Simpson's Rule

Simpson's rule is one of the formulas used to find the approximate value of a definite integral. A definite integral is an integral with lower and upper limits. Usually, to evaluate a definite integral, we first integrate (using the integration techniques) and then we use the fundamental theorem of calculus to apply the limits. But sometimes, we cannot apply any integration technique to solve an integral, and sometimes, we do not have a specific function to integrate, instead, we have some observed values (in case of experiments) of the function. In such cases, Simpson's rule helps in approximating the value of the definite integral.

Let us learn this Simpson's Rule and its formula along with its derivation and a few solved examples in the upcoming sections.

What is Simpson's Rule?

Simpson's rule is used to find the value of a definite integral (that is of the form $\int_a^b f(x)\,dx$) by approximating the area under the graph of the function f(x). While using the Riemann sum, we calculate the area under a curve (a definite integral) by dividing the area under the curve into rectangles whereas while using Simpson's rule, we evaluate the area under a curve is by dividing the total area into parabolas. Simpson's rule is also known as Simpson's 1/3 rule (which is pronounced as Simpson's one-third rule).

Simpson's Rule Formula

We have several numerical methods to approximate an integral, such as Riemann's left sum, Riemann's right sum, midpoint rule, trapezoidal rule, Simpson's 1/3 rule, etc. But among these, Simpson's rule gives the more accurate approximation of a definite integral. If we have f(x) = y, which is equally spaced between [a,b], the **Simpson's rule formula** is:

- $\int_a^b f(x)\,dx \approx (h/3)\,[f(x_0)+4\,f(x_1)+2\,f(x_2)+ \ldots +2\,f(x_{n-2})+4\,f(x_{n-1})+f(x_n)]$

Here,

- n is an even number which is the number of subintervals that the interval [a, b] should be divided into.
  (n is usually mentioned in the problem)
- $x_0 = a$ and $x_n = b$
- $h = [\ (b - a) / n]$
- $x_0, x_1, ..., x_n$ are the ends of the n subintervals.

Simpson's Rule Error Bound

Simpson's rule gives just an approximate value of the integral, not the exact value. So there is always an error that can be calculated using the following formula.

- Error bound in Simpson's rule = $\dfrac{M(b-a)^5}{180n^4}$,
  where $|f^{(4)}(x)| \leq M$

Simpson's Rule Formula

$$\Rightarrow \boxed{\int_a^b f(x)dx \approx \frac{\Delta x}{3}\left[f(x_0)+4f(x_1)+2f(x_2)+.....+2f(x_{n-2})+4f(x_{n-1})+f(x_n)\right]}$$

where, $\Delta x = \dfrac{b-a}{n}$

$x_0 = a$ and $x_n = b$

$x_0, x_0, ..., x_n$ are the ends of the n sub intervals

$$\Rightarrow \boxed{\text{Error bound} = \frac{M(b-a)^5}{180n^4}}$$

where $|f^{(4)}(x)| \leq M$

Simpson's 1/3 Rule Derivation

Let us derive Simpson's 1/3 rule where we are going to approximate the value of the definite integral $\int_a^b f(x)\,dx$ by dividing the area under the curve f(x) into parabolas. For this let us divide the interval [a, b] into n subintervals $[x_0, x_1], [x_1, x_2], [x_2, x_3], ..., [x_{n-2}, x_{n-1}], [x_{n-1}, x_n]$ each of width 'h', where $x_0 = a$ and $x_n = b$.

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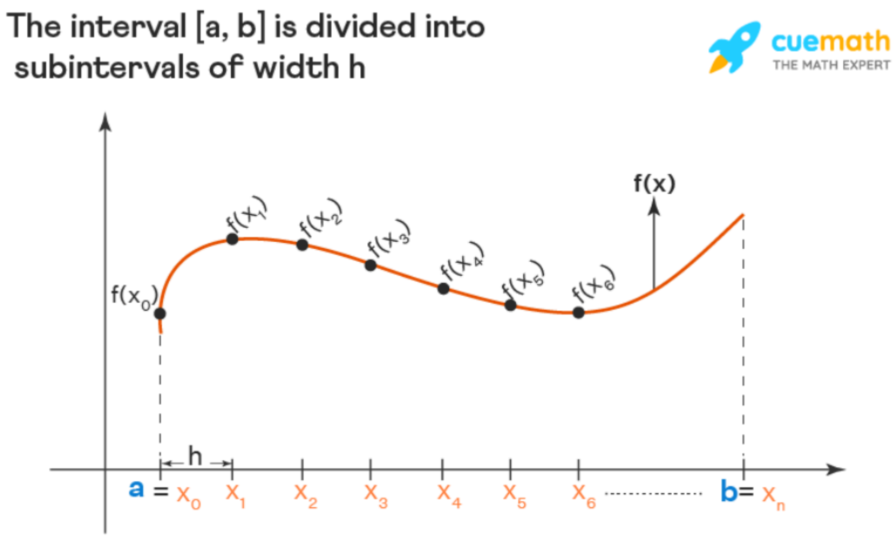
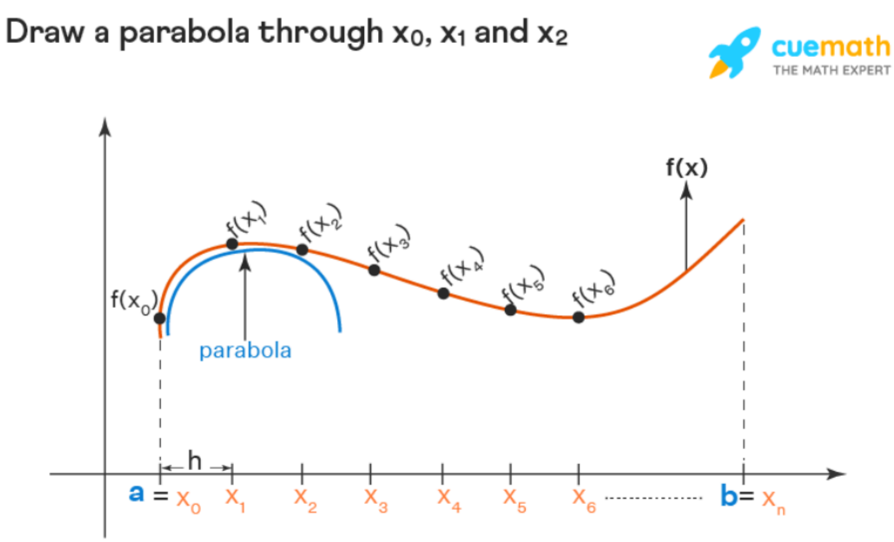
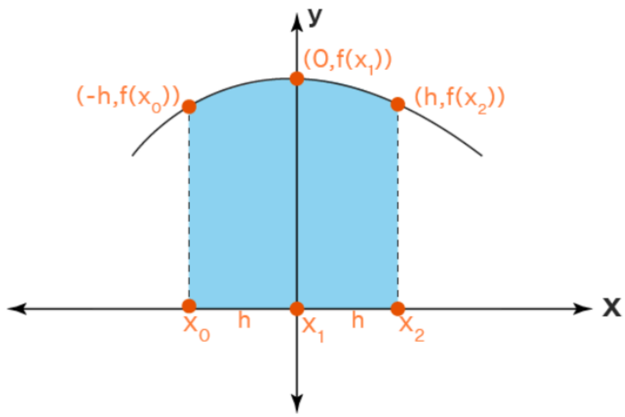


Now let us approximate the area under the curve by considering every 3 successive points to lie on a parabola. Let us approximate the area under the curve lying between x_0 and x_2 by drawing a parabola through the points x_0, x_1 and x_2. Of course, all three may not come on a single parabola. But let us try to draw an approximate parabola through these three points.



Let us make this parabola symmetric about the y-axis. Then it becomes something like this:

The parabola is made to be symmetric about y-axis



Let us assume that the equation of the parabola be $y = ax^2 + bx + c$. Then the area between x_0 and x_2 is approximated by the definite integral:

Area between x_0 and $x_2 \approx \int_{-h}^{h} (ax^2 + bx + c)\, dx$

$= (ax^3/3 + bx^2/2 + cx)\, \big|_{-h}^{h}$

$= (2ah^3/3 + 0 + 2ch)$

$= h/3\,(2ah^2 + 6c)$ ... (1)

Let us have another observation from the above figure.

- $f(x_0) = a(-h)^2 + b(-h) + c = ah^2 - bh + c$
- $f(x_1) = a(0)^2 + b(0) + c = c$
- $f(x_2) = a(h)^2 + b(h) + c = ah^2 + bh + c$

Now, $f(x_0) + 4f(x_1) + f(x_2) = (ah^2 - bh + c) + 4c + (ah^2 + bh + c) = 2ah^2 + 6c$.

Substitute this in (1):

Area between x_0 and $x_2 \approx h/3\,(f(x_0) + 4f(x_1) + f(x_2))$

Similarly, we can see that:

Area between x_2 and $x_4 \approx h/3\,(f(x_2) + 4f(x_3) + f(x_4))$

Calculating the other areas in a similar way, we get

$\int_a^b f(x)\, dx$

$= h/3\,(f(x_0) + 4f(x_1) + f(x_2))$

$+ h/3\,(f(x_2) + 4f(x_3) + f(x_4))$

$+ \ldots$