Numerical Analysis — Final Exam

Dec. 6, 2022. During this exam, you are welcome to use your copies of the HP-25 Owner's Handbook and the HP-25 Applications Programs book, and your notes, but not the internet. The relevant pages from the HP-25 Applications Programs book are attached, as are several copies of the HP-25 Program Form to help you document the programs you write in Problems 2 and 3. There are three problems. Budget 30 minutes for each of them.

Problem 1 — Statistics — Power Law Fit

In[25]:=
```
planetList = AstronomicalData["Planet"];
earthSemimajorAxis = AstronomicalData["Earth", "SemimajorAxis"];
earthPeriod = AstronomicalData["Earth", "OrbitPeriod"];
aList = AstronomicalData[#, "SemimajorAxis"] / earthSemimajorAxis & /@ planetList;
pList = AstronomicalData[#, "OrbitPeriod"] / earthPeriod & /@ planetList;
NumberForm[TableForm[{planetList, aList, pList}, TableHeadings → {{"", "Distance", "Period"}, None}], 4]
```

Out[30]//NumberForm=

	Mercury	Venus	Earth	Mars	Jupiter	Saturn	Uranus	Neptu
Distance	0.3871	0.7233	1.000	1.524	5.203	9.537	19.19	30.07
Period	0.2408	0.6152	1.000	1.881	11.86	29.45	84.02	164.8

Perhaps the most famous power law relationship of all time is captured in this table above. The upper row of data is the planet's distance from the Sun. This plays the role of x. The lower row is how long the planet takes to go around the Sun. This is called the period. It plays the role of y.

You might find it odd that the distance and period are both 1.000 for Earth, but that's because I have chosen(!) to measure distances using the Earth's distance from the Sun as the yardstick, and to measure times in the unit of Earth-years.

(a) Use the program on p. 99 of the Applications Programs book to enter this data (just as was done with the example data on p. 100). What does the program give for the a and b values in the formula $y = ax^b$?

HINT/NOTE: Since Earth has 1.000, 1.000 as its values, a good cross-check on your data entry is that a should come out extremely close — if not identically equal — to 1.000. b is what is really interesting. It is Kepler's Third Law.

(b) If an asteroid was found to be orbiting at a distance from the Sun of 3.700 in these units. What would its period be?

NOTE: Asteroids orbit the Sun following the same laws as planets. Asteroids are just smaller.

Problem 2 — Differential Equations — Euler's Method

There is a differential equation for approach of temperature T to an equilibrium temperature S called the Stefan-Boltzmann Law. It is:

$$T'(t) \equiv \frac{dT}{dt} \equiv \underset{\text{as teeny as } \Delta t \text{ can be made}}{\text{limit}} \frac{\Delta T}{\Delta t} = -k\left(T^4 - S^4\right)$$

In this equation, k and S are just numbers. We will choose $k = 1.20 \times 10^{-10}$ and $S = 300$. For this problem, use $h = 0.25$, and $T(0) = 2000$.

(a) For your program, put k in R5 and S in R6, and then write the program (with your starting point being the program on p. 83) that solves this problem using Euler's method.

(b) Fill in the table below.

NOTES: T is playing the role of y. Also, you should not use R2 for y, even though that is documented to contain y! (I could explain why, but we went through this program very carefully in class, so let's not go back into it.) Instead, for y, you must use the y that is in stack register Y.

In[32]:=
```
TableForm[Table[{t, If[t == 0, 2000, "    "]}, {t, 0, 5, 0.25}], TableHeadings → {None, {"t", "T"}}]
```

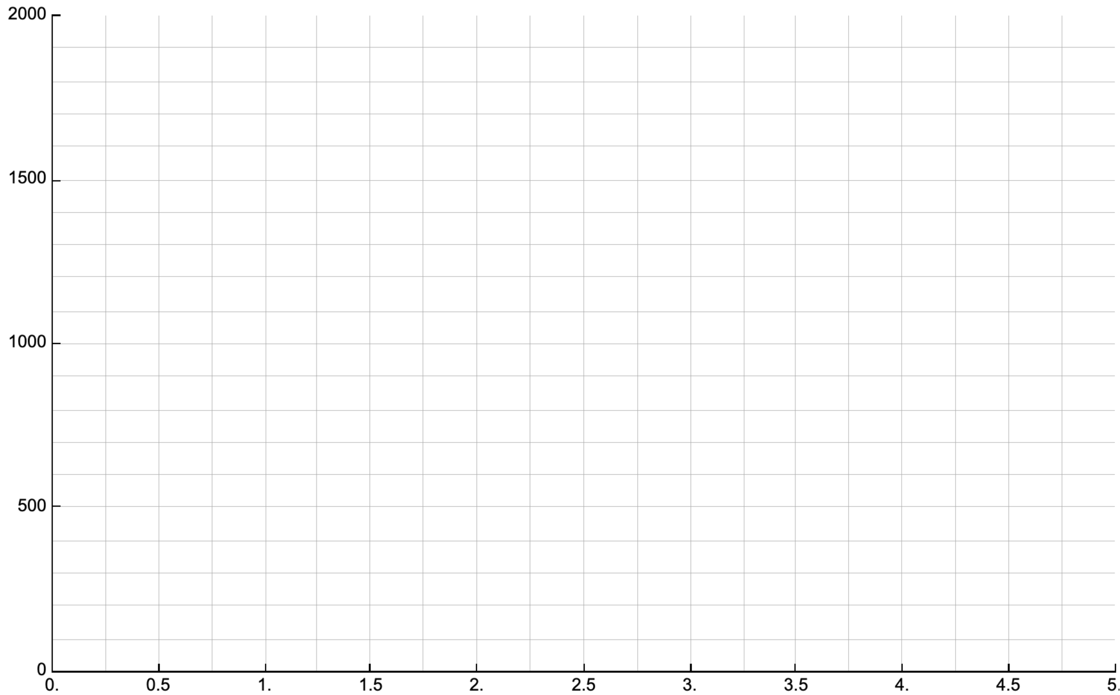
Out[32]//TableForm=

t	T
0.	2000
0.25	
0.5	
0.75	
1.	
1.25	
1.5	
1.75	
2.	
2.25	
2.5	
2.75	
3.	
3.25	
3.5	
3.75	
4.	
4.25	
4.5	
4.75	
5.	

(c) Graph your results:

In[54]:=
```
Plot[{}, {x, 0, 5}, PlotRange → {{0, 5}, {0, 2000}},
 Ticks → {Range[0, 5, 0.5], Range[0, 2000, 500]}, GridLines → {Range[0, 5, 1/4], Range[0, 2000, 100]}]
```

Out[54]=



Congratulations! You just found how something pulled out of a forge at 2000 Kelvin (that's about 3140° Fahrenheit) would cool towards room temperature (300 Kelvin) in its first five minutes.

Problem 3 — Interpolation — Using Quadratic Functions

Suppose you have three points with x-values at -1, 0, and 1, but the y-values can be anything. I'll write the three points as:

$(-1,\ y_{-1}),\ (0,\ y_0),$ and $(1,\ y_1)$.

There are formulas for the parabola, $y = a x^2 + b x + c$, that goes through these three points.

The formulas for a, b, and c are:

$a = \frac{1}{2}(y_1 + y_{-1}) - y_0$

$b = \frac{1}{2}(y_1 - y_{-1})$

$c = y_0$

Write a program that does two things:

(a) Assuming that y_{-1} is in R0, y_0 is in R1, and y_1 is in R2, have your program compute a and put it in R3 and b and put it in R4. You don't need to compute c, because it is just y_0 which is already in R1.

For part (a), your program just stops after it has done its work, but if the user puts an x-value into the X-register and hits R/S again, ...

(b) Your program should then compute $a\,x^2 + bx + c$ and then return to the same spot that it stopped before (not the beginning of the program) so that it can be ready to do this again with another x-value.

If you run out of time to do (b), bummer, but at least document what you wrote for part (a) so I can give you partial credit.

Then use your program to work through the following example:

(c) With the values:

```
In[∘]:= y₋₁ = 0.60653;
        y₀ = 1.00000;
        y₁ = 1.64872;
```

run your program and write down the a and b, it has calculated.

(d) Put your calculator into f FIX 5 mode (so that it shows 5 digits after the decimal place). Use the program you wrote in part (b) to fill in the following table:

```
In[∘]:= TableForm[Table[{x, If[x == -1, y₋₁, If[x == 0, "1.00000", If[x == 1, y₁, "    "]]]}, {x, Range[-1, 1, 0.2]}]]
```

Out[∘]//TableForm=

-1.	0.60653
-0.8	
-0.6	
-0.4	
-0.2	
0.	1.00000
0.2	
0.4	
0.6	
0.8	
1.	1.64872

HINT/NOTE: Of course, a good test of your program is that it works for the three values that were originally given, so don't skip them. As a note in passing, I'll say that the points I just had you do were to approximate the exponential $e^{x/2}$ by a parabola, which of course doesn't work perfectly, except at x of -1, 0, and 1, where we made the fit work, because an exponential is not a parabola. I checked all the values and the approximation works to better than 1%.

POWER CURVE FIT

This program fits a power curve

$$y = ax^b \qquad (a > 0)$$

to a set of data points

$$\{(x_i, y_i), i = 1, 2, ..., n\}$$

where $x_i > 0$, $y_i > 0$.

By writing this equation as

$$\ln y = b \ln x + \ln a$$

the problem can be solved as a linear regression problem.

Output statistics are:

1. Regression coefficients

$$b = \frac{\Sigma (\ln x_i)(\ln y_i) - \frac{(\Sigma \ln x_i)(\Sigma \ln y_i)}{n}}{\Sigma (\ln x_i)^2 - \frac{(\Sigma \ln x_i)^2}{n}}$$

$$a = \exp\left[\frac{\Sigma \ln y_i}{n} - b\,\frac{\Sigma \ln x_i}{n}\right]$$

2. Coefficient of determination

$$r^2 = \frac{\left[\Sigma (\ln x_i)(\ln y_i) - \frac{(\Sigma \ln x_i)(\Sigma \ln y_i)}{n}\right]^2}{\left[\Sigma (\ln x_i)^2 - \frac{(\Sigma \ln x_i)^2}{n}\right]\left[\Sigma (\ln y_i)^2 - \frac{(\Sigma \ln y_i)^2}{n}\right]}$$

3. Estimated value $\hat{y}$ for given x

$$\hat{y} = ax^b$$

Note:

n is a positive integer and $n \neq 1$.

DISPLAY LINE	DISPLAY CODE	KEY ENTRY
00		
01	14 07	f LN
02	31	↑
03	15 02	g x^2
04	23 51 02	STO + 2
05	22	R↓
06	21	x⇄y
07	14 07	f LN
08	25	Σ+
09	13 00	GTO 00
10	24 05	RCL 5
11	24 07	RCL 7
12	24 04	RCL 4
13	61	x
14	24 03	RCL 3
15	71	÷
16	41	–
17	24 06	RCL 6
18	24 07	RCL 7
19	15 02	g x^2
20	24 03	RCL 3
21	71	÷
22	41	–
23	71	÷
24	23 01	STO 1

DISPLAY LINE	DISPLAY CODE	KEY ENTRY
25	24 07	RCL 7
26	61	x
27	32	CHS
28	24 04	RCL 4
29	51	+
30	24 03	RCL 3
31	71	÷
32	15 07	g e^x
33	23 00	STO 0
34	74	R/S
35	24 01	RCL 1
36	74	R/S
37	21	x⇄y
38	22	R↓
39	61	x
40	24 02	RCL 2
41	24 04	RCL 4
42	15 02	g x^2
43	24 03	RCL 3
44	71	÷
45	41	–
46	71	÷
47	13 00	GTO 00
48		
49		

REGISTERS
R_0 a
R_1 b
R_2 $\Sigma (\ln y)^2$
R_3 n
R_4 $\Sigma \ln y$
R_5 $\Sigma (\ln x)(\ln y)$
R_6 $\Sigma (\ln x)^2$
R_7 $\Sigma \ln x$

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	Initialize		f	REG	f	PRGM	
3	Perform for i = 1,..., n:						
	Input x-value and y-value	x_i	↑				
		y_i	R/S				i
4	Compute constants		GTO	10	R/S		a*
			R/S				b*
5	Compute coefficient of deter-						
	mination		R/S				r^2
6	Input x-value and compute $\hat{y}$	x	RCL	1	f	y^x	
			RCL	0	x		$\hat{y}$
7	Perform step 6 as many times as						
	desired						
8	For new case, go to step 2.						
	* The stack must be maintained						
	at these points.						

Example:

x_i	10	12	15	17	20	22	25	27	30	32	35
y_i	0.95	1.05	1.25	1.41	1.73	2.00	2.53	2.98	3.85	4.59	6.02

Solution:

$a = .03$, $b = 1.46$

$y = .03x^{1.46}$

$r^2 = 0.94$

For $x = 18$, $\hat{y} = 1.76$

$x = 23$, $\hat{y} = 2.52$

NUMERICAL SOLUTION TO DIFFERENTIAL EQUATIONS

This program may be used to solve a wide variety of first order differential equations of the form

$$y' = f(x, y)$$

with initial values x_0, y_0.

The solution is a numerical solution which calculates y_i for $x_i = x_0 + ih$, where h is an increment specified by the user and $i = 1, 2, \ldots.$

The program uses a modified Euler method (predictor – corrector):

$$\hat{y}_{i+1} = y_i + h\, f(x_i, y_i) \qquad y_{i+1} = y_i + \frac{h}{2}\left[f(x_i, y_i) + f(x_{i+1}, \hat{y}_{i+1})\right]$$

f(x, y) is keyed into memory starting at line 18. The user has 13 program steps to write f(x, y); registers R_5, R_6, and R_7 are also available. The user should assume that x and y will be in the X- and Y-registers, respectively. The routine should return with the value of f(x, y) in the X-register and should end with a GTO 31.

DISPLAY		KEY ENTRY
LINE	**CODE**	
00		
01	34	CLX
02	23 04	STO 4
03	24 02	RCL 2
04	24 01	RCL 1
05	13 18	GTO 18
06	22	R↓
07	23 03	STO 3
08	24 00	RCL 0
09	61	x
10	24 02	RCL 2
11	51	+
12	24 01	RCL 1
13	24 00	RCL 0
14	51	+
15	01	1
16	23 04	STO 4
17	22	R↓
18		
19		
20		
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23		
24		

DISPLAY		KEY ENTRY
LINE	**CODE**	
25		
26		
27		
28		
29		
30		
31	24 04	RCL 4
32	15 71	g x=0
33	13 06	GTO 06
34	22	R↓
35	24 03	RCL 3
36	51	+
37	24 00	RCL 0
38	61	x
39	02	2
40	71	÷
41	24 02	RCL 2
42	51	+
43	23 02	STO 2
44	24 01	RCL 1
45	24 00	RCL 0
46	51	+
47	23 01	STO 1
48	14 74	f PAUSE
49	22	x⇄y

REGISTERS
R_0 h
R_1 x
R_2 y
R_3 y′
R_4 Flag
R_5
R_6
R_7

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in lines 1-17 of program						17 22
2	Key in function f(x, y)						
3	Key in branch to line 31		GTO	31			
4	Press SST repeatedly until dis-						
	play shows line 30						
5	Key in lines 31-49 of program						49 13 01
6	Switch to RUN						
7	Store increment	h	STO	0			
8	Store initial conditions	x_0	STO	1			
		y_0	STO	2	f	PRGM	
9	Display next x-value and cor-						
	responding y-value		R/S				(x_k)
							y_k
10	Repeat step 9 as often as desired						

Example:

Solve numerically the differential equation $y' = x\sqrt{y}$ with initial conditions $x_0 = 1, y_0 = 1$. Use a step size of $h = 0.1$.

Solution:

Key the function in as x⇄y f √x ×

x	1.0	1.1	1.2	1.3	1.4	1.5
y (by prgm)	1.0	1.1077	1.2319	1.3745	1.5372	1.7221
y (exact)	1.0	1.1078	1.2321	1.3748	1.5376	1.7227

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Switch to PRGM mode, press [f] [PRGM], then key in the program.

DISPLAY		KEY ENTRY	X	Y	Z	T	COMMENTS
LINE	CODE						
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REGISTERS
R_0
R_1
R_2
R_3
R_4
R_5
R_6
R_7

HEWLETT hp PACKARD

HP-25 Program Form

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Programmer____

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS

HP-25 Program Form

Title__________ Page____ of____

Switch to PRGM mode, press [f] [PRGM], then key in the program.

DISPLAY		KEY ENTRY	X	Y	Z	T	COMMENTS
LINE	CODE						
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REGISTERS
R_0
R_1
R_2
R_3
R_4
R_5
R_6
R_7

HP-25 Program Form

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Programmer____

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS

HP-25 Program Form

Title________________________________ Page_____ of_____

Switch to PRGM mode, press [f] [PRGM] , then key in the program.

DISPLAY		KEY ENTRY	X	Y	Z	T	COMMENTS
LINE	CODE						
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REGISTERS
R_0
R_1
R_2
R_3
R_4
R_5
R_6
R_7

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HP-25 Program Form

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Programmer____

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS

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Title______________________________ Page_____ of_____

Switch to PRGM mode, press [f] [PRGM], then key in the program.

DISPLAY		KEY ENTRY	X	Y	Z	T	COMMENTS
LINE	CODE						
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REGISTERS
R_0
R_1
R_2
R_3
R_4
R_5
R_6
R_7

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Programmer ______________________________

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS