Uniform Continuity Illustrations

For Chapter 8, Appendix, Problem 2, p. 144

2. (a) Prove that if f and g are uniformly continuous on A, then so is $f + g$.
(b) Prove that if f and g are uniformly continuous and bounded on A, then fg is uniformly continuous on A.
(c) Show that this conclusion does not hold if one of them isn't bounded.
(d) Suppose that f is uniformly continuous on A, that g is uniformly continuous on B, and that $f(x)$ is in B for all x in A. Prove that $g \circ f$ is uniformly continuous on A.

2(a) Sums

Prove that the sum of two uniformly continuous function is uniformly continuous.

Since f is uniformly continuous on A there is a δ_f such that for all $x$and y in A,

$|f(x) - f(y)| < \frac{\epsilon}{2}$,

provided that

$|x - y| < \delta_f$.

Similarly,

$|g(x) - g(y)| < \frac{\epsilon}{2}$,

provided that

$|x - y| < \delta_g$

Now choose δ to be the smaller of δ_f and δ_g. Then

$|f(x) + g(x) - (f(y) + g(y))| \leq |f(x) - f(y)| + |g(x) - g(y)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$

Voila.

2(b) Products

Prove that the product of two uniformly continuous function is uniformly continuous, provided that both functions are bounded on the interval in question.

As in 2(a), we will have a δ_f and δ_g. This time choose the ϵ for f to be $\frac{\epsilon}{2N}$. E.g., our δ_f will be sufficiently small such that

$$|f(x) - f(y)| < \frac{\epsilon}{2N}$$

where N is the bound on g. E.g., $|g(x)| < N$.

Similarly, choose the ϵ for g to be $\frac{\epsilon}{2M}$. E.g., our δ_g will be sufficiently small such that

$$|g(x) - g(y)| < \frac{\epsilon}{2M}$$

where M is the bound on f. E.g., $|f(x)| < M$.

Now we consider

$$\begin{aligned}
&|f(x)\,g(x) - f(y)\,g(y)| = \\
&\quad |f(x)\,g(x) - f(y)\,g(x) + f(y)\,g(x) - f(y)\,g(y)| \leq |f(x)\,g(x) - f(y)\,g(x)| + |f(y)\,g(x) - f(y)\,g(y)| = \\
&\qquad |f(x) - f(y)|\;|g(x)| + |f(y)|\;|g(x) - g(y)| < \\
&\qquad\quad |f(x) - f(y)|\,N + M\,|g(x) - g(y)| < \frac{\epsilon}{2N}\,N + M\,\frac{\epsilon}{2M} = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

Voila.

Counterexample asked for in 2(c)

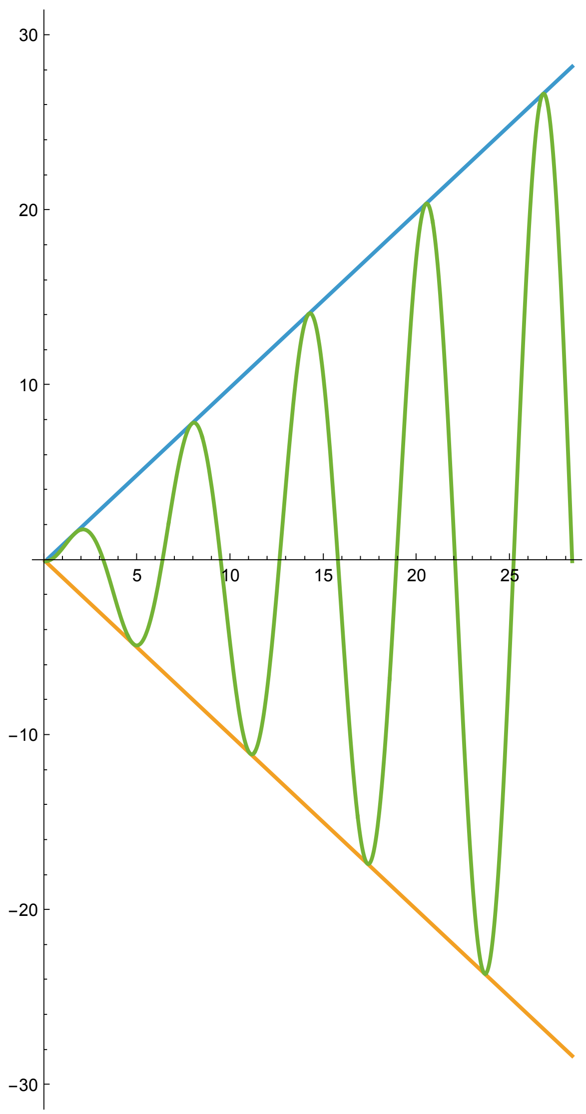
The product of two functions that are uniformly continuous may not be uniformly continuous if one of them is unbounded. Of course the only way this can happen is if the interval is unbounded, because a continuous function on a bounded interval is bounded.

Example: let $f(x) = \sin x$ and $g(x) = x$. Let the interval be $[0, \infty)$. Both f and g are uniformly continuous even though g is unbounded. That's because g's slope is constant.

Let's graph the product:

In[∘]:=
```
Plot[{x, -x, xSin[x]}, {x, 0, 18 Pi / 2}, AspectRatio → 2]
```

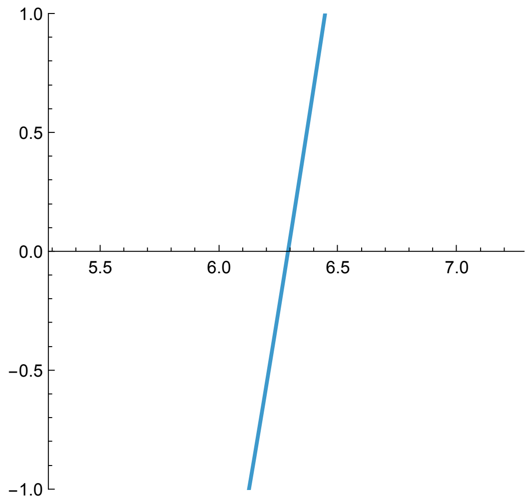
Out[∘]=



Let's consider $x = 2\,\pi$ and $x = 4\,\pi$. Let's consider $\epsilon = 1$. At $x = 2\,\pi$, that means $|x \sin x| < 1$. Let's blow up that region of the plot:

In[ ]:=
```
Plot[xSin[x], {x, 2 Pi - 1, 2 Pi + 1},
 PlotRange → {{2 Pi - 1, 2 Pi + 1}, {-1, 1}}, AspectRatio → 1]
```

Out[ ]=

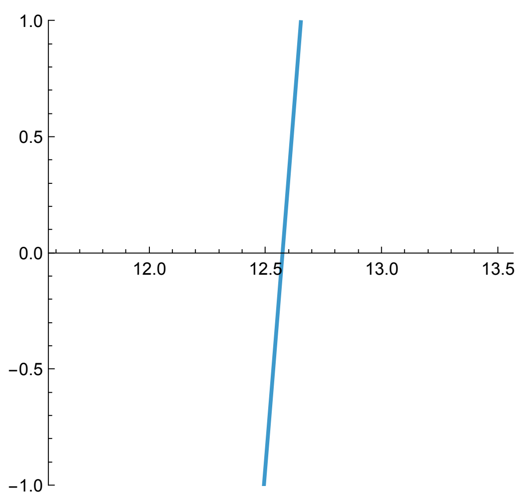


Eyeballing the plot, it appears that a δ of about 0.2 will do the job for ϵ=1.

Now let's make a similar plot but at $x = 4\,\pi$.

In[ ]:=
```
Plot[xSin[x], {x, 4 Pi - 1, 4 Pi + 1},
 PlotRange → {{4 Pi - 1, 4 Pi + 1}, {-1, 1}}, AspectRatio → 1]
```

Out[ ]=

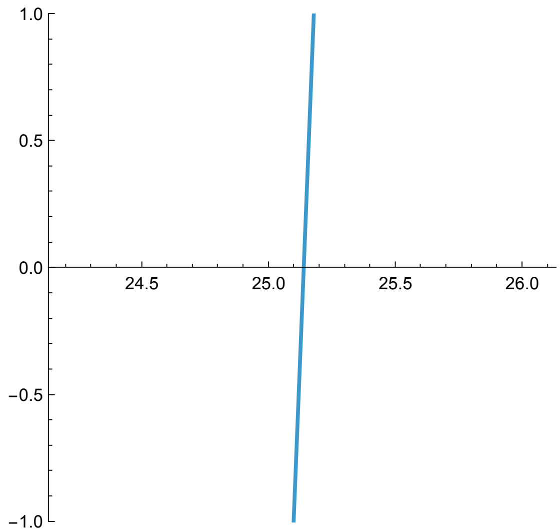


It is clear that the δ of about 0.2 will not do the job because the function is steeper here (twice as steep in fact).

You could do δ of about 0.1, but then if you go out twice as far again, to $x = 8\pi$, you'll need a δ of about 0.05 because the function is twice again as steep:

In[ ]:=
```
Plot[xSin[x], {x, 8 Pi - 1, 8 Pi + 1},
 PlotRange → {{8 Pi - 1, 8 Pi + 1}, {-1, 1}}, AspectRatio → 1]
```

Out[ ]=



2(d) Composition

Here again is the statement of Part (d):

Suppose that f is uniformly continuous on A, that g is uniformly continuous on B, and that $f(x)$ is in B for all x in A. Prove that $g \circ f$ is uniformly continuous on A.

The proof of this is going to have to go a lot like the proof of Theorem 2 of Chapter 6 on p. 115.

THEOREM 2 If g is continuous at a, and f is continuous at $g(a)$, then $f \circ g$ is continuous at a. (Notice that f is required to be continuous at $g(a)$, not at a.)

PROOF Let $\varepsilon > 0$. We wish to find a $\delta > 0$ such that for all x,

$$\text{if } |x - a| < \delta, \text{ then } |(f \circ g)(x) - (f \circ g)(a)| < \varepsilon,$$
$$\text{i.e., } |f(g(x)) - f(g(a))| < \varepsilon.$$

We first use continuity of f to estimate how close $g(x)$ must be to $g(a)$ in order for this inequality to hold. Since f is continuous at $g(a)$, there is a $\delta' > 0$ such that for all y,

$$(1) \qquad \text{if } |y - g(a)| < \delta', \text{ then } |f(y) - f(g(a))| < \varepsilon.$$

In particular, this means that

$$(2) \qquad \text{if } |g(x) - g(a)| < \delta', \text{ then } |f(g(x)) - f(g(a))| < \varepsilon.$$

We now use continuity of g to estimate how close x must be to a in order for the inequality $|g(x) - g(a)| < \delta'$ to hold. The number δ' is a positive number just like any other positive number; we can therefore take δ' as the ε (!) in the definition of continuity of g at a. We conclude that there is a $\delta > 0$ such that, for all x,

$$(3) \qquad \text{if } |x - a| < \delta, \text{ then } |g(x) - g(a)| < \delta'.$$

Combining (2) and (3) we see that for all x,

$$\text{if } |x - a| < \delta, \text{ then } |f(g(x)) - f(g(a))| < \varepsilon. \blacksquare$$

We just have to figure out what changes to the proof have to be made to carry it over from continuity to uniform continuity.