Brightness of Lunar and Planetary Phases

In this notebook, the normalization integral I didn't complete at the whiteboard is done correctly.

Preliminaries

Let L be the luminosity of the Sun (power, in mks units, Watts, in cgs units, ergs/second). Let r be the distance of the moon or planet being observed from the Sun. Let R be the radius of this planet, and a be its albedo. Then the amount of sunlight reflected off the planet is:

$$a \frac{L}{4\pi r^2} \pi R^2$$

To motivate what we are calculating, here is a picture of the waning gibbous moon (taken Wednesday, April 4th, a little after dawn):

Here is a better picture of the waning gibbous moon (Credit: Thomas Bresson):

Think of the gibbous moon as a simple two-dimensional shape without knowing that it is a round ball lit up from the side.

The area that is lit up is a half of a disk plus a half of an ellipse! (I put an explanation point here because I never knew that before Swihart helped us think about.)

The part that is half of a disk has area $\frac{1}{2}\pi R^2$. The rest, in class, we convincingly found was $\frac{1}{2}\pi R^2 \cos\alpha$, where α is how much you would have to rotate the moon to be squarely facing the fully lit side.

So the total amount of the moon that is lit is $\frac{1}{2}\pi R^2(1+\cos\alpha)$.

This works for the crescent moon, not just the gibbous moon. The full moon has $\alpha = 0$, the waxing and waning gibbous moon phases have $0 < \alpha < \frac{\pi}{2}$, the first and third quarter moon have $\alpha = \frac{\pi}{2}$, the waxing and waning crescent phases have $\frac{\pi}{2} < \alpha < \pi$ and the new moon has $\alpha = \pi$.

Swihart's big assumption that allows him to proceed from these preliminaries to an answer is that all of the lit up part is equally bright. As you can see from the picture, this assumption isn't great — the band near the terminator looks dimmer than the rest because the sunlight is falling on it at an extreme angle — but we will run with it.

Normalization

If we are s away from the moon, the amount of light per unit area landing on our eyes or telescope mirrors has a factor of $\frac{1}{4\,\pi s^2}$. This is the naive factor presuming that the light goes off evenly in all directions out to a giant sphere of radius s.

Here are all the factors we have so far:

$$\frac{1}{2}\,\pi R^2(1+\cos\alpha)\,\frac{1}{4\,\pi s^2}\,l$$

I put in a to-be-determined additional factor l. Normalization is how we are going to determine it. We know that if we integrate the above expression over a giant sphere of radius s (meaning all directions that the moon can be viewed) it must give the total amount of reflected light:

$$a\,\frac{L}{4\,\pi r^2}\,\pi R^2$$

Let's add an axial angle ϕ (because for every angle α, there are a range of $2\,\pi$ of angle ϕ from which the moon can be viewed. The equation that determines the normalization is:

$$a\,\frac{L}{4\,\pi r^2}\,\pi R^2 = \int_0^\pi \int_0^{2\,\pi} \frac{1}{2}\,\pi R^2(1+\cos\alpha)\,\frac{1}{4\,\pi s^2}\,l\,s^2\sin\alpha\,d\phi\,d\alpha$$

The only part of the integrand you might be skeptical about is $s^2\sin\alpha\,d\phi\,d\alpha$, and you realize that that is the infinitesimal bit of the surface area of a sphere of radius s.

The s^2 and the R^2 cancel. The ϕ integration just gives $2\,\pi$.

$$a\,\frac{L}{r^2} = 2\,\pi\,l\int_0^\pi \frac{1}{2}(1+\cos\alpha)\sin\alpha\,d\alpha$$

For some reason, Swihart suggests we fob that integration off on a grandparent, but I will fob it off onto Mathematica (because I need the practice using Mathematica more than I need the practice doing trig identities):

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Integrate[1/2 (1 + Cos[α]) Sin[α], {α, 0, π}]
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Now that was just awesomely convenient, and it means that:

$$a\,\frac{L}{r^2} = 2\,\pi\,l$$

or

$$l = \frac{1}{2\pi} a \frac{L}{r^2}$$

Put that into:

$$\frac{1}{2} \pi R^2 (1 + \cos\alpha) \frac{1}{4\pi s^2} l = \frac{1}{2} \pi R^2 (1 + \cos\alpha) \frac{1}{4\pi s^2} \frac{1}{2\pi} a \frac{L}{r^2} = \frac{1}{16\pi r^2 s^2} R^2 (1 + \cos\alpha) aL$$

Happily, this is exactly what Swihart got in equation 5.2-6 on p. 99.