
Cosmology — Problem Set 9 — Solution

Problem 1

(a) We want to discover what the metric $(\Delta s)^2 = \frac{L}{N}^2 [(\Delta x)^2 + (\Delta y)^2]$ looks like in these coordinates:

$$x = N \sin \phi$$

$$y = N \sin \psi$$

where ϕ and ψ each go from 0 to $\frac{\pi}{2}$. To get Δx and Δy in terms of $\Delta \phi$ and $\Delta \psi$, subtract equations:

$$\Delta x = x + \Delta x - x = N \sin(\phi + \Delta \phi) - N \sin \phi$$

$$\Delta y = y + \Delta y - y = N \sin(\psi + \Delta \psi) - N \sin \psi$$

Use

$$\sin(\phi + \Delta \phi) = \sin \phi \cos \Delta \phi + \cos \phi \sin \Delta \phi$$

$$\sin(\psi + \Delta \psi) = \sin \psi \cos \Delta \psi + \cos \psi \sin \Delta \psi$$

Make approximations valid only to first order in the small quantities:

$$\sin(\phi + \Delta \phi) = \sin \phi + \cos \phi \cdot \Delta \phi$$

$$\sin(\psi + \Delta \psi) = \sin \psi + \cos \psi \cdot \Delta \psi$$

Simplify:

$$\Delta x = N \cos \phi \cdot \Delta \phi$$

$$\Delta y = N \cos \psi \cdot \Delta \psi$$

(b) From Part (a) we have $\frac{\Delta x}{\Delta \phi} = N \cos \phi$ and $\frac{\Delta y}{\Delta \psi} = N \cos \psi$.

Put that into the expression for $(\Delta s)^2$:

$$(\Delta s)^2 = \left(\frac{L}{N}\right)^2 \left[\left(\frac{\Delta x}{\Delta \phi}\right)^2 (\Delta \phi)^2 + \left(\frac{\Delta y}{\Delta \psi}\right)^2 (\Delta \psi)^2 \right] = \left(\frac{L}{N}\right)^2 [(N \cos \phi)^2 (\Delta \phi)^2 + (N \cos \psi)^2 (\Delta \psi)^2]$$

Simplify:

$$(\Delta s)^2 = L^2 [\cos^2 \phi \cdot (\Delta \phi)^2 + \cos^2 \psi \cdot (\Delta \psi)^2]$$

Problems 2 and 3 — FRW Coordinate Transformations for the Cases of the Flat and Open Universe

See attachment.

Problem 4 — The Hubble Plot

(a) The correct units for the vertical axis are km/sec. The units of the horizontal axis are Mpc (abbreviation for Megaparsecs). The slope is therefore

km/sec/Mpc

You can measure it off the plot as about slope:

500 km/sec/Mpc

(b) If the vertical axis were in meters/second and the horizontal axis were in meters, the units of the slope would be 1/sec. The inverse of this would be seconds?

To convert from seconds to billions of years, we use

1 year = 365.25 * 24 * 60 * 60 seconds

So 1 billion years is:

$10^9 \cdot 365.25 \cdot 24 \cdot 60 \cdot 60 \text{ seconds} = 31\,557\,600 \times 10^9 \text{ seconds} = 3.15576 \times 10^{16} \text{ seconds}$

(c) The currently accepted value is $H_0 = 67 \text{ km/s / Mpc}$.

Find Hubble's estimate in his 1929 paper. You should find that it is shockingly different. In the text, Hubble says:

$H_0 = 530 \text{ km/sec/Mpc}$

(d) The parsec is $3.086 \times 10^{13} \text{ km}$. I probably should have had you calculate that too, but it is easy to look up. We use that and the conversion factor from seconds to billions of years:

$$H_0^{-1} = (67 \text{ km/sec / Mpc})^{-1} = \frac{1 \text{ Mpc}}{67 \text{ km}} \text{ sec} = \frac{10^6 \cdot 3.086 \cdot 10^{13} \text{ km}}{67 \text{ km}} \frac{1 \text{ billion years}}{3.15576 \cdot 10^{16} \text{ seconds}} = \frac{1000}{67} \times \frac{3.086}{3.15576} \text{ billion years}$$

That's 14.6 billion years. If the universe's expansion rate has not changed, then this is the age of the universe.

Problem 2

We follow the directions in the problem. First we write down Eqs. 1 and 5 from TWB Chapter 14:

$$(\Delta s)^2 = \frac{(\Delta r)^2}{1 - Kr^2} + r^2 (\Delta \phi)^2 \quad (1)$$

The $K=0$ case uses this transformation:

$$r = R\chi \quad \Delta r = R\Delta\chi \quad (5)$$

$$\begin{aligned} (\Delta s)^2 &= \frac{(R\Delta\chi)^2}{1 - \underset{\substack{\uparrow \\ K=0}}{0}} + (R\chi)^2 (\Delta\phi)^2 \\ &= R^2 [(\Delta\chi)^2 + \chi^2 (\Delta\phi)^2] \quad (6) \end{aligned}$$

We have derived Eq. 6. This is just a warm-up. Onward to Problem 3.

Problem 3

$$\begin{aligned} (a) \quad \sinh^2 \chi &= \left(\frac{e^\chi - e^{-\chi}}{2} \right)^2 \\ &= \frac{e^{2\chi} - 2 + e^{-2\chi}}{4} \end{aligned}$$

$$\begin{aligned} \cosh^2 \chi &= \left(\frac{e^\chi + e^{-\chi}}{2} \right)^2 \\ &= \frac{e^{2\chi} + 2 + e^{-2\chi}}{4} \end{aligned}$$

Compare these two formula. The only difference is that $\sinh^2 \chi$ has $\frac{-2}{4}$ in it, while $\cosh^2 \chi$ has $\frac{+2}{4}$ in it. Therefore

$$1 + \sinh^2 \chi = \cosh^2 \chi$$

$$\begin{aligned}
(6) \quad & \sinh(\chi + \Delta\chi) - \sinh \chi \\
&= \frac{e^{\chi + \Delta\chi} - e^{-(\chi + \Delta\chi)}}{2} - \frac{e^{\chi} - e^{-\chi}}{2} \\
&= \frac{e^{\chi + \Delta\chi} - e^{\chi}}{2} - \frac{e^{-(\chi + \Delta\chi)} - e^{-\chi}}{2} \\
&= \frac{e^{\chi}(e^{\Delta\chi} - 1)}{2} - \frac{e^{-\chi}(e^{-\Delta\chi} - 1)}{2} \\
&\approx \frac{e^{\chi} \Delta\chi - e^{-\chi}(-\Delta\chi)}{2} \\
&= \frac{e^{\chi} - e^{-\chi}}{2} \Delta\chi \\
&= \cosh \chi \cdot \Delta\chi
\end{aligned}$$

$$(c) \quad (\Delta s)^2 = \frac{(\Delta r)^2}{1 - Kr^2} + r^2(\Delta\phi)^2$$

Oops, should have $R = 1/\sqrt{-K}$

Case III, $K < 0$, $R = \sqrt{-K}^{-1}$

$$\begin{aligned}
r &= R \sinh \chi \\
r + \Delta r &= R \sinh(\chi + \Delta\chi)
\end{aligned}
\left. \begin{array}{l} \\ \end{array} \right\} \begin{array}{l} \text{subtract} \\ \text{first equation} \\ \text{from second} \end{array}$$

$$\begin{aligned}
\Delta r &= R [\sinh(\chi + \Delta\chi) - \sinh \chi] \\
&\approx R \cosh \chi \cdot \Delta\chi \leftarrow \text{by part (6)}
\end{aligned}$$

$$\begin{aligned}
\text{So } (\Delta s)^2 &= \frac{(R \cosh \chi \cdot \Delta\chi)^2}{1 - K(R \sinh \chi)^2} + (R \sinh \chi \cdot \Delta\phi)^2 \\
&= \frac{R^2 \cosh^2 \chi \cdot (\Delta\chi)^2}{(1 + \sinh^2 \chi)} + R^2 \sinh^2 \chi \cdot (\Delta\phi)^2 \\
&\quad \leftarrow \cosh^2 \chi \text{ by part (a)}
\end{aligned}$$

$$= R^2 [(\Delta\chi)^2 + \sinh^2 \chi \cdot (\Delta\phi)^2]$$

We have derived Eq. 16. Woo hoo!