Rain Frame Light Cones

Each coordinate system change sheds new light on the properties of the black hole. In Problem Set 15, you derived the rain-frame metric:

$$(\Delta\tau)^2 = \left(1 - \frac{2M}{r}\right)(\Delta t_{\text{rain}})^2 - 2\sqrt{\frac{2M}{r}}\,\Delta t_{\text{rain}}\,\Delta r - (\Delta r)^2 - r^2(\Delta\phi)^2$$

and you derived $\left(\frac{\Delta r}{\Delta t_{\text{rain}}}\right)_{\pm} = -\sqrt{\frac{2M}{r}} \pm 1$. Of course all these derivations were only valid for small displacements, so now we may finally take the limit that the displacements go to zero, and we have found:

$$\left(\frac{dr}{dt_{\text{rain}}}\right)_{\pm} = -\sqrt{\frac{2M}{r}} \pm 1.$$

Going back to p. 3-22 where we were calling τ the wristwatch time of the raindrop, rather than calling it t_{rain}, we learned that

$$\tau_2 - \tau_1 = -\int_{r_1}^{r_2} \frac{\sqrt{r}}{\sqrt{2M}}\,dr = -\frac{1}{\sqrt{2M}}\,\frac{2}{3}\left(r_2{}^{3/2} - r_1{}^{3/2}\right)$$

On Figure 5 on p. B-15, Taylor and Wheeler choose a raindrop that had $\tau_1 = 0$ at $r_1 = \frac{5}{2}M$. Let a new variable be $x = \frac{r_2}{M}$. Then plugging these three facts in, and recognizing that what we were calling τ_2 we are now calling t_{rain}, we have

$$t_{\text{rain}} - 0 = -\frac{1}{\sqrt{2M}}\,\frac{2}{3}\left(M^{3/2}\left(\frac{r_2}{M}\right)^{3/2} - \left(\frac{5}{2}M\right)^{3/2}\right) = -\frac{M}{\sqrt{2}}\,\frac{2}{3}\left(x^{3/2} - \left(\frac{5}{2}\right)^{3/2}\right)$$

or

$$\frac{t_{\text{rain}}}{M} = \frac{4}{3}\left[\left(\frac{5}{2}\right)^{3/2} - \left(\frac{x}{2}\right)^{3/2}\right]$$

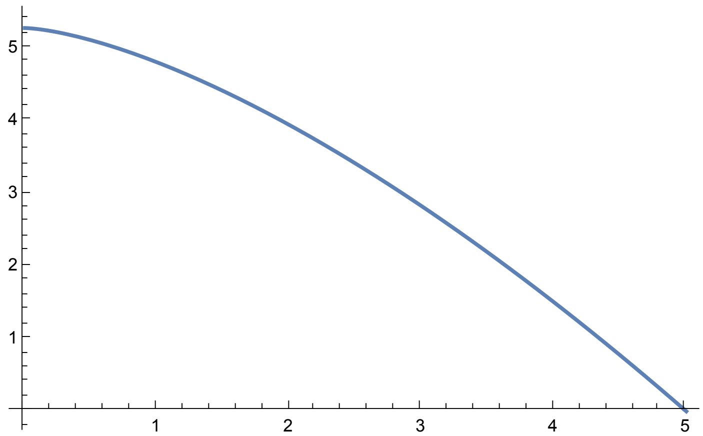
A bit of a mess, but let’s plot it

In[150]:=

```
tRainOverM[x_, zero_] := 4/3 ((zero/2)^(3/2) - (x/2)^(3/2));
Plot[tRainOverM[x, 5], {x, 0, 5}]
```

$$\text{tRainOverM}[x\_,\ \text{zero}\_] := \frac{4}{3}\left(\left(\frac{\text{zero}}{2}\right)^{3/2} - \left(\frac{x}{2}\right)^{3/2}\right);$$

Out[151]=

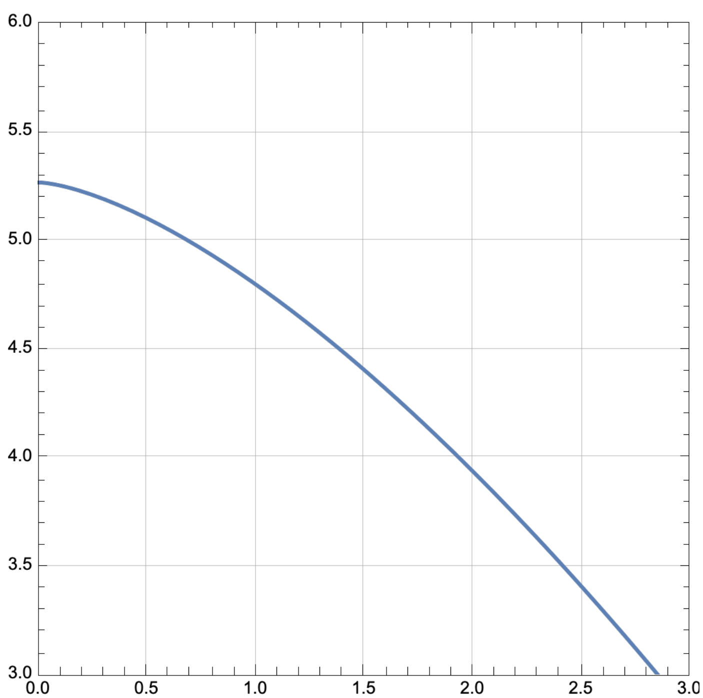


If we want it to be a little more obvious that the function we have plotted is exactly the same as theirs, we can adjust the plot range and the aspect ratio, and label the axes:

In[152]:=

```
plot[zero_] :=
 Plot[tRainOverM[x, zero], {x, 0, 3}, PlotRange → {{0, 3}, {3, 6}},
  AspectRatio → 1, AxesLabel → {r / M, t_rain / M}, GridLines → Automatic, Frame → True]
plot[5]
```

Out[153]=

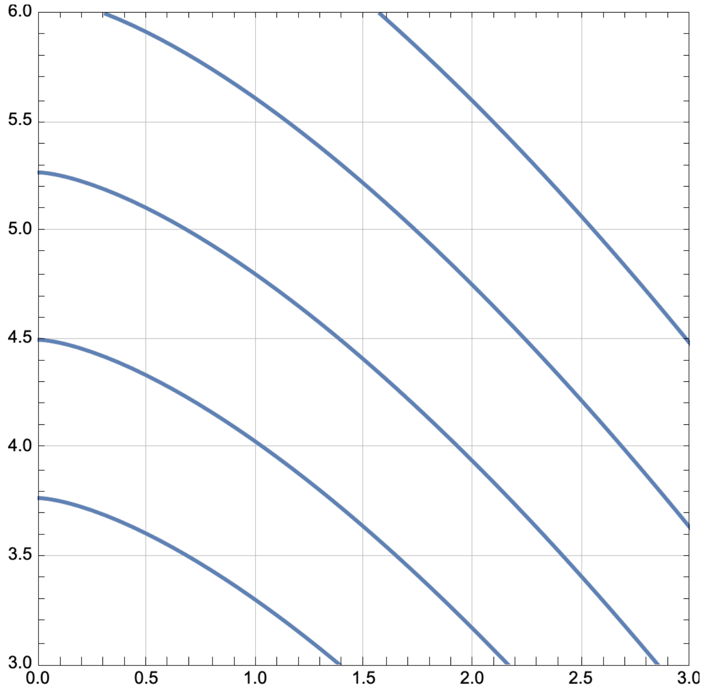


Let's add the paths for raindrops whose wristwatches were zero when they were at 4.0 M, 4.5 M, 5.5 M, and 6 M, respectively:

In[154]:=

```
Show[plot[4.0], plot[4.5], plot[5.0], plot[5.5], plot[6]]
```

Out[154]=



I apologize that the axes aren't labeled. There is some conflict between the labels and the frame. Sigh. The horizontal axis is r/M. The vertical axis is t_{rain}/M.

Ink in the following five points on the middle of the five lines:

In[155]:=

```
middle[x_] := Round[tRainOverM[x, 5], 0.01];
TableForm[{
  {0.5, middle[0.5]},
  {1.0, middle[1.0]},
  {1.5, middle[1.5]},
  {2.0, middle[2.0]},
  {2.5, middle[2.5]}
 }]
```

Out[156]//TableForm=

```
0.5   5.1
1.    4.8
1.5   4.4
2.    3.94
2.5   3.41
```

We computed the slopes, $\left(\frac{dr}{dt_{\text{rain}}}\right)_{\pm}$. Because the horizontal axis has r and the vertical axis has t, we really want the inverse of these slopes. In terms of $x = \frac{r}{M}$, the inverse slopes are:

In[157]:=

```
slopePlus[x_] := Round[1/(-1/Sqrt[x/2] + 1), 0.01];

slopeMinus[x_] := Round[1/(-1/Sqrt[x/2] - 1), 0.01];

TableForm[{
  {0.5, slopePlus[0.5], slopeMinus[0.5]},
  {1.0, slopePlus[1.0], slopeMinus[1.0]},
  {1.5, slopePlus[1.5], slopeMinus[1.5]},
  {2.0, slopePlus[2.000002], slopeMinus[2.0]},
  {2.5, slopePlus[2.5], slopeMinus[2.5]}
 }]
```

Out[159]//TableForm=

0.5	-1.	-0.33
1.	-2.41	-0.41
1.5	-6.46	-0.46
2.	$2. \times 10^6$	-0.5
2.5	9.47	-0.53

The slope that is $2 \text{x} 10^6$ is actually straight up. I just needed to avoid division by zero so I hacked the input a teeny bit.

You now have a total of 10 slopes, 2 for each point that you previously inked in. Add these slopes to the plot. Then draw in five pretty little light cones like Taylor and Wheeler did in Figure 5 on p. B-15. Yours won't be in quite the same spots as theirs.