Damped Oscillator and Damped Pendulum — Theory

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Damped Oscillator

We have contemplated an oscillator with the force law given by Hooke's Law for springs and a simple model of damping, and in the last notebook, we even included an external sinusoidal driving force. Without the external sinusoidal driving force, the force law is:

$F(x, v) = -k x - b v$

Divide through by the mass to find the acceleration. It is:

$a(x, v) = -\frac{k}{m} x - \frac{b}{m} v$

Let ${\omega_0}^2 = \frac{k}{m}$ and $\gamma = \frac{b}{2 m}$. The acceleration is then:

$a(x, v) = -{\omega_0}^2 x - 2 \gamma v$

Up until this point, we are still in the realm of problems that have exact solutions, and a solution of this equation I will just tell you is

$x(t) = A \cos \sqrt{{\omega_0}^2 - \gamma^2}\, t \cdot e^{-\gamma t}$

and the oddball frequency that shows up in the cosine is the somewhat slowed down oscillation rate due to the damping. It is necessary that γ (the decay rate due to damping) is less than ω_0, otherwise the oscillator is overdamped and does not oscillate. To summarize, there are these three important frequencies in the problem:

ω_0 the natural frequency of the oscillator in the absence of damping
γ the frequency that controls the rate of decay of the exponential
$\sqrt{{\omega_0}^2 - \gamma^2}$ the frequency of the oscillator including damping (which slows oscillation down somewhat)

Damped Pendulum

Angle, Angular Velocity, and Angular Acceleration

Pendulum Angular Acceleration per Newton’s Laws

The pendulum force law is:

$$F(\theta, \omega) = -m g \sin\theta - b l \omega$$

Divide through by the mass and the length of the pendulum, and the acceleration α is

$$\alpha(\theta, \omega) = -\frac{g}{l} \sin\theta - \frac{b}{m} \omega$$

Define

$$\omega_0{}^2 = \frac{g}{l}, \ \gamma = \frac{b}{2 m}$$

and you have

$$\alpha(\theta, \omega) = -\omega_0{}^2 \sin\theta - 2 \gamma \omega$$

Compare this with the acceleration for the damped harmonic oscillator which is

$$a(x, v) = -\omega_0{}^2 x - 2 \gamma v$$

and you’ll see that there are only two things different: (1) in angular problems, we use θ, ω, and α to describe the motion, instead of x, v, and a, and (2) there is a pesky $\sin\theta$ instead of a plain old θ. So you know the solution if it were

$$\alpha(\theta, \omega) = -\omega_0{}^2 \theta - 2 \gamma \omega$$

$$\theta(t) = A \cos \sqrt{\omega_0{}^2 - \gamma^2}\, t \cdot e^{-\gamma t}$$

and in fact, since $\sin\theta$ is very close to θ when θ is small, we can describe the pendulum very well using the theoretical solution, provided the oscillation is not wild. Let us summarize

ω_0	the natural frequency of the pendulum in the absence of damping, and when oscillation is small
γ	the frequency that controls the rate of decay of the exponential, when oscillation is small
$\sqrt{\omega_0{}^2 - \gamma^2}$	the frequency of the pendulum including damping (which slows it down a little), only valid

when oscillation is small

Phase Space