Cosmology — Problem Set 10

Problem 1 — Black-Body Radiation

The Black-Body Radiation formula was given in class. It follows from treating light as quanta which we call photons. We aren't going to derive it. We can do quantum mechanics next spring, if there is interest.

The first and most important thing is to remember what the formula represents: if you have a filter that allows a narrow range of light from ω to $\omega + \Delta\omega$ to pass through it, then the intensity of the light coming from the surface of the black body and passing through the filter will be:

$$\frac{\hbar\omega^3\,\Delta\omega}{\pi^2 c^2\left(e^{\hbar\omega/kT}-1\right)}$$

In this formula $\hbar$ is $\frac{h}{2\pi}$ and h is Planck's constant. It is a fundamental constant of nature. k is Boltzmann's constant. We now understand it as the conversion factor between T (the temperature) and energy. T is measured in Kelvin. Kelvin is abbreviated K. Don't confuse k with K.

In an ideal gas at temperature T, the kinetic energy of each atom in the gas is $\frac{3}{2}\,kT$. In an ideal gas, *the only energy* is kinetic energy. So we now understand that temperature is just measuring energy and k (and a factor of $\frac{3}{2}$) is the conversion factor that tells how much energy, but when they were defining the temperature scale, they didn't know that. The approximate values of these constants are:

$h = 6.626 \times 10^{34}\ \mathrm{J \cdot s}$

$k = 1.381 \times 10^{-23}\ \mathrm{J/K}$

The first formula above is usually written:

$$I_\omega\,\Delta\omega \quad\text{where}\quad I_\omega = \frac{\hbar\omega^3}{\pi^2 c^2\left(e^{\hbar\omega/kT}-1\right)}$$

(a) What are the units of $\hbar\omega$, ω, $\Delta\omega$, and c? Acceptable choices in the SI system are combinations of Joules (J), meters (m) and seconds (s).

(b) Using your answer from part (a), what are the units of $I_\omega\,\Delta\omega$?

(c) To get the luminosity, L_ω, of a black body in the range ω to $\omega + \Delta\omega$, you have to multiply by the surface area of the black-body. What are the units of $I_\omega\,\Delta\omega A$ where A is the surface area of the black body (typically a star). The surface area of a star is of course enormous.

DISCUSSION: If you want to the total intensity or luminosity at all frequencies, you have to integrate I_ω from $\omega = 0$ to $\omega = \infty$. I'm not going to derive that, because it requires some integration tricks, which people would forget almost as fast as I showed them. Want a couple of points of extra credit? Find a tool (like WolframAlpha) that can do the integral symbolically, record the answer and how you got the symbolic manipulation tool to do it, and show the rest of the class.

(d) The wavelength of orange light is 600nm. Divide this by 3×10^8 m/s to get the period and take the reciprocal to get the frequency. Finally, multiply by 2π to get ω. What is ω for orange light?

(e) Use a graphing tool to graph I_ω from $\omega = 0$ to ω equal to 3x whatever you got in Part d for our Sun, which has $T = 5777$ K. By hand, mark the position of orange light on the horizontal axis of the plot.

(f) Repeat (e) for a star of 10000K.

(g) Repeat (e) for a star of 3000K.

Problem 2 — Angular Momentum Conservation Derivation

In class on Tuesday, I derived Eq. 38. I followed exactly the methods I used in the handout titled "Conservation of Map Energy."

The authors say another conservation law can be derived, but they don't give the proof. It is Eq. 41. I'd write Eq. 41 as:

$$\frac{Q_\phi}{m} = R^2 S^2 \frac{\Delta\phi}{\Delta\tau} = RS\left(RS\frac{\Delta\phi}{\Delta\tau}\right) = RSp_\phi \text{ where } p_\phi \equiv RS\frac{\Delta\phi}{\Delta\tau}.$$

(a) In my proof of conservation of map energy, and in class on Tuesday when I proved that

$$\frac{Q_\chi}{m} = R^2 \frac{\Delta\chi}{\Delta\tau} = R\left(R\frac{\Delta\chi}{\Delta\tau}\right) = Rp_\chi \text{ (where } p_\chi \equiv R\frac{\Delta\chi}{\Delta\tau})$$

is conserved, at an intermediate step, I calculated $\tau_A + \tau_B$. In class on Tuesday, we had:

$$\tau_A + \tau_B = \sqrt{J_A - R(t_A)^2 (\chi_2 - \chi_1)^2} + \sqrt{J_B - R(t_B)^2 (\chi_3 - \chi_2)^2}$$

J_A was "junk not involving χ_2" and J_B was "other junk not involving χ_2".

In the derivation of Eq. 41, what would you have at this step for $\tau_A + \tau_B$? J_A would be "junk not involving ϕ_2" and J_B would be "other junk not involving ϕ_2".

(b) Write out J_A and J_B so that I know you have actually worked through how the proof is done.

(c) Take $\frac{d}{d\phi_2}$ of $\tau_A + \tau_B$ and set it equal to zero (you need to use the chain rule a few times, but it is otherwise just derivatives of powers). What is the resulting equation?

Problem 3 — The Magnitude System and Cepheid Variables

On p. 14-17, TWB introduce the terms “flux density” and “luminosity.”

Two stars with the same luminosity have different flux densities if they are at different distances. If I call the luminosities L_1 and L_2, the flux densities, I_1 and I_2 and the distances d_1 and d_2, then the formula relating them all is

$I_1 = L_1 / (4\pi d_1^2)$ and $I_2 = L_2 / (4\pi d_2^2)$

These are nice simple formulae. The light is escaping and therefore is spread out over the surface of a sphere of radius d_1 (and d_2) respectively. These formula have to be modified in curved, expanding spacetime, but let’s leave that aside for a moment.

In the above formulae, if you take the ratio of I_1 to I_2 you can of course cancel the factors of 4π. It is common to take the ratio of the two formula and take log base 100 and multiply by 5.

$$5 \log_{100} \frac{I_1}{I_2} = 5 \log_{100} \frac{L_1/d_1^2}{L_2/d_2^2}$$

Then you take I_2 to be the flux density of a standard star, Vega, and you define:

$$m = -5 \log_{100} \frac{I}{I_{\text{Vega}}}$$

This is the magnitude system and the authors say to blame the ancient Greeks for it. Well, that isn’t quite fair. Their system wasn’t nearly so precise. We didn’t get a precise magnitude system until the beginning of the 1900s, but it is true that it was set up to roughly agree with the very qualitative system that the Greeks had. The Greeks didn’t have photometers. They just looked at the stars with their eyes and assigned magnitudes from 1 to 5 with 1 being the brightest and 5 being the dimmest.

A consequence of the above equations is:

$$m = -5 \log_{100} \frac{L/(4\pi d^2)}{I_{\text{Vega}}}$$

The modification for curved expanding spacetime is that you have to replace d by "luminosity distance" d_L. So you have

$$m = -5 \log_{100} \frac{L/\left(4\,\pi\, d_L{}^2\right)}{I_{\text{Vega}}}$$

(a) Just to make sure you are paying attention to the formula, suppose a star has four times the luminosity, L, as Vega and it is twice as far away. What will its m be? Suppose a star has $I = L/\left(4\,\pi\, d_L{}^2\right)$ that is $3.4\, I_{\text{Vega}}$. What is its magnitude?

(b) IMAGINE, the star is at $d = 10$ pc instead of its actual distance. Then define

$$M = -5 \log_{100} \frac{L/(10\text{ pc})^2}{I_{\text{Vega}}}$$

Derive Eq. 42 for $m - M$ using the above definitions for m and M and miscellaneous properties of the logarithm.

NB: The formula TWB gave has $\log_{10}$ in it, not $\log_{100}$. That is not a typo.

(c) It is common to use the equation you just derived the other way. You have m and M, and you want to know d_L.

Solve the equation in part (b) for d_L.

(d) This is a good time to recollect the definition of the parsec. It is $1\text{ pc} \equiv 1\text{ AU}\,\frac{360}{2\,\pi} \cdot 60 \cdot 60$. The AU is the distance from the Earth to the Sun. Of course the Earth traces an ellipse, so "the distance from the Earth to the Sun" needs defining, and astronomers chose

1 AU = 149 597 870.7 km

So using these expressions, how much is a parsec?

(e) Convert this to light-years. You will need the speed of light, which is exactly 299,792,458 meters/second thanks to the fact that today the meter has been redefined in terms of the second (as we discussed many weeks ago), and you will need the definition of the year in terms of days, which isn't obvious, because like the Earth-Sun distance, it is just some astronomical accident. Astronomers define the year by 1 year = 365.25 days. If it were up to me, I would have chosen 365.24 days, because that is closer to the true value that it takes the Earth to go around the Sun once, which is measured to be 364.2422 days, but hey, it's just a definition, so we will use 1 year 365.25 days in order to get the correct conversion factor between parsecs and light-years.