Cosmology Term 5 Exam — Conservation Laws, Cosmology

Open book, but not open problem-set solution, so that I can re-assign some problems that are very similar to ones you have already done.

Use your own paper for everything. There isn't enough space on the exam for it to be your work and answer space. Staple and put your name on your work.

The problems are in chronological order of the way we studied this term. They get easier! Don't get bogged down on the first ones!

1. Conservation of Momentum in Flat Space (6 pts)

This is the most straightforward conservation law problem I can think of!

Consider the metric of flat space. Let us be super-simple and imagine there is just the time coordinate and a single space coordinate. Then the metric is just:

$(\Delta\tau)^2 = (\Delta t)^2 - (\Delta x)^2$

On Map A, a particle enters at $(t_1,\ x_1)$.

Then it crosses from Map A onto Map B at $(t_2,\ x_2)$.

Then it leaves Map B at (t_3, x_3).

(a) Write down $\tau_A + \tau_B$ which is the total proper time of the particle as it crosses both map A and map B.

(b) Imagining that everything but x_2 is fixed (meaning that t_1, x_1, t_2, t_3, and x_3 are all fixed), we will vary x_2 only and apply the principle of maximal aging.

Compute $\frac{d}{dx_2}(\tau_A + \tau_B)$.

ALTERNATE NON-CALCULUS Part (b) If your derivatives are still rusty, you can can compute $\tau_A + \tau_B$ replacing x_2 by $x_2 + \Delta x_2$ and then subtracting your original expression for $\tau_A + \tau_B$. You then need to make the usual small Δx_2 approximations and find the term that is linear in Δx_2.

(CONT'D ON NEXT PAGE)

(c) Set what you got in Part (b) equal to zero.

ALTERNATE NON-CALCULUS Part (c) If you did the alternate Part (b), then you set the coefficient of Δx_2 that you find in alternate part (b) equal to zero.

(d) Rewrite your result suggestively as follows: (1) Replace τ_A with $\Delta\tau_A$. (2) Replace τ_B with $\Delta\tau_B$. (3) Replace $x_2 - x_1$ with Δx_A. (4) Replace $x_3 - x_2$ as Δx_B. (5) Multiply both sides of your equation by m, the mass of the particle.

(e) Observe that your equation says that something is unchanged as the particle moves from map to map. What is the unchanging thing? This is momentum conservation in flat relativistic space!

2. Stone Falling Inward (6 pts)

On one of the problem sets, you found this formula for a stone falling toward a black hole:

$$v_{\text{shell}} = -\left(1 - \left(1 - \frac{2M}{\bar{r}}\right)\left(1 - \frac{2M}{r_0}\right)^{-1}\right)^{1/2}$$

r_0 is the r-coordinate that the stone was dropped from. $\bar{r}$ is the r-coordinate of the shell observer that the stone is zinging past.

(a) Plug this formula into the the shell observer's kinetic energy formula:

$$\text{KE} = \text{kinetic energy} = m\left(\frac{1}{\sqrt{1-v_{\text{shell}}^2}} - 1\right)$$

Simplify as much as you can.

(b) Now approximate your formula for KE using the formula $(1+\epsilon)^n \approx 1 + n\epsilon$.

To make the approximation, you need to assume that $\frac{2M}{\bar{r}}$ and $\frac{2M}{r_0}$ are both small.

DISCUSSION: You have just derived Newtonian conservation of energy from Einstein's more general and correct version.

3. Coordinate Transformation — Hyperbolic Case (5 pts)

This is essentially a repeat of a derivation you did on a problem set.

We are going to trade in the coordinate r for a coordinate χ. The transformation is

$$r = \frac{1}{\sqrt{-K}} \sinh\chi$$

This transformation only makes sense for $K < 0$.

(a) Use the properties of the exponential and $e^{\Delta\chi} \approx 1 + \Delta\chi$ to approximate

$$\sinh(\chi + \Delta\chi) = \frac{e^{\chi+\Delta\chi} - e^{-(\chi+\Delta\chi)}}{2}$$

You should be able to get a nice simple expression for

$$\Delta r = \frac{1}{\sqrt{-K}}[\sinh(\chi + \Delta\chi) - \sinh\chi]$$

To make it nice and simple you will also need $\cosh\chi = \frac{e^{\chi}+e^{-\chi}}{2}$

(b) Use the transformation $r = \frac{1}{\sqrt{-K}} \sinh\chi$ and your nice simple expression for Δr to rewrite

$$(\Delta\tau)^2 = (\Delta t)^2 - \frac{(\Delta r)^2}{1-Kr^2} - r^2(\Delta\phi)^2$$

You should get the Friedman-Robertson-Walker metric, hyperbolic case. To simplify, you will need the identity $\cosh^2\chi = 1 + \sinh^2\chi$.

4. Circumference vs. Radial Coordinate in an Expanding Universe (5 pts)

Consider the metric

$$(\Delta\tau)^2 = (\Delta t)^2 - f^2(t)\,(\Delta r)^2 - g^2(r)\,(\Delta\phi)^2 \quad \text{or} \quad (\Delta\sigma)^2 = -(\Delta t)^2 + f^2(t)\,(\Delta r)^2 + g^2(r)\,(\Delta\phi)^2$$

To keep things simple, the function f only depends on t and the function g only depends on r. *Make the usual small coordinate-difference approximations when answering.*

(a) How far apart are the points $(t,\ r,\ \phi)$ and $(t,\ r,\ \phi + \Delta\phi)$?

(b) Assuming the ϕ coordinate has the usual range $[0,\ 2\pi]$, what is the circumference of a circle at coordinate value r.

(c) How far apart are the points $(t,\ r,\ \phi)$ and $(t,\ r+\Delta r,\ \phi)$?

(d) Using your answer for (c), what is the distance from $(t,\ 0,\ \phi)$ to $(t,\ r,\ \phi)$?

(e) Call your answer to Part (d), $d(t)$. Assuming that r and ϕ are not changing with time. r and ϕ could represent the coordinates of a galaxy. What is the ratio of the the velocity of the galaxy to its distance?

Your answer will only involve $f(t)$, $\dot{f}(t)$, and r. HINT: You are just evaluating the ratio $\frac{\dot{d}(t)}{d(t)}$. We call this ratio $H(t)$.

5. Transition from a Radiation-Dominated to a Dark-Energy-Dominated Universe (3 pts)

Suppose there is no ordinary matter in the universe, so that there are only two contributions to the expression for $\frac{\rho_{\text{total}}(t)}{\rho_{\text{critical}}(t)}$, instead of three:

$$\frac{\rho_{\text{tot}}(t)}{\rho_{\text{crit},0}(t)} = \frac{\Omega_{\text{rad},0}}{a^4(t)} + \Omega_{\Lambda,0}.$$

Don't plug in any values until part (c).

(a) At what value of a, are the two contributions equal?

(b) What is $\frac{\rho_{\text{total}}(t)}{\rho_{\text{crit},0}}$ at this value of a?

(c) Now plug in to (b) using the actual values of $\Omega_{\text{rad},0}$ and $\Omega_{\Lambda,0}$ in Eqs. 17 and 18 on p. 15-11 of TWB.

EXTRA CREDIT (1 pt): Assuming the temperature of radiation today is 2.725K, and assuming Eq. 11 on TWB p. 15-9, what was the temperature when the transition from a radiation-dominated to a dark-energy-dominated universe occurred?

MORE EXTRA CREDIT (2 pts): Assuming that the average energy of a photon is $k_B\,T$ and assuming Eq. 12 on TWB p. 15-9 how many photons per cubic meter are there in the universe when the transition from a radiation-dominated to a dark-energy-dominated universe occurred. *Get a symbolic answer first. Then plug in.* To plug in, you will need $k_B = 1.38\times10^{23}$ J/ K, $h = 6.63\times10^{-34}$ J·s, $\hbar = \frac{h}{2\pi}$, and $c = 3.00\times10^{8}$ m/s.