賽局論 HW5

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Problem 1 (9.8.12)**.**

(a) Let $p = p_1, p_2 = 1 - p$.

$\because {p'_1}_{p_1=1} = 0, {p'_2}_{p_2=0} = 0$.

$\therefore (1, 0)$ (that is, $p = 1$) is always a rest point.

When $a \neq c$, for sufficiently small $\epsilon > 0$, for $p > 1 - \epsilon$, there is $p(a-c) + (1 - p)(b-d) > 0$ iff $a - c > 0$.

When $a = c$, there is $p(a-c) + (1-p)(b-d) > 0$ iff $b - d > 0$.

$\Rightarrow p'_1(p) > 0$ for any sufficiently small $\epsilon > 0$ iff $a - c > 0$ or ($a = c$ and $b - d > 0$).

$\Rightarrow 1 \in (1-\epsilon, 1] \subseteq \operatorname{int}(\{p : \lim_{t\to\infty} p(t) = 1\})$.

Otherwise, if $a - c < 0$ or $a = c$ and $b - d < 0$, then $p'_1(p) < 0$ for any sufficiently small $\epsilon > 0$, then $\lim_{t\to\infty} p \neq 1$.

Otherwise, if $a = c$ and $b = d$, then $p'_1(p) = 0$, and $\lim_{t\to\infty} p \neq 1$ if $p \neq 1$.

$\therefore$ it is an asymptotic attractor iff $a - c > 0$ or ($a = c$ and $b - d > 0$).

(d) Since if $p'_1 = p'_2 = 0$ but $p_1 \neq 0$ or 1, then $p_1 = \frac{d-b}{a-c+d-b}$.

$\Rightarrow (\frac{d-b}{a-c+d-b}, 1 - \frac{d-b}{a-c+d-b})$ is a rest point iff $\frac{d-b}{a-c+d-b} \in (0,1)$ iff $\frac{d-b}{a-c+d-b} > 0$ and $\frac{a-c}{a-c+d-b} > 0$ iff $(a-c)(d-b) \geq 0$ iff $a \geq c$ and $d \geq b$ or $a \leq c$ and $d \leq b$.

(e) Note that when $a = c$, $\tilde{p}_1 = 1$, which is the case of (a).

When $b = d$, $\tilde{p}_1 = 0$, which is the case of (b).

$\therefore$ in this problem, we can assume that $a \neq c$, and $b \neq d$.

If ($a > c$ and $d < b$) or ($a < c$ and $d > b$), then by (d), $(\tilde{p}_1, 1 - \tilde{p}_1)$ will not be an asymptotic attractor.

Otherwise there are two cases: case 1: $a < c$ and $d < b$; case 2: $a > c$ and $d > b$.

$p'_{1 p_1 = \tilde{p}_1 + \epsilon} = p_1(1-p_1)\{\epsilon(a-c) - \epsilon(b-d)\} = p_1(1-p_1)(a+b-c-d)\epsilon$.

If $a-c-b+d < 0$ (which is case 1), then $p'_1 < 0$ for $\epsilon > 0$, and $p'_1 > 0$ for $\epsilon < 0$.

$\Rightarrow (\tilde{p}_1 - \epsilon, \tilde{p}_1 + \epsilon) \subseteq \{p : \lim_{t\to\infty} p = \tilde{p}_1\}$ for some $\epsilon > 0$.

$\Rightarrow \tilde{p}_1$ is an asymptotic attractor in this case.

If $a-c-b+d > 0$ (which is case 2), then $p'_1 > 0$ for $\epsilon > 0$, and $p'_1 < 0$ for $\epsilon < 0$.

$\Rightarrow \lim_{t\to\infty} p \neq \tilde{p}_1$ for $p \neq p_1$.

$\therefore \tilde{p}_1$ is not an asymptotic attractor in this case.

$\therefore \tilde{p}_1$ is an asymptotic attractor iff $a < c$ and $d < b$.

Problem 2.

(a) (p^*, p^*) is NE.

$\Rightarrow \forall p,\ p^T A p^* \leq p^{*T} A p^*$. – (1)

Suppose that i is such that $(Ap^*)_i \geq (Ap^*)_j$ for all j.

Let e_i be the vector that is the i-th column of I.

If $p^*_j \neq 0$, then $p^{*T} A p^* \leq e_i^T A p^*$, the equation holds only if $(Ap^*)_i = (Ap^*)_j$.

But by (1), $e_i^T A p^* \leq p^{*T} A p^*$, and the equation should hold.

$\Rightarrow (Ap^*)_j = (Ap^*)_i,\ \forall j$ s.t. $p^*_j \neq 0$.

$\Rightarrow \forall j$ s.t. $p^*_j \neq 0$, $f_j(p^*) = (Ap^*)_j = (Ap^*)_i = \sum_{k, p^*_k \neq 0} p_k (Ap^*)_i = \sum_{k, p^*_k \neq 0} p_k (Ap^*)_k = \bar{f}(p^*)$.

$\therefore \forall j.$ either $p^*_j = 0$ or $f_j(p^*) - \bar{f}(p^*) = 0$.

$\Rightarrow p'_j = 0,\ \forall j$.

$\therefore p^*$ is a rest point.

(b) (e_1, e_1) is an asymptotic attractor.

$\Rightarrow$ for sufficiently small $\epsilon > 0$, there is $p'_1 \geq 0$ for $p = e_1(1-\epsilon) + e_i\epsilon$ for all $i \neq 1$.

$p'_1 = p_1(f_1(e_1) - \bar{f}(e_1)) = p_1(f_1(e_1) - (1-\epsilon)f_1(e_1) - \epsilon f_i(e_1)) \geq 0$.

$\because p_1 = 1 - \epsilon > 0$, there is $f_1(e_1) - (1-\epsilon)f_1(e_1) - \epsilon f_i(e_1) \geq 0$.

$\Rightarrow \epsilon(f_1(e_1) - f_i(e_1)) \geq 0$.

$\Rightarrow f_1(e_1) \geq f_i(e_1),\ \forall i$.

$\therefore e_1^T A e_1 = f_1(e_1) = \sum_{i=1}^{n} q_i f_1(e_1) \geq \sum_{i=1}^{n} q_i f_i(e_1) = q^T A e_1,\ \forall q$.

And $e_1^T A^T e_1 = \sum_{i=1}^{n}(e_1^T A^T)_1 q_i = \sum_{i=1}^{n} f_1(e_1) q_i \geq \sum_{i=1}^{n} f_i(e_1) q_i = \sum_{i=1}^{n}(e_1^T A^T)_i q_i = e_1^T A^t q,\ \forall q$.

$\therefore (e_1, e_1)$ is a NE.

Problem 3.

(a) Let $A = \begin{pmatrix} 0 & 6 & -4 \\ -3 & 0 & 5 \\ -1 & 3 & 0 \end{pmatrix}$.

One can see that $(Aq)_1 \leq (Aq)_2 \iff 6q_2 - 4q_3 \leq -3q_1 + 5q_3 \iff 3q_1 + 6q_2 \leq 9 - 9q_1 - 9q_2 \iff 4q_1 + 5q_2 \leq 3$.

$(Aq)_2 \leq (Aq)_3 \iff -3q_1 + 5q_3 \leq -1q_1 + 3q_2 \iff 2q_1 + 3q_2 \geq 5 - 5q_1 - 5q_2 \iff 7q_1 + 8q_2 \geq 5$.

$(Aq)_3 \leq (Aq)_1 \iff -1q_1 + 3q_2 \leq 6q_2 - 4q_3 \iff 3q_2 + q_1 \geq 4 - 4q_1 - 4q_2 \iff 5q_1 + 7q_2 \geq 4$.

When $(Aq)_1 = (Aq)_2, (Aq)_2 = (Aq)_3$, there is $q_1 = q_2 = \frac{1}{3}$.

$\therefore$ for NE that the row player does not play pure strategy, the column player plays $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

Similarly, for NE that the column player does not play pure strategy, the row player plays $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

$\therefore ((\frac{1}{3}, \frac{1}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}))$ is the only NE that some player does not play pure strategy.

When playing pure strategy, one can see that only (s_1, s_1) is NE because if a player plays s_3, then it will switch to s_2 if another plays s_3, switch to s_1 is another plays s_1, s_2.

$\Rightarrow$ no player will play s_3 in NE.

And (s_2, s_2) will be switch to (s_1, s_2) by the row player, $(s_1, s_2), (s_2, s_1)$ will be switch to (s_1, s_1) by the column player, row player, respectively.

$\therefore$ all NE are: $((1,0,0),(1,0,0)), ((\frac{1}{3}, \frac{1}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}))$.

(b) Let $e_1 = (1, 0, 0)$.

One can see that $Ae_1 = (0, -3, -1)$.

$\therefore \forall p \neq e_1,\ p^T A e_1 < e_1^T A e_1.$

$\therefore (e_1, e_1)$ is an evolutionarily stable strategy.

Let $u = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

One can see that $Au = (\frac{2}{3}, \frac{2}{3}, \frac{2}{3})$.

$\Rightarrow \forall p \neq u,\ p^T A u = 2 = u^T A u.$

However, consider $p = (0, \frac{1}{2}, \frac{1}{2})$.

$\Rightarrow p^T A p = 4 > u^T A p.$

$\therefore (u, u)$ is not an evolutionarily stable strategy. $\therefore$ the only evolutionarily stable strategy is (e_1, e_1).

Problem 4.

(a) (p_1, p_2, p_3) is a rest point.

$\Rightarrow p_1' = p_2' = p_3' = 0.$

If $p_1 = p_2 = p_3$, then $p_1' = p_2' = p_3' = 0$, which is a rest point.

Suppose that $p_1 = p_2 = p_3$ does not hold.

Without loss of generality, suppose that $p_1 \neq p_2$.

$p_3(p_1 - p_2) = p_3' = 0 \Rightarrow p_3 = 0.$

$\Rightarrow p_1 p_2 = p_1' = 0, p_2(-p_1) = p_2' = 0.$

$\Rightarrow p_1 = 0$ or $p_2 = 0$.

$\therefore (p_1, 0, 0), (0, p_2, 0), (0, 0, p_3)$ are also rest points.

$\therefore$ all rest points are $(p, p, p), (p, 0, 0), (0, p, 0), (0, 0, p)$.

(b) $\frac{d(p_1p_2p_3)}{dt} = p_1'p_2p_3 + p_1p_2'p_3 + p_1p_2p_3' = (p_1p_2p_3)\left(\frac{p_1'}{p_1} + \frac{p_2'}{p_2} + \frac{p_3'}{p_3}\right) = (p_1p_2p_3)(p_2 - p_3 + p_3 - p_1 + p_1 - p_2) = 0.$

(c) Since $\frac{d(p_1p_2p_3)}{dt} = 0$, $\lim_{t\to\infty}(p_1(t), p_2(t), p_3(t)) = (\sqrt[3]{p_1p_2p_3}, \sqrt[3]{p_1p_2p_3}, \sqrt[3]{p_1p_2p_3})$ for $p_1p_2p_3 \neq 0$ (where $p_i := p_i(0)$ for convenience).

$\lim_{t\to\infty}(p_1(t) + \epsilon, p_2(t), p_3(t)) \neq (\sqrt[3]{p_1p_2p_3}, \sqrt[3]{p_1p_2p_3}, \sqrt[3]{p_1p_2p_3})$ for any sufficiently small ϵ.

$\therefore \mathrm{int}((p, p, p)\text{'s BoA}) = \emptyset$ for $p \neq 0$.

Since $\forall p_1, p_2, p_3$ s.t. $p_1p_2p_3 = 0$, for any sufficiently small ϵ, $(p_1 + \epsilon)(p_2 + \epsilon)(p_3 + \epsilon) \neq 0$.

$\therefore \operatorname{int}((p,0,0)\text{'s BoA}) = \operatorname{int}((0,p,0)\text{'s BoA}) = \operatorname{int}((0,0,p)\text{'s BoA}) = \emptyset.$

$\therefore$ there is no asymptotic attractor.