# Information Theory HW5

許博翔

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**Problem 1.**

(a) (1) From Gaussian integral, we know that $\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$.

$\int xe^{-x^2}dx = \int \frac{1}{2}e^{-x^2}d(x^2) = -\frac{1}{2}e^{-x^2} + c.$

$\lim_{x\to\infty} xe^{-x^2} = \lim_{x\to\infty} \frac{x}{e^{x^2}} \overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{1}{2xe^{x^2}} = 0.$

$\lim_{x\to-\infty} xe^{-x^2} = \lim_{x\to-\infty} \frac{x}{e^{x^2}} \overset{\text{L'H}}{=} \lim_{x\to-\infty} \frac{1}{2xe^{x^2}} = 0.$

$\int_{-\infty}^{\infty} x^2e^{-x^2}dx = \int_{-\infty}^{\infty} xe^{-x^2}\cdot xdx = -\frac{1}{2}e^{-x^2}x\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} -\frac{1}{2}e^{-x^2}\cdot 1dx = 0 + \frac{1}{2}\sqrt{\pi} = \frac{1}{2}\sqrt{\pi}.$

$f(x) = \frac{1}{\sigma_1\sqrt{2\pi}}e^{-\frac{1}{2}(\frac{x-\mu_1}{\sigma_1})^2}, g(x) = \frac{1}{\sigma_2\sqrt{2\pi}}e^{-\frac{1}{2}(\frac{x-\mu_2}{\sigma_2})^2}.$

$$
\begin{aligned}
D(f\|g) &= \int_{-\infty}^{\infty} \frac{1}{\sigma_1\sqrt{2\pi}}e^{-\frac{1}{2}(\frac{x-\mu_1}{\sigma_1})^2} \log\left(\frac{1}{\sigma_1\sqrt{2\pi}}e^{-\frac{1}{2}(\frac{x-\mu_1}{\sigma_1})^2} / \frac{1}{\sigma_2\sqrt{2\pi}}e^{-\frac{1}{2}(\frac{x-\mu_2}{\sigma_2})^2}\right)dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sigma_1\sqrt{2\pi}}e^{-\frac{1}{2}(\frac{x-\mu_1}{\sigma_1})^2} \left(\log\left(\frac{\sigma_2}{\sigma_1}\right) + \frac{1}{2}\log e\left(-(\frac{x-\mu_1}{\sigma_1})^2 + (\frac{x-\mu_2}{\sigma_2})^2\right)\right)dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sigma_1\sqrt{2\pi}}e^{-\frac{1}{2}(\frac{x-\mu_1}{\sigma_1})^2} \left(\log\left(\frac{\sigma_2}{\sigma_1}\right) - \frac{1}{2}\log e(\frac{x-\mu_1}{\sigma_1})^2\right. \\
&\left. + \frac{1}{2}\log e(\frac{x-\mu_1}{\sigma_2})^2 + \log e(\frac{\mu_1-\mu_2}{\sigma_2})(\frac{x-\mu_1}{\sigma_2}) + \frac{1}{2}\log e(\frac{\mu_1-\mu_2}{\sigma_2})^2\right)dx \\
&= \log\left(\frac{\sigma_2}{\sigma_1}\right) - \frac{1}{2}\log e + \frac{1}{2}\log e\frac{\sigma_1^2}{\sigma_2^2} + \frac{1}{2}\log e(\frac{\mu_1-\mu_2}{\sigma_2})^2 \\
&= \log\left(\frac{\sigma_2}{\sigma_1}\right) + \frac{\log e}{2\sigma_2^2}(\sigma_1^2 - \sigma_2^2 + (\mu_1-\mu_2)^2).
\end{aligned}
$$

(2) $f(x) = \frac{1}{\sqrt{2}\sigma_1}e^{-\frac{\sqrt{2}|x-\mu_1|}{\sigma_1}}, g(x) = \frac{1}{\sqrt{2}\sigma_2}e^{-\frac{\sqrt{2}|x-\mu_2|}{\sigma_2}}.$

$\int xe^{x}dx = e^{x}x - \int e^{x}dx = (x-1)e^{x} + c.$

$\int xe^{-x}dx = -e^{-x}x - \int -e^{-x}dx = -(x+1)e^{-x} + c.$

$\lim_{x\to\infty} e^{-x}x = 0.$

$\int_{-\infty}^{\infty} |x-a|e^{-|x-b|}dx = \int_{-\infty}^{\infty} |x+b-a|e^{-|x|}dx.$

If $c := a-b \geq 0$, then $\int_{-\infty}^{\infty} |x+b-a|e^{-|x|}dx = \int_{-\infty}^{0} (c-x)e^{x}dx + \int_{0}^{c} (c-x)e^{-x} + \int_{c}^{\infty} (x-c)e^{-x}dx$

$= c+1+(-ce^{-c}+c)+((c+1)e^{-c}-1)+(c+1)e^{-c}-ce^{-c} = 2c+2e^{-c}.$

If $c < 0$, then $\int_{-\infty}^{\infty} |x-c|e^{-|x|}dx = \int_{-\infty}^{\infty} |x+c|e^{-|x|}dx = -2c+2e^{c}.$

$\therefore \int_{-\infty}^{\infty} |x-a|e^{-|x-b|}dx = 2|a-b| + 2e^{-|a-b|}.$

$D(f\|g) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2}\sigma_1} e^{-\frac{\sqrt{2}|x-\mu_1|}{\sigma_1}} \log\left(\frac{1}{\sqrt{2}\sigma_1} e^{-\frac{\sqrt{2}|x-\mu_1|}{\sigma_1}} / \frac{1}{\sqrt{2}\sigma_2} e^{-\frac{\sqrt{2}|x-\mu_2|}{\sigma_2}}\right) dx$

$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2}\sigma_1} e^{-\frac{\sqrt{2}|x-\mu_1|}{\sigma_1}} \left(\log\left(\frac{\sigma_2}{\sigma_1}\right) + \sqrt{2}\log e(\frac{|x-\mu_2|}{\sigma_2} - \frac{|x-\mu_1|}{\sigma_1})\right) dx$

$= \log\left(\frac{\sigma_2}{\sigma_1}\right) + \frac{\log e}{\sigma_1\sigma_2}(\frac{\sigma_1^2}{2})(2\cdot\frac{\sqrt{2}}{\sigma_1}|\mu_1-\mu_2| + 2e^{-\frac{\sqrt{2}}{\sigma_1}|\mu_1-\mu_2|}) - \frac{\log e}{\sigma_1^2}(\frac{\sigma_1^2}{2})2$

$= \log\left(\frac{\sigma_2}{\sigma_1}\right) + \frac{\sigma_1 \log e}{\sigma_2}(\frac{\sqrt{2}}{\sigma_1}|\mu_1-\mu_2| + e^{-\frac{\sqrt{2}}{\sigma_1}|\mu_1-\mu_2|}) - \log e.$

(b) The first KL divergence − the second KL divergence $= \frac{\log e}{2\sigma_2^2}(\sigma_1^2 - \sigma_2^2) - \frac{\sigma_1 \log e}{\sigma_2} - \log e = \frac{\log e}{2}\left((\frac{\sigma_1}{\sigma_2})^2 - 2(\frac{\sigma_1}{\sigma_2}) + 1\right) = \frac{\log e}{2}(\frac{\sigma_1}{\sigma_2} - 1)^2 \geq 0.$

$\therefore$ the first KL divergence $\geq$ the second KL divergence, the equation holds $\iff \sigma_1 = \sigma_2$.

(c) Let $x := |\mu_1 - \mu_2|$.

The first KL divergence − the second KL divergence $= \frac{\log e}{2}(\mu_1-\mu_2)^2 - \log e(\frac{\sqrt{2}}{\sigma_1}|\mu_1-\mu_2| + e^{-\frac{\sqrt{2}}{\sigma_1}|\mu_1-\mu_2|}) + \log e$

$= \frac{\log e}{2}x^2 - \log e(\frac{\sqrt{2}}{\sigma_1}x + e^{-\frac{\sqrt{2}}{\sigma_1}x}) + \log e$

$= \log e(\frac{1}{2}x^2 - \frac{\sqrt{2}}{\sigma_1}x - e^{-\frac{\sqrt{2}}{\sigma_1}x} + 1).$

$\therefore$ the first KL divergence is the larger $\iff \frac{1}{2}x^2 - \frac{\sqrt{2}}{\sigma_1}x - e^{-\frac{\sqrt{2}}{\sigma_1}x} + 1 \geq 0.$

**Problem 2.**

(a) $h(X) = E_{X\sim f_X}(\log\frac{1}{f_X(X)}) = \int_{-\infty}^{\infty} \frac{1}{2b}e^{-\frac{|x-\mu|}{b}}(\log(2b) + \log e\frac{|x-\mu|}{b})dx = \log(2b) + \log e \int_{\mu}^{\infty} \frac{1}{b}e^{-\frac{(x-\mu)}{b}} \frac{x-\mu}{b}dx = \log(2b) + \log e = \log(2be).$

(b) From Problem 1 (a)(2), we know that $\int_{-\infty}^{\infty} |x-a|e^{-|x-b|}dx = 2|a-b| + 2e^{-|a-b|}$.
$E(|X|) = \int_{-\infty}^{\infty} |x|\frac{1}{2b}e^{-\frac{|x-\mu|}{b}}dx = \frac{1}{2b}b^2(2|\mu| + 2e^{-|\mu|}) = b(|\mu| + e^{-|\mu|})$.
Let $g(y) := y + e^{-y}$.
$\Rightarrow g'(y) = 1 - e^{-y} > 0$ when $y > 0$.
$\Rightarrow g(y)$ is strictly increasing on $(0, \infty)$.
$\Rightarrow b(|\mu| + e^{-|\mu|}) = bg(|\mu|) \overset{(1)}{\geq} bg(0) = 2b$.
$\Rightarrow 2b \leq E(|X|) \leq B$.
$\Rightarrow b \overset{(2)}{\leq} \frac{B}{2}$.
$\Rightarrow h(X) = \log(2be) \leq \log Be$, and when the equation holds, the distribution of $X$ is $\text{Lap}(0, \frac{B}{2})$ since the equation in (1) holds $\iff$ $\mu = 0$, and the equation in (2) holds.

**Problem 3.**

(a) Consider $\tilde{b}(x) := \mathrm{E}[b(x,Y)] = \mathrm{E}_{P_{Y|X}}[b(x,Y)]$.
Since $\tilde{b}(x) = \sum_y P_{Y|X}(y|x)b(x,y)$ is a deterministic function of $x$, $\tilde{b}(x)$ is an input-only cost function.
$\because \frac{1}{n}\sum_{i=1}^{n} \mathrm{E}_{Y_i}[b(x_i, Y_i)] = \frac{1}{n}\sum_{i=1}^{n} \tilde{b}(x_i)$.
$\therefore$ the cost constraint becomes: $\frac{1}{n}\sum_{i=1}^{n} \tilde{b}(x_i) \leq B$.
Therefore, this problem is equivalent to the channel coding problem with input-cost only function $\tilde{b}$.
From Theorem 1 in Lecture 5, $C(B) = \max_{P_X:\mathrm{E}_{P_X}[\tilde{b}(X)]\leq B} I(X;Y)$
$= \max_{P_X:\mathrm{E}_{P_X}[\mathrm{E}_{P_{Y|X}}[b(X,Y)]]\leq B} I(X;Y) = \max_{P_X:\mathrm{E}_{P_X P_{Y|X}}[b(X,Y)]\leq B} I(X;Y)$.

(b) First, $P_{Y|X}(y|x) = P_Z(y-x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}(\frac{y-x}{\sigma})^2}$.
Let $b(x,y) := y^2$.
The cost constraint is $\frac{1}{n}\sum_{i=1}^{n} \mathrm{E}_{Y_i}[b(x_i, Y_i)] = \frac{1}{n}\sum_{i=1}^{n} \mathrm{E}_{Y_i}[Y_i^2] \leq B$.
From the formula in Problem 1 (a)(1):
$\tilde{b}(x) := \mathrm{E}[b(x,Y)] = \int_{-\infty}^{\infty} P_{Y|X}(y|x)b(x,y)dy = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}(\frac{y-x}{\sigma})^2}y^2dy$
$= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}(\frac{y-x}{\sigma})^2}((y-x)^2 + 2(y-x)x + x^2)dy$

$= \sigma^2 + 0 + x^2 = \sigma^2 + x^2.$

$\Rightarrow$ the cost constraint becomes $\frac{1}{n}\sum_{i=1}^{n}(\sigma^2 + x_i^2) = \frac{1}{n}\sum_{i=1}^{n}\tilde{b}(x_i) \leq B$, which is $\frac{1}{n}\sum_{i=1}^{n}|x_i|^2 \leq B - \sigma^2$.

From the example of Guasian channel capacity in Lecture 5, we get that $C(B) = \frac{1}{2}\log(1+\frac{B-\sigma^2}{\sigma^2}) = \frac{1}{2}\log(\frac{B}{\sigma^2})$.

**Problem 4.** In HW2, we know that if $\sum_i p_i = \sum_i q_i = 1$ where $p_i, q_i \geq 0$, then $\sum_i p_i \log\frac{1}{p_i} \leq \sum_i p_i \log\frac{1}{q_i}$. – (1)

(a) $D_{\min} = \min_{\mathbf{q}(s)} \mathrm{E}[d(S, \mathbf{q}(S))] = \min_{\mathbf{q}(s)} \mathrm{E}[\log\frac{1}{\mathbf{q}(S)}] = 0$ if $\mathbf{q}(s) = \mathbb{I}\{S = s\}$.

$D_{\max} = \max_{\mathbf{q}} \mathrm{E}[d(S, \mathbf{q})] = \min_{\mathbf{q}} \mathrm{E}[\log\frac{1}{\mathbf{q}(S)}]$.

$\because \mathrm{E}[\log\frac{1}{\mathbf{q}(S)}] = \sum_s P_S(s)\log\frac{1}{\mathbf{q}(S)} \overset{(1)}{\geq} \sum_s P_S(s)\log\frac{1}{P_S(s)} = H(S) = H(\pi)$, and the equation holds when $\mathbf{q}(s) = P_S(s)$.

$\therefore D_{\max} = H(\pi)$.

(b) $H(S|\mathbf{Q}) = \mathrm{E}_{(S,\mathbf{Q})\sim P}[\log\frac{1}{P_{S|\mathbf{Q}}}] = \sum_{\mathbf{q}} P_{\mathbf{Q}}(\mathbf{q})\sum_s P_{S|\mathbf{Q}}(s|\mathbf{q})\log\frac{1}{P_{S|\mathbf{Q}}(s|\mathbf{q})}$

$\overset{(1)}{\leq} \sum_{\mathbf{q}} P_{\mathbf{Q}}(\mathbf{q})\sum_s P_{S|\mathbf{Q}}(s|\mathbf{q})\log\frac{1}{\mathbf{q}(s)} = \mathrm{E}_{(S,\mathbf{Q})\sim P}\left[\log\frac{1}{\mathbf{Q}(S)}\right]$.

(c) $R(D) = \inf_{(S,\mathbf{Q})}\left\{I(S;\mathbf{Q})\,\middle|\,\mathrm{E}[\log\frac{1}{\mathbf{Q}(S)}] \leq D \text{ and } S \sim \pi\right\}$

$= \inf_{(S,\mathbf{Q})}\left\{I(S;\mathbf{Q})\,\middle|\,H(S|\mathbf{Q}) \leq \mathrm{E}[\log\frac{1}{\mathbf{Q}(S)}] \leq D \text{ and } S \sim \pi\right\}$

$\overset{(2)}{\leq} \inf_{(S,\mathbf{Q})}\{I(S;\mathbf{Q})\,|H(S|\mathbf{Q}) \leq D \text{ and } S \sim \pi\}$

$\overset{(3)}{\leq} \inf_{(S,\mathbf{Q})}\{I(S;\mathbf{Q})\,|H(S|\mathbf{Q}) \leq D \text{ and } S \sim \pi \text{ and } \mathbf{Q}(\hat{s}) = 1 \text{ for some } \hat{s} \in \mathcal{S}\}$

$= \min_{(S,\hat{S})}\left\{I(S;\hat{S})\,\middle|\,H(S|\hat{S}) \leq D \text{ and } S \sim \pi\right\}$.

(d) Let $\mathbf{q}_{\hat{s}}(s) := \mathbb{I}(s = \hat{s})$.

Consider the distribution $\mathbf{Q} = \mathbf{q}_S$:

The equation in (2) holds $\iff$ the equation in (1) holds $\iff$ $\forall s, \mathbf{q}$, $P_{S|\mathbf{Q}}(s|\mathbf{q}) = \mathbf{q}(s)$, which is true because $\forall \mathbf{q}$ with nonzero probability, $\mathbf{q} = \mathbf{q}_{\hat{s}}$ for some $\hat{s}$, and $\mathbf{q}_{\hat{s}}(s) = \mathbb{I}(s = \hat{s}) \overset{\mathbf{Q}=\mathbf{q}_S}{=} P_{S|\mathbf{Q}}(s|\mathbf{q}_{\hat{s}})$.

The equation in (3) holds since $\mathbf{q}_{\hat{s}} = 1$ for $\hat{s} \in S$.

$\therefore$ with this distribution, $R(D) = \min\limits_{(S,\hat{S})} \left\{ I(S;\hat{S}) \,\middle|\, H(S|\hat{S}) \leq D \text{ and } S \sim \pi \right\}$

$= \min\limits_{(S,\hat{S})} \left\{ H(S) - H(S|\hat{S}) \,\middle|\, H(S|\hat{S}) \leq D \text{ and } S \sim \pi \right\}$

$= \min\limits_{(S,\hat{S})} \left\{ H(\pi) - H(S|\hat{S}) \,\middle|\, H(S|\hat{S}) \leq D \text{ and } S \sim \pi \right\}$

$= H(\pi) - D \overset{0 \leq D \leq H(\pi) \text{ is given}}{=} \max(0, H(\pi) - D)$.