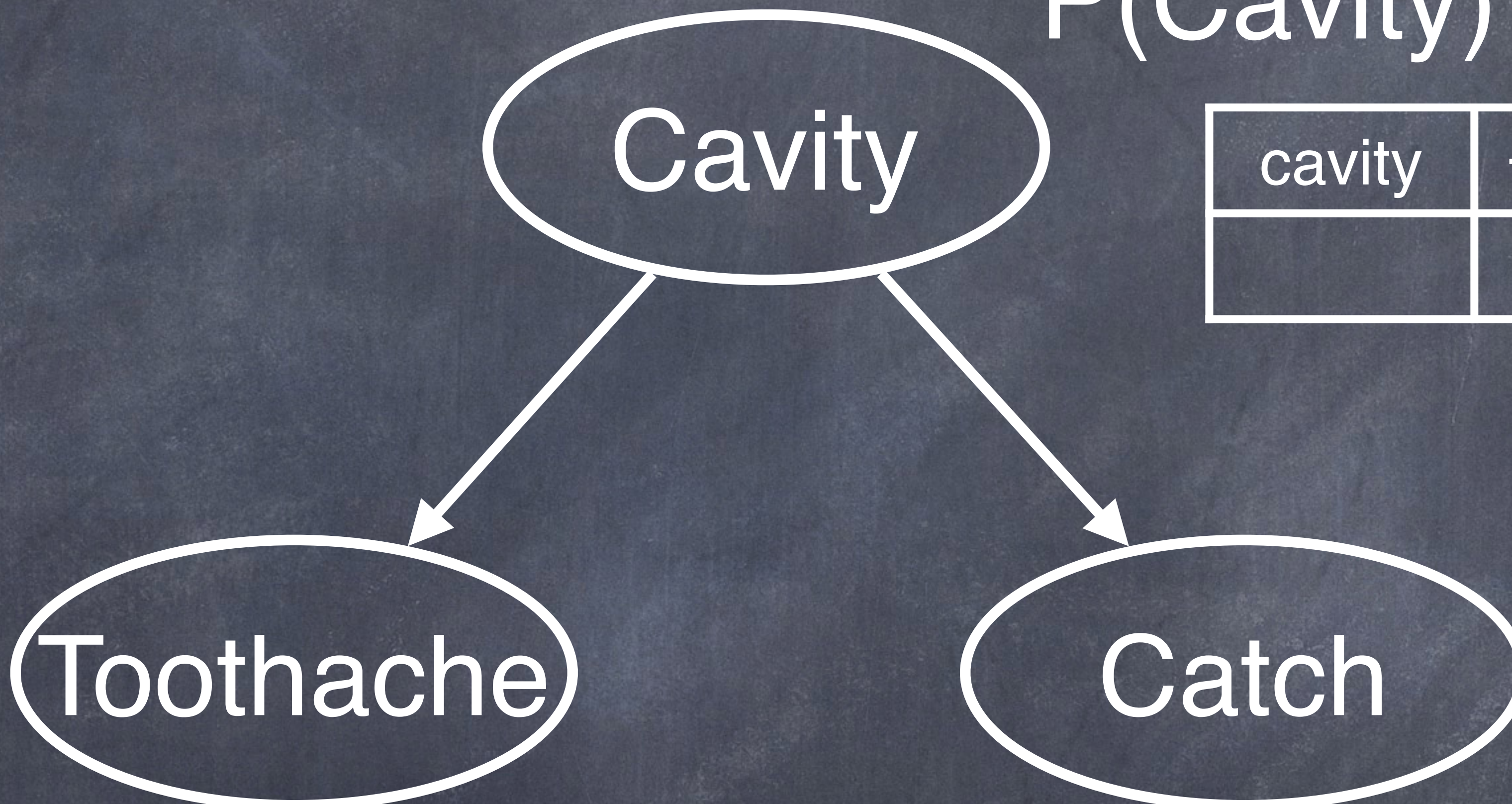
CSC442: Introduction to Artificial Intelligence

Lecture 3.4

Bayesian Networks

P(Cavity)

cavity	¬cavity



P(Toothache|Cavity)

Cavity	toothache	¬toothache
cavity		
¬cavity		

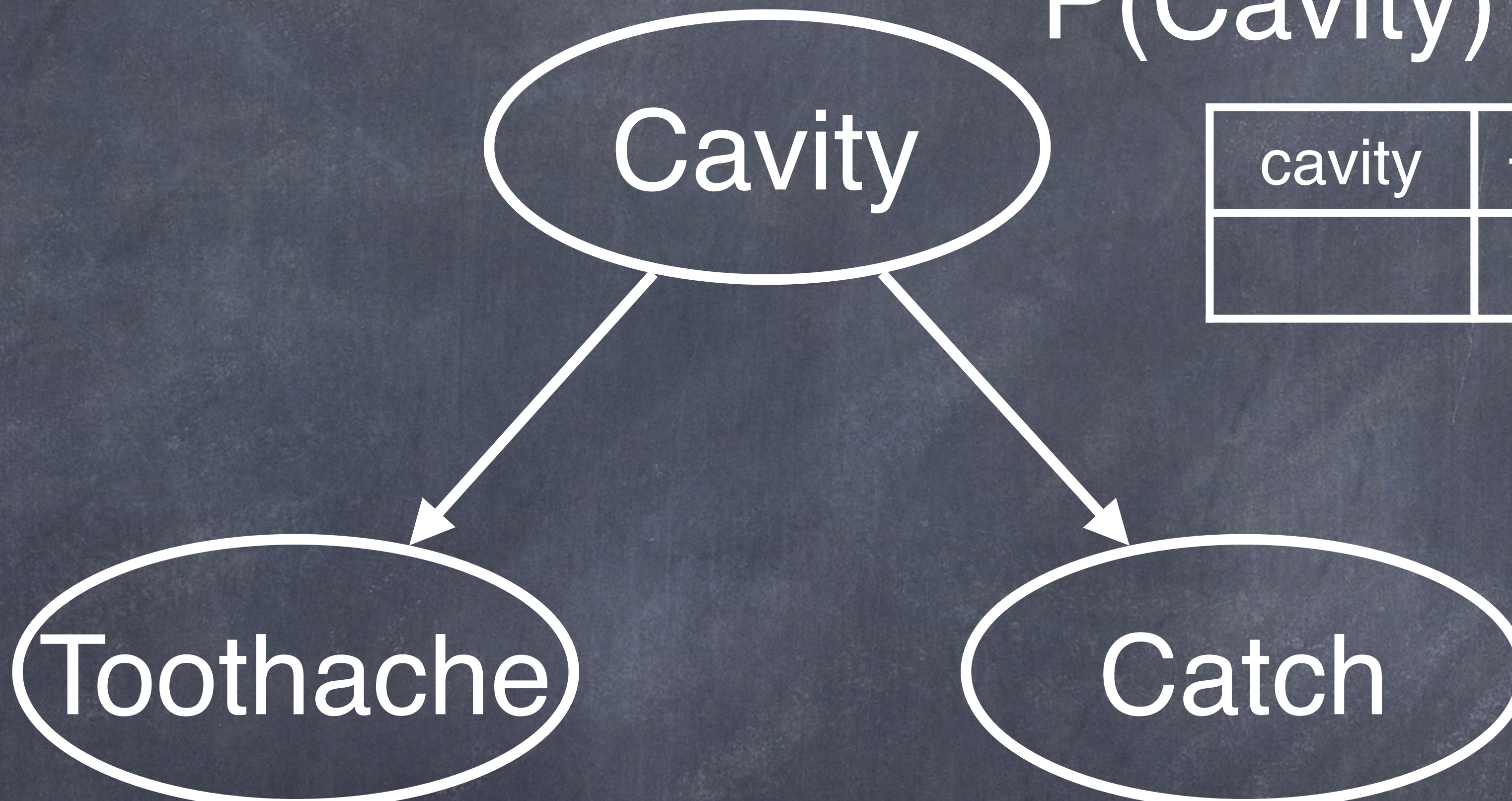
P(Catch|Cavity)

Cavity	catch	¬catch
cavity		
¬cavity		

$$P(toothache, cavity, catch) = P(toothache \mid cavity)P(catch \mid cavity)P(cavity)$$

P(Cavity)

cavity	¬cavity



P(Toothache|Cavity)

Cavity	toothache	¬toothache
cavity		
¬cavity		

P(Catch|Cavity)

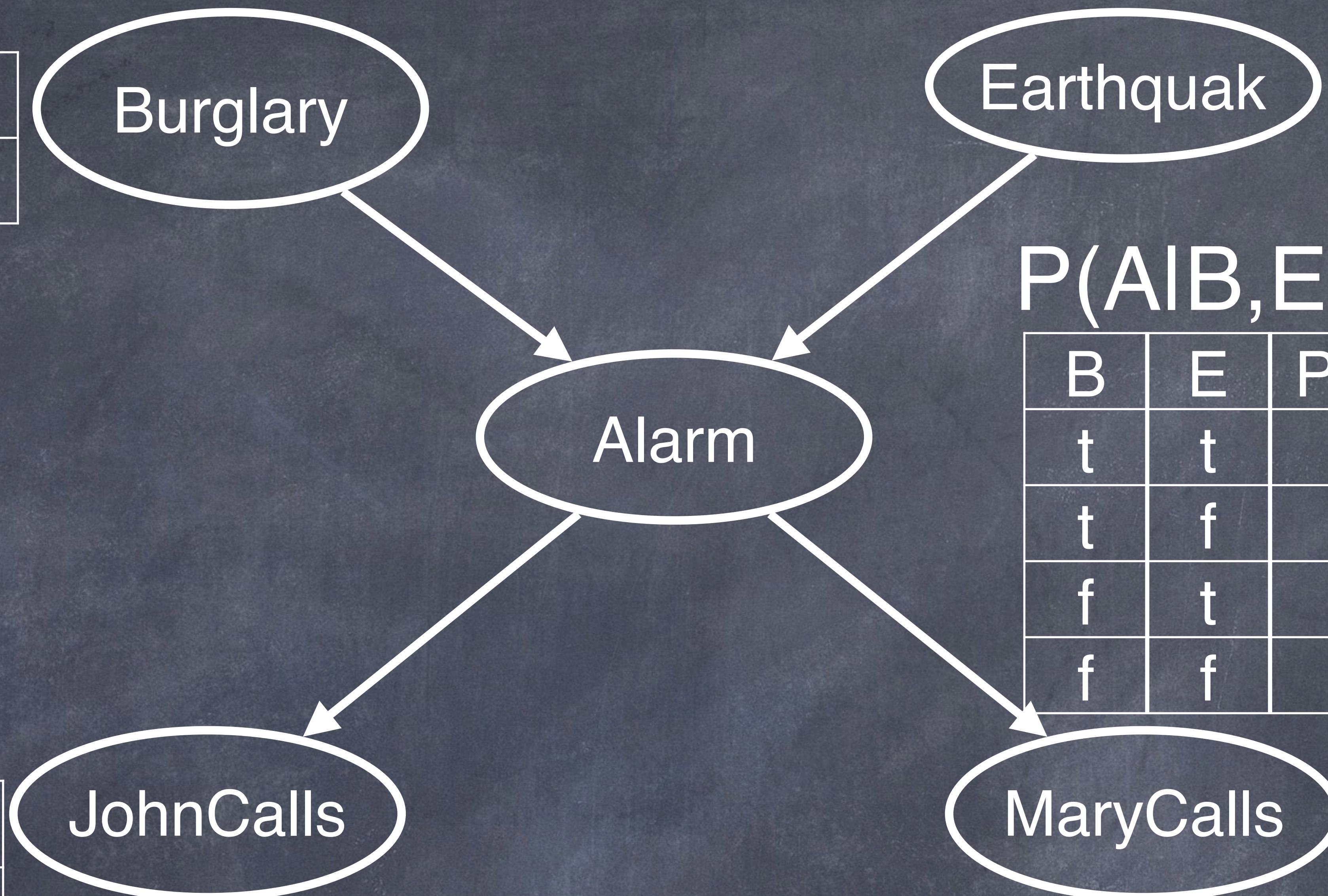
Cavity	catch	¬catch
cavity		
¬cavity		

P(B)

P(b)
0.001

P(E)

P(e)
0.002



$P(A \mid B,E)$

B	E	$P(a \mid B,E)$
t	t	0.95
t	f	0.94
f	t	0.29
f	f	0.001

$P(J \mid A)$

A	$P(j \mid A)$
t	0.9
f	0.05

$P(M \mid A)$

A	$P(m \mid A)$
t	0.7
f	0.01

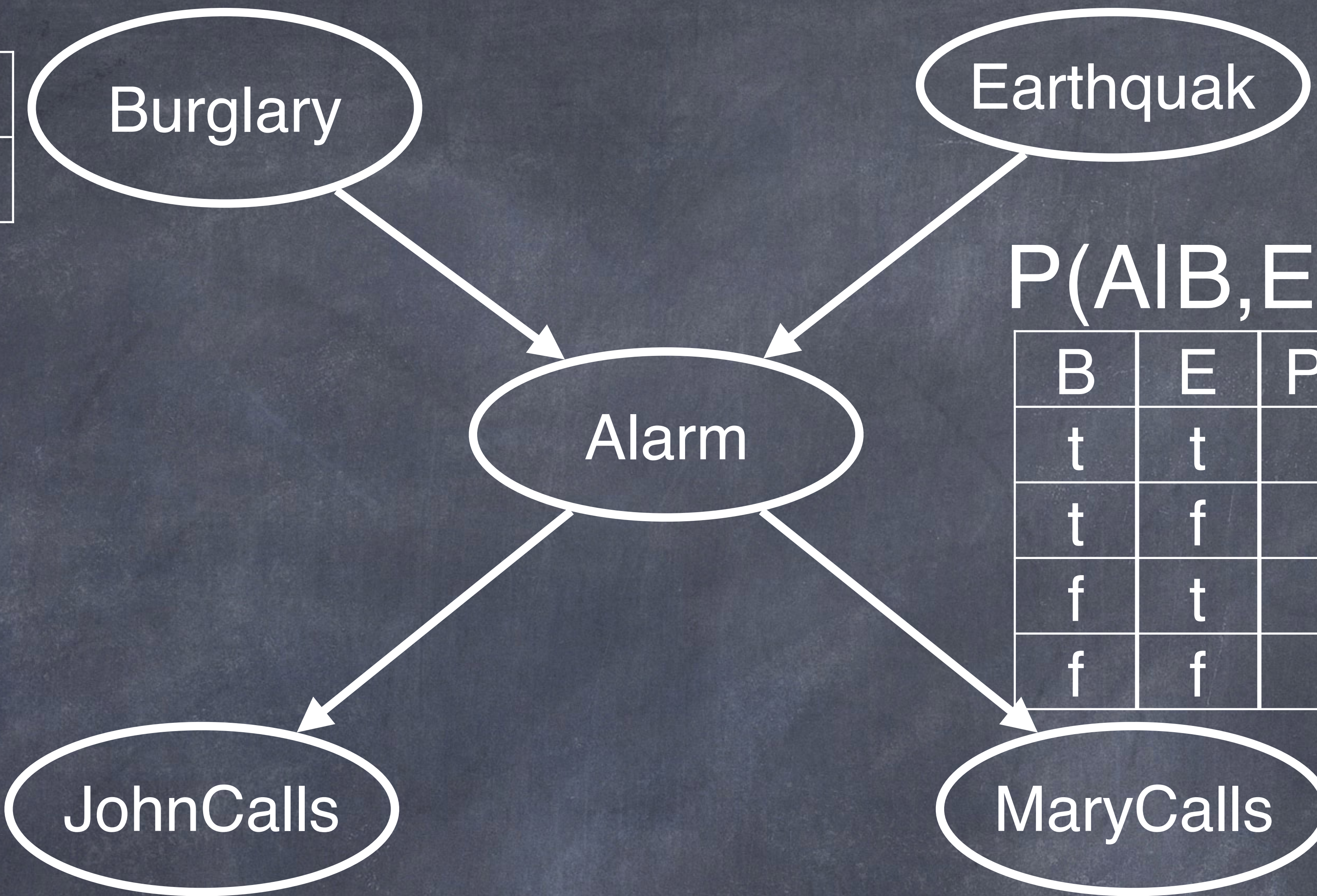
$$\mathbf{P}(B, E, A, J, M) = \alpha\, \mathbf{P}(B)\, \mathbf{P}(E)\, \mathbf{P}(A \mid B, E)\, \mathbf{P}(J \mid A)\, \mathbf{P}(M \mid A)$$

P(B)

P(b)
0.001

P(E)

P(e)
0.002



P(A|B,E)

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t	t	0.95
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f	t	0.29
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P(J|A)

A	P(j|A)
t	0.9
f	0.05

P(M|A)

A	P(m|A)
t	0.7
f	0.01

Full joint distribution: 2^5=32 entries

Bayesian network: 10 entries

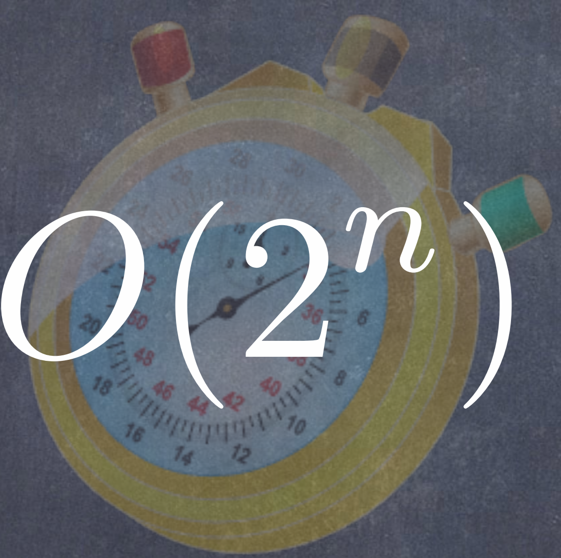
Assuming conditional independences encoded in the network

Inference in Bayesian Networks

$$\mathbf{P}(X \mid \mathbf{e}) = \alpha\, \mathbf{P}(X, \mathbf{e}) = \alpha \sum_{\mathbf{y}} \mathbf{P}(X, \mathbf{e}, \mathbf{y})$$
$$= \alpha \sum_{\mathbf{y}} \prod_{i=1}^{n} P(X_i \mid parents(X_i))$$

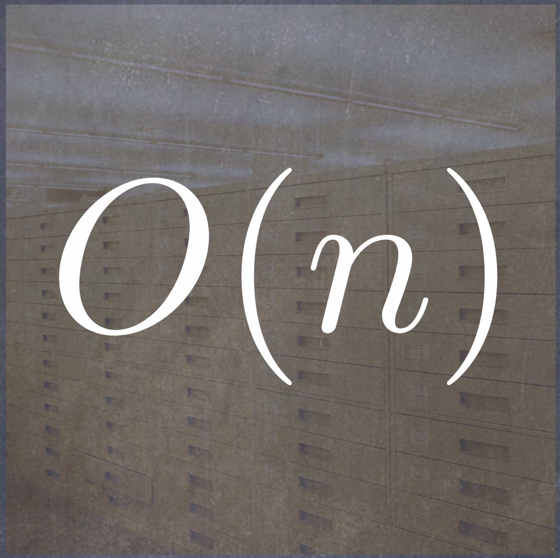
- "A query can be answered from a Bayesian Network by computing sums of products of conditional probabilities from the network."

Exact Inference in BNs



$$O(2^n)$$

Time Complexity



$$O(n)$$

Space Complexity

Approximate Inference in Bayesian Networks

Exact Inference

- Query variable X
- Evidence variables $E_1, \ldots, E_m$
  - Observed values: $e = \langle e_1, \ldots, e_m \rangle$
- Non-evidence, non-query ("hidden") variables: Y
- Compute: $P(X \mid e)$

Approximate Inference

- Query variable X
- Evidence variables $E_1, \ldots, E_m$
  - Observed values: $e = \langle e_1, \ldots, e_m \rangle$
- Non-evidence, non-query ("hidden") variables: Y
- Approximate (estimate): $P(X \mid e)$

Unconditional Approximation

- Query variable X
- Evidence variables $E_1, \ldots, E_m$
  - Observed values: $e = \langle e_1, \ldots, e_m \rangle$
- Non-evidence, non-query ("hidden") variables: Y
- Approximate (estimate): P(X)

Heads

Heads

Goal: P(Heads)

P(Heads=true)

P(heads)

Heads

# of flips: N

# of heads: N_{heads}

Heads

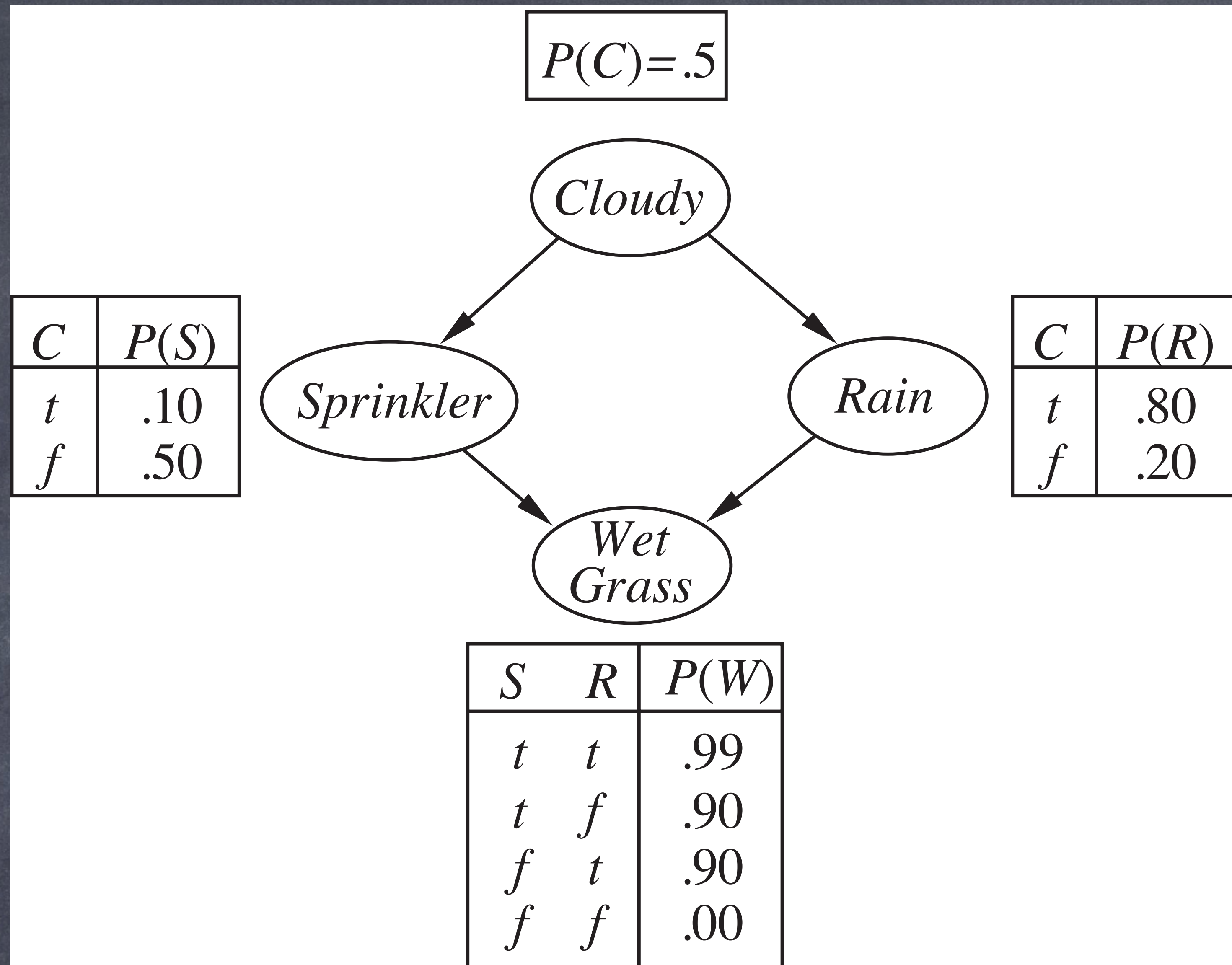
$$P(heads) \approx \frac{N_{heads}}{N}$$

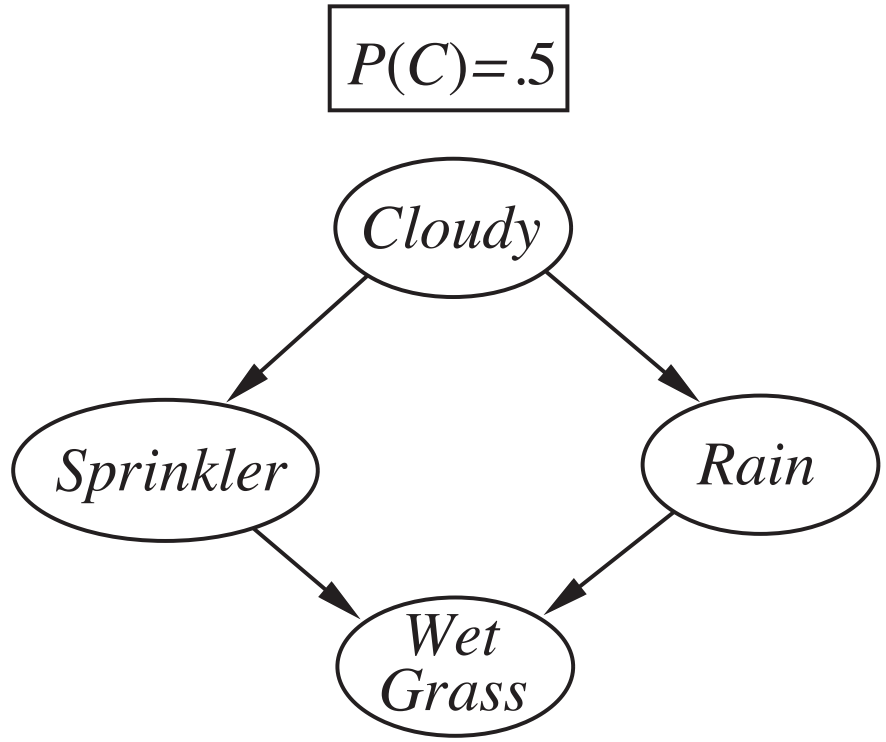
Heads

$$P(heads) = \lim_{N \to \infty} \frac{N_{heads}}{N}$$

Sampling

- Generating events (possible worlds) from a distribution
- Estimating probabilities as ratio of observed events to total events
  - Consistent estimate: becomes exact in the large-sample limit


P(C)=.5
Cloudy
Sprinkler
Rain
Wet Grass
C P(S)
t .10
f .50
C P(R)
t .80
f .20
S R P(W)
t t .99
t f .90
f t .90
f f .00



C	P(S)
t	.10
f	.50

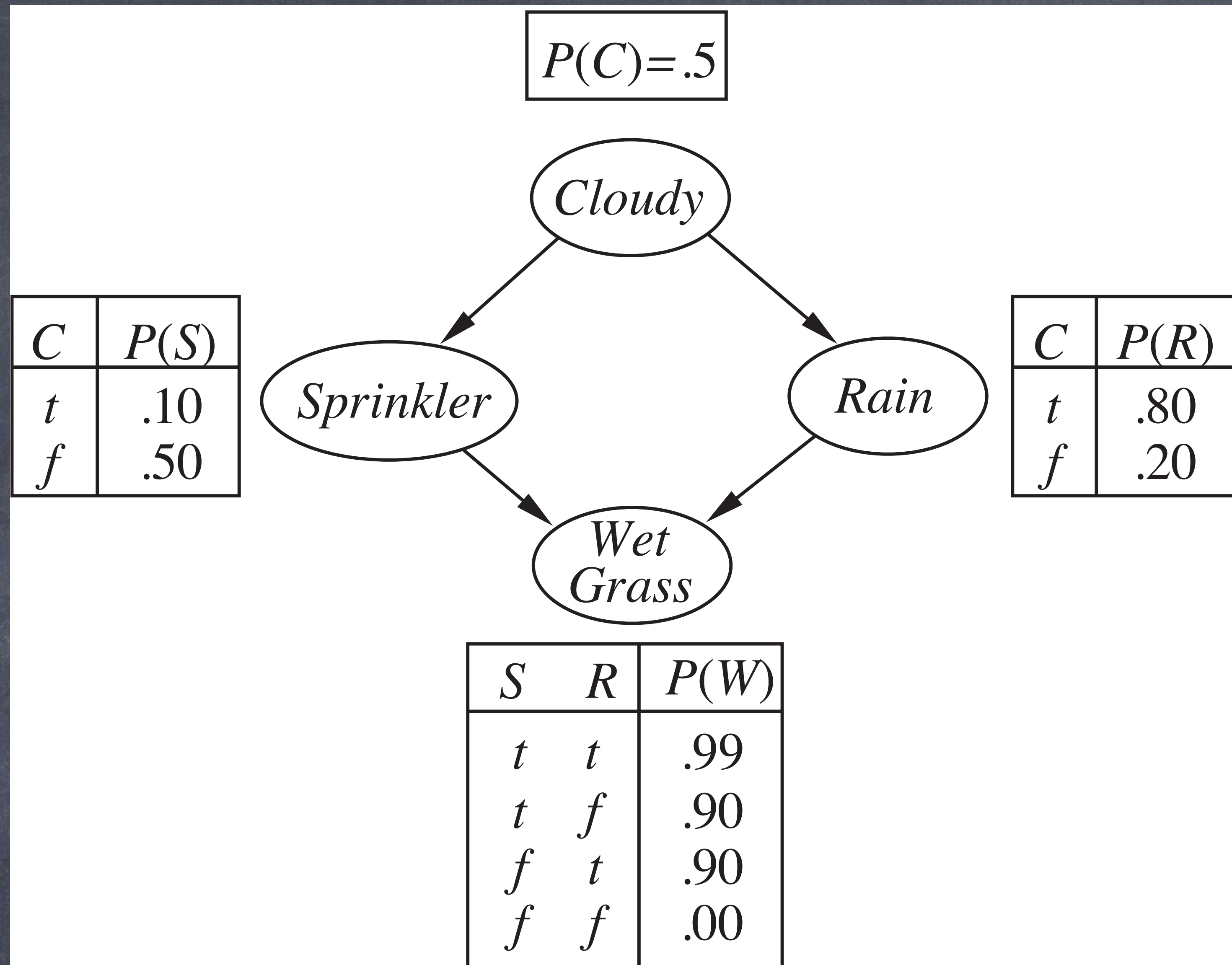
C	P(R)
t	.80
f	.20

S	R	P(W)
t	t	.99
t	f	.90
f	t	.90
f	f	.00

$$P(Rain = true)$$

Sampling

- Generate assignments of values to the random variables
- That are consistent with the full joint distribution encoded in the network
    - In the sense that in the limit, the frequency of occurrence of any event (possible world) is equal to its probability



⟨Cloudy=true, Sprinkler=false, Rain=true, WetGrass=true⟩

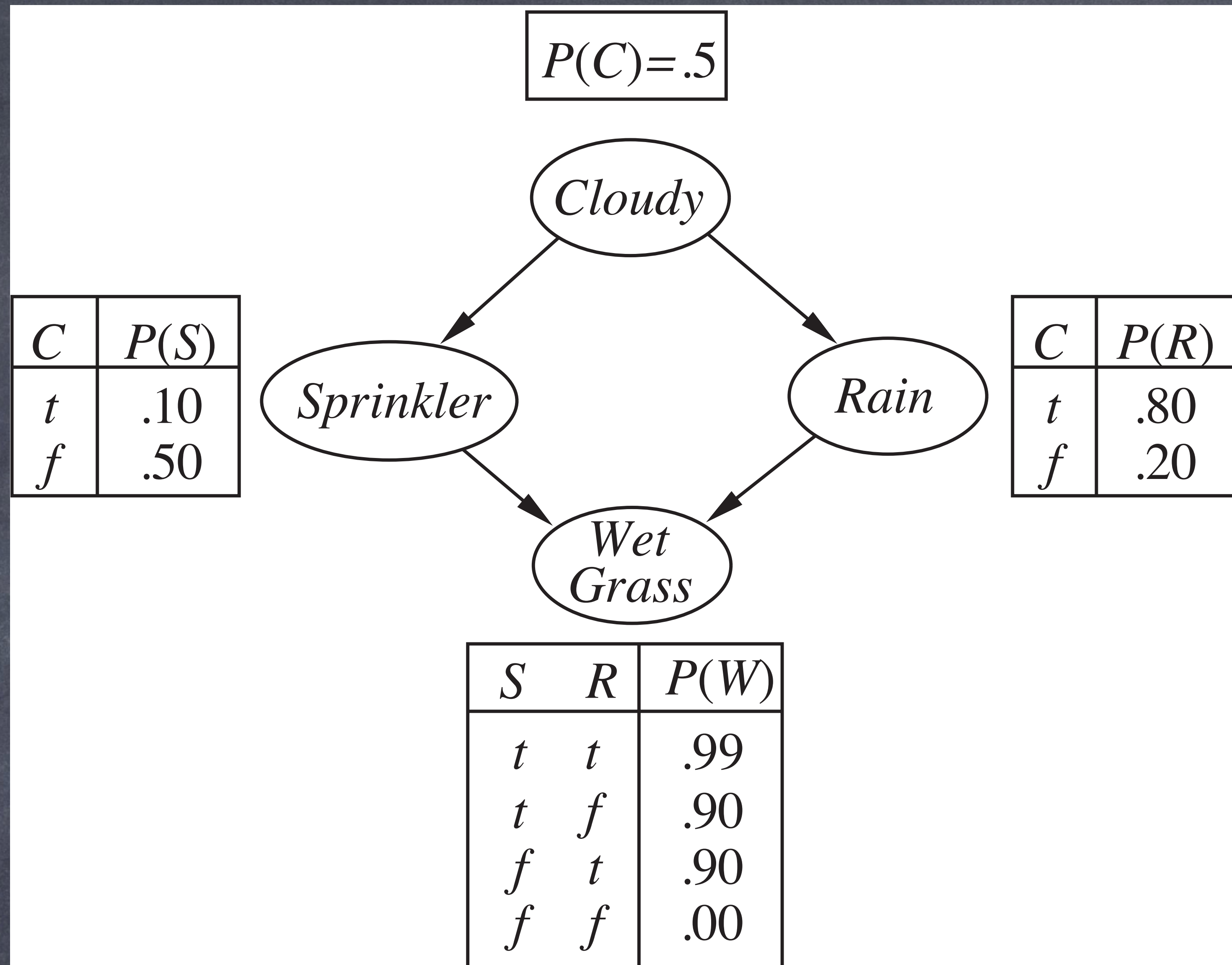
likelihood weighting :

$P(R \mid +C, +W)$

① C → evidence +C

② S → non-evidence −S

W = 1.0



⟨Cloudy=true, Sprinkler=false, Rain=true, WetGrass=true⟩

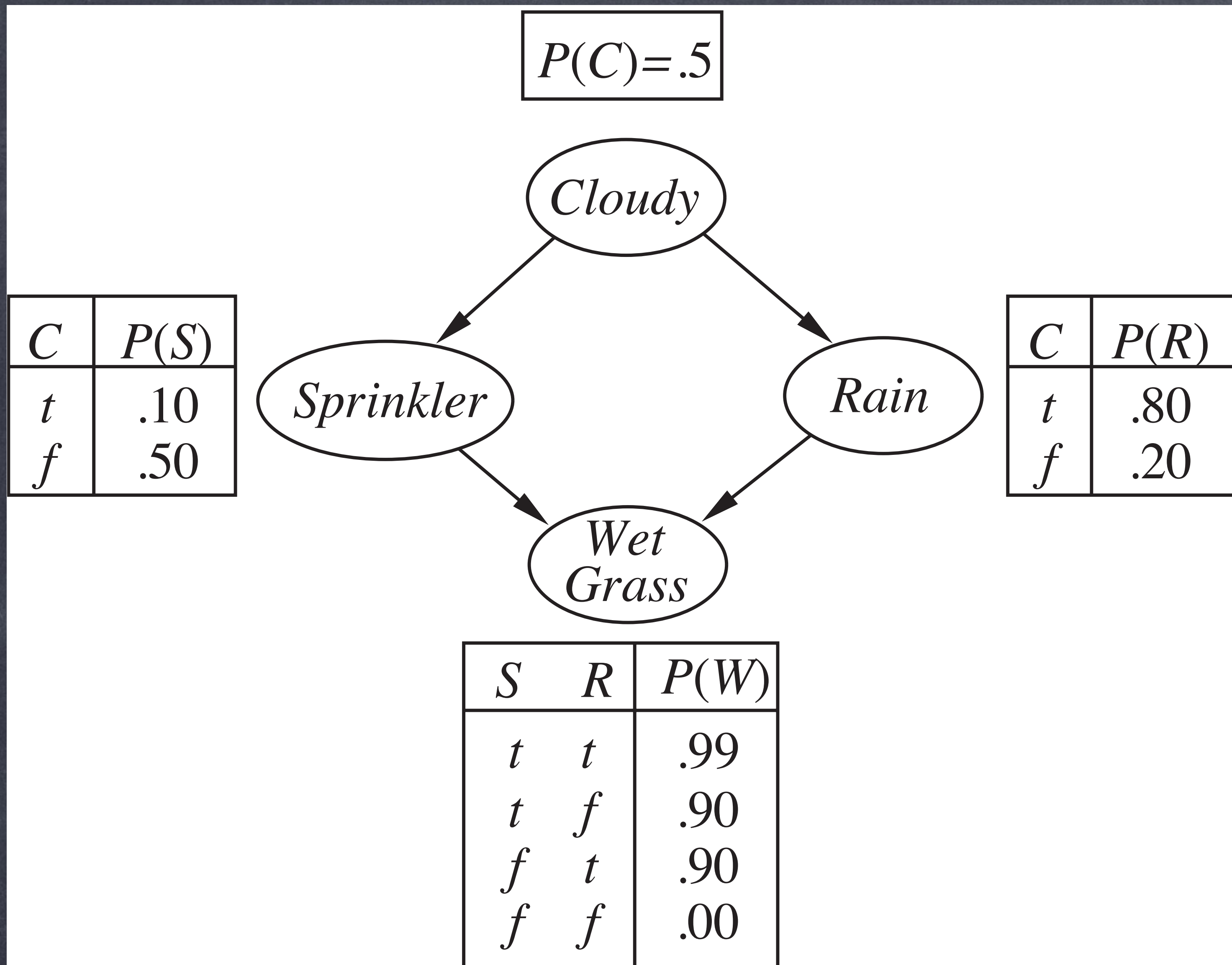
⟨Cloudy=false, Sprinkler=false, Rain=false, WetGrass=false⟩

⟨Cloudy=true, Sprinkler=true, Rain=true, WetGrass=true⟩

...

Generating Samples

- Sample each variable in topological order
  - Child appears after its parents
- Choose the value for that variable conditioned on the values already chosen for its parents



C	P(S)
t	.10
f	.50

C	P(R)
t	.80
f	.20

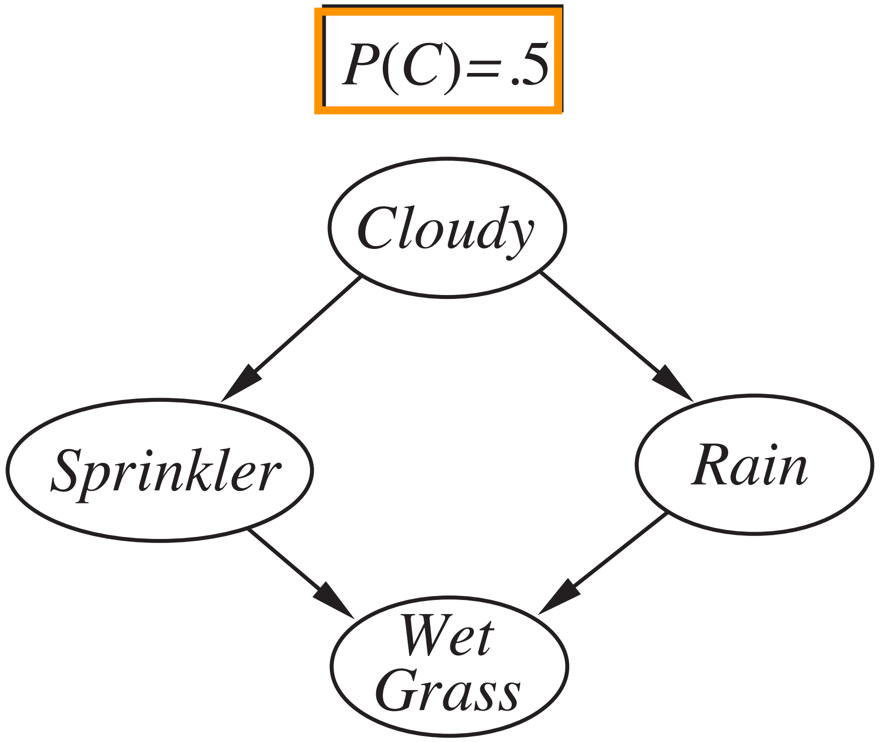
S	R	P(W)
t	t	.99
t	f	.90
f	t	.90
f	f	.00

Cloudy

Sprinkler

Rain

WetGrass



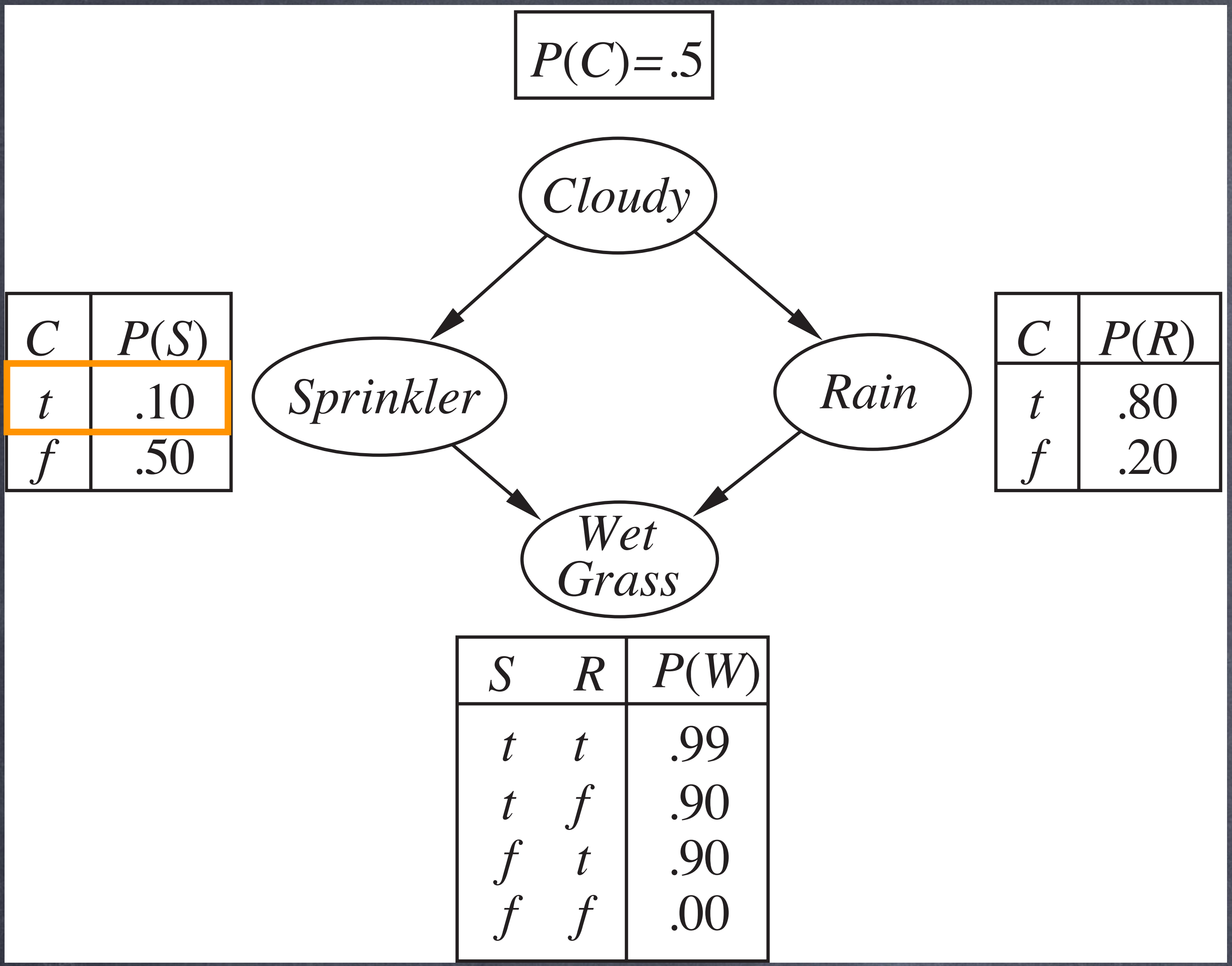
C	P(S)
t	.10
f	.50

C	P(R)
t	.80
f	.20

S	R	P(W)
t	t	.99
t	f	.90
f	t	.90
f	f	.00

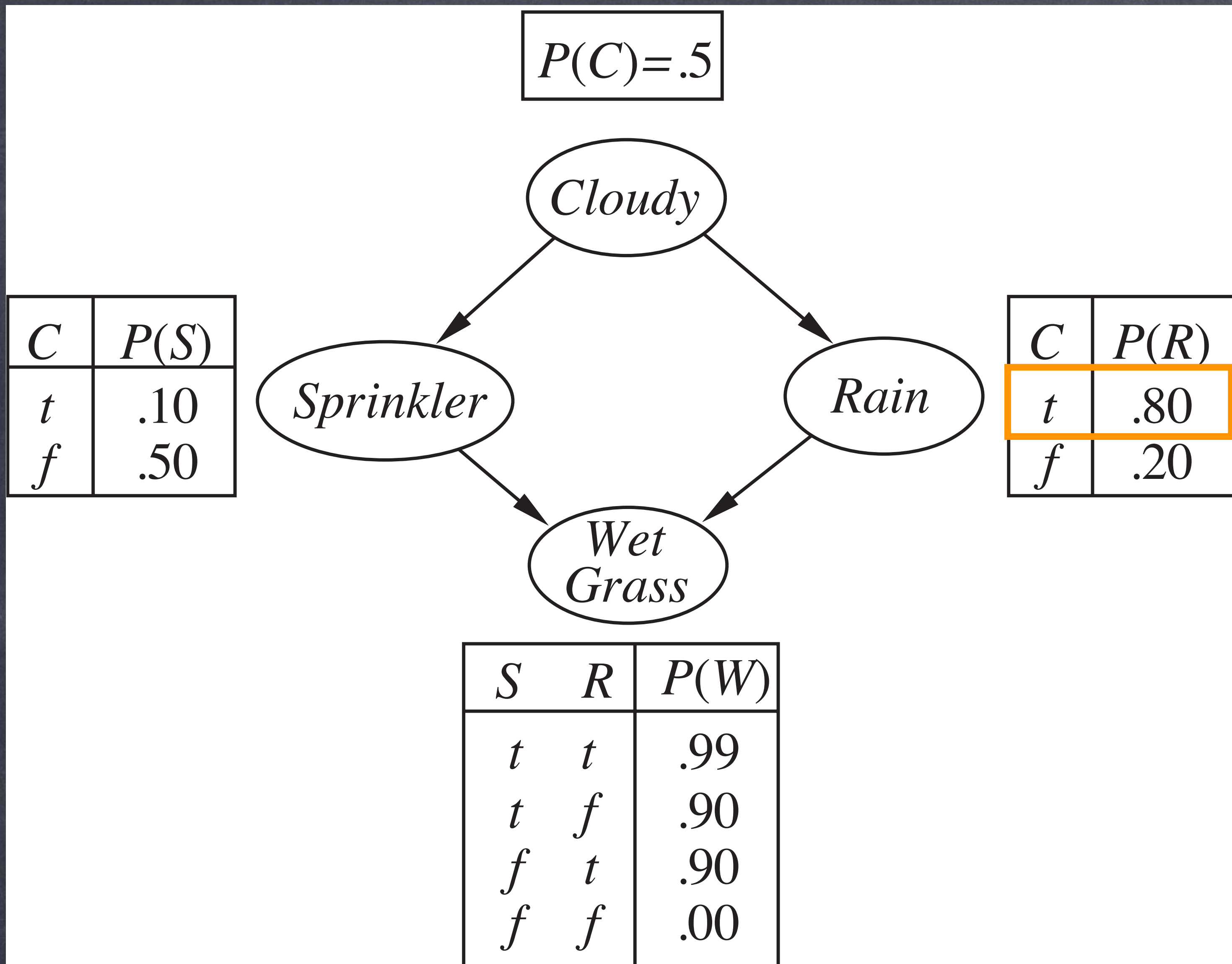
Cloudy	true
Sprinkler	
Rain	
WetGrass	

$$\mathbf{P}(Cloudy) = \langle 0.5, 0.5 \rangle$$



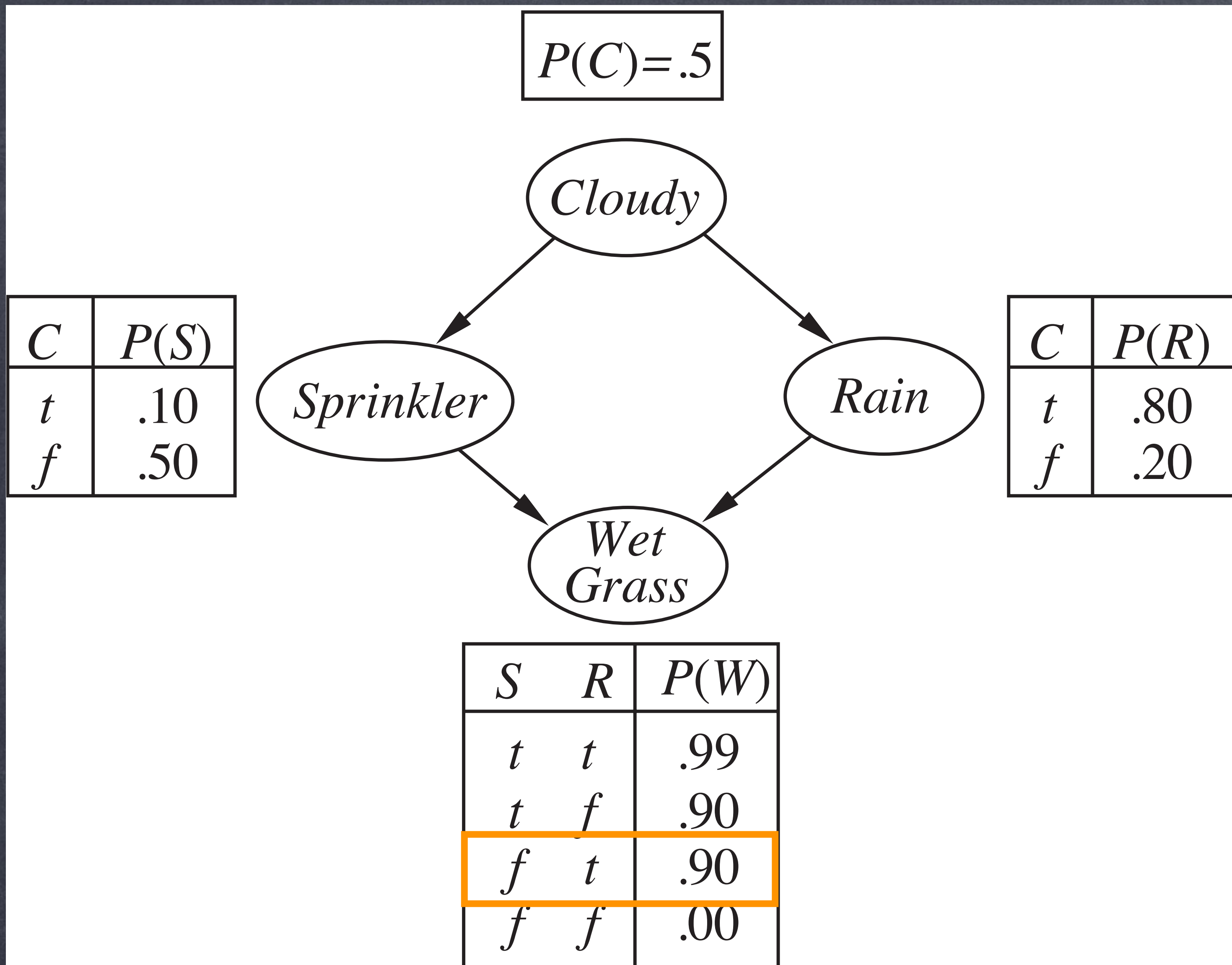
Cloudy	true
Sprinkler	false
Rain	
WetGrass	

$$\mathbf{P}(Sprinkler \mid Cloudy = true) = \langle 0.1, 0.9 \rangle$$



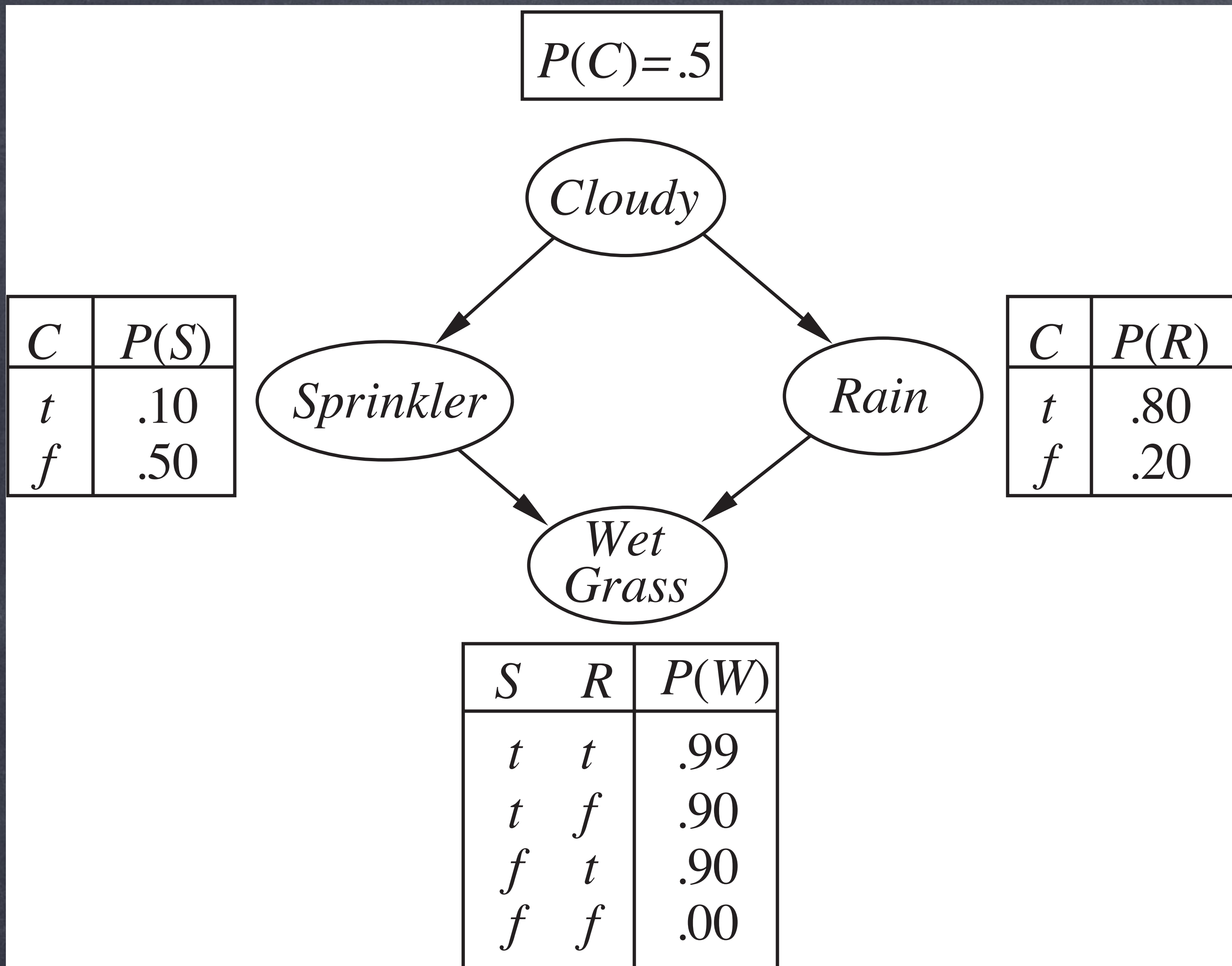
Cloudy	true
Sprinkler	false
Rain	true
WetGrass	

$$\mathbf{P}(Rain \mid Cloudy = true) = \langle 0.8, 0.2 \rangle$$



Cloudy	true
Sprinkler	false
Rain	true
WetGrass	true

$$\mathbf{P}(\,WetGrass \mid Sprinkler = false, Rain = true) = \langle 0.9, 0.1 \rangle$$



Cloudy	true
Sprinkler	false
Rain	true
WetGrass	true

⟨Cloudy=true, Sprinkler=false, Rain=true, WetGrass=true⟩

Guaranteed to be a consistent estimate (becomes exact in the large-sample limit)

Sampling

- Generate assignments of values to the random variables
- That are consistent with the full joint distribution encoded in the network
  - In the sense that in the limit, the frequency of occurrence of any event (possible world) is equal to its probability

Approximate Inference

- Query variable X
- Evidence variables $E_1, \ldots, E_m$
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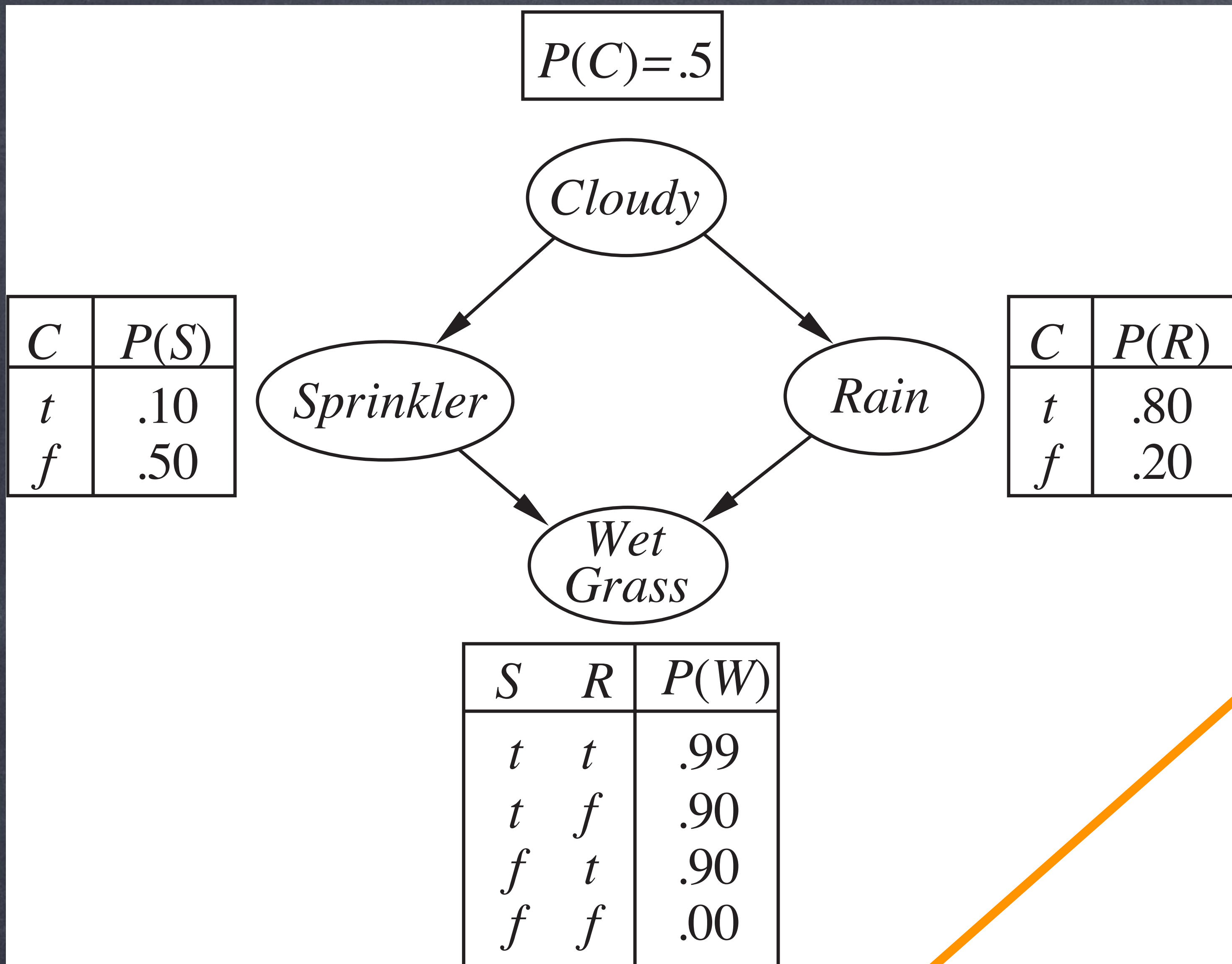
- Non-evidence, non-query ("hidden") variables: Y
- Approximate (estimate): $P(X \mid e)$

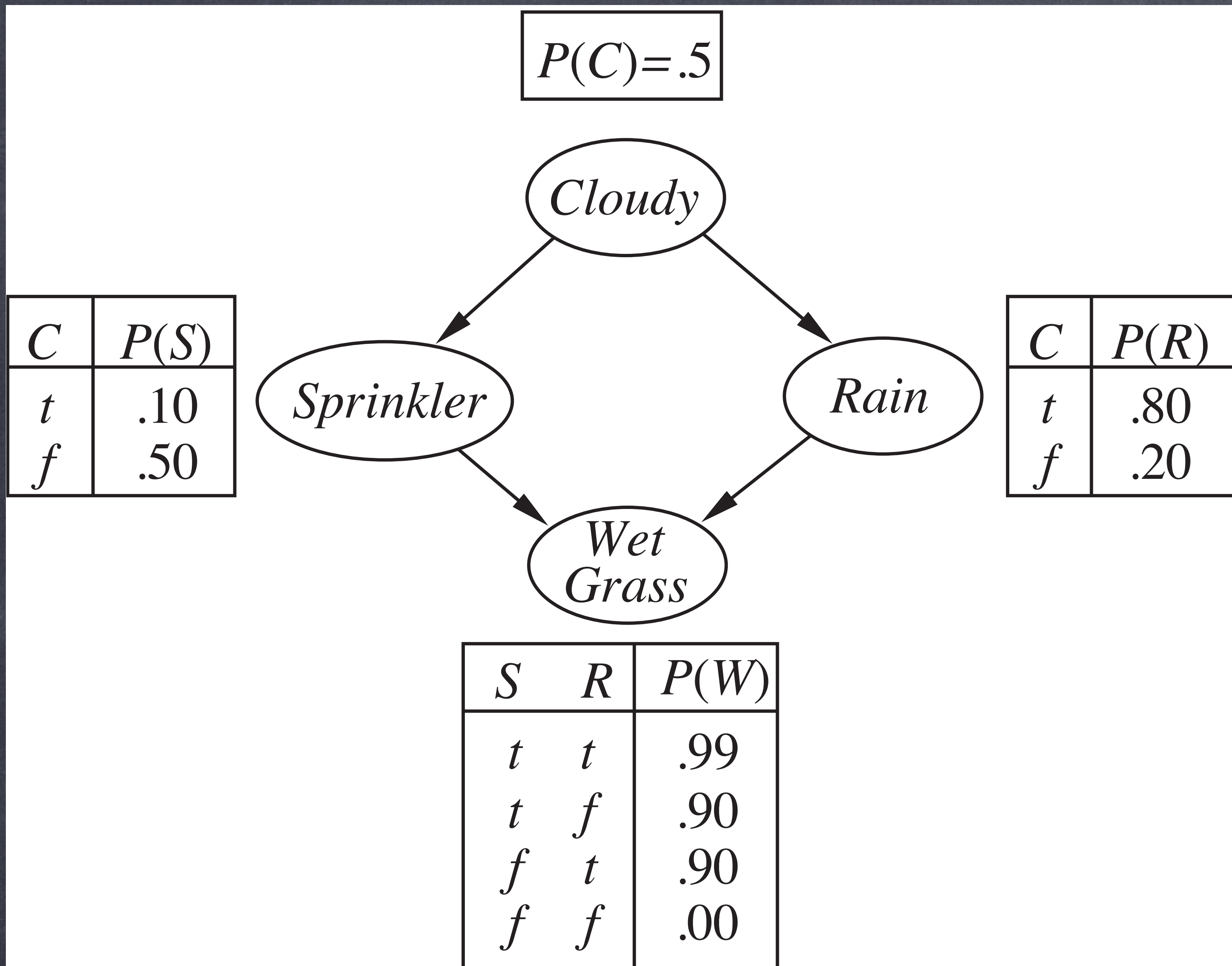


$\mathbf{P}(Rain \mid Sprinkler = true)$

⟨Cloudy=true, Sprinkler=false, Rain=true, WetGrass=true⟩

Rejection Sampling

- Generate sample from the prior distribution specified by the network
- Reject sample if inconsistent with the evidence
- Use remaining samples to estimate probability of event

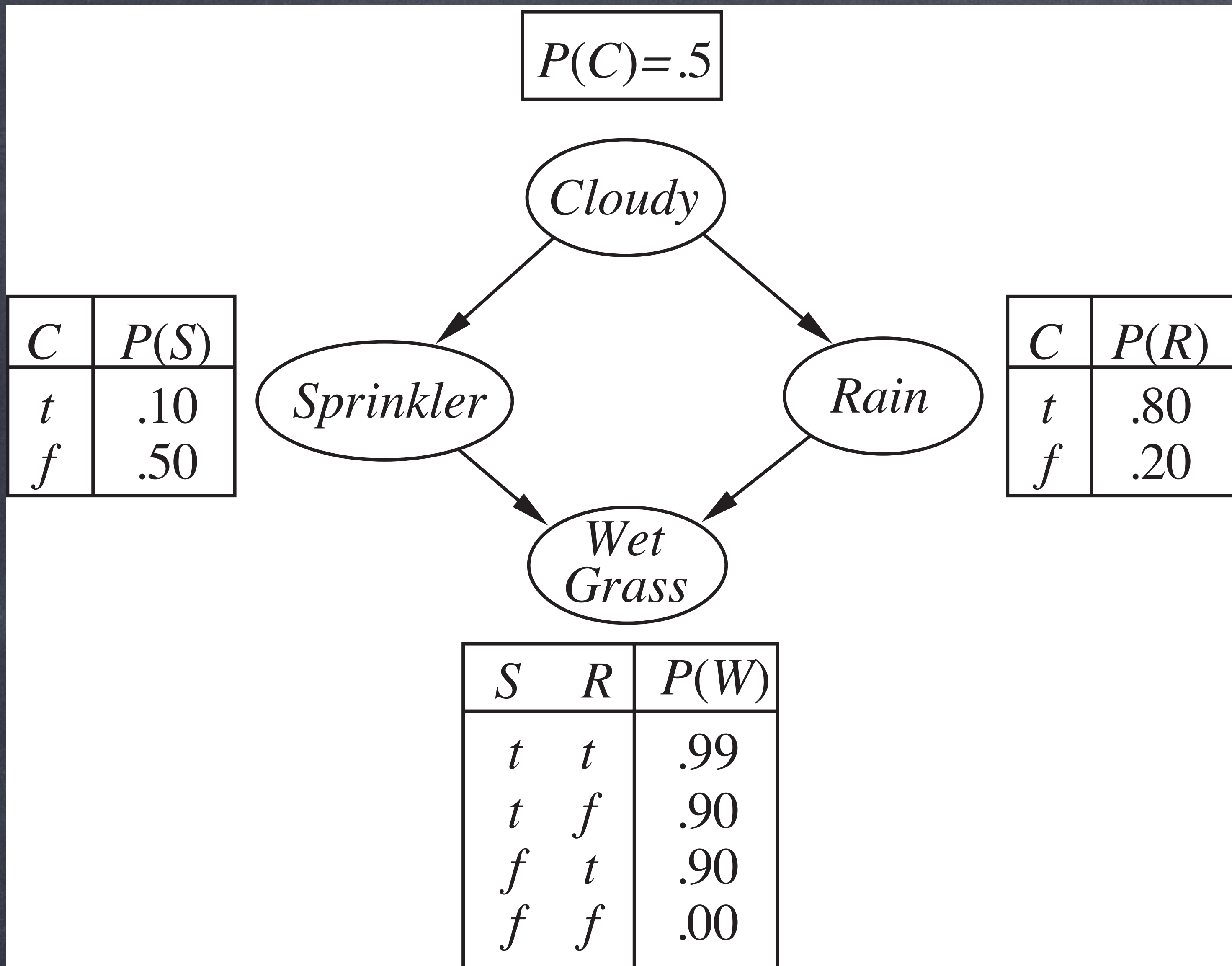


$\mathbf{P}(Rain \mid Sprinkler = true)$

100 samples

Sprinkler=false: 73

Sprinkler=true: 27



$\mathbf{P}(Rain \mid Sprinkler = true)$

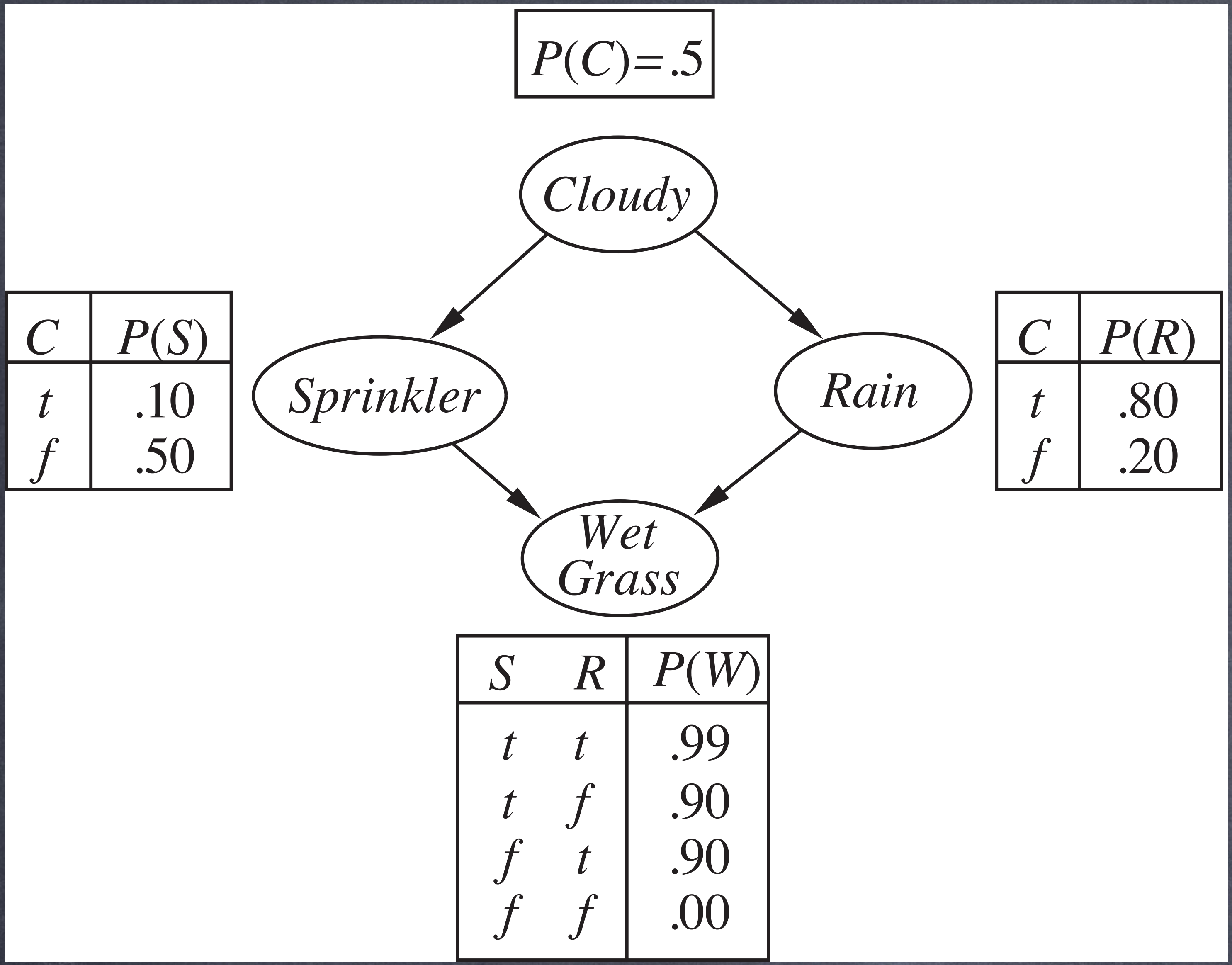
100 samples

Sprinkler=false: 73

Sprinkler=true: 27

Rain=true: 8

Rain=false: 19



$\mathbf{P}(Rain \mid Sprinkler = true)$

100 samples

Sprinkler=false: 73

Sprinkler=true: 27

Rain=true: 8

Rain=false: 19

$$\mathbf{P}(Rain \mid Sprinkler = true) \approx \alpha \, \langle \frac{8}{27}, \frac{19}{27} \rangle = \langle 0.296, 0.704 \rangle$$

Rejection Sampling

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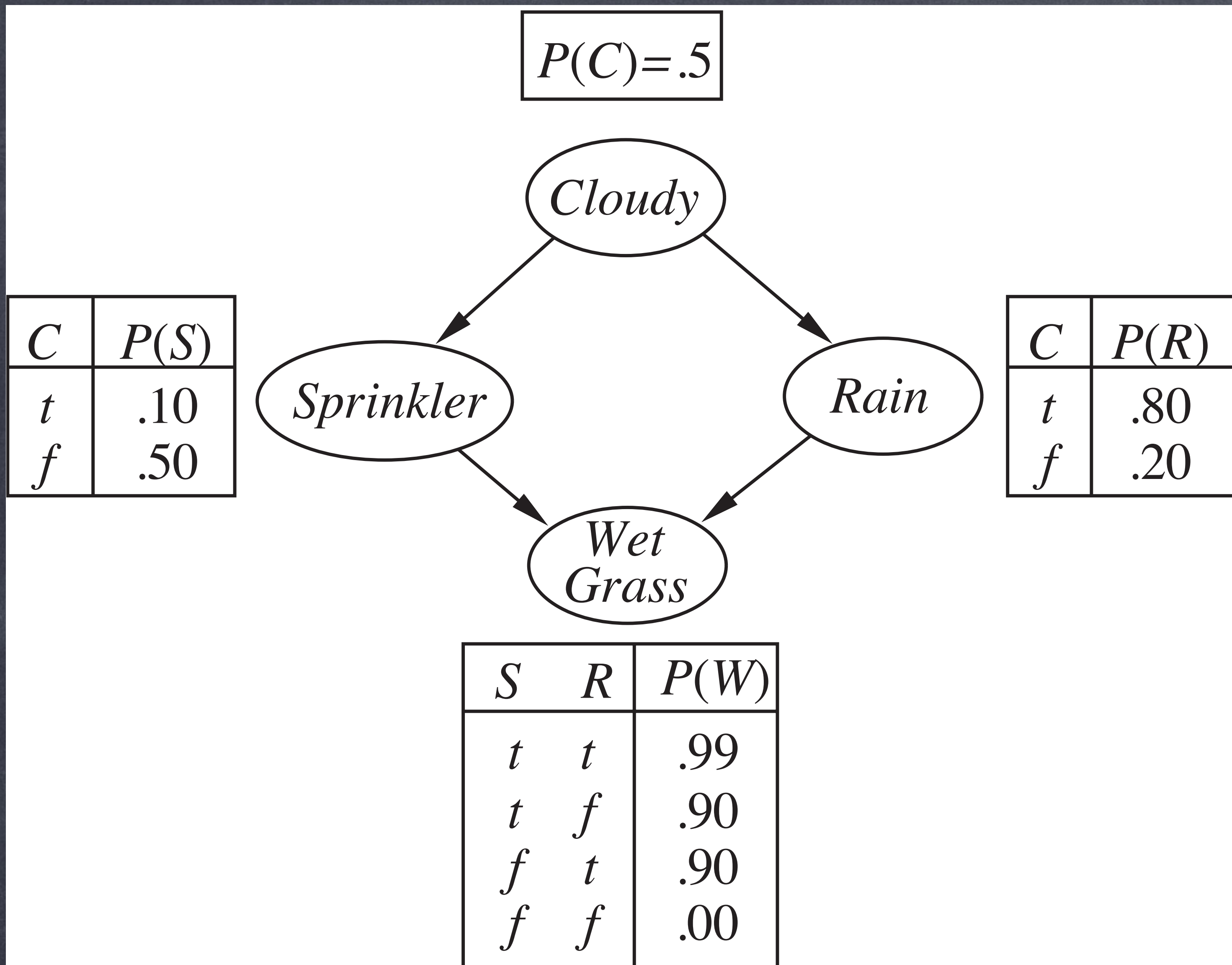
Rejection Sampling

- Generate sample from the prior distribution specified by the network
- Reject sample if inconsistent with the evidence
- Use remaining samples to estimate probability of event

Fraction of samples consistent with the evidence drops exponentially with number of evidence variables

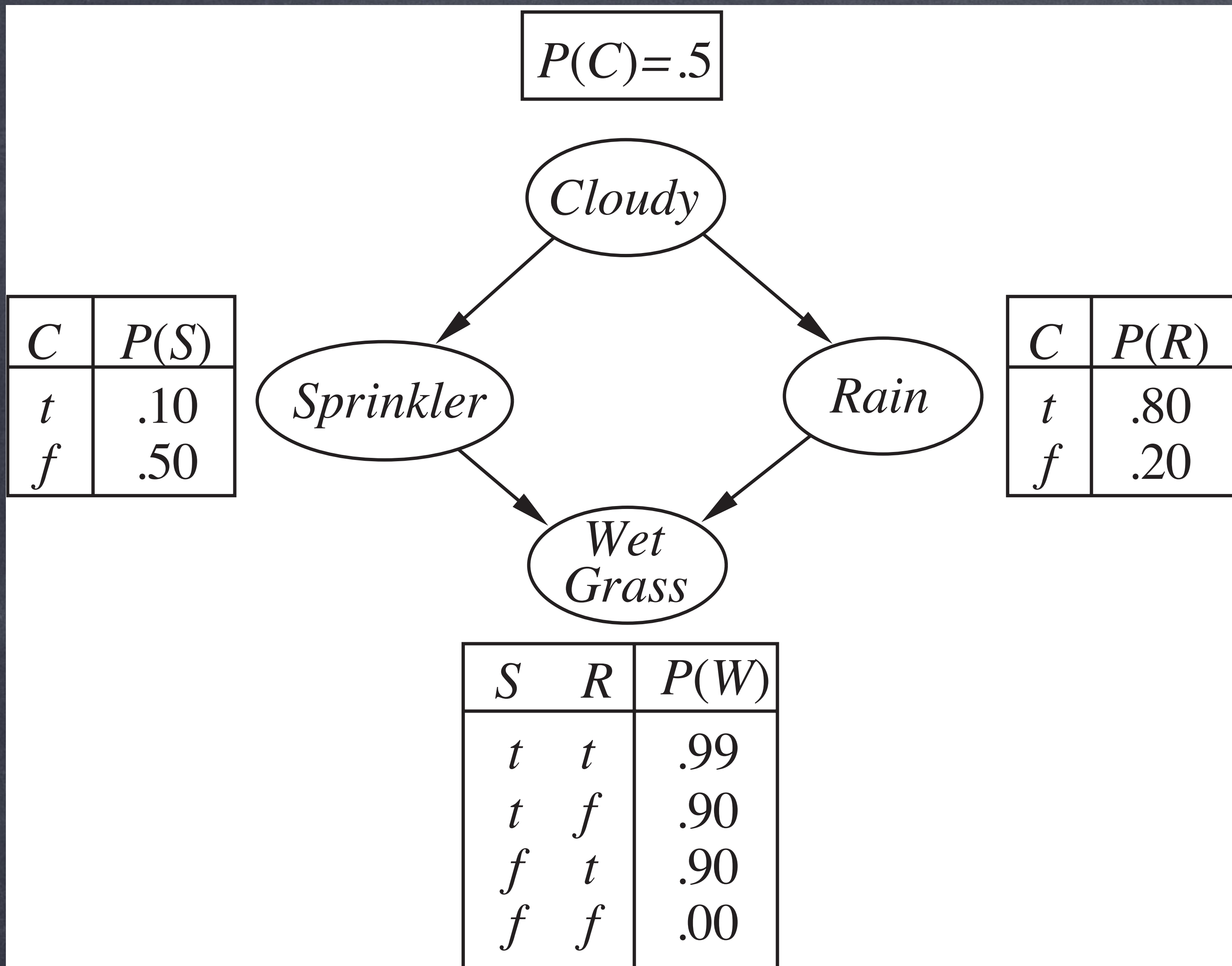


Cloudy

Sprinkler

Rain

WetGrass



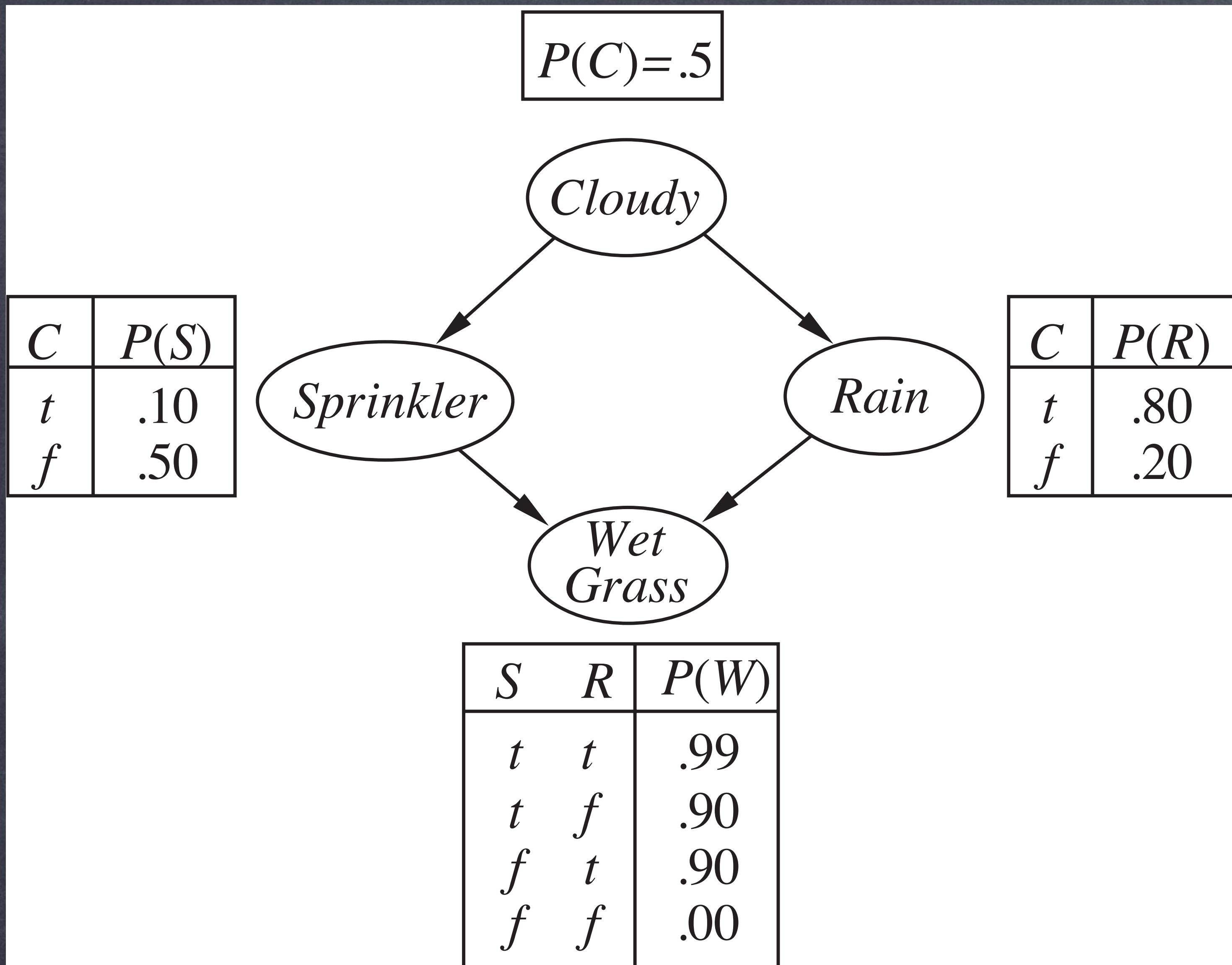
Cloudy

Sprinkler

Rain

WetGrass

$$\mathbf{P}(Rain \mid Cloudy = true, WetGrass = true)$$



Cloudy

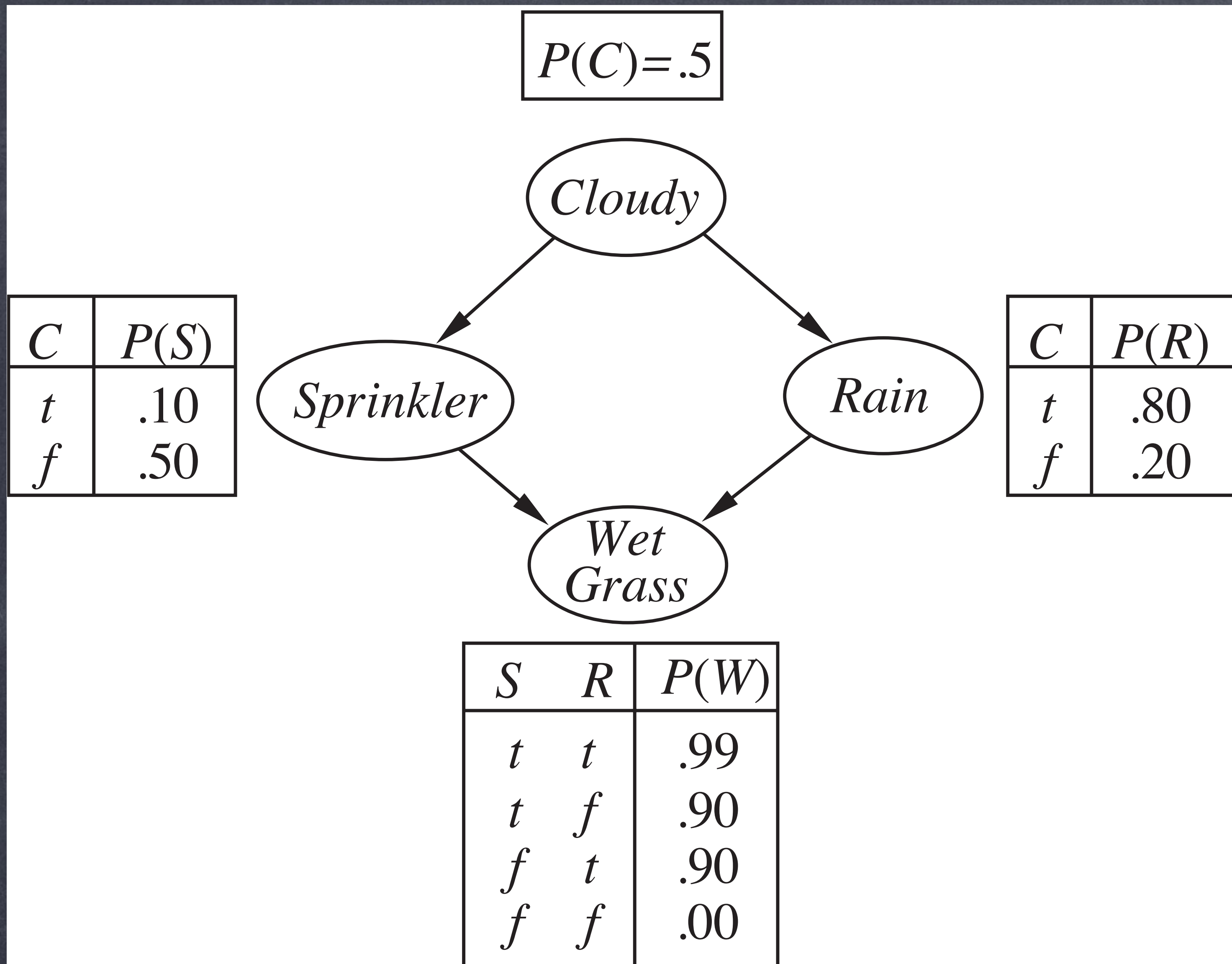
Sprinkler

Rain

WetGrass

w = 1.0

$$\mathbf{P}(Rain \mid Cloudy = true, WetGrass = true)$$



Cloudy true

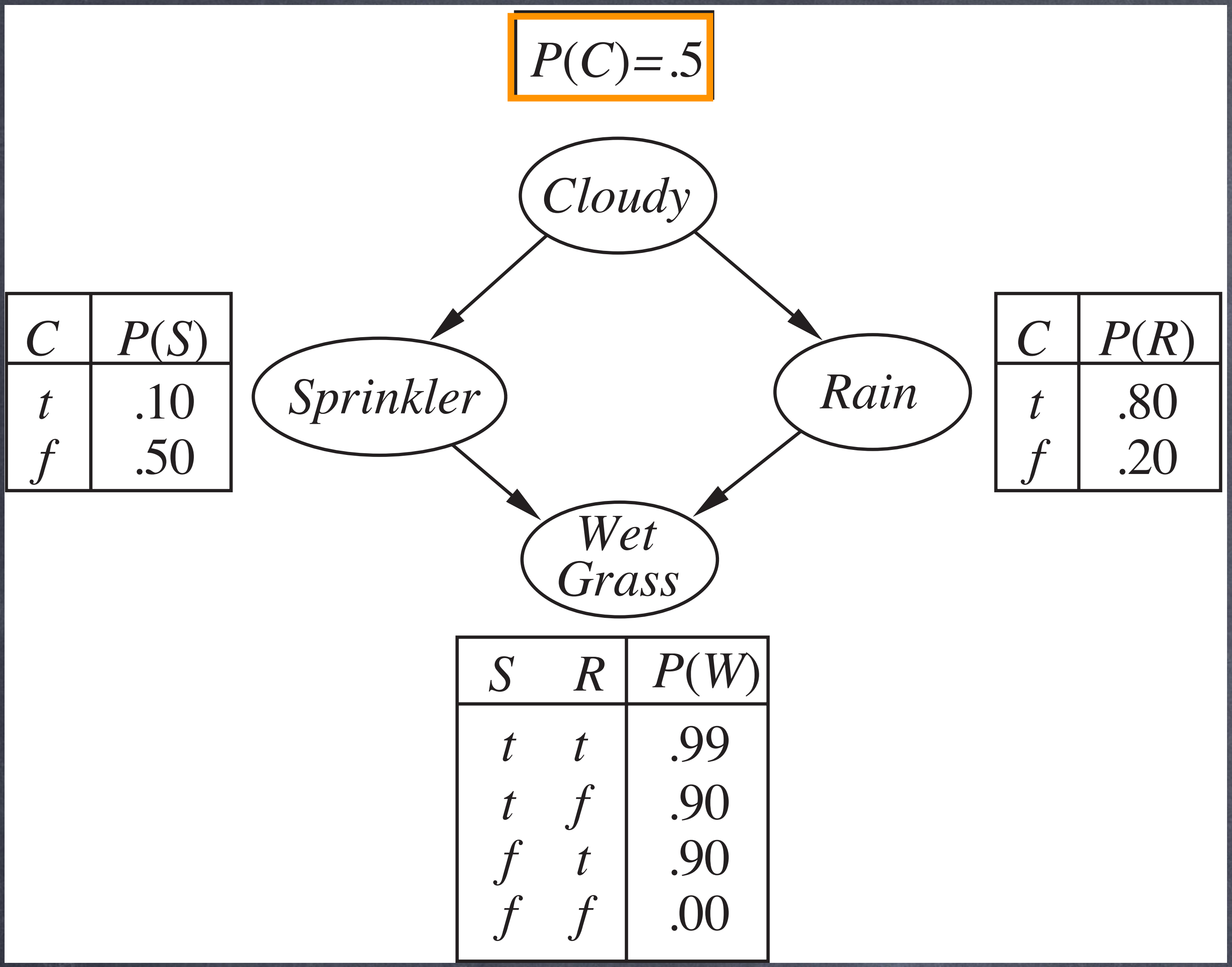
Sprinkler

Rain

WetGrass

w = 1.0

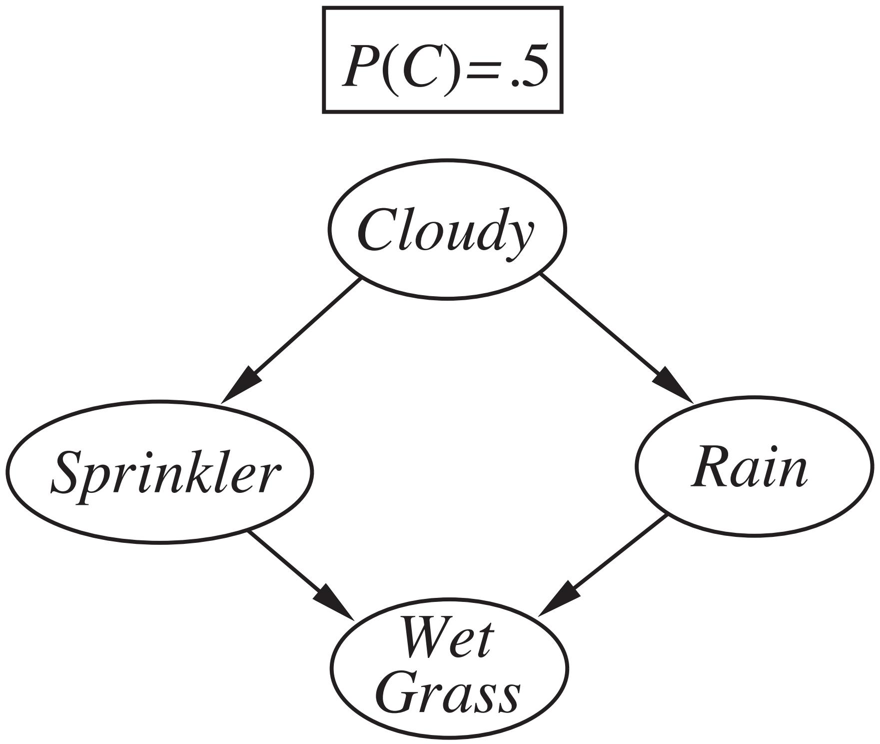
$$\mathbf{P}(Rain \mid Cloudy = true, WetGrass = true)$$



Cloudy	true
Sprinkler	
Rain	
WetGrass	

w = 0.5

$$\mathbf{P}(Rain \mid Cloudy = true, WetGrass = true)$$



C	P(S)
t	.10
f	.50

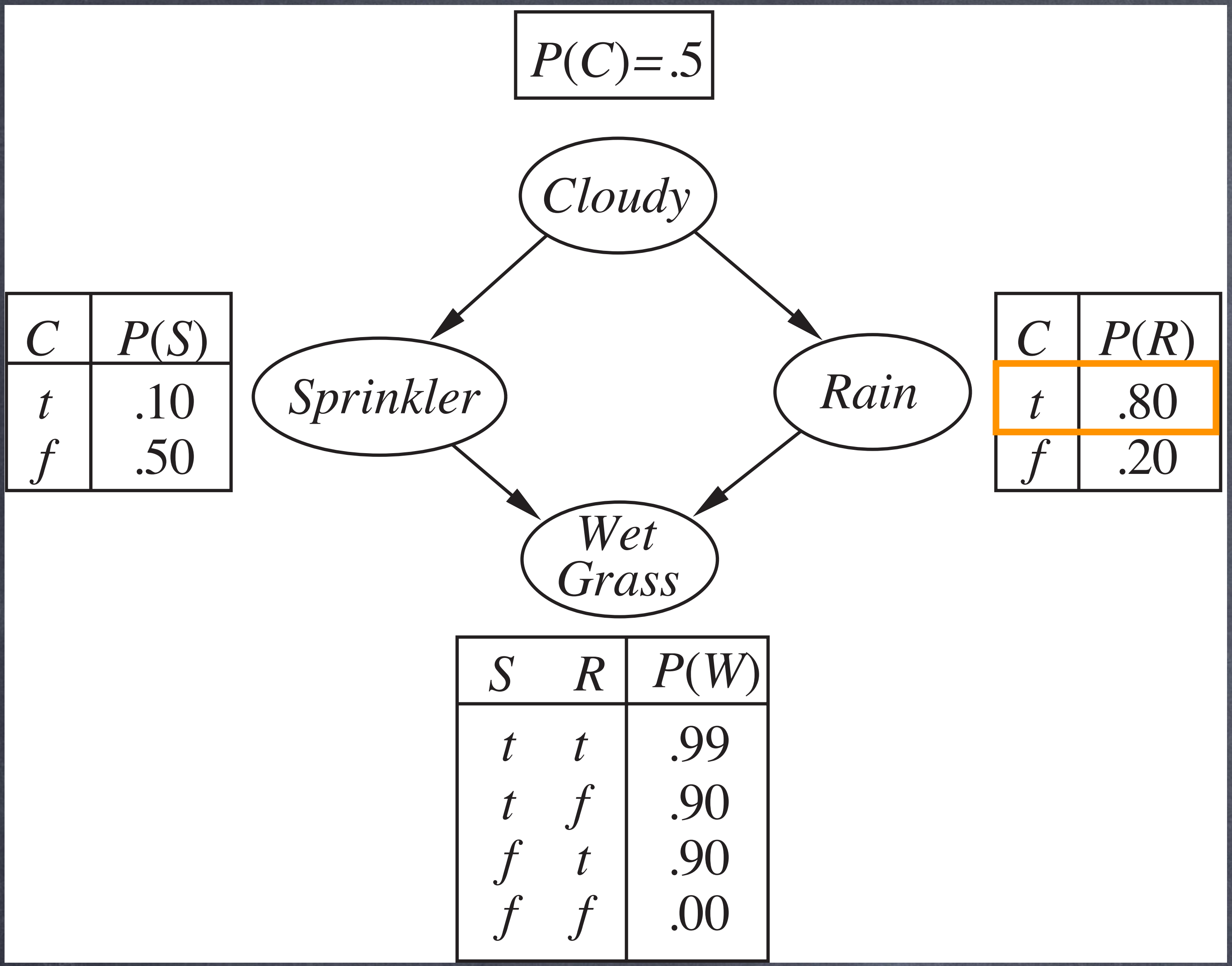
C	P(R)
t	.80
f	.20

S	R	P(W)
t	t	.99
t	f	.90
f	t	.90
f	f	.00

Cloudy	true
Sprinkler	false
Rain	
WetGrass	

w = 0.5

$$\mathbf{P}(Rain \mid Cloudy = true, WetGrass = true)$$



Cloudy true

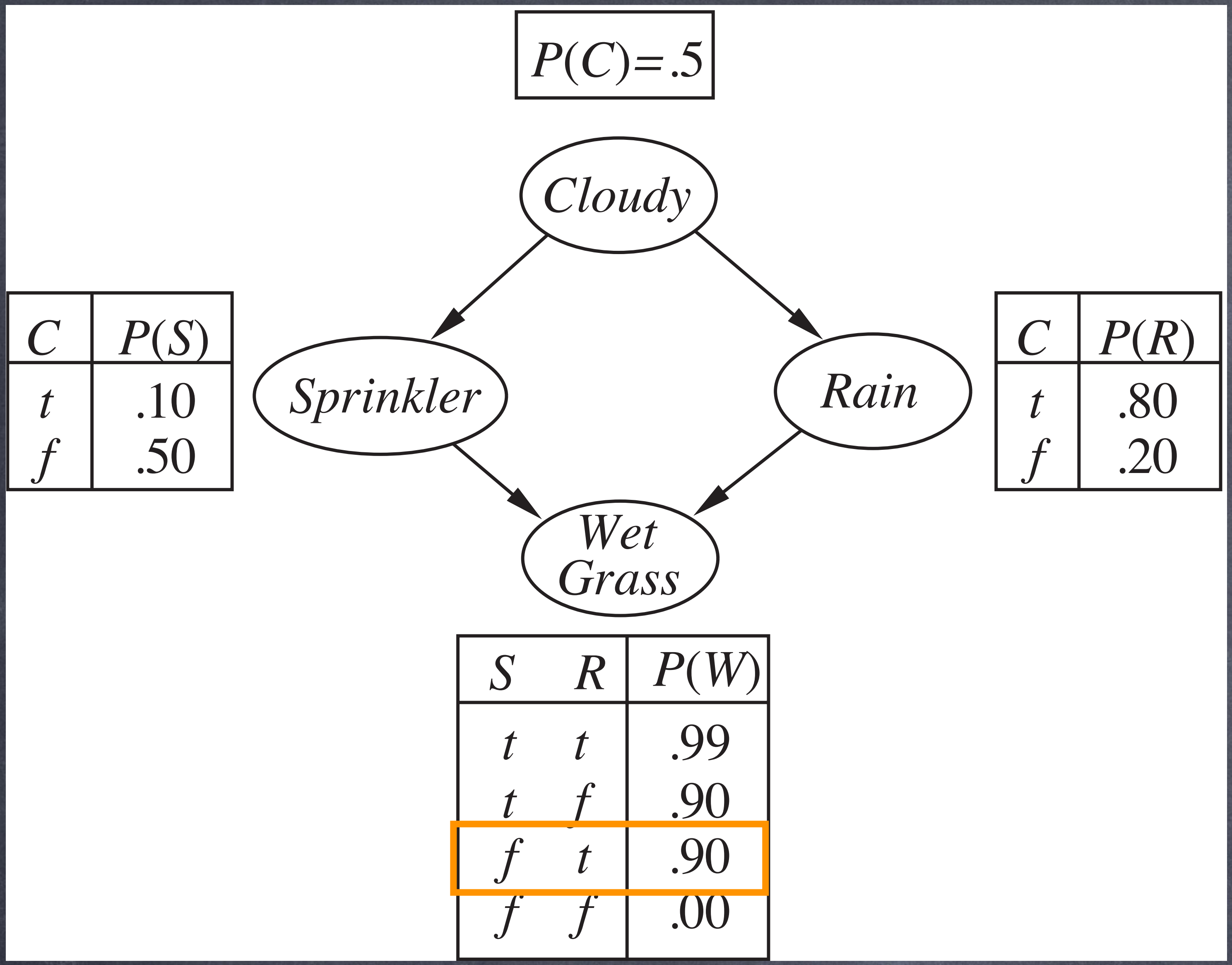
Sprinkler false

Rain true

WetGrass

w = 0.5

$$\mathbf{P}(Rain \mid Cloudy = true, WetGrass = true)$$



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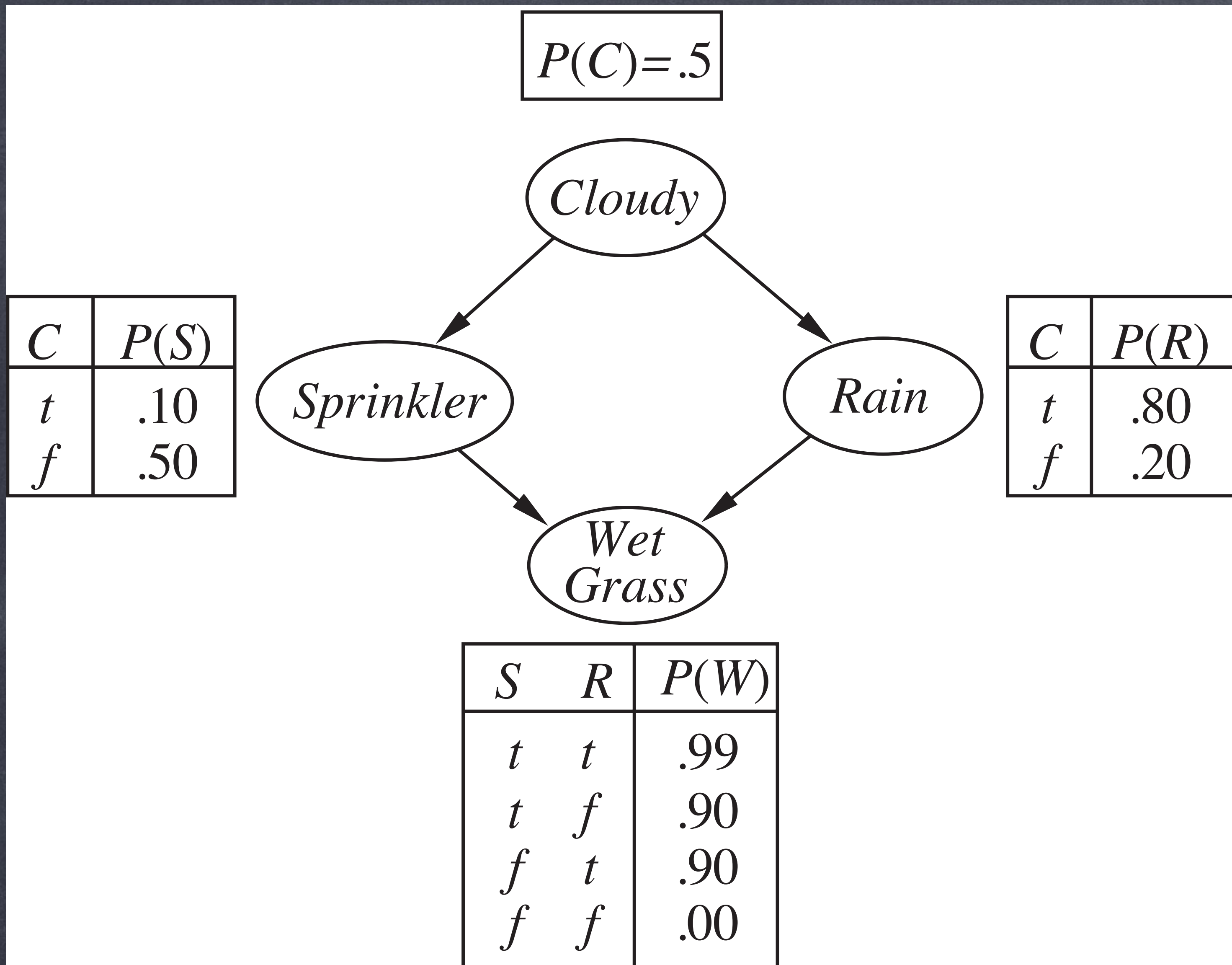
w = 0.5

$$\mathbf{P}(Rain \mid Cloudy = true, WetGrass = true)$$



Cloudy	true
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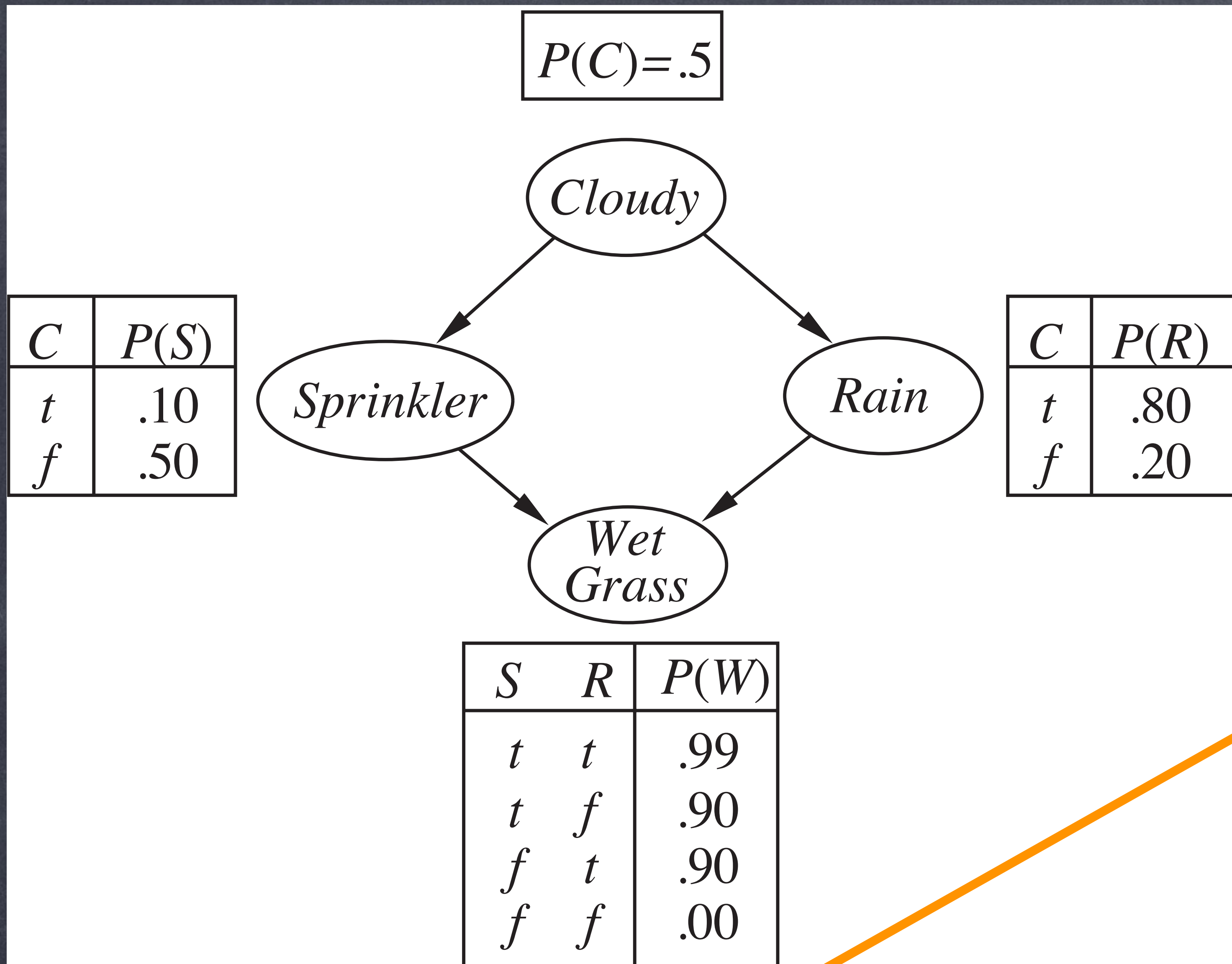
Rain	true
WetGrass	true

w = 0.45

$$\mathbf{P}(Rain \mid Cloudy = true, WetGrass = true)$$

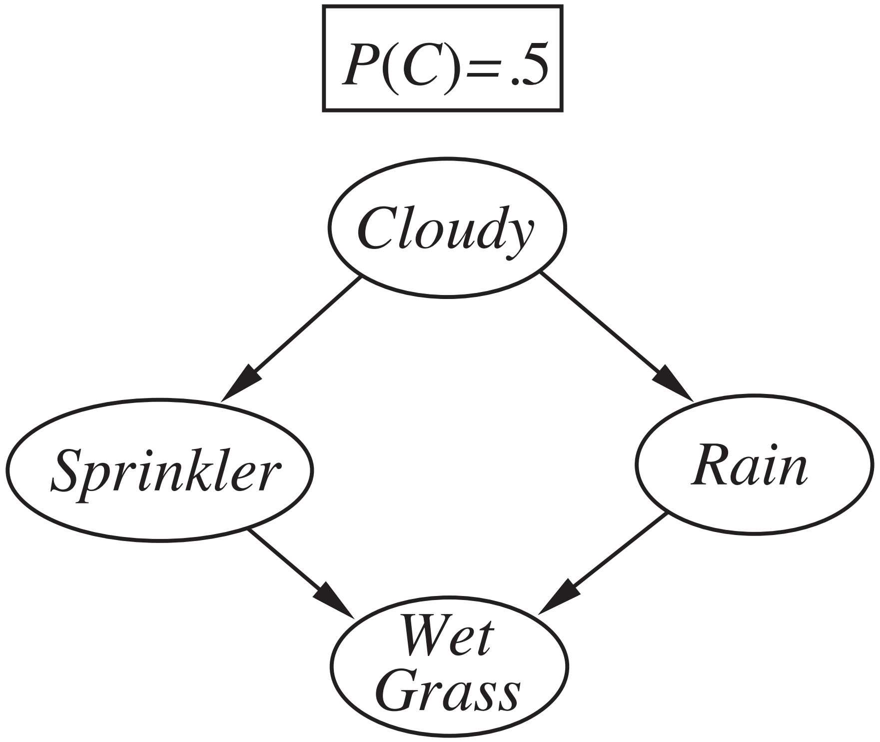


Cloudy	true
Sprinkler	false
Rain	true
WetGrass	true

w = 0.45

$$\mathbf{P}(Rain \mid Cloudy = true, WetGrass = true)$$

$N_{\text{Rain=true}}$ +=



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t	t	.99
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f	f	.00

Cloudy	true
Sprinkler	false
Rain	true
WetGrass	true

w = 0.45

$$\mathbf{P}(Rain \mid Cloudy = true, WetGrass = true)$$

$N_{Rain=true}$ += 0.45

Likelihood Weighting

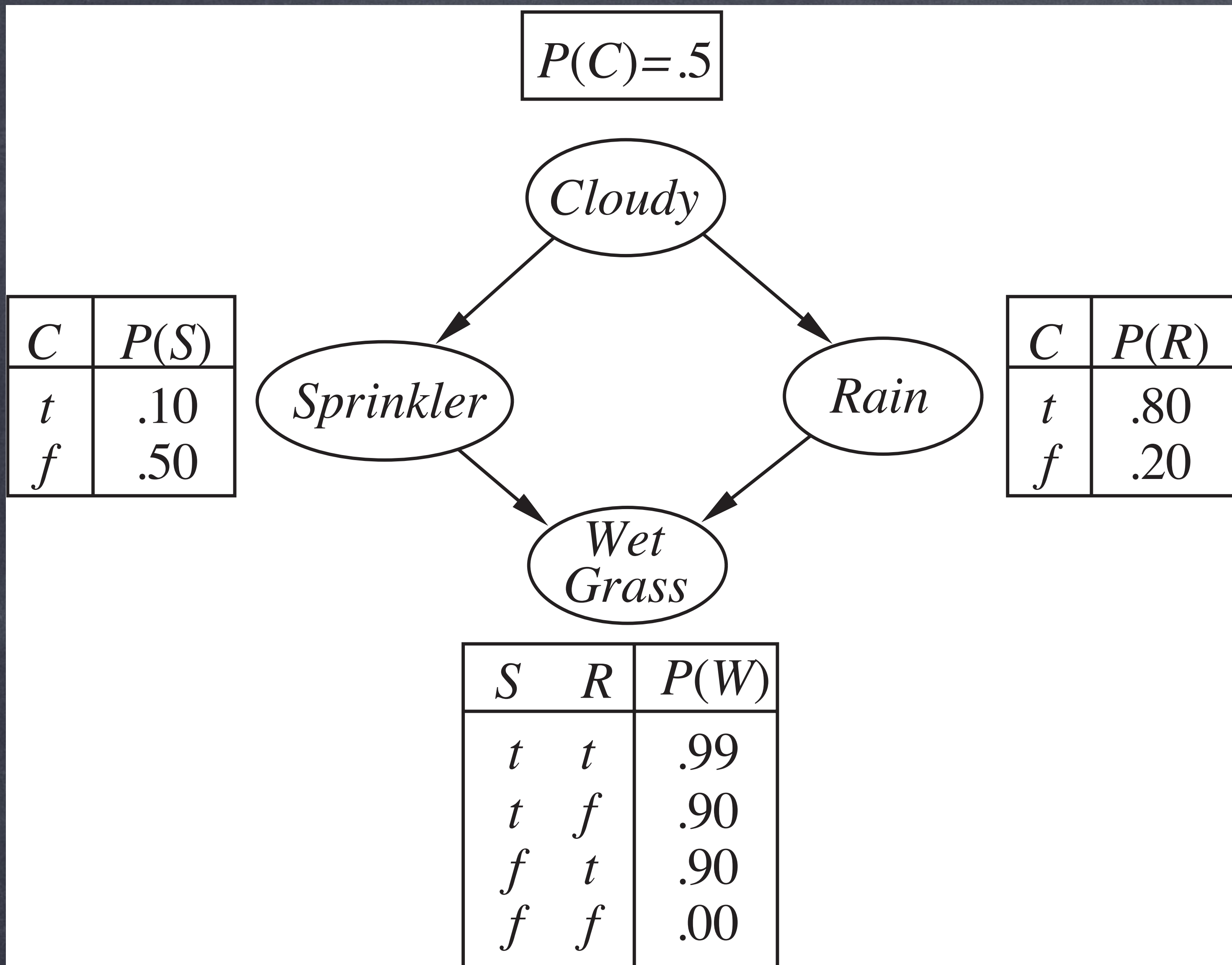
- Generate only samples consistent with the evidence
  - Generate sample using topological order
  - Evidence variable: Value fixed, update weight
  - Non-evidence variable: Sample value from network
- Compute weighted sum of samples for estimate

Likelihood Weighting

- Pros:
  - Doesn't reject any samples
- Cons:
  - More evidence $\Rightarrow$ lower weight
  - Affected by order of evidence vars in topsort (later = worse)

Approximate Inference in Bayesian Networks

- Rejection Sampling
- Likelihood Weighting



Cloudy	true
Sprinkler	false
Rain	true
WetGrass	true

$$\mathbf{P}(Rain \mid Sprinkler = true, WetGrass = true)$$

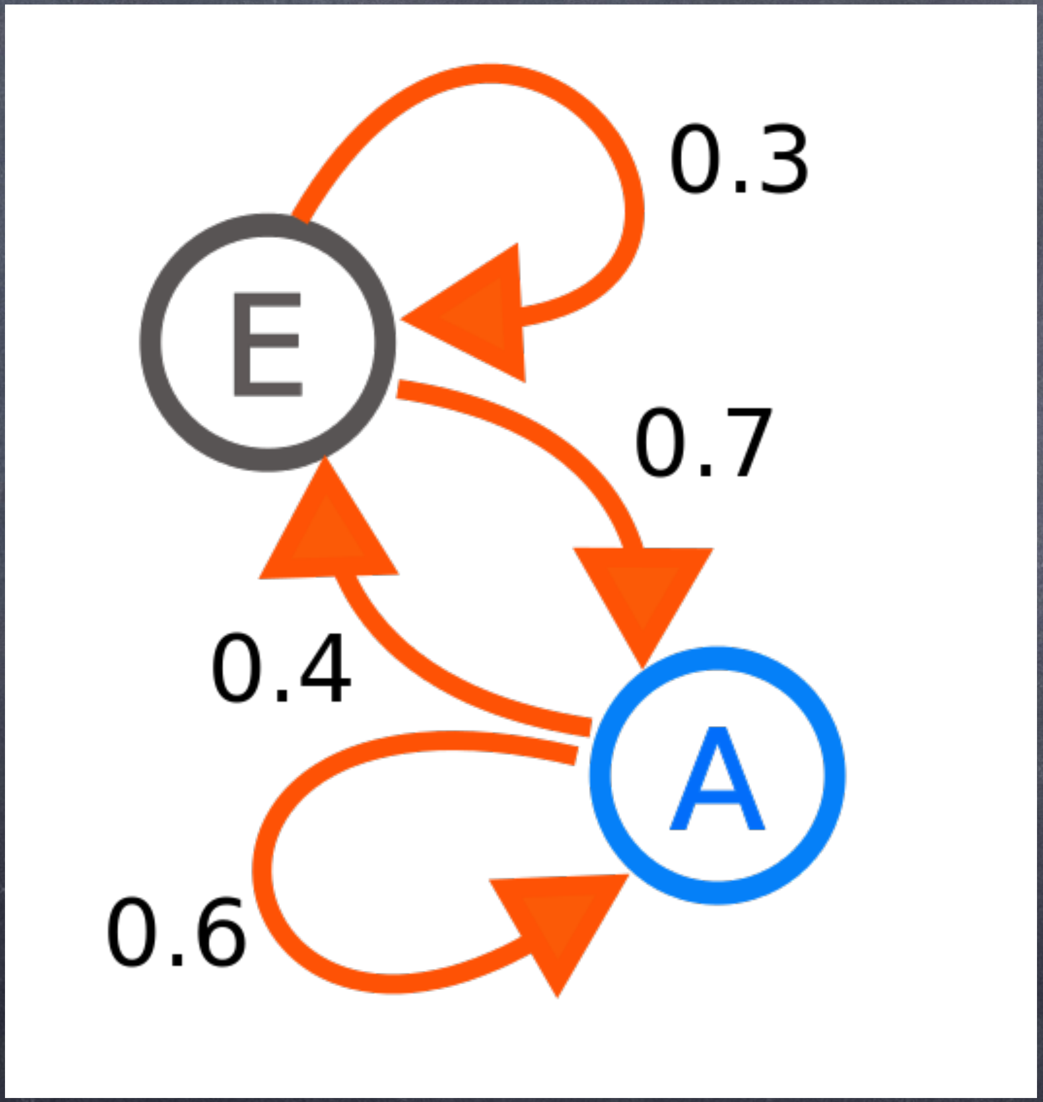
- To approximate: $P(X \mid e)$
- Generate a sequence of states
  - Values of evidence variables are fixed
  - Values of other variables appear in the right proportion given the distribution encoded by the network

Markov Chain Monte Carlo Simulation

- To approximate: $P(X \mid e)$
- Generate a sequence of states
  - Values of evidence variables are fixed
  - Values of other variables appear in the right proportion given the distribution encoded by the network

Monte Carlo

Markov Chain



Markov Chain Monte Carlo Simulation

- To approximate: $P(X \mid e)$
- Generate a sequence of states
  - Values of evidence variables are fixed
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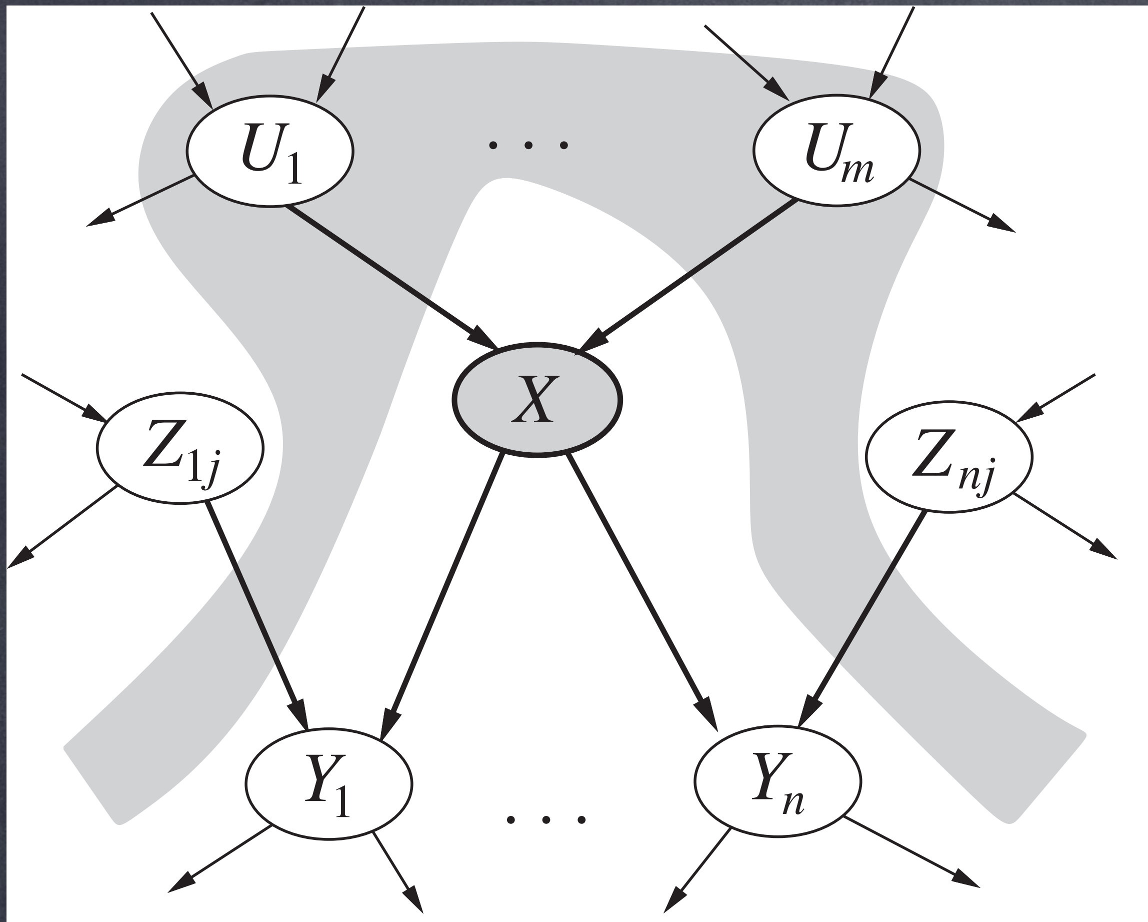
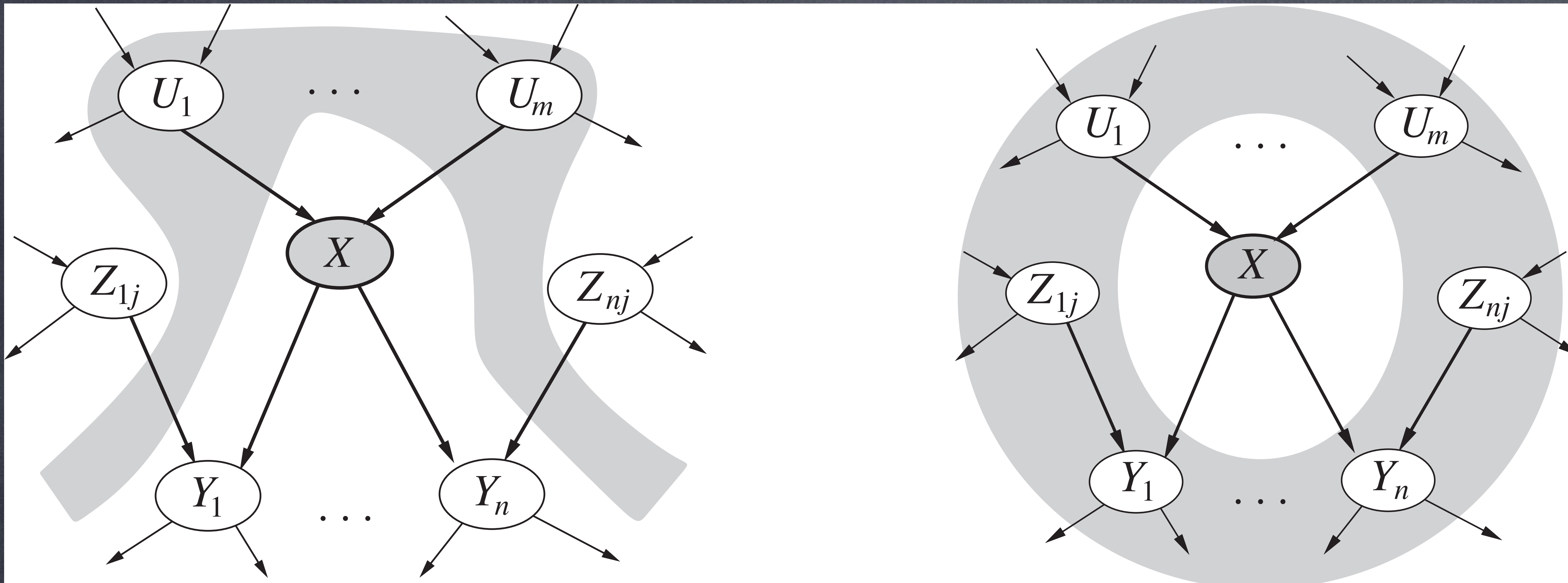


Conditional

Independence

X conditionally independent of Zs given Us



Conditional Independence

Markov Blanket

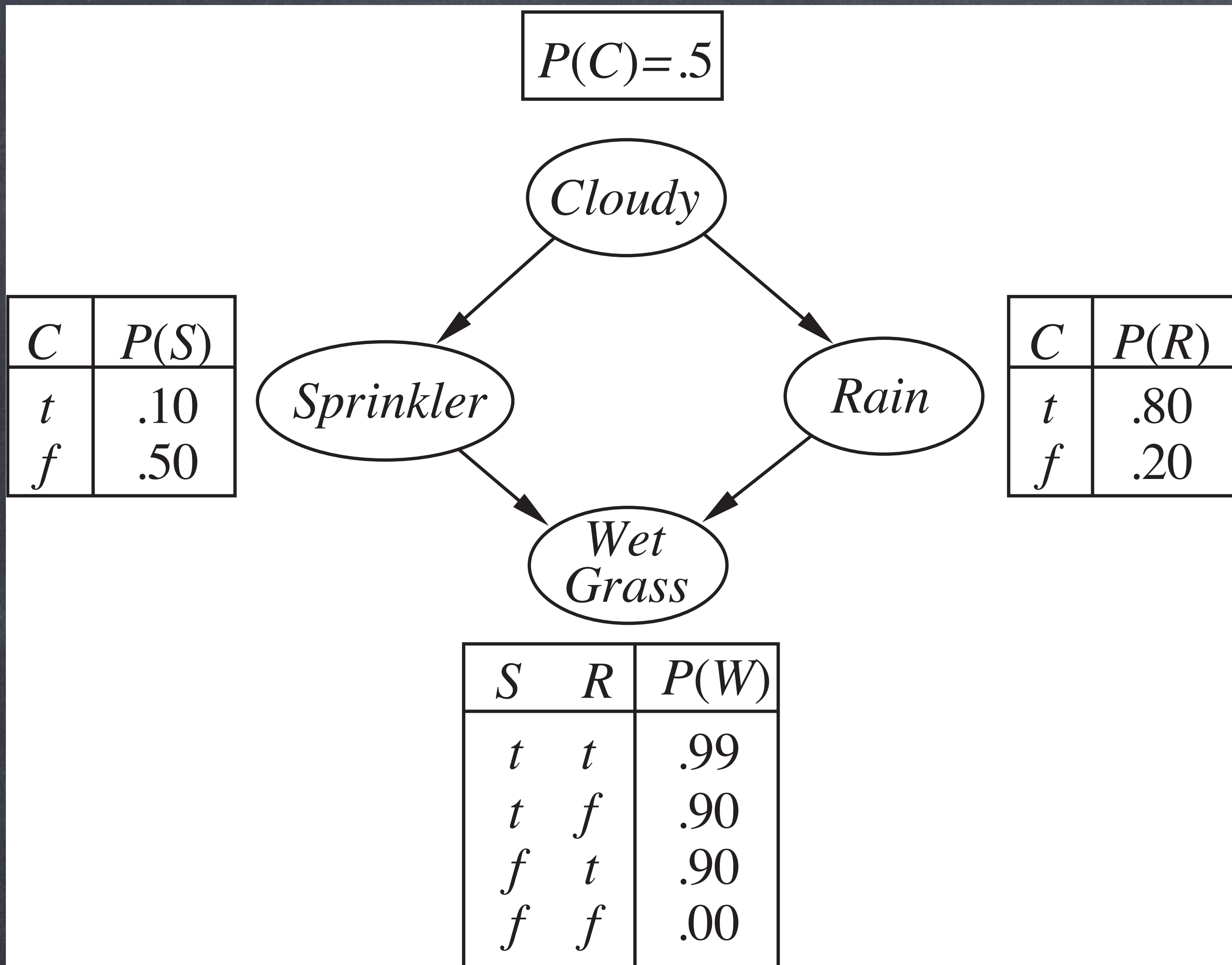
X conditionally independent of allVs given Us, Ys, and Zs

Markov Blanket

- The Markov Blanket of a node is its parents, its children, and its children's parents.
- A node is conditionally independent of all other nodes in the network given its Markov Blanket

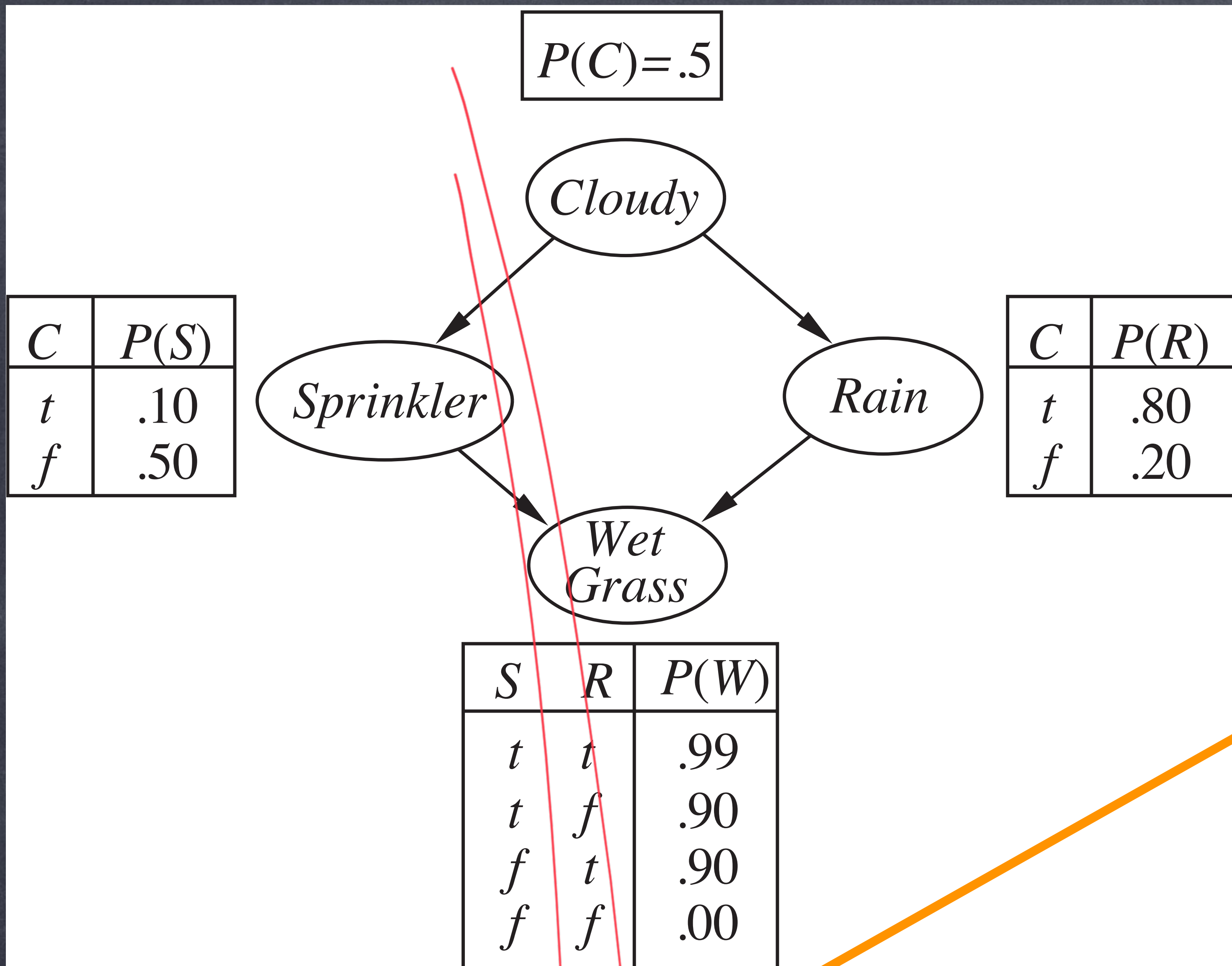
MCMC Simulation
Gibbs Sampling

- To approximate: P(X | e)
- Start in a state with evidence variables set to evidence values (others arbitrary)
- On each step, sample the non-evidence variables conditioned on the values of the variables in their Markov Blankets
    - Order irrelevant



Cloudy	true
Sprinkler	true
Rain	false
WetGrass	true

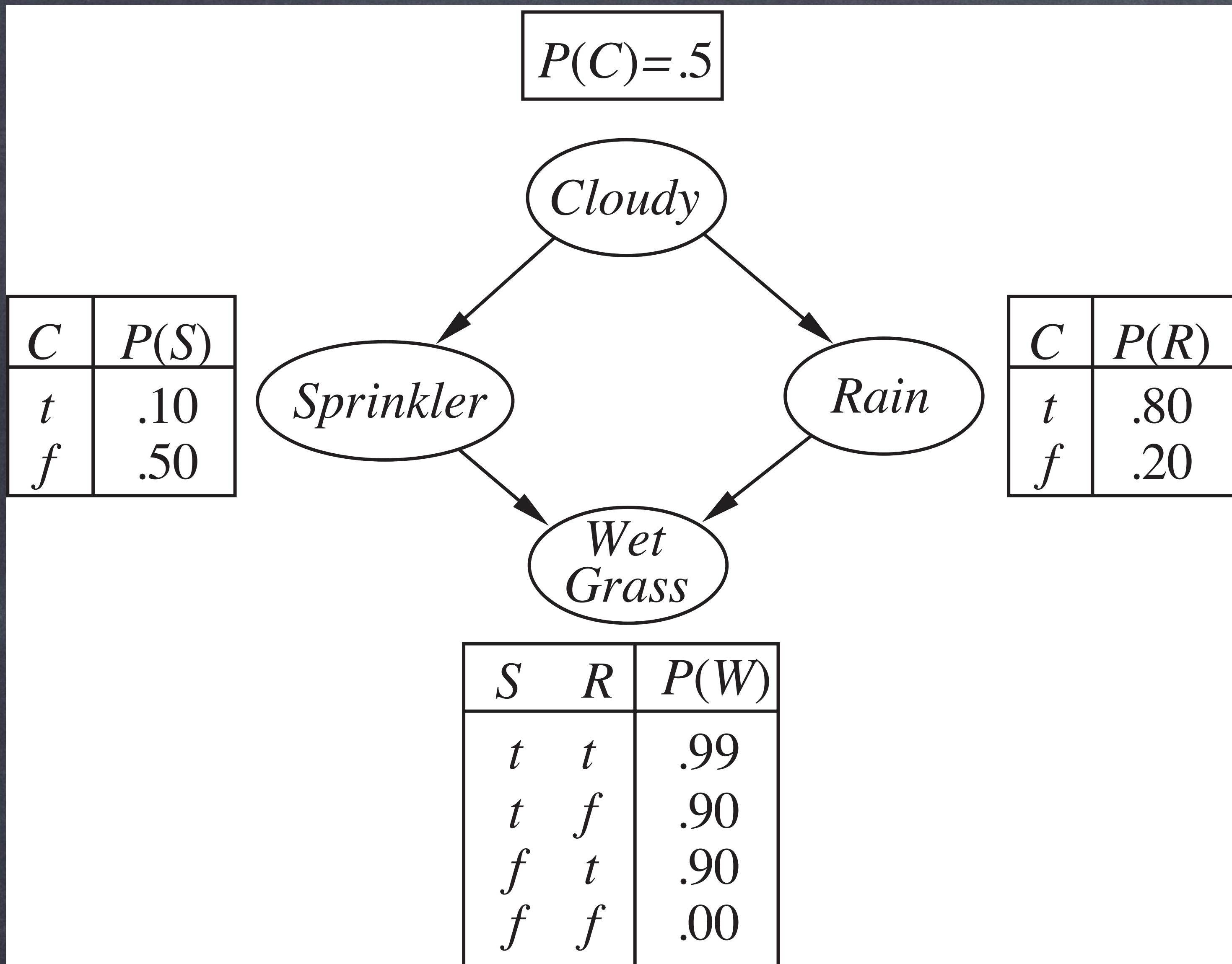
$$\mathbf{P}(Rain \mid Sprinkler = true, WetGrass = true)$$



Cloudy	true
Sprinkler	true
Rain	false
WetGrass	true

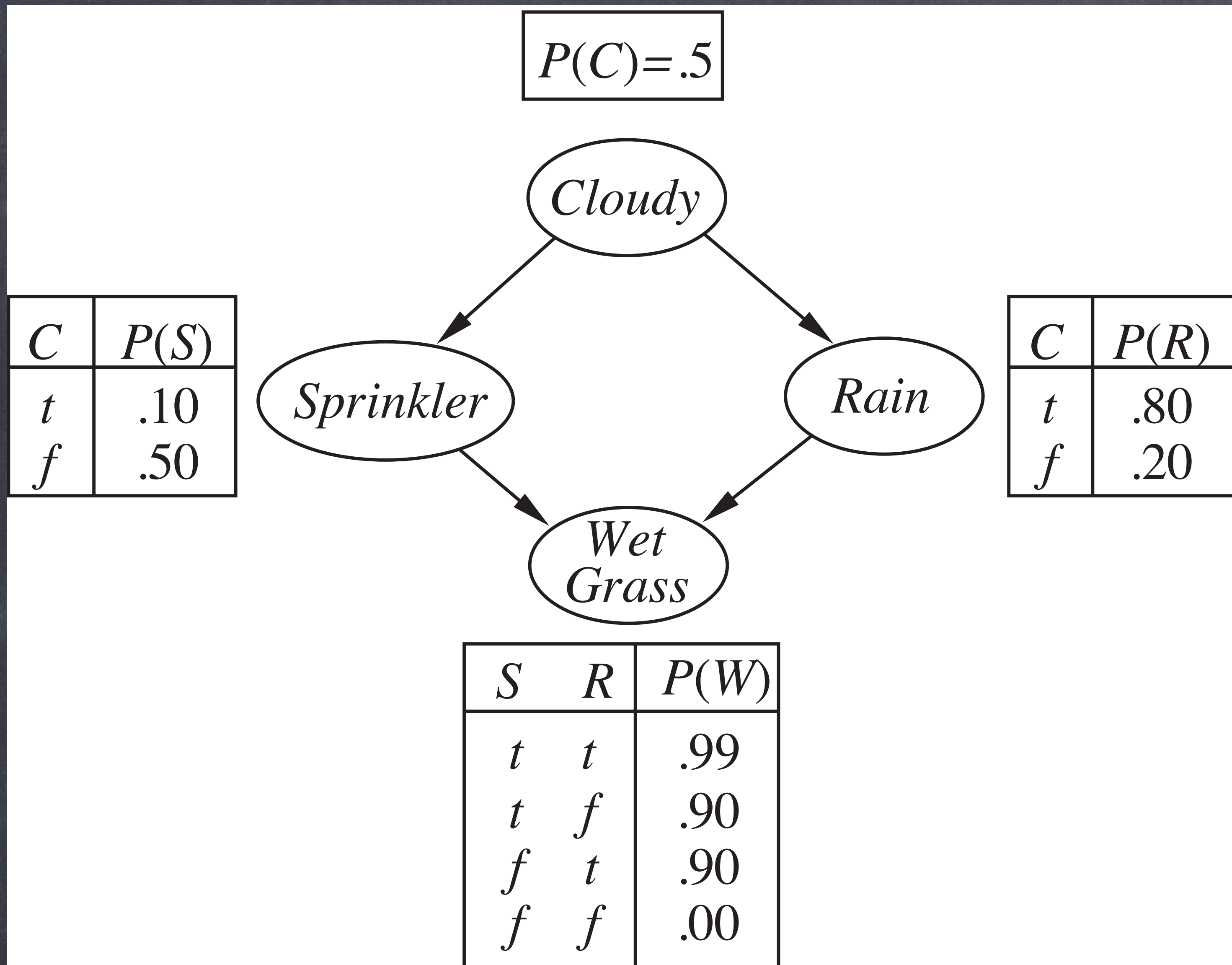
$$\mathbf{P}(Rain \mid Sprinkler = true, WetGrass = true)$$

$N_{\text{Rain=false}}$ += 1



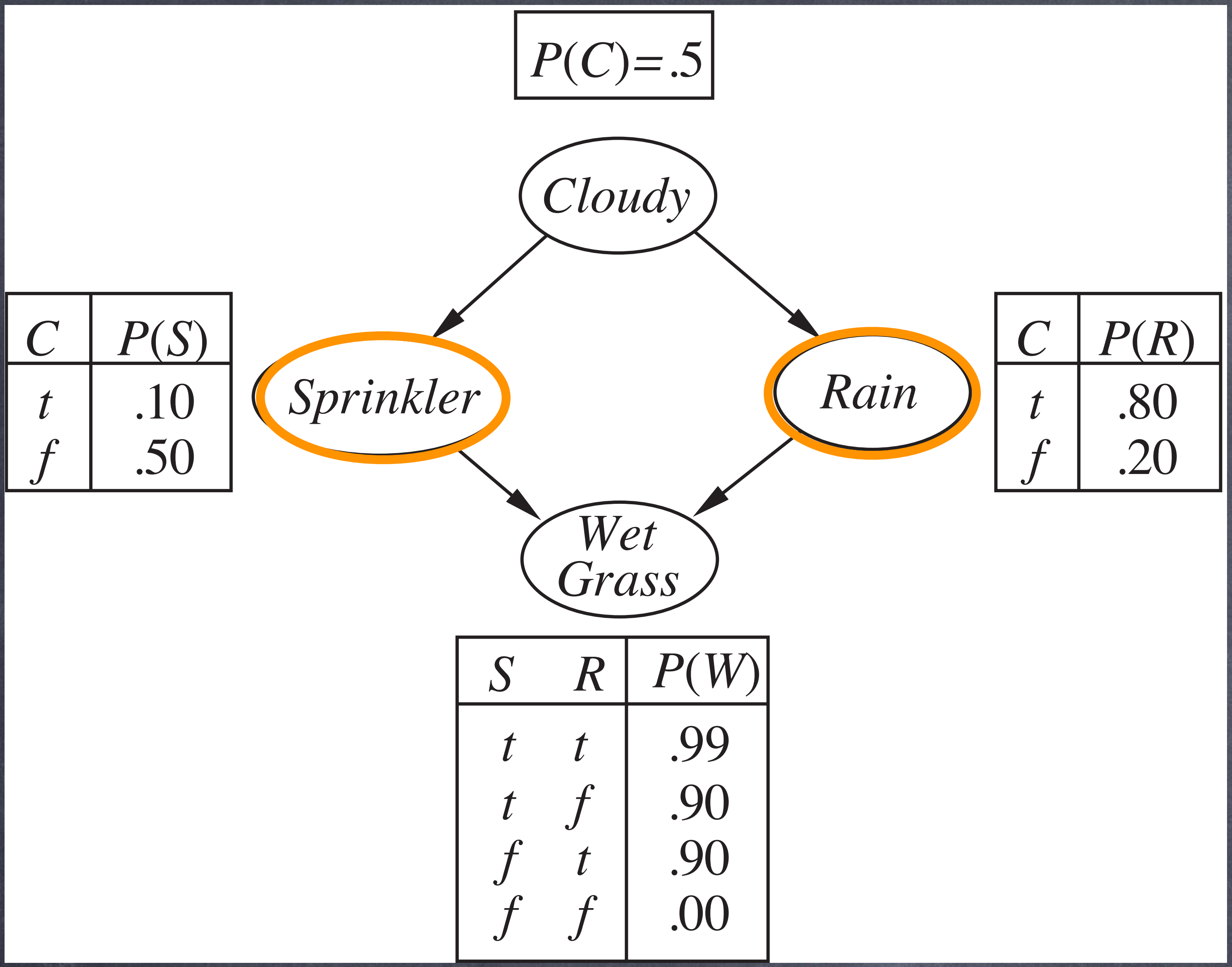
Cloudy	true
Sprinkler	true
Rain	false
WetGrass	true

$$\mathbf{P}(Rain \mid Sprinkler = true, WetGrass = true)$$



Cloudy	true
Sprinkler	true
Rain	false
WetGrass	true

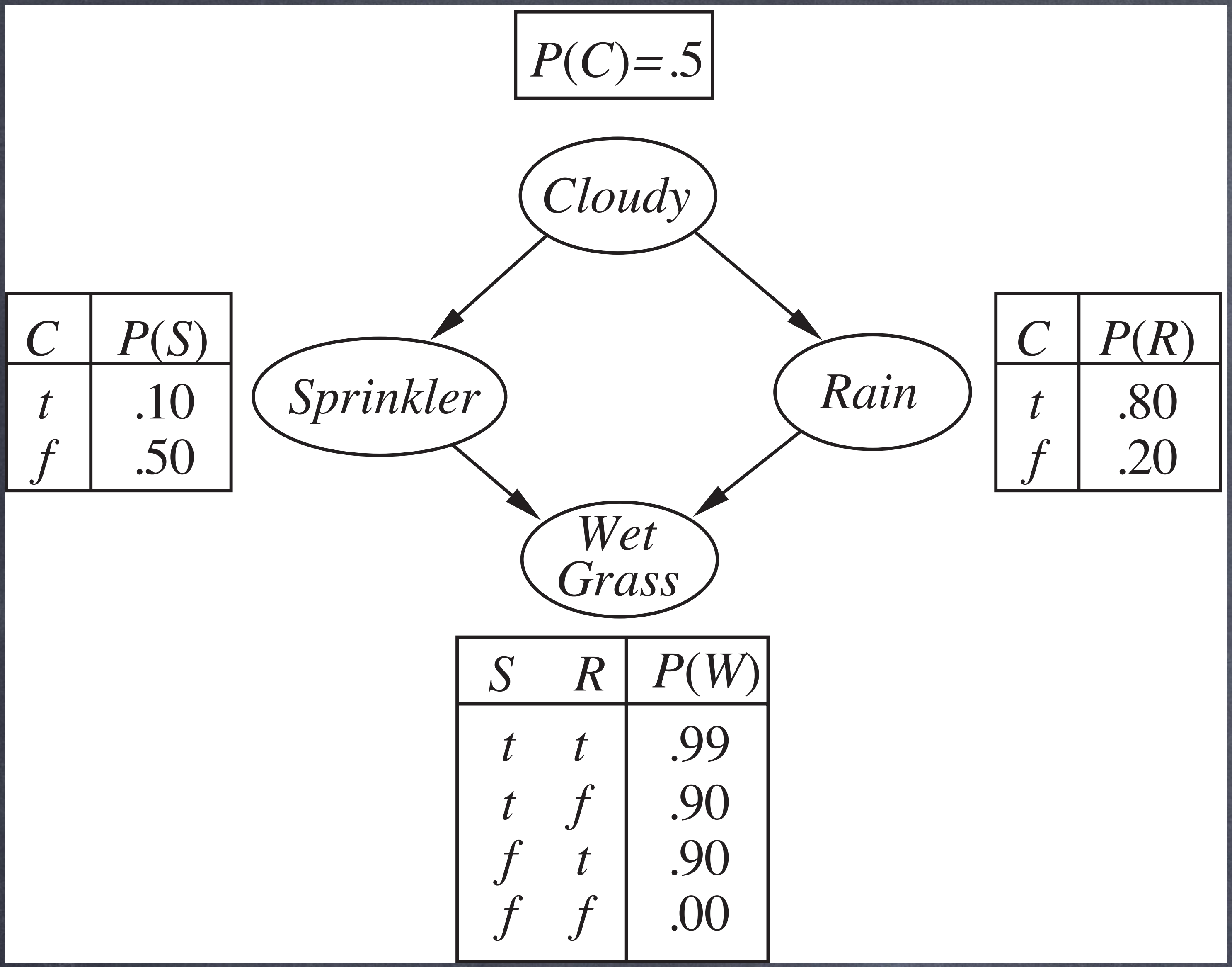
$$\mathbf{P}(Rain \mid Sprinkler = true, WetGrass = true)$$



Cloudy	true
Sprinkler	true
Rain	false
WetGrass	true

$$\mathbf{P}(Rain \mid Sprinkler = true, WetGrass = true)$$

P(Cloudy | Sprinkler=true, Rain=false)



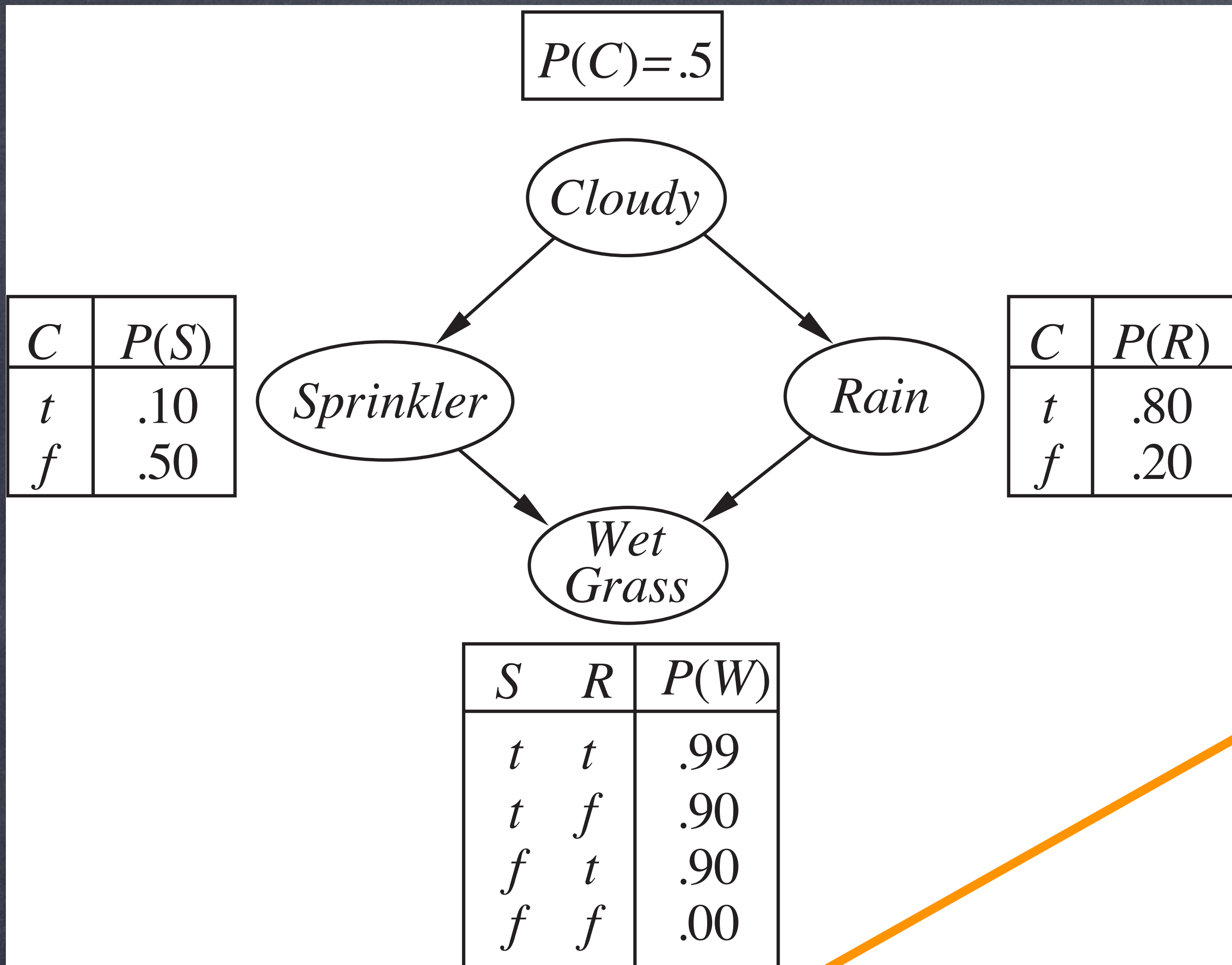
C	P(S)
t	.10
f	.50

C	P(R)
t	.80
f	.20

S	R	P(W)
t	t	.99
t	f	.90
f	t	.90
f	f	.00

Cloudy	false
Sprinkler	true
Rain	false
WetGrass	true

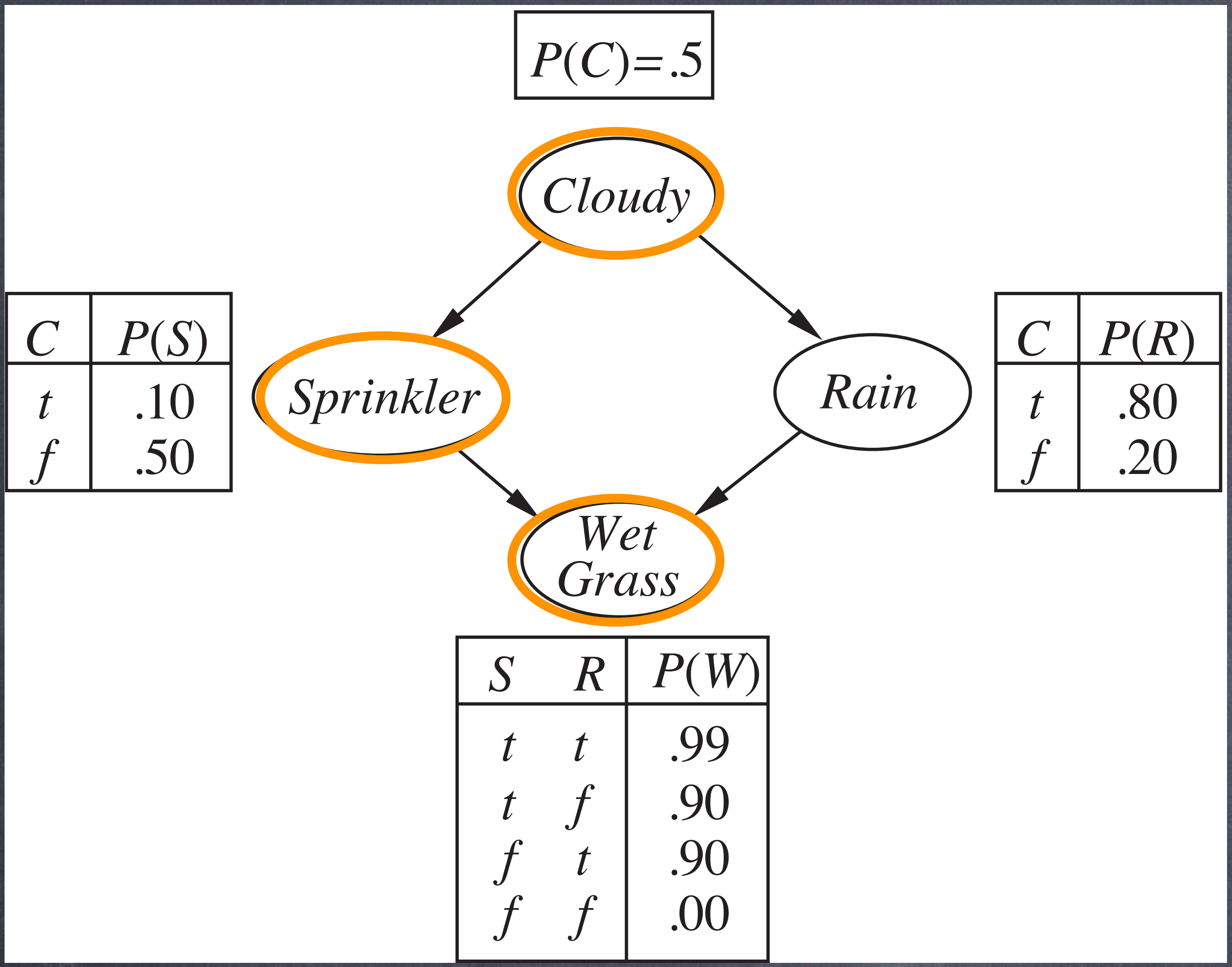
$$\mathbf{P}(Rain \mid Sprinkler = true, WetGrass = true)$$



Cloudy	false
Sprinkler	true
Rain	false
WetGrass	true

$$\mathbf{P}(Rain \mid Sprinkler = true, WetGrass = true)$$

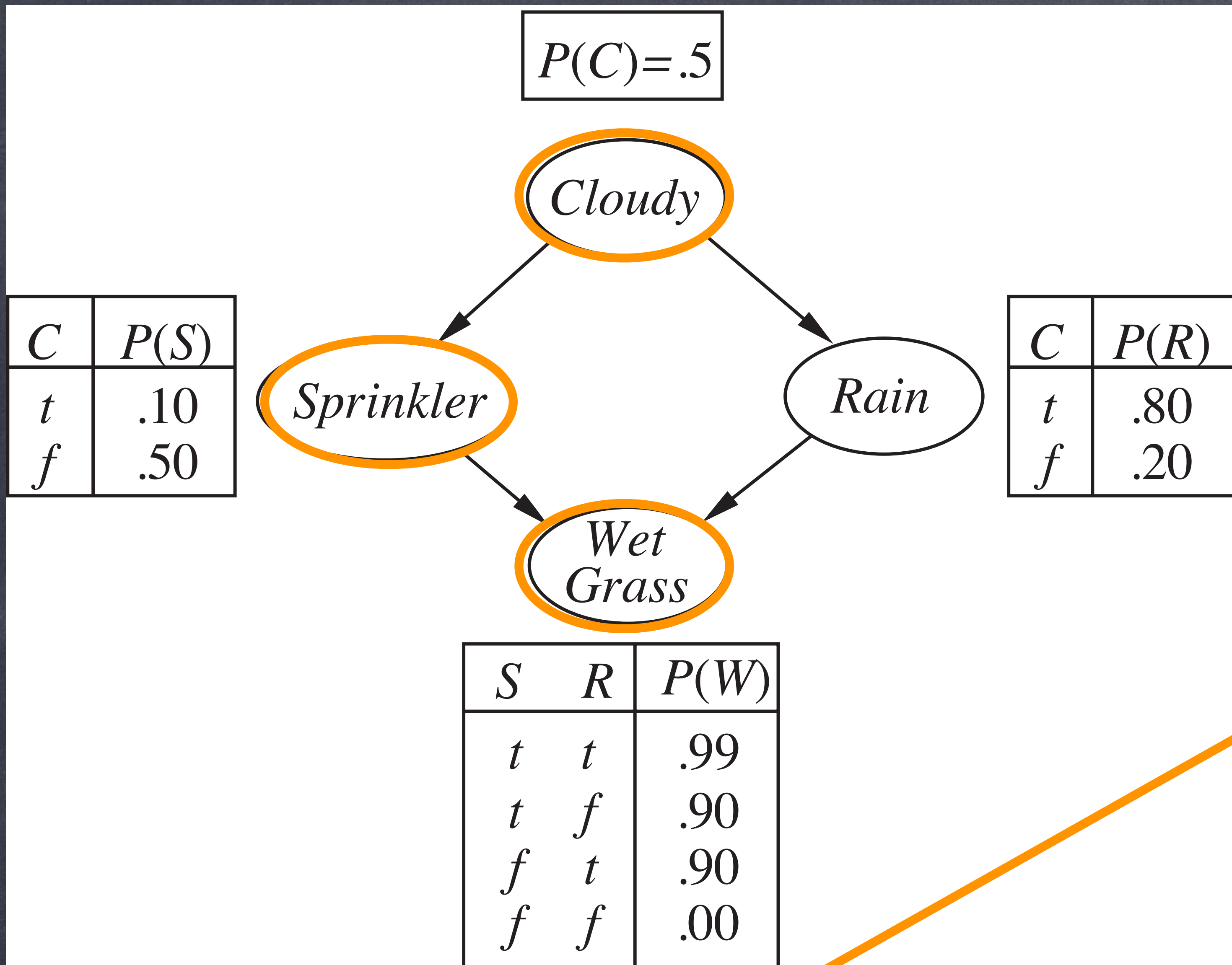
$N_{\text{Rain=false}}$ += 1



Cloudy	false
Sprinkler	true
Rain	false
WetGrass	true

$$\mathbf{P}(Rain \mid Sprinkler = true, WetGrass = true)$$

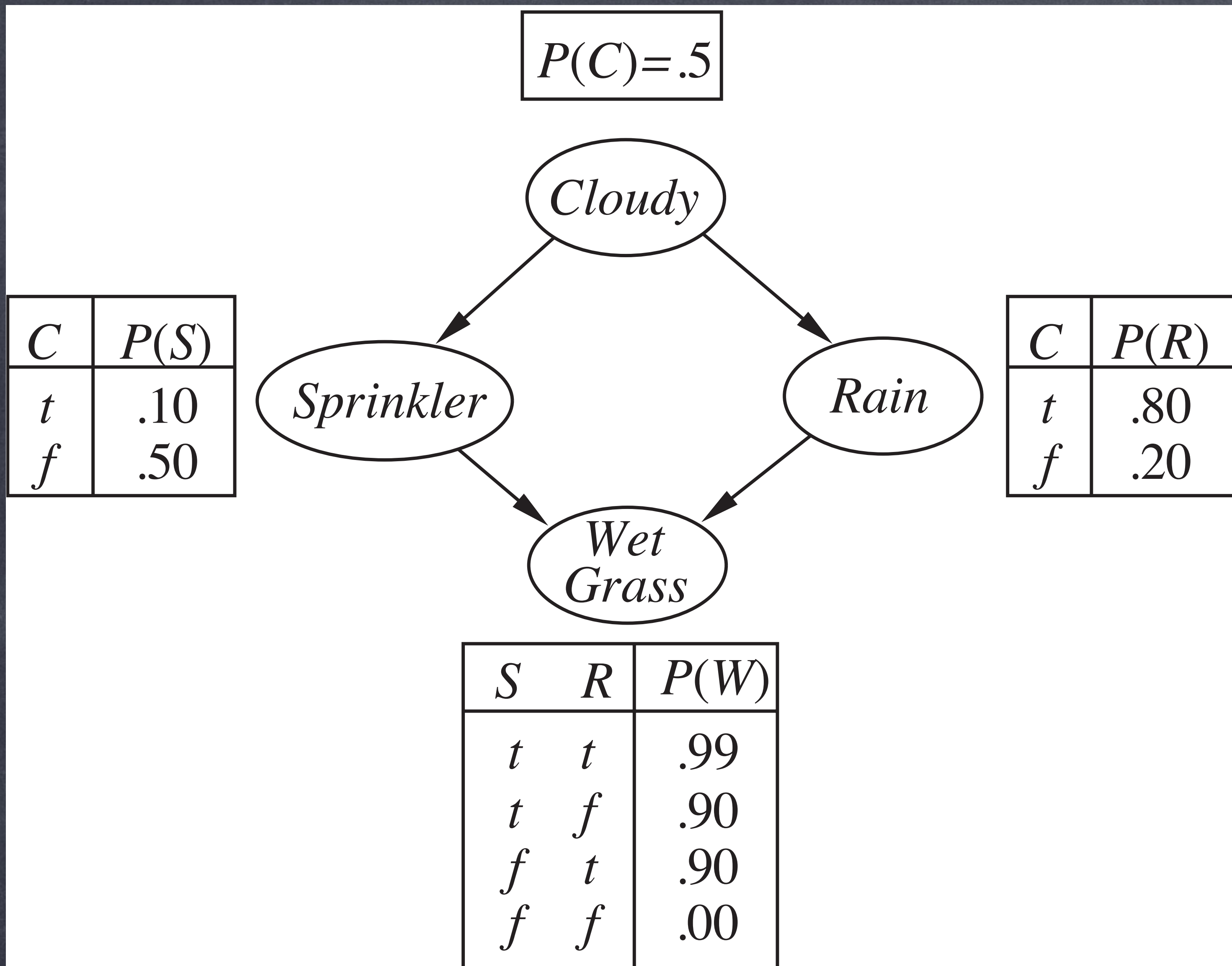
P(Rain | Sprinkler=true, Rain=false, Cloudy=false)



Cloudy	false
Sprinkler	true
Rain	true
WetGrass	true

$$\mathbf{P}(Rain \mid Sprinkler = true, WetGrass = true)$$

$N_{\text{Rain=true}} \mathrel{+}= 1$



Cloudy	false
Sprinkler	true
Rain	true
WetGrass	true

$$\mathbf{P}(Rain \mid Sprinkler = true, WetGrass = true)$$

Gibbs Sampling

- To approximate: $P(X \mid e)$
- Start in a state with evidence variables set to evidence values (others arbitrary)
- On each step, sample non-evidence variables conditioned on the values of the variables in their Markov Blanket
    - Order irrelevant
- A form of local search!

Exact Inference in Bayesian Networks

$$\mathbf{P}(X \mid \mathbf{e}) = \alpha\, \mathbf{P}(X, \mathbf{e}) = \alpha \sum_{\mathbf{y}} \mathbf{P}(X, \mathbf{e}, \mathbf{y})$$

$$= \alpha \sum_{\mathbf{y}} \prod_{i=1}^{n} P(X_i \mid parents(X_i))$$

- Intractable (#P-Hard)

Approximate Inference in Bayesian Networks

- Sampling consistent with a distribution
- Rejection Sampling: rejects too much
- Likelihood Weighting: weights get too small
- Gibbs Sampling: MCMC algorithm
  - Similar to local search
- All generate consistent estimates (equal to exact probability in the large-sample limit)