機率與統計 HW1

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Problem 1.

(a) $S \cap \emptyset = \emptyset$.

$\Rightarrow \Pr[S] = \Pr[S \cup \emptyset] \overset{\text{Axiom 3}}{=} \Pr[S] + \Pr[\emptyset]$.

$\Rightarrow \Pr[\emptyset] = 0$.

(b) $A \cap A^c = \emptyset$.

$\Rightarrow 1 \overset{\text{Axiom 2}}{=} \Pr[S] = \Pr[A \cup A^c] \overset{\text{Axiom 3}}{=} \Pr[A] + \Pr[A^c]$.

$\Rightarrow \Pr[A^c] = 1 - \Pr[A]$.

(c) Let $C := B \setminus A$.

$\Rightarrow A \cap C = \emptyset, A \cup C = B$.

$\Rightarrow \Pr[B] = \Pr[A \cup C] \overset{\text{Axiom 3}}{=} \Pr[A] + \Pr[C] \overset{\text{Axiom 1}}{\geq} \Pr[A]$.

$\Rightarrow \Pr[A] \leq \Pr[B]$.

Problem 2.

(a) 假設感染的人中就醫的比例爲 x、沒接種疫苗的人中感染率爲 y。

有接種疫苗且感染且確診的人佔全部的 $0.6 \times 0.05x = 0.03x$，沒接種疫苗且感染且確診的人佔全部的 $0.4yx$。

$\dfrac{0.03x}{0.03x + 0.4xy} = 0.15$.

$\Rightarrow \dfrac{0.03}{0.03 + 0.4y} = 0.15$.

$\Rightarrow 0.03 + 0.4y = 0.2$.

$\Rightarrow y = 0.425$.

$\therefore$ 全部人的感染率爲 $0.6 \times 0.05 + 0.4 \times 0.425 = 0.2 = 20\%$.

(b) 本題答案即上題的 $y = 0.425 = 42.5\%$.

Problem 3. $(A \cap B) \cap (A \cap B^c) \subseteq B \cap B^c = \emptyset$.
$(A \cup B) \cap (A \cup B^c) = A \cap (B \cup B^c) = A$.
$\Rightarrow$ by Axiom 3, $\Pr[A \cap B] + \Pr[A \cap B^c] = \Pr[A]$.
B implies $A \iff \Pr[A|B] > \Pr[A] \iff \dfrac{\Pr[A \cap B]}{\Pr[B]} > \Pr[A] \iff \Pr[A \cap B] > \Pr[A]\Pr[B] \iff \Pr[A] - \Pr[A \cap B^c] > \Pr[A](1 - \Pr[B^c]) \iff \Pr[A \cap B^c] < \Pr[A]\Pr[B^c] \iff \dfrac{\Pr[A \cap B^c]}{\Pr[B^c]} < \Pr[A] \iff \Pr[A|B^c] < \Pr[A] \iff B^c$ does not imply A.

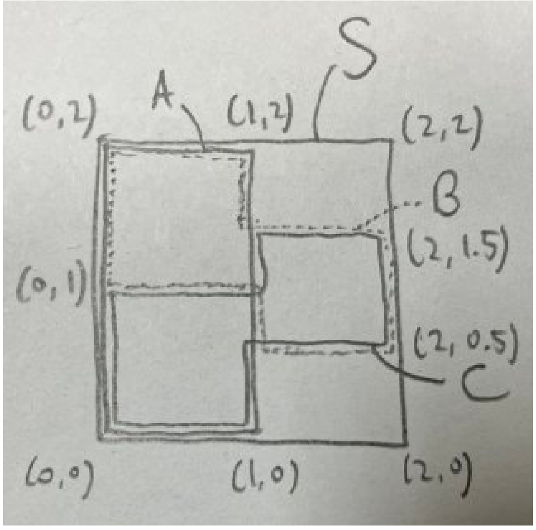
Problem 4.

(a) A and B are independent $\iff \Pr[A \cap B] = \Pr[A]\Pr[B] \iff \Pr[A \cap B^c] \overset{\text{proved in problem 3}}{=} \Pr[A] - \Pr[A \cap B] = \Pr[A] - \Pr[A]\Pr[B] = \Pr[A](1 - \Pr[B]) = \Pr[A]\Pr[B^c] \iff A$ and B^c are independent.

(b) A and B are independent $\iff \Pr[A \cap B] = \Pr[A]\Pr[B] \iff \Pr[A^c \cap B] \overset{\text{proved in problem 3}}{=} \Pr[B] - \Pr[A \cap B] = \Pr[B] - \Pr[A]\Pr[B] = \Pr[B](1 - \Pr[A]) = \Pr[B]\Pr[A^c] \iff A^c$ and B are independent.

(c) From (b), A and B are independent $\iff A^c$ and B are independent $\iff \Pr[A^c \cap B] = \Pr[A^c]\Pr[B] \iff \Pr[A^c \cap B^c] \overset{\text{proved in problem 3}}{=} \Pr[A^c] - \Pr[A^c \cap B] = \Pr[A^c] - \Pr[A^c]\Pr[B] = \Pr[A^c](1 - \Pr[B]) = \Pr[A^c]\Pr[B^c] \iff A^c$ and B^c are independent.

Problem 5.



Area of $S = 4$. Area of $A, B, C = 2$. Area of $A \cap B, B \cap C, C \cap A = 1$. Area of $A \cap B \cap C = 0$.

$\Rightarrow \Pr[A] = \Pr[B] = \Pr[C] = \frac{2}{4} = \frac{1}{2}, \Pr[A \cap B] = \Pr[B \cap C] = \Pr[C \cap A] = \frac{1}{4} = \Pr[A]\Pr[B] = \Pr[B]\Pr[C] = \Pr[C]\Pr[A]$.

$\Rightarrow A, B, C$ are pairwise independent.

$\Pr[A \cap B \cap C] = 0 \neq \frac{1}{8} = \Pr[A]\Pr[B]\Pr[C]$.

$\Rightarrow A, B, C$ are not independent.

Problem 6.

(a) Let X be the group of the selected kicker.
$\Pr[X = 1] = \frac{3}{3+6} = \frac{1}{3}, \Pr[X = 2] = \frac{6}{3+6} = \frac{2}{3}$.
$\Pr[K] = \Pr[X = 1]\frac{1}{1+1} + \Pr[X = 2]\frac{1}{2+1} = \frac{1}{6} + \frac{2}{9} = \frac{7}{18}$.

(b) From (a), $\Pr[K_1] = \Pr[K_2] = \frac{7}{18}$.
Let X_j be the group of the j-th selected kicker.
$\Pr[K_1 \cap K_2] = \Pr[X_1 = 1 \cap X_2 = 1]\frac{1}{4} + \Pr[X_1 = 1 \cap X_2 = 2]\frac{1}{6} + \Pr[X_1 = 2 \cap X_2 = 1]\frac{1}{6} + \Pr[X_1 = 2 \cap X_2 = 2]\frac{1}{9} = \frac{3 \times 2}{9 \times 8} \times \frac{1}{4} + \frac{3 \times 6}{9 \times 8} \times \frac{1}{6} + \frac{6 \times 3}{9 \times 8} \times \frac{1}{6} + \frac{6 \times 5}{9 \times 8} \times \frac{1}{9} = \frac{1}{72}(\frac{3}{2} + 3 + 3 + \frac{10}{3}) = \frac{65}{432} \neq (\frac{7}{18})^2 = \Pr[K_1]\Pr[K_2]$.
$\therefore K_1, K_2$ are not independent.

(c) Let X be the group of the selected kicker.
$\Pr[M = 5] = \Pr[X = 1]\binom{10}{5}(\frac{1}{2})^5(\frac{1}{2})^5 + \Pr[X = 2]\binom{10}{5}(\frac{1}{3})^5(\frac{2}{3})^5 = \frac{1}{3}\binom{10}{5}(\frac{1}{2})^{10} +$

$$\frac{2}{3}\binom{10}{5}(\frac{1}{3})^5(\frac{2}{3})^5.$$

Problem 7. The probability that Dr. Jones purchased a new umbrella on a day is the chance of raining and Dr. Jones visiting the library and the umbrella being stolen, which is $0.5 \times 0.8 \times 0.25 = 0.1$.
The probability that the 10-th time getting stolen happens right on the 20-th day is the probability that Dr. Jones got stolen for exactly 9 times in the 1-st to 19-th day, and got stolen in the 20-th day, which is $\binom{19}{9}0.9^{10}0.1^9 \times 0.1 = \binom{19}{9}0.09^{10}$.