General Runge-Kutta 2 — Best Estimate of Average

When we were trying to generalize our midpoint and trapezoid special cases, I wrote down this formula:

$$a_{\text{avg}} = \left(1 - \frac{1}{2\alpha}\right) a(t_i, x(t_i), v(t_i)) + \frac{1}{2\alpha} a(t^*, x^*, v^*)$$

The question was asked, why those coefficients? I didn't want to get into the algebra, because it would have taken me five minutes and I might have messed it up. Here is the derivation, beginning with an example.

Example

We'll think of y as a function of x rather than a as a function of t. We'll have two points (x_1, y_1) and (x_2, y_2) that determine a line:

```
x1 = 1; y1 = 4; x2 = 3; y2 = 6;
line[x_] := (y2 - y1)/(x2 - x1) (x - x1) + y1
```

Then we'll have some range of interest, x_i to x_f, over which we want the average value of this line:

```
xInitial = 1; xFinal = 5;
averageOverRangeOfInterest = (line[xInitial] + line[xFinal]) / 2
```

Out[140]=

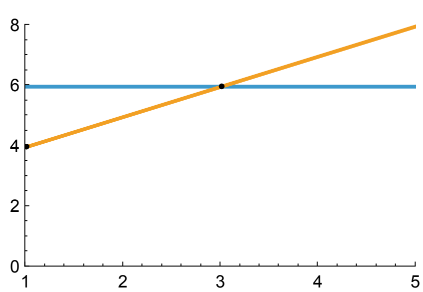
6

We can plot the points, the line, and the average, just to double-check that we haven't messed up:

In[146]:=

```
linePlot = Plot[{averageOverRangeOfInterest, line[x]},
   {x, xInitial, xFinal}, PlotRange → {{xInitial, xFinal}, {0, 8}}];
p1 = {x1, y1};
p2 = {x2, y2};
pointPlot = Graphics[Point[{p1, p2}]];
Show[{linePlot, pointPlot}]
```

Out[150]=



General Case

Having done a little example of (a) getting a line from two points, (x_1, y_1) and (x_2, y_2), and then (b) computing the average value of this line over a range of interest, we can do it fully generally. The line obeys:

$$y(x) = \frac{y_2-y_1}{x_2-x_1}(x - x_1) + y_1$$

A trapezoid approximation is a perfect estimate of a line's average, so we just evaluate $y(x)$ at x_i and x_f and compute the average:

$$y_{\text{avg}} = \frac{\frac{y_2-y_1}{x_2-x_1}(x_i-x_1)+y_1+\frac{y_2-y_1}{x_2-x_1}(x_f-x_1)+y_1}{2}$$

We have a nice simplification if we choose $x_1 = x_i$, which is in-line with what second-order Runge-Kutta does:

$$y_{\text{avg}} = \frac{y_1+\frac{y_2-y_1}{x_2-x_i}(x_f-x_i)+y_1}{2}$$

Also, if we rewrite $x_2 = x_i + \alpha\,(x_f - x_i)$, we get lots more simplification:

$$y_{\text{avg}} = \frac{y_1+\frac{y_2-y_1}{x_i+\alpha\,(x_f-x_i)-x_i}(x_f-x_i)+y_1}{2} = y_1 + \frac{y_2-y_1}{2\,\alpha} = \left(1 - \frac{1}{2\,\alpha}\right) y_1 + \frac{1}{2\,\alpha}\, y_2$$

Application to General Runge-Kutta 2

This is exactly what was claimed to be the best estimate of the average when we did General Second-Order Runge-Kutta.

Just put in for y_1 the value of $a(t_i, x(t_i), v(t_i))$, and for y_i the value of $a(t^*, x^*, v^*)$.

You get the best estimate of a_{avg} that you can get with only two evaluations of a:

$$a_{\text{avg}} = \left(1 - \frac{1}{2\,\alpha}\right) a(t_i, x(t_i), v(t_i)) + \frac{1}{2\,\alpha}\, a(t^*, x^*, v^*)$$

Our standard special cases are $\alpha = \frac{1}{2}$ which gives midpoint, and $\alpha = 1$ which gives a trapezoid.