UNIVERSITY OF TORONTO
Faculty of Arts and Science

DECEMBER EXAMINATIONS 2018

STA347H1 (L5101, L2501)

Examination Aids: Non-Programmable Calculators

Instructions

Please show all your work clearly in the space provided; partial credit will be awarded; you may use the back of the pages if necessary but you must remain organized.

There is choice on this exam: Answer any 4 questions of the 8 provided. All complete questions will be valued equally at (15) each but partial grades are shown in brackets to the right of each part.

Q1	**Q2**	**Q3**	**Q4**

Q5	**Q6**	**Q7**	**Q8**	**total**

Name SOLUTIONS ***Student Number***

1. Suppose $X|N \sim bin(N, 3/5)$ and $N \sim \text{geo}(2/3)$.

a) For the variable N obtain both EN and $EN(N-1)$

We have $P(N = n) \;=\; pq^{n-1}$ where $p = 2/3 = 1-q$.

$$EN \;=\; \sum_{n=1}^{\infty} npq^{n-1} \;=\; p\sum_{n=1}^{\infty} nq^{n-1} \;=\; p\frac{d}{dq}\left(\frac{1}{1-q}\right) \;=\; p\frac{1}{(1-q)^2} \;=\; \frac{1}{p} \;=\; \frac{3}{2}$$

$$EN(N-1) \;=\; \sum_{n=2}^{\infty} n(n-1)pq^{n-1} \;=\; pq\sum_{n=2}^{\infty} n(n-1)q^{n-2} \;=\; pq\frac{d^2}{dq^2}\left(\frac{1}{1-q}\right) \qquad (5)$$

$$\;=\; pq\frac{2}{(1-q)^3} \;=\; \frac{2q}{p^2} \;=\; \frac{2/3}{(2/3)^2} \;=\; \frac{3}{2}.$$

b) Hence or otherwise evaluate both EX and $\sigma(X)$

$$EX \;=\; EE(X|N) \;=\; EN(3/5) \;=\; \frac{3}{5}EN \;=\; \frac{3}{5}\times\frac{3}{2} \;=\; \frac{9}{10}$$

$$EX^2 \;=\; EE(X^2|N) \;=\; E\big(N(6/25) + N^2(9/25)\big) \;=\; \frac{6}{25}EN + \frac{9}{25}EN^2$$

$$\;=\; \frac{6}{25}\times\frac{3}{2} + \frac{9}{25}\times\frac{3}{1} \;=\; \frac{36}{25}$$

where (5)

$$EN^2 \;=\; EN(N-1) + EN \;=\; \frac{3}{2} + \frac{3}{2} \;=\; 3$$

And thus, altogether

$$var\,X \;=\; \frac{36}{25} - \frac{81}{100} \;=\; \frac{144}{100} - \frac{81}{100} \;=\; \frac{63}{100} \;\Rightarrow\; \sigma(X) \;=\; \frac{\sqrt{63}}{10} \;\overset{\text{or}}{=}\; \frac{3\sqrt{7}}{10}$$

c) Determine the *correlation coefficient* $\rho(X, N)$

$$EXN \;=\; EE(XN|N) \;=\; ENE(X|N) \;=\; EN^2(3/5) \;=\; \frac{9}{5}$$

$$cov(X, N) \;=\; \frac{9}{5} - \frac{9}{10}\times\frac{3}{2} \;=\; \frac{36}{20} - \frac{27}{20} \;=\; \frac{9}{20}$$

and since (5)

$$varN \;=\; EN^2 - (EN)^2 \;=\; 3 - \frac{9}{4} \;=\; \frac{3}{4}$$

thus, altogether

$$\rho(X, N) \;=\; \frac{cov(X, N)}{\sigma(X)\sigma(N)} \;=\; \sqrt{\frac{3}{7}}.$$

2. Suppose that $(T_n, n \in \mathbb{N})$ is a *poisson process* with $T_n \sim G(n, 5^{-1})$ and let $U = \dfrac{T_2}{T_5}$ & $V = \dfrac{T_2}{T_5 - T_2}$. Determine the following.

a) EU and $\sigma(U)$.

Since $U \sim beta(2,3) \;\Rightarrow\; U \overset{d}{=} \dfrac{Z}{T}$ $w.$ $T = Z + W$ $w.$ $\left.\begin{array}{l} Z \sim G(2) \\ W \sim G(3) \end{array}\right\} \perp\!\!\!\perp .$

Thus we get

$$EU = \frac{EZ}{ET} = \frac{2}{5}.$$

$$EU^2 = \frac{EZ^2}{ET^2} = \frac{6}{30} = \frac{1}{5}$$

so $\quad var\,U = \dfrac{1}{5} - \dfrac{4}{25} = \dfrac{1}{25}$ and $\sigma(U) = \dfrac{1}{5}.$

(5)

b) EV and $\sigma(V)$.

In this case $V \overset{d}{=} \dfrac{Z}{W}$

and thus

$$EV = EZEW^{-1} = 2 \cdot \frac{\Gamma(2)}{\Gamma(3)} = 2 \cdot \frac{1}{2} = 1.$$

$$EV^2 = EZ^2EW^{-2} = \frac{\Gamma(4)}{\Gamma(2)}\frac{\Gamma(1)}{\Gamma(3)} = \frac{6}{2} = 3$$

so $\quad var\,V = 3 - 1 = 2$ and $\sigma(V) = \sqrt{2}.$

(5)

c) $P(U > 1/2)$.

$$P(U > 1/2) = P(2T_2 > T_5) = P(T_2 > T_5 - T_2) = P(V > 1) = P(Z > W)$$

$$= EP(Z > W|W) = EP(T_2 > W) = EP(N_W < 2) \quad w.\ N_W \sim poisson(W)$$

$$= Ee^{-W}(1+W) = \frac{1}{2}\int_0^\infty w^2 e^{-2w}dw + \frac{1}{2}\int_0^\infty w^3 e^{-2w}dw$$

(5)

$$= \frac{1}{2^4}\int_0^\infty (2w)^2 e^{-2w} d2w + \frac{1}{2^5}\int_0^\infty (2w)^3 e^{-2w} d2w = \frac{2}{16} + \frac{3}{16} = \frac{5}{16}.$$

3. Suppose that $X \sim N(1,1)$, let $Y = X^3$, and consider the simple linear model

$$Y = \alpha + \beta X + W \quad w. \quad EW = 0 = \rho(X, W).$$

a) Evaluate the constants α and β.

Letting $X = 1+Z \quad w. \quad Z \sim N(0,1)$

we get

$$\begin{aligned}
\beta &= \frac{cov(X,Y)}{varX} = \frac{cov(X,X^3)}{varX} \\
&= \frac{cov\big(1+Z,(1+Z)^3\big)}{var(1+Z)} \\
&= \frac{cov\big(Z,1+3Z+3Z^2+Z^3\big)}{var(Z)} \\
&= 3varZ + 3cov(Z,Z^2) + cov(Z,Z^3) \\
&= 3varZ + 0 + cov(Z,Z^3) \\
&= 3 + EZ^4 = 3+3 = 6
\end{aligned}$$

(8)

$$\begin{aligned}
\alpha &= EY - \beta EX = EX^3 - 6EX = E(1+Z)^3 - 6E(1+Z) \\
&= 1 + 3EZ^2 - 6 = -2\,.
\end{aligned}$$

b) Determine the relative proximity of Y to its closest linear predictor

$$\frac{\|Y-(\alpha+\beta X)\|}{\|Y-EY\|}.$$

$$\begin{aligned}
&= \sqrt{1-\rho(X,Y)^2} = \sqrt{1-\frac{cov(X,Y)^2}{varX\,varY}} = \sqrt{1-\frac{6^2}{varY}} \\
&= \sqrt{1-\frac{6^2}{varY}} = \sqrt{1-\frac{6^2}{60}} = \sqrt{\frac{4}{10}}
\end{aligned}$$

(7)

where

$$\begin{aligned}
varY &= cov\big(1+3Z+3Z^2+Z^3, 1+3Z+3Z^2+Z^3\big) \\
&= 9varZ + 9varZ^2 + varZ^3 + 18cov(Z,Z^2) + 6cov(Z,Z^3) + 6cov(Z^2,Z^3) \\
&= 9 + 9(3-1) + 15 + 0 + 6\times 3 + 0 = 60\,. \quad \surd
\end{aligned}$$

4. Let $X \sim exp\,(1)$, $Y = e^{-X}$, and consider the simple linear model

$$Y \;=\; \alpha + \beta X + W \quad w. \quad EW = 0 = \rho(X, W).$$

a) Evaluate the constants α and β.

$$\begin{aligned}
\text{noting that } \; cov(X,Y) &= cov(X, e^{-X}) \;=\; EXe^{-X} - EXEe^{-X} \\
&= \int_0^\infty xe^{-x}e^{-x}dx \;-\; \int_0^\infty e^{-x}e^{-x}dx \\
&= \frac{1}{4}\int_0^\infty 2xe^{-2x}d2x \;-\; \frac{1}{2}\int_0^\infty e^{-2x}d2x \\
&= -\frac{1}{4}
\end{aligned}$$

$$\beta \;=\; \frac{cov(X,Y)}{varX} \;=\; \frac{cov(X,e^{-X})}{varX} \;=\; -\frac{1}{4}$$

(8)

$$\alpha \;=\; EY - \beta EX \;=\; Ee^{-X} + \frac{EX}{4} \;=\; \frac{1}{2} + \frac{1}{4} \;=\; \frac{3}{4}.$$

b) Determine the relative proximity of Y to its closest linear predictor

$$\frac{\|Y - (\alpha + \beta X)\|}{\|Y - EY\|}.$$

$$\begin{aligned}
&= \sqrt{1 - \rho(X,Y)^2} \;=\; \sqrt{1 - \frac{cov(X,Y)^2}{varX\, varY}} \\
&= \sqrt{1 - \frac{(1/4)^2}{varY}} \;=\; \sqrt{1 - \frac{12}{16}} \;=\; \frac{1}{2}
\end{aligned}$$

where

(7)

$$\begin{aligned}
varY &= var(e^{-X}) \;=\; Ee^{-2X} - (Ee^{-X})^2 \\
&= \int_0^\infty e^{-3x}dx - \left(\int_0^\infty e^{-2x}dx\right)^2 \;=\; \frac{1}{3} - \frac{1}{4} \;=\; \frac{1}{12}.
\end{aligned}$$

5. Suppose $WX\,|\,X \sim gamma\,(2)$ and $X \sim gamma\,(3)$.

a) First determine $P(W>0)$.

$$P(W>0) \;=\; EP\big(W>0\,\big|X\big) \;=\; EP\big(XW>0\,\big|X\big) \;=\; E1 \;=\; 1\,.$$

(3)

b) Obtain the *joint probability density function* $f(x,w)$ for (X,W).

$$xW\,|\,X=x \;\sim\; G(2) \quad\Rightarrow\quad f_{W|X}(w|x) \;=\; x^2we^{-xw} \quad w>0\,;x>0$$

$$X \sim gamma\,(3) \quad\Rightarrow\quad f_X(x) \;=\; \frac{x^2}{2}e^{-x} \quad x>0$$

and thus (4)

$$f(x,w) \;=\; f_{W|X}(w|x)f_X(x) \;=\; \frac{x^4w}{2}e^{-x(1+w)} \quad x>0, w>0\,.$$

c) Thus obtain the *marginal probability density function*, $f_W(w)$, of W.

$$\begin{aligned} f_W(w) \;=\; \int_0^\infty f(x,w)dx \;&=\; \frac{w}{2}\int_0^\infty x^4e^{-x(1+w)}dx \quad w>0 \\ &=\; \frac{w}{2(1+w)^5}\int_0^\infty z^4e^{-z}dx \quad w>0 \\ &=\; \frac{12w}{(1+w)^5} \qquad w>0\,. \end{aligned}$$

(4)

d) Hence determine the *distribution function* $F_W(t)$ of W.

$$\begin{aligned} F_W(t) \;=\; \int_0^t\frac{12w}{(1+w)^5}dw \;&=\; 12\left(\int_0^t\frac{1}{(1+w)^4}dw \;-\; \int_0^t\frac{1}{(1+w)^5}dw\right) \\ &=\; -12\left(\frac{1}{3(1+w)^3} \;-\; \frac{1}{4(1+w)^4}\right)\Bigg|_0^t \;=\; -\frac{4w+1}{(1+w)^4}\Bigg|_0^t \\ &=\; 1 \;-\; \frac{4t+1}{(1+t)^4} \qquad t>0\,. \end{aligned}$$

(4)

6. Suppose $WX \,|\, X \;\sim\; gamma\,(2)$ and $X \sim gamma\,(3)$.

a) For any value of $n \in \mathbb{N}$ determine $EX^n e^{-X}$.

$$\begin{aligned} EX^n e^{-X} &= \int_0^\infty x^n e^{-x} \frac{x^2 e^{-x}}{2} d \;=\; \frac{1}{2}\int_0^\infty x^{n+2} e^{-2x} dx \\ &= \frac{1}{2^{n+4}} \int_0^\infty (2x)^{n+2} e^{-2x} d2x \;=\; \frac{(n+2)!}{2^{n+4}} \,. \end{aligned} \tag{3}$$

b) Noting that $WX \perp\!\!\!\perp X$, obtain the *joint distribution* of $(1+W)X$ and W.

Since

$$WX \sim G(2)\,,\;\; X \sim G(3) \;\text{ and }\; WX \perp\!\!\!\perp X$$

it follows that

$$(1+W)X \sim G(5) \;,\;\; \frac{X}{(1+W)X} = \frac{1}{1+W} \sim beta(3,2)$$

$$\text{and} \quad (1+W)X \perp\!\!\!\perp W \tag{4}$$

where

$$\frac{1}{1+W} \stackrel{d}{=} \frac{X}{X+Y} \quad \Leftrightarrow \quad W \stackrel{d}{=} \frac{Y}{X} \;\; w. \quad \left.\begin{array}{l} X \sim G(3) \\ Y \sim G(2) \end{array}\right\} \perp\!\!\!\perp \,.$$

c) Hence, or otherwise, determine $E(X|W)$ and $\sigma(X|W)$

From b), it is clear that $(1+W)X \,|\, W \;\sim\; G(5)$

and therefore

$$\begin{aligned} E\big((1+W)X \,|\, W\big) \;=\; 5 &= var\big((1+W)X \,|\, W\big) \\ &= (1+W)^2 var\big(X \,|\, W\big) \end{aligned} \tag{4}$$

in which case

$$E(X|W) \;=\; \frac{5}{1+W} \quad \text{and} \quad \sigma(X|W) \;=\; \frac{\sqrt{5}}{1+W} \,.$$

d) Evaluate $P(W > 1)$

$$\begin{aligned} P(W>1) \;=\; P(Y>X) &= EP(Y>X|X) \\ &= EP(T_2 > X) \\ &= EP(N_X < 2) \quad \text{where } N_X \sim poisson(X) \\ &= E\,e^{-X}(1+X) \;=\; \frac{2}{2^4} + \frac{6}{2^5} \;=\; \frac{5}{16} \,. \end{aligned} \tag{4}$$

7. Suppose that $X \sim gamma(3)$, $Y \sim gamma(2)$ and $X \perp\!\!\!\perp Y$ and let $U = X/(X+Y)$ and $T = X+Y$. Determine the following

a) $E(X^{-1}|\, U)$ and $\sigma(X^{-1}|\, U)$.

We know that $U \perp\!\!\!\perp T$ and of course that $X = UT$ and that $T \sim gamma\,(5)$

$$\text{Thus } E(X^{-1}|\,U) \;=\; E(U^{-1}T^{-1}|\,U) \;=\; U^{-1}E(T^{-1}|\,U) \;=\; U^{-1}ET^{-1}$$
$$=\; \frac{1}{U}\frac{\Gamma(4)}{\Gamma(5)} \;=\; \frac{1}{4\,U}\,. \;\heartsuit$$

$$\text{and } E(X^{-2}|\,U) \;=\; E(U^{-2}T^{-2}|\,U) \;=\; U^{-2}E(T^{-2}|\,U) \;=\; U^{-2}ET^{-2}$$ (8)
$$=\; \frac{1}{U^2}\frac{\Gamma(3)}{\Gamma(5)} \;=\; \frac{1}{12\,U^2}$$

$$\text{whence } var\,(X^{-1}|\,U) \;=\; \frac{1}{U^2}\left(\frac{1}{12} - \frac{1}{16}\right) \;=\; \frac{1}{U^2}\frac{1}{48}$$

$$\text{and thence } \sigma(X^{-1}|\,U) \;=\; \frac{1}{4\sqrt{3}U} \;\overset{\text{or}}{=}\; \frac{\sqrt{3}}{12U}\,. \;\heartsuit$$

b) $E(U^{-1}\,|\,X)$ and $\sigma(U^{-1}\,|\,X)$.

$$E(U^{-1}|\,X) \;=\; E(X^{-1}T\,|\,X) \;=\; X^{-1}E(T\,|\,X) \;=\; X^{-1}\big(E(X\,|\,X) + E(Y\,|\,X)\big)$$
$$=\; \frac{X + EY}{X} \;=\; 1 + \frac{2}{X}\,. \;\heartsuit$$

$$E(U^{-2}|\,X) \;=\; E(X^{-2}T^2|\,X) \;=\; X^{-2}E(X^2{+}2XY{+}Y^2\,|\,X)$$
$$=\; \frac{X^2 + 2XEY + EY^2}{X^2} \;=\; 1 + \frac{4}{X} + \frac{6}{X^2}$$

$$\text{And thus } var\,(U^{-1}|\,X) \;=\; \frac{2}{X^2} \quad\text{and}\quad \sigma(U^{-1}|\,X) \;=\; \frac{\sqrt{2}}{X}\,. \;\heartsuit$$ (7)

8. Suppose $T\,|\,N \sim \textit{gamma}\,(N)$ and $N \sim \textit{geometric}\,(1/3)$.

a) State $P(N=n)$ and $P(T>t\,|\,N=n)$.

$$P(N=n) \;=\; (2/3)^{n-1}(1/3) \qquad n=1,2,\ldots$$

$$P(T>t\,|\,N=n) \;=\; P(N_t<n) \;=\; e^{-t}\sum_{k=0}^{n-1} t^k/k! \qquad t>0 \qquad \text{for each } n=1,2,\ldots \qquad (3)$$

$$\textbf{or} \quad = \int_t^\infty \frac{z^{n-1}}{(n-1)!}e^{-z}dz\,. \heartsuit$$

b) Use a) to find $P(T>t)$ as a function of $t>0$.

$$\begin{aligned} P(T>t) \;=\; EP(T>t\,|\,N) \;&=\; \sum_{n=1}^{\infty}(2/3)^{n-1}(1/3)\int_t^\infty \frac{z^{n-1}}{(n-1)!}e^{-z}dz \\ &= \frac{1}{3}\int_t^\infty \sum_{n=1}^{\infty}\frac{(2z/3)^{n-1}}{(n-1)!}e^{-z}dz \\ &= \frac{1}{3}\int_t^\infty e^{-z/3}\,dz \qquad (3) \\ &= \int_{t/3}^\infty e^{-w}\,dw \\ &= e^{-t/3}. \;\heartsuit \end{aligned}$$

c) Hence, or otherwise, determine ET and $\sigma(T)$.

We obviously have $T=3Z$ where $Z \sim \textit{exp}\,(1)$

Thus immediately $ET \;=\; 3$ $\heartsuit$, $\textit{var}T \;=\; 9$ and $\sigma(T) \;=\; 3\,.$ $\heartsuit$

(3)

d) Using the above, or otherwise, obtain the conditional distribution of $N|T$.

The joint distribution is given by $(2/3)^{n-1}(1/3)\dfrac{t^{n-1}}{(n-1)!}e^{-t}$

so the conditional probability mass function for $N|(T=t)$ is given by

$$\frac{(2/3)^{n-1}(1/3)t^{n-1}e^{-t}}{(n-1)!)(1/3)e^{-t/3}} \;=\; e^{-2t/3}\frac{(2t/3)^{n-1}}{(n-1)!)} \qquad n\in\mathbb{N} \;\heartsuit \qquad (3)$$

which means that $N-1\,|\,T \sim \textit{poisson}(2T/3)$ $\heartsuit$ and $T \sim \textit{exp}(3)$

e) Determine $E(N|T)$ and $\sigma(N|T)$

$$E(N|T) \;=\; 2T/3+1 \;\heartsuit$$

and $\textit{var}(N|T) \;=\; \textit{var}(N-1|T) \;=\; 2T/3 \;\Rightarrow\; \sigma(N|T) \;=\; \sqrt{2T/3}\,.$ $\heartsuit$

(3)