ΛCDM Cosmology

The Λ in ΛCDM is for Ω_Λ, the amount of dark energy in the universe.

The CDM is for "Cold Dark Matter," which in addition to the visible matter is the contribution to $\Omega_{matter,\,0}$ that we see today.

$\Omega_{radiation,\,0}$ is not in the acronym because it is negligible today.

The goals of this last set of notes is to get us from the results derived in TWB Section 15.6 to the results in TWB Section 15.9. We will not consider 15.7 and 15.8.

We will begin by summarizing results we have already obtained.

Starting Point

We know that

$t - t_0 = \int_1^{a(t)} \frac{db}{f(b)}$ where $f(b) = H_0\left[\Omega_{\text{rad},0}\left(\frac{1}{b^2} - 1\right) + \Omega_{\text{matter},0}\left(\frac{1}{b} - 1\right) + \Omega_{\Lambda,0}\, b^2\right]^{1/2}$.

This integral can be easily done in the radiation-dominated, the matter-dominated, and the dark-energy dominated cases, and you did as your first problem on Problem Set 12. The end result was that you had a formula for $a(t)$. Even though the integral cannot be easily done for other cases, it can *in principal* be done, and the result is again going to be a formula for $a(t)$. So given the Ω-values we get an $a(t)$.

Also on Problem Set 12, we derived a nice formula for the "look-back distance." We started with a result from Box 4 in Chapter 14

$d_0 = R(t_0)\,\chi_{\text{emit}} = R(t_0) \int_{t_{\text{emit}}}^{t_0} \frac{dt}{R(t)}$

Then we did some shenanigans with $1 + z(t) = \frac{1}{a(t)}$ to get

$d_0 = \int_0^z \frac{dw}{H(w)}$

See the solution to Problem Set 12 for the details.

One more result that I have frequently alluded to, and which was used to put the first equation above into the form you see it in, is that we believe the sum of the Ω-values is 1. This I am just not going to have time to review and explain to you. It comes from measurements of the cosmic microwave background radiation.

A Problem

We actually have a big problem, and the problem is, you can't go back in time and measure $a(t)$. So you aren't going to be able to figure out the best fit between the measurement of $a(t)$ and the Ω-values, because you don't have any direct measurement of $a(t)$!

What do you have?? Type Ia supernovae!!

Type Ia Supernovae

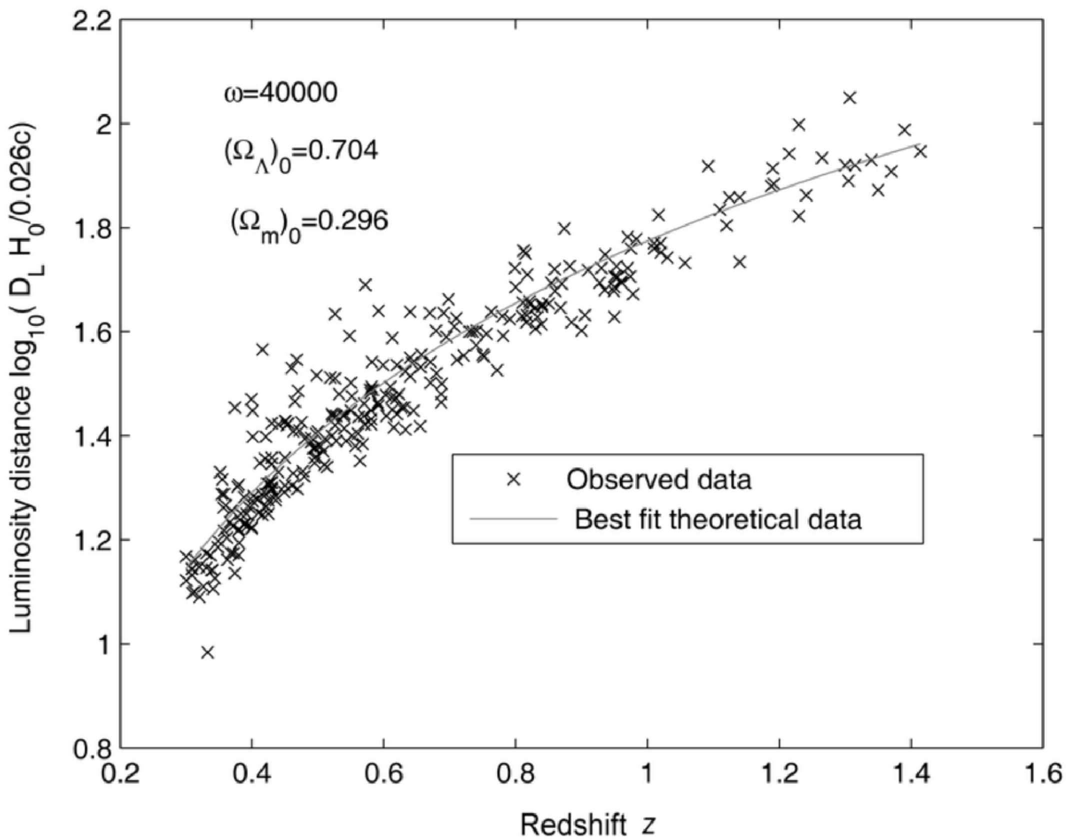
Thanks to the Hubble Space Telescope, we have lots of measurements of supernovae going off in faraway galaxies!! And a subset of these supernovae (called Type Ia supernovae) always have the same luminosity, L.

[Type Ia supernovae are special because they come from a binary star system where one star is a white dwarf, and it is accreting mass off the other star. The white dwarf eventually accretes so much mass that it collapses into a neutron star. The collapse always happens at the same mass! So it results in a cataclysmic flash with a known luminosity.]

For each supernova we have two measurements. On the one hand, we can determine each supernova's luminosity distance, d_L, where the luminosity distance is derived from the luminosity and the observed flux, F, by

$$F = \frac{L}{4 \pi d_L^2}$$

and on the other hand, we have each supernova's redshift, z, where that is measured spectroscopically. Here is a nice plot from a fairly recent theoretical paper on an alternative theory of gravity:



From: "Cosmological Parameters For Spatially Flat Dust Filled Universe In Brans-Dicke Theory," February 2017, *Research in Astronomy and Astrophysics* **17**(3):27, G. K. Goswami

The point of the plot is just that there are lots of supernovae to look at and to test theoretical fits to and there is a d_L vs. z curve.

The problem in the previous section is already emerging. The horizontal axis is z, not t. It is relatively straightforward to look at an object and figure out its redshift, but that doesn't tell us how long ago the light we are looking at left the object.

Luminosity Distance Almost to the Rescue

We know that the luminosity L of an object is spread out over a sphere of area $4\,\pi(R(t_0)\,S(\chi))^2$ and that its flux is decreased by $\frac{1}{(1+z)^2}$. Therefore its observed flux is

$$F = \frac{L}{4\,\pi(R(t_0)\,S(\chi))^2}\,\frac{1}{(1+z)^2}$$

Comparing this with the formula defining luminosity distance $F = \frac{L}{4\,\pi d_L^2}$ we have

$d_L = R(t_0)\, S(\chi)(1+\text{z})$

Remember that $S(\chi)$ can be χ (for a flat universe), sin χ (for a positive curvature universe), or sinh χ (for a negative curvature universe.

To keep life simple, we'll just do the flat universe case, in which case

$d_L = R(t_0)\, \chi \cdot (1 + z)$

So

$\frac{d_L}{1+z} = R(t_0)\, \chi_{\text{emit}} = R(t_0) \int_t^{t_0} \frac{dt}{R(t)} = \int_t^{t_0} \frac{dt}{a(t)}$

In the last step, I have used that $R(t)\, \text{d}\chi = dt$ which you will recall is true for light, and gives us a formula for χ_{emit} that goes all the way back to Eq. 22 on TWB p. 14-9.

Here comes some more integral calculus that you probably don't have at your disposal, but I need to use it. Differentiate both sides of the equation with respect to z.

On the LHS, we'll just leave it as $\frac{d}{dz} \frac{d_L}{1+z}$. On the RHS the only z-dependence is via the lower limit of integration, and that calls for the Fundamental Theorem of Calculus. Using that, we have:

$\frac{d}{dz} \frac{d_L}{1+z} = -\frac{dt}{dz} \frac{1}{a} = -\frac{dt}{dz} (1 + z)$

The minus sign comes in because the dependence was on the lower (not the upper) limit of integration. Anyway, think of this equation as:

$\frac{\Delta z}{1+z} \frac{d}{dz} \frac{d_L}{1+z} = -\Delta t$

Sum. The RHS is $-(t(z) - t_0)$. The LHS is $\int_0^z \frac{1}{1+z} \left[\frac{d}{\text{dz}} \frac{d_L}{1+z} \right] dz$. So

$t_0 - t(z) = \int_0^z \frac{1}{1+z} \left[\frac{d}{\text{dz}} \frac{d_L}{1+z} \right] dz$

This is perhaps a bit frustrating. Not only was it a lot of decently complicated applications of calculus, but TWB announce on p. 15-32 that we don't know $d_L(z)$ well enough, experimentally, to do the integral and discover $t(z)$. But we do have a beautiful thing here. We have a formula for $t(z)$, and that is why TWB derived it.

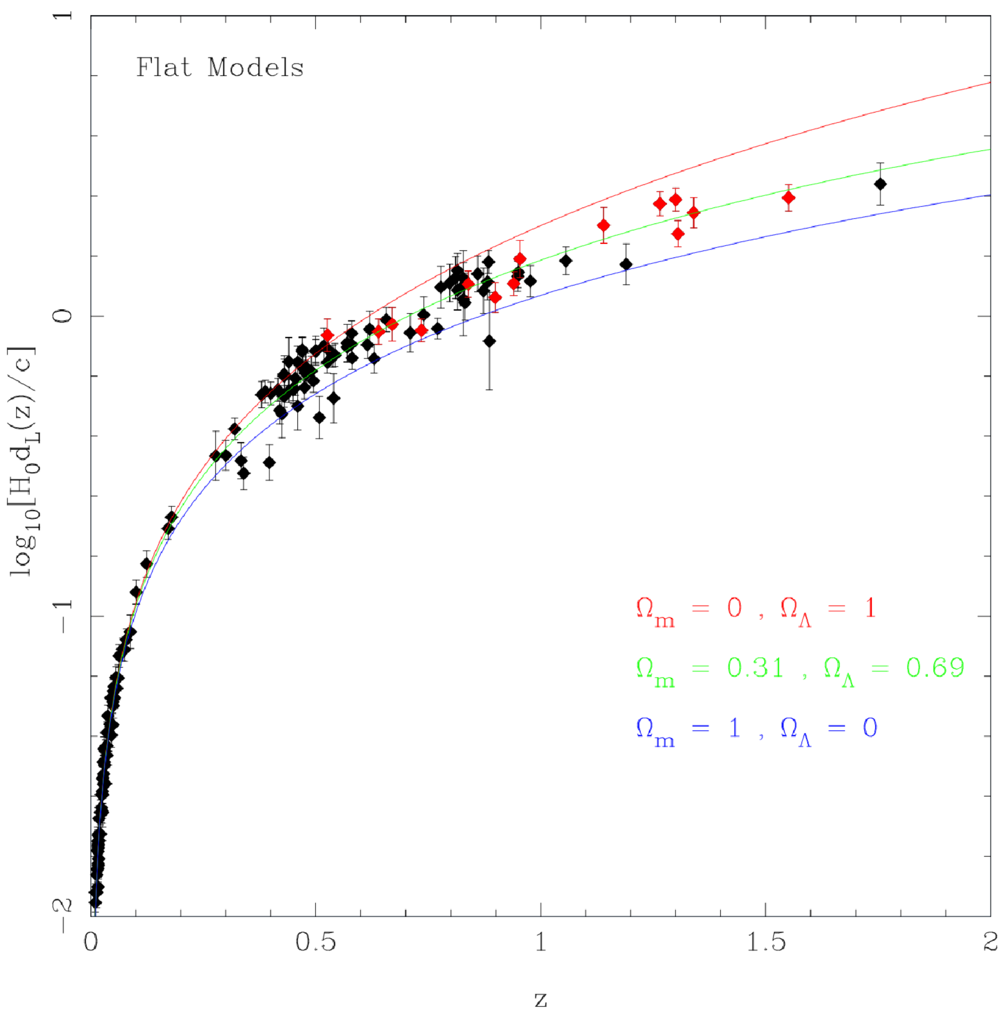
There is another relationship that gets you $t(z)$ and our authors also derive it and discover it is not useful. It is

$$t_0 - t(z) = \int_0^z \frac{dw}{(1+w)\,H(w)}$$

That one isn't useful because we know $H(t)$ from $H(t) = \frac{1}{a(t)}\,\frac{da(t)}{dt}$ but that doesn't tell us $H(w)$.

So What Analysis is Actually Done?

What is actually done is that you calculate $d_L(z)$ as a function of z for various values of $\Omega_{\text{mat},0}$ and $\Omega_\Lambda$usually with the constraint that their sum is 1 because that is known from the cosmic microwave background radiation.Then you put the supernovae data on the plot and see how good the fit is. Here is an example:

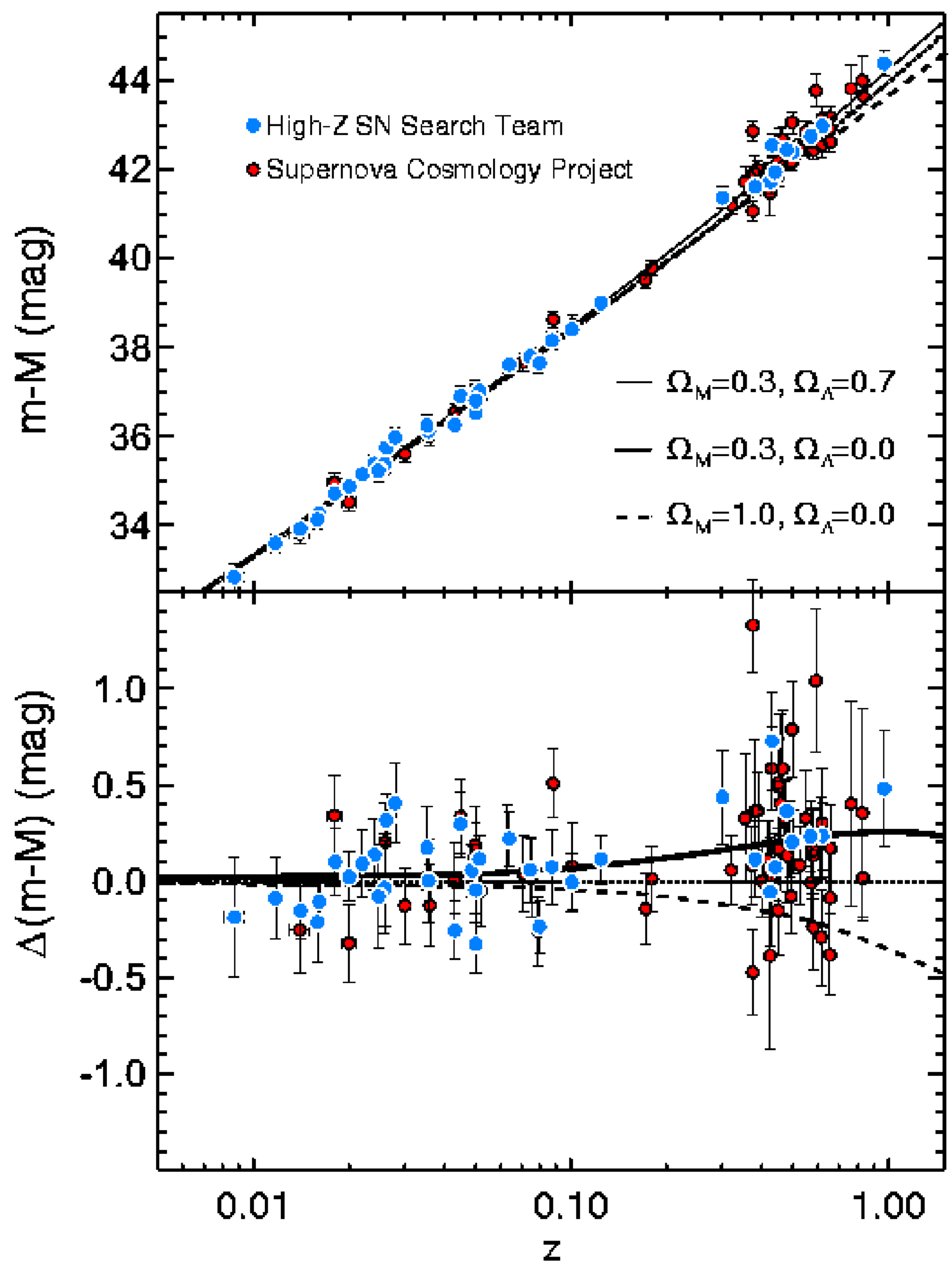


This analysis is from a relatively recent paper, "Cosmological parameters from supernova observations: A critical comparison of three data sets," *Astron. Astrophys.* **429** (2005) 807, T. Roy Choudhury and T.

Padmanabhan.

Goodness of Fit

Well, for each combination of the Ω-values, you get a goodness of fit. If the fit is poor, we claim that that combination of Ω-values is unlikely and if the fit is good, we say it is likely. Here is the actual best fit from the original 1997 papers:



“Measurements of the Cosmological Parameters Omega and Lambda from the FirstSeven Supernovae at z>= 0.35,” Perlmutter, S., *et al,* ApJ **483 (**1997) 565 and “Observational Evidence from Supernovae for an Accelerating Universe and a Cosmological Constant,” Riess, A.G. *et al,* AJ **116** (1998) 1009
© 1998.

The Expansion of the Universe is Accelerating

We can study the goodness of fit and the Ω at the end of TWB Chapter 15:

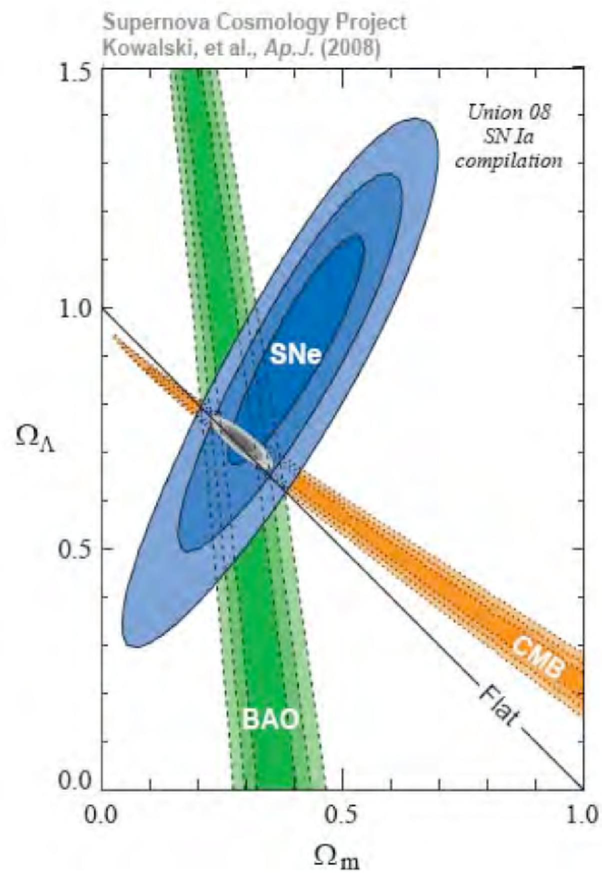


FIGURE 9 The Omega Diagram. Parameters Ω_m and Ω_Λ are called $\Omega_{mat,0}$ and $\Omega_{\Lambda,0}$ in this chapter. Relative amounts of matter and vacuum energy in the universe at present corresponds to the relatively tiny region of intersection of three sets of measurements: Type Ia supernovas (SNe), the cosmic microwave background radiation (CMB), and "baryon acoustic oscillations" (BAO). Darkest regions represent a statistical 68% confidence level and the lighter two represent statistical 95% and 99.78% confidence levels, respectively. The straight line represents conditions for a flat Universe.

The claim is that the universe is accelerating its expansion rate, and you know from solving the Friedmann equation, that that is caused by dark-energy domination of the universe. The best fit has $\Omega_\Lambda = 0.7$. Our book has been quoting 0.73 + / −0.03 as the best estimate.

However, as my closing explanation, rather than spending a lot of time on Figure 9, I'd like to see if I can convince you of the non-zero Ω_Λ just by looking at the Hubble plot above for the Type Ia supernovae.