Damped Oscillator and Damped Pendulum — Theory

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Damped Oscillator

We have contemplated an oscillator with the force law, and in the last notebook, we even included an external sinusoidal driving force. Without the external sinusoidal driving force, the force law is:

$F(x, v) = -kx - bv$

Divide through by the mass and the acceleration is:

$a(x, v) = -\frac{k}{m}x - \frac{b}{m}v$

Let ${\omega_0}^2 = \frac{k}{m}$ and $\gamma = \frac{b}{2m}$. The acceleration is then:

$a(x, v) = -{\omega_0}^2 x - 2\gamma v$

We are still in the realm of problems that have exact solutions, and a solution of this equation I will just tell you is

$x(t) = A\cos\sqrt{{\omega_0}^2 - \gamma^2}\,t \cdot e^{-\gamma t}$

where ${\omega_0}^2 = \frac{k}{m}$, $\gamma = \frac{b}{2m}$, and the oddball frequency that shows up in the cosine is a slightly slowed down oscillation rate relative to ω_0, provided γ, the decay rate due to damping is small.

ω_0	the natural frequency of the oscillator in the absence of damping
γ	the frequency that controls the rate of decay of the exponential
$\sqrt{{\omega_0}^2 - \gamma^2}$	the frequency of the oscillator including damping (which slows oscillation down a little)

Damped Pendulum

Angle, Angular Velocity, and Angular Acceleration

Pendulum Angular Acceleration per Newton’s Laws

The pendulum force law is:

$F(\theta, \omega) = -mg\sin\theta - bl\omega$

Divide through by the mass and the length of the pendulum, and the acceleration α is

$$\alpha(\theta, \omega) = -\frac{g}{l}\sin\theta - \frac{b}{m}\omega$$

Define

$$\omega_0{}^2 = \frac{g}{l}, \ \gamma = \frac{b}{2m}$$

and you have

$$\alpha(\theta, \omega) = -\omega_0{}^2 \sin\theta - 2\gamma\omega$$

Compare this with the acceleration for the damped harmonic oscillator which is

$$a(x, v) = -\omega_0{}^2 x - 2\gamma v$$

and you'll see that there are only two things different: (1) in angular problems, we use θ, ω, and α to describe the motion, instead of x, v, and a, and (2) there is a pesky $\sin\theta$ instead of a plain old θ. So you know the solution if it were

$$\alpha(\theta, \omega) = -\omega_0{}^2 \theta - 2\gamma\omega$$

$$\theta(t) = A\cos\sqrt{\omega_0{}^2 - \gamma^2}\, t \cdot e^{-\gamma t}$$

and in fact, since $\sin\theta$ is very close to θ when θ is small, we can describe the pendulum very well using the theoretical solution, provided the oscillation is not wild. Let us summarize

ω_0 the natural frequency of the pendulum in the absence of damping, and when oscillation is small
γ the frequency that controls the rate of decay of the exponential, when oscillation is small
$\sqrt{\omega_0{}^2 - \gamma^2}$ the frequency of the pendulum including damping (which slows it down a little), only valid when oscillation is small

Phase Space