Stretching of Space in the Radial Direction—Part II

We finish unpacking Sample Problem 1 on p. 3-16.

Summarizing What we Had in Stretching of Space—Part I

What we are looking for is the sum of the n little lengths:

distance = $\Delta\sigma_0 + \Delta\sigma_1 + \Delta\sigma_2 + \Delta\sigma_3 + \Delta\sigma_4 + \ldots + \Delta\sigma_{n-2} + \Delta\sigma_{n-1}$

In other words, we have n little steps each going a coordinate change of Δr, and those n little steps are going to get us from r_1 to r_2. So we have:

$$\Delta r = \frac{r_H - r_L}{n}$$

Furthermore, the little length $\Delta\sigma_k$ is given by:

$$\Delta\sigma_k = \frac{\Delta r}{\left(1 - \frac{2M}{r_L + k\Delta r}\right)^{1/2}}$$

And to make things precise, we are going to take a limit where n is huge and $\Delta r = \frac{r_H - r_L}{n}$ is tiny.

The Riemann Sum

It is clumsy writing lots of terms and using ... to stand for omitted terms. So first I want to introduce summation notation which uses a capital sigma, Σ. The summation has a lower limit and an upper limit and an integer (usually) that steps from the lower limit to the upper limit, in this case the lower limit is 0 and the upper limit is $n - 1$:

distance = $\Delta\sigma_0 + \Delta\sigma_1 + \Delta\sigma_2 + \Delta\sigma_3 + \Delta\sigma_4 + \ldots + \Delta\sigma_{n-2} + \Delta\sigma_{n-1} \equiv \sum_{k=0}^{n-1} \Delta\sigma_k$

The triple equals says that that is a definition. There are n terms (which might not be immediately obvious because I often like to count from 0 to $n - 1$ instead of 0 to n).

Now we substitute in what $\Delta\sigma_k$ is:

distance = $\sum_{k=0}^{n-1} \Delta\sigma_k = \sum_{k=0}^{n-1} \frac{\Delta r}{\left(1 - \frac{2M}{r_L + k\Delta r}\right)^{1/2}}$

I am going to take the entire mess $\frac{1}{\left(1-\frac{2M}{r_L+k\Delta r}\right)^{1/2}}$ and call it $f(r_k)$. Then we have:

distance = $\sum_{k=0}^{n-1} f(r_k)\,\Delta r$

where

$$f(r_k) = \frac{1}{\left(1-\frac{2M}{r_k}\right)^{1/2}}$$

and

$$r_k = r_L + k\Delta r$$

and

$$\Delta r = \frac{r_H - r_L}{n}$$

This is a Riemann sum and when we take n huge, it has a name and an interpretation!

The Interpretation of the Riemann Sum

The sum

$$\sum_{k=0}^{n-1} f(r_k)\,\Delta r$$

where

$$r_k = r_L + k\Delta r$$

and

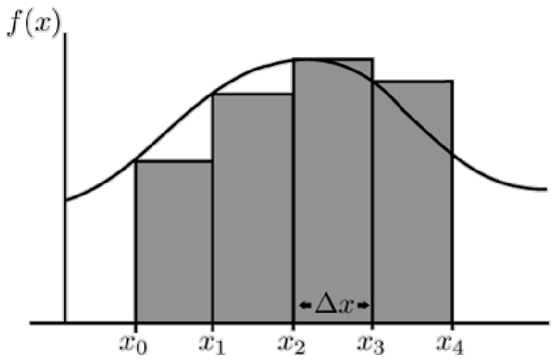
$$\Delta r = \frac{r_H - r_L}{n}$$

has the interpretation of being an area!

$$\sum_{k=0}^{n-1} \text{width} \cdot \text{height}_k$$

where the width is Δr and the kth height is $f(r_k)$.

You can easily find many pictures on the internet illustrating this. Here is one:



In this picture, $x_L = x_0$, $x_H = x_4$, $n = 4$, and $\Delta x = \frac{x_H - x_L}{n}$.

Because we chose $r_k = r_L + k \Delta r_k$ rather than $r_k = r_L + (k+1)\Delta r_k$ (or something in between), we have what is called the left-hand Riemann sum.

The Notation for the Riemann Sum when $n \to \infty$

Our entire procedure depends on taking locally flat patches. The more precision we want, the smaller the patches we have to take. This amounts to making n large. In the limit that $n \to \infty$, this is called the integral.

As I said in Part I, we are not going to dispel the fog of the 1700s by taking limits, but here I am talking about taking limits! Well, yes, but I am not doing it rigorously. There are occasionally problems when not doing limits rigorously, but you have to deliberately go looking for trouble to encounter those problems. So we are doing limits in a way that occasionally leads to trouble. Ryan's course is all about that!

There is a name and a notation for the sum in the limit that $n \to \infty$. The name is “the integral.” The notation is:

$$\int_{r_L}^{r_H} f(r)\, dr \equiv \lim_{n\to\infty} \sum_{k=0}^{n-1} f(r_L + k\Delta r)\,\Delta r \quad \text{where} \quad \Delta r = \frac{r_H - r_L}{n}$$

There are two important thing about this notation:

(1) It has an interpretation. The interpretation is the area under the curve of the function $f(r)$ between the r values of r_L and r_H.
(2) It is easy (nowadays) to use Mathematica, WolframAlpha, etc. to get the formula for the area under the curve for most any function $f(r)$.
(3) The answer depends on r_H and r_L. The answer always has the form $F(r_H) - F(r_L)$ where the function F is called “the integral of f.”

Because the answer always has the form $F(r_H) - F(r_L)$, there is a notation for that too:

$$F(r)\Big|_{r=r_L}^{r=r_H} \equiv F(r_H) - F(r_L)$$

We usually shorten the notation on the left-hand side a little to:

$$F(r)\Big|_{r_L}^{r_H}$$

Conclusion

We are done unpacking Sample Problem 1.

We have learned that the distance from r_L to r_H is:

$$\text{distance} = \int_{r_L}^{r_H} f(r)\, dr$$

and in this case $f(r) = \frac{1}{(1-2M/r)^{1/2}}$

Furthermore, the authors have done the integral for us in Sample Problem 1 and found:

distance $= \int_{r_L}^{r_H} f(r)\, dr = F(z)\Big|_{z_L}^{z_H} = F(z_H) - F(z_L)$ where $z_H = \sqrt{r_H}$ and $z_L = \sqrt{r_L}$.

The function $F(z)$ in their Eq. 18 is quite the mess:

$$F(z) = z\sqrt{z^2 - 2M} + 2M \ln\left(z + \sqrt{z^2 - 2M}\right)$$

Obviously their answer is only valid when $z_H = \sqrt{r_H}$ and $z_L = \sqrt{r_L}$ are both greater than $\sqrt{2M}$ because if those things weren't both true, you'd have negative numbers to take the square root of.