Position from Velocity — Theory

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Velocity

To use the concept of velocity, first we have to define it. If a particle is at x_1 at time t_1 and x_2 at time t_2, then the average velocity is *by definition* (note the triple-equals):

$$v_{1\to 2,\,\mathrm{avg}} \equiv \frac{\text{change in position}}{\text{change in time}} = \frac{x_2 - x_1}{t_2 - t_1}$$

Sometimes I will write $x(t_1)$ instead of x_1 and $x(t_2)$ instead of x_2. That is just notation. It isn't even worthy of the word "definition" to call such obvious notation a definition.

An Approximation

We can rearrange the definition:

$$x_2 - x_1 = v_{1\to 2,\,\mathrm{avg}} \cdot (t_2 - t_1)$$

or

$$x_2 = x_1 + v_{1\to 2,\,\mathrm{avg}} \cdot (t_2 - t_1)$$

That's just a rearrangement of the definition and we can't learn much if anything just by rearranging definitions, but *now we are going to make an approximation with consequences: we are going to assume that a good approximation for $v_{1\to 2,\,avg}$ is the value of v at the midpoint of the time interval.*

The midpoint of the time interval is $\frac{t_1+t_2}{2}$, and the value of v at the midpoint we'll denote $v\left(\frac{t_1+t_2}{2}\right)$, and so:

$$x_2 = x_1 + v\left(\frac{t_1+t_2}{2}\right) \cdot (t_2 - t_1)$$

I am going to introduce another definition, or another convenient notation:

$$\Delta t \equiv t_2 - t_1$$

Notice that with this new notation (or definition),

$$\frac{t_1+t_2}{2} = t_1 + \frac{t_2-t_1}{2} = t_1 + \frac{\Delta t}{2}$$

So with all of that, we have:

$$x(t_2) = x(t_1) + v\left(t_1 + \frac{\Delta t}{2}\right)\cdot\Delta t$$

Perhaps it is good to be explicit and also state the obvious, that we can get t_2 by rearranging the definition for Δt:

$$t_2 = t_1 + \Delta t$$

The Simplest Formulas

Perhaps you don’t see it yet, and even I can hardly believe that we have just gotten so much from so little, but rest assured, we have just derived some extremely powerful formulas and a procedure. The procedure is so powerful, I am just going to write it down again before discussing it:

$$t_2 = t_1 + \Delta t$$

$$x(t_2) = x(t_1) + v\left(t_1 + \frac{\Delta t}{2}\right)\cdot\Delta t$$

Δt is known as the time step and we are going to make it small. Note that these formulas only do us much good in the simple situation when the velocity is a known function of time, and in what follows I will repeatedly qualify my statements by saying “for the simple situation we are considering.” Rest assured, soon we will be considering more complex situations, but we have to get the simple one nailed first.

The first of the two formulas above needs no explanation. It just tells you how to get t_2 from t_1 and the small time step Δt. The second formula is not much more complex. On the right-hand side (RHS) is the position of the particle at time t_1. Also on the RHS is the velocity function evaluated at a particular time, the midpoint between t_1 and t_2. On the left-hand side (LHS) is the position of the particle at some later time t_2.

You might complain that we used an approximation to get the second formula, but for any reasonable velocity function, the approximation that the average velocity is the velocity at the midpoint gets better and better if you make Δt smaller and smaller. Since we have computers at our disposal, we can and will make the time steps as small as is needed to get accurate answers.

Making the Approximation Better

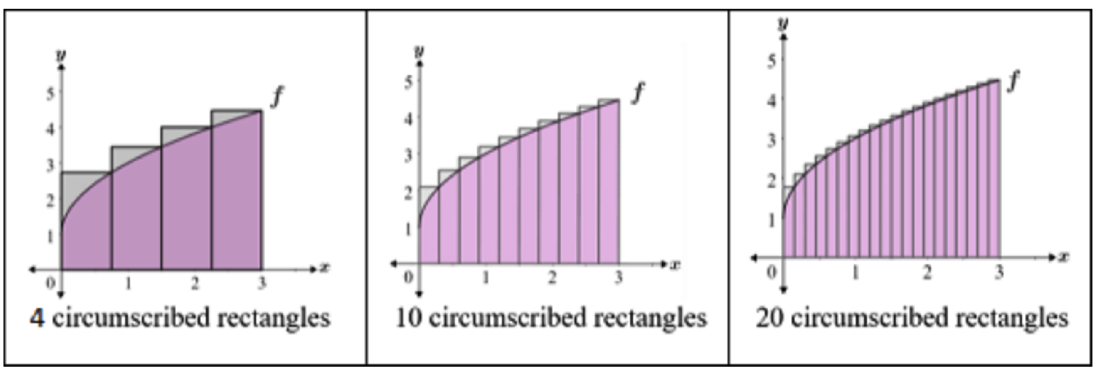
To make approximating using “small time steps” a little more visceral, let’s have an example. Suppose you made $\Delta t = 0.001$ and you wanted to learn the position of the particle at time $t_{\text{final}} = 3.5$ from the position of the particle at time $t_{\text{initial}} = 2.0$. Well, you’d just have to compound the procedure 1500 times and you’d work your way from $t_{\text{initial}} = 2.0$ to 2.001, to 2.002, etc., etc., all the way to 3.498, to

3.499, and finally to $t_{\text{final}} = 3.5$.

I'm not going to prove that the approximation can be made as good as you like in this write-up! Perhaps the Mathematical Analysis class will be able to prove it, given some assumptions and a lot of weeks of build-up.

There was a step in the above where I approximated $v_{1\to 2,\text{avg}}$ by $v\left(\frac{t_1+t_2}{2}\right)$. A poorer approximation would have been to approximate $v_{1\to 2,\text{avg}}$ by $v(t_1)$. This is what we called the Left Riemann Sum above and it is also known as Euler's Method for the simple situation we are considering. We could just as well have used $v(t_2)$ and that would be the Right Riemann Sum. Our midpoint choice you could call the Midpoint Riemann Sum.

Here is a drawing from MATH.net of some left Riemann sums that shows intuitively why the approximation gets better and better:



Please note that in the drawings I have taken off of the web, the horizontal axis is x, the function is $f(x)$, and the vertical axis is $y = f(x)$, as is common in mathematics. In physics, the horizontal axis is generally t, the function is often $v(t)$, and the vertical axis would be labeled v. It is extremely important to think about what the gray rectangular approximation to each lavender region represents, and why the approximation gets better and better as the width the the gray rectangles gets narrower and narrower.

For this class, you just have to believe what the drawing above makes quite plausible, which is that given any physically-reasonable velocity function, the procedure described works to any desired precision that is required, just as long as you make the time slices Δt sufficiently small.

If you needed to make Δt be 0.0001 to retain the desired precision at the final time, $t_{\text{final}} = 3.5$, well, then you make it that small and you then have to compound the procedure 15,000 times to get from t_{initial} to t_{final}.

Alternative Approximations

Instead of the Left Riemann Sum or the Right Riemann Sum, we used the midpoint because the midpoint is often a better approximation to $v_{1\to 2,\,\text{avg}}$ than $v(t_1)$ or $v(t_2)$. *The procedure works whether you take the Midpoint,*

$$x(t_2) = x(t_1) + v\left(t_1 + \frac{\Delta t}{2}\right)\cdot\Delta t$$

the Left,

$$x(t_2) = x(t_1) + v(t_1)\cdot\Delta t$$

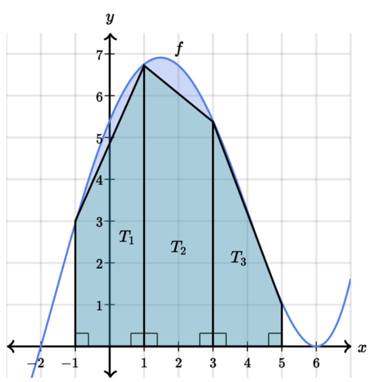
or the Right

$$x(t_2) = x(t_1) + v(t_2)\cdot\Delta t$$

Riemann Sum, but in practice, you don't have to make Δt as small to get good accuracy if you use the Midpoint Riemann Sum.

An alternative approximation which is generally as good or even better than $v\left(\frac{t_1+t_2}{2}\right)$ is $\frac{v(t_1)+v(t_2)}{2}$, and it is a significantly new approximation known as the Trapezoid Riemann Sum.

Here is a nice Khan Academy drawing of the trapezoid sum:



For the simple type of problem we are so far considering the Trapezoid Riemann Sum is equivalent to the Improved Euler Method.

Summary

These approximations are all just attempts to get some easy-to-calculate approximation to $v_{1\to 2,\,\text{avg}}$, which is what actually appears in the definition we started this write-up with. Every one of these approximations gets better and better if you make the time steps smaller and smaller.