and

$$S_{1,j} = \sum_{n=N}^{2N} \chi_{\mathcal{P}}(a_j n + b_j) \left(\sum_{\substack{d | \prod_i (a_i n + b_i) \\ p \nmid d \forall p \in \mathcal{P}}} \lambda_d \right)^2.$$

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Then $S = \sum_{j=1}^{k} S_{1,j} - \nu S_0$.
Choose the sieve weights λ_d so that S_0 is small and $S_{1,j}$ is big.
Estimate $S_{1,j}$, S_0: use BV estimate and Selberg diagonalization.

Applications

We now revisit two of Thorne's applications:

Bounded gaps between integers n with special properties:

1. The class group $\mathrm{Cl}(\mathbb{Q}(\sqrt{-n}))$ contains order 4 elements.
2. Given an elliptic curve $E/\mathbb{Q}$, the quadratic twist $E(n)$ has rank 0 and $L(E(n),1) \neq 0$.

Ideal class groups with order 4 elements

Thorne's main result leads to the corollary:

Corollary

There are infinitely many pairs of E_2 numbers m and n such that the class groups $\mathrm{Cl}(\mathbb{Q}(\sqrt{-m}))$ and $\mathrm{Cl}(\mathbb{Q}(\sqrt{-n}))$ each have elements of order 4, with

$$|m - n| \leq 64.$$

Ideal class groups with order 4 elements

Our main result yields:

Corollary

There are infinitely many pairs of square-free m and n such that the class groups $\mathrm{Cl}(\mathbb{Q}(\sqrt{-m}))$ and $\mathrm{Cl}(\mathbb{Q}(\sqrt{-n}))$ each have elements of order 4, with

$$|m - n| \leq 8.$$

Ideal class groups with order 4 elements

Theorem (Soundararajan)

For any positive square-free number $d \equiv 1 \bmod 8$ whose prime factors $\equiv \pm 1 \bmod 8$, the class group $\mathrm{Cl}(\mathbb{Q}(\sqrt{-d}))$ contains an element of order 4.

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Corollary

There are infinitely many square-free numbers n such that the class groups $\mathrm{Cl}(\mathbb{Q}(\sqrt{-n}))$ and $\mathrm{Cl}(\mathbb{Q}(\sqrt{-n-8}))$ each contain elements of order 4.

Quadratic twists of Elliptic curves

Thorne's main result leads to the corollary:

Corollary

Let $E/\mathbb{Q}$ be an elliptic curve without a $\mathbb{Q}$-rational torsion point of order 2. There is $C_E > 0$ and infinitely many pairs of square-free integers m and n for which:

(i) $L(E(m), 1) \cdot L(E(n), 1) \neq 0$,

(ii) $rank(E(m)) = rank(E(n)) = 0$,

(iii) $|m - n| \leq C_E$.

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The proof uses results by Ono and Murty-Murty.

Further improvement

Recall the main result:

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Using Maynard's new sieve method may improve our main result.