Magnetic Resonance I — Setting Up the Problem

Magnetic resonance is the process of getting a spin to flip by applying an oscillating magnetic field to it.

An important piece of technology combining magnetic resonance and nuclear physics is nuclear magnetic resonance imaging (various acronyms for this are NMR, NMRI, and MRI). The spin that is flipped is the proton which is the simple nucleus of the hydrogen atom. Hydrogen is abundant in water and fat molecules, so a magnetic resonance image is a map of the density of water and fat which varies from tissue to tissue.

To understand the basic, brilliant idea underlying nuclear magnetic resonance imaging, you need to understand how you flip a spin by applying an oscillating magnetic field, and that is going to take us right back to spin-1/2 systems. Protons and electrons are both spin-1/2 particles, and either can be flipped using magnetic resonance, but in medical MRI machines, the resonance is tuned to flip the proton spin.

Review of Magnetic Moment

Back in Chapter 6, we learned that if something is charged and it is spinning, then it is also a little magnet. We quantify the strength and direction of the little magnet, in a quantity we call the magnetic moment, which in symbols is $\vec{\mu}$ and the direction of $\vec{\mu}$ is parallel to the spin $\vec{S}$ if the particle is positively charged, and negative if it is negatively charged. The proportionality constant includes a flexible fudge factor g which account for the fact that dimensional analysis and vector analysis doesn't tell us the exact relation. For the proton, which is positively charged, we have

$$\vec{\mu} = g \frac{e}{2m} \vec{S}$$

Where $e = 1.6 \times 10^{-19}$ C is a (positive) constant, m is the mass of the proton, and g, the fudge factor, is 5.6 for the proton.

Torque on a Magnetic Moment in a Magnetic Field

The torque on a magnetic moment in a magnetic field aligns the moment with the field.

If the field is in the z-direction, the magnitude of the torque is

$$|\vec{\tau}| = |\vec{\mu}|\,|\vec{B}|\sin\theta$$

where θ is how much the magnetic moment is tipped away from the z-axis.

Potential Energy of a Magnetic Moment in a Magnetic Field

The potential energy that corresponds to the above torque is

$$V = -|\vec{\mu}|\,|\vec{B}|\cos\theta$$

The minus sign is because the potential energy is lowest when the magnetic moment is aligned with the field ($\theta = 0\,°$), and highest when the magnetic moment is anti-aligned with the field ($\theta = 180\,°$).

It will occur to you that the above is the same as the dot product, so we could write

$$V = -\vec{\mu}\cdot\vec{B}$$

and combining that with $\vec{\mu} = g\,\frac{e}{2m}\,\vec{S}$, we have

$$V = -g\,\frac{e}{2m}\,\vec{S}\cdot\vec{B}$$

This is the vector way of writing the equation for V, and we can use it no matter which way $\vec{B}$ points.

Schrodinger's Equation for a Magnetic Moment in a Magnetic Field

We aren't going to worry about the particle with the spin and the magnetic moment moving around from one place to another. We are just going to study how it reorients itself. In that case, $V = -g\,\frac{e}{2m}\,\vec{S}\cdot\vec{B}$ is everything we need to know about the particle's energy and we can write down Schrodinger's equation, it is

$$i\hbar\,\frac{d}{dt}\begin{pmatrix}\psi_+(t)\\ \psi_-(t)\end{pmatrix} = -g\,\frac{e}{2m}\,\vec{S}\cdot\vec{B}\begin{pmatrix}\psi_+(t)\\ \psi_-(t)\end{pmatrix}$$

To make this equation a little more intelligible, let's remember what it simplifies to when $\vec{B}$ has magnitude B_0 and points in the z-direction. In that case it is:

$$i\hbar\,\frac{d}{dt}\begin{pmatrix}\psi_+(t)\\ \psi_-(t)\end{pmatrix} = -g\,\frac{e}{2m}\,S_z\,B_0\begin{pmatrix}\psi_+(t)\\ \psi_-(t)\end{pmatrix} = -g\,\frac{e}{2m}\,B_0\,\frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\psi_+(t)\\ \psi_-(t)\end{pmatrix}$$

Let's take the mess $g\,\frac{e}{2m}\,B_0$ and give it the name, ω_0. It is after all, a frequency (how can you quickly deduce that?). The $\hbar$ cancels and we just have

$$i\,\frac{d}{dt}\begin{pmatrix}\psi_+(t)\\ \psi_-(t)\end{pmatrix} = -\omega_0\,\frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\psi_+(t)\\ \psi_-(t)\end{pmatrix}$$

The Resonance Idea

If the magnetic field pointed mostly in the z direction, *but also just a little in the x direction,* then we would have:

$$i\hbar \frac{d}{dt}\begin{pmatrix}\psi_+(t)\\ \psi_-(t)\end{pmatrix} = -g\,\frac{e}{2m}\,(S_z B_0 + S_x B_1)\begin{pmatrix}\psi_+(t)\\ \psi_-(t)\end{pmatrix}$$

with $B_1 \ll B_0$.

Here is the resonance idea. Instead of just letting the magnetic field point a little in the x direction, let the amount that it points in the x direction oscillate with angular frequency ω! Then we have:

$$i\hbar \frac{d}{dt}\begin{pmatrix}\psi_+(t)\\ \psi_-(t)\end{pmatrix} = -g\,\frac{e}{2m}\,(S_z B_0 + S_x B_1 \cos\omega t)\begin{pmatrix}\psi_+(t)\\ \psi_-(t)\end{pmatrix}$$

where B_0 and B_1 are constants and we still have $B_1 \ll B_0$, and the oscillation with angular frequency ω is accounted for in the $\cos\omega t$ factor.

Just like we took the mess $g\,\frac{e}{2m}\,B_0$ and gave it the name, ω_0, let's take the mess $g\,\frac{e}{2m}\,B_1$ and give it the name, ω_1. Then we have:

$$i\hbar \frac{d}{dt}\begin{pmatrix}\psi_+(t)\\ \psi_-(t)\end{pmatrix} = (S_z\,\omega_0 + S_x\,\omega_1 \cos\omega t)\begin{pmatrix}\psi_+(t)\\ \psi_-(t)\end{pmatrix}$$

You probably remembered that $S_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ when I used it in the previous section, but you may have forgotten that $S_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ and $S_y = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$. These matrices were forced upon us by the behavior of electrons in Stern-Gerlach apparatus.[1] These choices aren't unique, it turns out. They are just the standard ones. Other choices give the same answers, and we will just use these. Anyway, ...

Using the formulae for S_z and S_x, again the $\hbar$ cancels, and we have

$$i\,\frac{d}{dt}\begin{pmatrix}\psi_+(t)\\ \psi_-(t)\end{pmatrix} = -\omega_0\,\frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} - \omega_1 \cos\omega t\cdot\frac{1}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\psi_+(t)\\ \psi_-(t)\end{pmatrix} = -\frac{1}{2}\begin{pmatrix}\omega_0 & \omega_1\cos\omega t\\ \omega_1\cos\omega t & -\omega_0\end{pmatrix}\begin{pmatrix}\psi_+(t)\\ \psi_-(t)\end{pmatrix}$$

This is the differential equation we have to solve. It turns out we can't solve it exactly, even with good tricks.

Keeping it All Straight

It was a lot of work just to get to the differential equation we have to solve and this is a good place to stop! Let’s review so we know where we will start. The differential equation is:

$$i \frac{d}{dt} \begin{pmatrix} \psi_+(t) \\ \psi_-(t) \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} \omega_0 & \omega_1 \cos \omega t \\ \omega_1 \cos \omega t & -\omega_0 \end{pmatrix} \begin{pmatrix} \psi_+(t) \\ \psi_-(t) \end{pmatrix}$$

Let’s remind ourselves what is in it. $\psi_+(t)$ is the probability amplitude for finding the proton in the $+z$ state. If we solve the equation and find it, then $P_+(t) = |\psi_+(t)|^2$ will be the probability that the proton is in the $+z$ state (and similarly for $P_-(t) = 1 - P_+(t) = |\psi_-(t)|^2$).

The goal of magnetic resonance imaging is to get a lot of protons in your body first going to the $+z$ state and then to the $-z$ state and to then detect this by the radio signals that all these simultaneously oscillating protons collectively emit.

Part of keeping it all straight is to remember that there are three frequencies in the problem. The “zeroth” frequency came from the amount of magnetic field in the z direction, which is assumed to be large compared to the amount of magnetic field in the x direction. We had for the zeroth and first frequencies:

$$\omega_0 \equiv g \frac{e}{2m} B_0$$

$$\omega_1 \equiv g \frac{e}{2m} B_1$$

$$\omega_0 \gg \omega_1$$

The third frequency, ω, which has no subscript in our notation, is the angular frequency at which the magnetic field in the x direction oscillates.

Even if you can’t keep it all straight how we got to this point, you need to remember that this is where we are, that we need to solve this equation, at least approximately, and thereby learn the probability amplitudes $\psi_+(t)$ and $\psi_-(t)$, and that once we do, we will know $P_+(t)$ and $P_-(t)$.