Newton — Term 2 Exam

Tuesday, Oct. 11

1. An Arc in the Evanescent Limit (a riff on a problem posed by Ben)

On the curve below, the tangent at the point ($x = -1/4$, $y = -1$) is drawn.

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Plot[{-2 Sqrt[-x], 2 (x+1/4) - 1}, {x, -0.75, 0.29}, AspectRatio → 2, PlotRange → {{-0.75, 0.29}, {-2, 0.08}},
 GridLines → {Range[-0.7, 0.25, 0.05], Range[-2.0, 0.05, 0.05]}, Ticks → {Range[-0.7, 0.2, 0.1], Range[-2.0, 0.0, 0.1]}, Epilog → {PointSize[0.02],
   Point[{0.0, 0.0}], Text["E", {0, 0}, {-2, 2}],
   Point[{0.25, 0.0}], Text["X", {0.25, 0}, {-2, 2}],
   Point[{-0.25, -1.0}], Text["Z", {-0.25, -1.0}, {-2, 2}]}]
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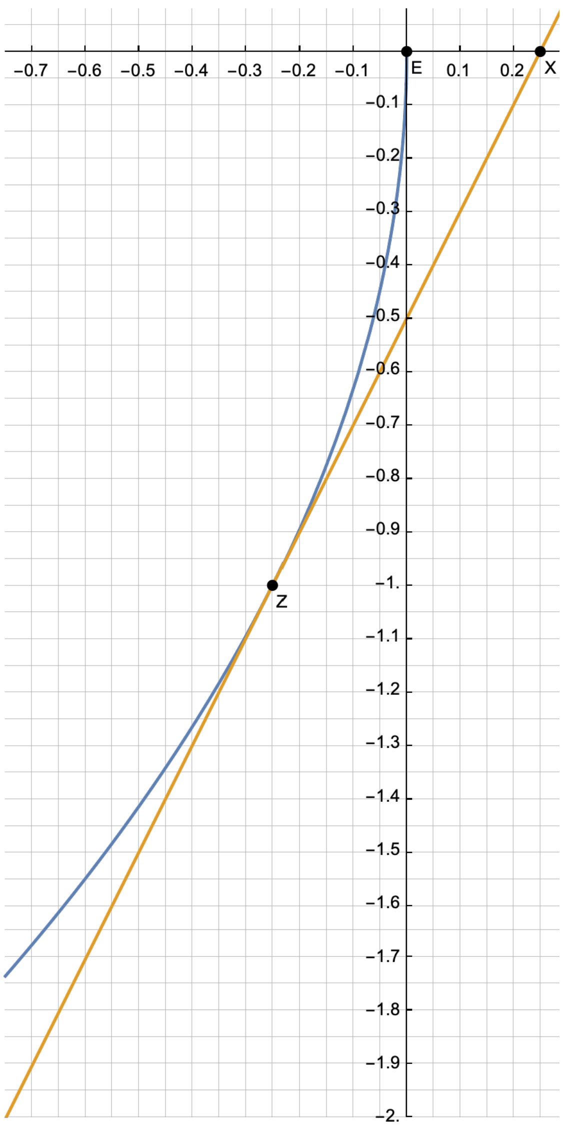
Out[ ]=



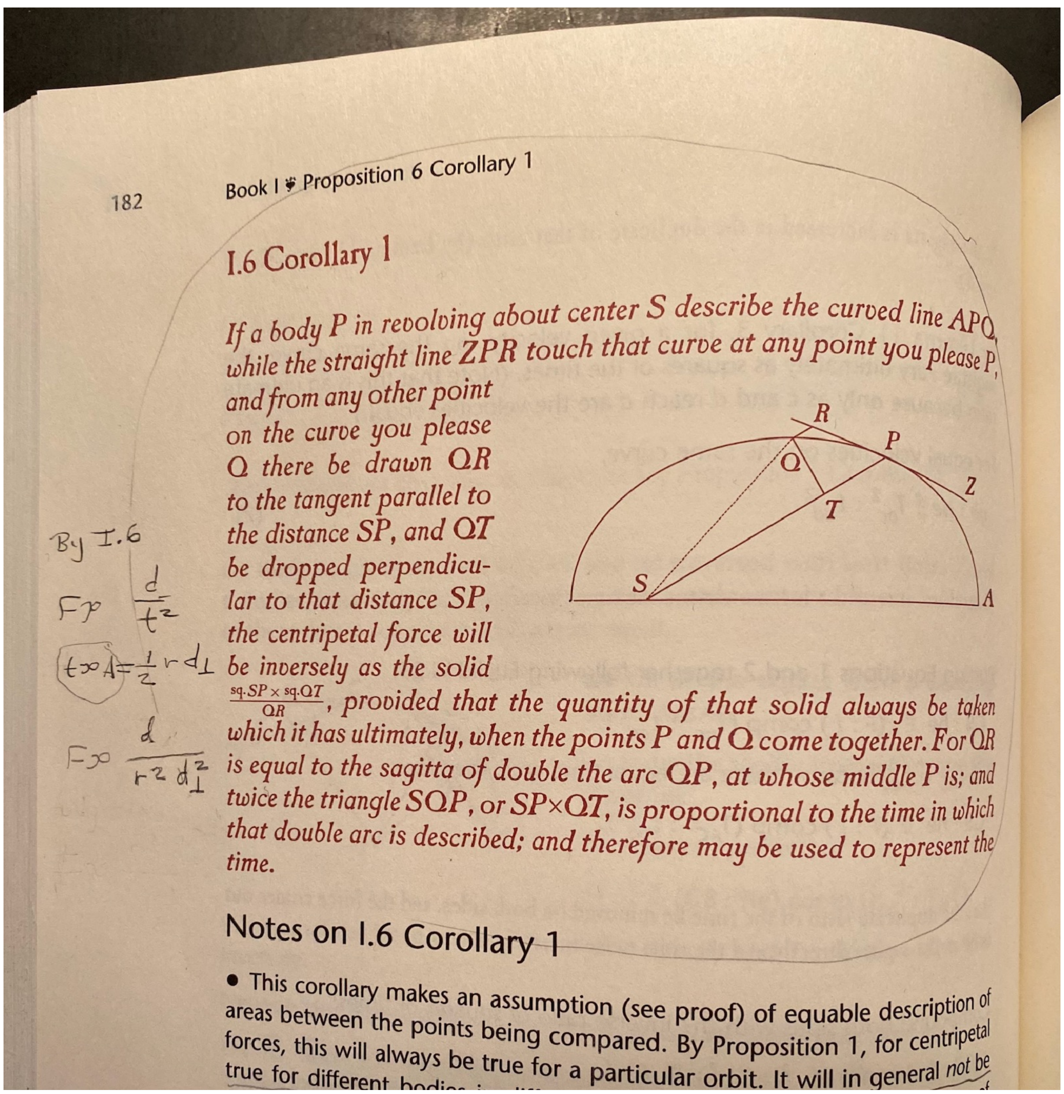
Assume that the centripetal force originating from S (not shown) is at a point far to the left of Z. For definiteness, let us place it at (−10.25, −1.0), so that it is 10 units of distance to the left of Z, and EX and

SZ are parallel. *Also assume that it is fine that anything you say when answering (a), (b), and (c) below are only approximate and only become exact in the evanescent limit where Z gets closer and closer to E.*

(a) What does Lemma 6 Corollary 1 tell us about the accelerative force at Z? Answer in terms of quantities such as EX (which refers to the length of segment EX). If necessary, introduce additional points and segments onto the drawing. An image of Lemma 6 Corollary 1 is below.

(b) The drawing is well-labeled with coordinate axes and a grid, so you can easily determine the lengths of the segments you found in (a). What does your whatever you said about the accelerative force in part (a) amount to — numerically — in terms of these lengths.

(c) Consider the three-sided figure EZX. It has some area. What can you say about the ratio of the area of EZ'X' to the area of EZX if Z' were at ($x = -1/16$, $y = -1/2$).



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I.6 Corollary 1

If a body P in revolving about center S describe the curved line APQ, while the straight line ZPR touch that curve at any point you please P, and from any other point on the curve you please Q there be drawn QR to the tangent parallel to the distance SP, and QT be dropped perpendicular to that distance SP, the centripetal force will be inversely as the solid $\frac{sq.SP \times sq.QT}{QR}$*, provided that the quantity of that solid always be taken which it has ultimately, when the points P and Q come together. For QR is equal to the sagitta of double the arc QP, at whose middle P is; and twice the triangle SQP, or SP×QT, is proportional to the time in which that double arc is described; and therefore may be used to represent the time.*

By I.6

$F \propto \frac{d}{t^2}$

$t \propto A = \frac{1}{2} r d_\perp$

$F \propto \frac{d}{r^2 d_\perp^2}$

Notes on I.6 Corollary 1

• This corollary makes an assumption (see proof) of equable description of areas between the points being compared. By Proposition 1, for centripetal forces, this will always be true for a particular orbit. It will in general not be true for different bodi…

2. A Frisbee is Thrown (a merger of two distinct problems posed by Luke and by Sofia)

Prof. Hill tosses a Frisbee whose tilt is down on the left side. Viewed from above, the Frisbee follows the arc below over the course of four seconds, starting from ($x = 60$, $y = 0$):

In[ ]:=
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ParametricPlot[{60 Cos[0.2 u^2], 40 Sin[0.2 u^2]}, {u, 0, 2}, PlotRange → {{30, 60.5}, {-0.5, 30}},
 AspectRatio → 1, GridLines → {Range[30, 60, 5], Range[0, 30, 5]}, Epilog → {PointSize[0.02],
   Point[{50.0, 5.0}], Text["Center of Force?", {50.0, 5.0}, {1.4, 0.0}],
   Point[{60.0, 0.0}], Text["0 s", {60, 0}, {3.7, -1.0}],
   Point[{60 Cos[0.8], 40 Sin[0.8]}], Text["4 s", {60 Cos[0.8], 40 Sin[0.8]}, {3.5, 0.8}]}]
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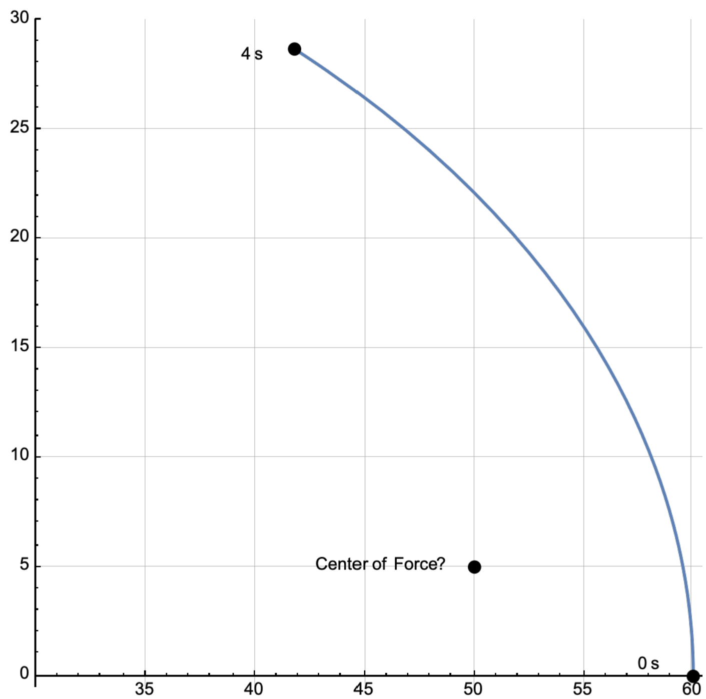
Out[ ]=



Ignore how the plot was constructed. All you have to go by is the arc that has been drawn.

(a) Someone who knows absolutely nothing about aerodynamics studies this arc and suggests that it could have been caused by a centripetal force whose center is at ($x = 50$, $y = 5$). Is Proposition 1 consistent with this?

(b) If your answer to (a) was no, demonstrate that it is impossible to place the intermediate points at $t = 1$ second, 2 seconds, and 3 seconds consistently with the equal area results of Proposition 1. If it was yes, demonstrate consistency with Proposition 1 over four equal time intervals by accurately placing the points $t = 1$ second, 2 seconds, and 3 seconds.

In case it is helpful, a good chunk of the statement and proof of Proposition 1 is on the next page.

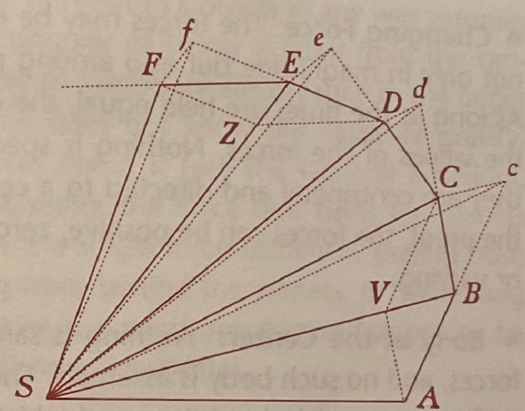
SECTION 2

On the Finding of Centripetal Forces

Proposition 1

The areas which bodies driven in orbits [gyros] *describe by radii drawn to an immobile center of forces, are contained in immobile planes and are proportional to the times.*

Let the time be divided into equal parts, and in the first part of the time let the body, by its inherent force, describe the straight line AB. In the second part of the time, the same body, if nothing were to impede it, would pass on by means of a straight line to c (by Law 1), describing the line Bc equal to AB, with the result that, radii AS, BS, cS being drawn to the center, the areas ASB, BSc would come out equal. But when the body comes to B, let the centripetal force act with an impulse that is single but great, and let it have the effect of making the body depart from the straight line Bc and continue in the straight line BC. Let cC be drawn parallel to BS, meeting BC at C; and, the second part of the time being completed, the body (by Corollary 1 of the Laws) will be located at C, in the same plane as the triangle ASB.

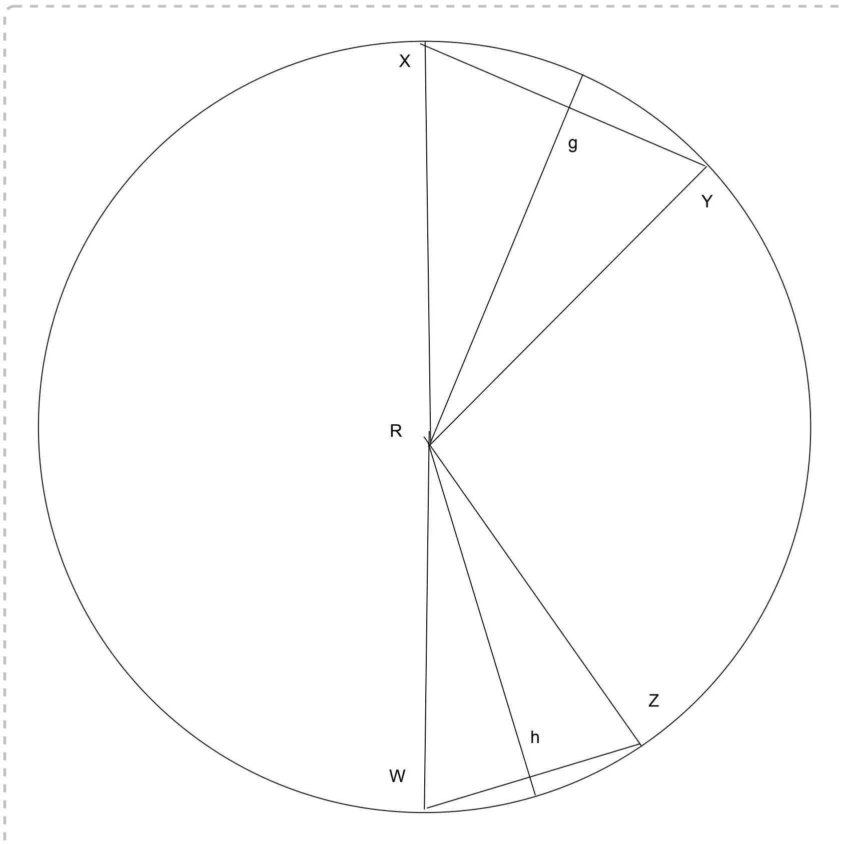


Connect SC, and, because of the parallels SB, Cc, triangle SBC will be equal to triangle SBc, and therefore also to triangle SAB. By a similar argument if the cen-

3. Two Particles on the Same Circle (a riff on a problem posed by Declan)

Particle *a* traverses XY on a circle in some time. Meanwhile, Particle *b* traverses WZ on the same circle in the same time.

In the figure, angle XRY = $\pi/4$, and angle ZRW = $\pi/6$. The radius of the circle, *r*, is 2. Point *g* is at the midpoint of the chord XY. Point *h* is at the midpoint of the chord WZ.



(a) What are the lengths of Rg and Rh?

(b) What are the lengths of the sagittae going from *g* to the circumference and from *h* to the circumference?

(c) What is the ratio of the accelerative force on Particle a to the accelerative force on Particle b?

(d) If Particle a is five times as massive as Particle b, what is the ratio of the impressed force on Particle a to the impressed force on Particle b?

4. Area Under a Curve (much like a problem on the 2nd problem set)

Consider the curve $y = \frac{e^x}{3}$ in the region $x = 0$ to $x = 2$:

In[ ]:=
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Plot[Exp[x]/3, {x, 0, 2}, PlotRange → {{0, 2}, {0, 2.6}},
 Ticks → {Range[0, 2, 0.2], Range[0.0, 2.6, 0.2]},
 GridLines → {Range[0, 2, 0.2], Range[0.0, 2.6, 0.2]},
 GridLinesStyle → Medium, Frame → True, AspectRatio → 2.6 / 2]
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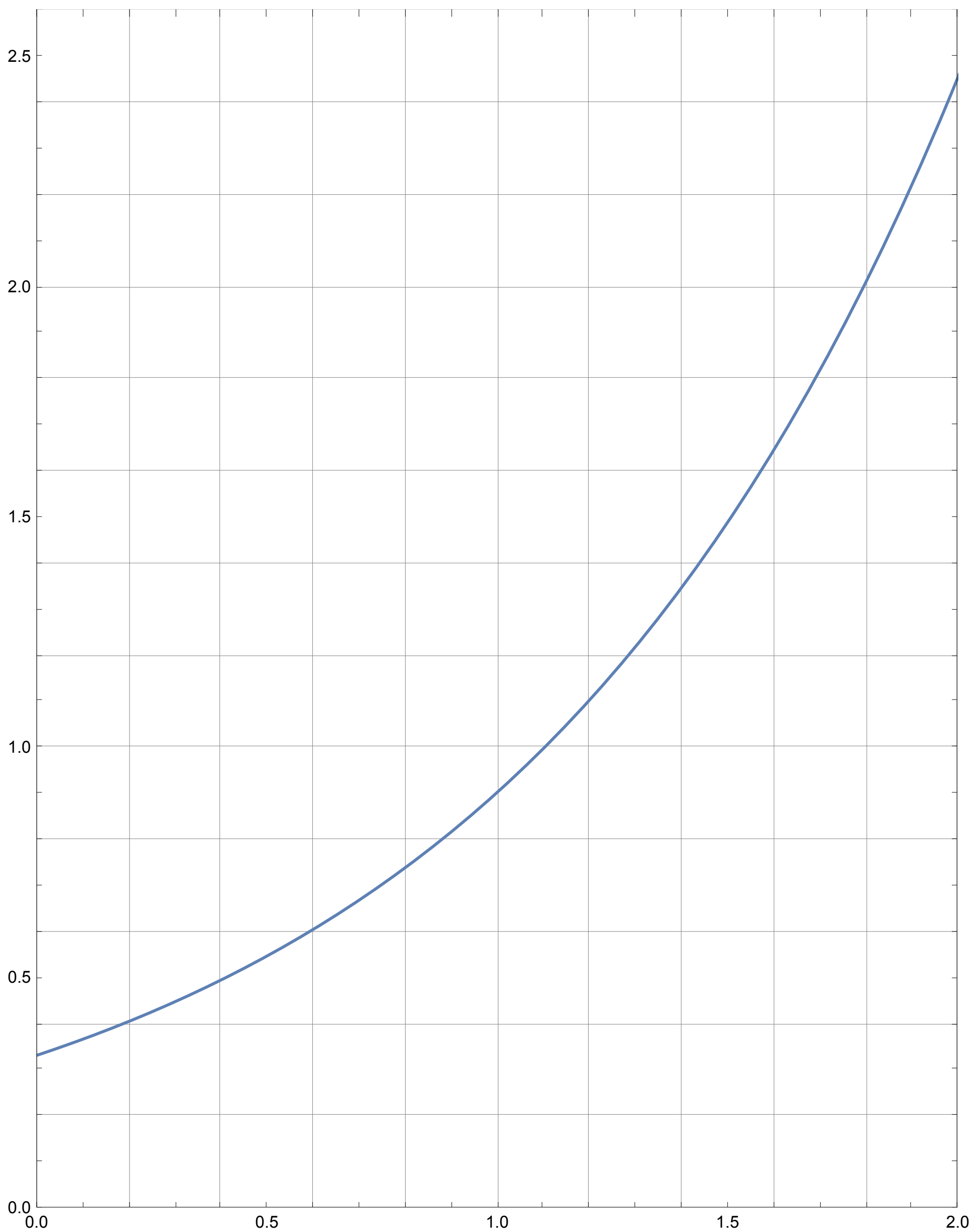
Out[ ]=



(a) What does each square on the plot represent in terms of area (in other words, what is the product of its width and height)? Count all the squares below the curve up (making an attempt to approximately account for partial squares). How many are there? Multiply by the individual area to get the (approximate) total area.

(b) Draw the rectangles that can be *inscribed under* this curve as Newton did back in Lemma 2. If you are not sure what I am looking for, the first one to draw is the rectangle going from $x = 0.0$ to $x = 0.2$ that is $\frac{1}{3} e^0 = \frac{1}{3}$ high. Also, I have reproduced part of Lemma 2 on the next page.

(c) Symbolically, the width 0.2 we can write as Δx. So we could have said that the quadrilateral going from $x = 0.0$ to $x = 0.2$, and was $\frac{1}{3}$ high went from $x = 0 \cdot \Delta x = 0$ to $x = 1 \cdot \Delta x = \Delta x$ and was $\frac{1}{3} e^0$ high. So it is $\Delta x \cdot \frac{1}{3} e^0$ in area. If we say that was the 0th rectangle, and we do the same thing for the other 9 rectangles you have inscribed, and we make k range from 0 to 9, we see that the areas can all be written as $\frac{1}{3} \Delta x \cdot e^{k \Delta x}$.

So the area is the sum of $\frac{1}{3} e^{k \Delta x} \Delta x$ from 0 to 9. Actually, be somewhat more general than that: write down the sum of $\frac{1}{3} e^{k \Delta x} \Delta x$ from 0 to $n - 1$. We know $n = 10$, but instead of plugging that in, leave it as a variable. Replace Δx everywhere it appears by $\frac{2}{n}$, since if we divide the range 0 to 2 into n equal parts each of them is $\frac{2}{n}$ wide. Do the sum. Here is an identity that is easy to prove, and that you need, but you might not know: $\sum_{k=0}^{n-1} \alpha^k = \frac{\alpha^n - 1}{\alpha - 1}$. To use this identity, you will need to correctly identify α.

(d) We made n a variable in part (c) instead of setting it to 10 because the final thing Newton wants us to do is to take the $n \to \infty$ limit of what you found. Do that. You will need another approximation to finish: the evanescent value as $n \to \infty$ of $n\left(e^{2/n} - 1\right)$ is just 2.

NB: If (a) and (d) do not approximately agree, find your errors!

Lemma 2

If in any figure you please AacE, contained by the straight lines Aa, AE, and the curve acE, there be inscribed any number of parallelograms Ab, Bc, Cd, and so on, contained beneath equal bases AB, BC, CD, and so on, and by the sides Bb, Cc, Dd, and so on, parallel to the side Aa of the figure; and if the parallelograms aKbl, bLcm, cMdn, and so on, be completed, and if afterward the breadth of the latter parallelograms be diminished and their number be increased in infinitum, I say that the ultimate ratios which the inscribed figure AKbLcMdD, the circumscribed figure AalbmcndoE, and the curvilinear figure AabcdE, have to each other are the ratios of equality.