Black Holes, Exam 3

Monday, Dec. 16, 2024 — Covering *Exploring Black Holes,* Chapters 3, B, and 4

Falling From Rest at Infinity

This is a derivation from Chapter 2, just recycled into a problem.

Once we had proven the conservation of energy formula in Schwarzschild coordinates, it didn't take much work to show that for an object falling from rest at infinity:

$$\frac{\Delta r}{\Delta t} = -\left(1 - \frac{2M}{r}\right)\sqrt{\frac{2M}{r}} \qquad \text{(from conservation of energy)}$$

Of course, this formula only becomes exact in the limit that Δr and Δt are small, but in Problem 1 you are going to manipulate Δr and Δt separately. *So I am not ready to take the limit and rewrite the left-hand side as a derivative.*

1. E_{shell} for a Plunging Object (4pts +1 EC)

A. Use the Black Hole Construction Company formula, $\Delta r_{\text{shell}} = \frac{1}{\sqrt{1-2M/r}}\Delta r$, and the Interstellar aging formula $\Delta t_{\text{shell}} = \sqrt{1-2M/r}\,\Delta t$ to get a nice tidy expression for $v_{\text{shell}} \equiv \frac{\Delta r_{\text{shell}}}{\Delta t_{\text{shell}}}$ in terms of $\frac{\Delta r}{\Delta t}$.

B. Now use the formula for $\frac{\Delta r}{\Delta t}$ we got from conservation of energy to rewrite v_{shell} again. HINT: It is such a pleasingly simple answer, you will remember it when you have it.

C. Use the usual formula for γ in terms of v_{shell}, and what you found in *B* to get a nice tidy expression for γ.

D. Now use $E_{\text{shell}} = m\gamma$ and what you got in C to get a nice tidy expression for E_{shell}.

E. Want a point of extra credit? Use an approximation to recover the Newtonian formula for the kinetic energy of an object of mass m at radius r falling from infinity toward an object of mass M.

Orbiting

When you did Problem 1 from Chapter 4, Taylor and Wheeler had a sneaky approach to the answer. I was thinking about a less sneaky approach that isn't any harder ....

2. Time of Orbit According to the Orbiter (4 pts)

In Chapter 4, Problem 1, Part B, you found **T_{orbiter}, the time for a full orbit, according to the orbiter.**

For an object in perfectly circular orbit, we have:

$(\Delta \tau)^2 = \left(1 - \frac{2M}{r}\right)(\Delta t)^2 - r^2 (\Delta \phi)^2$ (the metric applied to a perfectly circular orbit)

The above is just the Schwarzschild metric in the equatorial plane and with $\Delta r = 0$ (because we are in a perfectly circular orbit, not a quasi-elliptical one, so r is unchanging, so $\Delta r = 0$).

A. Use the same formula for Δt_{shell} that was recommended in Part 1A above to get rid of Δt in the formula for the metric applied to a perfectly circular orbit.

B. Now use $v_{\text{shell}} \equiv \frac{r \Delta\phi}{\Delta t_{\text{shell}}}$ to get rid of Δt_{shell} in the equation you found in A.

C. Usually we use the metric to evaluate small amounts of wristwatch time or measured distance. In one short sentence, why is it valid to stuff a big value $\Delta\phi = 2\pi$ into the equation you found in B.

D. So go ahead and stuff $\Delta\phi = 2\pi$ into the equation you found in B, but also remember the whole point is that we are trying to find T_{orbiter}, and what is the interpretation of $\Delta\tau$ once we have stuffed $\Delta\phi = 2\pi$ into the equation? HINT: It is bolded above.

E. Take the square root of the equation you found in D. You only need the positive square root, because T_{orbiter} is a period, and by convention we never report a negative period, regardless of the direction of orbit.

COMMENT: Your answer to E is exactly the same as what you found in Taylor and Wheeler's Problem 1 on p. 4-28. They used time dilation to get to the same place.

Two Conservation Law Problems

We never derived the conservation laws in the rain frame, and this yields two lovely results. But of course even these are not really new, but are merely expressions of the two conservation laws that you already know rewritten in the rain frame.

3. Conservation of Energy in the Rain Frame (5 pts)

A. The rain frame metric is

$$(\Delta\tau)^2 = \left(1 - \frac{2M}{r}\right)(\Delta t_{\text{rain}})^2 - 2\sqrt{\frac{2M}{r}}\,\Delta t_{\text{rain}}\,\Delta r - (\Delta r)^2 - r^2(\Delta\phi)^2$$

Consider two close-together points at the ends of segment A which have the coordinates

$t_{\text{rain}\,1}, r_1, \phi_1$

$t_{\text{rain}\,2}, r_2, \phi_2$

In terms of these six coordinates, what is $\Delta\tau_A$, where $\Delta\tau_A$ is the elapsed time on a watch carried by a person moving along segment A?

B. Repeat, but for two close-together points at the ends of segment B which have the coordinates

$t_{\text{rain}\,2}, r_2, \phi_2$

$t_{\text{rain}\,3}, r_3, \phi_3$

In other words, what is $\Delta\tau_B$, where $\Delta\tau_B$ is the elapsed time on a watch carried by an person moving between points 2 and 3?

C. Take $\frac{d(\Delta\tau_A + \Delta\tau_B)}{d\,t_{\text{rain}\,2}}$ and set it equal to 0, per the principle of extremal aging.

D. Now we need to simplify what you got in C and prepare to interpret it. Rewrite the equations in terms of $\Delta t_{\text{rain}\,A} = t_{\text{rain}\,2} - t_{\text{rain}\,1}$, $\Delta t_{\text{rain}\,B} = t_{\text{rain}\,3} - t_{\text{rain}\,2}$, etc.

E. Make the final conceptual step by recognizing that some terms in your equation only involve segment A, and some terms only involve segment B. Therefore each of these terms is separately a constant, which we will call $\frac{E_{\text{rain}}}{m}$. Declare that the terms in your equation that only involve segment A are this constant.

4. Conservation of Angular Momentum in the Rain Frame (3 pts)

Do exactly what you did in 3 C, D, and E, but this time by taking $\frac{d(\Delta\tau_A + \Delta\tau_B)}{d\,\phi_2}$ and setting it equal to 0. When you make your final conceptual leap, realize that it has exactly the same formula as angular momentum had in the Schwarzschild metric, so we can just call it $\frac{L}{m}$.

The Effective Potential

5. An Effective Potential You Haven’t Seen (4 pts)

Here is an effective potential you haven’t seen:

$V_{\text{eff}}(R) = -\frac{2M}{R}$

And here is the equation that effective potential appears in:

$1 = \left(\frac{dR}{dt}\right)^2 + V_{\text{eff}}(R)$

A. If $R = \frac{M}{4}$, what is must $\frac{dR}{dt}$ be? Don’t forget when you take the square root that there are two signs. For the moment, just take the positive square root and make that your answer.

B. If $R = \frac{2}{3}M$ what must $\frac{dR}{dt}$ be? Again stick with the + sign.

C. If $R = 2M$ what must $\frac{dR}{dt}$ be? <-------- PART C DID NOT DO WORK OUT TO WHAT I MEANT IT TO WORK OUT TO, SO WE SCRATCHED IT OFF THE EXAM.

D. Put in $R = \frac{M}{20\,000}$. Neglect the 1 in the equation $1 = \left(\frac{dR}{dt}\right)^2 + V_{\text{eff}}(R)$ because $V_{\text{eff}}(R)$ is now so huge. This time take the negative square root when you solve for $\frac{dR}{dt}$.

COMMENT: What you are approaching in D is called “the big crunch,” and it is a possible endpoint for the universe if there were enough mass in the universe to slow the expansion to a halt. All evidence is that there is not enough mass to slow the expansion, and the universe will expand forever.