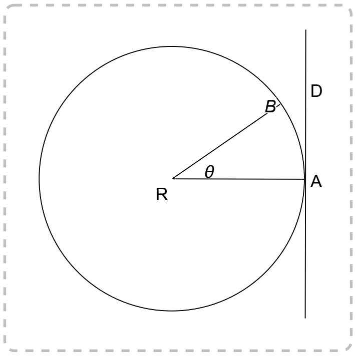
Newton — Problem Set 3 Solution

1. Problem 1

(a) By the definition of the radian, this is simply, $r\theta$, and since $r = 1$, len(arc AB) $= \theta$. The length of AD is given by the tangent: len(AD) $= \tan\theta$.

In[36]:=



(b) The chord we are looking for is twice half the chord. But half the chord (by the hint) is $\sin\frac{\theta}{2}$. So the full chord is $2\sin\frac{\theta}{2}$.

(c) For 1 radian: $\theta = 1$, $\tan\theta = 1.55740772465$, and $2\sin\frac{\theta}{2} = 0.9588510772$.

For 0.1 radian: $\theta = 0.1$, $\tan\theta = 0.10033467208$, and $2\sin\frac{\theta}{2} = 0.09995833854$. Oh my, this is looking promising.

For 0.01 radian: $\theta = 0.01$, $\tan\theta = 0.01000033334$, and $2\sin\frac{\theta}{2} = 0.00999995833$.

Let us do some ratios. For 1 radian: $\frac{\tan\theta}{\theta} = 1.557407724$, and $\frac{2\sin\frac{\theta}{2}}{\theta} = 0.95885107$.

For 0.1 radian: $\frac{\tan\theta}{\theta} = 1.003346720$, and $\frac{2\sin\frac{\theta}{2}}{\theta} = 0.999583385$.

For 0.01 radian: $\frac{\tan\theta}{\theta} = 1.000033334$, and $\frac{2\sin\frac{\theta}{2}}{\theta} = 0.999995833$.

So the ratios are improving (tending toward unity). In the first case, only the leading digit agrees with unity. In the second case, 3 or 4 digits agree with unity. In the third case, 5 or 6 digits agree with unity.

2. Arc Lengths and Chord Lengths of Parabolas

(a) The chord length from $(0,\ 0)$ to the point $(a,\ a^2)$ is $\sqrt{a^2 + a^4}$ by the Pythagorean theorem.

(b) The chord lengths and arc lengths for a = 3, a = 1, and a = 1/3 are below. I added $a = 1/10$ in my answer.

In[37]:=
```
pThree = N[{Sqrt[a^2 + a^4], 1/2 a Sqrt[1 + 4 a^2] + 1/4 Log[2 a + Sqrt[1 + 4 a^2]]} /. a → 3]
```
Out[37]=
```
{9.48683, 9.74709}
```

In[38]:=
```
pOne = N[{Sqrt[a^2 + a^4], 1/2 a Sqrt[1 + 4 a^2] + 1/4 Log[2 a + Sqrt[1 + 4 a^2]]} /. a → 1]
```
Out[38]=
```
{1.41421, 1.47894}
```

In[39]:=
```
pOneThird = N[{Sqrt[a^2 + a^4], 1/2 a Sqrt[1 + 4 a^2] + 1/4 Log[2 a + Sqrt[1 + 4 a^2]]} /. a → 1 / 3]
```
Out[39]=
```
{0.351364, 0.356595}
```

In[40]:=
```
pOneTenth = N[{Sqrt[a^2 + a^4], 1/2 a Sqrt[1 + 4 a^2] + 1/4 Log[2 a + Sqrt[1 + 4 a^2]]} /. a → 1 / 10]
```
Out[40]=
```
{0.100499, 0.100663}
```

In[41]:=
```
pThree[[1]] / pThree[[2]]
```
Out[41]=
```
0.973299
```

In[42]:=
```
pOne[[1]] / pOne[[2]]
```
Out[42]=
```
0.956233
```

In[43]:=
```
pOneThird[[1]] / pOneThird[[2]]
```
Out[43]=
```
0.985332
```

In[44]:=
```
pOneTenth[[1]] / pOneTenth[[2]]
```
Out[44]=
```
0.998371
```

The ratios are generally getting better, but the case $a = 1$ was an outlier.

3. Lemma 8 applied to Circles

(a) The area of the three-sided figure which has two of the sides as radii and one side as the arc is $\frac{\theta}{2\pi}$ of the area of the whole circle, so it is $\frac{\theta}{2\pi}\pi = \frac{\theta}{2}$. For the triangle, it is $\frac{1}{2}bh = \frac{1}{2}\tan\theta$.

(b) Using the same bisecting trick, we have two triangles, each of which has area $\frac{1}{2}bh$. So together they are bh. Their heights are $\sin\frac{\theta}{2}$. Their bases are $\cos\frac{\theta}{2}$. So the final result using the double-angle identity for the sine function is $\frac{\sin\theta}{2}$.

(c) For 1 radian: $\frac{\theta}{2} = 0.5$, $\frac{\tan\theta}{2} = 0.77870386232$, and $\frac{\sin\theta}{2} = 0.4207354924$.
For 0.1 radian: $\frac{\theta}{2} = 0.05$, $\frac{\tan\theta}{2} = 0.05016733604$, and $\frac{\sin\theta}{2} = 0.04991670832$.
For 0.01 radian: $\frac{\theta}{2} = 0.005$, $\frac{\tan\theta}{2} = 0.00500016667$, and $\frac{\sin\theta}{2} = 0.00499991666$.

Let us do some ratios. For 1 radian: $\frac{\tan\theta}{\theta} = 1.557407724$, and $\frac{\sin\theta}{\theta} = 0.8414709848$.
For 0.1 radian: $\frac{\tan\theta}{\theta} = 1.00334672085$, and $\frac{\sin\theta}{\theta} = 0.99833416646$.
For 0.01 radian: $\frac{\tan\theta}{\theta} = 1.00003333467$, and $\frac{\sin\theta}{\theta} = 0.99998333341$.

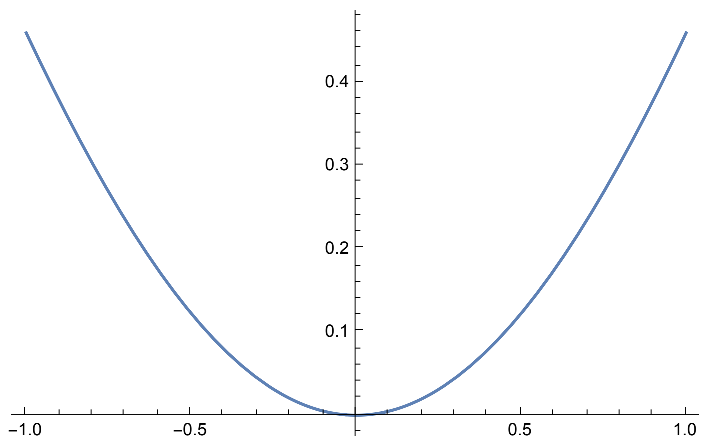
So the ratios of areas are improving (tending toward unity), the same way as they did for the ratios of lengths.

4. Lemma 10 — An Example

(a) Plot $x(t) = 1\text{ meter}\left(1 - \cos\frac{t\text{ radian}}{1\text{ second}}\right)$:

```
In[45]:= Plot[1 - Cos[t], {t, -1, 1}]
```

Out[45]=

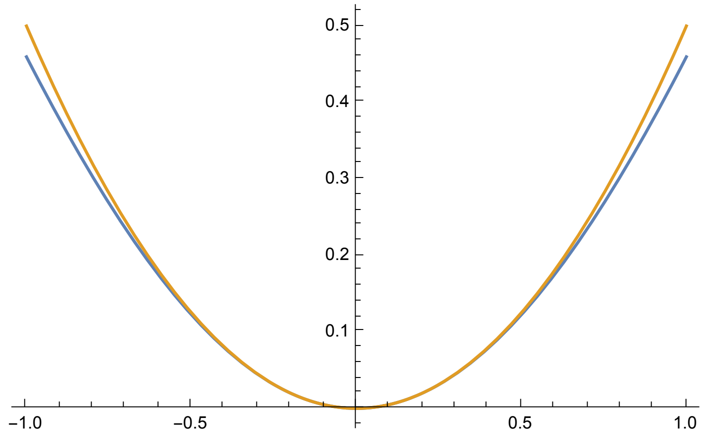


(b) Superimpose onto this graph a graph of the function $x(t) = \frac{1}{2}(1\text{ meter})\left(\frac{t}{1\text{ second}}\right)^2$:

In[46]:= `Plot[{1 - Cos[t], 1/2 t^2}, {t, -1, 1}]`

$$\text{Plot}\left[\left\{1-\text{Cos}[t],\ \frac{1}{2}t^2\right\},\ \{t,-1,1\}\right]$$

Out[46]=



(c) Evaluate and compare $x(t) = 1\ \text{meter}\left(1 - \cos\frac{t\ \text{radian}}{1\ \text{second}}\right)$ and $x(t) = \frac{1}{2}\,(1\ \text{meter})\left(\frac{t}{1\ \text{second}}\right)^2$ for $t = 1$ second, t = 0.5 seconds, t=0.25 seconds and t = 0.125 seconds.

In[47]:= `x1[t_] := 1 - Cos[t]`

In[48]:= $\text{x2}[t\_] := \frac{1}{2}t^2$

In[54]:= `MatrixForm[N[Table[{t, x1[t], x2[t], x1[t] / x2[t]}, {t, { 1, 0.5, 0.25, 0.125}}]]]`

Out[54]//MatrixForm=

$$\begin{pmatrix} 1. & 0.459698 & 0.5 & 0.919395 \\ 0.5 & 0.122417 & 0.125 & 0.97934 \\ 0.25 & 0.0310876 & 0.03125 & 0.994803 \\ 0.125 & 0.00780233 & 0.0078125 & 0.998699 \end{pmatrix}$$

(c) The last column in the table shows how the ratios are improving (tending toward unity).