The Characterisation, Subtraction, and Addition of Astronomical Images

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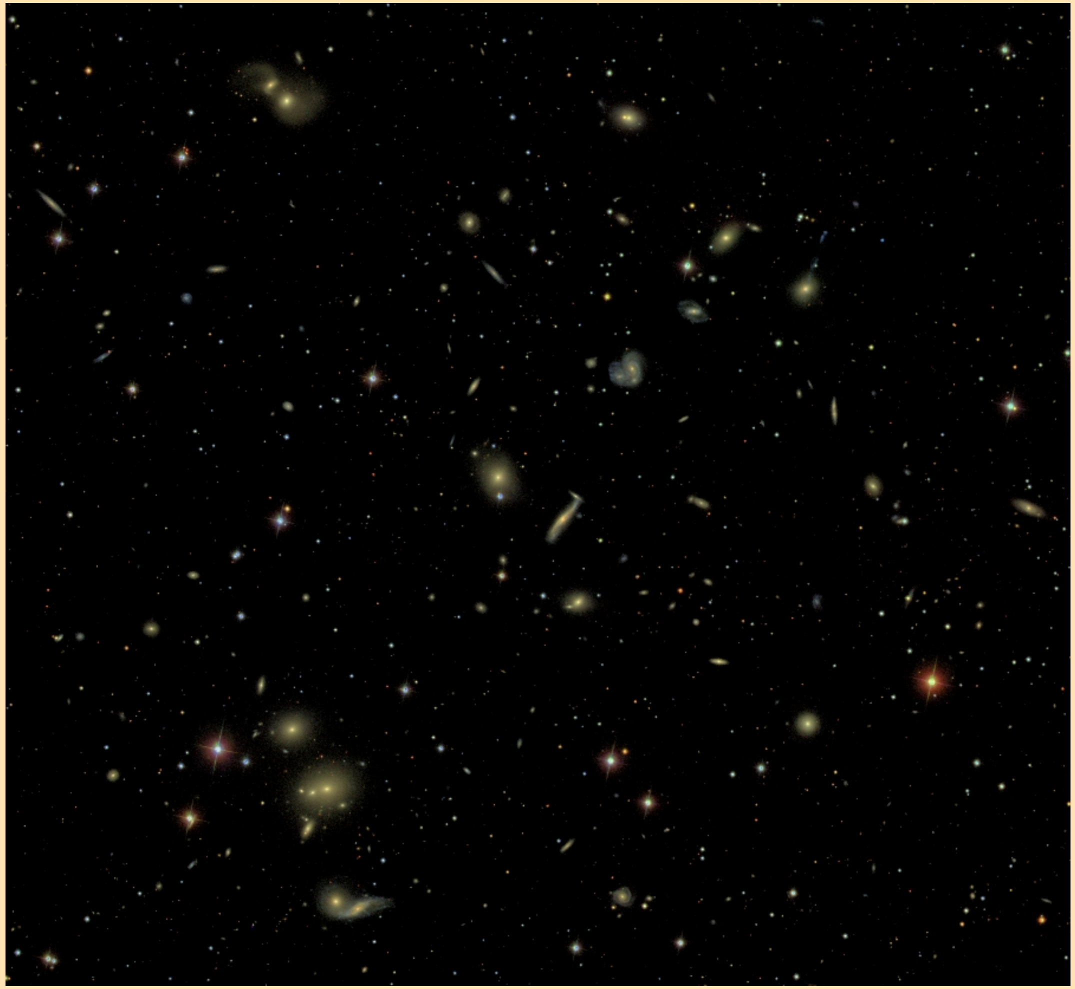
Introduction

Introduction

We can write down all of observational astronomy in terms of the model

$$I = \sum_{i \in \text{objects}} O_i \otimes \phi + \epsilon_i$$

where the O_i are our objects, ϕ is the PSF, and ϵ is the noise. If our objects are simple (e.g. if they're all stars, so $O_i = A_i \delta(x - x_i)$) then all that we have to do is estimate $3N$ numbers.



Astronomical PSFs

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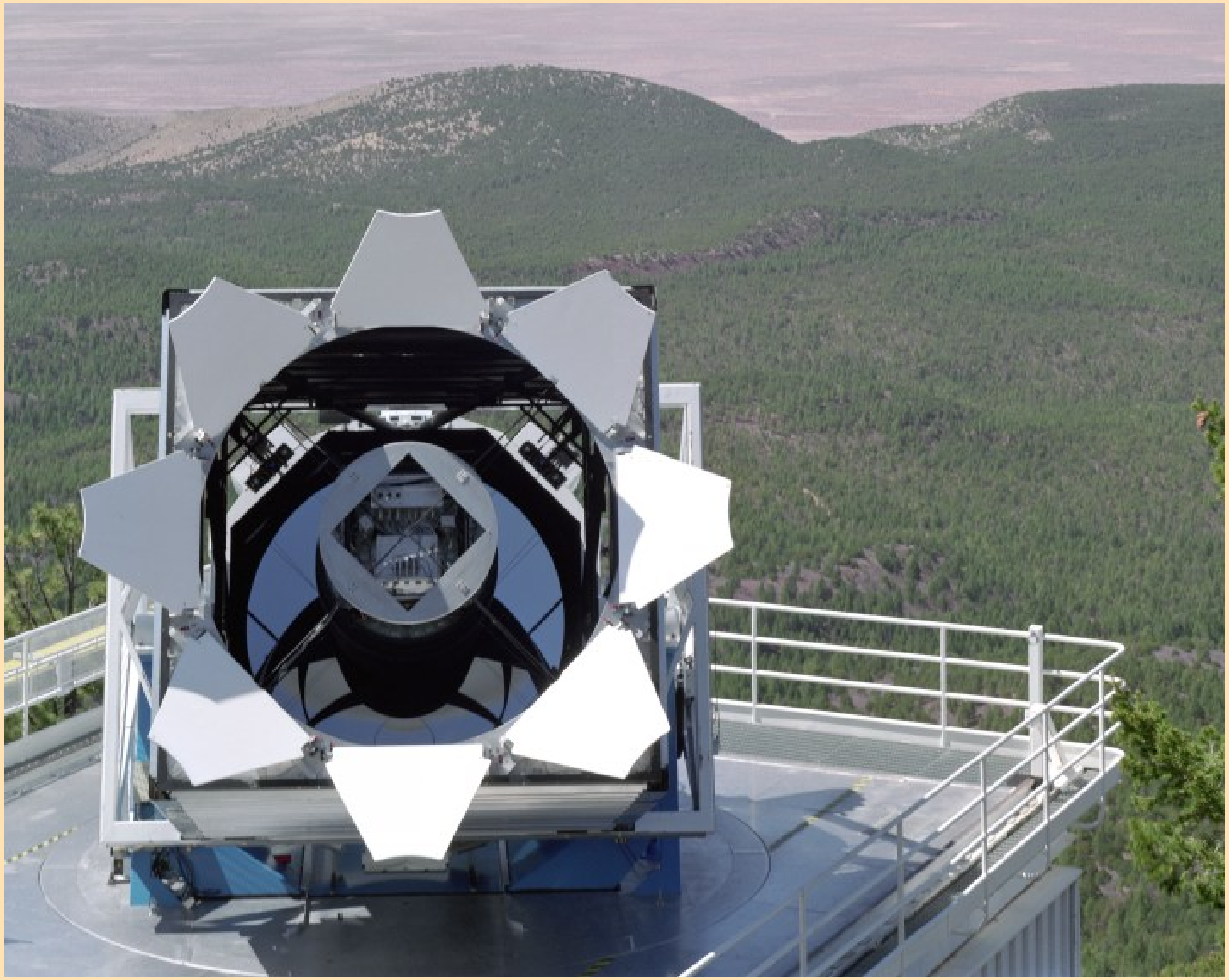
Critically sampling a PSF of rms size α requires a pixel of size $\sim \alpha$, so the pixel convolution increases the rms size of the PSF by c. $\sqrt{1 + 2/12} - 1 \sim 8\%$.

The telescope



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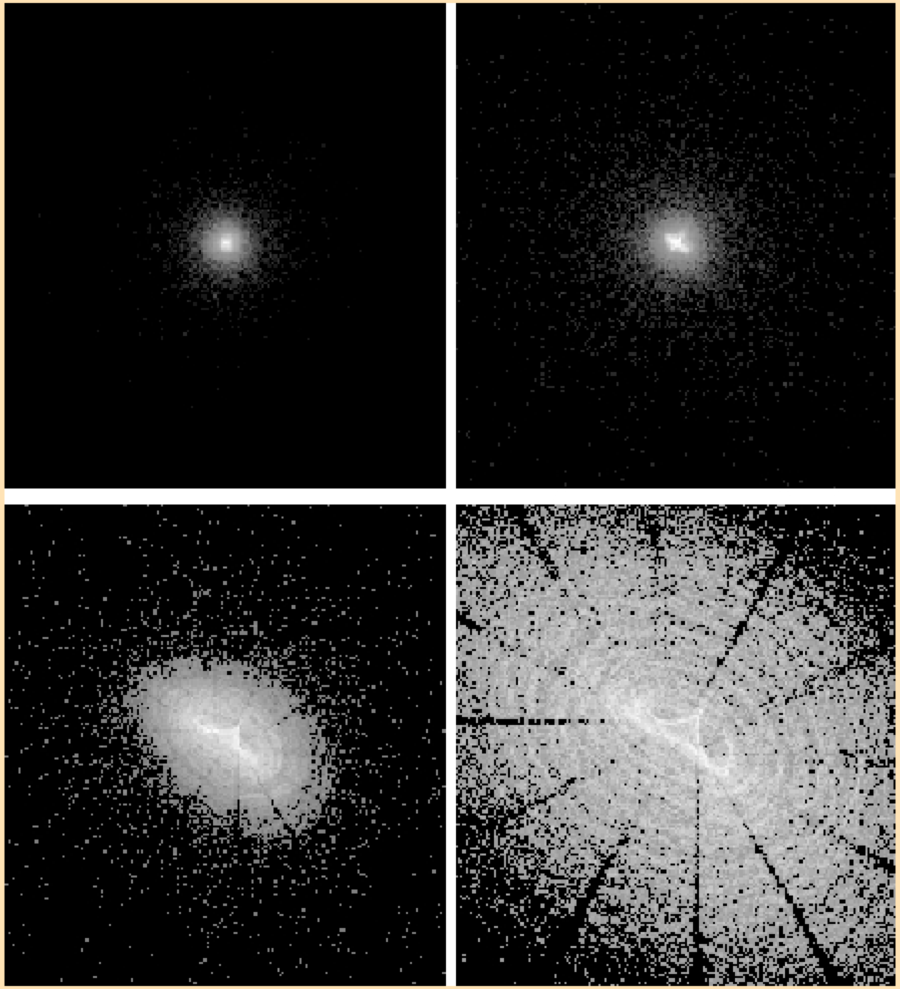

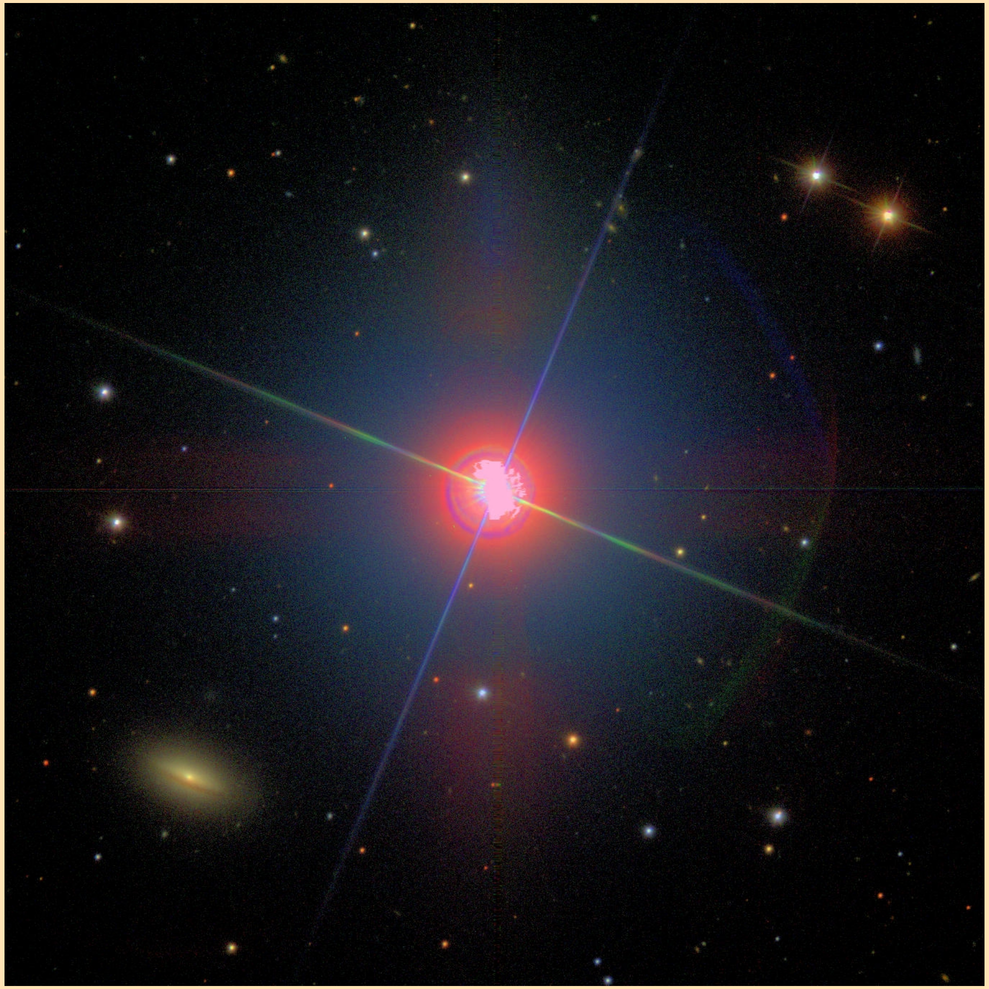

NGC442

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The Kolmogorov-dominated PSF's Fourier transform is given by

$$\exp\left(-6.8839\,(2\pi k\lambda/r_0))^{5/3}\,/2\right)$$

where $\lambda/r_0 \equiv$ FWHM$/0.976$

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A sum of two Gaussians, $G(\alpha) + 0.1G(2\alpha)$, is a convenient and reasonably accurate representation of (the core of) the PSF.

Chromatic Effects

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Sampling and Dithering

A telescope of diameter D band-limits the image of the sky at D/λ, and the atmosphere severly attenuates beyond r_0/λ; in general $r_0 \ll D$ and we design the instrumentation to sample a signal limited at the lower frequency r_0/λ but the signal has some power at higher (angular) frequencies.

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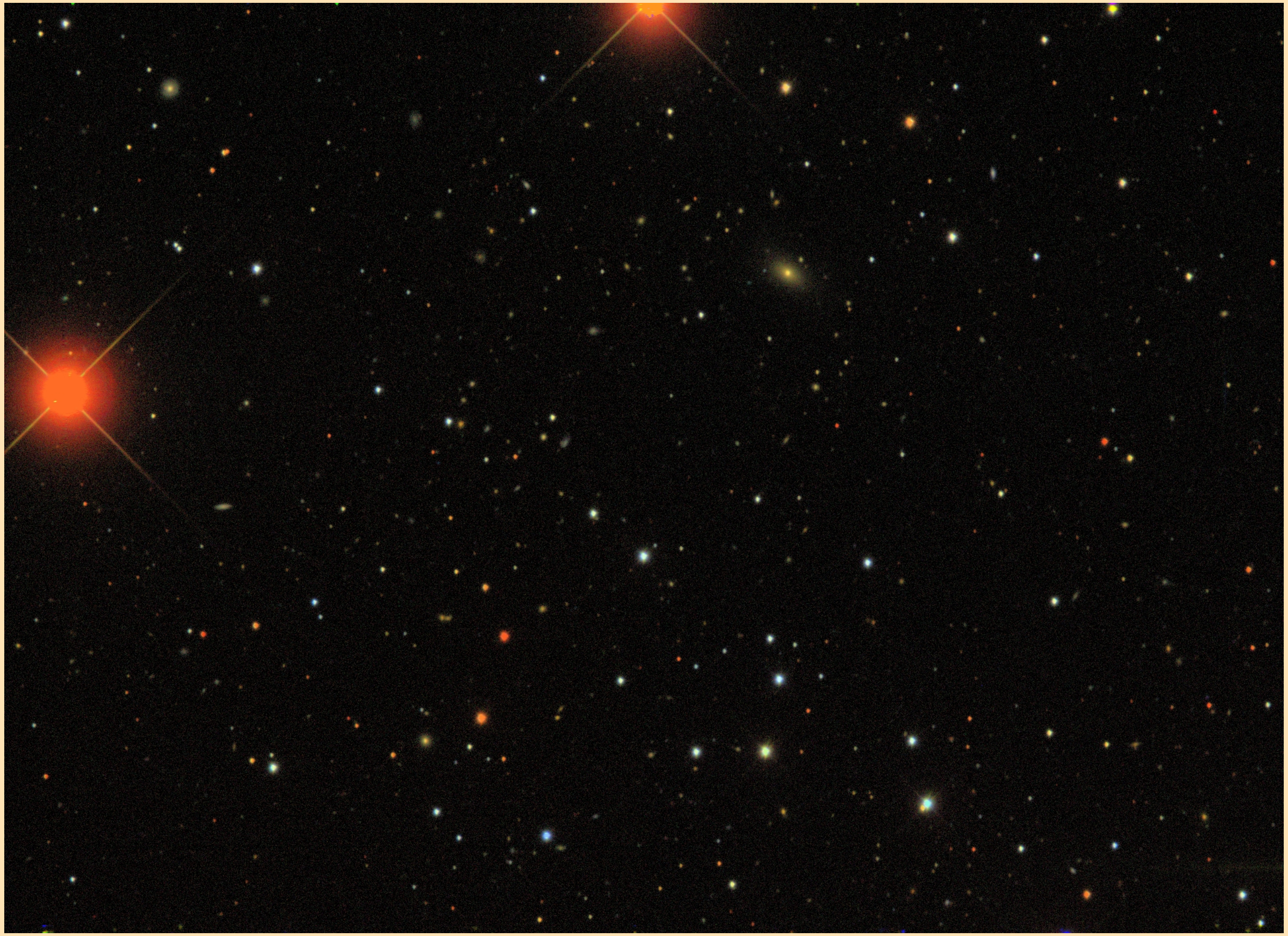
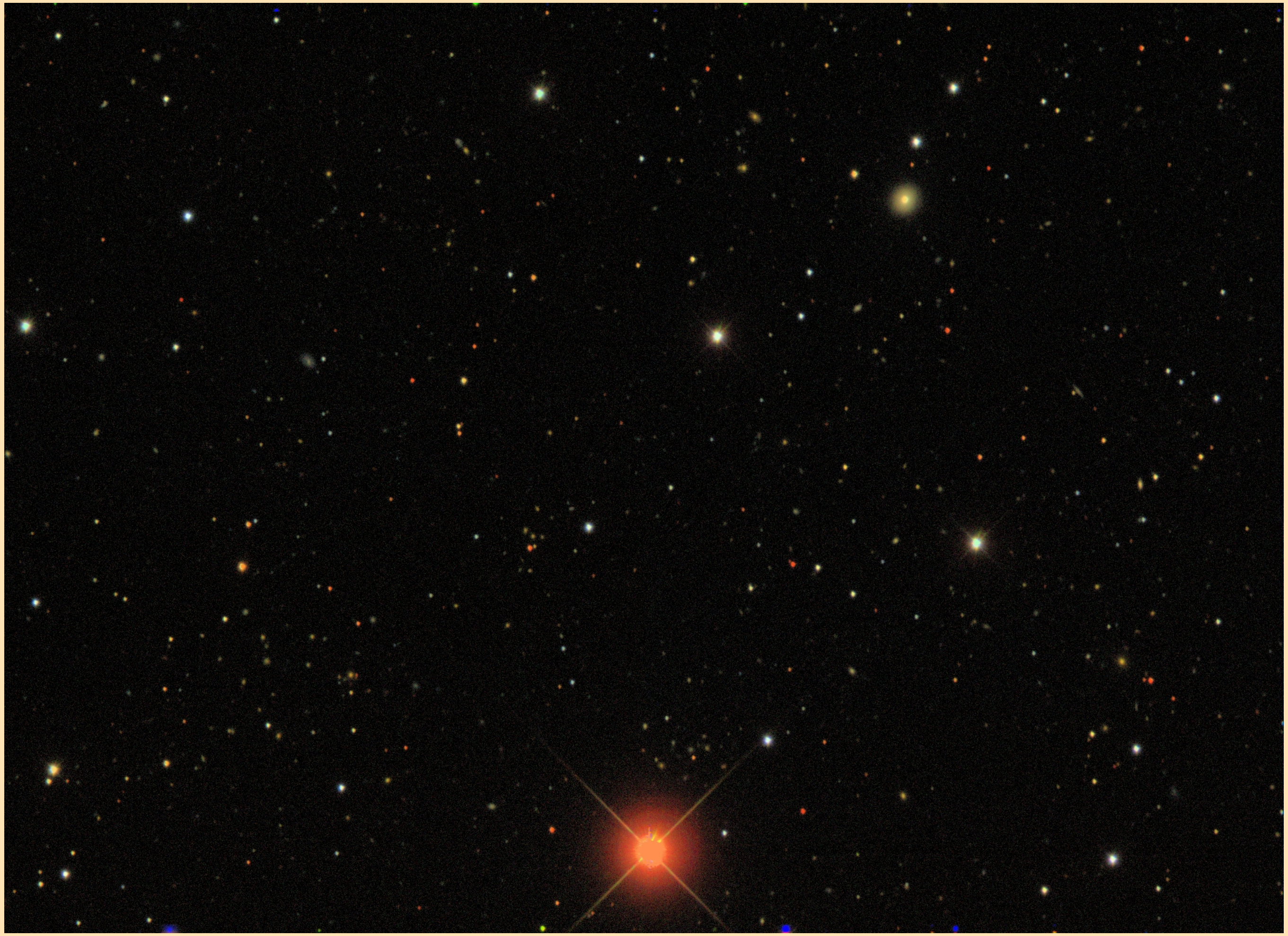
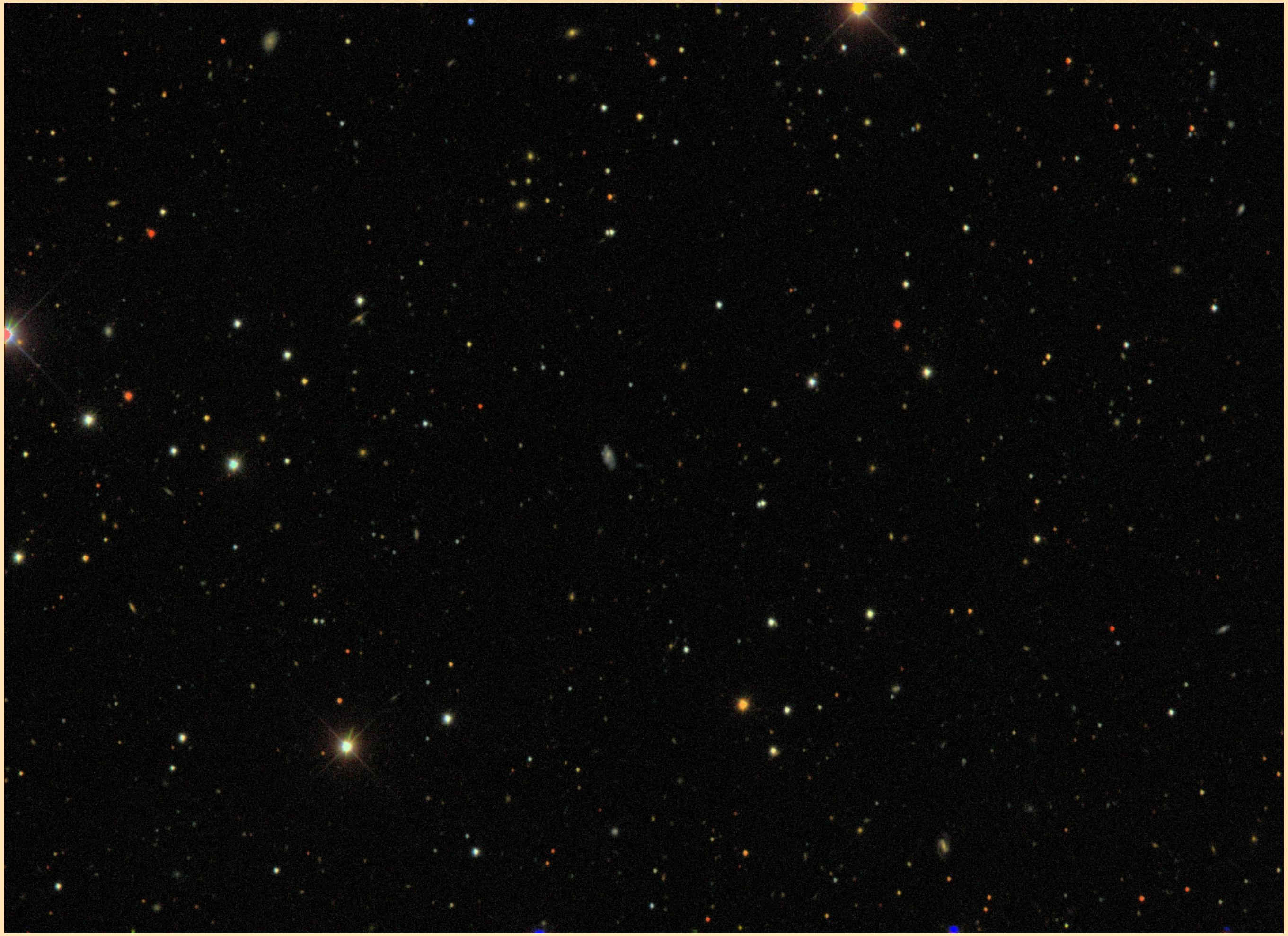
In many cases, multiple exposures with the telescope's pointing varied a little are taken to recover the fully sampled image.

Estimating the PSF







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Because the PSF has spatial structure, we have to also ask how to represent this in any of these approaches.

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To the best of my knowledge this has not been attempted, although Jarvis and Jain have modelled the PSF in an ensemble of images by studying its variation at a given point in the focal plane as a function of exposure number.

SDSS; KL Expansion of the PSF

We can use a set of objects identified as stars to form a KL basis, retaining the first n terms of the expansion:

$$P_{(i)}(u,v) = \sum_{r=1}^{r=n} a^r_{(i)} B_r(u,v)$$

where $P_{(i)}$ is the i^{th} PSF star, the B_r are the KL basis functions, and u, v are pixel coordinates relative to the origin of the basis functions.

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In determining the B_r, the $P_{(i)}$ are normalised to have equal peak value.

Once we know the B_r we can write

$$a^r_{(i)} \approx \sum_{l=m=0}^{l+m \leq N} b^r_{lm} x^l_{(i)} y^m_{(i)}$$

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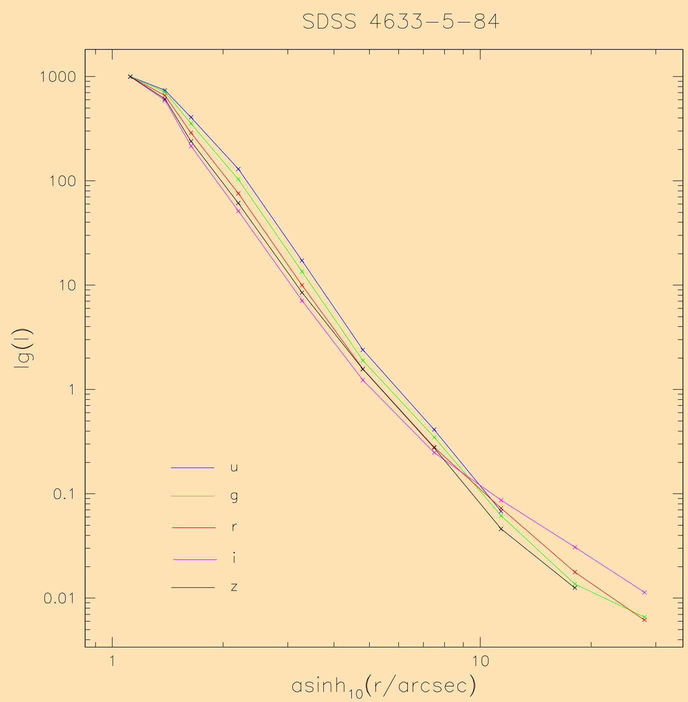
The b^r_{lm} are determined by minimising

$$\sum_i \left(\frac{P_{(i)}(u,v) - \sum_{r=1}^{r=n} a^r_{(i)} B_r(u,v)}{\sigma^2 + \Upsilon^2} \right)^2$$

The Outer Parts of the PSF

The KL model of the PSF extends to only about 7", but the outer parts contain a significant part of the total flux, and need to be modelled. We use partial radial profiles from stars of a range of magnitudes to construct a single profile in each filter.


SDSS 4633-5-84
1000
100
10
1
0.1
0.01
lg(I)
1
10
asinh$_{10}$(r/arcsec)
u
g
r
i
z

Application to SDSS data

For each CCD, in each band, there are typically 15-25 stars in a frame that we can use to determine the PSF. We usually take $n = 3$ and $N = 2$ (i.e. the PSF spatial variation is quadratic).

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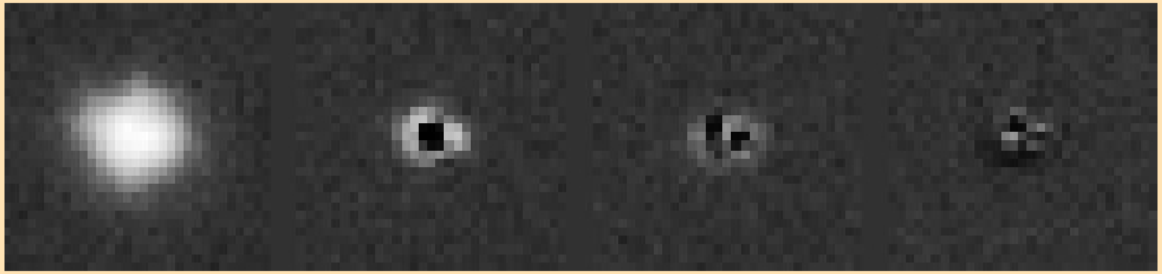

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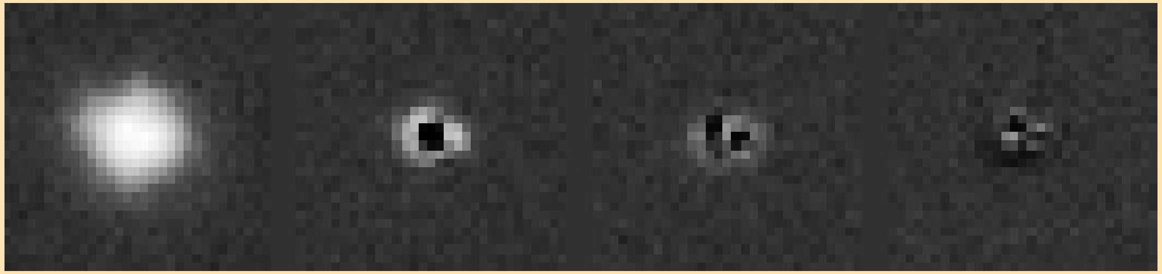
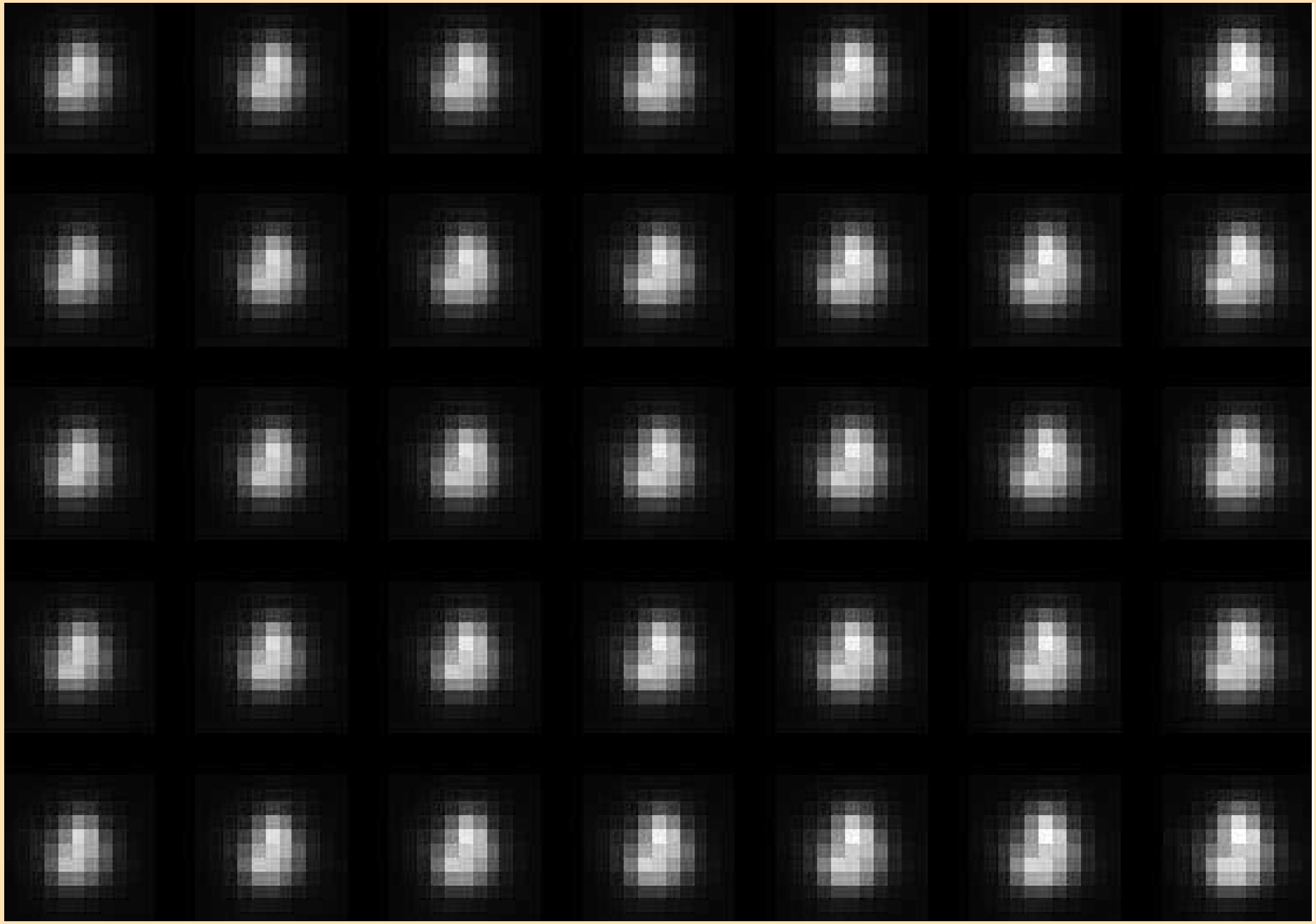


We force the KL basis images with $n > 0$ to have 0 mean in their outermost pixels, those more than approximately 7'' from the center of the star.



Reconstructed PSFs for 756-z6-700

Gauging the Success of the KL Expansion

We estimate the PSF at about 120 positions around the edge of the field, and also a few points in the interior. At each of these positions we calculate an aperture correction. The PSF model used in estimating the flux is based on a *local* Gaussian fit.

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We reject the PSF model if the maximum or minimum aperture correction seen differs from unity by more than 30%.

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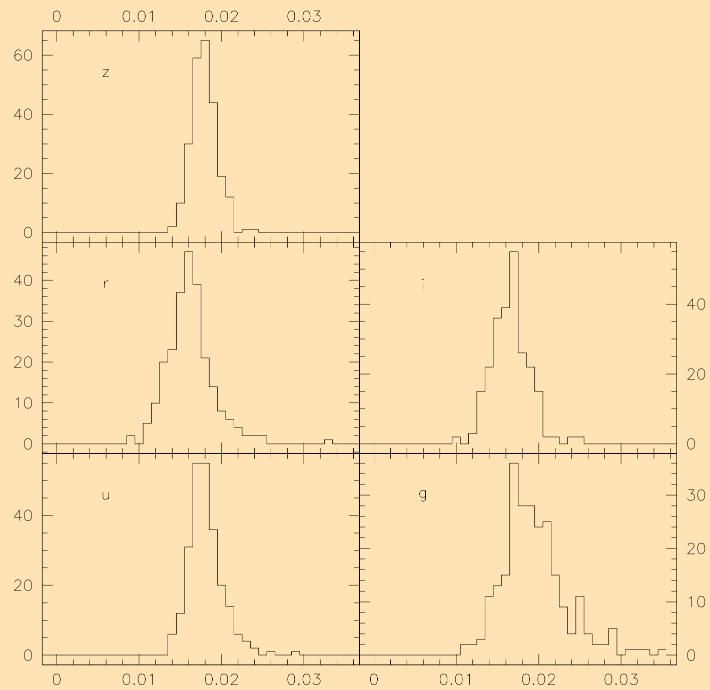

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The models used for spatial variability are generally polynomials. In the case of SDSS data this is probably not a very good basis (due to the coupling of time- and spatial-variability due to TDI).

For long exposures on large telescopes polynomials appear to work reasonably well, although it has been shown that, unsurprisingly, rational functions perform rather better.

Using an Ensemble of Exposures

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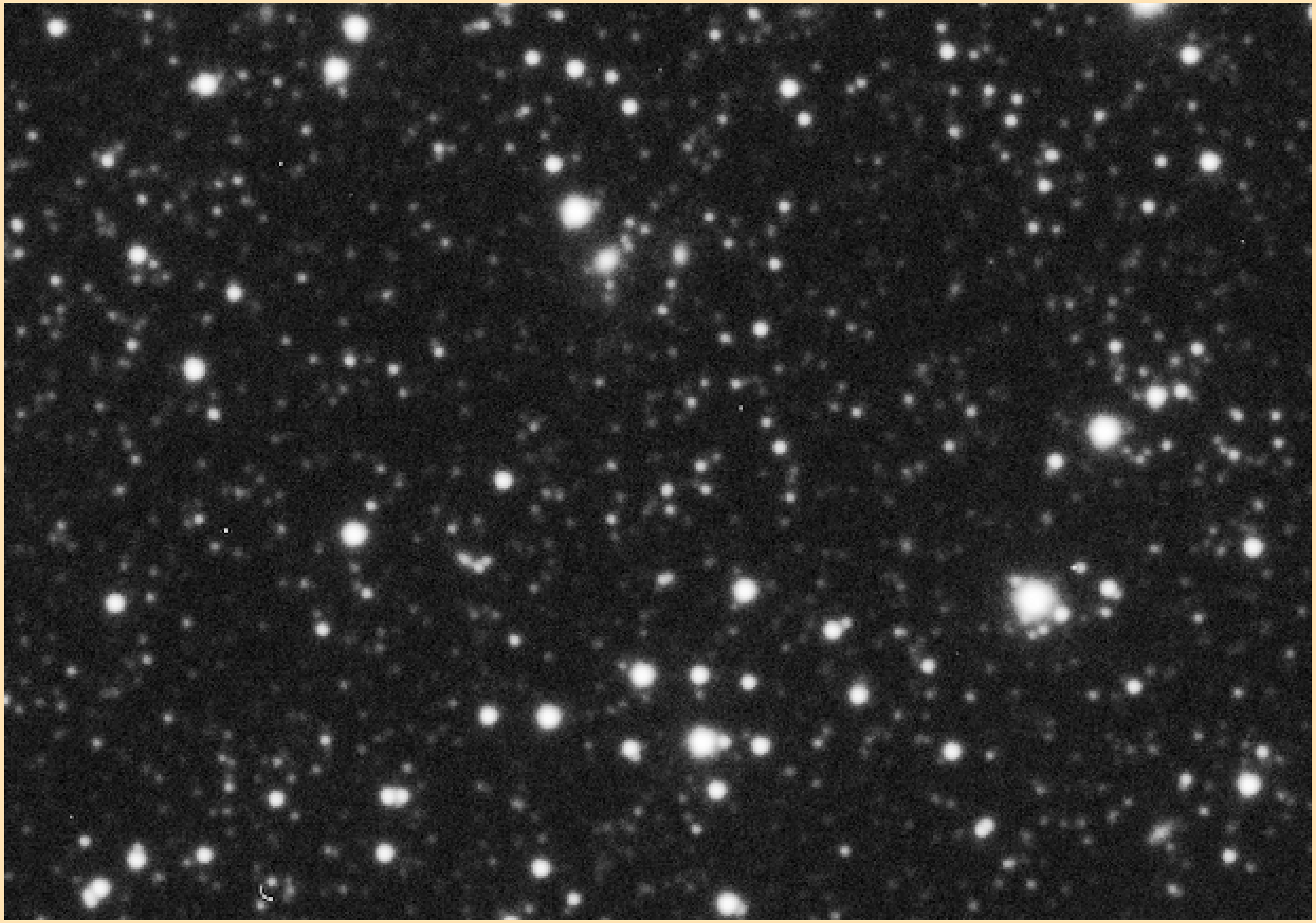
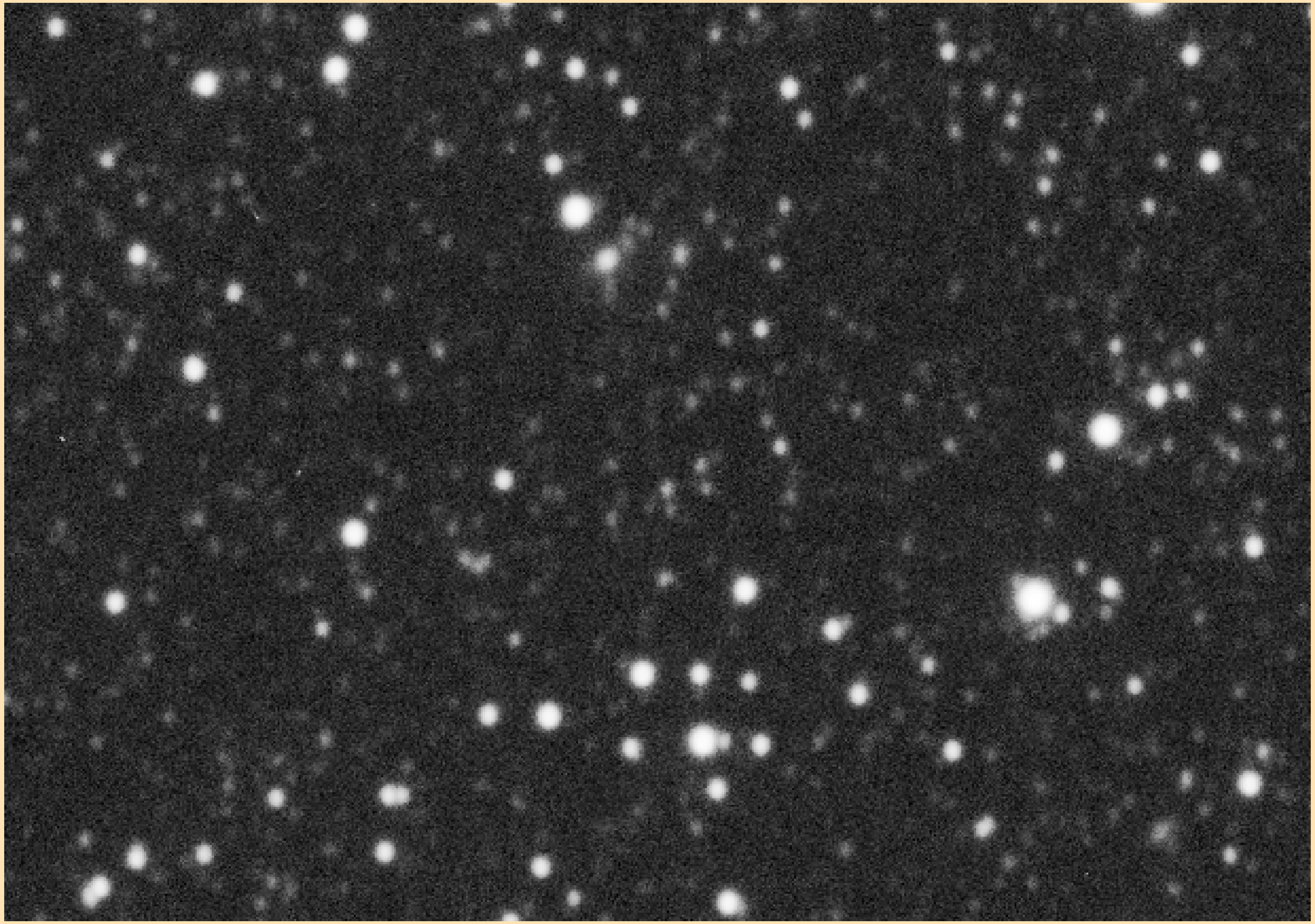
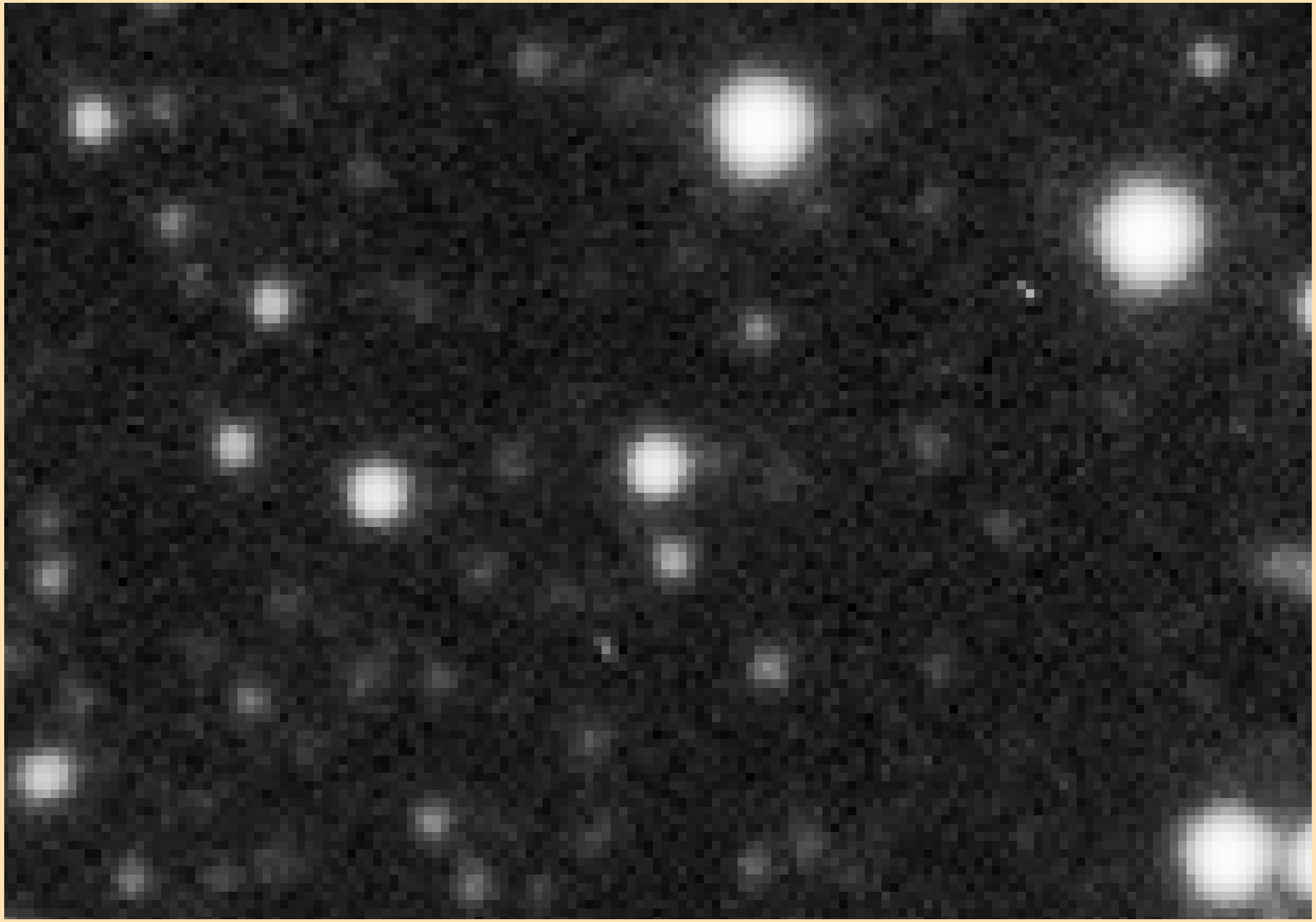
The parameters used to describe the PSF are not specified; for weak lensing studies it may be sufficient to measure a few weighted moments, but one could use a richer representation.

Image Subtraction

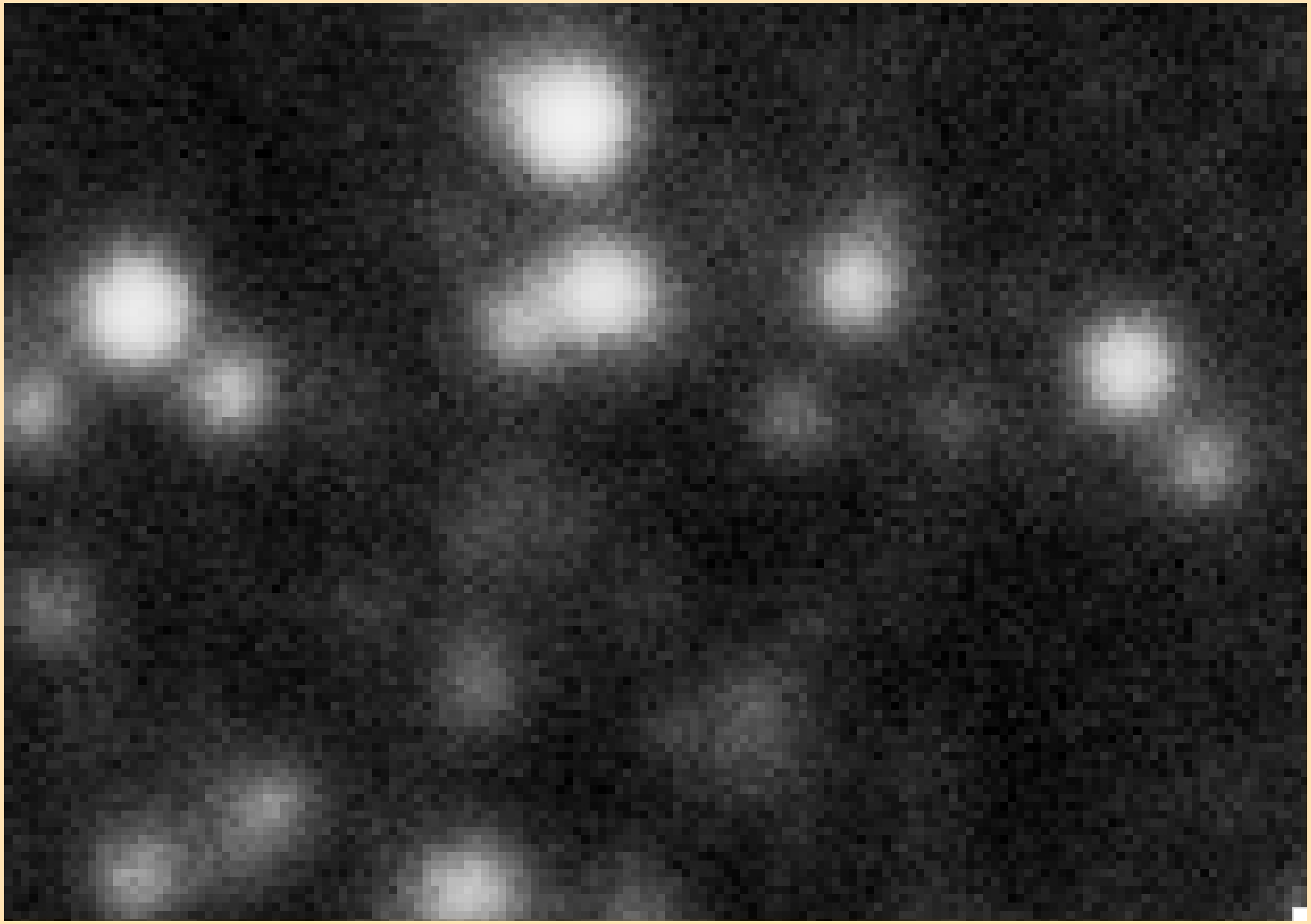
In many situations, only the time variable part of an image is of interest; the classic case is searching for gravitational micro-lensing in the direction of the Galactic bulge. In this case, we have two or more images of the same part of the sky, taken under different conditions; in particular the PSF will be different in the two exposures.

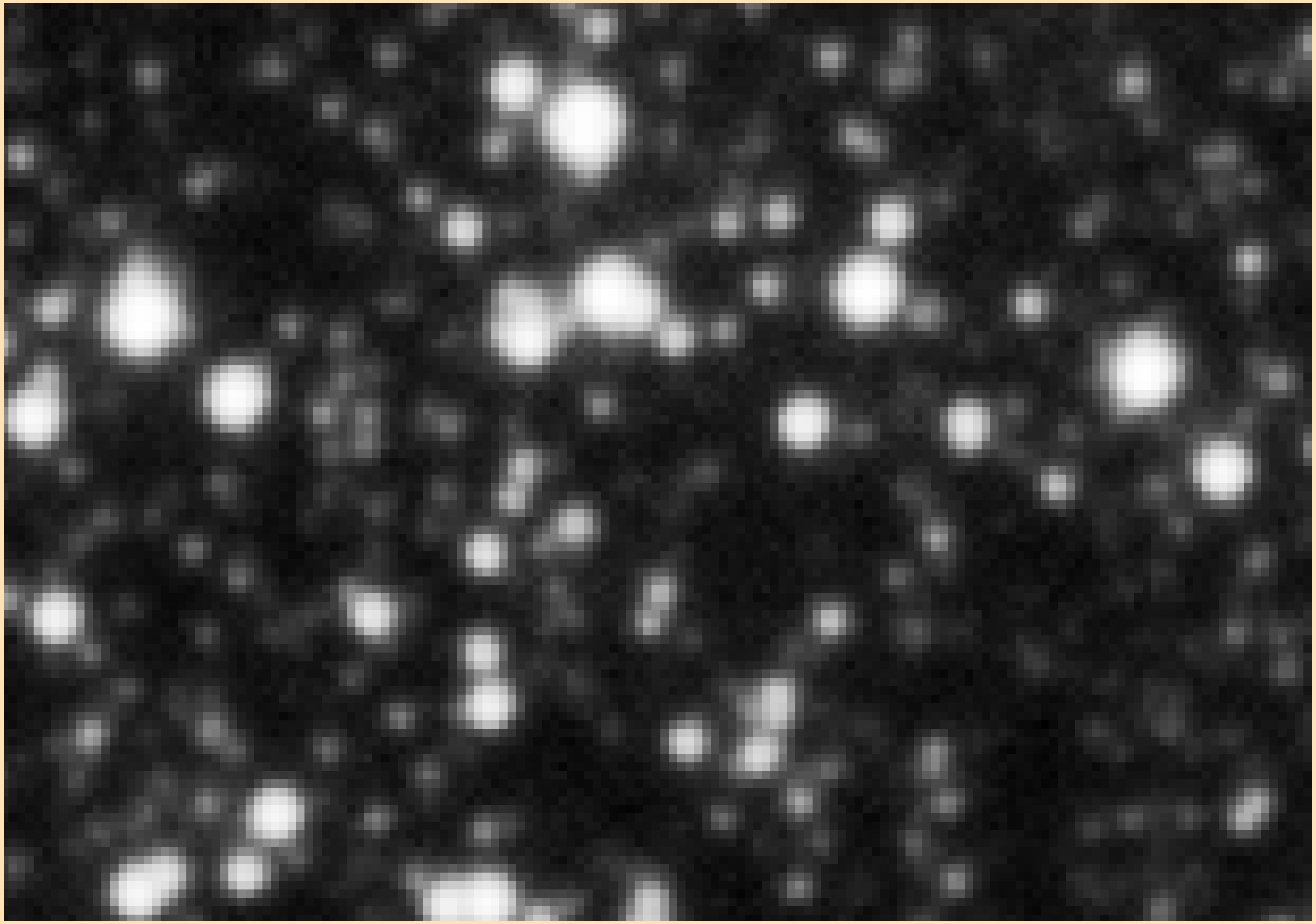






The classic solution to problem of searching for variability is to measure the brightness of each source in both images and compare the resulting catalogues.





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Unfortunately, it's difficult to measure the outer parts of the PSFs well enough to carry out this Fourier division.

What really matters is how well the subtraction worked, and that the the residuals left by subtracting objects that *hadn't* varied should be as small as possible; that is, we should find the kernel K such that

$$R \equiv ||I_i - K \otimes I_2||$$

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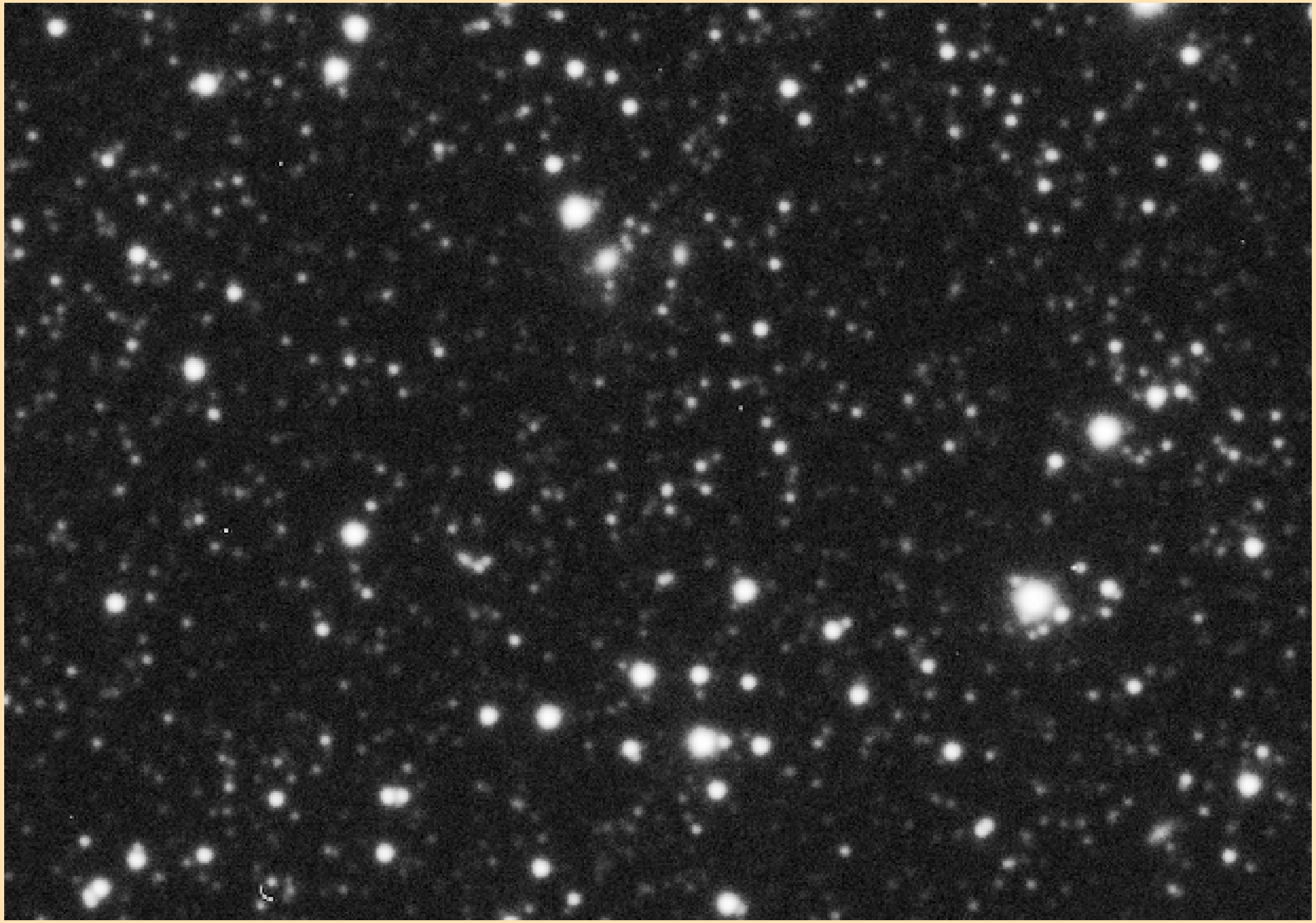
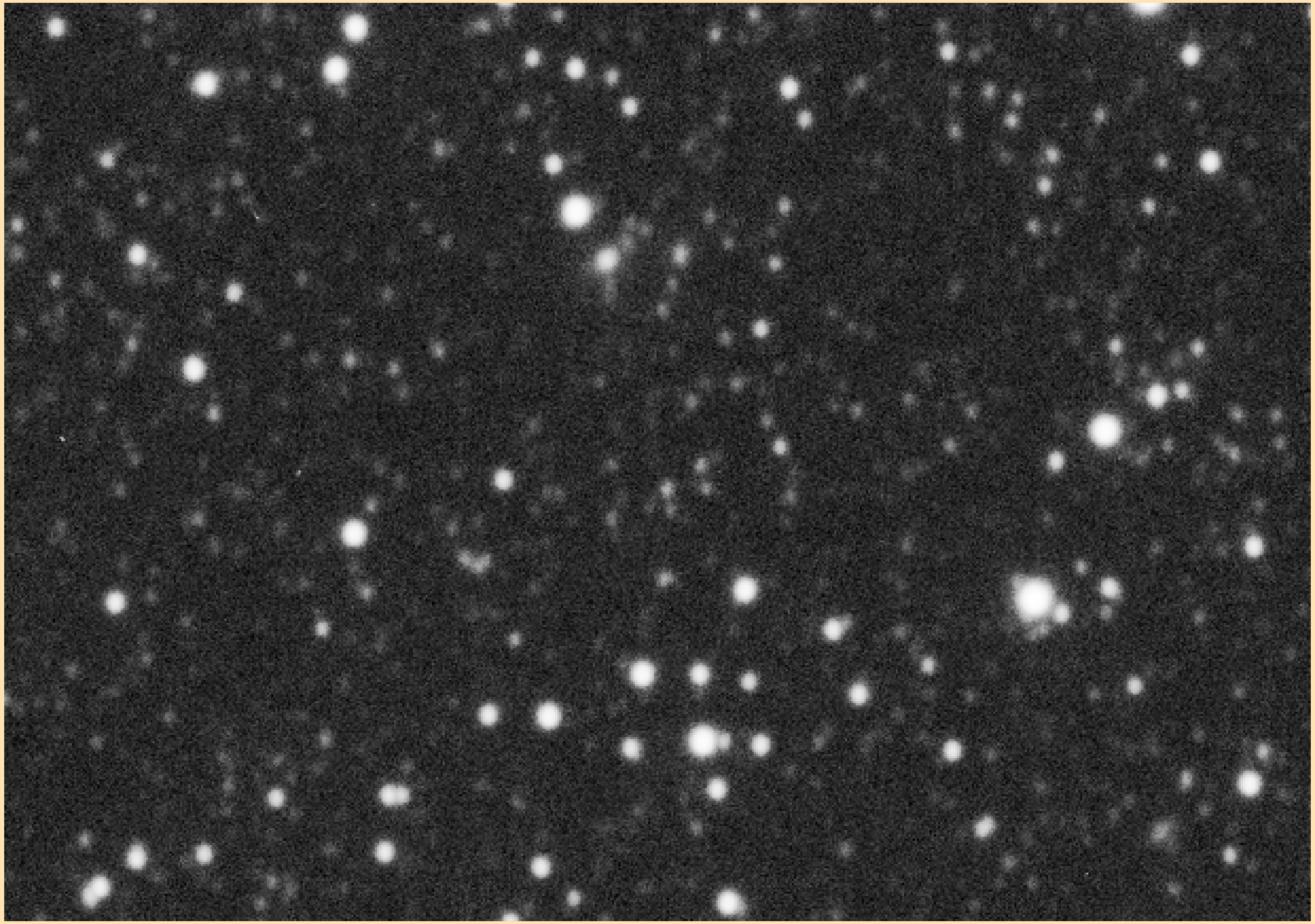
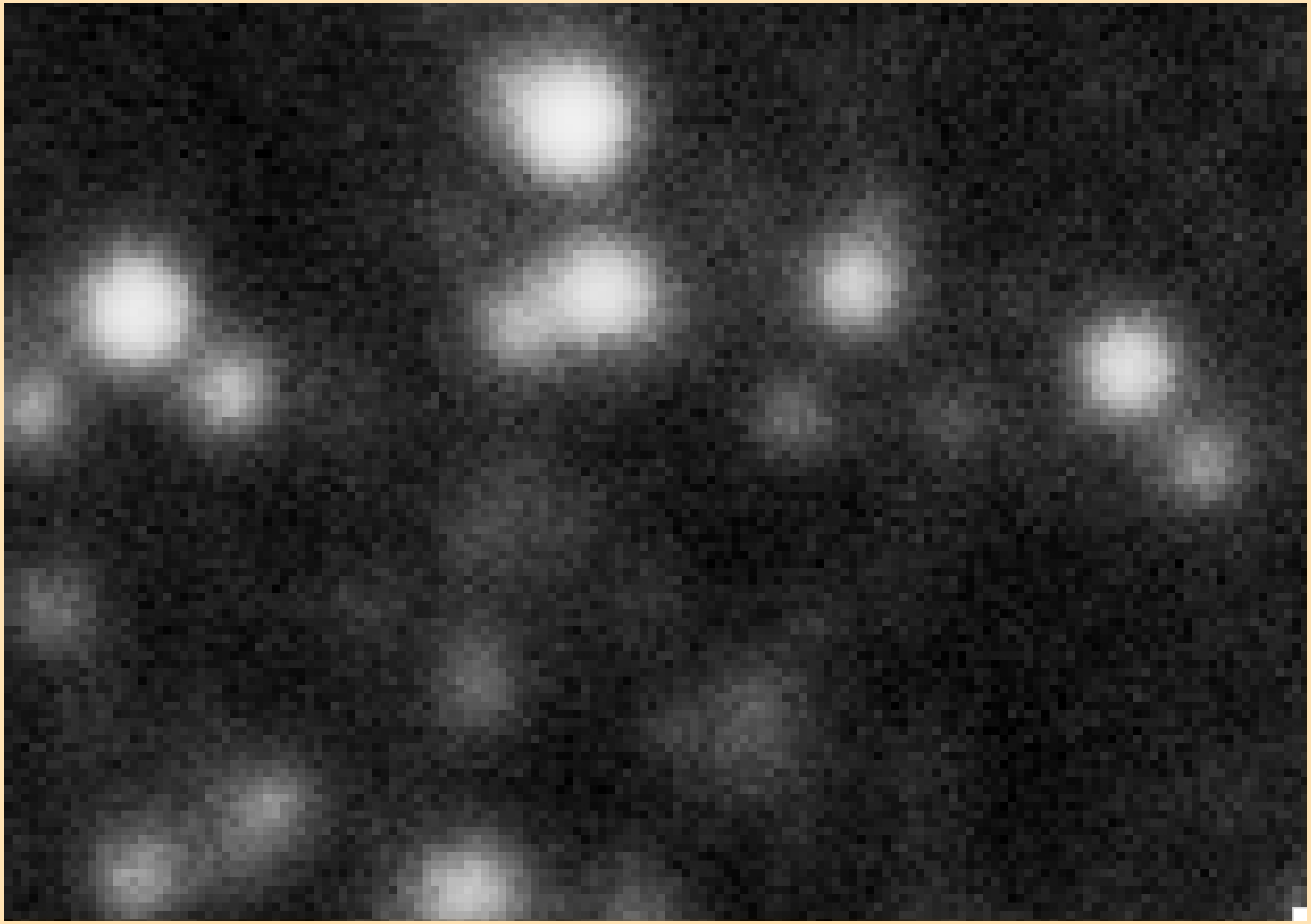
Let us write K as a sum of a set of basis functions:

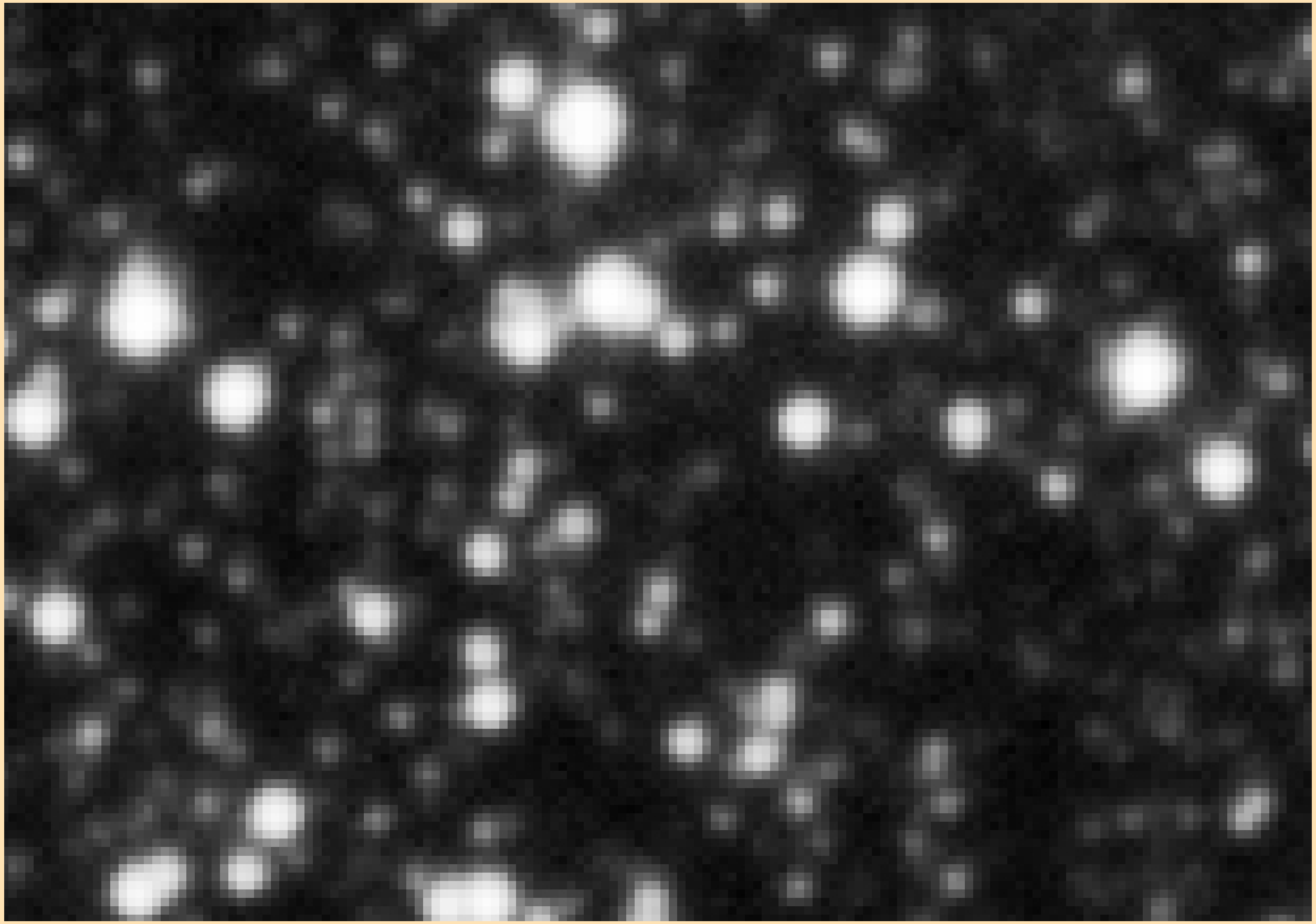
$$K(u, v) \equiv \sum_r a_r B_r(u, v).$$

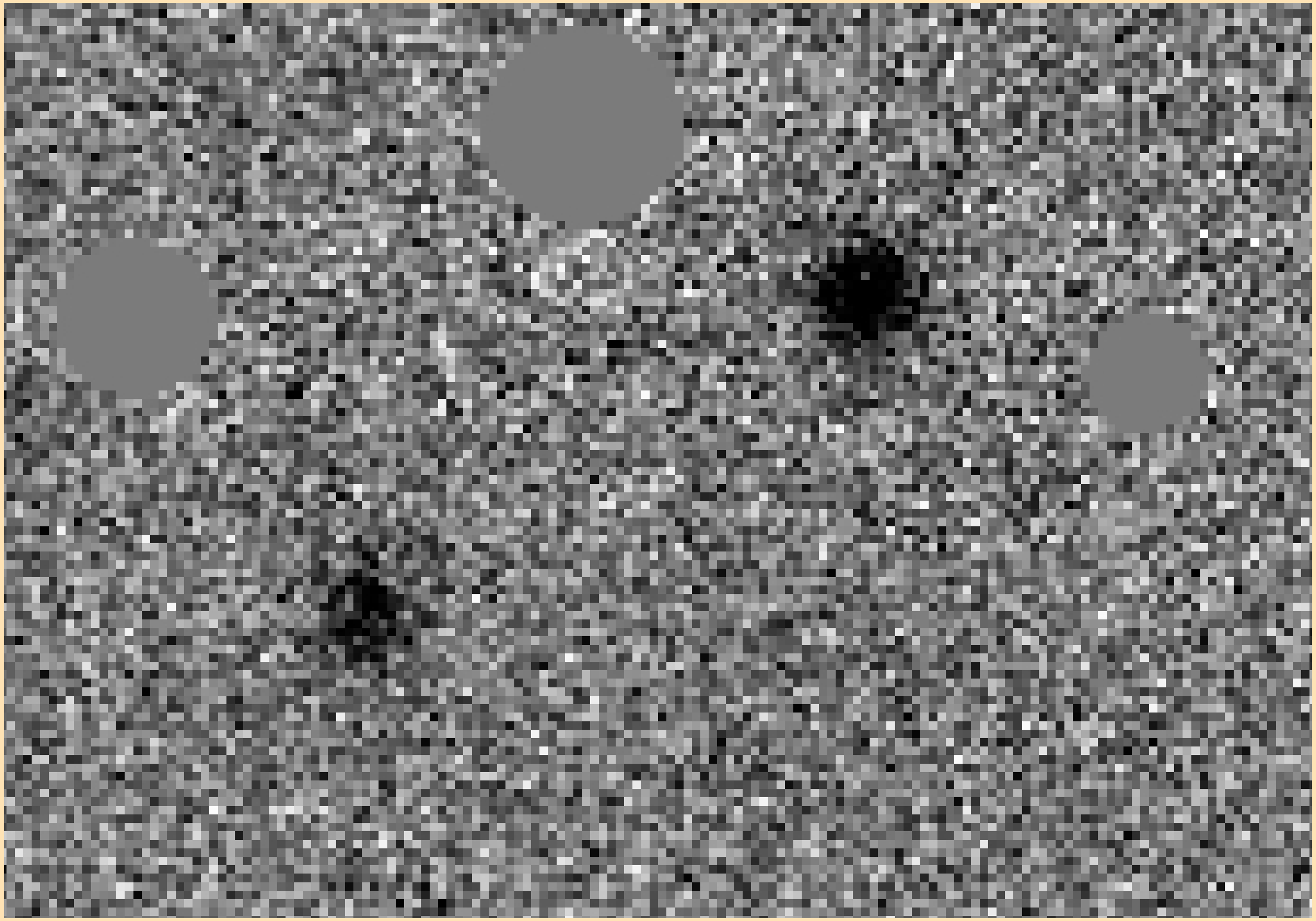
The task of subtracting two images is thus released to the problem of finding a set of a_r that minimise R; if we use an L_2 norm, this is a simple least-squares problem.











Spatially varying kernels can be handled by writing

$$K(u,v;x,y) = \sum_{r=1}^{r=n} \sum_{l=m=0}^{l+m\leq N} b^r_{lm} x^l_{(i)} y^m_{(i)} B_r(u,v)$$

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δ-function basis

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The resulting kernel is unacceptably noisy. We could add a cost function (e.g. the total variation?), but the problem would become non-linear.

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When the assembled Statistical Wisdom solves the spatially-varying PSF estimation problem, I'll be able to apply an identical algorithm to kernel estimation for image subtraction.

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The obvious remedy is to generate a fully sampled, defect free image from the ensemble of N images.

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A brute force approach would be to generate master templates at different airmasses. It seems probably that using the known colour distribution of the pixels in the image would be advantageous; but the obvious approaches (write the observed image as a linear combination of templates taken at different airmasses, or a template and a colour-template) do not appear to work, as we cannot write our data as a linear combination of the inputs.

Image Addition

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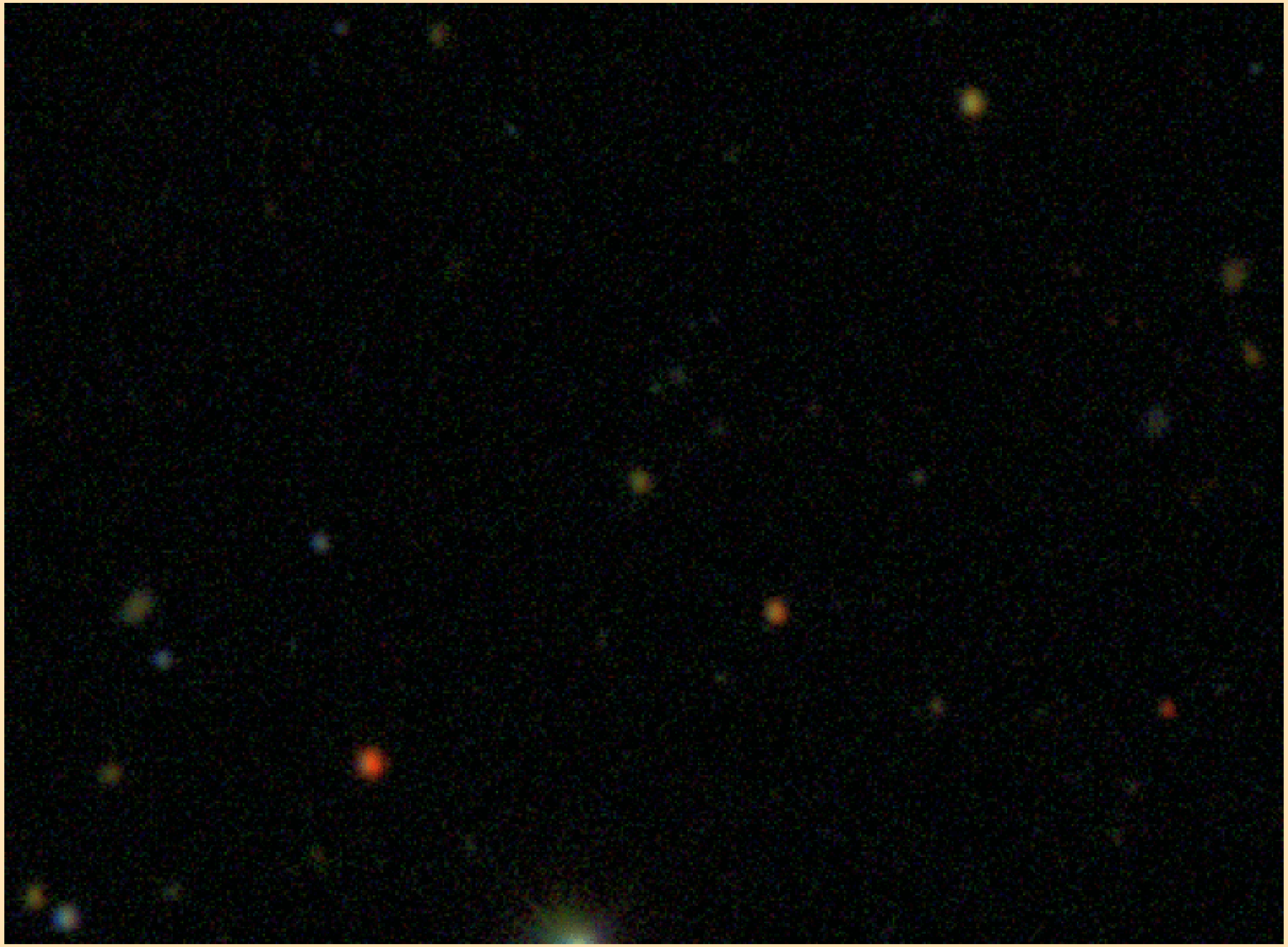

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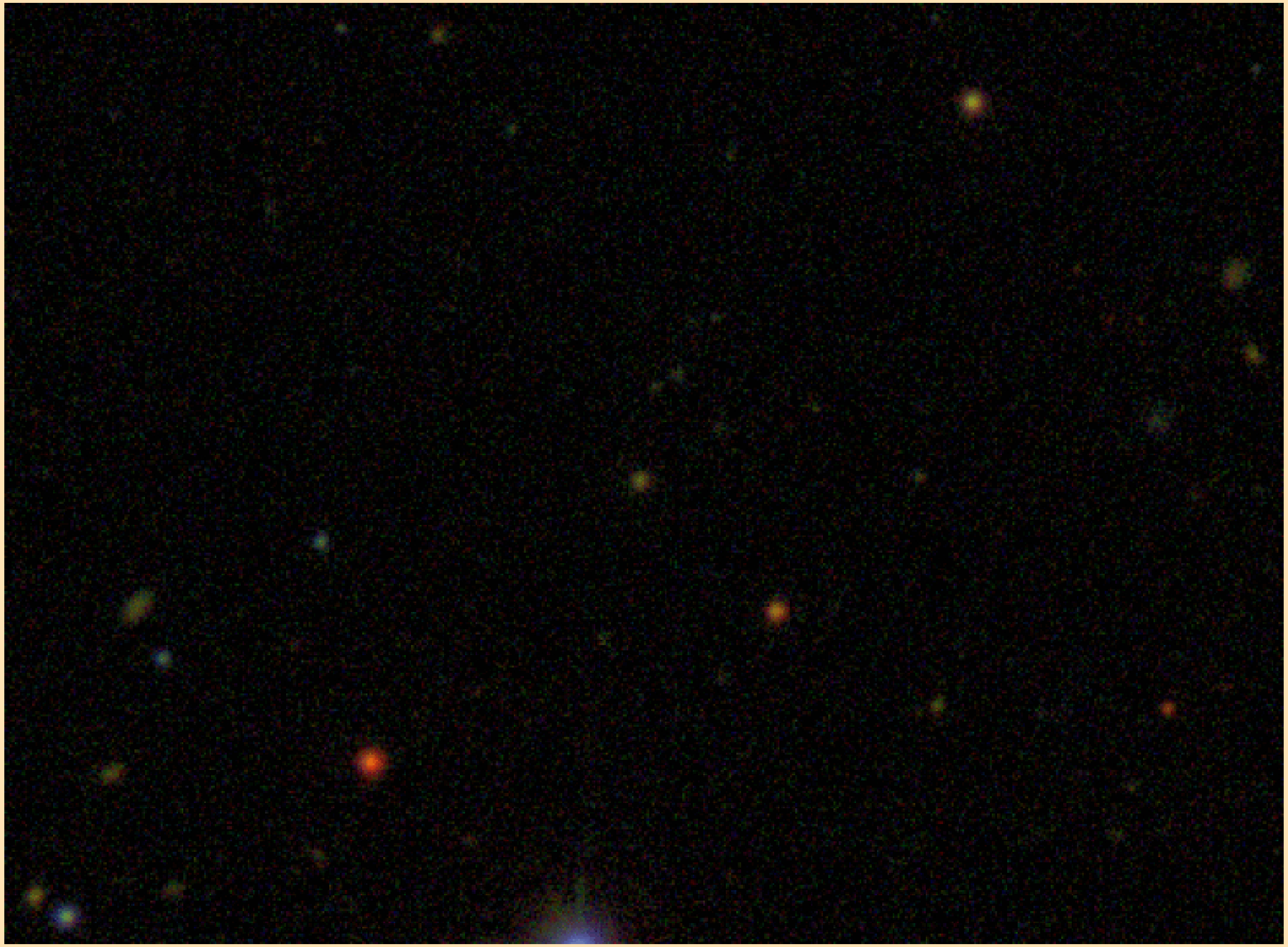

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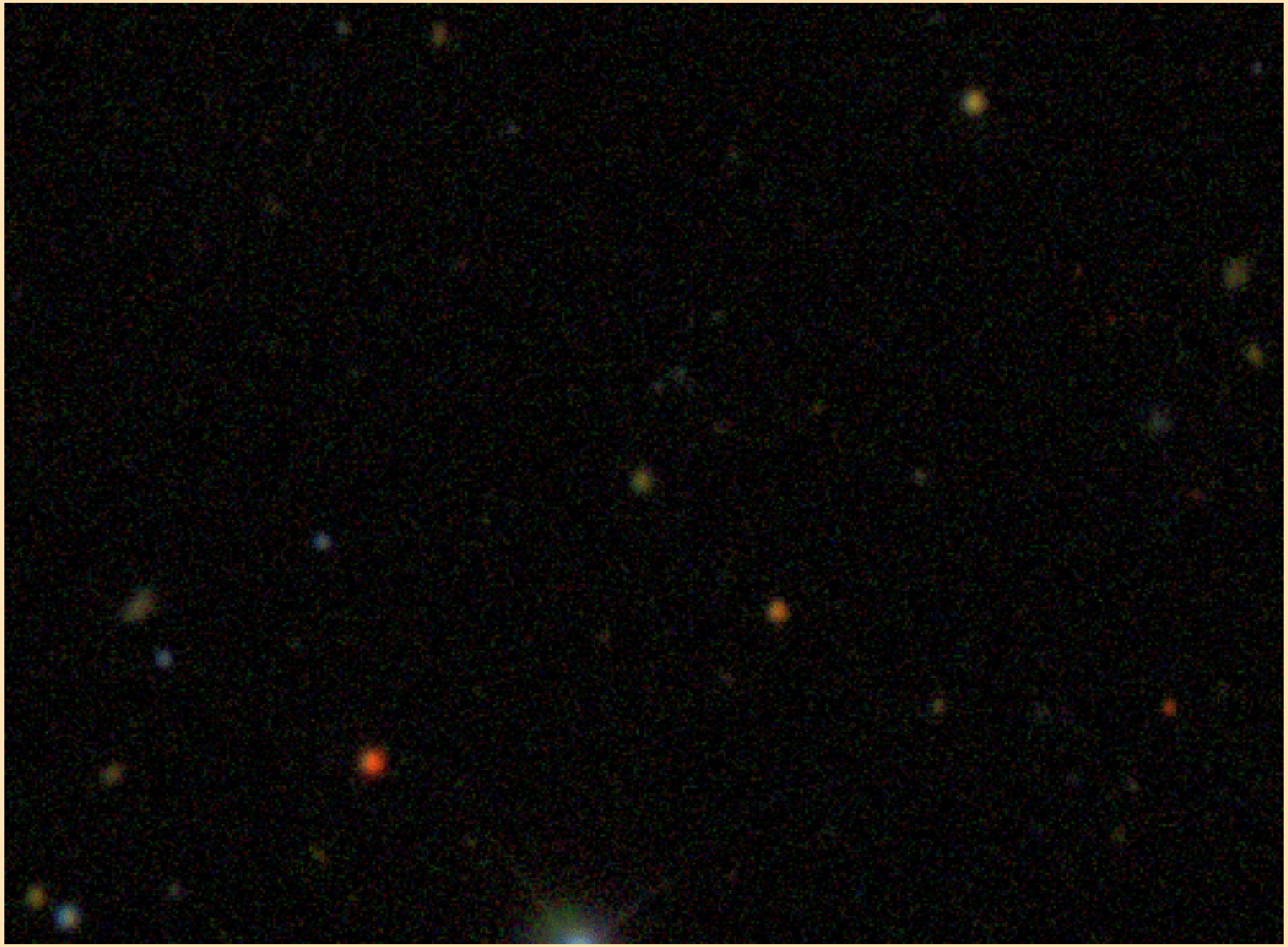

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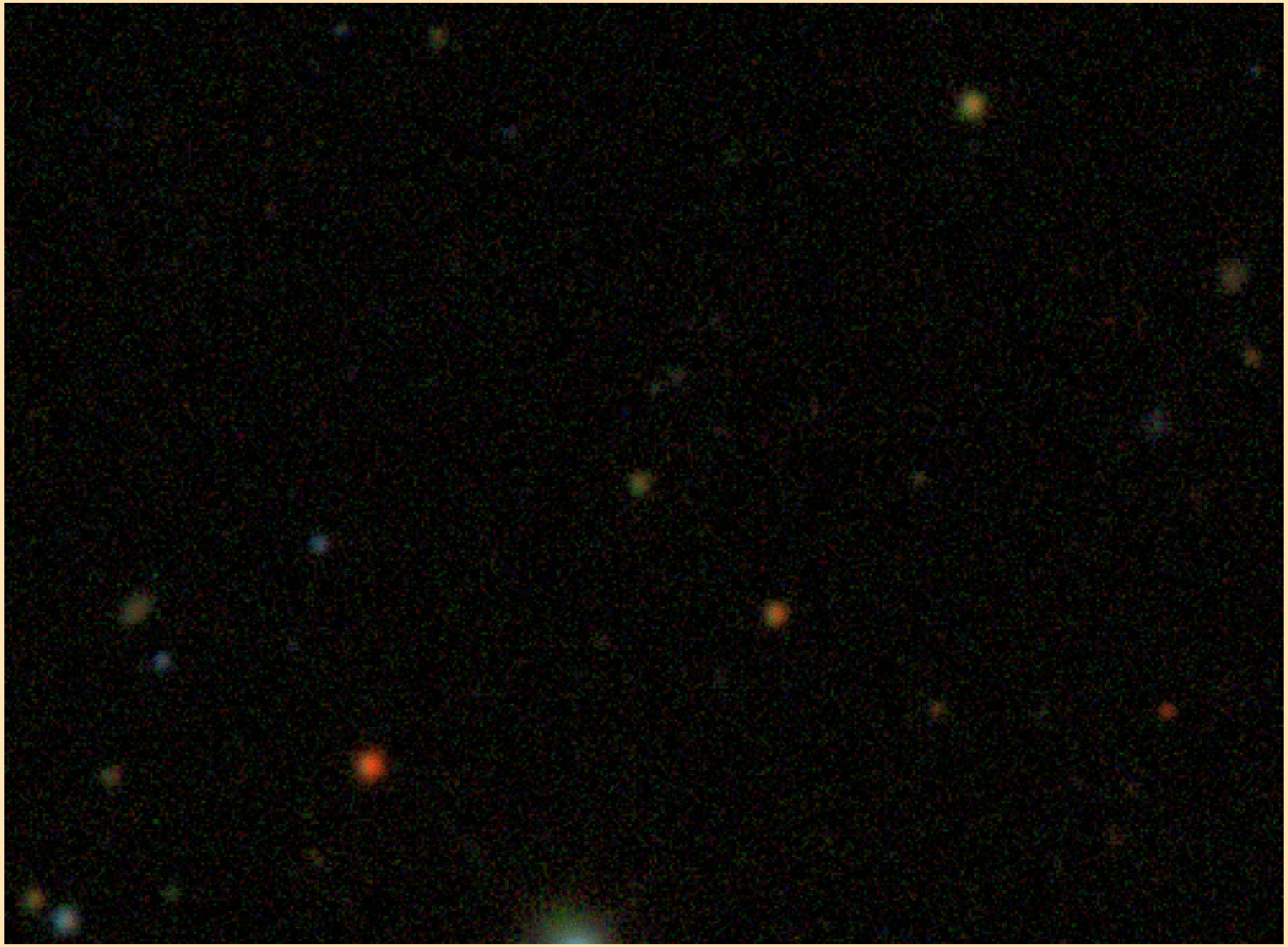

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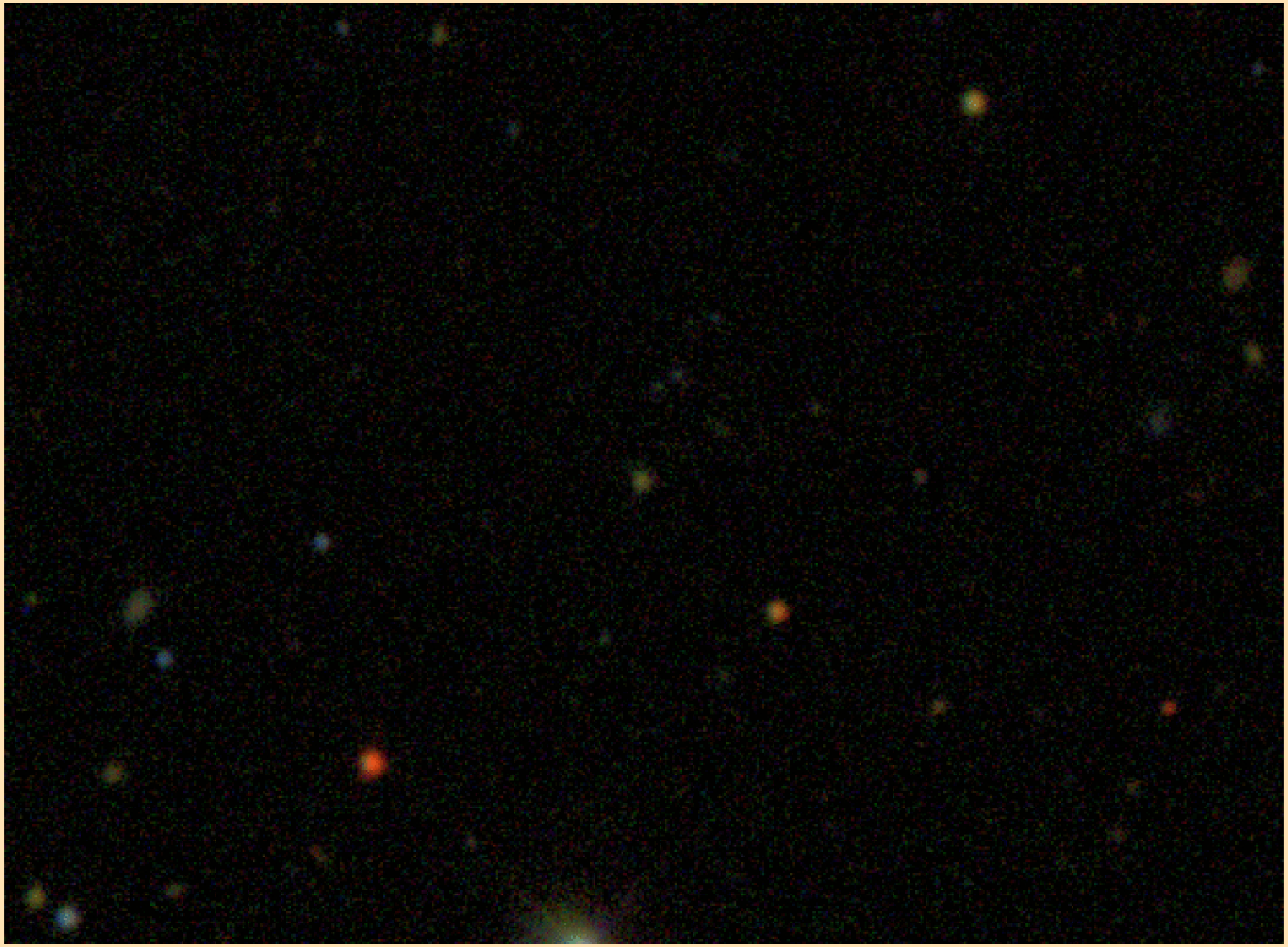

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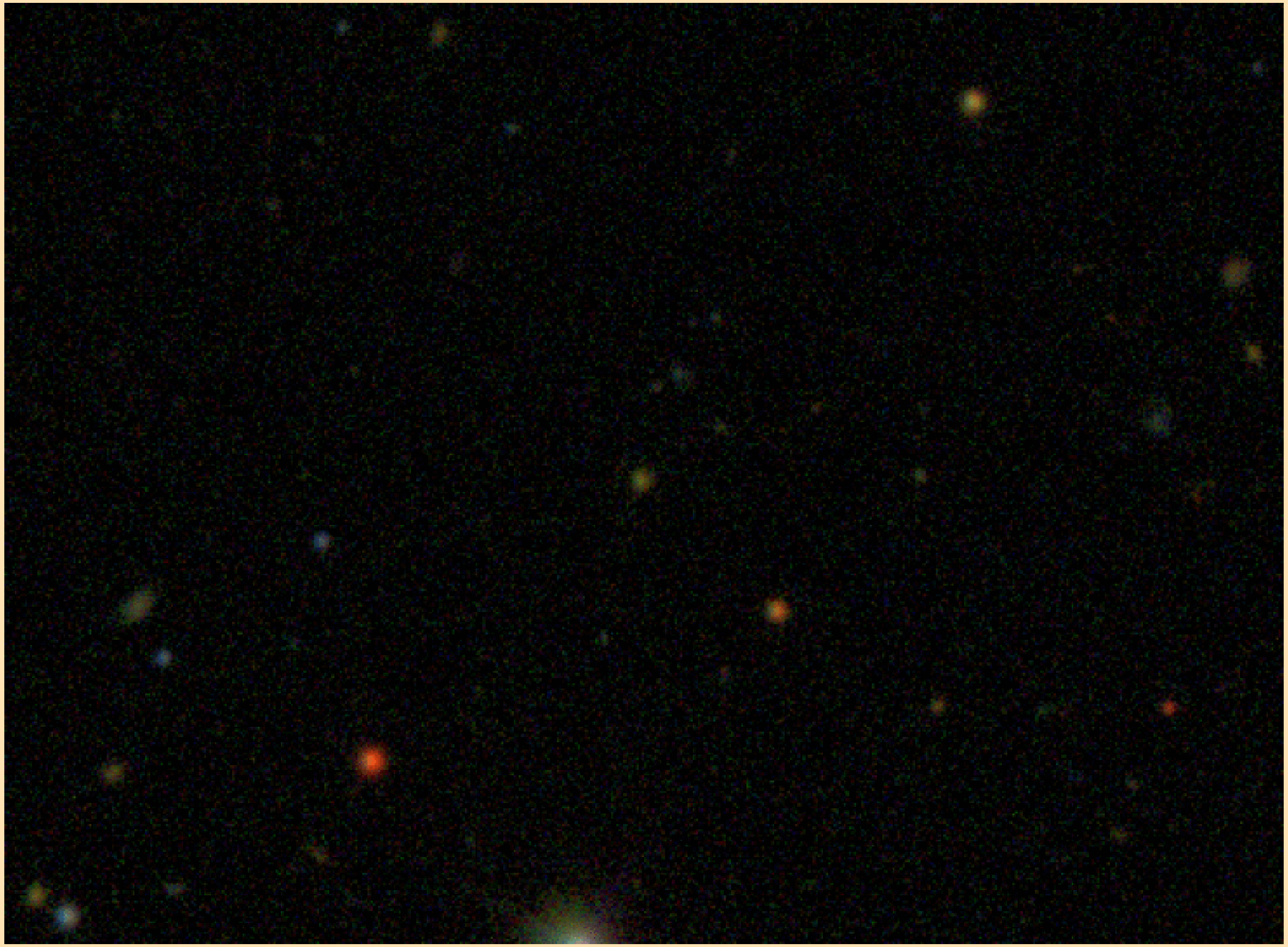

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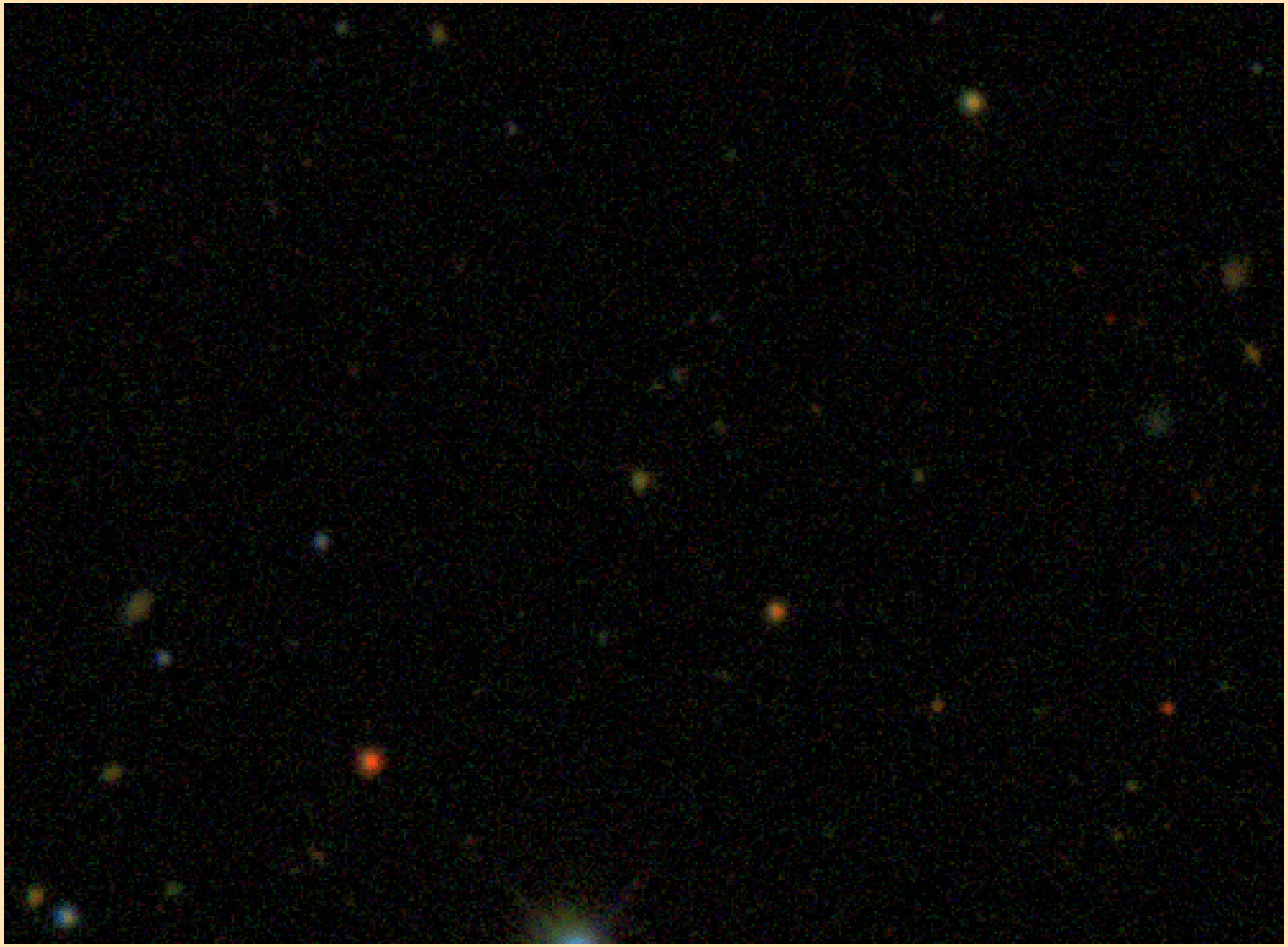

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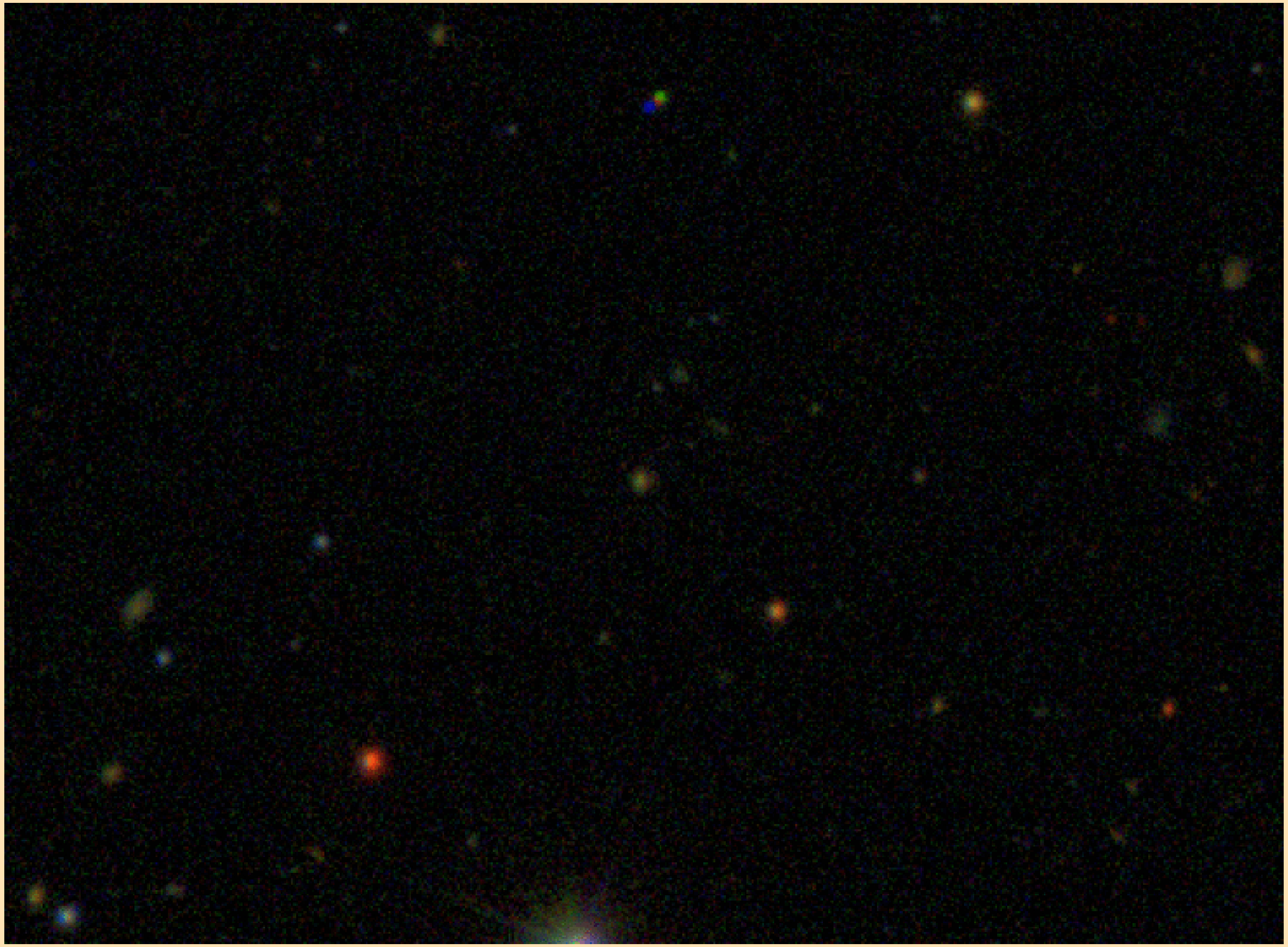

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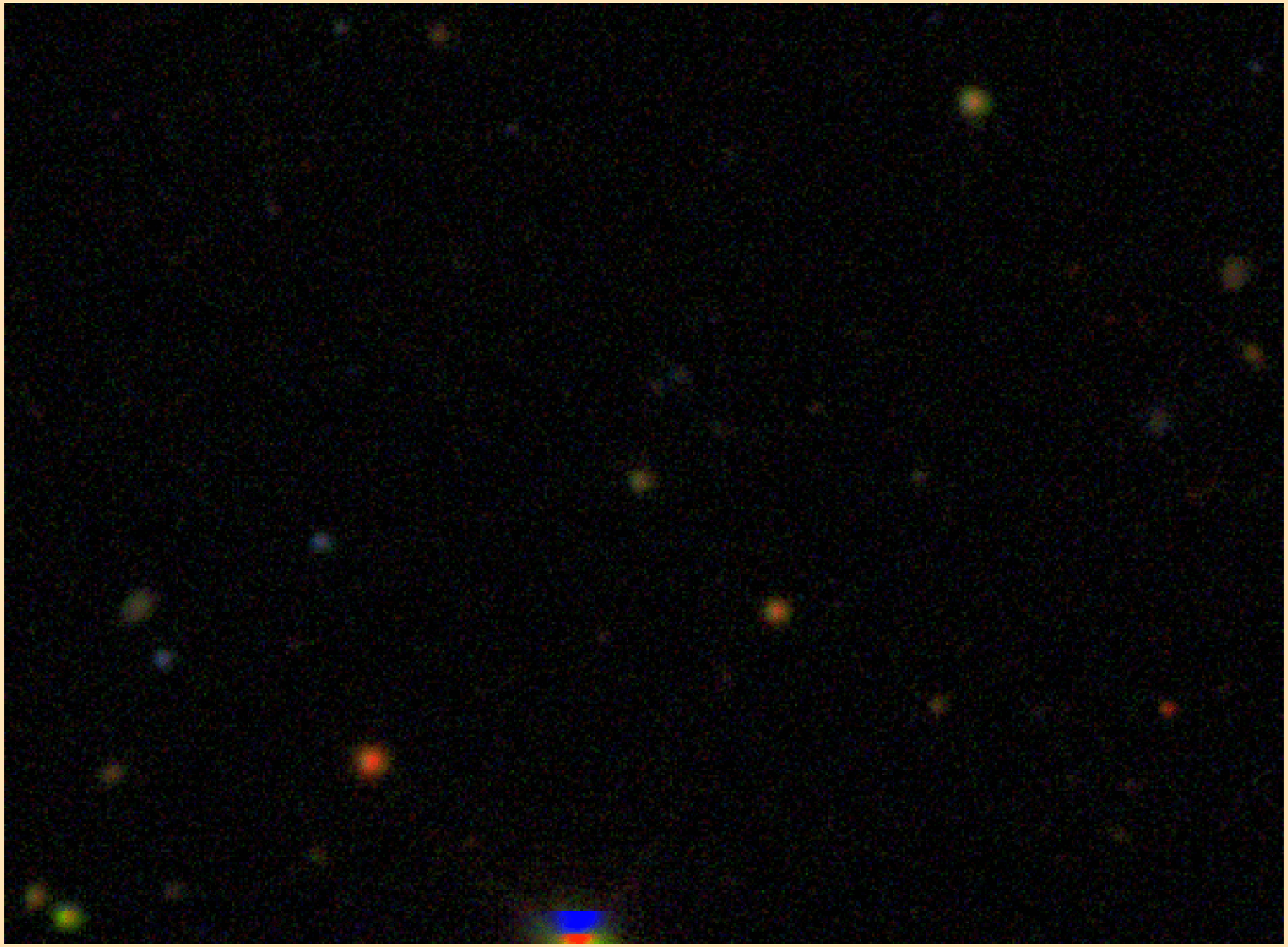

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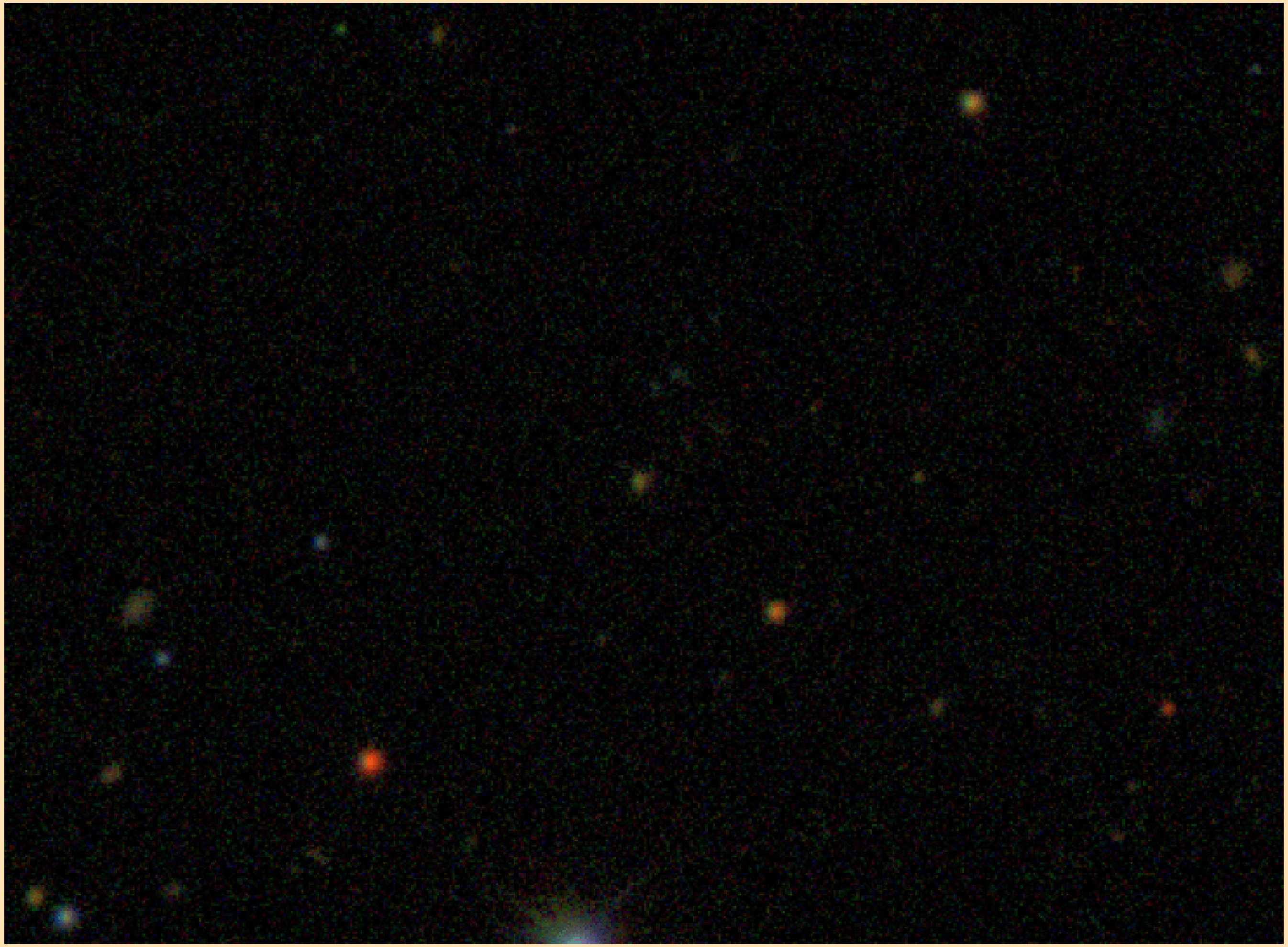

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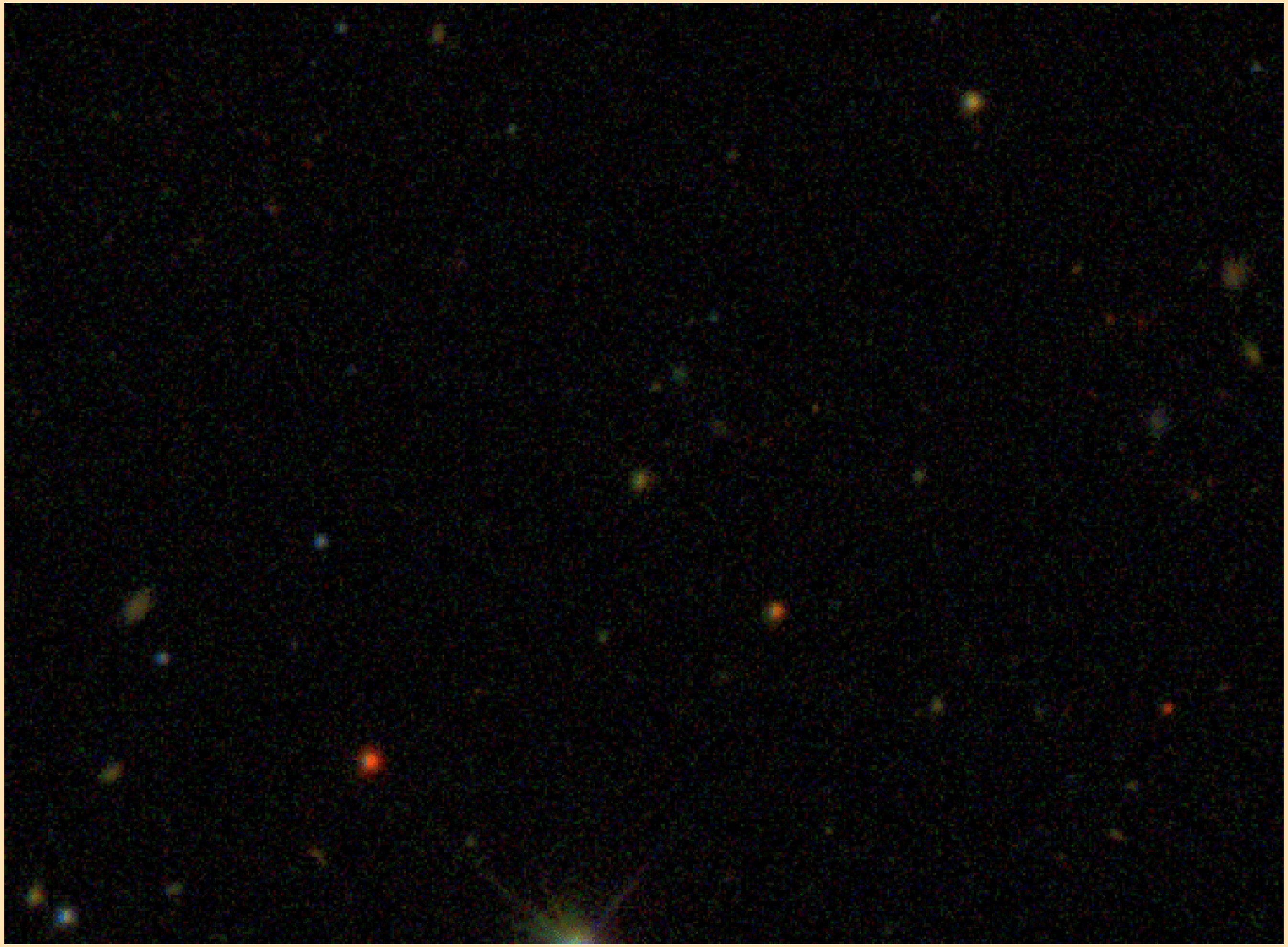

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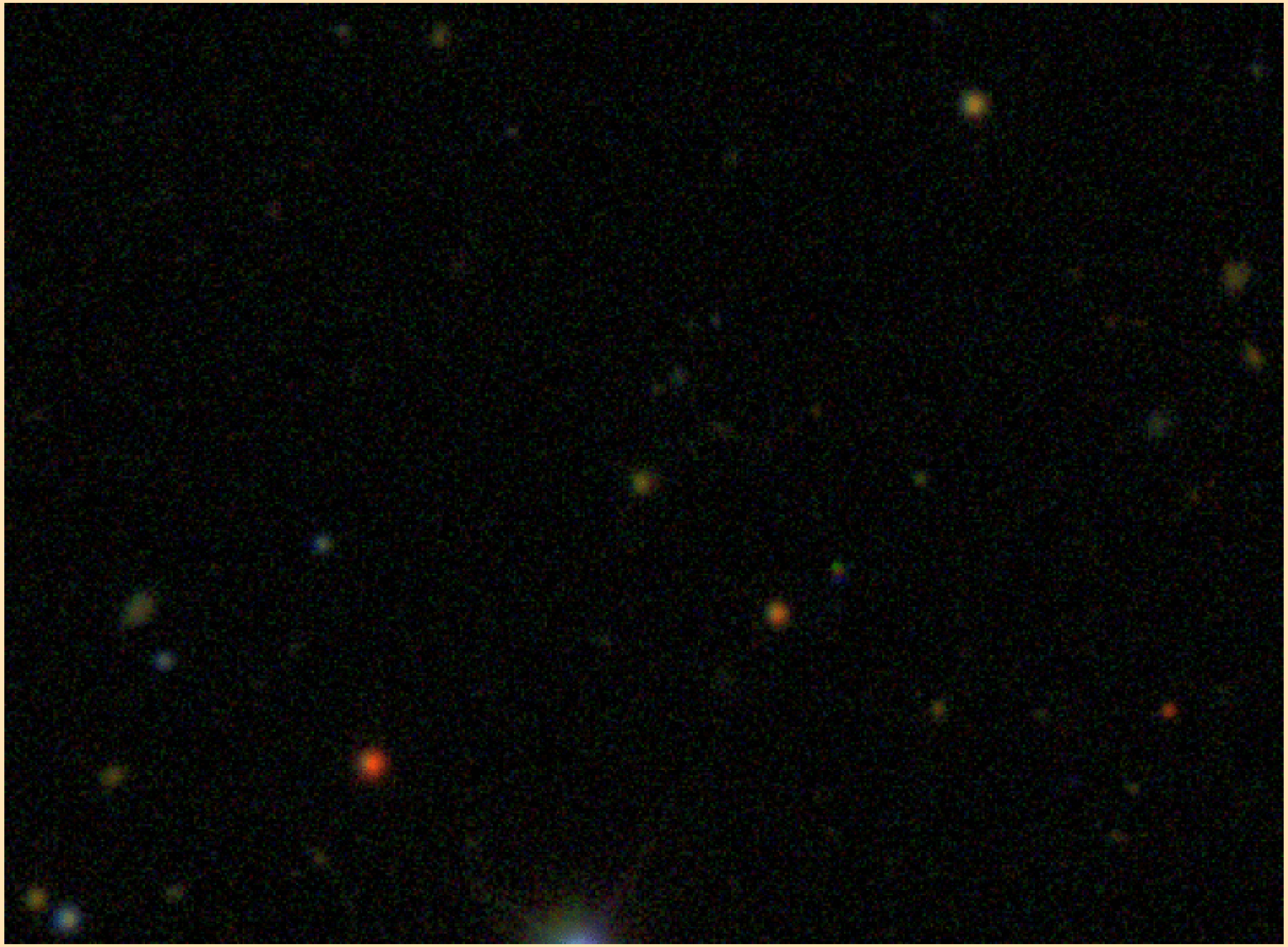

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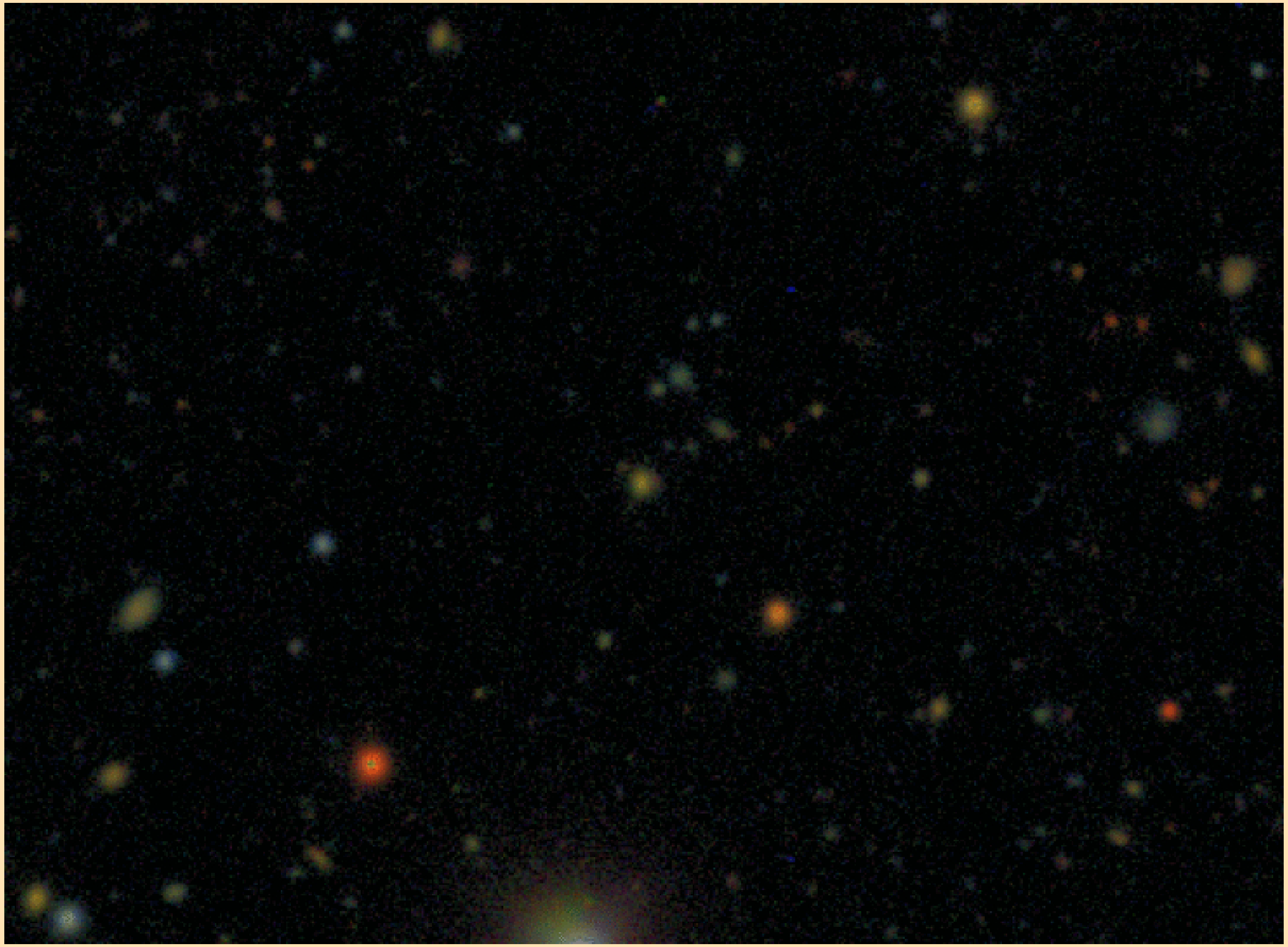

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In general each of these images has

1. a different (and possibly spatially varying) background level
2. its own footprint on the sky,
3. its own geometrical distortion
4. its own PSF structure.

Let us assume that the latter two are known.

Traditional Approaches; SWarp and Drizzle

Historically, most astronomers have used empirical approaches to adding images together (see also the poster by Jalobeanu et al.).

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For example, Bertin's SWarp :

1. Reads input images
2. Optionally reads corresponding variance maps
3. Constructs and removes a background image from each input
4. Resamples onto the desired output projection
5. Averages all of the resampled input images

Issues

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The input images may not be well sampled.

One popular strategy, especially for HST data, is to use Drizzle ; the basic idea here is to map the observed pixels onto a fully-sampled grid, but to pretend that all the flux in concentrated into the central p% (typically $p \sim 50$) of the physical pixel. This preserves surface brightness and has good noise properties, but fails to preserve the PSF.

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It is unclear how well these will scale for very large, noisy, datasets.

Estimation of the True Sky

An alternative approach that has not, so far as I know, been employed in anger was proposed by Kaiser.

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If an point in an image is given by f_p where the pixel index p runs over all the pixels in a set of images and the p^{th} pixel is associated with a PSF g_p, in the limit of uncorrelated Gaussian noise n_p, we may write down the likelihood of the (Fourier transform) true sky $\boldsymbol{I}$, thought of as a vector, as

$$\mathcal{L}(\boldsymbol{I}) = \boldsymbol{I}\boldsymbol{\phi} - \frac{1}{2}\boldsymbol{I}A^{-1}\boldsymbol{I} + \text{constant}$$

where $\boldsymbol{\phi} \equiv \sum_p f_p g_p / \sigma_p^2$ and $A \equiv \sum_p g_p g_p / \sigma_p^2$.

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This is minimised at the point

$$\bar{\boldsymbol{I}} = A^{-1}\boldsymbol{\phi}$$

As Kaiser points out, multiplying by A^{-1} is tantamount to deconvolving the orginal data, but nevertheless the data f_p enter into the estimator $\boldsymbol{I}$ only through the sum $\boldsymbol{\phi}$; in otherwords, $\boldsymbol{\phi}$ is a sufficient statistic for $\boldsymbol{I}$.

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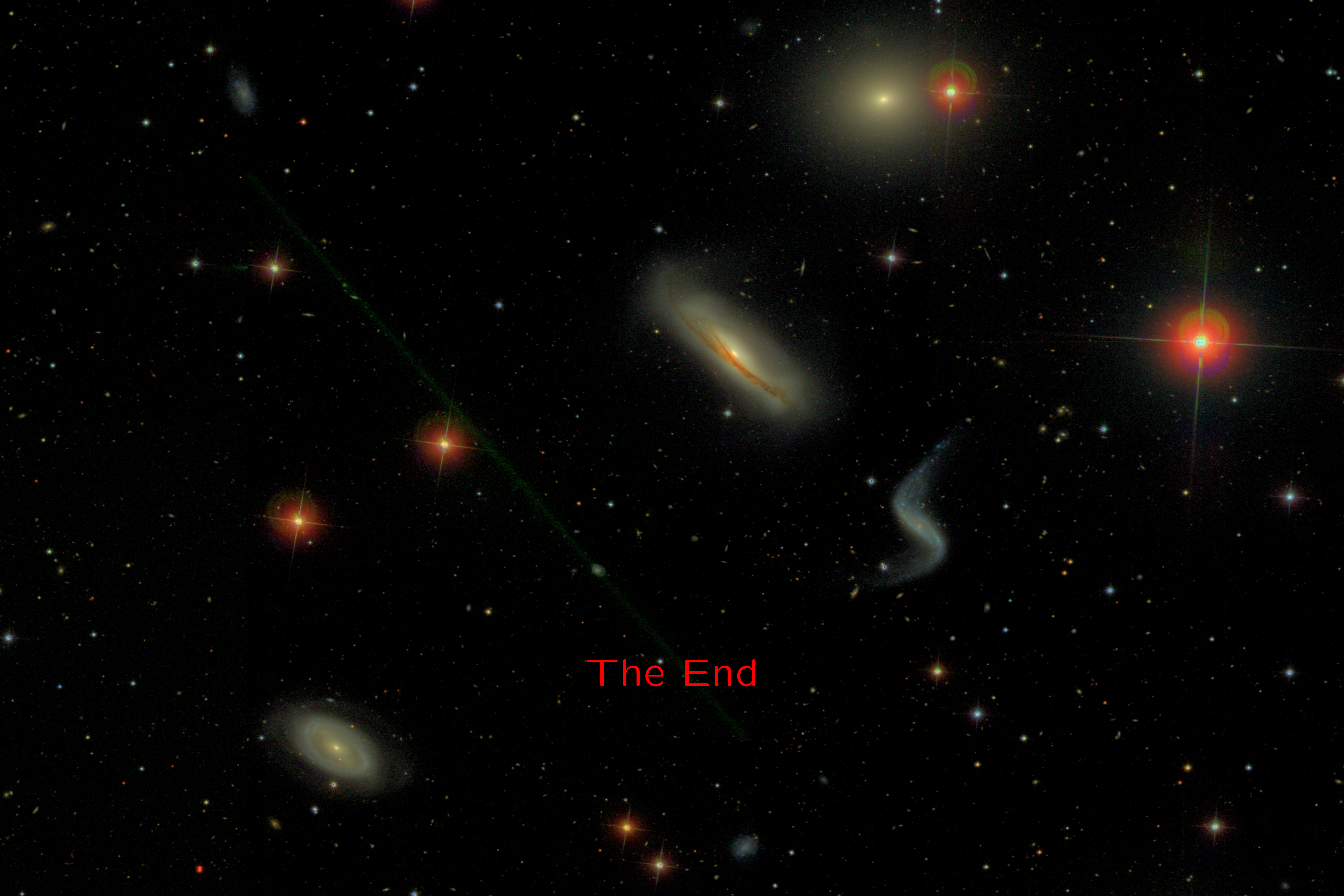
In real space,

$$\boldsymbol{\phi}(\boldsymbol{r}) = \sum_i \left(g_i^\dagger \otimes f_i\right)_{\boldsymbol{r}} / \sigma_i^2$$

which has effective PSF

$$\sum_i \left(g_i^\dagger \otimes g_i\right)_{\boldsymbol{r}} / \sigma_i^2$$


The End