Ch.2 Backing up

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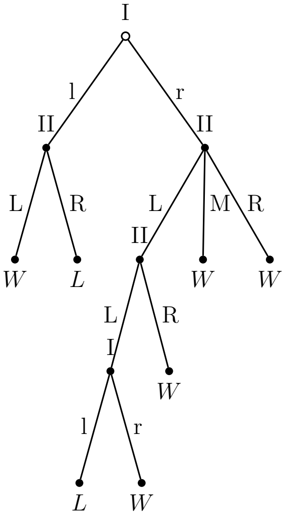
Consider a 2-person win-or-loss game with perfect information G:
$W \succ_I L$, $L \succ_{II} W$.

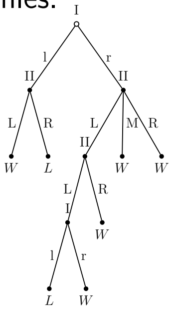


- In the previous slide, we have the *game tree* of G, or the *extensive form* of G.
- We could also present G's *strategic form*.
  - Def: Strategy of i is a statement that specifies an action at each of i's decision node.
  - Player I has 2 decision nodes, and I has 4 pure strategies: ll, lr, rl, rr.
  - How many pure strategies does player II have?

Strategic Form/Normal Form

We'll list 2 players' pure strategies in a matrix form and fill in the results of all strategy profiles.

		II		
		LLL	LLR	....
	ll	W	....	
I	lr	W	....	
	rl	L	....	
	rr	W	....	



Solution Concepts

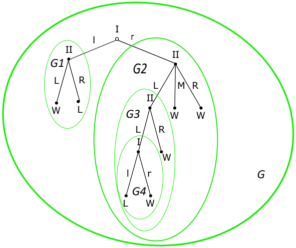
- extensive form: subgame perfect equilibrium (SPE)
- strategic form: Nash equilibrium (NE)
- An extensive form reveals more information about the game than a strategic form.
  $\Rightarrow$ More requirements can be imposed on players' choices in an extensive form than in a strategic form.
  $\Rightarrow \{\text{SPE}\} \subseteq \{\text{NE}\}$

Solution Concept: A Winning Strategy

- Def: If i has a strategy for game H that wins H whatever strategy i's opponent may use, then i's strategy is called his winning strategy.
- If a player has a winning strategy, he'll surely adopt it.
- Existence of a winning strategy
  - Zermelo developed an algorithm in 1912 to analyze Chess, and confirmed the existence.
- Who has a winning strategy?

Subgame and Value

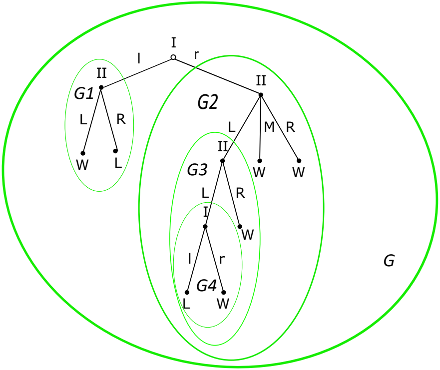
- Def: A subgame consists of a node x together with all of the game tree that follows x.
- 5 subgames of G:



- Def: Value of the subgame H, $v(H) = W$, if I has a winning strategy for H, and $v(H) = L$, if II has a winning strategy for H.

Subgame and Value

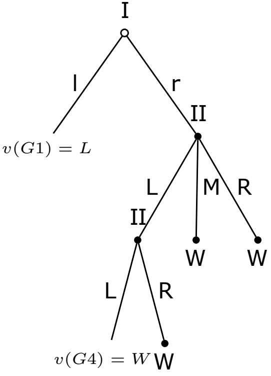
- Existence of a winning strategy $\Leftrightarrow$ $v(G)$ well-defined.



- $v(G) =?$, but ready answers when there remains only one decision node: $v(G1) = L$, $v(G4) = W$

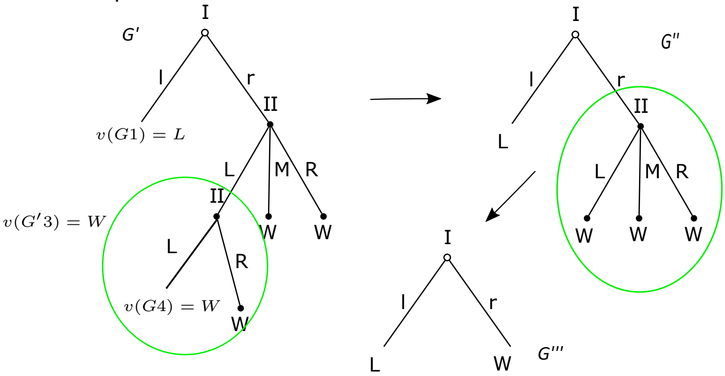
A Simplified Game, G'

- Game G' obtained from G by replacing $G1$ and $G4$ with their values



- Claim: If $v(G') = W$, then $v(G) = W$.
  Proof: If $v(G') = W$, player I has a winning strategy s' for G', i.e. player I can always direct the game to end w. a node labeled W. If the node corresponds to a subgame G_x of G, then $v(G_x) = W$, and player I has a winning strategy s_x in G_x.
  The winning strategy s for player I in G consists of playing s' all the way until the subgame G_x is reached, then playing s_x. $\square$
- Similarly, if player II has a winning strategy in G' ($v(G') = L$), then player II has a winning strategy in G ($v(G) = L$).
- In sum, if $v(G')$ is well defined, so is $v(G)$ and $v(G) = v(G')$.

- $v(G')$ well defined?
- further simplifications



- $W = v(G''') = V(G'') = v(G') = v(G)$
- A 2-person finite win-or-loss game with perfect information has a value.

Who has the winning strategy?

- Nim
  - There are several piles of matchsticks.
  - Two players alternate in moving.
  - When it is your turn to move, you must select one of the piles and remove at least one matchstick from that pile.
  - The last player to take a matchstick is the winner.
- Nim is a 2-person finite win-or-loss game with perfect information.
- Nim has a value.

- 3 piles of matchsticks: 5, 9, 4 sticks in each pile
- Who has the winning strategy?
- a binary representation of the game:

	8	4	2	1
5	0	1	0	1
9	1	0	0	1
4	0	1	0	0

- Def: Nim is *balanced* if each column has an even number of 1s and *unbalanced* otherwise.
- The example is unbalanced.

Claim: A player that starts with an unbalanced Nim has a winning strategy.

Proof: (a) A player that starts with a balanced Nim cannot win immediately.

(b) $\forall$ balanced Nim, every move converts the game back to an unbalanced one.

(c) $\forall$ unbalanced Nim, there is a move to convert it to a balanced one.

Reconsider the example:

	8	4	2	1
5	0	1	0	1
9	1	0	0	1
4	0	1	0	0

The starting player should take 8 sticks from the 2nd pile. $\square$

Chess

- It is not a win-or-loss game. It could end with a draw D.

$$W \succ_I D \succ_I L$$
$$W \prec_{II} D \prec_{II} L$$

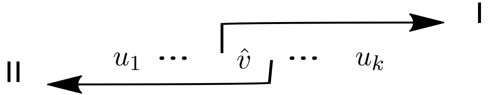
- Def: A 2-person game which has k different outcomes, $u_1, ...u_k$, is *strictly competitive*, if

$$u_k \succ_I ... \succ_I u_1$$
$$u_k \prec_{II} ... \prec_{II} u_1$$

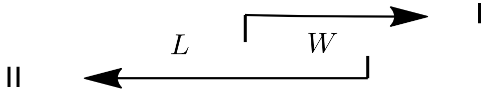
- Chess is strictly competitive.

Value of a strictly competitive game G

- Def: $v(G) = \hat{v}$, if two players can force an outcome no worse than $\hat{v}$ to themselves at the same time, i.e. player I can force an outcome in the set $W = \{u|u \succeq_I \hat{v}\}$ and player II can force an outcome in the set $L = \{u|u \succeq_{II} \hat{v}\}$.



- This extends the definition of the value in a win-loss game:



$v(G)$ exists

Lemma: Let T be any set of outcomes in a finite 2-person game with prefect information. Either player I can force an outcome in T, or player II can force an outcome in $\sim T$.

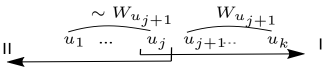
Proof: Relablel all outcomes in T with W, and all outcomes in $\sim T$ with L. Then it reduces to showing that any finite win-or-loss game has a value. $\square$

Theorem: Any 2-person finite strictly competitive game with perfect information w/o chance moves has a value.

Proof: Consider a game, G, with k different outcomes and $u_k \succ_I ... \succ_I u_1$.
Define W_{u_i} as follows:

$$\begin{aligned} W_{u_1} &= \{u_1, ..., u_k\} \\ W_{u_2} &= \{u_2, ..., u_k\} \\ &... \\ W_{u_k} &= \{u_k\} \end{aligned}$$

Player I can force an outcome in W_{u_1}.
Let W_{u_j} be the smallest set in which player I can force an outcome.
If $j = k$, $v(G) = u_k$. If $j \neq k$, then player I cannot force an outcome in $W_{u_{j+1}}$.
From the lemma, player II can force an outcome in $\sim W_{u_{j+1}}$



$v(G) = u_j$. $\square$

Corollary: Chess has a value.

Saddle Points

- The concept of value of the game was developed earlier than the concept of NE and SPE.
- They are different:
  - Value is about the outcome.
  - Equilibrium is about the strategy profile.
- But are they related?
  - We'll define a saddle point and find its relation with the value and NE, hence connect the value and NE.

- Def: A strategy pair (s,t) is a saddle point of the strategic form of a strictly competitive game if, for I, the outcome v of (s,t) is no worse than any outcome in column t and no better than any outcome in row s.
- $$\begin{array}{c|ccccc} & & & t & & \\ \hline & & & \preceq_1 v & & \\ & & & \vdots & & \\ s & \succeq_1 v & \ldots & v & \ldots & \succeq_1 v \\ & & & \vdots & & \\ & & & \preceq_1 v & & \end{array}$$
- Corollary: The strategic form of a finite, strictly competitive game of prefect information w/o chance moves has a saddle point (s,t).
  Proof: $\because$ value exists.
- (s,t) is a NE $\Leftrightarrow$ (s,t) is a saddle point.