Electromagnetics (I)
Chapter 1

Vectors and Fields

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Outline

- Vector algebra
- Cartesian coordinate system
- Cylindrical coordinate system
- Spherical coordinate system
- Scalar and vector fields
- Electric field
- Magnetic field
- Lorentz force equation

Homework 1

- Even numbers for even sections and odd numbers for odd sections

Outline

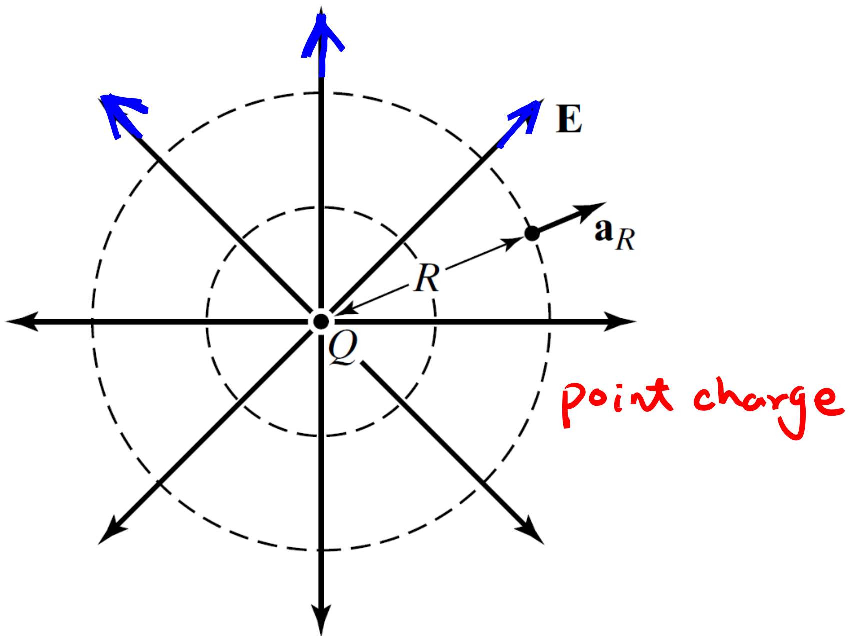
- Vector algebra <-- Learn fundamental math for electromagnetics
- Cartesian coordinate system
- Cylindrical coordinate system
- Spherical coordinate system
- Scalar and vector fields
- Electric field
- Magnetic field
- Lorentz force equation

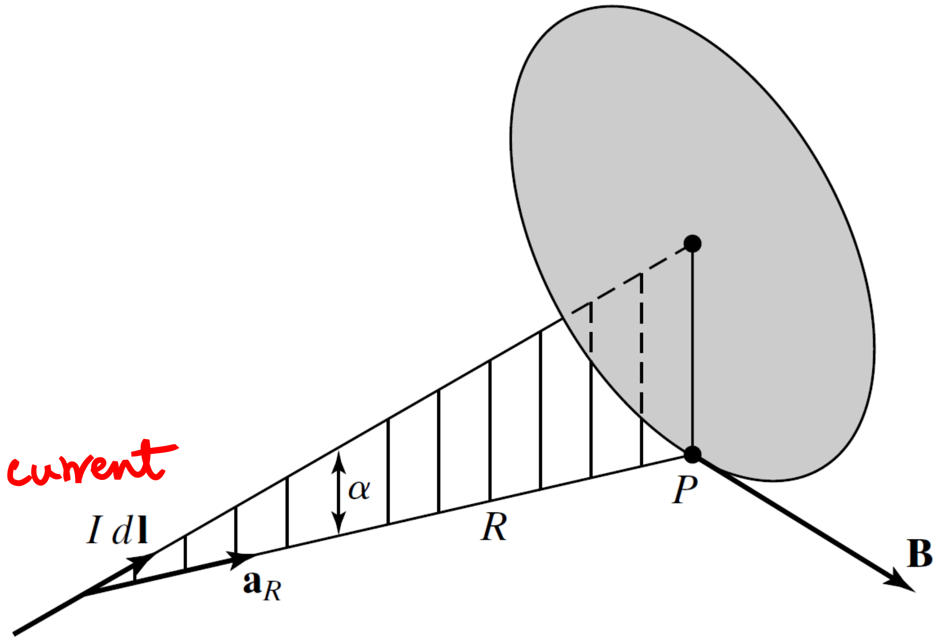
Electric and Magnetic Fields

- Vector quantities!
  - Need to learn basic rules about vectors.

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_0 R^2}\mathbf{a}_R$$

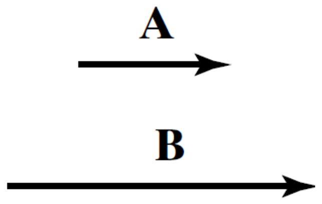
$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{I\,d\mathbf{l} \times \mathbf{a}_R}{R^2}$$



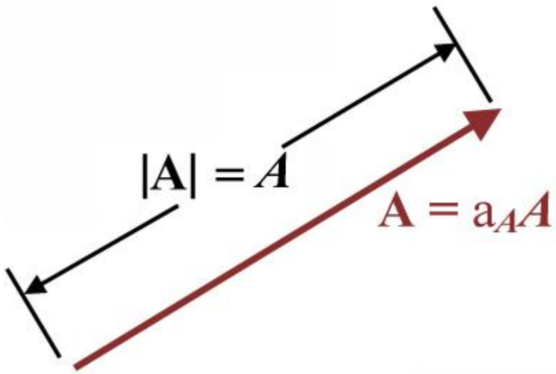


Vectors and Scalars

- Scalars
  - Characterized by magnitude only.
  - Examples: mass, temperature, charge, voltage, and current.
  - Lightface and italic, *A*.
- Vectors
  - Characterized by not only magnitude but also direction in space.
  - Examples: velocity, force, electric field, and magnetic field.
  - Boldface, **A**.



B = 2A

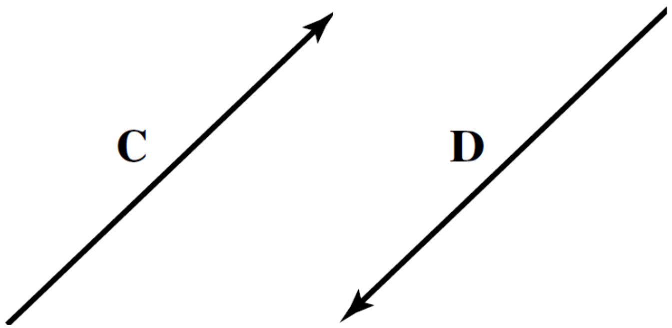


$$\mathbf{A} = \mathbf{a}_A A$$

$$A = |\mathbf{A}|$$

$$\mathbf{a}_A = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{\mathbf{A}}{A}$$ Unit vector

$|\vec{a}_A| = 1$



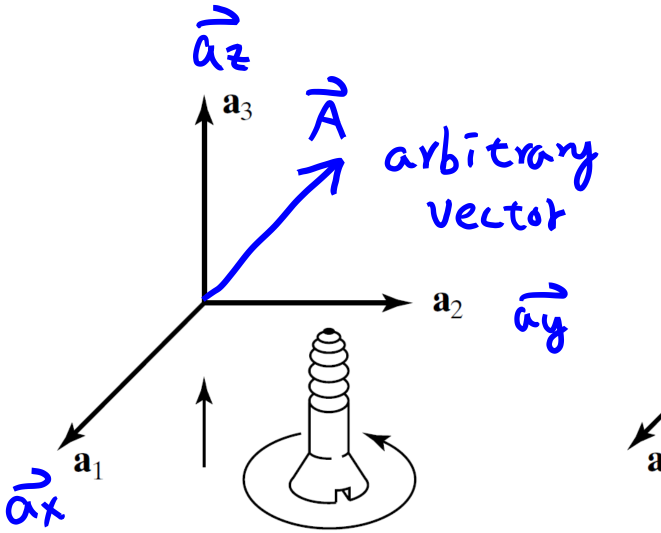
D = -C

Arbitrary Vector in 3D Space

- Use three orthogonal reference directions!

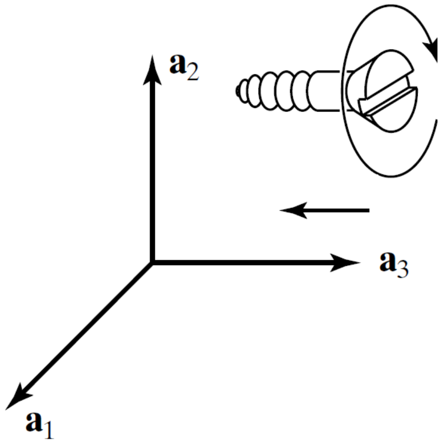
$\vec{a}_1, \vec{a}_2, \vec{a}_3$ unit vectors

$\rightarrow |\vec{a}_1| = |\vec{a}_2| = |\vec{a}_3| = 1$

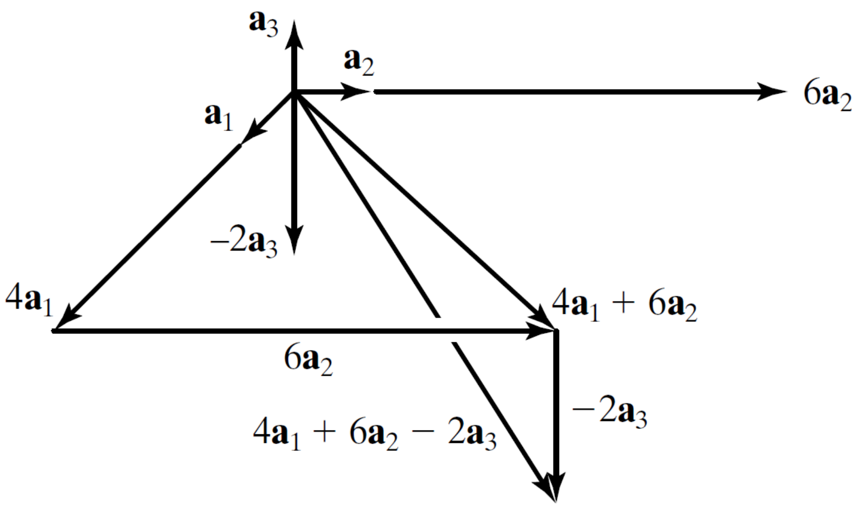


Right-handed system

Conventionally chosen



Left-handed system



$$\mathbf{A} = \underbrace{A_1}_{}\mathbf{a}_1 + A_2\mathbf{a}_2 + A_3\mathbf{a}_3$$

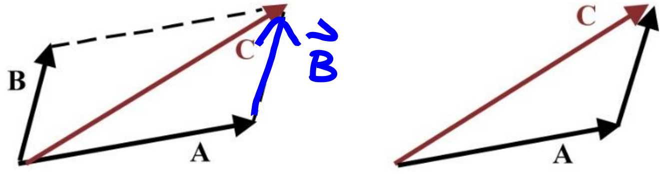
component of $\vec{A}$ along $\vec{a}_i$ direction

$$\vec{B} = B_1\vec{a}_1 + B_2\vec{a}_2 + B_3\vec{a}_3$$

Vector Algebra

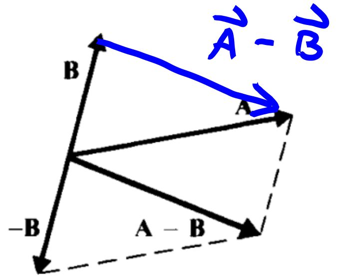
- Vector addition and subtraction

$$\mathbf{A} + \mathbf{B} = (A_1\mathbf{a}_1 + A_2\mathbf{a}_2 + A_3\mathbf{a}_3) + (B_1\mathbf{a}_1 + B_2\mathbf{a}_2 + B_3\mathbf{a}_3)$$
$$= \underline{(A_1 + B_1)}\mathbf{a}_1 + (A_2 + B_2)\mathbf{a}_2 + (A_3 + B_3)\mathbf{a}_3$$



(b) Parallelogram rule. (c) Head-to-tail rule.

$$\mathbf{B} - \mathbf{C} = \mathbf{B} + (-\mathbf{C})$$
$$= (B_1\mathbf{a}_1 + B_2\mathbf{a}_2 + B_3\mathbf{a}_3) + (-C_1\mathbf{a}_1 - C_2\mathbf{a}_2 - C_3\mathbf{a}_3)$$
$$= \underline{(B_1 - C_1)}\mathbf{a}_1 + (B_2 - C_2)\mathbf{a}_2 + (B_3 - C_3)\mathbf{a}_3$$



- Multiplication and division by a scalar

$$m\mathbf{A} = m(A_1\mathbf{a}_1 + A_2\mathbf{a}_2 + A_3\mathbf{a}_3) = mA_1\mathbf{a}_1 + mA_2\mathbf{a}_2 + mA_3\mathbf{a}_3$$

$$\frac{\mathbf{B}}{n} = \frac{1}{n}(\mathbf{B}) = \frac{B_1}{n}\mathbf{a}_1 + \frac{B_2}{n}\mathbf{a}_2 + \frac{B_3}{n}\mathbf{a}_3$$

- Magnitude of a vector

$$|\mathbf{A}| = |A_1\mathbf{a}_1 + A_2\mathbf{a}_2 + A_3\mathbf{a}_3| = \sqrt{A_1^2 + A_2^2 + A_3^2}$$

- Unit vector along **A**

$$\mathbf{a}_A = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{A_1}{|\mathbf{A}|}\mathbf{a}_1 + \frac{A_2}{|\mathbf{A}|}\mathbf{a}_2 + \frac{A_3}{|\mathbf{A}|}\mathbf{a}_3$$

Scalar or Dot Product of Vectors

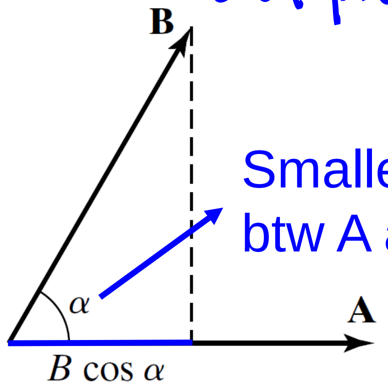
① $\vec{A}\cdot\vec{B}$ scalar

② Find the angle between $\vec{A}$ and $\vec{B}$

③ Find the amount of projection to the other vector

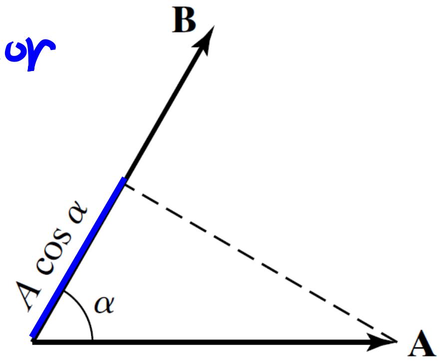
$$\mathbf{A}\cdot\mathbf{B} = |\mathbf{A}||\mathbf{B}|\cos\alpha = AB\cos = BA\cos\alpha = \mathbf{B}\cdot\mathbf{A} \qquad A = \sqrt[+]{\mathbf{A}\cdot\mathbf{A}}$$

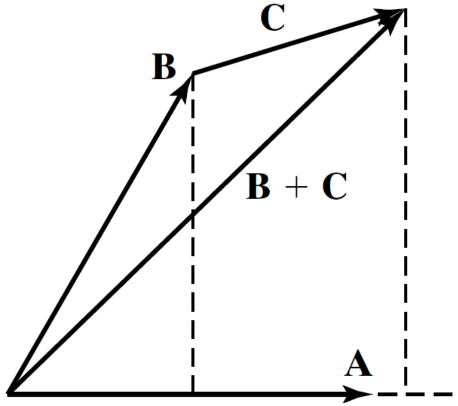
Scalar quantity Commutative Find magnitude



Smaller angle btw A and B

$$k\mathbf{A} = \mathbf{a}_A(kA)$$





$$\mathbf{a}_1\cdot\mathbf{a}_1 = 1 \qquad \mathbf{a}_1\cdot\mathbf{a}_2 = 0 \qquad \mathbf{a}_1\cdot\mathbf{a}_3 = 0$$
$$\mathbf{a}_2\cdot\mathbf{a}_1 = 0 \qquad \mathbf{a}_2\cdot\mathbf{a}_2 = 1 \qquad \mathbf{a}_2\cdot\mathbf{a}_3 = 0$$
$$\mathbf{a}_3\cdot\mathbf{a}_1 = 0 \qquad \mathbf{a}_3\cdot\mathbf{a}_2 = 0 \qquad \mathbf{a}_3\cdot\mathbf{a}_3 = 1$$

$$\begin{aligned}\mathbf{A}\cdot\mathbf{B} &= (A_1\mathbf{a}_1 + A_2\mathbf{a}_2 + A_3\mathbf{a}_3)\cdot(B_1\mathbf{a}_1 + B_2\mathbf{a}_2 + B_3\mathbf{a}_3)\\ &= A_1\mathbf{a}_1\cdot B_1\mathbf{a}_1 + A_1\mathbf{a}_1\cdot B_2\mathbf{a}_2 + A_1\mathbf{a}_1\cdot B_3\mathbf{a}_3\\ &\quad + A_2\mathbf{a}_2\cdot B_1\mathbf{a}_1 + A_2\mathbf{a}_2\cdot B_2\mathbf{a}_2 + A_2\mathbf{a}_2\cdot B_3\mathbf{a}_3\\ &\quad + A_3\mathbf{a}_3\cdot B_1\mathbf{a}_1 + A_3\mathbf{a}_3\cdot B_2\mathbf{a}_2 + A_3\mathbf{a}_3\cdot B_3\mathbf{a}_3\end{aligned}$$

$$\mathbf{A}\cdot\mathbf{B} = A_1B_1 + A_2B_2 + A_3B_3$$

$$\alpha = \cos^{-1}\frac{\mathbf{A}\cdot\mathbf{B}}{AB} = \cos^{-1}\frac{A_1B_1 + A_2B_2 + A_3B_3}{AB}$$

$$\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C}$$ Distributive

Associative law? A•B•C?

Vector or Cross Product of Vectors

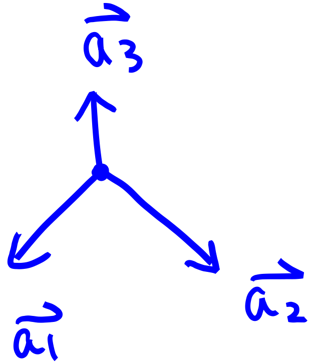
- Commutative law (X), associative law (X), and distributive law (O)

① $\vec{A} \times \vec{B}$ vector

② The direction is normal to the plane by $\vec{A}$ and $\vec{B}$

$$\mathbf{A} \times \mathbf{B} = |\mathbf{A}||\mathbf{B}| \sin \alpha \, \mathbf{a}_N = \underline{AB \sin \alpha \, \mathbf{a}_N}$$

Vector quantity

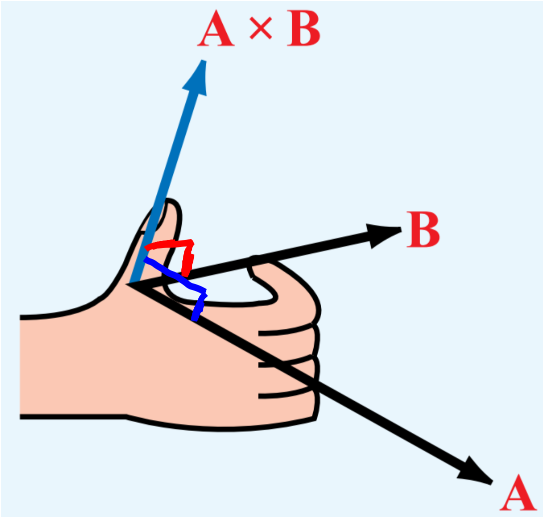


$$\begin{aligned} \mathbf{a}_1 \times \mathbf{a}_1 &= \mathbf{0} & \mathbf{a}_1 \times \mathbf{a}_2 &= \mathbf{a}_3 & \mathbf{a}_1 \times \mathbf{a}_3 &= -\mathbf{a}_2 \\ \mathbf{a}_2 \times \mathbf{a}_1 &= -\mathbf{a}_3 & \mathbf{a}_2 \times \mathbf{a}_2 &= \mathbf{0} & \mathbf{a}_2 \times \mathbf{a}_3 &= \mathbf{a}_1 \\ \mathbf{a}_3 \times \mathbf{a}_1 &= \mathbf{a}_2 & \mathbf{a}_3 \times \mathbf{a}_2 &= -\mathbf{a}_1 & \mathbf{a}_3 \times \mathbf{a}_3 &= \mathbf{0} \end{aligned}$$

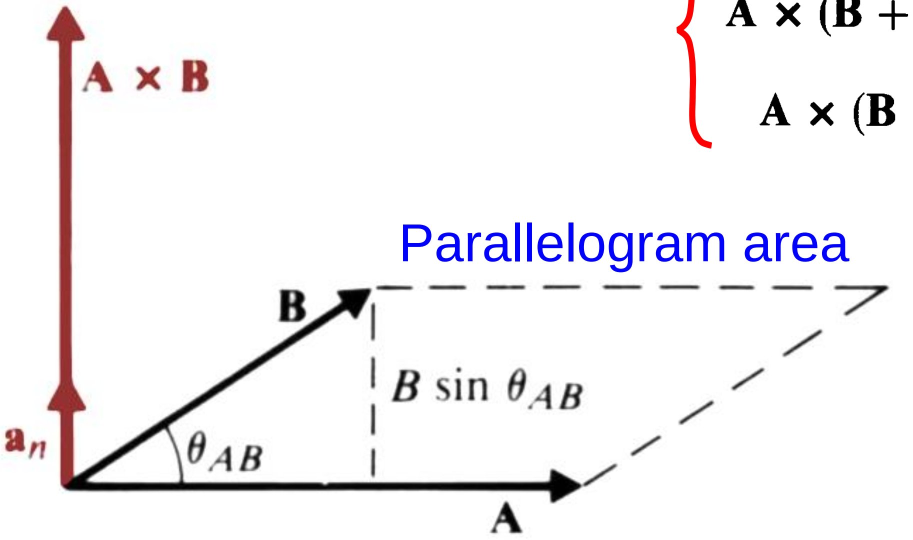
$1 \to 2 \to 3$

$$\left\{ \begin{aligned} &\mathbf{B} \times \mathbf{A} = -\mathbf{A} \times \mathbf{B} \\ &\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C} \\ &\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \neq (\mathbf{A} \times \mathbf{B}) \times \mathbf{C} \end{aligned} \right.$$

Right-hand rule



Parallelogram area



Vector or Cross Product of Vectors

$$\begin{aligned}\mathbf{A} \times \mathbf{B} &= (A_1\mathbf{a}_1 + A_2\mathbf{a}_2 + A_3\mathbf{a}_3) \times (B_1\mathbf{a}_1 + B_2\mathbf{a}_2 + B_3\mathbf{a}_3) \\ &= A_1\mathbf{a}_1 \times B_1\mathbf{a}_1 + A_1\mathbf{a}_1 \times B_2\mathbf{a}_2 + A_1\mathbf{a}_1 \times B_3\mathbf{a}_3 \\ &\quad + A_2\mathbf{a}_2 \times B_1\mathbf{a}_1 + A_2\mathbf{a}_2 \times B_2\mathbf{a}_2 + A_2\mathbf{a}_2 \times B_3\mathbf{a}_3 \\ &\quad + A_3\mathbf{a}_3 \times B_1\mathbf{a}_1 + A_3\mathbf{a}_3 \times B_2\mathbf{a}_2 + A_3\mathbf{a}_3 \times B_3\mathbf{a}_3 \\ &= A_1B_2\mathbf{a}_3 - A_1B_3\mathbf{a}_2 - A_2B_1\mathbf{a}_3 + A_2B_3\mathbf{a}_1 \\ &\quad + A_3B_1\mathbf{a}_2 - A_3B_2\mathbf{a}_1 \\ &= (A_2B_3 - A_3B_2)\mathbf{a}_1 + (A_3B_1 - A_1B_3)\mathbf{a}_2 \\ &\quad + (A_1B_2 - A_2B_1)\mathbf{a}_3 \\ &= \begin{vmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}\end{aligned}$$

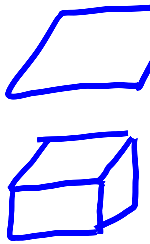
$1 \to 2 \to 3 \to 1 \to 2 \cdots$

$\vec{A} \cdot \vec{B} = A_1B_1 + A_2B_2 + A_3B_3$

- Very useful to find a unit vector normal to two given vectors at a point

$$\mathbf{a}_N = \frac{\mathbf{A} \times \mathbf{B}}{AB \sin \alpha} = \frac{\mathbf{A} \times \mathbf{B}}{|\mathbf{A} \times \mathbf{B}|}$$

Vector and Scalar Triple Products



- Vector triple product
  - Evaluation order critical.

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{a}_1 \times (\mathbf{a}_1 \times \mathbf{a}_2) = \mathbf{a}_1 \times \mathbf{a}_3 = -\mathbf{a}_2$$

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = (\mathbf{a}_1 \times \mathbf{a}_1) \times \mathbf{a}_2 = \mathbf{0} \times \mathbf{a}_2 = \mathbf{0}$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$ Back-cab rule

$$= \vec{B}(A_1C_1 + A_2C_2 + A_3C_3) - \vec{C}(A_1B_1 + A_2B_2 + A_3B_3)$$

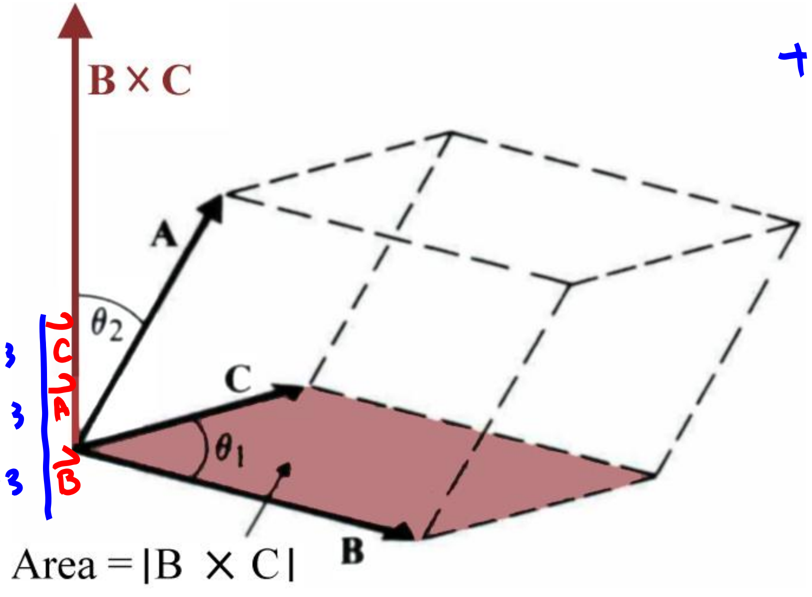
- Scalar triple product

$$\vec{A} \times \begin{vmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

$$\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = (A_1\mathbf{a}_1 + A_2\mathbf{a}_2 + A_3\mathbf{a}_3) \cdot \begin{vmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

$$= \begin{array}{cc} \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} & \begin{matrix} \vec{A} \\ \vec{B} \\ \vec{C} \end{matrix} \end{array} = \begin{array}{cc} \begin{vmatrix} B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \\ A_1 & A_2 & A_3 \end{vmatrix} & \begin{matrix} \vec{B} \\ \vec{C} \\ \vec{A} \end{matrix} \end{array} = \begin{array}{cc} \begin{vmatrix} C_1 & C_2 & C_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} & \begin{matrix} \vec{C} \\ \vec{A} \\ \vec{B} \end{matrix} \end{array}$$

$$= \mathbf{B} \cdot \mathbf{C} \times \mathbf{A} = \mathbf{C} \cdot \mathbf{A} \times \mathbf{B}$$



Volume = $|ABC \sin\theta_1 \cos\theta_2|$

Outline

- Vector algebra
- Cartesian coordinate system
- Cylindrical coordinate system
- Spherical coordinate system
- Scalar and vector fields
- Electric field
- Magnetic field
- Lorentz force equation

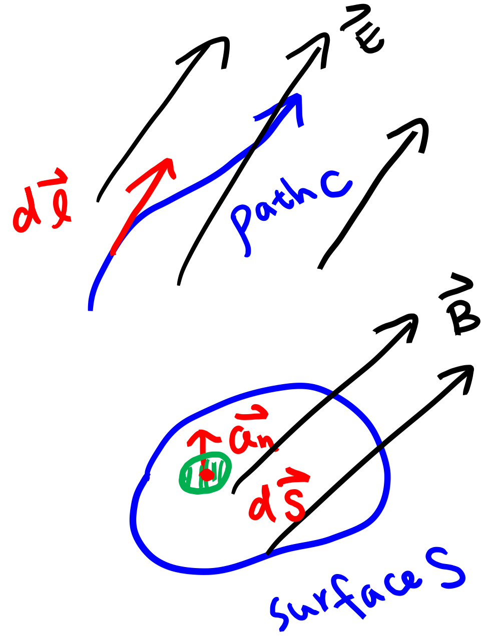
Maxwell's Equations in Integral Forms

- Vector calculus
  - Need to perform line, surface, and volume integrals.

vector algebra

① Need coordinate system

② math ← vector calculus



$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S}$$

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt}\int_S \mathbf{D} \cdot d\mathbf{S}$$

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho \, dv$$

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$$

Orthogonal Coordinate System

- Need to describe vectors (source position and point location) in 3D space

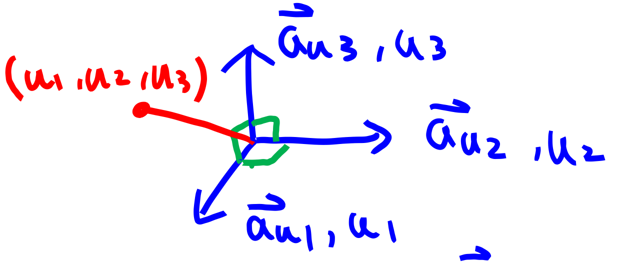
(u_1, u_2, u_3) (x, y, z)

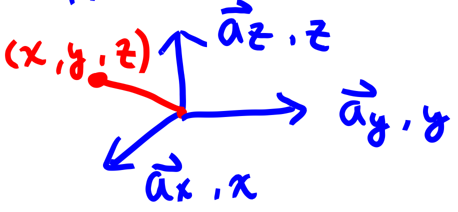
- A point in 3D space located as the intersection of three surfaces
  - u_1 = constant, u_2 = constant, and u_3 = constant.
  - These three surfaces are orthogonal to each other→ Orthogonal coordinate systems!
    - It may not be planes; it may be curved surfaces.
- Base vectors
  - $\mathbf{a}_{u1}$, $\mathbf{a}_{u2}$, and $\mathbf{a}_{u3}$ unit vectors in the three coordinate directions.
- Right-handed systems

Choose a coordinate system according to the geometry! (Symmetry!)

$$\mathbf{a}_{u_1} \times \mathbf{a}_{u_2} = \mathbf{a}_{u_3}$$
$$\mathbf{a}_{u_2} \times \mathbf{a}_{u_3} = \mathbf{a}_{u_1}$$
$$\mathbf{a}_{u_3} \times \mathbf{a}_{u_1} = \mathbf{a}_{u_2}$$

$$\mathbf{a}_{u_1} \cdot \mathbf{a}_{u_2} = \mathbf{a}_{u_2} \cdot \mathbf{a}_{u_3} = \mathbf{a}_{u_3} \cdot \mathbf{a}_{u_1} = 0$$
$$\mathbf{a}_{u_1} \cdot \mathbf{a}_{u_1} = \mathbf{a}_{u_2} \cdot \mathbf{a}_{u_2} = \mathbf{a}_{u_3} \cdot \mathbf{a}_{u_3} = 1$$



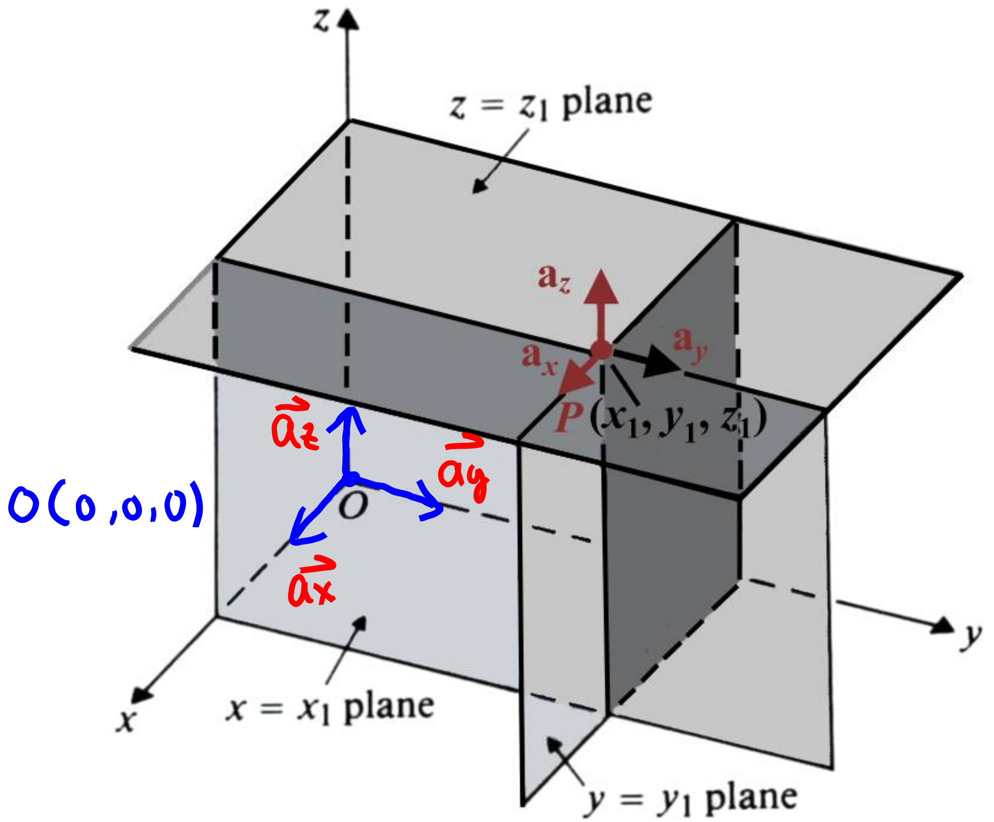


- Any vector **A**

$$\mathbf{A} = \mathbf{a}_{u_1} A_{u_1} + \mathbf{a}_{u_2} A_{u_2} + \mathbf{a}_{u_3} A_{u_3} \qquad A = |\mathbf{A}| = (A_{u_1}^2 + A_{u_2}^2 + A_{u_3}^2)^{1/2}$$

Orthogonal Coordinate System

- Express a vector at a point in space by a set of three mutually orthogonal directions defined by three mutually orthogonal unit vectors at that point

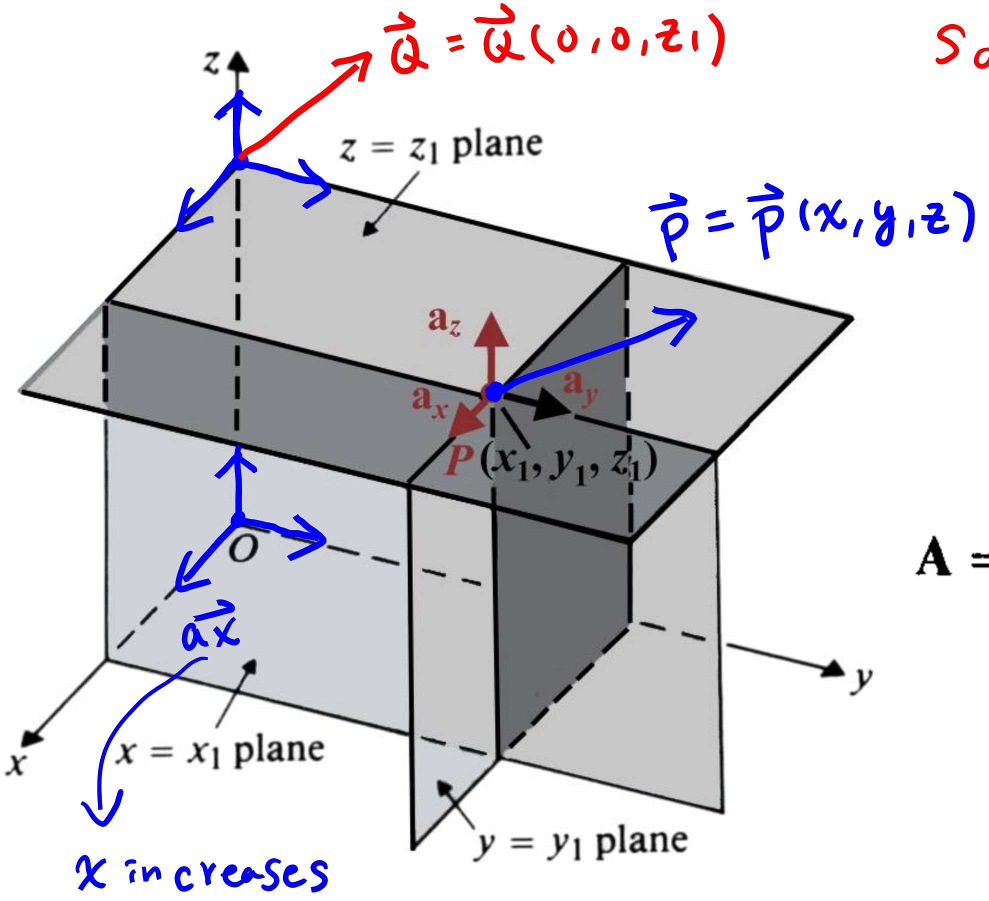


x = constant
y = constant
z = constant

Cartesian Coordinate System

(x, y, z)

- $u_1 = x$, $u_2 = y$, and $u_3 = z$, $\mathbf{a}_x$, $\mathbf{a}_y$, and $\mathbf{a}_z$: base vectors.



Same $\vec{a_x}, \vec{a_y}, \vec{a_z}$ → uniform coordinate system

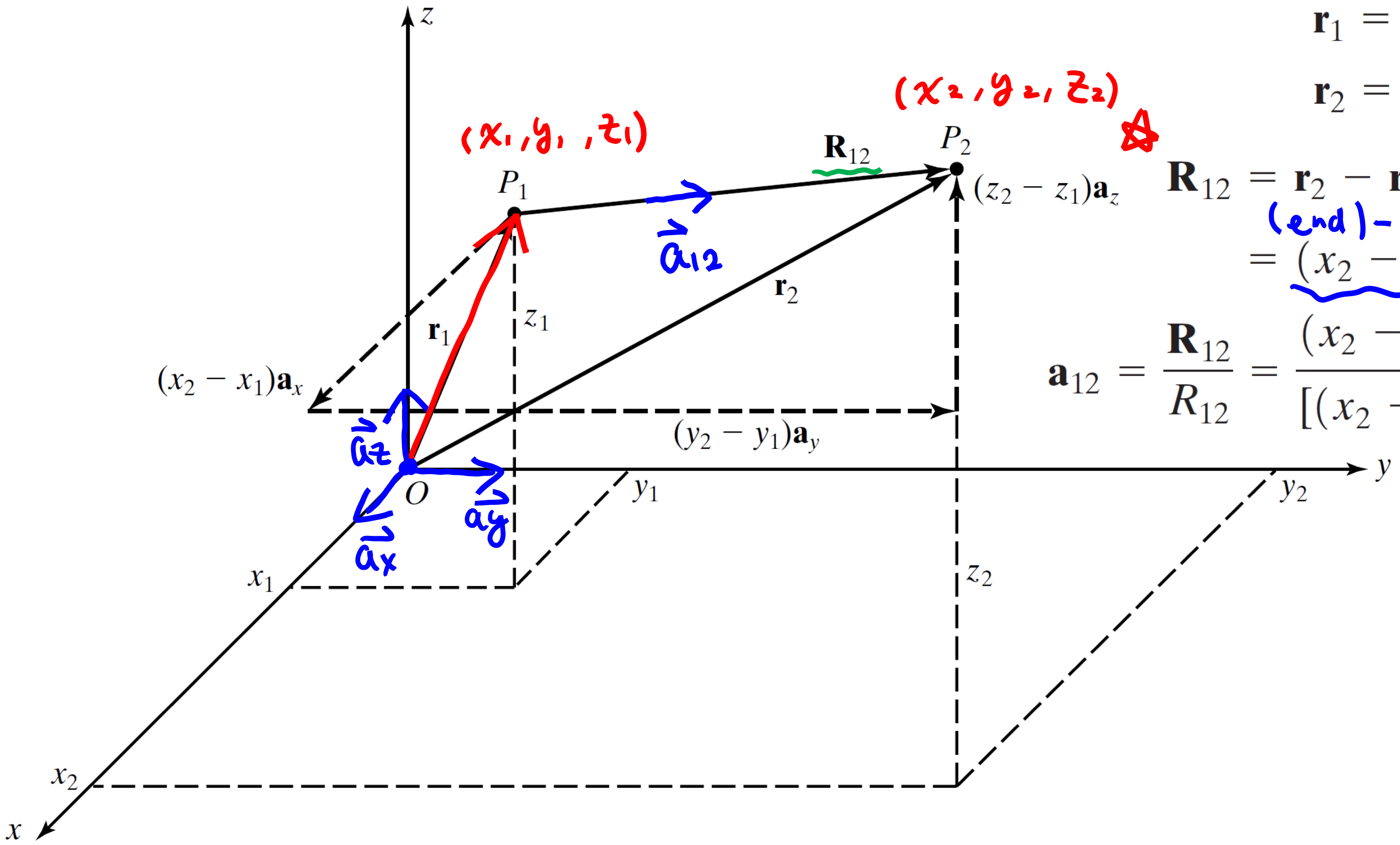
$$\mathbf{a}_x \times \mathbf{a}_y = \mathbf{a}_z$$
$$\mathbf{a}_y \times \mathbf{a}_z = \mathbf{a}_x$$
$$\mathbf{a}_z \times \mathbf{a}_x = \mathbf{a}_y$$

$\vec{a_x}$ normal to the $x = x_1 =$ constant plane

$$\mathbf{A} = \mathbf{a}_x A_x + \mathbf{a}_y A_y + \mathbf{a}_z A_z$$

Position Vectors

- Define the position of point P_1 relative to the origin set a reference point "O"
- With position vectors, we can describe any distance vector in 3D space



$$\mathbf{r}_1 = x_1\mathbf{a}_x + y_1\mathbf{a}_y + z_1\mathbf{a}_z$$

$$\mathbf{r}_2 = x_2\mathbf{a}_x + y_2\mathbf{a}_y + z_2\mathbf{a}_z$$

$$\mathbf{R}_{12} = \mathbf{r}_2 - \mathbf{r}_1 \leftarrow \vec{r}_1 + \vec{R}_{12} = \vec{r}_2$$

(end) − (begin)

$$= (x_2 - x_1)\mathbf{a}_x + (y_2 - y_1)\mathbf{a}_y + (z_2 - z_1)\mathbf{a}_z$$

$$\mathbf{a}_{12} = \frac{\mathbf{R}_{12}}{R_{12}} = \frac{(x_2 - x_1)\mathbf{a}_x + (y_2 - y_1)\mathbf{a}_y + (z_2 - z_1)\mathbf{a}_z}{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]^{1/2}}$$

$\vec{R}_{12}$

begin end

Maxwell's Equations in Integral Forms

- Vector calculus
  - Need to perform line, surface, and volume integrals.

differential length vector

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S}$$ Faraday's law

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt}\int_S \mathbf{D} \cdot d\mathbf{S}$$ Ampère's circuital law

differential volume

differential surface vector

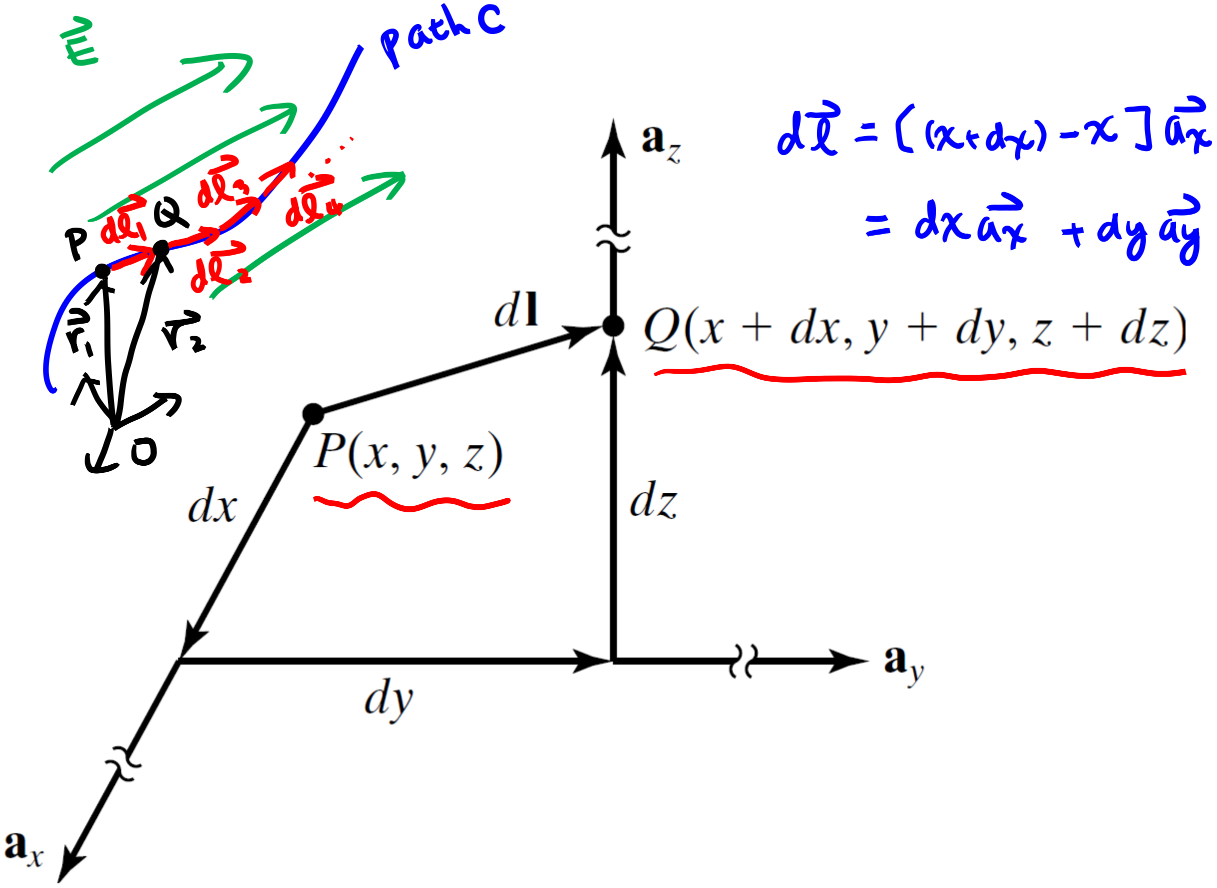
$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho \, dv$$

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$$

Gauss' laws

Differential Length Vector

- *dx, dy*, and *dz* not independent of each other
  - In the evaluation of line integrals, the integration is performed along a specified path.



$$d\vec{l} = [(x+dx)-x]\vec{a_x} + [(y+dy)-y]\vec{a_y} + [(z+dz)-z]\vec{a_z}$$

$$= dx\,\vec{a_x} + dy\,\vec{a_y} + dz\,\vec{a_z}$$

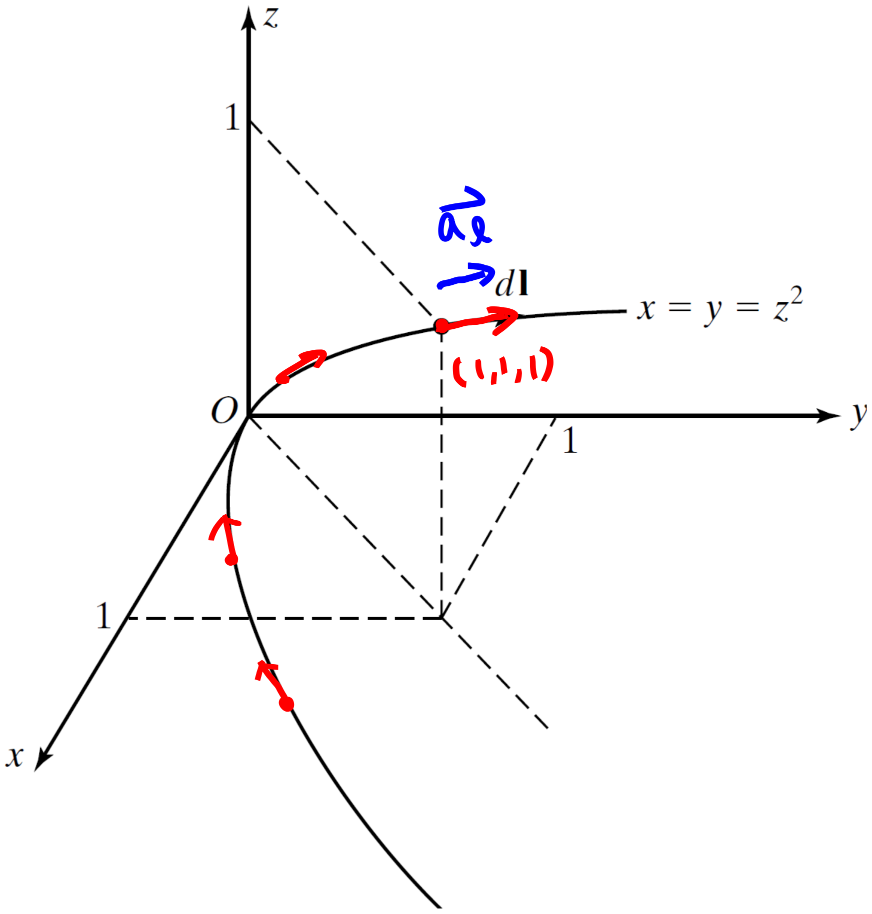
$$\oint_C \mathbf{E}\cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{S}$$ Faraday's law

☆ $$d\mathbf{l} = dx\,\mathbf{a}_x + dy\,\mathbf{a}_y + dz\,\mathbf{a}_z$$

$$\vec{E} = E_x\vec{a_x} + E_y\vec{a_y} + E_z\vec{a_z}$$

Example

- Let us consider the curve and obtain the expression for the differential length vector $d\mathbf{l}$ along the curve at the point (1, 1, 1).



$x = y = z^2$

$x = y \Rightarrow \frac{dy}{dx} = 1 \Rightarrow dy = dx$

$y = z^2 \Rightarrow \frac{dy}{dz} = 2z \Rightarrow dy = 2z\,dz$

$dy = dx = 2z\,dz$

$$x = y = z^2,\ dx = dy = 2z\,dz$$

$$\begin{aligned} d\mathbf{l} &= dx\,\mathbf{a}_x + dy\,\mathbf{a}_y + dz\,\mathbf{a}_z \\ &= 2\,dz\,\mathbf{a}_x + 2\,dz\,\mathbf{a}_y + dz\,\mathbf{a}_z \\ &= (2\mathbf{a}_x + 2\mathbf{a}_y + \mathbf{a}_z)\,dz \end{aligned}$$

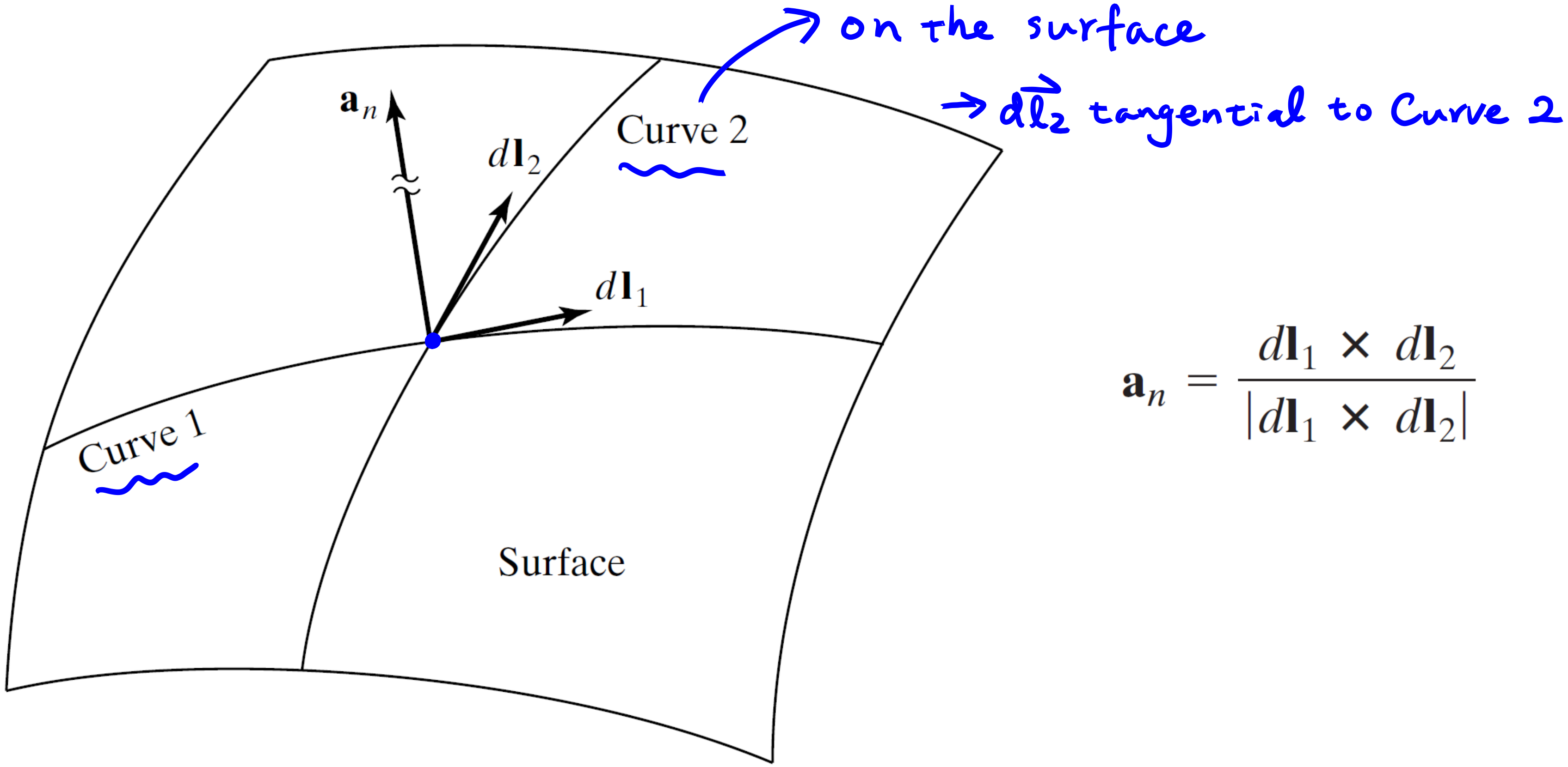
at the point (1,1,1)

$|d\vec{l}| = \sqrt{2^2 + 2^2 + 1^2}\,dz = 3\,dz$

$\therefore \vec{a_l} = (\frac{2}{3}\vec{a_x} + \frac{2}{3}\vec{a_y} + \frac{1}{3}\vec{a_z})$

Differential Length Vector

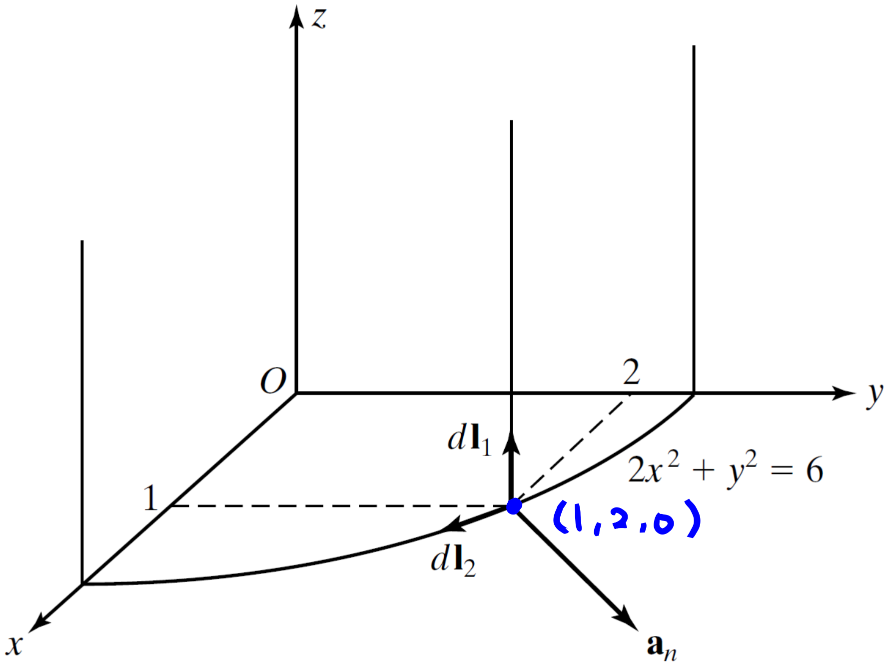
- Very useful to find the unit vector normal to a surface at a point on that surface



$$\mathbf{a}_n = \frac{d\mathbf{l}_1 \times d\mathbf{l}_2}{|d\mathbf{l}_1 \times d\mathbf{l}_2|}$$

Example

- Find the unit vector normal to the surface $2x^2+y^2 = 6$ at the point (1,2,0)



$$4x + 2y\frac{dy}{dx} = 0$$

$$\rightarrow 4x\,dx + 2y\,dy = 0$$

$$\rightarrow dy = -dx$$

$$d\mathbf{l}_1 = dz\,\mathbf{a}_z$$

$$d\mathbf{l}_2 = dx\,\mathbf{a}_x - dx\,\mathbf{a}_y = dx\,(\mathbf{a}_x - \mathbf{a}_y)$$

$$\mathbf{a}_n = \frac{dz\,\mathbf{a}_z \times dx\,(\mathbf{a}_x - \mathbf{a}_y)}{|dz\,\mathbf{a}_z \times dx\,(\mathbf{a}_x - \mathbf{a}_y)|}$$

$$= \frac{1}{\sqrt{2}}(\mathbf{a}_x + \mathbf{a}_y)$$

P1.12

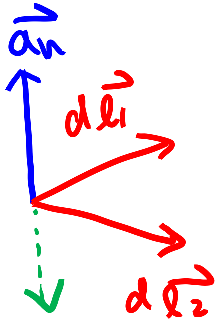
- Find the expression for the unit vector normal to the curve $x = y^2 = z^3$ at the point (1, 1, 1) and having no components along the line $x = y = z$

① $dx = 2y\,dy = 3z^2 dz$

at $(1,1,1)$

$dx = 2dy = 3dz$

$d\vec{l}_1 = (3\vec{a_x} + 1.5\vec{a_y} + \vec{a_z})dz$



② For $x = y = z$

$dx = dy = dz$

$d\vec{l}_2 = (\vec{a_x} + \vec{a_y} + \vec{a_z})dz$

$$\therefore \vec{a_n} = \pm \frac{d\vec{l_1} \times d\vec{l_2}}{|d\vec{l_1} \times d\vec{l_2}|} = \pm \frac{\vec{a_x} - 4\vec{a_y} + 3\vec{a_z}}{\sqrt{26}}$$

Example

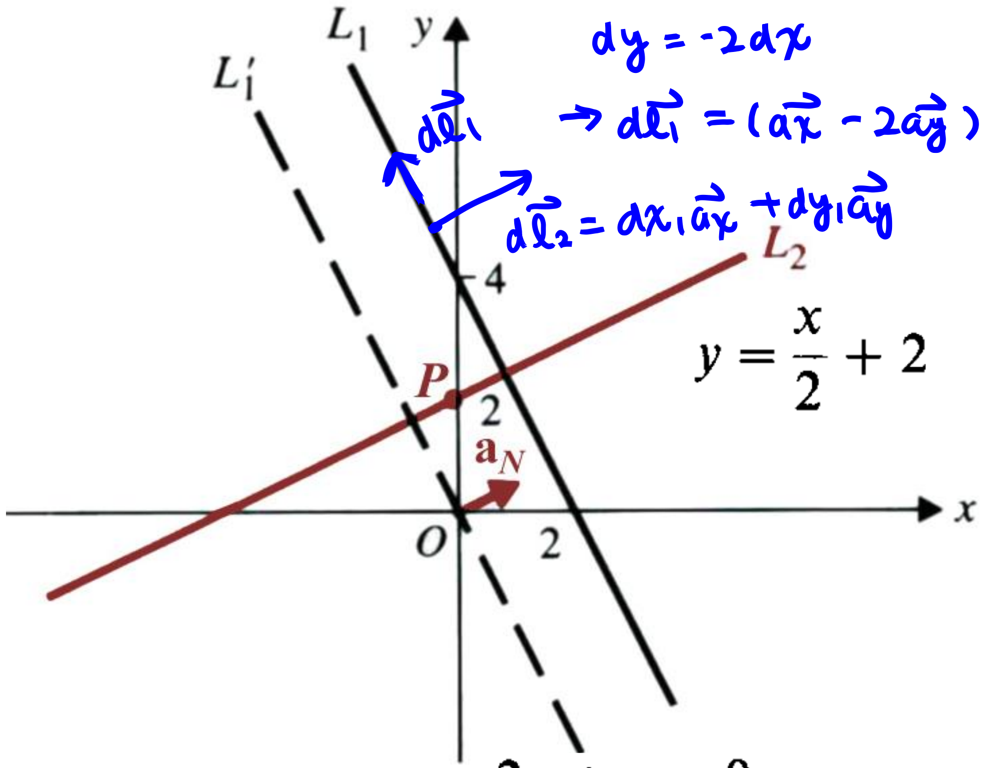
- The equation of a straight line in the *x*-*y* plane is given by $2x + y = 4$. (a) Find the vector equation of a unit normal from the origin to the line. (b) Find the equation of a line passing through the point $P(0,2)$ and perpendicular to the given line.

$2x + y = 4$ $\rightarrow y = -2x + 4$

$dy = -2dx$

$\rightarrow \vec{dl_1} = (\vec{a_x} - 2\vec{a_y})dx \xrightarrow{\because \perp} \vec{dl_1}\cdot\vec{dl_2} = 0 \rightarrow dx_1 - 2dy_1 = 0 \quad \therefore dx_1 = 2dy_1$

$\vec{dl_2} = dx_1\vec{a_x} + dy_1\vec{a_y}$



$2x + y = 0$

$$\mathbf{r} = \mathbf{a}_x x + \mathbf{a}_y y$$

$\therefore \vec{dl_2} = (2\vec{a_x} + \vec{a_y})dy_1$

$$\mathbf{N} = \mathbf{a}_x 2 + \mathbf{a}_y$$

$\therefore \vec{a_N} = \frac{1}{\sqrt{5}}(2\vec{a_x} + \vec{a_y})$

$$\mathbf{N} \cdot \mathbf{r} = 2x + y = 0$$

$$\mathbf{a}_N = \frac{\mathbf{N}}{|\mathbf{N}|} = \frac{1}{\sqrt{5}}(\mathbf{a}_x 2 + \mathbf{a}_y)$$

$\frac{dy_1}{dx_1} = \frac{1}{2} \rightarrow y_1 = \frac{1}{2}x_1 + c$

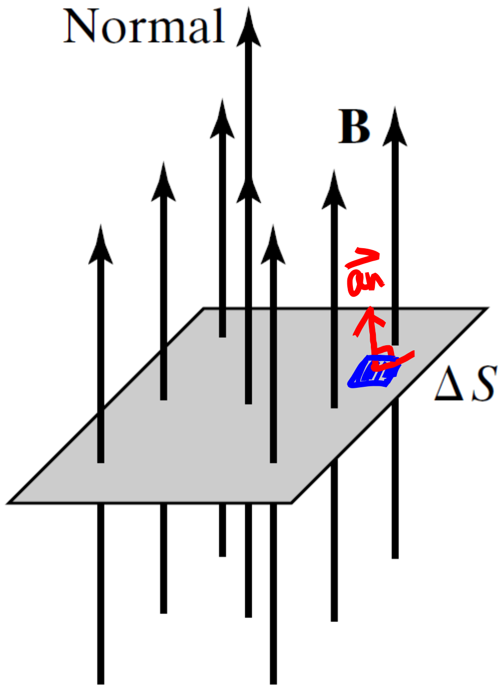
$P(0,2) \rightarrow c = 2 \quad \therefore y_1 = \frac{1}{2}x_1 + 2$

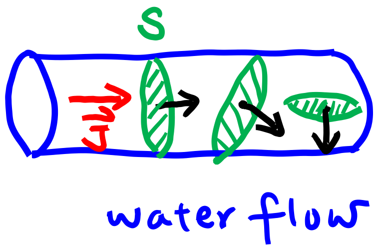
Differential Surface Vector

- Need to express current or flux through a differential area
  - Magnitude is the area *ds.*
  - Direction is normal to the differential surface.

$$\mathbf{ds} = \mathbf{a}_n\, ds$$

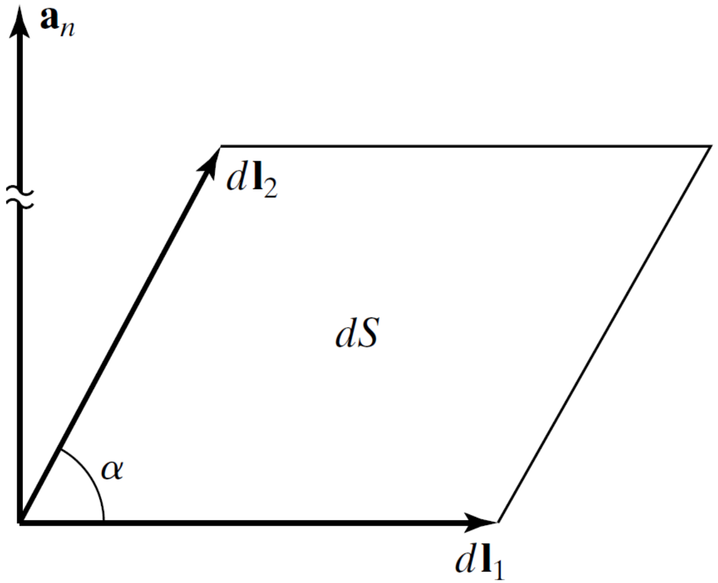
$$dI = \mathbf{J}\cdot\mathbf{ds} = \mathbf{J}\cdot\mathbf{a}_n\, ds$$



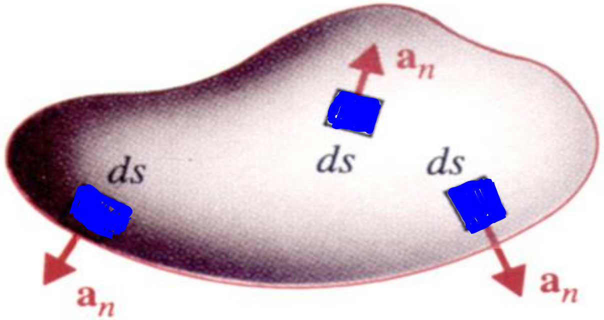


$$\oint_C \mathbf{E}\cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{S}$$ Faraday's law

Differential Surface Vector

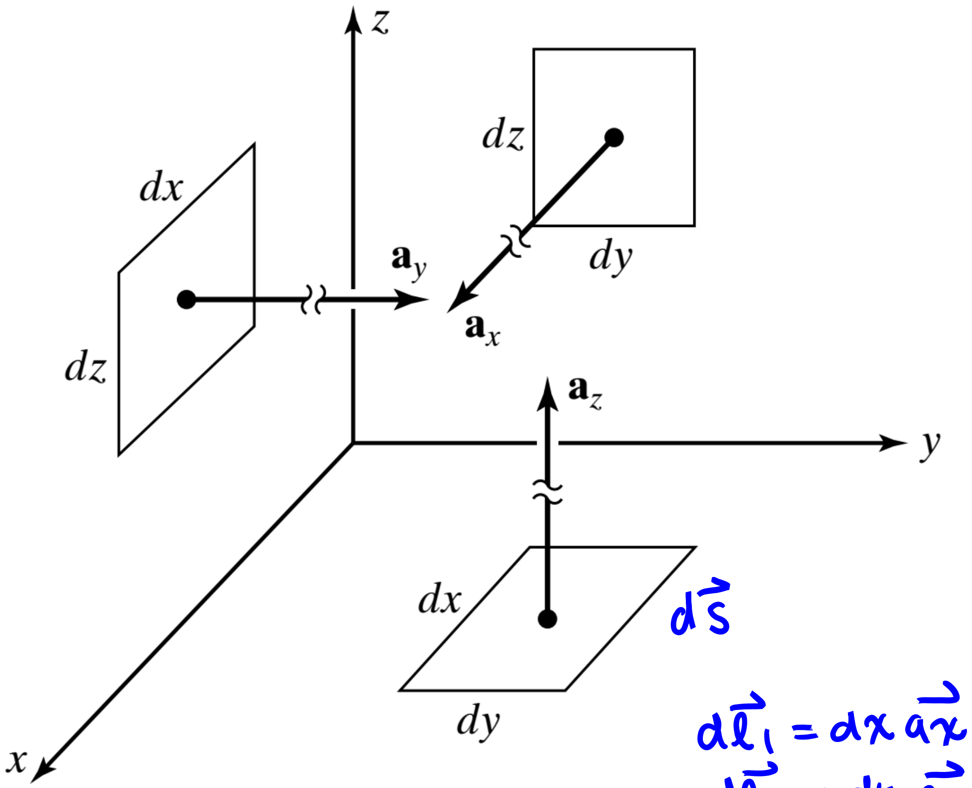


$$dS = dl_1\, dl_2 \sin\alpha = |d\mathbf{l}_1 \times d\mathbf{l}_2|$$



A closed surface.

$\vec{a_n}$ always outward!



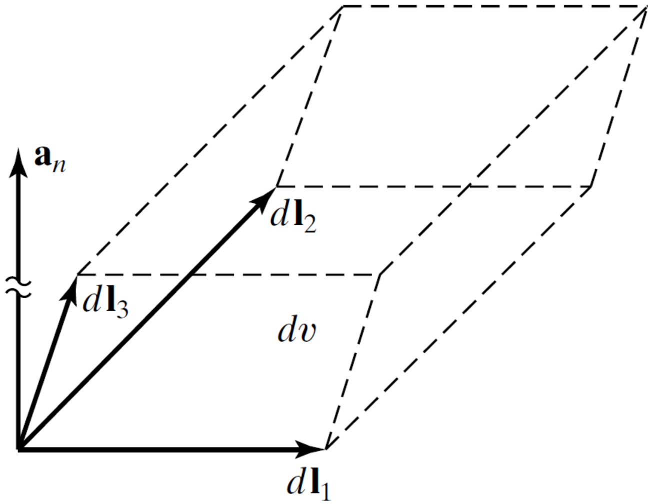
$\vec{dl_1} = dx\, \vec{a_x}$
$\vec{dl_2} = dy\, \vec{a_y}$
$\vec{dl_3} = dz\, \vec{a_z}$

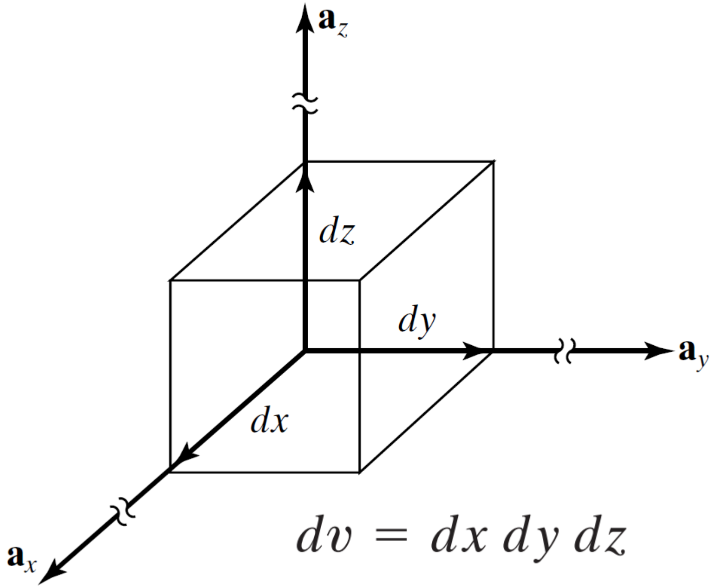
$$\pm dy\, \mathbf{a}_y \times dz\, \mathbf{a}_z = \pm dy\, dz\, \mathbf{a}_x$$
$$\pm dz\, \mathbf{a}_z \times dx\, \mathbf{a}_x = \pm dz\, dx\, \mathbf{a}_y$$
$$\pm dx\, \mathbf{a}_x \times dy\, \mathbf{a}_y = \pm dx\, dy\, \mathbf{a}_z$$

Differential Volume

- Differential volume

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho \, dv$$ Gauss' laws





$$dv = dx\,dy\,dz$$

$$\begin{aligned} dv &= \text{area of the base of the parallelepiped} \times \text{height of the parallelepiped} \\ &= |d\mathbf{l}_1 \times d\mathbf{l}_2||d\mathbf{l}_3 \cdot \mathbf{a}_n| \\ &= |d\mathbf{l}_1 \times d\mathbf{l}_2| \frac{|d\mathbf{l}_3 \cdot d\mathbf{l}_1 \times d\mathbf{l}_2|}{|d\mathbf{l}_1 \times d\mathbf{l}_2|} \\ &= |d\mathbf{l}_3 \cdot d\mathbf{l}_1 \times d\mathbf{l}_2| \end{aligned}$$

Cartisian

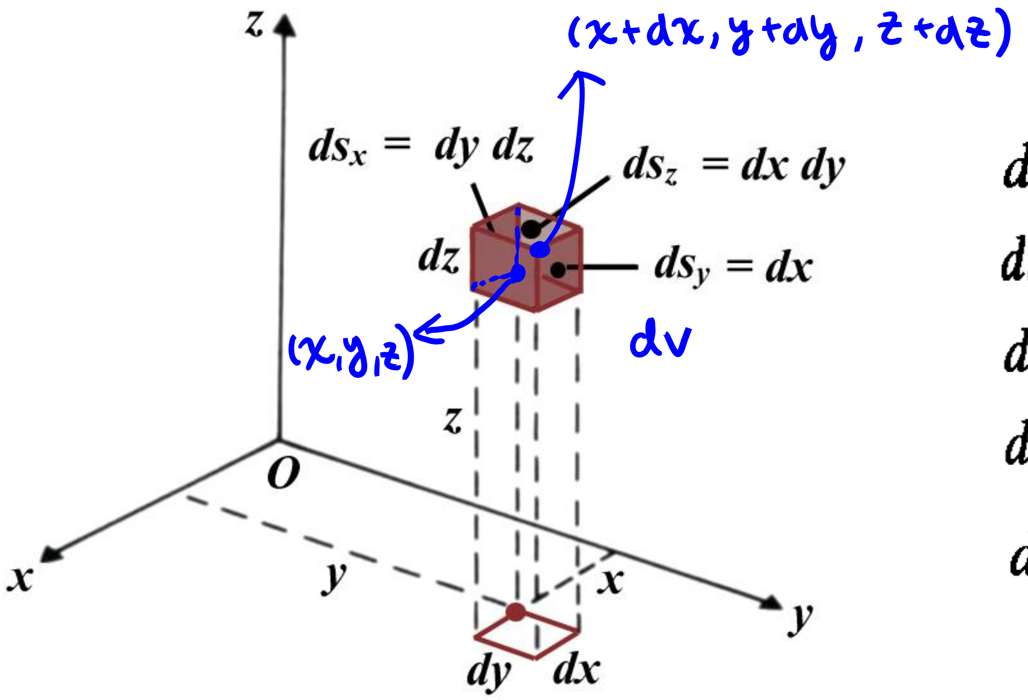
$$d\vec{l_1} = dx\,\vec{a_x}$$
$$d\vec{l_2} = dy\,\vec{a_y}$$
$$d\vec{l_3} = dz\,\vec{a_z}$$

Cartesian Coordinate System Summary

$$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z$$

$$\mathbf{A} \times \mathbf{B} = \mathbf{a}_x(A_y B_z - A_z B_y) + \mathbf{a}_y(A_z B_x - A_x B_z) + \mathbf{a}_z(A_x B_y - A_y B_x)$$

$$= \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$



$$d\ell = \mathbf{a}_x\,dx + \mathbf{a}_y\,dy + \mathbf{a}_z\,dz$$

$$ds_x = dy\,dz$$

$$ds_y = dx\,dz$$

$$ds_z = dx\,dy$$

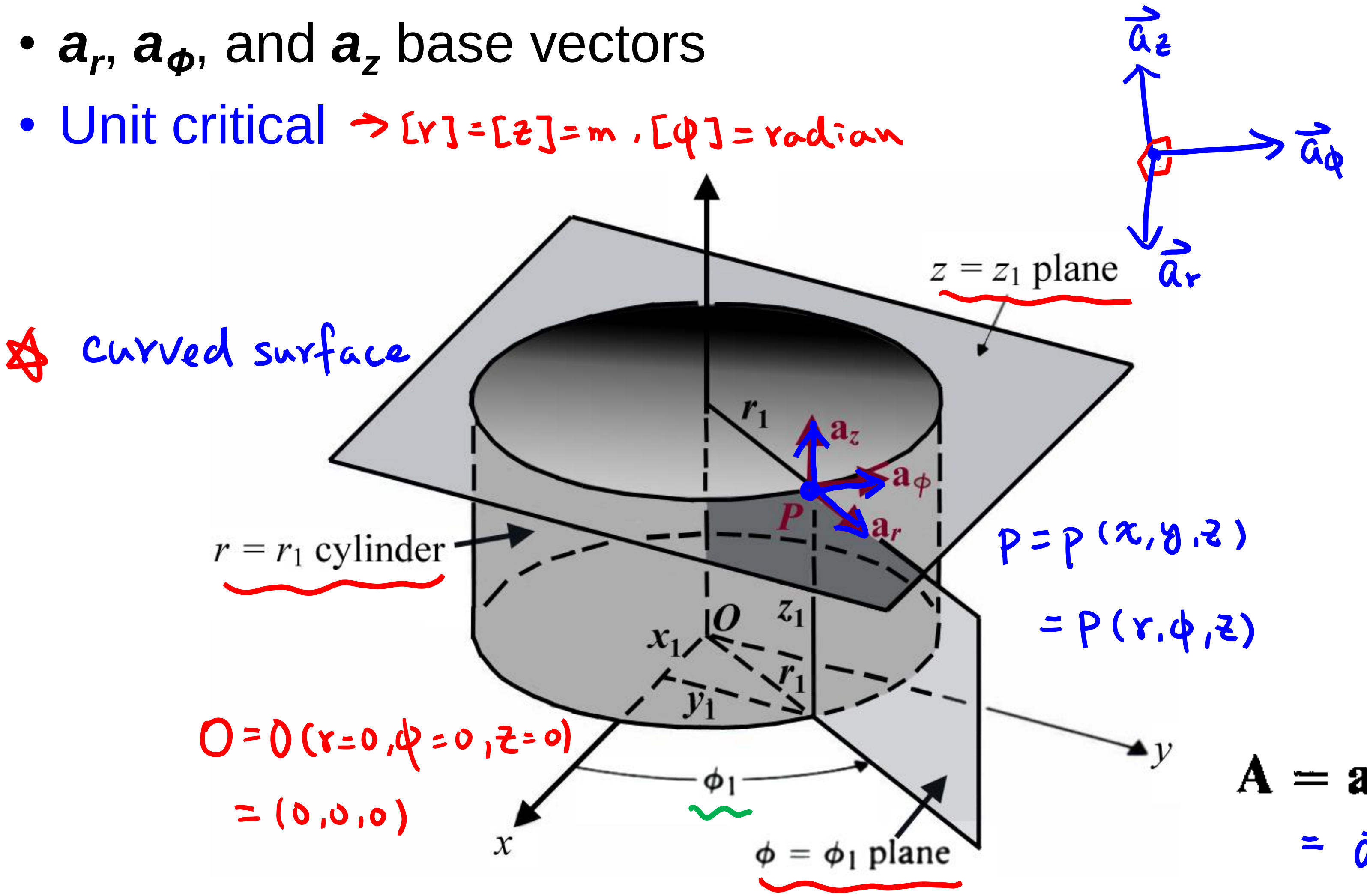
$$dv = dx\,dy\,dz$$

Outline

- Vector algebra
- Cartesian coordinate system
- Cylindrical coordinate system
- Spherical coordinate system
- Scalar and vector fields
- Electric field
- Magnetic field
- Lorentz force equation

Cylindrical Coordinate System (r, ϕ, z)

- r (0~∞), Φ (0~2π, angle with the x-z plane and measured from positive x-axis), and z (-∞ ~∞) independent variables
- $\boldsymbol{a}_r$, $\boldsymbol{a}_\phi$, and $\boldsymbol{a}_z$ base vectors
- Unit critical → $[r] = [z] = m$, $[\phi] = $ radian



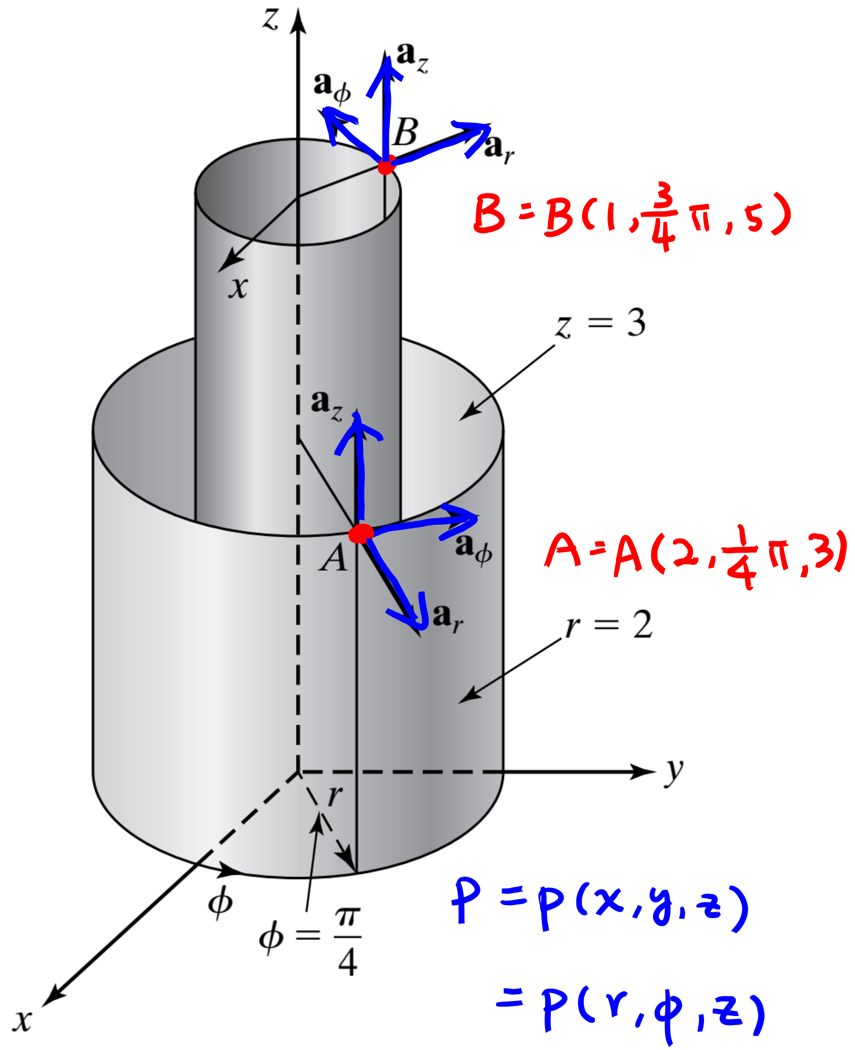
$$r = \text{constant}$$
$$\phi = \text{constant}$$
$$z = \text{constant}$$

$$\mathbf{a}_r \times \mathbf{a}_\phi = \mathbf{a}_z,$$ $r \to \phi \to z \to r \cdots$
$$\mathbf{a}_\phi \times \mathbf{a}_z = \mathbf{a}_r,$$
$$\mathbf{a}_z \times \mathbf{a}_r = \mathbf{a}_\phi.$$

$$\mathbf{A} = \mathbf{a}_r A_r + \mathbf{a}_\phi A_\phi + \mathbf{a}_z A_z$$
$$= \vec{a}_x A_x + \vec{a}_y A_y + \vec{a}_z A_z$$

Cylindrical Coordinate System

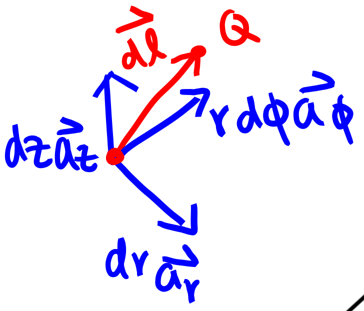
- Unit vectors except $\boldsymbol{a_z}$ do not have the same directions everywhere
  - Not uniform.
- Talk about the same thing, like language
  - Use English if you are talking to a person who speaks English.

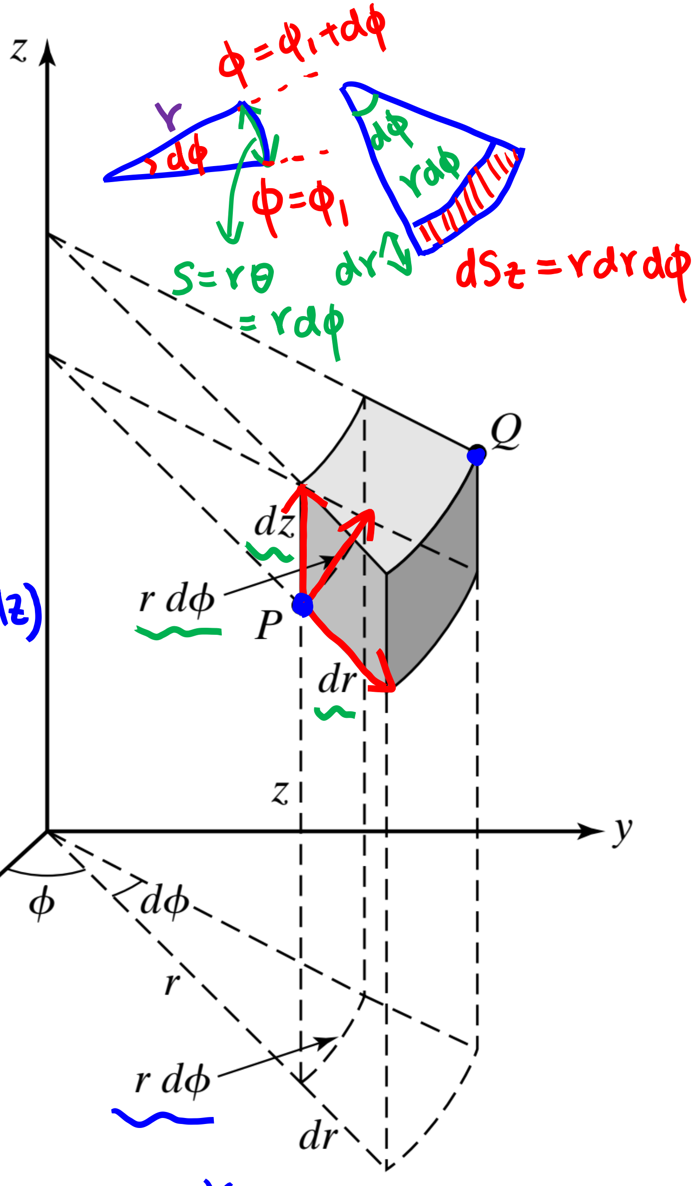


$d\mathbf{l}$, $d\mathbf{S}$, and dv

$P(r, \phi, z)$

$Q(r+dr, \phi+d\phi, z+dz)$





$r \to r_c$ cylindrical

- r stays on the x-y plane

$r \to r + dr$

$\phi \to \phi + d\phi$

$z \to z + dz$

$[r\,d\phi] = \text{meter}$

$$d\mathbf{l} = dr\,\mathbf{a}_r + r\,d\phi\,\mathbf{a}_\phi + dz\,\mathbf{a}_z$$

length $[d\vec{l}] = \text{meter}$

$$\pm r\,d\phi\,\mathbf{a}_\phi \times dz\,\mathbf{a} = \pm r\,d\phi\,dz\,\mathbf{a}_r$$

$$\pm dz\,\mathbf{a}_z \times dr\,\mathbf{a}_r = \pm dr\,dz\,\mathbf{a}_\phi$$

$$\pm dr\,\mathbf{a}_r \times r\,d\phi\,\mathbf{a}_\phi = \pm r\,dr\,d\phi\,\mathbf{a}_z$$

$$dv = (dr)(r\,d\phi)(dz) = r\,dr\,d\phi\,dz$$

$$d\vec{l} = dx\,\vec{a}_x + dy\,\vec{a}_y + dz\,\vec{a}_z$$

$$= dr\,\vec{a}_r + r\,d\phi\,\vec{a}_\phi + dz\,\vec{a}_z$$

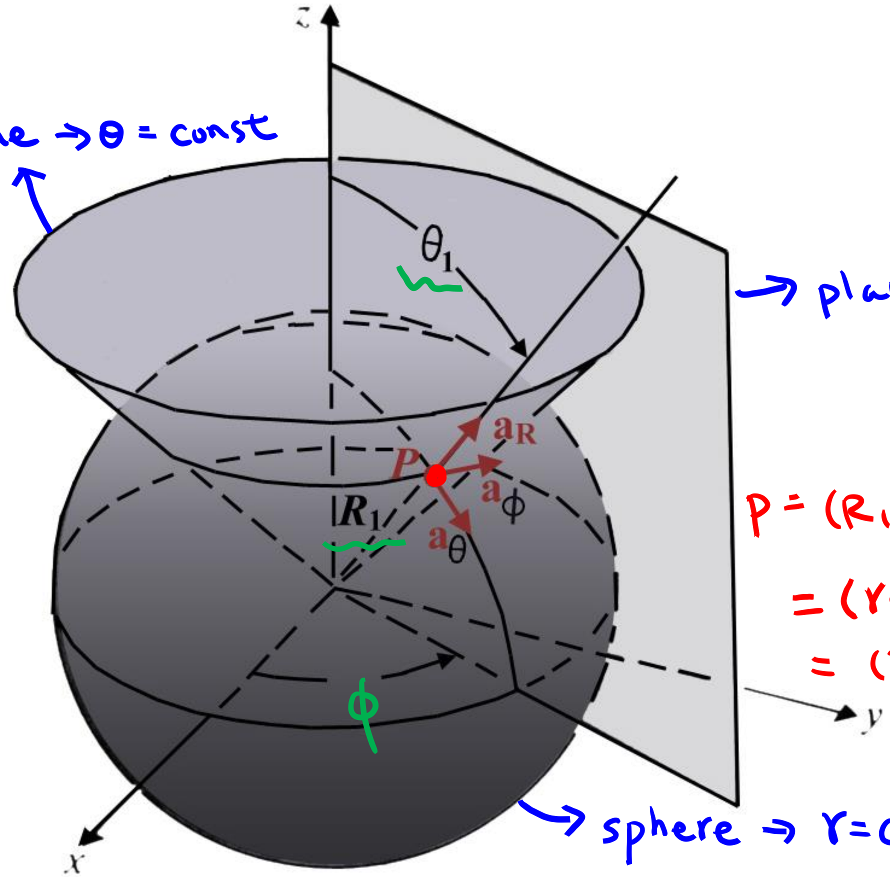
Outline

- Vector algebra
- Cartesian coordinate system
- Cylindrical coordinate system
- Spherical coordinate system
- Scalar and vector fields
- Electric field
- Magnetic field
- Lorentz force equation

Spherical Coordinates (R, θ, ϕ) or (r_s, θ, ϕ)

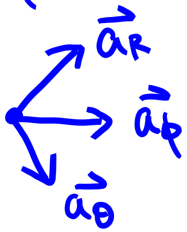
- r $(0\sim\infty)$, θ $(0\sim\pi$, angle with $+z$-axis), and Φ $(0\sim2\pi$, angle with the x-z plane and measured from positive x-axis)
- $\boldsymbol{a}_R$, $\boldsymbol{a}_\theta$, and $\boldsymbol{a}_\Phi$ base vectors
- r can be everywhere ☆ $[r] = m$, $[\theta] = [\phi] = \text{radian}$

cone → θ = const



→ plane → ϕ = const

$P = (R, \theta, \phi)$
$= (r_s, \theta, \phi) = (r_c, \phi, z)$
$= (x, y, z)$



→ sphere → r = const

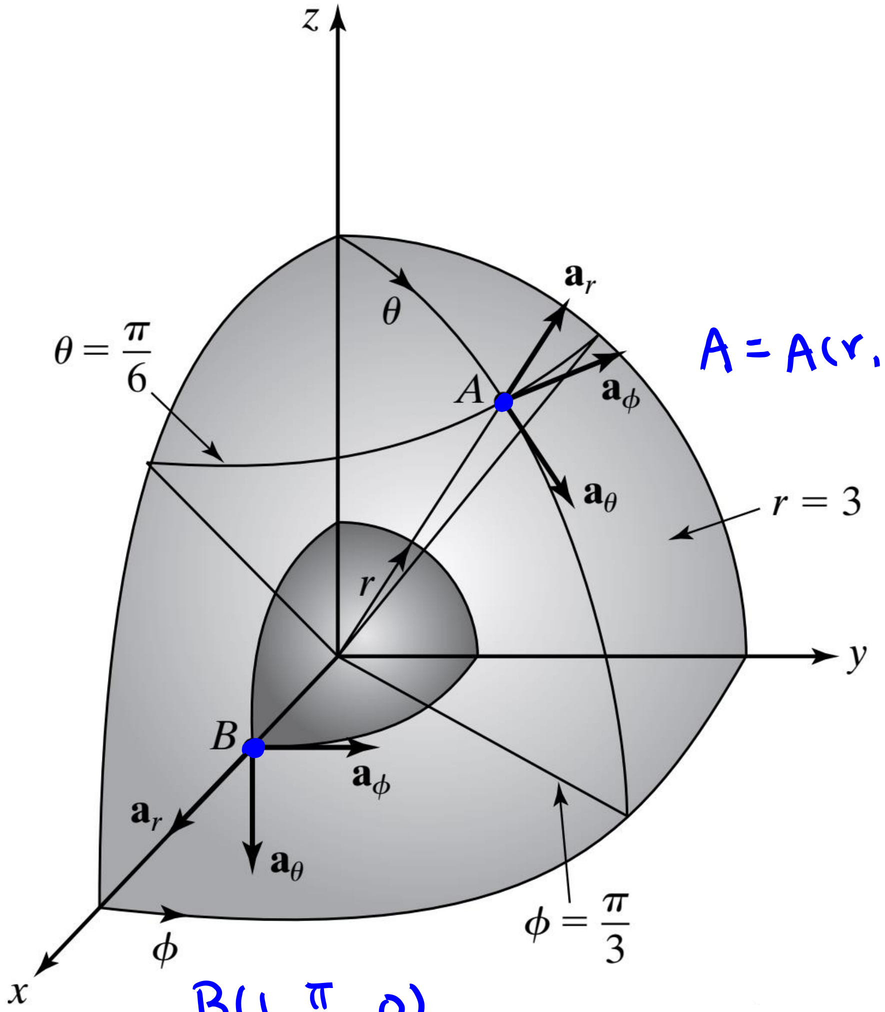
$$r = \text{constant} \rightarrow \text{sphere}$$
$$\theta = \text{constant} \rightarrow \text{cone}$$
$$\phi = \text{constant} \rightarrow \text{plane}$$

$$\mathbf{a}_R \times \mathbf{a}_\theta = \mathbf{a}_\phi,$$
$$\mathbf{a}_\theta \times \mathbf{a}_\phi = \mathbf{a}_R, \quad R \rightarrow \theta \rightarrow \phi \rightarrow R \cdots$$
$$\mathbf{a}_\phi \times \mathbf{a}_R = \mathbf{a}_\theta.$$

$$\mathbf{A} = \mathbf{a}_R A_R + \mathbf{a}_\theta A_\theta + \mathbf{a}_\phi A_\phi$$

Spherical Coordinates

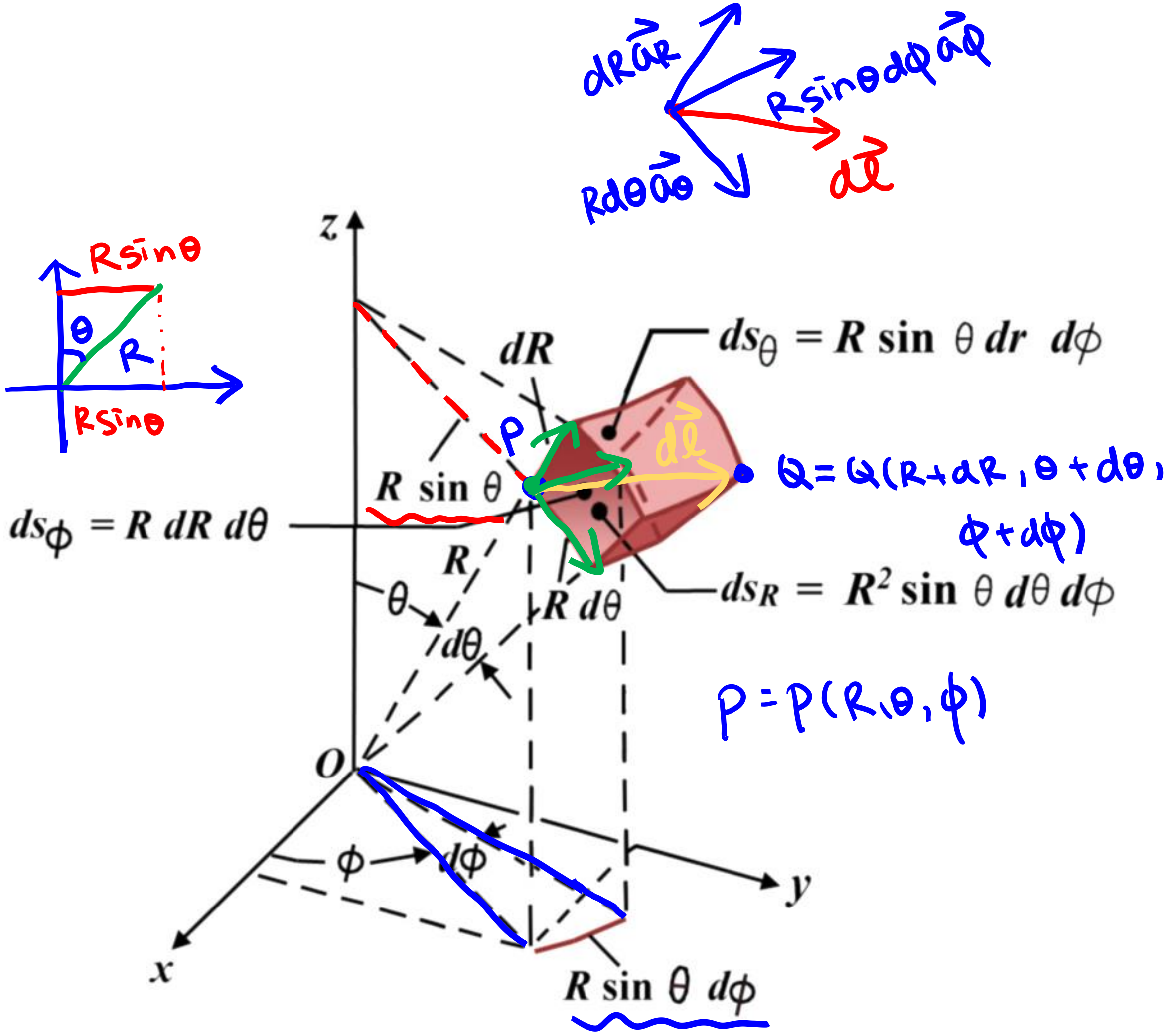
- Unit vectors do not have the same directions everywhere
  - Not uniform.



$A = A(r, \theta, \phi) = (3, \frac{1}{6}\pi, \frac{1}{3}\pi)$

$B(1, \frac{\pi}{2}, 0)$

$d\mathbf{l}$, $d\mathbf{S}$, and dv



$$d\mathbf{l} = dr\,\mathbf{a}_r + r\,d\theta\,\mathbf{a}_\theta + r\sin\theta\,d\phi\,\mathbf{a}_\phi$$

$$\pm r\,d\theta\,\mathbf{a}_\theta \times r\sin\theta\,d\phi\,\mathbf{a}_\phi = \pm r^2\sin\theta\,d\theta\,d\phi\,\mathbf{a}_r$$
$$\pm r\sin\theta\,d\phi\,\mathbf{a}_\phi \times dr\,\mathbf{a}_r = \pm r\sin\theta\,dr\,d\phi\,\mathbf{a}_\theta$$
$$\pm dr\,\mathbf{a}_r \times r\,d\theta\,\mathbf{a}_\theta = \pm r\,dr\,d\theta\,\mathbf{a}_\phi$$

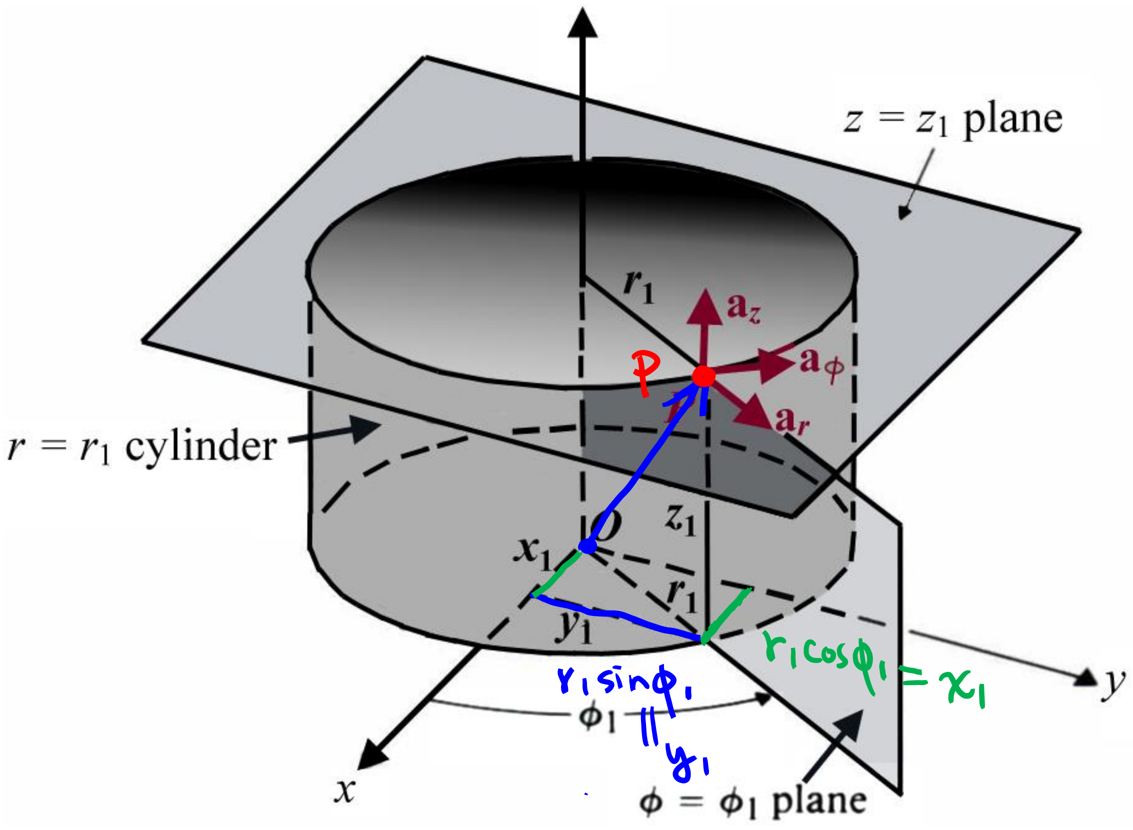
$$dv = (dr)(r\,d\theta)(r\sin\theta\,d\phi) = r^2\sin\theta\,dr\,d\theta\,d\phi$$

Cylindrical to Cartesian Coordinate Transformation

$\vec{a_r} = \vec{a_r}(x, y, z)$

$\vec{a_\theta} = \vec{a_\theta}(x, y, z)$

$\vec{a_\phi} = \vec{a_\phi}(x, y, z)$



$$\left.\begin{aligned} x &= r\cos\phi, \\ y &= r\sin\phi, \end{aligned}\right\} \quad x^2+y^2=r^2, \quad \frac{y}{x} = \frac{\sin\phi}{\cos\phi} = \tan\phi$$

$$z = z.$$

$$r = \sqrt{x^2+y^2},$$

$$\phi = \tan^{-1}\frac{y}{x},$$

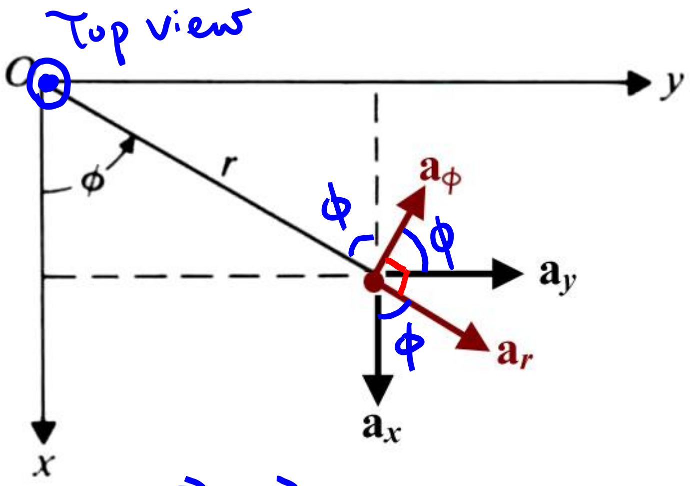
$$z = z.$$

Cylindrical to Cartesian Vector Transformation

$$\mathbf{A} = \mathbf{a}_r A_r + \mathbf{a}_\phi A_\phi + \mathbf{a}_z A_z$$

$$= \mathbf{a}_x A_x + \mathbf{a}_y A_y + \mathbf{a}_z A_z$$

- A_x, A_y, and A_z?



$$\begin{aligned} A_x &= \mathbf{A} \cdot \mathbf{a}_x \\ &= A_r \underline{\mathbf{a}_r \cdot \mathbf{a}_x} + A_\phi \underline{\mathbf{a}_\phi \cdot \mathbf{a}_x} \\ &= A_r \cos\phi - A_\phi \sin\phi \\ A_y &= \mathbf{A} \cdot \mathbf{a}_y \\ &= A_r \underline{\mathbf{a}_r \cdot \mathbf{a}_y} + A_\phi \underline{\mathbf{a}_\phi \cdot \mathbf{a}_y} \\ &= A_r \sin\phi + A_\phi \cos\phi \end{aligned}$$

$\vec{a_\phi} \cdot \vec{a_x} = \cos(90^\circ + \phi) = -\sin\phi$

$\vec{a_\phi} \cdot \vec{a_y} = \cos\phi$

$\vec{a_r} \cdot \vec{a_y} = \cos(90^\circ - \phi) = \sin\phi$

$\vec{a_r} \cdot \vec{a_x} = \cos\phi$

$$\mathbf{a}_r \cdot \mathbf{a}_x = \cos\phi$$

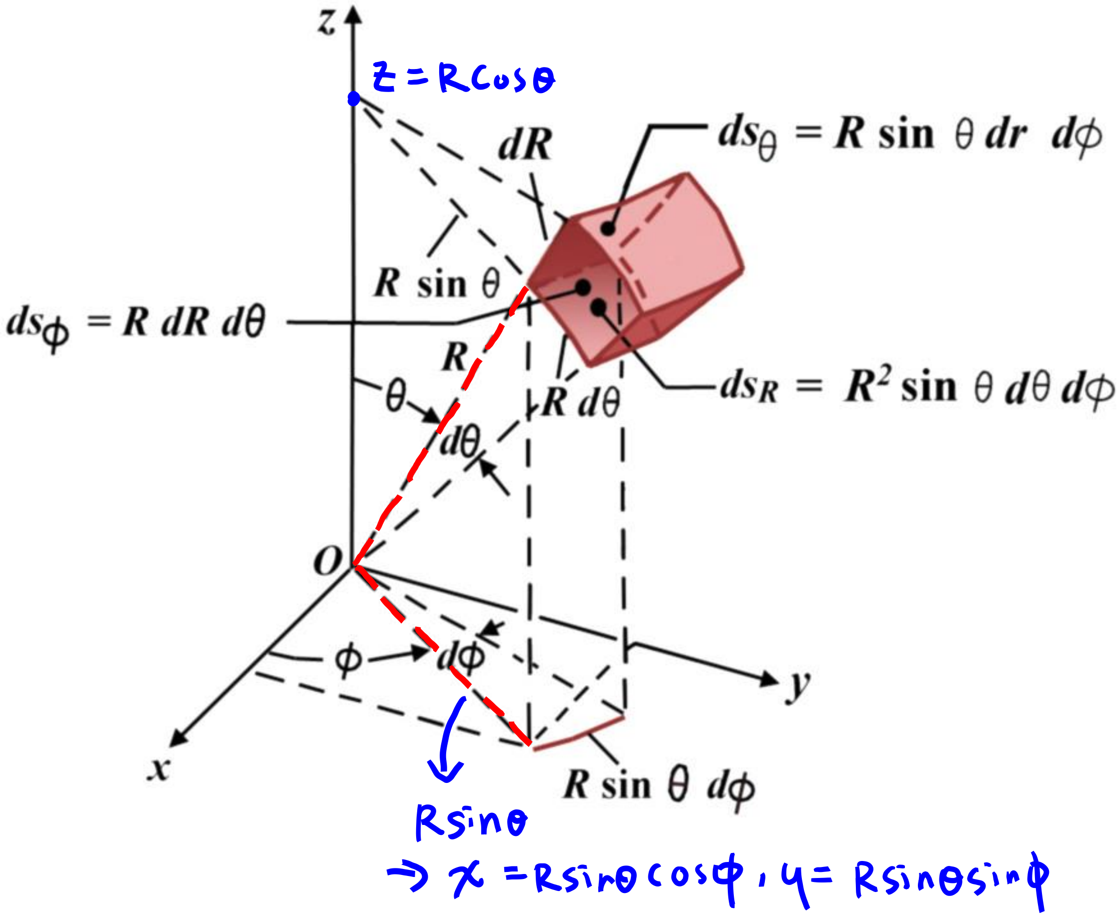
$$\mathbf{a}_\phi \cdot \mathbf{a}_y = \cos\phi$$

$$\mathbf{a}_\phi \cdot \mathbf{a}_x = \cos\left(\frac{\pi}{2} + \phi\right) = -\sin\phi$$

$$\mathbf{a}_r \cdot \mathbf{a}_y = \cos\left(\frac{\pi}{2} - \phi\right) = \sin\phi$$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix}$$

Spherical to Cartesian Coordinate Transformation



get ϕ

$$x = R \sin\theta \cos\phi,$$
$$y = R \sin\theta \sin\phi,$$

$x^2+y^2 = R^2\sin^2\theta$

$z^2 = R^2\cos^2\theta$

get θ

$$z = R \cos\theta.$$

$\rightarrow x^2+y^2+z^2 = R^2 \rightarrow$ get R

$$R = \sqrt{x^2 + y^2 + z^2},$$

$$\theta = \tan^{-1} \frac{\sqrt{x^2 + y^2}}{z},$$

$$\phi = \tan^{-1} \frac{y}{x}.$$

Spherical to Cartesian Vector Transformation

- A_x, A_y, and A_z?

$$\mathbf{A} = \mathbf{a}_R A_R + \mathbf{a}_\theta A_\theta + \mathbf{a}_\phi A_\phi$$
$$= \mathbf{a}_x A_x + \mathbf{a}_y A_y + \mathbf{a}_z A_z$$

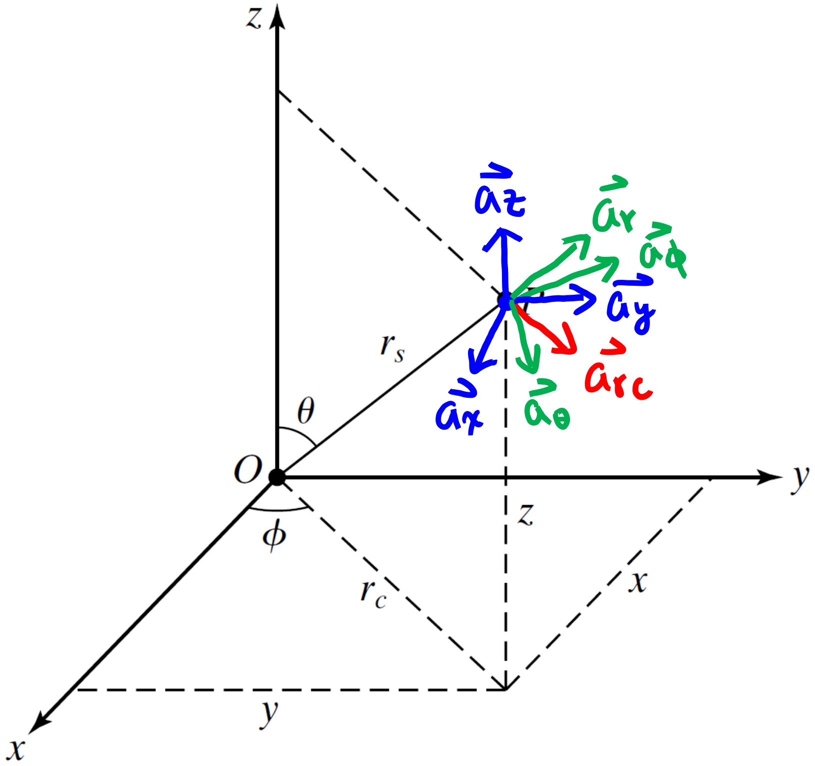
$$A_x = \mathbf{A} \cdot \mathbf{a}_x$$
$$= A_R \mathbf{a}_R \cdot \mathbf{a}_x + A_\theta \mathbf{a}_\theta \cdot \mathbf{a}_x + A_\phi \mathbf{a}_\phi \cdot \mathbf{a}_x$$

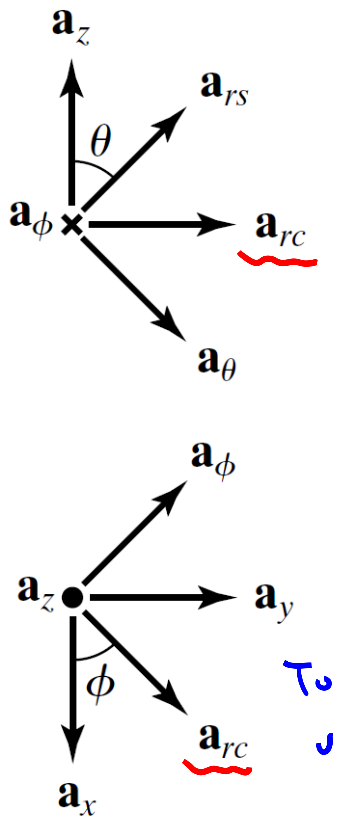
$\vec{a}_R = \vec{a}_{rs}$

$\vec{a}_{rs} \cdot \vec{a}_x = \sin\theta\cos\phi$

$\vec{a}_{rs} \cdot \vec{a}_{rc} = \sin\theta$

$\vec{a}_{rc} \cdot \vec{a}_x = \cos\phi$





Top views

project to $\vec{a}_{rc}$, then to $\vec{a}_x$ and $\vec{a}_y$

$$\mathbf{a}_{rs} \cdot \mathbf{a}_x = \sin\theta\cos\phi$$
$$\mathbf{a}_\theta \cdot \mathbf{a}_x = \cos\theta\cos\phi$$
$$\mathbf{a}_\phi \cdot \mathbf{a}_x = -\sin\phi$$

$$\mathbf{a}_{rs} \cdot \mathbf{a}_y = \sin\theta\sin\phi$$
$$\mathbf{a}_\theta \cdot \mathbf{a}_y = \cos\theta\sin\phi$$
$$\mathbf{a}_\phi \cdot \mathbf{a}_y = \cos\phi$$

$$\mathbf{a}_{rs} \cdot \mathbf{a}_z = \cos\theta$$
$$\mathbf{a}_\theta \cdot \mathbf{a}_z = -\sin\theta$$
$$\mathbf{a}_\phi \cdot \mathbf{a}_z = 0$$

Spherical to Cartesian Vector Transformation

$$\mathbf{a}_\theta \cdot \mathbf{a}_x = \cos\theta\cos\phi = \frac{xz}{\sqrt{(x^2+y^2)(x^2+y^2+z^2)}}$$

$$\mathbf{a}_R \cdot \mathbf{a}_x = \sin\theta\cos\phi = \frac{x}{\sqrt{x^2+y^2+z^2}}$$

$$\mathbf{a}_\phi \cdot \mathbf{a}_x = -\sin\phi = -\frac{y}{\sqrt{x^2+y^2}}$$

$$\begin{aligned} A_x &= \mathbf{A} \cdot \mathbf{a}_x \\ &= A_R \mathbf{a}_R \cdot \mathbf{a}_x + A_\theta \mathbf{a}_\theta \cdot \mathbf{a}_x + A_\phi \mathbf{a}_\phi \cdot \mathbf{a}_x \\ &= \frac{A_R x}{\sqrt{x^2+y^2+z^2}} + \frac{A_\theta xz}{\sqrt{(x^2+y^2)(x^2+y^2+z^2)}} - \frac{A_\phi y}{\sqrt{x^2+y^2}} \end{aligned}$$

$$\begin{aligned} A_y &= A_R \sin\theta\sin\phi + A_\theta\cos\theta\sin\phi + A_\phi\cos\phi \\ &= \frac{A_R y}{\sqrt{x^2+y^2+z^2}} + \frac{A_\theta yz}{\sqrt{(x^2+y^2)(x^2+y^2+z^2)}} + \frac{A_\phi x}{\sqrt{x^2+y^2}} \end{aligned}$$

$$A_z = A_R\cos\theta - A_\theta\sin\theta = \frac{A_R z}{\sqrt{x^2+y^2+z^2}} - \frac{A_\theta\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}$$

Example

- Let us consider the vector $3\mathbf{a}_x + 4\mathbf{a}_y + 5\mathbf{a}_z$ at the point (3, 4, 5) and convert it to one in spherical coordinates.

$$\begin{bmatrix} \mathbf{a}_x \\ \mathbf{a}_y \\ \mathbf{a}_z \end{bmatrix} = \begin{bmatrix} (\mathbf{a}_x \cdot \mathbf{a}_{rs}) & (\mathbf{a}_x \cdot \mathbf{a}_\theta) & (\mathbf{a}_x \cdot \mathbf{a}_\phi) \\ (\mathbf{a}_y \cdot \mathbf{a}_{rs}) & (\mathbf{a}_y \cdot \mathbf{a}_\theta) & (\mathbf{a}_y \cdot \mathbf{a}_\phi) \\ (\mathbf{a}_z \cdot \mathbf{a}_{rs}) & (\mathbf{a}_z \cdot \mathbf{a}_\theta) & (\mathbf{a}_z \cdot \mathbf{a}_\phi) \end{bmatrix} \begin{bmatrix} \mathbf{a}_{rs} \\ \mathbf{a}_\theta \\ \mathbf{a}_\phi \end{bmatrix}$$

$$= \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} \mathbf{a}_{rs} \\ \mathbf{a}_\theta \\ \mathbf{a}_\phi \end{bmatrix}$$

$$r_s = \sqrt{3^2 + 4^2 + 5^2} = 5\sqrt{2}$$

$$\theta = \tan^{-1}\frac{\sqrt{3^2 + 4^2}}{5} = \tan^{-1} 1 = 45^\circ$$

$$\phi = \tan^{-1}\tfrac{4}{3} = 53.13^\circ$$

$$\begin{bmatrix} \mathbf{a}_x \\ \mathbf{a}_y \\ \mathbf{a}_z \end{bmatrix} = \begin{bmatrix} 0.3\sqrt{2} & 0.3\sqrt{2} & -0.8 \\ 0.4\sqrt{2} & 0.4\sqrt{2} & 0.6 \\ 0.5\sqrt{2} & -0.5\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{a}_{rs} \\ \mathbf{a}_\theta \\ \mathbf{a}_\phi \end{bmatrix}$$

$$\vec{A} = 3\vec{a_x} + 4\vec{a_y} + 5\vec{a_z} = A_r\vec{a_{rs}} + A_\theta\vec{a_\theta} + A_\phi\vec{a_\phi}$$

$$A_r = \vec{A}\cdot\vec{a_{rs}}$$

$$= 3\vec{a_x}\cdot\vec{a_{rs}} + 4\vec{a_y}\cdot\vec{a_{rs}} + 5\vec{a_z}\cdot\vec{a_{rs}}$$

$$= 3\sin\theta\cos\phi + 4\sin\theta\sin\phi + 5\cos\theta$$

$$= 3\frac{\sqrt{2}}{2} + \cdots$$

$$\begin{aligned} 3\mathbf{a}_x + 4\mathbf{a}_y + 5\mathbf{a}_z &= 3(0.3\sqrt{2}\mathbf{a}_{rs} + 0.3\sqrt{2}\mathbf{a}_\theta - 0.8\mathbf{a}_\phi) \\ &\quad + 4(0.4\sqrt{2}\mathbf{a}_{rs} + 0.4\sqrt{2}\mathbf{a}_\theta + 0.6\mathbf{a}_\phi) \\ &\quad + 5(0.5\sqrt{2}\mathbf{a}_{rs} - 0.5\sqrt{2}\mathbf{a}_\theta) \\ &= 5\sqrt{2}\mathbf{a}_{rs} \end{aligned}$$

$$x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z = r_s\mathbf{a}_{rs}$$

Coordinate Transformation Summary

Transformation	Coordinate Variables	Unit Vectors	Vector Components
Cartesian to cylindrical	$r = \sqrt[+]{x^2 + y^2}$ $\phi = \tan^{-1}(y/x)$ $z = z$	$\hat{\mathbf{r}} = \hat{\mathbf{x}} \cos\phi + \hat{\mathbf{y}} \sin\phi$ $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}} \sin\phi + \hat{\mathbf{y}} \cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$A_r = A_x \cos\phi + A_y \sin\phi$ $A_\phi = -A_x \sin\phi + A_y \cos\phi$ $A_z = A_z$
Cylindrical to Cartesian	$x = r \cos\phi$ $y = r \sin\phi$ $z = z$	$\hat{\mathbf{x}} = \hat{\mathbf{r}} \cos\phi - \hat{\boldsymbol{\phi}} \sin\phi$ $\hat{\mathbf{y}} = \hat{\mathbf{r}} \sin\phi + \hat{\boldsymbol{\phi}} \cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$A_x = A_r \cos\phi - A_\phi \sin\phi$ $A_y = A_r \sin\phi + A_\phi \cos\phi$ $A_z = A_z$
Cartesian to spherical	$R = \sqrt[+]{x^2 + y^2 + z^2}$ $\theta = \tan^{-1}[\sqrt[+]{x^2 + y^2}/z]$ $\phi = \tan^{-1}(y/x)$	$\hat{\mathbf{R}} = \hat{\mathbf{x}} \sin\theta \cos\phi$ $\quad + \hat{\mathbf{y}} \sin\theta \sin\phi + \hat{\mathbf{z}} \cos\theta$ $\hat{\boldsymbol{\theta}} = \hat{\mathbf{x}} \cos\theta \cos\phi$ $\quad + \hat{\mathbf{y}} \cos\theta \sin\phi - \hat{\mathbf{z}} \sin\theta$ $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}} \sin\phi + \hat{\mathbf{y}} \cos\phi$	$A_R = A_x \sin\theta \cos\phi$ $\quad + A_y \sin\theta \sin\phi + A_z \cos\theta$ $A_\theta = A_x \cos\theta \cos\phi$ $\quad + A_y \cos\theta \sin\phi - A_z \sin\theta$ $A_\phi = -A_x \sin\phi + A_y \cos\phi$
Spherical to Cartesian	$x = R \sin\theta \cos\phi$ $y = R \sin\theta \sin\phi$ $z = R \cos\theta$	$\hat{\mathbf{x}} = \hat{\mathbf{R}} \sin\theta \cos\phi$ $\quad + \hat{\boldsymbol{\theta}} \cos\theta \cos\phi - \hat{\boldsymbol{\phi}} \sin\phi$ $\hat{\mathbf{y}} = \hat{\mathbf{R}} \sin\theta \sin\phi$ $\quad + \hat{\boldsymbol{\theta}} \cos\theta \sin\phi + \hat{\boldsymbol{\phi}} \cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{R}} \cos\theta - \hat{\boldsymbol{\theta}} \sin\theta$	$A_x = A_R \sin\theta \cos\phi$ $\quad + A_\theta \cos\theta \cos\phi - A_\phi \sin\phi$ $A_y = A_R \sin\theta \sin\phi$ $\quad + A_\theta \cos\theta \sin\phi + A_\phi \cos\phi$ $A_z = A_R \cos\theta - A_\theta \sin\theta$
Cylindrical to spherical	$R = \sqrt[+]{r^2 + z^2}$ $\theta = \tan^{-1}(r/z)$ $\phi = \phi$	$\hat{\mathbf{R}} = \hat{\mathbf{r}} \sin\theta + \hat{\mathbf{z}} \cos\theta$ $\hat{\boldsymbol{\theta}} = \hat{\mathbf{r}} \cos\theta - \hat{\mathbf{z}} \sin\theta$ $\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}$	$A_R = A_r \sin\theta + A_z \cos\theta$ $A_\theta = A_r \cos\theta - A_z \sin\theta$ $A_\phi = A_\phi$
Spherical to cylindrical	$r = R \sin\theta$ $\phi = \phi$ $z = R \cos\theta$	$\hat{\mathbf{r}} = \hat{\mathbf{R}} \sin\theta + \hat{\boldsymbol{\theta}} \cos\theta$ $\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}$ $\hat{\mathbf{z}} = \hat{\mathbf{R}} \cos\theta - \hat{\boldsymbol{\theta}} \sin\theta$	$A_r = A_R \sin\theta + A_\theta \cos\theta$ $A_\phi = A_\phi$ $A_z = A_R \cos\theta - A_\theta \sin\theta$

P1.20

- Determine if the vector $(\mathbf{a}_{rc} - \sqrt{3}\mathbf{a}_{\varphi} + 3\mathbf{a}_z)$ at the point (3, π/3, 5) in cylindrical coordinates is equal to the vector $(3\mathbf{a}_{rs} - \sqrt{3}\mathbf{a}_{\theta} - \mathbf{a}_{\varphi})$ at the point (1, π/3, π/6) in spherical coordinates

Convert to Cartisian coordinates

$$\begin{cases} \vec{a}_{rc} = \cos\phi\, \vec{a}_x + \sin\phi\, \vec{a}_y \\ \vec{a}_\phi = -\sin\phi\, \vec{a}_x + \cos\phi\, \vec{a}_y \\ \vec{a}_z = \vec{a}_z \end{cases}$$

$$\begin{cases} \vec{a}_{rs} = \sin\theta\cos\phi\, \vec{a}_x + \sin\theta\sin\phi\, \vec{a}_y + \cos\theta\, \vec{a}_z \\ \vec{a}_\theta = \cos\theta\cos\phi\, \vec{a}_x + \cos\theta\sin\phi\, \vec{a}_y - \sin\theta\, \vec{a}_z \\ \vec{a}_\phi = -\sin\phi\, \vec{a}_x + \cos\phi\, \vec{a}_y \end{cases}$$

$$\vec{a}_{rc} - \sqrt{3}\vec{a}_\phi + 3\vec{a}_z = (\cos\phi + \sqrt{3}\sin\phi)\vec{a}_x + (\sin\phi - \sqrt{3}\cos\phi)\vec{a}_y + 3\vec{a}_z$$

$$= 2\vec{a}_x + 3\vec{a}_z$$

$$3\vec{a}_{rs} - \sqrt{3}\vec{a}_\theta - \vec{a}_\phi$$

$$= 3(\sin\theta\cos\phi - \sqrt{3}\cos\theta\cos\phi + \sin\phi)\vec{a}_x + \cdots$$

$$= 2\vec{a}_x + 3\vec{a}_z$$

$\Rightarrow$ Same vectors

Differential Length Vector

- Differential length change

$$d\ell_i = h_i \, du_i$$

  - h_i: metric coefficient.
  - h_i may be a function of u_1, u_2, and u_3.
  - h_i is used to convert the differential variable to have the unit of length.
  - du_1, du_2, and du_3 are not independent of each other.

- Differential length vector

$$d\boldsymbol{\ell} = \mathbf{a}_{u_1} d\ell_1 + \mathbf{a}_{u_2} d\ell_2 + \mathbf{a}_{u_3} d\ell_3$$

$$d\boldsymbol{\ell} = \mathbf{a}_{u_1}(h_1 \, du_1) + \mathbf{a}_{u_2}(h_2 \, du_2) + \mathbf{a}_{u_3}(h_3 \, du_3)$$

$$\begin{aligned} d\ell &= [(d\ell_1)^2 + (d\ell_2)^2 + (d\ell_3)^2]^{1/2} \\ &= [(h_1 \, du_1)^2 + (h_2 \, du_2)^2 + (h_3 \, du_3)^2]^{1/2} \end{aligned}$$

Differential Surface Vector

- Differential area ds_1 normal to $\mathbf{a}_{\mathbf{u1}}$

$$ds_1 = d\ell_2 \, d\ell_3$$

$$ds_1 = h_2 h_3 \, du_2 \, du_3$$

- Differential area ds_2 normal to $\mathbf{a}_{\mathbf{u2}}$

$$ds_2 = h_1 h_3 \, du_1 \, du_3$$

- Differential area ds_3 normal to $\mathbf{a}_{\mathbf{u3}}$

$$ds_3 = h_1 h_2 \, du_1 \, du_2$$

Differential Volume

- Differential volume

$$dv = (d\ell_1\, d\ell_2\, d\ell_3) = h_1 h_2 h_3\, du_1\, du_2\, du_3$$

Coordinate Systems Summary

Coordinate System Relations		Cartesian Coordinates (x, y, z)	Cylindrical Coordinates (r, ϕ, z)	Spherical Coordinates (R, θ, ϕ)
Base vectors	$\mathbf{a}_{u_1}$	$\mathbf{a}_x$	$\mathbf{a}_r$	$\mathbf{a}_R$
	$\mathbf{a}_{u_2}$	$\mathbf{a}_y$	$\mathbf{a}_\phi$	$\mathbf{a}_\theta$
	$\mathbf{a}_{u_3}$	$\mathbf{a}_z$	$\mathbf{a}_z$	$\mathbf{a}_\phi$
Metric coefficients	h_1	1	1	1
	h_2	1	r	R
	h_3	1	1	$R \sin \theta$
Differential volume	dv	$dx\,dy\,dz$	$r\,dr\,d\phi\,dz$	$R^2 \sin \theta\,dR\,d\theta\,d\phi$

Coordinate Systems Summary

	Cartesian Coordinates	Cylindrical Coordinates	Spherical Coordinates
Coordinate variables	x, y, z	r, ϕ, z	R, θ, ϕ
Vector representation $\mathbf{A} =$	$\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$	$\hat{\mathbf{r}}A_r + \hat{\boldsymbol{\phi}}A_\phi + \hat{\mathbf{z}}A_z$	$\hat{\mathbf{R}}A_R + \hat{\boldsymbol{\theta}}A_\theta + \hat{\boldsymbol{\phi}}A_\phi$
Magnitude of A $\lvert\mathbf{A}\rvert =$	$\sqrt[+]{A_x^2 + A_y^2 + A_z^2}$	$\sqrt[+]{A_r^2 + A_\phi^2 + A_z^2}$	$\sqrt[+]{A_R^2 + A_\theta^2 + A_\phi^2}$
Position vector $\overrightarrow{OP_1} =$	$\hat{\mathbf{x}}x_1 + \hat{\mathbf{y}}y_1 + \hat{\mathbf{z}}z_1$, for $P(x_1, y_1, z_1)$	$\boxed{\hat{\mathbf{r}}r_1 + \hat{\mathbf{z}}z_1, \text{ for } P(r_1, \phi_1, z_1)}$	$\boxed{\hat{\mathbf{R}}R_1, \text{ for } P(R_1, \theta_1, \phi_1)}$
Base vectors properties	$\hat{\mathbf{x}}\cdot\hat{\mathbf{x}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{z}} = 1$ $\hat{\mathbf{x}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{z}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{x}} = 0$ $\hat{\mathbf{x}}\times\hat{\mathbf{y}} = \hat{\mathbf{z}}$ $\hat{\mathbf{y}}\times\hat{\mathbf{z}} = \hat{\mathbf{x}}$ $\hat{\mathbf{z}}\times\hat{\mathbf{x}} = \hat{\mathbf{y}}$	$\hat{\mathbf{r}}\cdot\hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{z}} = 1$ $\hat{\mathbf{r}}\cdot\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{z}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{r}} = 0$ $\hat{\mathbf{r}}\times\hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}$ $\hat{\boldsymbol{\phi}}\times\hat{\mathbf{z}} = \hat{\mathbf{r}}$ $\hat{\mathbf{z}}\times\hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}$	$\hat{\mathbf{R}}\cdot\hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\phi}} = 1$ $\hat{\mathbf{R}}\cdot\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{R}} = 0$ $\hat{\mathbf{R}}\times\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}$ $\hat{\boldsymbol{\theta}}\times\hat{\boldsymbol{\phi}} = \hat{\mathbf{R}}$ $\hat{\boldsymbol{\phi}}\times\hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}$
Dot product $\mathbf{A}\cdot\mathbf{B} =$	$A_xB_x + A_yB_y + A_zB_z$	$A_rB_r + A_\phi B_\phi + A_zB_z$	$A_RB_R + A_\theta B_\theta + A_\phi B_\phi$
Cross product $\mathbf{A}\times\mathbf{B} =$	$\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\phi}} & \hat{\mathbf{z}} \\ A_r & A_\phi & A_z \\ B_r & B_\phi & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{R}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\phi}} \\ A_R & A_\theta & A_\phi \\ B_R & B_\theta & B_\phi \end{vmatrix}$
Differential length $d\mathbf{l} =$	$\hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{r}}\,dr + \hat{\boldsymbol{\phi}}r\,d\phi + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{R}}\,dR + \hat{\boldsymbol{\theta}}R\,d\theta + \hat{\boldsymbol{\phi}}R\sin\theta\,d\phi$
Differential surface areas	$d\mathbf{s}_x = \hat{\mathbf{x}}\,dy\,dz$ $d\mathbf{s}_y = \hat{\mathbf{y}}\,dx\,dz$ $d\mathbf{s}_z = \hat{\mathbf{z}}\,dx\,dy$	$d\mathbf{s}_r = \hat{\mathbf{r}}r\,d\phi\,dz$ $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}\,dr\,dz$ $d\mathbf{s}_z = \hat{\mathbf{z}}r\,dr\,d\phi$	$d\mathbf{s}_R = \hat{\mathbf{R}}R^2\sin\theta\,d\theta\,d\phi$ $d\mathbf{s}_\theta = \hat{\boldsymbol{\theta}}R\sin\theta\,dR\,d\phi$ $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}R\,dR\,d\theta$
Differential volume $d\mathcal{V} =$	$dx\,dy\,dz$	$r\,dr\,d\phi\,dz$	$R^2\sin\theta\,dR\,d\theta\,d\phi$

Outline

- Vector algebra
- Cartesian coordinate system
- Cylindrical coordinate system
- Spherical coordinate system
- Scalar and vector fields
- Electric field
- Magnetic field
- Lorentz force equation

Before talking about **E** and **B**, understand the concept of the “field” first

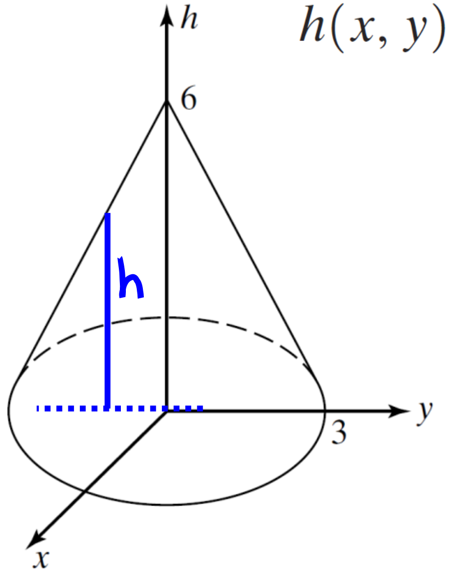
Field

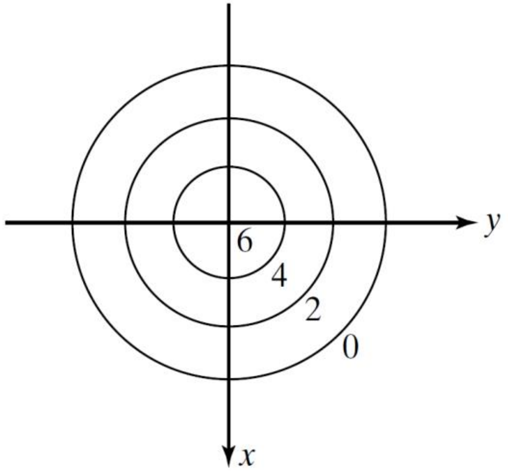
- A region in space where there is a physical phenomenon associated with points in that region
  - Spatial distribution of a quantity, which may or may not be a function of time.
  - Earth's gravitational field.
- A description, mathematical or graphical, of how the quantity varies from one point to another in the region of the field and with time
- Scalar fields or vector fields
- Static field or time-varying fields

Height, Distance, and Temperature Scalar Fields

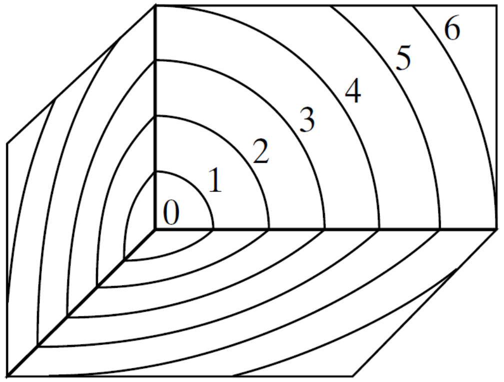
- Static fields

$$h(x, y) = 6 - 2\sqrt{x^2 + y^2}$$

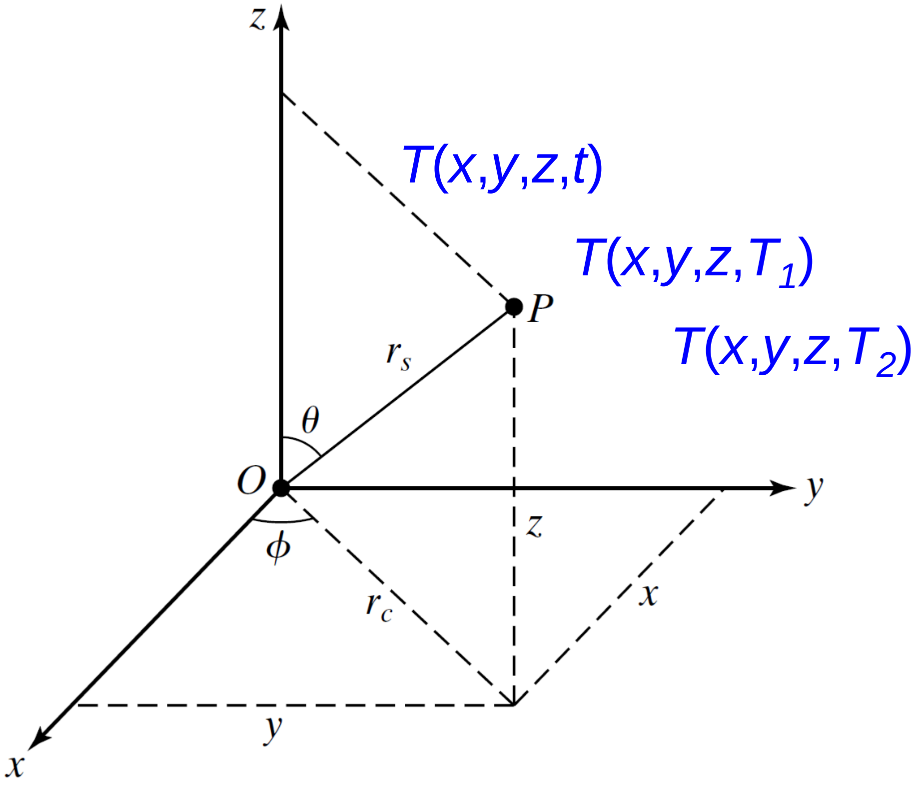




$$r(x, y, z) = \sqrt{x^2 + y^2 + z^2}$$



- Time-varying fields



Vector Fields

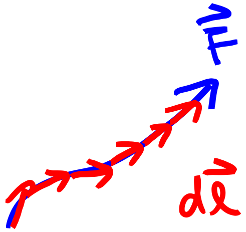
- Not only magnitude but also the direction
- Described by a set of direction lines, also known as streamlines and flux lines
- Direction lines
  - A curve is constructed such that the field is tangential to the curve for all points on the curve. (**F** and d**l** are parallel).

Vector field $\vec{F}$

$$\mathbf{F}(x, y, z, t) = F_x(x, y, z, t)\mathbf{a}_x + F_y(x, y, z, t)\mathbf{a}_y + F_z(x, y, z, t)\mathbf{a}_z$$

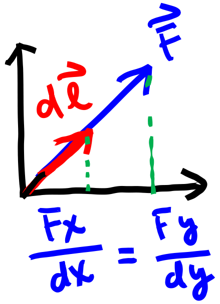
$$\mathbf{F}(r, \phi, z, t) = F_r(r, \phi, z, t)\mathbf{a}_r + F_\phi(r, \phi, z, t)\mathbf{a}_\phi + F_z(r, \phi, z, t)\mathbf{a}_z$$

$$\mathbf{F}(r, \theta, \phi, t) = F_r(r, \theta, \phi, t)\mathbf{a}_r + F_\theta(r, \theta, \phi, t)\mathbf{a}_\theta + F_\phi(r, \theta, \phi, t)\mathbf{a}_\phi$$



Find the equations for the direction lines

$\rightarrow d\vec{l} \parallel \vec{F}$



$\rightarrow$ Their components must have the same ratio

$$\frac{dx}{F_x} = \frac{dy}{F_y} = \frac{dz}{F_z} = a$$

$$d\vec{l} = dx\,\vec{a_x} + dy\,\vec{a_y} + dz\,\vec{a_z}$$

$$\frac{dr}{F_r} = \frac{r\,d\phi}{F_\phi} = \frac{dz}{F_z}$$

$$= dr\,\vec{a_{r_c}} + r\,d\phi\,\vec{a_\phi} + dz\,\vec{a_z}$$

$$\frac{dr}{F_r} = \frac{r\,d\theta}{F_\theta} = \frac{r\sin\theta\,d\phi}{F_\phi}$$

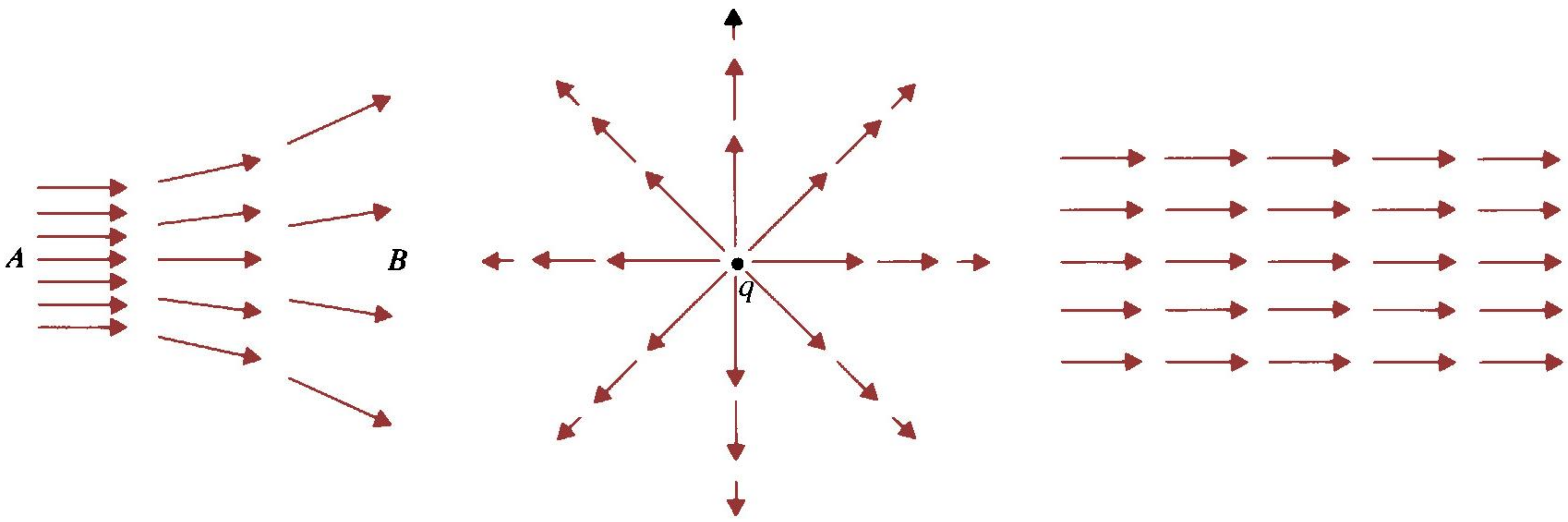
$$= dr\,\vec{a_{r_s}} + r\,d\theta\,\vec{a_\theta} + r\sin\theta\,d\phi\,\vec{a_\phi}$$

  - https://www.windy.com/

$\rightarrow d\vec{l} = a\vec{F}$

Flux Lines of Vector Fields

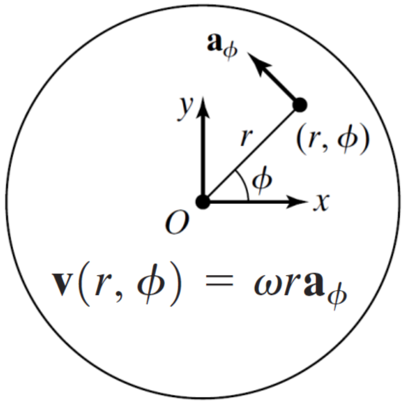
- Directed lines or curves that indicate at each point the direction of the vector field
  - – The magnitude of the field at a point is depicted either by the density or by the length of the directed lines in the vicinity of the point.

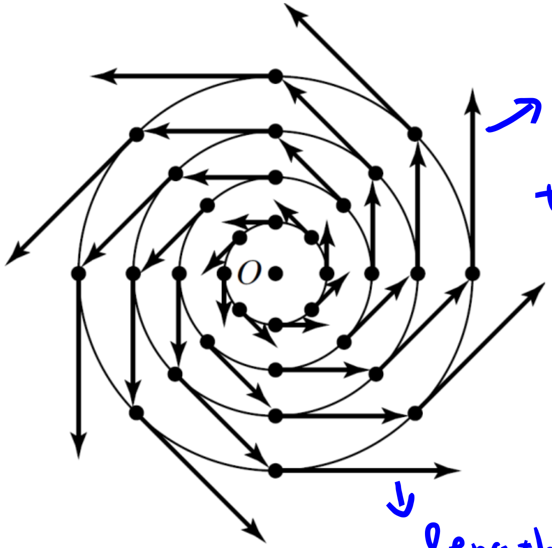


Example

- Consider a circular disk of radius *a* rotating with constant angular velocity ω about an axis normal to the disk and passing through its center. We wish to describe the linear velocity vector field associated with points on the rotating disk.

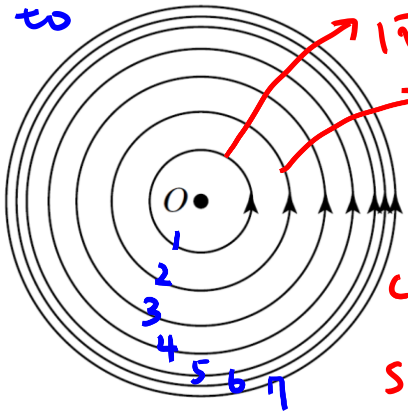
$\vec{v}(r,\phi) = \omega r \vec{a_\phi} = [(\vec{v}\cdot\vec{a_x})\vec{a_x} + (\vec{v}\cdot\vec{a_y})\vec{a_y} + (\vec{v}\cdot\vec{a_z})\vec{a_z}]\,\omega r$





→ tangential to the circle

↓ length ∝ magnitude



$|\vec{v}| = c_1$

$|\vec{v}| = c_2$

⋮

congested sketch

$$\frac{dr}{0} = \frac{r\,d\phi}{\omega r}$$
$$dr = 0$$
$$r = \text{constant}$$

$$\begin{aligned}\mathbf{v}(x, y) &= \omega r(\mathbf{a}_\phi \cdot \mathbf{a}_x)\mathbf{a}_x + \omega r(\mathbf{a}_\phi \cdot \mathbf{a}_y)\mathbf{a}_y \\ &= \omega r(-\sin\phi\,\mathbf{a}_x + \cos\phi\,\mathbf{a}_y) \\ &= \omega(-y\mathbf{a}_x + x\mathbf{a}_y)\end{aligned}$$

$$\frac{dx}{-y} = \frac{dy}{x}$$
$$x\,dx + y\,dy = 0$$
$$x^2 + y^2 = \text{constant}$$

Outline

- Vector algebra
- Cartesian coordinate system
- Cylindrical coordinate system
- Spherical coordinate system
- Scalar and vector fields
- Electric field
- Magnetic field
- Lorentz force equation

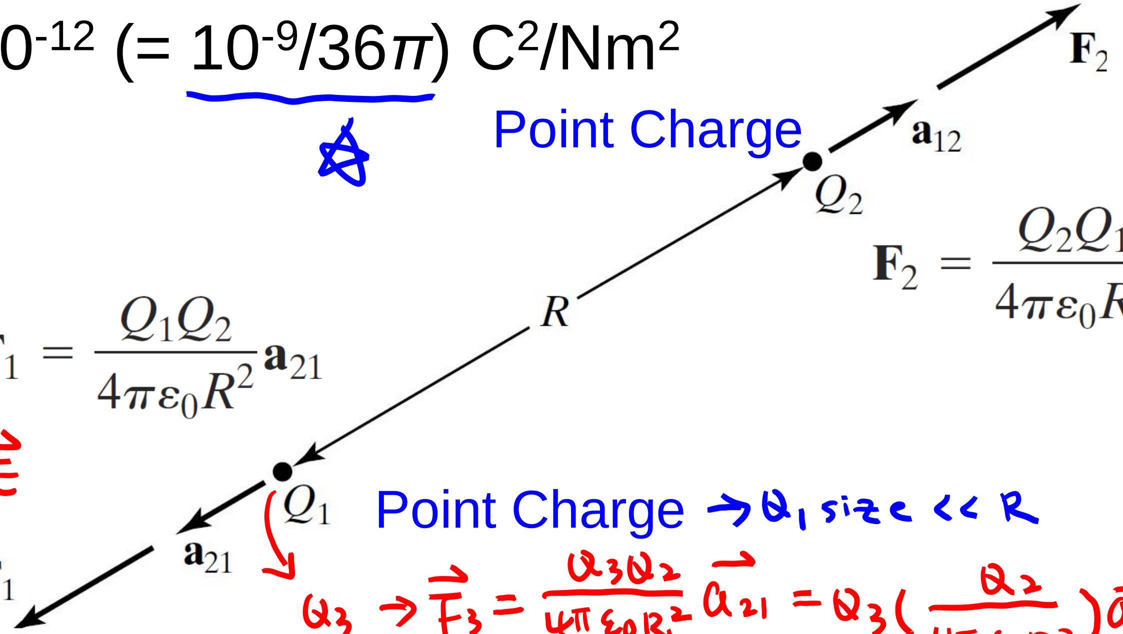
Coulomb's Experimental Law

- Experimental law ---> We do not need to prove it!
- $|\mathbf{F}| \propto |Q_1||Q_2|$
- $|\mathbf{F}| \propto 1/R^2$
- Different $|\mathbf{F}|$ in different medium
- **F**'s direction along the joining line
- Like charges repel, unlike charges attract
- ε_0: permittivity of free space= 8.854×10^{-12} (= $\underline{10^{-9}/36\pi}$) C^2/Nm^2
  - $[\varepsilon_0]$= C^2/Nm^2= Farads (F)/m. $[F] = C^2/m\,N$
- e= -1.6×10^{-19} C

$\vec{F} = K \dfrac{Q_1 Q_2}{R^2}\vec{a}_{21}$

$= \dfrac{1}{4\pi\varepsilon_0}\dfrac{Q_1Q_2}{R^2}\vec{a}_{21}$

$\varepsilon = \varepsilon_r\varepsilon_0$ → Forces depend on medium



$$\mathbf{F}_2 = \frac{Q_2 Q_1}{4\pi\varepsilon_0 R^2}\mathbf{a}_{12}$$

$$\mathbf{F}_1 = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R^2}\mathbf{a}_{21}$$

Point Charge → Q_1 size << R

$\dfrac{\vec{F}_1}{Q_1} = \dfrac{Q_2}{4\pi\varepsilon_0 R^2}a_{21} = \vec{E}$

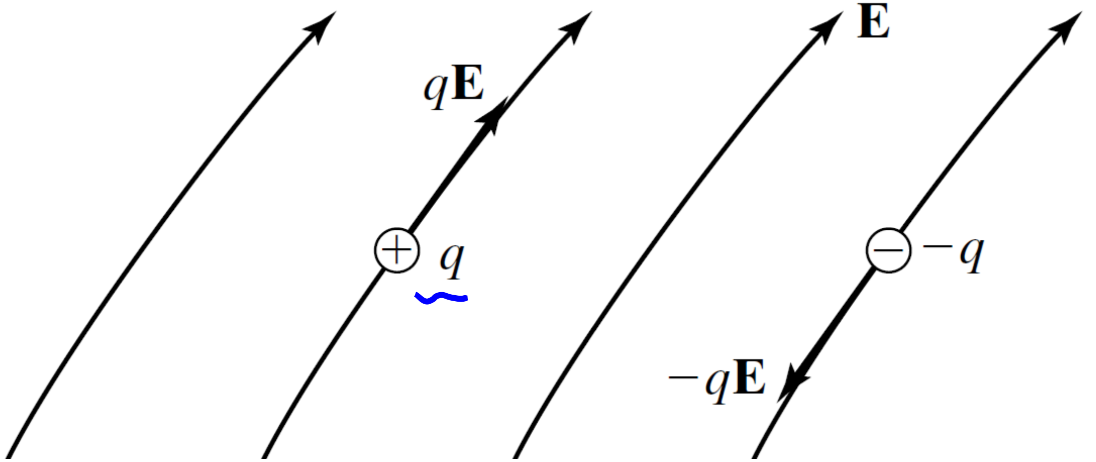
$= \dfrac{\vec{F}_3}{Q_3}$

$Q_3 \rightarrow \vec{F}_3 = \dfrac{Q_3Q_2}{4\pi\varepsilon_0 R^2}\vec{a}_{21} = Q_3\left(\dfrac{Q_2}{4\pi\varepsilon_0 R^2}\right)\vec{a}_{21}$

Electric Field Intensity

- [**E**]= N/C= V/m
  - The force per unit charge experienced by a small test charge placed in an electric field.
  - A test charge *q* experiences a force **F**.

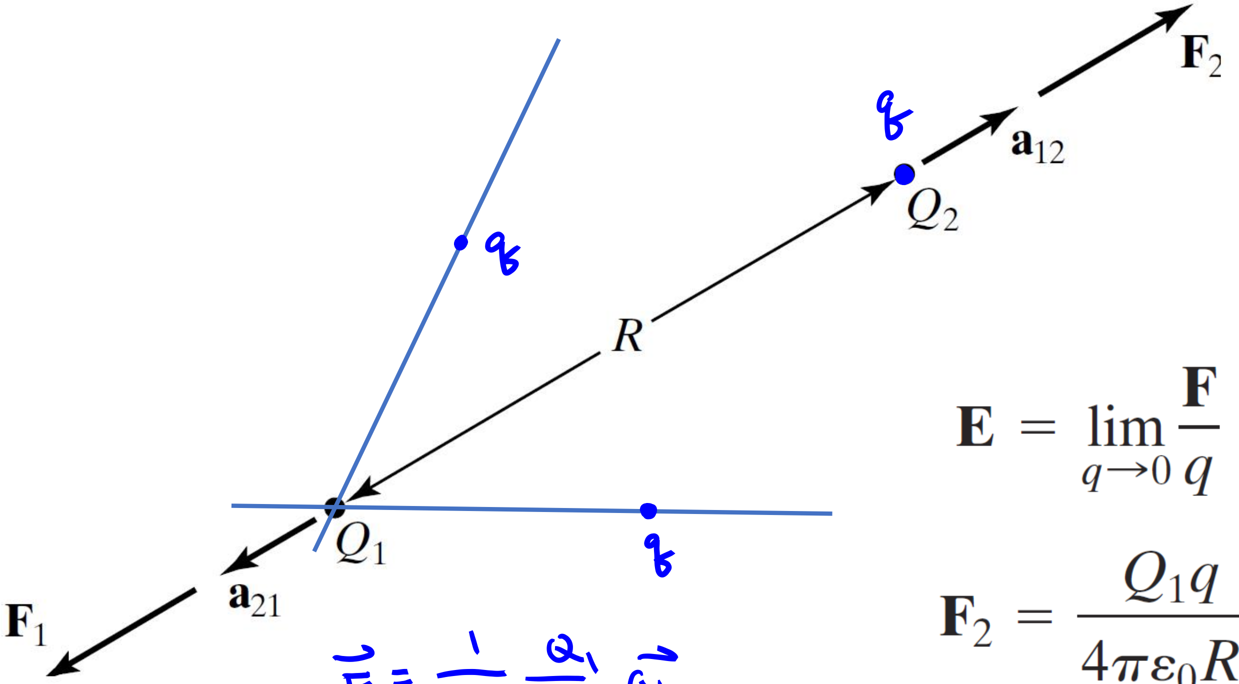
Q₂



$$\mathbf{E} = \frac{\mathbf{F}}{q} \qquad \mathbf{E} = \lim_{q \to 0} \frac{\mathbf{F}}{q}$$

Electric Field Intensity

- Q_2 a small test charge q
- Two kinds of charges
  - Positive and negative.



$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q_1}{R^2}\vec{a}_{12}$$

$$\mathbf{E} = \lim_{q \to 0} \frac{\mathbf{F}}{q}$$

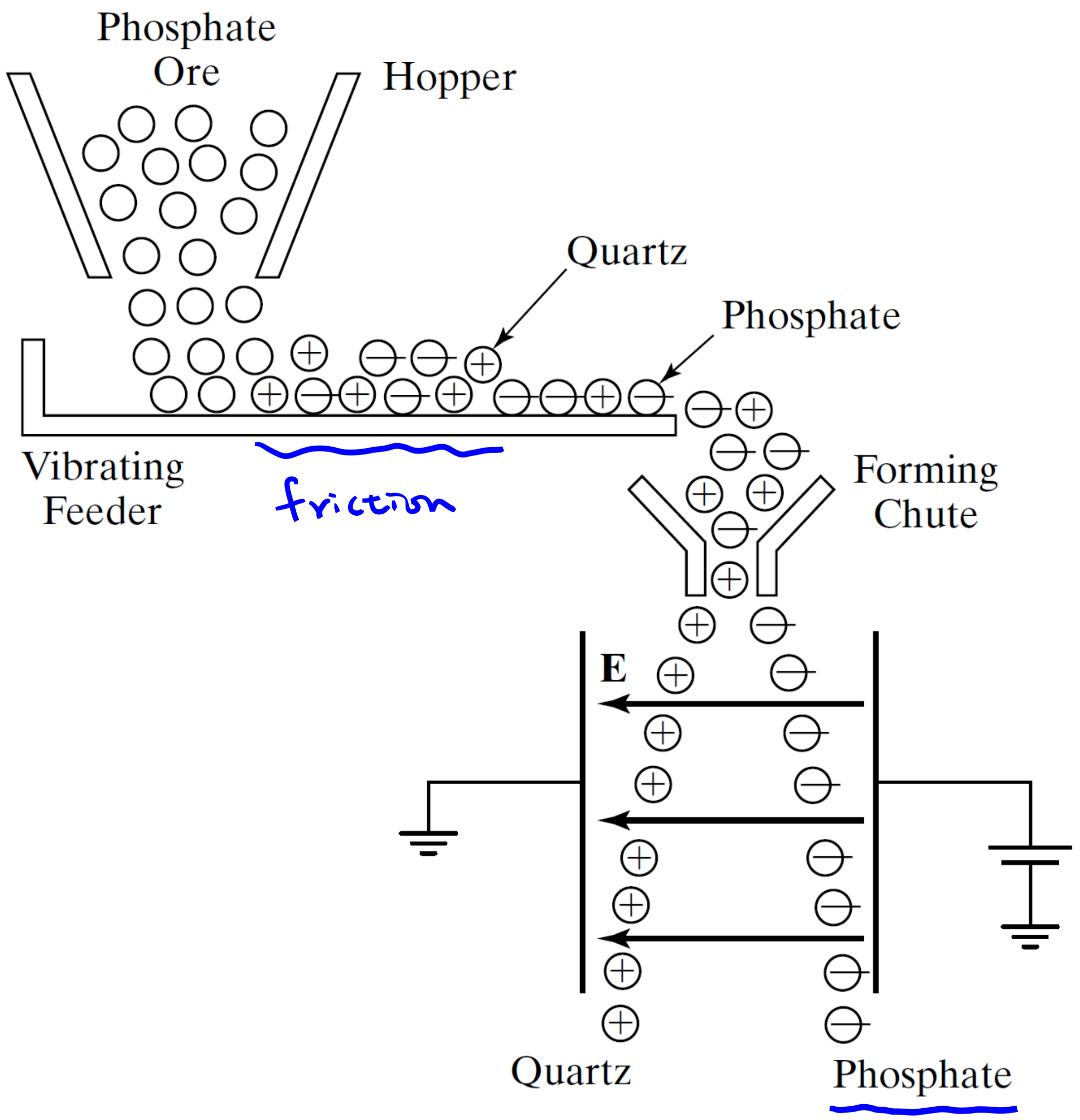
$$\mathbf{F}_2 = \frac{Q_1 q}{4\pi\varepsilon_0 R^2}\mathbf{a}_{12}$$

$$\mathbf{E}_2 = \frac{\mathbf{F}_2}{q} = \frac{Q_1}{4\pi\varepsilon_0 R^2}\mathbf{a}_{12}$$

Gravitational Force

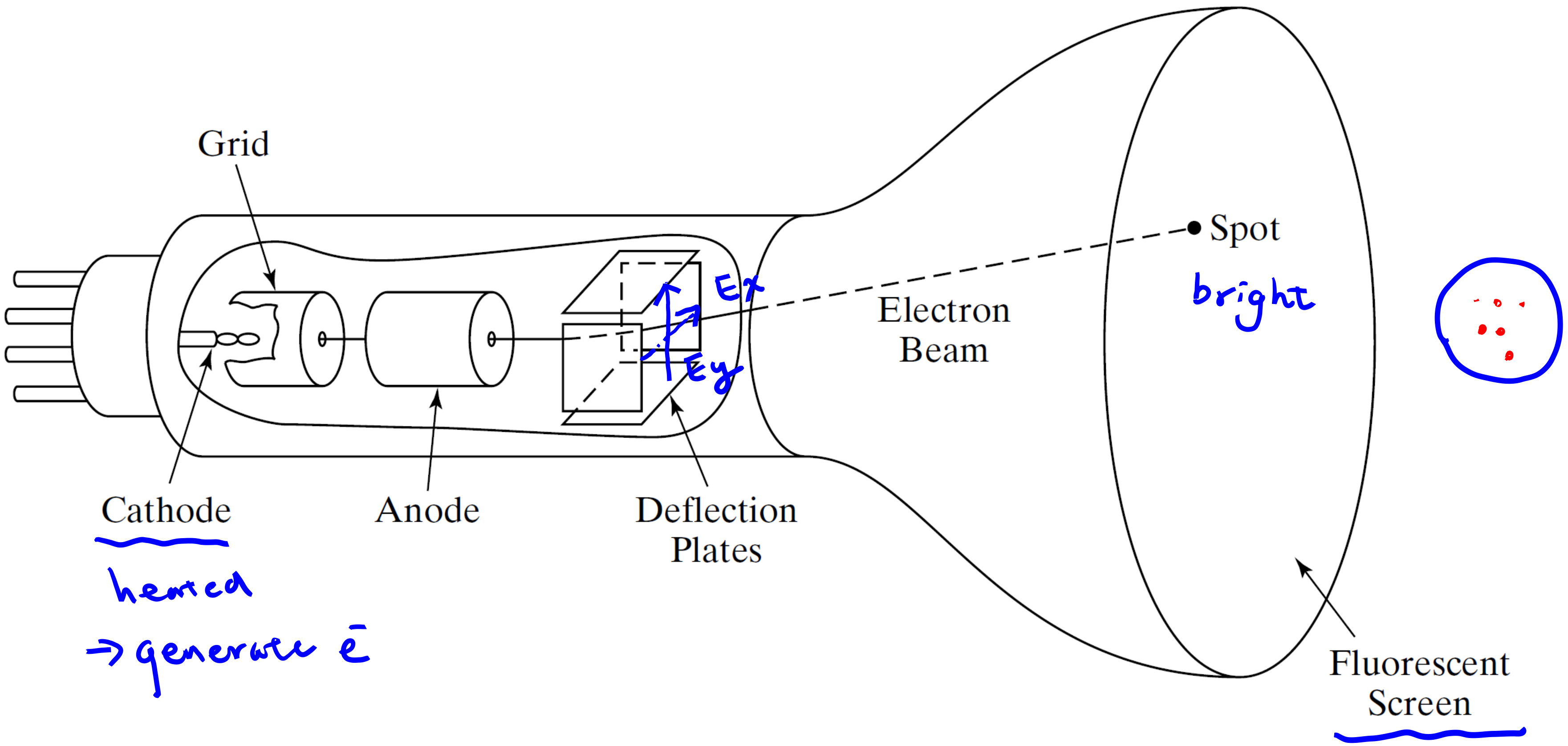
- $F_G = m_1 m_2 G/R^2$
  - Postulate the existence of a gravitational force field.
  - G: universal constant of gravitation= 6.67×10^{-11} $m^3Kg^{-1}s^{-2}$, m_1 and m_2: mass.
- Gravitational field intensity as the force per unit mass experienced by a small test mass placed in that field
- Communication and receiving signals from space?
  - Postulate the existence of fields.

Electrostatic Separation



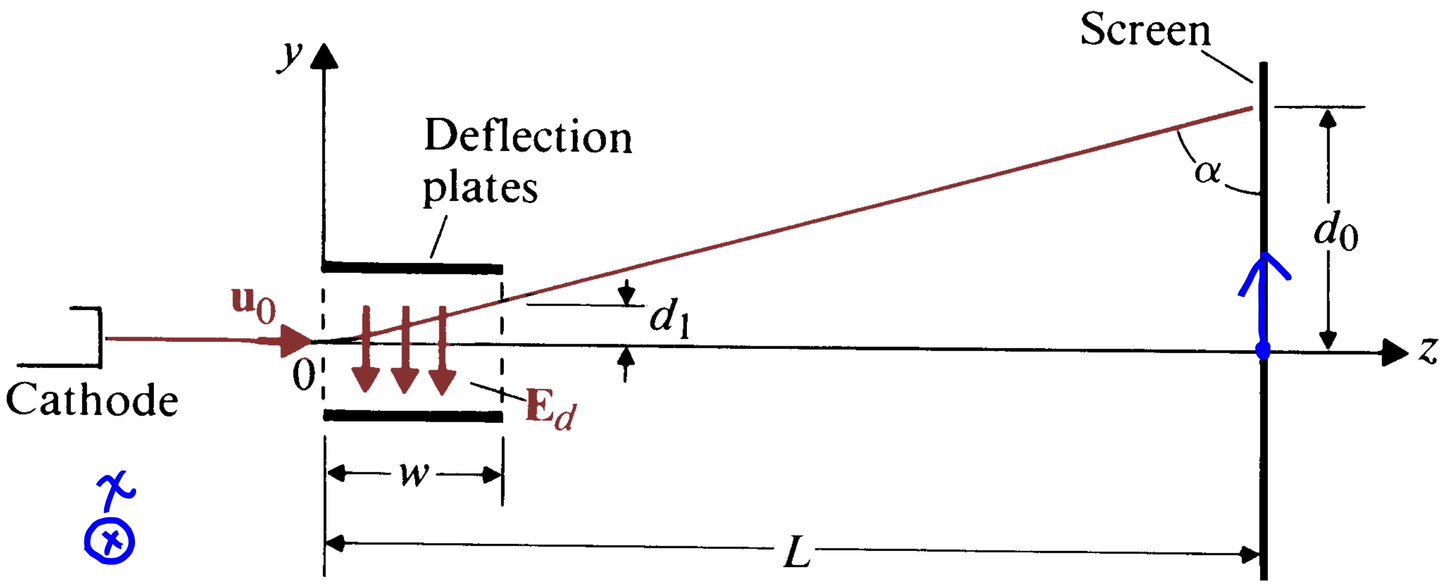
Cathode Ray Tube (CRT)

Replaced by LCD



Example

- The electrostatic deflection system of a cathode-ray oscilloscope is shown below. Electrons from a heated cathode are given an initial velocity $\mathbf{u}_0 = \mathbf{a}_z u_0$ by a positively charged anode (not shown). The electrons enter at $z = 0$ into a region of deflection plate where a uniform electric field $\mathbf{E}_d = -\mathbf{a}_y E_d$ is maintained over a width w. Ignoring gravitational effects, find the vertical deflection of the electrons on the fluorescent screen at $z = L$.



Example

$$\mathbf{F} = (-e)\mathbf{E}_d = \mathbf{a}_y e E_d \qquad u_y = \frac{dy}{dt} = \frac{e}{m} E_d t \quad u_y = 0 \text{ at } t = 0$$

$$m \frac{du_y}{dt} = eE_d \qquad y = \frac{e}{2m} E_d t^2 \quad y = 0 \text{ at } t = 0$$

- At the exit from the deflection plates, $t = w/u_0$,

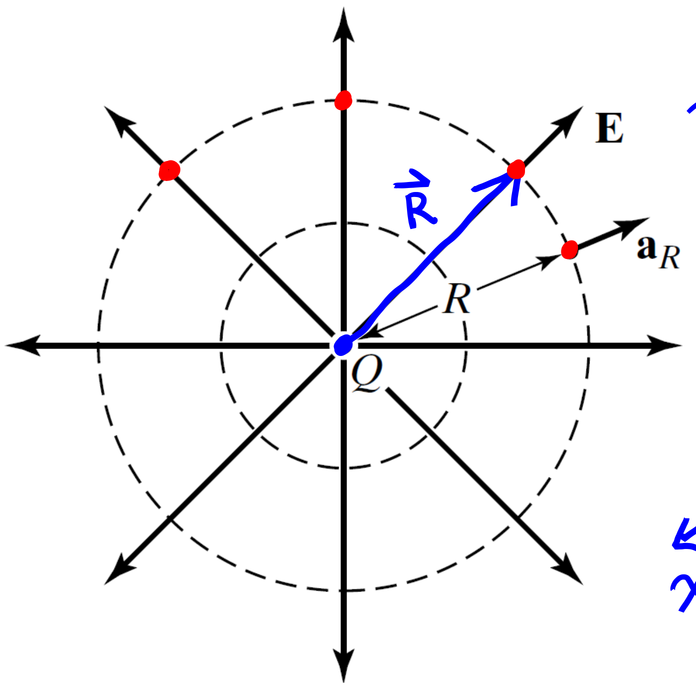
$$d_1 = \frac{eE_d}{2m}\left(\frac{w}{u_0}\right)^2 \qquad u_{y1} = u_y\left(t = \frac{w}{u_0}\right) = \frac{eE_d}{m}\left(\frac{w}{u_0}\right)$$

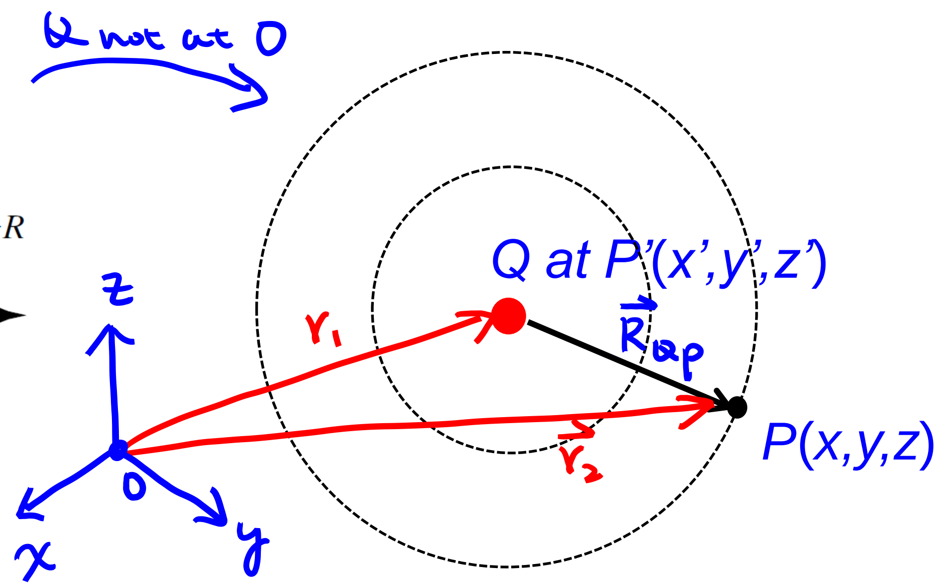
- Electrons have traveled a further horizontal distance of (L-w)

$$d_2 = u_{y1}\left(\frac{L-w}{u_0}\right) = \frac{eE_d}{m}\frac{w(L-w)}{u_0^2}$$

$$d_0 = d_1 + d_2 = \frac{eE_d}{mu_0^2} w\left(L - \frac{w}{2}\right)$$

Electric Field by Point Charge Q





$$\vec{R}_{Qp} = \vec{r}_2 - \vec{r}_1$$

$$= (x - x')\vec{a}_x + (y - y')\vec{a}_y + (z - z')\vec{a}_z$$

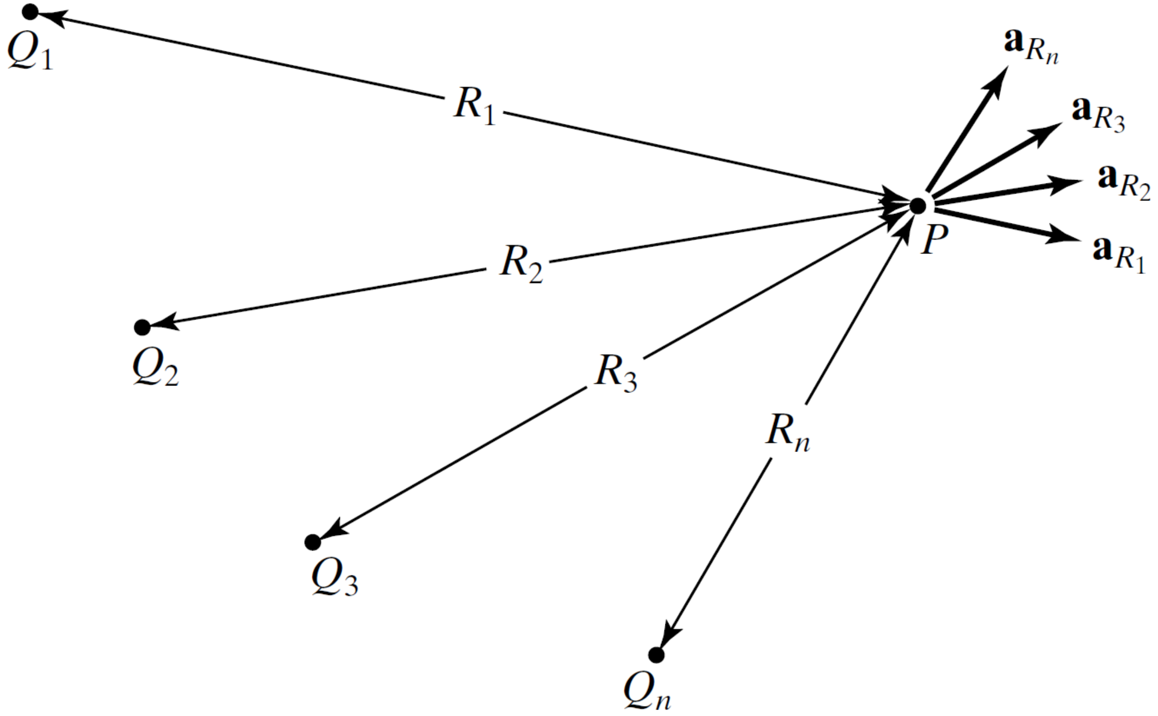
$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_0 R^2}\mathbf{a}_R \qquad \vec{a}_R = \frac{\vec{R}}{R}$$

$$\mathbf{E} = \frac{Q\mathbf{R}}{4\pi\varepsilon_0 R^3}$$

$$= \frac{Q}{4\pi\varepsilon_0}\frac{(x - x')\mathbf{a}_x + (y - y')\mathbf{a}_y + (z - z')\mathbf{a}_z}{[(x - x')^2 + (y - y')^2 + (z - z')^2]^{3/2}}$$

E Field by Multiple Charges

- Superposition applies



$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{R^2}\vec{a}_R \rightarrow \vec{E} \propto Q$$

linear wrt Q

$$\mathbf{E} = \frac{Q_1}{4\pi\varepsilon_0 R_1^2}\mathbf{a}_{R_1} + \frac{Q_2}{4\pi\varepsilon_0 R_2^2}\mathbf{a}_{R_2} + \cdots + \frac{Q_n}{4\pi\varepsilon_0 R_n^2}\mathbf{a}_{R_n}$$

Electric Dipole

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\left\{\frac{\mathbf{R}-\frac{\mathbf{d}}{2}}{\left|\mathbf{R}-\frac{\mathbf{d}}{2}\right|^3} - \frac{\mathbf{R}+\frac{\mathbf{d}}{2}}{\left|\mathbf{R}+\frac{\mathbf{d}}{2}\right|^3}\right\}$$

If $d \ll R$ (Far away from the point charge)

$$\left|\mathbf{R}-\frac{\mathbf{d}}{2}\right|^{-3} = \left[\left(\mathbf{R}-\frac{\mathbf{d}}{2}\right)\cdot\left(\mathbf{R}-\frac{\mathbf{d}}{2}\right)\right]^{-3/2}$$

$$= \left[R^2 - \mathbf{R}\cdot\mathbf{d} + \frac{d^2}{4}\right]^{-3/2} = \left[R^2\left(1-\frac{\vec{R}\cdot\vec{d}}{R^2}+\frac{d^2}{4R^2}\right)\right]^{-3/2}$$

$$\cong R^{-3}\left[1-\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\right]^{-3/2}$$

Taylor series expansion

$$\cong R^{-3}\left[1+\frac{3}{2}\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\right],$$

$$\left|\mathbf{R}+\frac{\mathbf{d}}{2}\right|^{-3} \cong R^{-3}\left[1-\frac{3}{2}\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\right]$$

$$\mathbf{E} \cong \frac{q}{4\pi\epsilon_0 R^3}\left[3\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\mathbf{R}-\mathbf{d}\right]$$

Define electric dipole moment p

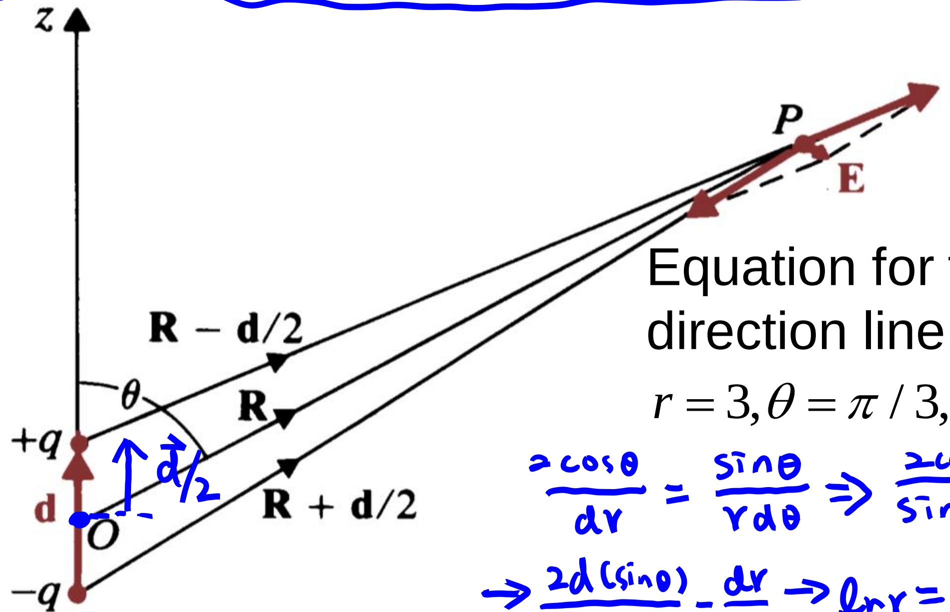
$$\mathbf{p} = q\mathbf{d}$$

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0 R^3}\left[3\frac{\mathbf{R}\cdot\mathbf{p}}{R^2}\mathbf{R}-\mathbf{p}\right]$$

$$\mathbf{p} = \mathbf{a}_z p = p(\mathbf{a}_R\cos\theta - \mathbf{a}_\theta\sin\theta)$$

$$\mathbf{R}\cdot\mathbf{p} = Rp\cos\theta$$

$$\mathbf{E} = \frac{p}{4\pi\epsilon_0 R^3}(\mathbf{a}_R 2\cos\theta + \mathbf{a}_\theta\sin\theta) \quad \text{(V/m)} \quad \propto 1/R^3$$



Equation for the direction line passing $r=3, \theta=\pi/3, \phi=0$? $r=4\sin^2\theta$

$$\frac{2\cos\theta}{dr} = \frac{\sin\theta}{r\,d\theta} \Rightarrow \frac{2\cos\theta}{\sin\theta}d\theta = \frac{1}{r}dr$$

$$\rightarrow \frac{2d(\sin\theta)}{\sin\theta} = \frac{dr}{r} \rightarrow \ln r = 2\ln\sin\theta + C$$

$$\rightarrow r = c'\sin^2\theta$$

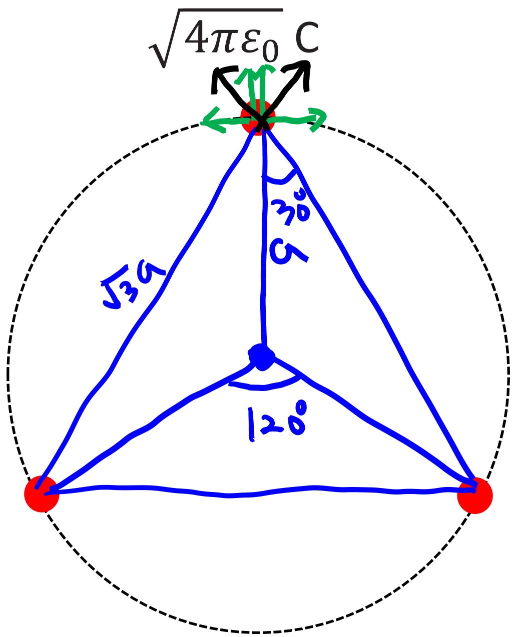
D1.13

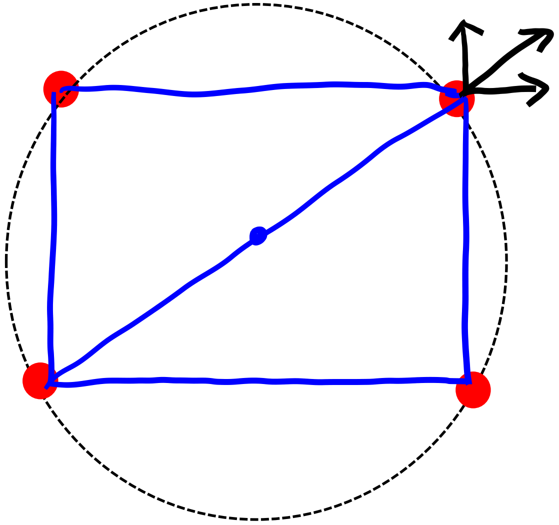
- Point charges, each of value $\sqrt{4\pi\varepsilon_0}$ C are located at the vertices of an n-sided regular polygon circumscribed by a circle of radius a. Find the electric force on each charge for (a) n= 3, (b) n= 4, and (c) n= 6.

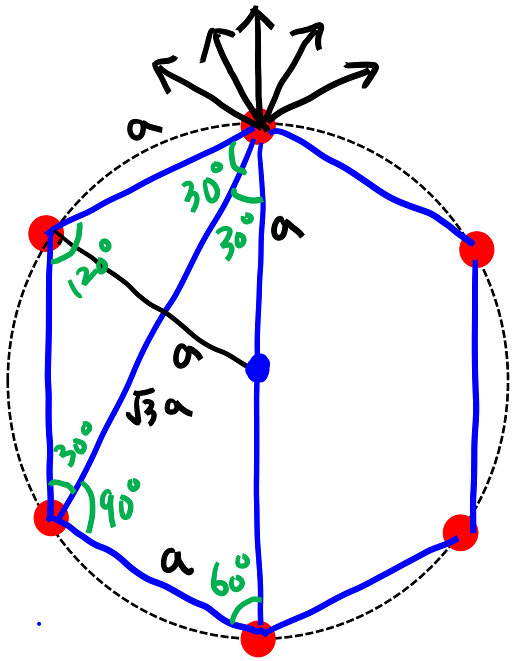
(a) $|\vec{F}| = 2\frac{1}{4\pi\varepsilon_0}\frac{4\pi\varepsilon_0}{(\sqrt{3}a)^2}\cos 30^\circ = \frac{0.577}{a^2}$ N

(b) $|\vec{F}| = 2\frac{1}{4\pi\varepsilon_0}\frac{4\pi\varepsilon_0}{(\sqrt{2}a)^2}\cos 45^\circ + \frac{1}{4\pi\varepsilon_0}\frac{4\pi\varepsilon_0}{(2a)^2} = \frac{0.957}{a^2}$ N

(c) $|\vec{F}| = \frac{1.83}{a^2}$ N



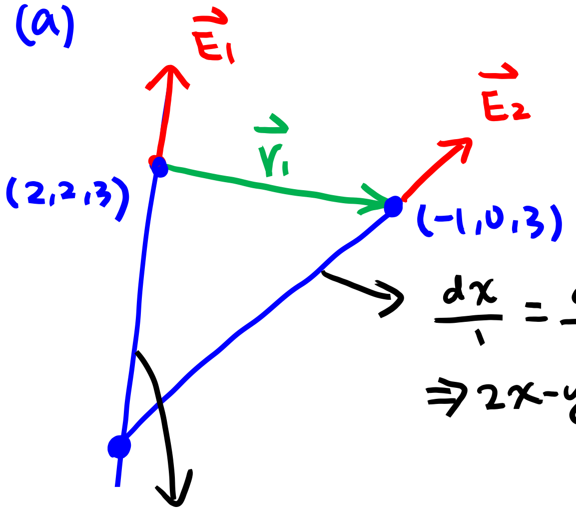




P1.31

- For each of the following pairs of electric field intensities, find, if possible, the location and the value of a point charge that produces both fields: (a) $\mathbf{E}_1 = (2\mathbf{a}_x + 2\mathbf{a}_y + \mathbf{a}_z)$ V/m at (2, 2, 3) and $\mathbf{E}_2 = (\mathbf{a}_x + 2\mathbf{a}_y + 2\mathbf{a}_z)$ V/m at (-1, 0, 3); and (b) $\mathbf{E}_1 = (2\mathbf{a}_x + 2\mathbf{a}_y + \mathbf{a}_z)$ V/m at (1, 1, 1) and $\mathbf{E}_2 = (2\mathbf{a}_x + \mathbf{a}_y + 2\mathbf{a}_z)$ V/m at (1, 2, 0)

(a)



$$(\vec{E}_1 \times \vec{E}_2) \cdot \vec{r}_1 = \begin{vmatrix} 3 & 2 & 0 \\ 2 & 2 & 1 \\ 1 & 2 & 2 \end{vmatrix} = 0$$

$$\Rightarrow \frac{dx}{1} = \frac{dy}{2} = \frac{dz}{2}$$

$$\Rightarrow 2x - y = -2,\ 2x - z = -5 \quad -①$$

$$\frac{dx}{2} = \frac{dy}{2} = \frac{dz}{1}$$

$$\Rightarrow x = y,\ x - 2z = -4 \quad -②$$

From ① and ②

Intersect at $(-2, -2, 1)$

$$\Rightarrow \vec{E}_1 = \frac{Q}{4\pi\varepsilon_0} \frac{4\vec{a}_x + 4\vec{a}_y + 2\vec{a}_z}{(16+16+4)^{1/2}} \rightarrow Q = 432\pi\varepsilon_0$$

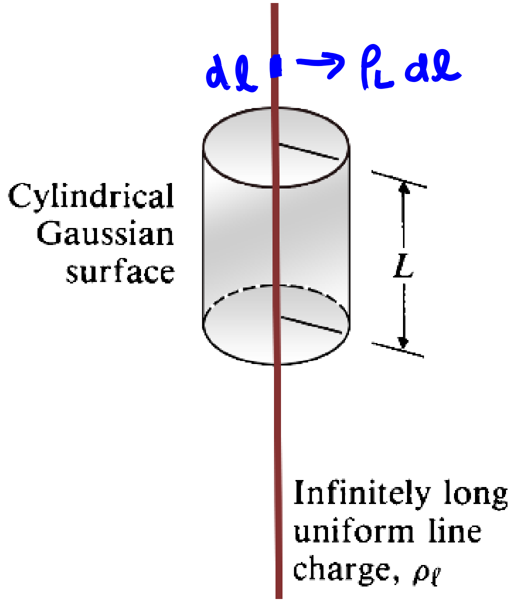
$$\vec{E}_2 \Rightarrow Q = 108\pi\varepsilon_0$$

$\Rightarrow$ No solution

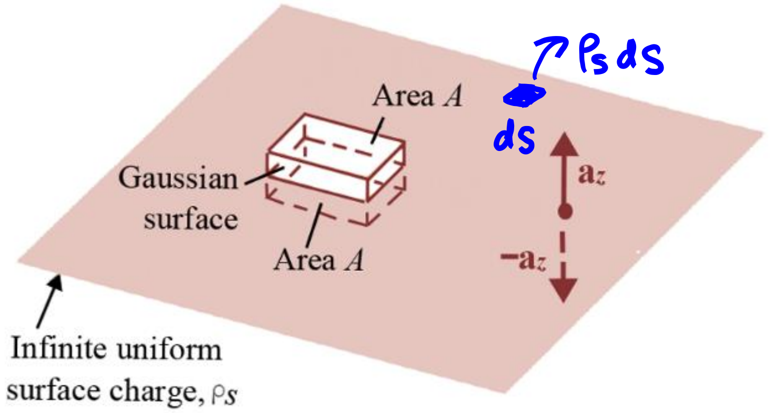
(b) $Q = -108\pi\varepsilon_0$ C at $(3,3,2)$

Continuous Charge Distribution

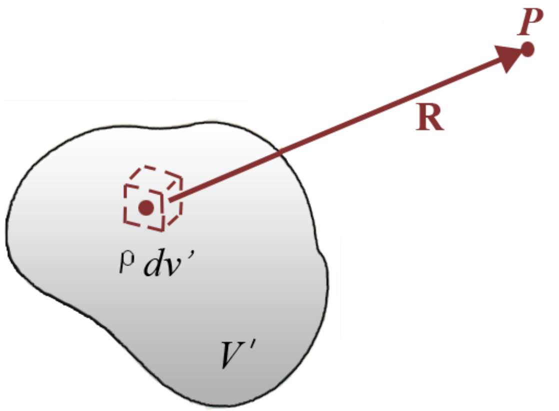
- Line charges
  - Like chalk powder along a thin line drawn on the blackboard.
  - Line charge density ρ_L (C/m).



- Surface charges
  - Like chalk powder on the erasing surface of a blackboard eraser.
  - Surface charge density ρ_S (C/m^2).



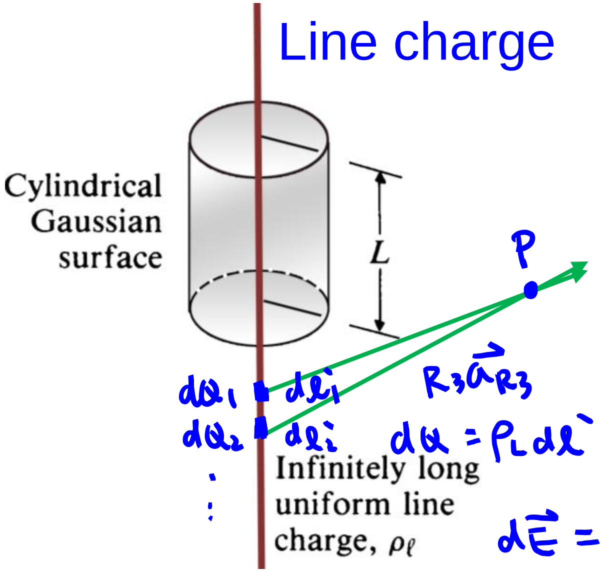
- Volume charges
  - Like chalk powder in the chalk itself.
  - Volume charge density ρ (C/m^3).



Electric Field by Continuous Charge Distribution

- Use superposition to find the total E field

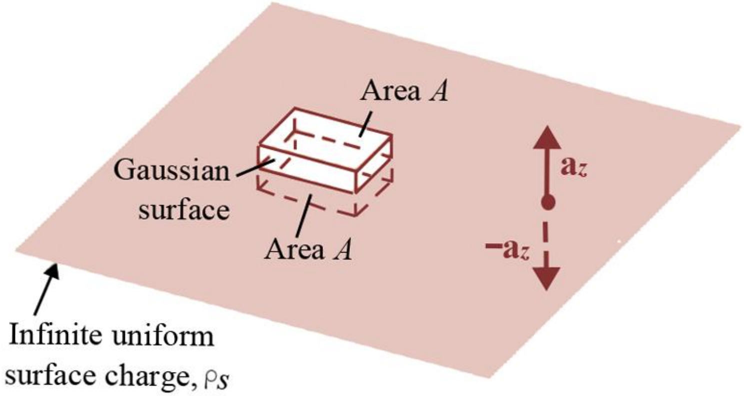
Line charge



$d\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{\rho_L dl'}{R^2}\vec{a}_R$

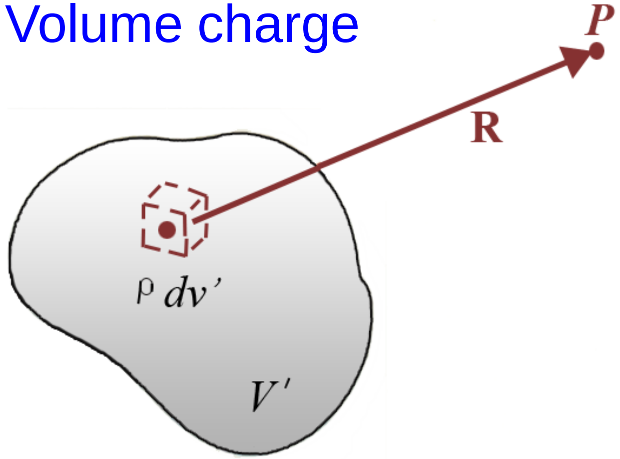
$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int_{L'} \mathbf{a}_R \frac{\rho_\ell}{R^2}\, d\ell'$$

Surface charge



$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int_{S'} \mathbf{a}_R \frac{\rho_s}{R^2}\, ds'$$

Volume charge



$$d\mathbf{E} = \mathbf{a}_R \frac{\rho\, dv'}{4\pi\epsilon_0 R^2}$$

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int_{V'} \mathbf{a}_R \frac{\rho}{R^2}\, dv'$$

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int_{V'} \rho \frac{\mathbf{R}}{R^3}\, dv'$$

$\vec{a}_R = \frac{\vec{R}}{R}$

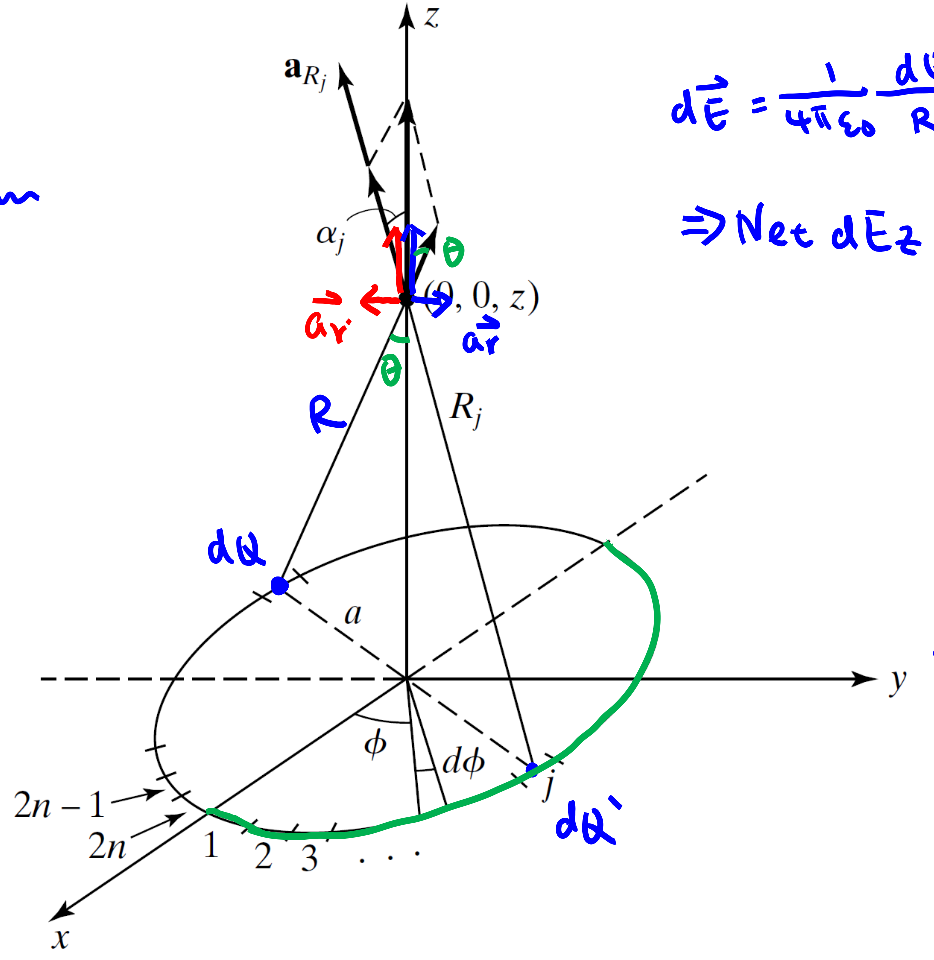
Example

- Charge Q C is distributed with uniform density along a circular ring of radius a lying in the xy-plane and having its center at the origin. Find the electric field intensity at a point on the z-axis.

① Check symmetric properties

② Choose coordinate system

→ Cylindrical



$$d\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{dQ}{R^2}\vec{a_R} = \frac{1}{4\pi\varepsilon_0}\frac{1}{R^2}\frac{Q}{2\pi a}\,a\,d\phi\,\vec{a_R}$$

$$\Rightarrow \text{Net } dE_z = \frac{2Qa}{4\pi\varepsilon_0 R^2 2\pi a}\cos\theta\,d\phi$$

$$= \frac{Q}{4\pi\varepsilon_0 R^2}\frac{z}{R}\,d\phi$$

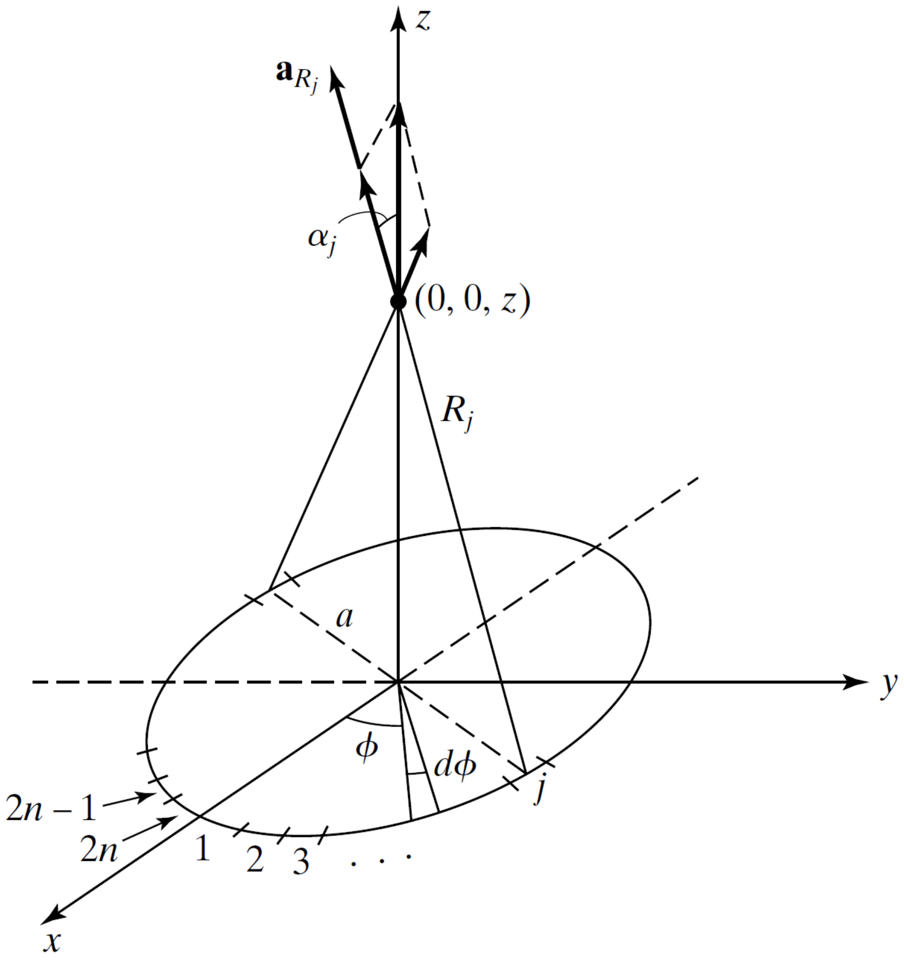
$$= \frac{Qz}{4\pi\varepsilon_0 R^3}\,d\phi$$

$$\therefore E_z = \frac{Qz}{4\pi\varepsilon_0 R^3}\int_0^{\pi} d\phi$$

$$= \frac{Qz}{4\pi\varepsilon_0}\frac{1}{(a^2+z^2)^{2/3}}$$

Example

- Use cylindrical coordinate system



$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int_{L'} \mathbf{a}_R \frac{\rho_\ell}{R^2}\, d\ell'$$

$$[E_z]_{(0,0,z)} = \int_{\phi=0}^{\pi} \underbrace{\frac{2(Q/2\pi a)a\, d\phi}{4\pi\varepsilon_0(a^2+z^2)}}_{} \overbrace{\frac{z}{(a^2+z^2)^{1/2}}}^{\cos\alpha}$$

dQ

$$= \frac{Qz}{4\pi^2\varepsilon_0(a^2+z^2)^{3/2}}\int_{\phi=0}^{\pi} d\phi$$

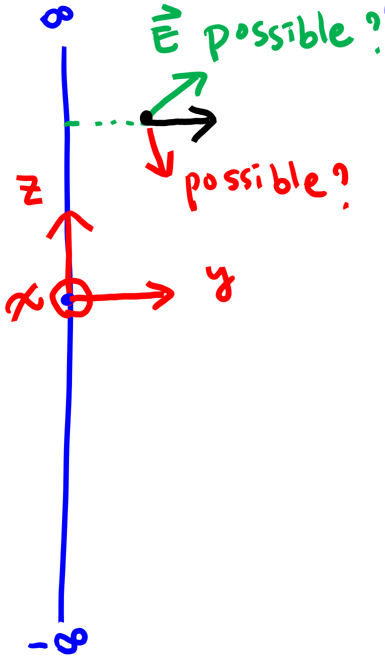
$$= \frac{Qz}{4\pi\varepsilon_0(a^2+z^2)^{3/2}}$$

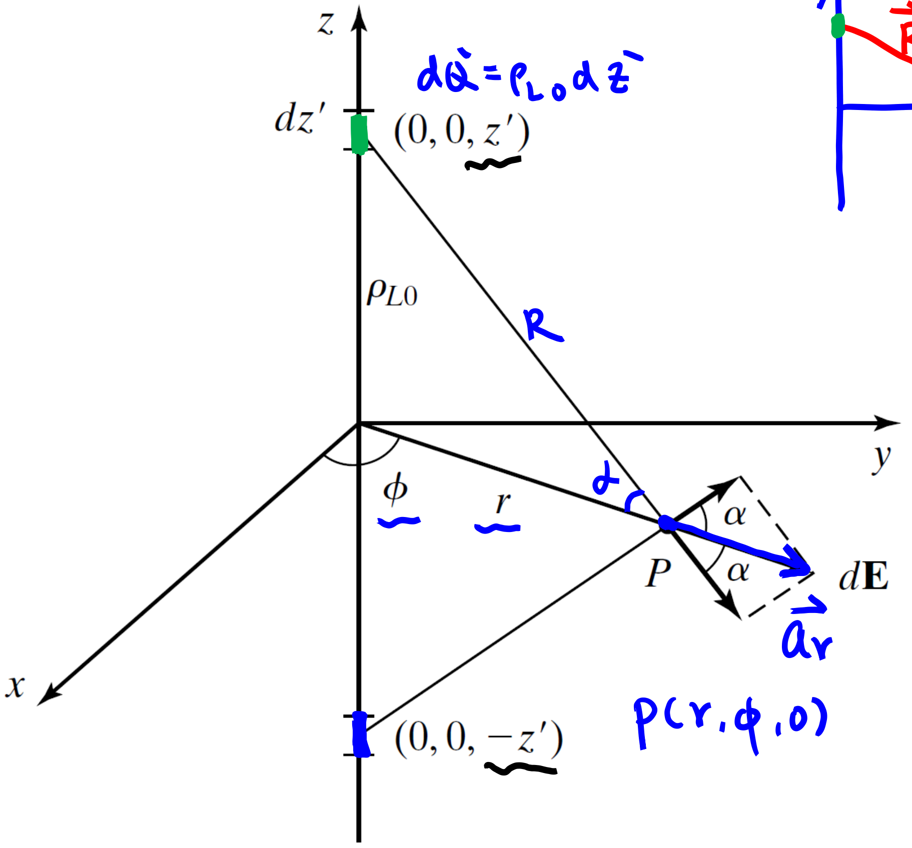
Example

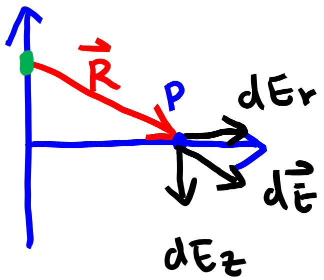
- An infinitely long line charge along the *z*-axis with uniform charge density ρ_{L0} (C/m) is shown below. Find the electric field intensity everywhere.

① Symmetry

② Coordinate system







$$\vec{R} = r\vec{a}_r - z'\vec{a}_z$$

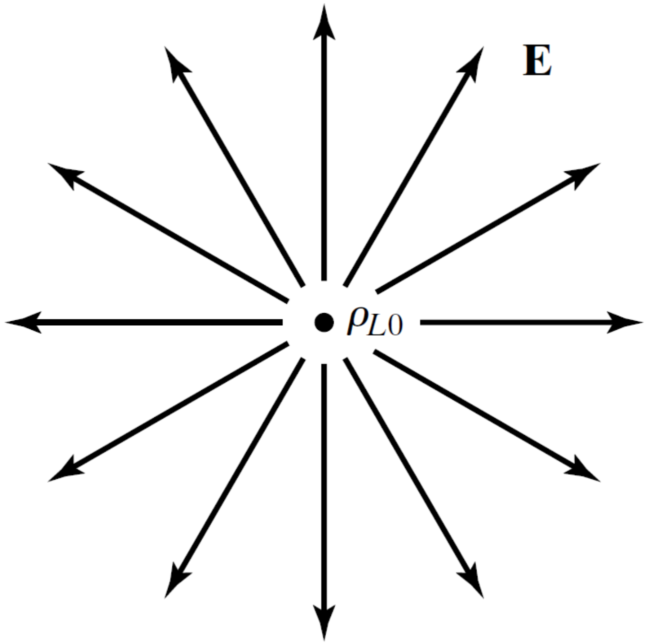
$$d\vec{E} = \frac{\rho_{L0}\,dz'}{4\pi\varepsilon_0}\,\frac{\vec{R}}{R^3} = \frac{\rho_{L0}\,dz'}{4\pi\varepsilon_0}\,\frac{r\vec{a}_r - z'\vec{a}_z}{(z'^2+r^2)^{3/2}} = dE_r\vec{a}_r + dE_z\vec{a}_z$$

$$\vec{R}' = r\vec{a}_r - (-z'\vec{a}_z) = r\vec{a}_r + z'\vec{a}_z$$

$$\Rightarrow dE_{z,net} = 2\,dE_r$$

Example

- Use cylindrical coordinate system



$$[d\mathbf{E}]_{(r,\,\phi,\,0)} = 2\frac{\rho_{L0}\,dz'}{4\pi\varepsilon_0[r^2 + (z')^2]}\cos\alpha\,\mathbf{a}_r$$

$$= \frac{\rho_{L0} r\,dz'}{2\pi\varepsilon_0[r^2 + (z')^2]^{3/2}}\mathbf{a}_r$$

$\cos\alpha = \frac{r}{R}$

$$[\mathbf{E}]_{(r,\,\phi,\,0)} = \int_{z'=0}^{\infty} [d\mathbf{E}]_{(r,\,\phi,\,0)}$$

$= \frac{\rho_{L0} r}{2\pi\varepsilon_0}\int_{z'=0}^{\infty}\frac{dz'}{[r^2+(z')^2]^{3/2}}\vec{a_r}$

$$= \int_{z'=0}^{\infty}\frac{\rho_{L0} r\,dz'}{2\pi\varepsilon_0[r^2 + (z')^2]^{3/2}}\mathbf{a}_r$$

Let $z' = r\tan a$

$dz' = r\sec^2 a\,da$

$z' = 0 \rightarrow a = 0$

$z' = \infty \rightarrow a = \frac{1}{2}\pi$

$\therefore = \frac{\rho_{L0} r}{2\pi\varepsilon_0}\int_0^{\frac{\pi}{2}}\frac{r\sec^2 a\,da}{[r^2+r^2\tan^2 a]^{3/2}}\vec{a_r}$

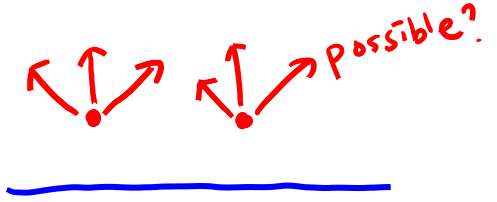
$= \frac{\rho_{L0} r}{2\pi\varepsilon_0}\frac{1}{r^2}\int\frac{da}{\sec a} = \frac{\rho_{L0}}{2\pi\varepsilon_0 r}\int_0^{\frac{\pi}{2}}\cos a\,da$

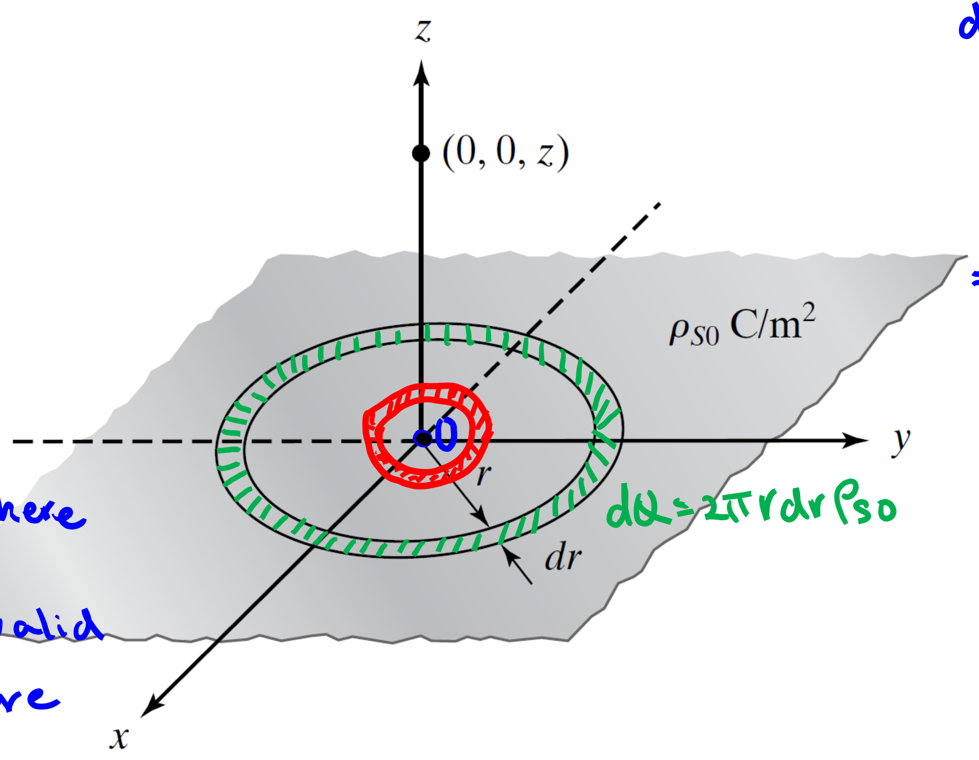
$$= \frac{\rho_{L0}}{2\pi\varepsilon_0 r}\int_{\alpha=0}^{\pi/2}\cos\alpha\,d\alpha$$

$$= \frac{\rho_{L0}}{2\pi\varepsilon_0 r}\mathbf{a}_r$$

Example

- An infinite plane sheet of charge in the *xy*-plane with uniform surface charge density ρ_{S0} (C/m^2) is shown below. Find the electric field intensity due to it everywhere.





$dQ = 2\pi r\,dr\,\rho_{s0}$

O can be everywhere on the plane

→ The result is valid everywhere

$$d\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Qz}{(r^2+z^2)^{3/2}}\vec{a}_z$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{2\pi r\,dr\,\rho_{s0} z}{(r^2+z^2)^{3/2}}\vec{a}_z$$

$$\Rightarrow \vec{E} = \frac{\rho_{s0} z}{2\varepsilon_0}\int_0^\infty \frac{r\,dr}{(r^2+z^2)^{3/2}}\vec{a}_z$$

$$= \frac{\rho_{s0} z}{2\varepsilon_0}(-1)\frac{1}{(r^2+z^2)^{1/2}}\Big|_0^\infty \vec{a}_z$$

$$= \frac{\rho_{s0} z}{2\varepsilon_0}\frac{1}{(z^2)^{1/2}}\vec{a}_z$$

If $z > 0$, $\vec{E} = \frac{\rho_{s0}}{2\varepsilon_0}\vec{a}_z$

If $z < 0$, $\vec{E} = \frac{\rho_{s0}}{2\varepsilon_0}(-\vec{a}_z)$

Example

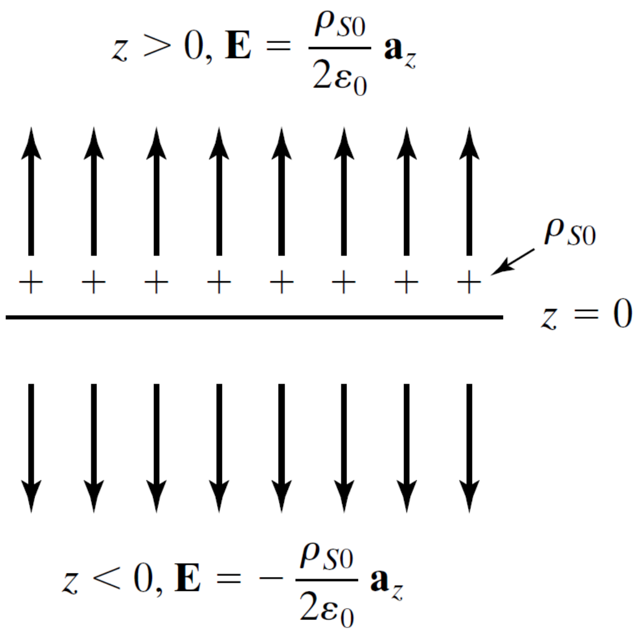
$$[d\mathbf{E}]_{(0,\,0,\,z)} = \frac{(\rho_{S0}2\pi r\,dr)z}{4\pi\varepsilon_0(r^2 + z^2)^{3/2}}\,\mathbf{a}_z$$

$$\begin{aligned}[\mathbf{E}]_{(0,\,0,\,z)} &= \int_{r=0}^{\infty}[d\mathbf{E}]_{(0,0,z)} \\ &= \int_{r=0}^{\infty}\frac{\rho_{S0} r z\,dr}{2\varepsilon_0(r^2 + z^2)^{3/2}}\,\mathbf{a}_z \\ &= \frac{\rho_{S0} z}{2\varepsilon_0}\left[-\frac{1}{\sqrt{r^2 + z^2}}\right]_{r=0}^{\infty}\mathbf{a}_z \\ &= \frac{\rho_{S0} z}{2\varepsilon_0|z|}\,\mathbf{a}_z\end{aligned}$$

$$\mathbf{E} = \pm\frac{\rho_{S0}}{2\varepsilon_0}\,\mathbf{a}_z \qquad \text{for} \quad z \gtrless 0$$

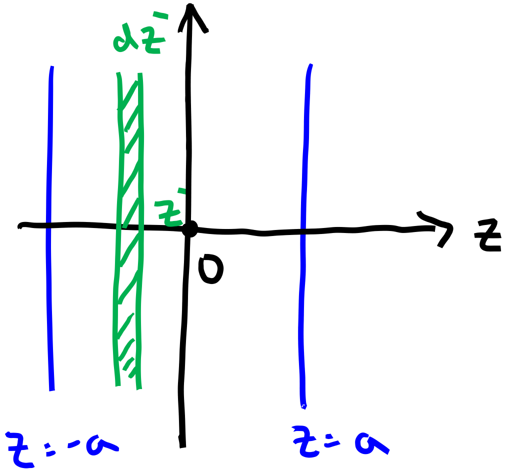
$$\mathbf{a}_n = \pm\mathbf{a}_z \qquad \text{for} \quad z \gtrless 0$$

$$\mathbf{E} = \frac{\rho_{S0}}{2\varepsilon_0}\mathbf{a}_n$$

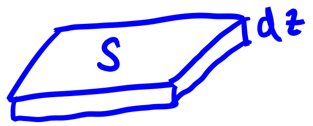


P1.38

- Consider the volume charge distributed uniformly with ρ_0 (C/m^3) density between the planes $z = -a$ and $z = a$. Using superposition in conjunction with the result of the last example, show that the electric field intensity due to the slab of charge is given by



$$\mathbf{E} = \begin{cases} -(\rho_0 a/\varepsilon_0)\mathbf{a}_z & \text{for} \quad z < -a \\ (\rho_0 z/\varepsilon_0)\mathbf{a}_z & \text{for} \quad -a < z < a \\ (\rho_0 a/\varepsilon_0)\mathbf{a}_z & \text{for} \quad z > a \end{cases}$$



dz very small

$Q_s \simeq Q_v$

$\rho_{s0} S = \rho_0 S \, dz$

$\Rightarrow \rho_{s0} = \rho_0 \, dz$

$z > z'$, $d\vec{E} = \dfrac{\rho_0 dz'}{2\varepsilon_0} \vec{a}_z$

$z < z'$, $d\vec{E} = \dfrac{\rho_0 dz'}{2\varepsilon_0} (-\vec{a}_z)$

$-a \leq z' \leq a$

$\Rightarrow$ ① $z < -a$, $\vec{E} = \displaystyle\int_{-a}^{a} \frac{\rho_0 dz'}{2\varepsilon_0} (-\vec{a}_z) = \frac{-\rho_0 a}{\varepsilon_0} \vec{a}_z$

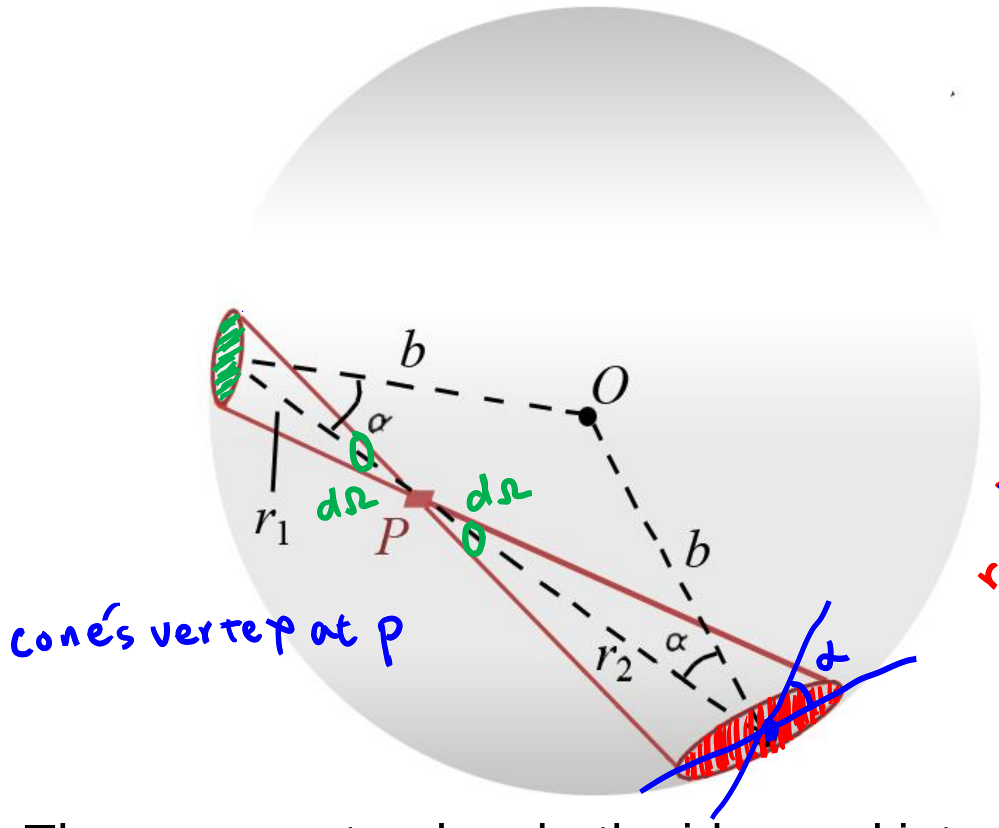
② $z > a$, $\vec{E} = \displaystyle\int_{-a}^{a} \frac{\rho_0 dz'}{2\varepsilon_0} (\vec{a}_z) = \frac{\rho_0 a}{\varepsilon_0} \vec{a}_z$

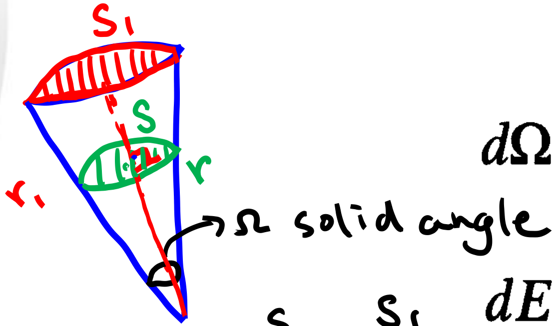
③ $-a < z < a$, $\vec{E} = \displaystyle\int_{-a}^{z} \frac{\rho_0 dz'}{2\varepsilon_0} \vec{a}_z + \int_{z}^{a} \frac{\rho_0 dz'}{2\varepsilon_0} (-\vec{a}_z)$
$= \dfrac{\rho_0 z}{\varepsilon_0} \vec{a}_z$

Example

- A total charge Q is put on a thin spherical shell of radius b. Determine the electric field intensity at an arbitrary point inside the shell.

No charge inside the shell
→ E= 0 according to
Gauss' law





$$\Omega = \frac{S}{r^2} = \frac{S_1}{r_1^2}$$

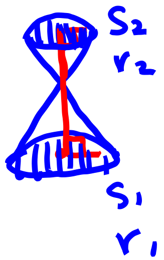
The cones extend on both sides and intersect the shell in areas of ds_1 and ds_2

$$\rho_s = \frac{Q}{4\pi b^2}$$

$$dE = \frac{\rho_s}{4\pi\epsilon_0}\left(\frac{ds_1}{r_1^2} - \frac{ds_2}{r_2^2}\right)$$

$$d\Omega = \frac{ds_1}{r_1^2}\cos\alpha = \frac{ds_2}{r_2^2}\cos\alpha$$

$$dE = \frac{\rho_s}{4\pi\epsilon_0}\left(\frac{d\Omega}{\cos\alpha} - \frac{d\Omega}{\cos\alpha}\right) = 0$$



$$\frac{S_1^2}{r_1} = \frac{S_2^2}{r_2}$$

If Coulomb's law not ∝ $1/r^2$, dE is not zero.

Outline

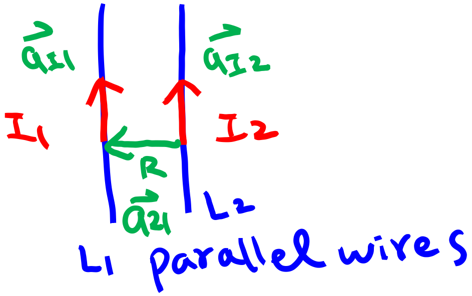
- Vector algebra
- Cartesian coordinate system
- Cylindrical coordinate system
- Spherical coordinate system
- Scalar and vector fields
- Electric field
- Magnetic field
- Lorentz force equation

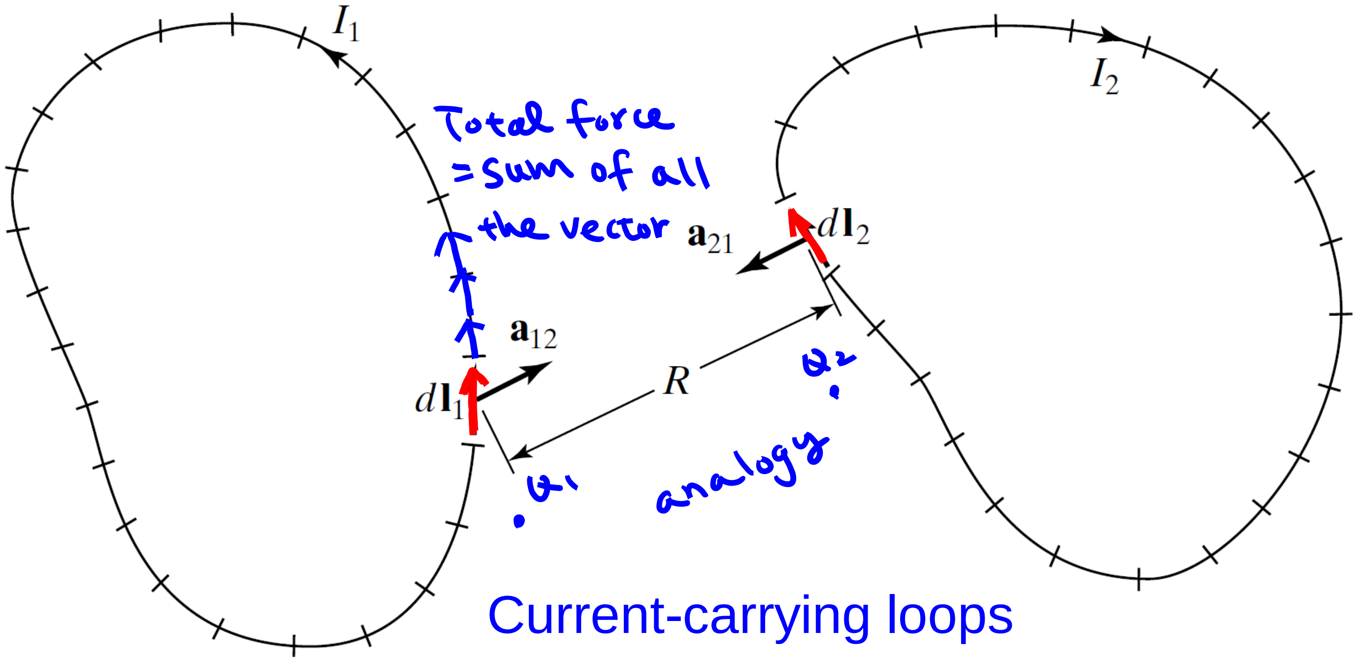
Ampère's Experimental law of force

- Experimental law ---> We do not need to prove it!
- $|\mathbf{F}| \propto |I_1||I_2|$, $|dI_1||dI_2|$, $1/R^2$
- Different $|\mathbf{F}|$ in different medium
- $d\mathbf{F}_1$'s direction along dI_1x (dI_2x $\boldsymbol{a}_{21}$)
- $d\mathbf{F}_2$'s direction along dI_2x (dI_1x $\boldsymbol{a}_{12}$)

$$d\mathbf{F}_1 = I_1\, d\mathbf{l}_1 \times \left(\frac{k I_2\, d\mathbf{l}_2 \times \mathbf{a}_{21}}{R^2} \right)$$

$$d\mathbf{F}_2 = I_2\, d\mathbf{l}_2 \times \left(\frac{k I_1\, d\mathbf{l}_1 \times \mathbf{a}_{12}}{R^2} \right)$$





Current-carrying loops

Ampère's Experimental law of force

- Magnetic forces with two loops of wire carrying currents
- *k* proportional constant
  - Equal to $\mu_0/4\pi$ in free space.
  - Depending on the medium. $\rightarrow \mu = \mu_r \mu_0$
  - μ_0 is called permeability, equal to $4\pi\times10^{-7}$ N/A^2 (= $\underbrace{\text{H}}_{\text{henrys}}$/m) in free space.

$I_2 d\mathbf{l}_2 = I_2 dy\mathbf{a}_y$ at (0,1,0)

$$d\mathbf{F}_1 = I_1\, d\mathbf{l}_1 \times \left(\frac{kI_2\, d\mathbf{l}_2 \times \mathbf{a}_{21}}{R^2}\right)$$

$$d\mathbf{F}_2 = I_2\, d\mathbf{l}_2 \times \left(\frac{kI_1\, d\mathbf{l}_1 \times \mathbf{a}_{12}}{R^2}\right)$$

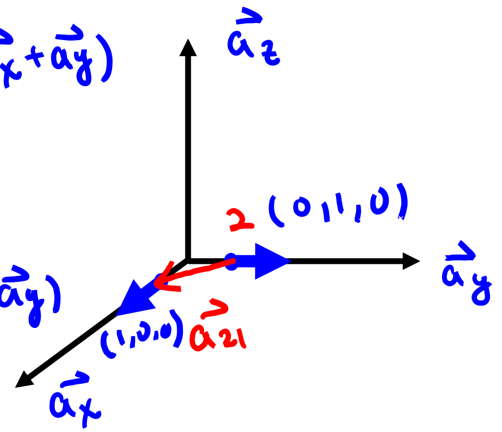
$\vec{a}_{12} = -\vec{a}_{21} = \frac{1}{\sqrt{2}}(-\vec{a}_x + \vec{a}_y)$

$R = \sqrt{2}$

$I_2 d\vec{l}_2 \times \vec{a}_{21}$

$= I_2 dy\, \vec{a}_y \times \frac{1}{\sqrt{2}}(\vec{a}_x - \vec{a}_y)$

$= \frac{I_2 dy}{\sqrt{2}}(-\vec{a}_z)$



$= k\frac{I_1 I_2 dx dy}{2\sqrt{2}}\vec{a}_y$

$I_1 d\mathbf{l}_1 = I_1 dx\mathbf{a}_x$ at (1,0,0)

$\therefore d\vec{F}_1$

$= k\frac{I_1 I_2 dx dy}{\sqrt{2}}\vec{a}_x \times (-\vec{a}_z) = \frac{I_1 I_2 dx dy}{\sqrt{2}}\vec{a}_y$

$k = \frac{\mu_0}{4\pi}$

Magnetic Flux Density Vector B

- Each current element acted on by a field which is due to the other current element
- Called magnetic field
  - Characterized by the magnetic flux density vector **B**.
  - [**B**]= N/A-m= Wb/m^2 (T: tesla).

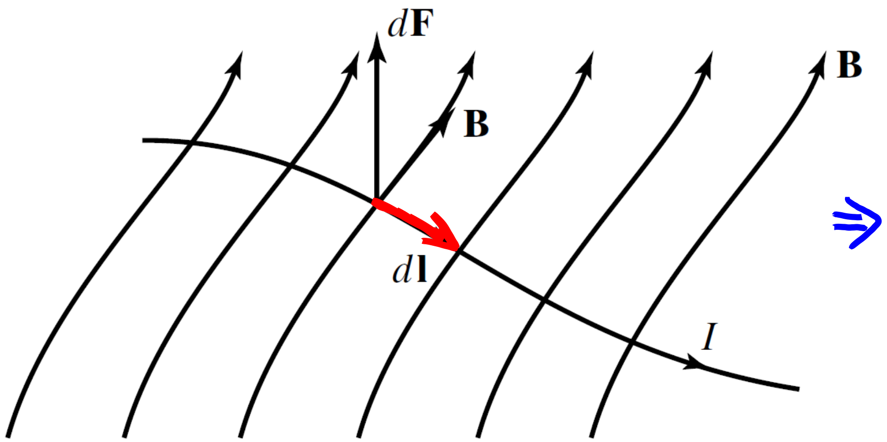
$$d\mathbf{F}_1 = I_1\, d\mathbf{l}_1 \times \left(\frac{kI_2\, d\mathbf{l}_2 \times \mathbf{a}_{21}}{R^2}\right) = I_1\, d\mathbf{l}_1 \times \mathbf{B}_2$$

$$d\mathbf{F}_2 = I_2\, d\mathbf{l}_2 \times \left(\frac{kI_1\, d\mathbf{l}_1 \times \mathbf{a}_{12}}{R^2}\right) = I_2\, d\mathbf{l}_2 \times \mathbf{B}_1$$

I_2 $d\vec{l}_2$ $\vec{B}_2$

I_1 $d\vec{l}_1$ $\vec{B}_1$

$$d\vec{F}_3 = I_3 d\vec{l}_3 \times \left(\frac{kI_1 d\vec{l}_1 \times \vec{a}_{31}}{R^2}\right)$$

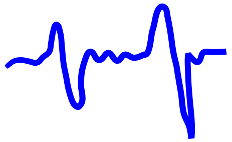


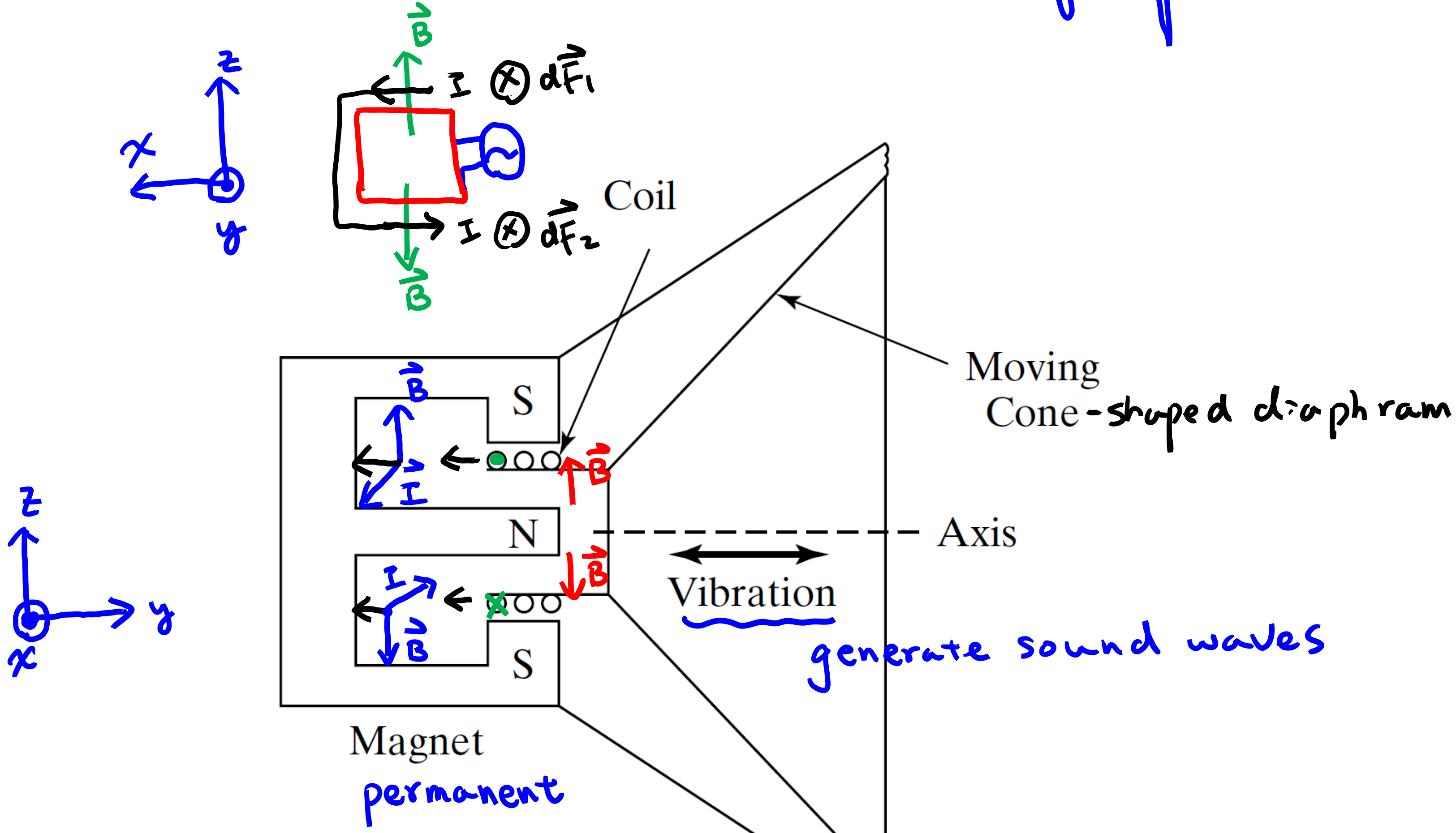
$$\Rightarrow d\mathbf{F} = I\, d\mathbf{l} \times \mathbf{B}$$

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{I d\vec{l} \times \vec{a}_R}{R^2}$$

Loudspeaker

- Current varies in accordance with the audio signal





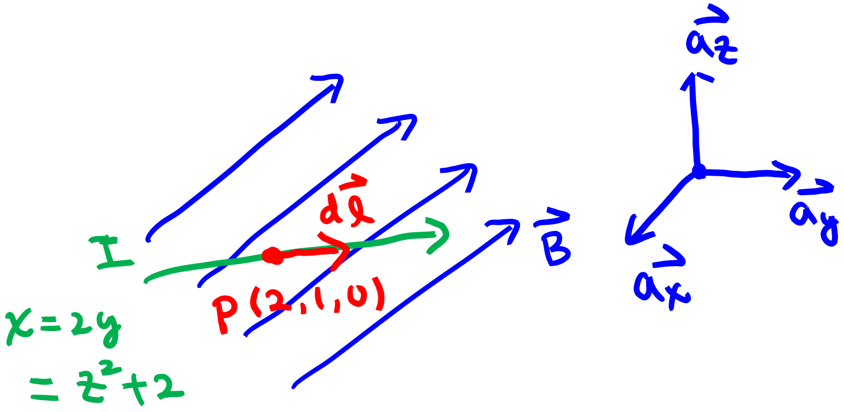
D1.18

- A current I flows in a wire along the curve $x= 2y= z^2+2$ and in the direction of increasing z. If the wire is situated in a magnetic field $\mathbf{B}= (y\mathbf{a}_x - x\mathbf{a}_y)/(x^2+y^2)$, find the magnetic force acting on an infinitesimal length of the wire having the projection dz on the z-axis at each of the following points: (a) (2, 1, 0); (b) (3, 1.5, 1); and (c) (6, 3, 2).

$x = 2y = z^2 + 2$

$\Rightarrow dx = 2dy = 2z\,dz$

$\therefore d\vec{l} = (2z\vec{a}_x + z\vec{a}_y + \vec{a}_z)\,dz$



(a) $P = P(2,1,0)$

$\Rightarrow d\vec{l} = dz\,\vec{a}_z$

$\vec{B} = \dfrac{\vec{a}_x - 2\vec{a}_y}{5}$

$d\vec{F} = I\,d\vec{l} \times \vec{B}$

$= I\,dz\,\vec{a}_z \times \left(\dfrac{\vec{a}_x - 2\vec{a}_y}{5}\right)$

$= \dfrac{I\,dz}{5}(\vec{a}_y + 2\vec{a}_x)$

(b) $\dfrac{I\,dz}{9.5}(2\vec{a}_x + \vec{a}_y - 5\vec{a}_z)$

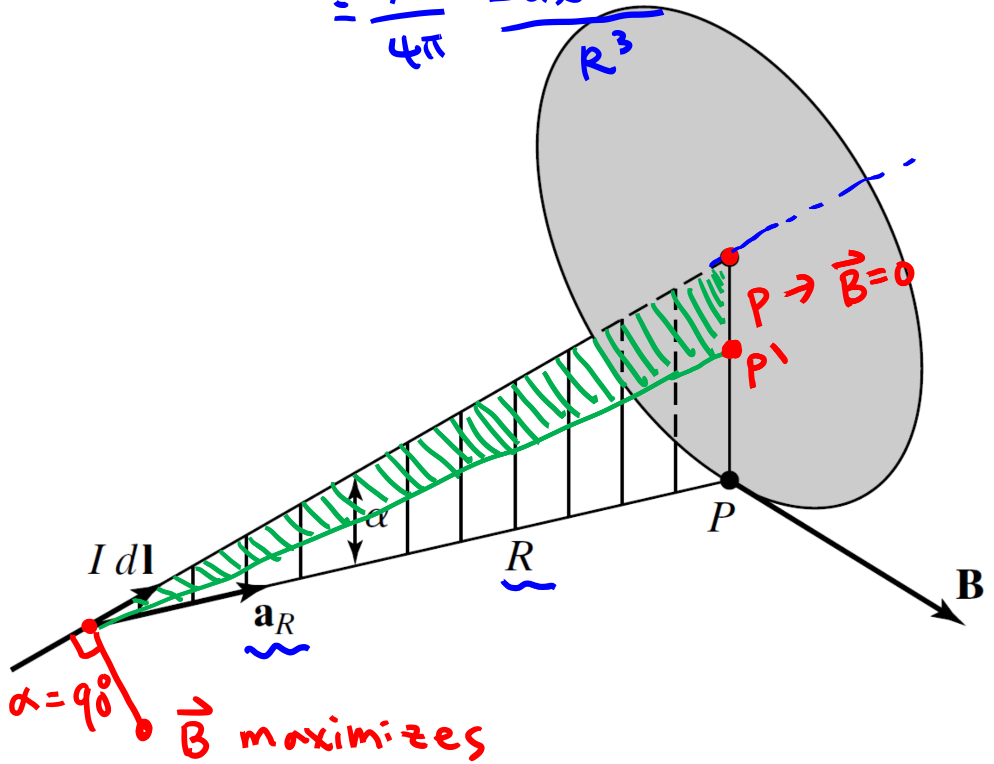
(c) $\dfrac{I\,dz}{15}(2\vec{a}_x + \vec{a}_y - 10\vec{a}_z)$

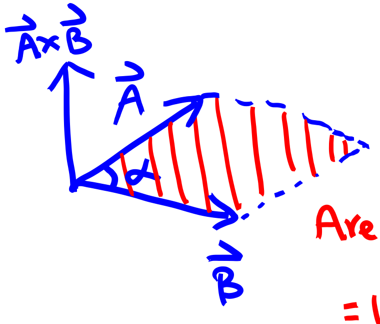
Biot-Savart Law

- Magnetic flux density due to an infinitesimal current element of length *d***l** and carrying current *I*

$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{I\,d\mathbf{l}\times\mathbf{a}_R}{R^2} \Rightarrow \vec{B} = \frac{\mu_0}{4\pi}\oint_c \frac{d\vec{l}'\times\vec{a}_R}{R^2}$$

$$= \frac{\mu_0}{4\pi}\frac{I\,d\vec{l}\times\vec{R}}{R^3}$$





$\text{Area} = |\vec{A}\times\vec{B}|$

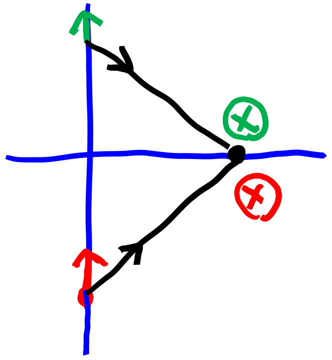
$= |\vec{A}||\vec{B}|\sin\alpha$

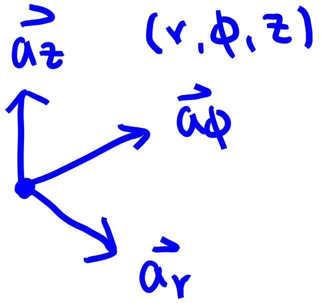
Example

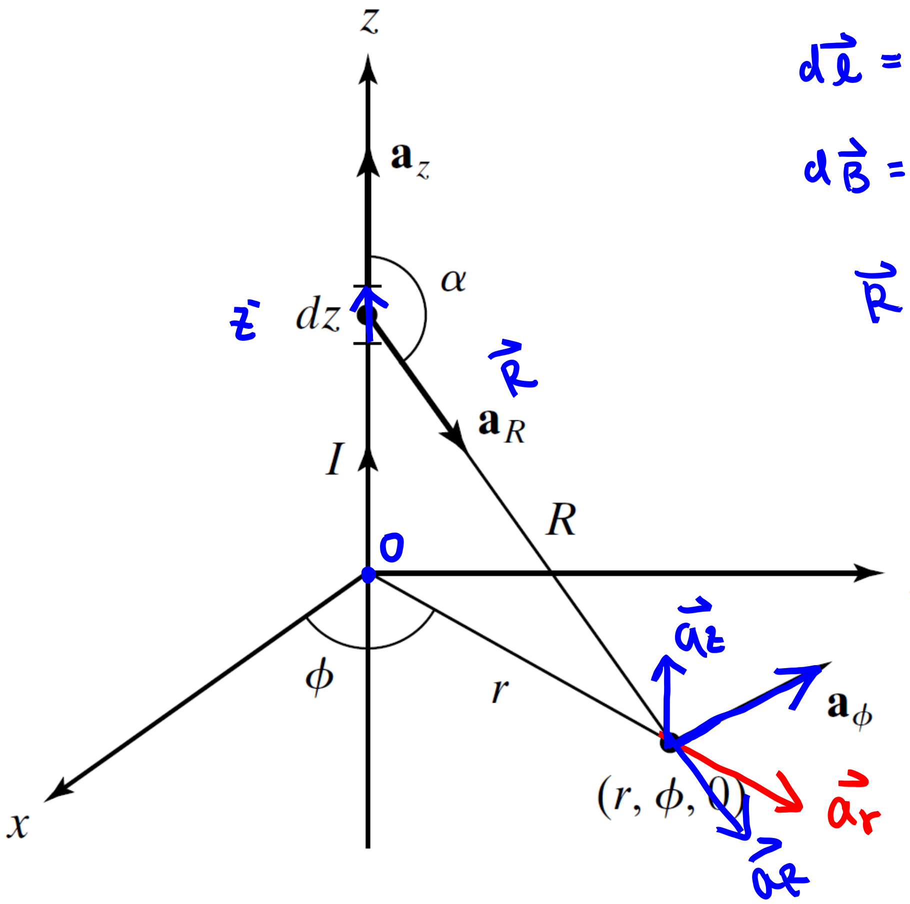
- An infinitely long, straight wire situated along the *z*-axis and carrying current *I* A in the +z direction is shown below. Find the magnetic flux density everywhere.

① symmetry

② Choose coordinate system







$d\vec{l} = dz\,\vec{a}_z$

$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I\,dz}{R^3}\,\vec{a}_z \times \vec{a}_R$

$\vec{R} = r\vec{a}_r - z'\vec{a}_z$

$\Rightarrow \vec{a}_z \times \vec{a}_R$

$= \vec{a}_z \times (r\vec{a}_r - z'\vec{a}_z)$

$= r\,\vec{a}_z \times \vec{a}_r$

$= r\,\vec{a}_\phi$

$\therefore d\vec{B} = \frac{\mu_0}{4\pi}\frac{I\,dz'}{R^3}\,r\,\vec{a}_\phi$

$\Rightarrow \vec{B} = \frac{\vec{a}_\phi \mu_0 I}{4\pi}\int_{-\infty}^{\infty}\frac{r\,dz'}{(z'^2+r^2)^{3/2}}$

Example

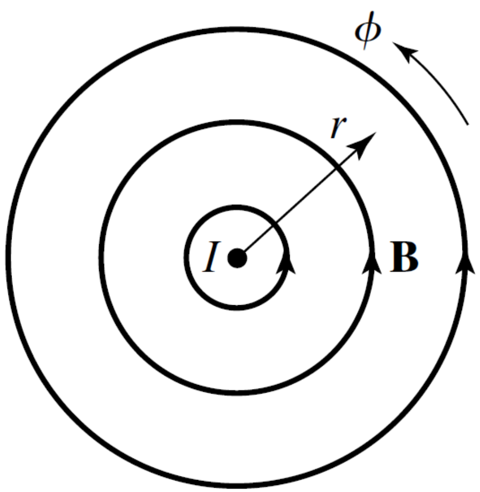
- Use cylindrical coordinate system

$$
\begin{aligned}
[d\mathbf{B}]_{(r,\phi,0)} &= \frac{\mu_0}{4\pi}\frac{I\,dz\,\mathbf{a}_z \times \mathbf{a}_R}{R^2} \\
&= \frac{\mu_0 I\,dz}{4\pi}\frac{\sin\alpha}{R^2}\mathbf{a}_\phi \\
&= \frac{\mu_0 I\,dz}{4\pi}\frac{r}{R^3}\mathbf{a}_\phi \\
&= \frac{\mu_0 Ir\,dz}{4\pi(z^2 + r^2)^{3/2}}\mathbf{a}_\phi
\end{aligned}
$$

$$
\begin{aligned}
[\mathbf{B}]_{(r,\phi,0)} &= \int_{z=-\infty}^{\infty} d\mathbf{B} \\
&= \int_{z=-\infty}^{\infty} \frac{\mu_0 Ir}{4\pi(z^2 + r^2)^{3/2}}dz\,\mathbf{a}_\phi \\
&= \frac{\mu_0 Ir}{4\pi}\left[\frac{z}{r^2\sqrt{z^2 + r^2}}\right]_{z=-\infty}^{\infty}\mathbf{a}_\phi \\
&= \frac{\mu_0 I}{2\pi r}\mathbf{a}_\phi
\end{aligned}
$$

Let $z = r\tan\alpha$

⋮

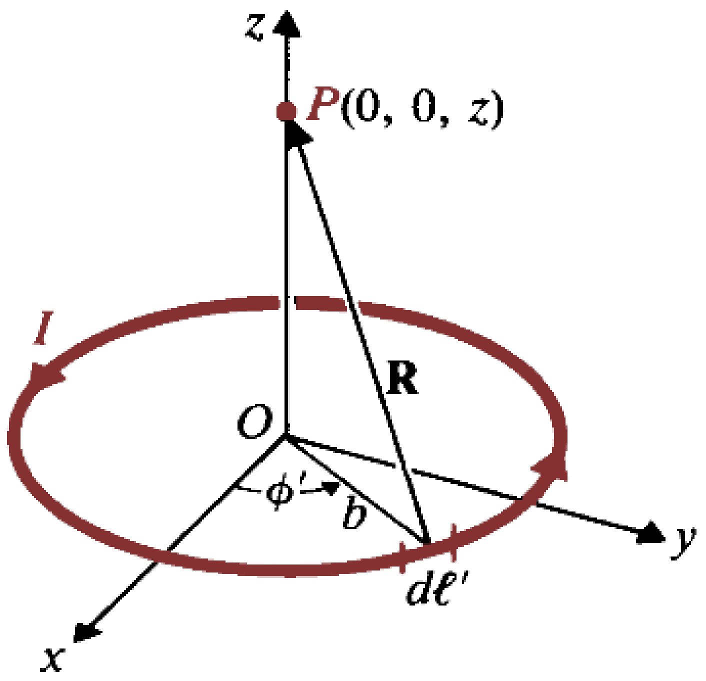


$\mathbf{B} = \frac{\mu_0 I}{2\pi r}\mathbf{a}_\phi$

z out of page

Example

- Find the magnetic flux density at a point on the axis of a circular loop of radius *b* carries a direct current *I*.



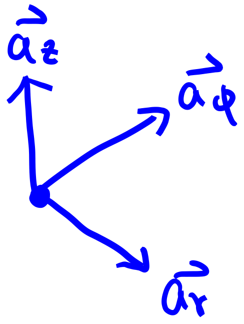
Example

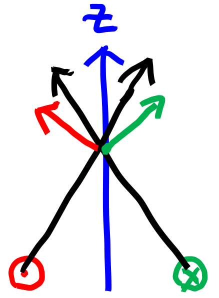
- Apply Biot-Savart law

$$d\boldsymbol{\ell}' = \mathbf{a}_\phi b\, d\phi'$$

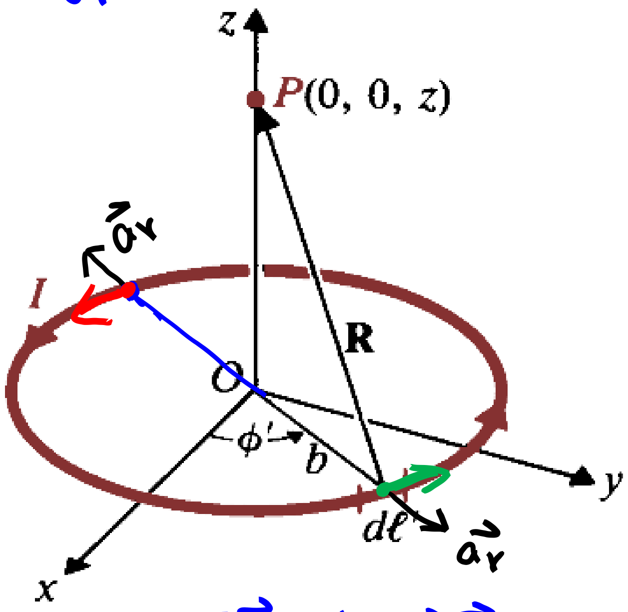
$$\mathbf{R} = \mathbf{a}_z z - \mathbf{a}_r b$$

$$R = (z^2 + b^2)^{1/2}$$





  – **R** is the vector from the source element *dl'* to the field point *P*.



$$d\boldsymbol{\ell}' \times \mathbf{R} = \mathbf{a}_\phi b\, d\phi' \times (\mathbf{a}_z z - \mathbf{a}_r b)$$

$$= \cancel{\mathbf{a}_r bz\, d\phi'} + \mathbf{a}_z b^2\, d\phi'$$

$\vec{dl'} = b\,d\phi'\,\vec{a_\phi}$

$R = \sqrt{b^2 + z^2}$

$\vec{R} = z\vec{a_z} - b\vec{a_r}$

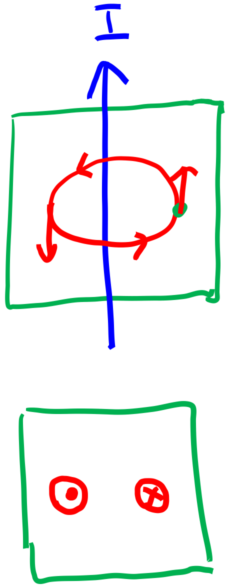
  – $\mathbf{a}_r$-component is canceled due to the symmetry.

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \mathbf{a}_z \frac{b^2\, d\phi'}{(z^2 + b^2)^{3/2}}$$

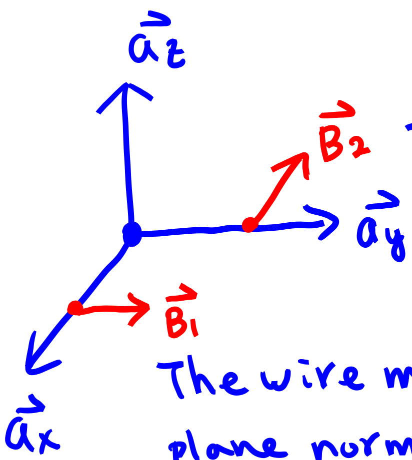
$$\mathbf{B} = \mathbf{a}_z \frac{\mu_0 I b^2}{2(z^2 + b^2)^{3/2}} \qquad \text{(T)}$$

P1.41

- For each of the following pairs of magnetic flux densities, find, if possible, the orientation of an infinitely long filamentary wire and the current in it required to produce
both fields: (a) $\mathbf{B}_1 = 10^{-7}\mathbf{a}_y$ Wb/m^2 at (3, 0, 0) and $\mathbf{B}_2 = -10^{-7}\mathbf{a}_x$ Wb/m^2 at (0, 4, 0) ; and (b) $\mathbf{B}_1 = 10^{-7}(\mathbf{a}_y - \mathbf{a}_z)$ Wb/m^2 at ($\sqrt{2}$, 0, 0, 0) and $\mathbf{B}_2 = -10^{-7}\mathbf{a}_x$ Wb/m^2 at (0, $\sqrt{2}$, 0).



①



$\vec B_2$ → x=0 plane

The wire must be on a plane normal to $\vec B_1$

⇒ y=0 plane

x=0 plane
y=0 plane } ⇒ Intersect along z-axis

$\vec B = \frac{\mu_0 I}{2\pi r}\vec a_\phi$

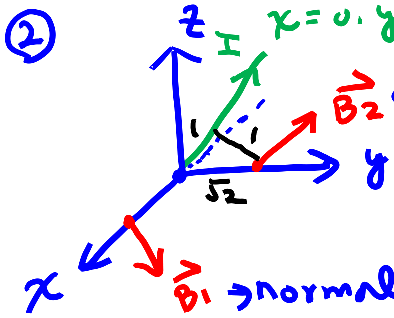
⇒ $\vec B_1 = \frac{\mu_0 I}{2\pi\, 3}\vec a_y = 10^{-7}\vec a_y$ → I = 1.5A

$\vec B_2 = \frac{\mu_0}{2\pi}\frac{I}{4}(-\vec a_x) = -10^{-7}\vec a_x$

→ I = 2A

∴ no solution

②



$\vec B_2$ → x=0 plane

$\vec B_1$ → normal to $\vec B_1$

→ y = z + c

→ c = 0, y = z

$\vec B_1 = \frac{\mu_0 I}{2\pi\sqrt{2}}\left(\frac{\vec a_y - \vec a_z}{\sqrt{2}}\right) = \frac{10^{-7}}{2}(\vec a_y - \vec a_z)$

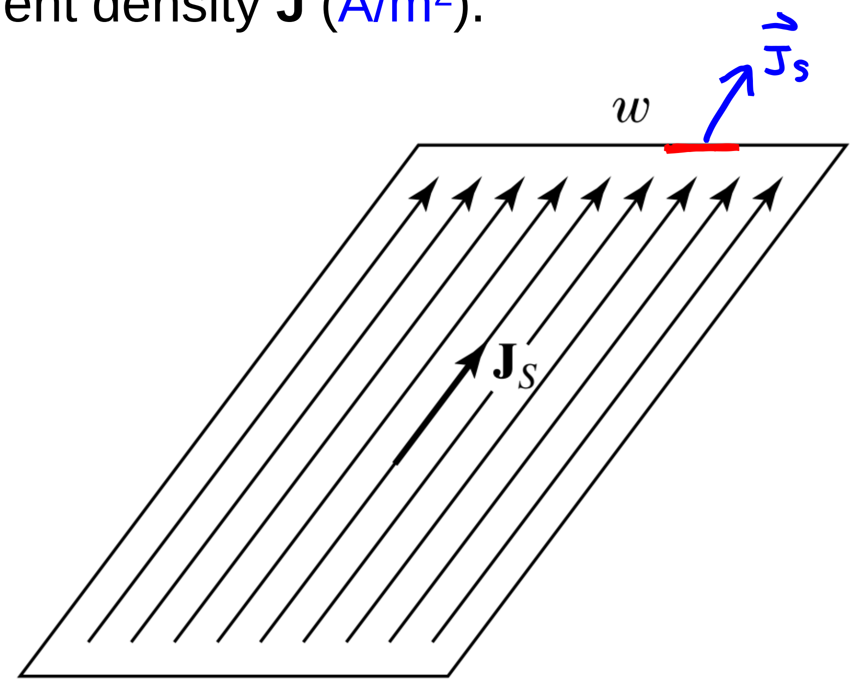
→ I = 0.5 A

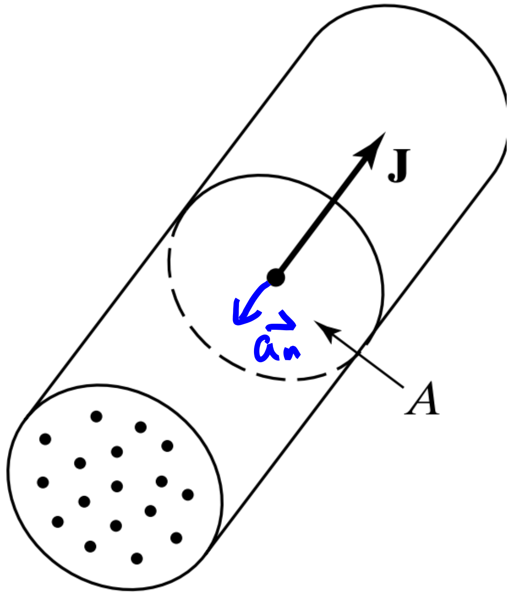
$\vec B_2 = \frac{\mu_0 I}{2\pi\, 1}(-\vec a_x) = -10^{-7}\vec a_x$

→ I = 0.5 A

Current Distribution

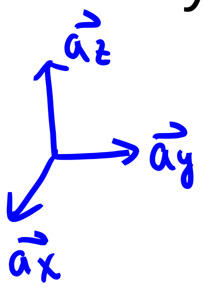
- Surface current
  - Like rainwater flowing down a smooth wall.
  - Surface current density $\mathbf{J_S}$ (A/m).
  - Vector!
- Volume current
  - Like rainwater flowing down a gutter downspout.
  - Volume current density $\mathbf{J}$ (A/m^2).
  - Vector!





Example

- An infinite plane sheet of current in the *xz*-plane with uniform surface current density $\mathbf{J}_S = \mathbf{J}_{S0}\mathbf{a}_z$ A/m is shown below. Find the magnetic flux density everywhere.



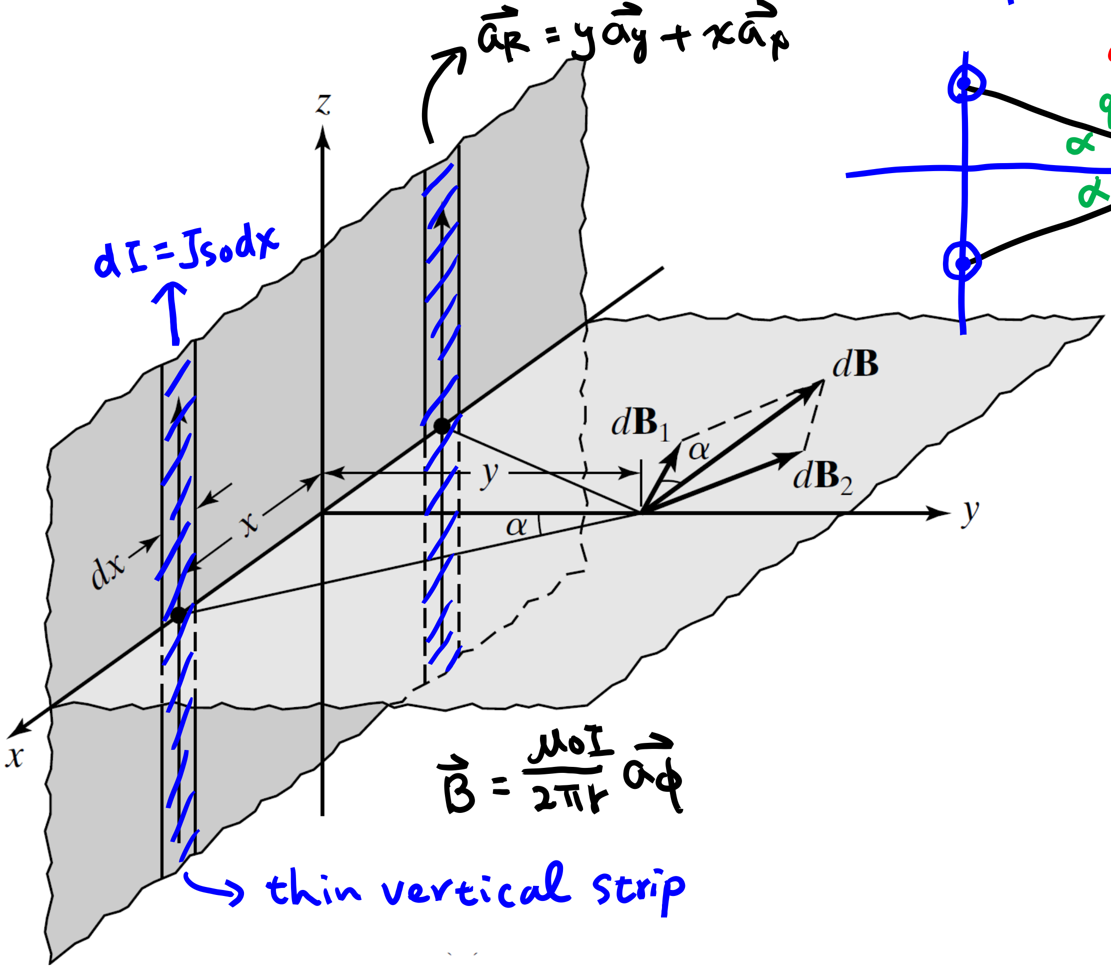
$d\vec{l} = dz\,\vec{a}_z$

$\vec{a}_R = y\vec{a}_y - x\vec{a}_x$

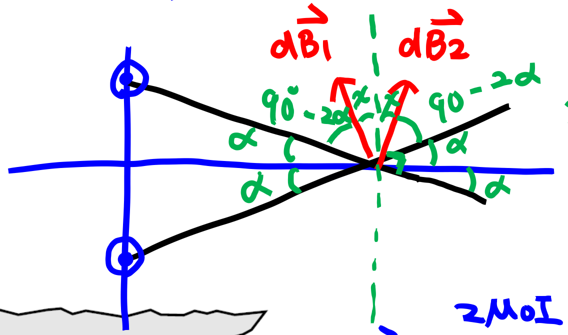
$\rightarrow d\vec{l} \times \vec{a}_R$

$= dz\,\vec{a}_z \times (y\vec{a}_y - x\vec{a}_x)$

$= dz(-\vec{a}_x) - x\,dz\,\vec{a}_y$



Top view



$2(90 - 2\alpha) + 2x = 180$

$\therefore x = \alpha$

$d\vec{B} = \frac{2\mu_0 I}{2\pi R}(-\vec{a}_x)\cos\alpha$

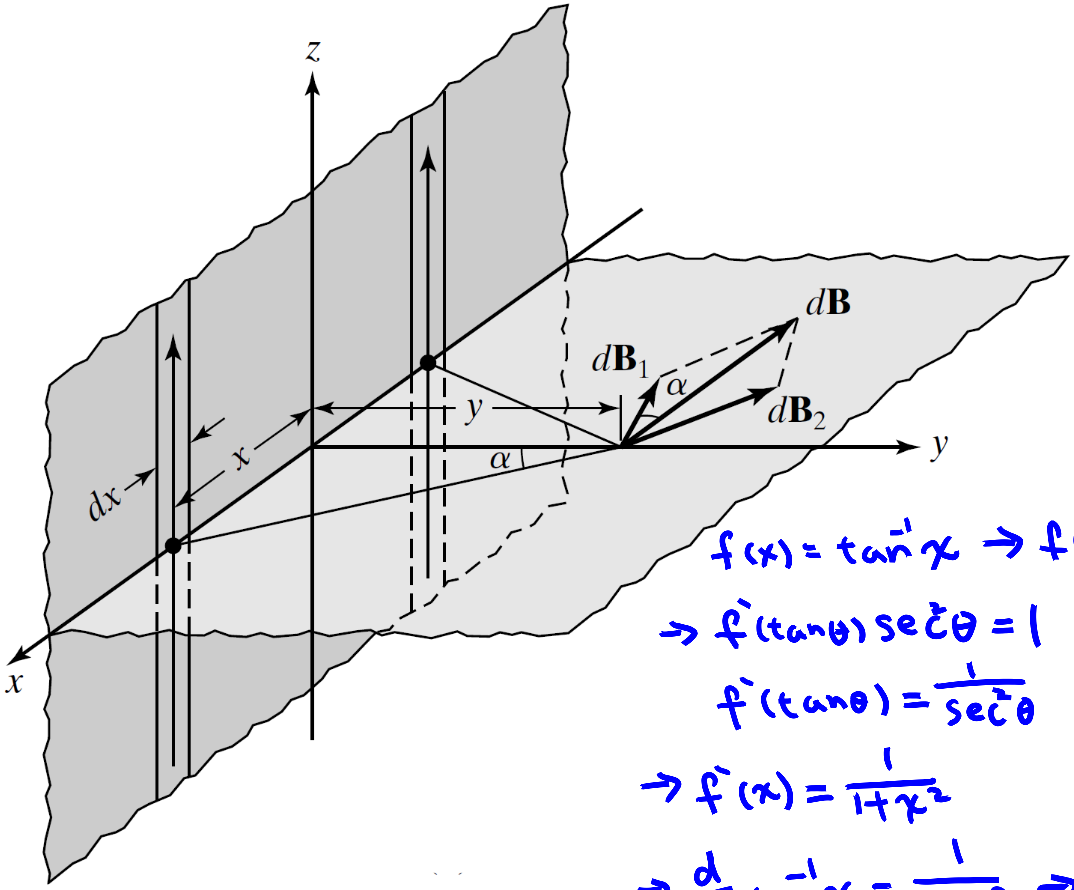
$= \frac{\mu_0 I}{\pi R}\cos\alpha(-\vec{a}_x)$

$= \frac{\mu_0 I}{\pi\sqrt{x^2+y^2}}\,\frac{y(-\vec{a}_x)}{\sqrt{x^2+y^2}}$

$dI = J_{s0}\,dx$

$\Rightarrow d\vec{B} = \frac{\mu_0 J_{s0}\, y}{\pi(x^2+y^2)}\,dx\,(-\vec{a}_x)$

Example



$$d\mathbf{B} = d\mathbf{B}_1 + d\mathbf{B}_2$$

$$= -2\,dB_1 \cos\alpha\, \mathbf{a}_x$$

$$= -2\frac{\mu_0 J_{S0}\, dx}{2\pi\sqrt{x^2+y^2}}\frac{y}{\sqrt{x^2+y^2}}\mathbf{a}_x$$

$$= -\frac{\mu_0 J_{S0} y\, dx}{\pi(x^2+y^2)}\mathbf{a}_x$$

$$[\mathbf{B}]_{(0,\,y,\,0)} = \int_{x=0}^{\infty} d\mathbf{B}$$

$$(-\vec{a}_x)\frac{\mu_0 J_{S0}}{\pi}\int_{-\infty}^{\infty}\frac{y}{x^2+y^2}dx$$

$$= -\int_{x=0}^{\infty}\frac{\mu_0 J_{S0} y}{\pi(x^2+y^2)}dx\,\mathbf{a}_x$$

$$= -\frac{\mu_0 J_{S0} y}{\pi}\left[\frac{1}{y}\tan^{-1}\frac{x}{y}\right]_{x=0}^{\infty}\mathbf{a}_x$$

$$= -\frac{\mu_0 J_{S0}}{2}\mathbf{a}_x \qquad \text{for} \quad y > 0$$

$f(x) = \tan^{-1}x \rightarrow f(\tan\theta) = \theta$

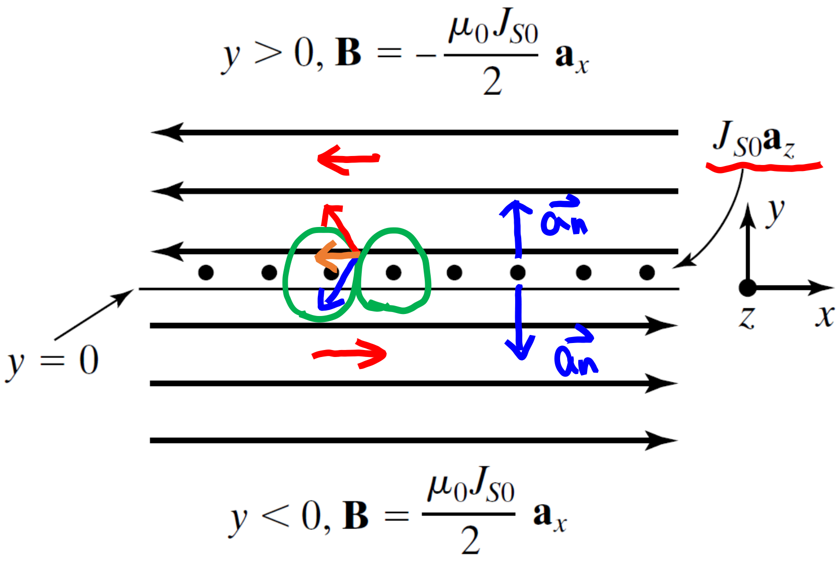
$\rightarrow f'(\tan\theta)\sec^2\theta = 1$

$f'(\tan\theta) = \frac{1}{\sec^2\theta}$, $x = \tan\theta$

$\rightarrow f'(x) = \frac{1}{1+x^2}$

$\rightarrow \frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2} \rightarrow \frac{1}{x^2+d^2} = \frac{1}{d^2}\,\frac{1}{1+(x/d)^2}$

Example



$$[\mathbf{B}]_{(0,\,y,\,0)} = \frac{\mu_0 J_{S0}}{2}\mathbf{a}_x \qquad \text{for} \quad y < 0$$

$$\mathbf{B} = \mp\frac{\mu_0 J_{S0}}{2}\mathbf{a}_x \qquad \text{for} \quad y \gtrless 0$$

$$\mathbf{a}_n = \pm\mathbf{a}_y \qquad \text{for} \quad y \gtrless 0$$

$$\mathbf{B} = \frac{\mu_0}{2}(J_{S0}\mathbf{a}_z) \times (\pm\mathbf{a}_y) \text{ for} \quad y \gtrless 0$$

$$\mathbf{B} = \frac{\mu_0}{2}\mathbf{J}_S \times \mathbf{a}_n$$

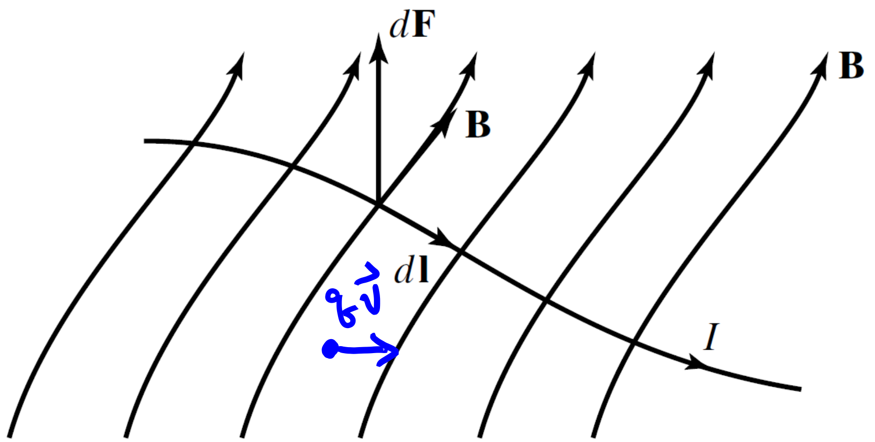
Analogy between E and B

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_0 R^2}\mathbf{a}_R \qquad \leftrightarrow \qquad \mathbf{B} = \frac{\mu_0 I}{4\pi R^2} d\mathbf{l} \times \mathbf{a}_R$$

$$\mathbf{E} = \frac{\rho_{L0}}{2\pi\varepsilon_0 r}\mathbf{a}_r \qquad \leftrightarrow \qquad \mathbf{B} = \frac{\mu_0 I}{2\pi r}\mathbf{a}_\phi = \frac{\mu_0 I}{2\pi r}\mathbf{a}_z \times \mathbf{a}_r$$

$$\mathbf{E} = \frac{\rho_{L0}}{2\varepsilon_0}\mathbf{a}_n \qquad \leftrightarrow \qquad \mathbf{B} = \frac{\mu_0}{2}\mathbf{J}_S \times \mathbf{a}_n$$

Magnetic Force in Terms of Charge



For a test charge q

$$d\mathbf{F} = I\, d\mathbf{l} \times \mathbf{B} \quad , \; I = \frac{dq}{dt}$$

$$d\mathbf{F} = \frac{dq}{dt}\mathbf{v}\, dt \times \mathbf{B} = dq\, \mathbf{v} \times \mathbf{B}$$

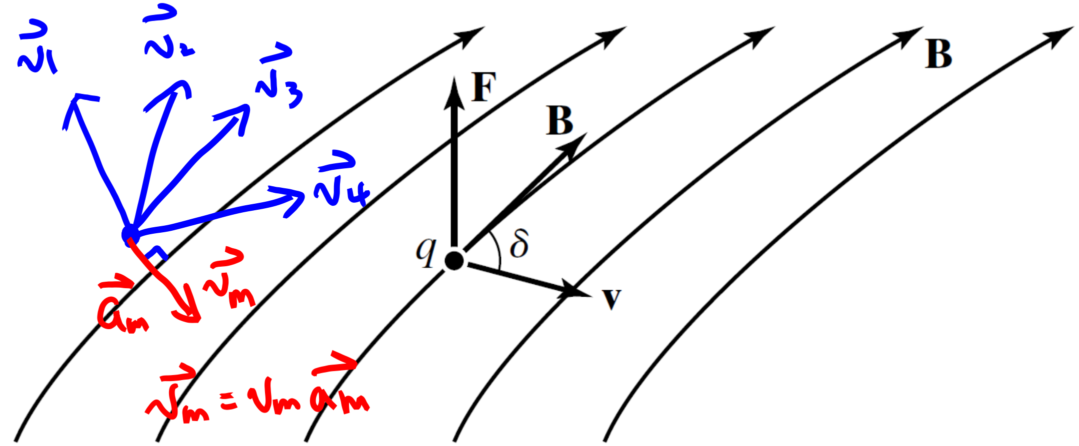
velocity

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}$$

B Defined in Terms of Charge

- Magnetic force magnitude= $qvB\sin\delta$ $|\vec{F}| = q|\vec{v}\times\vec{B}| = qvB\sin\delta$
  - Need to find the max magnetic force to define **B**.
- If this maximum force is $\mathbf{F}_m$ and it occurs for a velocity $v\mathbf{a}_m$

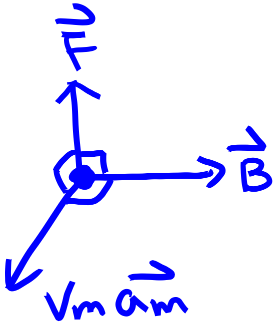
$\vec{F} = q\vec{v}\times\vec{B}$



$\vec{F} = q\vec{v}\times\vec{B}$

$\vec{F}_m = q v_m \vec{a}_m \times \vec{B}$

$\Rightarrow \vec{B} = \dfrac{\vec{F}_m \times \vec{a}_m}{q v_m}$



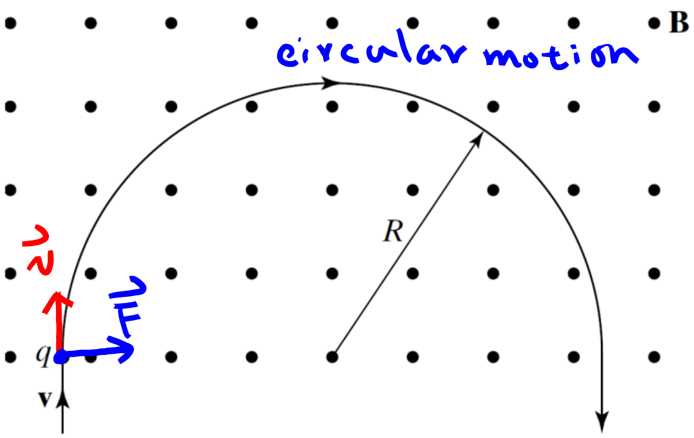
$$\mathbf{B} = \frac{\mathbf{F}_m \times \mathbf{a}_m}{qv}$$

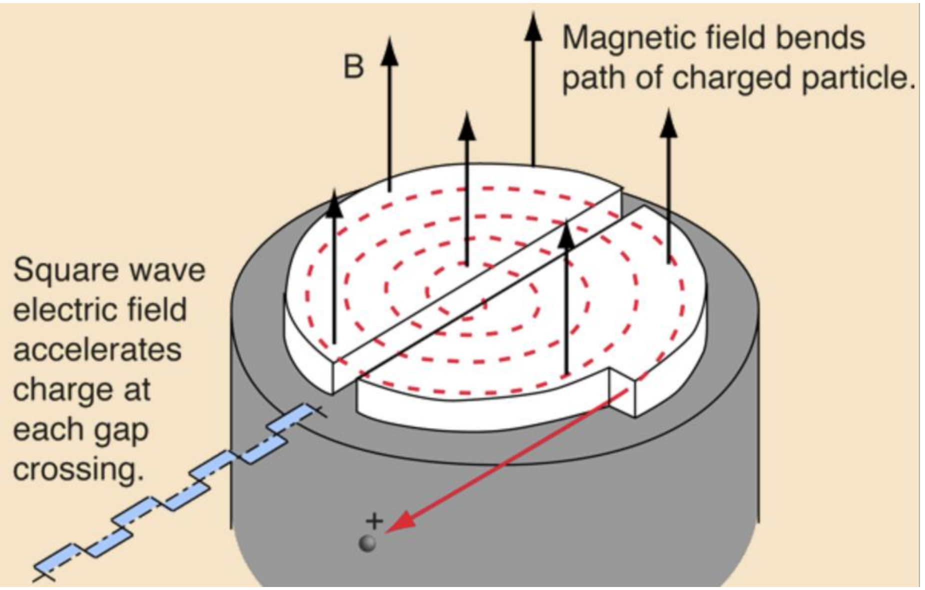
$$\mathbf{B} = \lim_{qv\to 0} \frac{\mathbf{F}_m \times \mathbf{a}_m}{qv}$$

Cyclotron: Particle Accelerator

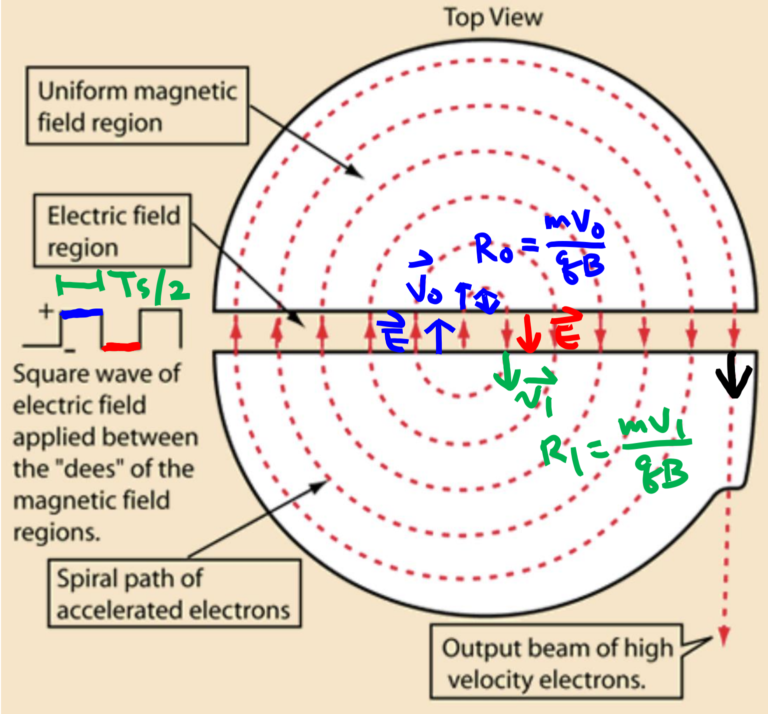
- $qvB= mv^2/R$ Magnetic force = centripetal force
- $R= mv/qB$ $v = R\omega \rightarrow \omega = \frac{v}{R} = \frac{qBv}{mv} = \frac{qB}{m}$
  (m/q) ratio, isotopes
- Cyclotron frequency= $\omega_C= qB/m$ → only depend on B
- Final speed $v= \omega_C R$, $T = \frac{2\pi}{\omega_c}$

  $\therefore T = T_s$





signal source



$R = \frac{mv_F}{qB}$

$\rightarrow v_F = \frac{qBR}{m}$

$= \omega_c R$

Ref: HyperPhysics

Outline

- Vector algebra
- Cartesian coordinate system
- Cylindrical coordinate system
- Spherical coordinate system
- Scalar and vector fields
- Electric field
- Magnetic field
- Lorentz force equation

Lorentz Force Equation

- A test charge q placed in an electric field of intensity **E** experiences a force

$$\mathbf{F}_E = q\mathbf{E} \qquad \vec{B} = 0$$

- A test charge q moving with a velocity **v** in a magnetic field of flux density **B** experiences a force

$$\mathbf{F}_M = q\mathbf{v} \times \mathbf{B} \qquad \vec{E} = 0$$

- A test charge q moving with velocity **v** in a region with **E** and **B** experiences a force

$$\mathbf{F} = \mathbf{F}_E + \mathbf{F}_M = q(\underline{\mathbf{E}} + \mathbf{v} \times \underline{\mathbf{B}}) \qquad \vec{B} \neq 0, \vec{E} \neq 0$$

  - Lorentz force equation.
  - **F** can be used to find **E** and **B** by giving the charge different **v**.

Example

- The forces experienced by a test charge q for three different velocities at a point in a region of electric and magnetic fields are given below where and are constants. Find **E** and **B** at that point.

$$
\begin{aligned}
\mathbf{F}_1 &= qE_0\mathbf{a}_x && \text{for} \quad \mathbf{v}_1 = v_0\mathbf{a}_x \\
\mathbf{F}_2 &= qE_0(2\mathbf{a}_x + \mathbf{a}_y) && \text{for} \quad \mathbf{v}_2 = v_0\mathbf{a}_y \\
\mathbf{F}_3 &= qE_0(\mathbf{a}_x + \mathbf{a}_y) && \text{for} \quad \mathbf{v}_3 = v_0\mathbf{a}_z
\end{aligned}
$$

Example

$$qE + qv_0\mathbf{a}_x \times \mathbf{B} = qE_0\mathbf{a}_x \quad ①$$

$$q\mathbf{E} + qv_0\mathbf{a}_y \times \mathbf{B} = q(2E_0\mathbf{a}_x + E_0\mathbf{a}_y) \quad ②$$

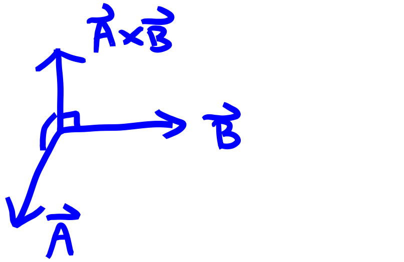
$$q\mathbf{E} + qv_0\mathbf{a}_z \times \mathbf{B} = q(E_0\mathbf{a}_x + E_0\mathbf{a}_y) \quad ③$$

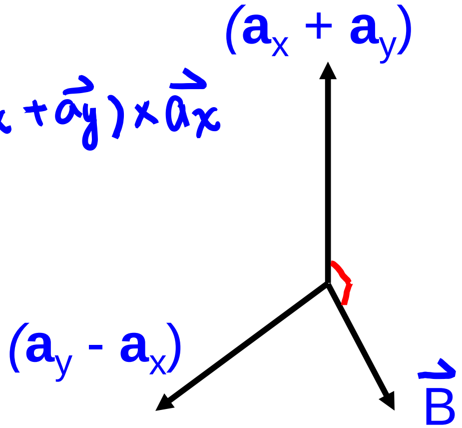
① − ②

$$v_0(\mathbf{a}_y - \mathbf{a}_x) \times \mathbf{B} = E_0(\mathbf{a}_x + \mathbf{a}_y) \rightarrow \vec{B} \perp (\vec{a_x} + \vec{a_y})$$

$$v_0(\mathbf{a}_y - \mathbf{a}_z) \times \mathbf{B} = E_0\mathbf{a}_x \rightarrow \vec{B} \perp \vec{a_x}$$

③ − ①





$\vec{B}$ perpendicular to $(\vec{a_x} + \vec{a_y})$ and $\vec{a_x}$ → $\vec{B}$'s direction $(\vec{a_x} + \vec{a_y}) \times \vec{a_x}$

$$\Rightarrow \mathbf{B} = C(\mathbf{a}_x + \mathbf{a}_y) \times \mathbf{a}_x = -C\mathbf{a}_z$$

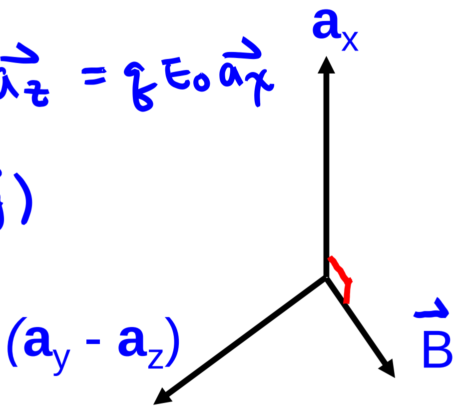
$$v_0(\mathbf{a}_y - \mathbf{a}_z) \times (-C\mathbf{a}_z) = E_0\mathbf{a}_x$$

$$-v_0C\mathbf{a}_x = E_0\mathbf{a}_x$$

$\therefore C = E_0/v_0 \rightarrow \vec{B} = \frac{E_0}{v_0}\vec{a_z}$ — ④

④ → ① $\quad q\vec{E} + qv_0\vec{a_x} \times \frac{E_0}{v_0}\vec{a_z} = qE_0\vec{a_x}$

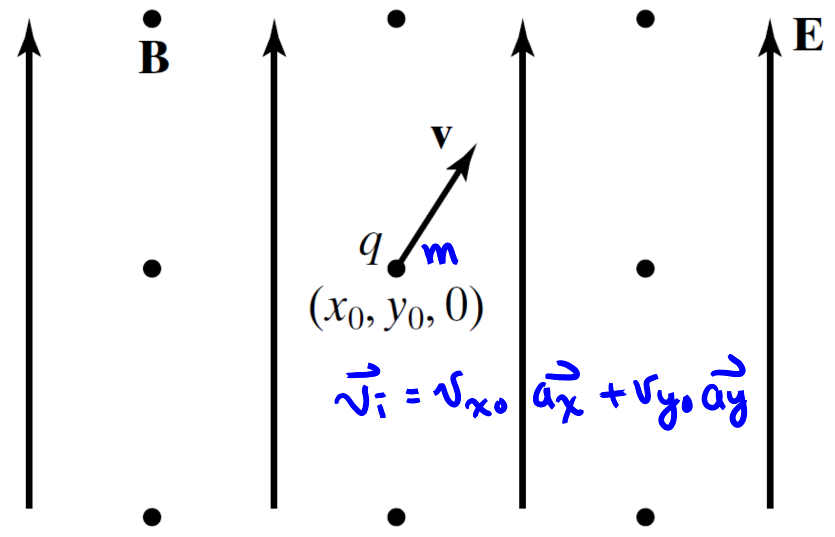
$\therefore \vec{E} = E_0(\vec{a_x} + \vec{a_y})$



$$\begin{cases} \mathbf{B} = \dfrac{E_0}{v_0}\mathbf{a}_z \\ \mathbf{E} = E_0(\mathbf{a}_x + \mathbf{a}_y) \end{cases}$$

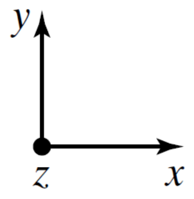
Charged Particle Tracing

- Initial conditions: $v_z = 0$, $z=0$ at $t= 0$, $v_x= v_{x0}$ and $v_y= v_{y0}$ at $t= 0$, and , $x= x_0$ and $y= y_0$ at $t= 0$
- Applications
  - Cathode tubes.
  - Ink-jet printers.
  - Electron microscopes.

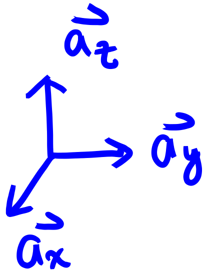


Assume charges are confined on the *x*-*y* plane for simplicity

$\rightarrow \vec{E} = E_0\,\vec{a}_y$, $\vec{B} = B_0\,\vec{a}_z$



$$
\begin{aligned}
\mathbf{F} &= q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \\
&= qE_0\mathbf{a}_y + q\underbrace{(v_x\mathbf{a}_x + v_y\mathbf{a}_y + v_z\mathbf{a}_z)}_{=\vec{v}} \times B_0\mathbf{a}_z \\
&= qB_0v_y\mathbf{a}_x + (qE_0 - qB_0v_x)\mathbf{a}_y \\
&= ma
\end{aligned}
$$



Charged Particle Tracing

$$\mathbf{F} = qB_0 v_y \mathbf{a}_x + (qE_0 - qB_0 v_x)\mathbf{a}_y = ma = m a_x \vec{a_x} + m a_y \vec{a_y}$$

$$= m\frac{dV_x}{dt}\vec{a_x} + m\frac{dV_y}{dt}\vec{a_y} + 0\frac{dV_z}{dt}$$

$$\Rightarrow \begin{cases} \dfrac{dv_x}{dt} = \dfrac{qB_0}{m}v_y & -① \\ \dfrac{dv_y}{dt} = \dfrac{qE_0}{m} - \dfrac{qB_0}{m}v_x & -② \\ \dfrac{dv_z}{dt} = 0 \end{cases}$$

From ① $\dfrac{d^2V_x}{dt^2} = \dfrac{qB_0}{m}\dfrac{dV_y}{dt}$ $-③$

$$② \rightarrow ③ \Rightarrow \frac{d^2v_x}{dt^2} + \left(\frac{qB_0}{m}\right)^2 v_x = \left(\frac{q}{m}\right)^2 B_0 E_0$$

$$x(t) = \underbrace{x_c(t)}_{\text{complementary solution}} + \underbrace{x_p(t)}_{\text{particular solution}}$$

Ke^{st}

$V_{x,p}(t) = C$

$$\Rightarrow \left(\frac{qB_0}{m}\right)^2 C = \left(\frac{q}{m}\right)^2 B_0 E_0$$

$$\Rightarrow C = \frac{E_0}{B_0}$$

$$\frac{d^2V_x}{dt^2} + \left(\frac{qB_0}{m}\right)^2 V_x = 0$$

$$V_{x,c}(t) = K_1 e^{j\omega_c t} + K_2 e^{-j\omega_c t}$$

$$\begin{cases} v_x = \dfrac{E_0}{B_0} + C_1 \cos \omega_c t + C_2 \sin \omega_c t \\ v_y = -C_1 \sin \omega_c t + C_2 \cos \omega_c t \end{cases}$$

$$\omega_c = \frac{qB_0}{m}$$

P1.50

Charged Particle Tracing

- Time average constant velocity $\mathbf{v}_{avg}$ of the test charge?

$\vec{v}(t) = \vec{v}_{avg} + \vec{v}'(t)$

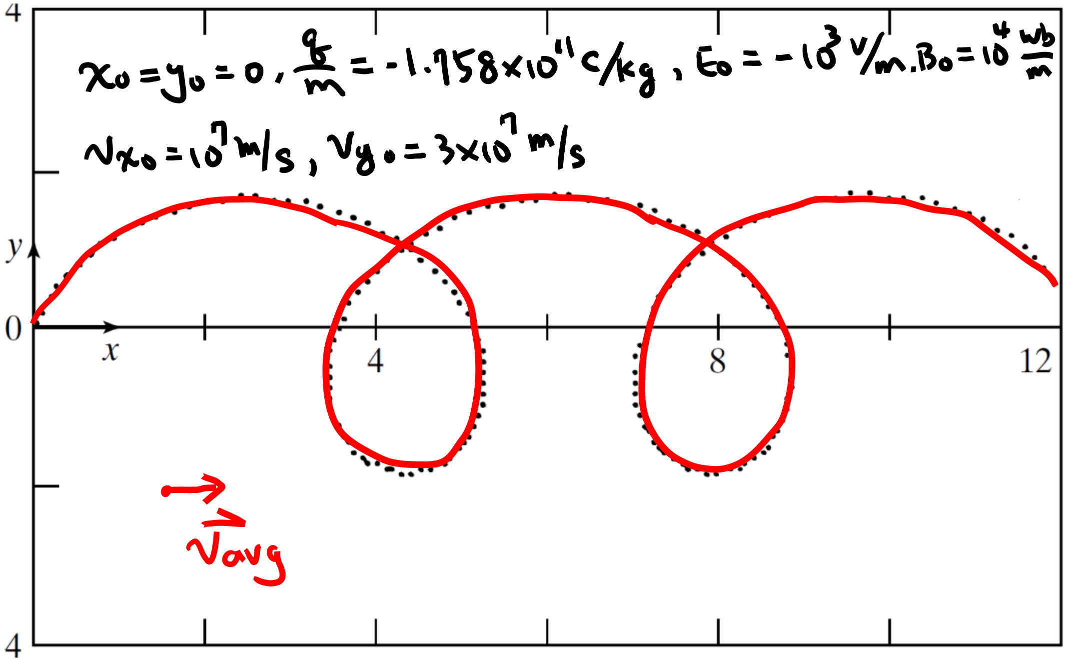
$\vec{E}$ force $+ \vec{B}$ force $= 0$

$\Rightarrow q\vec{E} + \vec{v}_{avg} \times \vec{B} = 0$

$\to q\vec{E} \times \vec{B} + (\vec{v}_{avg} \times \vec{B}) \times \vec{B} = 0$

$\to \vec{E} \times \vec{B} = \vec{B} \times (\vec{v}_{avg} \times \vec{B}) = \vec{v}_{avg}(\vec{B} \cdot \vec{B}) - \vec{B}(\vec{B} \cdot \vec{v}_{avg})$ BAC − CAB rule

$= \vec{v}_{avg}|\vec{B}|^2 \quad \therefore \vec{v}_{avg} = \dfrac{\vec{E} \times \vec{B}}{|\vec{B}|^2} \to \vec{v}_1 = \dfrac{\vec{E} \times \vec{B}}{|\vec{B}|^2} \to q(\vec{E} + \vec{v}_1 \times \vec{B}) = 0$



$$\begin{cases} v_x = \dfrac{E_0}{B_0} + \left(v_{x0} - \dfrac{E_0}{B_0}\right)\cos\omega_c t + v_{y0}\sin\omega_c t \\ v_y = -\left(v_{x0} - \dfrac{E_0}{B_0}\right)\sin\omega_c t + v_{y0}\cos\omega_c t \\ x = x_0 + \dfrac{E_0}{B_0}t + \dfrac{1}{\omega_c}\left(v_{x0} - \dfrac{E_0}{B_0}\right)\sin\omega_c t + \dfrac{v_{y0}}{\omega_c}(1 - \cos\omega_c t) \\ y = y_0 - \dfrac{1}{\omega_c}\left(v_{x0} - \dfrac{E_0}{B_0}\right)(1 - \cos\omega_c t) + \dfrac{v_{y0}}{\omega_c}\sin\omega_c t \end{cases}$$

$v_x = \dfrac{dx}{dt}$

$\to dx = v_x\, dt$

$B_0 = 0,\ \omega_c = 0$

$x = x_0 + v_{x0} t \quad , \quad v_x = v_{x0}$

$y = y_0 + v_{y0} t + \dfrac{qE_0}{2m} t^2 \quad , \quad v_y = v_{y0} + \dfrac{qE_0}{m} t^2$