Position from Velocity — Theory

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Velocity

To use the concept of velocity, first we have to define it. If a particle is at x_1 at time t_1 and x_2 at time t_2, then the average velocity is ***by definition*** (note the triple-equals):

$$v_{1\text{ to }2,\text{ avg}} \equiv \frac{\text{change in position}}{\text{change in time}} = \frac{x_2 - x_1}{t_2 - t_1}$$

Sometimes I will write $x(t_1)$ instead of x_1 and $x(t_2)$ instead of x_2. That is just notation. It isn't even worthy of calling it a definition.

Position from Velocity and an Approximation

We can rearrange the definition:

$$x_2 - x_1 = v_{1\text{ to }2,\text{ avg}} * (t_2 - t_1)$$

or

$$x_2 = x_1 + v_{1\text{ to }2,\text{ avg}} * (t_2 - t_1)$$

That's just a rearrangement of the definition and we can't learn much if anything just by rearranging definitions, but ***now we are going to make an approximation with consequences: we are going to assume that a good approximation for $v_{1\text{ to }2,\text{ avg}}$ is the value of v at the midpoint of the time interval.***

The midpoint of the time interval is $\frac{t_1+t_2}{2}$, and the value of v at this midpoint we'll denote $v\left(\frac{t_1+t_2}{2}\right)$, so we have:

$$x_2 = x_1 + v\left(\frac{t_1+t_2}{2}\right) * (t_2 - t_1)$$

I am going to introduce another definition, or another convenient notation:

$$\Delta t \equiv t_2 - t_1$$

Notice that

$\frac{t_1+t_2}{2} = t_1 + \frac{\Delta t}{2}$

So with all of that, we have:

$x(t_2) = x(t_1) + v\left(t_1 + \frac{\Delta t}{2}\right) * \Delta t$

Perhaps it is good to be explicit and state the obvious, that we can get t_2 by rearranging the definition for Δt:

$t_2 = t_1 + \Delta t$

Numerical Integration — The Formulas

Perhaps you don’t see it yet, and even I can hardly believe that we have gotten so much from so little, but rest assured, we have just derived an extremely powerful formula and procedure. It is so powerful, I am just going to write it down again, and then later discuss it:

$t_2 = t_1 + \Delta t$

$x(t_2) = x(t_1) + v\left(t_1 + \frac{\Delta t}{2}\right) * \Delta t$

Numerical Integration — Discussion

The first of the two formulas above needs no explanation. It just tells you how to get t_2 from t_1 and Δt.

The second formula is not much more complex. On the right-hand side (RHS) is the position of the particle at time t_1. Also on the RHS is the velocity function evaluated at a particular time, the midpoint between t_1 and t_2. On the left-hand side (LHS) is the position of the particle at some later time t_2.

You might complain that we used an approximation to get the second formula, but for any reasonable velocity function, the approximation that the average velocity is the velocity at the midpoint gets better and better if you make Δt smaller and smaller. Since we have computers at our disposal, we can and will make the time steps as small as is needed to get accurate answers.

To make this business of “small time steps” a little more visceral, let’s have an example. Suppose you made $\Delta t = 0.001$ and you wanted to learn the position of the particle at time 3.5 from the position of the particle at time 2.0. Well, you’d just have to compound this equation 1500 times and you’d work your way from 2.0 to 2.001, to 2.002, etc., etc., all the way to 3.498, to 3.499, and finally to 3.5.

I’m not going to prove that the approximation can be made as good as you like in this write-up! Perhaps the Math Analysis class will be able to prove it, given some assumption.

In this class, you just have to believe what is quite plausible, which is that for any physically reasonable velocity function, the procedure described works to any desired precision required of the final position, just as long as you make Δt sufficiently small. If you needed to make Δt be 0.0001 to retain the desired precision at 3.5, well, then you make it that small and you compound the procedure 15,000 times.

For the simple situation we have been considering, the entire procedure has a fancy name. It is called “numerical integration,” and the approximation we are using is called the Midpoint Riemann Sum. It is used instead of the Left Riemann Sum or the Right Riemann Sum, because the midpoint is often a better approximation to $v_{1\text{ to }2,\text{ avg}}$ than $v(t_1)$ or $v(t_2)$. The procedure works whether you take the Middle, the Left, or the Right Riemann Sum, but in practice, you don’t have to make Δt as small to get good accuracy if you use the Middle Riemann Sum.

Alternative Approximations

There was a step in the above where I approximated $v_{1\text{ to }2,\text{ avg}}$ by $v\left(\frac{t_1+t_2}{2}\right)$. A poorer approximation would have been to approximate $v_{1\text{ to }2,\text{ avg}}$ by $v(t_1)$. This is what we called the Left Riemann Sum above and it is also known as Euler’s Method. The two ideas (Left Riemann Sum and Euler’s Method) are synonymous in the simple situation we are investigating. An equally poor approximation to $v(t_1)$ would be $v(t_2)$, and I don’t think it has a name other than Right Riemann Sum. The main one we used earlier, $v\left(\frac{t_1+t_2}{2}\right)$ is the Midpoint Riemann Sum. We can make drawings of these things so that you can see why they have the names that they do.

An alternative approximation which is as good or better than $v\left(\frac{t_1+t_2}{2}\right)$ is $\frac{v(t_1)+v(t_2)}{2}$. This is a significantly new approximation and it is called the Trapezoid, and I believe it is also known as the Improved Euler’s Method. They are all just attempts to get some easy-to-calculate approximation to $v_{1\text{ to }2,\text{ avg}}$. Every last one of these is an approximation, and every last one of these gets better and better if you make the time steps smaller and smaller.