Quantum Mechanics — Exam 1

Monday, Feb. 2, 2026

This exam has two sections. The first section consists of the first three problems and is on complex variables. The second section is on classical and quantum-mechanical wave interference.

THIS EXAM MAY BE TOO LONG. DO NOT DESPAIR. JUST DO WHAT YOU CAN BETWEEN 9:30 and 10:50. IF YOU BOTH CAN STAY LONGER, AND YOU BOTH AGREE TO DO SO AT 10:45, THEN GO UNTIL 11:15. TO COMPENSATE FOR THE LENGTH, THE SIXTH PROBLEM IS EXTRA CREDIT.

1. Double and Triple Angle Formulas

(a) Starting with $\left(e^{i\theta}\right)^2 = e^{2i\theta}$ show that $\cos^2\theta - \sin^2\theta = \cos 2\theta$ and that $2\cos\theta\sin\theta = \sin 2\theta$.

(b) Use what you just showed in (a) to show that $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$ and $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$. Of course, you can use the defining fact about the cosine and sine, which is that they locate the x and y coordinates on the unit circle, $x^2 + y^2 = 1$, and therefore $\cos^2\theta + \sin^2\theta = 1$.

(c) Use what you just showed in (b) to argue that $\cos^2\theta$ actually has period π, rather than 2π, and then give its average value over one full period. Also, give the average value of $\sin^2\theta$ over one full period.

(d) Very similarly to what you did in (a), start with $\left(e^{i\theta}\right)^3 = e^{3i\theta}$ and find some combination of things that is $\cos 3\theta$ and another that is $\sin 3\theta$.

2. Trigonometric Identities Used in Interference

(a) Give an expression for A and another expression for B that only involves their average, $m \equiv \frac{A+B}{2}$, and their difference, $d \equiv A - B$.

(b) Use what you got in (a) to give four expressions for (i) e^{iA}, (ii) e^{-iA}, (iii) e^{iB}, (iv) e^{-iB}. Each of your four expressions will have combinations of m and d in it.

(c) Since $\cos A = \frac{e^{iA}+e^{-iA}}{2}$ and $\cos B = \frac{e^{iB}+e^{-iB}}{2}$, you can use what you derived in (b) to get an expression for $\cos A + \cos B$ in terms of m and d. You should be able to tidy your expression so that it only involves trig functions, not exponentials.

(d) Let $A = \omega t + \phi_1$ and $B = \omega t + \phi_2$. Notice that ωt disappears from the difference d, which we now will call $\Delta\phi$. Let's call the average $\omega t + \phi_{\text{avg}}$.

3. Hyperbolic Trigonometry

Thanks to the exponential function now being able to take complex numbers in the exponent, we have new expressions for $\cos\theta$ and $\sin\theta$, namely: $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$ and $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$.

There is a similar set of functions for "hyperbolic trigonometry." The defining fact about the new functions cosh (pronounced like it sounds) and sinh (pronounced like "cinch"), is that they locate x and y coordinates on the unit hyperbola, $x^2 - y^2 = 1$, so we must have $\cosh^2\alpha - \sinh^2\alpha = 1$.

(a) Show that $\cosh\alpha \equiv \frac{e^{\alpha}+e^{-\alpha}}{2}$ and $\sinh\alpha \equiv \frac{e^{\alpha}-e^{-\alpha}}{2}$ satisfy $\cosh^2\alpha - \sinh^2\alpha = 1$.

(b) For the ordinary trig functions, the derivatives are $\frac{d}{d\theta}\cos\theta = -\sin\theta$ and $\frac{d}{d\theta}\sin\theta = \cos\theta$. What are the corresponding derivatives for the hyperbolic trig functions?

(c) With ordinary trig functions we define $\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$ and $\sec\theta \equiv \frac{1}{\cos\theta}$. We can use chain rule and the trig derivatives mentioned in (b) to get $\frac{d}{d\theta}\tan\theta = \sec^2\theta$. For hyperbolic trig functions, we define $\tanh\alpha \equiv \frac{\sinh\alpha}{\cosh\alpha}$ and $\mathrm{sech}\,\alpha \equiv \frac{1}{\cosh\alpha}$. What is $\frac{d}{d\alpha}\tanh\alpha$?

(d) Derive an expression for $\cosh(\alpha_1 + \alpha_2)$ that is a lot like $\cos(\theta_1 + \theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$.

4. Classical Interference of Four Radiators

Do Feynman Problem 21.7, reproduced below. To discover the maximum, you may have to take $\frac{d}{da}$ and $\frac{d}{db}$ of your expression for the intensity, set both those derivatives to 0, and solve the two resulting equations in two unknowns. Don't take my word for it though. That's just how I would likely start.

21.7 Four vertical dipoles are located at the corners of a horizontal rectangle of sides a, b as shown in Fig. 21-5. If they are driven in phase, at wavelength λ, what minimum values (> 0) should a, b have to produce maximum intensity in the direction $\theta = 30°$ far from the charges?

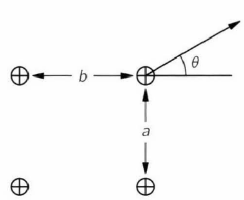

Figure 21-5

5. Double Slit Experiment with Different-Sized Slits

Do Feynman Problem 70.8, reproduced below.

70.8 Surprisingly, large interference effects can occur even when one of the interfering possibilities is not very probable. In the two hole diffraction experiment, if one hole is stopped down so that the probability of getting through is reduced by a factor of 100, show that the arrival probability at a maximum of the pattern is still about 50 percent higher than at a minimum.

Since we don't do the unequal-sized slit very often, I will remind you that

$$P(\theta) = A_1{}^2 + A_2{}^2 + 2\,A_1\,A_2\cos\Delta\phi$$

and Feynman is just asking you to make some statements about this function for the case $A_2 = \frac{1}{10}\,A_1$, and for $\Delta\phi$ of 0 at a maximum, or π at a minimum.

6. Brown and Twiss Intensity Correlation (EXTRA CREDIT MAX 2)

Do Feynman Problem 70.9. The diagram is below and the problem statement is on the next page.

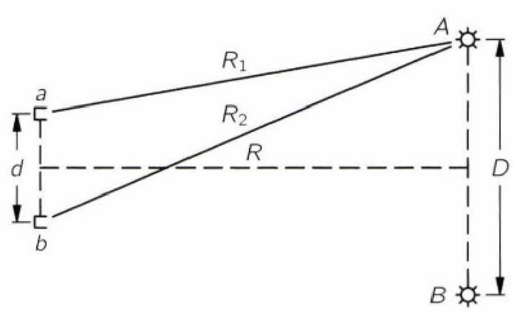

Figure 70-4

70.9 The diameter of the nearest stars is too small to be "seen" with the best telescopes (the angle subtended is less than the resolution of the telescope). The diameter of a star was first measured by Michelson using an optical interferometer. The method just barely works for the nearest stars. In *Nature* **178**, 1046 (1956), Brown and Twiss proposed a new method, called "intensity correlation," for such measurements, and tested their method on the star Sirius. They took two parabolic reflectors (old searchlight mirrors) each with a photomultiplier tube at the focus. The outputs of the multipliers were fed by coax cables to a circuit that measured the average value of the *product* of the two currents (a so-called "correlator"). From the variation of this product with the separation of the two mirrors they determined the angle subtended by the star.

There were at the time many physicists who said that the method couldn't work. The argument was that since light came in photons which went either to one mirror or the other, there could be no correlation in the two currents. You can show that this argument is wrong by considering the following idealized experiment. There are two small sources—say two light bulbs—A and B, at a large distance from two photomultiplier tubes a and b with the geometry shown in Fig. 70-4. Counters are attached to the detectors a and b that measure the number of photons per second p_1 and p_2 arriving at each counter. The counters a and b are also connected to a "coincidence" circuit that measures p_{12} the counting rate for the *simultaneous* appearance (within some small time τ) of two photoelectrons.

Let $\langle a \mid A \rangle$ be the amplitude for a photon to arrive at a from A in any particular resolving time interval. Then $\langle a \mid A \rangle$ is $ce^{i\alpha_1}$ where c is a complex constant and α_1 is k times the distance R_1 from A to a. Similarly $\langle b \mid A \rangle = ce^{i\alpha_2}$ with $\alpha_2 = kR_2$ where R_2 equals the distance from A to b.

(a) Show that the coincidence counting rate p_{12} is proportional to

$$2 + \cos 2k(R_2 - R_1).$$

(b) How can this result be used to measure D if R is known? Ignore the fact that the actual process must be represented by a superposition of such models because light comes from all areas on the star's surface and not just from two points on the limbs.