Cosmology — Problem Set 5 — Solution

1. Taylor-Wheeler-Bertschinger (TWB) Problem 3.1 on p. 3-35

A. Make estimates using $\Delta\sigma = \dfrac{\Delta r}{(1 - 2\,M/r)^{1/2}}$.

M is in km so let's convert that to $M = 5000$ m. The location of the event horizon is $2\,M = 10\,000$ m. For cases (a)-(e), the values in meters are 50000, 15000, 10100, 10010, and 10001, respectively. Δr is 1m in all cases.

In[6]:= $\texttt{Table}\!\left[\left\{\texttt{r}, \texttt{N}\!\left[\dfrac{1}{(1 - 10\,000\,/\,\texttt{r})^{1/2}}\right]\right\}, \{\texttt{r}, \{50\,000, 15\,000, 10\,100, 10\,010, 10\,001\}\}\right]$

Out[6]= $\{\{50\,000,\ 1.11803\},\ \{15\,000,\ 1.73205\},\ \{10\,100,\ 10.0499\},\ \{10\,010,\ 31.6386\},\ \{10\,001,\ 100.005\}\}$

So at (a) $r = 50\,000$ m, $\Delta\sigma = 1.12$ m, (b) $r = 15\,000$ m, $\Delta\sigma = 1.73$ m, (c) $r = 10\,100$ m, $\Delta\sigma = 10.05$ m, (d) $r = 10\,010$ m, $\Delta\sigma = 31.63$ m, and (e) $r = 10\,001$ m, $\Delta\sigma = 100.01$ m.

B. Make estimates using Eq. 18 on p. 3-16:

$$\sigma = z_H\,\sqrt{z_H{}^2 - 2\,M} + 2\,M\,\ln\!\left(z_H + \sqrt{z_H{}^2 - 2\,M}\right) - z_L\,\sqrt{z_L{}^2 - 2\,M} + 2\,M\,\ln\!\left(z_L + \sqrt{z_L{}^2 - 2\,M}\right)$$

where $z_H = \sqrt{r_H}$ and $z_L = \sqrt{r_L}$ and $r_H = r_L + 1$. Everything is in meters and our answer will be in meters.

In[14]:= $\texttt{f[z\_] := z\ Sqrt}\!\left[\texttt{z}^2 - 10\,000\right] + 10\,000\ \texttt{Log}\!\left[\texttt{z + Sqrt}\!\left[\texttt{z}^2 - 10\,000\right]\right]$

In[15]:= $\texttt{Table}\!\left[\{\texttt{r}, \texttt{N}[\texttt{f}[\texttt{Sqrt}[\texttt{r} + 1]] - \texttt{f}[\texttt{Sqrt}[\texttt{r}]]]\}, \{\texttt{r}, \{50\,000, 15\,000, 10\,100, 10\,010, 10\,001\}\}\right]$

Out[15]=
$\{\{50\,000,\ 1.11803\},\ \{15\,000,\ 1.73199\},\ \{10\,100,\ 10.0251\},\ \{10\,010,\ 30.8856\},\ \{10\,001,\ 82.8488\}\}$

So at (a) $r = 50\,000$ m, $\Delta\sigma = 1.12$ m, (b) $r = 15\,000$ m, $\Delta\sigma = 1.73$ m, (c) $r = 10\,100$ m, $\Delta\sigma = 10.03$ m, (d) $r = 10\,010$ m, $\Delta\sigma = 30.89$ m, and (e) $r = 10\,001$ m, $\Delta\sigma = 82.85$ m.

C. To three significant figures, (a), (b), and (c) agree. Then (d) is deviating even at two significant figures (32m vs 31m), and (e) is deviating even in the leading digit (100m vs. 83m).

If one wants an approximation that is good to much better than a fraction f (the authors are suggesting three significant figure accuracy, which is $f = 0.001$), then

$\dfrac{1}{(1 - 10\,000/r_L)^{1/2}} - \dfrac{1}{(1 - 10\,000/(r_L + 1))^{1/2}}$ needs to be much less than f. If $r_L = 10\,000 + d$ then the criterion for acceptable accuracy becomes:

$$\frac{1}{(1 - 10\,000/(10\,000 + d))^{1/2}} - \frac{1}{(1 - 10\,000/(10\,000 + d + 1))^{1/2}} \ll f$$

d

$$\frac{1}{(1-10\,000/r_L)^{1/2}} - \frac{1}{(1-10\,000/(r_L+1))^{1/2}}$$

f $r_L = 10\,000 + d$

I am going to discuss the criterion and rearrange that to a statement about *d* in class, since it requires a bit of algebra and some good arguments that will be unlikely to rub off if it is buried in a problem set solution.

Anybody who succeeds in pushing this through convincingly given your current experience level with approximations deserves some extra credit :)

2. TWB Problem 3.2 on p. 3-35

The statement the authors are asking you to verify is actually in Section 3.3 not 3.4.

The statement was: for a shell at 695980km and another shell at 695981km:

"the directly measured distance between the two would be not 1 kilometer, but 2 millimeters *more* than 1 kilometer."

The hint was to use this approximation from inside the front cover:

$$(1 + \epsilon)^n \approx 1 + n\epsilon$$

The authors want us to apply that to:

$$\Delta\sigma = \frac{\Delta r}{(1-2\,M/r)^{1/2}}$$

with $n = -\frac{1}{2}$ and $\epsilon = -\frac{2\,M}{r}$.

So,

$$\Delta\sigma = \left(1 + \frac{1}{2}\,\frac{2\,M}{r}\right)\Delta r = \Delta r + \frac{M}{r}\,\Delta r$$

So we have the 1km, that is the first time Δr. The additional 2mm in "2 millimeters *more* than 1 kilometer" must come from:

$$\frac{M}{r}\,\Delta r = \frac{1.45\,\text{km}}{695\,980\,\text{km}}\,1\,\text{km} = \frac{1\,450\,000}{695\,980} \times 1\,\text{mm} = 2.1\,\text{mm}$$

Exactly as claimed.

3. Inward Falling Light — Outside the Event Horizon

Eq. 26 on p. 3-25 is:

$$t - t_1 = \pm\left(r - r_1 + 2\,M\ln\frac{r/M-2}{r_1/M-2}\right)$$

(a) If we put in $t_1 = 0$ and $r_1 = 3\,M$, the equation simplifies to:

$$t = \pm(r - 3\,M + 2\,M\ln(r/M - 2))$$

(b) There is a ± sign in your answer to Part (a). If the flashlight is directed inward then increasing t—which since we started with $t_1 = 0$ means positive t—must correspond to decreasing r. So which sign is going to give you positive times for decreasing r? Simplify the formula to have just that sign.

We need the minus sign if decreasing r is going to correspond to increasing t:

$$t = -(r - 3\,M + 2\,M\ln(r/M - 2))$$

(c) Plug in $r = 2.5\,M$ to double-check:

$$t = -(2.5\,M - 3\,M + 2\,M\ln(2.5\,M/M - 2)) = -M(-0.5 + 2\ln 0.5) = 1.886\,M$$

(d) Take the equation $t = -(r - 3\,M + 2\,M\ln(r/M - 2))$ and divide both sides through by M:
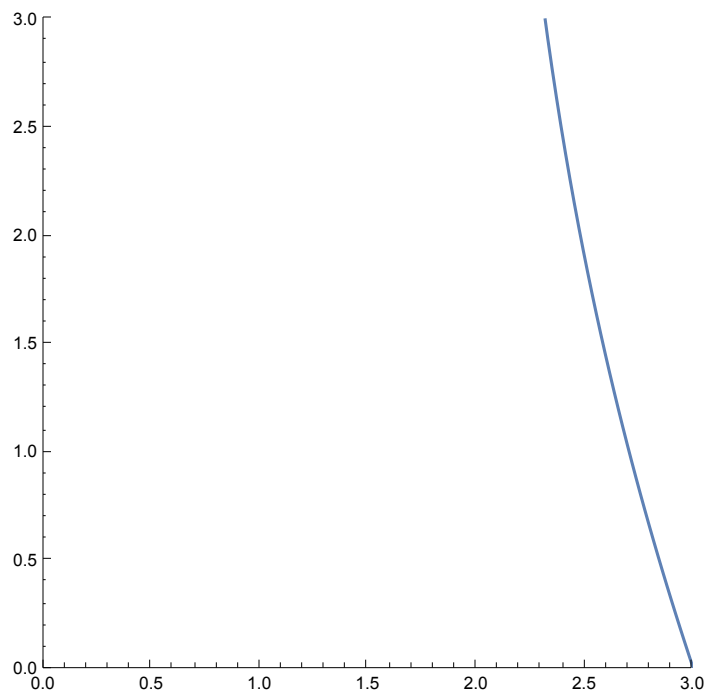
$$t/M = -(r/M - 3 + 2\ln(r/M - 2))$$

Oh, isn't that lovely. We see that if we think of t/M as a y-value and r/M as an x-value, then the equation is:

$$y = -(x - 3 + 2\ln(x - 2))$$

and that is something we can graph!

In[ ]:= `youtside[x_] := - (x - 3 + 2 Log[x - 2]);`
`Plot[youtside[x], {x, 2, 3}, PlotRange → {{0, 3}, {0, 3}}, AspectRatio → 1]`

Out[ ]=

4. Inward Falling Light — Inside the Event Horizon

At $r = 2\,M$ is the event horizon, and our coordinates behave poorly there. So we will skip over that zone and use the same procedure again, but this time using $t_1 = 0$ and $r_1 = 1.5\,M$.

(a) If we put in $t_1 = 0$ and $r_1 = 1.5\,M$, the equation simplifies to:

$$t = \pm\left(r - 1.5\,M + 2\,M \ln \frac{r/M-2}{-0.5}\right) = \pm(r - 1.5\,M + 2\,M \ln(4 - 2\,r/M))$$

(b) Looks like we need the minus sign again:

$$t = -(r - 1.5\,M + 2\,M \ln(4 - 2\,r/M))$$

(c) Plug in $r = 0.5\,M$ to double-check:

$$t = -(0.5\,M - 1.5\,M + 2\,M \ln(4 - 2 \cdot 0.5\,M/M)) = -M(-1 + 2\ln 3) = -1.197\,M$$

OOPS! I guess we needed the plus sign. Good thing we double-checked. The problem was that the 2 ln3 overwhelmed the -1. So, for (b) we should have had:

$$t = r - 1.5\,M + 2\,M \ln(4 - 2\,r/M)$$

and then for (c) we would have gotten
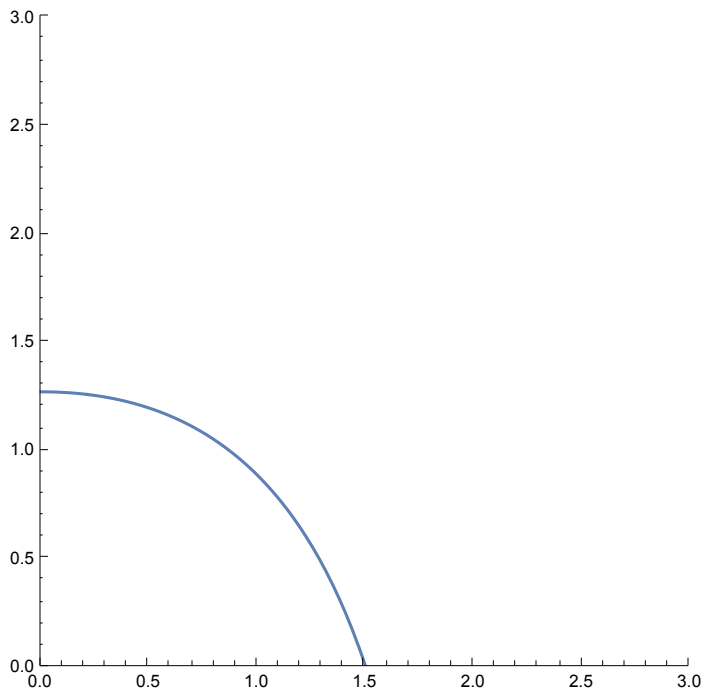
$$t = 1.197\,M$$

(d) Same as 3(d), except the region to make the graph in will be $0 < r/M < 1.5$.

In[●]:= `yinside[x_] := x - 1.5 + 2 Log[4 - 2 x];`
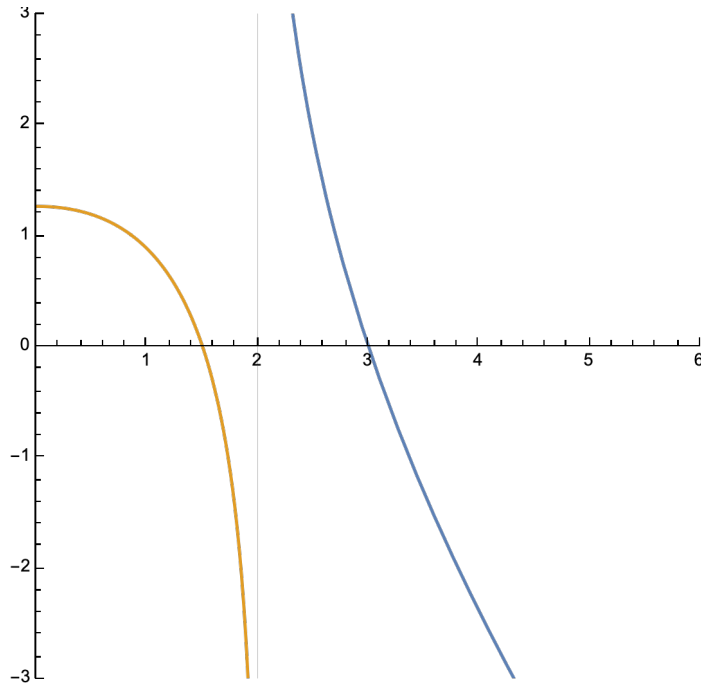`Plot[yinside[x], {x, 0, 1.5}, PlotRange → {{0, 3}, {0, 3}}, AspectRatio → 1]`

Out[●]=

I want to see more of both graphs and pile them onto the same graph, so I can more readily compare with Fig. 8 on p. 3-24:

In[ ]:= `Plot[{youtside[x], yinside[x]}, {x, 0, 8},`
  `PlotRange → {{0, 6}, {-3, 3}}, GridLines → {{2.0}, {}}, AspectRatio → 1]`

Out[ ]=

5. TWB Problem 3.5 on p. 3-37

A. We need $\Delta g / \Delta r$. We had that on Problem 2-8(a) on Problem Set 1. The answer was:

$$\frac{\Delta g}{\Delta r} = -\frac{2 M}{r^3}$$

The minus sign tells you that decreasing r is increasing g. In our study session, we said you might start to notice this when Δg is somewhere between 1 and 5% of Earth's gravity. But our authors are suggesting you get uncomfortable (feeling like you might be torn apart) when $\Delta g = g_E$ where $g_E = -9.8 \text{ m}/\text{s}^2$, the gravity at the surface of Earth. To make things rounder, let's just take $g_E = -10 \text{ m}/\text{s}^2$.

So we have:

$$\frac{g_E}{\Delta r} = \frac{-2 M}{r_{\text{ouch}}^3}$$

where Δr is the distance from your middle to your head.

So $r_{\text{ouch}} = \left(\frac{-2 M \Delta r}{g_E} \right)^{1/3}$

B. Our authors say to take $\Delta r = 1$ m and to take $M = 10\,000$ m in the crazy units where mass is measured in meters (and they say that $10\,000$ m is about 7 times the mass of our Sun).

So we put all the numbers in:

$$r_{\text{ouch}} = \left(\frac{2 \cdot 10\,000 \text{ m} \cdot 1 \text{ m}}{10 \text{ m}/\text{s}^2} \right)^{1/3}$$

Oops, we have a problem. We need to convert the seconds to meters and the exact conversion factor is $299\,792\,458$ m $= 1$ s, which we can round to 3×10^8 m $= 1$ s.

$$r_{\text{ouch}} = \left(\frac{2000 \text{ m}}{1/(3 \times 10^8 \text{ m})^2} \right)^{1/3} = \left(\frac{18\,000}{10^{-16}} \right)^{1/3} \text{ m} = \left(180 \times 10^{18} \right)^{1/3} \text{ m} = 5.6 \times 10^6 \text{ m}$$

Or, since our authors are also using km a lot, $r_{\text{ouch}} = 5600$ km.