Circular Drumhead Redux

Completed and Analyzed in class, April 15, 2025

This is our twentieth notebook. It builds on the techniques in the nineteenth notebook. The big change is the introduction of polar coordinates, which make it possible to analyze circular drumheads.

Circular Drumhead — Theory

To do the circular drumhead back in the fifteenth notebook, we introduced polar coordinates. The second derivative with respect to both directions takes a distinctly different form in polar coordinates. See https://www.math.ucdavis.edu/~saito/courses/21C.w11/polar-lap.pdf if you want a derivation, and I can tell you that, you almost certainly don't. Just believe the bottom line that our partial differential equation in the interior of the drumhead becomes

$$\frac{\partial^2 z}{\partial t^2} = v_0{}^2\left(\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\,\frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2}\,\frac{\partial^2 z}{\partial \theta^2}\right)$$

Yet More Partial Derivatives

Now that we have polar coordinates, we will have things like this:

```
1
── Derivative[0, 0, 2][z][t, r, θ] // TraditionalForm
r²
```

Out[20]//TraditionalForm=

$$z^{(0,0,2)}(t, r, \theta)$$

The Circular Drumhead Differential Equation

Recopying what was above, you can give Mathematica this differential equation:

$$\frac{\partial^2 z}{\partial t^2} = v_0{}^2\left(\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\,\frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2}\,\frac{\partial^2 z}{\partial \theta^2}\right)$$

```
Module[{v0 = 1}, {Derivative[2, 0, 0][z][t, r, θ] ==
    v0² (Derivative[0, 2, 0][z][t, r, θ] + 1/r Derivative[0, 1, 0][z][t, r, θ] +
       1/r² Derivative[0, 0, 2][z][t, r, θ])}] // TraditionalForm
```

Out[22]//TraditionalForm=

$$z^{(2,0,0)}(t, r, \theta) = \frac{z^{(0,0,2)}(t, r, \theta)}{r^2} + \frac{z^{(0,1,0)}(t, r, \theta)}{r} + z^{(0,2,0)}(t, r, \theta)$$

Adding the Boundary Conditions

For the specification of boundary conditions in polar coordinates, I consulted “Symbolic Solutions of PDEs,” https://reference.wolfram.com/language/tutorial/SymbolicSolutionsOfPDEs.html.en. Specifically, the section titled “The Vibrating Drumhead.”

In[23]:=
```
drumheadRadius = 1;

Module[{v0 = 1},
  {Derivative[2, 0, 0][z][t, r, θ] == v0^2 (Derivative[0, 2, 0][z][t, r, θ] +
        1/r Derivative[0, 1, 0][z][t, r, θ] + 1/r^2 Derivative[0, 0, 2][z][t, r, θ]),
   z[t, drumheadRadius, θ] == 0}] // TraditionalForm
```

Out[24]//TraditionalForm=

$$\left\{z^{(2,0,0)}(t,r,\theta)=\frac{z^{(0,0,2)}(t,r,\theta)}{r^2}+\frac{z^{(0,1,0)}(t,r,\theta)}{r}+z^{(0,2,0)}(t,r,\theta),\,z(t,1,\theta)=0\right\}$$

Adding the Initial Conditions

We are also missing any specification of the initial motion of the drumhead. It isn’t just going to start vibrating by itself. Here is an initial displacement function:

```
amplitude = 1;
radialMode = 2;
angularMode = 1;

f[r_, theta_] := amplitude
```

As you can see, we can specify a mode number in either axis. Add that to the equations and also have the initial velocity be zero: