Portfolio Optimization
投資組合最佳化

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Outline

- Math background
  - Linear combination of random variables
- Portfolio optimization
  - Problem definition
  - Objective functions
  - Matrix formulas
  - Efficient frontier
- References

Linear Combination of Random Variables

- Given two random variables X and Y
  - Definition
    - Mean: $\mu_X = E(X)$
    - Variance: $\sigma_X^2 = V(X) \triangleq E((X-\mu_X)^2) = E(X^2) - \mu_X^2$
    - Covariance: $\sigma_{XY} = \sigma_{YX} = E((X-\mu_X)(Y-\mu_Y)) = E(XY) - \mu_X\mu_Y$
  - Basic formulas for a single variable
    - $E(aX) = aE(X) \Rightarrow \mu_{aX} = a\mu_X$
    - $V(aX) = a^2V(X) \Rightarrow \sigma_{aX}^2 = a^2\sigma_X^2$
  - Extension to two variables (not necessarily independent)
    - $E(aX+bY) = aE(X) + bE(Y)$
      $\Rightarrow \mu_{aX+bY} = a\mu_X + b\mu_Y$
    - $V(aX+bY) = a^2V(X) + b^2V(Y) + 2abE((X-\mu_X)(Y-\mu_Y))$
      $\Rightarrow \sigma_{aX+bY}^2 = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\sigma_{xy} = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\sigma_X\sigma_Y\rho_{XY}$

https://hackmd.io/@rogerjang/BJwCkf0LK, with exercise!

Portfolio Optimization (PO)

- Goal
  - To maximize the overall returns or minimize the overall variance of a portfolio of n assets based on each individual expected return and variance (aka risk or volatility)
- Facts
  - Introduced in a 1952 doctoral thesis by Harry Markowitz (awarded Nobel Memorial Prize in Economic Science in 1990)
  - Also known as mean-variance model or Markowitz model, which is foundational to Modern Portfolio Theory (MPT)
- Assumptions
  - Risk or volatility is equivalent to standard deviation.
  - No consideration for taxes, transaction fees, etc.

PO for Two Assets: Combined Mean and Variance

Given two risky assets as follows:

$$\begin{cases} \text{Asset 1: } \mu = \mu_1, \sigma^2 = \sigma_1^2 \\ \text{Asset 2: } \mu = \mu_2, \sigma^2 = \sigma_2^2 \end{cases}$$

We can use a weight vector $[w_1, w_2]^T$, with $w_1 + w_2 = 1$ to allocate these two assets to have mean return μ and risk (variance) σ^2:

μ: Overall mean return

σ: Overall risk or volatility

$$\begin{cases} \mu & = & w_1\mu_1 + w_2\mu_2 \\ \sigma^2 & = & w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_{12} \end{cases}$$

We can put the above equations into a matrix form (with $\sigma_{12} = \sigma_{21}$):

$$\begin{cases} \mu & = & \begin{bmatrix} \mu_1 & \mu_2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \\ \sigma^2 & = & \begin{bmatrix} w_1 & w_2 \end{bmatrix} \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \end{cases}$$

PO for Two Assets: Efficient Frontier

Instead of using two parameters in the above expression, we can use only a single parameter w, with $w_1 = w$ and $w_2 = 1 - w$:

$$\begin{cases} \mu & = & w\mu_1 + (1-w)\mu_2 \\ \sigma^2 & = & w^2\sigma_1^2 + (1-w)^2\sigma_2^2 + 2w(1-w)\sigma_{12} \end{cases}$$

From the first expression, we have

$$w = \frac{\mu_2 - \mu}{\mu_2 - \mu_1}, 1 - w = \frac{\mu - \mu_1}{\mu_2 - \mu_1}$$

Therefore

$$\begin{aligned} \sigma^2 &= \left(\frac{\mu_2-\mu}{\mu_2-\mu_1}\right)^2\sigma_1^2 + \left(\frac{\mu-\mu_1}{\mu_2-\mu_1}\right)^2\sigma_2^2 + 2\left(\frac{\mu_2-\mu}{\mu_2-\mu_1}\right)\left(\frac{\mu-\mu_1}{\mu_2-\mu_1}\right)\sigma_{12} \\ &= \frac{1}{(\mu_2-\mu_1)^2}[(\mu^2 - 2\mu_2\mu + \mu_2^2)\sigma_1^2 + (\mu^2 - 2\mu_1\mu + \mu_1^2)\sigma_2^2 - 2(\mu^2 - (\mu_1+\mu_2)\mu + \mu_1\mu_2)\sigma_{12}] \\ &= \frac{1}{(\mu_2-\mu_1)^2}[(\sigma_1^2 + \sigma_2^2 - 2\sigma_{12})\mu^2 - 2(\mu_2\sigma_1^2 + \mu_1\sigma_2^2 - (\mu_1+\mu_2)\sigma_{12})\mu + \mu_2^2\sigma_1^2 + \mu_1^2\sigma_2^2 - 2\mu_1\mu_2\sigma_{12}] \end{aligned}$$

Since $\sigma_1^2 + \sigma_2^2 - 2\sigma_{12} \geq 2(\sigma_1\sigma_2 - \sigma_{12}) \geq 0$, the above equation is a hyperbola on the $\sigma - \mu$ plane. It can reduce to a parabola if $\sigma_1^2 + \sigma_2^2 - 2\sigma_{12} = 0$.

https://hackmd.io/@rogerjang/SJa16FaLt, with exercise!

PO for n Assets: Problem Definition

In general, for n assets, we can combine them to the overall return μ and risk σ:

$$\begin{cases} \mu & = & \boldsymbol{\mu}^T \mathbf{w} \\ \sigma^2 & = & \mathbf{w}^T \Sigma \mathbf{w} \end{cases}$$

where $\mathbf{w} = [w_1, \ldots, w_n]^T$, $\boldsymbol{\mu} = [\mu_1, \ldots, \mu_n]^T$, and Σ is the covariance matrix of these n assets.

Suppose we want to minimize risk with fixed return, as follows.

$$\min_{\mathbf{w}} \sigma^2 = \mathbf{w}^T \Sigma \mathbf{w}$$

$$s.t. \begin{cases} \mathbf{1}^T \mathbf{w} = 1 \\ \boldsymbol{\mu}^T \mathbf{w} = \mu_0 \end{cases}$$

where $\mathbf{1} = [1, \ldots, 1]^T$.

Objective Functions for PO (1/2)

- Minimize risk with fixed return: Given a return μ, find the weights to minimize the overall variance σ^2. (給定預期報酬值，最佳投資組合將產生最小風險。)

$$\min_{\mathbf{w}} \sigma^2 = \mathbf{w}^T \Sigma \mathbf{w}$$

$$s.t. \begin{cases} \mathbf{w}^T \mathbf{1} = 1 \\ \boldsymbol{\mu}^T \mathbf{w} = \mu_0 \end{cases}$$

where $\mathbf{1} = [1, \ldots, 1]^T$.

- Maxmimize return with fixed risk: Given a variance σ^2, find the weights to maximize the overall return μ. (給定風險下，最佳投資組合將產生預期報酬最大值。)

$$\max_{\mathbf{w}} \mu = \boldsymbol{\mu}^T \mathbf{w}$$

$$s.t. \begin{cases} \mathbf{w}^T \mathbf{1} = 1 \\ \mathbf{w}^T \Sigma \mathbf{w} = \sigma_0^2 \end{cases}$$

Objective Functions for PO (2/2)

- Minimize risk regardless of return: Find the weights to minimize the overall variance σ^2 regardless of the return. (讓最佳投資組合將產生最小風險，而完全不看報酬。)

$$\min_{\mathbf{w}} \sigma^2 = \mathbf{w}^T \Sigma \mathbf{w}$$
$$s.t.\ \mathbf{w}^T \mathbf{1} = 1.$$

- Maximize the Sharpe ratio

$$\max_{\mathbf{w}} \frac{\mu - \mu_0}{\sigma}$$
$$s.t.\ \mathbf{w}^T \mathbf{1} = 1$$

- Maximize the difference between returm and risk

$$\max_{\mathbf{w}} \mu - \beta\sigma$$
$$s.t.\ \mathbf{w}^T \mathbf{1} = 1$$

In fact, there are a lot more objective functions and constraints in practice!

$PO_{n=2}$: $\rho_{12}=1$

$$\begin{cases} \text{Asset 1: } \mu = \mu_1, \sigma = \sigma_1 \\ \text{Asset 2: } \mu = \mu_2, \sigma = \sigma_2 \end{cases} \xrightarrow{w_1 + w_2 = 1} \begin{cases} \mu &= w_1\mu_1 + w_2\mu_2 \\ \sigma^2 &= w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_1\sigma_2\rho_{12} \end{cases}$$

When $\rho_{12} = 1$, we have

$$\sigma^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_1\sigma_2 = (w_1\sigma_1 + w_2\sigma_2)^2 \Rightarrow \begin{cases} \mu &= w_1\mu_1 + w_2\mu_2 \\ \sigma &= |w_1\sigma_1 + w_2\sigma_2| \end{cases}$$

As w_1 is changing from 0 to 1, the above equations represent a line connecting (σ_1, μ_1) (when $w_1 = 1$ and $w_2 = 0$) and (σ_2, μ_2) (when $w_1 = 0$ and $w_2 = 1$). So the minimum variance is $\min(\sigma_1^2, \sigma_2^2)$.

$PO_{n=2}$: ρ_{12}=0

When $\rho_{12} = 0$, we have

$$\sigma^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2$$

By using Cauchy-Schwartz inequality, we have

$$(w_1^2\sigma_1^2 + w_2^2\sigma_2^2)(\sigma_1^{-2} + \sigma_2^{-2}) \geq (w_1 + w_2)^2 = 1$$

Therefore the minimum variance can be derived as follows:

$$\sigma^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 \geq (\sigma_1^{-2} + \sigma_2^{-2})^{-1} = \frac{\sigma_1^2\sigma_2^2}{\sigma_1^2 + \sigma_2^2}$$

The equality holds when

$$w_1^2\sigma_1^2/\sigma_1^{-2} = w_2^2\sigma_2^2/\sigma_2^{-2} \Rightarrow w_1 = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}, w_2 = \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2}$$

$PO_{n=2}$: ρ_{12}=-1

When $\rho_{12} = -1$, we have

$$\sigma^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 - 2w_1w_2\sigma_1\sigma_2 = (w_1\sigma_1 - w_2\sigma_2)^2$$

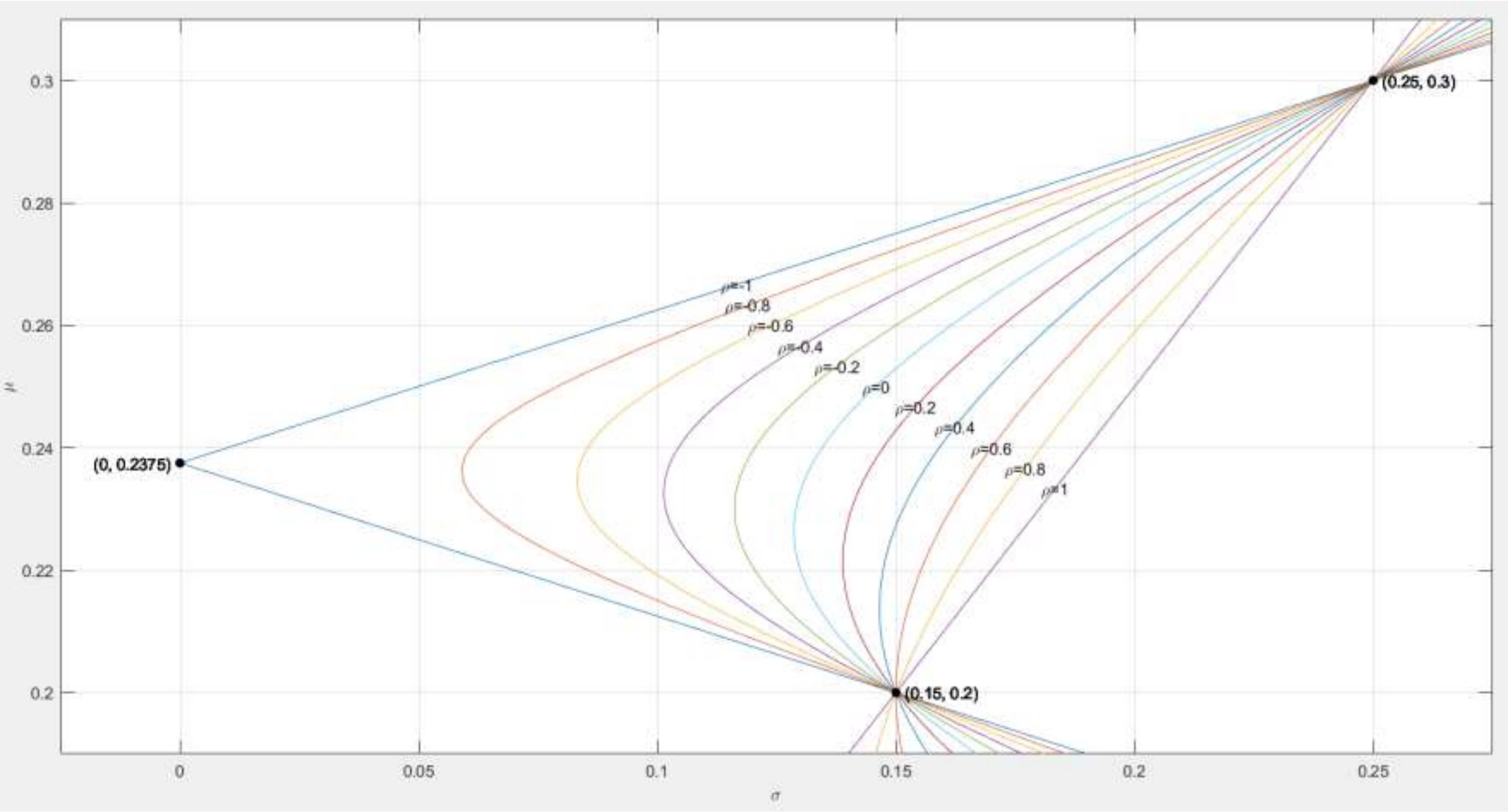
In this case, we can achieve zero risk by setting

$$w_1\sigma_1 = w_2\sigma_2 \Rightarrow w_1 = \frac{\sigma_2}{\sigma_1 + \sigma_2}, w_2 = \frac{\sigma_1}{\sigma_1 + \sigma_2}$$

Therefore the minimum variace is 0, and the corresponding return is $\frac{\sigma_2\mu_1+\sigma_1\mu_2}{\sigma_1+\sigma_2}$.

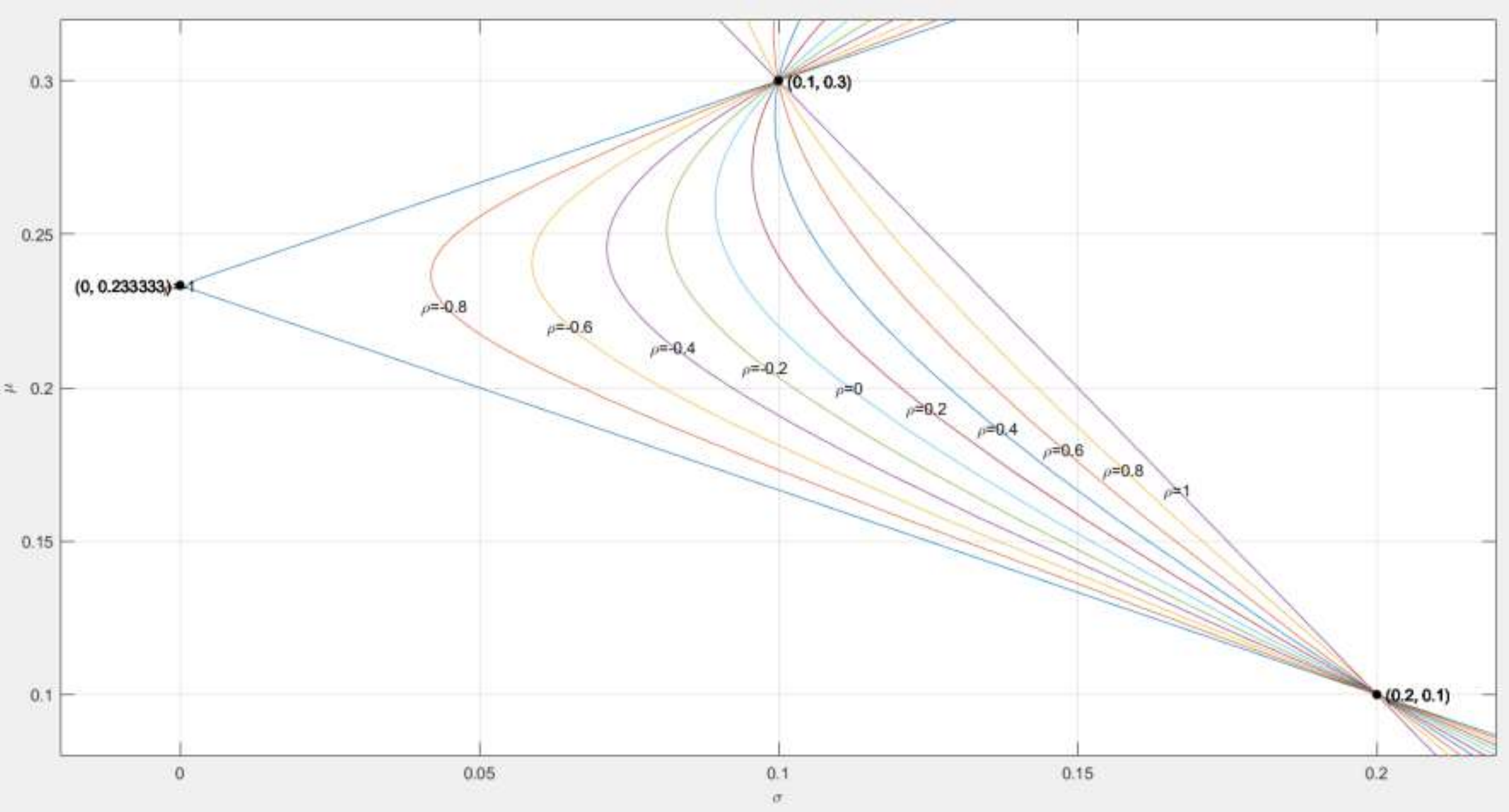
$PO_{n=2}$: Efficient Frontier with Varying ρ_{12} (1/2)

- n=2: $(\sigma_1, \mu_1)=(0.15, 0.2)$ and $(\sigma_2, \mu_2)=(0.25, 0.3)$



$PO_{n=2}$: Efficient Frontier with Varying ρ_{12} (2/2)

- n=2: (σ_1, μ_1)=(0.2, 0.1) and (σ_2, μ_2)=(0.1, 0.3)



PO_n: Min. Variance Only

In general, for n assets, we can combine them to the overall return μ and risk σ:

$$\begin{cases} \mu & = & \boldsymbol{\mu}^T \mathbf{w} \\ \sigma^2 & = & \mathbf{w}^T \Sigma \mathbf{w} \end{cases}$$

where $\mathbf{w} = [w_1, \ldots, w_n]^T$, $\boldsymbol{\mu} = [\mu_1, \ldots, \mu_n]^T$, and Σ is the covariance matrix of these n assets.

Suppose we want to minimize the overall risk regardless of the overall return, then the problem can be formulated as follows:

$$\min_{\mathbf{w}} \sigma^2 = \mathbf{w}^T \Sigma \mathbf{w}$$
$$\text{s.t. } \mathbf{1}^T \mathbf{w} = 1$$

PO_n: Min. Variance Only (Block-form Solution)

To find the solution to this constrained optimization problem, we can formulate a new objective function using the Lagrange multiplier:

$$\max_{\mathbf{w},\lambda} J(\mathbf{w},\lambda) = \mathbf{w}^T\Sigma\mathbf{w} + \lambda(\mathbf{1}^T\mathbf{w} - 1).$$

By taking the gradient and set it to zero, we have

$$\nabla_{\boldsymbol{w}} J(\boldsymbol{w},\lambda) = 2\Sigma\mathbf{w} + \mathbf{1}\lambda = 0 \Rightarrow \mathbf{w} = -\frac{1}{2}\Sigma^{-1}\mathbf{1}\lambda$$

Since $\mathbf{1}^T\mathbf{w} = 1$, we have

$$-\frac{1}{2}\mathbf{1}^T\Sigma^{-1}\mathbf{1}\lambda = 1 \Rightarrow \lambda = -\frac{2}{\mathbf{1}^T\Sigma^{-1}\mathbf{1}}.$$

Therefore

$$\mathbf{w} = \frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}^T\Sigma^{-1}\mathbf{1}}$$

PO_n: Min. Var. with Fixed Return

To solve this problem, we can use the Lagrange multiplier to form a new objective function:

$$\max_{\mathbf{w},\lambda_1,\lambda_2} J(\mathbf{w},\lambda_1,\lambda_2) = \mathbf{w}^T\Sigma\mathbf{w} + \lambda_1(\mathbf{1}^T\mathbf{w} - 1) + \lambda_2(\boldsymbol{\mu}^T\mathbf{w} - \mu_0),$$

We can take the gradient and set it to zero to have the following equations:

$$\begin{cases} 2\Sigma\mathbf{w} + \mathbf{1}\lambda_1 + \boldsymbol{\mu}\lambda_2 & = & \mathbf{0} \\ \mathbf{1}^T\mathbf{w} & = & 1 \\ \boldsymbol{\mu}^T\mathbf{w} & = & \mu_0 \end{cases}$$

(Note that we omit the use of "hat" to keep simplicity.)

When n=3

$$\begin{bmatrix} 2\Sigma_{11} & 2\Sigma_{12} & 2\Sigma_{13} & 1 & \mu_1 \\ 2\Sigma_{21} & 2\Sigma_{22} & 2\Sigma_{23} & 1 & \mu_2 \\ 2\Sigma_{31} & 2\Sigma_{32} & 2\Sigma_{33} & 1 & \mu_3 \\ 1 & 1 & 1 & 0 & 0 \\ \mu_1 & \mu_2 & \mu_3 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ \mu_0 \end{bmatrix}$$

PO_n: Min. Var. with Fixed Return (Block-form Solution)

If we put the above equations into the block form:

$$\begin{bmatrix} 2\Sigma & B \\ B^T & \mathbf{0}_{2\times 2} \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{n\times 2} \\ \mathbf{k} \end{bmatrix}$$

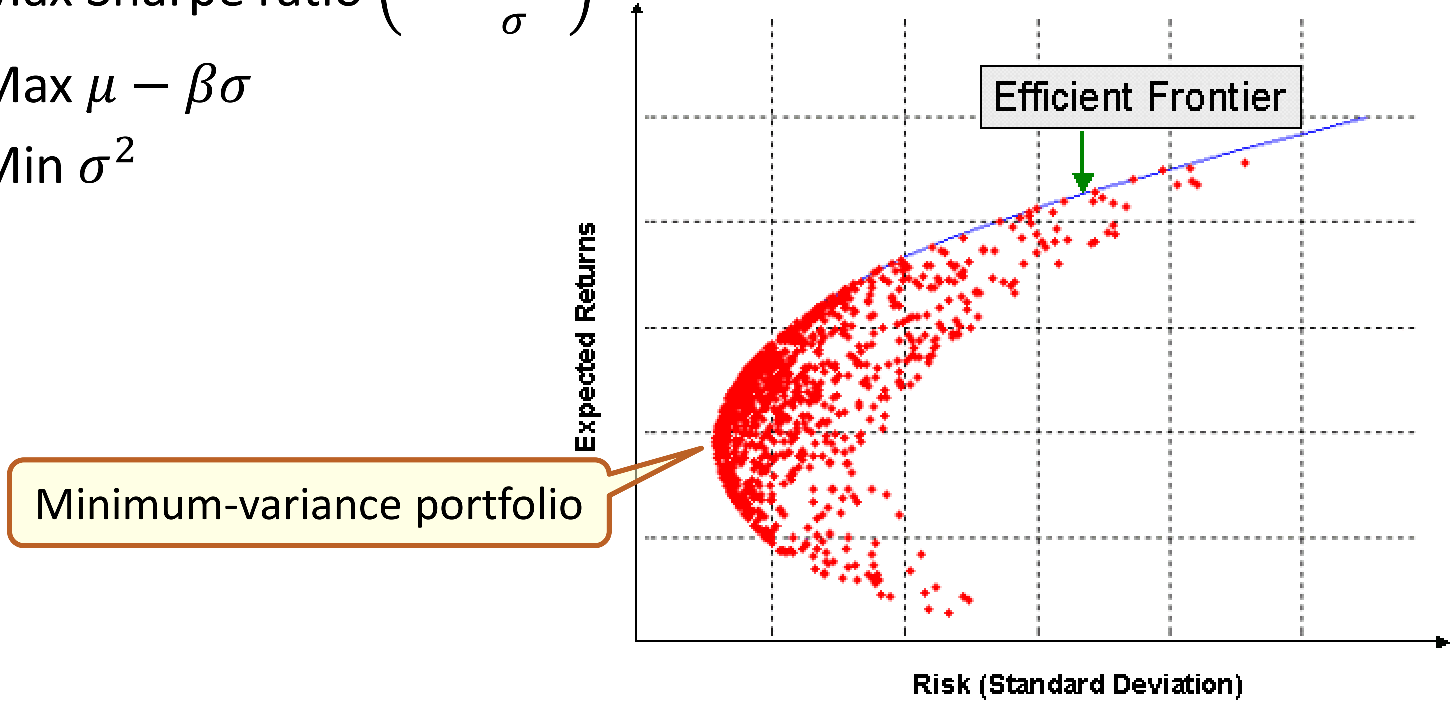
where $B = \begin{bmatrix} 1 & \mu_1 \\ 1 & \mu_2 \\ \vdots & \vdots \\ 1 & \mu_n \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$, $\boldsymbol{\lambda} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$, and $\mathbf{k} = \begin{bmatrix} 1 \\ \mu_0 \end{bmatrix}$.

By direct matrix manipulation, we can obtain the solution as follows:

$$\begin{cases} \mathbf{w} & = & \Sigma^{-1}B(B^T\Sigma^{-1}B)^{-1}\mathbf{k} \\ \lambda & = & -2(B^T\Sigma^{-1}B)^{-1}\mathbf{k} \end{cases}$$

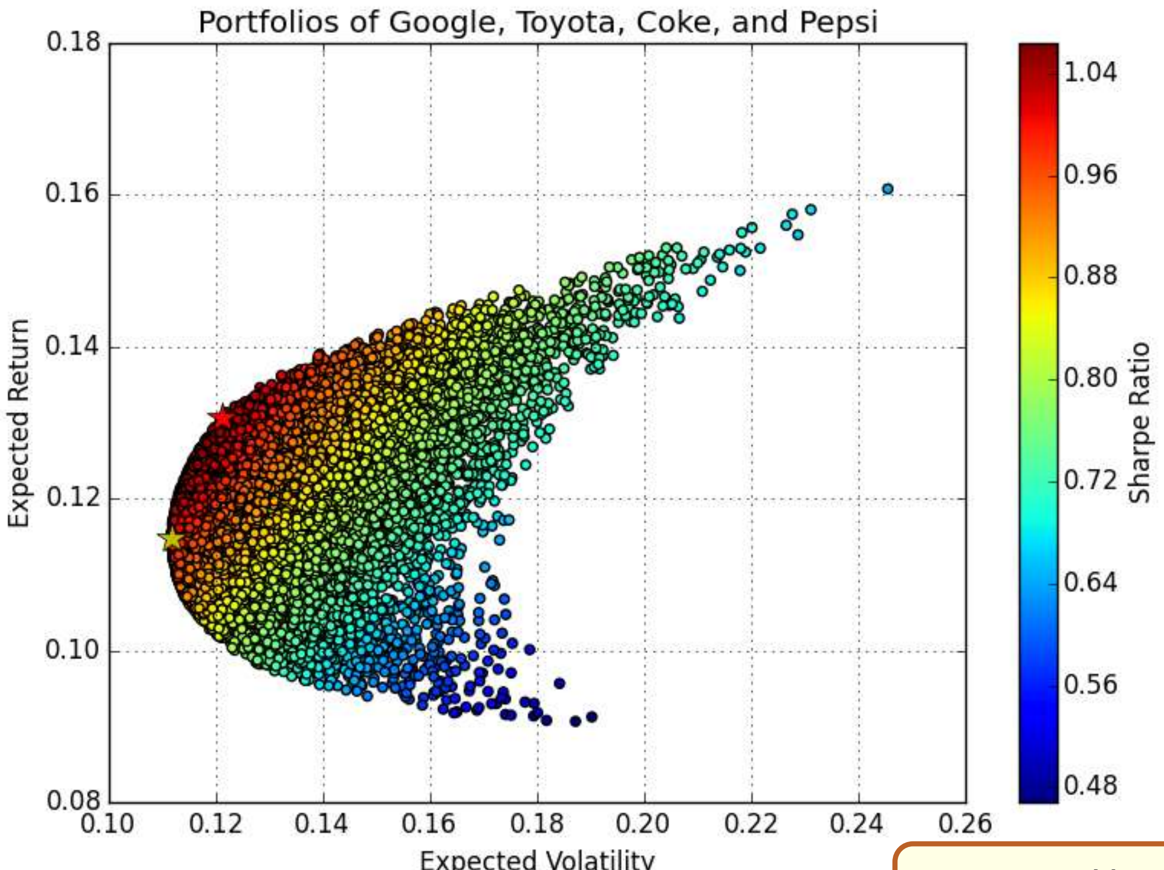
Efficient Frontier

- Efficient frontier for portfolio optimization
  - Max μ (return) with fixed σ^2 (risk)
  - Min σ^2 (risk) with fixed μ (return)
  - Max Sharpe ratio $\left(=\frac{\mu-\mu_f}{\sigma}\right)$
  - Max $\mu - \beta\sigma$
  - Min σ^2



Resources

- Investment Portfolio Optimization (with Python code)



More references:
https://hackmd.io/@rogerjang/SJN4FQbvF

Other Things to Consider

- How to compute $\boldsymbol{\mu}$ (returns) and $\sum$ (covariance matrix)?
- When to rebalance the assets?
- Other constraints
  - Max. value of n
  - Max. number of changes in assets
  - Conversion of individual risk attributes to objective functions
  - Conversion of individual preferences to objection functions

Exercises (1/2)

1. In PO of $n=2$, when will the efficient frontier reduce to a straight line?
2. In PO of $n=2$, when will the efficient frontier reduce to a parabola?
3. In PO of $n=2$, when will the overall risk go to zero? What are the weights when this happens?
4. In PO of $n=2$, can you derive the general formula for minimum-variance portfolio?
   - What is the minimum variance?
   - What is the corresponding return and weights?
5. Given two risky assets as follows:

$$\begin{cases} \text{Asset 1: } \mu = 0.2, \sigma = 0.1 \\ \text{Asset 2: } \mu = 0.3, \sigma = 0.2 \end{cases}$$

   Under the following conditions, what are the corresponding minimum variances when we achieve minimum-variance portfolio?
   - $\rho_{12} = 1$
   - $\rho_{12} = 0$
   - $\rho_{12} = -1$

Exercises (2/2)

6. Given two risky assets as follows:

$$\begin{cases} \text{Asset 1: } \mu = 0.2, \sigma = 0.1 \\ \text{Asset 2: } \mu = 0.3, \sigma = 0.2 \end{cases}$$

And the correlation coefficient of these two assets is $\rho_{12} = 0$. We want to perform portfolio optimization with investment weigthing of w_1 and w_2 for assets 1 and 2, respectively.

- What are the overall μ (return) and σ (volatility) when $w_1 = 0.4$ and $w_2 = 0.6$?
- What are the overall μ, overall σ, and w_1 for achieving the minimum-variance portfolio?

7. Given two risky assets as follows:

$$\begin{cases} \text{Asset 1: } \mu = 0.2, \sigma = 0.1 \\ \text{Asset 2: } \mu = 0.3, \sigma = 0.2 \end{cases}$$

And the correlation coefficient of these two assets is $\rho_{12} = 0.4$. We want to perform portfolio optimization with investment weigthing of w_1 and w_2 for assets 1 and 2, respectively.

- What are the overall μ (return) and σ (volatility) when $w_1 = 0.4$ and $w_2 = 0.6$?
- What are the overall μ, overall σ, and w_1 for achieving the minimum-variance portfolio?

References

- References at hackmd (with detailed math formula)
  - Intro to portfolio optimization
  - Objective functions for portfolio optim.
  - Portfolio for 2 assets
  - Portfolio optim.: Min. risk only
  - Portfolio optim.: Min. risk with fixed return
  - References