Cosmology — Problem Set 9 — Solution

Problem 1

(a) We want to discover what the metric $(\Delta s)^2 = \frac{L}{N}^2\left[(\Delta x)^2 + (\Delta y)^2\right]$ looks like in these coordinates:

$$x = N\sin\phi$$
$$y = N\sin\psi$$

where ϕ and ψ each go from 0 to $\frac{\pi}{2}$. To get Δx and Δy in terms of $\Delta\phi$ and $\Delta\psi$, subtract equations:

$$\Delta x = x + \Delta x - x = N\sin(\phi + \Delta\phi) - N\sin\phi$$
$$\Delta y = y + \Delta y - y = N\sin(\psi + \Delta\psi) - N\sin\psi$$

Use

$$\sin(\phi + \Delta\phi) = \sin\phi\cos\Delta\phi + \cos\phi\sin\Delta\phi$$
$$\sin(\psi + \Delta\psi) = \sin\psi\cos\Delta\psi + \cos\psi\sin\Delta\psi$$

Make approximations valid only to first order in the small quantities:

$$\sin(\phi + \Delta\phi) = \sin\phi + \cos\phi \cdot \Delta\phi$$
$$\sin(\psi + \Delta\psi) = \sin\psi + \cos\psi \cdot \Delta\psi$$

Simplify:

$$\Delta x = N\cos\phi \cdot \Delta\phi$$
$$\Delta y = N\cos\psi \cdot \Delta\psi$$

(b) From Part (a) we have $\frac{\Delta x}{\Delta\phi} = N\cos\phi$ and $\frac{\Delta y}{\Delta\phi} = N\cos\psi$.

Put that into the expression for $(\Delta s)^2$:

$$(\Delta s)^2 = \left(\frac{L}{N}\right)^2\left[\left(\frac{\Delta x}{\Delta\phi}\right)^2(\Delta\phi)^2 + \left(\frac{\Delta y}{\Delta\psi}\right)^2(\Delta\psi)^2\right] = \left(\frac{L}{N}\right)^2\left[(N\cos\phi)^2(\Delta\phi)^2 + (N\cos\psi)^2(\Delta\psi)^2\right]$$

Simplify:

$$(\Delta s)^2 = L^2\left[\cos^2\phi\cdot(\Delta\phi)^2 + \cos^2\psi\cdot(\Delta\psi)^2\right]$$

Problems 2 and 3 — FRW Coordinate Transformations for the Cases of the Flat and Open Universe

See attachment.

Problem 4 — The Hubble Plot

(a) The correct units for the vertical axis are km/sec. The units of the horizontal axis are Mpc (abbreviation for Megaparsecs). The slope is therefore

km/sec/Mpc

You can measure it off the plot as about slope:

500 km/sec/Mpc

(b) If the vertical axis were in meters/second and the horizontal axis were in meters, the units of the slope would be 1/sec. The inverse of this would be seconds?

To convert from seconds to billions of years, we use

1 year = 365.25 * 24 * 60 * 60 seconds

So 1 billion years is:

$10^9 \cdot 365.25 * 24 * 60 * 60$ seconds = $31\,557\,600 \times 10^9$ seconds = 3.15576×10^{16} seconds

(c) The currently accepted value is $H_0 = 67$ km/s / Mpc.

Find Hubble's estimate in his 1929 paper. You should find that it is shockingly different. In the text, Hubble says:

$H_0 = 530$ km/sec/Mpc

(d) The parsec is 3.086×10^{13} km. I probably should have had you calculate that too, but it is easy to look up. We use that and the conversion factor from seconds to billions of years:

$$H_0^{-1} = (67\text{ km/sec / Mpc})^{-1} = \frac{1\text{ Mpc}}{67\text{ km}}\text{ sec} = \frac{10^6 \times 3.086 \times 10^{13}\text{ km}}{67\text{ km}} \frac{1\text{ billion years}}{3.15576 \times 10^{16}\text{ seconds}} = \frac{1000}{67} \times \frac{3.086}{3.15576}\text{ billion years}$$

That's 14.6 billion years. If the universe's expansion rate has not changed, then this is the age of the universe.

Problem 2

We follow the directions in the problem. First we write down Eqs. 1 and 5 from TWB Chapter 14:

$$(\Delta s)^2 = \frac{(\Delta r)^2}{1-Kr^2} + r^2(\Delta\phi)^2 \qquad (1)$$

The $K=0$ case uses this transformation:

$$r = R\chi \qquad \Delta r = R\Delta\chi \qquad (5)$$

$$(\Delta s)^2 = \frac{(R\Delta\chi)^2}{1-\underset{K=0}{0}} + (R\chi)^2(\Delta\phi)^2$$

$$= R^2\left[(\Delta\chi)^2 + \chi^2(\Delta\phi)^2\right] \qquad (6)$$

We have derived Eq. 6. This is just a warm-up. Onward to Problem 3.

Problem 3

(a)
$$\sinh^2\chi = \left(\frac{e^{\chi}-e^{-\chi}}{2}\right)^2$$
$$= \frac{e^{2\chi}-2+e^{-2\chi}}{4}$$

$$\cosh^2\chi = \left(\frac{e^{\chi}+e^{-\chi}}{2}\right)^2$$
$$= \frac{e^{2\chi}+2-e^{-2\chi}}{4}$$

Compare these two formula. The only difference is that $\sinh^2\chi$ has $\frac{-2}{4}$ in it, while $\cosh^2\chi$ has $\frac{+2}{4}$ in it. Therefore

$$1+\sinh^2\chi = \cosh^2\chi$$

(b)

$$\sinh(\chi+\Delta\chi)-\sinh\chi$$

$$= \frac{e^{\chi+\Delta\chi}-e^{-(\chi+\Delta\chi)}}{2} - \frac{e^{\chi}-e^{-\chi}}{2}$$

$$= \frac{e^{\chi+\Delta\chi}-e^{\chi}}{2} - \frac{e^{-(\chi+\Delta\chi)}-e^{-\chi}}{2}$$

$$= \frac{e^{\chi}(e^{\Delta\chi}-1)}{2} - \frac{e^{-\chi}(e^{-\Delta\chi}-1)}{2}$$

$$\approx \frac{e^{\chi}\Delta\chi}{2} - \frac{e^{-\chi}(-\Delta\chi)}{2}$$

$$= \frac{e^{\chi}-e^{-\chi}}{2}\Delta\chi$$

$$= \cosh\chi\cdot\Delta\chi$$

(c) $$(\Delta s)^2 = \frac{(\Delta r)^2}{1-Kr^2} + r^2(\Delta\phi)^2$$

Case III, $K<0$, $R=\sqrt{-K}$

$$\left.\begin{aligned} r &= R\sinh\chi \\ r+\Delta r &= R\sinh(\chi+\Delta\chi) \end{aligned}\right\} \text{Subtract first equation from second}$$

$$\Delta r = R[\sinh(\chi+\Delta\chi)-\sinh\chi]$$

$$\approx R\cosh\chi\cdot\Delta\chi \longleftarrow \text{by part (b)}$$

So $$(\Delta s)^2 = \frac{(R\cosh\chi\cdot\Delta\chi)^2}{1-K(R\sinh\chi)^2} + (R\sinh\chi\cdot\Delta\phi)^2$$

$$= \frac{R^2\cancel{\cosh^2\chi}\cdot(\Delta\chi)^2}{\cancel{1+\sinh^2\chi}} + R^2\sinh^2\chi\cdot(\Delta\phi)^2 \quad \longleftarrow \cosh^2\chi \text{ by part (a)}$$

$$= R^2\left[(\Delta\chi)^2+\sinh^2\chi\cdot(\Delta\phi)^2\right]$$

We have derived Eq. 16. Woo hoo!