Double Pendulum — Theory

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Simple Pendulum — Angular Acceleration — Recap

The pendulum force law (without damping) was:

$F(\theta, \omega) = -mg \sin\theta$

Technically, this is the "tangential force." Divide through by the mass, m, and you get the tangential acceleration. Also divide through by the length, l, and then you get the angular acceleration:

$\alpha(\theta, \omega) = -\frac{g}{l} \sin\theta$

Define ${\omega_0}^2 = \frac{g}{l}$, and you have:

$\alpha(\theta, \omega) = -{\omega_0}^2 \sin\theta$

Compare this with the acceleration for the harmonic oscillator (without damping):

$a(x, v) = -{\omega_0}^2 x$

You'll see that there are only two things different: (1) in angular problems, we use angles to describe the motion, and (2) there is a pesky $\sin\theta$ in the pendulum formula.

Double Pendulum — Angular Accelerations

I took the following equations from Louisiana State University Phys 7221 (https://www.phys.lsu.edu/faculty/gonzalez/Teaching/Phys7221/) which is a first-year graduate course in mechanics. These were Eqs. 69 and 70 that they got to at the end of the first week of the class:

$$\begin{aligned}
(m_1 + m_2) l_1 \ddot{\theta}_1 + m_2 l_2 \ddot{\theta}_2 \cos(\theta_2 - \theta_1) &= m_2 l_2 \dot{\theta}_2^2 \sin(\theta_2 - \theta_1) - (m_1 + m_2) g \sin\theta_1 \\
l_2 \ddot{\theta}_2 + l_1 \ddot{\theta}_1 \cos(\theta_2 - \theta_1) &= -l_1 \dot{\theta}_1^2 \sin(\theta_2 - \theta_1) - g \sin\theta_2
\end{aligned}$$

It's just Newton's Laws again, but for a system that is much harder to analyze than any of the others we have dealt with, and I am not going to attempt the derivation. It is enough for us to trust that these equations describe the system.

Second-Order Runge-Kutta — Formulas for Two Particles — Recap

When we first did two coupled harmonic oscillators, we had two positions, two velocities, and two formulas for acceleration:

$t_{i+1} = t_i + \Delta t$

$x_1{}^* = x_1(t_i) + v_1(t_i) \cdot \frac{\Delta t}{2}$

$x_2{}^* = x_2(t_i) + v_2(t_i) \cdot \frac{\Delta t}{2}$

$v_1(t_{i+1}) = v_1(t_i) + a_1(x_1{}^*, x_2{}^*) \cdot \Delta t$

$v_2(t_{i+1}) = v_2(t_i) + a_2(x_1{}^*, x_2{}^*) \cdot \Delta t$

$x_1(t_{i+1}) = x_1(t_i) + (v_1(t_i) + v_1(t_{i+1})) \frac{\Delta t}{2}$

$x_2(t_{i+1}) = x_2(t_i) + (v_2(t_i) + v_2(t_{i+1})) \frac{\Delta t}{2}$

Second-Order Runge-Kutta — Pendulum Recap

Here is a recap of the second-order Runge-Kutta formula for the pendulum formula, with $\lambda = \frac{1}{2}$, and removing the possibility of a damping force and removing explicit dependence on time. Those three assumptions make the equations a little easier:

$t_{i+1} = t_i + \Delta t$

$\theta^* = \theta(t_i) + \omega(t_i) \cdot \frac{\Delta t}{2}$

$\omega(t_{i+1}) = \omega(t_i) + \alpha(\theta^*) \cdot \Delta t$

$\theta(t_{i+1}) = \theta(t_i) + (\omega(t_i) + \omega(t_{i+1})) \frac{\Delta t}{2}$

Second-Order Runge-Kutta — Double Pendulum

Now we just have to combine the previous two ideas! We have angles, angular velocity, and angular acceleration as in the previous section. But we have two of them, as in the section before that.

$t_{i+1} = t_i + \Delta t$

$\theta_1{}^* = \theta_1(t_i) + \omega_1(t_i) \cdot \frac{\Delta t}{2}$

$\theta_2{}^* = \theta_2(t_i) + \omega_2(t_i) \cdot \frac{\Delta t}{2}$

$\omega_1(t_{i+1}) = \omega_1(t_i) + \alpha_1(\theta_1{}^*, \theta_2{}^*) \cdot \Delta t$

$\omega_2(t_{i+1}) = \omega_2(t_i) + \alpha_2(\theta_1{}^*, \theta_2{}^*) \cdot \Delta t$

$$\theta_1(t_{i+1}) = \theta_1(t_i) + (\omega_1(t_i) + \omega_1(t_{i+1})) \frac{\Delta t}{2}$$

$$\theta_2(t_{i+1}) = \theta_2(t_i) + (\omega_2(t_i) + \omega_2(t_{i+1})) \frac{\Delta t}{2}$$