Random walks on surfaces and homogeneous spaces: Measure rigidity, equidistribution and orbit closures

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Thesis defense

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Measure rigidity, equidistribution and orbit closures

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start with $x_0 \in M$, at each stage pick a random element $g_n \in \Gamma$ w.r.t. μ, and get a new point $x_{n+1} := g_n x_n$.

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Question 3: (Equidistribution) How does the sequence $\{x_n\}$ distribute on M?

Example 1

Say $M = \mathbb{T}^2$, and $\Gamma = \langle f, g \rangle \subset SL_2(\mathbb{Z})$ with $f = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, $g = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, which acts on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ by left multiplication, i.e.

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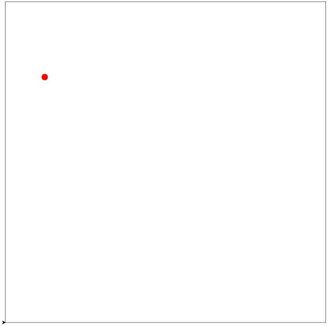

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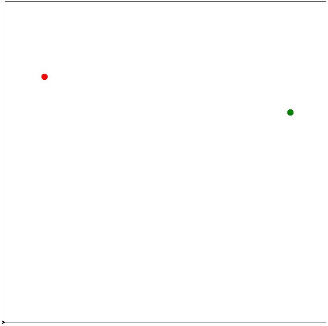

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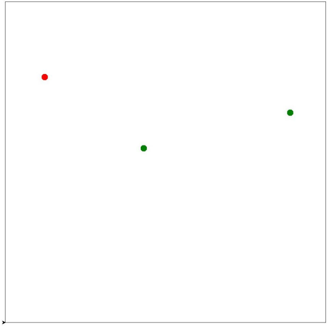

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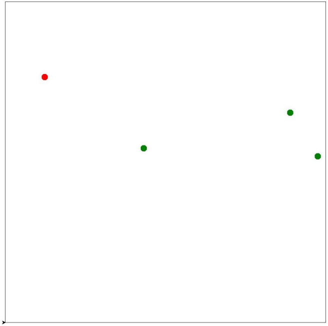

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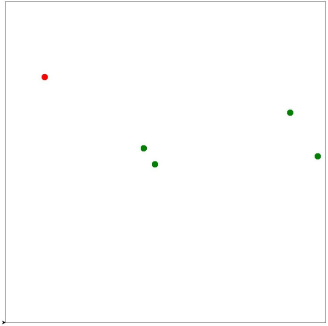

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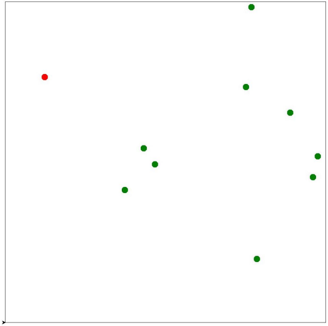

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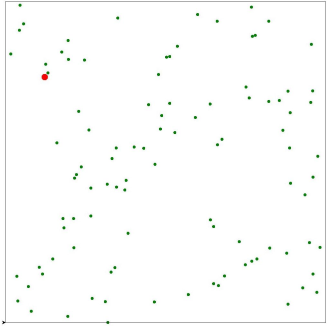

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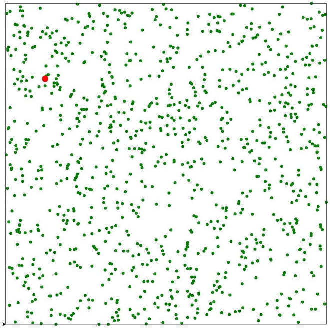

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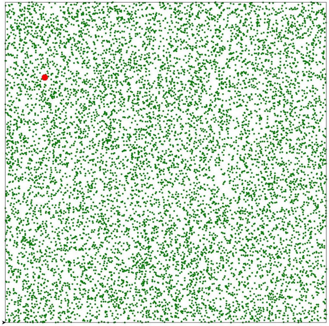

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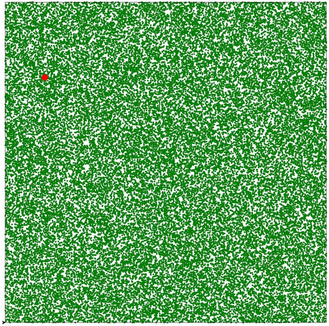

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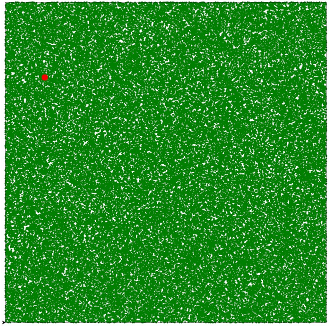

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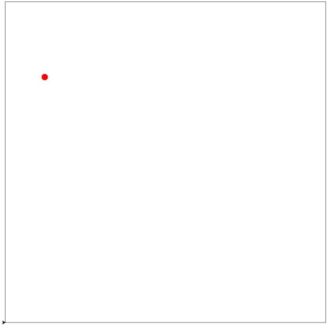

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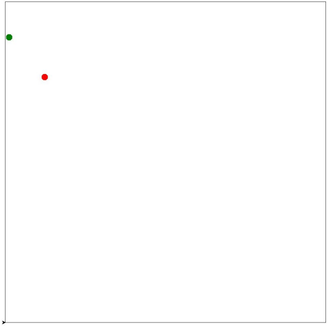

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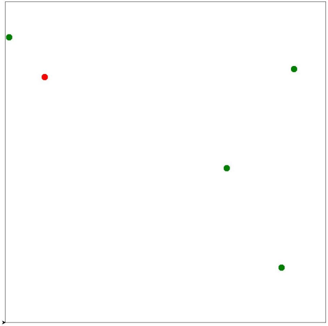

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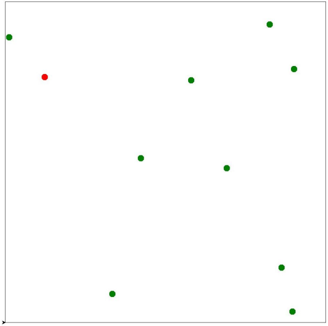

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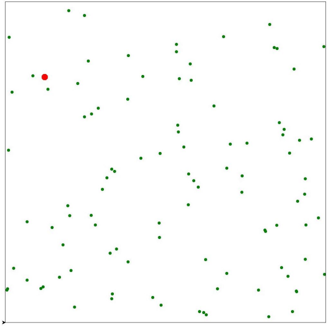

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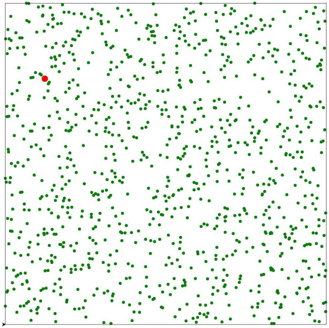

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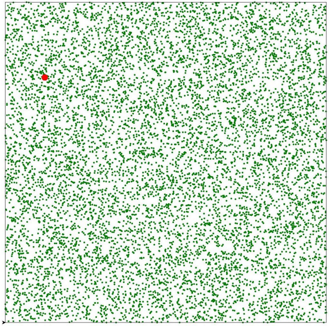

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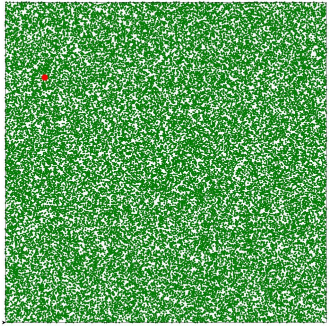

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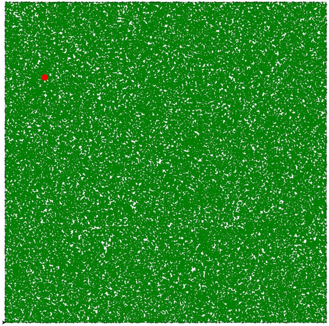

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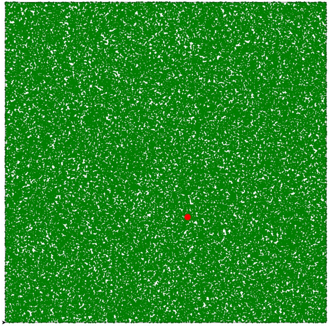

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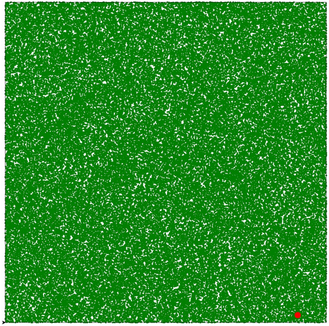

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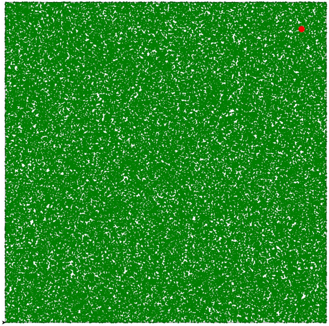

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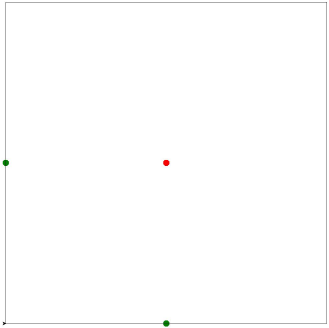

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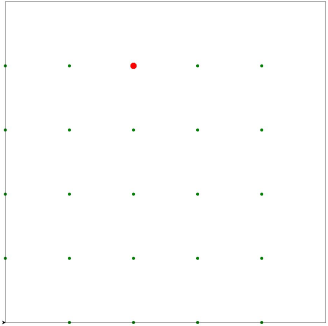

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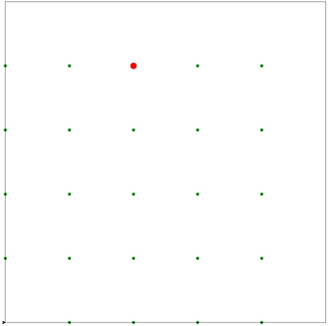

$n = 100000$

This will be the prototypical example of Part I.

Example 2

Say $M = \mathbb{R}^2$, and for suitable $\Gamma = \langle f_1, f_2, f_3, f_4 \rangle \subset \mathrm{Aff}(\mathbb{R}^2)$,

$n = 1$

Example 2

Say $M = \mathbb{R}^2$, and for suitable $\Gamma = \langle f_1, f_2, f_3, f_4 \rangle \subset \mathrm{Aff}(\mathbb{R}^2)$,

$n = 2$

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$n = 3$

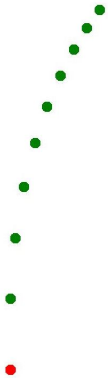
Example 2

Say $M = \mathbb{R}^2$, and for suitable $\Gamma = \langle f_1, f_2, f_3, f_4 \rangle \subset \mathrm{Aff}(\mathbb{R}^2)$,

$n = 5$

Example 2

Say $M = \mathbb{R}^2$, and for suitable $\Gamma = \langle f_1, f_2, f_3, f_4 \rangle \subset \mathrm{Aff}(\mathbb{R}^2)$,



$n = 10$

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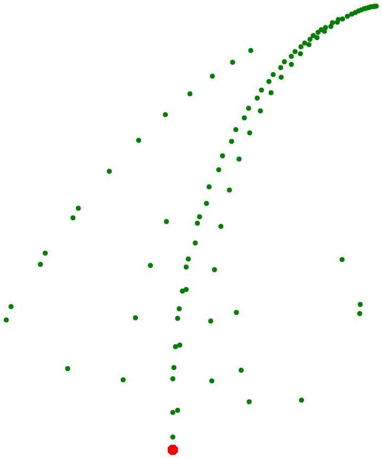

$n = 100$

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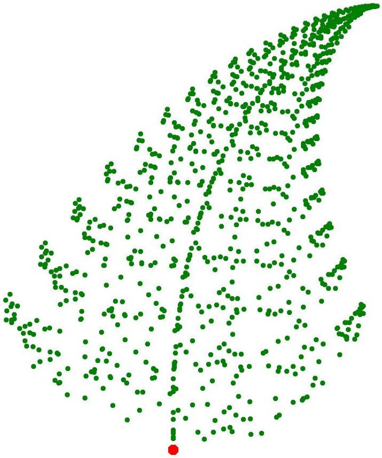

$n = 1000$

Example 2

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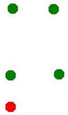

$n = 5$

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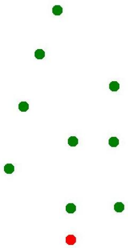

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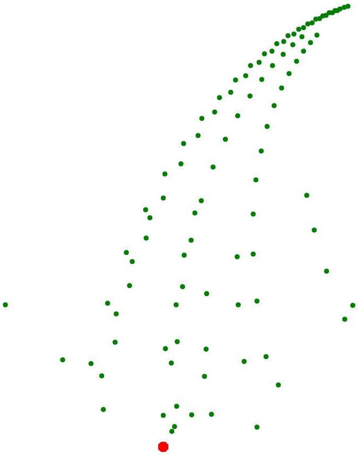

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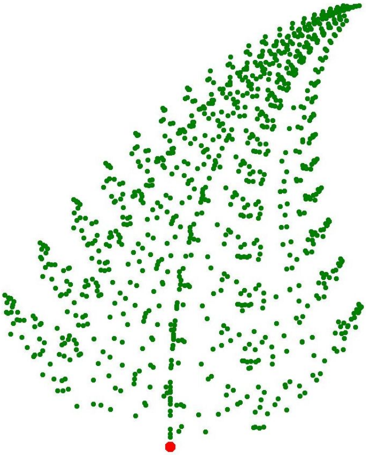

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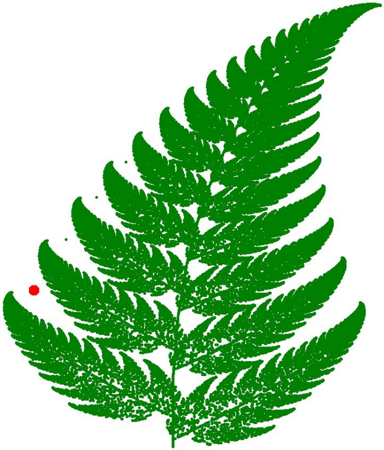

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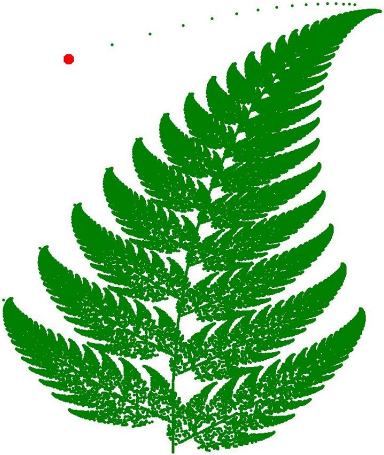

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This will be the prototypical example of Part II.

Example 3

Say $M = SL_2(\mathbb{R})/SL_2(\mathbb{Z})$, $\Gamma = \langle f, g\rangle$ with $f = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, $g = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, which acts on $SL_2(\mathbb{R})/SL_2(\mathbb{Z})$ by left multiplication.

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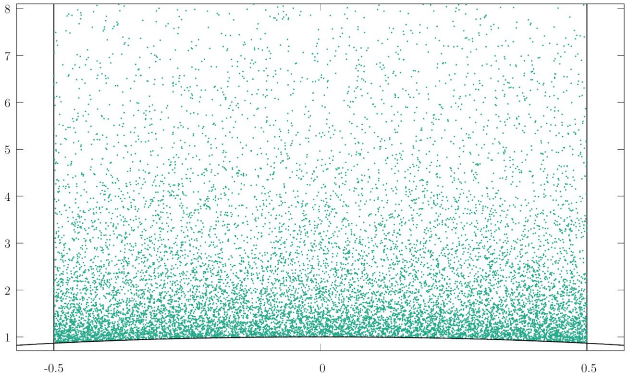

Source: *Dynamics on the space of* 2*-lattices in* 3*-space* by Oliver Sargent and Uri Shapira

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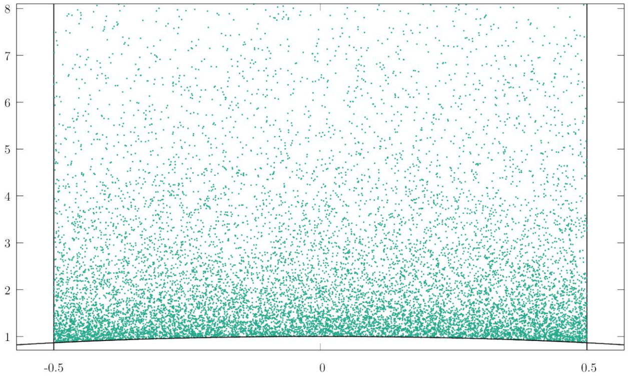

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This will be the prototypical example of Part III.

Stationary measure

It turns out that studying these three questions amounts to understanding the μ-stationary measures on M.

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Let X be a metric space, G be a group acting continuously on X. Let μ be a probability measure on G.

Definition

A measure ν on X is μ-stationary if

$$\nu = \mu * \nu := \int_G g_* \nu \, d\mu(g).$$

i.e. ν is "invariant on average" under the random walk driven by μ.
Note: Every Γ-invariant measure is μ-stationary.

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$$\mu = \frac{1}{2}\left(\delta_A + \delta_B\right), \quad \text{where} \quad A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

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Equidistribution of random orbits

Theorem (Breiman's law of large numbers)

Suppose G acts on a compact metric space X, and μ is a probability measure on G. Then for every $x \in X$ and $\mu^{\mathbb{N}}$-a.e. $(g_1, g_2, \ldots) \in G^{\mathbb{N}}$, any weak-$*$ limit of the sequence of empirical measures

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Main Question: When can we classify the μ-stationary measures?

Part I: Compact surfaces

Free group action on 2-torus

Let $M = \mathbb{T}^2$, and $\Gamma = \langle f, g \rangle$ with

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Nonetheless the situation is more rigid if we use both f and g.

Theorem (Bourgain-Furman-Lindenstrauss-Mozes, 2007)

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Theorem (Bourgain-Furman-Lindenstrauss-Mozes, Benoist-Quint)

Let μ be a compactly supported probability measure on $SL_d(\mathbb{Z})$.
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- The Zariski density assumption is necessary since the theorem is false for say cyclic Γ generated by a hyperbolic element in $SL_d(\mathbb{Z})$.
- The second conclusion implies that under the given assumptions, every μ-stationary measure is Γ-invariant (i.e. stiffness).

Non-homogeneous setting

Let M be a closed manifold with (normalized) volume measure vol,
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Under what condition on μ and/or Γ do we have that

- For all $x \in M$, $\mathrm{Orbit}(x, \Gamma)$ is either finite or dense.
- Every ergodic μ-stationary probability measure ν on M is either finitely supported or vol.
- Every point with infinite orbit equidistributes to vol on M a.s.?

Uniform expansion

Definition

Let M be a Riemannian manifold, μ be a probability measure on $\mathrm{Diff}^2_{\mathrm{vol}}(M)$. We say that μ is uniformly expanding if there exists $C > 0$ and $N \in \mathbb{N}$ such that for all $x \in M$ and nonzero $v \in T_x M$,

$$\int_{\mathrm{Diff}^2_{\mathrm{vol}}(M)} \log \frac{\|D_x f(v)\|}{\|v\|} d\mu^{(N)}(f) > C > 0.$$

Here $\mu^{(N)} := \mu * \mu * \cdots * \mu$ is the N-th convolution power of μ.

In other words, the random walk w.r.t. μ expands every vector $v \in T_x M$ at every point $x \in M$ on average (might be contracted by a specific word though!)

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Remark: Uniform expansion is an open condition, expected to be generic.

C^2-action on compact Riemannian surfaces

Theorem (C.)

Let M be a closed 2-manifold with volume measure vol. *Let μ be a compactly supported probability measure on $\mathrm{Diff}^2_{\mathrm{vol}}(M)$ that is uniformly expanding, and $\Gamma := \langle \mathrm{supp}\ \mu \rangle$. Then*

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- *For $M = \mathbb{T}^2$ and μ supported on $SL_2(\mathbb{Z})$, if $\Gamma = \langle \mathrm{supp}\ \mu \rangle$ is Zariski dense in $SL_2(\mathbb{R})$, then μ is uniformly expanding.*

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Remark

- *For $M = \mathbb{T}^2$ and μ supported on $SL_2(\mathbb{Z})$, if $\Gamma = \langle \mathrm{supp}\ \mu \rangle$ is Zariski dense in $SL_2(\mathbb{R})$, then μ (and perturbations) are uniformly expanding.*
- *The proof applies the recent deep work of Brown and Rodriguez Hertz, and some ideas from Dolgopyat-Krikorian and Eskin-Margulis.*

Verify uniform expansion

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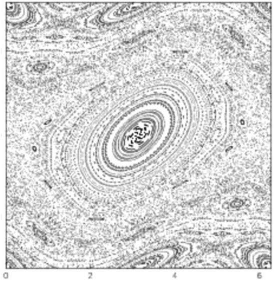
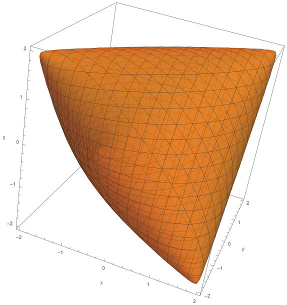
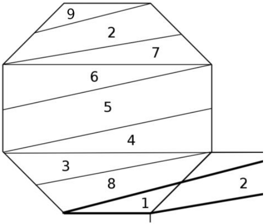
1. Discrete perturbation of the standard map (verified by hand)
2. $\mathrm{Out}(F_2)$-action on the character variety $\mathrm{Hom}(F_2, \mathrm{SU}(2)) /\!\!/ \mathrm{SU}(2)$ (verified numerically).
3. Related: Veech group action on individual Veech surfaces.







Source: (i) Wolfram Mathworld (iii) *Cutting sequences, regular polygons, and the Veech group* by Diana Davis

Part II: Linear action on vector spaces

Linear action on vector spaces

Given a probability measure μ on $\mathrm{GL}(V)$, we can attempt to classify μ-stationary measures on V. There are a few features which distinguish this from the compact manifold case:

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2. V is highly noncompact, so there might not be any other stationary measures, or the stationary measures may be supported on a low dimensional subspace.
3. there might be fractal stationary measures, but unlike the single element case, their supports have only finitely many possible Hausdorff dimensions.

Linear action on vector spaces

Example: $V = \mathbb{R}^2, \Gamma = \langle \mathrm{supp}\ \mu \rangle = \langle A, B \rangle \subset \mathrm{GL}(V)$,

$$\mu = \frac{1}{2}\left(\delta_A + \delta_B\right), \quad \text{where} \quad A = \begin{pmatrix} 1/3 & 1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1/3 & -1 \\ 0 & 1 \end{pmatrix}.$$

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Explicitly, the action is given by

$$A : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x/3 + y \\ y \end{pmatrix}, \qquad B : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x/3 - y \\ y \end{pmatrix}.$$

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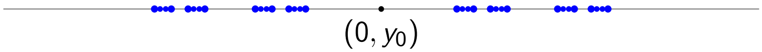
Then each horizontal line $y = y_0$ is an invariant set, and on each line there is a μ-stationary probability measure ν_{y_0}:



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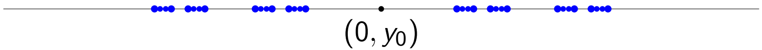

It turns out that this is the typical situation.

Linear action on vector spaces

Theorem (C.)

Let $V \neq 0$ be a real vector space, μ be a finitely supported probability measure on $\mathrm{GL}(V)$ and $\Gamma = \langle \mathrm{supp}\ \mu \rangle \subset \mathrm{GL}(V)$. Then there exist Γ-invariant vector subspaces $W' \subset W \subset V$ such that

1. *every μ-stationary probability measure on V is supported on W,*
2. *either $W' = 0$ or the top Lyapunov exponent of μ on W' is negative,*
3. *Γ acts compactly on W/W'.*

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Example: For the μ-action on $\mathbb{R}^3$ with

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Remark: The proof mainly follows from some linear algebra, a result of Bougerol, and understanding of the Furstenberg measures ν_b.

Linear action on vector spaces

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Let $W' \subsetneq W \subset V$ from the previous theorem. Then the map $\nu \mapsto \mathrm{supp}\, \pi_* \nu$ gives a one-to-one correspondence between

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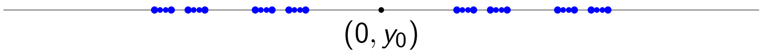
The statement implies that on each horizontal line $y = y_0$, there is exactly one ergodic μ-stationary probability measure ν_{y_0}:



And each point (x, y_0) equidistribute to ν_{y_0}.

Part III: Action on homogeneous spaces

Simple action on G/Λ

The theorem of Benoist-Quint also works for homogeneous spaces G/Λ. The statement is similar to the case of $\mathbb{T}^d$ when G is simple.

Theorem (Benoist-Quint, 2011)

Let G be a connected simple real Lie group, Λ be a lattice in G, μ be a compactly supported probability measure on G. If $\Gamma = \langle \mathrm{supp}\ \mu \rangle$ is a Zariski dense subsemigroup of G, then

- *For all $x \in G/\Lambda$, $\mathrm{Orbit}(x, \Gamma)$ is either finite or dense.*
- *Every ergodic μ-stationary probability measure ν on G/Λ is either finitely supported or the Haar measure.*
- *Every $x \in G/\Lambda$ with infinite orbit equidistributes to Haar on G/Λ a.s.*

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Example: $X = SL_2(\mathbb{R})/SL_2(\mathbb{Z}), G = SL_2(\mathbb{R}), \Gamma = \langle \mathrm{supp}\ \mu \rangle \subset G$,

$$\mu = \frac{1}{2}(\delta_A + \delta_B), \quad \text{where} \quad A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

Semisimple action on G/Λ

In the setting when the Zariski closure of Γ is semisimple, an analogous statement holds, but with more possibilities (though still "nice").

Theorem (Benoist-Quint, 2013)

Let G be a real Lie group, Λ be a lattice in G, μ be a compactly supported probability measure on G. If the Zariski closure of $\Gamma = \langle \mathrm{supp}\ \mu \rangle$ is semisimple, Zariski connected and has no compact factors, then

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Here homogeneity of closed sets and probability measures are in the sense of Ratner.

The semisimple assumption is further relaxed by Eskin-Lindenstrauss. Here's the measure rigidity statement:

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1. *There exists a closed unimodular subgroup $H \subset G$ with $\dim(H) > 0$, an H-homogeneous probability measure ν_0 on G/Γ, and a μ-stationary probability measure λ on G/H such that*

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In particular if λ is a delta mass (e.g. if $H = G$), ν is homogeneous.

The conclusion of Eskin-Lindenstrauss prompts one to investigate:

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  $$L := \{g \in N_G(H^\circ) \mid h \mapsto ghg^{-1} \text{ preserves Haar on } H^\circ\}.$$

  Then for any ergodic μ-stationary measure ν on G/H, there exists $g \in G$ such that $\Gamma \subset gLg^{-1}$ and ν is supported on gL/H.
  (Note that if H is discrete, $L = G$.)

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Now we consider G/H as the total space of the fiber bundle

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- apply the technique of Eskin-Lindenstrauss for G/Λ to obtain extra invariance in the L/H fiber direction,
- apply the knowledge about stationary measures on V for the projection of ν to the base G/L (which is also stationary),
- The main assumption is a "fiberwise" version of uniform expansion.

We thus obtain a statement analogous to Eskin-Lindenstrauss for G/H.

Theorem (C.)

Let G be a real algebraic Lie group and $H \subset G$ be a unimodular subgroup of G and let $L := N_G^1(H^\circ)$. Suppose that μ is a probability measure on G with finite first moment, and μ is "uniformly expanding on L/H".
Let ν be an ergodic μ-stationary probability measure on G/H.
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2. *The measure ν can be decomposed into $\nu = \int_{G/L} \nu_x \, d\nu_{G/L}(x)$ where ν_x is finitely supported for $\nu_{G/L}$-a.e. x, and $\nu_{G/L}$ is a "generalized Bernoulli convolution".*

Example

To illustrate a nontrivial case of the statement, one can take $G = SL_4(\mathbb{R})$,

$$H = \begin{bmatrix} SL_2(\mathbb{R}) & * \\ 0 & SL_2(\mathbb{Z}) \end{bmatrix}, \quad H^\circ = \begin{bmatrix} SL_2(\mathbb{R}) & * \\ 0 & I \end{bmatrix}, \quad L = \begin{bmatrix} SL_2(\mathbb{R}) & * \\ 0 & SL_2(\mathbb{R}) \end{bmatrix},$$

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$$A = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 4 & 0 & 1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ -4 & 0 & 1 & 2 \end{bmatrix}.$$

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Then using the theorem, one can show that any ergodic μ-stationary measure ν on G/H is of the form

$$\nu = \int_{G/L} \nu_x d\nu_{G/L}(x)$$

where $\nu_{G/L}$ is a μ-stationary measure on G/L, and for $\nu_{G/L}$-a.e. $x \in G/L$, either ν_x is the Haar measure or supported on k points on xL/H.

Thank you!