Drumheads Redux

Completed and Analyzed in class, April 15, 2025

This is our nineteenth notebook. It builds on the techniques in the eighteenth notebook. The difference between a guitar string and a drumhead is just the number of dimensions. After graduating to two dimensions, it will (I hope) be fairly apparent how to go to three dimensions.

I am hopeful that we can cover both the rectangular and circular drumhead in one class. If we only did the rectangular one, that would be too modest a goal. It is hardly any different than the guitar string.

Rectangular Drumhead — Theory

Back in the fourteenth notebook we had these acceleration formulas for drumheads:

$$a_{j,k} = v_0{}^2\left(z_{j,k+1} + z_{j,k-1} + z_{j+1,k} + z_{j-1,k} - 4\,z_{j,k}\right)$$

The continuum version of this equation is:

$$\frac{\partial^2 z}{\partial t^2} = v_0{}^2\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right)$$

Again, let's be clear about the dependent and independent variables. The time t, is most definitely an independent variable. The drumhead is stretched along the x- and y-axes. Those are the independent variables. The dependent variable is the displacement, which we are putting in the z-direction. To summarize, the dependent variable is z, and we want to find it as a function of three independent variables, so we are looking for $z(t, x, y)$.

More Partial Derivatives

Now that we have three independent variables, we will have things like this:

In[20]:= `Derivative[0, 0, 2][z][t, x, y] // TraditionalForm`

Out[20]//TraditionalForm=
$$z^{(0,0,2)}(t, x, y)$$

The Rectangular Drumhead Differential Equation

Recopying what was above, you can give Mathematica this differential equation:

$$\frac{\partial^2 z}{\partial t^2} = v_0{}^2\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right)$$

In[21]:= `Module[{v0 = 1}, Derivative[2, 0, 0][z][t, x, y] ==`
`    v0² (Derivative[0, 2, 0][z][t, x, y] + Derivative[0, 0, 2][z][t, x, y])]`

Out[21]=

$z^{(2,0,0)}[t, x, y] == z^{(0,0,2)}[t, x, y] + z^{(0,2,0)}[t, x, y]$

Adding the Boundary Conditions

We are missing the boundary conditions at the edge of the drumhead.

In[22]:= `lengthX = 2;`
`lengthY = 3 / 2;`

`Module[{v0 = 1}, {Derivative[2, 0, 0][z][t, x, y] ==`
`    v0² (Derivative[0, 2, 0][z][t, x, y] + Derivative[0, 0, 2][z][t, x, y]),`
`   z[t, x, 0] == 0, z[t, 0, y] == 0, z[t, x, lengthY] == 0,`
`   z[t, lengthX, y] == 0}] // TraditionalForm`

Out[24]//TraditionalForm=

$$\left\{z^{(2,0,0)}(t, x, y) = z^{(0,0,2)}(t, x, y) + z^{(0,2,0)}(t, x, y), z(t, x, 0) = 0, z(t, 0, y) = 0, z\left(t, x, \frac{3}{2}\right) = 0, z(t, 2, y) = 0\right\}$$

Adding the Initial Conditions

We are also missing any specification of the initial motion of the drumhead. It isn't just going to start vibrating by itself. Here is an initial displacement function:

In[25]:= `amplitude = 1;`
`modeX = 2;`
`modeY = 1;`

`lengthX = 2;`
`lengthY = 3 / 2;`

`f[x_, y_] := amplitude Sin[modeX Pi x / lengthX] Sin[modeY Pi y / lengthY]`

As you can see, we can specify a mode number in either axis. Add that to the equations and also have the initial velocity be zero:

In[31]:= `rectangularDrumheadProblem = Module[{v0 = 1}, {Derivative[2, 0, 0][z][t, x, y] ==`
`    v0² (Derivative[0, 2, 0][z][t, x, y] + Derivative[0, 0, 2][z][t, x, y]),`
`   z[t, x, 0] == 0, z[t, 0, y] == 0, z[t, x, lengthY] == 0, z[t, lengthX, y] == 0,`
`   z[0, x, y] == f[x, y], Derivative[1, 0, 0][z][0, x, y] == 0}];`

In[32]:= `rectangularDrumheadProblem // TraditionalForm`

Out[32]//TraditionalForm=

$$\left\{ z^{(2,0,0)}(t, x, y) = z^{(0,0,2)}(t, x, y) + z^{(0,2,0)}(t, x, y), z(t, x, 0) = 0, z(t, 0, y) = 0, \right.$$

$$\left. z\left(t, x, \frac{3}{2}\right) = 0, z(t, 2, y) = 0, z(0, x, y) = \sin(\pi x) \sin\left(\frac{2\pi y}{3}\right), z^{(1,0,0)}(0, x, y) = 0 \right\}$$

Making Mathematica Solve the Rectangular Problem

This problem has an exact solution. So we can use NDSolve[] rather than NDSolve[]:

In[33]:= `rectangularDrumheadSolutionRule = NDSolve[`
`    rectangularDrumheadProblem, z, {t, 0, 10}, {x, 0, lengthX}, {y, 0, lengthY}]⟦1⟧`

Out[33]=

$$\left\{ z \rightarrow \text{InterpolatingFunction}\left[\boxed{\begin{array}{l} \text{Domain: } \{\{0., 10.\}, \{0., 2.\}, \{0., 1.5\}\} \\ \text{Output: scalar} \end{array}} \right] \right\}$$

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In[34]:= `rectangularDrumheadSolution[t_, x_, y_] =`
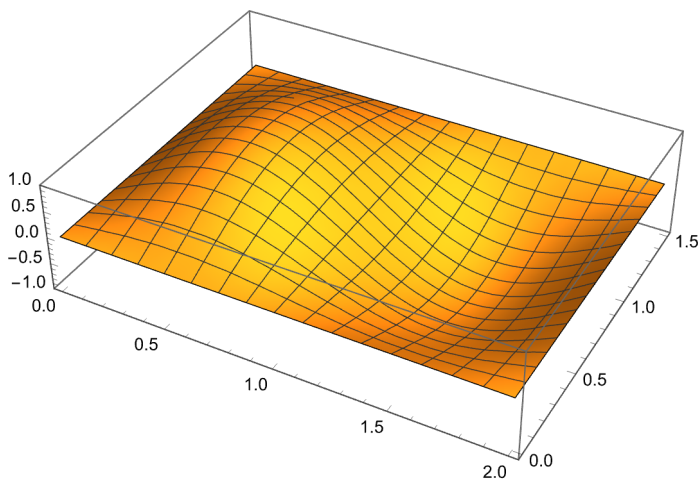`    z[t, x, y] /. rectangularDrumheadSolutionRule`

Out[34]=

$$\text{InterpolatingFunction}\left[\boxed{\begin{array}{l} \text{Domain: } \{\{0., 10.\}, \{0., 2.\}, \{0., 1.5\}\} \\ \text{Output: scalar} \end{array}} \right] [t, x, y]$$

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Plotting the Rectangular Solution at $t = 1/8$

In[58]:= `Plot3D[rectangularDrumheadSolution[1/8, x, y], {x, 0, lengthX}, {y, 0, lengthY},`
`    PlotRange → {Automatic, Automatic, {-1.0, 1.0}}, BoxRatios → {lengthX, lengthY, 0.5}]`
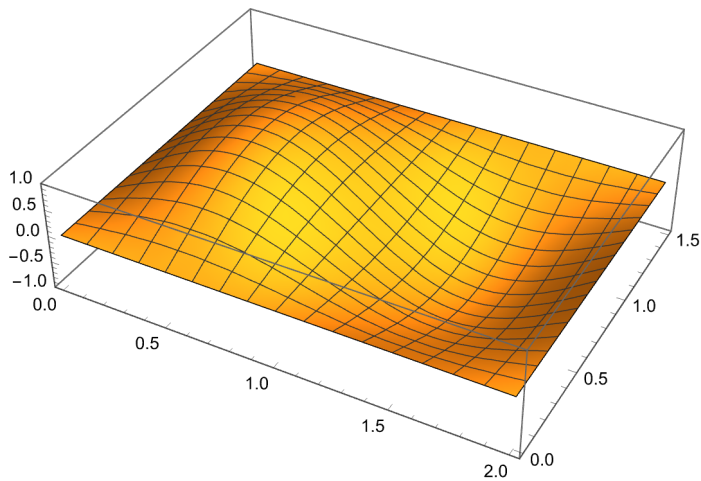
Out[58]=

Animating the Rectangular Solution

In[59]:= ```plots = Table[Plot3D[rectangularDrumheadSolution[t / 15, x, y], {x, 0, lengthX},
    {y, 0, lengthY}, PlotRange → {Automatic, Automatic, {-1.0, 1.0}},
    BoxRatios → {lengthX, lengthY, 0.5}], {t, 0, 150}];```

In[60]:= ```plots[[3]]```

Out[60]=

In[61]:= ```Animate[plots[[step]], {step, 1, 151, 1},
    DefaultDuration → 10, AnimationRunning → False, RefreshRate → 15]```

Out[61]=