Forced Pendulum — With Phase Space Plots

Done in class, February 11, 2025.

This is our seventh numerical methods notebook.

Forced Oscillation

Angular Acceleration α

In[266]:=

```
omega1 = 1;
drivingForce[t_] := 100 Sin[omega1 t]; (* tangential driving force *)
mass = 5;
gravity = 9.80665;
(* the value of gravity in units of meters / seconds-squared *)
length = 0.24840;
(* A pendulum whose length is 9.7795 inches converted to meters  *)
(* The natural frequency of such a
 pendulum provided the swings are not large: *)
omega0 = Sqrt[gravity / length];
gamma = 0.03;
(* A real pendulum swinging in air typically has a small gamma. *)
period = 2 Pi / omega0;
(* The length was chosen so that the period is 1 second. To be *)
(* precise, 2 Pi / omega0 = 0.999989,
and 2 Pi / Sqrt[omega0^2-gamma^2] = 1.000000. *)
α[t_, theta_, omega_] := -omega0^2 Sin[theta] - 2 gamma omega + drivingForce[t] / mass;
```

Simulation Parameters

In[ ]:=

```
tInitial = 0.0;
tFinal = 50.0;
steps = 200 000;
deltaT = (tFinal - tInitial) / steps;
```

Initial Angle and Angular Velocity

Let’s let the pendulum be initially at rest and see what the driving force does to it:

```
thetaInitial = 0;
omegaInitial = 0;
initialConditions = {tInitial, thetaInitial, omegaInitial};
```

General Second-Order Runge-Kutta — Damped Pendulum Theory Recap

So you don't have to flip back to the damped pendulum theory handout, I'll recapitulate:

$t^* = t + \lambda \Delta t$

$\theta^* = \theta(t_i) + \omega(t_i) \cdot \lambda \Delta t$

$\omega^* = \omega(t_i) + \alpha(t_i, \theta(t_i), \omega(t_i)) \cdot \lambda \Delta t$

$t_{i+1} = t_i + \Delta t$

$\omega(t_{i+1}) = \omega(t_i) + \left(\left(1 - \frac{1}{2\lambda}\right) \alpha(t_i, \theta(t_i), \omega(t_i)) + \frac{1}{2\lambda} \alpha(t^*, \theta^*, \omega^*)\right) \cdot \Delta t$

$\theta(t_{i+1}) = \theta(t_i) + (\omega(t_i) + \omega(t_{i+1})) \frac{\Delta t}{2}$

We got this by mindlessly making the replacements:

$x \to \theta$
$v \to \omega$
$a \to \alpha$

General Second-Order Runge-Kutta — Implementation

The implementation of the damped pendulum is almost the same as the damped oscillator. Finish the implementation.

```
lambda = 1;
rungeKutta2[cc_] := (
  (* Extract time, angle, and angular velocity from the list *)
  curTime = cc[[1]];
  (* Compute tStar, xStar, vStar *)
  tStar = curTime + lambda deltaT;
  (* Implement General Second-Order Runge-Kutta *)
  newTime = curTime + deltaT;
  newAngularVelocity = dog;
  newAngle = pony;
  {newTime, newAngle, newAngularVelocity}
 )
N[rungeKutta2[initialConditions]]
(* Test the rungeKutta2 function you just wrote. *)
(* The output just below should be {0.0025,0.174531,-0.00694977} *)
```

Displaying Oscillation

Nest the procedure, transpose the results, and produce a plot of the angle θ as a function of time:

```
In[ ]:= rk2Results = NestList[rungeKutta2, initialConditions, steps];
rk2ResultsTransposed = Transpose[rk2Results];
times = rk2ResultsTransposed[[1]];
thetas = rk2ResultsTransposed[[2]];
thetaPlot = ListPlot[Transpose[{times, thetas}]];
(* the theoretical solution is approximately known,
provided the angle remains small *)
(* let's plot the envelope of the theoretical solution *)
envelopeFunction[t_] := thetaInitial Exp[-gamma t]
approximateTheoreticalEnvelope =
  Plot[{envelopeFunction[t], -envelopeFunction[t]}, {t, tInitial, tFinal}];
Show[{thetaPlot, approximateTheoreticalEnvelope}]
```

In the preceding plot, the theoretical solution is approximately known, provided the angle remains small, and so I added the envelope of the theoretical solution to the plot.

Displaying Theory

In the following plot, I have included the theoretical oscillation, not just the envelope (but the same approximation that the angle must remain small still applies):

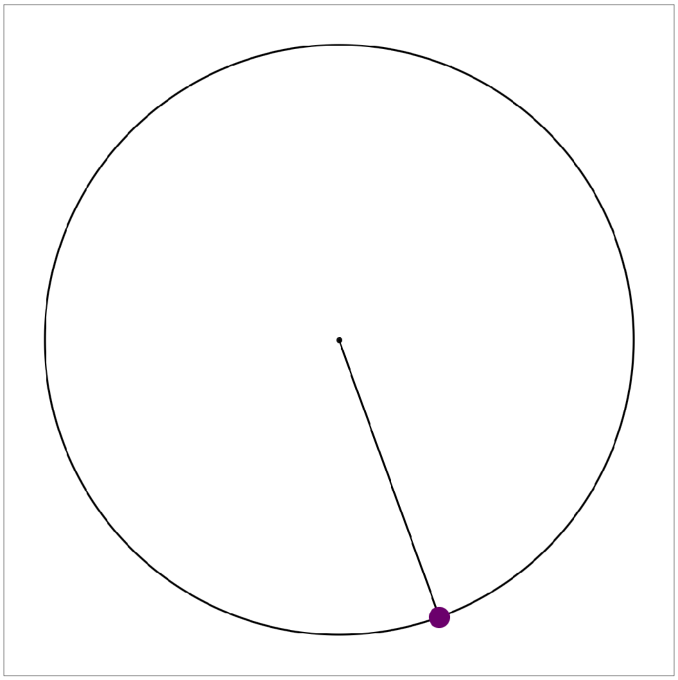
```
approximateTheoreticalSolutionPlot =
  Plot[{envelopeFunction[t], -envelopeFunction[t],
    envelopeFunction[t] Cos[Sqrt[omega0^2 - gamma^2] t]}, {t, tInitial, tFinal}];
Show[{thetaPlot, approximateTheoreticalSolutionPlot}]
```

Drawing a Pendulum with Coordinates and Graphics

To do a legible job of this, you may need to review Section 14 of *EIWL3*. The goal is to finish implementing the function below so that you get a picture something like the one I have pasted in.

```
In[ ]:= pendulumGraphic[angle_] := Graphics[{
    EdgeForm[Thin], White,
    RegularPolygon[{0.0, 0.0}, 0.4, 4],
    Black,
    Circle[{0, 0}, length],
    (* all I left for you to add is two points and a line *)
   }]
pendulumGraphic[20 °]
```

The pendulum graphic you are trying for (when the function is passed in 20° for the angle, and of course your function should do the right thing for any other angle):



Animating the Graphics

It’s also nice to have an animation, arranged so that the default duration of the animation is the actual duration of the animation:

```
In[ ]:= Animate[pendulumGraphic[thetas[[step]]],
 {step, 0, steps, 1}, DefaultDuration → tFinal - tInitial]
```