Electromagnetics (I)
Chapter 2

Ch1 { vector algebra
vector calculus
$\vec{E}$ and $\vec{B}$ fields

→ Ready to study Maxwell's eqs

Maxwell's Equations in Integral Form

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Outline

- Inductive and deductive approaches
- Line integral, surface integral, and volume integral
- Faraday’s law
- Ampère’s circuital law
- Gauss’ laws
- Law of conservation of charge
- Applications to static fields

Homework 2

- Even numbers for even sections and odd numbers for odd sections

Outline

- Inductive and deductive approaches
- Line integral, surface integral, and volume integral
- Faraday’s law
- Ampère’s circuital law
- Gauss’ laws
- Law of conservation of charge
- Applications to static fields

Development of Scientific Subject

- Inductive approach
  - Follow the historical development, starting with observing some simple experiments and inferring from them laws and theorems.

Example: electromagnetics

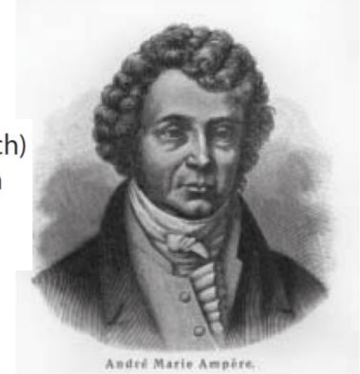
Electromagnetics in the Classic Era

1733 **Charles-François du Fay** (French) discovers that electric charges are of two forms, and that like charges repel and unlike charges attract.

1785 **Charles-Augustin de Coulomb** (French) demonstrates that the electrical force between charges is proportional to the inverse of the square of the distance between them.

$$\mathbf{F}_1 = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R^2}\mathbf{a}_{21}$$

1820 **Hans Christian Oersted** (Danish) demonstrates the interconnection between electricity and magnetism through his discovery that an electric current in a wire causes a compass needle to orient itself perpendicular to the wire.



1820 **Andre-Marie Ampère** (French) notes that parallel currents in wires attract each other and opposite currents repel.

1820 **Jean-Baptiste Biot** (French) and **Félix Savart** (French) develop the Biot-Savart law relating the magnetic field induced by a wire segment to the current flowing through it.

$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{I\,d\mathbf{l} \times \mathbf{a}_R}{R^2}$$

1827 **Georg Simon Ohm** (German) formulates Ohm's law relating electric potential to current and resistance.

1827 **Joseph Henry** (American) introduces the concept of inductance, and builds one of the earliest electric motors. He also assisted Samual Morse in the development of the telegraph.

1831 **Michael Faraday** (English) discovers that a changing magnetic flux can induce an electromotive force.

1835 **Carl Friedrich Gauss** (German) formulates Gauss's law relating the electric flux flowing through an enclosed surface to the enclosed electric charge.

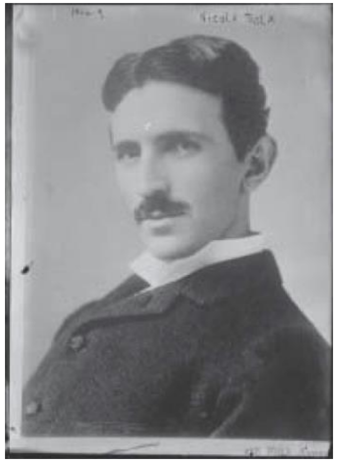
Ref: Umberto, Fundamentals of Applied Electromagnetics, 7-th Ed.

Electromagnetics in the Classic Era

1873 **James Clerk Maxwell** (Scottish) publishes his Treatise on Electricity and Magnetism in which he unites the discoveries of Coulomb, Oersted, Ampère, Faraday, and others into four elegantly constructed mathematical equations, now known as Maxwell's Equations.

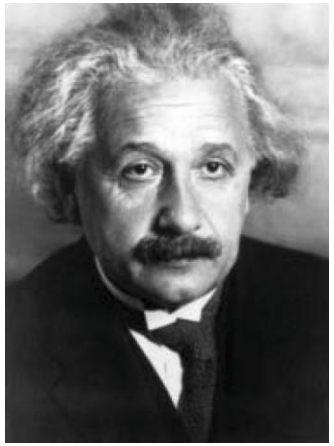
1887 **Heinrich Hertz** (German) builds a system that can generate electromagnetic waves (at radio frequencies) and detect them.

1888 **Nikola Tesla** (Croatian-American) invents the ac (alternating current) electric motor.



1897 **Joseph John Thomson** (English) discovers the electron and measures its charge-to-mass ratio. [1906 Nobel prize in physics.]

1905 **Albert Einstein** (German-American) explains the photoelectric effect discovered earlier by Hertz in 1887. [1921 Nobel prize in physics.]



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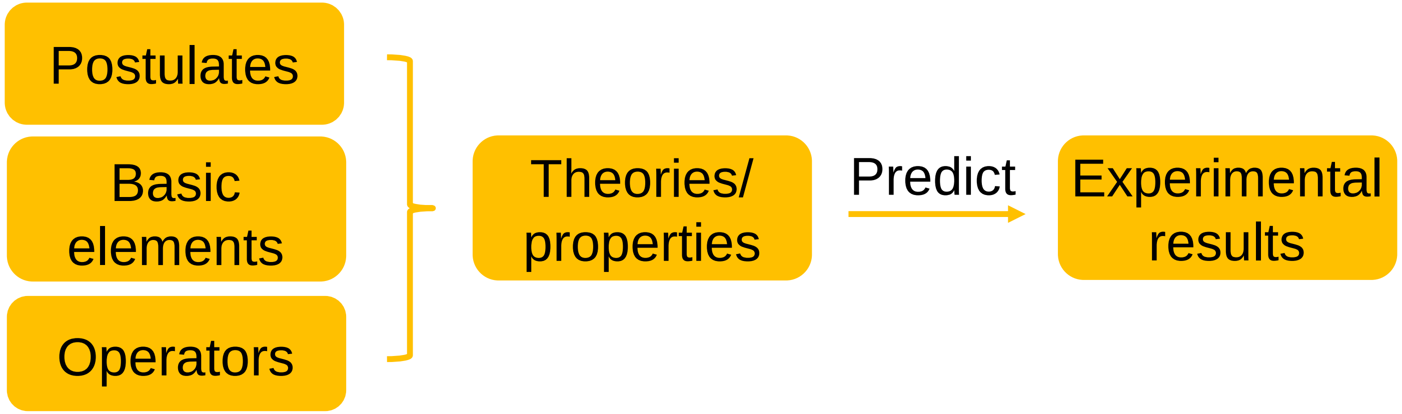
Development of Scientific Subject

✓ Deductive approach

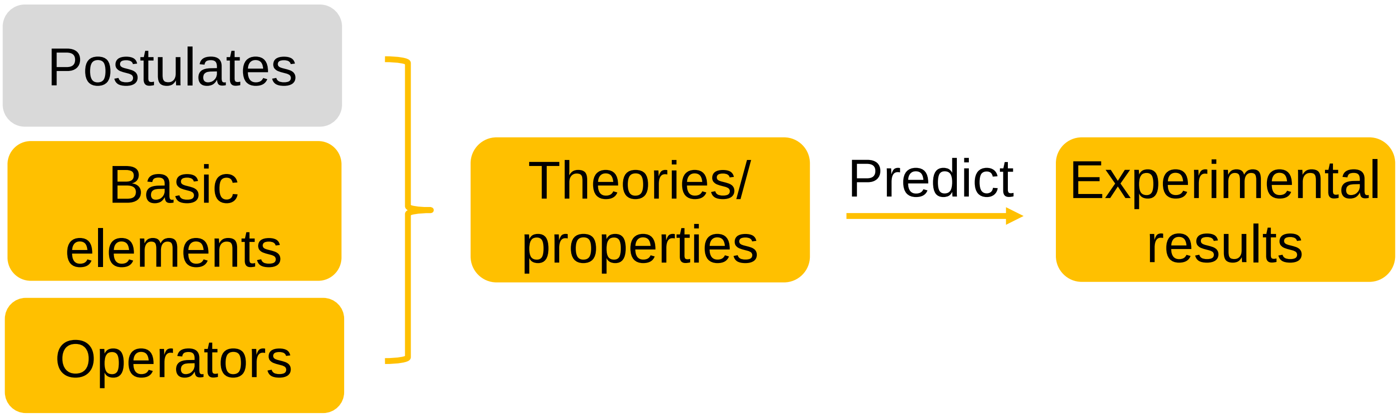
Examples: electromagnetics, Boolean algebra

1. elements: 0, 1
2. operators: +, ·
3. postulates $x+0=x$, $x \cdot 1 = x$, ⋮

- Postulate a few fundamental relations (axioms) for an idealized model.
- Particular laws and theorems can be derived from the postulates.
- The validity of the model and axioms is verified by their ability to predict consequences that can check with experimental results.
- 1. Some basic quantities are defined, 2. Rules of operations of the quantities are specified, 3. Fundamental relations are postulated (acquired from experimental observations and brilliant minds).
- Example: 1. R, L, C, 2. Algebra, ordinary differential equations, and Laplace transformation, 3. KCL and KVL.



Electromagnetic Model – Postulates



- Postulates
  - Principle of conservation of charge. (can neither be created nor destroyed.)
  - Maxwell's equations.

$$\oint_C \mathbf{E}\cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{S}$$

$$\oint_C \mathbf{H}\cdot d\mathbf{l} = \int_S \mathbf{J}\cdot d\mathbf{S} + \frac{d}{dt}\int_S \mathbf{D}\cdot d\mathbf{S}$$

$$\oint_S \mathbf{D}\cdot d\mathbf{S} = \int_V \rho\, dv$$

$$\oint_S \mathbf{B}\cdot d\mathbf{S} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\nabla \cdot \mathbf{D} = \rho$$

$$\nabla \cdot \mathbf{B} = 0$$

Principle of Conservation of Charge

- Electric charge conserved
  - It can neither be created nor destroyed.
- Law of nature and cannot be derived from other principles or relations
  - Never been questioned or doubted in practice.
- Must be satisfied at all times and under any circumstances
- Equation of continuity
- Example: KCL.

Maxwell's Equations

- Integral forms
  - Govern the interdependence of certain fields and source quantities associated with regions in space, that is, contours, surfaces, and volumes.
- Differential forms
  - Relate the characteristics of the field vectors at a given point to one another and to the source densities at that point.

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S}$$

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt}\int_S \mathbf{D} \cdot d\mathbf{S}$$

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho \, dv$$

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$$

$\vec{E}(x, y, z, t)$ Time

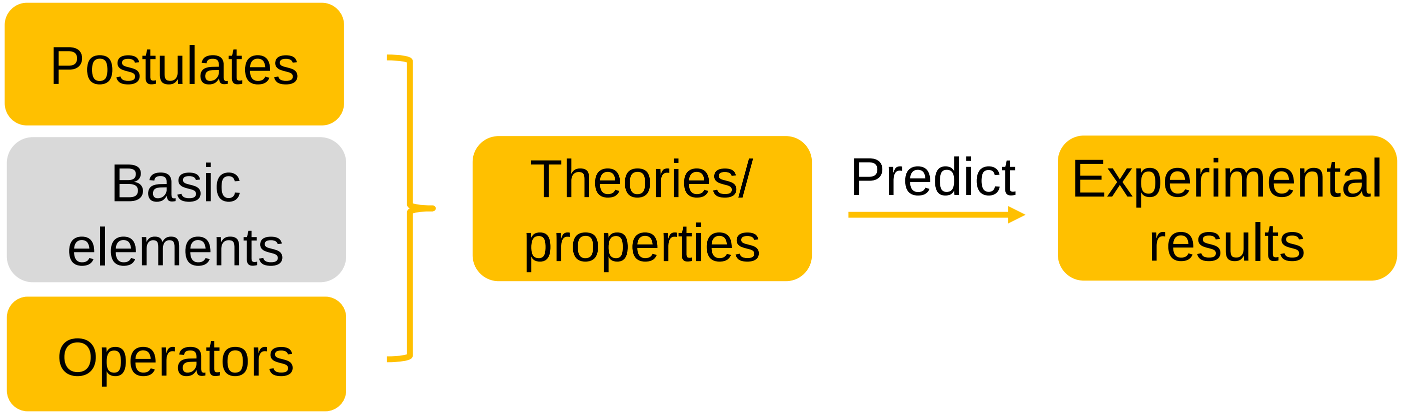
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$ $\vec{B}(x, y, z, t)$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\nabla \cdot \mathbf{D} = \rho$$

$$\nabla \cdot \mathbf{B} = 0$$

Electromagnetic Model – Quantities



- Quantities and units
  - Source quantities: electric charges and electric currents. (1 electron charge (e)= -1.6×10^{-19} C.)
  - Field quantities: **E**, **D**, **B,** and **H**. (The source of the electromagnetic field is electric charge at rest or in motion.)
  - Units: c, μ_0, and ε_0.

Source Quantities

- Electric charge
  - Electric charge exists at a point in a discrete manner in a microscopic sense.
  - Smooth-out average density functions yield very good results in a macroscopic sense.

$$\rho = \lim_{\Delta v \to 0} \frac{\Delta q}{\Delta v} \qquad (\mathrm{C/m^3})$$

$$\rho_s = \lim_{\Delta s \to 0} \frac{\Delta q}{\Delta s} \qquad (\mathrm{C/m^2})$$

$$\rho_\ell = \lim_{\Delta \ell \to 0} \frac{\Delta q}{\Delta \ell} \qquad (\mathrm{C/m})$$

- Electric currents
  - Volume current density **J** (A/m^2).
  - Surface current density $\mathbf{J}_S$ (A/m).

$$I = \frac{dq}{dt} \qquad (\mathrm{C/s\ or\ A})$$

Field Quantities and Units

Symbols and Units for Field Quantities	Field Quantity	Symbol	Unit
Electric	Electric field intensity	**E**	V/m
	Electric flux density (Electric displacement)	**D** $= \varepsilon_0 \vec{E}$	C/m^2
Magnetic	Magnetic flux density	**B**	T
	Magnetic field intensity	**H** $= \vec{B}/\mu_0$	A/m

$T = wb/m^2$

Universal Constants	Symbol	Value	Unit
Velocity of light in free space	c	3×10^8	m/s
Permeability of free space	μ_0	$4\pi \times 10^{-7}$	H/m
Permittivity of free space	ϵ_0	$\frac{1}{36\pi} \times 10^{-9}$	F/m

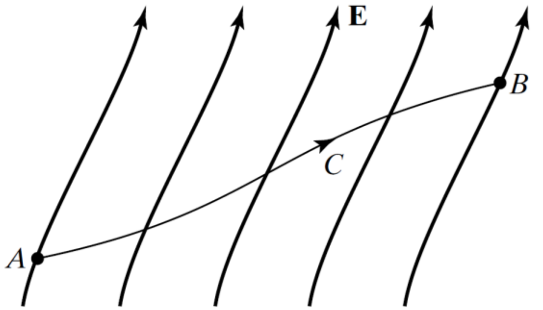
Electromagnetic Model – Operators

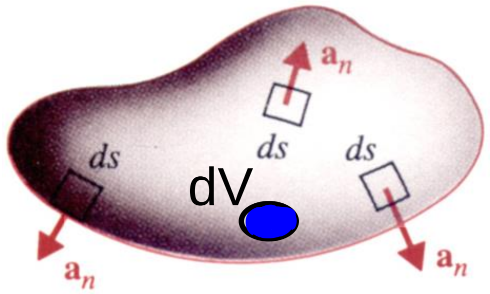
Postulates
Basic elements
Operators

Theories/ properties $\xrightarrow{\text{Predict}}$ Experimental results

- Operators
  - Vector algebra and calculus.
  - Line, surface, and volume integrals.

$$\oint_C \mathbf{E} \cdot d\mathbf{l} \qquad \oint_S \mathbf{B} \cdot d\mathbf{S} = 0 \qquad \int_V \rho \, dv$$





Closed surface

Electromagnetic Model – Theories and Properties

Postulates

Basic elements

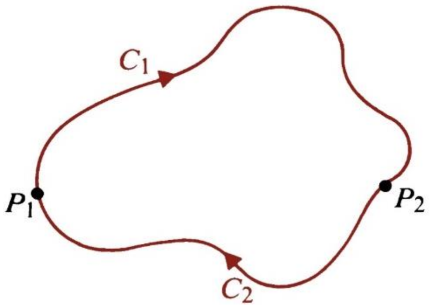
Operators

Theories/ properties

Predict

Experimental results

- Theories and properties
  - Conservation of energy.
  - Poynting’s theorem.
  - Vector potential.
  - Etc.



$$\underset{\text{Along } C_1}{\int_{P_1}^{P_2}} \mathbf{E} \cdot d\boldsymbol{\ell} = \underset{\text{Along } C_2}{\int_{P_1}^{P_2}} \mathbf{E} \cdot d\boldsymbol{\ell}$$

$$\frac{dW}{dt} = -\frac{d}{dt}\int_{\mathcal{V}} \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right) d\tau - \frac{1}{\mu_0}\oint_{\mathcal{S}} (\mathbf{E} \times \mathbf{B}) \cdot d\mathbf{a}$$

Electromagnetic Model – Predict Experimental Results

Postulates

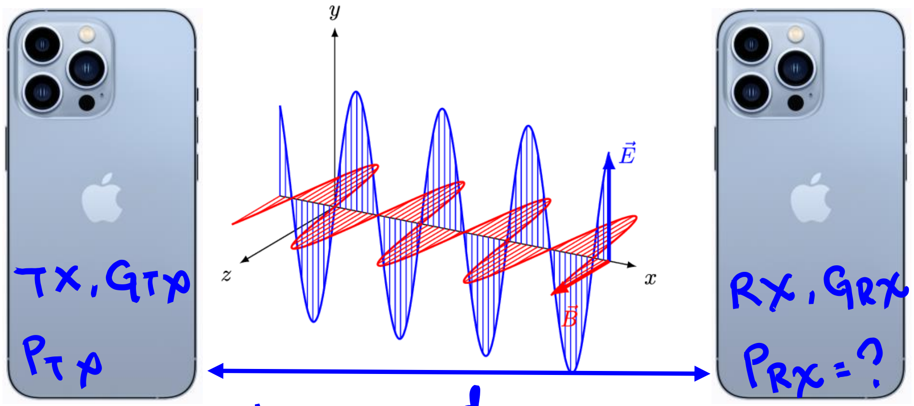
Basic elements

Operators

→ Theories/ properties → Predict → Experimental results

- Predict experiment results

Electromagnetic wave propagation



TX, G_{TX}

P_{TX}

$= 0\,dBm = 1\,mW$

d

RX, G_{RX}

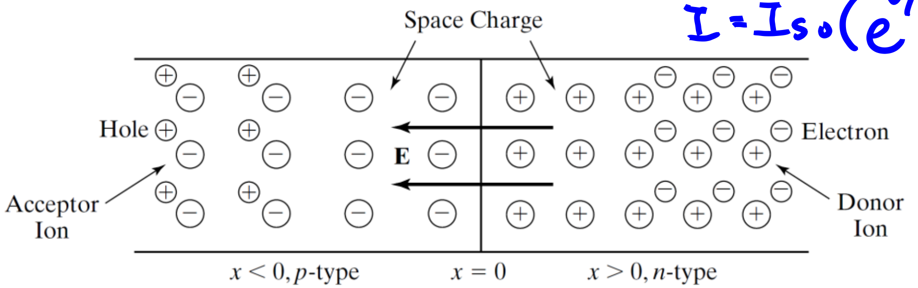
$P_{RX} = ?$

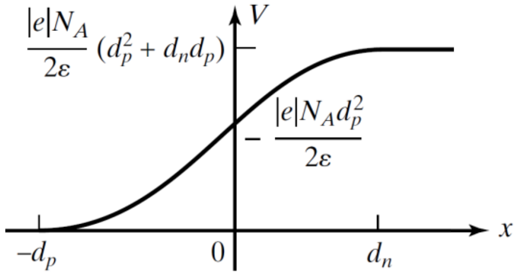
$-100\,dBm = 10^{-10}\,mW$

$$P_{RX} = G_{TX} G_{RX} \left(\frac{\lambda}{4\pi d}\right)^2 P_{TX}$$

PN Junction — diode

$$I = I_{S0}\left(e^{V/V_T} - 1\right)$$





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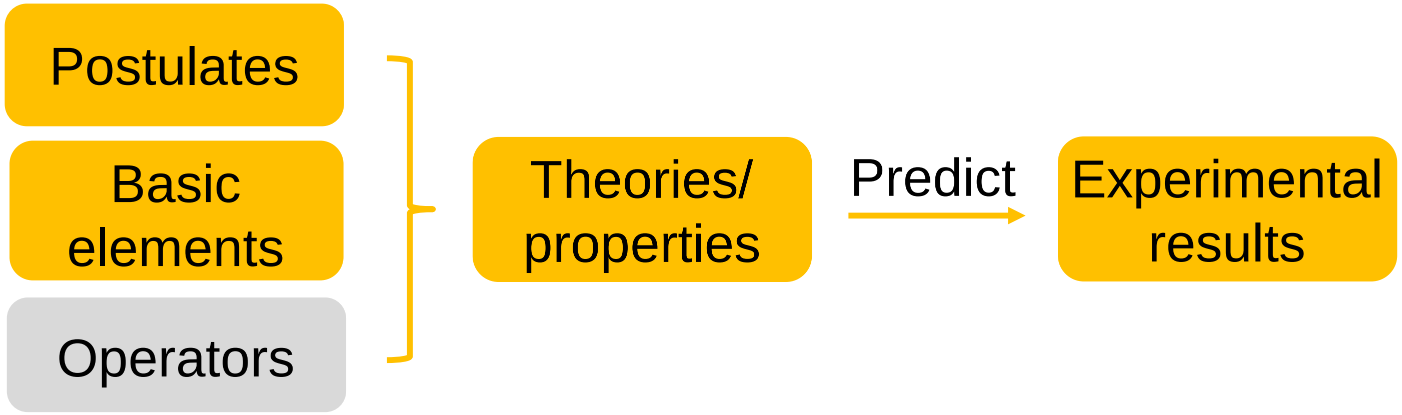
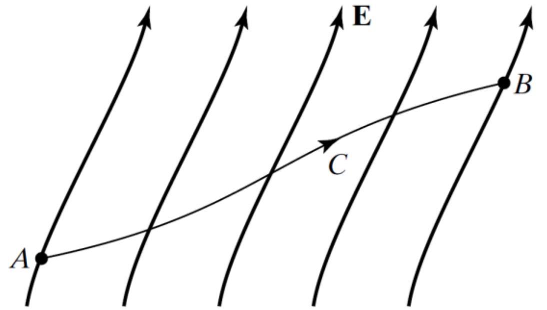
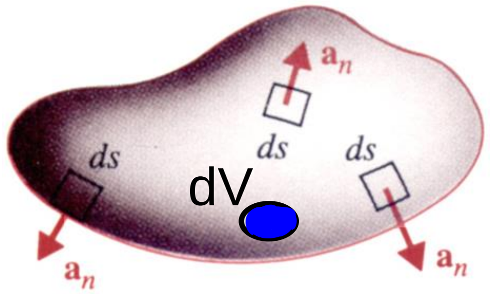
Electromagnetic Model – Operators



- Operators
  - Vector algebra and calculus.
  - Line, surface, and volume integrals.

$$\oint_C \mathbf{E} \cdot d\mathbf{l} \qquad \oint_S \mathbf{B} \cdot d\mathbf{S} = 0 \qquad \int_V \rho \, dv$$



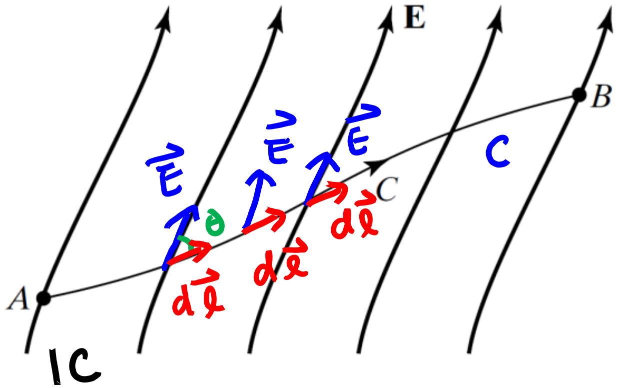


Closed surface

Line Integral

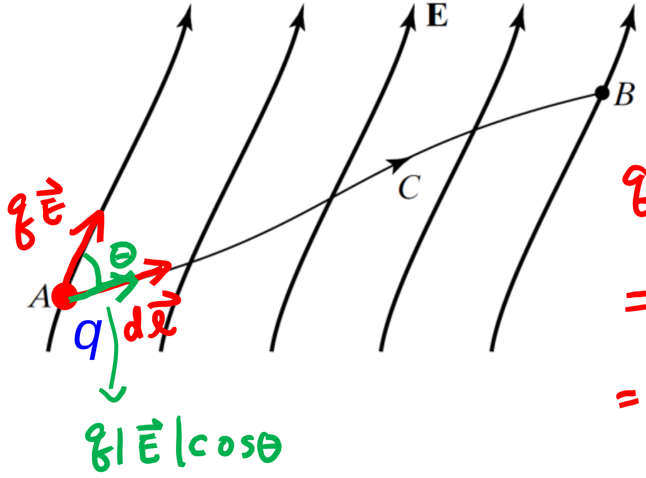
$\vec{E}\cdot d\vec{l} = |\vec{E}||d\vec{l}|\cos\theta$

$$\oint_C \mathbf{E}\cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{S}$$



$$\int_A^B \vec{E}\cdot\vec{dl}$$

C



$$q\int \vec{E}\cdot\vec{dl}$$

$$= \int q\vec{E}\cdot d\vec{l}$$

$$= \int \underbrace{q|\vec{E}|\cos\theta}_{F}\, \underbrace{|d\vec{l}|}_{|d\vec{l}|}$$

$[F\cdot S]$ = work

$$W_{AB} = q\int_A^B \vec{E}\cdot d\vec{l}$$

$$\frac{W_{AB}}{q} = V_{AB}$$

- Physical meaning
  - Move a test charge q from the point A to the point B along the path C.
  - Total work done by **E**

$$W_{AB} = q\int_{A\ (C)}^{B} \mathbf{E}\cdot d\mathbf{l}$$

potential difference

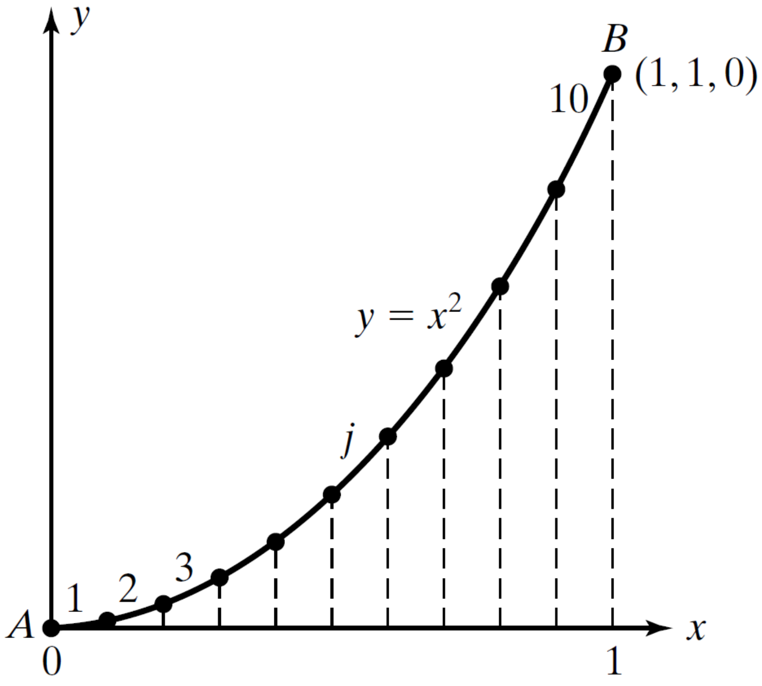
- Voltage between A and B along the specified path
  - Work per unit charge done by the field in moving the test charge from A to B.
  - $[V]$= volts.

$$V_{AB} = \int_{A\ (C)}^{B} \mathbf{E}\cdot d\mathbf{l} \qquad W_{AB} = qV_{AB}$$

Example

- Determine the work done by the electric field in the movement of a 3 μC charge from the point *A*(0, 0, 0) to the point *B*(1, 1, 0) along the parabolic path *y*= *x*^{2} and *z*= 0.

$$\mathbf{E} = y\mathbf{a}_y$$



$\vec{E} = y\vec{a}_y$

$$W_{AB} = q\int_A^B \mathbf{E}\cdot d\mathbf{l}$$

$$dy = 2x\,dx \qquad dz = 0$$

$$d\mathbf{l} = dx\,\mathbf{a}_x + dy\,\mathbf{a}_y + dz\,\mathbf{a}_z$$

$$= dx\,\mathbf{a}_x + 2x\,dx\,\mathbf{a}_y$$

$$\mathbf{E}\cdot d\mathbf{l} = y\mathbf{a}_y\cdot(dx\,\mathbf{a}_x + 2x\,dx\,\mathbf{a}_y)$$

$$= x^2\mathbf{a}_y\cdot(dx\,\mathbf{a}_x + 2x\,dx\,\mathbf{a}_y)$$

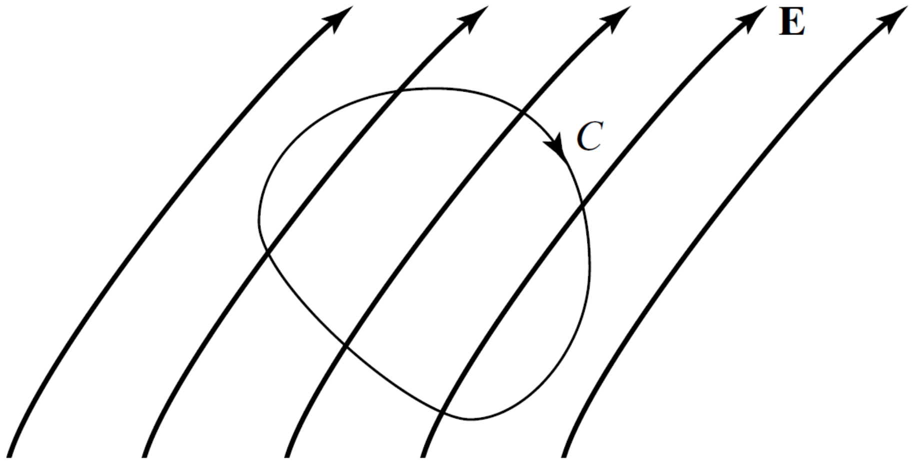
$$= 2x^3\,dx$$

$$W_{AB} = q\int_{(0,0,0)}^{(1,1,0)} \mathbf{E}\cdot d\mathbf{l} = 3\times10^{-6}\int_0^1 2x^3\,dx$$

$$= 3\times10^{-6}\left[\frac{2x^4}{4}\right]_0^1 = 1.5\ \mu\text{J}$$

Line Integral – Closed Path

- Called vector circulation of that vector



$$\underbrace{\oint_C \mathbf{E}\cdot d\mathbf{l}}_{\neq 0} = -\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{S}$$

[emf] = volts

$\oint_C \mathbf{E}\cdot d\mathbf{l}$

lc

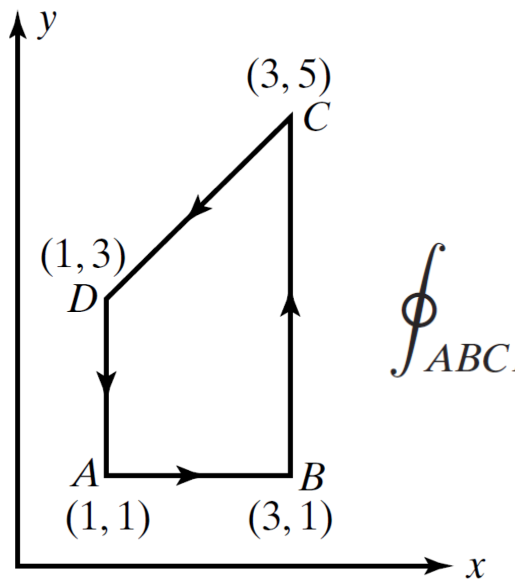
– Work per unit charge done by the field in moving a test charge around the closed path.

– Voltage around the closed path, also known as the electromotive force.

emf

Example

- Let us consider the force field $\mathbf{F} = x\mathbf{a}_y$ and evaluate $\oint_C \mathbf{F} \cdot d\mathbf{l}$ where *C* is the closed path *ABCDA* as shown below.



$$\oint_{ABCDA} \mathbf{F} \cdot d\mathbf{l} \ ?$$

Example

$$\oint_{ABCDA} \mathbf{F}\cdot d\mathbf{l} = \int_A^B \mathbf{F}\cdot d\mathbf{l} + \int_B^C \mathbf{F}\cdot d\mathbf{l} + \int_C^D \mathbf{F}\cdot d\mathbf{l} + \int_D^A \mathbf{F}\cdot d\mathbf{l}$$

$$d\mathbf{l} = dx\,\mathbf{a}_x + dy\,\mathbf{a}_y$$

$$\boxed{\oint_{ABCDA} \mathbf{F}\cdot d\mathbf{l} = 0 + 12 - 4 - 2 = 6}$$

$y = 2 + x, \qquad dy = dx, \qquad d\mathbf{l} = dx\,\mathbf{a}_x + dx\,\mathbf{a}_y$

$\mathbf{F}\cdot d\mathbf{l} = (x\mathbf{a}_y)\cdot(dx\,\mathbf{a}_x + dx\,\mathbf{a}_y) = x\,dx$

$\vec{a}_y \cdot \vec{a}_y = 1$

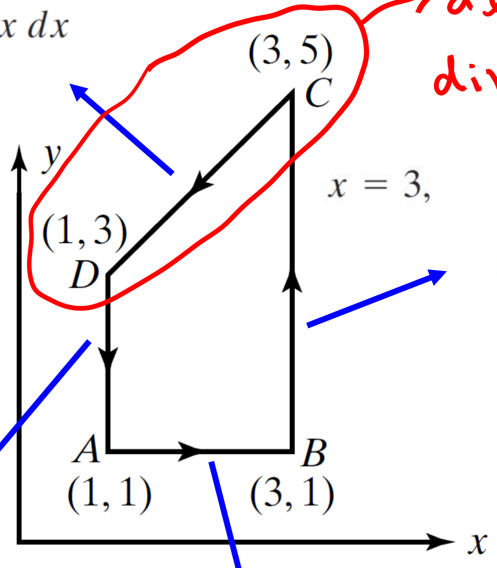
$\vec{a}_y \cdot \vec{a}_x = 0$

$$\int_C^D \mathbf{F}\cdot d\mathbf{l} = \int_3^1 x\,dx = -4$$

$\vec{F} = x\vec{a}_y$

$d\vec{l} = ?$

direction ✱



$x = 3, \qquad dx = 0, \qquad d\mathbf{l} = (0)\mathbf{a}_x + dy\,\mathbf{a}_y = dy\,\mathbf{a}_y$

$\mathbf{F}\cdot d\mathbf{l} = (3\mathbf{a}_y)\cdot(dy\,\mathbf{a}_y) = 3\,dy$

$$\int_B^C \mathbf{F}\cdot d\mathbf{l} = \int_1^5 3\,dy = 12$$

$x = 1, \qquad dx = 0, \qquad d\mathbf{l} = (0)\mathbf{a}_x + dy\,\mathbf{a}_y$

$\mathbf{F}\cdot d\mathbf{l} = (\mathbf{a}_y)\cdot(dy\,\mathbf{a}_y) = dy$

$$\int_D^A \mathbf{F}\cdot d\mathbf{l} = \int_3^1 dy = -2$$

$y = 1, \qquad dy = 0, \qquad d\mathbf{l} = dx\,\mathbf{a}_x + (0)\mathbf{a}_y = dx\,\mathbf{a}_x$

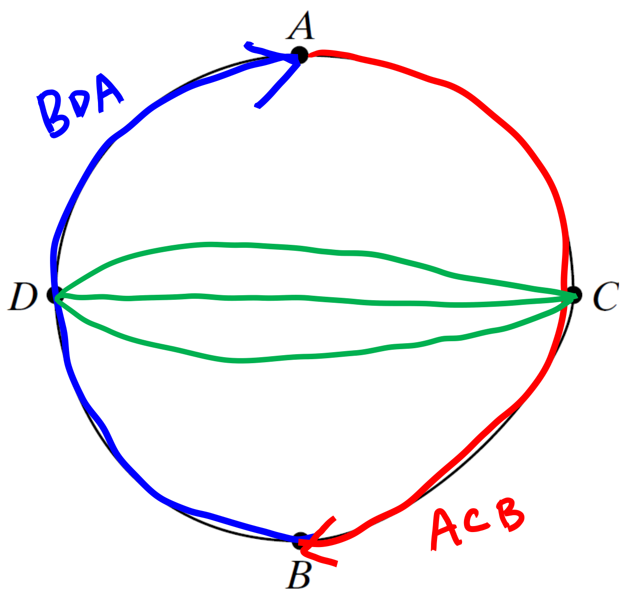
$\mathbf{F}\cdot d\mathbf{l} = (x\mathbf{a}_y)\cdot(dx\,\mathbf{a}_x) = 0$

$$\int_A^B \mathbf{F}\cdot d\mathbf{l} = 0$$

Conservative and Nonconservative Fields

$\oint \vec{E} \cdot d\vec{l} = -\frac{d}{dt}\int_S \vec{B} \cdot d\vec{a} = 0$ for static field

$\frac{d}{dt} = 0$



$$\oint_{ACBDA} \mathbf{F} \cdot d\mathbf{l} = \int_{ACB} \mathbf{F} \cdot d\mathbf{l} + \int_{BDA} \mathbf{F} \cdot d\mathbf{l}$$

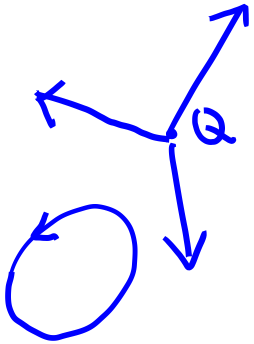
$$= \int_{ACB} \mathbf{F} \cdot d\mathbf{l} - \int_{ADB} \mathbf{F} \cdot d\mathbf{l}$$

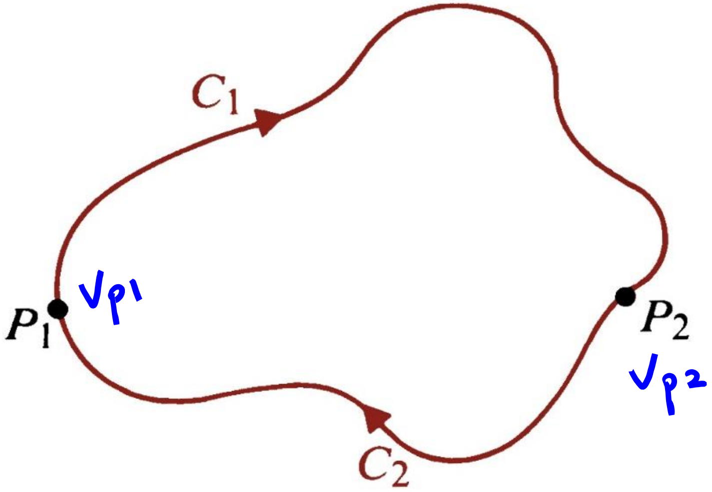
$= 0$

- Nonconservative field (closed loop line integral ≠ 0)
  - Line integral depends on the path. $\vec{B} = \vec{B}(t)$
  - Example: time-varying electric field. $\vec{E} = \vec{E}(t)$
- Conservative field (closed loop line integral = 0)
  - Example: Earth's gravitational field and the static electric field.

Static Electric Field

$$\oint_C \mathbf{E}\cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{S} = 0$$





$$\int_{C_1} \mathbf{E}\cdot d\ell + \int_{C_2} \mathbf{E}\cdot d\ell = 0$$

$$\underset{\text{Along } C_1}{\int_{P_1}^{P_2}} \mathbf{E}\cdot d\ell = -\underset{\text{Along } C_2}{\int_{P_2}^{P_1}} \mathbf{E}\cdot d\ell$$

$$\underset{\text{Along } C_1}{\int_{P_1}^{P_2}} \mathbf{E}\cdot d\ell = \underset{\text{Along } C_2}{\int_{P_1}^{P_2}} \mathbf{E}\cdot d\ell$$

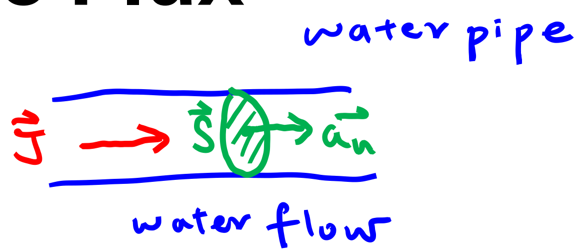
uniquely defined because indep of path

Potential difference= $V_{P1} - V_{P2}$

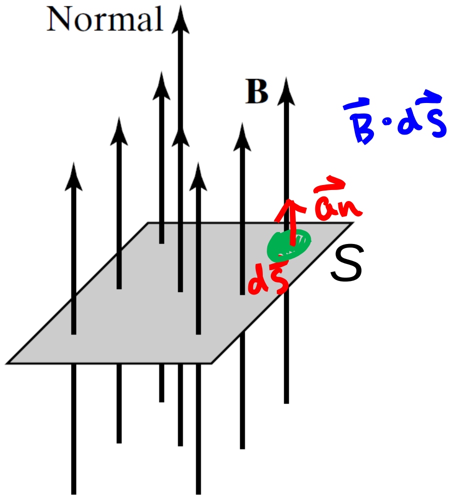
- – **E** field irrotational. → conservative
- – Line integral independent of the path (conservation of energy in a static electric field).
- Kirchhoff's voltage law (KVL)
  - – The algebraic sum of the voltage drops around any closed circuit is zero.

Surface Integral – Magnetic Flux

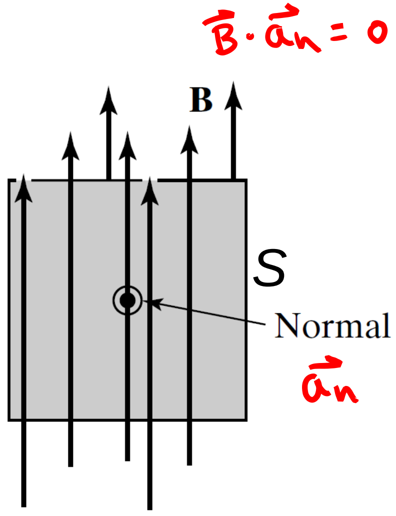
$$\oint_C \mathbf{E}\cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{S}$$



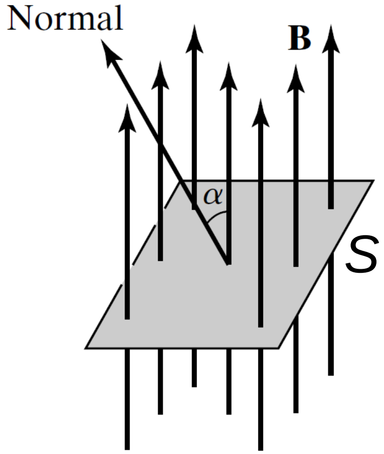
– [B]= Wb/m^2, [Magnetic flux]= Wb.



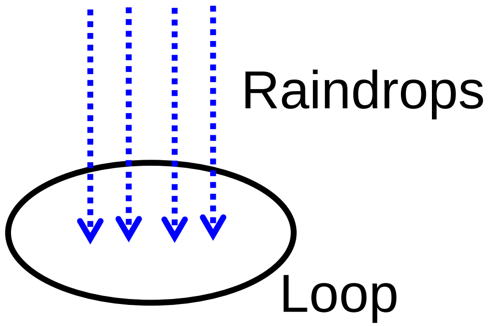
Magnetic flux= $B\ \Delta S$

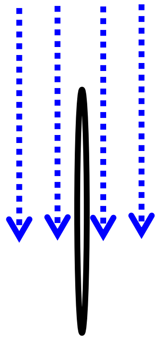


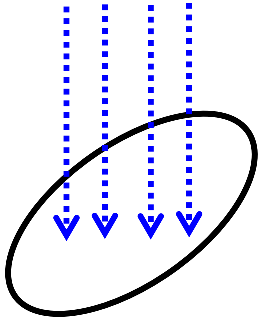
Magnetic flux= 0

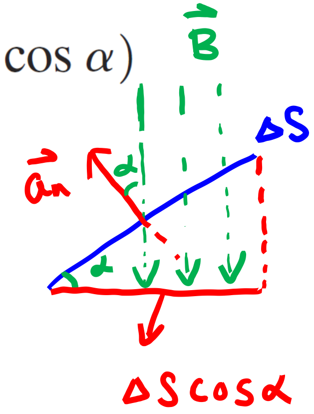


Magnetic flux= $B(\Delta S\cos\alpha)$



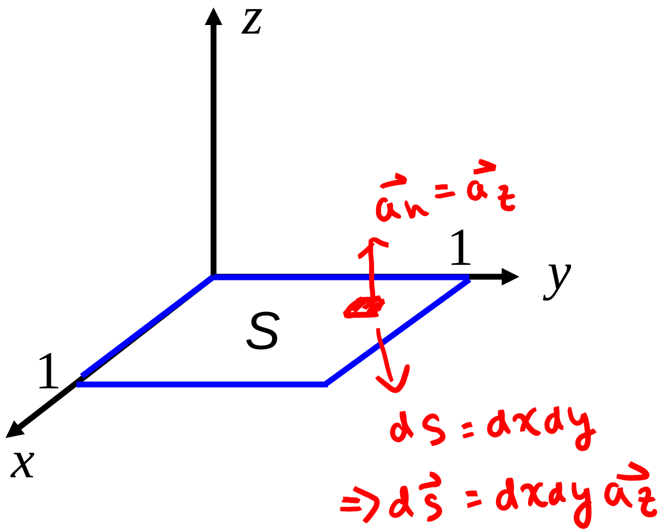






Example

- Consider the magnetic field $\mathbf{B} = 3xy^2\ \mathbf{a}_z$ Wb/m². Determine the magnetic flux crossing the portion of the xy-plane lying between $x = 0$, $x = 1$, $y = 0$, and $y = 1$.



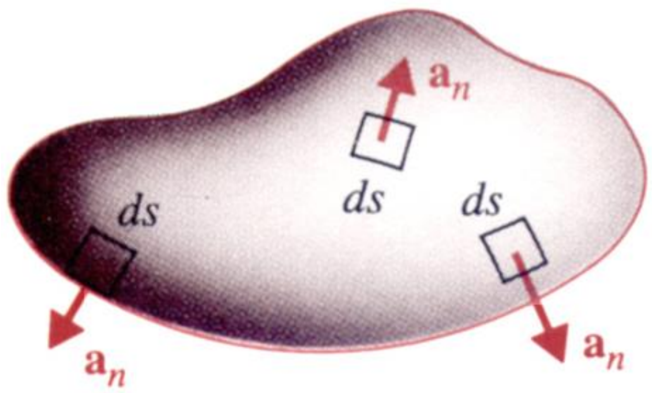
$\vec{B} = 3xy^2\,\vec{a}_z$

$$\begin{aligned} \mathbf{B} \cdot d\mathbf{S} &= 3xy^2\,\mathbf{a}_z \cdot \underline{dx\,dy\,\mathbf{a}_z} \\ &= 3xy^2\,dx\,dy \end{aligned}$$

$$\begin{aligned} [\psi]_S &= \int_S \mathbf{B} \cdot d\mathbf{S} \\ &= \int_{x=0}^{1} \int_{y=0}^{1} 3xy^2\,dx\,dy = 0.5 \text{ Wb} \end{aligned}$$

*d*s Vector Directions

$\oint_S \mathbf{B} \cdot d\mathbf{S}$



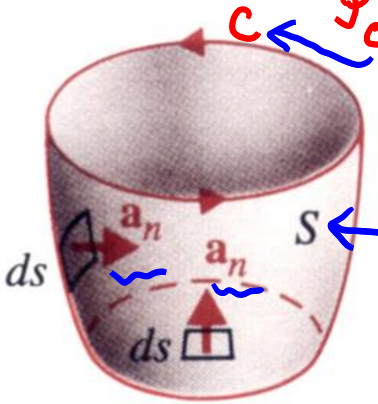
① A closed surface.

Always outward !

Point out of the volume

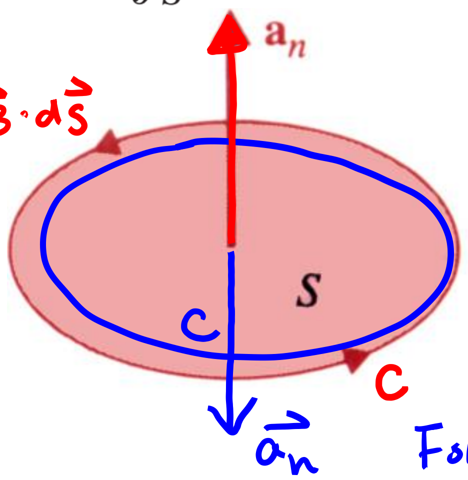
$\int_S \mathbf{B} \cdot d\mathbf{S}$

$\oint_C \vec{E} \cdot d\vec{l} = -\frac{d}{dt}\int_S \vec{B} \cdot d\vec{S}$



② An open surface.

$\int_S \mathbf{B} \cdot d\mathbf{S}$



A disk.

Follow the right-hand rule

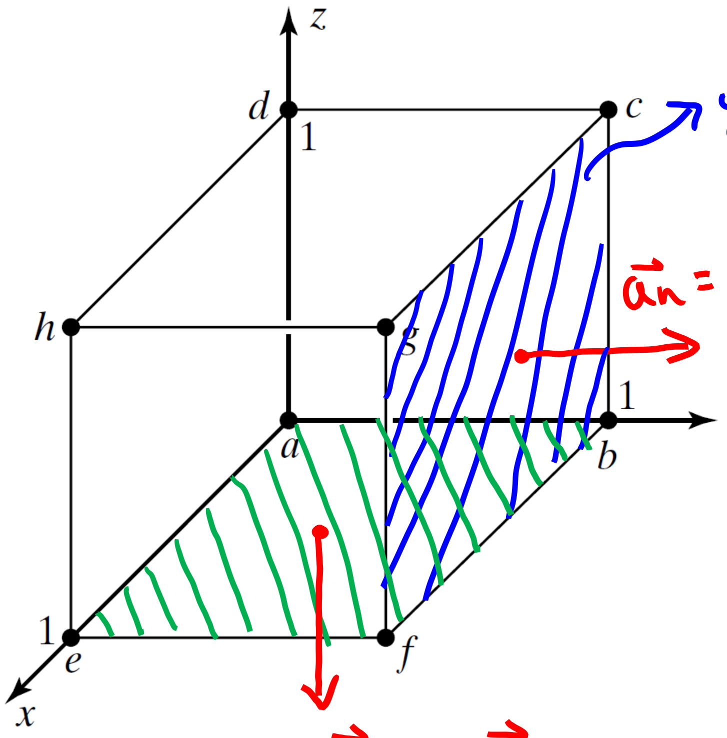
– Closed surface encloses a volume.

Example

- Evaluate $\oint_S \mathbf{B} \cdot d\mathbf{S}$ where S is the surface of the cubical box bounded by the planes

$$\oint \vec{B} \cdot d\vec{S} = 0$$

$$\begin{cases} x = 0 \quad x = 1 \\ y = 0 \quad y = 1 \\ z = 0 \quad z = 1 \end{cases} \qquad \mathbf{B} = (x + 2)\mathbf{a}_x + (1 - 3y)\mathbf{a}_y + 2z\mathbf{a}_z$$



$y = 1,$

$\vec{a}_n = \vec{a}_y$

$\vec{a}_n = -\vec{a}_z$

$$\vec{B} = (x+2)\vec{a}_x - 2\vec{a}_y + 2z\vec{a}_z$$

$$d\vec{S} = dx\,dz\,\vec{a}_y$$

$$\Rightarrow \vec{B} \cdot d\vec{S} = -2\,dx\,dz$$

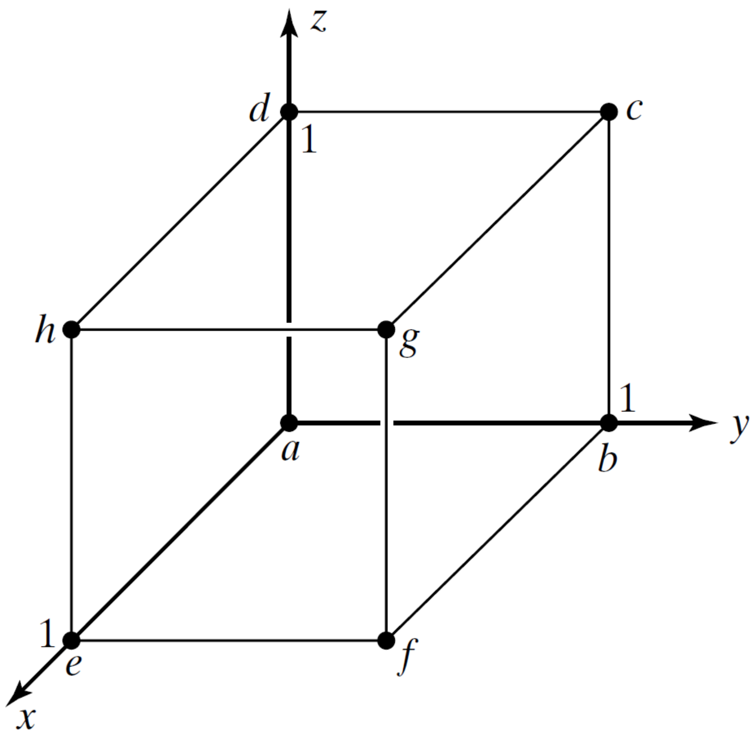
$$\Rightarrow \int_{bfgh} \vec{B} \cdot d\vec{S} = \int_{x=0}^{x=1} \int_{z=0}^{z=1} (-2)\,dx\,dz$$

$$= -2$$

Example

- Surface integral over a closed surface
  - A closed surface encloses a volume.
  - The surface integral of **B** over the closed surface *S* is simply the magnetic flux emanating from the volume bounded by the surface.

$$\oint_S \mathbf{B}\cdot d\mathbf{S} = \int_{abcd} \mathbf{B}\cdot d\mathbf{S} + \int_{efgh} \mathbf{B}\cdot d\mathbf{S} + \int_{adhe} \mathbf{B}\cdot d\mathbf{S} + \int_{bcgf} \mathbf{B}\cdot d\mathbf{S} + \int_{aefb} \mathbf{B}\cdot d\mathbf{S} + \int_{dhgc} \mathbf{B}\cdot d\mathbf{S}$$



abcd

$$x = 0, \qquad \mathbf{B} = 2\mathbf{a}_x + (1 - 3y)\mathbf{a}_y + 2z\mathbf{a}_z, \qquad d\mathbf{S} = -dy\,dz\,\mathbf{a}_x$$

$$\mathbf{B}\cdot d\mathbf{S} = -2\,dy\,dz$$

$$\int_{abcd} \mathbf{B}\cdot d\mathbf{S} = \int_{z=0}^{1}\int_{y=0}^{1} (-2)\,dy\,dz = -2$$

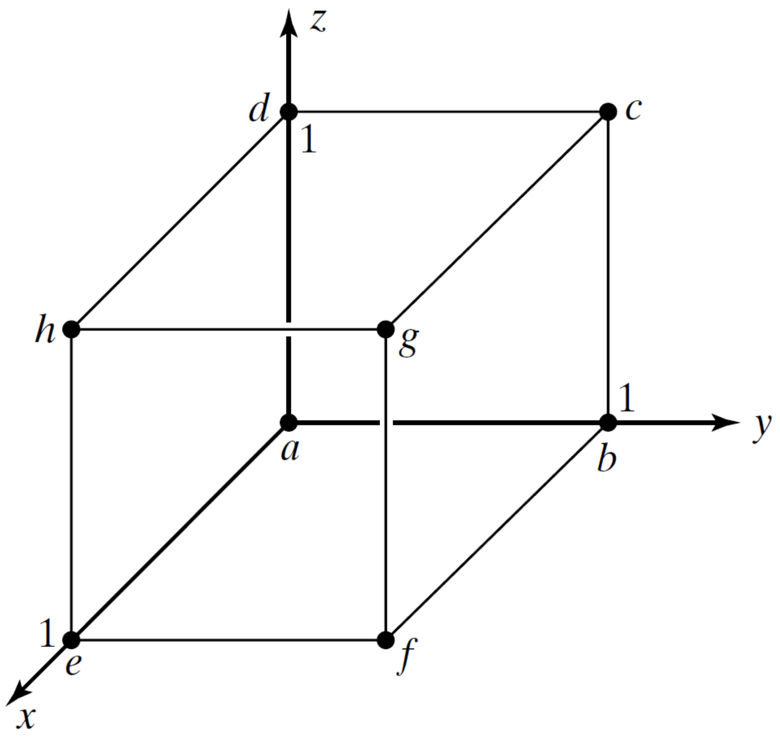
efgh

$$x = 1, \qquad \mathbf{B} = 3\mathbf{a}_x + (1 - 3y)\mathbf{a}_y + 2z\mathbf{a}_z, \qquad d\mathbf{S} = dy\,dz\,\mathbf{a}_x$$

$$\mathbf{B}\cdot d\mathbf{S} = 3\,dy\,dz$$

$$\int_{efgh} \mathbf{B}\cdot d\mathbf{S} = \int_{z=0}^{1}\int_{y=0}^{1} 3\,dy\,dz = 3$$

Example



adhe

$$y = 0, \quad \mathbf{B} = (x + 2)\mathbf{a}_x + 1\mathbf{a}_y + 2z\mathbf{a}_z, \quad d\mathbf{S} = -dz\,dx\,\mathbf{a}_y$$

$$\mathbf{B}\cdot d\mathbf{S} = -dz\,dx$$

$$\int_{aehd} \mathbf{B}\cdot d\mathbf{S} = \int_{x=0}^{1}\int_{z=0}^{1} (-1)\,dz\,dx = -1$$

bcgf

$$y = 1, \quad \mathbf{B} = (x + 2)\mathbf{a}_x - 2\mathbf{a}_y + 2z\mathbf{a}_z, \quad d\mathbf{S} = dz\,dx\,\mathbf{a}_y$$

$$\mathbf{B}\cdot d\mathbf{S} = -2\,dz\,dx$$

$$\int_{bfgc} \mathbf{B}\cdot d\mathbf{S} = \int_{x=0}^{1}\int_{z=0}^{1} (-2)\,dz\,dx = -2$$

aefb

$$z = 0, \quad \mathbf{B} = (x + 2)\mathbf{a}_x + (1 - 3y)\mathbf{a}_y + 0\mathbf{a}_z, \quad d\mathbf{S} = -dx\,dy\,\mathbf{a}_z$$

$$\mathbf{B}\cdot d\mathbf{S} = 0$$

$$\int_{aefb} \mathbf{B}\cdot d\mathbf{S} = 0$$

dhgc

$$z = 1, \quad \mathbf{B} = (x + 2)\mathbf{a}_x + (1 - 3y)\mathbf{a}_y + 2\mathbf{a}_z, \quad d\mathbf{S} = dx\,dy\,\mathbf{a}_z$$

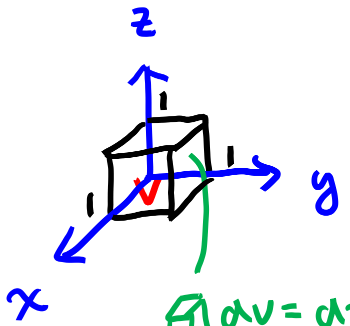
$$\mathbf{B}\cdot d\mathbf{S} = 2\,dx\,dy$$

$$\int_{dhgc} \mathbf{B}\cdot d\mathbf{S} = \int_{y=0}^{1}\int_{x=0}^{1} 2\,dx\,dy = 2$$

$$\oint_S \mathbf{B}\cdot d\mathbf{S} = -2 + 3 - 1 - 2 + 0 + 2 = 0$$

Volume Integral

- The cubical volume V is bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$. The volume charge density is $\rho = (x + y + z)$ C/m^3. Find the total charge Q contained within the cubical volume.



$dv = dx\,dy\,dz$

$= r\,dr\,d\phi\,dz$

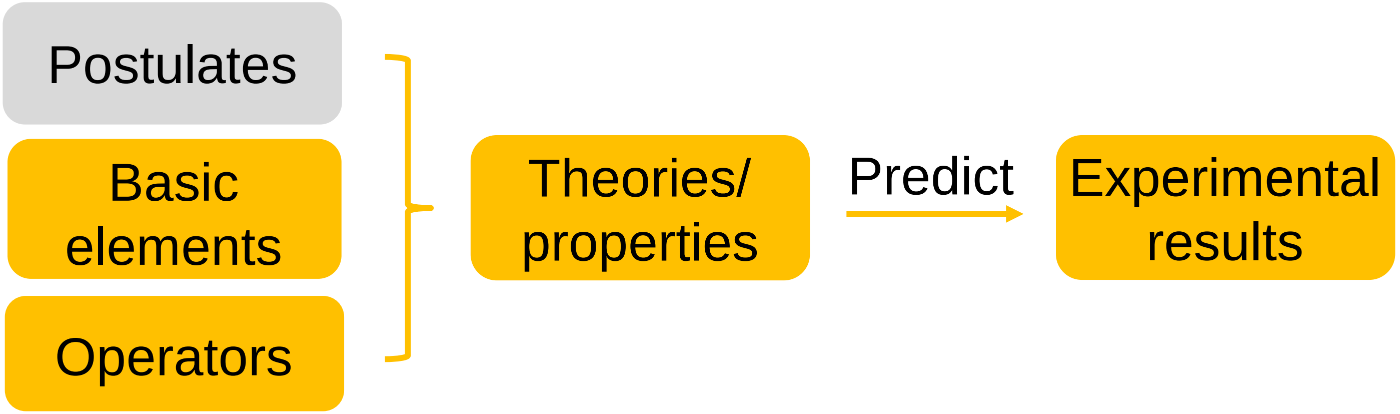
$= R^2 \sin\theta\,dR\,d\theta\,d\phi$

$$
\begin{aligned}
Q = \int_V \rho\,dv &= \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1}(x + y + z)\,dx\,dy\,dz \\
&= \int_{x=0}^{1}\int_{y=0}^{1}\left[xz + yz + \frac{z^2}{2}\right]_{z=0}^{1} dx\,dy \\
&= \int_{x=0}^{1}\int_{y=0}^{1}\left(x + y + \frac{1}{2}\right) dx\,dy \\
&= \int_{x=0}^{1}\left[xy + \frac{y^2}{2} + \frac{y}{2}\right]_{y=0}^{1} dx \\
&= \int_{x=0}^{1}(x + 1)\,dx \\
&= \left[\frac{x^2}{2} + x\right]_{x=0}^{1} \\
&= \tfrac{3}{2}\ \mathrm{C}
\end{aligned}
$$

Outline

- Inductive and deductive approaches
- Line integral, surface integral, and volume integral
- Faraday’s law
- Ampère’s circuital law
- Gauss’ laws
- Law of conservation of charge
- Applications to static fields

Electromagnetic Model – Postulates



- Postulates
  - Principle of conservation of charge. (can neither be created nor destroyed.)
  - Maxwell's equations.

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{S}$$

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt} \int_S \mathbf{D} \cdot d\mathbf{S}$$

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho \, dv$$

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\nabla \cdot \mathbf{D} = \rho$$

$$\nabla \cdot \mathbf{B} = 0$$

Maxwell's Equations

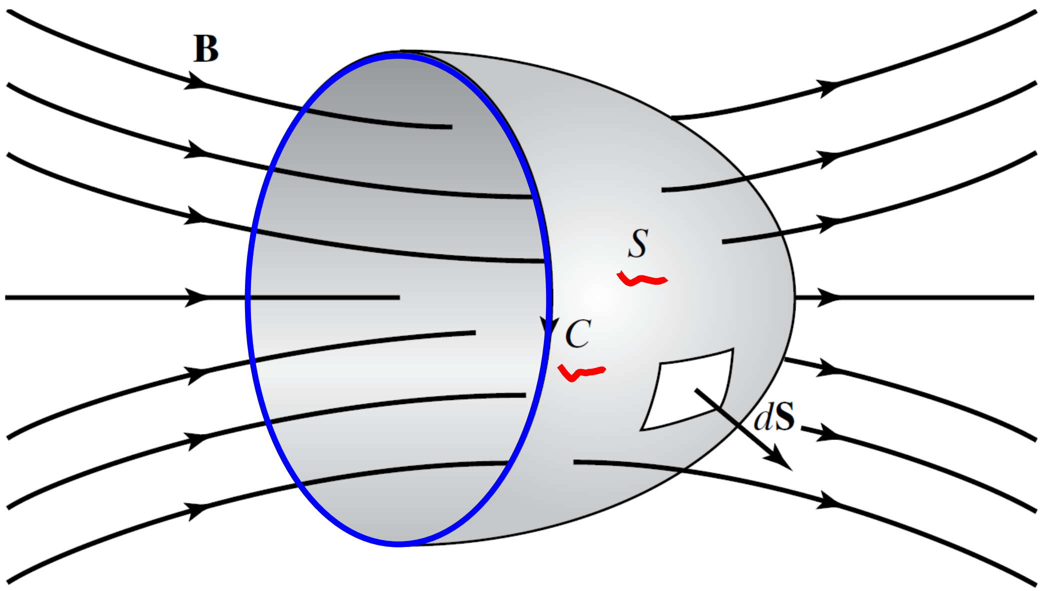
$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S}$$ Faraday's law

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt}\int_S \mathbf{D} \cdot d\mathbf{S}$$ Ampère's circuital law

$$\left.\begin{aligned} &\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho \, dv \\ &\oint_S \mathbf{B} \cdot d\mathbf{S} = 0 \end{aligned}\right\} \text{Gauss' laws}$$

Faraday's Law

- Experimental finding by Michael Faraday in 1831
  - When magnetic flux enclosed by a loop of wire changes with time, a current is produced in the loop.
  - A voltage or an electromotive force (emf) is induced around the loop.



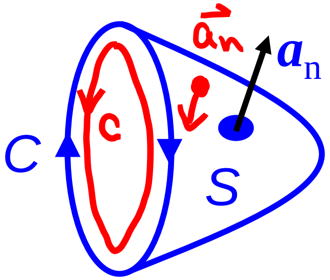
$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S}$$

emf $\neq 0$

S is the surface bounded by the closed path "C"

Faraday's Law Properties

$$\oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{s}$$



Follow right-hand rule

2

$$\mathscr{V} = \oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = \text{emf induced in circuit with contour } C$$

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{s} = \text{magnetic flux crossing surface } S$$

$$\mathscr{V} = -\frac{d\Phi}{dt}$$

1

– *S* is a surface bounded by *C*.

– The electromotive force induced in a stationary closed circuit is equal to the negative rate of increase of the magnetic flux linking the circuit.

– Faraday's law of electromagnetic induction.

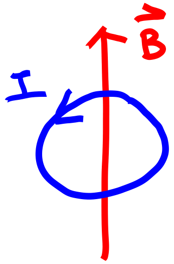
Lenz's Law

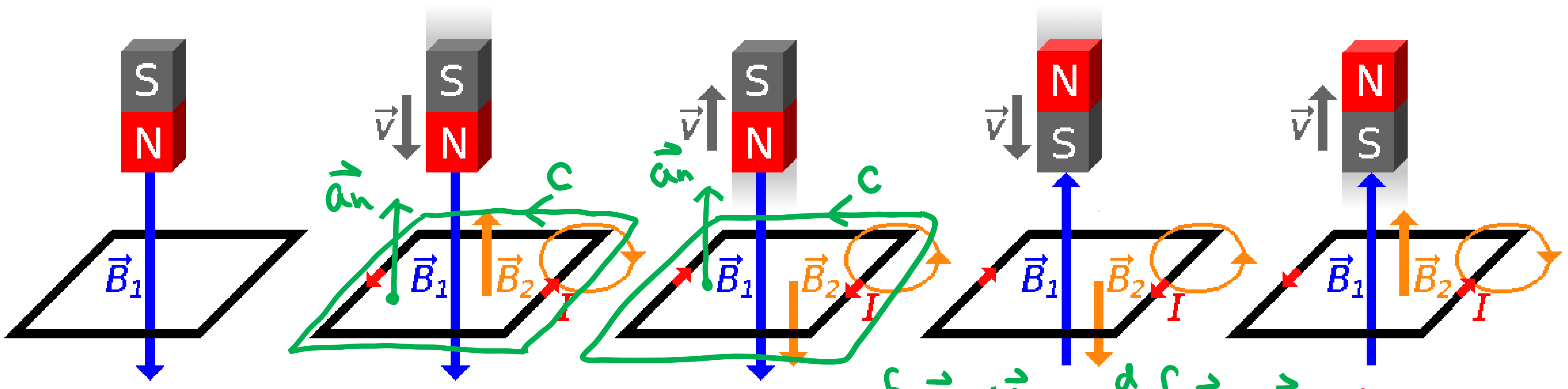
① Qualitative law

② Can specify the direction of the currents, but not the magnitude

- Induced emf causing a current to flow in the closed loop in such a direction to oppose the change in the linking magnetic flux
  - Satisfied by a minus sign and right-hand rule.

$$\mathcal{V} = -\frac{d\Phi}{dt}$$





$$\oint_C \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int_S \vec{B}\cdot d\vec{S}$$

(+) (+)

$$\oint_C \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int_S \vec{B}\cdot d\vec{S} = (-)$$

(-) (-)

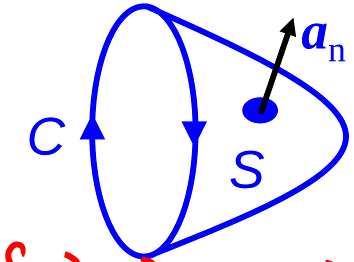
Ref: WiKi

Faraday's Law Properties

- Any surface S bounded by C can be employed
- Closed path C can be an imaginary contour
  - Time-varying magnetic flux induces an electric field in the region, and an emf is induced around the closed path.

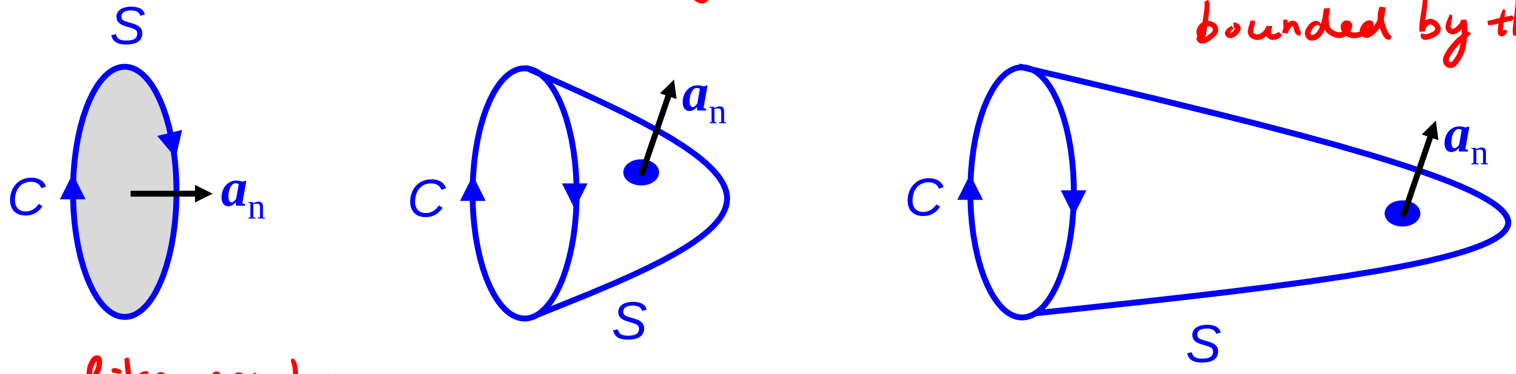
choose the surface that you can easily evaluate the surface integral

$$\oint_C \mathbf{E} \cdot d\ell = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{s}$$



Follow right-handed rule

C fixed, S arbitrary $\Rightarrow \int_S \vec{B} \cdot d\vec{s}$ magnetic flux is the same through all surfaces bounded by the same path "C"



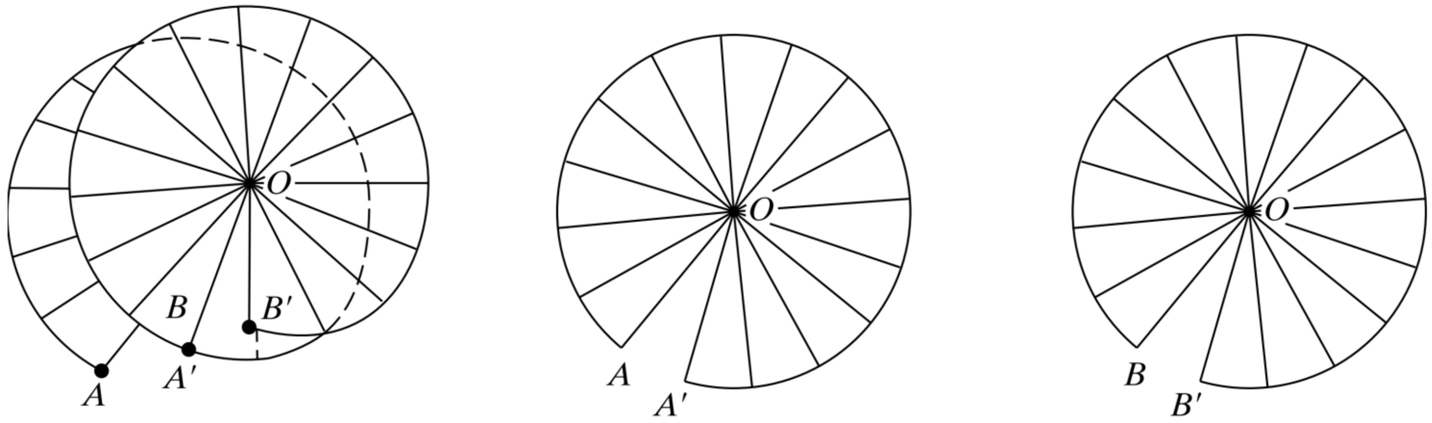
like membrance

Faraday's Law Properties

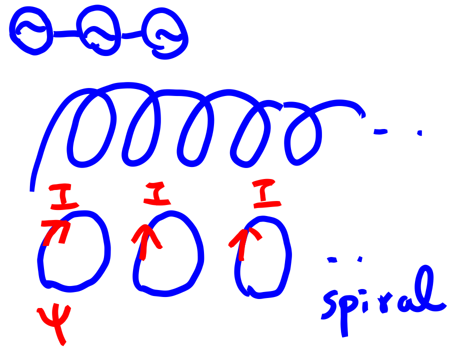
- Loop with multiple turns

$$\text{emf} = -\underset{\sim}{N}\frac{d\psi}{dt}$$

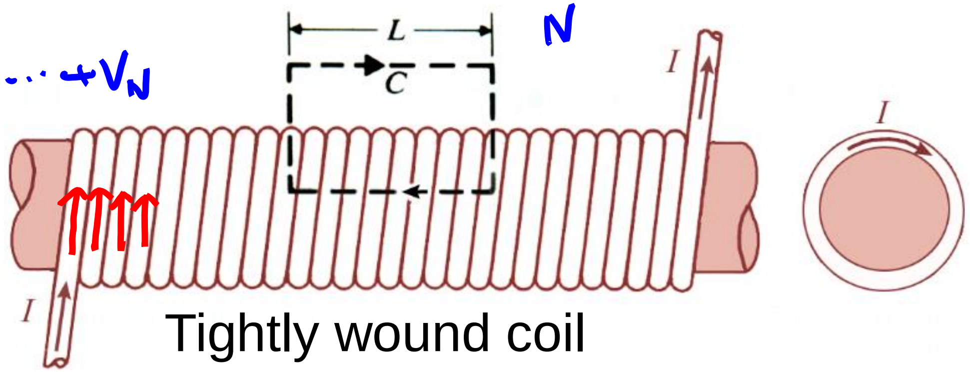
, ψ the magnetic flux through a one-turn coil



$-\frac{d\psi}{dt}$
$\parallel$
$V_1 \quad V_2 \quad V_3 \quad \rightarrow V_T = V_1 + V_2 + \cdots + V_N$



spiral ramp surface



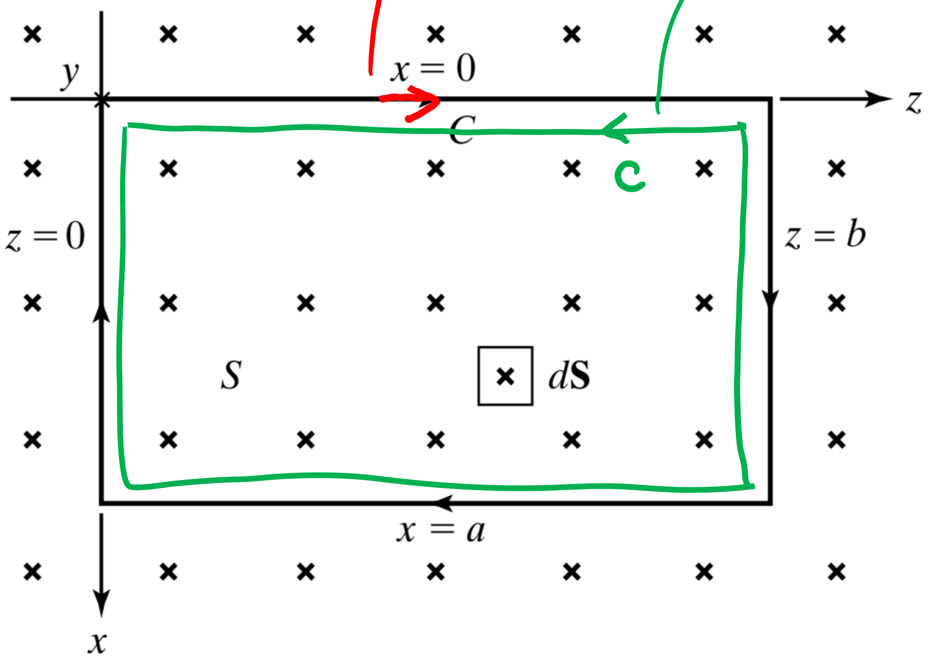
Tightly wound coil

Example

- A time-varying magnetic field is given by $\mathbf{B} = B_0 \cos \omega t \, \mathbf{a}_y$ where B_0 is a constant. Find the induced emf around the rectangular loop C in the xz-plane bounded by the lines $x = 0$, $x = a$, $z = 0$, and $z = b$, as shown below.

$\oint d\vec{s}$ direction follows the right-hand rule

$d\vec{s} = dxdz\,\vec{a}_y$ $\quad$ $d\vec{s} = dxdz\,(-\vec{a}_y$



$\text{emf} = \oint_c \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int_s \vec{B}\cdot d\vec{s} = -\frac{d\psi}{dt}$

$$\psi = \int_S \mathbf{B} \cdot d\mathbf{S}$$

$$= \int_{z=0}^{b} \int_{x=0}^{a} B_0 \cos \omega t \, \mathbf{a}_y \cdot dx \, dz \, \mathbf{a}_y$$

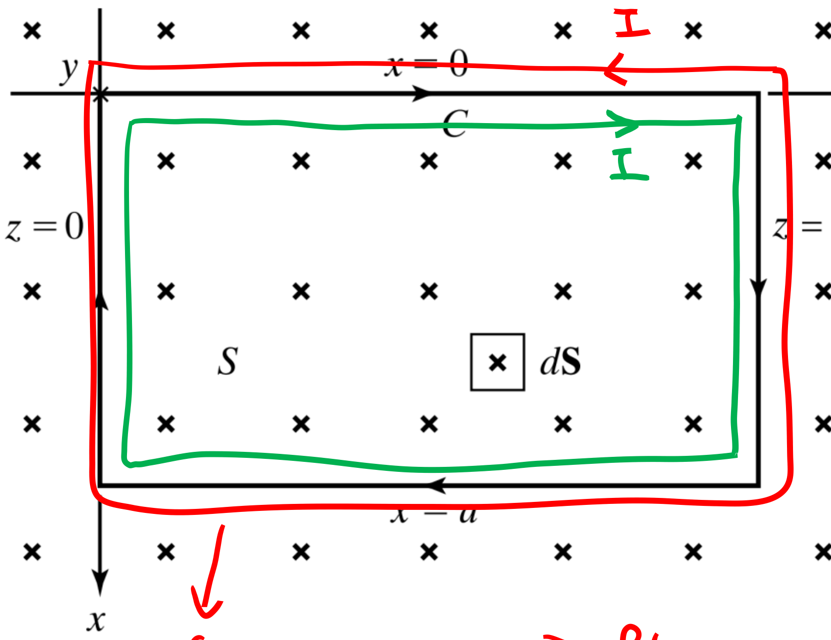
$$= abB_0 \cos \omega t$$

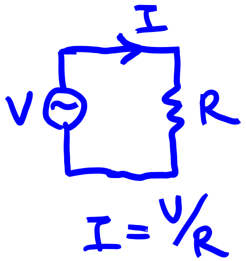
$$\text{emf} = \oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S}$$

$$= abB_0 \omega \sin \omega t$$

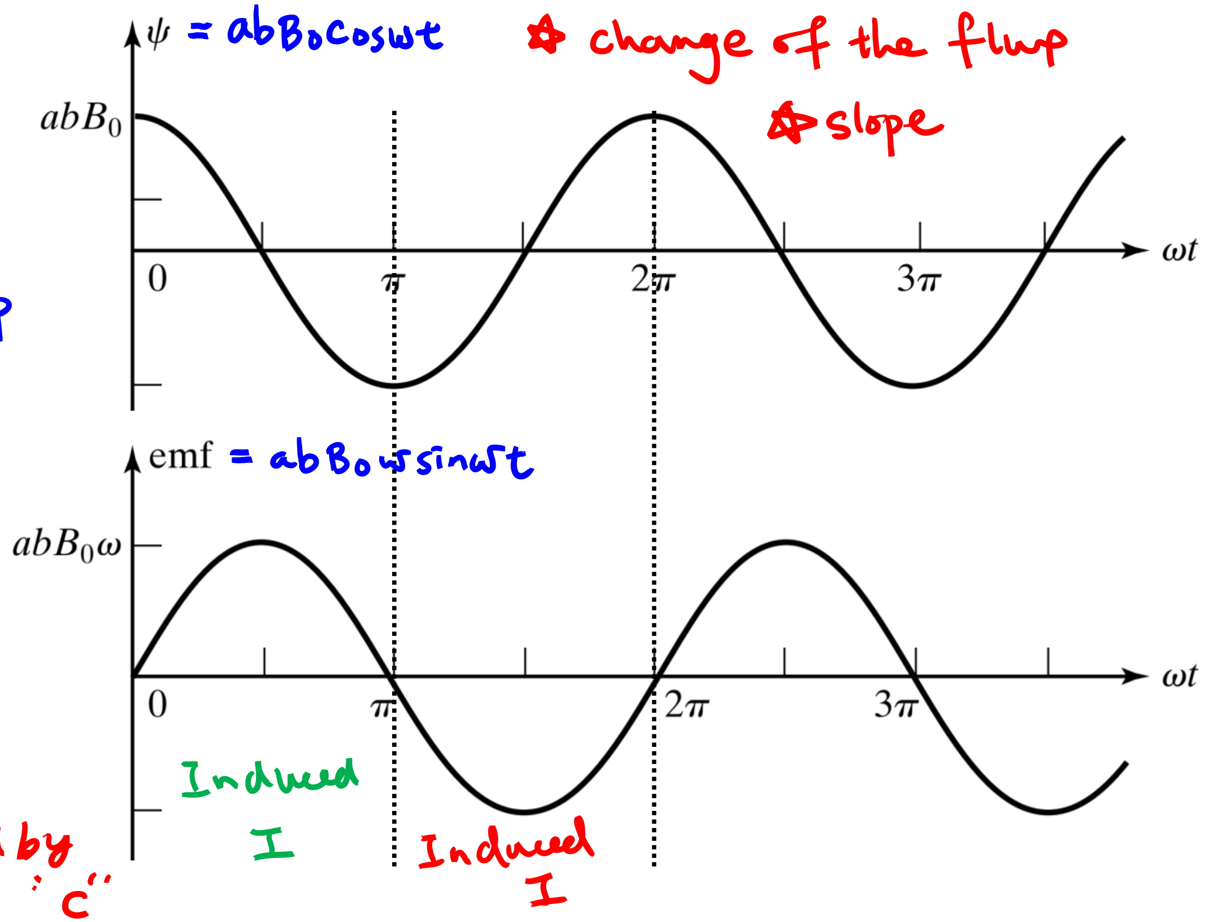
Example

- Consistent with Len's law





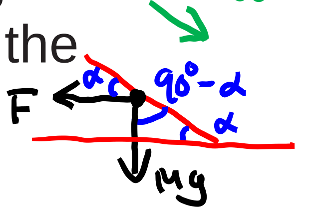
Generate magnetic flux to oppose the increase of the magnetic flux enclosed by "C"



P2.14

- A rigid conducting bar of length L, mass M, and electrical resistance R rolls without friction down two parallel conducting rails that are inclined at an angle α with the horizontal, as shown below. The rails are of negligible resistance and are joined at the bottom by another conductor, also of negligible resistance, so that the total resistance of the loop formed by the rolling bar and the three other sides is R. The entire arrangement is situated in a region of uniform static magnetic field $\mathbf{B}=B_0\mathbf{a}_z$ Wb/m^2, directed vertically downward. Assume the bar to be rolling down with uniform velocity $\mathbf{v}$ parallel to the rails under the influence of Earth's gravity (acting in the positive z-direction) and the magnetic force due to the current in the loop produced by the induced emf. Show that v is equal to $(MgR/B_0^2L^2)\tan\alpha\sec\alpha$.

$F_{net} = 0$



net force = 0

① $\oint_C \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int_S \vec{B}\cdot d\vec{S}$

$= -\frac{d}{dt}(B_0 \cos\alpha\, wL)$

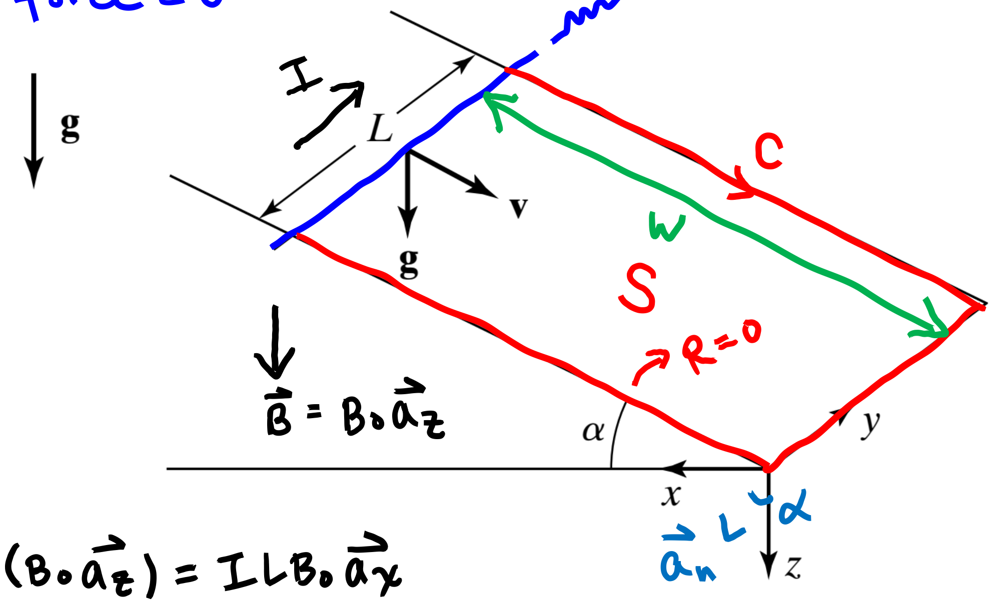
$= -B_0 L\cos\alpha \frac{dw}{dt}$ $\quad \frac{dw}{dt} = -v$

$= B_0 L v\cos\alpha$

$\Rightarrow I = \text{emf}/R = \frac{B_0 L v}{R}\cos\alpha$

② Magnetic force on the bar

$\vec{F} = I d\vec{l}\times\vec{B} = IL\vec{a}_y \times (B_0\vec{a}_z) = ILB_0\vec{a}_x$



③ Net force = 0

$\Rightarrow Mg\cos(90°-\alpha) = ILB_0\cos\alpha$

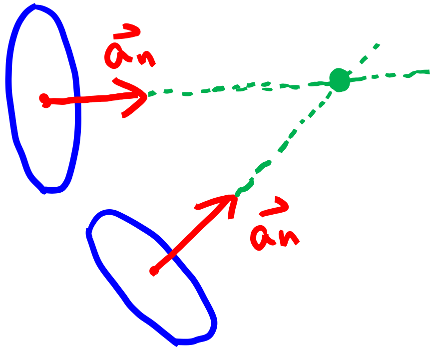
$\Rightarrow Mg\sin\alpha = \frac{B_0 L v\cos\alpha}{R} B_0 L\cos\alpha$

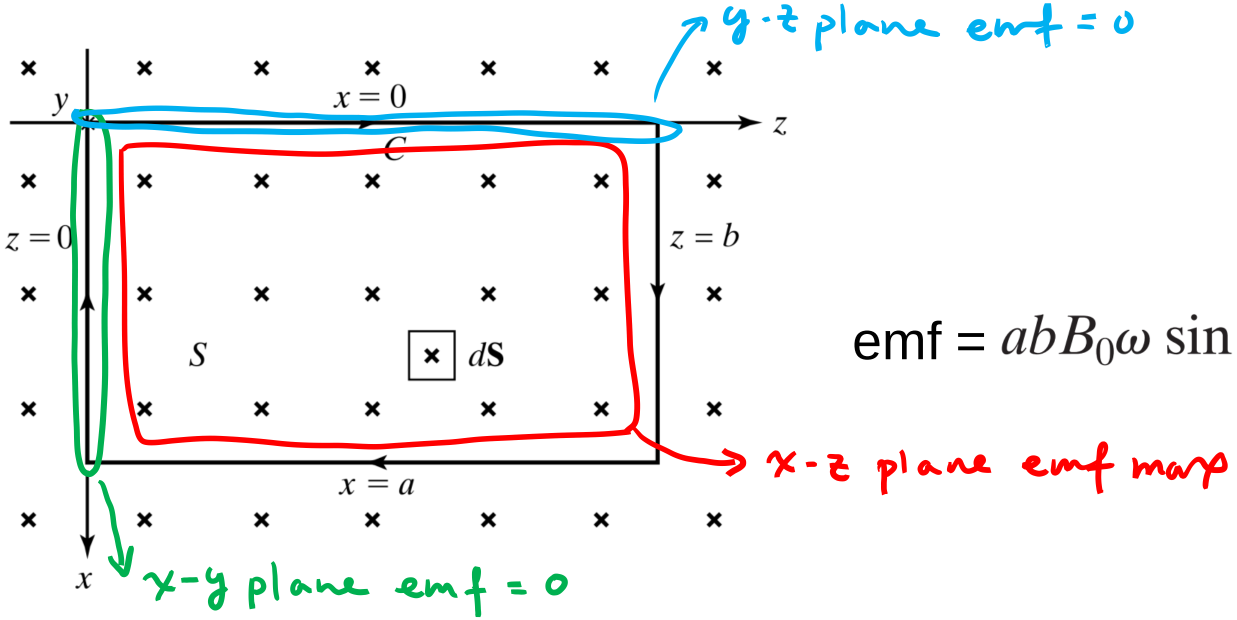
$\Rightarrow v = \frac{MgR\sin\alpha}{B_0^2 L^2 \cos^2\alpha}$

$= \frac{MgR}{B_0^2L^2}\tan\alpha\sec\alpha$

Faraday's Law Applications

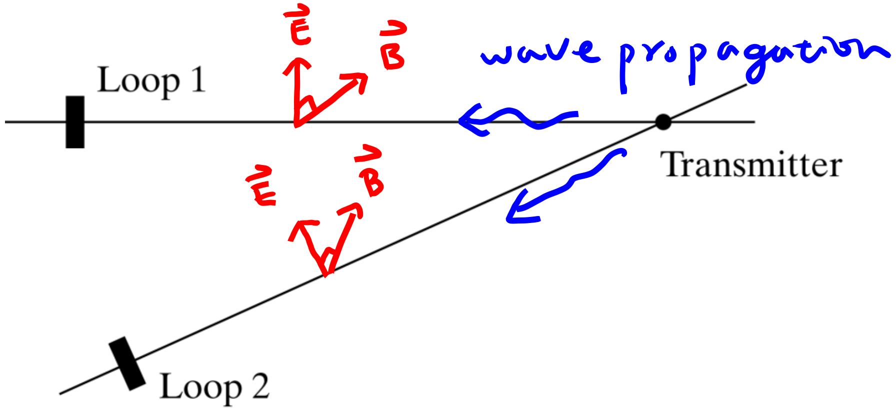
- Voltage induced varies as the orientation of the loop changed
  - Loop antennas for pocket AM radios and TV.
  - Locating transmitter.

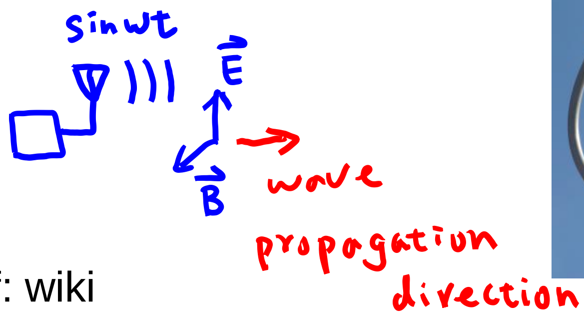




emf = $abB_0\omega \sin \omega t$ → emf ∝ ω

Loop Antenna

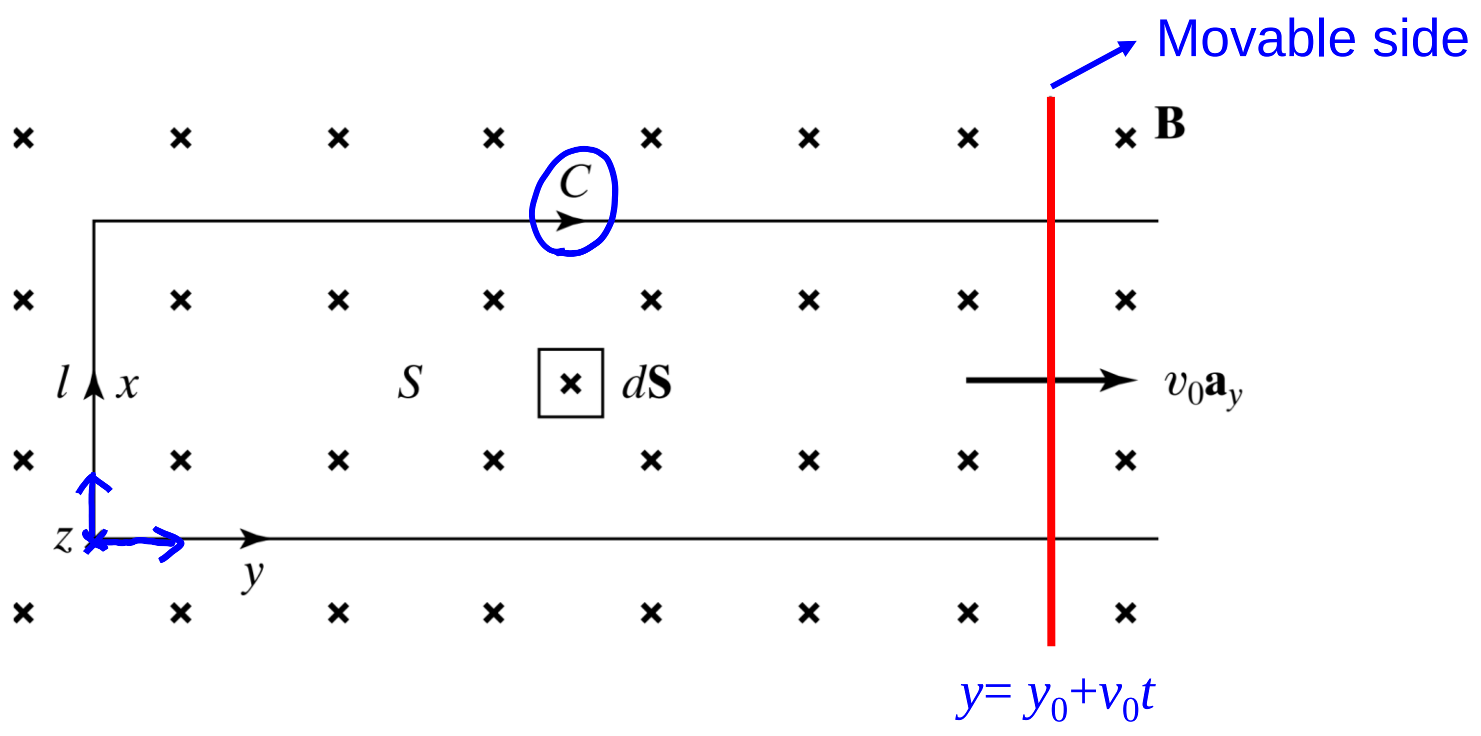




Ref: wiki

Example

- A rectangular loop of wire with three sides fixed and the fourth side movable is situated in a plane perpendicular to a uniform magnetic field **B**= $B_0\mathbf{a_z}$ as show below. The movable side consists of a conducting bar moving with a velocity v_0 in the y-direction. Find the emf induced around the closed path C of the loop.



Example

- Consistent with Len's law

$$\psi = \int_S \mathbf{B}\cdot d\mathbf{S} = \int_S B_0\mathbf{a}_z \cdot dx\,dy\,\mathbf{a}_z$$

$$= \int_{x=0}^{l}\int_{y=0}^{y_0+v_0t} B_0\,dx\,dy$$

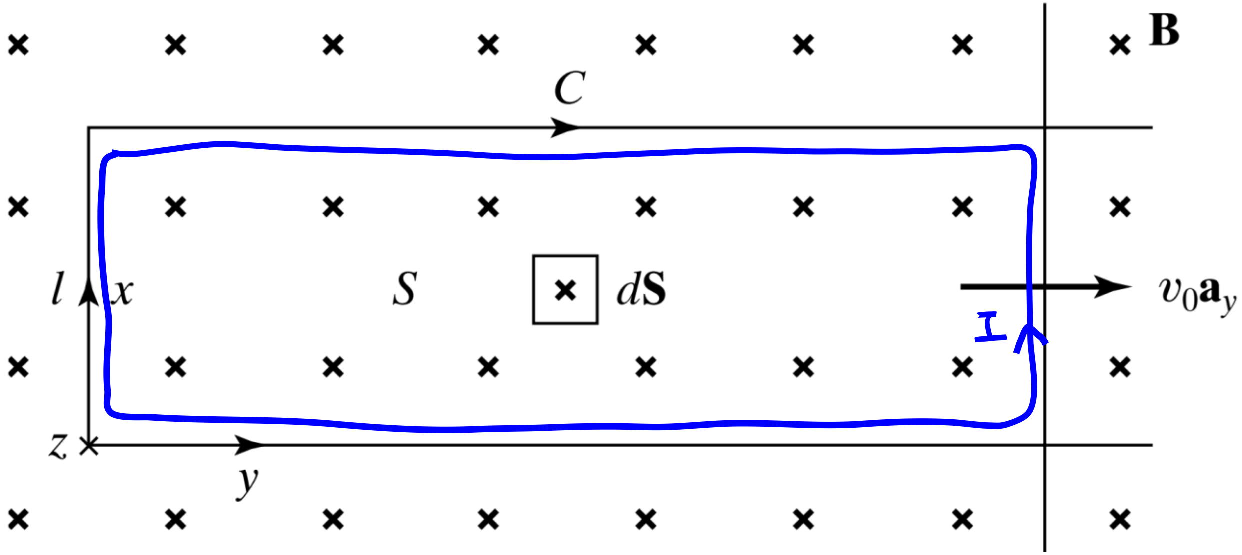
$$= B_0 l(y_0 + v_0 t)$$

$$\text{emf} = \oint_C \mathbf{E}\cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{S}$$

$$= -\frac{d}{dt}[B_0 l(y_0 + v_0 t)]$$

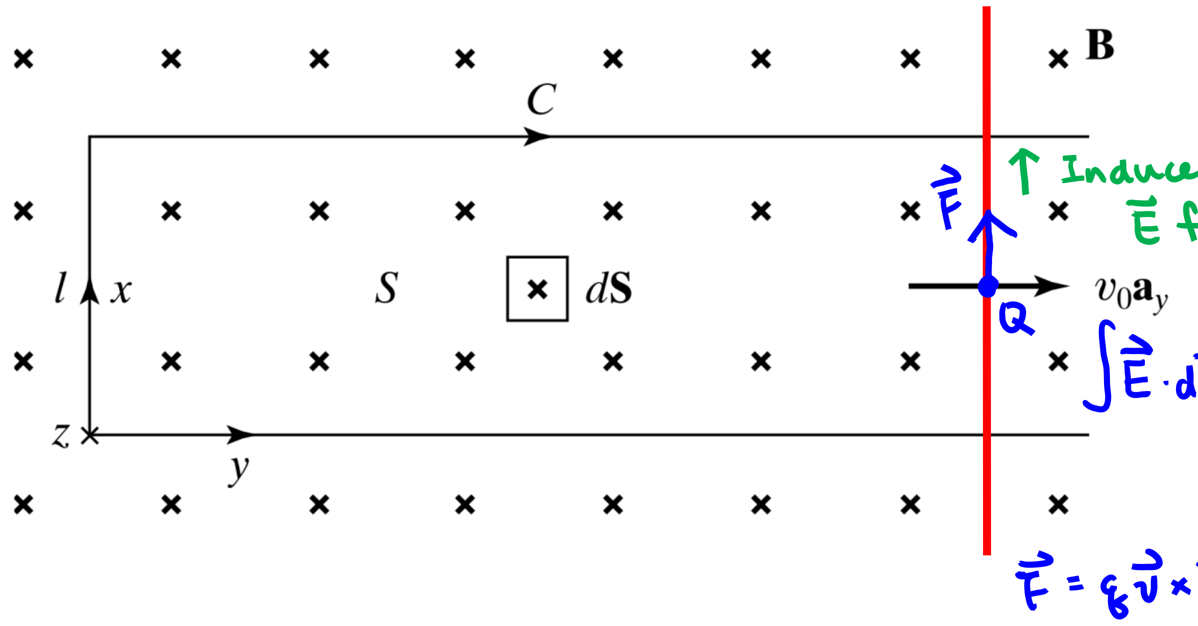
$$= -B_0 l v_0$$

Produce a current opposite to "C"



Another Interpretation

- Being due to the electric field induced in the moving bar by its motion perpendicular to the magnetic field
  - Motional emf concept.



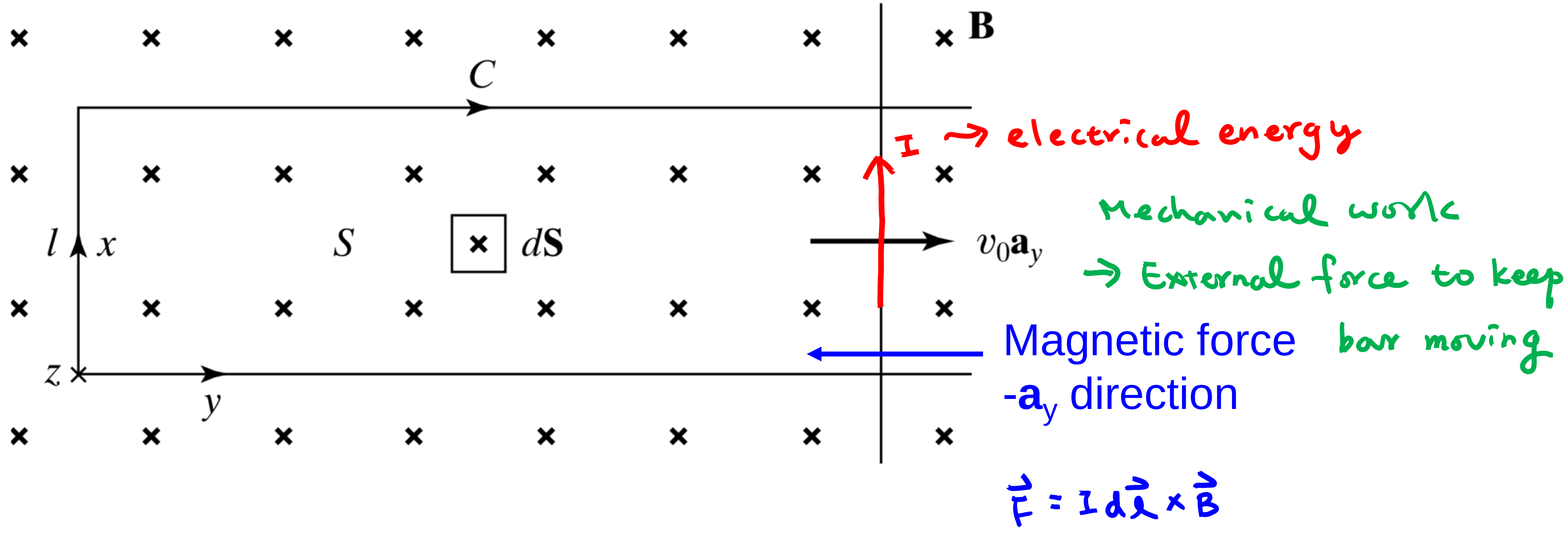
$$\mathbf{F} = Q\mathbf{v} \times \mathbf{B} = Qv_0\mathbf{a}_y \times B_0\mathbf{a}_z = Qv_0B_0\mathbf{a}_x = Q\vec{E}_{eq}$$

$$\Rightarrow \vec{E}_{eq} = \mathbf{F}/Q = v_0B_0\mathbf{a}_x$$

$$\int \vec{E}\cdot d\vec{l} = \int_{x=0}^{l} v_0B_0\mathbf{a}_x \cdot dx\,\mathbf{a}_x = \int_{x=0}^{l} v_0B_0\,dx = \underbrace{v_0B_0l}_{\text{induced emf}}$$

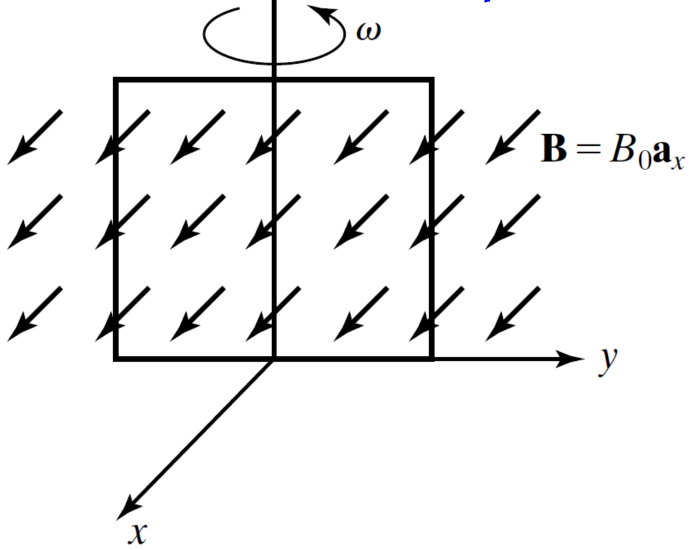
Faraday's Law Applications

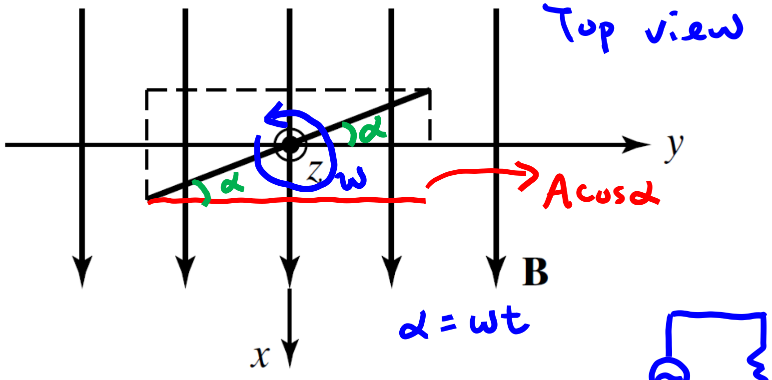
- Electromechanical Energy Converter
- External force must be exerted in the $+\mathbf{a}_y$ direction to keep the bar moving
  - Mechanical work converted into electrical energy in the loop.



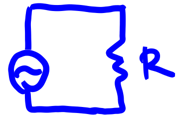
Rotating Power Generator

→ keep rotating by wind, water, ...



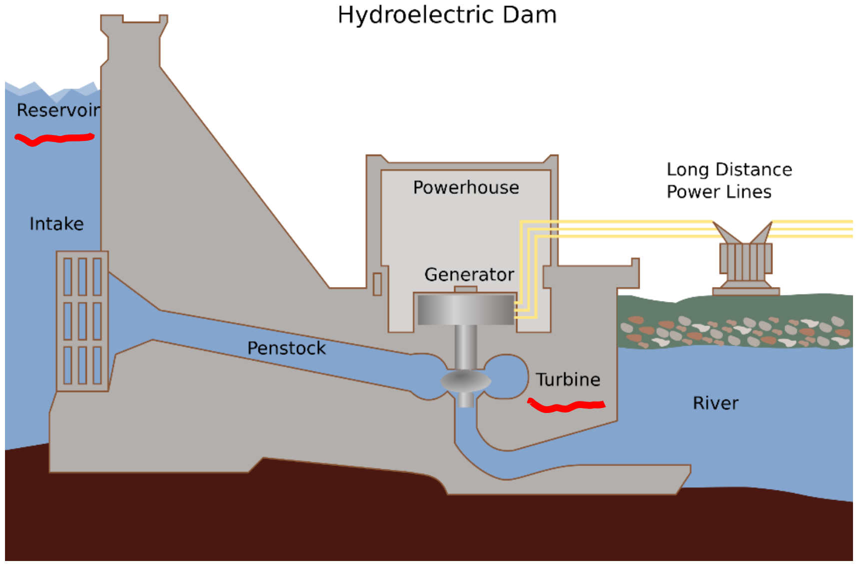


$$\psi = B_0 A \cos \omega t$$
$$\text{emf} = -d\psi/dt$$
$$= \omega B_0 A \sin \omega t$$ ac voltage



- Rotor and stator


Rotor (lower left) and stator (upper right) of an electric motor

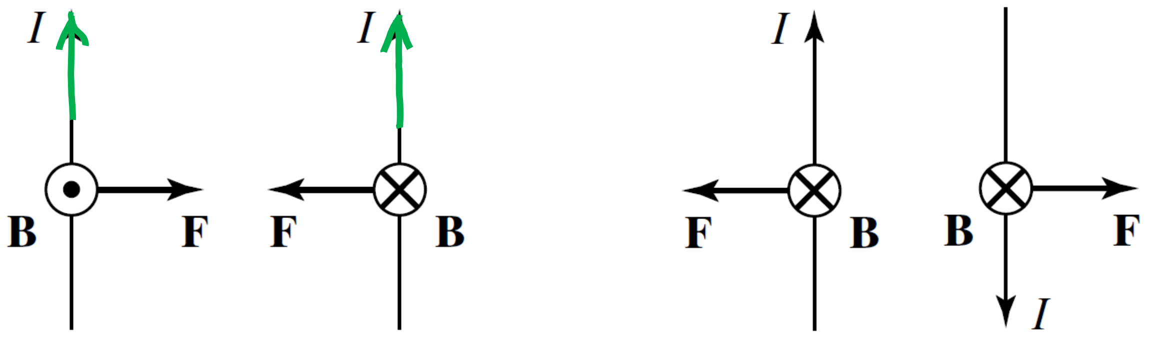


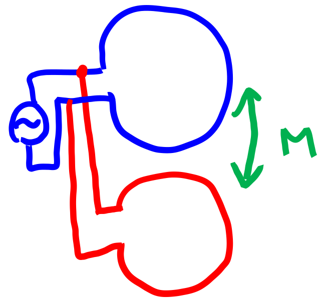
Ref: wiki

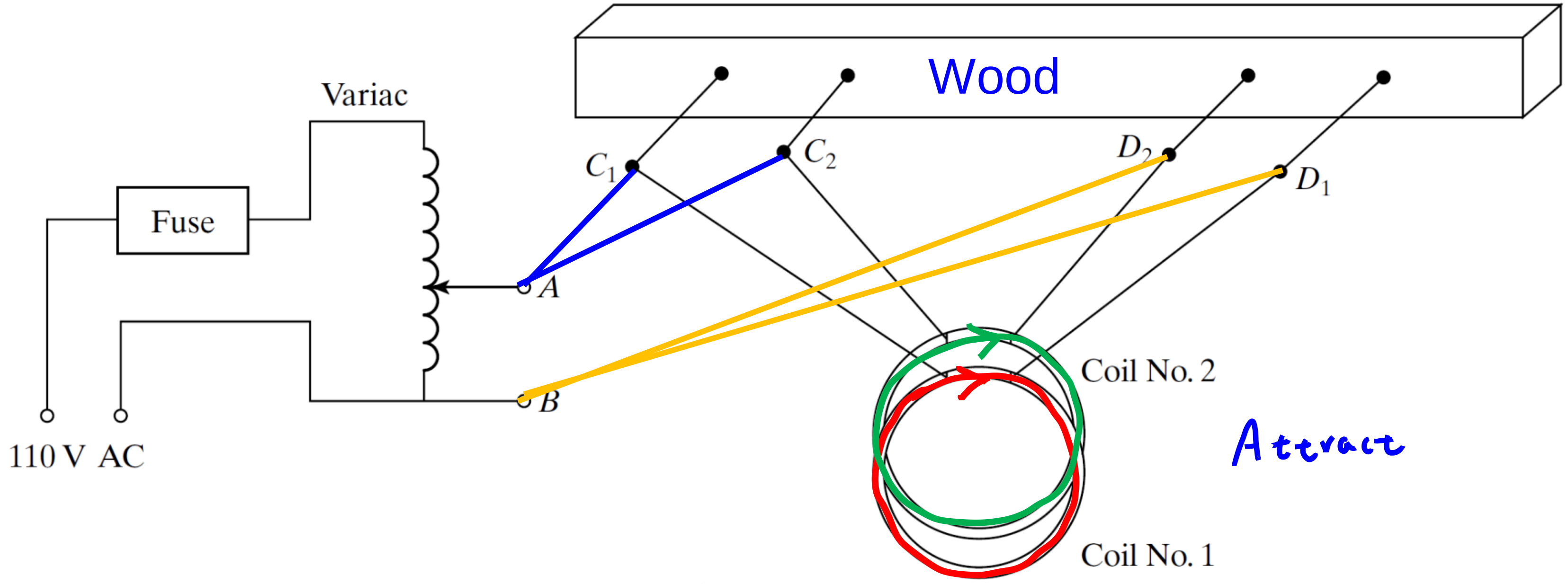
Faraday’s Law

- Demonstrate Faraday’s law and Ampère’s force law.
  - Applications: Maglev train.

Magnetic levitation

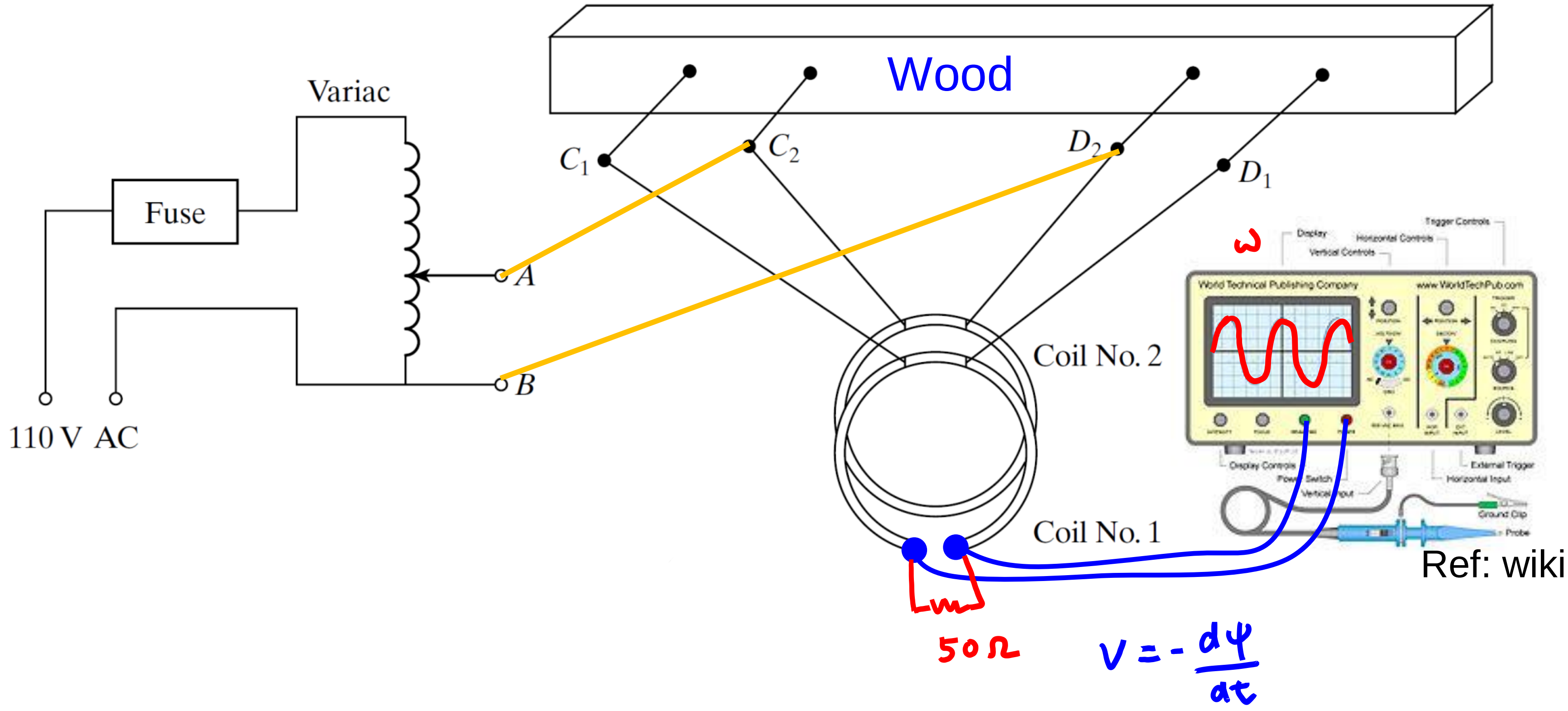






Faraday's Law

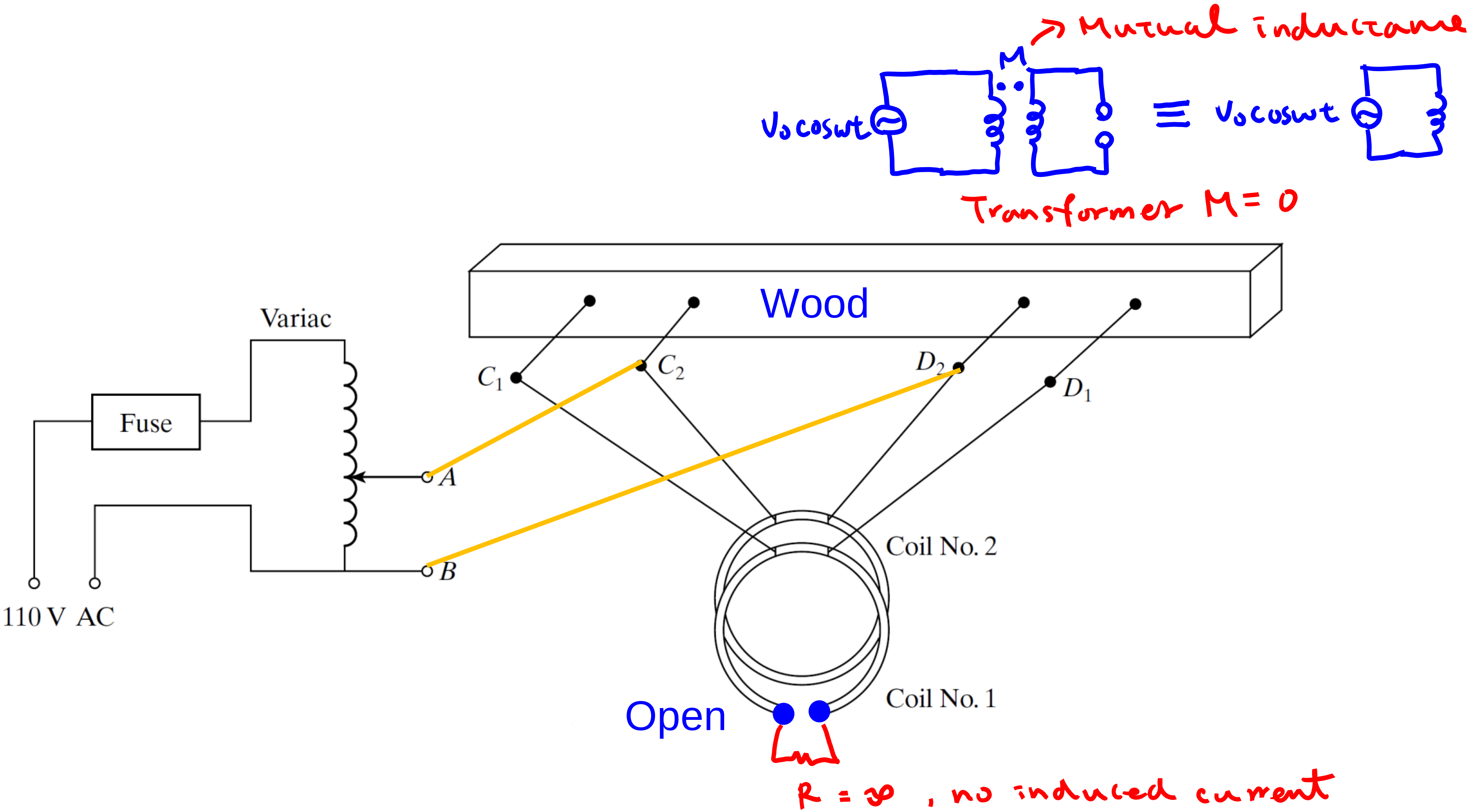
- Connect coil No. 2 to the variac and coil No. 1 to an oscilloscope to observe the induced voltage in coil No. 1



Ref: wiki

Faraday's Law

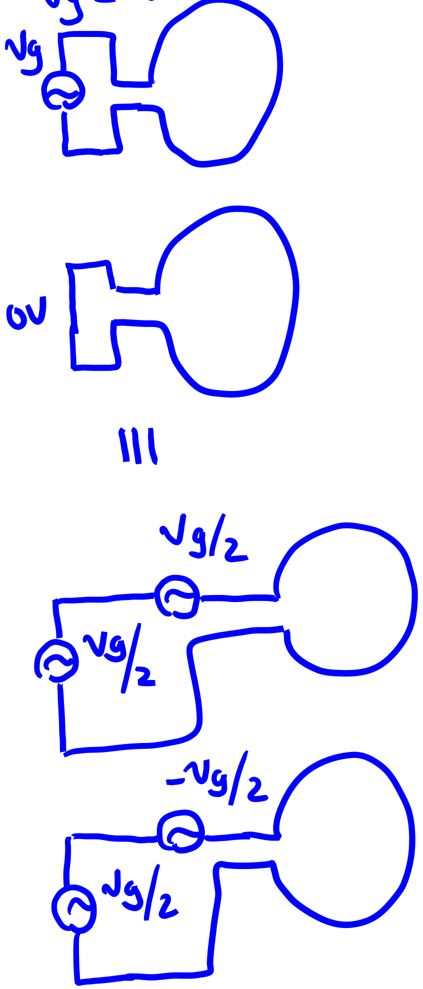
- Connect coil No. 2 to the variac and leave coil No. 1 open-circuited

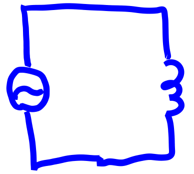


Faraday's Law

- Connect coil No. 2 to the variac and leave coil No. 1 short-circuited
  - Demonstrate Faraday's law and Ampère's force law.

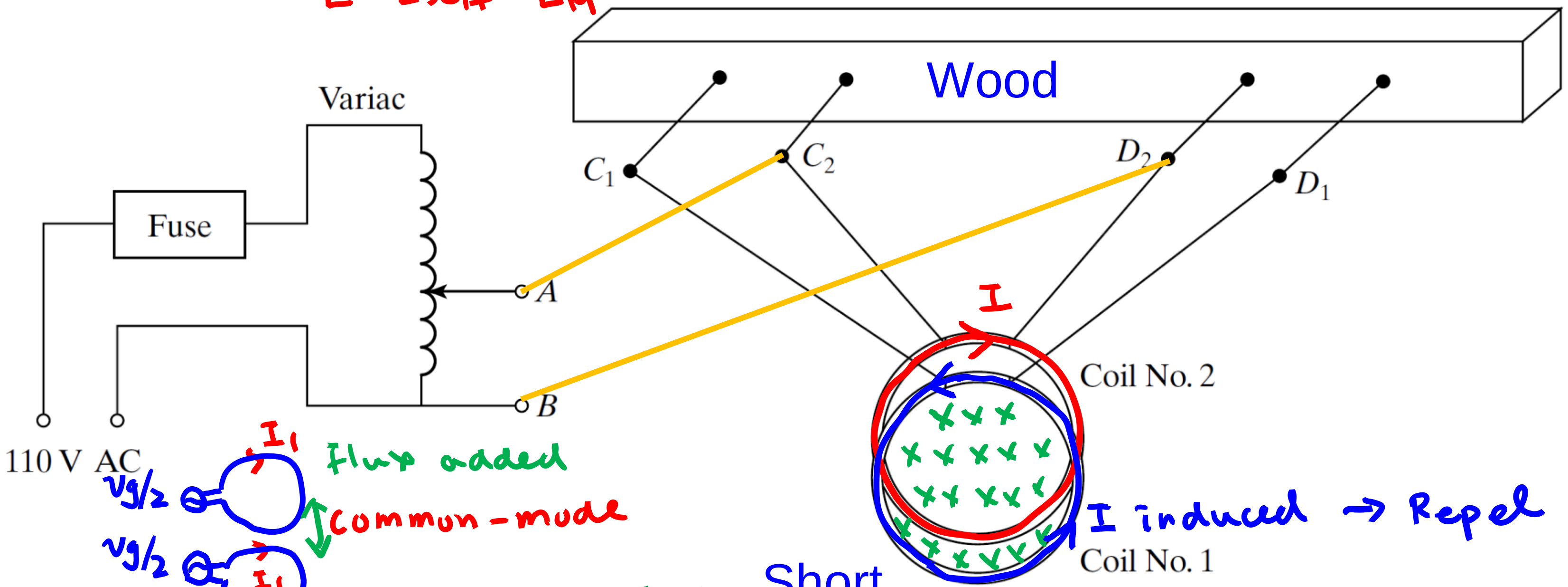
$v_g = V_0 \cos\omega t$





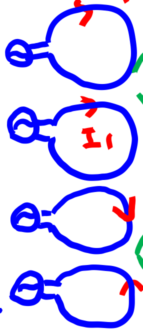
$L = L_{self} + L_M$, $L = \phi/I$, $V = L\frac{di}{dt}$

$L = L_{self} - L_M$



I induced → Repel

≡



Flux added

common-mode

Flux reduced

differential-mode

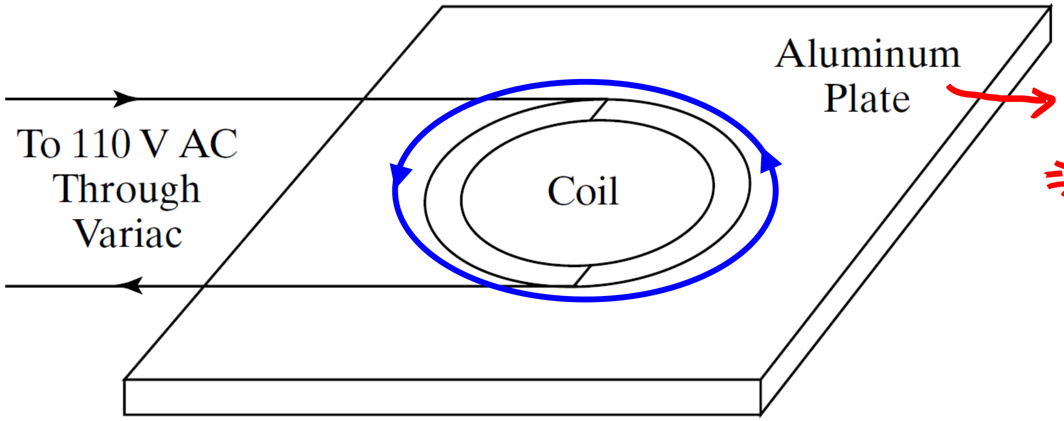
$I_2 > I_1$

→ I_2 must be increased to keep the same voltage $v_g/2$

$F_T = F_{CM} + F_{DM}$

Magnetic Levitation

- Levitation due to the repulsive action of the coil current and the induced currents in the metallic plate



heavy

⇒ won't move

Ref: Amazon

Outline

- Inductive and deductive approaches
- Line integral, surface integral, and volume integral
- Faraday’s law
- Ampère’s circuital law
- Gauss’ laws
- Law of conservation of charge
- Applications to static fields

Maxwell's Equations

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S}$$ Faraday's law

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt}\int_S \mathbf{D} \cdot d\mathbf{S}$$ Ampère's circuital law

$$\left.\begin{aligned} &\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho \, dv \\ &\oint_S \mathbf{B} \cdot d\mathbf{S} = 0 \end{aligned}\right\} \text{Gauss' laws}$$

Ampère’s Circuital Law

- Combination of
  - Experimental finding by Oersted that electric currents generate magnetic fields.
  - Mathematical contribution of Maxwell that time-varying electric fields give rise to magnetic fields.

$$\oint_C \frac{\mathbf{B}}{\mu_0} \cdot d\mathbf{l} = [I_c]_S + \underbrace{\frac{d}{dt}\int_S \varepsilon_0 \mathbf{E} \cdot d\mathbf{S}}_{\text{Added by Maxwell}}$$

Ampère's Circuital Law

$$\oint_C \overset{=\mathbf{H}}{\boxed{\frac{\mathbf{B}}{\mu_0}}} \cdot d\mathbf{l} = \overset{=\int_S \mathbf{J}\cdot d\mathbf{S}}{\boxed{[I_c]_S}} + \underline{\frac{d}{dt}\int_S \overset{=\mathbf{D}}{\boxed{\varepsilon_0\mathbf{E}}}\cdot d\mathbf{S}}$$

$[\vec{D}\cdot d\vec{s}]$ = charge

Displacement current → Mathematically equivalent to a current across S
→ Not due to flow of charges

- *S* is a surface bounded by *C*.
- Magnetic force doesn't do work in the movement of the charge.
- $\vec{H} = \vec{B}/\mu_0$ – $\mathbf{B}/\mu_0$: magnetic field intensity vector, denoted as **H**, [**B**]= [(permeability)(current)(length)] per [(distance)^2], [**H**]= current per unit distance or <u>amp/m.</u>

  ↙ μ magnetic polarization

- Electromotive force $\underline{\oint_C \mathbf{E}\cdot d\mathbf{l}}$, magnetomotive force (mmf) $\underline{\oint_C \mathbf{H}\cdot d\mathbf{l}}$.

  work

  no physical meaning of work

  $\vec{F} = q\vec{v}\times\vec{B}$

- Current due to flow of free <u>charges</u> crossing the surface *S*.

  Conduction current due to motion of charges in a conductor

$$[I_c]_S = \int_S \mathbf{J}\cdot d\mathbf{S}$$

↗ ε electrical polarization

$\vec{D} = \varepsilon_0\vec{E}$

- $\varepsilon_0\mathbf{E}$: Displacement vector or the displacement flux density vector, denoted by the symbol **D**, [**E**]= charge per [(permittivity)(distance)^2], [**D**]= charge per area or <u>C/m^2</u>.

Fundamental Quantities Review

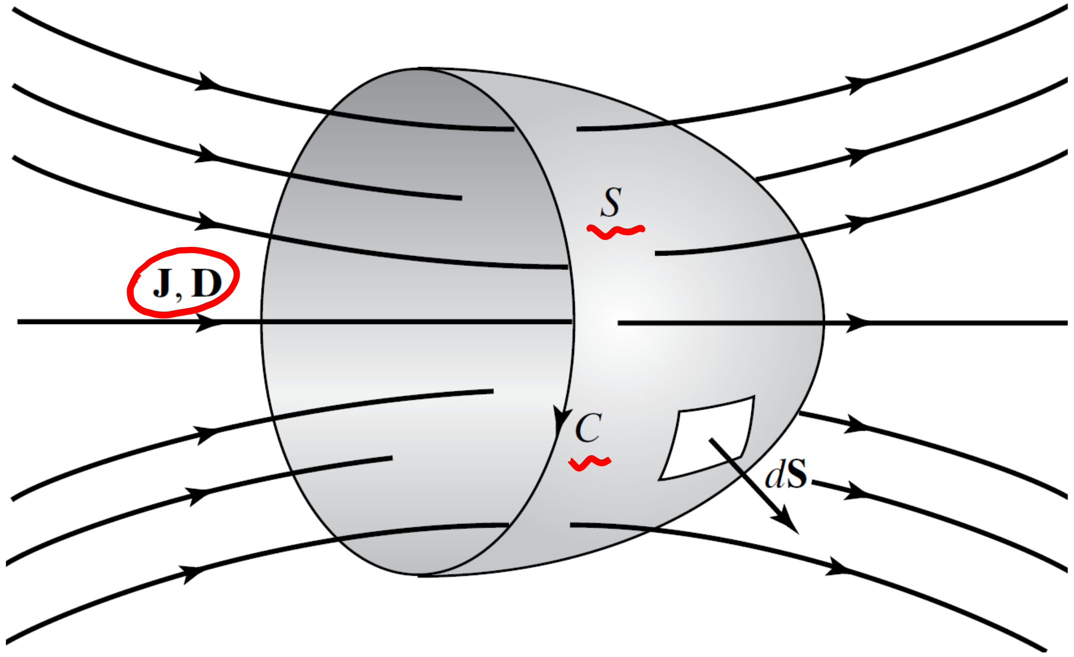
Symbols and Units for Field Quantities	Field Quantity	Symbol	Unit
Electric	Electric field intensity	**E**	V/m
	Electric flux density (Electric displacement)	**D**	C/m^2
Magnetic	Magnetic flux density	**B**	T $= \frac{wb}{m^2}$
	Magnetic field intensity	**H**	A/m

Ampère's Circuital Law

$$\oint_C \frac{\mathbf{B}}{\mu_0} \cdot d\mathbf{l} = [I_c]_S + \frac{d}{dt}\int_S \varepsilon_0 \mathbf{E} \cdot d\mathbf{S}$$

$$\rightarrow \oint_C \vec{H} \cdot d\vec{l} = \int_S \vec{J} \cdot d\vec{S} + \frac{d}{dt}\int_S \vec{D} \cdot d\vec{S}$$

– mmf around a closed path *C* equals the sum of the current due to the flow of charges and the displacement current bounded by *C*.

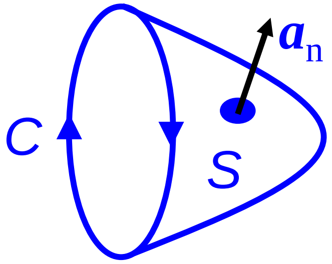


Ampère’s Circuital Law

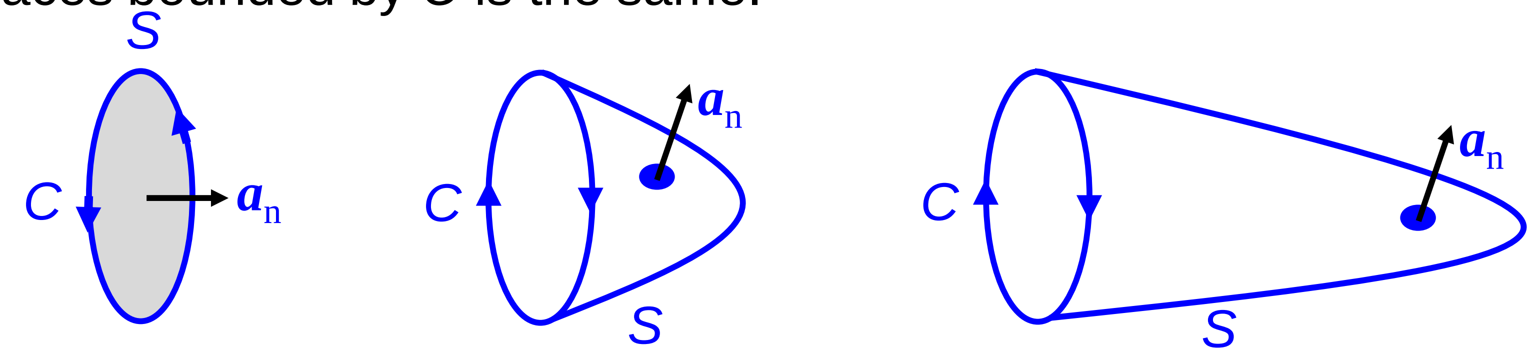
$$\oint_C \frac{\mathbf{B}}{\mu_0}\cdot d\mathbf{l} = [I_c]_S + \frac{d}{dt}\int_S \varepsilon_0 \mathbf{E}\cdot d\mathbf{S} \rightarrow \oint_C \vec{H}\cdot d\vec{\ell} = \int_S \vec{J}\cdot d\vec{S} + \frac{d}{dt}\int_S \vec{D}\cdot d\vec{S}$$

should be the same surface

- Follow right-hand rule

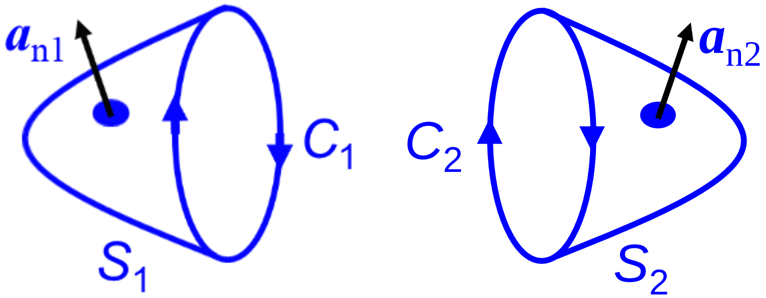


- Any surface S bounded by C can be employed $\rightarrow \oint_C \vec{H}\cdot d\vec{\ell}$ = mmf is unique
  - The sum of the current due to the flow of charges and displacement current through all possible surfaces bounded by C is the same.

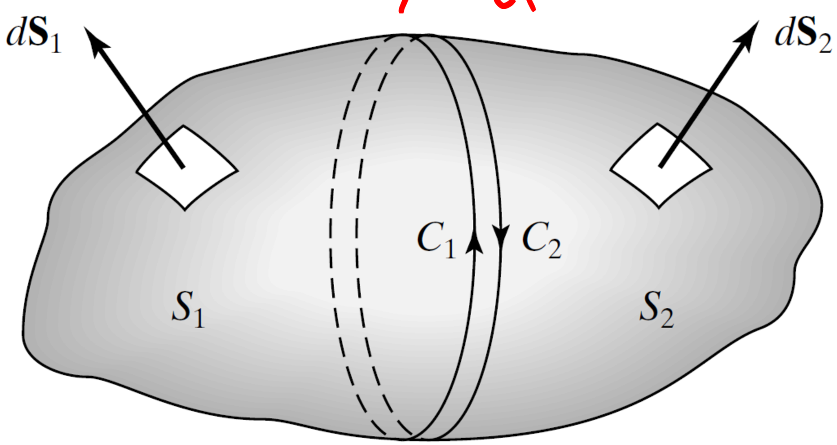


Ampère's Circuital Law

- S_1 and S_2 form a closed surface



$\oint_{C_1} \vec{H}\cdot d\vec{l} = -\oint_{C_2} \vec{H}\cdot d\vec{l}$



closed surface

$S = S_1 + S_2$

$$\oint_{C_1} \mathbf{H}\cdot d\mathbf{l} = \int_{S_1} \mathbf{J}\cdot d\mathbf{S}_1 + \frac{d}{dt}\int_{S_1} \mathbf{D}\cdot d\mathbf{S}_1$$

$+$

$$\oint_{C_2} \mathbf{H}\cdot d\mathbf{l} = \int_{S_2} \mathbf{J}\cdot d\mathbf{S}_2 + \frac{d}{dt}\int_{S_2} \mathbf{D}\cdot d\mathbf{S}_2$$

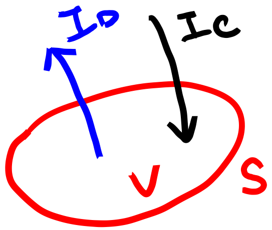
$$\underset{\smile}{0} = \oint_{\underbrace{S_1+S_2}_{= S}} \mathbf{J}\cdot d\mathbf{S} + \frac{d}{dt}\oint_{\underbrace{S_1+S_2}_{= S}} \mathbf{D}\cdot d\mathbf{S}$$

Ampère's Circuital Law

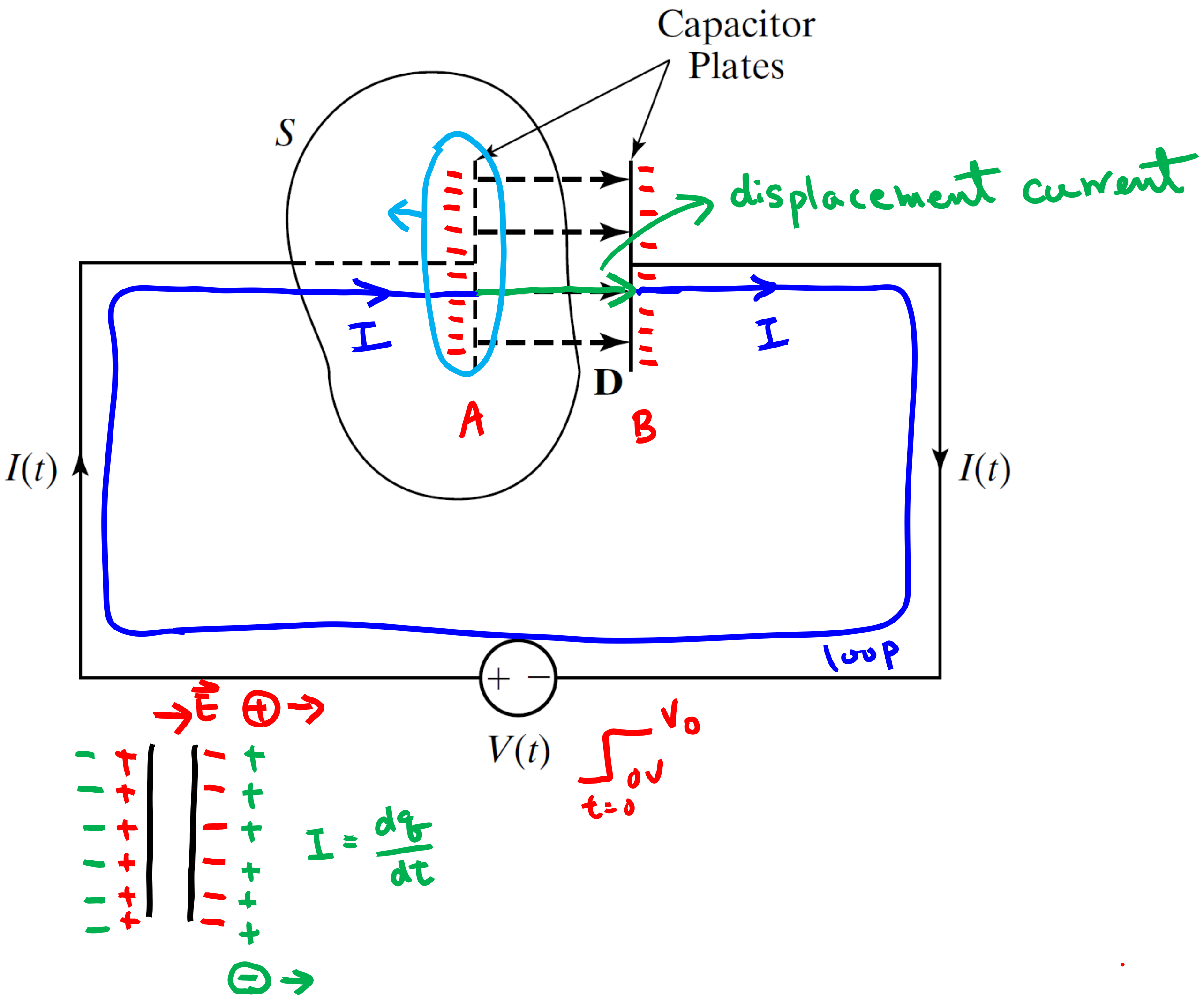
$$\oint_S \mathbf{J}\cdot d\mathbf{S} + \frac{d}{dt}\oint_S \mathbf{D}\cdot d\mathbf{S} = 0$$

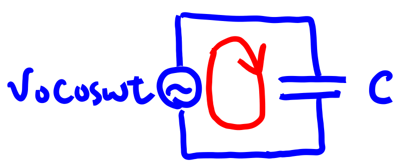
$$\frac{d}{dt}\oint_S \mathbf{D}\cdot d\mathbf{S} = -\oint_S \mathbf{J}\cdot d\mathbf{S}$$

– Displacement current emanating from a closed surface equals the current due to charges flowing into the volume bounded by that closed surface.



Capacitor Circuit





$i_c = C\frac{dV}{dt}$

$i_c = 0$ for static case

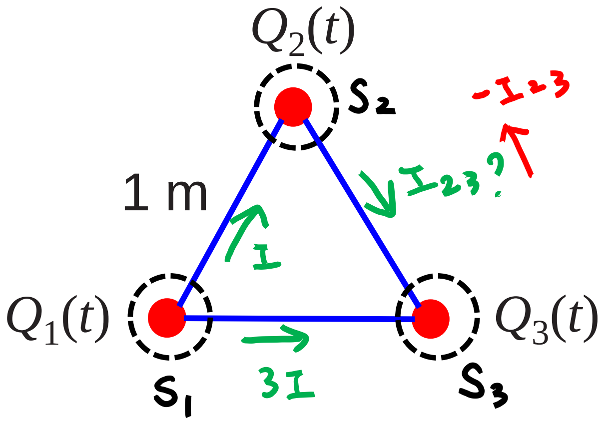
$$\frac{d}{dt}\oint_S \mathbf{D}\cdot d\mathbf{S} = -\oint_S \mathbf{J}\cdot d\mathbf{S}$$

$$\frac{d}{dt}\oint_S \mathbf{D}\cdot d\mathbf{S} = I(t)$$

$$\frac{d}{dt}\oint_S \mathbf{D}\cdot d\mathbf{S} = \frac{d}{dt}(DA) = I(t)$$

D2.9

- Three point charges $Q_1(t)$, $Q_2(t)$, and $Q_3(t)$ situated at the corners of an equilateral triangle of sides 1 m are connected to each other by wires along the sides of the triangle. Currents of I A and $3I$ A flow from Q_1 to Q_2 and Q_1 to Q_3, respectively. The displacement current emanating from a spherical surface of radius 0.1 m and centered at Q_2 is $-2I$ A. Find the following: (a) the current flowing from Q_2 to Q_3; (b) the displacement current emanating from the spherical surface of radius 0.1 m and centered at Q_1; and (c) the displacement current emanating from the spherical surface of radius 0.1 m and centered at Q_3.



(a) $I + (-I_{23}) = -2I$

$\Rightarrow I_{23} = 3I$

(b) $(-I) + (-3I) = -4I$

(c) $3I + 3I = 6I$

Ampère’s Circuital Law + Faraday’s Law

$$\begin{cases} \oint_C \mathbf{E}\cdot d\mathbf{l} = -\dfrac{d}{dt}\displaystyle\int_S \mathbf{B}\cdot d\mathbf{S} \\ \oint_C \mathbf{H}\cdot d\mathbf{l} = \displaystyle\int_S \mathbf{J}\cdot d\mathbf{S} + \dfrac{d}{dt}\displaystyle\int_S \mathbf{D}\cdot d\mathbf{S} \end{cases}$$

Time-varying **B** → **E**

Interdependent

If static, independent

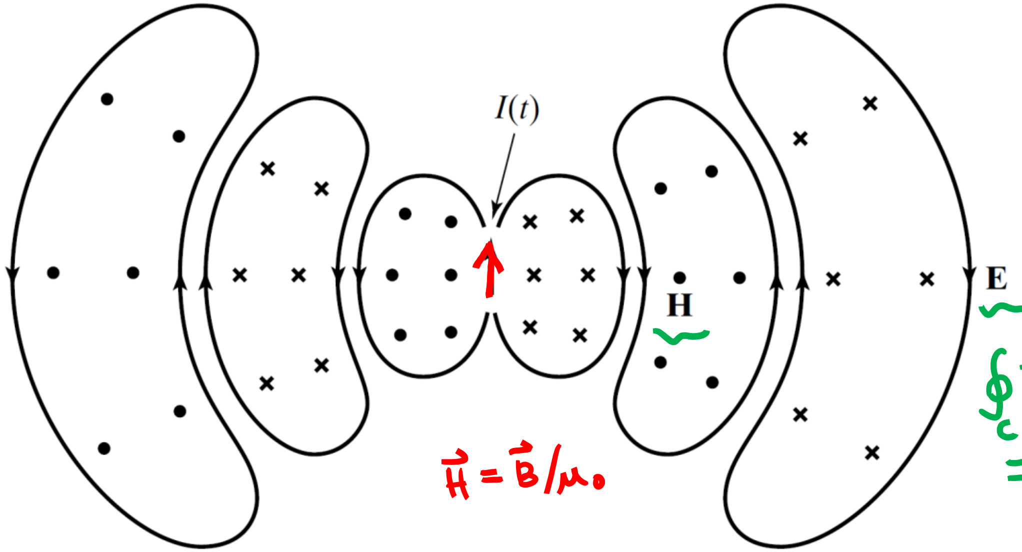
Time-varying **E** → **B**
Current → **B**

$\vec{B}/\mu_0 = \vec{H}$ $\quad \vec{D} = \varepsilon_0 \vec{E}$

– Basis for the phenomena of electromagnetic waves.

Electromagnetic Wave Propagation

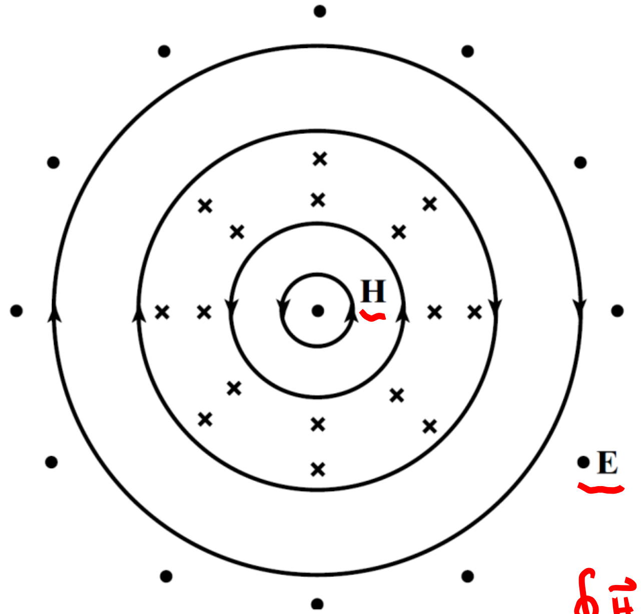
- Antenna radiation



$\vec{H} = \vec{B}/\mu_0$

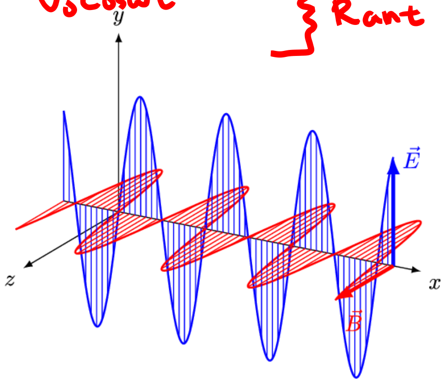
$\vec{P} = \vec{E} \times \vec{H}$

$$\oint_C \vec{E} \cdot d\vec{l} = -\frac{d}{dt}\int_S \vec{B} \cdot d\vec{S}$$



$$\oint_C \vec{H} \cdot d\vec{l} = \int_S \vec{J} \cdot d\vec{S} + \frac{d}{dt}\int_S \vec{D} \cdot d\vec{S}$$

$V_0 \cos\omega t$ $\downarrow I(t)$ R_{ant}



Outline

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- Faraday’s law
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$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt}\int_S \mathbf{D} \cdot d\mathbf{S}$$ Ampère’s circuital law

$$\left.\begin{aligned} \oint_S \mathbf{D} \cdot d\mathbf{S} &= \underbrace{\int_V \rho \, dv}_{Q} \\ \oint_S \mathbf{B} \cdot d\mathbf{S} &= 0 \end{aligned}\right\} \text{Gauss' laws}$$

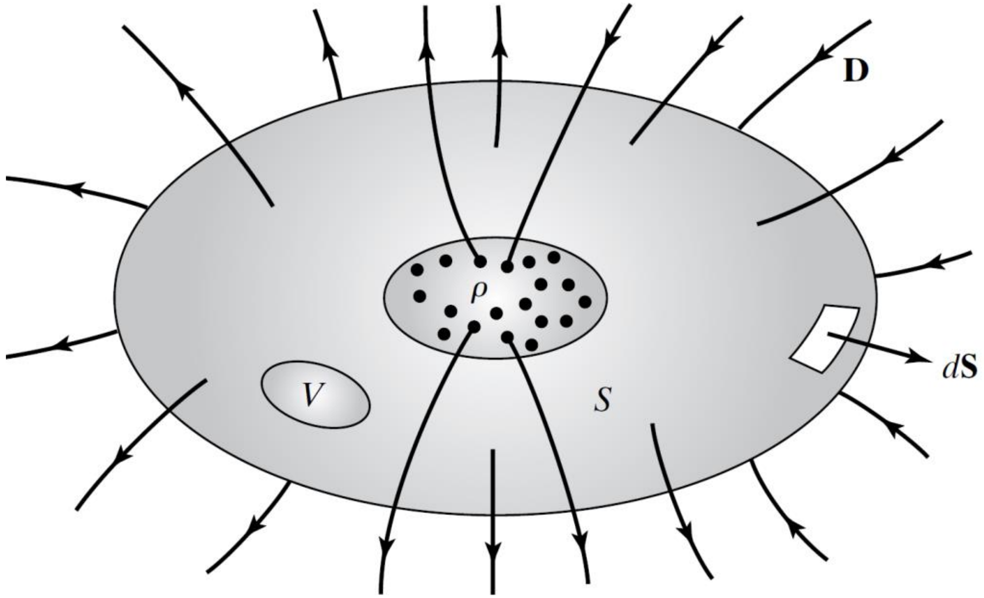
Gauss' Laws for E Field

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = [Q]_V$$

$$[Q]_V = \int_V \rho \, dv$$

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_{\underbrace{V}_{\text{volume bounded by "S"}}} \rho \, dv$$

$$\mathbf{D} = \varepsilon_0 \mathbf{E}$$



– Displacement flux emanating from a closed surface *S* is equal to the free charge contained within the volume *V* bounded by that surface.

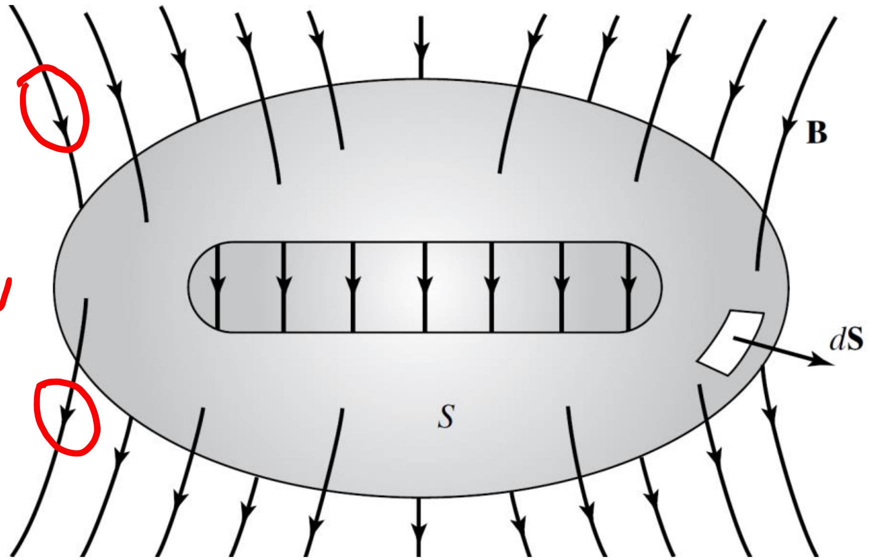
– Very useful in determining the E-field of charge distributions with some symmetric conditions.

Gauss' Laws for B Field

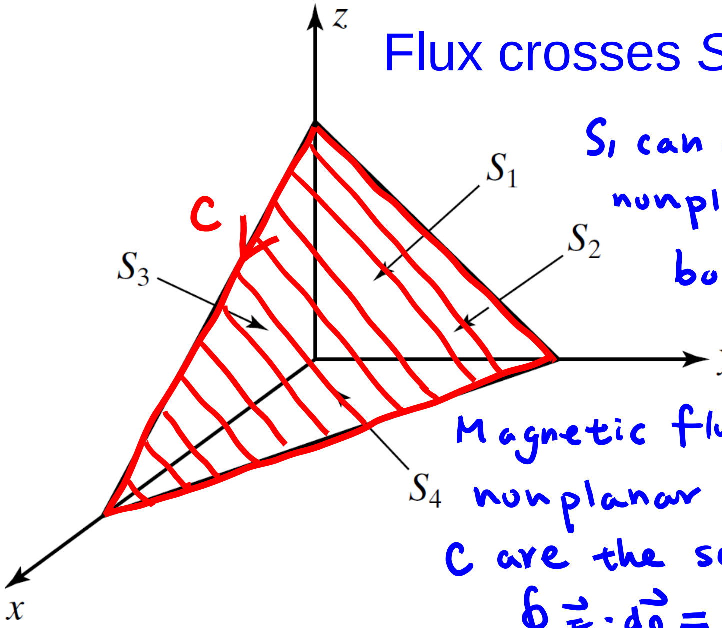
$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$$

- Magnetic flux emanating from a closed surface equal to zero.
- Magnetic charges do not exist.
- Magnetic flux lines are closed.
- Law of conservation of magnetic flux.

$$\Rightarrow \underbrace{\int_{S_1} \vec{B}\cdot d\vec{S}}_{\text{fixed too}} + \underbrace{\int_{S_2} \vec{B}\cdot d\vec{S} + \int_{S_3} \vec{B}\cdot d\vec{S} + \int_{S_4} \vec{S}\cdot d\vec{S}}_{\text{fixed}} = 0$$



Flux crosses S_1?



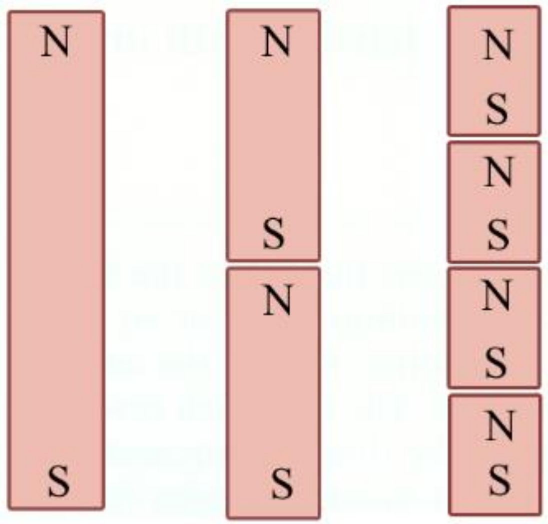
S_1 can be any nonplanar surface bounded by "C"

Magnetic flux through any nonplanar S_1 bounded by C are the same

$$\oint_C \vec{E}\cdot d\vec{\ell} = -\frac{d}{dt}\int_S \vec{B}\cdot d\vec{S}$$

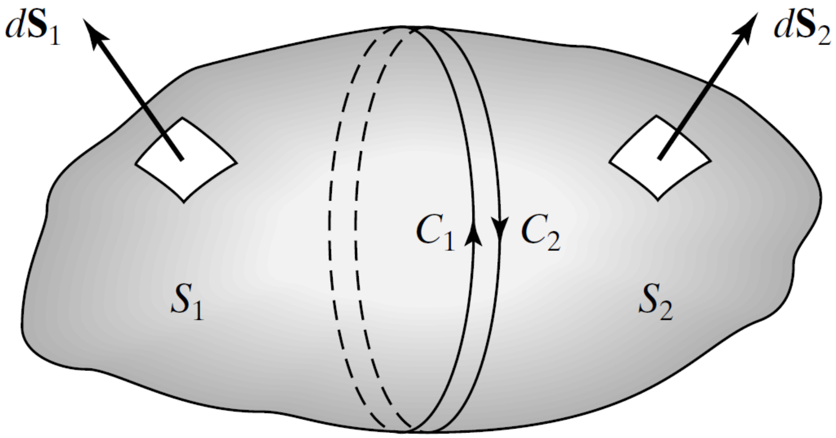
Permanent Bar Magnet

- New south and north poles always exist as you cut the magnet
  - Magnetic poles cannot be isolated.
  - Magnetic charges do not exist.



Gauss' Law and Faraday's Law Relationship

- Interdependence



Faraday's law

$$\oint_{C_1} \mathbf{E}\cdot d\mathbf{l} = -\frac{d}{dt}\int_{S_1} \mathbf{B}\cdot d\mathbf{S}_1$$

+

$$\oint_{C_2} \mathbf{E}\cdot d\mathbf{l} = -\frac{d}{dt}\int_{S_2} \mathbf{B}\cdot d\mathbf{S}_2$$

$$0 = -\frac{d}{dt}\oint_{S_1+S_2} \mathbf{B}\cdot d\mathbf{S}$$

$$\oint_{S_1+S_2} \mathbf{B}\cdot d\mathbf{S} = \text{constant with time}$$

$$\oint_{S} \mathbf{B}\cdot d\mathbf{S} = \underline{0}$$

Experimental results

(no magnetic charge)

D2.11

- Magnetic fluxes of absolute values φ_1, φ_2, and φ_3 cross three surfaces S_1, S_2, and S_3, respectively, constituting a closed surface S. If φ_1 + φ_2 + φ_3= φ_0, find the smallest of φ_1, φ_2, and φ_3 for each of the following cases: (a) φ_1, φ_2, and φ_3 are in arithmetic progression; (b) $1/\varphi_1$, $1/\varphi_2$, and $1/\varphi_3$ are in arithmetic progression; and (c) $\ln\varphi_1$, $\ln\varphi_2$, and $\ln\varphi_3$ are in arithmetic progression.
- Arithmetic progression
  - A sequence (= an ordered series of numbers) in which the numbers get bigger or smaller by the same amount, such as 3, 6, 9..., or 9, 6, 3.

D2.11

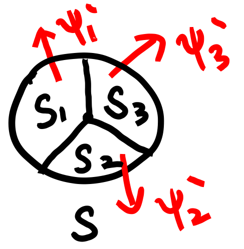
$$\psi_1 + \psi_2 + \psi_3 = \psi_0$$

Assume ψ_1 smallest

(a) $\psi_2 = \psi_1 + a,\ \psi_3 = \psi_2 + a = \psi_1 + 2a \rightarrow \psi_3$ biggest

$\Rightarrow \psi_1 + \psi_1 + a + \psi_1 + 2a = \psi_0 \quad \therefore 3\psi_1 = \psi_0 - 3a$

$$\oint_S \vec{B} \cdot d\vec{S} = 0$$



$$\begin{cases} \psi_1' + \psi_2' + \psi_3' = 0, \quad |\psi_1'| + |\psi_2'| + |\psi_3'| = \psi_0 \\ \psi_3 \text{ biggest} \end{cases}$$

$\Rightarrow \psi_1 + (\psi_1 + a) - (\psi_1 + 2a) = 0$

$\Rightarrow \psi_1 - a = 0 \quad \therefore \psi_1 = a$

$3\psi_1 = \psi_0 - 3a = \psi_0 - 3\psi_1$

$\therefore \psi_1 = \psi_0 / 6$

(b) $\dfrac{1}{\psi_2} = \dfrac{1}{\psi_1} - a,\ \dfrac{1}{\psi_3} = \dfrac{1}{\psi_1} - 2a,\ a > 0$

$\psi_1 + \psi_2 + \psi_3 = \psi_1 + \dfrac{\psi_1}{1 - a\psi_1} + \dfrac{\psi_1}{1 - 2a\psi_1} = \psi_0$

$\psi_1' + \psi_2' + \psi_3' = 0 \rightarrow \psi_1 + \dfrac{\psi_1}{1 - a\psi_1} - \dfrac{\psi_1}{1 - 2a\psi_1} = 0 \Rightarrow a\psi_1 = 1 \pm \dfrac{1}{\sqrt{2}}$

$\psi_1 + \dfrac{\psi_1}{\mp\sqrt{2}} + \dfrac{\psi_1}{-1 \mp \sqrt{2}} = \psi_0$

$\psi_1 + \left(1 \mp \sqrt{2} - \dfrac{1}{1 \pm \sqrt{2}}\right) = \psi_0$

$\psi_1 > 0 \quad \therefore \psi_1 = \dfrac{\psi_0}{2 + 2\sqrt{2}}$

(c) $\ln\psi_2 = \ln\psi_1 + \ln a$

$\ln\psi_3 = \ln\psi_1 + 2\ln a$

$\Rightarrow \psi_2 = \psi_1 a,\ \psi_3 = \psi_1 a^2$

$\psi_1 + a\psi_1 + a^2\psi_1 = \psi_0$

$\psi_1 + a\psi_1 - a^2\psi_1 = 0$

$\Rightarrow a = \dfrac{1 + \sqrt{5}}{2}$

$\Rightarrow \psi_1 = \dfrac{1}{3 + \sqrt{5}}\psi_0$

Outline

- Inductive and deductive approaches
- Line integral, surface integral, and volume integral
- Faraday’s law
- Ampère’s circuital law
- Gauss’ laws
- Law of conservation of charge
- Applications to static fields

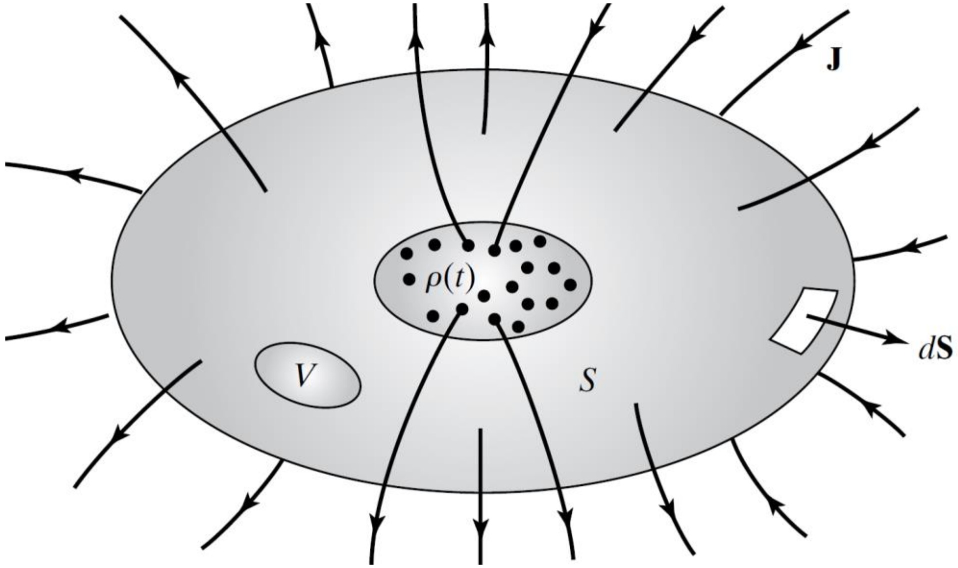
Charge Conservation Law

- Electric charge conserved
  - Electric charges may not be created or destroyed.
  - The algebraic sum of the positive and negative charges in a closed (isolated) system remains unchanged.
- Equation of continuity
  - The net current due to the flow of charges emanating from a closed surface *S* is equal to the time rate of decrease of the charge within the volume V bounded by *S*.

charge conserved at any time

$\vec{J}$ flows in, $\rho(t)$ increases with time

$\vec{J}$ flows out, $\rho(t)$ decreases with time

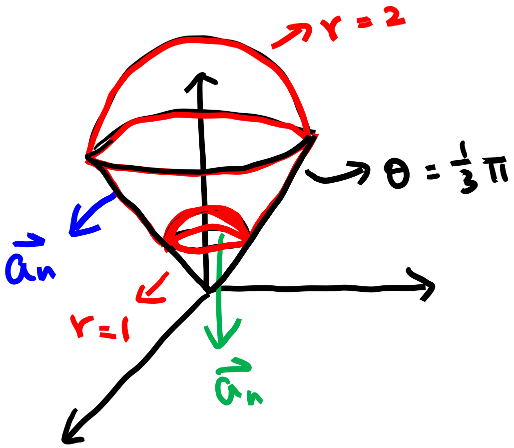


$$\oint_S \mathbf{J} \cdot d\mathbf{S} = -\frac{d}{dt}\int_V \rho \, dv$$

$-\frac{d}{dt} Q(t)$

P2.24(c)

- Given **J**= (x**a**$_x$ + y**a**$_y$ + z**a**$_z$) A/m^2. Find the time rate of decrease of the charge contained within the volume bounded by the spherical surfaces r= 1 and r= 2 and the conical surface θ= π/3.



$$\oint_S \vec{J}\cdot d\vec{S} = -\frac{d}{dt}\int_V \rho\, dv \quad , \quad \vec{J} = (x\vec{a_x} + y\vec{a_y} + z\vec{a_z}) = r_s \vec{a_{r_s}}$$

$$d\vec{S} = r_s^2 \sin\theta\, dr\, d\theta\, d\phi\, \vec{a_{r_s}}$$

$$\Rightarrow \oint_S \vec{J}\cdot d\vec{S}$$

$$= \int_{\theta=0}^{\theta=\frac{\pi}{3}}\int_{\phi=0}^{\phi=2\pi} \overset{=8}{r_s^3} \sin\theta\, dr\, d\theta\, d\phi$$

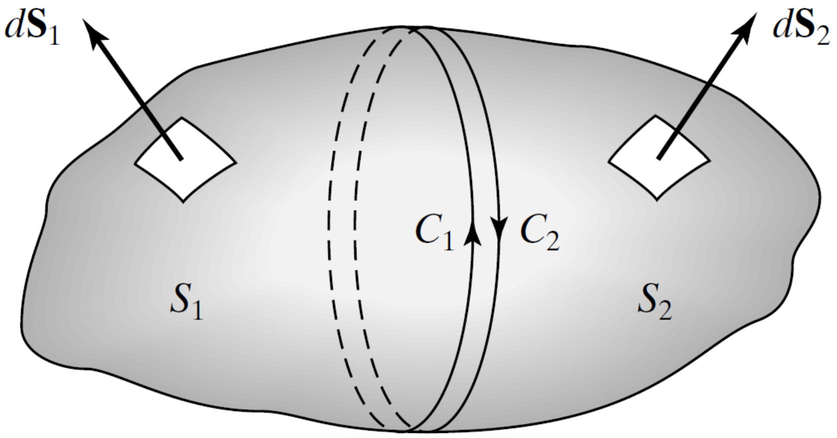
$$- \int_{\theta=0}^{\theta=\frac{\pi}{3}}\int_{\phi=0}^{\phi=2\pi} \overset{=1}{r_s^3} \sin\theta\, dr\, d\theta\, d\phi \quad -\vec{a_{r_s}}$$

$$= 7\pi$$

$$\Rightarrow -\frac{d}{dt}\int_V \rho\, dv = \oint_S \vec{J}\cdot d\vec{S} = 7\pi \text{ A}$$

Gauss' Law and Ampère's Circuital Law Interdependence

- Through the law of charge conservation
- Ampère's circuital law + law of charge conservation → Gauss' law for E field



$$\frac{d}{dt}\oint_S \mathbf{D}\cdot d\mathbf{S} = -\oint_S \mathbf{J}\cdot d\mathbf{S}$$

$$\Rightarrow \frac{d}{dt}\oint_S \mathbf{D}\cdot d\mathbf{S} = \frac{d}{dt}\int_V \rho\, dv$$

$$\frac{d}{dt}\left(\oint_S \mathbf{D}\cdot d\mathbf{S} - \int_V \rho\, dv\right) = 0$$

$$\oint_S \mathbf{D}\cdot d\mathbf{S} - \int_V \rho\, dv = \underbrace{\text{constant}}_{} = 0$$

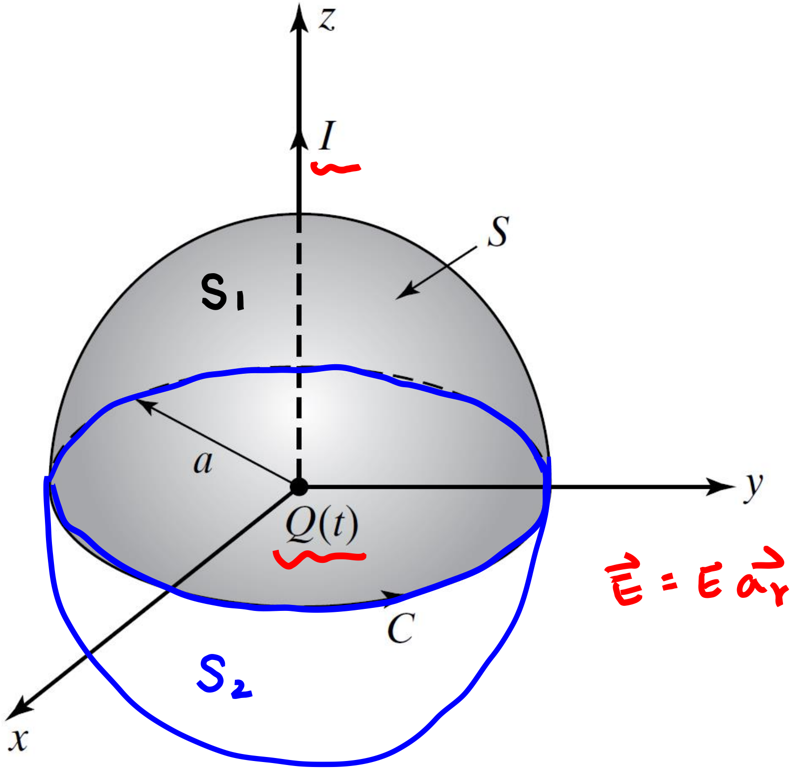
Experimental results

$$\oint_S \mathbf{D}\cdot d\mathbf{S} = \int_V \rho\, dv$$

Example

Combined applications of serveral Maxwell's equations in integral form

- Let us consider current I A flowing from a point charge $Q(t)$ at the origin to infinity along a semi-infinitely long straight wire occupying the positive z-axis. Find $\oint_C \mathbf{H} \cdot d\mathbf{l}$ where C is a circular path of radius a lying in the xy-plane and centered at the point charge, as shown below.



$\vec{E} = E\,\vec{a_r}$

$S = S_1 + S_2$

$$\oint_S \vec{J} \cdot d\vec{S} = -\frac{d}{dt}\int_V \rho\, dv$$

$$= -\frac{d}{dt} Q(t)$$

Example

$$① \quad \oint_C \mathbf{H}\cdot d\mathbf{l} = \underset{\oint_S \vec{J}\cdot d\vec{S}}{\overset{}{I}} + \underline{\frac{d}{dt}\int_{S=S_1+S_2} \mathbf{D}\cdot d\mathbf{S}} \quad Q/2$$ Ampère's circuital law

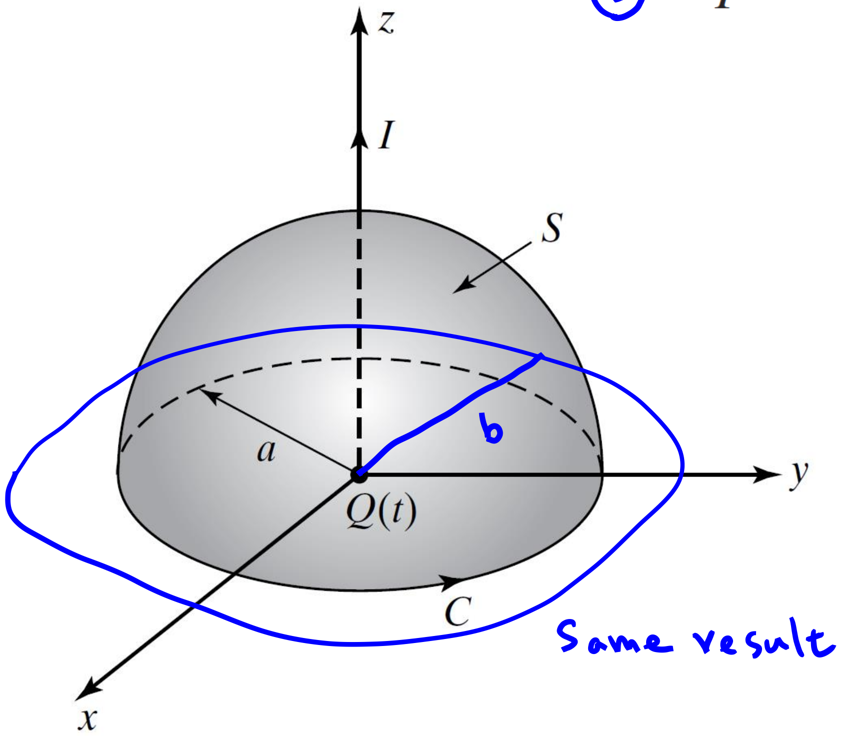
$$\oint_S \vec{D}\cdot d\vec{S} = \int_{S_1}\vec{D}\cdot d\vec{S} + \int_{S_2}\vec{D}\cdot d\vec{S}$$

$$② \quad \int_S \mathbf{D}\cdot d\mathbf{S} = \frac{Q}{2}$$ Gauss' law

$$= 2\int_{S_1}\vec{D}\cdot d\vec{S} = Q$$

$$\therefore \int_{S_1}\vec{D}\cdot d\vec{S} = Q/2$$

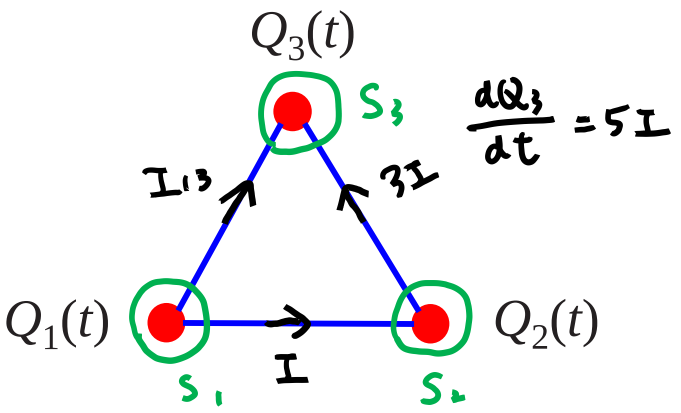
$$③ \quad I = -\frac{dQ}{dt}$$ Charge Conservation Law



$$\oint_C \mathbf{H}\cdot d\mathbf{l} = I + \frac{d}{dt}\left(\frac{Q}{2}\right)$$
$$= I + \frac{1}{2}\frac{dQ}{dt}$$
$$= I + \frac{1}{2}(-I)$$
$$= \frac{I}{2}$$

D2.12

- Three point charges $Q_1(t)$, $Q_2(t)$, and $Q_3(t)$ are situated at the vertices of a triangle and are connected by means of wires carrying currents. A current I A flows from Q_1 to Q_2 and $3I$ A flows from Q_2 to Q_3. The charge Q_3 is increasing with time at the rate of $5I$ C/s. Find the following: (a) dQ_1/dt; (b) dQ_2/dt; and (c) the current flowing from to Q_1 to Q_3.



(a) $I_{13} + 3I = \frac{dQ_3}{dt} = 5I$

$\Rightarrow I_{13} = 2I$

$I_{13} + I = -\frac{dQ_1}{dt}$

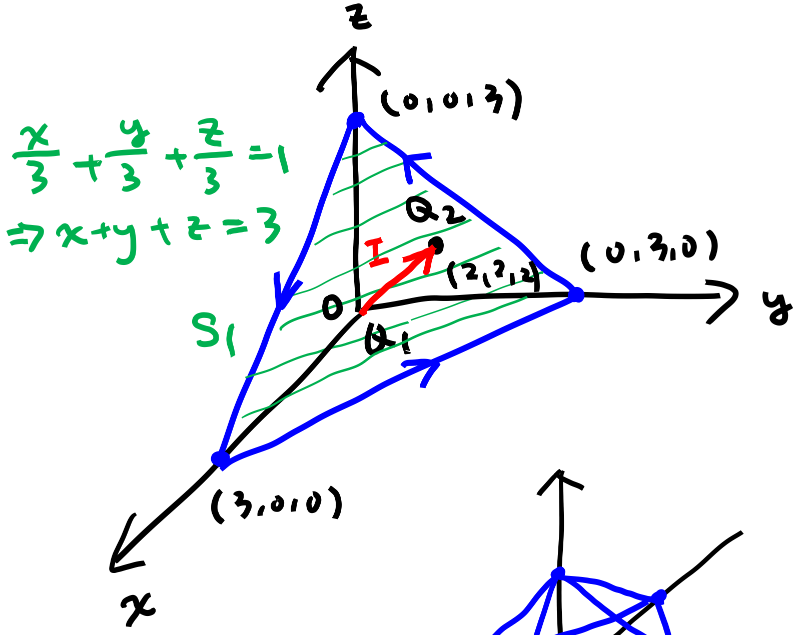
$\therefore \frac{dQ_1}{dt} = -3I$

(b) $\frac{dQ_2}{dt} = I - (3I) = -2I$

(c) $I_{13} = 2I$ C/s

P2.26

- Current I flows along a straight wire from a point charge $Q_1(t)$ at the origin to a point charge $Q_2(t)$ at the point (2, 2, 2). Find the line integral of **H** around the triangular closed path having the vertices at (3, 0, 0), (0, 3, 0), and (0, 0, 3) and traversed in that order.



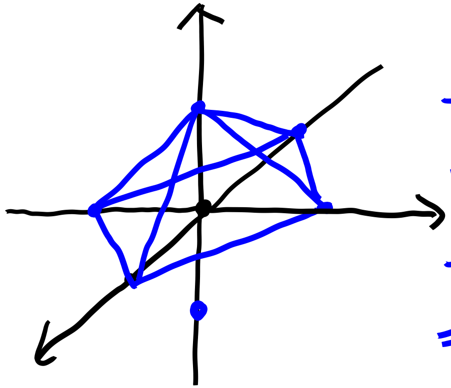
P = (1,1,1) on the slanted surface

$\vec{PO} = \vec{a}_x + \vec{a}_y + \vec{a}_z$ $\Rightarrow \vec{PO} = \vec{Q_2P}$

$\vec{Q_2P} = \vec{a}_x + \vec{a}_y + \vec{a}_z$

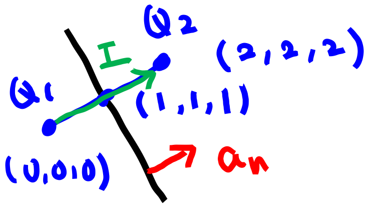
$\Rightarrow$ (2,2,2) and (0,0,0) located symmetrically with respect to S_1

$\therefore \int_{S_1} \vec{D}_{Q_2}\cdot d\vec{S} = -\frac{1}{8}Q_2$



Top half = 4 tetrahedron

Total = 8 tetrahedron



$\oint_C \vec{H}\cdot d\vec{l} = \int_S \vec{J}\cdot d\vec{S} + \frac{d}{dt}\int_S \vec{D}\cdot d\vec{S}$

$\int_S \vec{J}\cdot d\vec{S} = I$

$\therefore \oint_S \vec{D}_{Q_1}\cdot d\vec{S} = Q_1$

$\Rightarrow \int_{S_1} \vec{D}_{Q_1}\cdot d\vec{S} = \frac{1}{8}Q_1$

Charge conservation

$I = -\frac{dQ_1}{dt}$, $I = \frac{dQ_2}{dt}$

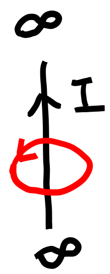
$\therefore \oint_C \vec{H}\cdot d\vec{l} = I + \frac{d}{dt}\left(\frac{Q_1}{8} - \frac{Q_2}{8}\right)$

$= I - \frac{1}{8}I - \frac{1}{8}I$

$= \frac{3}{4}I$

Example

- As shown below, current I flows along a straight wire from a point charge $Q_1(t)$ located at (0, 0, -1) to a point charge $Q_2(t)$ located at (0, 0, 0). Find the line integral of $\boldsymbol{H}$ along the square closed path C *(mmf)* having the vertices at (1, 1, 0), (-1, 1, 0), (-1, -1, 0), and (1, -1, 0) and traversed in that order. (a) Please solve it by using Ampere's law in integral form and considering the plane surface S bounded by C except for a slight upward bulge at the origin to avoid $Q_2(t)$ as shown below. (b) If $Q_2(t)$ is moved to the point (0, 0, 1), please find the mmf again. (c) If $Q_2(t)$ is moved to positive infinity along z-axis, please find the mmf again. (d) If $Q_1(t)$ is moved to negative infinity and $Q_2(t)$ is moved to positive infinity along z-axis, please find the mmf again.

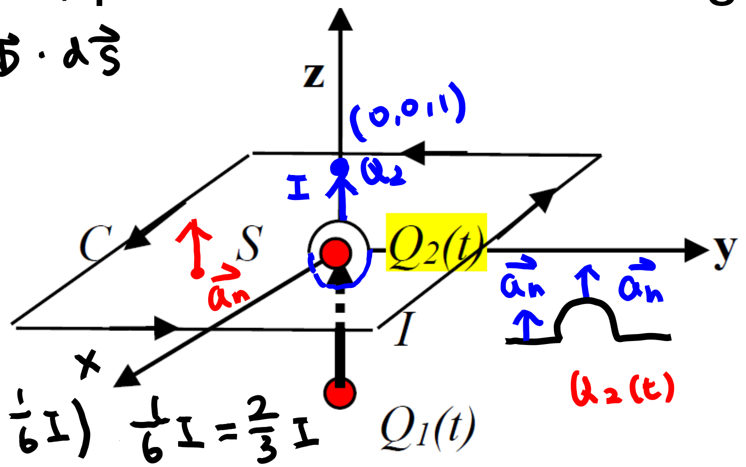


(a) $\text{mmf} = \oint_C \vec{H}\cdot d\vec{l} = \int_S \vec{J}\cdot d\vec{S} + \frac{d}{dt}\int_S \vec{D}\cdot d\vec{S}$

$I = \frac{dQ_1}{dt} = \frac{dQ_2}{dt}$

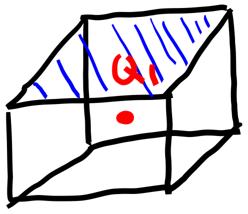
$\Rightarrow \oint_C \vec{H}\cdot d\vec{l} = 0 + \frac{d}{dt}\left(\frac{Q_1}{6} + \frac{Q_2}{2}\right)$

$= -\frac{1}{6}I + \frac{1}{2}I = \frac{1}{3}I$



(c) $\vec{D} = \varepsilon_0\vec{E}$, $|\vec{E}| \propto \frac{1}{R^2}$

$\therefore \text{mmf} = I + 0 = I$



$\oint_S \vec{D}\cdot d\vec{S} = 6\int_{S_1}\vec{D}\cdot d\vec{S}$

$= Q_1$

(b) $\text{mmf} = I + \frac{d}{dt}\left(\frac{1}{6}Q_1 - \frac{1}{6}Q_2\right) = I + \left(-\frac{1}{6}I\right) - \frac{1}{6}I = \frac{2}{3}I$

Outline

- Inductive and deductive approaches
- Line integral, surface integral, and volume integral
- Faraday’s law
- Ampère’s circuital law
- Gauss’ laws
- Law of conservation of charge
- Applications to static fields

Maxwell's Equations + Continuity Equation

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S}$$ Faraday's law

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt}\int_S \mathbf{D} \cdot d\mathbf{S}$$ Ampère's circuital law

$$\left.\begin{aligned} &\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho \, dv \\ &\oint_S \mathbf{B} \cdot d\mathbf{S} = 0 \end{aligned}\right\} \text{Gauss' laws}$$

$$\oint_S \mathbf{J} \cdot d\mathbf{S} = -\frac{d}{dt}\int_V \rho \, dv$$ Charge Conservation Law

Static Case (*d*/*dt* = 0)

- **E** and **B** no interdependence in static fields

$$\oint_C \mathbf{E}\cdot d\mathbf{l} = \cancelto{0}{-\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{S}}$$

$$\oint_C \mathbf{H}\cdot d\mathbf{l} = \int_S \mathbf{J}\cdot d\mathbf{S} + \cancelto{0}{\frac{d}{dt}\int_S \mathbf{D}\cdot d\mathbf{S}}$$

$$\oint_S \mathbf{D}\cdot d\mathbf{S} = \int_V \rho\, dv$$

$$\oint_S \mathbf{B}\cdot d\mathbf{S} = 0$$

$$\oint_S \mathbf{J}\cdot d\mathbf{S} = \cancelto{0}{-\frac{d}{dt}\int_V \rho\, dv}$$

Static field

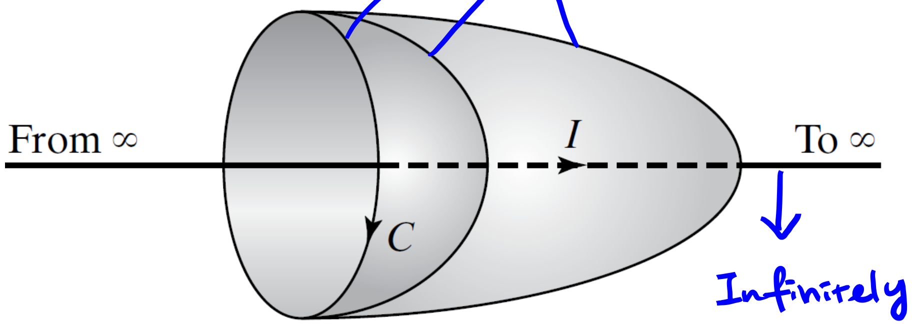
$$d/dt = 0$$

$$\left\{\begin{aligned}
&\oint_C \mathbf{E}\cdot d\mathbf{l} = 0\\
&\oint_C \mathbf{H}\cdot d\mathbf{l} = \int_S \mathbf{J}\cdot d\mathbf{S}\\
&\oint_S \mathbf{D}\cdot d\mathbf{S} = \int_V \rho\, dv\\
&\oint_S \mathbf{B}\cdot d\mathbf{S} = 0\\
&\oint_S \mathbf{J}\cdot d\mathbf{S} = 0
\end{aligned}\right.$$

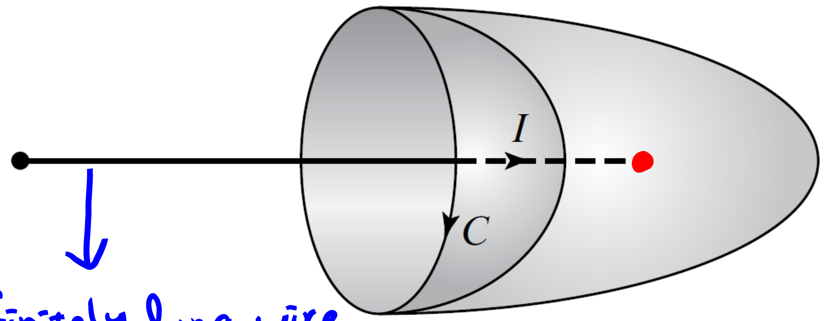
Static Case ($d/dt = 0$)

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S}$$

$\int_S \vec{J}\cdot d\vec{S} = I$ → Current crossing all surfaces is I



Infinitely long wire



finitely long wire

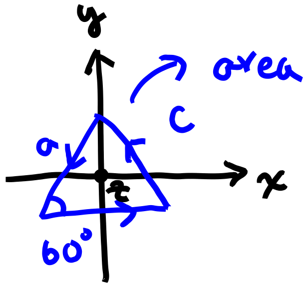
→ Must have time-varying charges at the two ends

$$\oint_C \vec{H}\cdot d\vec{\ell} = \int_S \vec{J}\cdot d\vec{S} + \underline{\frac{d}{dt}\int_S \vec{D}\cdot d\vec{S}}$$

D2.14

- The cross section of an infinitely long solid wire having the z-axis as its axis is a regular polygon of sides a. Current flows in the wire with uniform density $J_0\mathbf{a}_z$ A/m^2. Find the line integral of **H** along one side of the polygon and traversed in the sense of increasing φ for each of the following shapes of the polygon: (a) equilateral triangle; (b) square; and (c) octagon.

(a)



$\text{area} = \frac{1}{2} a a \sin 60° = \frac{\sqrt{3}}{4} a^2$

$$\oint_c \vec{H}\cdot d\vec{l} = \int_s \vec{J}\cdot d\vec{s} = J_0 \frac{\sqrt{3}}{4} a^2$$

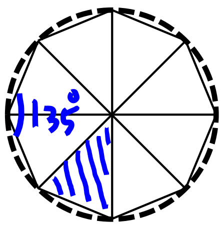
$$\Rightarrow \int_{c/3} \vec{H}\cdot d\vec{l} = \frac{1}{3}\oint_c \vec{H}\cdot d\vec{l} = \frac{\sqrt{3}}{12} a^2 J_0$$

(b)

$\square$ $\text{area} = a^2$

$$\int_{c/4} \vec{H}\cdot d\vec{l} = \frac{1}{4}\oint_c \vec{H}\cdot d\vec{l} = \frac{1}{4} J_0 a^2$$

(c)



area

$= \frac{a}{2}\,\frac{a}{2}\tan 67.5$

$= 0.6a^2$

$$\int_{c/8} \vec{H}\cdot d\vec{l} = \frac{1}{8}\oint_c \vec{H}\cdot d\vec{l} = \frac{1}{8} J_0 \times 8 \times 0.6 a^2 = 0.6 J_0 a^2$$

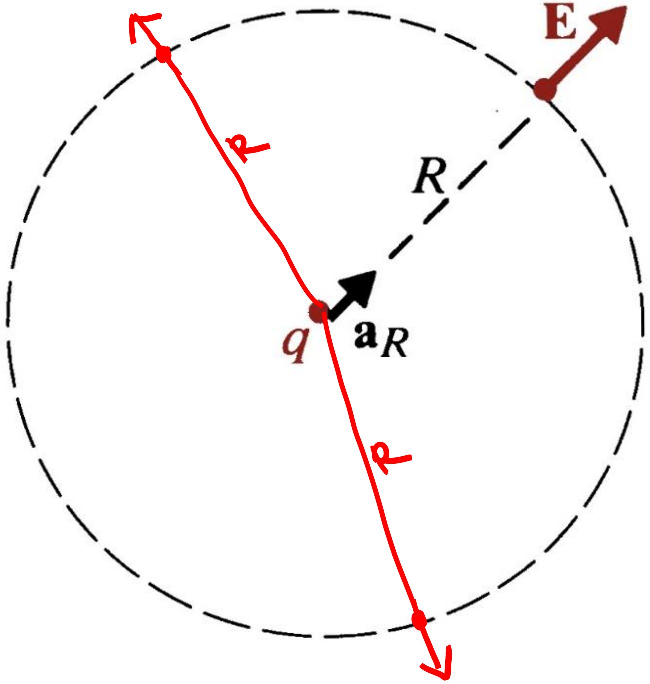
Example: E Field of Point Charge

Gauss law for $\vec{E}$ field

① choose coordinate system
→ spherical

② symmetric properties → $\vec{E} = E_R \vec{a}_R$

$$\mathbf{E} = \mathbf{a}_R E_R$$



③ $$\oint_S \underbrace{\mathbf{D}}_{\vec{D} = \varepsilon_0 \vec{E}} \cdot d\mathbf{S} = \underbrace{\int_V \rho \, dv}_{= Q} \rightarrow \varepsilon_0 \oint_S \vec{E} \cdot d\vec{s} = Q$$

$$\therefore \oint_S \vec{E} \cdot d\vec{s} = Q/\varepsilon_0$$

$$\oint_S \mathbf{E} \cdot d\mathbf{s} = \oint_S (\mathbf{a}_R E_R) \cdot \mathbf{a}_R \, ds = \frac{q}{\epsilon_0}$$

$$= \int_0^{\pi} \int_0^{2\pi} E_R(R) \vec{a}_R \cdot \vec{a}_R R^2 \sin\theta \, d\theta \, d\phi$$

$$E_R \oint_S ds = E_R(4\pi R^2) = \frac{q}{\epsilon_0}$$

$$= \int_0^{\pi} \int_0^{2\pi} R^2 \sin\theta \, d\theta \, d\phi = 2\pi R^2 \int_0^{\pi} \sin\theta \, d\theta = 2\pi R^2 (-\cos\theta)\Big|_0^{\pi} = 4\pi R^2$$

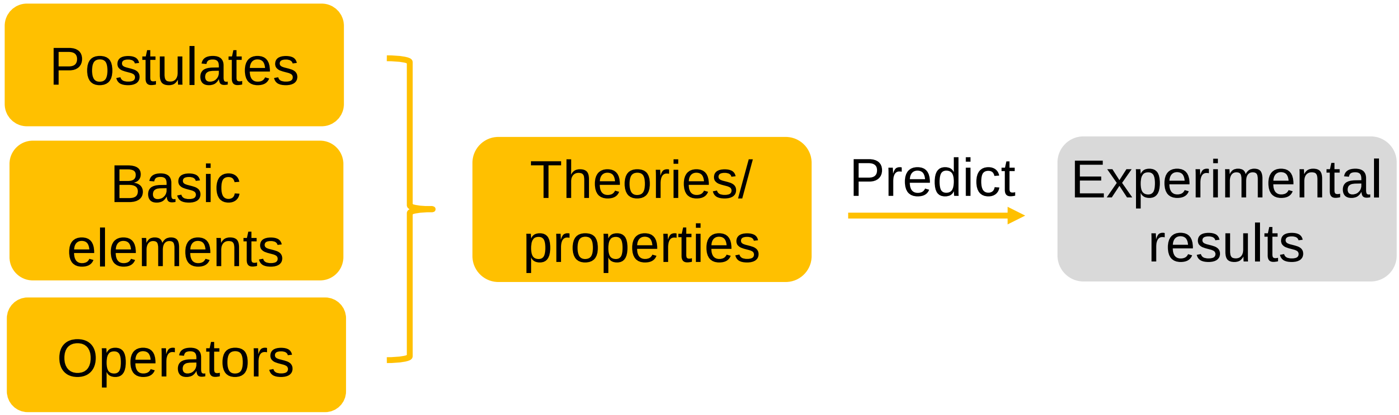
$$\mathbf{E} = \mathbf{a}_R E_R = \mathbf{a}_R \frac{q}{4\pi\epsilon_0 R^2}$$

exactly "2"

Proportional constant

$$K = \frac{1}{4\pi\varepsilon_0}$$

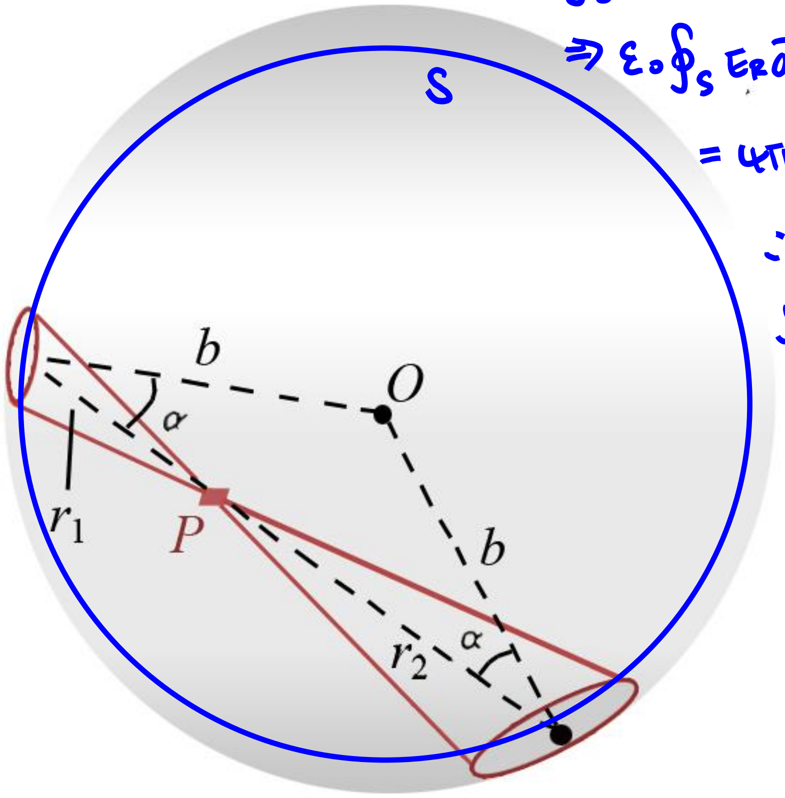
Electromagnetic Model – Predict Experimental Results



- Predict Coulomb’s law !

Example

- A total charge Q is put on a thin spherical shell of radius b. Determine the electric field intensity at an arbitrary point inside the shell.



$$\oint_S \vec{D} \cdot d\vec{s} = Q = 0$$

$$\Rightarrow \varepsilon_0 \oint_S E_R \vec{a}_R \cdot \vec{a}_R \, ds$$

$$= 4\pi R^2 \varepsilon_0 E_R = 0$$

$$\therefore E_R = 0$$

$$\therefore \vec{E} = 0$$

No charge inside the shell
→ **E**= 0 according to Gauss' law

$$\rho_s = \frac{Q}{4\pi b^2}$$

$$dE = \frac{\rho_s}{4\pi\epsilon_0}\left(\frac{ds_1}{r_1^2} - \frac{ds_2}{r_2^2}\right)$$

$$d\Omega = \frac{ds_1}{r_1^2}\cos\alpha = \frac{ds_2}{r_2^2}\cos\alpha$$

$$dE = \frac{\rho_s}{4\pi\epsilon_0}\left(\frac{d\Omega}{\cos\alpha} - \frac{d\Omega}{\cos\alpha}\right) = 0$$

If Coulomb's law not $\propto 1/r^2$, dE is not zero.

Example

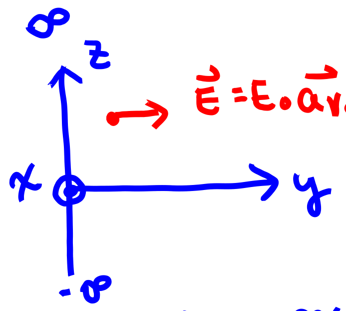
- Consider charge distributed uniformly with density ρ_{L0} C/m along the z-axis and find the electric field due to the infinitely long line charge.

① Choose coordinate system

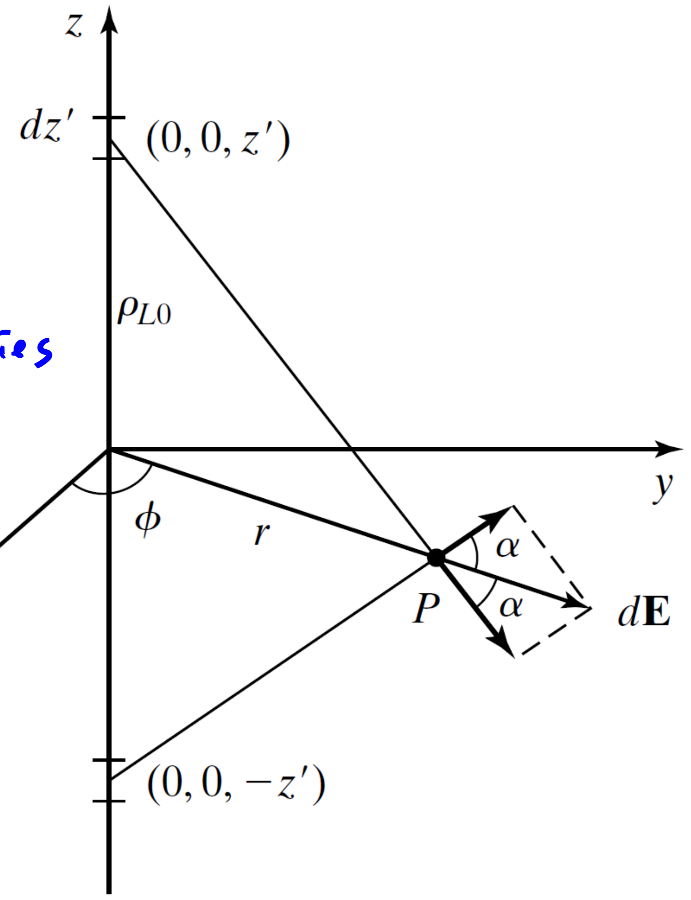
→ Cylindrical

② Symmetric properties

→ $\vec{E} = E_0 \vec{a}_{rc}$, $\vec{D} = D_0 \vec{a}_{rc}$



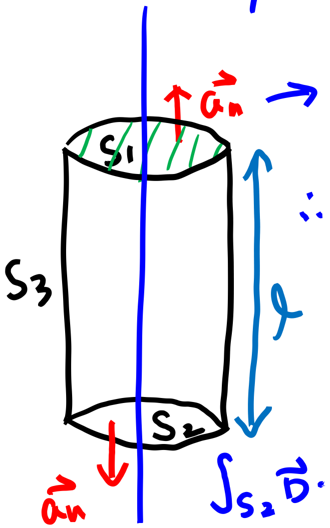
→ $\vec{E} = E_0 \vec{a}_{rc}$



$$\mathbf{D} = D_r(r)\mathbf{a}_r$$

③ Choose surface according to its symmetric properties

→ cylinder



$\vec{a}_n$ → $\int_{S_1} \vec{D} \cdot d\vec{S} = 0$

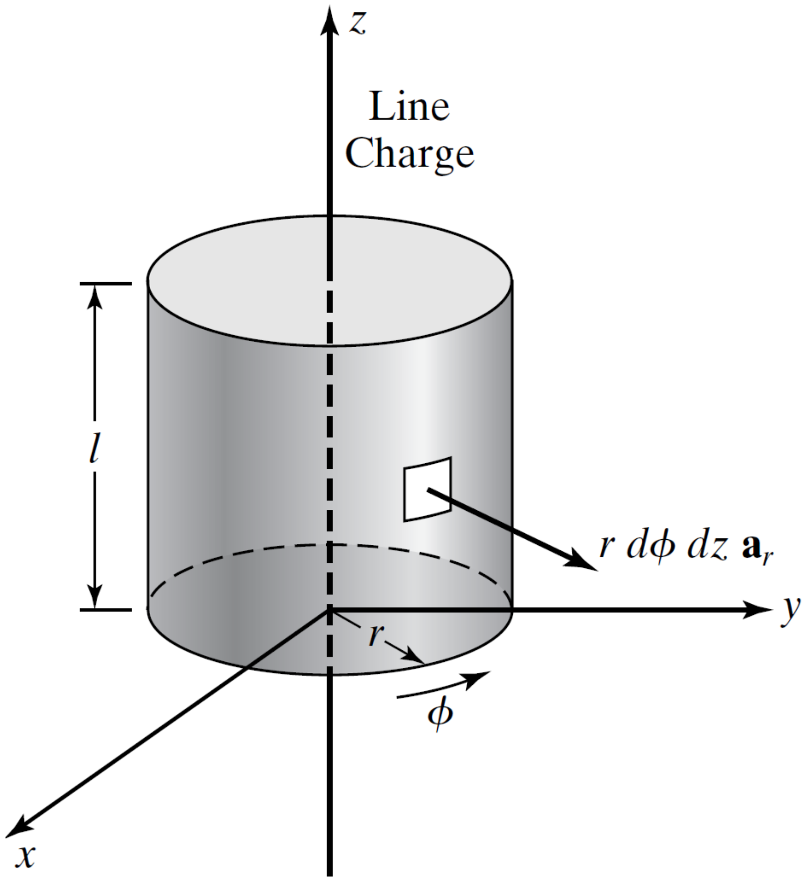
$\therefore \oint_S \vec{D} \cdot d\vec{S} = \int_{S_3} \vec{D} \cdot d\vec{S}$

$= \int_{S_3} D_0 \vec{a}_{rc} \cdot (r\, d\phi\, dz\, \vec{a}_{rc})$

$= \int_{S_3} D_0\, r\, d\phi\, dz = D_0\, 2\pi r l$

$\vec{a}_n$ $\quad \int_{S_2} \vec{D} \cdot d\vec{S} = 0$ $\quad = \int \rho_l\, dl = \rho_{L0}\, l$ $\quad \therefore D_0 = \dfrac{\rho_{L0}}{2\pi r l}$ $\quad \therefore \vec{D} = \dfrac{\rho_{L0}}{2\pi r l}\vec{a}_{rc} = \varepsilon_0 \vec{E}$

Example



$$\oint_S \mathbf{D}\cdot d\mathbf{S} = \rho_{L0}l$$

$$\mathbf{D} = D_r(r)\mathbf{a}_r$$

$$\oint_S \mathbf{D}\cdot d\mathbf{S} = \int_{\phi=0}^{2\pi}\int_{z=0}^{l} D_r(r)\mathbf{a}_r \cdot r\,d\phi\,dz\,\mathbf{a}_r$$

$$= 2\pi r l D_r(r)$$

$$2\pi r l D_r(r) = \rho_{L0}l$$

$$D_r(r) = \frac{\rho_{L0}}{2\pi r}$$

$$\mathbf{D} = \frac{\rho_{L0}}{2\pi r}\mathbf{a}_r$$

Example

- Determine **E** and **D** field caused by a spherical cloud of electrons with a volume charge density $\rho = \rho_0$ for $0 \le R \le b$ (both ρ_0 and b are positive) and $\rho = 0$ for $R > b$.

① Choose coordinate system

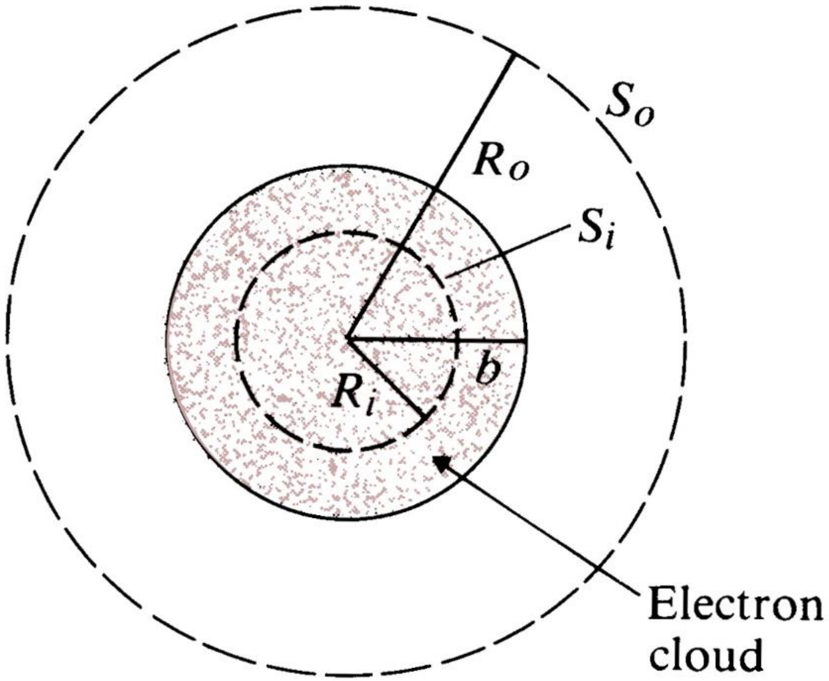
→ Spherical

② Check symmetric properties

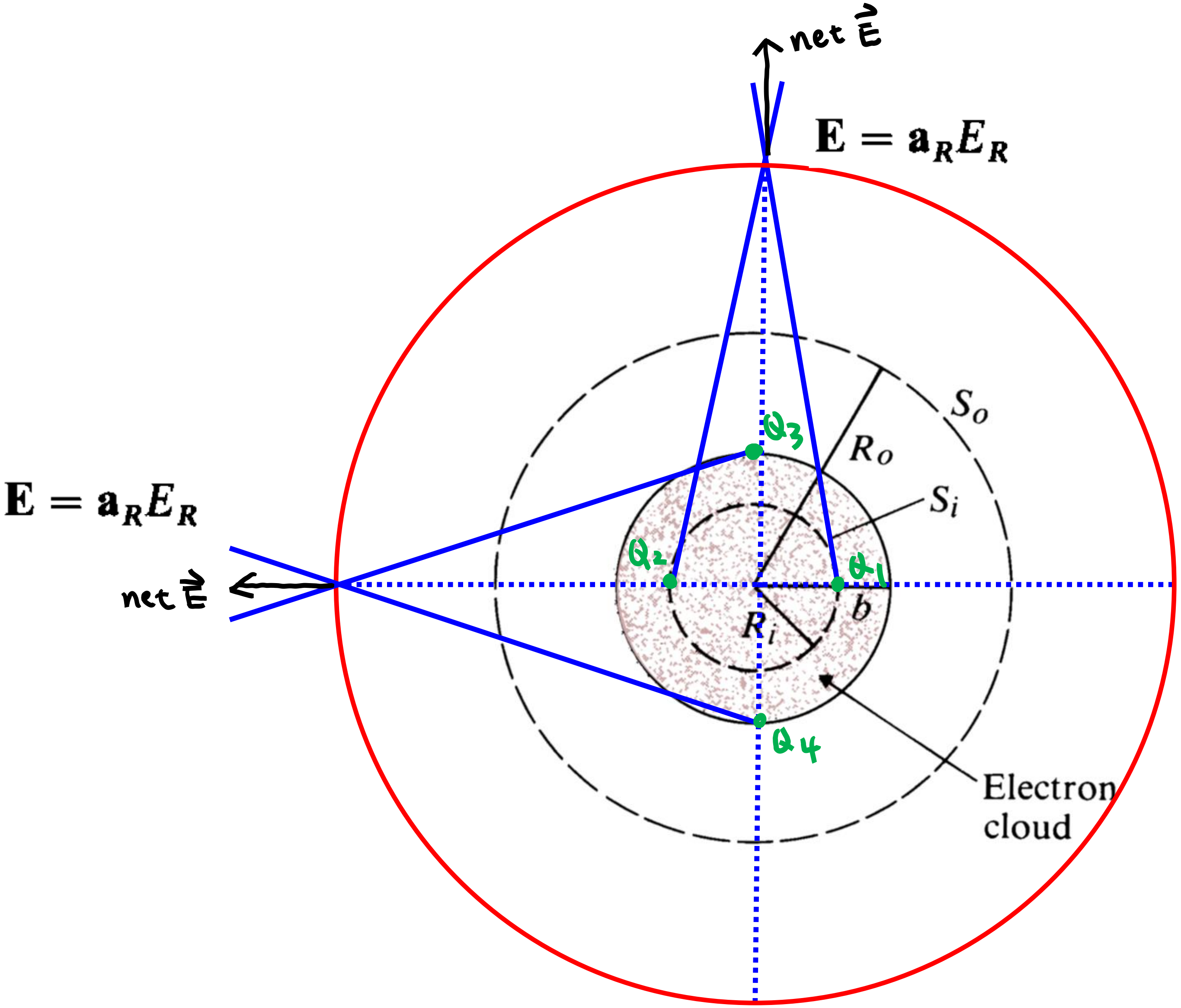
→ $\vec{E} = E_R \vec{a_R}$

③ Choose Gauss surface

→ Sphere

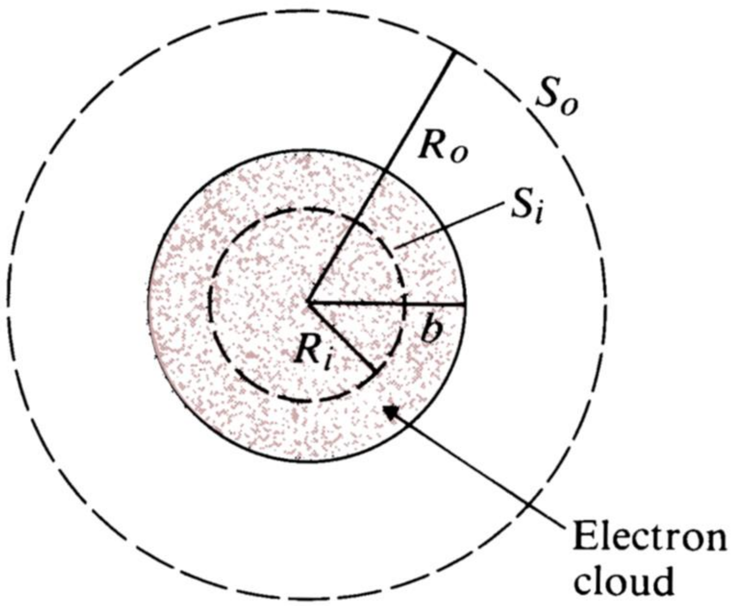


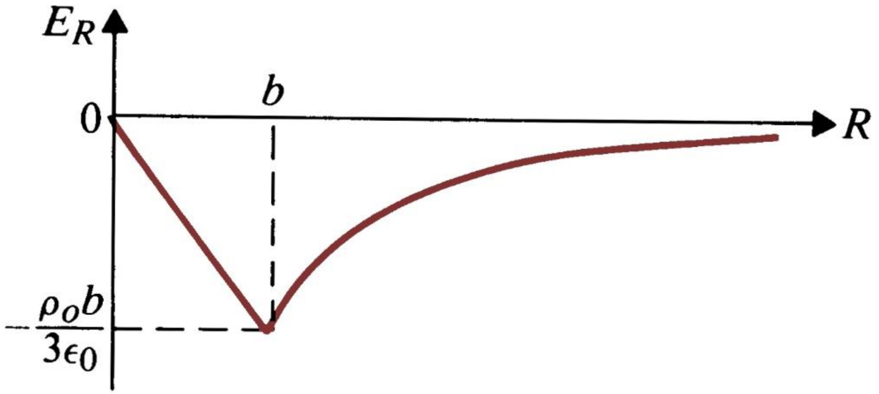
Example



net $\vec{E}$
$\mathbf{E} = \mathbf{a}_R E_R$
$\mathbf{E} = \mathbf{a}_R E_R$
net $\vec{E}$
Q_3
S_o
R_o
S_i
Q_2
Q_1
b
R_i
Q_4
Electron
cloud

Example





For $0 \leq R \leq b$

$$\mathbf{E} = \mathbf{a}_R E_R$$

$$d\mathbf{s} = \mathbf{a}_R \, \underbrace{ds} \quad = 4\pi R^2$$

$$\oint_{S_i} \mathbf{E} \cdot d\mathbf{s} = E_R \int_{S_i} ds = E_R 4\pi R^2$$

$$Q = \int_V \rho \, dv$$

$$\int_0^R \int_0^\pi \int_0^{2\pi} R^2 \sin\theta \, dR \, d\theta \, d\phi = \frac{4}{3}\pi R^3$$

$$= -\rho_o \int_V dv = -\rho_o \frac{4\pi}{3} R^3$$

$$\mathbf{E} = -\mathbf{a}_R \frac{\rho_o}{3\epsilon_0} R, \qquad 0 \leq R \leq b$$

For $R \geq b$

$$Q = -\rho_o \frac{4\pi}{3} b^3$$

• Q $\quad \frac{1}{4\pi\epsilon_0} \frac{Q}{R^2}$

$$\mathbf{E} = -\mathbf{a}_R \frac{\rho_o b^3}{3\epsilon_0 R^2}, \qquad R \geq b$$

like a point charge located at the origin

Example

- Consider current flowing with uniform density $\mathbf{J} = J_0\mathbf{a}_z$ A/m^2 in an infinitely long solid cylindrical wire of radius a with its axis along the z-axis, as shown by the cross-sectional view in the figure below. Find the magnetic field everywhere.

① Choose coordinate system

→ cylindrical

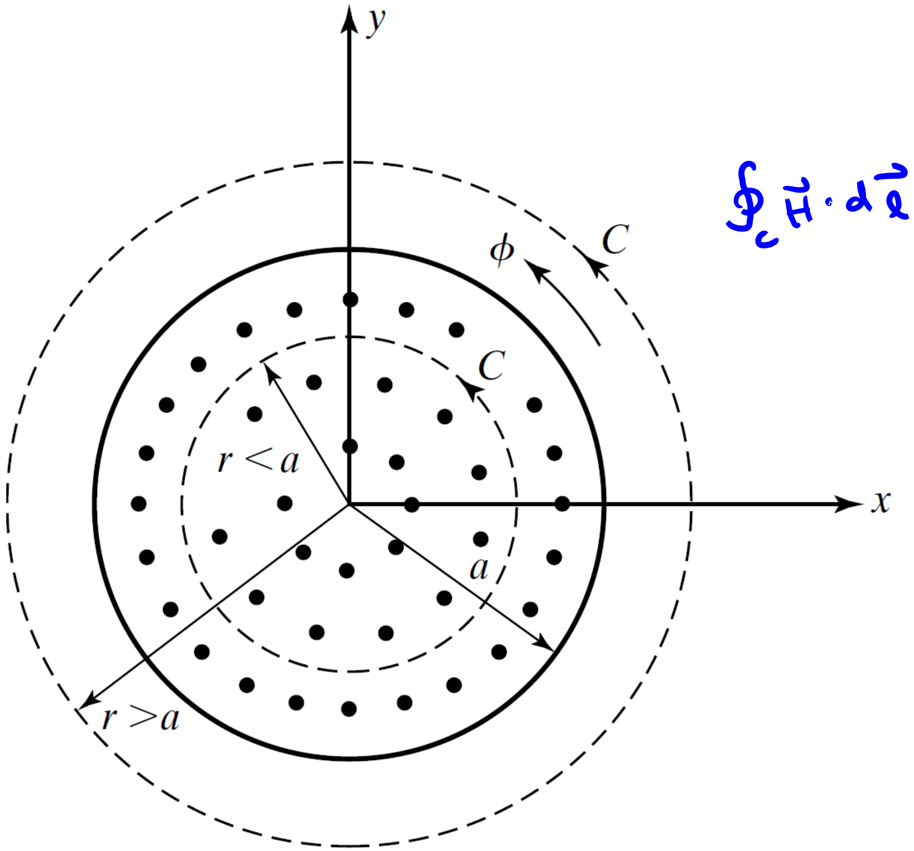
② check symmetric properties

$\vec{B} = B_\phi \vec{a}_\phi$, $\vec{B} = \mu_0 \vec{H}$

$\vec{H} = H_\phi \vec{a}_\phi$

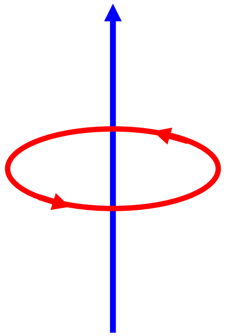
③ Choose closed path c

→ circle

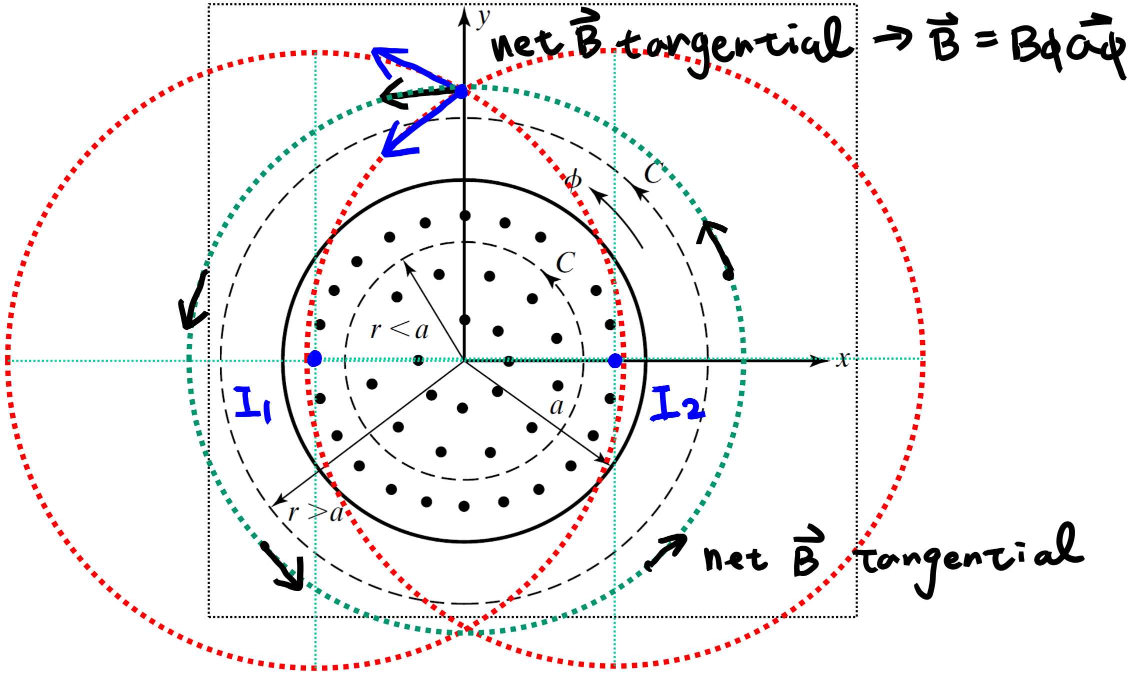


$\oint_c \vec{H} \cdot d\vec{l} = \int_s \vec{J} \cdot d\vec{s}$

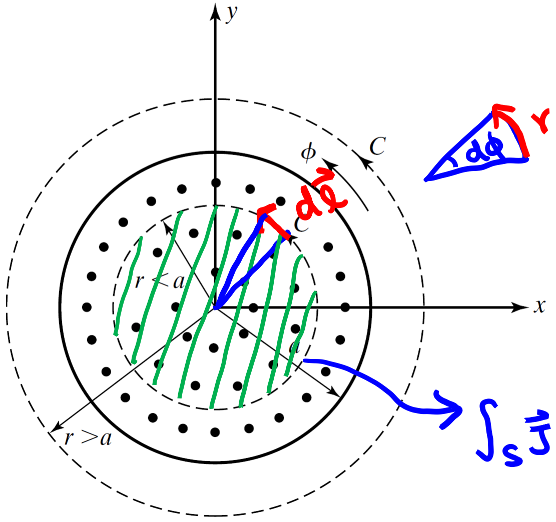
Example



$$\mathbf{H} = H_\phi(r)\mathbf{a}_\phi$$



Example



$r\,d\phi\,\vec{a}_\phi$

$r\,d\phi\,\vec{a}_\phi$

$$\oint_C \mathbf{H}\cdot d\mathbf{l} = \int_S \mathbf{J}\cdot d\mathbf{S}$$

$$\mathbf{H} = H_\phi(r)\mathbf{a}_\phi$$

$\int_S \vec{J}\cdot d\vec{S} = \int_S J_0\vec{a}_z\cdot(r\,dr\,d\phi\,\vec{a}_z)$

$= J_0\int_S r\,dr\,d\phi$

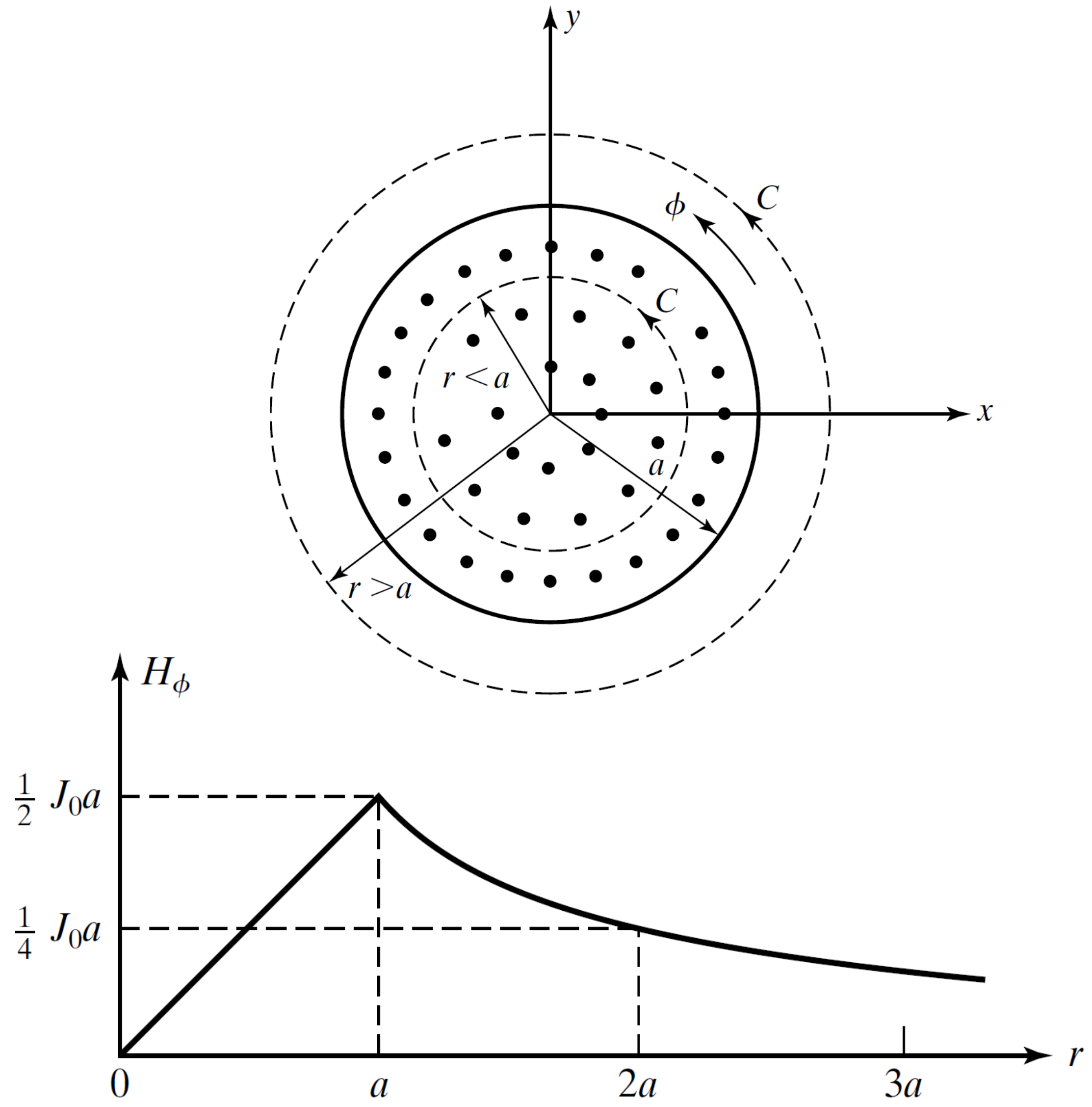
$$\oint_C \mathbf{H}\cdot d\mathbf{l} = \int_{\phi=0}^{2\pi} H_\phi(r)\mathbf{a}_\phi\cdot r = 2\pi r H_\phi(r)$$

$d\phi\,\vec{a}_\phi$

$$\int_S \mathbf{J}\cdot d\mathbf{S} = \begin{cases} \displaystyle\int_{r=0}^{r}\int_{\phi=0}^{2\pi} J_0\mathbf{a}_z\cdot r\,dr\,d\phi\,\mathbf{a}_z & \text{for} \quad r \le a \\ \displaystyle\int_{r=0}^{a}\int_{\phi=0}^{2\pi} J_0\mathbf{a}_z\cdot r\,dr\,d\phi\,\mathbf{a}_z & \text{for} \quad r \ge a \end{cases}$$

$$= \begin{cases} J_0\pi r^2 & \text{for } r \le a \\ J_0\pi a^2 & \text{for } r \ge a \end{cases}$$

Example



$$2\pi r H_\phi = \begin{cases} J_0\pi r^2 & \text{for} \quad r \le a \\ J_0\pi a^2 & \text{for} \quad r \ge a \end{cases}$$

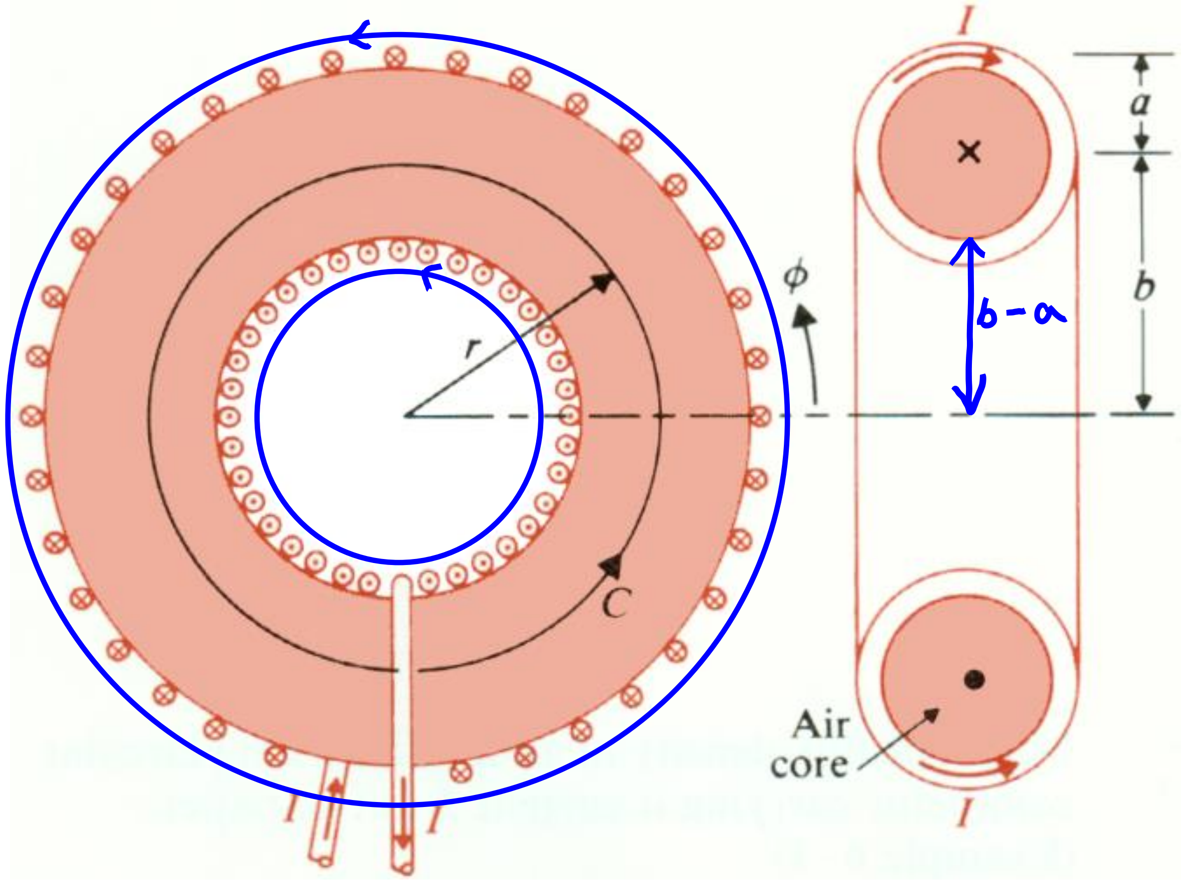
$$H_\phi = \begin{cases} \dfrac{J_0 r}{2} & \text{for} \quad r \le a \\[2ex] \dfrac{J_0 a^2}{2r} & \text{for} \quad r \ge a \end{cases}$$

$$\mathbf{H} = \begin{cases} \dfrac{J_0 r}{2}\mathbf{a}_\phi & \text{for} \quad r \le a \\[2ex] \dfrac{J_0 a^2}{2r}\mathbf{a}_\phi & \text{for} \quad r \ge a \end{cases}$$

Example

- Determine the magnetic flux density inside a closely wound toroidal coil with an air core having N turns and carrying a current I. The toroid has a mean radius b, and the radius of each turn is a.

Ref: Digi-Key



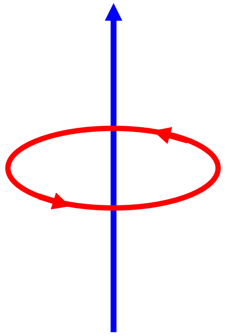
Example

① Choose coordinate system

→ cylindrical

② Check symmetric properties

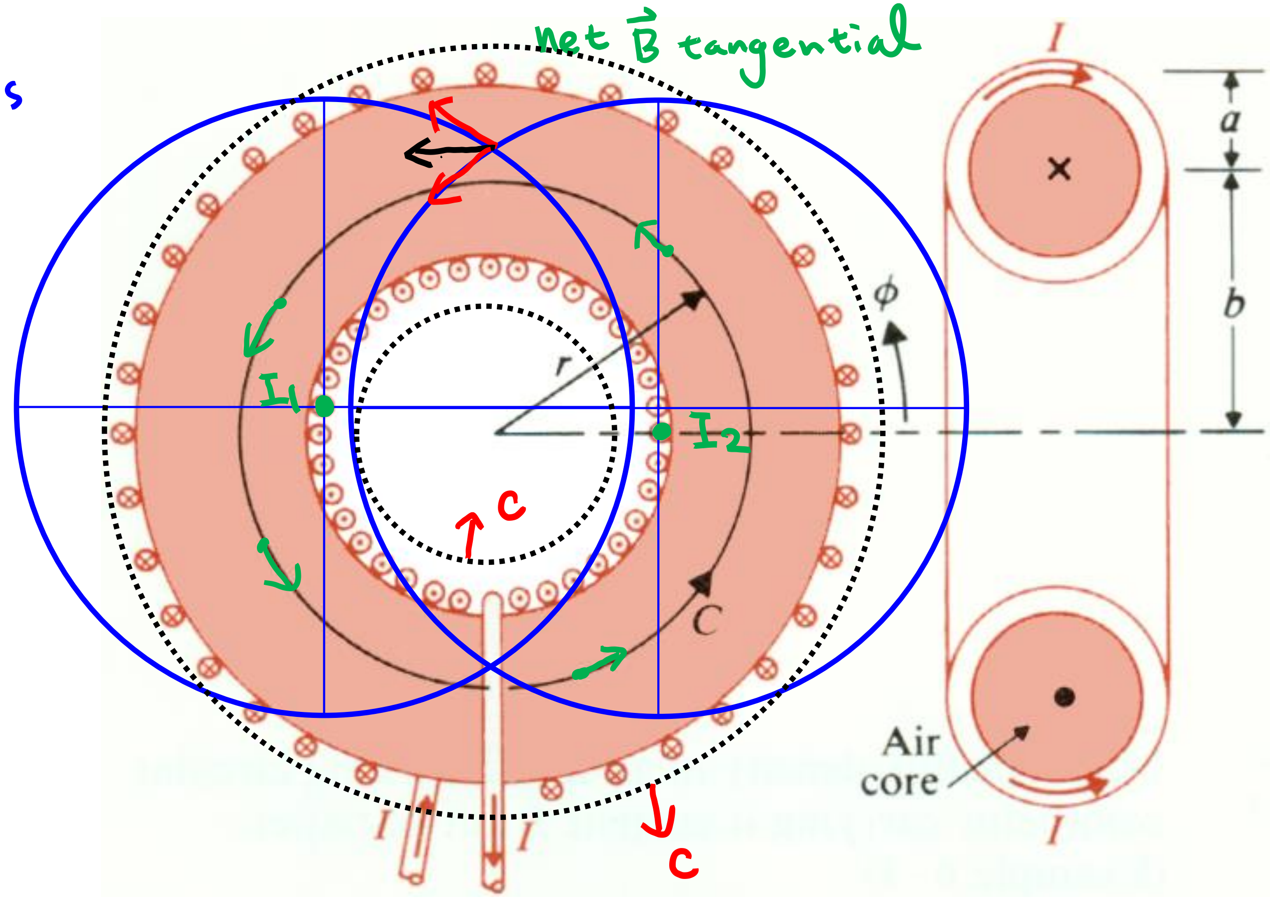
→ $\vec{B} = B_\phi \vec{a}_\phi$



③ Choose closed path c

→ circle

$$\mathbf{B} = \mathbf{a}_\phi B_\phi$$



Example

- B has only a Φ component
- For (b-a) < r < b+a

$$\oint_c \vec{H} \cdot d\vec{\ell} = \int_s \vec{J} \cdot d\vec{s} = NI$$

$$\vec{B} = \mu_0 \vec{H}$$

$$\oint \mathbf{B} \cdot d\ell = 2\pi r B_\phi = \mu_0 NI$$

$$\Rightarrow \oint_c \frac{1}{\mu_0} B_\phi \vec{a}_\phi \cdot (r\,d\phi\,\vec{a}_\phi) = 2\pi r B_\phi = NI$$

$$\therefore B_\phi = \frac{\mu_0 NI}{2\pi r}$$

$$\mathbf{B} = \mathbf{a}_\phi B_\phi = \mathbf{a}_\phi \frac{\mu_0 NI}{2\pi r}, \qquad (b-a) < r < (b+a).$$

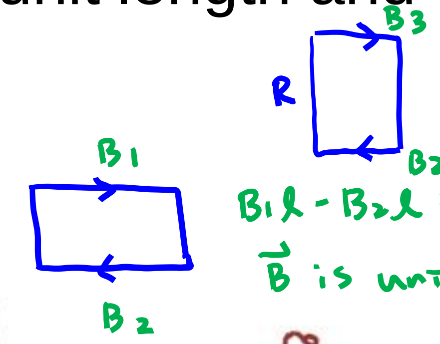
$$\therefore \vec{B} = \frac{\mu_0 NI}{2\pi r} \vec{a}_\phi$$

$$\mathbf{B} = \mathbf{a}_\phi B_\phi = \mathbf{a}_\phi \frac{\mu_0 NI}{2\pi r}, \qquad (b-a) < r < (b+a).$$

- For r < (b-a) and r > (b+a)
  - The net total current enclosed by a contour is zero.
  - **B** = 0.

Example

- Determine the magnetic flux density inside an infinitely long solenoid with air core having n closely wound turns per unit length and carrying a current I.



go to infinity $\rightarrow B_3 = 0 \rightarrow B_2 = 0$

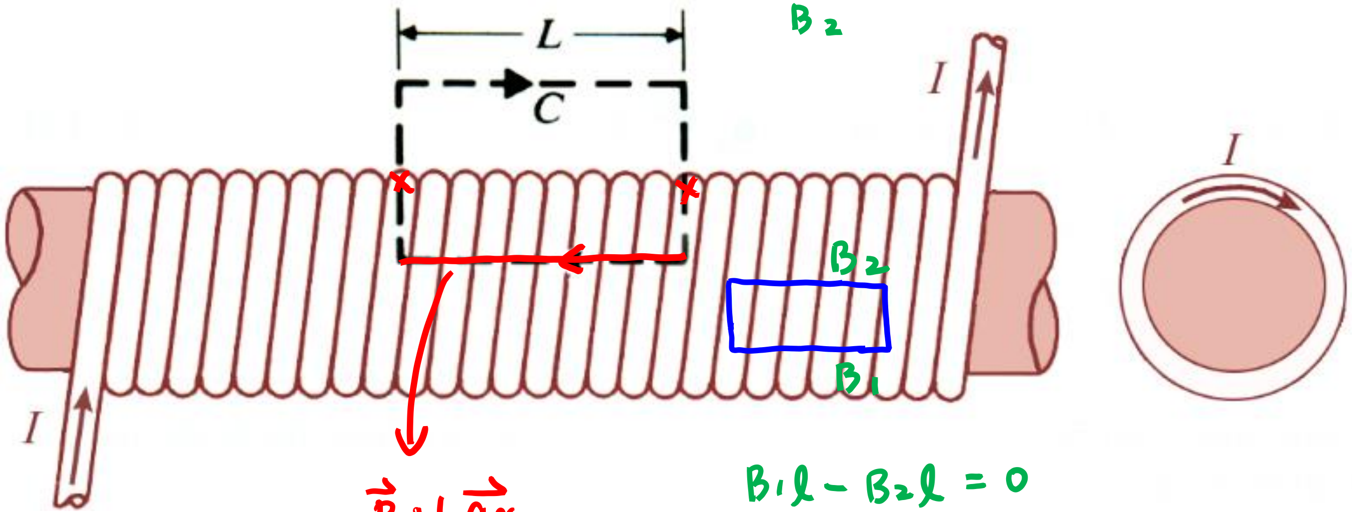
$R = \infty$

$\therefore B_1 = B_2 = B_3 = 0$

$\vec{B}$ is zero outside the solinoid

$B_1 \ell - B_2 \ell = 0 \quad \therefore B_1 = B_2$

$\vec{B}$ is uniform outside the solinoid



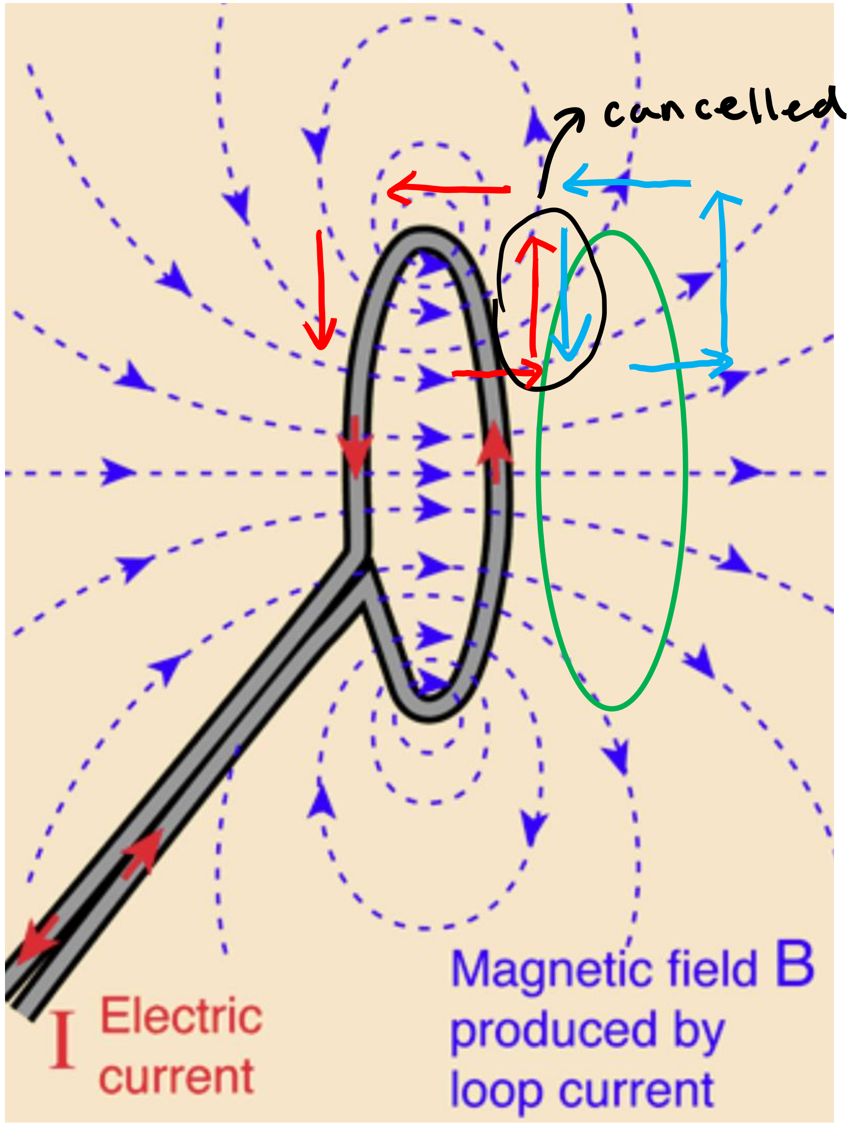
$\vec{B} \cdot L \vec{a_x}$

$B_1 \ell - B_2 \ell = 0$

$\therefore B_1 = B_2 \Rightarrow \vec{B}$ is uniform inside the solinoid

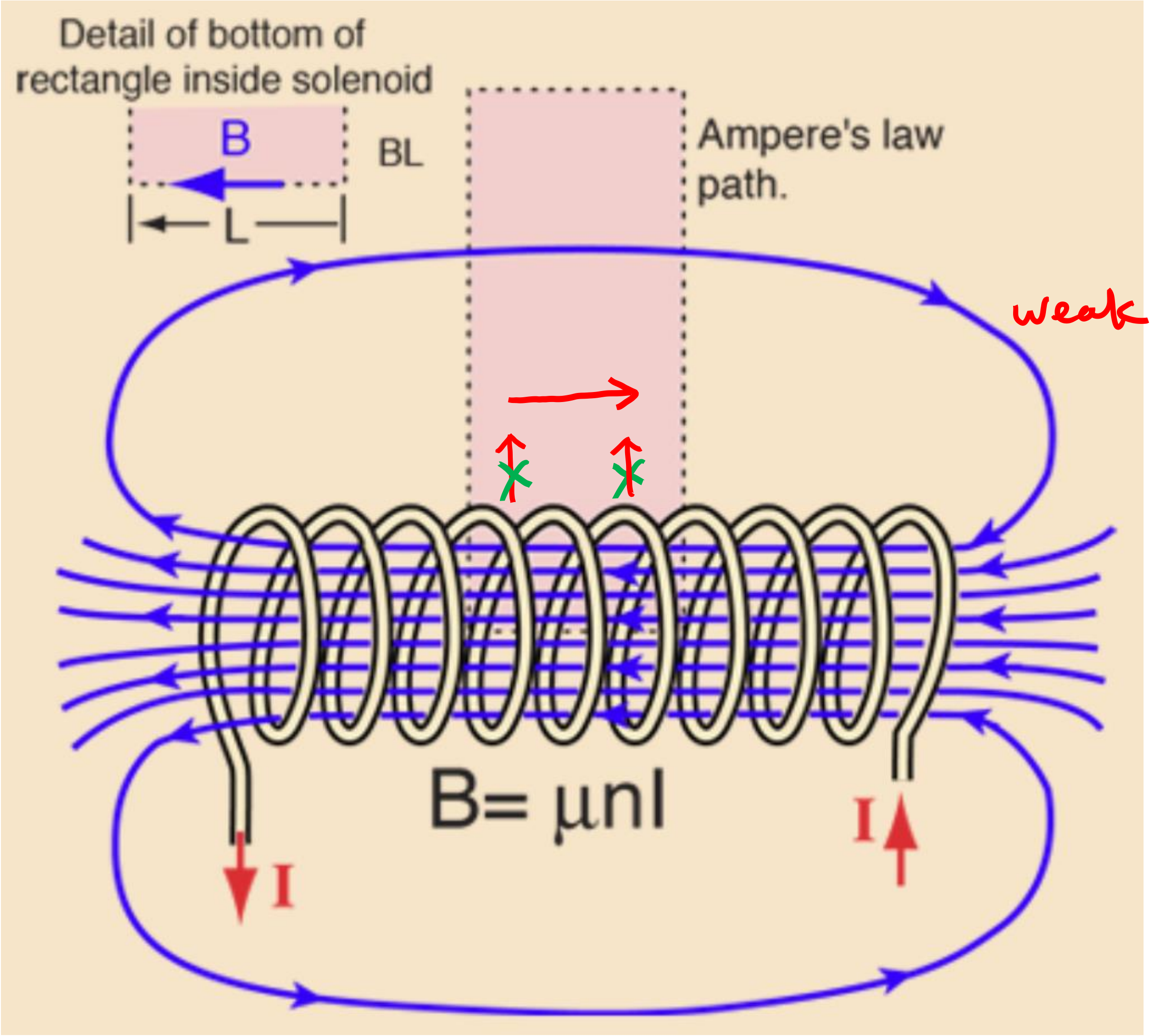
$\Rightarrow \vec{B} = B \vec{a_x}$

Example



Ref: HyperPhysics

Practical Case



Ref: HyperPhysics

Example

- No magnetic field outside the solenoid
- Inside the solenoid
  - **B** field inside must be parallel to the axis because of the symmetry.

$$BL = \mu_0 nLI$$

N = nL

$$B = \mu_0 nI.$$

- Let *b* → ∞ toroidal coil

$$\mathbf{B} = \mathbf{a}_\phi B_\phi = \mathbf{a}_\phi \frac{\mu_0 NI}{2\pi r}, \qquad (b-a) < r < (b+a).$$

$$B = \mu_0 \underbrace{\left(\frac{N}{2\pi b}\right)}_{n} I = \mu_0 nI$$