Many Harmonic Oscillators

Completed and Analyzed in class, February 21, 2025

This is the ninth notebook for you to complete.

Five Oscillators — Formulas for the Accelerations

At the end of the *Coupled Harmonic Oscillators - Theory* writeup, I pointed to where we were going to go after doing two oscillators. For five equal masses connected to each other and the walls by six identical springs, we have:

$$a_1 = -\omega_0^2(x_1 - 0) + \omega_0^2(x_2 - x_1)$$

$$a_2 = -\omega_0^2(x_2 - x_1) + \omega_0^2(x_3 - x_2)$$

$$a_3 = -\omega_0^2(x_3 - x_2) + \omega_0^2(x_4 - x_3)$$

$$a_4 = -\omega_0^2(x_4 - x_3) + \omega_0^2(x_5 - x_4)$$

$$a_5 = -\omega_0^2(x_5 - x_4) + \omega_0^2(0 - x_5)$$

where

$$\omega_0^2 \equiv k/m$$

As you can see, I have done some small but suggestive rewrites to make the five equations appear more similar to each other. The left-most and right-most oscillators have equations that don't quite fit the pattern due to their connection to the walls. The walls can't move. Their lack of movement is captured in the 0's that I have suggestively placed into the amount the two end springs are stretched.

Why Stop at Five? — Formulas for the Accelerations of Many Oscillators

Let's have n masses and $n + 1$ springs. For starters, let's do $n = 5$, but later on, we can make n whatever we like. Let's also define ω_0 while we are at it:

In[ ]:=
```
n = 5;
omega0 = 2 Pi;
```

It is most definitely not going to cut it to have to write out an equation for the acceleration of each of the n masses. Sure we could write out all five equations for all five masses if there are only five, but if we don't want to exhaust ourselves doing error-prone copy-and-paste operations when we get to larger values of n, we are going to have to be craftier.

The trick is to number the masses and have that number be part of the acceleration equations. We will index the oscillators by *j* where *j* can take on any value from 1 to *n* (I'd use *i* again, but we are already using *i* for the time steps, and it would be confusing to re-use that letter). Can you use those ideas to make a provisional first stab at the equations for acceleration?

To make what I am asking you to do more concrete, look at a_3 above. What is the pattern? How are you going to code that?! It will help to imagine that you have a variables called **`j`** and **`allPositions`**:

```
wrongEquationsForAcceleration[] := injured pony
```

We need to handle the left-most and right-most oscillators (the ones that are next to the walls) specially. In other words, the pattern that a_3 followed works for a_2 and a_4 but not a_1 and a_5. That's going to require some **`If[]`** statements:

```
acceleration[j_, allPositions_] := recovered pony
```

Actually, if we don't want the code to have 5 hard-coded into it, we should be more general and note that the pattern is broken for a_1 and a_n.

Initial Conditions

First set up the duration. Let's also define **`steps`** and **`deltaT`** while we are at it::

In[ ]:=
```
tInitial = 0.0;
tFinal = 10.0;
steps = 2000;
deltaT = (tFinal - tInitial) / steps;
```

Cool Initial Conditions, but Ignore them for Now

Here are five sets of initial conditions (these are known as eigenvectors and the omegas are known as eigenvalues) **that are very particular to the case *n* = 5**:

In[ ]:=
```
iA5 = {
    {1/(2 Sqrt[3]), 1/2, 1/Sqrt[3], 1/2, 1/(2 Sqrt[3])},
    {1/2, 1/2, 0, -1/2, -1/2},
    {1/Sqrt[3], 0, -1/Sqrt[3], 0, 1/Sqrt[3]},
    {1/2, -1/2, 0, 1/2, -1/2},
    {1/(2 Sqrt[3]), -1/2, 1/Sqrt[3], -1/2, 1/(2 Sqrt[3])}
  };
omega5 = {Sqrt[1/(2 + Sqrt[3])], 1/Sqrt[3], 1/Sqrt[2], 1, Sqrt[1/(2 - Sqrt[3])]} omega0;
```

Those of you that had Ryan's linear algebra class last spring might like to know that these special initial conditions came from diagonalizing a 5x5 matrix.

Simple-Minded Initial Conditions

We'll use the cool initial conditions after we get the notebook working. For now, just implement these more simple-minded initial condition:

```
pluck = -0.5;
(* Make pluck the second mass's displacement and all the others zero. *)
initialPositions = rat;
(* Make all the initial velocities zero *)
initialVelocities = kangaroo;
```

!! A List Containing Lists !!

We have a decision to make on how we are packing the time, the positions, and the velocities into the initial conditions list, but here is the obvious choice:

In[ ]:=
```
initialConditions = {tInitial, initialPositions, initialVelocities};
```

In particular, initialPositions and initialVelocities are lists, so the list we have just created contains lists!

Euler's Method — Formulas for *n* Particles

Now there are enough things going on that you have to start keeping your wits about you. Let's back off on the complexity for a minute by going back to Euler's method. After we nail that, we'll get back to second-order Runge-Kutta.

$t_{i+1} = t_i + \Delta t$

$x_j(t_{i+1}) = x_j(t_i) + v_j(t_i) \cdot \Delta t \quad (j \text{ goes from 1 to n})$

$v_j(t_{i+1}) = v_j(t_i) + a_j(\text{that depends on neighboring } x \text{ — see formulas for the accelerations above}) \cdot \Delta t$

Euler's Method — Implementation

Implement Euler's method:

```
eulersMethod[cc_] := (
  doggo likes walks
 )
eulersMethod[initialConditions]
(* This is what I get: *)
(* {0.005, {, -0.5, 0., 0., 0.}, {-0.098696, 0.197392, -0.098696, 0., 0.}} *)
```

Second-Order Runge-Kutta — Formulas for *n* Particles

$t_{i+1} = t_i + \Delta t$

$x_j^* = x_j(t_i) + v_j(t_i) \cdot \frac{\Delta t}{2} \quad (j \text{ goes from 1 to n})$

$v_j(t_{i+1}) = v_j(t_i) + a_j(\text{that depends on neighboring } x^* \text{ — see formulas for the accelerations above}) \cdot \Delta t$

$x_j(t_{i+1}) = x_j(t_i) + \left(v_j(t_i) + v_j(t_{i+1})\right) \frac{\Delta t}{2}$

Second-Order Runge-Kutta — Implementation

Your turn to put it all together into the real thing:

```
rungeKutta2[cc_] := (
  doggo likes treats
 )
(* Test your implementation on the initial conditions *)
N[rungeKutta2[initialConditions]]
(* This is what I get: *)
(* {0.005, {-0.00024674, -0.499507, -0.00024674, 0., 0.},
 {-0.098696, 0.197392, -0.098696, 0., 0.}} *)
```

Using NestList[] to Repeatedly Apply rungeKutta2[]

In[ ]:=
```
rk2Results = NestList[rungeKutta2, initialConditions, steps];
```

Transposing to Get Positions We Can Put into Graphics

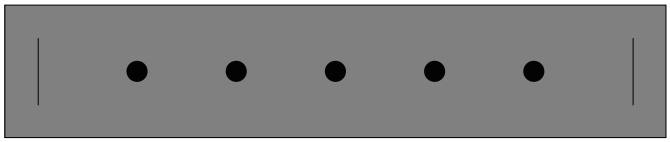
In[ ]:=
```
rk2ResultsTransposed = Transpose[rk2Results];
positionLists = rk2ResultsTransposed[[2]];
```

A Graphic

The graphics work is straightforward but a little time-consuming, and not terribly instructive, so it is already all done:

```
coupledOscillatorGraphic[positionList_] := Graphics[{
    width = 10;
    buffer = 0.5;
    netWidth = width - 2 buffer;
    wallHeight = 1.0;
    numberOfSprings = n + 1;
    (* the next line makes a gray rectangle *)
    {EdgeForm[Thin], Gray, Polygon[{{0, -1}, {width, -1}, {width, 1}, {0, 1}}]},
    (* the next two lines make the walls *)
    Line[{{buffer, -wallHeight / 2}, {buffer, wallHeight / 2}}],
    Line[{{width - buffer, -wallHeight / 2}, {width - buffer, wallHeight / 2}}],
    (* finally we draw all the points *)
    Table[Style[Point[{positionList[[j]] + netWidth / numberOfSprings j + buffer, 0.0}],
      PointSize[0.03]], {j, n}]
  }]
(* A little test to see if the code at least draws equally-
 spaced points when the positions are all zero: *)
coupledOscillatorGraphic[Table[0.0, n]]
```

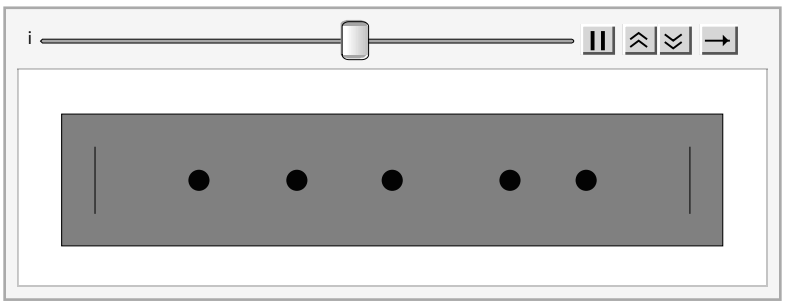
Out[•]=



Animating The Graphics

In[•]:=
```
Animate[coupledOscillatorGraphic[positionLists[[i]]],
 {i, 1, steps, 1}, DefaultDuration → tFinal - tInitial]
```

Out[•]=



Things to Try Once the Notebook is Working

Try Changing the Initial Conditions

Go back up to the initial conditions section, and where it said,

```
initialPositions={0.0,-pluck,0.0,0.0,0.0};
```

change that line to ***one*** of the special initial conditions:

```
initialPositions=iA5[[1]];
initialPositions=iA5[[2]];
initialPositions=iA5[[3]];
initialPositions=iA5[[4]];
initialPositions=iA5[[5]];
```

All of these are interesting!

When you are done, change the code back to what it originally was.

Try Changing the Graphics

Make the masses bob up and down instead of left to right.

You might view this as an improvement and just leave it that way, even though it no longer represents springs moving masses to the left and right.

Try Messing with the Acceleration Formula — Introducing "Periodic Boundary Conditions"

I hope you had this (or something equivalent):

```
acceleration[j_, allPositions_] :=
 -omega0^2 (allPositions[[j]] - If[j == 1, 0, allPositions[[j - 1]]]) +
  omega0^2 (If[j == n, 0, allPositions[[j + 1]]] - allPositions[[j]])
```

The special cases of the two 0's are due to the walls.

The thing to change is the first 0 to allPositions[[n]] and the second 0 to allPositions[[1]].

Physically, what that is doing is eliminating the walls and connecting the right-most mass to the left-most mass. This is called “periodic boundary conditions.”

When you are done, put the walls back in by putting the zeros back where they were.

Experimentation I am Doing but You Can Skip

I have been experimenting around trying to make other more regular-looking patterns, but I have been unsuccessful. The code I was trying had special values for the initial velocities, and it looked like:

```
innerProducts = Table[Dot[initialPositions,iA5[[k]]],{k,5}];
weightediA5=-Table[iA5[[k]]innerProducts[[k]]omega5[[k]],{k,5}];
initialVelocities=N[Total[weightediA5]]
```

Since I have been unsuccessful, I recommend skipping to my next and last idea that follows.

Taking Much Larger Values of *n*

If your initial positions code looks like this:

```
initialPositions={0.0,pluck,0.0,0.0,0.0};
```

then it assumed $n = 5$, and it has to be rewritten for other values of n. It is a small but good exercise to ***make that code work for any n.*** Don’t just copy-and-paste 999 zeros if you want to get $n = 1000$. Even for $n = 100$ that is a sloppy way to get something going.

A good amount to try is $n = 100$.

Drop the point size from 0.3 to 0.005 so the dots aren’ t crowding each other, increase the animation final time from 10.0 to 40.0, and increase steps from 2000 to 8000.

Finally, I’d recommend combining this with the idea of making the masses bob up and down instead of left to right because otherwise they overlap each other in an illegible way.

If you do all that, you’ll finally see something wave-like appear. It is admittedly still jumbled, but there are very good physical reasons for this! I could talk some about how there is an analogy to plucking a guitar string that is relevant, and it is because of that analogy that I called the parameter in the initial condition “pluck,” but we can save some of that discussion for early in Term 5.

Looking Ahead

We’ve done enough with coupled harmonic oscillators. We’ll leave harmonic oscillators and do two coupled pendula as our last notebook in Term 4 and switch systems to torsion pendula after the Term

4-5 break.

Although the physics and the equations of torsion pendula are identical to harmonic oscillators, they look quite a bit different.

With many coupled torsion pendula, we'll take the time to start understanding and minimizing the jumbling so that we can more clearly see waves emerging.

Please do not confuse coupled pendula with coupled torsion pendula. Coupled pendula have all the complexity of pendula. Coupled torsion pendula are no more complex than harmonic oscillators, and they have the added benefit that they are easier to visualize.