Black Holes, Exam 2

Monday, Nov. 18, 2024 — Covering *Spacetime Physics,* Chapters 4-7 and *Exploring Black Holes,* Chapters 1 and 2

EXAM-TAKING STRATEGY: The first four problems are quickies! Knock those out in a few minutes each and get on to the ones that will make you think a little more. All of them take longer to read than to answer.

Invariance of the Interval and Causality

1. Relationships Among Events

EXERCISE 6-2

LABORATORY COORDINATES OF THREE EVENTS

	t (years)	x (years)	y (years)
Event 0	0	0	0
Event 1	3	4	0
Event 2	6	5	0
Event 3	8	8	3

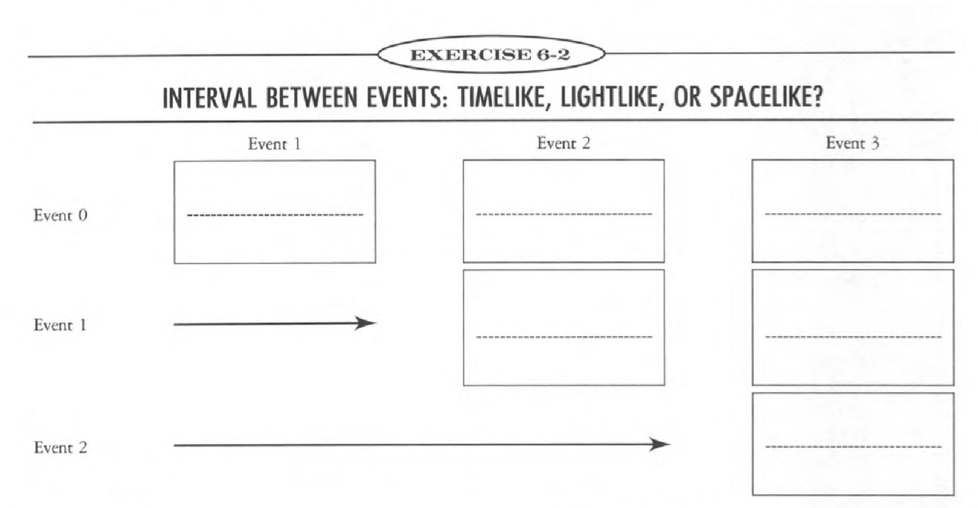
Using the table above, fill in the table on the next page as follows for each pair of the events numbered 0 to 3: in the top of each box, write the nature of the interval: “timelike,” “lightlike,” or “spacelike.” In the bottom of each box, write “yes” if either event could have caused the other, or “no” if that is impossible.

SCRATCH SPACE TO ORGANIZE YOUR THOUGHTS

1-0
2-0
3-0

2-1
3-1

3-2



2. Events in Different Frames and Causality

(a) Referring to Events 2 and 3 in Problem 1, is there a frame in which these two events occur at the same time?

(b) If the answer to (a) is yes, what is the distance, $\Delta\sigma$, between these events in such a frame? If the answer to (a) is no, what is the proper time, $\Delta\tau$, between these two events in a frame in which they occur at the same place?

NOTE: For this part, feel free to leave a square root in your answer. ***Include units.*** Also, feel free to leave the units of both space and time in years.

(c) Is one of these events able to cause the other? If so which can be the cause of the other?

3. Events in Different Frames and Causality Again

(a) Referring to Events 1 and 2 in Problem 1, is there a frame in which these two events occur at the same time?

(b) If the answer to (a) is yes, what is the distance, $\Delta\sigma$, between these events in such a frame? If the answer to (a) is no, what is the proper time, $\Delta\tau$, between these two events in a frame in which they occur at the same place?

NOTE: Also this part, feel free to leave a square root in your answer. ***Include units.*** Also, feel free to leave the units of both space and time in years.

(c) Is one of these events able to cause the other? If so which can be the cause of the other?

Momentum and Energy

4. Force of Sunlight on a Picnic Table

Each picnic table at Deep Springs is about 3 m^2, and sunlight rains down upon the surface of the Earth at an intensity of about 1000 W/m^2. So about 3000W of power is warming the surface of the table, assuming full overhead sun.

***We won't use the numbers until part (e). For parts (a) to (d), just call the power P and don't put in that P = 3000* W.**

(a) If each photon carries an energy, E, what is the rate, r, at which photons arrive?

HINT: Part (a) is super-easy, but if you don't know what to write down, perhaps it would help to think about it to imagine that it was MnM's arriving at rate r measured in MnM's per minute, and that in total the MnM's were piling up at a rate P measured in grams per minute, and that each MnM weighs E grams.

(b) Consider a single photon raining downward, in a coordinate system with the z-direction upward. There is an incredibly simple relation between E and p_z. Assuming the photon is moving ***in the negative-z direction,*** what is p_z in terms of E?

(c) The force on the picnic table due to all these arriving photons depositing their momentum on it is also ***in the negative z-direction***, and its magnitude is $r\,|p_z|$. So using what you found in (b), what is the force in terms of P?

$F_z =$

(d) Ok, let's finally put in the numbers, but before we do, let's restore the conversion factors required to get to conventional units. Specifically, whatever formula you found in (c), the left-hand side has conventional MKS units of Newtons which is $kg \cdot m/s^2$ and the right-hand side has conventional MKS units of Joules/second which is $kg \cdot m^2/s^3$ so you are forced to introduce $c = 3 \times 10^8$ m/s somewhere. What is the conventional equation?

(e) Now stick in the numbers and find the downward force in Newtons.

Reduced Radius

5. The Equatorial Construction Company

The same lawyer that works for the North Pole Construction Company also works for the Equatorial Construction Company. The Equatorial Construction Company has been asked to build concentric circles around the Earth at latitude 6°, latitude 12°, latitude 18°, latitude 24°, and latitude 30°.

(a) Leaving the radius of the Earth, R, and 2π in your answer, and not bothering to grab a calculator to evaluate simplify cos 6°, cos 12°, etc. what is the total length of these five circles.

(b) The contract actually said to build 5 concentric circles and end up "1/3 of the way to the North Pole from the Equator." The lawyer considers whether these circles could have circumferences given by:

$$\left(1-\frac{1}{15}\right)2\,\pi R,\ \left(1-\frac{2}{15}\right)2\,\pi R,\ \left(1-\frac{3}{15}\right)2\,\pi R,\ \left(1-\frac{4}{15}\right)2\,\pi R,\ \text{and}\ \left(1-\frac{5}{15}\right)2\,\pi R$$

Note that the last circle has circumference $\frac{2}{3}\cdot 2\,\pi R$ and arguably it is 1/3 of the way to the North Pole. To put it slightly differently, with this alternative contract interpretation, the circles are spaced equally in "reduced" radius, r, and $r_i = \left(1-\frac{i}{15}\right)R$, where $i = 1,\ 2,\ 3,\ 4,$ or 5. The question is what is the total length of these five circles? As usual, leave $2\,\pi R$ in your answer.

(c) Really not sure what to recommend to the Equatorial Construction Company, the lawyer fires up Mathematica and has it put out the following table:

In[ ]:=
```
trigTable = Table[{θ Degree, Round[Cos[θ Degree], 0.001]}, {θ, 6, 30, 6}];
TableForm[trigTable, TableHeadings → {None, {"latitude", "cosine"}}]
```

Out[ ]//TableForm=

latitude	cosine
6 °	0.995
12 °	0.978
18 °	0.951
24 °	0.914
30 °	0.866

He also asks Mathematica what the total of these five cosines is, and gets 4.704.

So, would the company be better off building concentric circles that are equally-spaced in latitude as outlined in (a), or equally-spaced in the reduced r variable as outlined in (b)?

Schwarzschild Metric

6. Fly-By of Gargantua

In *Interstellar,* Cooper is forced to spend a few minutes doing a fly-by of the black hole Gargantua. For definiteness and to make things come out round lets say Cooper spent $\Delta\tau$=2.6 minutes of his time near Gargantua. During this time, his daughter who is far from the black hole, ages 51 years which is Δt=2.6$*10^7$minutes. Those numbers are appropriate for the movie, but we aren't actually going to use them until part (d).

Knowing that the Schwarzschild metric contains the term

$(\Delta\tau)^2 = \left(1 - \frac{2M}{r}\right)(\Delta t)^2$

Define $x = \frac{r}{2M}$, so $(\Delta\tau)^2 = (1 - x)(\Delta t)^2$

(a) Solve for x in terms of $\frac{\Delta\tau}{\Delta t}$.

(b) You know an approximation valid to first order in ϵ for $(1+\epsilon)^n$ is $1 + n\epsilon$ (provided only that ϵ is small). Using that approximation and the numbers above, what is x in terms of $\frac{\Delta\tau}{\Delta t}$?

(c) Now put in that $x = \frac{r}{2M}$ and solve for r. Subtract $2M$ and answer the question, how much more than $2M$ was r?

(d) Now put in that $2M = 10\,000$ m and $\frac{\Delta\tau}{\Delta t} = 10^{-7}$ into your formula in (c).

7. Proper Distance Between Spherical Shells

A black hole has mass in Taylor and Wheeler's favorite units, given by $2M = 10\,000$ m (which makes it a little more than three times the mass of our Sun). Two concentric spherical shells surround this black hole. The inner shell is at $r = 50\,000$ m and the outer shell is at $r = 50\,001$ m.

Knowing that the Schwarszchild metric contains the term

$$(\Delta\sigma)^2 = \frac{1}{\left(1 - \frac{2M}{r}\right)} (\Delta r)^2$$

and using the same approximation as was suggested in 6(b), and feeling good about assuming that $\frac{1}{5}$ is small enough to make that approximation valid, what is $\Delta\sigma$?

Name ______________________________

1. / 3

2. / 2

3. / 2

4. / 4

5. / 4

6. / 3

7. / 2

TOTAL

/ 20