Ch.7 Zero-Sum Games

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Zero-Sum Games

Minimax and Maximin

Mixed Strategies

Minimax Theorem

An Example

- Two players: I and II
- The sum of their payoffs is always 0.
- In the payoff matrix, only I's payoff is shown.
- It's a strictly competitive game.
- von Neumann: Does the value exist?

Security Level

- $$M = \begin{array}{c|ccc} & t_1 & t_2 & t_3 \\ \hline s_1 & 1 & 6 & 0 \\ s_2 & 2 & 0 & 3 \\ s_3 & 3 & 2 & 4 \end{array}$$
- Def: A player's security level is the largest expected payoff he can guarantee to himself whatever the other player may do.
- For player I,
  - s_1: $\min\{1, 6, 0\} = 0$
  - s_2: $\min\{2, 0, 3\} = 0$
  - s_3: $\min\{3, 2, 4\} = 2$
  - I's security level $= \max\{0, 0, 2\} = 2$
- $S \equiv \{s_1, s_2, s_3\}$, $T \equiv \{t_1, t_2, t_3\}$
- $\pi(s, t)$: I's payoff for the strategy profile (s, t)
- I's security level $= \max_{s \in S} \min_{t \in T} \pi(s, t) \equiv \underline{m}$
- $\underline{m}$ is also called the maximin value of matrix M

Security Level of II

- $$M = \begin{array}{c|ccc} & t_1 & t_2 & t_3 \\ \hline s_1 & 1 & 6 & 0 \\ s_2 & 2 & 0 & 3 \\ s_3 & 3 & 2 & 4 \end{array}$$
- For player II,
  - t_1: $\max\{1, 2, 3\} = 3$
  - t_2: $\max\{6, 0, 2\} = 6$
  - t_3: $\max\{0, 3, 4\} = 4$
  - II's security level $= \min\{3, 6, 4\} = 3$
- II's security level $= \min_{t \in T} \max_{s \in S} \pi(s, t) \equiv \overline{m}$
- $\overline{m}$ is also called the minimax value of matrix M

Minimax and Maximin

- In this example, $\overline{m} = 3 > 2 = \underline{m}$.
- In general, $\overline{m} \geq \underline{m}$
- Proof: Consider $\underline{m}$ is in the i-th row, and $\overline{m}$ is in the j-th column of matrix M.
  Consider the element m_{ij} of M.
  We have $\underline{m} \leq m_{ij} \leq \overline{m}$. $\square$
- What happens if $\underline{m} = \overline{m}$?

- Let (σ, τ) denote a strategy profile and let $\pi(\sigma, \tau)$ denote the payoff to I.
- Def: (σ, τ) is a saddle point of M if

$$\pi(\sigma, \tau) = \max_{s \in S} \pi(s, \tau) = \min_{t \in T} \pi(\sigma, t)$$

- Theorem: (a) $\underline{m} = \overline{m} \Leftrightarrow \exists$ saddle point
  (b) If (σ, τ) is a saddle point, then $\pi(\sigma, \tau) = \underline{m} = \overline{m}$

Proof: (a) Suppose $\underline{m} = \overline{m}$.
Let σ be I's security strategy, i.e. $\min_{t \in T} \pi(\sigma, t) = \underline{m}$.
Let τ be II's security strategy, i.e. $\max_{s \in S} \pi(s, \tau) = \overline{m}$.
$\Rightarrow \overline{m} \geq \pi(\sigma, \tau) \geq \underline{m}$
$\Rightarrow \overline{m} = \pi(\sigma, \tau) = \underline{m}$, since $\underline{m} = \overline{m}$
$\Rightarrow \pi(\sigma, \tau) = \max_{s \in S} \pi(s, \tau) = \min_{t \in T} \pi(\sigma, t)$, (σ, τ) is a saddle point.

On the other hand, if (σ, τ) is a saddle point,

$$\begin{aligned} \pi(\sigma, \tau) &= \min_{t \in T} \pi(\sigma, t) \leq \underline{m} \\ \pi(\sigma, \tau) &= \max_{s \in S} \pi(s, \tau) \geq \overline{m} \end{aligned}$$

So, $\underline{m} \geq \overline{m}$. But we know that $\underline{m} \leq \overline{m}$, so $\underline{m} = \overline{m}$.

(b) If (σ, τ) is a saddle point, from above,

$$\overline{m} \leq \pi(\sigma, \tau) \leq \underline{m}.$$

Since $\underline{m} = \overline{m}$ from (a), $\pi(\sigma, \tau) = \underline{m} = \overline{m}$. $\square$

- In our example, $\overline{m} > \underline{m}$, so saddle point does not exist.
- Von Neumann pressed to seek a saddle point in the product space of mixed strategies.
- Def: $P \equiv \{(p_1, p_2, p_3) : \sum p_i = 1, p_i \geq 0\}$ as I's strategy space.
- Def: $Q \equiv \{(q_1, q_2, q_3) : \sum q_i = 1, q_i \geq 0\}$ as II's strategy space.
- Def: Payoff funciton $\pi : P * Q \to R$
- $\pi(p, q) = pMq^T$
- $$\text{Ex. } M = \begin{pmatrix} 1 & 6 & 0 \\ 2 & 0 & 3 \\ 3 & 2 & 4 \end{pmatrix}, \quad p = (1/4, 1/4, 1/2), \quad q = (1/2, 1/2, 0)$$
- With p, when II uses t_1, t_2, or t_3, I's expected payoff is:

$$(1/4, 1/4, 1/2) \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \frac{9}{4}, \quad (1/4, 1/4, 1/2) \begin{pmatrix} 6 \\ 0 \\ 2 \end{pmatrix} = \frac{5}{2}, \quad (1/4, 1/4, 1/2) \begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix} = \frac{11}{4}$$

- Consider q, I's expected payoff is:

$$\left(\frac{9}{4}, \frac{5}{2}, \frac{11}{4}\right) \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix} = 2\frac{3}{8}$$

Déjà vu

- Def: I's security level $\underline{v} \equiv \max_{p \in P} \min_{q \in Q} \pi(p, q)$
- Def: II's security level $\overline{v} \equiv \min_{q \in Q} \max_{p \in P} \pi(p, q)$
- Let $\tilde{p}$, $\tilde{q}$ be I's and II's security strategy.

$$\underline{v} = \min_{q \in Q} \pi(\tilde{p}, q), \quad \overline{v} = \max_{p \in P} \pi(p, \tilde{q})$$

- $\Rightarrow \underline{v} \leq \pi(\tilde{p}, \tilde{q}) \leq \overline{v}$

Déjà vu

- Def: $(\tilde{p}, \tilde{q})$ is a saddle point for $\pi(.,.)$ if $\pi(\tilde{p}, q) \geq \pi(\tilde{p}, \tilde{q}) \geq \pi(p, \tilde{q}), \forall p \in P, \forall q \in Q$.
- Theorem: (a) $\underline{v} = \overline{v} \Leftrightarrow \exists$ saddle point
  (b) If $(\tilde{p}, \tilde{q})$ is a saddle point, then $\pi(\tilde{p}, \tilde{q}) = \underline{v} = \overline{v}$

- Minimax Theorem: $\underline{v} = \overline{v}$.
- Proof: When M is 1*1, clearly $\underline{v} = \overline{v}$.
  Mathematical induction: suppose the theorem holds for all submatrices M' of M_{m*n}, we'll show that it then also holds for M.
  Suppose not, we then have:
  $$\underline{v} < \overline{v}.$$
  Let $\tilde{p}$ and $\tilde{q}$ denote I's and II's security strategies:
  $$\underline{v} \leq \pi(\tilde{p}, \tilde{q}) \leq \overline{v}.$$
  These inequalities implies one of the following must be true:
  $$\underline{v} < \pi(\tilde{p}, \tilde{q}), \quad \pi(\tilde{p}, \tilde{q}) < \overline{v}.$$
  We'll assume the latter is true. The proof is similar if the former is true.
  $$\begin{aligned} \pi(\tilde{p}, \tilde{q}) &= \tilde{p}(M\tilde{q}^T) \\ &= \tilde{p}_1 M_{1.}\tilde{q}^T + \ldots + \tilde{p}_m M_{m.}\tilde{q}^T. \end{aligned}$$
  $\pi(\tilde{p}, \tilde{q}) < \overline{v}$ implies $\exists M_{k.}$, s.t. $M_{k.}\tilde{q}^T < \overline{v}$.
  Let $s = \overline{v} - M_{k.}\tilde{q}^T > 0$.

Let M' be M w/o $M_{k.}$. Let p' and q' be the security strategies of M'. By assumption M''s value v' exists.
Note p' is $1*(m-1)$. To proceed smoothly, we'll expand p' to $1*m$ by inserting 0 to be its k-th element.
Consider $r \in (0,1)$ and define:

$$\begin{aligned} p'' &= (1-r)\tilde{p} + rp', \\ q'' &= (1-r)\tilde{q} + rq'. \end{aligned}$$

$$\begin{aligned} \min_j p''M_{.j} &\geq \min_j (1-r)\tilde{p}M_{.j} + \min_j rp'M_{.j} \\ &= (1-r)\underline{v} + rv'. \qquad (1) \\ \max_{i\neq k} M_{i.}q''^T &\leq \max_{i\neq k}(1-r)M_{i.}\tilde{q} + \max_{i\neq k} rM_{i.}q' \\ &\leq (1-r)\overline{v} + rv'. \end{aligned}$$

Consider a very large number $a = \max_{ij} M_{ij} + 1$.

$$\begin{aligned} M_{k.}q''^T &< (1-r)(\overline{v}-s) + ra \\ &= (1-r)\overline{v} + rv' - (1-r)s + r(a-v'). \end{aligned}$$

Note $-(1-r)s + r(a-v') \uparrow$ in r, and it is 0 when $r = s/[(a-v')+s] \in (0,1)$.
Consider $r \in (0, s/[(a-v')+s])$, then

$$\max_i M_{i.}q''^T \leq (1-r)\overline{v} + rv'. \qquad (2)$$

From (1) and (2),

$$\begin{aligned} \min_j p'' M_{.j} &\geq (1-r)\underline{v} + rv', \\ \max_i M_{i.} q''^T &\leq (1-r)\overline{v} + rv'. \end{aligned}$$

Thus

$$\begin{aligned} \max_i M_{i.} q''^T - \min_j p'' M_{.j} &\leq (1-r)(\overline{v} - \underline{v}) \\ &< \overline{v} - \underline{v} \end{aligned}$$

At least one of the following inequality must hold:

$$\max_i M_{i.} q''^T < \overline{v} \text{ or } \min_j p'' M_{.j} > \underline{v}.$$

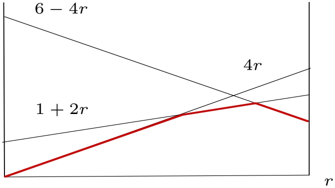
But, neither can be true. □

- In the following example, we have shown that $\overline{m} = 3 > 2 = \underline{m}$.

$$M = \begin{array}{c|ccc} & t_1 & t_2 & t_3 \\ \hline s_1 & 1 & 6 & 0 \\ s_2 & 2 & 0 & 3 \\ s_3 & 3 & 2 & 4 \end{array}$$

- Minimax Thm. tells us that $\overline{v} = \underline{v}$. What is the value?
- What is I's mixed security strategy?
  - I plays s_2 w.p.0.
  - I's security strategy is: $p = (1-r, 0, r)$.
- Given r, let's imagine the worst situation for I.
  - If II plays t_1, $\pi = (1-r) + 3r = 1 + 2r$.
  - .... t_2, $\pi = 6(1-r) + 2r = 6 - 4r$.
  - .... t_3, $\pi = 0 + 4r = 4r$.
- Given r, I's worst expected payoff is $\min\{1+2r, 6-4r, 4r\}$.
- Do we have to consider II's mixed strategy?

- The red line in following graph shows $\min\{1+2r, 6-4r, 4r\}$ for different r.



- What is I's security strategy?
- $6-4r = 1+2r \Rightarrow r = 5/6$
- $\underline{v} = 8/3$

- $$M = \begin{array}{c|ccc} & t_1 & t_2 & t_3 \\ \hline s_1 & 1 & 6 & 0 \\ s_2 & 2 & 0 & 3 \\ s_3 & 3 & 2 & 4 \end{array}$$
- From the Minimax Them., we know II's security level $\overline{v} = 8/3$.
- What is II's security strategy, $q = (q_1, q_2, q_3)$?
  - If I plays s_1, $\pi = q_1 + 6q_2$.
  - .... s_3, $\pi = 3q_1 + 2q_2 + 4q_3$
  - Do we have to consider s_2?
- Given q, the worst situation for II is that I receives $\max\{q_1 + 6q_2, 3q_1 + 2q_2 + 4q_3\}$, or $\max\{q_1 + 6q_2, 4 - q_1 - 2q_2\}$.
- II wishes to:
$$\min_{q_1, q_2} \max\{q_1 + 6q_2, 4 - q_1 - 2q_2\}$$
- The textbook shows the a 3D graph of II's solution.
- We'll solve it algebraically.

II's security strategy

- $\min_{q_1,q_2} \max\{q_1 + 6q_2, 4 - q_1 - 2q_2\}$=?
- The previous theorem states that $\underline{v} = \overline{v} \Leftrightarrow \exists$ saddle point.
- And in the proof, it is shown when $\underline{v} = \overline{v}$, two players' security strategies form a saddle point.
- From the Minimax Thm., two players' security strategies form a saddle point, or an NE.
  - II's equilibrium strategy will make I indifferent between s_1 and s_3:

$$q_1 + 6q_2 = 4 - q_1 - 2q_2.$$

  - We need 1 more equation to solve q_1 and q_2.