The Schwarzschild Metric and Stretching of Space in the Radial Direction—Part I

By the end of this handout, we will have made major headway unpacking Sample Problem 1 on p. 3-16. Some unpacking is good for everyone and is essential for those who have little or no calculus.

The Schwarzschild Metric

On pages 3-2 and 3-3 is the Schwarzschild metric:

$$(\Delta\tau)^2 = \left(1 - \tfrac{2M}{r}\right)(\Delta t)^2 - \frac{(\Delta r)^2}{1-\frac{2M}{r}} - r^2(\Delta\lambda)^2 - r^2\cos^2\lambda\,(\Delta\phi)^2$$

$$(\Delta\sigma)^2 = -\left(1 - \tfrac{2M}{r}\right)(\Delta t)^2 + \frac{(\Delta r)^2}{1-\frac{2M}{r}} + r^2(\Delta\lambda)^2 + r^2\cos^2\lambda\,(\Delta\phi)^2$$

You usually use the first equation if you have time-like separations and the second one if you have space-like separations.

A comment on infinitesimals (aka “differentials”): Whenever possible, I will use small coordinate differences instead of infinitesimals. Humanity wandered around in a fog for all of the 1700s before limits put infinitesimals on a rigorous footing in the 1800s, and this class is not the place to dispel that fog. Cauchy and Weierstrass were especially important in that work. The subject is called real analysis.

Working with small coordinate differences sidesteps the problem of defining infinitesimals. We will still have to take limits to get precise answers.

Our authors set $\lambda = 0$ and $\Delta\lambda = 0$ in their equations. They warned you of that when they said “think of two adjacent events that lie on our equatorial r, ϕ, plane.” If you start on the equator then $\lambda = 0$. If you never leave the equator, then $\Delta\lambda = 0$. So put that in above and you get:

$$(\Delta\tau)^2 = \left(1 - \tfrac{2M}{r}\right)(\Delta t)^2 - \frac{(\Delta r)^2}{1-\frac{2M}{r}} - r^2(\Delta\phi)^2$$

$$(\Delta\sigma)^2 = -\left(1 - \tfrac{2M}{r}\right)(\Delta t)^2 + \frac{(\Delta r)^2}{1-\frac{2M}{r}} + (\Delta\phi)^2$$

Distances in the Radial Direction

The next simplification is to set $\Delta t = 0$ and $\Delta\phi = 0$. We are only going to consider separations in the Δr direction (for starters). We have to start somewhere! We still have two equations to choose from:

$$(\Delta\tau)^2 = -\frac{(\Delta r)^2}{1-\frac{2M}{r}}$$

$$(\Delta\sigma)^2 = \frac{(\Delta r)^2}{1-\frac{2M}{r}}$$

Let's also imagine that we are far from the black hole. How far? How about $r > 2M$? Then $1 - \frac{2M}{r}$ is a positive number. Then we see that the equation for $\Delta\tau$ has $(\Delta\tau)^2 < 0$, which says that this is not a time-like separation that we are investigating — you can't put on a wristwatch and move in the Δr direction alone. So it is more appropriate to use the second equation. I am also going to take the square root of it:

$$\Delta\sigma = \frac{\Delta r}{\left(1-\frac{2M}{r}\right)^{1/2}}$$

Really I ought to have absolute values around the Δr, but I am only going to consider moving radially outward (to larger r), so my little changes Δr are going to be positive.

Summing up a Lot of Small Changes from r_1 to r_2

Now we are going to want to imagine stitching a lot of small patches together. We go from:

r_1 to $r_1 + \Delta r$
$r_1 + \Delta r$ to $r_1 + 2\Delta r$
$r_1 + 2\Delta r$ to $r_1 + 3\Delta r$
$r_1 + 3\Delta r$ to $r_1 + 4\Delta r$
$r_1 + 4\Delta r$ to $r_1 + 5\Delta r$

all the way to:

$r_2 - 2\Delta r$ to $r_2 - \Delta r$

and:

$r_2 - \Delta r$ to r_2

I am going to call the little distances traversed in each of these steps:

$\Delta\sigma_0$
$\Delta\sigma_1$
$\Delta\sigma_2$
$\Delta\sigma_3$
$\Delta\sigma_4$

all the way to:

$\Delta\sigma_{n-2}$

and:

$\Delta\sigma_{n-1}$

What we are looking for is the sum of the n little lengths:

distance $= \Delta\sigma_0 + \Delta\sigma_1 + \Delta\sigma_2 + \Delta\sigma_3 + \Delta\sigma_4 + \ \ldots\ + \Delta\sigma_{n-2} + \Delta\sigma_{n-1}$

So, to summarize, we have n little steps each going a coordinate change of Δr, and those n little steps are going to get us from r_1 to r_2. That means that we must have:

$$\Delta r = \frac{r_2 - r_1}{n}$$

Furthermore, the little length $\Delta\sigma_k$ is given by:

$$\Delta\sigma_k = \frac{\Delta r}{\left(1 - \frac{2M}{r}\right)^{1/2}}$$

The only question remains is what to use for the variable r that is down in the denominator. We want to use an r that is good on the locally flat patch from $r_1 + k\Delta r$ to $r_1 + (k+1)\,\Delta r$. We could use $r = r_1 + k\Delta r$ or $r = r_1 + (k+1)\,\Delta r$ or actually, anything in between!

The reason it doesn't matter is that there is already a factor of Δr in the numerator, and so the difference between using $r = r_1 + k\Delta r$ or $r = r_1 + (k+1)\,\Delta r$ is only going to affect $\Delta\sigma_k$ by $(\Delta r)^2$. The final answer is going to have n terms in it, and n terms proportional to $(\Delta r)^2$ is something we can neglect.

On the other hand, n terms proportional to Δr is most definitely not something we can neglect, because $\Delta r = \frac{r_2 - r_1}{n}$ which means $n\Delta r = r_2 - r_1$ and that is obviously non-zero even as we take the little steps Δr to be tiny while simultaneously taking $n \to \infty$.

We aren’t done. That is as far as we got on Thursday.

Those of you who have had integrals know that I am very close to deriving the integral as the Riemann sum of a bunch of little pieces, and then I am going to take the width of the little pieces to zero as I take the number of little pieces to ∞.

Stay tuned for Stretching of Space Part II.