Jensen's Inequality
(AM-GM Inequality)

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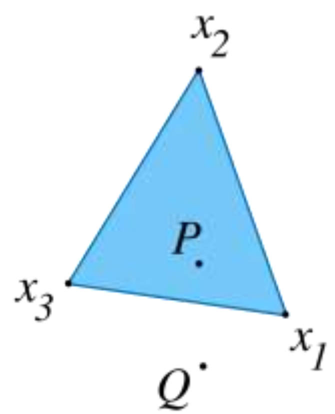
Convex Combination

- Convex combination of n points ($\mathbf{x}_i, i = 1\sim n$) in a d-dim space is $\sum_{i=1}^{n} \lambda_i \mathbf{x}_i$, with $\lambda_i \geq 0$ and $\sum_{i=1}^{n} \lambda_i = 1$.

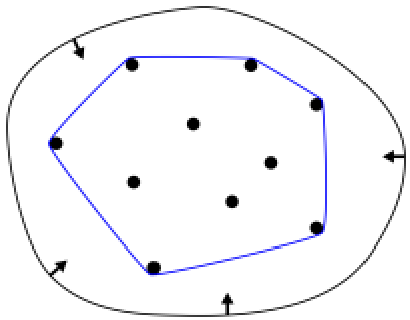
- n=2
  ➔ $\mathrm{P} = \lambda_1 \mathbf{x}_1 + \lambda_2 \mathbf{x}_2$
- n=3
  ➔ $\mathrm{P} = \lambda_1 \mathbf{x}_1 + \lambda_2 \mathbf{x}_2 + \lambda_3 \mathbf{x}_3$
- n=4
  ➔ $\mathrm{P} = \sum_{i=1}^{4} \lambda_i \mathbf{x}_i$



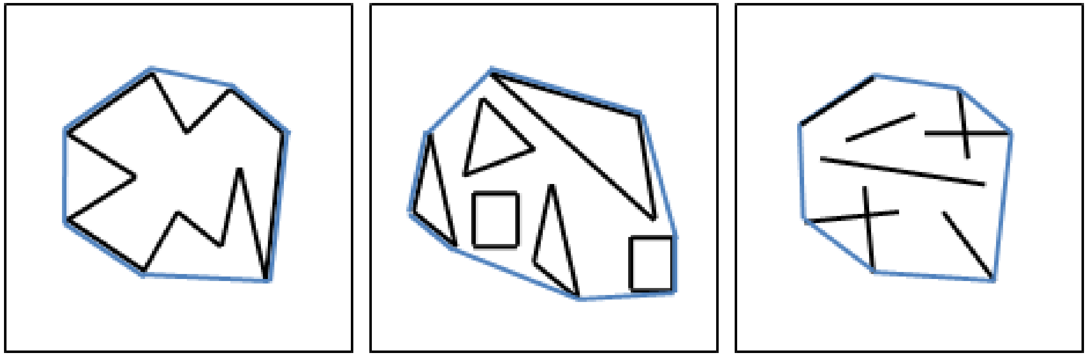
Convex Hull

- Given any n points in a set X, the convex hull (or convex set) of X is the convex combination of these n points.

Rubber band analogy



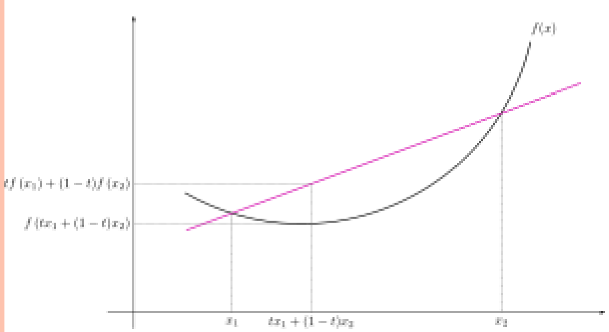
Various sets of X

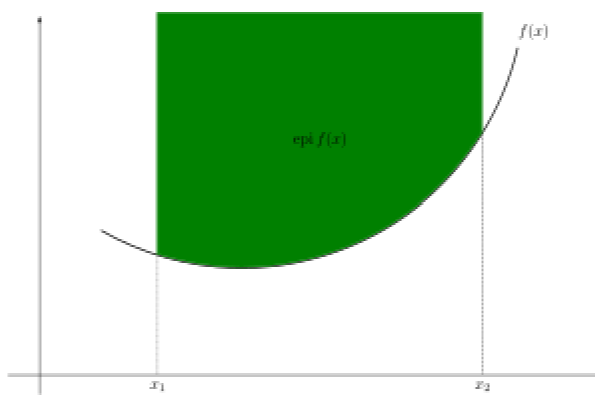


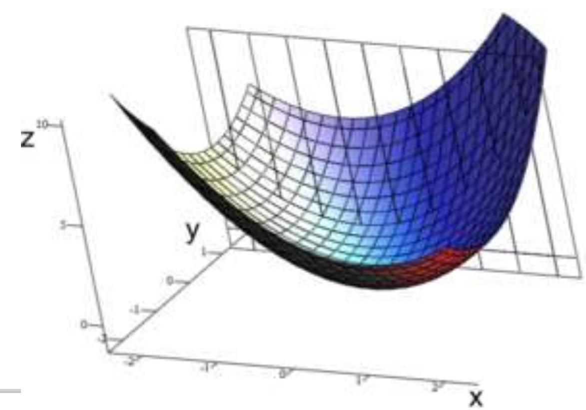
http://web.ntnu.edu.tw/~algo/ConvexHull.html

Convex Functions

- A convex function
  - A line segment connecting two points on the function lies above the function.
  - The function's second derivative is nonnegative.
  - The sets of points on or above the function is a convex set.
- Examples of convex functions
  - $y = x^2$ or $y = e^x$

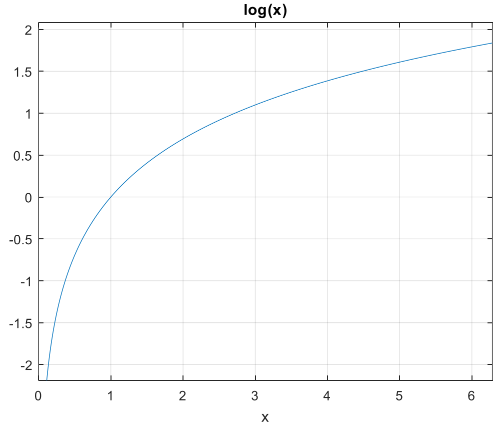






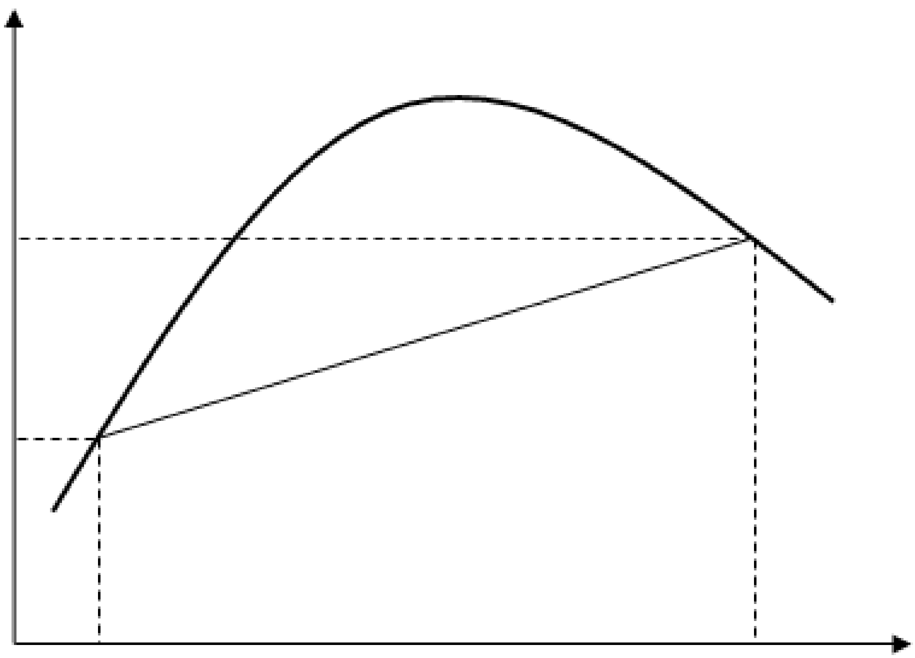
Concave Functions

- A concave function
  - A line segment connecting two points on the function lies below the function.
  - The function's second derivative is nonpositive.
  - The sets of points on or below the function is a convex set.
- Example of concave functions
  - $y = \ln(x) \rightarrow y' = \frac{1}{x} \rightarrow y'' = -\frac{1}{x^2} < 0$



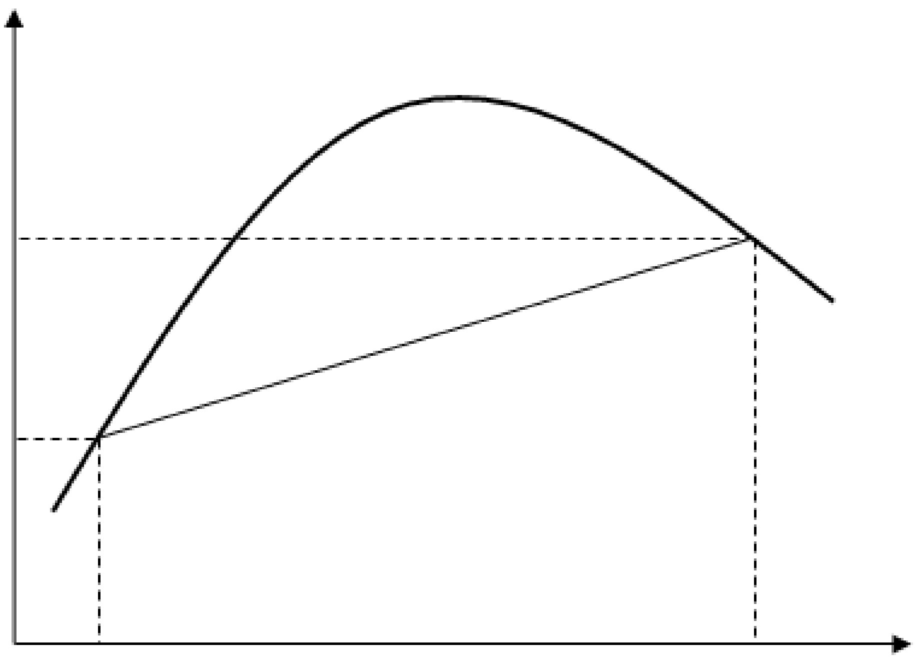
Jensen's Inequality when n=2

- If f(x) is a concave function, then
  - $f(\lambda_1 x_1 + \lambda_2 x_2) \geq \lambda_1 f(x_1) + \lambda_2 f(x_2)$, with $\lambda_1 + \lambda_2 = 1$ and $\lambda_1, \lambda_2 \geq 0$.



Jensen's Inequality in General

- If f(x) is a concave function, then
  - $f(\sum_{i=1}^{n} \lambda_i x_i) \geq \sum_{i=1}^{n} \lambda_i f(x_i)$, with $\sum_{i=1}^{n} \lambda_i = 1$ and $\lambda_i \geq 0, \forall i$.



Inequality of Arithmetic and Geometric Means

- AM-GM inequality

Quiz!

$$\frac{\sum_{i=1}^{n} x_i}{n} \geq \left(\prod_{i=1}^{n} x_i \right)^{1/n}, \; with\ x_i \geq 0, \forall i$$

The equality holds only when $x_1 = x_2 = \cdots = x_n$.

- Proof by Wikipedia

Cumbersome!

- Proof by Jensen's inequality
  - Take $f(x) = ln(x)$ and $\lambda_i = \frac{1}{n}, \forall i$

$$\Rightarrow \ln\left(\frac{\sum_{i=1}^{n} x_i}{n}\right) \geq \frac{1}{n}\sum_{i=1}^{n} \ln(x_i) = \ln\left(\left(\prod_{i=1}^{n} x_i\right)^{1/n}\right)$$ Q.E.D.

Proof by Induction: $\frac{\sum_{i=1}^{n} x_i}{n} \geq \left(\prod_{i=1}^{n} x_i\right)^{1/n}$

$n=1 \Rightarrow x_1 \geq x_1$

$n=2 \Rightarrow \ln\left(\frac{x_1+x_2}{2}\right) \geq \frac{\ln x_1 + \ln x_2}{2}$. (Or you can start with $\left(\sqrt{x_1}-\sqrt{x_2}\right)^2 \geq 0$)

$$n=3 \Rightarrow \ln\left(\frac{x_1+x_2+x_3}{3}\right) = \ln\left(\frac{2\left(\frac{x_1+x_2}{2}\right)+x_3}{3}\right) \geq \frac{2\ln\left(\frac{x_1+x_2}{2}\right)+\ln x_3}{3} \geq \frac{\ln x_1+\ln x_2+\ln x_3}{3}$$

$$n=k \text{ holds by assumption } \Rightarrow \ln\left(\frac{\sum_{i=1}^{k} x_i}{k}\right) \geq \left(\frac{\sum_{i=1}^{k} \ln x_i}{k}\right)$$

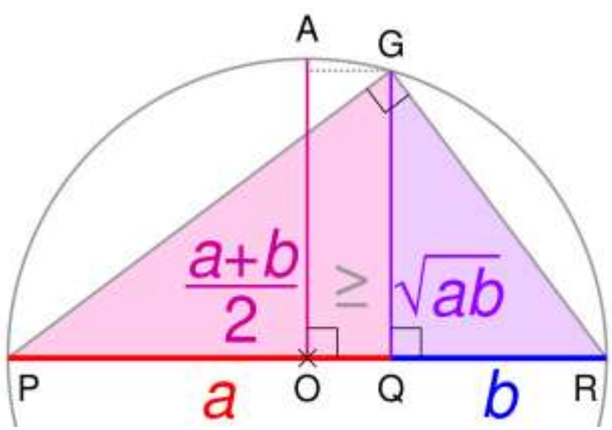
$$n=k+1 \Rightarrow \ln\left(\frac{\sum_{i=1}^{k} x_i + x_{k+1}}{k+1}\right) = \ln\left(\frac{k\frac{\sum_{i=1}^{k} x_i}{k}+x_{k+1}}{k+1}\right) \geq \frac{k\ln\left(\frac{\sum_{i=1}^{k} x_i}{k}\right)+\ln x_{k+1}}{k+1} \geq \frac{k\left(\frac{\sum_{i=1}^{k} \ln x_i}{k}\right)+\ln x_{k+1}}{k+1} = \frac{\sum_{i=1}^{k+1} \ln x_i}{k+1}$$

Q.E.D.

Summary

- AM-GM inequality can be derived by Jensen's inequality.
- Jensen's inequality can be proved by convex combination. ➔ Seeing the insight is the key to math!

$$\frac{a+b}{2} \geq \sqrt{ab}$$



$$(x+y)^2 \geq 4xy$$