The non-calculus and calculus ways of computing $\frac{1}{(r+\Delta r)^2} - \frac{1}{r^2}$

First let us dispense with the need for calculus in part (a). The question is what is the change in $\frac{1}{r^2}$ if we change it to $\frac{1}{(r+\Delta r)^2}$ where $\Delta r << r$. We want to know (new value minus old value is what we're computing)

$$\frac{1}{(r+\Delta r)^2} - \frac{1}{r^2} = \frac{1}{r^2}\left[\frac{1}{\left(1+\frac{\Delta r}{r}\right)^2} - 1\right]$$

common factor of $\frac{1}{r^2}$, then common denominator, then FOIL ing

$$= \frac{1}{r^2}\,\frac{1-\left(1+\frac{\Delta r}{r}\right)^2}{1+\left(\frac{\Delta r}{r}\right)^2} = \frac{1}{r^2}\,\frac{-2\frac{\Delta r}{r}-\left(\frac{\Delta r}{r}\right)^2}{1+\left(\frac{\Delta r}{r}\right)^2}$$

$$\approx -2\frac{\Delta r}{r^3}$$

this step requires you to think hard about what can and cannot be neglected — perhaps it helps to be concrete and imagine $\frac{\Delta r}{r} = 0.01$ or some other small number

Exactly what you'd get if you threw the full power of derivatives at the problem instead of algebra and approximations as follows:

What is the calculus method? We use

$$f(x+\Delta x) - f(x) \approx f'(x)\Delta x$$

and apply it to $g(r) = \frac{1}{r^2}$. $g'(r) = -\frac{2}{r^3}$.

$$g(r+\Delta r) - g(r) \approx g'(r)\Delta r = \frac{-2\Delta r}{r^3}$$