Compounding
複利

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複利的故事

- 故事
  - 國王和棋盤
  - 一莫耳的新台幣有多少？
  - 為什麼領18%是老賊？
- 信用卡循環利率
  - George & Mary現金卡廣告（救急免利卡）
  - 2006年，台灣有70萬人淪為卡奴，平均欠款數100萬新台幣… (wiki)
  - 2010年：最新信用卡循環利率 最低2.74%最高20% (link)
  - 如果你有卡債100萬、18%計息
    - 月付2萬，約4年可還清 ➔ 錯！需要八年！
    - 月付1萬，約8年可還清 ➔ 錯！一輩子也還不完！

複利基本公式

- Basic formula

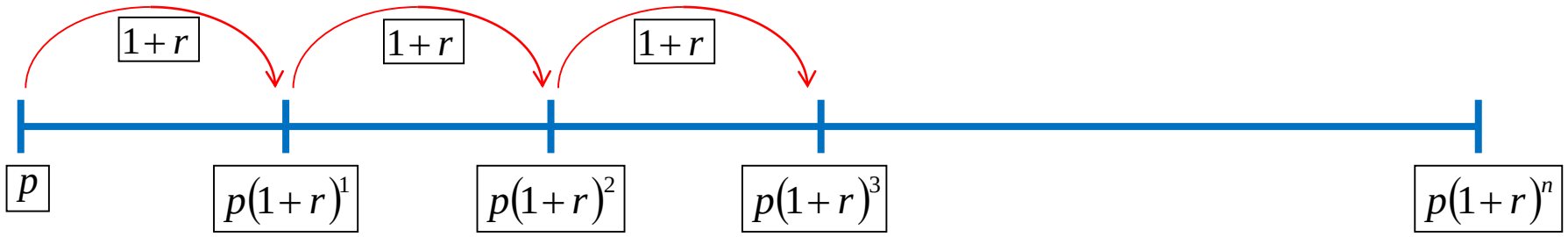
Quiz!

$f = p(1+r)^n$
p = present value (principal)
f = final value
r = interest rate per time period
n = no. of time periods

- Example
  - Initial investment: 1000
  - Annual interest: 3%
  - No. of periods: 20 years

$f = 1000 * (1 + 3/100)^{20}$
$= 1806.11$

This is NOT the formula used by banks!



定期複利（Periodic Compounding）

- 公式

$$f = p\left(1 + \frac{r}{n}\right)^{tn}$$, r: 年利率，n: 每年期數

- 範例
  - 本金1000，利率3%，計算20年後的本利和
    - 年複利

$$f = 1000 * (1 + 3\%)^{20} = 1806.11$$

    - 月複利

$$f = 1000 * \left(1 + \frac{3\%}{12}\right)^{20*12} = 1820.75$$

    - 日複利

$$f = 1000 * \left(1 + \frac{3\%}{365}\right)^{20*365} = 1822.07$$

- 提醒：
  - 我們一般所講的利率指的是年利率（r），但是
    - 銀行計算房貸是以月利率（r/12）為主
    - 信用卡循環利率計算是以日利率（r/365）為主

連續複利（Continuous Compounding）

- 公式

$$f = p * \lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{tn} = p * \left(\lim_{n/r\to\infty}\left(1+\frac{1}{n/r}\right)^{n/r}\right)^{rt} = pe^{rt}$$

Euler's number, see next page.

- 範例

$$f = 1000 * e^{20*3\%} = 1822.12$$

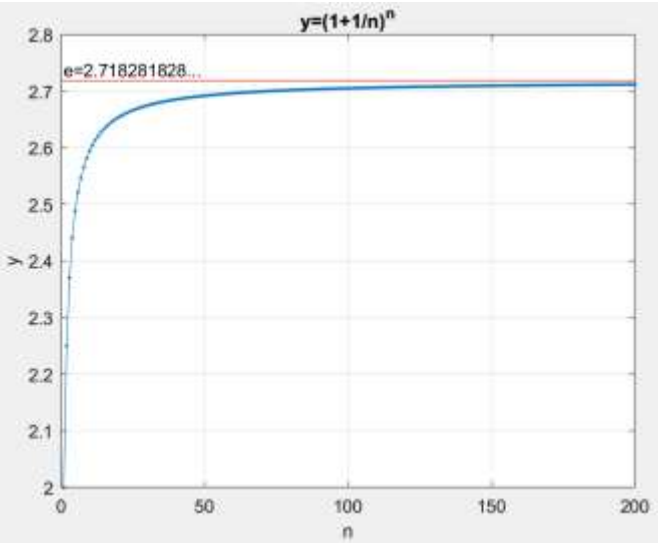
Euler's Number

- Definition

Quiz!

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{k!} = \frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots$$

=2.718281828…



- Proof by binomial theorem (二項式定理)

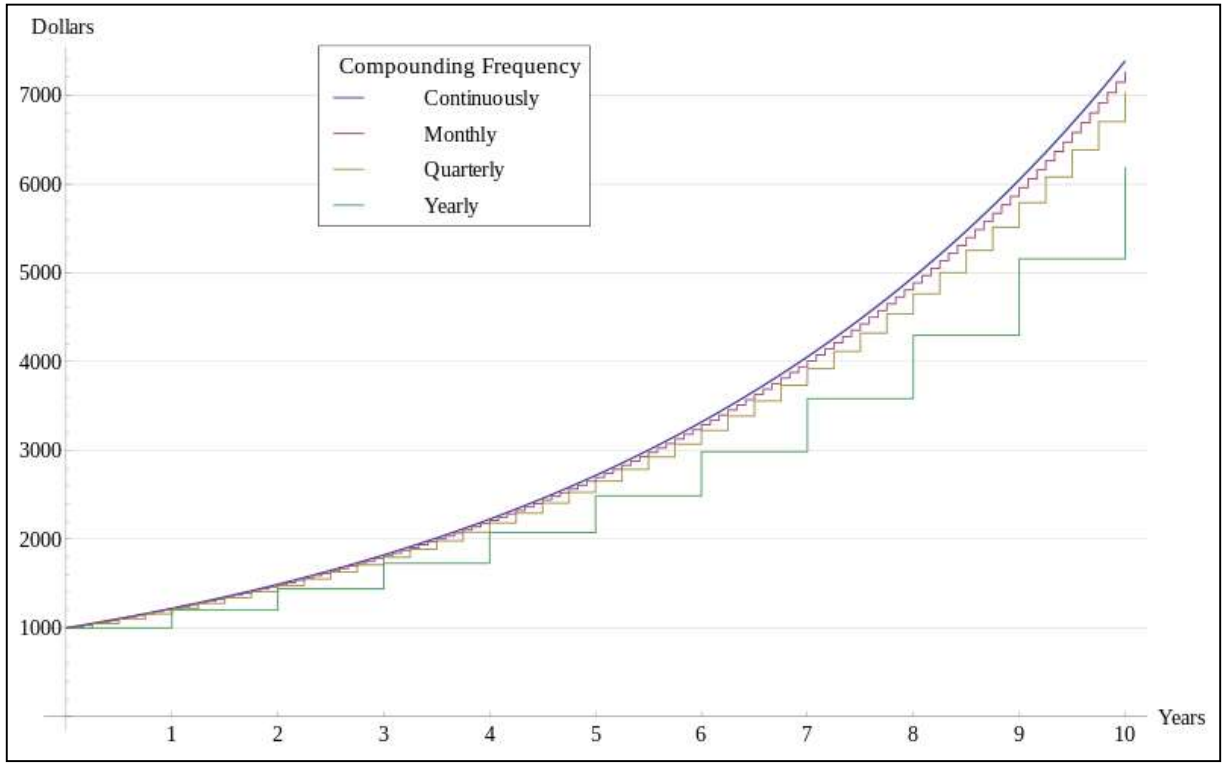
Quiz!

$$(x+y)^n = \sum_{k=0}^{n} C_k^n x^{n-k}y^k = \sum_{k=0}^{n}\frac{n!}{(n-k)!\,k!}x^{n-k}y^k$$

$$= x^n + nx^{n-1}y + \frac{n(n-1)}{2}x^{n-2}y^2 + \frac{n(n-1)(n-2)}{6}x^{n-3}y^3 + \cdots + y^n$$

Comparison

- Effect of compounding at various frequencies, with an initial investment of 1000 and 20% annual interest. (wiki)



Rule of 70

- Goal
  - To estimate the number of years it would take for an investment to double
- Also known as
  - Rule of 72
  - Rule of 69
- Reference
  - Wiki
- Formula
  - T*r=70

Quiz!

    - T=no. of year to double an investment
    - r=annual interest rate (%)
- Example:
  - r=1% ➔ T=70
  - r=3% ➔ T=23.3
  - r=18% ➔ T=3.9
  - r=20% ➔ T=3.5

Proof of Rule of 70

- 3 Ways of compounding to derive the rule

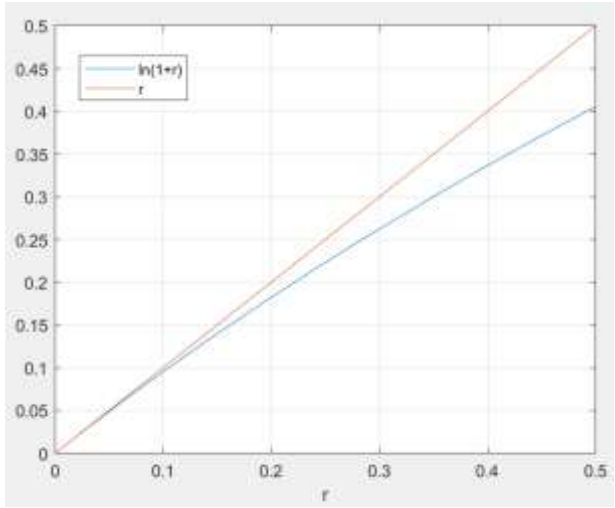
Yearly compounding: $f/p = 2 = (1+r)^T \Rightarrow T_1 = \dfrac{\ln(2)}{\ln(1+r)} \approx \dfrac{\ln(2)}{r} = \dfrac{0.6931}{r} \approx \dfrac{0.7}{r}$

Monthly compounding: $f/p = 2 = \left(1+\dfrac{r}{12}\right)^{12T} \Rightarrow T_2 = \dfrac{\ln(2)/12}{\ln(1+r/12)} \approx \dfrac{\ln(2)/12}{r/12} = \dfrac{0.6931}{r} \approx \dfrac{0.7}{r}$

Continuous compounding: $f/p = 2 = e^{rT} \Rightarrow T_3 = \dfrac{\ln(2)}{r} = \dfrac{0.6931}{r} \approx \dfrac{0.7}{r}$

- Examples

$$\mathrm{r} = 3\% \Rightarrow \begin{cases} \mathrm{T}_1 = 23.45 \approx 70/3 = 23.33 \\ \mathrm{T}_2 = 23.13 \approx 70/3 = 23.33 \\ \mathrm{T}_3 = 23.10 \approx 70/3 = 23.33 \end{cases}$$

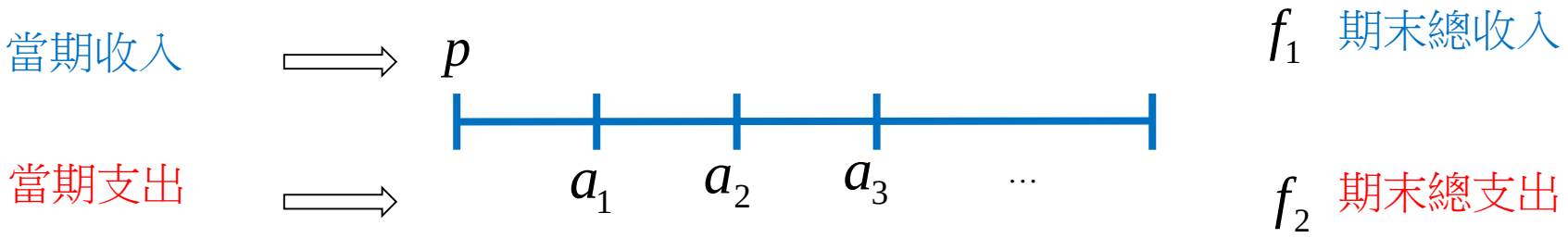


Quiz!

Prove that $\lim_{r\to 0} \dfrac{\ln(1+r)}{r} = 1.$

貨幣時間價值

- 貨幣時間價值 (Time value of money)
  - 若投資得當，金錢的價值通常會隨著時間的推移而增加。(The value of money usually increases over time if invested properly.)
- 現金流量表 (Cash flow statement)
  - 在時間軸上秀出收入及支出，並以期末（或期初）總值來判斷投資是否划算



房貸計算

- 問題定義
  - 銀行貸款100萬，20年還清，利率固定為3%，請問每個月還款金額？
- 銀行傳統
  - 利率 ➔ 年利率
  - 複利計算方式 ➔ 以「月」為單位來計算複利
- 銀行政策
  - 因為有不動產擔保品（房子），所以利率特別低
    - 若屋主無法按時繳款，銀行可以收回房子並拍賣
  - 信用貸款則屬於無擔保品的貸款，風險較高，所以利率也高
    - 若貸款方無法按時繳款，銀行可以扣此人的薪水

房貸攤還的方式

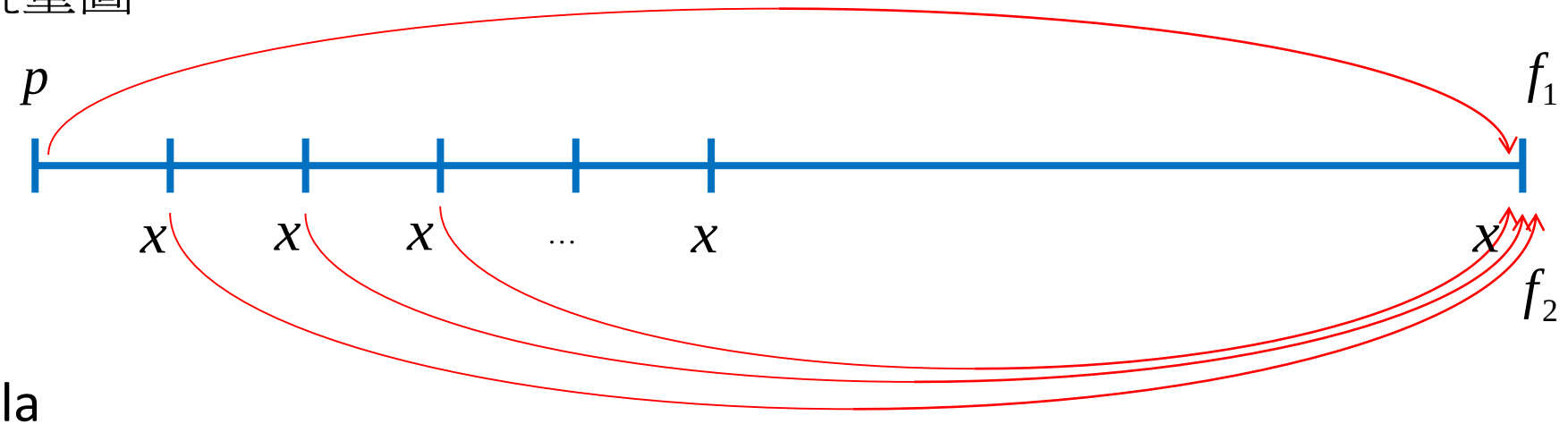
- 兩種房貸攤還的方式
  - 本息平均攤還
  - 本金平均攤還
- 相關資訊
  - 房貸試算器
  - 買房前，這些貸款名詞要知道！
  - 房貸怎麼還最有利？等額本金VS等額本息選哪個好？

等額本息與等額本金還款比較表（以貸款1萬元1年月利率為3.45‰為例）

貸款年限	等額本息			等額本金		
	還款本金	利息	本息合計	還款本金	利息	本息合計
1	817.64	34.50	852.14	833.33	34.50	867.83
2	820.46	31.68	852.14	833.33	31.63	864.96
3	823.29	28.85	852.14	833.33	28.75	862.08
4	826.13	26.01	852.14	833.33	25.88	859.21
5	828.98	23.16	852.14	833.33	23.00	856.33
6	831.84	20.30	852.14	833.33	20.13	853.46
7	834.71	17.43	852.14	833.33	17.25	850.58
8	837.59	14.55	852.14	833.33	14.38	847.71
9	840.48	11.66	852.14	833.33	11.50	844.83
10	843.38	8.76	852.14	833.33	8.63	841.96
11	846.29	5.85	852.14	833.33	5.75	839.08
12	849.21	2.93	852.14	833.37	2.88	836.25
合計	10000.00	225.68	10225.68	10000.00	224.28	10224.28

本息平均攤還：如何計算月付額

- 現金流量圖



- Formula

$$\begin{cases} f_1 = p(1+r/12)^{12*20} \\ f_2 = x + x(1+r/12) + x(1+r/12)^2 + \cdots + x(1+r/12)^{239} = x\dfrac{(1+r/12)^{240}-1}{r} \end{cases}$$

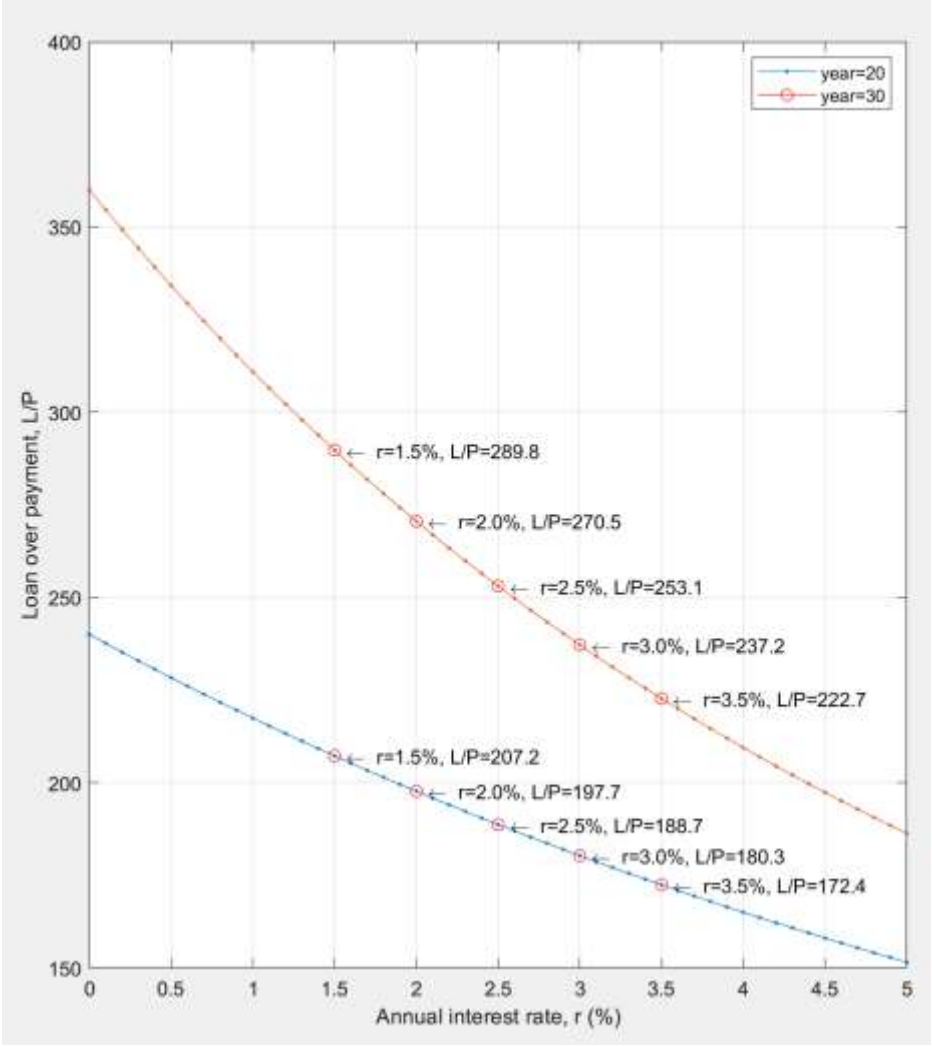
$$f_1 = f_2 \Rightarrow x = \frac{pr(1+r/12)^{240}}{(1+r/12)^{240}-1}$$

Rule of 200 (Roger's Formula)

- Loan/payment vs. r

$$L\left(1+\frac{r}{12}\right)^{12*20} = P\frac{(1+r/12)^{240}-1}{r/12}$$
$$\Rightarrow \frac{L}{P} = \frac{(1+r/12)^{240}-1}{(r/12)(1+r/12)^{240}} = \frac{1-(1+r/12)^{-240}}{r/12}$$

- Rule of 200
  - 假設20年房貸，利率2%
    ➔ 每月應付款 = 房貸總額/200
  - 延伸：若是30年房貸
    ➔ 每月應付款 = 房貸總額/270



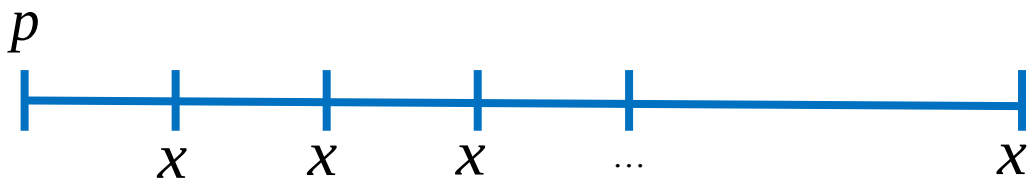
References

- References
  - Wiki: Time value of money
  - Wiki: Compound interest

Appendix

本息平均攤還：計算月付額的方法之二

- 現金流量圖



- Formula

Let b_i be the total unpaid amount at period i. Then
$b_0 = p$
$b_1 = p(1+r) - x$
$b_2 = (p(1+r) - x)(1+r) - x$
…
$\Rightarrow b_i = b_{i-1}(1+r) - x,$ with $b_0 = p$.

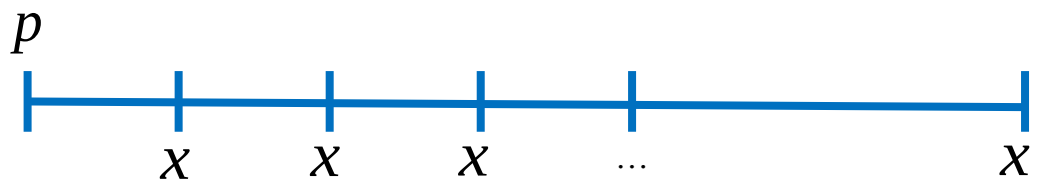
- Quiz

(a) Find the close−form solution of b_i.
(b) Set $b_n = 0$ to find the value of x.

hint: $b_i - x/r = (1+r)(b_{i-1} - x/r)$

本息平均攤還：計算月付額的方法之三

- 現金流量圖



- 本金及利息每期分攤

Let $x = p_i + q_i$,where p_i is the principal and $q_i$is the interest returned. Then

$q_1 = pr, p_1 = x - q_1$

$q_2 = (p - p_1)r, p_2 = x - q_2$

$q_3 = (p - p_1 - p_2)r, p_3 = x - q_3$

…

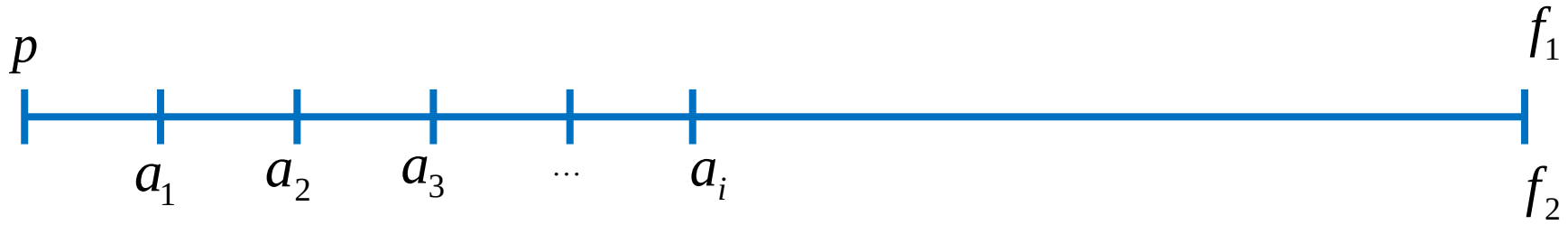
$$\Rightarrow q_i = \left(p - \sum_{k=1}^{i-1} p_k\right) r, p_i = x - q_i$$

- Quiz

(a) Find the recurrent formula of q_i.
(b) Find the close−form expression of q_i.

本金平均攤還法：計算月付額的方法之一

- 現金流量圖



$$
\begin{aligned}
a_1 &= p/n + pr \\
a_2 &= p/n + (p - p/n)r \\
a_3 &= p/n + (p - 2p/n)r \\
&\cdots \\
a_i &= p/n + p\left(1 - \frac{i-1}{n}\right)r
\end{aligned}
$$

Quiz:

$$
\begin{cases}
f_1 = p(1+r)^n \\
f_2 = \sum_{i=1}^{n} a_i (1+r)^{n-i}
\end{cases}
\Rightarrow \text{How to prove } f_1 = f_2 \text{ ?}
$$

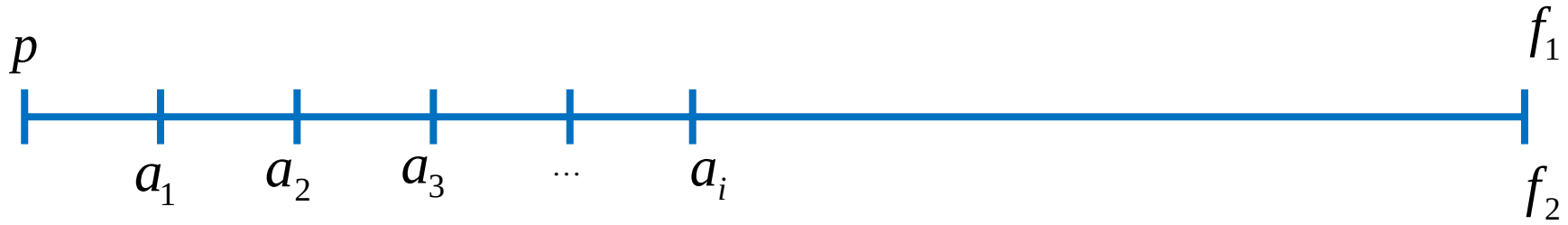
Hints:

$$S_0 = \sum (1+r)^{-i} = (1 - (1+r)^{-n})/r$$

$$S_1 = \sum i(1+r)^{-i} = (1+r)S_0/r - n(1+r)^{-n}/r$$

本金平均攤還法：計算月付額的方法之二

- 現金流量圖



Let $a_i = p_i + q_i$ be the periodic payment,
where p_i and q_i are principal and interest components, respectively. Then:

$p_1 = p_2 = \cdots = p_n = p/n$

$q_1 = pr$

$q_2 = (p - p/n)r$

$q_3 = (p - 2p/n)r$

$\cdots$

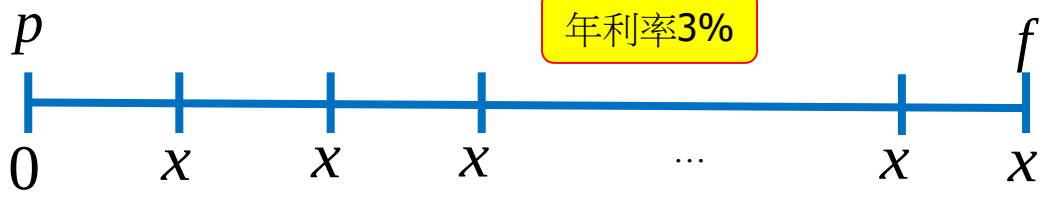
$q_i = (p - (i-1)p/n)r$

Useful Functions

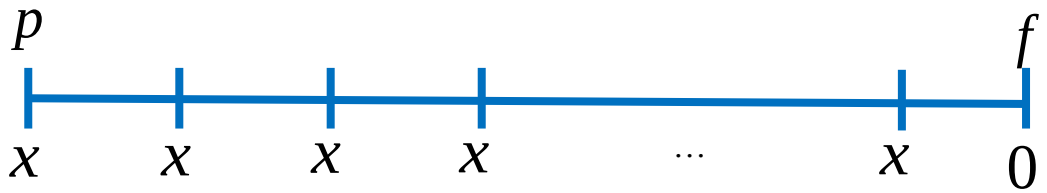
- Two MATLAB functions (utility toolbox) with self demo
  - loan.m
    - Monthly payment ➔ x=loan(p, 3/100, 20)

為期20年

年利率3%



  - saving.m
    - Present value ➔ p=saving(x, 3/100, 20, 'initial');
    - Final value ➔ f=saving(x, 3/100, 20, 'final');



Rule of 270

- 社會新鮮人的困惑
  - 若月收入3萬，房貸約佔1/3（1萬），為期30年 ➔ 台北市哪裡去找270萬的房子？
- 方法是人想出來的！
  - 從已無貸款的房子來進行增貸
    - 請和父母保持好關係，早晚問安、不時共餐