Graph Theory HW6

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Problem 1. Consider the following graph:

$V = \{v_1, v_2, \ldots, v_{2k}\}, E = \{v_{2i-1}v_{2j} | i, j \in [k], i \neq j\}$ with coloring order $v_1, v_2, \ldots, v_{2k}$.

Let's have an induction on l to prove that $\forall i \leq l, v_{2i-1}, v_{2i}$ are colored in the i-th color. – (1)

For $l = 1$, (1) holds because v_1, v_2 are not neighbors and none of their neighbors are colored before them.

Suppose for $l = l'$, (1) holds.

For $l = l' + 1$, since $v_{2l'+1}$'s neighbors $v_2, v_4, \ldots, v_{2l'}$ are colored in the $1, 2, \ldots, l'$-th color, respectively, and the neighbors $v_{2l'+4}, v_{2l'+6}, \ldots$ are colored after it, $v_{2l'+1}$ will be colored in the $l'+1$-th color; since $v_{2l'+2}$'s neighbors $v_1, v_3, \ldots, v_{2l'-1}$ are colored in the $1, 2, \ldots, l'$-th color, respectively, and the neighbors $v_{2l'+3}, v_{2l'+5}, \ldots$ are colored after it, $v_{2l'+2}$ will be colored in the $l' + 1$-th color.

$\therefore$ by induction, the greedy algorithm uses k colors on this graph.

Problem 3.

(a) Let f_1 be G_1's proper $\chi(G_1)$-coloring, f_2 be G_2's proper $\chi(G_2)$-coloring.

Consider the $\chi(G_1) + \chi(G_2)$-coloring f on H:

$$f(v) := \begin{cases} f_1(v), \text{ if } v \in V_1 \\ f_2(v) + \chi(G_1), \text{ if } v \in V_2 \end{cases}.$$

One can see that if $uv \in E_1$, then $f(u) = f_1(u) \neq f_1(v) = f(v)$; if $uv \in E_2$, then $f(u) = f_2(u) + \chi(G_1) \neq f_2(v) + \chi(G_1) = f(v)$; if $uv \in \{u'v' : u' \in V_1, v' \in V_2\}$, WLOG suppose that $u \in V_1$, then $f(u) = f_1(u) \leq \chi(G_1) < \chi(G_1) + 1 \leq$

$f_2(v) + \chi(G_1) = f(v)$.

$\therefore f$ is a proper $\chi(G_1) + \chi(G_2)$-coloring, which means $\chi(H) \leq \chi(G_1) + \chi(G_2)$.

On the other hand, if one use at most $\chi(G_1) + \chi(G_2) - 1$, since the vertices in V_1 will be colored in at least $\chi(G_1)$ colors, and the vertices in V_2 will be colored in at least $\chi(G_2)$ colors, there exists $u \in V_1$ and $v \in V_2$ such that u, v have the same color. But $uv \in E(H)$, which is not a proper coloring.

$\therefore \chi(H) \geq \chi(G_1) + \chi(G_2)$.

$\therefore \chi(H) = \chi(G_1) + \chi(G_2)$.

(b) Let f_1 be G_1's proper $\chi(G_1)$-coloring, f_2 be G_2's proper $\chi(G_2)$-coloring.

Consider the $\chi(G_1)\chi(G_2)$-coloring f on H:

$f(v) := f_1(v)\chi(G_2) + f_2(v)$.

Since $f_2(v) \in [\chi(G_2)]$, there is $f(u) = f(v) \iff (f_1(u), f_2(u)) = (f_1(v), f_2(v))$.

If $uv \in E(H)$, then $uv \in E_1$ or $uv \in E_2$ must hold.

$\Rightarrow f_1(u) \neq f_1(v)$ or $f_2(u) \neq f_2(v)$ must hold.

$\Rightarrow f(u) \neq f(v)$ must hold, which means f is a proper coloring.

$\therefore \chi(H) \leq \chi(G_1)\chi(G_2)$.

Problem 5.

(a) Consider $G = K_{n,n}$. $dg(G) \geq \delta(G) \geq n$, but $\chi(G) = 2$.

$\forall f,\ f(2)$ is a constant, hence $\exists n$ s.t. $dg(G) \geq n > f(\chi(G))$.

$\therefore \rho_1$ is not ρ_2-bounded.

(b) Consider G where $V(G) = \{v_1, v_2, \ldots, v_{n(n-1)}\}$ and $E(G) = \{v_iv_j | 1 \leq i < j \leq n\}$.

Since the induced subgraph of $\{v_1, \ldots, v_n\}$ is K_n, $\chi(G) \geq n$.

$\bar{d}(G) = \dfrac{(n-1) \times n}{n(n-1)} = 1$.

$\forall f,\ f(1)$ is a constant, hence $\exists n$ s.t. $\chi(G) \geq n > f(\bar{d}(G))$.

$\therefore \rho_1$ is not ρ_2-bounded.

(c) Consider $G = K_{n,n}$. $\bar{d}(G) = n$, but $\chi(G) = 2$.

$\forall f,\ f(2)$ is a constant, hence $\exists n$ s.t. $\bar{d}(G) = n > f(\chi(G))$.

$\therefore \rho_1$ is not ρ_2-bounded.

(d) By problem 2 of HW2, there exists a subgraph of G whose minimum degree is at least $\frac{1}{2}\bar{d}$.

$\therefore dg(G) \geq \frac{1}{2}\bar{d}(G)$.

$\Rightarrow$ consider $f(x) = 2x$ (which is increasing), since $\bar{d}(G) \leq 2dg(G)$ always holds, ρ_1 is ρ_2-bounded.

(e) Consider G where $V(G) = \{v_1, v_2, \ldots, v_{n(n-1)}\}$ and $E(G) = \{v_iv_j | 1 \leq i < j \leq n\}$.

Since the induced subgraph of $\{v_1, \ldots, v_n\}$ is K_n, $dg(G) \geq n-1$.

$\bar{d}(G) = \dfrac{(n-1) \times n}{n(n-1)} = 1$.

$\forall f,\ f(1)$ is a constant, hence $\exists n$ s.t. $dg(G) \geq n-1 > f(\bar{d}(G))$.

$\therefore \rho_1$ is not ρ_2-bounded.