Ch.12 Getting the Message

September 15, 2023

Dirty Faces

- A, B and C travel together in a railway carriage and each one has a dirty face.
- If one is conscious of appearing in public with a dirty face, she will surely blush.
- Nobody blushes. $\Rightarrow$ None of them knows that her own face is dirty, although each can clearly see the dirty faces of the others.
- A clergyman who always tells the truth enters the carriage and announces that some one has a dirty face.
- A blushes and why?

- A: Suppose that my face were clean. Then B would reason as follows:
- B: I see that A's face is clean. Suppose that my face were also clean. Then C would reason as follows:
- C: I see that A's and B's faces are clean. If my face were clean, nobody's face would be dirty. But the clergyman announced..., so my face is dirty, and I must blush.
- B: Since C hasn't blushed, my face is dirty. So I must blush.
- A: Since B hasn't blushed, my face is dirty. So I must blush.

Knowledge

- state of world, ω
- knowledge operator, K
- possibility operator, P
- axioms about K and P
- truism

State of World

- In the example of the dirty faces, there are 8 possible states of world.
- Let C denote a clean face and let D denote a dirty face. The 8 states of worlds are:

	1	2	3	4	5	6	7	8
A	C	D	C	C	D	D	C	D
B	C	C	D	C	D	C	D	D
C	C	C	C	D	C	D	D	D

- state $\omega = 1$: everyone has a clean face.
- universe $\Omega \equiv \{1, 2, 3, 4, 5, 6, 7, 8\}$
- event E is a subset of Ω
  - The event that A has a dirty face, $D_A = \{2, 5, 6, 8\}$.
- If the true state is an element of an event E, then E has occurred.
  - If $\omega = 8$, then D_A has occurred.

Knowledge Operator, K

	1	2	3	4	5	6	7	8
A	C	D	C	C	D	D	C	D
B	C	C	D	C	D	C	D	D
C	C	C	C	D	C	D	D	D

- KE is the set of states of the world in which A knows that E has occurred.
- $D_A = \{2, 5, 6, 8\}$
- The event that A knows her own face being dirty, $KD_A =?$
  - After the clergyman's announcement, and before A reaches any deep induction, she looks at two other faces and only when two other ladies both have clean faces, could A know she herself has a dirty face. $KD_A = \{2\}$.
  - But, $\omega = 8 \notin KD_A$, so KD_A does not occur and A does not know her face is dirty yet.

Possibility Operator, P

- PE means the event that A considers the event E possible.
- $PE \equiv \sim K \sim E$

Axioms

(K0) $K\Omega = \Omega$

(K1) $K(E \cap F) = KE \cap KF$

(K2) $KE \subseteq E$[1]

(K3) $KE \subseteq K(KE) \equiv K^2E$

(K4) $PE \subseteq KPE$

(P0) $P\emptyset = \emptyset$

(P1) $P(E \cup F) = PE \cup PF$

(P2) $E \subseteq PE$

(P3) $P^2E \subseteq PE$

(P4) $PKE \subseteq KE$

- There appear to be 10 different axioms.
- But since K axioms $\Leftrightarrow P$ axioms, there are only 5 effective axioms.
- For instance, we can derive (P0) from (K0):

$$\begin{aligned} P\emptyset &\equiv \sim K \sim \emptyset \\ &= \sim K\Omega \\ &= \sim \Omega \quad \text{(from (K0))} \\ &= \emptyset \quad \square \end{aligned}$$

[1] $P \subseteq Q$ means if P then Q.

Truism

- Def: The event T is a truism for A, if T can't be true w/o A knowing it, i.e. $T \subseteq KT$.
- Recall (K2) $KE \subseteq E$, so for a truism T, we have $T = KT$.

Theorems and Possibility Sets

- We shall establish 3 theorems that lead us to characterize when A knows her face is dirty.
- During the course, we need to define a new concept, possibility set.

Theorem 1: A knows E has occurred iff a truism that implies E has occurred.

Proof: First we'll establish $\Rightarrow$.

For this purpose, we have to show $\exists T$, $KE \subseteq T$ where T is a truism and $T \subseteq E$.

Define $T \equiv KE$.

Clearly, $KE \subseteq T$.

From (K3), $T \subseteq KT$, T is a truism. ((K3) $KE \subseteq K(KE)$)

From (K2), $T \subseteq E$. ((K2) $KE \subseteq E$)

We'll now establish $\Leftarrow$.

For that, we have to show $\forall$ truism $T \subseteq E$, $T \subseteq KE$.

We'll first establish that if $G \subseteq H$, then $KG \subseteq KH$.

Recall from (K1), $K(G \cap H) = KG \cap KH$.

If $G \subseteq H$, we have $KG = KG \cap KH$ which implies $KG \subseteq KH$.

From this, $T \subseteq E \Rightarrow KT \subseteq KE$.

Recall that T is a truism, $T \subseteq KT \subseteq KE$. $\square$

Theorem 2: $P\{\omega\}$ is the smallest truism containing ω.

Lemma 1: If $E \subseteq F$, $PE \subseteq PF$.
Proof: From (P1), $P(E \cup F) = PE \cup PF$
If $E \subseteq F$, $P(E \cup F) = PF = PE \cup PF$, $PE \subseteq PF$. $\square$

Proof of Theorem 2: From (P2), $E \subseteq PE$, so $\{\omega\} \subseteq P\{\omega\}$, i.e. $P\{\omega\}$ contains ω.
From (K4), $PE \subseteq KPE$, i.e. $\forall E$, PE is a truism.
Lastly, consider any truism T containing ω, we'll show $P\{\omega\} \subseteq T$.
From lemma 1, $\{\omega\} \subseteq T \Rightarrow P\{\omega\} \subseteq PT = PKT \subseteq KT = T$.
The equalities are due to T being a truism; the $\subseteq$ is due to (P4). $\square$

Possibility Set

- The last theorem involves possibility sets.
- Def: The possibility set $P(\omega)$ is the set of states that A considers possible when the true state is ω.
- Distinction between $P\{\omega\}$ and $P(\omega)$:
  - $a \in P(\omega) \Leftrightarrow \{\omega\} \subseteq P\{a\}$
  - $a \in P\{\omega\} \Leftrightarrow \{a\} \subseteq P\{\omega\}$
- But, $P\{\omega\} = P(\omega)$!
- To prove this, from above, it's sufficient to show that $\forall$ states c, d, $\{c\} \subseteq P\{d\} \Rightarrow \{d\} \subseteq P\{c\}$

Lemma 2: $\{c\} \subseteq P\{d\} \Rightarrow \{d\} \subseteq P\{c\}$.

Proof: Suppose $\{c\} \subseteq P\{d\}$, but $\{d\} \nsubseteq P\{c\}$.
Then $\{d\} \subseteq \sim P\{c\} \equiv K \sim \{c\}$.
Applying lemma 1, $\{c\} \subseteq P\{d\} \subseteq PK \sim \{c\} \subseteq K \sim \{c\}$[2] $\equiv \sim P\{c\}$.
It contradicts (P2). □

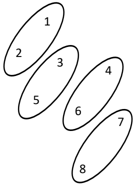
[2] (P4)

Possibility Set

- Def: The possibility set $P(\omega)$ is the set of states that A considers possible when the true state is ω.
- Recall that there are 8 states in the example of dirty faces.

	1	2	3	4	5	6	7	8
A	C	D	C	C	D	D	C	D
B	C	C	D	C	D	C	D	D
C	C	C	C	D	C	D	D	D

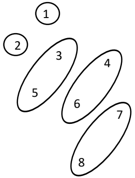
- For A, what are $P_A(1)$, ... $P_A(8)$ before the clergyman appears?
- $P_A(1) = \{1, 2\} = P_A(2)$ ... which are expressed in the following figure:



- Possibility sets change when new information arrives.
- The clergyman will announce that someone has a dirty face iff it is true.
- What are A's possibility sets right after the clergyman arrives?

	1	2	3	4	5	6	7	8
A	C	D	C	C	D	D	C	D
B	C	C	D	C	D	C	D	D
C	C	C	C	D	C	D	D	D

- $P_A(1) = \{1\}$, $P_A(2) = \{2\}$ and all other possibility sets remain the same.



- According to this figure, does A know $D_A (= \{2, 5, 6, 8\})$ has occurred?

- Theorem 1: A knows E has occurred iff a truism that implies E has occurred.
- Theorem 2: $P\{\omega\}$ is the smallest truism containing ω.
- Theorem 3: A knows that E has occurred in state ω iff $P(\omega) \subseteq E$.
  Proof: $\Rightarrow$
  If A knows that E has occurred in state ω, from Theorem 1, $\exists$ truism T, $\omega \in T \subseteq E$.
  From Theorem 2, $\omega \in P\{\omega\} = P(\omega) \subseteq T \subseteq E$.

  $\Leftarrow$
  From Theorem 2, in state ω, the truism $P(\omega)$ occurs.
  If $P(\omega) \subseteq E$, from Theorem 1, A knows that E has occurred in state ω. $\square$
- Back to the previous slide, because $P_A(8) = \{7, 8\} \not\subseteq D_A = \{2, 5, 6, 8\}$, according to Theorem 3, A does not know that she has a dirty face right after the clergyman's announcement.

Possibility Set

- Def: C is a partition of Ω, if C is a collection of non-empty subsets of Ω such that every state in Ω is in exactly one of these subsets.
- ex. $C = \{\{1,2\},\{3,4,5\},\{6,7,8\}\}$
- Lemma 3: $\forall \omega' \in P(\omega), P(\omega') = P(\omega)$

  Proof: Suppose $\omega' \in P(\omega)$.
  We'll first show that $\forall \omega'' \in P(\omega'), \omega'' \in P(\omega)$.
  By the definition of a possibility set, $\omega'' \in P(\omega) \Leftrightarrow \{\omega\} \subseteq P\{\omega''\}$.
  We have:

  $$\{\omega\} \subseteq P\{\omega'\} \subseteq PP\{\omega''\} \subseteq P\{\omega''\}$$

  The 1st $\subseteq$ follows from $\omega' \in P(\omega)$; the 2nd from $\omega'' \in P(\omega')$ which means $\{\omega'\} \subseteq P\{\omega''\}$, and lemma 1; the 3rd from (P3).

  We still need to show that $\forall \omega'' \in P(\omega), \omega'' \in P(\omega')$, or to show that $\{\omega'\} \subseteq P\{\omega''\}$. This part is left as your exercise.

Possibility Set

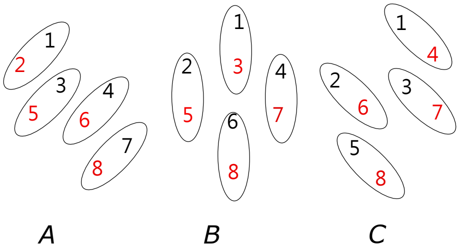
Theorem 4: $\{P(\omega)|\omega \in \Omega\}$ is a partition of Ω.

Proof: From Theorem 2, $\omega \in P(\omega)$, hence $\bigcup_{\omega\in\Omega} P(\omega) = \Omega$.
Next, we'll show that $\forall \omega_1, \omega_2, P(\omega_1) \neq P(\omega_2)$, $P(\omega_1) \cap P(\omega_2) = \emptyset$.
Suppose $P(\omega_1) \neq P(\omega_2)$ and $\exists \omega_3 \in P(\omega_1) \cap P(\omega_2)$.
From lemma 3, $P(\omega_1) = P(\omega_3) = P(\omega_2)$ which contradicts the presumption. $\square$

Def: Consider 2 partitions of Ω, C and D. C is a refinement of D if each set in C is a subset of a set in D.

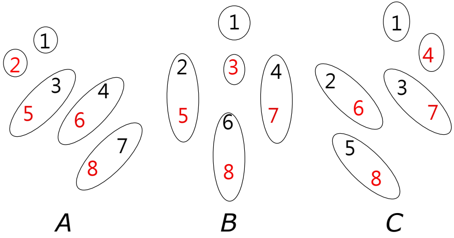
Ex. A's information partition becomes refined after the clergyman's announcement.

Information Partitions before the Clergyman's Announcement



	1	2	3	4	5	6	7	8
A	C	D	C	C	D	D	C	D
B	C	C	D	C	D	C	D	D
C	C	C	C	D	C	D	D	D

After the Clergyman's Announcement



No one knows that her face is dirty because $P_i(8) \not\subseteq D_i, i = A, B, C$.

Taking Turns to Blush: $A \to B \to C \to A \quad \ldots$