Amath 586: Homework 1

Due on April 9, 2015

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Problem 1

Prove that the ODE

$$u'(t) = \frac{1}{t^2 + u(t)^2}, \quad \text{for } t \geq 1$$

has a unique solution for all time from any initial value $u(1) = \eta$.

Solution:
Let $f(u,t) = u'(t)$, $D = \mathbb{R} \times [1, \infty)$, and observe that $0 \leq f(u,t) \leq 1$ for $t \geq 1$ as $t^2 + u(t)^2 > 1$ for $(u,t) \in D$. Also notice that for $(u,t) \in D$

$$\frac{\partial f}{\partial u} = f_u = \frac{-2u}{(t^2 + u(t)^2)^2},$$

so

$$|f_u| = \frac{2|u|}{(t^2 + u(t)^2)^2} \leq \frac{2|u|}{(1 + u(t)^2)^2} \leq \frac{3\sqrt{3}}{8}.$$

The bound above can be obtained by locating the local maxima and minima of $g(u) = \frac{2u}{(1+u^2)^2}$.
Since f is differentiable with respect to u for any $(u,t) \in D$, and f_u is bounded for $(u,t) \in D$ we can apply the result from Section 5.2.1 which states that if f is Lipschitz continuous over a domain D, then there is a unique solution to the initial value problem above at least up to time $T^* = min(t_1, t_0 + a/S)$, where

$$S = \max_{(u,t) \in D} |f(u,t)| \leq 1,$$

and $a,\ t_0$, and t_1 appear in the definition of D as

$$D \ = \ \{(u,t) : \ |u - \eta| \leq a, \ t_0 \leq t \leq t_1\}$$

(they specify the domain over which f is Lipschitz continuous). But the calculations above show that $t_0 = 1$ and that a and t_1 can be taken to be arbitrarily large. It follows that $T^* = \infty$, i.e. that there exists a unique solution to the provided initial value problem for any η, and for all $t \geq 1$.

Problem 2

Consider the system of ODEs

$$u_1' = 3u_1 + 4u_2,$$
$$u_2' = 5u_1 - 6u_2.$$

Determine the best possible Lipschitz constant for this system in the max-norm $\|\cdot\|_\infty$ and the 1-norm $\|\cdot\|_1$. (See Appendix A.3.)

Solution:
This system can be written as $\mathbf{u}' = A\mathbf{u}$ where

$$A = \begin{bmatrix} 3 & 4 \\ 5 & -6 \end{bmatrix}, \ \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

Recall that for any vector norm $\|\cdot\|$ inducing a submultiplicative matrix norm we have

$$\|A\mathbf{x} - A\mathbf{y}\| = \|A(\mathbf{x} - \mathbf{y})\| \leq \|A\| \ \|\mathbf{x} - \mathbf{y}\|$$

for any $\mathbf{x}, \mathbf{y}$. The (matrix) 1-norm and ∞-norm are just the maximum absolute column and row sums, respectively, so $\|A\|_1 = 4 + 6 = 10$ and $\|A\|_\infty = 5 + 6 = 11$. Since the 1-norm and ∞-norm induce submultipliative matrix norms,

$$\|A\mathbf{u} - A\mathbf{u}^*\|_1 \leq \|A\|_1 \ \|\mathbf{u} - \mathbf{u}^*\|_1 = 10\|\mathbf{u} - \mathbf{u}^*\|_1,$$

$$\|A\mathbf{u} - A\mathbf{u}^*\|_\infty \leq \|A\|_\infty \ \|\mathbf{u} - \mathbf{u}^*\|_\infty = 11\|\mathbf{u} - \mathbf{u}^*\|_\infty.$$

Thus 10 and 11 are Lipschitz constants for the system with respect to the 1 and infinity norms, respectively. To see that they are the best possible Lipshitz constants, notice that equality is attained in the above two equations for

$$\mathbf{u} - \mathbf{u}^* = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ \ \mathbf{u} - \mathbf{u}^* = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

which are unit vectors with respect to the 1 and infinity norms, respectively.

Problem 3

The initial value problem

$$v''(t) = -4v(t), \qquad v(0) = v_0, \quad v'(0) = v_0'$$

has the solution $v(t) = v_0 \cos(2t) + \frac{1}{2} v_0' \sin(2t)$. Determine this solution by rewriting the ODE as a first order system $u' = Au$ so that $u(t) = e^{At}u(0)$ and then computing the matrix exponential using (D.30) in Appendix D.

Solution:
Letting $x(t) = v(t)$ and $y(t) = v'(t)$, we can rewrite the initial value problem as the following system

$$\begin{aligned} x' &= y, \\ y' &= -4x \end{aligned}$$

with initial conditions $x(0) = v_0, y(0) = v_0'$. In turn, we can represent this system as the matrix equation $\mathbf{u}' = A\mathbf{u}$, with

$$\mathbf{u} = \begin{bmatrix} x \\ y \end{bmatrix}, A = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}.$$

Then the solution to the system is given by

$$\mathbf{u}(t) = e^{At}\mathbf{u}(0).$$

In order to compute the matrix exponential e^{At} we will need the Jordan decomposition of A.
To this end $det(A - \lambda I) = \lambda^2 + 4$, so the eigenvalues of A are simply $\pm 2i$. It can easily be shown that the eigenvectors of A are scalar multiples of

$$v_1 = \begin{bmatrix} i \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} -i \\ 2 \end{bmatrix}.$$

Letting $R = [v_1,\ v_2]$ we see that $A = R\Lambda R^{-1}$, where

$$\Lambda = \begin{bmatrix} -2i & 0 \\ 0 & 2i \end{bmatrix}.$$

Hence

$$\begin{aligned} e^{At} &= e^{R\Lambda R^{-1}} \\ &= Re^{\Lambda t}R^{-1} \\ &= \begin{bmatrix} i & -i \\ 2 & 2 \end{bmatrix} \begin{bmatrix} e^{-2it} & 0 \\ 0 & e^{2it} \end{bmatrix} \begin{bmatrix} \frac{-i}{2} & \frac{1}{4} \\ \frac{i}{2} & \frac{1}{4} \end{bmatrix} \\ &= \begin{bmatrix} \cos(2t) & \frac{1}{2}\sin(2t) \\ -2\sin(2t) & \cos(2t) \end{bmatrix}. \end{aligned}$$

Therefore,

$$\mathbf{u} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} v \\ v' \end{bmatrix} = e^{At}\mathbf{u}(0) = \begin{bmatrix} \cos(2t) & \frac{1}{2}\sin(2t) \\ -2\sin(2t) & \cos(2t) \end{bmatrix} \begin{bmatrix} v_0 \\ v_0' \end{bmatrix} = \begin{bmatrix} v_0\cos(2t) + \frac{v_0'}{2}\sin(2t) \\ -2v_0\sin(2t) + v_0'\cos(2t) \end{bmatrix},$$

from which we get $v(t) = v_0\cos(2t) + \frac{v_0'}{2}\sin(2t)$.

Problem 4

Compute the leading term in the local truncation error of the following methods:

(a) the trapezoidal method (5.22),

(b) the 2-step Adams-Bashforth method,

(c) the Runge-Kutta method (5.32).

Solution:

Throughout this problem I will use u to denote $u(t_n)$, f for $f(u(t_n))$, etc. when it does not add confusion to do so. The following Taylor expansions will be useful later in the problem:

$$u(t+k) = u(t) + ku'(t) + \frac{k^2}{2}u''(t) + \frac{k^3}{3!}u'''(t) + O(k^4) \tag{1}$$

$$u(t-k) = u(t) - ku'(t) + \frac{k^2}{2}u''(t) - \frac{k^3}{3!}u'''(t) + O(k^4) \tag{2}$$

$$f(u(t+k)) = u'(t) + ku''(t) + \frac{k^2}{2}u'''(t) + \frac{k^3}{3!}u^{(4)}(t) + O(k^4) \tag{3}$$

$$f(u(t-k)) = u'(t) - ku''(t) + \frac{k^2}{2}u'''(t) - \frac{k^3}{3!}u^{(4)}(t) + O(k^4) \tag{4}$$

(a) The local truncation error for the trapezoidal method (5.22) is given by

$$\tau^n = \frac{u(t_n+k) - u(t_n)}{k} - \frac{1}{2}(f(u(t_n)) + f(u(t_n+k))).$$

Substituting (1) and (3) for $u(t_n+k)$ and $f(u(t_n+k))$ and cancelling some terms yields

$$\begin{aligned}
\tau^n &= \frac{ku' + \frac{k^2}{2}u'' + \frac{k^3}{3!}u''' + O(k^4)}{k} - \frac{1}{2}(2u' + ku'' + \frac{k^2}{2}u''' + O(k^3)) \\
&= u' + \frac{k}{2}u'' + \frac{k^2}{3!}u''' - (u' + \frac{k}{2}u'' + \frac{k^2}{4}u''') + O(k^3) \\
&= \frac{-k^2}{12}u'''(t_n) + O(k^3).
\end{aligned}$$

(b) The local truncation error for the 2-step Adams-Bashforth method is given by

$$\tau^n = \frac{u(t_n+k) - u(t_n)}{k} - \frac{1}{2}(3f(u(t_n)) - f(u(t_n-k))).$$

Problem 4 continued on next page...

Substituting (1) and (4) for $u(t_n + k)$ and $f(u(t_n - k))$ and cancelling some terms gives

$$\begin{aligned}
\tau^n &= \frac{ku' + \frac{k^2}{2}u'' + \frac{k^3}{3!}u''' + O(k^4)}{k} - \frac{1}{2}(2u' + ku'' - \frac{k^2}{2}u''') + O(k^3) \\
&= u' + \frac{k}{2}u'' + \frac{k^2}{3!}u''' - (u' + \frac{k}{2}u'' - \frac{k^2}{4}u''') + O(k^3) \\
&= \frac{5}{12}k^2 u'''(t_n) + O(k^3).
\end{aligned}$$

(c) The local truncation error for the 2-stage Runge-Kutta method (5.32) is given by

$$\tau^n = \frac{u(t_n + k) - u(t_n)}{k} - f\left(u(t_n) + \frac{k}{2}f(u(t_n), t_n),\ t_n + \frac{k}{2}\right).$$

In this problem I use f to denote $f(u(t), t)$, f_t to denote $\frac{\partial f}{\partial t}(u(t), t)$, etc. when doing so does not add ambiguity. We need to use slightly different Taylor series expansions than before to find the leading term of the local truncation error for this method. First, we have

$$\begin{aligned}
u''(t) &= \frac{d}{dt}(f(u, t)) = f_t + f f_u, \\
u'''(t) &= \frac{d}{dt}(u''(t)) \\
&= \frac{d}{dt}(f_t + f f_u) \\
&= f_{tt} + 2f f_{ut} + f_{uu}f^2 + (f_u)^2 f + f_u f_t.
\end{aligned}$$

Using these and the multidimensional Taylor theorem we obtain

$$\begin{aligned}
u(t + k) &= u + ku' + \frac{k^2}{2}u'' + \frac{k^3}{3!}u''' + O(k^4) \\
&= u(t) + kf + \frac{k^2}{2}(f_t + f f_u) \\
&\quad + \frac{k^3}{3!}(f_{tt} + 2f f_{ut} + f_{uu}f^2 + (f_u)^2 f + f_u f_t) + O(k^4).
\end{aligned}$$

$$f\left(u(t) + \frac{k}{2}f(u(t), t),\ t + \frac{k}{2}\right) =$$

$$f + \frac{k}{2}f f_u + \frac{k}{2}f_t + \frac{1}{2}(\frac{k}{2}f)^2 f_{uu} + \frac{k^2}{8}f_{tt} + \frac{k}{2}(\frac{k}{2}f)f_{ut} + O(k^3).$$

Substituting the above Taylor expansions into the expression for the local truncation error and cancelling some terms gives

$$\begin{aligned}
\tau^n &= \frac{kf + \frac{k^2}{2}(f_t + ff_u) + \frac{k^3}{3!}(f_{tt} + 2ff_{ut} + f_{uu}f^2 + (f_u)^2 f + f_u f_t)}{k} \\
&- (f + \frac{k}{2}ff_u + \frac{k}{2}f_t + \frac{1}{2}(\frac{k}{2}f)^2 f_{uu} + \frac{k^2}{8}f_{tt} + \frac{k}{2}(\frac{k}{2}f)f_{ut}) + O(k^3) \\
&= f + \frac{k}{2}(f_t + ff_u) + \frac{k^2}{3!}(f_{tt} + 2ff_{ut} + f_{uu}f^2 + (f_u)^2 f + f_u f_t) \\
&- (f + \frac{k}{2}(ff_u + f_t) + \frac{k^2}{8}(f^2 f_{uu} + f_{tt} + 2ff_{ut})) + O(k^3) \\
&= \frac{k^2}{6}(\frac{1}{4}f_{tt} + \frac{1}{2}ff_{ut} + \frac{1}{4}fuuf^2 + (f_u)^2 f + f_u f_t) + O(k^3).
\end{aligned}$$

Problem 5

Determine the coefficients $\beta_0,\ \beta_1,\ \beta_2$ for the third order, 2-step Adams-Moulton method. Do this in two different ways:

(a) Using the expression for the local truncation error in Section 5.9.1,

(b) Using the relation

$$u(t_{n+2}) = u(t_{n+1}) + \int_{t_{n+1}}^{t_{n+2}} f(u(s))\,ds.$$

Interpolate a quadratic polynomial $p(t)$ through the three values $f(U^n),\ f(U^{n+1})$ and $f(U^{n+2})$ and then integrate this polynomial exactly to obtain the formula. The coefficients of the polynomial will depend on the three values $f(U^{n+j})$. It's easiest to use the "Newton form" of the interpolating polynomial and consider the three times $t_n = -k$, $t_{n+1} = 0$, and $t_{n+2} = k$ so that $p(t)$ has the form

$$p(t) = A + B(t+k) + C(t+k)t$$

where $A,\ B$, and C are the appropriate divided differences based on the data. Then integrate from 0 to k. (The method has the same coefficients at any time, so this is valid.)

Solution:

(a) The expression for the local truncation error for a 2-step Adams-Moulton method given in section 5.9.1 is

$$\begin{aligned}
\tau(t_{n+2}) = \frac{1}{k}\left(\sum_{j=0}^{2}\alpha_j\right)u(t_n) + \left(\sum_{j=0}^{2}(j\alpha_j - \beta_j)\right)u'(t_n) \\
+ k\left(\sum_{j=0}^{2}(j^2\alpha_j - j\beta_j)\right)u''(t_n) + k^2\left(\sum_{j=0}^{2}(j^3\alpha_j - j^2\beta_j)\right)u'''(t_n) + O(k^3).
\end{aligned}$$

Problem 5 continued on next page...

We already know $\alpha_0 = 0, \alpha_1 = -1, \alpha_2 = 1$ (hence the $O(\frac{1}{k})$ term drops out), so we must choose $\beta_i,\ i = 1,2,3$ so that the $O(1),\ O(k)$, and $O(k^2)$ terms vanish (if we try to eliminate the $O(k^3)$ term we end up with an overdetermined system of equations which we cannot solve). Each sum gives an equation:

$$\begin{aligned}
\left(\sum_{j=0}^{2}(j\alpha_j - \beta_j)\right) u'(t_n) &= (-\beta_0 - \beta_1 - \beta_2 + 1)u'(t_n) = 0 &&\implies \beta_0 + \beta_1 + \beta_2 = 1\\
k\left(\sum_{j=0}^{2}(j^2\alpha_j - j\beta_j)\right) u''(t_n) &= k(-\beta_1 - 2\beta_2 + \frac{3}{2})u''(t_n) = 0 &&\implies \beta_1 + 2\beta_2 = \frac{3}{2}\\
k^2\left(\sum_{j=0}^{2}(j^3\alpha_j - j^2\beta_j)\right) u'''(t_n) &= k^2(-\frac{1}{2}\beta_1 - 2\beta_2 + \frac{7}{6})u'''(t_n) = 0 &&\implies \frac{1}{2}\beta_1 + 2\beta_2 = \frac{7}{6}.
\end{aligned}$$

Solving these equations simultaneously for β_0, β_1, and β_2 yields $\beta_0 = -\frac{1}{12},\ \beta_1 = \frac{2}{3}$, and $\beta_2 = \frac{5}{12}$, giving the 2-step Adams-Moulton method:

$$U^{n+2} = U_{n+1} + \frac{k}{12}(-f(U^n) + 8f(U^{n+1}) + 5f(U_{n+2})).$$

(b) Taking $t_n = -k,\ t_{n+1} = 0$, and $t_{n+2} = k$, the "Newton form" (and using divided differences notation) of the quadratic polynomial interpolating $f(U^n),\ f(U^{n+1})$ and $f(U^{n+2})$ is given by

$$p(t) = [f(U^n)] + [f(U^n), f(U^{n+1})](t+k) + [f(U^n), f(U^{n+1}), f(U^{n+1})](t+k)t.$$

The divided differences are

$$\begin{aligned}
[f(U^n)] &= f(U^n),\\
[f(U^n), f(U^{n+1})] &= \frac{f(U^{n+1}) - f(U^n)}{k},\\
[f(U^n), f(U^{n+1}), f(U^{n+2})] &= \frac{[f(U^{n+1}), f(U^{n+2})] - [f(U^n), f(U^{n+1})]}{2k} = \frac{f(U^{n+2}) + f(U^n) - 2f(U^{n-1})}{2k^2}.
\end{aligned}$$

Integrating $p(t)$ from 0 to k gives

$$\begin{aligned}
\int_0^k p(t)dt &= kf(U^n) + \frac{f(U^{n+1}) - f(U^n)}{k}(\frac{1}{2}t^2 + kt)\Big|_0^k + \frac{f(U^{n+2}) + f(U^n) - 2f(U^{n-1})}{2k^2}(\frac{1}{3}t^3 + \frac{1}{2}t^2k)\Big|_0^k\\
&= kf(U^n) + (f(U^{n+1}) - f(U^n))\frac{3k}{2} + (f(U^{n+2}) + f(U^n) - 2f(U^{n+1}))\frac{5k}{12}\\
&= \frac{k}{12}(-f(U^n) + 8f(U^{n+1}) + 5f(U^{n+2})).
\end{aligned}$$

From this it follows that the coefficients $\beta_0,\ \beta_1,\ \beta_2$ are $\beta_0 = \frac{-1}{12},\ \beta_1 = \frac{2}{3},\ \beta_2 = \frac{5}{12}$, as before.

Problem 6

The initial value problem

$$\begin{aligned}
u'(t) &= u(t)^2 - \sin(t) - \cos^2(t),\\
u(0) &= 1
\end{aligned} \tag{5}$$

has the solution $u(t) = \cos(t)$.
Write a computer code (preferably in Python or Matlab) to solve problem (5) up to time $T = 8$ with various different time steps $\Delta t = T/N$, with

$$N = 25,\ 50,\ 100,\ 200,\ 400,\ 800,\ 1600,\ 3200.$$

Do this using two different methods:

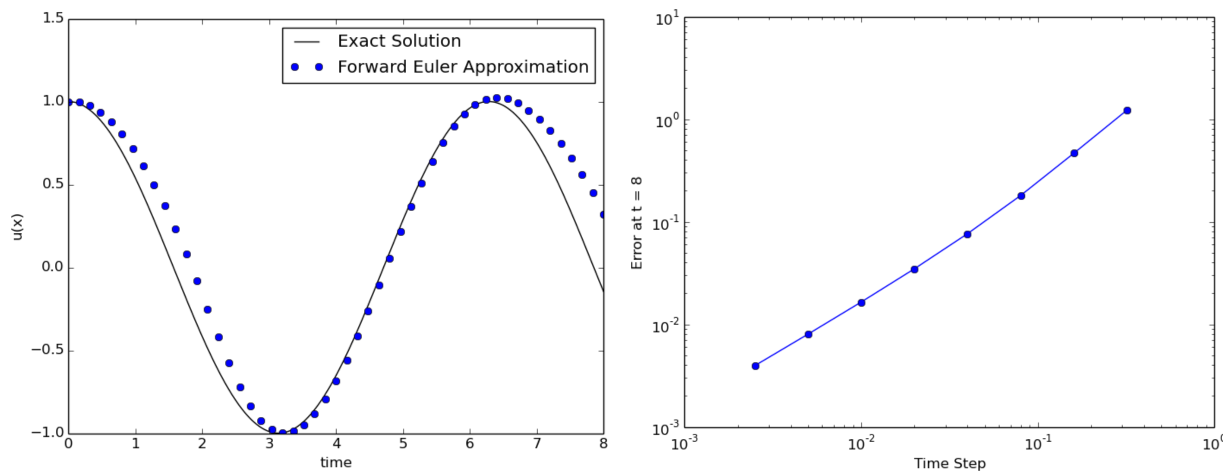
(a) Forward Euler

(b) The Runge-Kutta method (5.32). Note that this should be second order accurate for sufficiently small Δt. If not, then you might have a bug.

Produce a log-log plot of the errors versus Δt, with both plots in the same figure.

Solution:

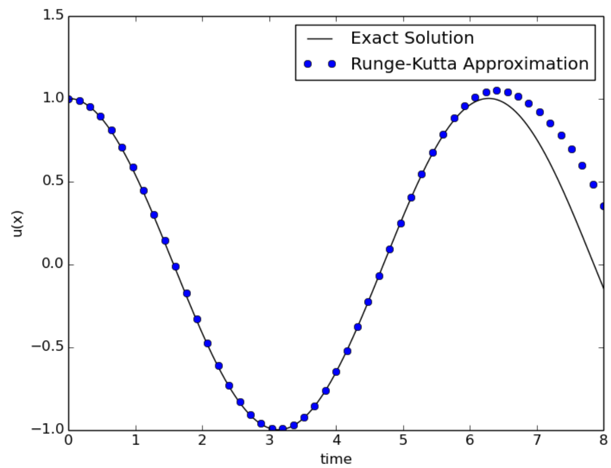
Figure 1: Forward Euler method



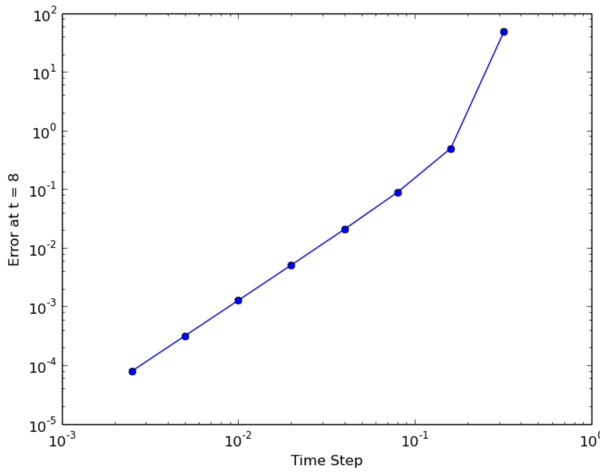
(a) The Forward Euler solution with $N = 50$ (b) Log-log plot of the error in Forward Euler

As the timestep is decreased in both cases, the error at $t = 8$ begins decreasing as h for the Forward Euler method and as h^2 for the Runge-Kutta method.

Figure 2: 2-stage Runge-Kutta method



(a) The Runge-Kutta solution with $N = 50$



(b) Log-log plot of the error in 2-stage Runge-Kutta