Graph Theory 1-HW3

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Exercise. (2)

The edges of G can be partitioned into k trails if and only if G has at most $2k$ vertices of odd degree.

Proof:

$\Rightarrow$:

Every trail consists of 2 odd degree vertices and some even degree vertices, and adding even numbers doesn't create odd number, therefore k trails add up to at most $2k$ odd degree vertices. $\Leftarrow$:

Let $2m$ be the number of odd degree vertices.

Because every connected graph has an even total degree, the number of odd degree vertices must be even. For every odd degree vertex in graph G, we can start walking along any connected edge until we encounter another originally odd degree vertex (such a vertex must exist because the number of odd degree vertices must be even), and by removing this trail, we decrease the total number of odd degree vertices by 2. By doing the above step m times, the resulting graph will consist of some even degree vertices, which means each connected component must be a Euler circuit. Since the original graph G is connected, each connected component must be connected to at least one trail, and we can add the Euler circuit to one of the trails. Therefore, every edge would belong to one of the k trails (if $m < k$, there would be $k - m$ empty trails).

Exercise. (4)

First, we consider G', because G' is connected and all degrees in G' are even, G' must be a Euler circuit, thus in its line graph, the Euler circuit would map to a Hamiltonian path.

Next, because $V(G') = V(G)$, every edge in $E(G) \setminus E(G')$ connects 2 vertices in $V(G')$. Every vertex v maps to the edges in a $K_{d(v)}$, so if there are some edges in $E(G) \setminus E(G')$ connected to v, we can add the edges into the Hamiltonian path in the $L(G)$ by adding the corresponding vertices in any order between 2 originally neighboring (in the original Hamiltonian path) vertices. Every such edge in the original graph connects 2 vertices, and we only need to choose one of them to operate the previously mentioned step.

This way, we can we can always find a Hamiltonian path in any connected graph G.

Exercise. (5)

In the case of $k = 1$, the statement is incorrect because cycles of length 2 does not exist, so we only consider $k \geq 2$. For the proof below, if we say there is a cycle of length at least n, it indicates we have the list of n of the vertices in the cycle.

We can use Mathematical Induction to prove the statement:

For any k-connected graph G ($k \geq 2$), we can choose 2 arbitrary edges with distinct vertices, and according to Menger's Theorem (let the 2 vertices on an edge be $A = a_1,\ a_2$ and those on the other edge be $B = b_1,\ b_2$), we can find 2 disjoint paths connecting the 2 sets, which together with the 2 edges itself forms a cycle of length≥ 4.

If there exists a cycle of length at least $2(m-1)$ ($3 \leq m \leq k$), then there exists a cycle of length at least $2m$. Proof:

We choose 2 arbitrary vertices $v_1,\ v_2$ which are not in the current list of vertices to add into the cycle, and add them one by one.

For the first vertex v_1, if it is already in the cycle (but not in the current list), then we simply add that vertex to the list. Otherwise, we can find at least m distinct paths from v_1 to a vertex on the cycle. Since $m > \frac{2(m-1)}{2}$, there exists 2 of the m vertices on the cycle corresponding each path that are adjacent. Thus, we can find distinct vertex-disjoint (apart from the starting vertex) paths (we will call such paths "distinct paths" below) from v_1 to 2 adjacent vertices $w_1,\ w_2$ on the cycle, remove the edge connecting w_1 and w_2, and insert v_1 and the 2 paths into the cycle. Such paths must exist because of the following:

Vertex v must be connected to at least m other vertices (the graph is m-connected as $m \leq k$). We consider 2 special cases first. Firstly, if at least m vertices are conveniently on the cycle, then it's trivial. Secondly, if only $m-1$ of the vertices are on the cycle, there exists a path from the another vertex to any not-chosen vertex on the cycle, because the graph should still be connected after removing the $m-1$ vertices, thus we have m distinct paths.

As for general cases where there are n vertices on the cycle that are adjacent to v_1 and $n \leq m-2$, we let the set of the n vertices and $m-n$ other vertices adjacent to v_1 be A, and let the set of any m vertices in the current cycle but not in A be B (such set can be found since $n \leq m-2 \Rightarrow 2(m-1) \geq m+n$). By Menger's Theorem, we can find a system of k vertex disjoint A, B-paths $\Rightarrow$ the $m-n$ vertices in A that are not in the cycle each has a disjoint paths to a different vertex on the cycle (if an A, B-path includes some vertices on the cycle but not in A nor B, we can replace the endpoint with the first of such vertices encountered during the path from A to B, in order to meet our requirements). Thus, we show that there are at least m distinct paths from v_1 to the cycle.

Therefore, we can compose a cycle with a cycle of at least $2m-1$.

Similarly, we add the second vertex v_2 to the cycle as well (a cycle of length at least $2m-1$ can be treated as that of $2m-2$ and therefore we can apply the method above.)

By Mathematical Induction, there exists a cycle of length at least $2k$ in a k-connected graph G with at least $2k$ vertices.