Graph Theory HW3

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Problem 1.

(a) Let $X := \{a \in A : |N(a) \cap Y| < (d-\epsilon)|Y|\}$.
$e(X,Y) = \sum_{a \in X} |N(a) \cap Y| < \sum_{a \in X} (d-\epsilon)|Y| = (d-\epsilon)|X||Y|$.
$\Rightarrow d(X,Y) = \frac{e(X,Y)}{|X||Y|} < d - \epsilon = d(A,B) - \epsilon$.
If $X \geq \epsilon|A|$, then by the definition of ϵ-regular, $|d(X,Y) - d(A,B)| \leq \epsilon$, which contradicts to $d(X,Y) < d(A,B) - \epsilon$.
$\therefore X < |\{a \in A : |N(a) \cap Y| < (d-\epsilon)|Y|\}| \, \epsilon|A|$.

(b) For any $C \subseteq X, D \subseteq Y$ with $|C| \geq \epsilon'|X|$, $|D| \geq \epsilon'|Y|$, there is $|C| \geq \epsilon'|X| \geq \frac{\epsilon}{\alpha}|X| \geq \epsilon|A|$, $|D| \geq \epsilon'|Y| \geq \frac{\epsilon}{\alpha}|Y| \geq \epsilon|B|$.
Since $\{A, B\}$ is an ϵ-regular pair, there is $|d(C,D) - d(A,B)| \leq \epsilon$.
Note that $|X| \geq \alpha|A| > \epsilon|A|, |Y| \geq \alpha|B| > \epsilon|B|$, there is $|d(X,Y) - D(A,B)| \leq \epsilon$.
$\Rightarrow |d(C,D) - d(X,Y)| \leq |d(C,D) - d(A,B)| + |d(X,Y) - d(A,B)| \leq \epsilon + \epsilon \leq 2\epsilon \leq \epsilon'$.
$\therefore$ by the definition, $\{X, Y\}$ is an ϵ'-regular pair.

Problem 2. Let's prove that $W(k,r) \leq k^{HJ(k,r)}$.

For every coloring $c : [k^{HJ(k,r)}] \to [r]$, consider the coloring $c' : [k]^{HJ(k,r)} \to [r]$ where $c'(a_1, a_2, \ldots, a_{HJ(k,r)}) := c\left(1 + \sum_{i=1}^{HJ(k,r)} (a_i - 1)k^{i-1}\right)$.

By the Hales-Jewett Theorem, there is a monochromatic combinatorial line in the coloring c'.

That is, there is a set $S \neq \emptyset$ and $a_{ij} (1 \leq i \leq k, 1 \leq j \leq HJ(k,r))$, where $a_{ij} =$

$\begin{cases} i, \text{ if } j \in S \\ a_{1j}, \text{ otherwise} \end{cases}$, such that $c'(a_{i1}, a_{i2}, \ldots, a_{i,HJ(k,r)})$ are the same for all $i \in [k]$.

$\Rightarrow c\left(1 + \sum_{j=1}^{HJ(k,r)} (a_{ij} - 1)k^{j-1}\right)$ are the same for all $i \in [k]$.

$\Rightarrow c\left(1 + \sum_{j \in S}(i-1)k^{j-1} + \sum_{j \notin S}(a_{1j} - 1)k^{j-1}\right)$ are the same for all $i \in [k]$.

$\Rightarrow c\left(1 + \sum_{j \notin S}(a_{1j} - 1)k^{j-1} + (i-1)\sum_{j \in S} k^{j-1}\right)$ are the same for all $i \in [k]$.

$\because 1 + \sum_{j \notin S}(a_{1j} - 1)k^{j-1}, \sum_{j \in S} k^{j-1}$ are constants with respect to i,

$\therefore \left\{1 + \sum_{j \notin S}(a_{1j} - 1)k^{j-1} + (i-1)\sum_{j \in S} k^{j-1} | i \in [k]\right\}$ is a k-AP.

$\Rightarrow$ we find a monochromatic k-AP.

$\Rightarrow W(k,r) \leq k^{HJ(k,r)}$, which proves Van der Waerden's Theorem.