Rain Frame Light Cones

Each coordinate system change sheds new light on the properties of the black hole. In Problem Set 15, you derived the rain-frame metric:

$$(\Delta \tau)^2 = \left(1 - \frac{2M}{r}\right)(\Delta t_{\text{rain}})^2 - 2\sqrt{\frac{2M}{r}}\,\Delta t_{\text{rain}}\,\Delta r - (\Delta r)^2 - r^2(\Delta\phi)^2$$

and you derived $\left(\frac{\Delta r}{\Delta t_{\text{rain}}}\right)_{\pm} = -\sqrt{\frac{2M}{r}} \pm 1$. Of course all these derivations were only valid for small displacements, so now we may finally take the limit that the displacements go to zero, and we have found:

$$\left(\frac{dr}{dt_{\text{rain}}}\right)_{\pm} = -\sqrt{\frac{2M}{r}} \pm 1.$$

Going back to p. 3-22 where we were calling τ the wristwatch time of the raindrop, rather than calling it t_{rain}, we learned that

$$\tau_2 - \tau_1 = -\int_{r_1}^{r_2} \frac{\sqrt{r}}{\sqrt{2M}}\,dr = -\frac{1}{\sqrt{2M}}\,\frac{2}{3}\left(r_2{}^{3/2} - r_1{}^{3/2}\right) = -\frac{4}{3}\,M\left(\left(\frac{r_2}{2M}\right)^{3/2} - \left(\frac{r_1}{2M}\right)^{3/2}\right)$$

On Figure 5 on p. B-15, Taylor and Wheeler choose a raindrop that had $\tau_1 = 0$ at $r_1 = 5\,M$. Let a new variable be $x = \frac{r_2}{M}$. Then plugging these three facts in, and recognizing that what we were calling τ_2 we are now calling t_{rain}, we have

$$t_{\text{rain}} - 0 = -\frac{4}{3}\,M\left(\left(\frac{x}{2}\right)^{3/2} - \left(\frac{5}{2}\right)^{3/2}\right)$$

or

$$\frac{t_{\text{rain}}}{M} = \frac{4}{3}\left[\left(\frac{5}{2}\right)^{3/2} - \left(\frac{x}{2}\right)^{3/2}\right]$$

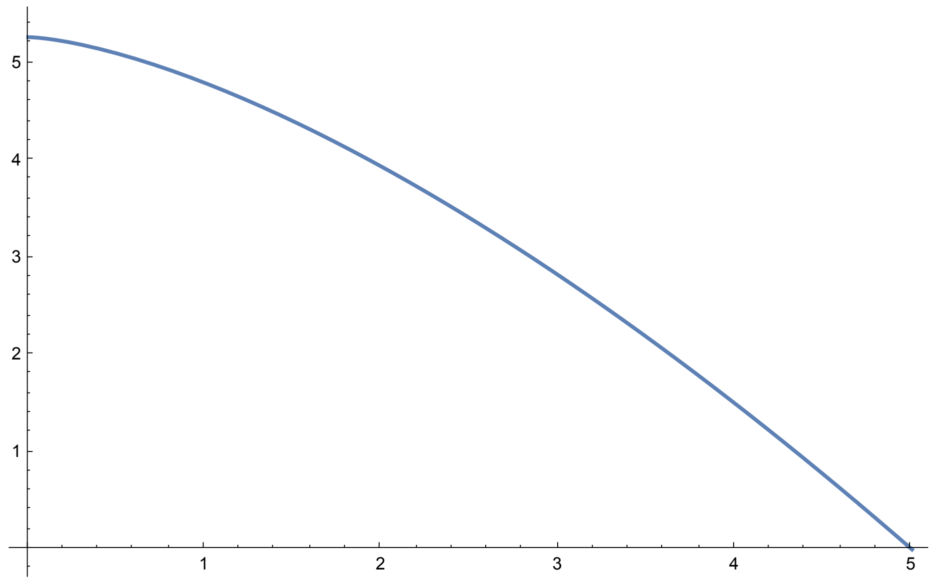
The 3/2 power is a a bit of a mess, so let's plot it:

In[164]:=

$$\texttt{tRainOverM[x\_, rZero\_, tOffset\_] := tOffset} + \frac{4}{3}\left(\left(\frac{\texttt{rZero}}{2}\right)^{3/2} - \left(\frac{\texttt{x}}{2}\right)^{3/2}\right);$$

```
Plot[tRainOverM[x, 5, 0], {x, 0, 5}]
```

Out[165]=

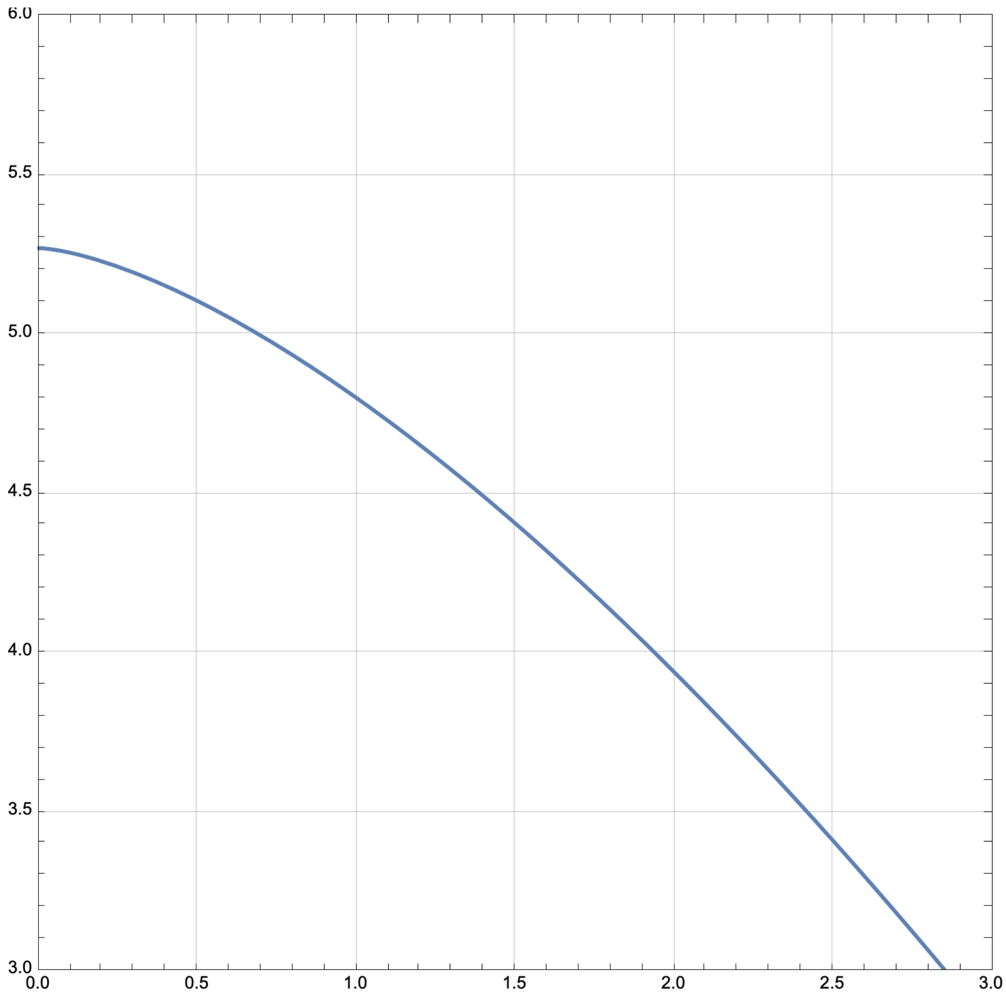


If we want it to be a little more obvious that the function we have plotted is exactly the same as theirs, we can adjust the plot range and the aspect ratio, and label the axes:

In[166]:=

```
plot[zero_] :=
 Plot[tRainOverM[x, zero, 0], {x, 0, 3}, PlotRange → {{0, 3}, {3, 6}},
  AspectRatio → 1, AxesLabel → {r / M, t_rain / M}, GridLines → Automatic, Frame → True]
plot[5]
```

Out[167]=

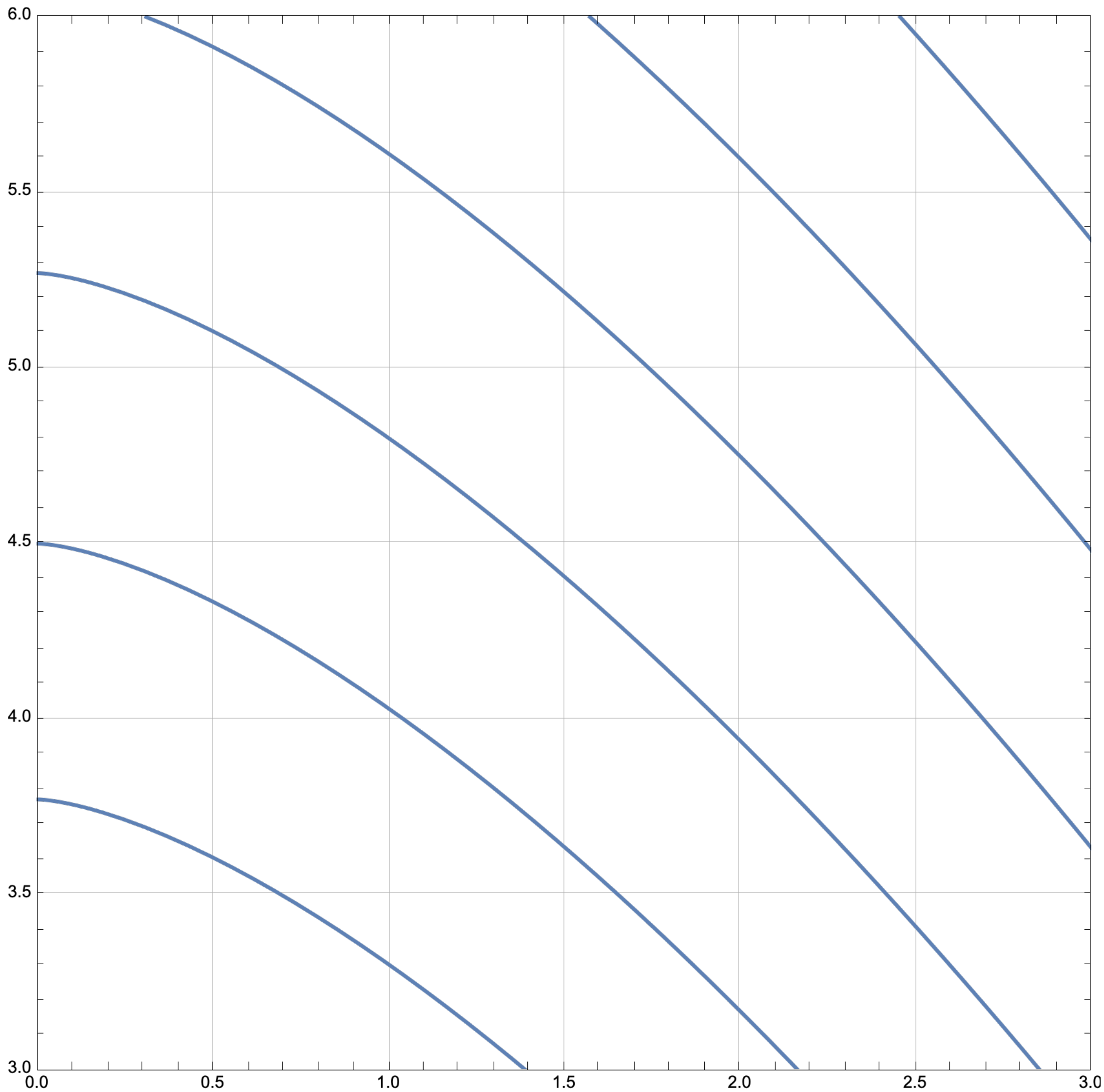


Let’s add the paths for raindrops whose wristwatches were zero when they were at 4.0 M, 4.5 M, 5.5 M, 6M, and 6.5 M, respectively:

In[168]:=

```
Show[plot[4.0], plot[4.5], plot[5.0], plot[5.5], plot[6], plot[6.5]]
```

Out[168]=



I apologize that the axes aren’t actually labeled. There is some conflict between the labels and the frame. Sigh. The horizontal axis is r/M. The vertical axis is t_{rain}/M.

Ink in the following five points on the original of the six lines:

In[169]:=

```
middle[x_] := Round[tRainOverM[x, 5, 0], 0.01];
TableForm[{
  {0.5, middle[0.5]},
  {1.0, middle[1.0]},
  {1.5, middle[1.5]},
  {2.0, middle[2.0]},
  {2.5, middle[2.5]}
 }]
```

Out[170]//TableForm=

0.5	5.1
1.	4.8
1.5	4.4
2.	3.94
2.5	3.41

We computed the slopes, $\left(\frac{dr}{dt_{\text{rain}}}\right)_{\pm}$. Because the horizontal axis has r and the vertical axis has t, we really want the inverse of these slopes. In terms of $x = \frac{r}{M}$, the inverse slopes are:

In[171]:=

```
slopePlus[x_] := Round[1/ (-1/Sqrt[x/2] + 1), 0.01];
slopeMinus[x_] := Round[1/ (-1/Sqrt[x/2] - 1), 0.01];
TableForm[{
  {0.5, slopePlus[0.5], slopeMinus[0.5]},
  {1.0, slopePlus[1.0], slopeMinus[1.0]},
  {1.5, slopePlus[1.5], slopeMinus[1.5]},
  {2.0, slopePlus[2.000002], slopeMinus[2.0]},
  {2.5, slopePlus[2.5], slopeMinus[2.5]}
 }]
```

Out[173]//TableForm=

0.5	-1.	-0.33
1.	-2.41	-0.41
1.5	-6.46	-0.46
2.	$2. \times 10^{6}$	-0.5
2.5	9.47	-0.53

The slope that is $2 \text{ x} 10^6$ is actually straight up. I just needed to avoid division by zero so I hacked the input a teeny bit.

You now have a total of 10 slopes, 2 for each point that you previously inked in. Add these slopes to the plot. Then draw in five pretty little light cones like Taylor and Wheeler did in Figure 5 on p. B-15. Yours won't be in quite the same spots as theirs.

A Different Family of Curves

While we have successfully reproduced all the features of Figure 5 on p. B-15, I am actually not that happy with the figure. The reason is that it doesn't make it obvious that successive raindrops just come in with later wristwatch time. Other than that, there paths are identical. Here is a different plot that makes that more obvious.

$$t_{\text{rain}} = t_{\text{offset}} - \frac{4}{3} M\left(\left(\frac{x}{2}\right)^{3/2} - \left(\frac{5}{2}\right)^{3/2}\right)$$

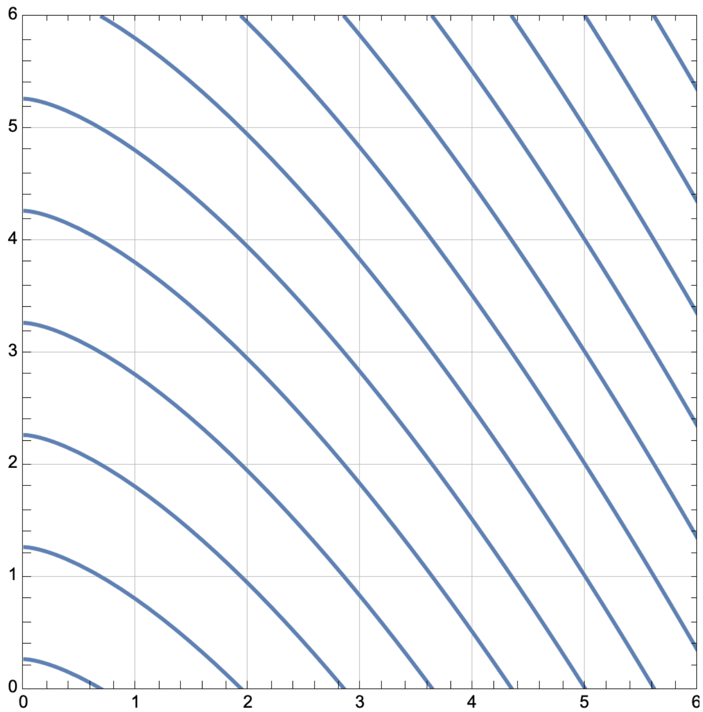
or

$$\frac{t_{\text{rain}}}{M} = \frac{t_{\text{offset}}}{M} + \frac{4}{3}\left[\left(\frac{5}{2}\right)^{3/2} - \left(\frac{x}{2}\right)^{3/2}\right]$$

In[174]:=

```
newPlot[tOffset_] :=
 Plot[tRainOverM[x, 5, tOffset], {x, 0, 6}, PlotRange → {{0, 6}, {0, 6}},
  AspectRatio → 1, AxesLabel → {r / M, train / M}, GridLines → Automatic, Frame → True]
Show[newPlot[-5], newPlot[-4], newPlot[-3], newPlot[-2], newPlot[-1], newPlot[0],
 newPlot[1], newPlot[2], newPlot[3], newPlot[4], newPlot[5], newPlot[6], newPlot[7]]
```

Out[175]=



I hope you understand the difference between these two families of curves. If not, take a closer look at the figure immediately above. Look how regular the crossings of any given x-line are. There was no such regularity in the original family of curves.