Cosmology — Assignment 6 — Solution

Problem 1 — Whacky Units

Cosmologists set $G = c = 1$. In conventional SI units, $G = 6.67 \times 10^{-11} \frac{\text{N m}^2}{\text{kg}^2}$ and

$c = 299\,792\,458 \,\frac{\text{m}}{\text{s}} = 3 \times 10^8 \,\frac{\text{m}}{\text{s}}$. But the Newton, N, is a derived unit, $1\text{ N} = \frac{\text{kg m}}{\text{s}^2}$, so $G = 6.67 \times 10^{-11} \frac{\text{m}^3}{\text{kg s}^2}$.

(a) If $G = c = 1$ then $1 = \frac{G}{c^2} = \frac{6.67 \times 10^{-11} \frac{\text{m}^3}{\text{kg s}^2}}{\left(3 \times 10^8 \frac{\text{m}}{\text{s}}\right)^2} = \frac{6.67}{9} \times 10^{-11-16} \frac{\text{m}}{\text{kg}} = 7.41 \times 10^{-28} \frac{\text{m}}{\text{kg}}$

So apparently $1\text{ kg} = 7.41 \times 10^{-28}$ m.

(b) $1 = c = 3\text{x}10^8 \text{ m/s} = \frac{3 \times 10^8 \text{ m/s}^2}{1/\text{s}}$ so the frequence of 1Hz corresponds to an acceleration of $3 \times 10^8 \frac{\text{m}}{\text{s}}$.

(c) The diameter of the Earth is 12.74×10^6 m. Divide this by $c = 3\text{x}10^8$ m/s, and get 0.0425 s.

(d) The diameter of the Sun is 1.393×10^9 m. Divide this by $c = 3\text{x}10^8 \frac{\text{m}}{\text{s}}$, and get 4.64 s.

(e) The mass of the Sun is 1.989×10^{30} kg. But $1\text{ kg} = 7.41 \times 10^{-28}$ m by part (a). So The mass of the Sun is

$1.989 \times 10^{30}\text{ kg} \, \frac{7.41 \times 10^{-28}\text{ m}}{1\text{ kg}} = 1474\text{ m}$.

If the Sun were a black hole, its event horizon would be at $2M = 2878$ m.

Problem 2 — Astronaut Stretching

(a) If we put $r = \infty$ and $v = 0$ into, $K = \frac{1}{2} m_{\text{astronaut}} v^2$, $U = -\frac{G m_{\text{astronaut}} M}{r}$, then $K = U = 0$.

(b) $E = 0$.

(c) $0 = \frac{1}{2} m_{\text{astronaut}} v^2 - \frac{G m_{\text{astronaut}} M}{r}$

Well, we can cancel $m_{\text{astronaut}}$ out and rearrange a little, giving

$\frac{1}{2} v^2 = \frac{GM}{r}$ or $v^2 = \frac{2\,GM}{r}$ or $v = \sqrt{\frac{2\,GM}{r}}$. So at r_{ouch}, $v_{\text{ouch}} = \sqrt{\frac{2\,GM}{r_{\text{ouch}}}}$. We are working in the whacky units where $G = 1$, so I guess I could just write $v_{\text{ouch}} = \sqrt{\frac{2M}{r_{\text{ouch}}}}$.

(d) TWB Chapter 3, Problem 5, p. 3-38, Part D. I called the speed we found in Part (c), v_{ouch}. As a fraction of the speed of light,

$$\frac{v_{\text{ouch}}}{c} = \sqrt{\frac{2\,GM/c^2}{r_{\text{ouch}}}},$$

but again, we are working in units where $G = c = 1$, so as a fraction of the speed of light, we could just as well have left the answer as:

$$v_{\text{ouch}} = \sqrt{\frac{2\,M}{r_{\text{ouch}}}}.$$

(e) TWB Chapter 3, Problem 5, p. 3-38, Part E. Your period of trauma would be at most,

$$T_{\text{ouch}} = \frac{r_{\text{ouch}}}{v_{\text{ouch}}} = r_{\text{ouch}} \Big/ \sqrt{\frac{2M}{r_{\text{ouch}}}} = \sqrt{\frac{r_{\text{ouch}}^3}{2M}}$$

It's time we substitute in what we learned in Problem Set 5, Problem 5, specifically, $r_{\text{ouch}} = \left(\frac{2\,M\Delta r}{|g_E|}\right)^{1/3}$.

So,

$$T_{\text{ouch}} = \sqrt{\frac{\frac{2M\Delta r}{|g_E|}}{2M}} = \sqrt{\frac{\Delta r}{|g_E|}}$$

How fabulously simple! T_{ouch} does not depend on the mass of the black hole! Nor does it depend on G or c.

(f) TWB Chapter 3, Problem 5, p. 3-38, Part F.

Well, finally we will plug in some numbers, and find out what T_{ouch} is in seconds:

$$T_{\text{ouch}} = \sqrt{\frac{1\text{ m}}{9.8\text{ m}/\text{s}^2}} = 0.32\text{ s}$$

Problem 3 — Embedding the $[r, \phi]$-Slice Outside the Event Horizon

(a) We start with the metric in Eq. (6): $(\Delta\sigma)^2 = -(1 - 2\,M/r)\,(\Delta t)^2 + \frac{(\Delta r)^2}{1-2\,M/r} + r^2(\Delta\phi)^2$, and put $\Delta t = 0$, and get:

$$(\Delta\sigma)^2 = \frac{(\Delta r)^2}{1-2\,M/r} + r^2(\Delta\phi)^2.$$

(b) Also put $\Delta\phi = 0$ and get $(\Delta\sigma)^2 = \frac{(\Delta r)^2}{1-2\,M/r}$.

(c) Take the square root and get $\Delta\sigma = \frac{|\Delta r|}{\sqrt{1-2\,M/r}}$, or if you prefer, $\Delta\sigma = \frac{\pm\Delta r}{\sqrt{1-2\,M/r}}$.

(d) If we want $\Delta\sigma$ positive when Δr is negative, we need:

$$\Delta\sigma = \frac{-\Delta r}{\sqrt{1-2\,M/r}}$$

(e) Substitute in $M\Delta s = \Delta\sigma$, $Mx = r$, and $M\Delta x = \Delta r$. Then

$$M\,\Delta s = \frac{-M\Delta x}{\sqrt{1-2\,M/(Mx)}} \text{ or } \Delta s = \frac{-\Delta x}{\sqrt{1-2/x}}$$

(f) We start at $x = 7.0$ and finish at $x = 2.2$. We do that in 12 steps of $\Delta x = -0.4$ each.

In[ ]:=
```
table = Table[{i, 7.0 - 0.4 i, -0.4, "    -10",
    If[False, "", Round[0.4 / Sqrt[1 - 2 / (7 - 0.4 i)], 0.01]], If[False, "", Round[25 × 0.4 / Sqrt[1 - 2 / (7 - 0.4 i)]]]}, {i, 0, 11, 1}];
tableWithHeader = PrependTo[table, {"i", "xᵢ", " Δx ", "25*Δx (for mm) " , " Δsᵢ      ", "25*Δsᵢ (for mm)"}];
TableForm[tableWithHeader]
```

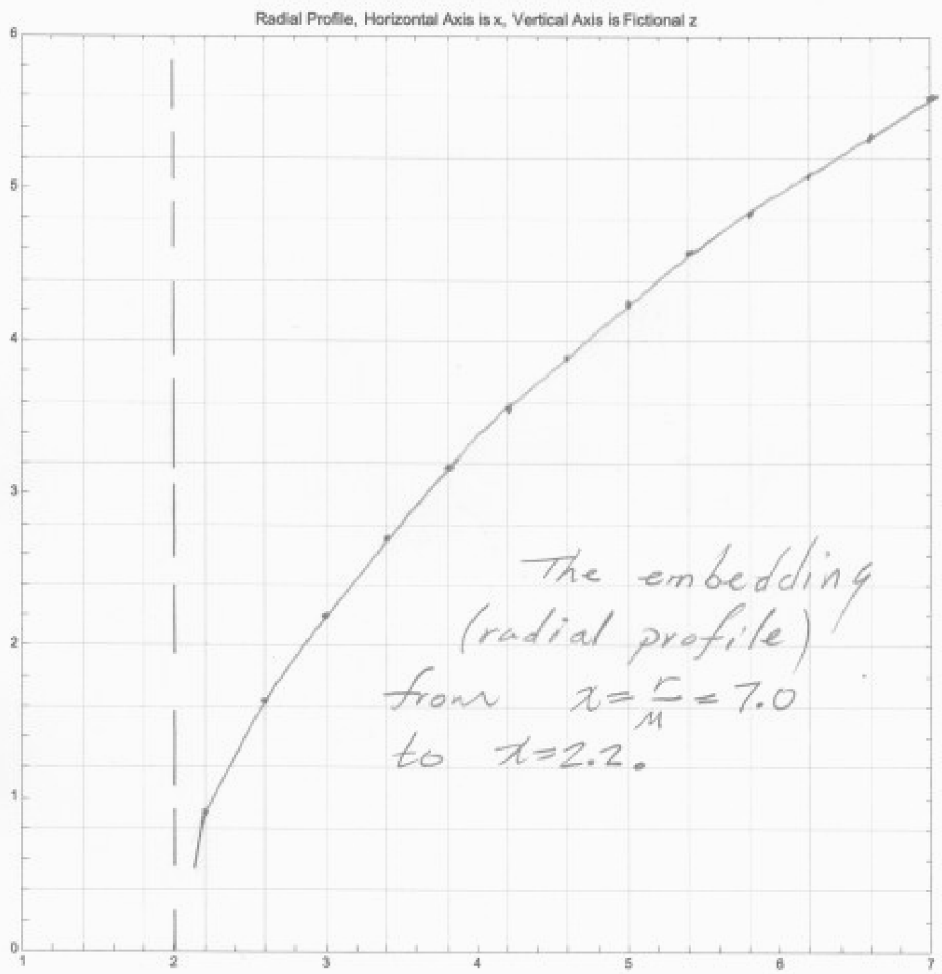
Out[ ]//TableForm=

i	x_i	Δx	$25*\Delta x$ (for mm)	Δs_i	$25*\Delta s_i$ (for mm)
0	7.	-0.4	-10	0.47	12
1	6.6	-0.4	-10	0.48	12
2	6.2	-0.4	-10	0.49	12
3	5.8	-0.4	-10	0.49	12
4	5.4	-0.4	-10	0.5	13
5	5.	-0.4	-10	0.52	13
6	4.6	-0.4	-10	0.53	13
7	4.2	-0.4	-10	0.55	14
8	3.8	-0.4	-10	0.58	15
9	3.4	-0.4	-10	0.62	16
10	3.	-0.4	-10	0.69	17
11	2.6	-0.4	-10	0.83	21

(g) The beautiful radial profile. If you were going to restore ϕ and make this symmetric, you'd need more 3-d artistry skills than we are applying here. See Figure 12 on p. 3-30.

```
Plot[{}, {x, 0, 8}, PlotRange → {{1, 7}, {0, 6}}, AspectRatio → 1, GridLines → {Range[1, 7, 0.4], Range[0, 6, 0.4]},
 Frame → True, PlotLabel → "Radial Profile, Horizontal Axis is x, Vertical Axis is Fictional z"]
```

Out[ ]=



Problem 4 — Exploring Light-Like Worldlines of the $[r, \phi]$-Slice Inside the Event Horizon

(a) We start with the metric in Eq. (5): $(\Delta\tau)^2 = (1 - 2M/r)(\Delta t)^2 - \frac{(\Delta r)^2}{1-2M/r} - r^2(\Delta\phi)^2$, but because $1 - 2M/R$ is always negative in the region we are interested in for this problem (inside the event horizon), it is probably clearer to write:

$$(\Delta\tau)^2 = -(2M/r - 1)(\Delta t)^2 + \frac{(\Delta r)^2}{2M/R-1} - r^2(\Delta\phi)^2$$

(b) Set $\Delta t = 0$ and simplify:

$$(\Delta\tau)^2 = \frac{(\Delta r)^2}{2M/R-1} - r^2(\Delta\phi)^2$$

(c) $0 = \frac{(\Delta r)^2}{2M/R-1} - r^2(\Delta\phi)^2$ or $r^2(\Delta\phi)^2 = \frac{(\Delta r)^2}{2M/R-1}$ or $\Delta\phi = \frac{\pm\Delta r}{r\sqrt{2M/R-1}}$.

If we want decreasing Δr to result in increasing $\Delta\phi$, then we need: $\Delta\phi = \frac{-\Delta r}{r\sqrt{2M/R-1}}$

(e) Introduce $x = r/M$, $\Delta x = \Delta r/M$:

$$\Delta\phi = \frac{-M\Delta x}{Mx\sqrt{2M/(Mx)-1}} = \frac{-\Delta x}{x\sqrt{2/x-1}}$$

(f) Fill in the final three columns of the table table below using $\Delta\phi_i = \frac{-\Delta x}{x_i \sqrt{2/x_i - 1}}$.

bleForm=

i	x_i	Δx		$\Delta\phi_i$	$\frac{360}{2\pi} * \Delta\phi_i$ (degrees)	Cumulative change in ϕ
0	1.5	−0.1	0.115		6.6	6.6
1	1.4	−0.1	0.109		6.3	12.9
2	1.3	−0.1	0.105		6.	18.9
3	1.2	−0.1	0.102		5.8	24.7
4	1.1	−0.1	0.101		5.8	30.5
5	1.	−0.1	0.1		5.7	36.2
6	0.9	−0.1	0.101		5.8	42.0
7	0.8	−0.1	0.102		5.8	47.8
8	0.7	−0.1	0.105		6.	53.8
9	0.6	−0.1	0.109		6.3	60.1
10	0.5	−0.1	0.115		6.6	66.7
11	0.4	−0.1	0.125		7.2	73.9
12	0.3	−0.1	0.14		8.	81.9
13	0.2	−0.1	0.167		9.5	91.4
14	0.1	−0.1	0.229		13.1	104.5

(g) Finally, it is time to make a beautiful world line:

Worldline for a Light-Like Particle on an [r,ϕ]-Slice Inside the Event Horizon