Cosmology — Term 4 Exam — Solution

1. John Tries to Graffiti Mary's Ship

(a) Length contraction by $\gamma \equiv \frac{1}{\sqrt{1-v_{\text{rel}}^2}}$ says that Mary's ship is shortened by a factor of γ. So,

$$\text{MF} = \left(0,\ \frac{L}{2\gamma}\right)$$

$$\text{MB} = \left(0,\ -\frac{L}{2\gamma}\right)$$

(b) Plugging in the coordinates of GF, which are $t = 0$ and $x = \frac{L}{2}$, into

The y and z components are respectively equal in both frames, as before. Then the **inverse Lorentz transformation equations** become

$$\begin{aligned} t' &= -v_{\text{rel}}\gamma x + \gamma t \\ x' &= \gamma x - v_{\text{rel}}\gamma t \\ y' &= y \\ z' &= z \end{aligned} \qquad \text{(L-11a)}$$

we find

$$\text{GF} = \left(-v_{\text{rel}}\,\gamma\,\frac{L}{2},\ \gamma\,\frac{L}{2}\right)$$

in Mary's coordinates.

(c) And we find

$$\text{GB} = \left(v_{\text{rel}}\,\gamma\,\frac{L}{2},\ -\gamma\,\frac{L}{2}\right)$$

in Mary's coordinates.

(d) (i) Mary forgot about the Relativity of Simultaneity. (ii) The paint bomb ahead of her ship went off $v_{\text{rel}}\,\gamma\,\frac{L}{2}$ too early, and the paint bomb at the back of her ship went off $v_{\text{rel}}\,\gamma\,\frac{L}{2}$ too late.

(e) $\frac{L}{2} = 20\text{ m}$, $v_{\text{rel}} = \frac{3}{5}$, $\gamma = \frac{5}{4}$, so $\text{MF} = \left(0,\ \frac{L}{2\gamma}\right) = \left(0,\ 20\text{ m}\cdot\frac{4}{5}\right) = (0,\ 16\text{ m})$ and $\text{MB} = (0,\ -16\text{ m})$, which is well within the ends of his lab.

(f) $\text{GF} = (-15\text{ m},\ 25\text{ m})$ and $\text{GB} = (15\text{ m},\ -25\text{ m})$ in Mary's coordinates.

Problem 2 — The Famous Equation, $E = mc^2$

(a) Because wristwatch time and coordinate time are the same for a particle at rest in a coordinate system, in that system $E = m\,\frac{\Delta t}{\Delta \tau} = m$.

(b) $\Delta x = v \Delta t$

(c) $\Delta \tau = \sqrt{(\Delta t)^2 - (\Delta x)^2}$.

(d) $\Delta \tau = \sqrt{(\Delta t)^2 - (\Delta x)^2} = \sqrt{(\Delta t)^2 - (v \Delta t)^2} = \sqrt{1 - v^2}\,\Delta t = \Delta t / \gamma$.

So,

$m\,\frac{\Delta t}{\Delta \tau} = \mathrm{m}\gamma$.

(e) Just cancel the m:

$\frac{\Delta t}{\Delta \tau} = \gamma$

(f) $p = m\,\frac{\Delta x}{\Delta \tau} = m\,\frac{\Delta t}{\Delta \tau}\,\frac{\Delta x}{\Delta \mathrm{t}} = m\gamma\,\frac{\Delta x}{\Delta \mathrm{t}} = m \gamma v$.

(g) $E = m\gamma = m\,\frac{1}{\sqrt{1-0.995^2}} = m\,\frac{1}{\sqrt{1-0.99}}\;m\,\frac{1}{\sqrt{0.01}} = 10\,m$

(h) $E = m\gamma = m\,\frac{1}{\sqrt{1-0.99995^2}} = m\,\frac{1}{\sqrt{1-0.9999}}\;m\,\frac{1}{\sqrt{0.0001}} = 100\,m$

Problem 3 — Inward-Going Photons on the [t, r]-Slice Outside the Event Horizon

We start with the metric in Eq. (5) of TWB Chapter 3: $(\Delta\tau)^2 = (1 - 2\,M/r)\,(\Delta t)^2 - \frac{(\Delta r)^2}{1-2\,M/r} - r^2(\Delta\phi)^2$.

(a) $(\Delta\tau)^2 = (1 - 2\,M/r)\,(\Delta t)^2 - \frac{(\Delta r)^2}{1-2\,M/r}$.

(b) $0 = (1 - 2\,M/r)\,(\Delta t)^2 - \frac{(\Delta r)^2}{1-2\,M/r}$

or

$(1 - 2\,M/r)\,(\Delta t)^2 = \frac{(\Delta r)^2}{1-2\,M/r}$

or

$(\Delta t)^2 = \frac{(\Delta r)^2}{(1-2\,M/r)^2}$.

So,

$|\Delta t| = \frac{|\Delta r|}{1-2\,M/r}$.

(c) $\Delta t = \frac{-\Delta r}{1-2\,M/r}$

(d) $\Delta y = \frac{-\Delta x}{1-2/x}$

(e) We will start at $x = 4.0$ and finish at $x = 2.4$ and do that in 8 steps of $\Delta x = -0.2$ each. Fill in the final two columns of the table below using the formula you found in (d).

```
In[•]:= table2 = Table[
   {i, 4.0 - 0.2 i, "     -0.2          ",
    If[True, "", Round[0.2 / (1 - 2 / (4.0 - 0.2 i)), 0.01]]}, {i, 0, 7, 1}];
  table2WithHeader =
   PrependTo[table2, {"i", " xᵢ", "      Δx", "Δyᵢ    ", "Cumulative change in y"}];
  TableForm[table2WithHeader]
Out[•]//TableForm=
```

i	x_i	Δx	Δy_i	Cumulative change in y
0	4.	-0.2	0.40	0.40
1	3.8	-0.2	0.42	0.82
2	3.6	-0.2	0.45	1.27
3	3.4	-0.2	0.49	1.76
4	3.2	-0.2	0.53	2.29
5	3.	-0.2	0.60	2.89
6	2.8	-0.2	0.70	3.59
7	2.6	-0.2	0.87	4.46

In the spirit of continuing to not need a calculator, below is a table of miscellaneous combinations, some of which are relevant.

```
In[•]:= TableForm[
   PrependTo[Table[{Round[x, 0.1], Round[0.2 / x, 0.01], Round[0.2 / (1 - 2 / x), 0.01],
      Round[0.2 / Sqrt[1 - 2 / x], 0.01]}, {x, 4.0, 2.6, -0.2}],
    {"xᵢ", "0.2/xᵢ", "0.2/(1-2/xᵢ)", "0.2/sqrt(1-2/xᵢ)"}]]
Out[•]//TableForm=
```

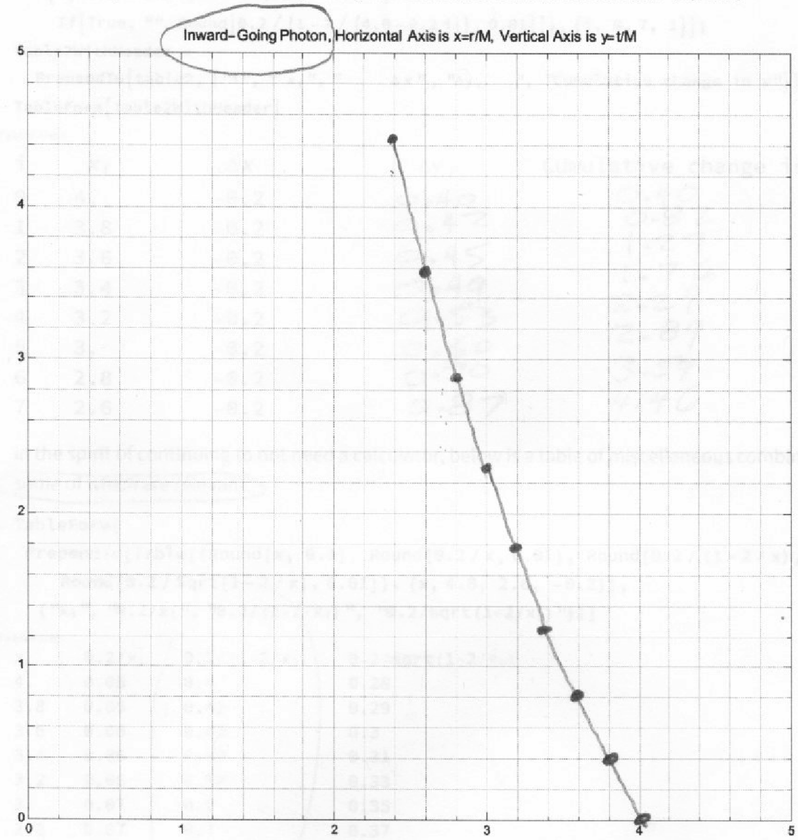
x_i	$0.2/x_i$	$0.2/(1-2/x_i)$	$0.2/\text{sqrt}(1-2/x_i)$
4.	0.05	0.4	0.28
3.8	0.05	0.42	0.29
3.6	0.06	0.45	0.3
3.4	0.06	0.49	0.31
3.2	0.06	0.53	0.33
3.	0.07	0.6	0.35
2.8	0.07	0.7	0.37
2.6	0.08	0.87	0.42

This is the only column that is relevant.

(g) Graph y vs. x starting with $x = 4.0$, $y = 0.0$, and working your way inward. The last row of the table is the row that goes from $x = 2.6$ to $x = 2.4$.

In[ ]:=
```
Plot[{}, {x, 0, 5}, PlotRange → {{0, 5}, {0, 5}}, AspectRatio → 1, GridLines → {Range[0, 5, 0.2], Range[0, 5, 0.2]},
 Frame → True, PlotLabel → "Inward-Going Photon, Horizontal Axis is x=r/M, Vertical Axis is y=t/M"]
```

Out[ ]=



This is a numerical solution to a graph that you made on Problem Set 5 (see Problem 3(d)).

4. Hanging Out Near Gargantuan

(Gargantuan is the black hole in *Interstellar.*)

(a) Again, we start with the metric in Eq. (5) of TWB Chapter 3,

$(\Delta\tau)^2 = (1 - 2M/r)(\Delta t)^2 - \frac{(\Delta r)^2}{1-2M/r} - r^2(\Delta\phi)^2$, and put in $\Delta r = \Delta\phi = 0$ to get:

$(\Delta\tau)^2 = (1 - 2M/r)(\Delta t)^2$

(b) Take the square root and make the assumptions suggested:

$\Delta\tau = \sqrt{1 - 2M/r}\,\Delta t$

(c) For the clocks at either of the two positions, we use the Part (b) equation:

$T = \Delta\tau_L = \sqrt{1 - 2M/r_L}\,\Delta t_L$

or

$\Delta t_L = T\Big/\sqrt{1 - 2M/r_L}\,.$

$T = \Delta\tau_H = \sqrt{1 - 2M/r_H}\,\Delta t_H$

or

$\Delta t_H = T\Big/\sqrt{1 - 2M/r_H}\,.$

(d) With $r_H = \infty$, $\Delta t_H = T$, and $\Delta t_L = T\Big/\sqrt{1 - 2M/r_L}$ is larger than Δt_H, because it has a number less than 1 in the square root of the denominator.

(e) Per part (d), for a given amount of coordinate time, the person at r_L is aging less. Put in $r_L = \frac{200M}{99}$.

$\Delta t_L = T\Big/\sqrt{1 - 2M/(200M/99)} = T\Big/\sqrt{1 - 99/100} = T\Big/\sqrt{0.01} = T/0.1 = 10\,T.$

If a person hangs out for 7 years of wristwatch time at r_L, 70 years of coordinate time and proper time will have elapsed for the person at $r_H = \infty$!

("This little maneuver is going to cost us 63 years." Apologies to Cooper. It was 51 years in the movie.)