Three Harmonic Oscillators

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Three Oscillators — Formulas for the Accelerations

Three equal masses connected to each other and the walls by six identical springs, we have:

$$a_1 = -\omega_0{}^2\, x_1 + \omega_0{}^2 (x_2 - x_1)$$

$$a_2 = -\omega_0{}^2 (x_2 - x_1) + \omega_0{}^2 (x_3 - x_2)$$

$$a_3 = -\omega_0{}^2 (x_3 - x_2) + \omega_0{}^2 (-x_3)$$

where

$$\omega_0{}^2 \equiv k / m$$

Why Stop at Three? — Formulas for the Accelerations of Many Oscillators

Let’s have n masses and $n + 1$ springs. We’ll set $n = 3$, but if we are careful with the code, we can later make n whatever we like. Let’s also define ω_0 while we are at it:

```
In[74]:= n = 3;
         omega0 = 2 Pi;
```

We need to handle the left-most and right-most oscillators (the ones that are next to the walls) specially, because they are each attached to a wall on one side instead of having other oscillators on both sides. That’s going to require some `If[]` statements:

```
In[76]:= acceleration[j_, allPositions_] :=
          -omega0^2 (allPositions⟦j⟧ - If[j == 1, 0, allPositions⟦j - 1⟧]) +
           omega0^2 (If[j == n, 0, allPositions⟦j + 1⟧] - allPositions⟦j⟧)
```

Initial Conditions

First set up the duration. Let’s also define `steps` and `deltaT` while we are at it::

```
In[77]:= tInitial = 0.0;
         tFinal = 10.0;
         steps = 2000;
         deltaT = (tFinal - tInitial) / steps;
```

Cool Initial Conditions, but Ignore them for Now

Here are three sets of initial conditions (these are known as eigenvectors and the omegas are known as eigenvalues) **that are very particular to the case *n* = 3**:

In[81]:=
```
iA3 = {
    {1/2, 1/Sqrt[2], 1/2},
    {-1/Sqrt[2], 0, 1/Sqrt[2]},
    {1/2, -1/Sqrt[2], 1/2}
  };
omega3 = {Sqrt[2 - Sqrt[2]], Sqrt[2], Sqrt[2 + Sqrt[2]]} omega0;
```

Simple-Minded Initial Conditions

For now, just implement these more simple-minded initial condition:

In[83]:=
```
pluck = -0.5;
(* Make pluck the first mass's displacement and all the others zero. *)
initialPositions = {pluck, 0.0, 0.0};
(* Make all the initial velocities zero *)
initialVelocities = Table[0.0, n];
```

Try Changing the Initial Conditions

Go back up to the initial conditions section, and where it said,

```
initialPositions={-pluck, 0.0, 0.0};
```

change that line to ***one*** of the special initial conditions:

```
initialPositions=iA3[[1]];
initialPositions=iA3[[2]];
initialPositions=iA3[[3]];
```

All of these are interesting!

In[86]:=
```
initialPositions = iA3⟦3⟧;
```

We have a decision to make on how we are packing the time, the positions, and the velocities into the initial conditions list, but here is the obvious choice:

In[87]:=
```
initialConditions = {tInitial, initialPositions, initialVelocities};
```

In particular, initialPositions and initialVelocities are lists, so the list we have just created contains lists!

Second-Order Runge-Kutta — Formulas for *n* Particles

$t_{i+1} = t_i + \Delta t$

$x_j^* = x_j(t_i) + v_j(t_i) \cdot \frac{\Delta t}{2}$ $\quad$ (j goes from 1 to n)

$v_j(t_{i+1}) = v_j(t_i) + a_j(\text{that depends on neighboring } x^* \text{ — see formulas for the accelerations above}) \cdot \Delta t$

$x_j(t_{i+1}) = x_j(t_i) + \left(v_j(t_i) + v_j(t_{i+1})\right) \frac{\Delta t}{2}$

Second-Order Runge-Kutta — Implementation

Your turn to put it all together into the real thing:

```
In[88]:= rungeKutta2[cc_] := (
           curTime = cc⟦1⟧;
           curPositions = cc⟦2⟧;
           curVelocities = cc⟦3⟧;
           newTime = curTime + deltaT;
           positionsStar = curPositions + curVelocities deltaT / 2;
           accelerations = acceleration[#, positionsStar] & /@ Range[n];
           newVelocities = curVelocities + accelerations deltaT;
           newPositions = curPositions + (curVelocities + newVelocities) deltaT / 2;
           {newTime, newPositions, newVelocities}
          )
         (* Test your implementation on the initial conditions *)
         N[rungeKutta2[initialConditions]]
         (* This is what I get: *)
         (* {0.005, {-0.00024674, -0.499507, -0.00024674, 0., 0.},
          {-0.098696, 0.197392, -0.098696, 0., 0.}} *)
Out[89]=
         {0.005, {0.499158, -0.705915, 0.499158}, {-0.336969, 0.476547, -0.336969}}
```

Using NestList[] to Repeatedly Apply rungeKutta2[]

```
In[90]:= rk2Results = NestList[rungeKutta2, initialConditions, steps];
```

Transposing to Get Points We Can Put in ListLinePlot[]

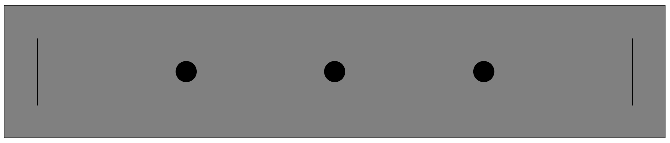
```
In[91]:= rk2ResultsTransposed = Transpose[rk2Results];
         positionLists = rk2ResultsTransposed⟦2⟧;
```

A Graphic

The graphics work is straightforward but a little time-consuming, and not terribly instructive, so it is already all done:

In[93]:=
```
coupledOscillatorGraphic[positionList_] := Graphics[{
    width = 10;
    buffer = 0.5;
    netWidth = width - 2 buffer;
    wallHeight = 1.0;
    numberOfSprings = n + 1;
    (* the next line makes a gray rectangle *)
    {EdgeForm[Thin], Gray, Polygon[{{0, -1}, {width, -1}, {width, 1}, {0, 1}}]},
    (* the next two lines make the walls *)
    Line[{{buffer, -wallHeight / 2}, {buffer, wallHeight / 2}}],
    Line[{{width - buffer, -wallHeight / 2}, {width - buffer, wallHeight / 2}}],
    (* finally we draw all the points *)
    Table[Style[Point[{positionList[[j]] + netWidth/numberOfSprings j + buffer, 0.0}],
      PointSize[0.03]], {j, n}]
  }]
(* A little test to see if our code at least draws equally-
 spaced points when the positions are all zero: *)
coupledOscillatorGraphic[Table[0.0, n]]
```

Out[94]=



Animating The Graphics

In[95]:=
```
Animate[coupledOscillatorGraphic[positionLists[[i]]],
 {i, 1, steps, 1}, DefaultDuration → tFinal - tInitial]
```

Out[95]=