Black Holes, Problem Set 7 for Thursday, Oct. 3

Reading from Spacetime Physics

Finish Chapter 5.

Presentations

Will volunteered to do Problem 5-7 on p. 168 as a presentation. Thank you, Will! He can handle it by himself, but maybe one of the rest of you will let him practice the presentation on you.

For Problem Set 7

Problem 1 — Three Reference Frames and the Twin Paradox

A way I like to analyze the twin paradox is to use three reference frames *that are coincident at the turnaround.* We'll call those three frames, the Earth frame, the outgoing rocket frame, and the returning rocket frame.

We'll call the leaving-Earth-event, Event E, we'll call turning-around-at-Canopus-event, Event B, and we'll call the arriving-back-to-Earth-event, Event R. (If those letters seem familiar, it is because those were the letters we used for emission, bounce, and reception of a light flash in a different problem, but this is not that problem.)

In the Earth frame, the outbound rocket has speed v in the $+x$ direction, and the returning rocket has speed v but in the $-x$ direction. The distance to Canopus in the Earth frame, we'll just call L. In all three reference frames, we'll make Event B have coordinates (0, 0).

(a) Where do Events E, B, and R, occur in the Earth frame? I have filled in three table entries to get you started.

Out[84]=

	Event E	Event B	Event R
t	-L/v	0	
x		0	

(b) To continue with the rest of this problem, you can use the heavy-handed machinery of the Lorentz transformation, or you can just use invariance of the interval! What is the invariant interval from E to B using the table in (a)? And what is the invariant interval from B to R using the table in (a)? Simplify your answers for these intervals interval by using $\gamma \equiv 1 \big/ \sqrt{1 - v^2}$.

(c) Where do Events E and B occur in the outgoing rocket frame? Again, I have filled in three table entries to get you started. Fill in the fourth using what you learned in (b).

(b) To continue with the rest of this problem, you can use the heavy-handed machinery of the Lorentz transformation, or you can just use invariance of the interval! What is the invariant interval from E to B using the table in (a)? And what is the invariant interval from B to R using the table in (a)? Simplify your answers for this interval by using $\gamma \equiv 1 \big/ \sqrt{1-v^2}$.

(c) Where do Events E and B occur in the outgoing rocket frame? Again, I have filled in three table entries to get you started. Fill in the fourth using what you learned in (b).

Out[88]=

	Event E	Event B
t'		0
x'	0	0

(d) Where do Events B and R occur in the returning rocket frame? Again, I have filled in three table entries to get you started.

Out[92]=

	Event B	Event R
t''	0	
x''	0	0

If you look at the table from part (a), you will see that $2\,L/v$ of time has elapsed for the twin that stayed on Earth from Event E to Event R. Let's call that $t_{\text{old twin}}$.

(d) Add together the time elapsed from E to B in the outgoing frame, and the time elapsed from B to R in the returning frame. How much time in total, we'll call it $t_{\text{young twin}}$, has elapsed for the twin that rode the outgoing rocket from frame E to B and the returning rocket from B to R?

(e) What is the ratio, $t_{\text{old twin}} \big/ t_{\text{young twin}}$?

(f) By comparing your answers to (a) and (c) you can see that there is a major disagreement between Earth-frame clocks and outgoing-rocket-frame clocks about when Event E occurred. Similarly by comparing your answers to (a) and (d), you can see that there is an equally-large disagreement between Earth-frame clocks and returning-rocket-frame clocks about when Event R occurred. Add the two disagreements together.

(g) Keep in mind that both of the rocket frame observers would say that the Earth-frame clocks are ticking too slowly, so only $2\,L/\gamma v$ of time elapsed according to the rocket frame observers while the Earth frame observers recorded that $2\,L/v$ of time elapsed. Now add the sum of the two disagreements you found in (f) to $2\,L/\gamma v$. Simplify. You should find that this sum is the same as $t_{\text{old twin}}$!

Problem 2 — A Summer Evening's Fantasy, Problem 5-6 on p. 168

This should be fun.

Problem 3 — Making Contact with Integrals

(a) Assume a particle follows a path defined by the function $x = f(t)$ from $t = 0$, to $t = t_f$. Break the time from $t = 0$ to $t = t_f$ up into N equal and tiny intervals Δt. What is the (approximate) expression for the amount of proper time elapsed in the ith interval? Factor out Δt so that it is abundantly clear that your answer is directly proportional to Δt.

HINT: it is going to involve $f'(t_i)$. Go back to your notes on what it involved when we were computing ordinary path lengths in ordinary space, when we had $y = f(x)$ instead of $x = f(t)$, and follow the same steps.

(b) In (a) you calculated the small amount of proper time elapsed in the ith interval, and now you can add all N of them up, and you will get a sum that will look like:

$$\sum_{i=0}^{N-1} [\text{whatever you got in part } (a)] \qquad \text{where} \qquad N\,\Delta t = t_f$$

Fill in whatever you got in part (a) and write out your answer for the sum.

(c) Now take the limit that $N \to \infty$ and $\Delta t \to 0$, while holding $N\Delta t = t_f$. This is the definition of a left-handed sum for a Riemann integral. What integral is it?

NOTE: I am happy to help people who don't remember much calculus with part (c), but first you should make a full attempt at parts (a) and (b), for which you don't need any calculus. You are just adding up small amounts of proper time in exactly the same way as we added up small amounts of distance in class.