NUMERICAL INTEGRATION WITH STEP REFINEMENT

1. Purpose

To evaluate the integral

$$I := \int_0^b f(x)\, dx$$

numerically for an arbitrary function f that can be evaluated by a program requiring no more than 16 instructions. The general integral $\int_a^b$ is reduced to the above by the substitution $x' = x - a$.

2. Method

We approximate I by a sequence of approximate values I_k, $k = 0, 1, 2, \ldots$. The approximation I_k is obtained by dividing $[0, b]$ into 2^k congruent subintervals and evaluating the integral on each subinterval by the midpoint formula. (The trapezoidal rule is avoided because integrands frequently have singularities at the endpoints of the interval. Although harmless for the existence of the integral, these may cause the program to fail; see example $\boxed{2}$.) Writing

$$h_k := 2^{-k}b ,$$

we have

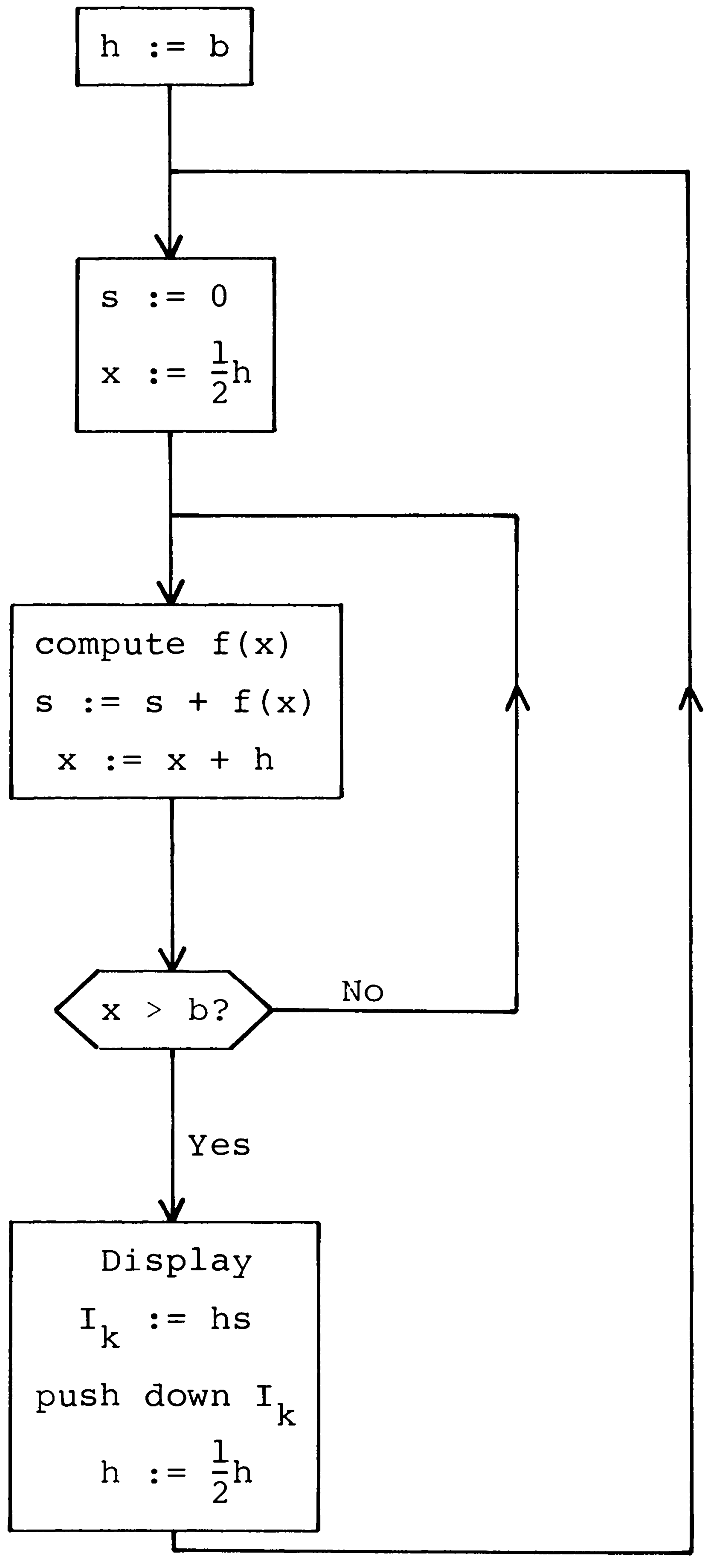
$$I_k = h_k \sum_{m=0}^{2^m-1} f((m + \tfrac{1}{2})h_k) .$$

For any continuous f,

$$\lim_{k\to\infty} I_k = I .$$

If f is sufficiently smooth, the convergence of the sequence $\{I_k\}$ may be sped up by the Romberg algorithm. To this end the program saves the five currently most recent values of I_k in locations in which they can be used as input for the Romberg acceleration program (see the following program) without external storage.

3. Flow Diagram



4. Storage and Program

R_0	R_1	R_2	R_3	R_4	R_5	R_6	R_7
I_0	I_1	I_2	I_3	I_4	h	x	$\boxed{b}$

```
   00
   01   RCL 7
   02   STO 5
→  03     2
   04     ÷
   05   STO 6
   06    CLX
   07   STO 4
   08   GTO 33
→  09   STO+4
   10   RCL 5
   11   STO+6
   12   RCL 7
   13   RCL 6
   14   x < y
   15   GTO 33
   16   RCL 5
   17   RCL 4
   18     *
   19   STO 4
   20   PAUSE (R/S)
   21     2
   22   STO÷5
   23   RCL 1
   24   STO 0
   25   RCL 2
   26   STO 1
   27   RCL 3
   28   STO 2
   29   RCL 4
   30   STO 3
   31   RCL 5
   32   GTO 03
→  33
   34
   35
   36
   37
   38
   39
   40
   41
   42
   43
   44
   45
   46
   47
   48
   49
```

The program to compute f(x) should be in locations 33 through 49, assuming x in R_6. Only the stack may be used for temporary storage. The last instruction must be GTO 09. At this point, f(x) must be in the X register.

5. Operating Instructions

Load the main program. Load the program for computing f into locations 33 through 49; the last instruction must be GTO 09. Switch to RUN. Select a mode of displaying numbers, for instance

FIX 8

If f involves trigonometric functions with argument in radians, press

RAD

To start computation, press

PRGM
R/S

The calculator will pause briefly while displaying each I_k. If $k \geq 0$, and if

R/S

is pressed during pause, the values I_{k-m} will be found in R_{4-m}, m = 0, 1, 2, 3, 4.

6. Examples and Timing

$\boxed{1}$ b = 1, $f(x) := \dfrac{4}{1 + x^2}$. Program to compute f:

33	4
34	RCL 6
35	x^2
36	1
37	+
38	÷
39	GTO 09

The following values of I_k are obtained:

k	I_k		$I_4^{(k)}$
0	3.20000000		3.14191817
1	3.16235294		3.14159265
2	3.14680052	$\xrightarrow{\text{Romberg}}$	3.14159264
3	3.14289473		3.14159266
4	3.14191817		3.14159264

Time required 35 sec. Subjecting the values I_k to the Romberg algorithm produces the exact value π = 3.14159265 with an error of one unit in the last digit due to rounding effects.

$\boxed{2}$ $b = 1$, $f(x) := \dfrac{\mathrm{Log}(1 + x)}{x}$. Program to compute f:

```
33    1
34    RCL 6
35    +
36    ln
37    RCL 6
38    ÷
39    GTO 09
```

Values obtained:

k	I_k		$I_4^{(k)}$
0	0.81093022		0.82241711
1	0.81936429		0.82246693
2	0.82167416	$\xrightarrow{\text{Romberg}}$	0.82246702
3	0.82226766		0.82246702
4	0.82241711		0.82246702

Exact value: $\dfrac{\pi^2}{12} = 0.82246703$. Time required to generate $I_0, \ldots, I_4$ about 40 sec.