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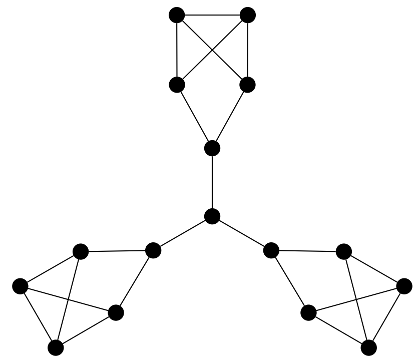


6 Planar Graphs

In this course so far, we have seen that an effective way to communicate a graph is to draw it. However, for large graphs with many vertices and edges, this can get quite messy. To prevent any confusion, we would ideally like the edges of the graph to only meet at common vertices. In this chapter we will explore this idea, seeing for which graphs this is possible, and then using those results to classify very regular three-dimensional bodies that were classically studied by the Ancient Greeks.

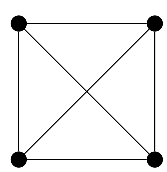
6.1 Definitions and Classification

We begin by formally defining what we mean when we speak of “drawing a graph."

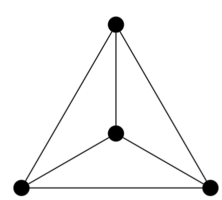
Definition 6.1. A *drawing* of a graph $G = (V, E)$ consists of an injective map $\varphi : V(G) \to \mathbb{R}^2$ and, for each edge $e = \{u, v\} \in E(G)$, a continuous curve $\gamma_e : [0, 1] \to \mathbb{R}^2$ with endpoints $\{\gamma_e(0), \gamma_e(1)\} = \{\varphi(u), \varphi(v)\}$.

Given edges e and e', a *crossing* is a point that lies on both γ_e and $\gamma_{e'}$ that is not a common endpoint of the two curves. A graph is *planar* if it has a drawing without crossings, and such a drawing is called a *planar embedding*. A *plane graph* is a particular crossing-free drawing of a planar graph in the plane.

Example 6.2.



K_4 with a crossing



planar drawing

Remark 6.3. We get the same class of graphs if we require that the images of edges be piecewise-linear, since such arcs can approximate continuous curves to an arbitrary degree of precision. If you want to prove things about planar graphs very formally, that can be a useful change to make, since piecewise-linear curves can be a bit easier to work with than arbitrary continuous curves.

Definition 6.4. An *open set* in the plane is a set $U \subset \mathbb{R}^2$ such that for every $p \in U$, all points within some small distance from p belong to U. A *region* is an open set U that contains a continuous curve between u and v for every pair $u, v \in U$ (that is, it is "path-connected"). The *faces* of a plane graph are the maximal regions of the plane that are disjoint from the drawing.

Of course, a face is a geometric notion, and so when one proves results about the faces of a plane graph, one must use theorems and ideas from geometry. As this is not a course in geometry, we shall not be fully rigorous in this chapter, and shall instead rely on some intuition (hopefully with sufficient justification). For a more formal treatment of the subject matter, you are invited to consult Diestel's "Graph Theory." One of the main tools used in these proofs is the following seemingly obvious fact, which states that the image of a cycle separates the plane into two regions.

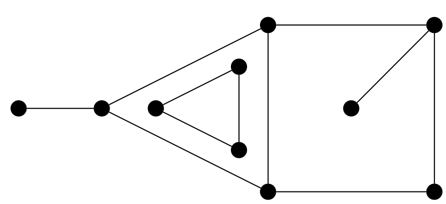
Theorem 6.5 (Jordan curve theorem)**.** *A non-self-intersecting closed continuous curve $\varphi : [0,1] \to \mathbb{R}^2$, with $\varphi(0) = \varphi(1)$ and φ injective on $[0,1)$, partitions the plane into exactly two faces, each having the image of φ as boundary.*

Remark 6.6. This is not true in three dimensions. In $\mathbb{R}^3$ there is a surface called the Möbius band which has only one side.

Remark 6.7. The faces of G are pairwise disjoint (they are separated by the edges of G). Two points are in the same face if and only if there is a continuous curve between them which does not cross an edge of G. Also, note that a finite graph has a single unbounded face (the area "outside" of the graph).

Definition 6.8. The *length* of the face f in a planar embedding of G is the number of edges in the boundary of f, where an edge is counted twice if it is not contained in the boundary of any other face.

Example 6.9. The following graph has 4 faces of lengths 6,6,3 and 7.



For example, the outer face has length 7; the leftmost horizontal edge is counted twice as it only touches the outer face and no other.

Proposition 6.10. *If $l(f_i)$ denotes the length of a face f_i in a plane graph G, then $2e(G) = \sum l(f_i)$.*

Proof. In the sum $\sum l(f_i)$, every edge is counted exactly twice. □

The following theorem provides an important invariant, showing that the vertices, edges and faces of any connected planar graph must satisfy this simple relation.

Theorem 6.11 (Euler's formula, 1758)**.** *If a connected plane graph G has exactly v vertices, e edges and f faces, then $v - e + f = 2$.*

Proof. We use induction on the number of edges in G. If $e = e(G) = 0$ then, since G is connected, we must have $G = K_1$, for which we have 1 vertex, 0 edges and 1 (unbounded) face, and thus the formula holds.

Otherwise, suppose $e(G) \geq 1$. If G contains a cycle, then let g be an edge on the cycle. The image of the cycle in the plane graph is a simple closed curve, and thus by the Jordan Curve Theorem it follows that the cycle divides the plane into two regions. Hence the edge g lies on the boundary of two different faces. If we let $G' = G - \{g\}$, then these two faces will be merged into one. G' thus has the same number of vertices as G, but one fewer edge and one fewer face. We thus have

$$v - e + f = v' - (e' + 1) + (f' + 1) = v' - e' + f' = 2,$$

where v', e' and f' are the number of vertices, edges and faces of G' respectively, and the final equation follows from the induction hypothesis.

If G does not contain a cycle, then as a connected acyclic graph, G is a tree. Hence, G has a leaf u, incident to a pendant edge g. This time, let G' be G after removing both u and g. We thus have $v' = v - 1$ and $e' = e - 1$. As g is a pendant edge, it does not lie on the boundary of two separate faces, and so removing g does not decrease the number of faces; that is, $f' = f$. Then again, we have

$$v - e + f = (v' + 1) - (e' + 1) + f' = v' - e' + f' = 2,$$

completing the proof. □

Remark 6.12. The fact that deleting an edge in a cycle decreases the number of faces by one can be proved formally using the Jordan curve theorem.

Using Euler's Formula, we can derive a strong bound on the size of planar graphs.

Theorem 6.13. *If G is a planar graph with at least three vertices, then $e(G) \leq 3v(G) - 6$.*

Proof. Fix a planar embedding of G. We may assume G is connected, as otherwise we can add edges between connected components while maintaining planarity. By Proposition 6.10, we have $2e(G) = \sum \ell(f_i)$, where the sum ranges over the faces of the embedding.

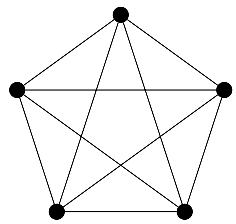
If the boundary of a face contains a cycle, then we must have $\ell(f_i) \geq 3$. Otherwise, G must be a tree with at least two edges, and each edge is counted twice in the length of the (unique, unbounded) face. So in this case we also have $\ell(f_i) \geq 3$. Thus, $2e(G) = \sum \ell(f_i) \geq 3f$, for $f \leq \frac{2}{3}e(G)$.

Substituting this into Euler's Formula, we have

$$2 = v - e + f \leq v - e + \tfrac{2}{3}e.$$

Solving for $e = e(G)$, we find $e \leq 3v - 6$, as required. □

Corollary 6.14. *K_5 is not planar.*



K_5

Proof. K_5 is a non-planar graph since $e = 10 > 9 = 3v - 6$. □

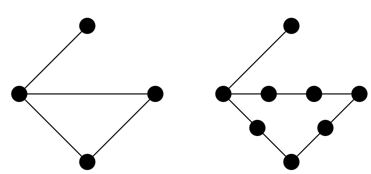
Remark 6.15 (Maximal planar graphs / triangulations)**.** The proof of Theorem 6.13 shows that having $3n - 6$ edges in a simple n-vertex planar graph requires $2e = 3f$, meaning that every face is a triangle. If G has some face that is not a triangle, then we can add an edge between non-adjacent vertices on the boundary of this face to obtain a larger plane graph. Hence the simple plane graphs with $3n - 6$ edges, the triangulations, and the *maximal* plane graphs are all the same family.

Remark 6.16. Using Theorem 6.13, we would not be able to show $K_{3,3}$ is not planar. Indeed, for $G = K_{3,3}$, we have $v = 6$ but $e = 9 < 12 = 3v - 6$. However, the previous remark shows that for the $3v - 6$ bound to hold, every face must be a triangle, while $K_{3,3}$ is bipartite and therefore triangle-free. On the homework, you will show that triangle-free planar graphs satisfy the stronger bound of $e \leq 2v - 4$, from which one can deduce that $K_{3,3}$ is indeed not planar.

The above results show that K_5 and $K_{3,3}$ are not planar. We close this section by stating a remarkable theorem of Kuratowski, which shows that these two graphs are, in some sense, present in all non-planar graphs. The proof of this theorem is beyond the scope of this course, but some notes will be provided separately.

Definition 6.17. A *subdivision* of a graph H is a graph obtained from H by replacing the edges of H by internally vertex disjoint paths of non-zero length with the same endpoints.

Example 6.18.



H Subdivision of H

Remark 6.19. Any subdivision H of a non-planar graph G must itself be non-planar. Indeed, given a drawing of H in the plane, we could 'forget' the internal vertices in the subdivided paths to obtain a drawing of G.

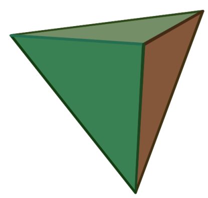
Theorem 6.20 (Kuratowski, 1930)**.** *A graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$ as a subgraph.*

Remark 6.21. If you need to convince your friend that a graph is planar, you can always show them a plane drawing of the graph. It is then easy for your friend to verify that there are no crossing edges. However, from the definition, it is not clear how you would show that a graph is *not* planar — you would need to show that all possible drawings contain a crossing. Kuratowski's Theorem gives us a way to convince our friend of a graph's non-planarity, as we can simply exhibit a K_5- or $K_{3,3}$-subdivision, which can again be quickly verified.

6.2 Application: Platonic solids

Definition 6.22. A *polytope* is a solid in 3 dimensions with flat faces, straight edges and sharp corners. Faces of a polytope are joined at the edges. A polytope is *convex* if the line connecting any two points of the polytope lies inside the polytope.

Example 6.23. The tetrahedron:

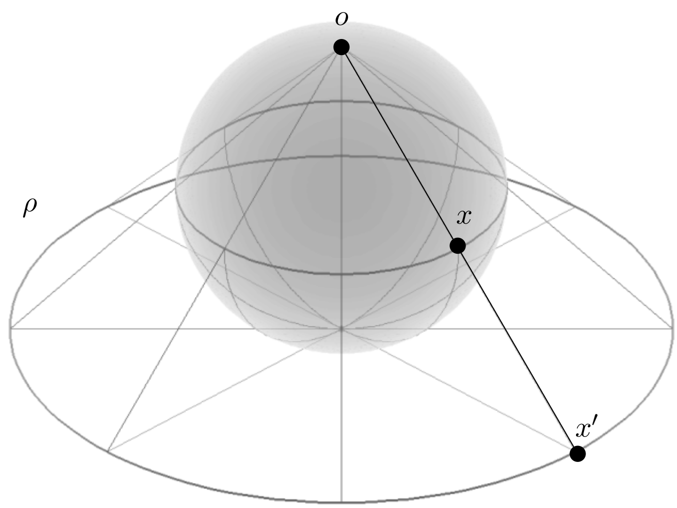


Definition 6.24. A *regular* or *Platonic* solid is a convex polytope which satisfies the following:

1. all of its faces are congruent regular polygons,
2. all vertices have the same number of faces adjacent to them.

We will now characterise all Platonic solids. The first step is to convert a convex polytope into a planar graph. To do this, we place the considered polytope inside a sphere. Then we project the polytope onto the sphere (imagine that the edges of the polytope are made from wire and we place a tiny lamp in the center). This yields a graph drawn on the sphere without edge crossings.

Now let us show that planar graphs are exactly graphs that can be drawn on the sphere. This becomes quite obvious if we use the *stereographic projection*. We place the sphere in the 3-dimensional space in such a way that it touches the considered plane ρ. Let o denote the point of the sphere lying farthest from ρ, the 'north pole'.



Then the stereographic projection maps each point $x \neq o$ of the sphere to a point x', where x' is the intersection of the line ox with the plane ρ. (For the point o, the projection is undefined.) This defines a bijection between the plane and the sphere without the point o. Given a drawing of a graph G on the sphere without edge crossings, where the point o lies on no arc of the drawing (which we may assume by a suitable choice of o), the stereographic projection yields a planar drawing of G. Conversely, from a planar drawing we get a drawing on the sphere by the inverse projection.

Corollary 6.25. *If K is a convex polytope with v vertices, e edges and f faces then $v - e + f = 2$.*

Suppose K is a Platonic solid. All its faces are congruent; assume that they have k vertices (and, thus, k edges). Let us assume moreover that each vertex is adjacent to ℓ faces (and, thus, it has ℓ edges adjacent to it). Since each edge is adjacent to exactly two faces,

$$2e = kf. \tag{3}$$

Moreover, each edge is adjacent to two vertices, and one vertex belongs to ℓ edges, thus

$$\ell v = 2e. \tag{4}$$

Expressing v and f in terms of e, and substituting to Euler's formula, we obtain that $\frac{2e}{\ell} - e + \frac{2e}{k} = 2$. Rearranging, we arrive at

$$\frac{1}{k} + \frac{1}{\ell} = \frac{1}{2} + \frac{1}{e}.$$

Note that since K is a 3-dimensional polytope, each of its faces is a polygon and thus has at least 3 vertices; that is, $k \geq 3$. Moreover, at each vertex, there are at least three faces meeting; $\ell \geq 3$. On the other hand, since $e \geq 1$, we must have

$$\frac{1}{k} + \frac{1}{\ell} > \frac{1}{2}. \tag{5}$$

These conditions do not leave too much leeway; there are only five possible (k, ℓ) pairs for which the above inequality holds. These are $(3,3), (3,4), (3,5), (4,3), (5,3)$.

A Platonic solid corresponds to each of these pairs. We list them below.

- **Tetrahedron**. Here $k = 3$ and $\ell = 3$. Thus, (5) yields that $e = 6$. By (4), $v = 4$, and by (3), $f = 4$. There are 4 vertices and 4 faces of the tetrahedron; the faces are regular triangles, and the vertices are adjacent to 3 edges.
- **Octahedron.** Here $k = 3$ and $\ell = 4$. Thus, (5) yields that $e = 12$. By (4), $v = 6$, and by (3), $f = 8$. There are 8 vertices and 8 faces of the octahedron; the faces are regular triangles, and the vertices are adjacent to 4 edges.
- **Icosahedron.** Here $k = 3$ and $\ell = 5$. Thus, (5) yields that $e = 30$. By (4), $v = 12$, and by (3), $f = 20$. There are 12 vertices and 20 faces of the icosahedron; the faces are regular triangles, and the vertices are adjacent to 5 edges.
- **Cube.** Here $k = 4$ and $\ell = 3$. Thus, (5) yields that $e = 12$. By (4), $v = 8$, and by (3), $f = 6$. There are 8 vertices and 6 faces of the cube; the faces are squares, and the vertices are adjacent to 3 edges.
- **Dodecahedron.** Here $k = 5$ and $\ell = 3$. Thus, (5) yields that $e = 30$. By (4), $v = 20$, and by (3), $f = 12$. There are 20 vertices and 12 faces of the dodecahedron; the faces are regular pentagons, and the vertices are adjacent to 3 edges.