Graph Theory HW1

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Problem 2. First, let's prove that $R(3,4) \leq 9$. That is, for any red/blue-edge-coloring of K_9, there exists either a red K_3 or a blue K_4.

Let G be a K_9, and $c : E(G) \to \{R, B\}$ be an arbitrary coloring.

Let $H_R := (V(G), \{e \in E(G) : c(e) = R\}), H_B := (V(G), \{e \in E(G) : c(e) = B\})$.

If for all $v \in V(G)$, $2 \nmid \deg_{H_R}(v)$, then $\sum_{v \in V(G)} d_{H_R} \equiv 9 \times 1 \equiv 1 \pmod 2$, which contradicts to the handshake lemma.

$\therefore \exists v \in V(G)$ s.t. $2 \mid \deg_{H_R}(v)$.

There are two cases:

Case 1: $\deg_{H_R}(v) \geq 4$.

Let $S := N_{H_R}(v)$.

If there are distinct $u, w \in S$ with $c(uw) = R$, then $\{u, v, w\}$ forms a red K_3 as $c(uv) = c(wv) = R$.

Otherwise, for all $u, w \in S$, $c(uw) = B$, then S forms a blue $K_{|S|} = K_{\deg_{H_R}(v)}$, which contains a blue K_4 as a subgraph.

Case 2: $\deg_{H_R}(v) \leq 2$.

$\Rightarrow \deg_{H_B}(v) = \deg_G(v) - \deg_{H_R}(v) = 8 - \deg_{H_R}(v) \geq 6$.

Let $T := N_{H_B}(v)$.

Since $R(3,3) = 6$, T contains a monochromatic K_3.

If T does not contain a red K_3, then it contains a blue K_3. Let U be the set of vertices that form the blue K_3 in T. Since $c(uv) = B$, $\forall u \in T$, $U \cup \{v\}$ forms a blue K_4.

$\therefore$ either a red K_3 or a blue K_4 exists in a red/blue-edge-coloring of K_9.

From Exercise 1(a), $R(4,4) \leq R(3,4) + R(4,3) = R(3,4) + R(3,4) \leq 18$.