Cosmology — Problem Set 9 — Coordinate Transformations and Hubble's 1929 Discovery

Do not use infinitesimal notation in Problems 1-3. Use Δ's as I do in the attached derivation. If you use infinitesimals, I will give you 0s Why? Because I don't think anyone is genuinely ready for infinitesimals. It is just meaningless copying to use them.

Problem 1

The Farm Manager has gridded a field into 40 equal parts in each direction. Mathematically, we have an x-coordinate that runs from 0 to 40, and a y-coordinate that runs from 0 to 40. Let's say the field in physical units is 200 feet by 200 feet. Here is the metric for the field

$$(\Delta s)^2 = W^2\left[(\Delta x)^2 + (\Delta y)^2\right]$$

where $W = \frac{L}{N}$, $L = 200$ feet, and $N = 40$. $\frac{L}{N}$ can be thought of a scale factor. It relates the total size of the field (the universe) to whatever measuring device the Farm Manager used when gridding the field.

Ignore the numbers in the above description. Those were just to help you visualize the situation. Use the variables x, y, L, N, ϕ, and ψ in what follows.

(a) A very mathematical student joins the farm team and likes to measure everything in radians, whether or not that makes sense in any given situation. The very mathematical student grids off the field in terms of new variables ϕ and ψ according the following formulae:

$$x = N\sin\phi$$
$$y = N\sin\psi$$

The variables ϕ and ψ each go from 0 to $\frac{\pi}{2}$.

Using all the tricks we used in class (e.g., $\sin(A + B) = \sin A\cos B + \cos A\sin B$, approximations, etc.), use the above equations for x and y along with these equations,

$$x + \Delta x = N\sin(\phi + \Delta\phi)$$
$$y + \Delta y = N\sin(\psi + \Delta\psi)$$

to find relationships valid to first order between Δx and $\Delta\phi$, and between Δy and $\Delta\psi$.

(b) Another of the tricks we used in class was to write

$$(\Delta s)^2 = W^2\left[(\Delta x)^2 + (\Delta y)^2\right]$$

as

$$(\Delta s)^2 = \left(\frac{L}{N}\right)^2\left[\left(\frac{\Delta x}{\Delta\phi}\right)^2 (\Delta\phi)^2 + \left(\frac{\Delta y}{\Delta\psi}\right)^2 (\Delta\psi)^2\right]$$

Then use what you found in Part (a) to replace $\left(\frac{\Delta x}{\Delta\phi}\right)^2$ and $\left(\frac{\Delta y}{\Delta\psi}\right)^2$.

Discussion: In these new coordinates, it is no longer obvious that the field is a flat surface embedded in ordinary three-dimensional Euclidean space, that the Pythagorean theorem holds on the field, and that the relationship between the circumference of a circle and its radius is the usual $C = 2\,\pi r$. Hence we are not likely to use these coordinates. They have obscured the simple nature of the field. This is therefore a contrived coordinate transformation and a contrived problem. However, coordinate changes often reveal properties of a space, and this is a good warmup problem.

Problem 2 — Case I of the Friedman-Robertson-Walker Metric ($K = 0$)

Study the attached derivation of Case II of the Friedman-Robertson-Walker Metric ($K > 0$). Using Eqs. 1 and 5 on p. TWB pp. 14-3 and 14-4, derive Eq. 6.

This should be a quick and easy problem. After all, $K = 0$, so the metric in Eq. 1 is quite simple, and Eq. 5 is just a linear transformation.

Problem 3 — Case III of the Friedman-Robertson-Walker Metric ($K < 0$)

In the attached derivation of Case II of the Friedman-Robertson-Walker Metric ($K > 0$), I outlined how to do Case III ($K < 0$). Follow that outline to get from Eq. 1 on p. 14-3 to Eq. 16 on p. 14-6.

Let's call your final answer, part (c). As warmups to part (c), which you will need do the following:

(a) Use Eq. 14 on p. 14-6 to derive a remarkably simple relationship between $\sinh^2\chi$ and $\cosh^2\chi$.

(b) Use $e^{x+\Delta x} = e^x\, e^{\Delta x}$ and $e^{\Delta x} = 1 + \Delta\chi$ + negligible terms of order $(\Delta\chi)^2$ to get a simple expression for

$$\sinh(\chi + \Delta\chi) - \sinh\chi$$

valid to first order in $\Delta\chi$.

(c) Now you have some identities which you can use in obtaining the final answer.

Problem 4 — The Hubble Plot

You have Hubble's 1929 paper. In the original Hubble plot the vertical axis should be km/s. Not km. It is just a mislabeling.

(a) The slope of the line in the Hubble plot is what we mean by the expansion rate of the universe. What are the units of the slope of the line in Hubble's plot? What is the value of the slope? This is called the Hubble constant.

(b) If the vertical axis were in meters/second and the horizontal axis were in meters, what would the units of the slope be? If you take the inverse of that slope what would the units be? This is called the Hubble time. What is the conversion factor between seconds and billions of years? Use that 1 year = 365.25 * 24 * 60 * 60 seconds (this amount of time is called a Julian year). The Hubble time is usually quoted in billions of years.

Discussion: If the expansion rate of the universe is constant, the Hubble time is the age of the universe.

(c) The Hubble constant we observe today is called H_0. It is the current expansion rate of the universe. The 0 means current. The expansion rate may be decreasing, increasing, or staying the same. The currently accepted value is H_0 = 67 km/s per Mpc. Find Hubble's estimate in his 1929 paper. You should find that it is shockingly different.

Discussion: Rarely is an initial quantitative measurement of a correctly formulated concept so far off from a modern estimate. Obviously it hasn't changed in the 90 or so years since Hubble did his measurements. It's just that the measurement errors were large.

(d) Convert the currently-accepted value of $H_0{}^{-1}$ into the more convenient (for humans) units of billions of years.