機率與統計 HW5

許博翔

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Problem 1.

(a) If $z \geq 0$, then $x \geq y \geq 0$.

$$\Rightarrow f_Z(z) = \int_0^\infty \lambda e^{-\lambda(y+z)} \mu e^{-\mu y} dy$$

$$= \int_0^\infty \lambda\mu e^{-(\lambda+\mu)y-\lambda z} dy$$

$$= \frac{\lambda\mu}{-\lambda-\mu} e^{-(\lambda+\mu)y-\lambda z}\Big|_0^\infty$$

$$= \frac{\lambda\mu}{\lambda+\mu} e^{-\lambda z}.$$

If $z \leq 0$, then $y \geq x \geq 0$.

$$\Rightarrow f_Z(z) = \int_0^\infty \lambda e^{-\lambda x} \mu e^{-\mu(x-z)} dx$$

$$= \int_0^\infty \lambda\mu e^{-(\lambda+\mu)x+\mu z} dx$$

$$= \frac{\lambda\mu}{-\lambda-\mu} e^{-(\lambda+\mu)x+\mu z}\Big|_0^\infty$$

$$= \frac{\lambda\mu}{\lambda+\mu} e^{\mu z}.$$

$$\therefore f_Z(z) = \begin{cases} \frac{\lambda\mu}{\lambda+\mu} e^{-\lambda z}, \text{ if } z \geq 0 \\ \frac{\lambda\mu}{\lambda+\mu} e^{\mu z}, \text{ otherwise} \end{cases}.$$

(b) Clearly, for $w < 0$, $f_W(w) = 0$.

For $w \geq 0$:

$W = |X - Y| = |Z|$.

$$\Rightarrow f_W(w) = f_Z(w) + f_Z(-w) = \frac{\lambda\mu}{\lambda+\mu} e^{-\lambda w} + \frac{\lambda\mu}{\lambda+\mu} e^{-\mu w} = \frac{\lambda\mu}{\lambda+\mu}(e^{-\lambda w} + e^{-\mu w}).$$

$$\therefore f_W(w) = \begin{cases} \frac{\lambda\mu}{\lambda+\mu}(e^{-\lambda w} + e^{-\mu w}), \text{ if } w \geq 0 \\ 0, \text{ otherwise} \end{cases}.$$

Problem 2. $e^{2e^s-2} = e^{2(e^s-1)}$ is the MGF of Poisson(2).
$(\frac{3}{4}e^s + \frac{1}{4}) = (1 - \frac{3}{4} + \frac{3}{4}e^s)^{10}$ is the MGF of Binominal(10, $\frac{3}{4}$).
$\therefore X \sim$ Poisson(2) and $Y \sim$ Binominal(10, $\frac{3}{4}$).
$\Rightarrow P_X(x) = \frac{2^x e^{-2}}{x!}$ for $x = 0, 1, 2, \ldots$, $P_Y(y) = \binom{10}{y}(\frac{3}{4})^y(\frac{1}{4})^{10-y}$ for $y = 0, 1, \ldots, 10$.

(a) $\because X, Y \geq 0$

$\therefore \Pr[X+Y=2] = P_X(0)P_Y(2)+P_X(1)P_Y(1)+P_X(2)P_Y(0) = e^{-2}45 \times \frac{9}{16}(\frac{1}{4})^8 + 2e^{-2}10 \times \frac{3}{4}(\frac{1}{4})^9 + 2e^{-2}(\frac{1}{4})^{10}$.

Problem 3.

Problem 4.

Problem 5.

Problem 6.

Problem 7.

Problem 8.

Problem 9.

Problem 10.

Problem 11.