賽局論 HW4

許博翔

November 3, 2023

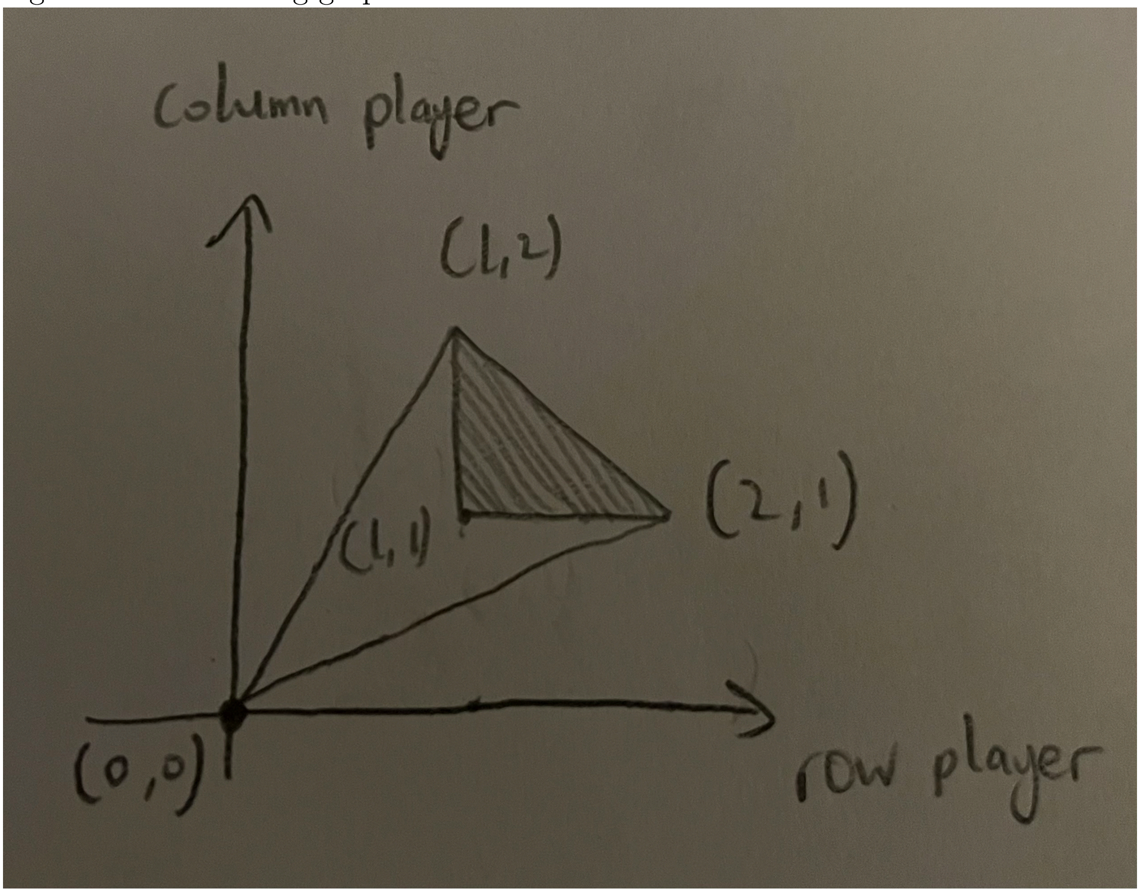
Problem 1.

(a) $\overline{m_1} = \min_{t \in T} \max_{s \in S} \pi_1(s,t) = \min(1,2) = 1.$

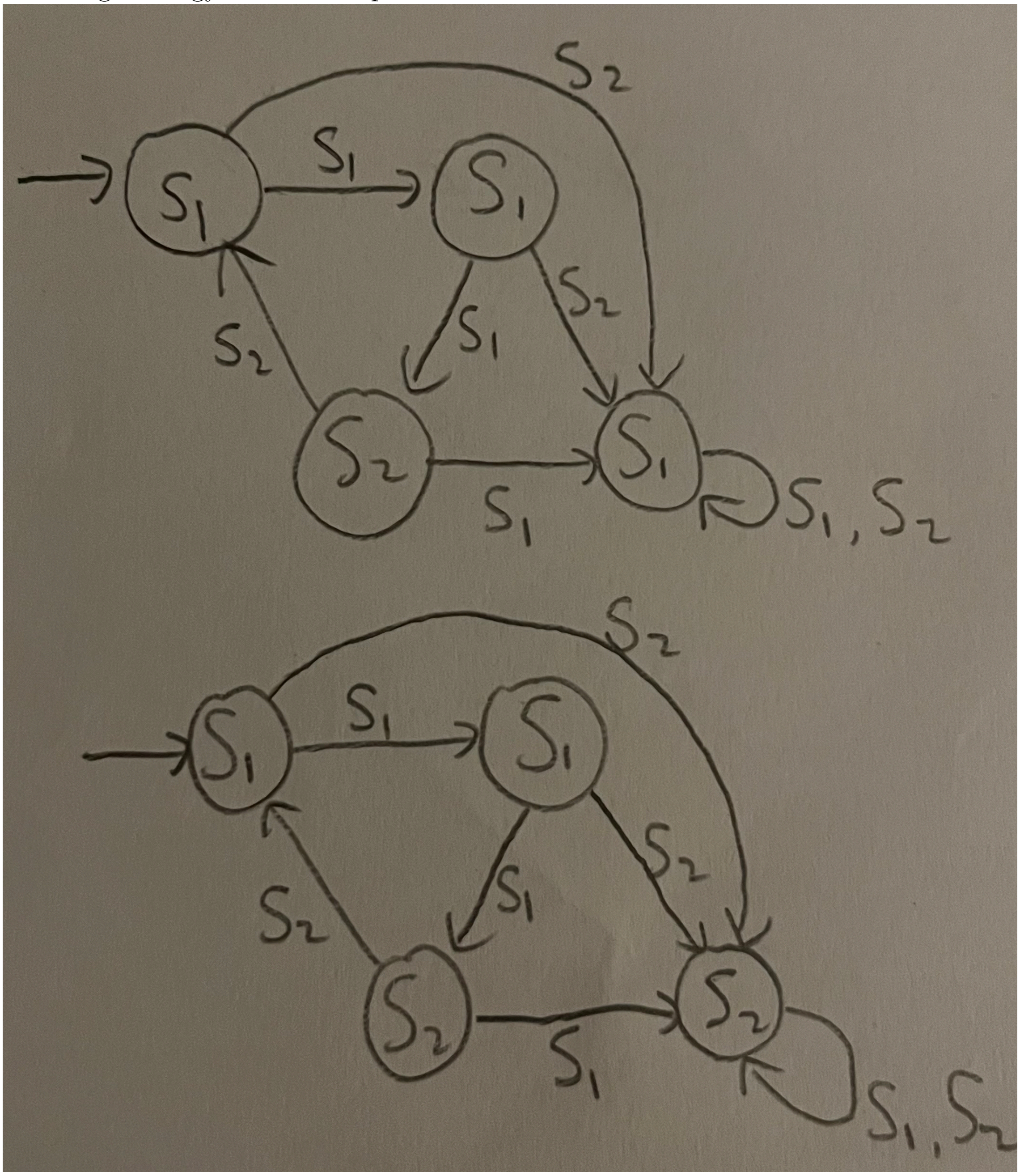
$\overline{m_2} = \min_{s \in S} \max_{t \in T} \pi_2(s,t) = \min(2,1) = 1.$

$\therefore$ by Folk theorem, the payoffs they could achieve in a Nash equilibrium is larger than or equal to $(\overline{m_1}, \overline{m_2}) = (1,1)$.

Therefore, the payoffs they could achieve in a Nash equilibrium is the colored region in the following graph:

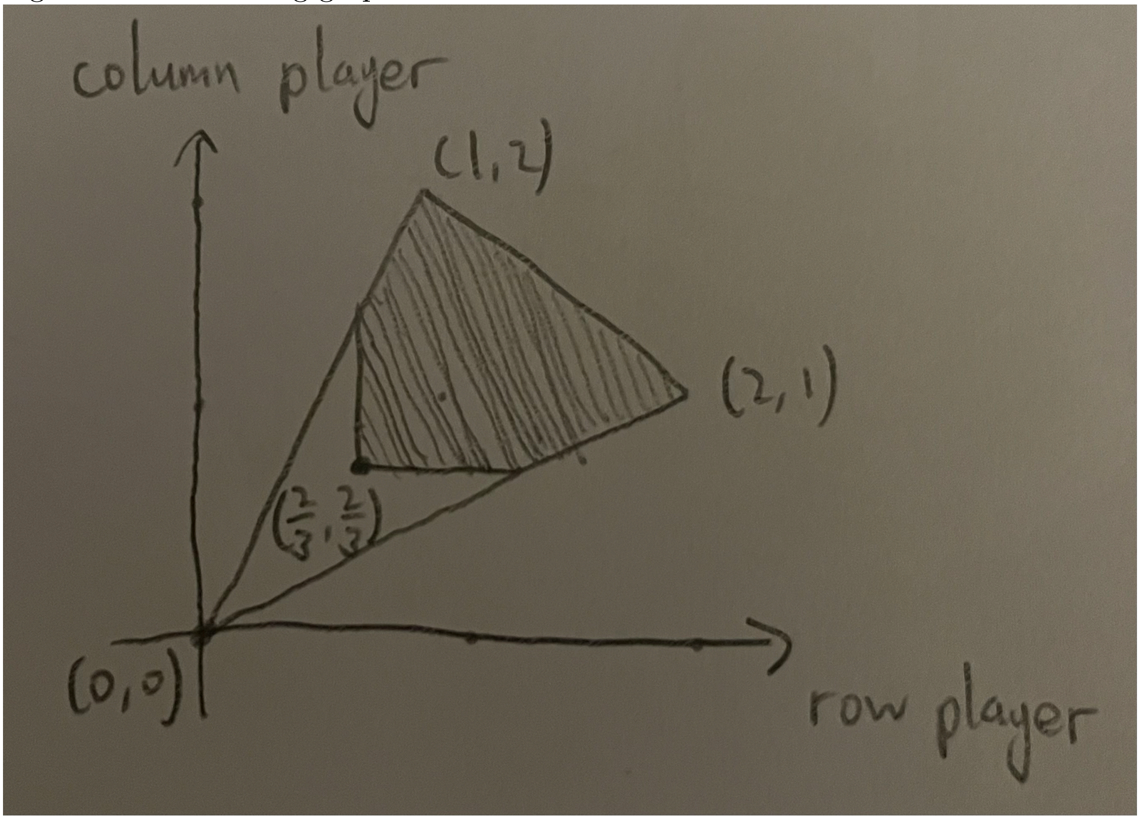


(b) The upper one is the row player's strategy, while the bottom one is the columns player's strategy. The strategy is to play $(s_1, s_1), (s_1, s_1), (s_2, s_2)$ cyclically. Since if one of them change the strategy, their long-run average payoff will decrease to 1, no one can change the strategy to achieve a better payoff. Therefore, the following strategy is a Nash equilibrium.



(c) Suppose that $(p, 1-p)$ is the row player's mixed strategy.
$\underline{v_1} = \max_p \min_{t \in T} \mathbb{E}(\pi_1(s,t)) = \max_p \min(2p, 1-p) = \frac{2}{3}$, where $p = \frac{1}{3}$.
Similarly, $\underline{v_2} = \frac{2}{3}$.

$\therefore$ by Folk theorem, the payoffs they could achieve in a Nash equilibrium is larger than or equal to $(\frac{2}{3}, \frac{2}{3})$.
Therefore, the payoffs they could achieve in a Nash equilibrium is the colored region in the following graph:



Problem 2 (11.9.5). Let A, B be the space of the strategy of the row, column player, respectively.
Suppose that the 3 Nash equilibria are $(a_1, b_1), (a_2, b_2), (a_3, b_3)$.
Let $p_i(s), q_i(t)$ denote the probability that the row, column player plays s, t when using strategy a_i, b_i, respectively.
Let $f : T \to A, g : S \to B$ denote the strategy of the row, column player when playing the game for the second time, where f_i, g_i denote the constant function $f_i(t) = a_i, g_i(s) = b_i$, respectively.
Let $u(a, b), v(a, b)$ denote the expected payoff of the row, column player when the row player uses strategy a and the column player uses strategy b, respectively.
Claim: $\forall i, j,\ (a_i f_j, b_i g_j)$ are Nash equilibria.
Proof: $\forall a \in A,\ f : T \to A,\ u(af, b_i g_j) \overset{\forall s,\ g_j(s)=b_j}{=} u(a, b_i) + \sum_{t \in T} q_i(t) u(f(t), b_j) \overset{(a_j,b_j) \text{ is a Nash equilibria}}{\leq}$

$u(a,b_i)+\sum_{t\in T} q_i(t)u(a_j,b_j) \overset{\sum_{t\in T} q_i(t)=1}{=} u(a,b_i)+u(a_j,b_j) \overset{(a_i,b_i)\text{ is a Nash equilibria}}{\leq} u(a_i,b_i)+u(a_j,b_j) = u(a_i f_j, b_i g_j)$.

Similarly, $\forall b \in B,\ g : S \to B,\ v(a_i f_j, bg) \leq v(a_i f_j, b_i g_j)$.

$\therefore$ by the definition, $(a_i f_j, b_i g_j)$ is a Nash equilibria.

$\therefore$ the claim holds. Since there are 9 ways to choose i, j, there are at least 9 Nash equilibria.

Problem 3 (11.9.28)**.**

(a) The expected payoff of s playing with s is $2 \times 99 + \frac{1}{4}(2+0+3-1) = 199$.

The maximum expected payoff of a strategy that chooses the different action (from s when playing with s) in the i-th ($1 \leq i \leq 99$) stage is $2(i-1)+3 \leq 2 \times 98 + 3 = 199$.

The maximum expected payoff of a strategy that chooses the same action (as s when playing with s) in the first 99 stages, and its probability of choosing slow in the last stage is p is $2 \times 99 + \frac{1}{2}p(2+0) + \frac{1}{2}(1-p)(3-1) = 199$.

$\therefore$ changing s to another strategy won't increase the expected payoff, and therefore (s, s) is a Nash equilibrium.

(b) Claim: For all $1 \leq i \leq 100$, the subgame that starts from the i-th stage using the suffix strategy of s (that is, do what s will do in the $j+i-1$-th stage when playing the j-th stage) denote as s^*, has a Nash equilibrium (s^*, s^*).

Proof: The expected payoff of s^* playing with s^* is $2 \times (100-i) + \frac{1}{4}(2+0+3-1) = 201 - 2i$.

The maximum expected payoff of a strategy that chooses the different action (from s^* when playing with s^*) in the j-th ($1 \leq j \leq 100-i$) stage is $2(j-1)+3 \leq 2 \times (100-i-1)+3 = 201-2i$.

The maximum expected payoff of a strategy that chooses the same action (as s^* when playing with s^*) in the first $100-i$ stages, and its probability of choosing slow in the last stage is p is $2 \times (100-i) + \frac{1}{2}p(2+0) + \frac{1}{2}(1-p)(3-1) = 201-2i$.

$\therefore$ changing s^* to another strategy won't increase the expected payoff, and therefore (s^*, s^*) is a Nash equilibrium.

By the definition of subgame-perfect equilibrium, since (s, s) plays a rational strategy (which is (s^*, s^*)) in every subgame, (s, s) is a subgame-perfect equilibrium.

(c) $\underbrace{-1, \ldots, -1}_{i}, \underbrace{2, \ldots, 2}_{99-i}, 1$ for any $0 \leq i \leq 66$ can be supported by both players using the strategy that tells you always to choose speed up until the $(i+1)$-th stage, to choose slow up until the 100th stage, and to use slow and speed with equal probabilities at the 100th stage—unless the two players have failed to use the same actions at every preceding stage. If such a coordination failure has occurred in the past, the strategy tells a player to look for the first stage at which differing actions were used and then always to use speed.

(d) $\underline{v_1} = \max_p \min_t \mathbb{E}(\pi(s, t)) = \max_p \min(2p + 3(1-p), 0 - (1-p)) = 0$ when $p = 1$.
Similarly, $\underline{v_2} = 0$.
$\Rightarrow$ strings that have expected payoff ≥ 0 for any subgame can be supported as equilibrium outcomes by folk theorem.