CHAPTER

6

Diving

Edmund Bertschinger & Edwin F. Taylor *

Many historians of science believe that special relativity could have been developed without Einstein; similar ideas were in the air at the time. In contrast, it's difficult to see how general relativity could have been created without Einstein – certainly not at that time, and maybe never.

—David Kaiser

6.1 ■ GO STRAIGHT: THE PRINCIPLE OF MAXIMAL AGING IN GLOBAL COORDINATES

"Go straight!" spacetime shouts at the stone.
The stone's wristwatch verifies that its path is straight.

Section 5.7 described how an observer passes through a sequence of local inertial frames, making each measurement in only one of these local frames. Special relativity describes motion in each such local inertial frame. The observer is just a stone that acts with purpose. Now we ask how a (purposeless!) free stone moves in global coordinates.

Section 1.6 introduced the Principle of Maximal Aging that describes motion in a single inertial frame. To describe global motion, we need to extend this principle to a *sequence* of adjacent local inertial frames. Here, without proof, is the simplest possible extension, to a *single adjacent pair* of local inertial frames.

Definition: **Principle of Maximal Aging** in curved spacetime

DEFINITION 1. Principle of Maximal Aging (curved spacetime)
The *Principle of Maximal Aging* states that a free stone follows a worldline through spacetime such that its wristwatch time (aging) is a maximum when summed across every adjoining pair of local inertial frames along its worldline.

*Draft of Second Edition of *Exploring Black Holes: Introduction to General Relativity*

Box 1. What Then Is Time?

What then is time? If no one asks me, I know what it is. If I wish to explain it to him who asks me, I do not know.

****

The world was made, not in time, but simultaneously with time. There was no time before the world.

—St. Augustine (354–430 C.E.)

Time takes all and gives all.

—Giordano Bruno (1548–1600 C.E.)

Everything fears Time, but Time fears the Pyramids.

—Anonymous

Philosophy is perfectly right in saying that life must be understood backward. But then one forgets the other clause—that it must be lived forward.

—Søren Kierkegaard

As if you could kill time without injuring eternity.

****

Time is but the stream I go a-fishing in.

—Henry David Thoreau

Although time, space, place, and motion are very familiar to everyone, . . . it is useful to distinguish these quantities into absolute and relative, true and apparent, mathematical and common.

—Isaac Newton

Time is defined so that motion looks simple.

—Misner, Thorne, and Wheeler

Nothing puzzles me more than time and space; and yet nothing troubles me less, as I never think about them.

—Charles Lamb

Either this man is dead or my watch has stopped.

—Groucho Marx

“What time is it, Casey?”

“You mean right now?”

—Casey Stengel

It’s good to reach 100, because very few people die after 100.

—George Burns

The past is not dead. In fact, it’s not even past.

—William Faulkner

Time is Nature’s way to keep everything from happening all at once.

—Graffito, men’s room, Pecan St. Cafe, Austin, Texas

What time does this place get to New York?

—Barbara Stanwyck, during trans-Atlantic crossing on the steamship *Queen Mary*

Objection 1. *Now you have gone off the deep end! In Chapter 1, Speeding, you convinced me that the Principle of Maximal Aging was nothing more than a restatement of Newton’s First Law of Motion, the observation that in flat spacetime the free stone moves at constant speed along a straight line in space. But in curved spacetime the stone’s path will obviously be curved. You have violated your own Principle.*

On the contrary, we have changed the Principle of Maximal Aging as little as possible in order to apply it to curved spacetime. We require the free stone to move along a straight worldline across *each one* of the pair of adjoining local inertial frames, as demanded by the special relativity Principle of Maximal Aging in each frame. We allow the stone only the choice of one map coordinate of the event, at the boundary between these two frames. That single generalization extends the Principle of Maximal Aging from flat to curved spacetime. And the result is a single kink in the worldline. When we shrink all adjoining inertial frames along the worldline to the calculus limit, then the result is what you predict: a curved worldline in global coordinates.

Now we can use the more general Principle of Maximal Aging to discover a constant of motion for a free stone, what we call its *map energy*.

6.2 ■ MAP ENERGY FROM THE PRINCIPLE OF MAXIMAL AGING

The global metric plus the Principle of Maximal Aging leads to map energy as a constant of motion.

Map energy: a constant of motion

This section uses the Principle of Maximal Aging from Section 6.1, plus the Schwarzschild global metric to derive the expression for map energy of a free stone near a nonspinning black hole. For a free stone, map energy is a constant of motion; its value remains the same as the stone moves. Our derivation uses a stone that falls inward along the r-direction, but at the end we show that the resulting expression for map energy also applies to a stone moving in any direction; energy is a *scalar*, which has no direction.

?

Objection 2. *Here is a fundamental objection to the Principle of Maximal Aging: You nowhere derive it, yet you expect us readers to accept this arbitrary Principle. Why should we believe you?*

!

Guilty as charged! Our major tool in this book is the metric, which—along with the topology of a spacetime region—tells us everything we can know about the shape of spacetime in that region. But the shape of spacetime revealed by the metric tells us nothing whatsoever about how a free stone moves in this spacetime. For that we need a second tool, the Principle of Maximal Aging which, like the metric, derives from Einstein's field equations. In this book the metric plus the Principle of Maximal Aging—both down one step from the field equations—are justified by their immense predictive power. Until we derive the metric in Chapter 22, we apply the slogan, "Handsome is as handsome does!"

Find maximal aging: find natural motion.

The Principle of Maximal Aging maximizes the stone's total wristwatch time across *two adjoining* local inertial frames. Figure 1 shows the Above Frame A (of average map coordinate $\bar{r}_{\mathrm{A}}$) and adjoining Below Frame B (of average map coordinate $\bar{r}_{\mathrm{B}}$). The stone emits initial flash 1 as it enters the top of Frame A, emits middle flash 2 as it transits from Above Frame A to Below Frame B, and emits final flash 3 as it exits the bottom of Below Frame B. We use the three *flash emission events* to find maximal aging.

Outline of the method: Fix the r- and ϕ-coordinates of all three flash emissions and fix the t-coordinates of upper and lower events 1 and 3. Next vary the t-coordinate of the middle flash emission 2 to maximize the total *wristwatch time* (aging) of the stone across both frames.

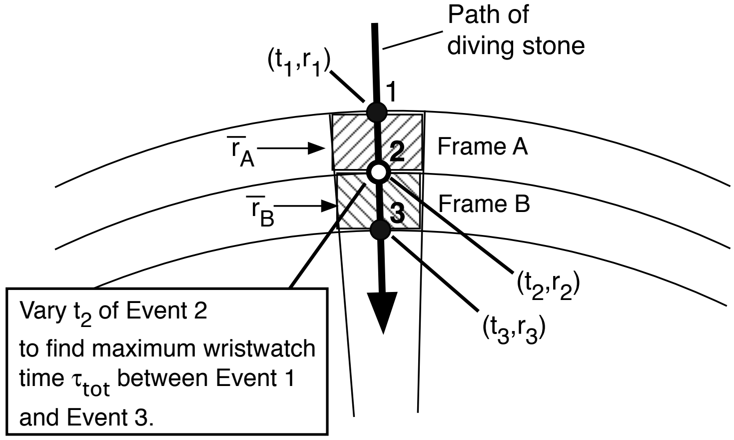

FIGURE 1 Use the Principle of Maximal Aging to derive the expression for Schwarzschild map energy. The diving stone first crosses the Above Frame A, then crosses the Below Frame B, emitting flashes at events 1, 2, and 3. Fix all three coordinates of events 1 and 3; but fix only the r- and ϕ-coordinates of intermediate event 2. Then vary the t-coordinate of event 2 to maximize the *total wristwatch time* (aging) across both frames between fixed end-events 1 and 3. This leads to expression (8) for the stone's map energy, a constant of motion.

So much for t-coordinates. How do we find *wristwatch times* across the two frames? The Schwarzschild metric ties the increment of wristwatch time to changes in r- and t-coordinates for a stone that falls inward along the r-coordinate. Write down the approximate form of the global metric twice, first for Above frame A (at average $\bar{r}_A$) and second for the Below frame B (at average $\bar{r}_B$). Take the square root of both sides:

Approximate the Schwarzschild metric for each frame.

$$\tau_A \approx \left[\left(1 - \frac{2M}{\bar{r}_A}\right)(t_2 - t_1)^2 + (\text{terms without } t\text{-coordinate})\right]^{1/2} \qquad (1)$$

$$\tau_B \approx \left[\left(1 - \frac{2M}{\bar{r}_B}\right)(t_3 - t_2)^2 + (\text{terms without } t\text{-coordinate})\right]^{1/2} \qquad (2)$$

We are interested only in those parts of the metric that contain the map t-coordinate, because we take derivatives with respect to that t-coordinate. To prepare for the derivative that leads to maximal aging, take the derivative of τ_A with respect to t_2 of the intermediate event 2. The denominator in the resulting derivative is just τ_A:

$$\frac{d\tau_A}{dt_2} \approx \left(1 - \frac{2M}{\bar{r}_A}\right)\frac{(t_2 - t_1)}{\tau_A} \qquad (3)$$

The corresponding expression for $d\tau_B/dt_2$ is:

$$\frac{d\tau_{\rm B}}{dt_2} \approx -\left(1 - \frac{2M}{\bar{r}_{\rm B}}\right)\frac{(t_3 - t_2)}{\tau_{\rm B}} \tag{4}$$

Add the two wristwatch times to obtain the summed wristwatch time $\tau_{\rm tot}$ between first and last events 1 and 3:

$$\tau_{\rm tot} = \tau_{\rm A} + \tau_{\rm B} \tag{5}$$

Maximize aging summed across both frames.

Recall that we keep constant the total t-coordinate separation across both frames. To find the maximum total wristwatch time, take the derivative of both sides of (5) with respect to t_2, substitute from (3) and (4), and set the result equal to zero in order to find the maximum:

$$\frac{d\tau_{\rm tot}}{dt_2} = \frac{d\tau_{\rm A}}{dt_2} + \frac{d\tau_{\rm B}}{dt_2} \approx \left(1 - \frac{2M}{\bar{r}_{\rm A}}\right)\frac{(t_2 - t_1)}{\tau_{\rm A}} - \left(1 - \frac{2M}{\bar{r}_{\rm B}}\right)\frac{(t_3 - t_2)}{\tau_{\rm B}} \approx 0 \tag{6}$$

From the last approximate equality in (6),

$$\left(1 - \frac{2M}{\bar{r}_{\rm A}}\right)\frac{(t_2 - t_1)}{\tau_{\rm A}} \approx \left(1 - \frac{2M}{\bar{r}_{\rm B}}\right)\frac{(t_3 - t_2)}{\tau_{\rm B}} \tag{7}$$

The expression on the left side of (7) depends only on parameters of the stone's motion across the Above Frame A; the expression on the right side depends only on parameters of the stone's motion across the Below Frame B. Hence the value of either side of this equation must be independent of *which* adjoining pair of frames we choose to look at: this pair can be *anywhere* along the worldline of a stone. Equation (7) displays a quantity that has the same value on *every* local inertial frame along the worldline. We have found the expression for a quantity that is a constant of motion.

Map energy of a stone in Schwarzschild coordinates

Now shrink differences $(t_2 - t_1)$ and $(t_3 - t_2)$ in (7) to their differential limits. In this process the average r-coordinate becomes exact, so $\bar{r} \to r$. Next use the result to *define* the stone's **map energy per unit mass**:

$$\boxed{\frac{E}{m} \equiv \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau} \qquad \text{(map energy of a stone per unit mass)}} \tag{8}$$

Far from the black hole, map energy takes special relativity form.

Why do we call the expression on the right side of (8) *energy* (per unit mass)? Because when the mass M of the center of attraction becomes very small—or when the stone is very far from the center of attraction—the limit $2M/r \to 0$ describes a stone in flat spacetime. That condition reduces (8) to $E/m = dt/d\tau$, which we recognize as equation (23) in Section 1.7 for E/m in flat spacetime. Hence we take the right side of (8) to be the general-relativistic generalization, near a nonspinning black hole, of the special relativity expression for E/m.

Map energy E same unit as m

Note that the right side of (8) has no units; therefore both E and m on the left side must be expressed in the *same* unit, a unit that we may choose for our convenience. *Both* numerator and denominator in E/m may be expressed in kilograms or joules or electron-volts or the mass of the proton, or any other common unit.

Map energy expression valid for *any* motion of the stone.

Our derivation of map energy employs only the t-coordinate in the metric. It makes no difference in the outcome for map energy—expression (8)—whether dr or $d\phi$ is zero or not. This has an immediate consequence: The expression for map energy in Schwarzschild global coordinates is valid for a free stone moving on *any* orbit around a spherically symmetric center of attraction, not just along the inward r-direction. We will use this generality of (8) to predict the general motion of a stone in later chapters.

6.3 ■ UNICORN MAP ENERGY VS. MEASURED SHELL ENERGY

Map energy is like a unicorn: a mythical beast

Map energy E/m is a unicorn: a mythical beast.

The expression on the right side of equation (8) is like a unicorn: a mythical beast. Nobody measures directly the r- or t-coordinates in this expression, which are Schwarzschild global map coordinates: entries in the mapmaker's spreadsheet or accounting form. Nobody measures E/m on the left side of (8) either; the map energy is also a unicorn. If this is so, why do we bother to derive expression (8) in the first place? Because E/m has an important virtue: It is a constant of motion of a free stone in Schwarzschild global coordinates; it has the same value at every event along the global worldline of a stone. The value of E/m helps us to predict its global motion (Chapters 8 and 9). But it does not tell us the value of the energy measured by an observer in a local inertial frame.

What is the stone's energy measured by the shell observer? The special relativity energy expression is valid for the shell observer. Equation (9) in Section 5.7 gives us:

$$\Delta t_{\text{shell}} = \left(1 - \frac{2M}{\bar{r}}\right)^{1/2} \Delta t \tag{9}$$

Then:

$$\frac{E_{\text{shell}}}{m} = \lim_{\Delta\tau \to 0} \frac{\Delta t_{\text{shell}}}{\Delta\tau} = \lim_{\Delta\tau \to 0} \left(1 - \frac{2M}{\bar{r}}\right)^{1/2} \frac{\Delta t}{\Delta\tau} \tag{10}$$

As we shrink increments to the differential calculus limit, the average r-coordinate becomes exact: $\bar{r} \to r$. The result is:

$$\frac{E_{\text{shell}}}{m} = \left(1 - \frac{2M}{r}\right)^{1/2} \frac{dt}{d\tau} \quad \text{(shell energy of a stone per unit mass)} \tag{11}$$

Into this equation substitute expression (8) for the stone's map energy to obtain:

$$\boxed{\frac{E_{\text{shell}}}{m} = \frac{1}{(1-v_{\text{shell}}^2)^{1/2}} = \left(1-\frac{2M}{r}\right)^{-1/2}\frac{E}{m}} \qquad (12)$$

Shell energy

where we have added the special relativity expression (23) in Section 1.7. Equation (12) tells us how to use the map energy—a unicorn—to predict the frame energy directly measured by the shell observer as the stone streaks past.

Expression (12) for shell energy E_{shell} applies to a stone moving in any direction, not just along the r-coordinate. Why? Energy—including map energy E—is a *scalar*, a property of the stone independent of its direction of motion.

Different shell observers compute same map energy.

The shell observer knows only his local shell frame coordinates, which are restricted in order to yield a local inertial frame. He observes a stone zip through his local frame and disappear from that frame; he has no global view of the stone's path. However, equation (12) is valid for a stone in *every* local shell frame and for *every* direction of motion of the stone in that frame. The shell observer uses this equation and his local r—stamped on every shell—to compute the map energy E/m, then radios his result to every one of his fellow shell observers, for example, "The green-colored free stone has map energy $E/m = 3.7$." A different shell observer, at different map r, measures a different value of shell energy E_{shell}/m of the green stone as it streaks through his own local frame, typically in a different direction. However, armed with (12), every shell observer verifies the constant value of map energy of the green stone, for example $E/m = 3.7$.

In brief, each local shell observer carries out a real measurement of shell energy; from this result plus his knowledge of his r-coordinate he derives the value of the map energy E/m, then uses this map energy—a constant of motion—to predict results of shell energy measurements made by shell observers distant from him. The result is a multi-shell account of the entire orbit of the stone.

The entire scheme of shell observers depends on the existence of local shell frames, which cannot be built inside the event horizon. Now we turn to the experience of the diver who passes inward across the event horizon.

6.4 ■ RAINDROP CROSSES THE EVENT HORIZON

Convert t-coordinate to raindrop wristwatch time.

How to get inside the event horizon?

The Schwarzschild metric satisfies Einstein's field equations everywhere in the vicinity of a nonrotating black hole (except on its singularity at $r = 0$). Map coordinates alone may satisfy Schwarzschild and Einstein, but they do not satisfy us. We want to make every measurement in a local inertial frame. Shell frames serve this purpose nicely outside the event horizon, but we cannot construct stationary shells inside the event horizon. Moreover, the expression $(1-2M/r)^{-1/2}$ in energy equation (12) becomes imaginary inside the event horizon, which provides one more indication that shell energy does not apply there.

Yet everyone tells us that an unfortunate astronaut who crosses inward through the event horizon at $r = 2M$ inevitably arrives at the lethal central singularity $r = 0$. In the following chapter we build a local frame around a falling astronaut. To prepare for such a local diving frame, we start here as simply as possible: We ask the stone wearing a wristwatch that began our study of relativity (Section 1.1) to take a daring dive, to drop from initial rest far from the black hole and plunge inward to $r = 0$. We call this diving, wristwatch-wearing stone a **raindrop**, because on Earth raindrops fall from rest at a great height. By definition, the raindrop has no significant spatial extent; it has no frame, it is just a stone wearing a wristwatch.

Raindrop defined: stone dropped from rest far away

DEFINITION 2. Raindrop

A **raindrop** is a stone wearing a wristwatch, that freely falls inward starting from initial rest far from the center of attraction.

Map energy of a raindrop

Examine the map energy (8) of a raindrop. Far from the black hole $r \gg 2M$ so that $(1 - 2M/r) \to 1$. For a stone at rest there, $dr = d\phi = 0$ and the Schwarzschild metric tells us that $d\tau \to dt$. As a result, (8) becomes:

$$\frac{E}{m} \equiv \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau} \to 1 \quad \text{(raindrop: released from rest at } r \gg 2M) \qquad (13)$$

The raindrop, released from rest far from the black hole, must fall inward along a radial line. In other words, $d\phi = 0$ along the raindrop worldline. Formally we write:

$$\frac{d\phi}{d\tau} = 0 \qquad \text{(raindrop)} \qquad (14)$$

The raindrop-stone, released from rest at a large r map coordinate, begins to move inward, gradually picks up speed, finally plunges toward the center. As the raindrop hurtles inward, the value of $E/m\,(= 1)$ remains constant. Equation (12) then tells us that as r decreases, $2M/r$ increases, and so E_{shell} must also increase, implying an increase in v_{shell}. The local shell observer measures this increased speed directly. Equation (12) with $E/m = 1$ for the raindrop yields:

Shell energy of the raindrop

$$\frac{E_{\text{shell}}}{m} = \left(1 - v_{\text{shell}}^2\right)^{-1/2} = \left(1 - \frac{2M}{r}\right)^{-1/2} \qquad \text{(raindrop)} \qquad (18)$$

It follows immediately that:

$$v_{\text{shell}} = -\left(\frac{2M}{r}\right)^{1/2} \qquad \text{(raindrop shell velocity)} \qquad (19)$$

where the negative value of the square root describes the stone's inward motion. Equation (19) shows that the shell-measured speed of the

Box 2. Slow speed + weak field $\Longrightarrow$ Mass + Newtonian KE and PE

"If you fall, I'll be there."—Floor

The map energy E/m may be a unicorn in general relativity, but it is a genuine race horse in Newtonian mechanics. We show here that the map energy E/m of a stone moving at non-relativistic speed in a weak gravitational field reduces to the mass of the stone plus the familiar Newtonian energy (kinetic + potential). Rearrange (12) to read:

$$\frac{E}{m} = \left(1 - \frac{2M}{r}\right)^{1/2} \left(1 - v_{\text{shell}}^2\right)^{-1/2} \qquad (15)$$

For $r \gg 2M$ (weak gravitational field) and $v_{\text{shell}}^2 \ll 1$ (non relativistic stone speed) use the approximation inside the front cover twice:

$$\left(1 - \frac{2M}{r}\right)^{1/2} \approx 1 - \frac{M}{r} \qquad (r \gg 2M) \qquad (16)$$

$$\left(1 - v_{\text{shell}}^2\right)^{-1/2} \approx 1 + \frac{1}{2} v_{\text{shell}}^2 \qquad (v_{\text{shell}}^2 \ll 1)$$

Substitute these into (15) and drop the much smaller product $(M/2r)v_{\text{shell}}^2$. The result is an approximation:

$$E \approx m + \frac{1}{2} m v_{\text{shell}}^2 - \frac{Mm}{r} \qquad (17)$$

$$\left(r \gg 2M, \;\; v_{\text{shell}}^2 \ll 1\right)$$

In this equation $-Mm/r$ is the gravitational potential energy of the stone. (In conventional mks units it would read $-GM_{\text{kg}}m_{\text{kg}}/r$.) We recognize in (17) Newtonian's kinetic energy (KE) plus his potential energy (PE) of a stone, with the added stone's mass m.

As a jockey in curved spacetime, you must beware of riding the unicorn map energy E/m; gravitational potential energy is a fuzzy concept in general relativity. Dividing energy into separate kinetic and potential forms works only under special conditions, such as those given in equation (16).

Except for these special conditions, we expect the map constant of motion E to differ from E_{shell}: The local shell frame is inertial and excludes effects of curved spacetime. In contrast, map energy E—necessarily expressed in map coordinates—includes curvature effects, which Newton attributes to a "force of gravity."

The approximation in (17) is quite profound. It reproduces a central result of Newtonian mechanics without using the concept of force. In general relativity, we can always eliminate gravitational force (see inside the back cover).

raindrop—the magnitude of its velocity—increases to the speed of light at the event horizon. This is a limiting case, because we cannot construct a shell—even in principle—at the exact location of the event horizon.

?

Objection 3. *I am really bothered by the idea of a material particle such as a stone traveling across the event horizon as a particle. The shell observer sees it moving at the speed of light, but it takes light to travel at light speed. Does the stone—the raindrop—become a flash of light at the event horizon?*

!

No. Be careful about limiting cases. No shell can be built *at* the event horizon, because the initial gravitational acceleration increases without limit there (Section 6.7). An observer on a shell just outside the event horizon clocks the diving stone to move with a speed *slightly less* than the speed of light. Any directly-measured stone speed less than the speed of light is perfectly legal in relativity. So there is no contradiction.

Raindrop dr/dt

Compare the shell velocity (19) of the raindrop with the value of dr/dt at a given r-coordinate. To derive dr/dt, solve the right-hand equation in (13) for $d\tau$ and substitute the result into the Schwarzschild metric with $d\phi = 0$. Result for the raindrop:

Sample Problems 1. The Neutron Star Takes an Aspirin

Neutron Star Gamma has a total mass 1.4 times that of our Sun and a map $r_0 = 10$ kilometers. An aspirin tablet of mass one-half gram falls from rest at a large r coordinate onto the surface of the neutron star. An advanced civilization converts into useful energy the entire kinetic energy of the aspirin tablet, measured in the local surface rest frame. Estimate how long this energy will power a 100-watt bulb. Repeat the analysis to find the useful energy for the case of an aspirin tablet falling from a large r coordinate onto the surface of Earth.

SOLUTION

From the value of the mass of our Sun (inside the front cover), the mass of the neutron star is $M \approx 2 \times 10^3$ meters. Hence $2M/r_0 \approx 2/5$. Far from the neutron star the total map energy of the aspirin tablet equals its rest energy, namely its mass, hence $E/m = 1$. From (18), the shell energy of the aspirin tablet just before it hits the surface of the neutron star rises to the value

$$\frac{E_{\text{shell}}}{m} = \left(1 - \frac{2M}{r_0}\right)^{-1/2} \approx 1.3 \qquad \text{(Neutron Star)} \qquad (20)$$

The shell *kinetic energy* of the half-gram aspirin tablet is 0.3 of its rest energy. The rest energy is $m = 0.5$ gram = 5 $\times 10^{-4}$ kilogram or $mc^2 = 4.5 \times 10^{13}$ joules. The fraction 0.3 of this is 1.35×10^{13} joules. One watt is one joule/second; a 100-watt bulb consumes 100 joules per second. At that rate, the bulb can burn for 1.35×10^{11} seconds on the kinetic energy of the aspirin tablet. One year is about 3×10^7 seconds. Result: The kinetic energy of the half-gram aspirin tablet falling to the surface of Neutron Star Gamma from a large r coordinate provides energy sufficient to light a 100-watt bulb for approximately 4500 years!

What happens when the aspirin tablet falls from a large r coordinate onto Earth's surface? Set the values of M and r_0 to those for Earth (inside front cover). In this case $2M \ll r_{\text{E}}$, so equation (20) becomes, to a very good approximation:

$$\frac{E_{\text{shell}}}{m} \approx \left(1 + \frac{M}{r_0}\right) \approx 1 + 6.97 \times 10^{-10} \qquad \text{(Earth)} \qquad (21)$$

Use the same aspirin tablet rest energy as before. The lower fraction of kinetic energy yields 3.14×10^4 joules. At 100 joules per second the kinetic energy of the aspirin tablet will light the 100-watt bulb for 314 seconds, or a little more than 5 minutes.

$$\frac{dr}{dt} = -\left(1 - \frac{2M}{r}\right)\left(\frac{2M}{r}\right)^{1/2} \qquad \text{(raindrop)} \qquad (22)$$

Raindrop dr/dt: a unicorn!

Equation (22) shows an apparently outrageous result: as the raindrop reaches the event horizon at $r = 2M$, its Schwarzschild dr/dt drops to zero. (This result explains the strange spacing of event-dots along the orbit approaching the event horizon in Figure 3.6.) Does any local observer witness the stone coasting to rest? No! Repeated use of the word "map" reminds us that map velocities are simply spreadsheet entries for the Schwarzschild mapmaker and need not correspond to direct measurements by any local observer. Figure 2 shows plots of both shell speeds and map $dr/d\tau$ of the descending raindrop. Nothing demonstrates more clearly than the diverging lines in Figure 2 the radical difference between (unicorn) map entries and the results of direct measurement.

Does the raindrop cross the event horizon or not? To answer that question we need to track the descent with its directly-measured wristwatch time, not the global t-coordinate. Use equation (13) to convert global coordinate differential dt to wristwatch differential $d\tau$. With this substitution, (22) becomes:

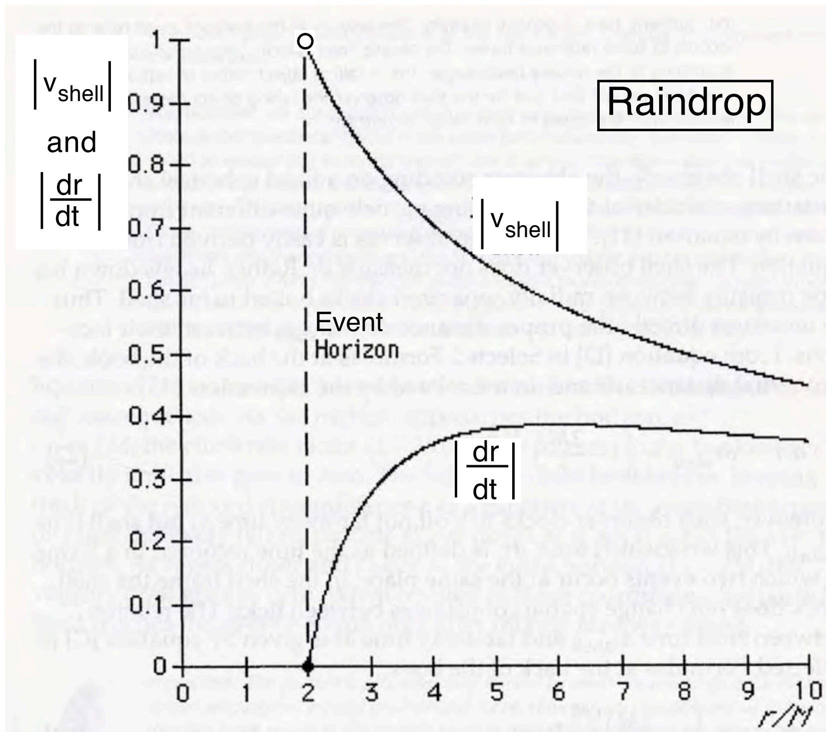

FIGURE 2 Computer plot of the speed $|v_{\rm shell}|$ of a raindrop directly measured by shell observers at different r-values, from (19), and its Schwarzschild map speed $|dr/dt|$ from (22). Far from the black hole the raindrop is at rest, so both speeds are zero, but both speeds increase as the raindrop descends. Map speed $|dr/dt|$ is not measured but computed from spreadsheet records of the Schwarzschild mapmaker. At the event horizon, the measured shell speed rises to the speed of light, while the computed map speed drops to zero. The upper open circle at $r = 2M$ reminds us that this is a limiting case, since no shell can be constructed *at* the event horizon. (Why not? See the Appendix, Section 6.7.)

$$\frac{dr}{d\tau_{\rm raindrop}} = -\left(\frac{2M}{r}\right)^{1/2} \qquad (23)$$

Raindrop crosses the event horizon.

Expression (23) combines a map quantity dr with the differential advance of the wristwatch $d\tau_{\rm raindrop}$. It shows that the raindrop's r-coordinate decreases as its wristwatch time advances, so the raindrop passes inward through the event horizon. Indeed, inside the event horizon the magnitude of $dr/d\tau_{\rm raindrop}$ becomes greater than one, and increases without limit as $r \to 0$. But this need not worry us: Both r and dr are map quantities, so $dr/d\tau$ is just an entry on the mapmaker's spreadsheet, not a quantity measured by anyone.

Comment 1. How do we find the value of dr inside the event horizon?
The numerator dr on the left side of (23) has a clear meaning only *outside* the

Box 3. Newton Predicts the Black Hole?

It's remarkable how well much of Newton's mechanics works—sort of—on the stage of general relativity. One example is that Newton appears to predict the r-coordinate of the event horizon $r = 2M$. Yet the meaning of that barrier is strikingly different in the two pictures of gravity, as the following analysis shows.

A stone initially at rest far from a center of attraction drops inward. Or a stone on the surface of Earth or of a neutron star is fired outward along r, coming to rest at a large r coordinate. In either case, Newtonian mechanics assigns the same total energy (kinetic plus potential) to the stone. We choose the gravitational potential energy to be zero at the large r coordinate, and the stone out there does not move. From (17), we then obtain

$$\frac{E}{m} - 1 = \frac{v^2}{2} - \frac{M}{r} = 0 \quad \text{(Newton)} \tag{24}$$

From (24) we derive the diving (or rising) speed at any r-coordinate:

$$|v| = \left(\frac{2M}{r}\right)^{1/2} \quad \text{(Newton)} \tag{25}$$

which is the same as equation (19) for the shell speed of the raindrop. One can predict from (25) the r-value at which the speed reaches one, the speed of light, which yields $r = 2M$, the black hole event horizon. For Newton the speed of light is the **escape velocity** from the event horizon.

Newton assumes a single universal inertial reference frame and universal time, whereas (19) applies only to shell separation divided by shell time. A quite different expression (22) describes dr/dt—map differential dr divided by map differential dt—for raindrops.

Does Newton correctly describe black holes? No. Newton predicts that a stone launched radially outward from the event horizon with a speed less than that of light will rise to higher r, slow, stop without escaping, then fall back. In striking contrast, Einstein predicts that nothing, not even light, can be successfully launched outward from inside the event horizon, and that light launched outward *exactly at* the event horizon hovers there, balanced as on a knife-edge (Box 4).

event horizon, where every shell displays the stamped value of r. Box 7 in Section 7.8 describes one practical method by which a descending rain observer can measure map r, both outside and inside the event horizon.

6.5 ■ GRAVITATIONAL MASS

A new way to measure total energy

Mass m of the stone

This book uses the word *mass* in two different ways. Symbol m in equations (8) and (11) represents the inertial mass of a test particle, which we call a *stone*. By definition, the mass of a stone is too small to curve spacetime by a detectable amount. Expression (8) measures the stone's map energy E and mass m in the same units.

Mass M of the center of attraction

The mass M of the center of attraction is quite different: It is the gravitational mass that curves spacetime, as reflected in the global metric expression $(1 - 2M/r)$.

Drop a stone of mass m into a star of mass M.

What happens when a stone of mass m falls into a black hole of mass M? Does the swallowed mass m increase the black hole's mass? Our new understanding of energy helps us to calculate how much the mass of a black hole grows when it swallows matter—and yields a surprising result. To begin, start with a satellite orbiting close to a star. How can we measure the total gravitational mass of the star-plus-satellite system? We make this measurement using the initial acceleration of a distant test particle so remote