5/8 in Class – Runge Kutta for Chaotic Systems
Systems of DE's

In class, I showed my version of the book's Lorenz System (section 14.6.2). You can find it on the class website under today's date on the calendar, if you'd like to play with it. The book's version, and mine, used MATLAB's `ode45` function to solve the 4th order R-K approximation of the following system of DE's:

$$\begin{aligned}
\dot{x} &= 10(y - x) & \iff \quad \dot{x}_1 &= 10(x_2 - x_1) \\
\dot{y} &= -xz + 28x - y & \iff \quad \dot{x}_2 &= -x_1 x_3 + 28x_1 - x_2 \\
\dot{z} &= xy - 8z/3 & \iff \quad \dot{x}_3 &= x_1 x_2 - 8x_3/3
\end{aligned}$$

where the version on the left and right use differing notation for vectors as follows:

$$\vec{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \iff \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The version on the RHS is often more easily adaptable to programming, ie x_1 becomes $x(1)$, as in my code or even, possibly $x(:,1)$ for the first column of a matrix of positions if you will have rows and rows of positions.

1. Go to the white board and write a structure plan for how you will: Write a script that does a 4th order Runge Kutta approximation of this system the hard way. Your script should also make the following plots:

   (a) On figure(1): Plot y vs t (which will probably be $x(:,2)$ vs t) for several slightly different initial conditions.

   (b) On figure(2), Plot y vs x (which will probably be $x(:,2)$ vs $x(:,1)$) for the book's initial conditions.

2. Exercise 8.2 from Newman (shown on the back). You may do it either way.

Exercise 8.2: The Lotka–Volterra equations

The Lotka–Volterra equations are a mathematical model of predator–prey interactions between biological species. Let two variables x and y be proportional to the size of the populations of two species, traditionally called "rabbits" (the prey) and "foxes" (the predators). You could think of x and y as being the population in thousands, say, so that $x = 2$ means there are 2000 rabbits. Strictly the only allowed values of x and y would then be multiples of 0.001, since you can only have whole numbers of rabbits or foxes. But 0.001 is a pretty close spacing of values, so it's a decent approximation to treat x and y as continuous real numbers so long as neither gets very close to zero.

In the Lotka–Volterra model the rabbits reproduce at a rate proportional to their population, but are eaten by the foxes at a rate proportional to both their own population and the population of foxes:

$$\frac{dx}{dt} = \alpha x - \beta xy,$$

where α and β are constants. At the same time the foxes reproduce at a rate proportional to the rate at which they eat rabbits—because they need food to grow and reproduce—but also die of old age at a rate proportional to their own population:

$$\frac{dy}{dt} = \gamma xy - \delta y,$$

where γ and δ are also constants.

a) Write a program to solve these equations using the fourth-order Runge–Kutta method for the case $\alpha = 1$, $\beta = \gamma = 0.5$, and $\delta = 2$, starting from the initial condition $x = y = 2$. Have the program make a graph showing both x and y as a function of time on the same axes from $t = 0$ to $t = 30$. (Hint: Notice that the differential equations in this case do not depend explicitly on time t—in