Basics about Random Variables

J.-S. Roger Jang (張智星)
MIR Lab, CSIE Dept.
National Taiwan University
jang@mirlab.org, http://mirlab.org/jang

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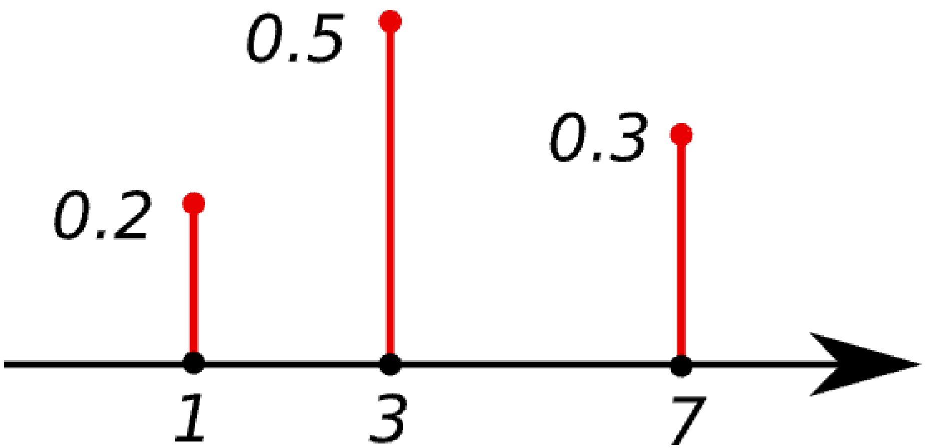
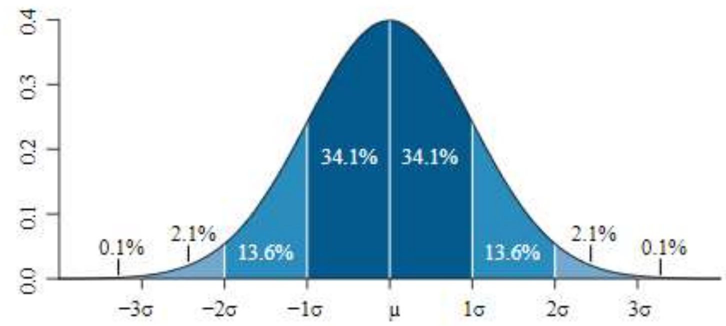
Math Background: Random Variables

- Discrete



Discrete Probability Disrtibution: This shows the probability mass function of a discrete probability distribution. The probabilities of the singletons {1}, {3}, and {7} are respectively 0.2, 0.5, 0.3. A set not containing any of these points has probability zero.

- Continuous



Probability Density Function: The image shows the probability density function (pdf) of the normal distribution, also called Gaussian or "bell curve", the most important continuous random distribution. As notated on the figure, the probabilities of intervals of values corresponds to the area under the curve.

https://courses.lumenlearning.com/boundless-statistics/chapter/discrete-random-variables/

Linear Combination of Random Variables

- Given two random variables X and Y
  - Definition
    - Mean: $\mu_X = E(X)$
    - Variance: $\sigma_X^2 = V(X) \triangleq E((X-\mu_X)^2) = E(X^2) - \mu_X^2$
    - Covariance: $\sigma_{XY} = \sigma_{YX} = E((X-\mu_X)(Y-\mu_Y)) = E(XY) - \mu_X\mu_Y$
  - Basic formulas for a single variable
    - $E(aX) = aE(X) \Rightarrow \mu_{aX} = a\mu_X$
    - $V(aX) = a^2V(X) \Rightarrow \sigma_{aX}^2 = a^2\sigma_X^2$
  - Extension to two variables (not necessarily independent)
    - $E(aX+bY) = aE(X) + bE(Y)$
      $\Rightarrow \mu_{aX+bY} = a\mu_X + b\mu_Y$
    - $V(aX+bY) = a^2V(X) + b^2V(Y) + 2abE((X-\mu_X)(Y-\mu_Y))$
      $\Rightarrow \sigma_{aX+bY}^2 = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\sigma_{XY} = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\sigma_X\sigma_Y\rho_{XY}$

https://hackmd.io/@rogerjang/BJwCkf0LK, with exercise!

Proof of Variance after Combination

- Proof of $\sigma^2_{aX+bY} = a^2\sigma^2_X + b^2\sigma^2_Y + 2ab\sigma_{XY} = a^2\sigma^2_X + b^2\sigma^2_Y + 2ab\sigma_X\sigma_Y\rho_{XY}$

$$
\begin{aligned}
V(aX+bY) &= E((aX+bY-E(aX+bY))^2) \\
&= E((aX+bY-a\mu_X-b\mu_Y))^2) \\
&= E((a(X-\mu_X)+b(Y-\mu_Y))^2) \\
&= E(a^2(X-\mu_X)^2+b^2(Y-\mu_Y)^2+2ab(X-\mu_X)(Y-\mu_Y)) \\
&= a^2E((X-\mu_X)^2)+b^2E((Y-\mu_Y)^2)+2abE((X-\mu_X)(Y-\mu_Y)) \\
&= a^2\sigma^2_X+b^2\sigma^2_Y+2ab\sigma_{XY} \\
&= a^2\sigma^2_X+b^2\sigma^2_Y+2ab\sigma_X\sigma_Y\rho_{XY}
\end{aligned}
$$

where $\rho_{XY} = \sigma_{XY}/(\sigma_X\sigma_Y)$ is the correlation coefficient between X and Y, with $-1 \le \rho_{XY} \le 1$.

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