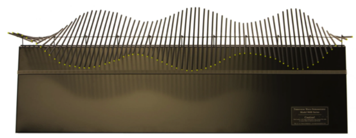
The Second Derivative

March 21, 2025

It is time to introduce the second derivative, and torsion waves are a great place to do it. Here again is our apparatus:



Each rod has an angle, and it has a new angle at every time step. I am going to try to be consistent and write $\theta_j(t_i)$. In other words, if there are 72 rods, j is going to be the index that through the range 1 to 72. Meanwhile, i is going to be the index that goes through the time steps.

Derivatives With Respect to Time

Here is why we defined the average angular velocity for rod j from time t_i to t_{i+1}:

$$\omega_{j,i\to i+1,\text{avg}} \equiv \frac{\theta_j(t_{i+1})-\theta_j(t_i)}{t_{i+1}-t_i}$$

Another way you might write this using $t_{i+1} = t_i + \Delta t$ is $\omega_{j,i\to i+1,\text{avg}} = \frac{\theta_j(t_i+\Delta t)-\theta_j(t_i)}{\Delta t}$.

and it is helpful to think of this loosely as "$\omega_j(t_{i+1/2})$". We also have: $\omega_{j,i-1\to i,\text{avg}} = \frac{\theta_j(t_i)-\theta_j(t_i-\Delta t)}{\Delta t}$.

and it is helpful to think of that loosely as "$\omega_j(t_{i-1/2})$". If we want to get the average angular acceleration, we compute the rate of change of angular velocity as follows:

$$\alpha_{j,\text{avg}}(t_i) \equiv \frac{"\omega_j(t_{i+1/2})"-"\omega_j(t_{i-1/2})"}{\Delta t} = \frac{\frac{\theta_j(t_i+\Delta t)-\theta_j(t_i)}{\Delta t}-\frac{\theta_j(t_i)-\theta_j(t_i-\Delta t)}{\Delta t}}{\Delta t} = \frac{\theta_j(t_i+\Delta t)-2\,\theta_j(t_i)+\theta_j(t_i-\Delta t)}{(\Delta t)^2}$$

Equivalently, I could have written:

$$\alpha_{j,\text{avg}}(t_i) = \frac{\theta_j(t_{i+1})-2\,\theta_j(t_i)+\theta_j(t_{i-1})}{(\Delta t)^2}$$

Of course, this is an average velocity, and if you are in a calculus course, you learn how to take the limit that $\Delta t \to 0$ at which point the right-hand side becomes the second derivative, and the left-hand side is what we call instantaneous acceleration.

The second derivative here is a second derivative with respect to the time coordinate, t.

Derivatives With Respect to Space

Let’s inspect more closely, just as an example, the formula α_3 we had in the “Torsion Saves—Theory” derivation:

$$\alpha_3 = -\omega_0{}^2(\theta_3 - \theta_2) + \omega_0{}^2(\theta_4 - \theta_3)$$

Remember that $\omega_0{}^2 = \frac{\kappa}{I}$ where κ is the proportionality constant that says how much torque the stainless steel wire creates in opposition to twist, and I is the moment of inertia of the rod. So putting some of these things back in, and letting $j = 3$ so that we have a formula that is true for any rod (except for the end ones that are a little bit special), we have

$$\alpha_j(t_i) = -\tfrac{\kappa}{I}\left(\theta_j(t_i) - \theta_{j-1}(t_i)\right) + \tfrac{\kappa}{I}\left(\theta_{j+1}(t_i) - \theta_j(t_i)\right)$$

I also put back in the times so that you don’t forget how much complexity is hidden in the notation.

Now I need to argue two more things: (1) That if you have a piece of stainless steel wire, and you cut it in half, it will resist twisting twice as much; (2) That if you have a sheet of material and you cut it into rods that are half as wide, each rod will have half the moment of inertia.

In other words, if you have a continuous sheet, and you are trying to imagine that it has a backbone on which the continuous sheet is carried, and you are imagining that this backbone is cut up into little bits of length Δx, then the κ is actually inversely proportional to Δx and the I is actually proportional to Δx. In other words, if we want to capture the correct transition to an infinite number of infinitely narrow rods, we should have written:

$$\alpha_j(t_i) = -\tfrac{\kappa/\Delta x}{I \Delta x}\left(\theta_j(t_i) - \theta_{j-1}(t_i)\right) + \tfrac{\kappa/\Delta x}{I \Delta x}\left(\theta_{j+1}(t_i) - \theta_j(t_i)\right)$$

Now let’s re-introduce $\omega_0{}^2 = \frac{\kappa}{I}$ and simplify, and we see

$$\alpha_j(t_i) = \omega_0{}^2\,\frac{\theta_{j+1}(t_i) - 2\,\theta_j(t_i) + \theta_{j-1}(t_i)}{(\Delta x)^2}$$

If you look closely at the right-hand side, you will see that the combination of θ’s is shockingly similar to the second derivative with respect to time, which was

$$\frac{\theta_j(t_{i+1}) - 2\,\theta_j(t_i) + \theta_j(t_{i-1})}{(\Delta t)^2}$$

The difference is that the index j is changing instead of the index i and what is in the denominator is the rod-spacing squared instead of the time-step squared. This new combination is called the second derivative with respect to the space coordinate, x.

Putting the Equations Together

We have two expressions for $\alpha_j(t_i)$ and putting them equal to each other, we see that what we are having Mathematica do is doing is solving these equations:

$$\frac{\theta_j(t_{i+1})-2\,\theta_j(t_i)+\theta_j(t_{i-1})}{(\Delta t)^2} = \omega_0{}^2\ \frac{\theta_{j+1}(t_i)-2\,\theta_j(t_i)+\theta_{j-1}(t_i)}{(\Delta x)^2}$$

These equations tell how to get the θ values at a later time from the θ values at the current and previous times. If you don't see this, let me re-arrange so that you can see that this is really a way of stepping forward in time:

$$\theta_j(t_{i+1}) = 2\,\theta_j(t_i) - \theta_j(t_{i-1}) + \omega_0{}^2(\Delta t)^2\ \frac{\theta_{j+1}(t_i)-2\,\theta_j(t_i)+\theta_{j-1}(t_i)}{(\Delta x)^2}$$

Runge-Kutta 2 is a tad fancier because it introduces a time t^* which is part-way from t_i to t_{i+1}, but it is only doing that for sake of efficiency, and it is essentially doing this work.

That's enough on second derivatives for now, except I guess I should introduce the very fancy notation that is used when the time steps Δt and the spacing Δx is taken to zero. Then the way to write

$$\lim_{\Delta t\to 0}\ \frac{\theta_j(t_{i+1})-2\,\theta_j(t_i)+\theta_j(t_{i-1})}{(\Delta t)^2} = \omega_0{}^2\lim_{\Delta x\to 0}\ \frac{\theta_{j+1}(t_i)-2\,\theta_j(t_i)+\theta_{j-1}(t_i)}{(\Delta x)^2}$$

is

$$\frac{\partial^2\theta}{\partial t^2} = \omega_0{}^2\ \frac{\partial^2\theta}{\partial x^2}$$

This means no more and no less than the equations you are already familiar with, but the notation is becoming ever more abstract and compact.

When you see physicists and mathematicians write down "wave equations" this is the notation they use.