賽局論 HW3

許博翔

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Problem 1 (7.11.24)**.**

(a) $\min(7,2,1,2,7)=1, \min(2,6,2,6,2)=2, \min(5,4,3,4,5)=3, \min(2,6,2,6,2)=2, \min(7,2,1,2,7)=1$, so $\underline{m}=\max(1,2,3,2,1)=3$.

$\max(7,2,5,2,7)=7, \max(2,6,4,6,2)=6, \max(1,2,3,2,1)=3, \max(2,6,4,6,2)=6, \max(7,2,5,2,7)=7$, so $\overline{m}=\min(7,6,3,6,7)=3$.

Since from 4b, we get that $\underline{v}=\overline{v}=\underline{m}=3$.

$\therefore$ the value of this game is 3, and the row player's strategy is $p=(0,0,1,0,0)$; the column player's strategy is $q=(0,0,1,0,0)$.

(b) $$\begin{pmatrix} 1 & 3 & 2 & 5 \\ 0 & -1 & 6 & 7 \\ 3 & 4 & 2 & 3 \\ -7 & 2 & 2 & 1 \end{pmatrix} \overset{c_4\geq c_1 \text{ and } r_4\leq r_1}{\Rightarrow} \begin{pmatrix} 1 & 3 & 2 \\ 0 & -1 & 6 \\ 3 & 4 & 2 \end{pmatrix} \overset{r_1\leq r_3}{\Rightarrow} \begin{pmatrix} 0 & -1 & 6 \\ 3 & 4 & 2 \end{pmatrix}.$$

Suppose the strategy of the row player is $(0,p_2,p_3,0)$, and the strategy of the column player is $(q_1,q_2,q_3,0)$.

$\underline{v}=\max\limits_{p_2,p_3}\min(3p_3,-p_2+4p_3,6p_2+2p_3)=\max\limits_{p_2}\min(3-3p_2,4-5p_2,2+4p_2)$.

The maximum occurs when $3-3p_2=4-5p_2$ or $3-3p_2=2+4p_2$ or $4-5p_2=2+4p_2$. We take these solutions $p_2=\frac{1}{2},\frac{1}{7},\frac{2}{9}$, calculate the correspond maximin and we get that when $p_2=\frac{1}{7}$, the maximin is maximized.

$\therefore \underline{v}=\min(\frac{18}{7},\frac{23}{7},\frac{18}{7})=\frac{18}{7}$.

On the other hand, $\overline{v}=\min\limits_{q_1,q_2,q_3}\max(-q_2+6q_3,3q_1+4q_2+2q_3)=\max(-q_2+6q_3,3+q_2-q_3)$.

The minimum occurs when $-q_2+6q_3=3+q_2-q_3$.

$\Rightarrow q_2 = \dfrac{7q_3 - 3}{2}$.

Since $q_2 \in [0,1], q_3 \in [0,1], q_2 + q_3 \in [0,1]$, we get the bound of $q_3 = [\frac{3}{7}, \frac{5}{7}] \cap [0,1] \cap [\frac{1}{3}, \frac{5}{9}] = [\frac{3}{7}, \frac{5}{9}]$.

$\Rightarrow -q_2 + 6q_3 = \frac{3}{2} + \frac{5}{2}q_3$, which has minimum when $q_3 = \frac{3}{7}$.

$\therefore \overline{v} = \frac{3}{2} + \frac{5}{2} \times \frac{3}{7} = \frac{18}{7}$.

$\therefore$ the value of this game is $\frac{18}{7}$, and the row player's strategy is $p = (0, \frac{1}{7}, \frac{6}{7}, 0)$; the column player's strategy is $q = (\frac{4}{7}, 0, \frac{3}{7}, 0)$.

Problem 2. The following is the payoff Colonel Blotto gets for each strategy:

Colonel Blotto \ Count Baloney	$(2,1)$	$(1,2)$
$(3,1)$	$2+0$	$1-1$
$(2,2)$	$0+1$	$1+0$
$(1,3)$	$-1+1$	$0+2$

Since Count Baloney gets exactly the opposite of the payoff that Clonel Blotto gets, the above value is what Colonel Blotto wants to maximize and Count Baloney wants to minimize.

$\min(2,0) = 0, \min(1,1) = 1, \min(0,2) = 0$, so for Clonel Blotto, the security value is $\max(0,1,0) = 1$, and the strategy is $(2,2)$.

$\max(2,1,0) = 2, \max(0,1,2) = 2$, so for Count Baloney, the security is $\min(2,2) = 2$, and the strategy is $(2,1)$ or $(1,2)$.

Since $2 > 1$, the saddle point does not exist by the theorem in the powerpoint of minimax and maximin.

Suppose that Count Baloney's mixed strategy is $q = (q_1, q_2)$, if Colonel Blotto plays:

$(3,1)$, $\pi = 2q_1$

$(2,2)$, $\pi = q_1 + q_2$

$(1,3)$, $\pi = 2q_2$.

$\max(2q_1, q_1+q_2, 2q_2) \overset{q_1+q_2=\frac{2q_1+2q_2}{2}}{=} \max(2q_1, 2q_2) = \max(2q_1, 2-2q_1) \geq \dfrac{2q_1 + 2 - 2q_1}{2} = 1$, the equation holds $\iff q_1 = \frac{1}{2}$.

$\therefore q = (\frac{1}{2}, \frac{1}{2})$ is Count Baloney's strategy.

Suppose that Colonel Blotto's mixed strategy is $p = (p_1, p_2, p_3)$, if Count Baloney's plays:

$(2,1)$, $\pi = 2p_1 + p_2$

$(1,2)$, $\pi = p_2 + 2p_3$.

The maximum of $\min(2p_1 + p_2, p_2 + 2p_3)$ occurs when $2p_1 + p_2 = p_2 + 2p_3$, that is, $p_1 = p_3$.

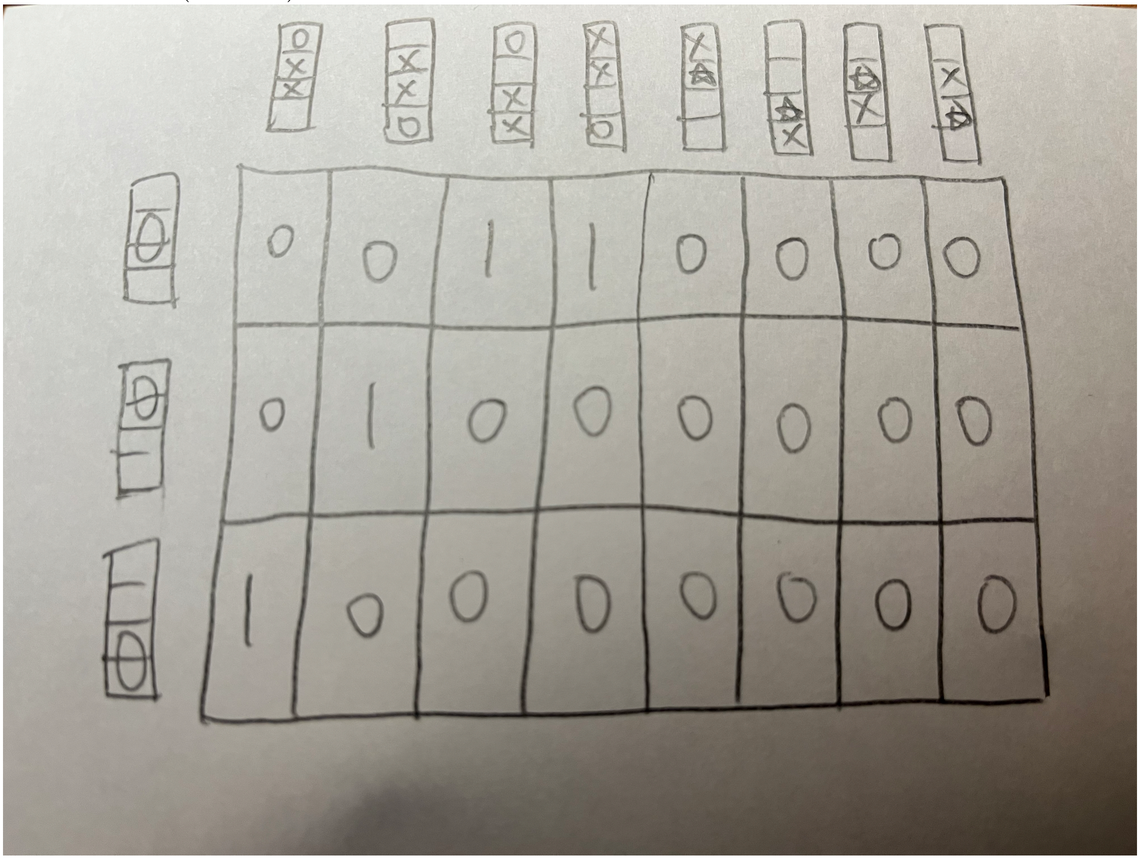
$\Rightarrow p_2 = 1 - p_1 - p_3 = 1 - 2p_1$.

$\Rightarrow \max\min(2p_1 + p_2, p_2 + 2p_3) = \max\min(1,1) = 1$.

$\therefore p = (p_1, 1-2p_1, p_1)$ for all $0 \le p_1 \le \dfrac{1}{2}$ is Colonel Blotto strategy.

$\therefore p = (p_1, 1-2p_1, p_1)$ for $0 \le p_1 \le \dfrac{1}{2}$, $q = (\dfrac{1}{2}, \dfrac{1}{2})$ is a mixed-strategy Nash equilibrium.

Problem 3 (7.11.32)**.**



1 represents that the starship is survived, and 0 represents that the starship is destroyed.

$\Rightarrow$ Captain Kirk's aim is to maximize the value, while Mr. Spock's aim is to minimize the value.

One can see that the intersection of the first row and the 8-th column is a saddle point since $0 \leq 0, 0 \leq 1, 0 \geq 0$, and therefore the starship will be destroyed if both players play optimally.

Problem 4. Suppose that M is an $n \times k$ matrix, and let M_{ij} denote the value of the intersection of the i-row and the j-column of M.

(a) $\underline{m} = \max\limits_{i=1,2,\ldots,n} \min\limits_{j=1,2,\ldots,k} M_{ij}$.

Let $S = \{(a_1, a_2, \ldots, a_n) \in [0,1]^n : a_1 + a_2 + \cdots + a_n\}$.

Let $p = (p_1, p_2, \ldots, p_n)$ be the mixed strategy of the row player.

$\underline{v} = \max\limits_{(p_1,p_2,\ldots,p_n) \in S} \min\limits_{j=1,2,\ldots,k} (M_{1j}p_1 + M_{2j}p_2 + \cdots + M_{nj}p_n)$.

Let $e_i := (e_{i1}, e_{i2}, \ldots, e_{in})$, where $e_{ij} = \mathbb{I}\{i = j\}$.

Clearly, $e_1, e_2, \ldots, e_n \in S$.

$\Rightarrow \underline{v} = \max\limits_{(p_1,p_2,\ldots,p_n)\in S} \min\limits_{j=1,2,\ldots,k} (M_{1j}p_1 + M_{2j}p_2 + \cdots + M_{nj}p_n) \geq \max\limits_{(p_1,p_2,\ldots,p_n)\in\{e_1,e_2,\ldots,e_n\}} \min\limits_{j=1,2,,k} (M_{1j}p_1 + M_{2j}p_2 + \cdots + M_{nj}p_n) = \max\limits_{i=1,2,\ldots,n} \min\limits_{j=1,2,\ldots,k} (M_{1j}e_{i1} + M_{2j}e_{i2} + \cdots + M_{nj}e_{in}) = \max\limits_{i=1,2,\ldots,n} \min\limits_{j=1,2,\ldots,k} (M_{ij}) = \underline{m}$.

(b) Since the row player and the column player are symmetric.

$\therefore$ from the above, similarly, $\overline{v} \leq \overline{m}$.

$\Rightarrow \underline{m} = \overline{m} \geq \overline{v} \geq \underline{v} \geq \underline{m}$.

$\Rightarrow \overline{v} = \underline{m} = \underline{v}$.

Problem 5.

Decision variables: $p_1, p_2, \ldots, p_m$.

The objective function: maximize $\underline{v}$.

Constrained:

$\forall j,\ \underline{v} \leq p_1\pi_{1,j} + p_2\pi_{2,j} + \cdots + p_m\pi_{m,j}$, $(\because \underline{v} = \min\limits_j (p_1\pi_{1,j} + p_2\pi_{2,j} + \cdots + p_m\pi_{m,j}))$

$p_1, p_2, \ldots, p_m \in [0, 1]$,

$p_1 + p_2 + \cdots + p_m = 1$.