Manhattan Project — Assignment 3 — Half Lives and Decays

Let’s recap the equations on p. 26 before doing any problems.

Decay Rate Derivation

I derived the equation for decay rate without using any calculus. I had to use some properties of the exponential though. One of the properties was this one that you may or may not be familiar with:

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots$$

If you are curious and know factorials, the denominators are 0!=1, 1!=1, 2!=2, 3!=6, etc. I will use the above formula to make an approximation below. Another property of the exponential that I used is:

$$e^{x+y} = e^x\, e^y$$

That’s actually true whatever base is being exponentiated. For example:

$$10^{x+y} = 10^x \cdot 10^y$$

Finally, I used $e^0 = 1$. At least I'm not using any calculus!

So we start with the claim that

$$N(t) = N_0\, e^{-\lambda t}$$

describes radioactive decay. If you have N_0 atoms at time $t = 0$, this formula is the one that tells you how much you have at any later time.So it certainly tells you how much you have at both time t and time $t + \Delta t$ where Δt is a small amount of time. We have:

$$N(t+\Delta t) = N_0\, e^{-\lambda(t+\Delta t)} = N_0\, e^{-\lambda t}\, e^{-\lambda \Delta t} = N(t)\, e^{-\lambda \Delta t} = N(t)\left[1 - \lambda \Delta t + \frac{(\lambda \Delta t)^2}{2} - \frac{(\lambda \Delta t)^3}{6} + \ldots\right]$$

Here comes the tricky part! If $\lambda \Delta t$ is small (think of something like 0.01), then every term in the infinite series is 0.01 times as small as the previous one. Let's neglect all but the first two!

$$N(t+\Delta t) = N(t)[1 - \lambda \Delta t] = N(t) - \lambda \Delta t N(t)$$

Rearrange:

$$N(t+\Delta t) - N(t) = -\lambda \Delta t N(t)$$

Rearrange more:

$$\frac{N(t+\Delta t)-N(t)}{\Delta t} = -\lambda N(t)$$

What we have on the left side is what Reed calls $R(t)$ in equation 2.3. It is the rate that the number of particles is changing. The right-hand-side of the equation has a minus sign because the number of particles is decreasing. Let's summarize before we move on:

$$R(t) = -\lambda N(t)$$

Relationship Between λ and $t_{1/2}$

There is another thing we derived in class that I want to re-derive here: the relationship between λ and $t_{1/2}$.

$t_{1/2}$ is the time at which you have half as many particles. So on the left-hand side of,

$$N(t) = N_0\, e^{-\lambda t}$$

we put $N_0/2$ and for t on the right-hand side, we put $t_{1/2}$,

$$N_0/2 = N_0\, e^{-\lambda t_{1/2}}$$

The N_0 on each side cancels, leaving:

$$1/2 = e^{-\lambda t_{1/2}}$$

Now take the reciprocal of each side of the equation:

$$2 = e^{\lambda t_{1/2}}$$

Finally take the natural log of each side. The natural log is by definition the function that undoes the exponential:

$$\ln 2 = \lambda t_{1/2}$$

We have Reed’s equation 2.2:

$$\lambda = \frac{\ln 2}{t_{1/2}}$$

1. Using $R(t) = -\lambda N(t)$ and $\lambda = \frac{\ln 2}{t_{1/2}}$

(a) Convert 138 days to seconds.

(b) If you start off with an Avogadro's number of Polonium-210 atoms $\left(N_A \approx 6.02 \cdot 10^{23}\right)$ and the half life of Polonium-210 is $t_{1/2} = 138$ days, what number of atoms will be decaying per second.

(c) A Curie (abbreviated Ci) is $3.7 \cdot 10^{10}$ decays / second. Convert your answer in (b) to Ci.

2. Alpha Decay

Polonium-210 alpha decays. The reaction is:

$${}^{210}_{84}\text{Po} \rightarrow {}^{4}_{2}\text{He} + {}^{A}_{Z}\text{X}$$

(a) What must A and Z be?

(b) Use the Table of Isotopic Masses and Natural Abundances you have. What element has the Z you found in (a)?

3. β^- and β^+ Decay

(a) Suppose Polonium-210 did a β^- decay. Consult Fig. 2.12 to find out what the N and Z value of the resulting nucleus would be. (N is the number of neutrons and $N = A - Z$.)

$${}^{210}_{84}\text{Po} \rightarrow e^- + {}^{A}_{Z}\text{X}$$

In addition to reporting N and Z of the new nucleus, what is the A value of the new nucleus?

(b) Suppose Polonium-210 did a β^+ decay. Consult Fig. 2.12 to find out what the N and Z value of the resulting nucleus would be.

$${}^{210}_{84}\text{Po} \rightarrow e^+ + {}^{A}_{Z}\text{X}$$

In addition to reporting N and Z of the new nucleus, what is the A value of the new nucleus?

4. Energy Released in Fission

Returning to the actual Polonium-210 alpha decay that you found in Problem 2:

$${}^{210}_{84}\text{Po} \rightarrow {}^{4}_{2}\text{He} + {}^{A}_{Z}\text{X}$$

Look up the mass of each atom involved in your the Table of Isotopic Masses and Natural Abundances. Actually, Polonium-210 isn't stable enough to be in our table, so I'll just tell you that it has mass 209.98286u.

(a) What is the total mass on the left-hand side, what is the total mass on the right-hand side, and what is the difference? Keep all six decimal places.

(b) Convert the difference (this is the energy released) to MeV using the handy-dandy conversion factor between mass units and MeV in the middle of p. 34 of Reed.