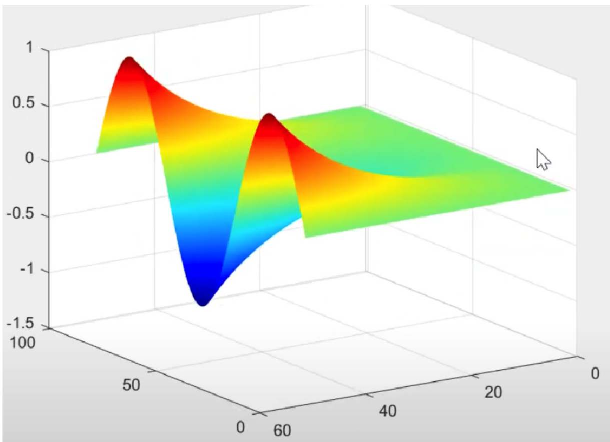
Diffusion in Two Dimensions — Steady State

Completed and Analyzed in class, April 18, 2025

This is our twenty-first notebook. We just did one-dimensional diffusion. Now we are graduating to two-dimensional diffusion. Here is an example (this is a two-dimensional problem, not a three-dimensional one—the third dimension in the plot is just helping to visualize the temperature):



This is a solution of the type of equation that we are currently studying. This is a plate that has alternating regions of hot and cold along one side, and is at an intermediate temperature along the other three sides. The full reference is here: https://youtu.be/2aJ3fFwET68.

We are going to specify the equation on the interior and the boundary conditions, and Mathematica is going to figure out the steady-state solution in the interior.

If you want to see more of the wide variety of equations that can be solved using the techniques we are using, try https://www.cfm.brown.edu/people/dobrush/am34/Mathematica/ch6/parabolic.html.

Two-Dimensional Diffusion — Steady-State Theory

In one dimension, we had:

$$c(T, x)\,\frac{dT}{dt} = \sigma(x)\,\frac{d^2 T}{dx^2}$$

The obvious generalization (I won’t bore you with another derivation of combinations of second derivatives) to two dimensions is:

$$c(T, x, y)\,\frac{dT}{dt} = \sigma(x, y)\left(\frac{d^2 T}{dx^2} + \frac{d^2 T}{dy^2}\right)$$

It can be very interesting to see the time-dependent way that the system settles down to equilibrium, but now I want to make a huge simplification, which is that the system has already settled down to an equilibrium.

As long as the external sources of heat are supplied in a steady (unchanging) way, the system will eventually settle down. Once it settles down, $\frac{dT}{dt} = 0$, and this is true everywhere in x and y. So for the steady state, our equations become:

$$0 = \sigma(x, y)\left(\frac{d^2 T}{dx^2} + \frac{d^2 T}{dy^2}\right)$$

and T, whatever it is, is unchanging, and so it is no longer a function of t. It is only a function of x and y. We just have this as what we specify to Mathematica:

In[85]:=
```
Module[{sigma = 1},
  { 0 == sigma (Derivative[2, 0][temp][x, y] + Derivative[0, 2][temp][x, y])}] //
 TraditionalForm
```

Out[85]//TraditionalForm=

$$\{0 = \text{temp}^{(0,2)}(x, y) + \text{temp}^{(2,0)}(x, y)\}$$

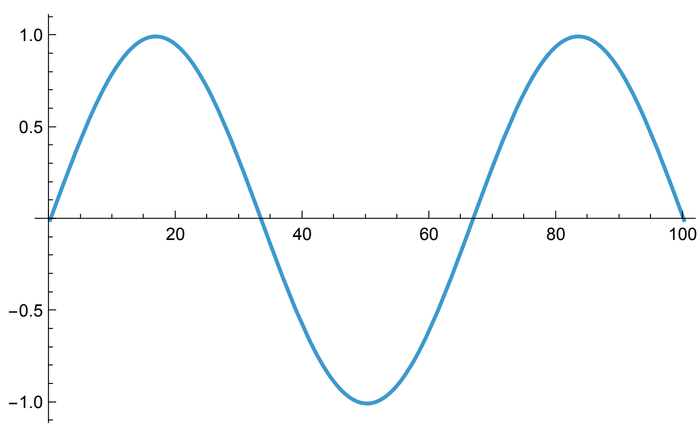
Adding the Boundary Conditions

Let’s make boundary conditions like the one pictured:

In[86]:=
```
lengthX = 100;
lengthY = 60;
amplitude = 1;

mode = 3;
f[x_] := amplitude Sin[mode Pi x / lengthX]
Plot[f[x], {x, 0, lengthX}]
```

Out[91]=



In[92]:=

In[93]:=
```
Module[{sigma = 1},
  { 0 == sigma (Derivative[2, 0][temp][x, y] + Derivative[0, 2][temp][x, y]),
   temp[x, 0] == f[x], temp[x, lengthY] == 0, temp[0, y] == 0, temp[lengthX, y] == 0}] ;
```

No Initial Conditions

This is a steady-state problem. Time is no longer involved. We do not need to specify any initial conditions.

In[94]:=
```
twoDimensionalDiffusionProblem = Module[{sigma = 1},
  { 0 == sigma (Derivative[2, 0][temp][x, y] + Derivative[0, 2][temp][x, y]),
   temp[x, 0] == f[x], temp[x, lengthY] == 0, temp[0, y] == 0, temp[lengthX, y] == 0}]
```

Out[94]=

$$\left\{0 == \text{temp}^{(0,2)}[x, y] + \text{temp}^{(2,0)}[x, y],\ \text{temp}[x, 0] == \text{Sin}\left[\frac{3\pi x}{100}\right],\right.$$
$$\left.\text{temp}[x, 60] == 0,\ \text{temp}[0, y] == 0,\ \text{temp}[100, y] == 0\right\}$$

Making Mathematica Numerically Solve the Problem

In[95]:=
```
twoDimensionalDiffusionSolutionRule =
 NDSolve[twoDimensionalDiffusionProblem, temp, {x, 0, lengthX}, {y, 0, lengthY}][[1]]
```

Out[95]=

{temp → InterpolatingFunction[Domain: {{0., 100.}, {0., 60.}} Output: scalar]}

Convert the rule to a function:

In[96]:=
```
twoDimensionalDiffusionSolution[t_, x_] =
 temp[x, y] /. twoDimensionalDiffusionSolutionRule
```

Out[96]=

InterpolatingFunction[Domain: {{0., 100.}, {0., 60.}} Output: scalar][x, y]

Plotting the Numerical Solution

In[97]:=
```
Plot3D[twoDimensionalDiffusionSolution[x, y], {x, 0, lengthX}, {y, 0, lengthY}]
```

Out[97]=