Black Holes, Exam 1

Thursday, Sept. 26, 2024 — Covering Spacetime Physics, Chapters 1, 2, 3, and L

DIRECTIONS: Include units in your answers. But make your life easy! In a few problems you can measure both space and time in meters. If space and time are both measured in meters, then there is no need to convert between meters and seconds!

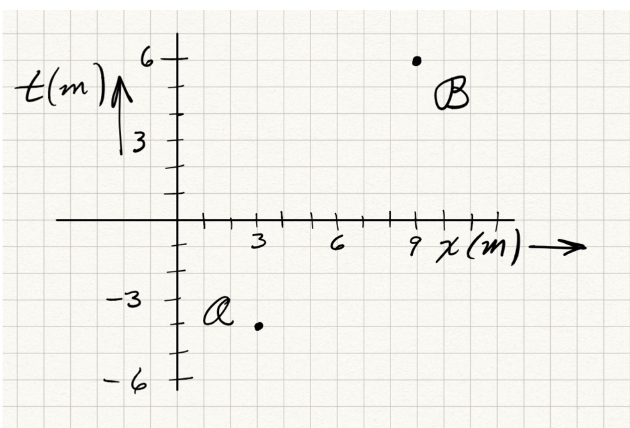
EXAM-TAKING STRATEGY: There are six problems. If a problem is taking you more than 5-7 minutes, move on. Your first pass through will then take only 40 minutes. Then when you make a second pass through the exam to catch errors and omissions, any steps that you got stuck on may suddenly seem obvious.

HARD STOP: So people can get to the lower ranch for horsemanship, we have a hard stop at 9:20am.

Coordinates and Intervals

1. Coordinates and Speed (2 pts)

Below is a spacetime diagram showing two events $\mathscr{A}$ and $\mathscr{B}$ in the lab frame. The ticks on both the x and t axes represent 1m of space and 1m of time, respectively.



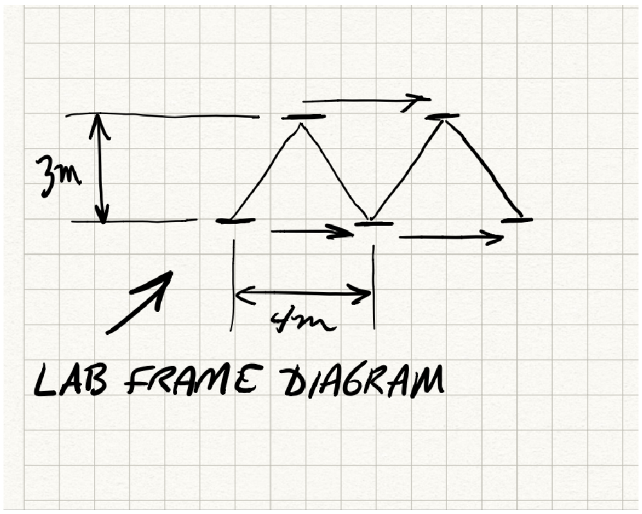
(a) Write down the coordinates $(t_A,\ x_A)$ and $(t_B,\ x_B)$ of the two points. NB: Re-read the directions at the top of the exam. They included directions you need to follow (like including units), and also suggestions for making your life easy.

(b) Imagine a rocket that passes $\mathscr{A}$ just as that event occurs, and passes $\mathscr{B}$ just as that event occurs. How fast is this rocket going in the lab frame? I'm just asking you to compute $v = \Delta x/\Delta t$.

(c) What is the interval-squared — I'm just asking you to calculate $(\Delta t)^2 - (\Delta x)^2$ — for these two points? Include units — which are m^2 (or meters^2 if you prefer to write them out). Then take the square root of what you to get the interval. Include units — which are m (or meters if you prefer to write them out).

(d) According to a clock carried by the rocket in part (b), what is the elapsed time going from $\mathcal{A}$ to $\mathcal{B}$? (Include units — which are most conveniently meters).

2. Moving Mirrors (3 pts)



Above is a diagram of two mirrors moving to the right. Light is bouncing between them as they move.

Two round trips are shown. In the following, analyze one round trip.

(a) What is the total distance traveled by a photon in one round trip (ordinary distance, not interval). It is fine to leave a square root in your answer.

(b) How long does one round trip take? (It is fine to leave your answer in meters.)

(c) Knowing that the mirrors traveled 4m in the time you found in b, what must be the speed of the mirrors in the lab frame? Again, it is fine to leave a square root in your answer.

(d) How far has the photon gone in one round trip in the mirrors' rest frame?

3. Meson Decay (2 pts)

A meson lives a time τ in its own frame. It is moving at speed v and lives a time t in the lab frame, during which it moves a distance vt. This is summarized in the following table:

	time	distance
lab frame	t	vt
meson frame	τ	0

(a) Write down a relationship between the quantities in the table using invariance of the interval.

(b) Solve this relationship for t.

4. Doppler Shift (3 pts)

For this problem, answer all results to three decimal places. This will be easy without a calculator thanks to approximations!

A particle going 0.1 (in the natural units where the speed of light is 1) emits flashes 2 seconds apart in its frame. The time dilation formula says these flashes are this much time apart as measured by a lattice of clocks in the lab frame:

$$\frac{1}{\sqrt{1-0.1^2}}\ 2\,\text{s}$$

(a) Using $\frac{1}{\sqrt{1-x^2}} \approx 1+\frac{1}{2}x^2$ when x is small, write down an approximation for

$$\frac{1}{\sqrt{1-0.1^2}} \approx$$

(b) Now multiply by 2s.

$$\frac{1}{\sqrt{1-0.1^2}}\ 2\,\text{s} \approx$$

(c) For definiteness, assume the particle is moving away from a person watching the flash, who is at rest in the center of the lab coordinates. Use the speed v and your answer to (b) to determine how much farther in the lab frame from the origin the particle is after each successive flash?

(d) What is the time needed for light to traverse the additional distance you found in (c)?

(e) Add your answers to (b) and (d) together to find the difference in the reception times between successive flashes. What is the difference between successive reception times?

Lorentz Transformations

5. Lorentz Transformation Derivation (2 pts)

When we derived the Lorentz transformation in class, we started at the same place as Taylor and Wheeler:

$x = B\,x' + D\,t'$
$t = G\,x' + H\,t'$

DISCUSSION: The first of four major steps was demanding that the origin $(t', 0)$ of the rocket frame moves with velocity v_{rel} through the lab frame. So first I am going to have you re-do that.

(a) Using the equations above, what does $(t', 0)$ in the rocket frame transform to in the lab frame? Now also use $x = v_{\text{rel}}\,t$. What does this tell you about a relation between D and H?

Summarize the results of (a) here:

$x =$
$t =$

DISCUSSION: Our second and third major steps were to fix B and H using Lorentz contraction and time dilation (both effects that we had long-since derived by contemplating clocks made of mirrors). Then our last major step was to demand that the origin $(t, 0)$ of the lab frame moves with velocity $-v_{\text{rel}}$ through the rocket frame. Now I am going to instead have you do what was the last major step as the second step.

(b) Put $x = 0$ into the first of your two equations above. Also put $x' = -v_{\text{rel}}\,t'$ into that equation. What does this tell you about a relation between B and H?

Summarize the results of (b) here:

$x =$
$t =$

DISCUSSION: That's all for Problem 5, because time is short, but you can hopefully see how you would now finish the derivation of the Lorentz transformation by using time dilation and Lorentz contraction to fix the two remaining constants.

6. John (in the Lab) tries to Graffiti Mary’s Rocket (3 pts)

There is only one speed in this problem v_{rel}, the speed of Mary’s rocket, which is moving in the positive x-direction according to John. Mary’s rocket is L long (according to Mary). John’s lab is L long (according to John). The center of Mary’s rocket reaches the center of John’s lab at $t' = t = 0$, where t' is Mary’s time coordinate and t is John’s time coordinate. To make life simple, let’s measure John’s x-coordinate from the center of his lab, and Mary’s x'-coordinate from the center of her rocket. At $t = 0$, John discharges two paint bombs at each end of his lab in an attempt to graffiti the front and back of Mary’s rocket.

(a) John forgot about length contraction! He thought that graffiti bomb GF, that goes off at coordinates GF = $\left(0, \frac{L}{2}\right)$, would graffiti the front of Mary’s rocket, and graffiti bomb GB, that goes off at coordinates GB = $\left(0, -\frac{L}{2}\right)$, would graffiti the back of Mary’s rocket. Just using length contraction, and you can even use the shorthand $\gamma \equiv \frac{1}{\sqrt{1-v_{\text{rel}}^2}}$ when you write down your answer, what are the correct values, in John’s coordinates, of MF (the front) and MB (the back) of Mary’s rocket at $t = 0$?

MF = (0,)

MB = (0,)

(b) Despite John’s incompetence, Mary is extremely concerned about her rocket’s paint job. She has not forgotten about length contraction, and according to her, it is John’s lab that is length-contracted! So at $t' = 0$, both ends of her rocket are sticking out of his shortened lab. Perhaps the paint bombs will graffiti her rocket after all!? Use the inverse Lorentz transformation, Eq. L-11a (reproduced on the next page), to discover where event GF = $\left(0, \frac{L}{2}\right)$, occurs in Mary’s coordinates.

GF = (,)

(c) Use the transformation again, to discover where GB = $\left(0, -\frac{L}{2}\right)$ occurs in Mary’s coordinates.

GB = (,)

(d) Plugging in $L = 40$ m, $\frac{L}{2} = 20$ m, $v_{\text{rel}} = \frac{3}{5}$, and $\gamma \equiv \frac{1}{\sqrt{1-v_{\text{rel}}^2}} = \frac{5}{4}$, what are the coordinates MF and MB (in John’s coordinates) that you found in Part (a)? Of course, include units in this part and the next.

MF = (0,)
MB = (0,)

(e) Plugging in these same values, what are GF and GB in Mary’s coordinates that you found in (b) and (c)?

GF = (,)
GB = (,)

The Inverse Lorentz Transformation:

The following is screenshot from p. 103 of *Spacetime Physics:*

The y and z components are respectively equal in both frames, as before. Then the **inverse Lorentz transformation equations** become

$$\begin{aligned} t' &= -v_{\text{rel}}\gamma x + \gamma t \\ x' &= \gamma x - v_{\text{rel}}\gamma t \\ y' &= y \\ z' &= z \end{aligned} \qquad \text{(L-11a)}$$

Name ______________________________

1. / 2
2. / 3
3. / 2
4. / 3
5. / 2
6. / 3

TOTAL

/ 15