Graph Theory 1-HW6

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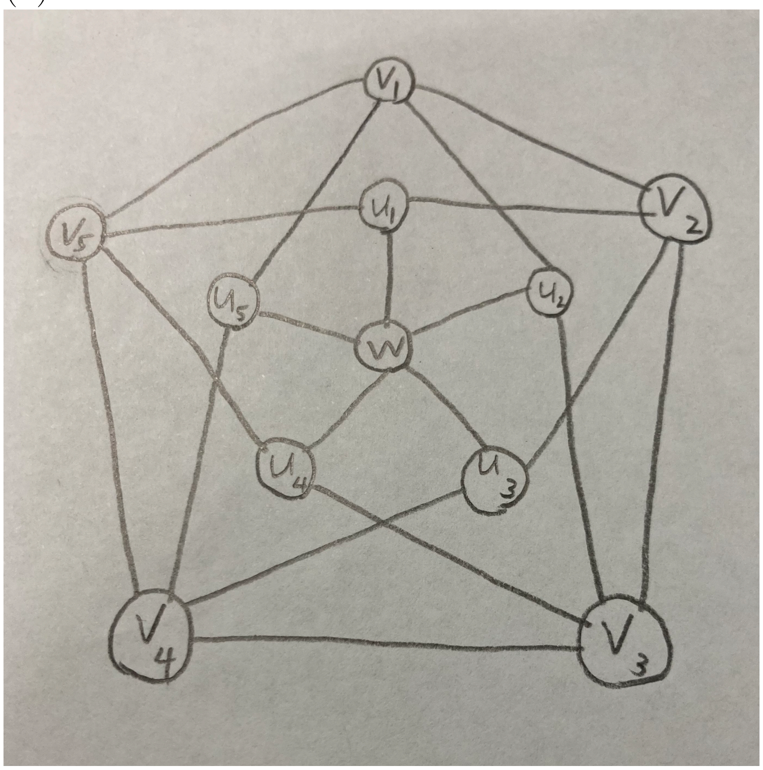
Exercise. (2)

(a)



(b)

$\omega(M(G)) = \omega(G)$:

The case where $\omega(G) = 1$ does not make sense (E should not be empty) so we only consider $\omega(G) \geq 2$.

If there is a new clique larger than the original max clique, then its size must be at least 3, and each of its vertices must be in at least one triangle.

w won't be in any triangle because it's only adjacent to vertices in U and no pair $u_i,\ u_j (i, j \in [n])$ are adjacent to each other.

There can't be more than 1 vertex in U in the same triangle because no pair $u_i,\ u_j(i, j \in [n])$ are adjacent.

Thus, if there is a new clique larger than the original maximum, then it must contain exactly one $u_i(i \in [n])$, but u_i can't be in the same clique with v_i (because they're not adjacent) and we can always get another clique with the same size by replacing u_i with v_i, therefore no new clique can have a size larger than $\omega(G)$, and thus $\omega(M(G)) = \omega(G)$.

$\chi(M(G)) = \chi(G) + 1$:

For any valid coloring of $M(G)$, if there exists a $i \in [n]$ such that the colors of v_i and u_i are different, then we can replace the color of v_i with that of u_i, because their neighbors are the same.

If V is colored with $\chi(G)$ colors and the set of colors that appeared in U is less than $\chi(V)$, then for every $i \in [n]$ with different v_i and u_i colors, we can change the colors one by one, eventually all pairs of v_i and u_i will have the same color and the number of colors in V will decrease. But this contradicts the fact that V is colored with $\chi(M(G))$ colors, therefore U must contain the same number of colors as V, which is $\chi(G)$, and have the same set too (This case exists, one example being the color of u_i and v_i are the same for all $i \in [n]$).

The color of w must be different from all other vertices because it's adjacent to every vertex in U.

Therefore $\chi(M(G)) = \chi(G) + 1$

(c)

Since $\omega(M(G)) = \omega(G)$, if a graph G is triangle-free ($\omega = 2$), then $M(G)$ must be triangle-free.

Now, consider the graph C_4, $\omega(C_4) = 2,\ \chi(C_4) = 2$, we can increase its χ by 1 every time we do the M operation to it, so for every $k \geq 2$, the graph required is $M^{k-2}(C_4)$

Exercise. (4)

Let graph G have n vertices and we label them $v_1,\ v_2, \cdots,\ v_n$.

$\chi(G) = k \Rightarrow$ there exists a way to color each vertex with one of the k colors. We can color

the vertices in the order of $v_1,\ v_2, \cdots,\ v_n$ and let the first appearances of each color (smallest index) be at $v_{u_1},\ v_{u_2}, \cdots,\ v_{u_k}$ (U is an increasing sequence), then for every i, v_{u_i} must have at least $i-1$ different colored adjacent vertices that have smaller indices (which is why it needs a new color), and each of these adjacent relations can map to distinct edges.
If we compute the number of all those edges, we get $0+1+2+3+\cdots+(k-1) = \binom{k}{2}$, therefore $\min\{e(G) : G \in \mathcal{G}_k\} = \binom{k}{2}$

Exercise. (6)

(a)

If we look at the 12 regions one by one, we can see that each of them is adjacent to all 11 other regions, so all 12 of them can't have same colors.

(b)

If there exists a map where it's not possible to color with only 12 colors, there must exists a case where there are 13 countries adjacent to each other on a planar graph.
Let the number of faces be f, the number of vertices (intersections of edges) be v and the number of edges be e.
If 13 regions (each composed of two connected regions) are adjacent to each other, by Euler's formula, $f = 26,\ e = \binom{13}{2} = 78 \Rightarrow v = 2 - 26 + 78 = 54$
But $2e \geq 3v$ (every vertex must be connected to at least 3 edges, and each edge connects 2 vertices), and in this case $2 \cdot 78 < 3 \cdot 54$, therefore such case does not exist and any map can be properly colored with at most 12 colors.