General Second-Order Runge-Kutta — Damped Oscillation

Done in class, January 31, 2025

This is the fourth notebook for you to finish in-class.

Damped Oscillator

Problem Description

In[ ]:=
```
springConstant = 20;
force[x_, v_] := -springConstant x - v
mass = 5;
a[x_, v_] := force[x, v] / mass;
tInitial = 0;
tFinal = 10 Pi;
steps = 4800;
deltaT = (tFinal - tInitial) / steps;
```

Initial Conditions

Let’s stretch this spring to $x_{\text{initial}} = 25$ and let it go with no initial velocity, so $v_{\text{initial}} = 0.0$.

In[ ]:=
```
xInitial = 25.0;
vInitial = 0.0;
initialConditions = {tInitial, xInitial, vInitial};
```

General Second-Order Runge-Kutta — Theory — Summary

This is a more general version of Second-Order Runge-Kutta, which has a parameter α, typically chosen as $\alpha = \frac{1}{2}$ or $\alpha = 1$:

$$t^* = t + \alpha \Delta t$$

$$x^* = x(t_i) + v(t_i) \cdot \alpha \Delta t$$

$$v^* = v(t_i) + a(t_i, x(t_i), v(t_i)) \cdot \alpha \Delta t$$

$$t_{i+1} = t_i + \Delta t$$

$$v(t_{i+1}) = v(t_i) + \left(\left(1 - \frac{1}{2\alpha}\right) a(t_i, x(t_i), v(t_i)) + \frac{1}{2\alpha} a(t^*, x^*, v^*)\right) \cdot \Delta t$$

$$x(t_{i+1}) = x(t_i) + (v(t_i) + v(t_{i+1})) \frac{\Delta t}{2}$$

General Second-Order Runge-Kutta — Implementation

```
alpha = 1;
rungeKutta2[cc_] := (
  (* Extract time, position, and velocity from the list. *)
  {newTime, newPosition, newVelocity}
 )
```

Displaying Damped Oscillation

Nest the procedure and then transpose the results to produce position and velocity plots:

```
In[◦]:= rk2Results = NestList[rungeKutta2, initialConditions, steps];
rk2ResultsTransposed = Transpose[rk2Results];
positionPlot = ListPlot[Transpose[rk2ResultsTransposed[[{1, 2}]]]]
```

```
In[◦]:= positions = rk2ResultsTransposed[[2]];
```

```
In[◦]:= Animate[NumberLinePlot[positions[[step]], PlotRange → {-25, 25}], {step, 0, steps, 1}]
```

Conclusion / Commentary

Our oscillator now has the force law $F(x) = -20\,x - v$. Nowhere did we put sines or cosines or decaying exponential functions into the problem! Damped oscillation (for which the position is the product of a cosine and a decaying exponential) has emerged from the force law.