5/1 in Class – more 2nd and 4th Order Runge-Kutta Approximations to ODE's and MATLAB's built-in versions

1. Last time we did exponential decay. Today, I showed an exponential growth example modeled on the book's version. Here's the (slightly edited) problems 2 and 4 from last time, with MATLAB function `ode45` added in:

   Use 2nd Order Runge-Kutta approximation to solve this DE: $\dot{N} = -\Gamma N$ where $\Gamma = 3.6\times10^{-4}$s. At $t = 0$, $N = 1000$. For the first pass, go from t=0 to 15,000 seconds with only 10 time steps. (That's just for debugging.) Once you get it working, up the number of time steps hugely.

   (a) Write the equations you will use to do both second and fourth order Runge Kutta approximations.
   (b) Write your own code to calculate the 2nd order R-K approximation.
   (c) Graph your approximation of N and the exact N as functions of time. Compare them. Use a legend. You may need to zoom in to see differences.
   (d) Play around. Decrease the time step by a lot (if you haven't already). What happens?
   (e) Get the two values of N at $t = 5,000$ seconds. How accurate is your approximation?
   (f) What happens if you change the time step?
   (g) Now calculate and write your own fourth order R-K approximation and add it to your code and graph. You may have to zoom in to see the differences if any.
   (h) How accurate is this method–for the same number of time steps–at $t = 5,000$ seconds?
   (i) Now write your own function (just like the book's with a negative sign) and use MATLAB's function `ode45` to get MATLAB's approximation.
   (j) Graph it on top of your previous N's.
   (k) Compare MATLAB's answer for $N(5000)$ to the exact and to your 4th order number.

2. **Simple Pendulum as a Simple Harmonic Oscillator:** A simple pendulum consists of a mass on the end of a string that is free to swing back and forth. The mass of the string must be small compared to hanging mass. (If it's not, it's called a physical pendulum.) When we need numbers, let's say: $m = 1.5$kg on the end of a $\ell = .75$m string. We will start it at $\theta = 25$ degrees and and let it run for five full periods.

   Recall the equation for a SHO is:

   $$\ddot{\bigcirc} = \omega^2 \bigcirc$$

   where $\bigcirc$ is any position variable. For the simple pendulum, you might choose either θ or s as your position variable, where θ is the angle measured from the vertical to the string, and s is the path followed by the mass. (s would be a circle if you let it swing all the way around.) Recall also that $s = \ell\theta$. So that we are all consistent, let's choose θ as our variable in the end. (You can derive in s if you prefer, then divide by ℓ.)

   (a) Go to the white board and derive the equation for $\ddot{\theta} = \cdots$. Box the exact equation and save it. (I plan to use it later.) This is not a SHO.

(b) Show that for small angles, where $\sin\theta \approx \theta$, ($\theta$ must be in radians) the simple pendulum is a SHO and find the period, τ, for it.

(c) To use Runge-Kutta, you will need to break this second order DE into two first order DE's. Then you can do R-K twice–once on each DE.

(d) Write your own 4th order Runge-Kutta approximations for $\dot{\theta}$ and θ as a functions of t.

(e) Write two functions so that you can use MATLAB's `ode45`.

(f) Also calculate the "exact" solution to your DE. (I put that in quotes because the exact solution to the DE for SHO is an approximation of the motion of the actual pendulum.)

(g) Plot your velocities and positions as functions of time. Keep all the velocities on one graph, and all positions on another.

(h) How accurate are the various approximations of the position when $t = 5\tau$?

3. Do 4th order R-K on the exact second order DE of the pendulum to find the position of the mass as a function of time. (The one with the $\sin\theta$ in it.) Plot five periods of the motion for the same mass and length given above. Compare to the "exact" solution of the SHO version.

   Try one where it goes all the way around!