Graph Theory HW5

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November 28, 2023

Problem 1.

($\Rightarrow$):

Lemma 1: Let G be a graph, and G_1, G_2 be induced subgraph of G such that $V(G_1), V(G_2)$ is a partition of $V(G)$. Then $q(G) \leq q(G_1) + q(G_2)$.

Proof: $q(G_1) + q(G_2)$ is the total number of odd components of G_1 and G_2, which is the number of odd components after removing all edges between $V(G_1)$ and $V(G_2)$ in G. Let C be an odd component of G. It will be decomposed into several components (denote this set of components as D_C) after removing all edges between $V(G_1)$ and $V(G_2)$, since the sum of several even numbers is even, there should be at least one component in D_C whose size is odd.

$\Rightarrow q(G) = \sum_{C \text{ is an odd component in } G} 1 \leq \sum_{C \text{ is an odd component in } G}$ the number of odd components in $D_C = q(G_1) + q(G_2)$, which finishes the proof of Lemma 1.

Suppose that G has a matching M covering all but at most k vertices. Let U denote the induced subgraph of G of the set of vertices that are not covered by M, and $H := G \setminus U$.

By the definition, $|U| \leq k$.

By Tutte's theorem, $\forall T \subseteq V(H)$, $q(H \setminus T) \leq |T|$. − (1)

$\Rightarrow \forall S \subseteq V(G)$, let $T := S \cap H$, $q(G \setminus S) = q((H \cup U) \setminus S) = q((H \setminus S) \cup (U \setminus S)) = q((H \setminus T) \cup (U \setminus S)) \overset{\text{Lemma 1}}{\leq} q(H \setminus T) + q(U \setminus S) \overset{(1)}{\leq} |T| + q(U \setminus S) \overset{U \setminus S \text{ has at most } |U| \text{ components}}{\leq} |T| + |U| \leq |T| + k = |S \setminus U| + k \leq |S| + k$.

($\Leftarrow$):

Suppose that $\forall S \subseteq V(G)$, there is $q(G \setminus S) \leq |S| + k$.

Let H be a graph where $V(H) = V(G) \cup \{v_1, v_2, \ldots, v_k\}$ ($v_1, v_2, \ldots, v_k \notin V(G)$, and $E(H) = E(G) \cup \{v_iv_j | 1 \leq i < j \leq k\} \cup \{uv_i | u \in V(G), 1 \leq i \leq k\}$.
Let $T \subseteq V(H)$, and $S = T \setminus \{v_1, v_2, \ldots, v_k\}$.

Case 1: $\{v_1, v_2, \ldots, v_k\} \subseteq T$.
$q(H \setminus T) = q(G \setminus S) \leq |S| + k = |T|$.
Case 2: $\exists i$ s.t. $v_i \notin T$.
Since v_i connects to all vertices in H by definition, it connects to all vertices in $H \setminus T$, which implies $q(H \setminus T) \leq 1 \leq |T|$.
Therefore, $q(H \setminus T) \leq T$ always holds.
By Tutte's theorem, $\exists$ a perfect matching M of H.
Let $M' = M \cap G$, and we can see that a vertex u is not covered by M' in $G \iff u$ matches to v_i in M for some i.
Since there is at most k vertices in G matches to $\{v_1, v_2, \ldots, v_k\}$ in M, in M' there is at most k vertices in G that are not covered, which finishes the proof.

Problem 4. By section 6.2, the vertices and the edges on the football form a planar graph.
Suppose that it contains a pentagons, b hexagons, v vertices, e edges.
By proposition 6.10, $2e = 5a + 6b$.
By Euler's formula, $v - e + a + b = 2 \Rightarrow 6v - 6e + 6a + 6b = 12$.
By handshake lemma, $3v = 2e$ since it's a 3-regular graph.
$6a + 6b - 2e = 4e - 6e + 6a + 6b = 6v - 6e + 6a + 6b = 12$.
$\Rightarrow 6a + 6b - (5a + 6b) = 12$.
$\Rightarrow a = 12$.
$\therefore$ there are 12 pentagons.

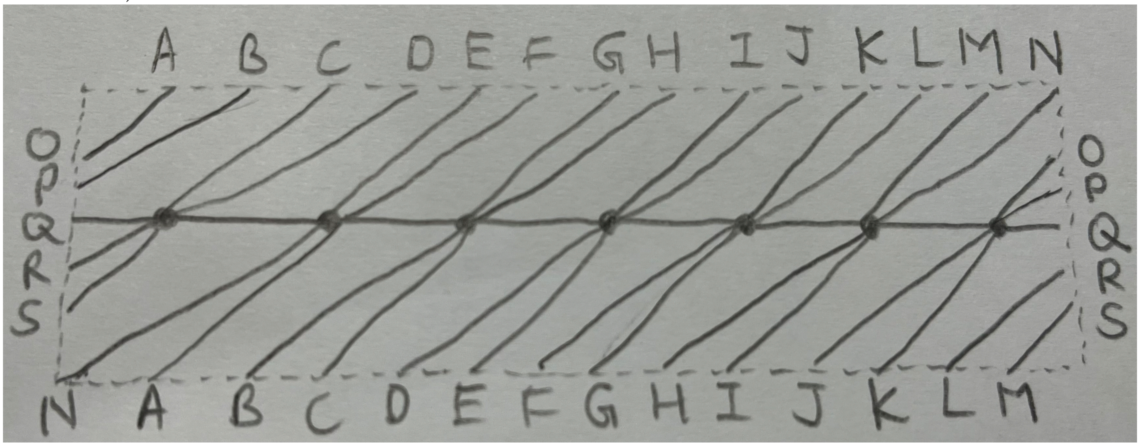
Problem 5.

(a) If a graph is planar, we can draw it on a plane without crossing edges, and then scale the graph so that it can be put in the unit square $\{(x, y) | 0 < x < 1, 0 < y < 1\}$.

Since it does not have crossing edges, after setting $(0, y) = (1, y), (x, 0) = (x, 1)$, it still doesn't contain crossing edges. Therefore, it is toroidal.

$\therefore$ every planar graph is toroidal.

(b) We can draw K_7 like the following: (the points marked with the same letter are the same)



(c) Suppose the opposition that K_8 is toroidal.

$v =$ the number of vertices in $K_8 = 8$, $e =$ the number of edges in $K_8 = \binom{8}{2} = 28$.

If the boundary of a face contains a cycle, then the length of the cycle ≥ 3.

Otherwise, G must be a tree (but K_8 is not).

So for all faces f, $l(f) \geq 3$.

Since each edge is counted exactly twice in the sum $\sum l(f_i)$, there is $2e = \sum l(f_i) \geq 3f$.

$v - e + f \leq v - e + \frac{2}{3}e = 8 - 28 + \frac{56}{3} = -\frac{4}{3} < 0$, which contradicts to $v - e + f = 0$ required in a toroidal graph.

Therefore, K_8 is not toroidal.