Graph Theory HW3

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Problem 1. Let's have an induction on r to prove the following claim:
Claim: For every k, there exists a least integer $n = n(k, r)$ such that whenever $[n]$ is r-coloured, there is a monochromatic k-AP $a_0, a_1, \ldots, a_{k-1}$ whose common difference $d = a_1 - a_0$ is also the same colour.
For $r = 1$, $1, 2, \ldots, k$ along with 1 are monochromatic, the claim holds.
Suppose for all $r < r'$, the claim holds.
For $r = r'$:
By Van der Waerden's Theorem: for all positive integers x and y, there exists a least integer $W = W(x, y)$ such that any y-coloring of $[W]$ contains a x-term monochromatic arithmetic progression.
Let $m = n(k, r-1)$, by Van der Waerden's Theorem, there exists $a > 0, d > 0$ such that $S = \{a + id | i \in \{0, 1, \ldots, m(k-1)\}\}$ is monochromatic.
By the induction hypothesis, every $(r-1)$-coloring of $[m]$ contains a monochromatic k-AP along with its common difference, and so is every $(r-1)$-coloring of $\{di | i \in [m]\}$.
$\therefore$ either $\{di | i \in [m]\}$ contains r different colors, or a monochromatic k-AP along with its common difference. The claim holds for the latter case.
If $\{di | i \in [m]\}$ contains r different colors, let dj have the same color as S.
$\because a + dj(k-1) \leq a + dm(k-1)$.
$\therefore \{a + idj | i \in \{0, 1, \ldots, k-1\}\} \subseteq S$.
$\Rightarrow \{a + idj | i \in \{0, 1, \ldots, k-1\}\}$ along with dj have the same color.
$\Rightarrow$ the claim holds for this case.
$\therefore$ by induction, the claim holds for all k, r, and this finishes the proof of this problem.

Problem 2. Let's prove that $W(k,r) \le k^{HJ(k,r)}$.

For every coloring $c : [k^{HJ(k,r)}] \to [r]$, consider the coloring $c' : [k]^{HJ(k,r)} \to [r]$ where $c'(a_1, a_2, \dots, a_{HJ(k,r)}) := c\left(1 + \sum_{i=1}^{HJ(k,r)} (a_i - 1)k^{i-1}\right)$.

By the Hales-Jewett Theorem, there is a monochromatic combinatorial line in the coloring c'.

That is, there is a set $S \neq \emptyset$ and $a_{ij} (1 \le i \le k, 1 \le j \le HJ(k,r))$, where $a_{ij} = \begin{cases} i, \text{ if } j \in S \\ a_{1j}, \text{ otherwise} \end{cases}$, such that $c'(a_{i1}, a_{i2}, \dots, a_{i,HJ(k,r)})$ are the same for all $i \in [k]$.

$\Rightarrow c\left(1 + \sum_{j=1}^{HJ(k,r)} (a_{ij} - 1)k^{j-1}\right)$ are the same for all $i \in [k]$.

$\Rightarrow c\left(1 + \sum_{j \in S} (i-1)k^{j-1} + \sum_{j \notin S} (a_{1j} - 1)k^{j-1}\right)$ are the same for all $i \in [k]$.

$\Rightarrow c\left(1 + \sum_{j \notin S} (a_{1j} - 1)k^{j-1} + (i-1)\sum_{j \in S} k^{j-1}\right)$ are the same for all $i \in [k]$.

$\because 1 + \sum_{j \notin S} (a_{1j} - 1)k^{j-1}, \sum_{j \in S} k^{j-1}$ are constants with respect to i,

$\therefore \left\{1 + \sum_{j \notin S} (a_{1j} - 1)k^{j-1} + (i-1)\sum_{j \in S} k^{j-1} \middle| i \in [k]\right\}$ is a k-AP.

$\Rightarrow$ we find a monochromatic k-AP.

$\Rightarrow W(k,r) \le k^{HJ(k,r)}$, which proves Van der Waerden's Theorem.