Coupled Torsion Pendula — Introduction to Second Derivatives

March 18, 2025

THIS THEORY ISN'T FLESHED OUT YET

Position from Velocity — Recap

On January 17, we carefully went though how you get position from velocity.

Let's quickly recap how we got position from velocity because we are going to follow an extremely similar procedure for getting velocity from acceleration.

Definitions

We defined:

$$v_{i \to i+1,\text{avg}} \equiv \frac{\text{change in position}}{\text{change in time}} = \frac{x(t_{i+1}) - x(t_i)}{\Delta t}$$

where

$$\Delta t \equiv t_{i+1} - t_i$$

We rearranged these two definitions to get:

$$t_{i+1} = t_i + \Delta t$$

$$x(t_{i+1}) = x(t_i) + v_{i \to i+1,\text{avg}} \cdot \Delta t$$

Approximations

You can't accomplish anything just by making and rearranging definitions, so we considered several approximations to $v_{i \to i+1,\text{avg}}$:

left: $v_{i \to i+1,\text{avg}} \approx v(t_i)$
right: $v_{i \to i+1,\text{avg}} \approx v(t_{i+1})$
midpoint: $v_{i \to i+1,\text{avg}} \approx v\left(\frac{t_i + t_{i+1}}{2}\right)$

trapezoid: $v_{i \to i+1,\text{avg}} \approx \frac{v(t_i)+v(t_{i+1})}{2}$

I am using the squiggly equals sign to emphasize that an approximation is being made.

Example

In a worksheet done in-class on 2025-01-17, from the above approximations, we chose the midpoint approximation, we chose $\Delta t = 0.4$, we chose $v(t) = 6 \cdot t$, and we iterated from $t_{\text{initial}} = 0.0$ to $t_{\text{final}} = 6.0$ by applying these formulas 15 times:

$$t_{i+1} = t_i + \Delta t$$

$$x(t_{i+1}) \approx x(t_i) + v\left(t_i + \frac{\Delta t}{2}\right) \cdot \Delta t$$

I have made yet another slight improvement to the equations by noticing that $t_i + \frac{\Delta t}{2}$ is just another way of writing the midpoint, $\frac{t_i + t_{i+1}}{2}$.

The index i in our example ranged from 0 to 15 with $t_0 = t_{\text{initial}}$ and $t_{15} = t_{\text{final}}$. This gave us 15 times steps and 16 points (counting both the initial and final ones).

Velocity from Acceleration

We copy the above procedure, but using velocity and acceleration instead of position and velocity.

Definitions

We define:

$$a_{i \to i+1,\,\text{avg}} \equiv \frac{\text{change in velocity}}{\text{change in time}} = \frac{v(t_{i+1}) - v(t_i)}{\Delta t}$$

We rearrange this definition and add the new equation to the other two we already had to get:

$$t_{i+1} = t_i + \Delta t$$

$$x(t_{i+1}) = x(t_i) + v_{i \to i+1,\,\text{avg}} \cdot \Delta t$$

$$v(t_{i+1}) = v(t_i) + a_{i \to i+1,\,\text{avg}} \cdot \Delta t$$

We now have three equations that tell us how to get everything we care about at step $i + 1$ from everything we know at step i.

Approximations

But we still can't get anywhere without knowing $v_{i\to i+1,\text{ avg}}$ and $a_{i\to i+1,\text{ avg}}$, and the only way to make headway on those quantities is to do approximations. Just as we did for velocity, we will quite often use the midpoint or trapezoid approximation for $a_{i\to i+1,\text{avg}}$:

midpoint: $a_{i\to i+1,\text{avg}} \approx a\left(t_i + \frac{\Delta t}{2}\right)$
trapezoid: $a_{i\to i+1,\text{avg}} \approx \frac{a(t_i)+a(t_i+\Delta t)}{2}$

Then we put one of those choices into the equation for $v(t_{i+1})$. If we do midpoint, we get:

$$v(t_{i+1}) \approx v(t_i) + a\left(t_i + \frac{\Delta t}{2}\right)\cdot\Delta t$$

If we do trapezoid, we get:

$$v(t_{i+1}) \approx v(t_i) + \frac{a(t_i)+a(t_i+\Delta t)}{2}\cdot\Delta t$$

What Next!? Newton's 2nd Law of Motion!

This is more than enough theory to start coding up some interesting motion. There is only one piece of the puzzle left....

1. We learned how to get position from velocity from last time.
2. We just reviewed and then repeated that exact same set of ideas to get velocity from acceleration.
3. But how are we going to get acceleration!? From the most famous equation in physics (other than perhaps $E = mc^2$)! Physicists call this Newton's Second Law of Motion:

$$F = ma$$

On the left is the force. On the right is the mass times the acceleration. If we know the force, then we get the acceleration using

$$a = \frac{F}{m}$$

The critical among you will say, so how do we get the force? Well, that depends on the system under study. Forces have been catalogued, quantified, and systematized for lots of interesting systems. For example, the Coulomb repulsive force between two like-sign charges is easy to write down, or the Coulomb attractive force between two opposite-sign charges is the same thing but with a minus sign that reverses the force's direction. We will start with forces that are even easier to write down, like the force of a spring on a mass, and soon we will be observing oscillation.

A Complication — Why Numerical Analysis is Not Just Riemann Integration

In the formula

$$v(t_{i+1}) \approx v(t_i) + a\left(t_i + \frac{\Delta t}{2}\right) \cdot \Delta t$$

or with Newton's 2nd Law substituted in,

$$v(t_{i+1}) \approx v(t_i) + \frac{F\left(t_i + \frac{\Delta t}{2}\right)}{m} \cdot \Delta t$$

there is actually a hidden complication! You see, we assumed *F* was given as a function of time, and that isn't realistic most of the time.

Yes, for a few problems, we can treat *F* as a function of time. But those problems are only a small fraction of the interesting problems we will want to study.

The complicating fact is that *F* is just as often or more often a function of *x* and *v* rather than being a function of *t*.

Summary

We will eventually introduce and use Newton's Second Law properly in full generality by making *F* a function of all three of *t*, *x*, and *v*, but for the moment it is going to be easier to become more familiar with numerical methods by assuming that the $a = \frac{F}{m}$ is just a given function of time.

In the simple case we will be considering, the iterative procedure we will generally use is:

$$t_{i+1} = t_i + \Delta t$$

$$x(t_{i+1}) \approx x(t_i) + \frac{v(t_i) + v(t_{i+1})}{2} \cdot \Delta t$$

$$v(t_{i+1}) \approx v(t_i) + a\left(t_i + \frac{\Delta t}{2}\right) \cdot \Delta t$$

Notice that this is an oddball hybrid of trapezoid and midpoint. Specifically, we are using midpoint for getting velocity from acceleration and we are using trapezoid for getting position from velocity.