Double Pendulum — Theory

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My apologies for muddying the waters by previously calling this the "Compound Pendulum." That name is reserved for something else, that frankly is not a very interesting variation on the simple pendulum.

Simple Pendulum — Angular Acceleration — Recap

The pendulum force law (with damping) was:

$F(\theta, \omega) = -mg\sin\theta - bl\omega$

Technically, this is the "tangential force." Divide through by the mass, m, and you get the tangential acceleration. Also divide through by the length, l, and then you get the angular acceleration:

$\alpha(\theta, \omega) = -\frac{g}{l}\sin\theta - \frac{b}{m}\,\omega$

Define $\omega_0{}^2 = \frac{g}{l}$, and $\gamma = \frac{b}{2\,m}$ and you have:

$\alpha(\theta, \omega) = -\omega_0{}^2\sin\theta - 2\,\gamma\,\omega$

Double Pendulum — Angular Accelerations

The equations below are from https://www.phys.lsu.edu/faculty/gonzalez/Teaching/Phys7221/ which is the course notes for LSU Phys 7221, a first-year graduate course in mechanics for physicists. It is a course in advanced mechanics. The following were Eqs. 69 and 70 that they got to at the end of the first week of class:

$$\begin{aligned}(m_1 + m_2)l_1\ddot{\theta}_1 + m_2 l_2\ddot{\theta}_2\cos(\theta_2 - \theta_1) &= m_2 l_2\dot{\theta}_2^2\sin(\theta_2 - \theta_1) - (m_1 + m_2)g\sin\theta_1 \\ l_2\ddot{\theta}_2 + l_1\ddot{\theta}_1\cos(\theta_2 - \theta_1) &= -l_1\dot{\theta}_1^2\sin(\theta_2 - \theta_1) - g\sin\theta_2\end{aligned}$$

It's just Newton's Laws again, but for a system that is much harder to analyze than any of the others we have dealt with, and unlike all the other systems I have shown you, for this system I am not going to attempt to give you a derivation that these are in fact the accelerations.

The reason I am doing such a challenging system as our final system in Term 4 is that it exhibits chaos — but with no external driving force! At the beginning of Term 5, we will get right back to equations that are less challenging to derive the theory for.

To turn these equations into something you will recognize, I have to mention that the θ's with one dot over them are what we have been calling the ω's, and the θ's with two dots over them are what we have been calling the α's. That's just a notational difference.

If I make the notational changes, divide the top equation through by $m_1 + m_2$, and change the lengths l_1 and l_2 to L_1 and L_2 just for legibility, we have

$$L_1\,\alpha_1 + \frac{m_2}{m_1+m_2}\,L_2\,\alpha_2\cos(\theta_2-\theta_1) = \frac{m_2}{m_1+m_2}\,L_2\,\omega_2^2\sin(\theta_2-\theta_1) - g\sin\theta_1$$

$$L_2\,\alpha_2 + L_1\,\alpha_1\cos(\theta_2-\theta_1) = -L_1\,\omega_1^2\sin(\theta_2-\theta_1) - g\sin\theta_2$$

Now let's divide the top equation through by L_1 and the bottom equation through by L_2:

$$\alpha_1 + \frac{m_2}{m_1+m_2}\,\frac{L_2}{L_1}\,\alpha_2\cos(\theta_2-\theta_1) = \frac{m_2}{m_1+m_2}\,\frac{L_2}{L_1}\,\omega_2^2\sin(\theta_2-\theta_1) - \frac{g}{L_1}\sin\theta_1$$

$$\alpha_2 + \frac{L_1}{L_2}\,\alpha_1\cos(\theta_2-\theta_1) = -\frac{L_1}{L_2}\,\omega_1^2\sin(\theta_2-\theta_1) - \frac{g}{L_2}\sin\theta_2$$

Now I have never coded up the double pendulum before, and perhaps it is not so wise to do the general case on the first try. So let's make some simplifying choices!

We'll take $m_2 = \frac{1}{3}\,m_1$ and $L_2 = \frac{1}{4}\,L_1$ so we have a third of the mass on a quarter-length rod relative to the main mass and the main rod. In that case $\frac{m_2}{m_1+m_2} = \frac{1/3}{1+1/3} = \frac{1/3}{4/3} = \frac{1}{4}$ and $\frac{L_2}{L_1} = \frac{1}{4}$.

You know how I often take $\sqrt{\frac{g}{L}} = 2\,\pi$ so that the natural period of the system is 1 second? Well, I am going to do that here too, so $\frac{g}{L_1} = 4\,\pi^2$ and $\frac{g}{L_2} = \frac{L_1}{L_2}\,\frac{g}{L_1} = 16\,\pi^2$.

With all those simplifying choices,

$$\alpha_1 + \frac{1}{16}\,\alpha_2\cos(\theta_2-\theta_1) = +\frac{1}{16}\,\omega_2^2\sin(\theta_2-\theta_1) - 4\,\pi^2\sin\theta_1$$

$$\alpha_2 + 4\,\alpha_1\cos(\theta_2-\theta_1) = -4\,\omega_1^2\sin(\theta_2-\theta_1) - 16\,\pi^2\sin\theta_2$$

Note that α_1 appears in the equation for α_2 and vice versa, so we aren't going to be able to code that up. We need to solve for at least one of the α's. Then once we have that α, we can put it in the equation for the other.

Solving

Multiply the bottom equation by $\frac{1}{16}\cos(\theta_2-\theta_1)$ and subtract that from the top equation. First multiply:

$$\alpha_2\,\frac{1}{16}\cos(\theta_2-\theta_1)+\frac{1}{4}\,\alpha_1\cos^2(\theta_2-\theta_1)=-\frac{1}{16}\,\omega_1^{\,2}\sin(\theta_2-\theta_1)\cos(\theta_2-\theta_1)-\pi^2\sin\theta_2\cos(\theta_2-\theta_1)$$

Then subtract:

$$\alpha_1-\frac{1}{4}\,\alpha_1\cos^2(\theta_2-\theta_1)=$$
$$+\frac{1}{16}\,\omega_2^{\,2}\sin(\theta_2-\theta_1)-4\,\pi^2\sin\theta_1+\frac{1}{16}\,\omega_1^{\,2}\sin(\theta_2-\theta_1)\cos(\theta_2-\theta_1)+\pi^2\sin\theta_2\cos(\theta_2-\theta_1)$$

You see that α_2 is gone (which is of course why I did that particular multiplication and subtraction). The only thing left to do is divide through by $1-\frac{1}{4}\cos^2(\theta_2-\theta_1)$. We get

$$\alpha_1$$
$$=\left[\frac{1}{16}\,\omega_2^{\,2}\sin(\theta_2-\theta_1)-4\,\pi^2\sin\theta_1+\frac{1}{16}\,\omega_1^{\,2}\sin(\theta_2-\theta_1)\cos(\theta_2-\theta_1)+\pi^2\sin\theta_2\cos(\theta_2-\theta_1)\right]\Big/\left(1-\frac{1}{4}\cos^2(\theta_2-\theta_1)\right)$$

Well, that is going to be messy to code, but at least we have an answer.

Second-Order Runge-Kutta — Formulas for Two Particles — Recap

For two oscillators, we had two positions, two velocities, and two acceleration formulas:

$$t_{i+1}=t_i+\Delta t$$

$$x_1{}^*=x_1(t_i)+v_1(t_i)\cdot\frac{\Delta t}{2}$$

$$x_2{}^*=x_2(t_i)+v_2(t_i)\cdot\frac{\Delta t}{2}$$

$$v_1{}^*=v_1(t_i)+a_1(x_1,\,v_1,\,x_2,\,v_2)\cdot\frac{\Delta t}{2}$$

$$v_2{}^*=v_2(t_i)+a_2(x_1,\,v_1,\,x_2,\,v_2)\cdot\frac{\Delta t}{2}$$

$$v_1(t_{i+1})=v_1(t_i)+a_1(x_1{}^*,\,v_1{}^*,\,x_2{}^*,\,v_2{}^*)\cdot\Delta t$$

$$v_2(t_{i+1})=v_2(t_i)+a_2(x_1{}^*,\,v_1{}^*,\,x_2{}^*,\,v_2{}^*)\cdot\Delta t$$

$$x_1(t_{i+1})=x_1(t_i)+\,(v_1(t_i)+v_1(t_{i+1}))\,\frac{\Delta t}{2}$$

$$x_2(t_{i+1})=x_2(t_i)+\,(v_2(t_i)+v_2(t_{i+1}))\,\frac{\Delta t}{2}$$

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Second-Order Runge-Kutta — Double Pendulum

So now our job is just to translate this for the double pendulum, using mindless substitutions:

$x_1 \to \theta_1 \quad x_2 \to \theta_2$
$v_1 \to \omega_1 \quad v_2 \to \omega_2$
$a_1 \to \alpha_1 \quad a_2 \to \alpha_2$

$t_{i+1} = t_i + \Delta t$

$\theta_1{}^* = \theta_1(t_i) + \omega_1(t_i) \cdot \frac{\Delta t}{2}$

$\theta_2{}^* = \theta_2(t_i) + \omega_2(t_i) \cdot \frac{\Delta t}{2}$

$\omega_1{}^* = \omega_1(t_i) + \alpha_1(\theta_1, \omega_1, \theta_2, \omega_2) \cdot \frac{\Delta t}{2}$

$\omega_2{}^* = \omega_2(t_i) + \alpha_2(\theta_1, \omega_1, \theta_2, \omega_2) \cdot \frac{\Delta t}{2}$

That takes care of the starred values.

Now for the final values:

$\omega_1(t_{i+1}) = \omega_1(t_i) + \alpha_1(\theta_1{}^*, \omega_1{}^*, \theta_2{}^*, \omega_2{}^*) \cdot \Delta t$

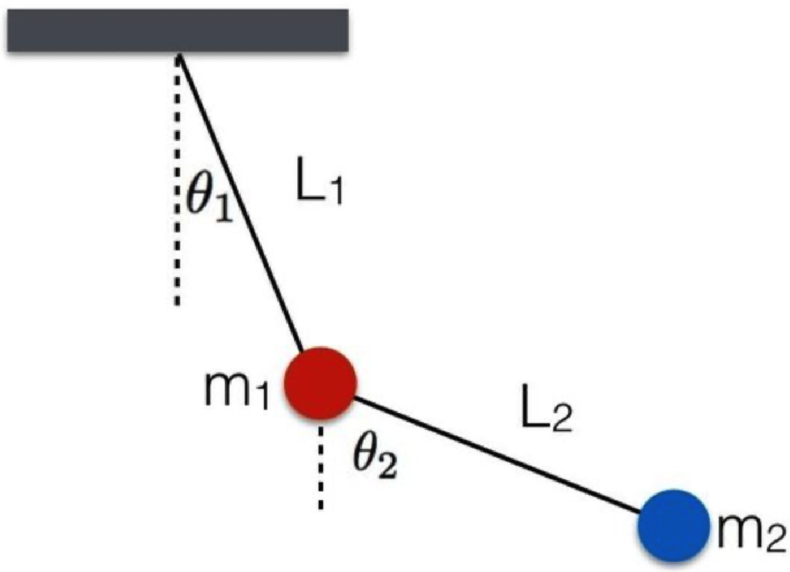
$\omega_2(t_{i+1}) = \omega_2(t_i) + \alpha_2(\theta_1{}^*, \omega_1{}^*, \theta_2{}^*, \omega_2{}^*) \cdot \Delta t$

$\theta_1(t_{i+1}) = \theta_1(t_i) + (\omega_1(t_i) + \omega_1(t_{i+1})) \frac{\Delta t}{2}$

$\theta_2(t_{i+1}) = \theta_2(t_i) + (\omega_2(t_i) + \omega_2(t_{i+1})) \frac{\Delta t}{2}$

A Visual Summary

In case you have lost the picture of what we are trying to do, here is a visual summary:



We have chosen the blue mass to be one-third the weight of the red mass, and L_2 to be one-quarter the length of L_1.