Black Holes, Worksheet I for Thursday, Oct. 10

The Runner-on-the-Train Paradox (Taylor and Wheeler, Problem 5-7)

Statement of the Paradox

A runner is running backwards through a moving train and the runner carries a clock that ticks with period T (according to the runner). The runner runs with speed v with respect to the train. (Let's make that the $-x$ direction, so if you use the whole Lorentz transformation machinery to solve this problem, you need to put the runner's frame as moving with $v_{\text{rel}} = -v$ with respect to the train.) The train is moving with speed v (in the $+x$ direction) relative to the Earth.

The paradox is the following argument:

(i) The runner's clock appears to tick with period $\gamma\, T$ according to observers moving with the train (this is just time dilation).
(ii) We also know that a train-frame clock with period T' appears to tick with period $\gamma\, T'$ according to observers on the Earth (this is just time dilation applied again).
(iii) That means something like the runner's clock, which the train-frame observers measure as ticking with period $T' = \gamma\, T$, should tick with period $\gamma^2\, T$ according to the Earth-frame observers (time dilation compounded).
(iv) But that can't be right! The runner is stationary with respect to the Earth, so of course the Earth observers say that the runner's clock ticks with period T, not $\gamma^2\, T$.
(v) So relativity is inconsistent and should be tossed!

Resolution of the Paradox

The first two steps in the argument above are fine. The resolution of the paradox is that the third step is invalid. To make that clearer, let's reword the second step.

(ii) An object that is is at rest for train-frame observers and ticks with period T' should tick with period $\gamma\, T'$ according to the Earth-frame observers.

Now it is obvious that the third step is invalid; the runner's clock is not at rest for the train-frame observers.

1. A Valid Version of the Argument

(i) The runner’s clock appears to tick with period $\gamma\, T$ according to observers moving with the train.
(ii) An object that is is at rest for train-frame observers and ticks with period T' should tick with period $\gamma\, T'$ according to the Earth-frame observers.
(iii) The runner’s clock is not at rest with respect to the train frame clocks. In a period T' it has moved through the clocks by an amount $L' = v\, T'$ in the $-x$ direction.
(iv) An Earth-frame observer says that the line of clocks that the train-frame observers have synchronized are not actually synchronized. In fact, a clock that is L' to the left another clock is $\gamma\, L'\, v$ behind.

(a) Finish the argument. Substitute in $L' = v\, T'$ and $T' = \gamma\, T$ to find out what $\gamma\, L'\, v$ is in terms of T.

(b) The clock to the left is not synchronized, and therefore the tick rate that was found in the paradoxical and wrong argument, $\gamma^2\, T$, has to be corrected by the amount found in (a). Subtract whatever you got in (a) from $\gamma^2\, T$ and simplify.

2. Putting in Some Numbers

Let’s say $T = 16$ s and $v = 3/5$. Use those numbers to get answers for each part below.

(a) What is γ?

(b) What is $T' = \gamma\, T$?

(c) What is $L' = v\, T'$? (Of course, be convenient, and state your answer for L' in seconds.)

(d) What is $\gamma\, v\, L' =$

(e) What is $\gamma^2\, T - \gamma\, v\, L'$

3. Making a Drawing

Earth-frame observers say that clocks that are L' apart and synchronized in the train frame are only L'/γ apart.

(a) Continue plugging in numbers to find L'/γ.

(b) In the large amount of space below, make a drawing of seven train-frame clocks as viewed in the Earth frame. Have each clock just be a blank circle. Put some whooshing streaks behind each clock. Show the distance between the clocks according to the Earth observers.

(c) On the drawing you started in (b), draw in the second hand on the center of the seven clocks, and have it point up.

(d) On the drawing you started in (b) and (c), draw in the second hand on the clock to the left of the center clock using your answer to 2(d).

(e) Finish the drawing you made in (b), (c) and (d), by putting in the second hand on the remaining five clocks.