機率與統計 HW5

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May 23, 2024

Problem 1.

(a) If $z \geq 0$, then $x \geq y \geq 0$.
$\Rightarrow f_Z(z) = \int_0^\infty \lambda e^{-\lambda(y+z)} \mu e^{-\mu y} dy$
$= \int_0^\infty \lambda\mu e^{-(\lambda+\mu)y-\lambda z} dy$
$= \frac{\lambda\mu}{-\lambda-\mu} e^{-(\lambda+\mu)y-\lambda z}\Big|_0^\infty$
$= \frac{\lambda\mu}{\lambda+\mu} e^{-\lambda z}$.
If $z \leq 0$, then $y \geq x \geq 0$.
$\Rightarrow f_Z(z) = \int_0^\infty \lambda e^{-\lambda x} \mu e^{-\mu(x-z)} dx$
$= \int_0^\infty \lambda\mu e^{-(\lambda+\mu)x+\mu z} dx$
$= \frac{\lambda\mu}{-\lambda-\mu} e^{-(\lambda+\mu)x+\mu z}\Big|_0^\infty$
$= \frac{\lambda\mu}{\lambda+\mu} e^{\mu z}$.
$\therefore f_Z(z) = \begin{cases} \frac{\lambda\mu}{\lambda+\mu} e^{-\lambda z}, \text{ if } z \geq 0 \\ \frac{\lambda\mu}{\lambda+\mu} e^{\mu z}, \text{ otherwise} \end{cases}$.

(b) Clearly, for $w < 0$, $f_W(w) = 0$.

For $w \geq 0$:

$W = |X - Y| = |Z|$.

$\Rightarrow f_W(w) = f_Z(w) + f_Z(-w) = \frac{\lambda\mu}{\lambda+\mu} e^{-\lambda w} + \frac{\lambda\mu}{\lambda+\mu} e^{-\mu w} = \frac{\lambda\mu}{\lambda+\mu}(e^{-\lambda w} + e^{-\mu w})$.

$\therefore f_W(w) = \begin{cases} \frac{\lambda\mu}{\lambda+\mu}(e^{-\lambda w} + e^{-\mu w}), \text{ if } w \geq 0 \\ 0, \text{ otherwise} \end{cases}$.

Problem 2. $e^{2e^s-2} = e^{2(e^s-1)}$ is the MGF of Poisson(2).
$(\frac{3}{4}e^s + \frac{1}{4}) = (1 - \frac{3}{4} + \frac{3}{4}e^s)^{10}$ is the MGF of Binominal$(10, \frac{3}{4})$.
$\therefore X \sim$ Poisson(2) and $Y \sim$ Binominal$(10, \frac{3}{4})$.
$\Rightarrow P_X(x) = \frac{2^x e^{-2}}{x!}$ for $x = 0, 1, 2, \ldots$, $P_Y(y) = \binom{10}{y}(\frac{3}{4})^y(\frac{1}{4})^{10-y}$ for $y = 0, 1, \ldots, 10$.

(a) $\because X, Y \geq 0$
$\therefore \Pr[X + Y = 2] \overset{\because X,Y \text{ are independent}}{=} P_X(0)P_Y(2) + P_X(1)P_Y(1) + P_X(2)P_Y(0) = e^{-2}45 \times \frac{9}{16}(\frac{1}{4})^8 + 2e^{-2}10 \times \frac{3}{4}(\frac{1}{4})^9 + 2e^{-2}(\frac{1}{4})^{10} = \frac{1}{4^{10}e^2}(405+60+2) = \frac{467}{1048576e^2}$.

(b) $\Pr[XY = 0] = \Pr[X = 0 \vee Y = 0] = \Pr[X = 0] + \Pr[Y = 0] - \Pr[X = 0 \wedge Y = 0] \overset{\because X,Y \text{ are independent}}{=} P_X(0) + P_Y(0) - P_X(0)P_Y(0) = e^{-2} + (\frac{1}{4})^{10} - e^{-2}(\frac{1}{4})^{10} = e^{-2} + \frac{1}{1048576} - \frac{1}{e^2 1048576}$.

(c) $\mathrm{E}[XY] \overset{\because X,Y \text{ are independent}}{=} \mathrm{E}[X]\mathrm{E}[Y] = 2 \times 10 \times \frac{3}{4} = 15$.

Problem 3.

(a) $\phi_X(s) \int_{-\infty}^{\infty} \frac{1}{2}e^{-|x|}e^{sx}dx$
$= \int_{-\infty}^{0} \frac{1}{2}e^{(s+1)x}dx + \int_{0}^{\infty} \frac{1}{2}e^{(s-1)x}dx$
$= \frac{1}{2(s+1)}e^{(s+1)x}\Big|_{-\infty}^{0} + \frac{1}{2(s-1)}e^{(s-1)x}\Big|_{0}^{\infty}$
$= \frac{1}{2(s+1)} - \frac{1}{2(s-1)}$, where the limits exist $\iff \lim_{x\to-\infty} e^{(s+1)x}, \lim_{x\to\infty} e^{(s-1)}x$ exist $\iff s + 1 > 0$ and $s - 1 < 0 \iff -1 < s < 1$.
$\therefore \phi_X(s) = \frac{1}{2(1+s)} + \frac{1}{2(1-s)}$, and it converges iff $|s| < 1$.

(b) $\frac{1}{2(1+s)} = \frac{1}{2}(1 - s + s^2 - s^3 + s^4 - \cdots)$, $\frac{1}{2(1-s)} = \frac{1}{2}(1 + s + s^2 + s^3 + s^4 + \cdots)$.
$\therefore \mathrm{E}[X^{2n}] =$ the coefficient of $s^{2n} = \frac{1}{2} + \frac{1}{2} = 1$.

Problem 4.

Lemma 4.1. Let $n \geq 0$ be an integer, and $c \in \mathbb{R}^+$.

$$f(n) := \int_0^\infty x^n e^{-cx}dx = (\frac{1}{c})^{n+1}n!$$

Proof. Let's have an induction on n to prove this.
For $n=0$, $f(n) = \int_0^\infty e^{-cx}dx = -\frac{1}{c}e^{-cx}\Big|_0^\infty = \frac{1}{c}$, **Lemma (4.1)** holds.
Suppose for $n=m$, $f(m) = (\frac{1}{c})^{m+1}m!$.
For $n=m+1$, $f(m+1) = \int_0^\infty x^{m+1}e^{-cx}dx$

$$= x^{m+1}(-\frac{1}{c}e^{-cx})\Big|_0^\infty + \int_0^\infty (m+1)x^m \cdot \frac{1}{c}e^{-cx}dx$$

$$= \lim_{x\to\infty} \frac{x^{m+1}}{-ce^{cx}} + \frac{m+1}{c}f(m)$$

$$\overset{\text{L'Hospital for } m+1 \text{ times}}{=} \lim_{x\to\infty} \frac{(m+1)!}{-c^{m+2}e^{cx}} + \frac{m+1}{c}f(m)$$

$$= \frac{m+1}{c}f(m) = \frac{m+1}{c}\cdot(\frac{1}{c})^{m+1}m! = (\frac{1}{c})^{m+2}(m+1)!.$$

$\therefore$ by induction, **Lemma (4.1)** holds for all $n \geq 0$. ■

(a) $f_T(t) = \alpha e^{-\alpha t}$.

Clearly, for $n<0$, $P_N(n) = 0$.
For $n \geq 0$, $P_N(n) = \mathrm{E}\left[\frac{(\beta T)^n e^{-\beta T}}{n!}\right]$

$$= \int_0^\infty f_T(t)\frac{(\beta T)^n e^{-\beta T}}{n!}dt$$

$$= \int_0^\infty \alpha e^{-\alpha t}\frac{(\beta T)^n e^{-\beta T}}{n!}dt$$

$$= \frac{\alpha\beta^n}{n!}\int_0^\infty t^n e^{-(\alpha+\beta)t}dt$$

$$\overset{\textbf{Lemma (4.1)}}{=} \frac{\alpha\beta^n}{n!}\cdot\frac{1}{(\alpha+\beta)^{n+1}}n!$$

$$= \frac{\alpha\beta^n}{(\alpha+\beta)^{n+1}}.$$

$$\therefore P_N(n) = \begin{cases} \frac{\alpha\beta^n}{(\alpha+\beta)^{n+1}}, \text{ if } n = 0,1,2,\dots \\ 0, \text{ otherwise} \end{cases}.$$

(b) $\because P_N(n) = \frac{\alpha}{\alpha+\beta}(1-\frac{\alpha}{\alpha+\beta})^n = P_G(n+1)$, where $G \sim \text{Geometric}(\frac{\alpha}{\alpha+\beta})$.
$\therefore N = G-1$.
$\Rightarrow \mathrm{E}[N] = \mathrm{E}[G-1] = \mathrm{E}[G]-1 = \frac{\alpha+\beta}{\alpha} - 1 = \frac{\beta}{\alpha}$.
$\mathrm{Var}(N) = \mathrm{Var}(G-1) = \mathrm{Var}(G) = \frac{1-\frac{\alpha}{\alpha+\beta}}{(\frac{\alpha}{\alpha+\beta})^2} = \frac{\beta(\alpha+\beta)}{\alpha^2}$.

Problem 5.

(a) Let N be the number of games played.

Given $N = n$, there is $\Pr[X_i = 1] = \Pr[X_i = 2] = \dfrac{1}{2}$ for $i < n$.
Also, since each game loses with probability $\dfrac{1}{3}$, there is $N \sim \text{Geometric}(\dfrac{1}{3})$.
$\Rightarrow Y = Z_1 + Z_2 + \cdots + Z_{N-1}$, where $N \sim \text{Geometric}(\dfrac{1}{3})$ and Z_i is an uniform distribution on $\{1, 2\}$.
Note that $Z_1, Z_2, \ldots, Z_{N-1}, N$ are independent.
$\phi_N(s) = \dfrac{\frac{1}{3}e^s}{1 - \frac{2}{3}e^s}$, and $\phi_{N-1}(s) = e^{-s}\phi_N(s) = \dfrac{\frac{1}{3}}{1 - \frac{2}{3}e^s} = \dfrac{1}{3 - 2e^s}$.
By theorem 6.12 in the textbook, $\phi_Y(s) = \phi_{N-1}(\ln \phi_{Z_i}(s)) = \phi_{N-1}(\ln(\dfrac{1}{2}(e^s + e^{2s}))) = \dfrac{1}{3 - 2 \times \frac{1}{2}(e^s + e^{2s})} = \dfrac{1}{3 - e^s - e^{2s}}$.

(b) $\text{E}[Y] = \phi'_Y(0)$
$= \left(-\dfrac{-e^s - 2e^{2s}}{(3 - e^s - e^{2s})^2}\right)\Bigg|_{s=0}$
$= \left(\dfrac{e^s + 2e^{2s}}{(3 - e^s - e^{2s})^2}\right)\Bigg|_{s=0}$
$= \dfrac{3}{1^2} = 3.$
$\text{E}[Y^2] = \phi''_Y(0)$
$= \left(\dfrac{e^s + 4e^{2s}}{(3 - e^s - e^{2s})^2} - 2 \cdot \dfrac{(e^s + 2e^{2s})(-e^s - 2e^{2s})}{(3 - e^s - e^{2s})^3}\right)\Bigg|_{s=0}$
$= \dfrac{5}{1^2} - 2 \cdot \dfrac{3 \times (-3)}{1^3} = 5 + 18 = 23.$
$\text{Var}(Y) = \text{E}[Y^2] - (\text{E}[Y])^2 = 23 - 3^2 = 14.$

Problem 6. Note that $\text{E}[K] = np$, and $\text{E}[K^2] = \text{Var}(K) + (\text{E}[K])^2 = np(1-p) + n^2p^2 = np(1 - p + np)$.
Given there are $X_1 + \cdots + X_n = k$, the expected number of 1 in $X_1, X_2, \ldots, X_k$ is $\dfrac{k^2}{n}$.
$\therefore \text{E}[X_1 + \cdots + X_k | K = k] = \dfrac{k^2}{n}$.
$\text{E}[U] = \sum_{k=0}^{n} \Pr_K(k)\text{E}[X_1 + X_2 + \cdots + X_k | K = k] = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \dfrac{k^2}{n} = \dfrac{1}{n}\text{E}[K^2] = p(1 - p + np)$.
$\text{E}[V] = \sum_{m=0}^{n} \Pr_M(m)\text{E}[X_1 + X_2 + \cdots + X_m | M = m] = \sum_{m=0}^{n} \Pr_M(m)\text{E}[X_1 + X_2 + \cdots + X_m] = \sum_{m=0}^{n} \binom{n}{m} p^m (1-p)^{n-m} pm = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} pk = p\text{E}[K] = np^2$.
$\because np^2 = p(1 - p + np)$ doesn't always hold.

$\therefore$ E$[U]$ and E$[V]$ are not the same.

Problem 7. $\mathrm{E}[X_i^2] = \mathrm{Var}(X_i) + (\mathrm{E}[X_i])^2 = 1.$

$\Rightarrow \phi_{X_i}(s) = 1 + 0 \cdots s + \frac{1}{2}s^2 + \cdots.$

Let $\phi_{X_i}(s) = 1 + \frac{1}{2}s^2 + a_3s^3 + a_4s^4 + \cdots.$

$\phi_{W_n}(s) = (\phi_{X_i}(\frac{s}{\sqrt{n}}))^n = (1 + \frac{1}{2n}s^2 + \frac{a_3}{n^{3/2}}s^3 + \frac{a_4}{n^2}s^4 + \cdots)^n.$

Suppose that $\phi_{W_n}(s) = b_0 + b_1s + b_2s^2 + b_3s^3 + b_4s^4 + \cdots.$

Since $s = 1^{n-1}s$, and the coefficient of s in $\phi_{X_i}(s)$ is 0.

$\therefore b_1 = 0.$

Since $s^2 = 1^{n-2}s^2 = 1^{n-1}(s^2)$, and the coefficient of s in $\phi_{X_i}(s)$ is 0.

$\therefore b_2 = \binom{n}{1} \cdot 1^{n-1}\frac{1}{2n} = \frac{1}{2}.$

Since the coefficient of s in $\phi_{X_i}(s)$ is 0, the only possible combination is $s^3 = 1^{n-1}(s^3)$.

$\therefore b_3 = \binom{n}{1} \cdot 1^{n-1}\frac{a_3}{n^{3/2}} = \frac{a_3}{\sqrt{n}}.$

$\Rightarrow \lim_{n\to\infty} b_3 = 0.$

Since the coefficient of s in $\phi_{X_i}(s)$ is 0, the only possible combinations are $s^4 = 1^{n-2}(s^2)^2, 1^{n-1}(s^4)$.

$\therefore b_4 = \binom{n}{2} \cdot 1^{n-2}(\frac{1}{2n})^2 + \binom{n}{1}1^{n-1}\frac{a_4}{n^2} = \frac{n(n-1)}{8n^2} + \frac{a_4}{n}.$

$\Rightarrow \lim_{n\to\infty} b_4 = \frac{1}{8}.$

$\therefore \mathrm{E}[W_n] = b_1 = 0, \mathrm{E}[W_n^2] = 2b_2 = 1, \mathrm{E}[W_n^3] = 6b_3 = 0, \mathrm{E}[W_n^4] = 24b_4 = 3.$

Problem 8. $\phi_{X_i}(s) = e^{s\cdot 0 + s^2 1^2/2} = e^{s^2/2}.$

$\phi'_{X_i}(s) = se^{s^2/2}.$

$\phi^{(2)}_{X_i}(s) = e^{s^2/2} + s^2e^{s^2/2}.$

$\phi^{(3)}_{X_i}(s) = se^{s^2/2} + 2se^{s^2/2} + s^3e^{s^2/2}.$

$\phi^{(4)}_{X_i}(s) = 3e^{s^2/2} + 3s^2e^{s^2/2} + 3s^2e^{s^2/2} + s^4e^{s^2/2}.$

$\therefore \mathrm{E}[X_i^4] = \phi^{(4)}_{X_i}(0) = 3.$

$\mathrm{E}[X_i^2] = \mathrm{Var}(X_i) + (\mathrm{E}[X_i])^2 = \mathrm{Var}(X_i) = 1.$

$\mathrm{Var}(X_i^2) = \mathrm{E}[X_i^4] - (\mathrm{E}[X_i^2])^2 = 3 - 1 = 2.$

By Central Limit Theorem, for i.i.d. $Y_1, Y_2, \ldots, Y_n$, $F_{Y_1+Y_2+\cdots+Y_n}(t) \approx \Phi(\frac{t - n\mu_Y}{\sqrt{n\sigma_Y^2}})$.

$\therefore \lim_{n\to\infty} \Pr[X_1^2 + \cdots + X_n^2 \le n + c\sqrt{n}] = \Phi(\frac{n + c\sqrt{n} - n\mathrm{E}[X_i^2]}{\sqrt{n\mathrm{Var}(X_i^2)}} = \Phi(\frac{c\sqrt{n}}{\sqrt{2n}}) = \Phi(\frac{c}{\sqrt{2}}).$

Problem 9. $\Pr[M_n(X) \geq c] \overset{\text{Chernoff bound}}{\leq} \min_{s\geq 0} e^{-sc}\phi_{M_n(X)}(s)$
$\overset{X_1,X_2,\ldots,X_n \text{ are independent}}{=} \min_{s\geq 0} e^{-sc}\phi_X(\frac{s}{n})^n$
$\overset{t=\frac{s}{n}}{=} \min_{t\geq 0} e^{-tnc}\phi_X(t)^n$
$= \min_{t\geq 0}(e^{-tc}\phi_X(t))^n$
$= (\min_{s\geq 0} e^{-sc}\phi_X(s))^n.$

Problem 10.

(a) X_i is the sum of 10 independent Bernoulli trials, so $X_i \sim \text{Binomial}(10, 0.8)$.
$\Rightarrow P_{X_i}(x) = \binom{10}{x} 0.8^x 0.2^{1-x}$.

(b) X is the average of 100 independent Bernoulli trials (denote them by $Y_1, Y_2, \ldots, Y_{100}$), so $X = 0.01Z$, and $Z \sim \text{Binomial}(100, 0.8)$.
By Central Limit Theorem, $\Pr[X \leq x] = \Pr[Z \leq 100x] \approx \Phi(\frac{100x - 100\mu_{Y_i}}{\sqrt{100\sigma_{Y_i}^2}}) = \Phi(\frac{100x-80}{\sqrt{100 \times 0.8 \times 0.2}}) = \Phi(\frac{100x-80}{4})$.
$\therefore \Pr[A] = \Pr[X \geq 0.9] = 1 - \Pr[X < 0.9] \approx 1 - \Phi(\frac{10}{4}) = 1 - \Phi(2.5) \approx 0.0062.$

(c) (Use the notations and results in (b)).
$\Pr[A] = \Pr[X' \geq 0.9] = \Pr[\frac{10n + \sum_{i=1}^{10} X_i}{10n+100} \geq 0.9] = \Pr[\sum_{i=1}^{10} X_i \geq 0.9(10n+100) - 10n] = \Pr[Z \geq 90 - n] \approx 1 - \Phi(\frac{90-n-80}{4}) = 1 - \Phi(\frac{10-n}{4})$.

(d) It will affect the letter grade $\iff$ in the following 90 questions, exactly $89-8, 79-8, 69-8, 59-8$ are correct.
$\therefore$ the probability $= \binom{90}{81}(0.8)^{81}(0.2)^9 + \binom{90}{71}(0.8)^{71}(0.2)^{19} + \binom{90}{61}(0.8)^{61}(0.2)^{29} + \binom{90}{51}(0.8)^{51}(0.2)^{39} \approx 0.1064.$