Position from Velocity

January 16, 2025

Study Sections 4-6 of *EIWL3* before working through this notebook.

Associations

A limitation of Nest and NestList may have already caught your eye: the function you are compounding can only take one argument. When we are studying particle motion, we needs lots of arguments: time, position, velocity. When we are studying the motion of multiple particles in three dimensions, we need positions and velocities for each particle and each of these positions and velocities themselves will have three components! So there is no way we are going to be able to get by with functions that take and return only one number (like the total number of heads in the “Heads or Tails” notebook).

There are lots of ways around this, but the easiest way I can think of is to use Associations. Here is an association that holds the current time and the current position of a particle moving in one dimension:

In[82]:=
```
state = <|"time" → 3600, "position" → 10 000|>
```
Out[82]=
```
<|time → 3600, position → 10 000|>
```

This is a single variable that holds two things! In this association, “time” and “position” are keys, and 3600 and 10000 are their respective values.

Of course I could have used a list like this:

In[83]:=
```
stateAsAList = {3600, 10 000}
```
Out[83]=
```
{3600, 10 000}
```

But then you have to remember that the first element of the list is the time and the second element is the position. The nice thing about associations is that you are forced by the notation to always be specific about which key-value pair of the association you are accessing.

Here is a function that adds 1 to the time and 1 to the position:

In[84]:=
```
addOne[a_] := (
  newTime = a[["time"]] + 1;
  newPosition = a[["position"]] + 1;
  <|"time" → newTime, "position" → newPosition|>
 )
```

Let’s see if it works:

In[85]:=
```
addOne[<|"time" → 3600, "position" → 10 000|>]
```

Out[85]=
```
<|time → 3601, position → 10 001|>
```

So we are now able to use Nest and NestList with functions that have multiple numbers in their arguments and return values, except that the numbers are bundled up in an association to get around the one-argument limitation:

In[86]:=
```
NestList[addOne, <|"time" → 3600, "position" → 10 000|>, 5]
```

Out[86]=
```
{<|time → 3600, position → 10 000|>, <|time → 3601, position → 10 001|>,
 <|time → 3602, position → 10 002|>, <|time → 3603, position → 10 003|>,
 <|time → 3604, position → 10 004|>, <|time → 3605, position → 10 005|>}
```

Velocity

To use the concept of velocity, first we have to define it. If a particle is at x_1 at time t_1 and x_2 at time t_2, then the average velocity is by definition

$$v_{1\text{ to }2,\text{avg}} \equiv \frac{\text{change in position}}{\text{change in time}} = \frac{x_2 - x_1}{t_2 - t_1}$$

Position from Velocity and an Approximation

We can rearrange the definition and learn something:

$$x_2 - x_1 = v_{1\text{ to }2,\text{avg}} * (t_2 - t_1)$$

or

$$x_2 = x_1 + v_{1\text{ to }2,\text{avg}} * (t_2 - t_1)$$

That’s just a rearrangement of the definition, but ***now we are going to make an approximation with consequences: we are going to assume that a good approximation for $v_{1\text{ to }2,\text{avg}}$ is the value of v at the midpoint of the time interval.*** Then we have:

$x_2 = x_1 + v\left(\frac{t_1+t_2}{2}\right) * (t_2 - t_1)$

I am going to introduce another definition, which is just a convenient notation:

$\Delta t \equiv t_2 - t_1$

Notice that

$\frac{t_1+t_2}{2} = t_1 + \frac{\Delta t}{2}$

$x(t_2) = x(t_1) + v\left(t_1 + \frac{\Delta t}{2}\right) * \Delta t$

In the last rearrangement, I introduced a tad more notation:

$x_1 = x(t_1)$

and

$x_2 = x(t_2)$

Numerical Integration

Perhaps you don't see it yet, but this is an extremely powerful equation. It is so powerful, I am just going to write it down again, and then discuss it:

$x(t_2) = x(t_1) + v\left(t_1 + \frac{\Delta t}{2}\right) * \Delta t$

On the right-hand side (RHS) of this equation is the position of the particle at time t_1. Also on the RHS is the velocity function evaluated at a particular time, the midpoint. On the left-hand side (LHS) of this equation is the position of the particle at some later time t_2.

You might complain that we used an approximation to get this equation, but for any reasonable velocity function, this approximation gets better and better if you make Δt smaller and smaller. Since we have computers at our disposal, we can make the time steps as small as we like.

For example, suppose you made $\Delta t = 0.001$ and you wanted to learn the position of the particle at time 3.5 from the position of the particle at time 2.0. Well, you'd just have to compound this equation 1500 times and you'd work your way from 2.0 to 2.001, to 2.002, etc., etc., all the way to 3.448, to 3.449, and finally to 3.5.

I'm not going to prove that the approximation can be made as good as you like in this write-up! You just have to believe: for any reasonable velocity function, this works to any desired precision required of the final position, just as long as you make Δt sufficiently small. If you needed to make it 0.0001 to get your desired precision at 3.5, well, then you'd have to compound the procedure 15,000 times.

This procedure has a fancy name. It is called "numerical integration," and the approximation we are using is called the Midpoint Riemann Sum. It is used instead of the Left Riemann Sum or the Right Riemann Sum, because the midpoint is often a better approximation to $v_{1 \text{ to } 2, \text{avg}}$ than $v(t_1)$ or $v(t_2)$.

Constant Velocity

By far the simplest example is constant velocity. Let's just have a constant velocity of 3 and a time step of 0.01.

```
In[87]:= move[a_] := (newTime = a[["time"]] + 0.1;
   newPosition = a[["position"]] + 3 * 0.1;
   <|"time" → newTime, "position" → newPosition|>
  )
```

By far the simplest example is constant velocity. Let's just have a constant velocity of 3 and a time step of 0.1. Then we need to do 15 steps to get from 2.0 to 3.5. Where should we start the particle? How about at time 2.0 we say it was at position -2.0?

```
In[88]:= NestList[move, <|"time" → 2.0, "position" → -2.0|>, 15]

Out[88]=
{<|time → 2., position → -2.|>, <|time → 2.1, position → -1.7|>,
 <|time → 2.2, position → -1.4|>, <|time → 2.3, position → -1.1|>,
 <|time → 2.4, position → -0.8|>, <|time → 2.5, position → -0.5|>,
 <|time → 2.6, position → -0.2|>, <|time → 2.7, position → 0.1|>,
 <|time → 2.8, position → 0.4|>, <|time → 2.9, position → 0.7|>,
 <|time → 3., position → 1.|>, <|time → 3.1, position → 1.3|>,
 <|time → 3.2, position → 1.6|>, <|time → 3.3, position → 1.9|>,
 <|time → 3.4, position → 2.2|>, <|time → 3.5, position → 2.5|>}
```

By far the simplest example is constant velocity. Let's just have a constant velocity of 3 and a time step of 0.1. Then we need to do 15 steps to get from 2.0 to 3.5. Where should we start the particle? How about at time 2.0 we say it was at position -2.0?

A Bounce

Let's have the particle go at constant velocity of 6 until the time is 3.0, and then bounce and go −6 after that:

In[89]:=
```
bounce[a_] := (
  deltaT = 0.01;
  time = a⟦"time"⟧;
  position = a⟦"position"⟧;
  newTime = time + deltaT;
  midpointTime = time + deltaT / 2;
  midpointVelocity = If[midpointTime < 3.0, 6, -6];
  newPosition = position + midpointVelocity * deltaT;
  <|"time" → newTime, "position" → newPosition|>
 )
```

I cranked down the Δt to 0.1, and so I am going to have to crank up the number of steps to 150, and I am going to suppress the output:

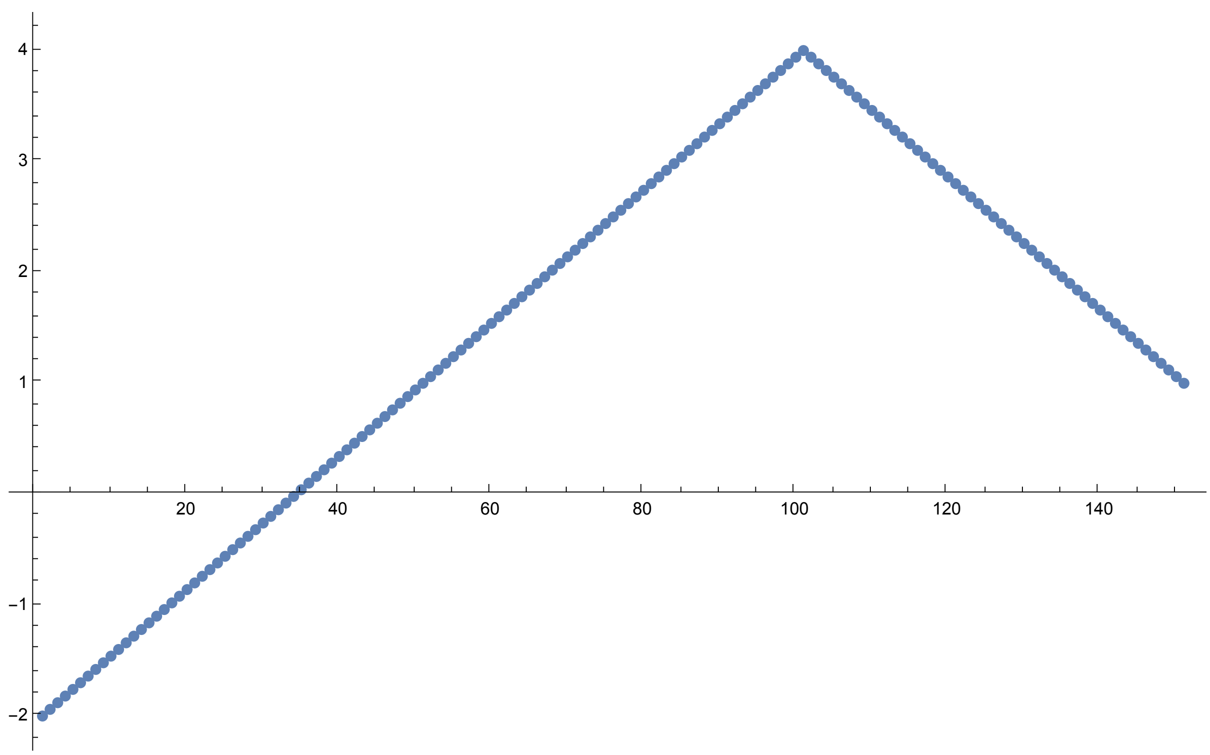
In[90]:=
```
lotsaPositions = NestList[bounce, <|"time" → 2.0, "position" → -2.0|>, 150];
```

Displaying the Bounce

```
allThePositions = lotsaPositions⟦All, "position"⟧;
```

In[96]:=
```
ListPlot[allThePositions]
```

Out[96]=



The bounce at $t = 3.0$ occurs at the 100th time step. The plot might look a little jaggy if viewed at low resolution, but it is entirely smooth when enlarged.