Cosmology — Term 4 Exam — Special Relativity, Black Holes

Open book, but not open problem-set solution, so that I can re-assign some problems that are very similar to ones you have already done. Except for the table and graph, please use your own paper.

1. John Tries to Graffiti Mary's Ship

There is only one speed in this problem v_{rel}, the speed of Mary's ship, which is moving in the positive x-direction according to John. Mary's rocket is L long (according to Mary). John's lab is L long (according to John). The center of Mary's rocket reaches the center of John's lab at $t' = t = 0$, where t' is Mary's time coordinate and t is John's time coordinate. To make life simple, let's measure John's x-coordinate from the center of his lab, and Mary's x'-coordinate from the center of her ship.

At $t = 0$, John discharges two paint bombs at each end of his lab in an attempt to graffiti the front and back of Mary's ship.

(a) John was naive. He forgot about length contraction. He thought that paint bomb 1, that goes off at coordinates $\text{GF} = \left(0, \frac{L}{2}\right)$, would graffiti the front of Mary's ship, and paint bomb 2, that goes off at coordinates $\text{GB} = \left(0, -\frac{L}{2}\right)$, would graffiti the back of Mary's ship. What are the correct values, in John's coordinates, of MF (the front) and MB (the back) of Mary's ship at $t = 0$?

(b) Despite John's incompetence, Mary is extremely concerned about her ship's paint job. *She has not forgotten* about length contraction, and according to her, it is John's lab that is length-contracted! So at $t' = 0$, both ends of her rocket are sticking out of his shortened lab. Perhaps the paint bombs will graffiti her ship after all!?

Use the inverse Lorentz transformation, Eq. L-11a (reproduced at top of next page), to discover where event $\text{GF} = \left(0, \frac{L}{2}\right)$, occurs in Mary's coordinates.

(c) Use the transformation again, to discover where $\text{GB} = \left(0, -\frac{L}{2}\right)$ occurs in Mary's coordinates.

(d) In two short sentences, (i) explain what concept Mary forgot, and (ii) explain why her paint job is fine.

(e) Plugging in $L = 40$ m and $v_{rel} = \frac{3}{5}$, what are the coordinates MF and MB (in John's coordinates) that you found in Part (a)?

(f) Plugging in these same values, what are GF and GB in Mary's coordinates that you found in (b) and (c)?

The Inverse Lorentz Transformation, for your convenience:

The y and z components are respectively equal in both frames, as before. Then the **inverse Lorentz transformation equations** become

$$\begin{aligned} t' &= -v_{\text{rel}}\gamma x + \gamma t \\ x' &= \gamma x - v_{\text{rel}}\gamma t \\ y' &= y \\ z' &= z \end{aligned} \qquad \text{(L-11a)}$$

Problem 2 — The Famous Equation, $E = mc^2$

$E = mc^2$ is probably the most famous equation in physics. However, the equation is only correct for a particle at rest!

Instead of $E = mc^2$, the correct formula, when a particle is not at rest, is:

$E = mc^2 \frac{\Delta t}{\Delta \tau}$

In this class, we usually set $c = 1$. In that case, the formula is just:

$E = m \frac{\Delta t}{\Delta \tau}$

(a) In a short sentence, why does $E = m \frac{\Delta t}{\Delta \tau}$ simplify to $E = m$ if a particle is at rest?

(b) In a time Δt how far, Δx, does a particle moving at speed v in the $+x$-direction go? (This is super-simple. Don't overthink it. Just write down the equation for Δx in terms of v and Δt.)

(c) Take the square root of the Lorentz metric, $(\Delta \tau)^2 = (\Delta t)^2 - (\Delta x)^2$ to solve for $\Delta \tau$. You can be fast and loose with the minus sign when taking the square root, and assume that $(\Delta t)^2 - (\Delta x)^2 > 0$, and that $\Delta \tau > 0$. (This is also super-simple. Don't overthink it.)

(d) Substitute your answer for Part (b) into your answer for Part (c). Simplify, including using $\gamma \equiv \frac{1}{\sqrt{1-v_{\text{rel}}^2}}$ to make your answer tidier.

(e) Alternatively, we could have left m out of this and just computed a nice equation for $\frac{\Delta t}{\Delta \tau}$. What is that equation?

(f) In (e), you found a nice simple equation for $\frac{\Delta t}{\Delta \tau}$. Use that and $p = m\,\frac{\Delta x}{\Delta \tau}$ to get the correct answer for p, the relativistic momentum. HINT: For Part (f), instead of repeating a bunch of work, how about using $p = m\,\frac{\Delta x}{\Delta \tau} = m\,\frac{\Delta t}{\Delta \tau}\,\frac{\Delta x}{\Delta \mathrm{t}}$ and the nice simple expression you found for $\frac{\Delta t}{\Delta \tau}$ in part (e). You might also need what you got in Part (b).

(g) Put in $v = 0.995$ into your equation. So that you don't have to use a calculator, I'll tell you that $(0.995)^2$ is very close to 0.99. What is E?

(h) Put in $v = 0.99995$ into your equation. Again, so that you don't have to use a calculator, I'll tell you that $(0.99995)^2$ is very close to 0.9999. What is E now?

DISCUSSION: The speed only increased by 0.00495 going from (g) to (h), but the energy increased by $90\,m$ (if you did (g) and (h) right). Perhaps it is already becoming obvious that a finite amount of energy can never bring a massive particle to the speed of light.

Problem 3 — Inward-Going Photons on the [*t*, r]-Slice Outside the Event Horizon

We start with the metric in Eq. (5) of TWB Chapter 3: $(\Delta\tau)^2 = (1 - 2\,M/r)\,(\Delta t)^2 - \frac{(\Delta r)^2}{1-2\,M/r} - r^2(\Delta\phi)^2$.

(a) Outside the event horizon, 1-2M/r is positive. We are going to restrict ourselves to the [t, r]-slice.

So set $\Delta\phi = 0$ and simplify!

(b) We want to study the behavior of light-like world-lines. For light, $\Delta\tau = 0$. Make that simplification too.

Then rearrange and then do a *careful* taking of square roots.

(c) We are going to focus on inward-going photons, not outward-going ones.

So simplify what you got in Part (b) under the assumption that $\Delta t > 0$ and $\Delta r < 0$.

(d) We are going to introduce a new dimensionless global coordinate $x = r/M$, and of course it follows that $\Delta x = \Delta r/M$. Also introduce $t = y/M$, and $\Delta t = \Delta y/M$.

Rewrite what you got in (c) using these new coordinates and simplify.

(e) We will start at $x = 4.0$ and finish at $x = 2.4$ and do that in 8 steps of $\Delta x = -0.2$ each. Fill in the final two columns of the table below using the formula you found in (d).

In[ ]:=
```
table2 = Table[
    {i, 4.0 - 0.2 i, "    -0.2         ", If[True, "", Round[0.2 / (1 - 2 / (4.0 - 0.2 i)), 0.01]]}, {i, 0, 7, 1}];
table2WithHeader = PrependTo[table2, {"i", " xi ", "     Δx ", "Δyi    ", "Cumulative change in y"}];
TableForm[table2WithHeader]
```

Out[ ]//TableForm=

i	x_i	Δx	Δy_i	Cumulative change in y
0	4.	−0.2		
1	3.8	−0.2		
2	3.6	−0.2		
3	3.4	−0.2		
4	3.2	−0.2		
5	3.	−0.2		
6	2.8	−0.2		
7	2.6	−0.2		

In the spirit of continuing to not need a calculator, below is a table of miscellaneous combinations, *some* of which are relevant.

In[ ]:=
```
TableForm[
 PrependTo[Table[{Round[x, 0.1], Round[0.2 / x, 0.01], Round[0.2 / (1 - 2 / x), 0.01],
    Round[0.2 / Sqrt[1 - 2 / x], 0.01]}, {x, 4.0, 2.6, -0.2}],
  {"xi", "0.2/xi", "0.2/(1-2/xi)", "0.2/sqrt(1-2/xi)"}]]
```

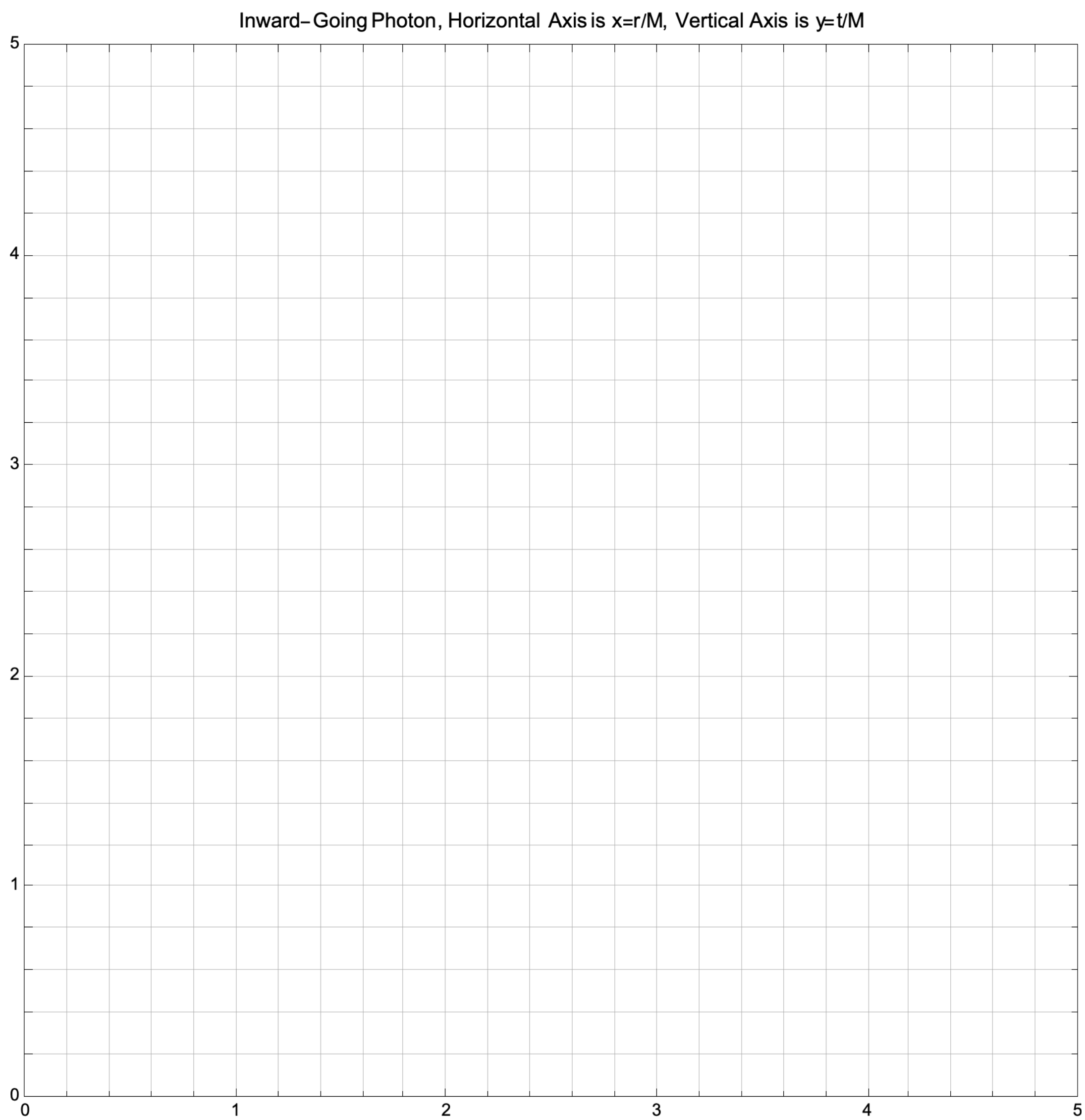
Out[ ]//TableForm=

x_i	$0.2/x_i$	$0.2/(1-2/x_i)$	$0.2/\text{sqrt}(1-2/x_i)$
4.	0.05	0.4	0.28
3.8	0.05	0.42	0.29
3.6	0.06	0.45	0.3
3.4	0.06	0.49	0.31
3.2	0.06	0.53	0.33
3.	0.07	0.6	0.35
2.8	0.07	0.7	0.37
2.6	0.08	0.87	0.42

(g) Graph y vs. x starting with $x = 4.0$, $y = 0.0$, and working your way inward. The last row of the table is the row that goes from $x = 2.6$ to $x = 2.4$.

In[ ]:=
```
Plot[{}, {x, 0, 5}, PlotRange → {{0, 5}, {0, 5}}, AspectRatio → 1, GridLines → {Range[0, 5, 0.2], Range[0, 5, 0.2]},
 Frame → True, PlotLabel → "Inward-Going Photon, Horizontal Axis is x=r/M, Vertical Axis is y=t/M"]
```

Out[ ]=



4. Hanging Out Near Gargantuan

(Gargantuan is the black hole in *Interstellar.*)

Again, we start with the metric in Eq. (5) of TWB Chapter 3: $(\Delta\tau)^2 = (1 - 2M/r)(\Delta t)^2 - \frac{(\Delta r)^2}{1-2M/r} - r^2(\Delta\phi)^2$.

(a) For this problem, we are not going to move in r, nor are we going to move in ϕ. We are just going to sit at a fixed position r, ϕ. Time will elapse and proper time will elapse.

Simplify Eq. (5) for this situation.

(b) Take the square root, and go ahead and be sloppy by assuming that (i) we are outside the event horizon so that 1-2M/r > 0, and that $\Delta\tau > 0$, and also that $\Delta t > 0$.

(c) Imagine two people who possess two identical clocks each of which ticks with interval $\Delta\tau = T$. One clock is closer to Gargantuan, at $r = r_L$, and the other is farther from Gargantuan, at $r = r_H$.

Write down formulas for Δt_L and Δt_H where those are the elapsed coordinate times between ticks of the two clocks and then use $\Delta\tau_L = \Delta\tau_H = T$ to simplify the formulae.

(d) Put in $r_H = \infty$ to the formula for Δt_H. For it, in that case, you should get something really simple.

Examining the formula for Δt_L, is Δt_L larger or smaller than Δt_H?

(e) You should have found that more coordinate time is elapsing for the clock at r_L. That doesn't mean much, until these people someday meet. Wherever they meet, they will have the same coordinate time. But for a given amount of coordinate time, fewer ticks will have elapsed for the person who hung out at r_L. So they will have aged less!! Put in r_L just barely bigger than $2M$. So that the numbers work out nicely and you don't need a calculator, put in $r_L = \frac{200M}{99}$. If a person hangs out for 7 years of wristwatch time at this position, how much coordinate time and proper time will have elapsed for the person at $r_H = \infty$?