A Guitar String — Blasting into the Stratosphere

Completed and Analyzed in class, April 8, 2025

This is the seventeenth notebook for you to finish in class.

We are leaving behind the crutch of thinking of continuous systems as chunks. We are going to describe continuous systems as continuous systems, which means imagining the limit that the sizes of the chunks goes to zero while the number of chunks goes to infinity. This is exactly the limit that turns our differences of differences into second derivatives.

But first, we need to get familiar with how Mathematica expects continuous systems to be described. Just as we divided space into chunks, even before that in the course, we learned to divide time into steps and then used Euler's Method or Second Order Runge-Kutta to march forward through the steps of time.

In reality, time is also continuous. Let's see how we describe time-dependent equations to Mathematica without resorting to time steps. Mathematica will then solve them for us.

So before we get to the guitar string, we are going all the way back to our fourth notebook, the damped harmonic oscillator, and doing it a new way.

But First, Back to the Earth — The Harmonic Oscillator

Back in the fourth notebook, we considered this force law:

$F = -20\,x - v$

We combine this with Newton's Law $F = ma$ with $m = 5$ and then we had

$5a = -20\,x - v$

Then we got a bit fancier and more general and divided through by the 5. In fact, we could put all the terms on the left at the same time:

$a + \frac{20}{5}\,x + \frac{1}{5}\,v = 0$

The ratio $\frac{20}{5}$ is the ratio of the spring constant to the mass and we called that ${\omega_0}^2$. So for these constants, $\omega_0 = \sqrt{\frac{20}{5}}$ =2. The ratio $\frac{1}{5}$ is the ratio of the damping coefficient to the mass and we called that combination 2γ. So for these constants $2\,\gamma = \frac{1}{5}$ or $\gamma = \frac{1}{10}$. Our equation is now:

$a + 2\,\gamma v + \omega_0^{\,2}\, x = 0$

When $\omega_0 > \gamma$ as it is here (2 is definitely greater than $\frac{1}{10}$), the system is "underdamped." The greater the ratio of $\frac{\omega_0}{\gamma}$ the more oscillations occur for each halving (or $\frac{1}{e}$-folding) of the envelope of the oscillation as its motion damps toward nothing. We went through all this many weeks back, and I am giving you a quick refresher, but now we want to force Mathematica to do the hard work.

We have one more step, which is to introduce the notation of derivatives.

The velocity v is the rate of change (with respect to time) of the position x. Meanwhile the acceleration a is the rate of change (with respect to time) of the velocity v. The rate of change is called the derivative. To say that you want to take a derivative of $x(t)$ with respect to t in Mathematica, you write:

In[1]:=
```
Derivative[1][x][t];
```

This says take one derivative of the function x with respect to its argument, and evaluate the resulting function at time t. So far we haven't done anything at all with our symbolic expression. One thing we can do with it is just display it in a pretty form:

In[2]:=
```
Derivative[1][x][t] // TraditionalForm
```

Out[2]//TraditionalForm=

$x'(t)$

The afterthought of TraditionalForm says that you would like to see the result as it would be likely to be typeset in a mathematics or physics textbook. Note that physicists (and older mathematicians) often use a different notation for the first derivative, which is known as Leibniz notation. We have our hands full. Let's not add yet more notations into the mix.

The acceleration is the rate of change of the velocity, and that is the second derivative, and here is how you tell Mathematica you want to take a second derivative of x with respect to t:

In[3]:=
```
Derivative[2][x][t] // TraditionalForm
```

Out[3]//TraditionalForm=

$x''(t)$

Here is how you write the whole harmonic oscillator equation, $a + 2\,\gamma v + \omega_0^{\,2}\, x = 0$:

In[4]:=
```
Derivative[2][x][t] + 2 gamma Derivative[1][x][t] + omega0^2 x[t] == 0 // TraditionalForm
```

Out[4]//TraditionalForm=

$2\,\text{gamma}\, x'(t) + \text{omega0}^2\, x(t) + x''(t) = 0$

Notice extremely carefully the use of == rather than = in the equation for the derivative. We are not making an assignment! We are making a conditional test, and Mathematica is going to do its best to approximately satisfy that conditional test when it goes nuts (deep under the hood) applying some solver like Euler, or Runge-Kutta Second Order, or maybe Runge-Kutta Fourth Order (that I never got around to introducing, because it is a mess, even for me).

Now let’s define omega0 and gamma, and also add some initial conditions on the position and velocity:

In[5]:=
```
Module[{omega0 = 2, gamma = 1/10},
  {Derivative[2][x][t] + 2 gamma Derivative[1][x][t] + omega0^2 x[t] == 0,
   x[0] == 0, Derivative[1][x][0] == 3}] // TraditionalForm
```

Out[5]//TraditionalForm=

$$\left\{x''(t) + \frac{x'(t)}{5} + 4\,x(t) = 0,\, x(0) = 0,\, x'(0) = 3\right\}$$

I set the initial position at $t = 0$ as 0 and I set the initial velocity as 3. Again, it is extremely important to use == rather than =. If you screw that up, it is quit and restart time.

We have failed to do one thing. We have set up a pile of equations, but we haven’t given this pile of equations any name, so we can’t use them anywhere else. Let’s give them a name:

In[8]:=
```
harmonicOscillatorProblem = Module[{omega0 = 2, gamma = 1/10},
   {Derivative[2][x][t] + 2 gamma Derivative[1][x][t] + omega0^2 x[t] == 0,
    x[0] == 0, Derivative[1][x][0] == 3}] ;
```

In[9]:=
```
harmonicOscillatorProblem // TraditionalForm
```

Out[9]//TraditionalForm=

$$\left\{x''(t) + \frac{x'(t)}{5} + 4\,x(t) = 0,\, x(0) = 0,\, x'(0) = 3\right\}$$