Graph Theory I

Math 7703

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Acknowledgements

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Some of the material in these notes is based on the books *Graph Theory* by Reinhard Diestel, *Introduction to Graph Theory* by Douglas West, and *Proofs from the Book* by Martin Aigner and Günter Ziegler.

Contents

1 Fundamentals

1.1 Graphs

Definition 1.1. A (simple) *graph* G is a pair $G = (V, E)$ where V is a set of *vertices* and $E \subseteq \binom{V}{2}$ is a set of unordered pairs of vertices. The elements of E are called *edges*. We write $V(G)$ for the set of vertices and $E(G)$ for the set of edges of a graph G. Also, the *order* of a graph $|G| = |V(G)|$ is the number of vertices it contains, while the *size* $e(G) = |E(G)|$ denotes the number of edges.

Remark 1.2. One can also consider *directed graphs*, where the edges $E \subseteq V \times V$ are ordered pairs of vertices, or *multigraphs*, where there may be mutliple edges between a pair of vertices. However, in this course, unless explicitly stated otherwise, we shall only concern ourselves with simple graphs.

Definition 1.3. Some further terminology concerning relations between vertices and edges:

- Vertices u, v are *adjacent* in G if $\{u, v\} \in E(G)$.
- An edge $e \in E(G)$ is *incident* to a vertex $v \in V(G)$ if $v \in e$.
- Edges e, e' are *incident* if $e \cap e' \neq \varnothing$.
- If $\{u, v\} \in E$ then v is a *neighbour* of u.

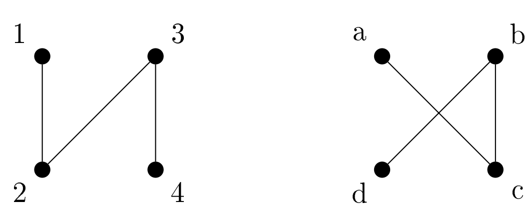
Example 1.4. Any symmetric relation between objects can be modelled with a graph. For example:

- let V be the set of people in a room, and let E be the set of pairs of people who met for the first time today;
- let V be the set of cities in a country, and let the edges in E correspond to roads connecting them;
- the internet: let V be the set of computers, and let the edges in E correspond to the links connecting them.

1.2 Graph isomorphism

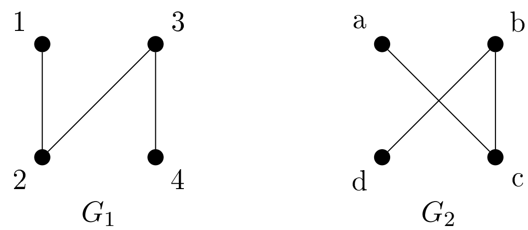
Although graphs are formally defined as a set of vertices and a set of edges, simply listing all the edges of a graph is not a very convenient representation for a human to understand. The usual way to picture a graph is to put a dot for each vertex and to join adjacent vertices with lines. Bear in mind, though, that the specific drawing is irrelevant, and all that matters is which pairs are adjacent.

Question 1.5. Consider the two graphs drawn below. How similar can they be said to be?



Definition 1.6. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs. An *isomorphism* $\phi : V_1 \to V_2$ is a bijection (a one-to-one correspondence) from V_1 to V_2 such that $\{u, v\} \in E_1$ if and only if $\{\phi(u), \phi(v)\} \in E_2$. We say G_1 is *isomorphic* to G_2 if there is an isomorphism between them.

Example 1.7. Recall the graphs in Question 1.5:



The function $\phi : G_1 \to G_2$ given by $\phi(1) = a$, $\phi(2) = c$, $\phi(3) = b$, $\phi(4) = d$ is an isomorphism.

Remark 1.8. Isomorphism is an equivalence relation of graphs. This means that

- Any graph is isomorphic to itself
- if G_1 is isomorphic to G_2 then G_2 is isomorphic to G_1
- If G_1 is isomorphic to G_2 and G_2 is isomorphic to G_3, then G_1 is isomorphic to G_3.

Definition 1.9. An *unlabelled graph* is an isomorphism class of graphs. In the previous example G_1 and G_2 are different labelled graphs, but since they are isomorphic, they represent the same unlabelled graph.

Since we are only interested in graphs up to isomorphism (that is, we only care about unlabelled graphs), the actual labels of the vertices do not matter, and so we will generally take $V = [n] = \{1, 2, \ldots, n\}$.

1.3 The adjacency and incidence matrices

Although drawings of graphs are a convenient human representation, they suffer from two major drawbacks:

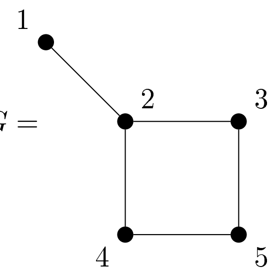
- they can be hard to read when there are many vertices and even more edges, and
- they are not very computer-friendly.

We therefore often represent graphs in matrix form, as we describe in this section.

Definition 1.10. Let $G = (V, E)$ be a graph with $V = [n]$. The *adjacency matrix* $A = A(G)$ is the $n \times n$ symmetric matrix defined by

$$a_{ij} = \begin{cases} 1 & \text{if } \{i, j\} \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Example 1.11.

$G =$ 

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

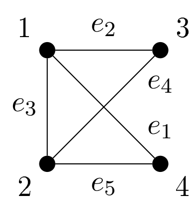
Remark 1.12. Any adjacency matrix A is real and symmetric, hence the spectral theorem proves that A has an orthogonal basis of eigenvalues with real eigenvectors. This important fact allows us to use *spectral methods* in graph theory. Indeed, there is a large subfield of graph theory called *spectral graph theory.*

Aside from the adjacency matrix, we can also represent graphs using the incidence matrix, defined below.

Definition 1.13. Let $G = (V, E)$ be a graph with $V = \{v_1, \dots, v_n\}$ and $E = \{e_1, \dots, e_m\}$. Then the *incidence matrix* $B = B(G)$ of G is the $n \times m$ matrix defined by

$$b_{ij} = \begin{cases} 1 & \text{if } v_i \in e_j, \\ 0 & \text{otherwise.} \end{cases}$$

Example 1.14.

$G =$ 

$$B = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Remark 1.15. Every column of B has $|e| = 2$ entries 1.

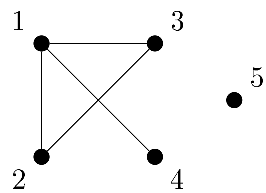
1.4 Degrees

It is often important to take a more local view of a graph, and study what goes on in the immediate surroundings of an individual vertex. The first step in this direction is to look at how many edges a vertex is incident to.

Definition 1.16. Given $G = (V, E)$ and a vertex $v \in V$, we define the *neighbourhood* $N(v) = \{u \in V : \{u, v\} \in E(G)\}$ of v to be the set of neighbours of v. Let the *degree* $d(v)$ of v be $|N(v)|$, the number of neighbours of v. A vertex v is *isolated* if it has no neighbours, that is, if $d(v) = 0$.

Remark 1.17. $d(v)$ is the number of ones in the row corresponding to v in the adjacency matrix $A(G)$ or the incidence matrix $B(G)$.

Example 1.18.



$d(1) = 3,\ d(2) = 2,\ d(3) = 2,\ d(4) = 1,\ d(5) = 0;$

5 is isolated.

Question 1.19. For any graph G on the vertex set $[n]$ with adjacency and incidence matrices A and B, show that $BB^T = D + A$, where

$$D = \begin{pmatrix} d(1) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & d(n) \end{pmatrix}.$$

The degree of a vertex is a local statistic, but looking at the extreme values it takes over all vertices of the graph gives rise to some very important graph parameters.

Notation 1.20. The *minimum degree* of a graph G is denoted

$$\delta(G) = \min\{d(v) : v \in V(G)\},$$

the *maximum degree* is denoted

$$\Delta(G) = \max\{d(v) : v \in V(G)\},$$

and the *average degree* is

$$\bar{d}(G) = \frac{\sum_{v \in G} d(v)}{|V(G)|}.$$

Note that we trivially have $\delta(G) \leq \bar{d}(G) \leq \Delta(G)$ for all graphs G.

Definition 1.21. We say a graph G is *d-regular* if all vertices have degree exactly d.

Although graphs are very flexible structures, it turns out that there are restrictions on what degrees a graph can have. For instance, consider the following question.

Question 1.22. Is there a 3-regular graph on 9 vertices?

To answer the question, we will prove our first result of this course, a simple yet very useful equation between the degrees and the size of a graph.

Proposition 1.23 (Handshake Lemma)**.** *For every graph $G = (V, E)$, $\sum_{v \in V} d(v) = 2e(G)$.*

Proof. We double-count the incidences between vertices and edges; that is, we count the set $I = \{(v, e) : v \in V(G), e \in E(G), v \in e\}$. Every edge $e = \{u, v\}$ is counted twice, once in the pair (u, e) and once in the pair (v, e). Thus $|I| = 2|E| = 2e(G)$. On the other hand, a vertex $v \in V(G)$ is counted once for every edge it is incident to, which is precisely its degree $d(v)$. Thus $|I| = \sum_{v \in V(G)} d(v)$, and the result follows. $\square$

Corollary 1.24. *Every graph has an even number of vertices of odd degree.*

This shows that the answer to Question 1.22 is "no".

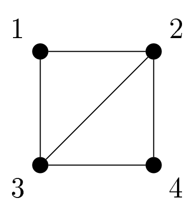
1.5 Subgraphs

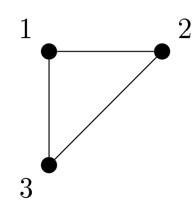
The degree or neighbourhood of a vertex is a very local part of a graph. Another very important concept is that of a subgraph, which is again part of a graph, although potentially involving several, if not all, vertices.

Definition 1.25. A graph $H = (U, F)$ is a *subgraph* of a graph $G = (V, E)$ if $U \subseteq V$ and $F \subseteq \binom{U}{2} \cap E$. If $U = V$ then H is called *spanning.*

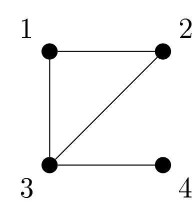
Definition 1.26. Given $G = (V, E)$ and $U \subseteq V$ $(U \neq \varnothing)$, let $G[U]$ denote the graph with vertex set U and edge set $E(G[U]) = \{e \in E(G) : e \subseteq U\}$. (We include all the edges of G which have both endpoints in U.) Then $G[U]$ is called the subgraph of G *induced* by U.

Example 1.27.

$G =$ 



induced subgraph



not induced
but spanning

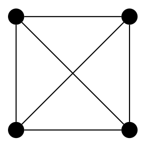
1.6 Special graphs

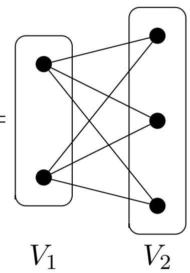
We have so far seen a few arbitrary examples of graphs. Here we collect a few important families of graph that will appear repeatedly in our course.

- $K_n = \left([n], \binom{[n]}{2}\right)$ is the *complete graph*, or a *clique.* Take n vertices and all possible edges connecting them.
- An *empty graph* $G = (V, \emptyset)$ has no edges.

- $G = (V, E)$ is *bipartite* if there is a partition $V = V_1 \cup V_2$ into two disjoint sets such that each $e \in E(G)$ intersects both V_1 and V_2 (equivalently, V_1 and V_2 induce empty subgraphs).
- $K_{n,m}$ is the *complete bipartite graph.* Take $n + m$ vertices partitioned into a set A of size n and a set B of size m, and include every possible edge between A and B.

Example 1.28.

$K_4 =$ 

$K_{2,3} =$ 

1.7 Walks, paths and cycles

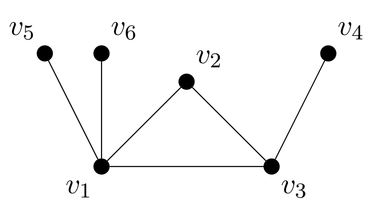
If a graph represents a network, then one crucial consideration is how we can travel from one vertex to another. This is captured through the notion of walks, paths and cycles.

Definition 1.29. A *walk* in G is a sequence of vertices $v_0, v_1, v_2, \ldots, v_k$, and a sequence of edges $\{v_i, v_{i+1}\} \in E(G)$. A walk is a *path* if all v_i are distinct. If for such a path with $k \geq 2$, $\{v_0, v_k\}$ is also an edge in G, then $v_0, v_1, \ldots, v_k, v_0$ is a *cycle.*

Remark 1.30. The above definition treats paths and cycles as having a start point and an endpoint (so reversing a path technically gives a different path). However, in practice, when we use language like "there is a unique path between u and v", we mean that the path or cycle is unique up to this choice.

Definition 1.31. The *length* of a path, cycle or walk is the number of edges in it.

Example 1.32.



$v_5v_1v_3v_4 \equiv$ path of length 3;

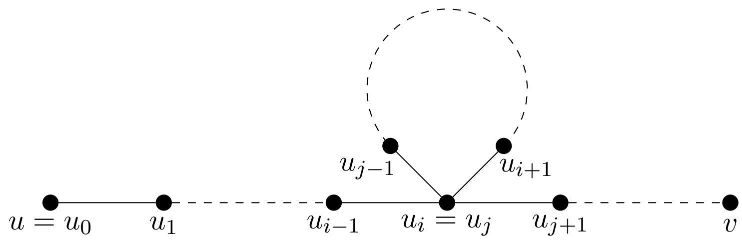
$v_1v_2v_3v_1 \equiv$ cycle of length 3;

$v_5v_1v_2v_3v_1v_6 \equiv$ walk of length 5.

Proposition 1.33. *Every walk from u to v in G contains a path between u and v.*

Proof. By induction on the length ℓ of the walk $u = u_0, u_1, \ldots, v_\ell = v$.

If $\ell = 1$ then our walk is just an edge, which is also a path. Otherwise, if our walk is not a path there is $u_i = u_j$ with $i < j$, then $u = u_0, \ldots, u_i, u_{j+1}, \ldots, v$ is a shorter walk from u to v. We can use induction to conclude that the walk contains a path.

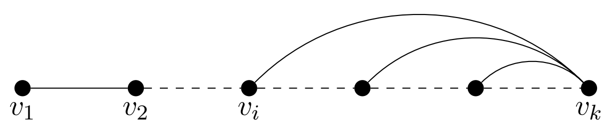


□

In graph theory, we typically ask questions of the following form: what conditions can we impose on a graph to force it to contain certain subgraphs? The following result is an example of this kind of problem.

Proposition 1.34. *Every G with minimum degree $\delta(G) = \delta \geq 2$ contains a path of length δ and a cycle of length at least $\delta + 1$.*

Proof. Let $v_1, \ldots, v_k$ be a longest path in G. Then all neighbours of v_k belong to $v_1, \ldots, v_{k-1}$, as otherwise we could extend the path. So $k - 1 \geq \delta$ and $k \geq \delta + 1$, and our path has at least δ edges. Let i $(1 \leq i \leq k - 1)$ be the minimum index such that $(v_i, v_k) \in E(G)$. Then the neighbours of v_k are among $v_i, \ldots, v_{k-1}$, so $k - i \geq \delta$. Then $v_i, v_{i+1}, \ldots, v_k$ is a cycle of length at least $\delta + 1$.



□

Remark 1.35. Note that we have also proved that a graph with minimum degree $\delta \geq 2$ contains cycles of at least $\delta - 1$ different lengths. This fact, and the statement of Proposition 1.34, are both best possible; to see this, consider the complete graph $G = K_{\delta+1}$.

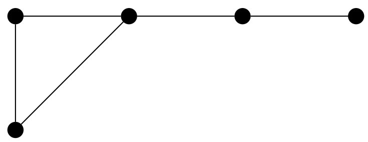
1.8 Connectivity

Paths allow us to travel from one endpoint to another along the edges of the graph. The concept of connectivity concerns whether such paths are guaranteed to exist.

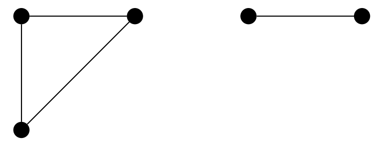
Definition 1.36. A graph G is *connected* if for all pairs $u, v \in V(G)$, there is a path in G from u to v.

Note that it suffices for there to be a walk from u to v, by Proposition 1.33.

Example 1.37.



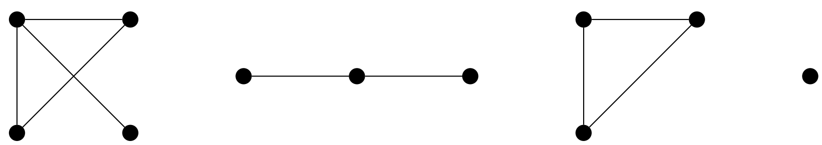
connected



not connected

Definition 1.38. A *(connected) component* of G is a connected subgraph that is maximal by inclusion. G is *connected* if and only if it has one connected component.

Example 1.39.

$G =$ 

has 4 connected components.

Proposition 1.40. *A graph G with n vertices and m edges has at least $n-m$ connected components.*

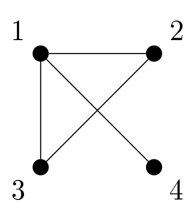
Proof. We start with the empty graph on n vertices, and form G by adding its edges one-by-one, keeping track of the number of components during the process. At the beginning, in the empty graph, each individual vertex is a connected component, so we have n components.

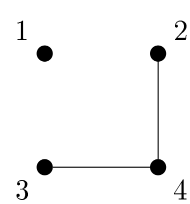
Now, if we add the edge $e = \{u, v\}$, we merge the components of u and v, and so each edge can decrease the number of components by at most one. Hence, after adding all m edges, we have at least $n - m$ connected components. $\square$

1.9 Graph operations and parameters

Definition 1.41. Given $G = (V, E)$, the *complement* $\overline{G}$ of G has the same vertex set V and $(u, v) \in E\big(\overline{G}\big)$ if and only if $\{u, v\} \notin E(G)$.

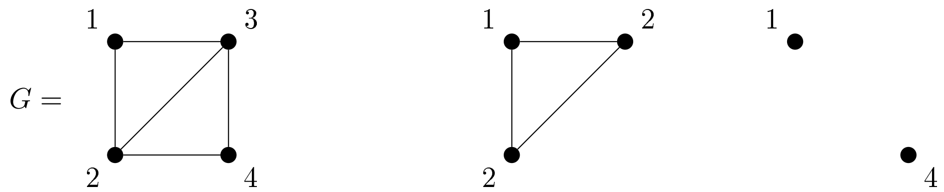
Example 1.42.

$G =$ 

$\overline{G} =$ 

Definition 1.43. A *clique* in G is a complete subgraph in G. An *independent set* is an empty induced subgraph in G.

Example 1.44.



clique in G independent set in G

Definition 1.45. Let $G = (V, E)$ be a graph. The *clique number* $\omega(G)$ denotes the maximum number of vertices in a clique in G. The *independence number* let $\alpha(G)$ denote the number of vertices in a maximum-size independent set in G.

Example 1.46. In Example 1.44, $\omega(G) = 3$ and $\alpha(G) = 2$.

Claim 1.47. A vertex set $U \subseteq V(G)$ is a clique if and only if $U \subseteq V\left(\overline{G}\right)$ is an independent set.

Corollary 1.48. *We have* $\omega(G) = \alpha\left(\overline{G}\right)$ *and* $\alpha(G) = \omega\left(\overline{G}\right)$.