# Circular Drumhead

Completed and Analyzed in class, March 28, 2025

THIS IS A WORK IN PROGRESS THAT SHOULD BE READY FOR THE FRIDAY MARCH 28 CLASS.

This is the fifteenth notebook for you to complete. It is our second notebook that has a two-dimensional network of masses. This time the network is disk-shaped, and we will use the coordinates $r$ and $\theta$ to describe the disk. As with the rectangular two-dimensional network in the previous notebook, this disk of masses will oscillate vertically (in the $z$ direction).

## Initial Conditions

Set up the duration, **steps,** and **deltaT**:

```
In[ ]:= tInitial = 0.0;
tFinal = 10.0;
steps = 5000;
deltaT = (tFinal - tInitial) / steps;
```

Set up the size of the grid (enough masses so that the grid looks like a drumhead, but not so many that we tax our computers):

```
In[ ]:= nr = 4;
nθ = 8;
```

We are going to make initial conditions a product of Bessel functions and sine functions. What product is specified by the modes.

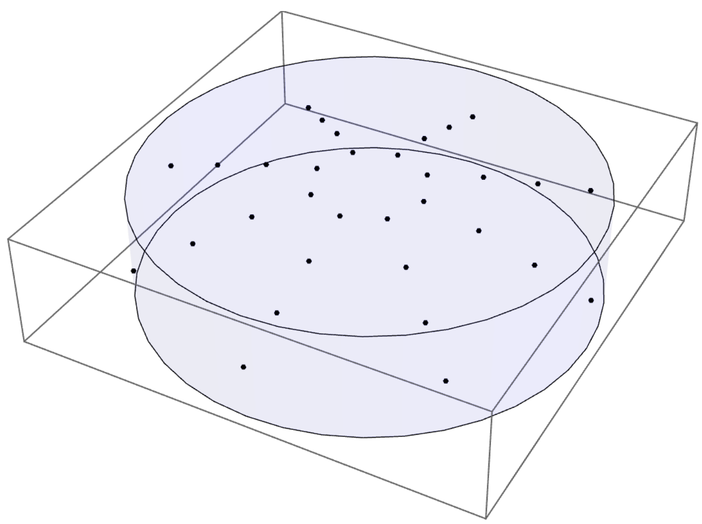
```
In[ ]:= maxz = 1.0;
initialzs =
  Table[maxz BesselJ[0, BesselJZero[0, 1] r / nr], {r, 1, nr}, {θ, 0, nθ - 1}];
initialvs = Table[0, {r, 1, nr}, {θ, 0, nθ - 1}];
initialConditions = {tInitial, initialzs, initialvs};
```

## 3D Graphics

We need a graphics implementation with $n_r\, n_\theta$ masses.

In[ ]:=
```
halfHeight = 1;
radius = 4;
rspacing = radius / nr;
θspacing = 360 ° / nθ;
cylinder = {FaceForm[{Blue, Opacity[0.04]}],
   Cylinder[{{0, 0, -halfHeight}, {0, 0, halfHeight}}, radius]};
drumheadGraphic[zs_] := Graphics3D[Flatten[{
      {cylinder},
      Table[
       Point[{rspacing r Cos[θspacing θ], rspacing r Sin[θspacing θ], zs[[r, θ + 1]]}],
       {r, nr}, {θ, 0, nθ - 1}]
     }, 1]];
drumheadGraphic[initialzs]
```

Out[ ]=



## Formulas for the Accelerations — Theory

The formula for the accelera-
tions,

$$a_{r,\theta} = \frac{n_\theta n_r}{2\pi r} v_0{}^2\left[\frac{2\pi r}{n_r}\left(\frac{n_\theta}{2\pi r n_r}\right)^2\left(z_{j,\theta+1} + z_{j,\theta-1} - 2 z_{r,\theta}\right) + \frac{2\pi r}{n_\theta}\frac{1}{n_r{}^2}\left(z_{r+1,\theta} + z_{r-1,\theta} - 2 z_{r,\theta}\right)\right]$$

is valid except for the perimeter and the center.

### Fixed Perimeter

A round drumhead is normally fixed at its perimeter, and we are going to deal with the perimeter by just freezing the perimeter masses to have $z_{n_r,\theta} = 0$. We can capture this in the initial conditions, and we can preserve it by having $a_{n_r,\theta}$ be 0 in the acceleration formula.

### An Issue at the Center

There is also an issue at the center. The position of the center, $z_{0,\theta}$, is referenced in the term $z_{r-1,\theta}$ when $r = 1$. You can even tell there is something screwy about $z_{0,\theta}$ because there is only one center, and so its $z$ value cannot depend on $\theta$. A way to handle this reference is to interpret $z_{0,\theta}$ as the average of all of the $z_{1,\theta}$. Essentially, there is negligible mass in the center, and its $z$ value must track the average z values of the points in the surrounding ring.

## Implementing the Accelerations

```
In[∘]:= v0 = 4 Pi;
averageInnerRing[allzs_] := 1/nθ Total[allzs⟦1, All⟧]
a[r_, θ_, allzs_] := v0^2 If[r == nr,
   0, (* no acceleration at the edges *)
   allzs⟦r, Mod[θ, nθ] + 1⟧ + allzs⟦r, Mod[θ - 1, nθ]⟧ + allzs⟦r + 1, θ⟧ +
    If[r == 1, averageInnerRing[allzs], allzs⟦r - 1, θ⟧] - 4 allzs⟦r, θ⟧
  ]
```

## Second-Order Runge-Kutta — Implementation

```
In[∘]:= rungeKutta2[cc_] := (
   curTime = cc⟦1⟧;
   curzs = cc⟦2⟧;
   curvs = cc⟦3⟧;
   newTime = curTime + deltaT;
   zsStar = curzs + curvs deltaT / 2;
   as = Table[a[r, θ, zsStar], {r, 1, nr}, {θ, 0, nθ - 1}];
   newvs = curvs + as deltaT;
   newzs = curzs + (curvs + newvs) deltaT / 2;
   {newTime, newzs, newvs}
  )

rk2Results = NestList[rungeKutta2, initialConditions, steps];

rk2ResultsTransposed = Transpose[rk2Results];
zs = rk2ResultsTransposed⟦2⟧;

Out[∘]=
$Aborted
```

### Animating the 3D Graphics

The default duration of the animation is the duration of our simulation:

In[ ]:=
```
Animate[drumheadGraphic[zs〚step〛],
 {step, 0, steps, 1}, DefaultDuration → tFinal - tInitial]
```

Out[ ]=

step

```
drumheadGraphic[zs〚3772〛]
```