# 機率與統計 HW2

許博翔

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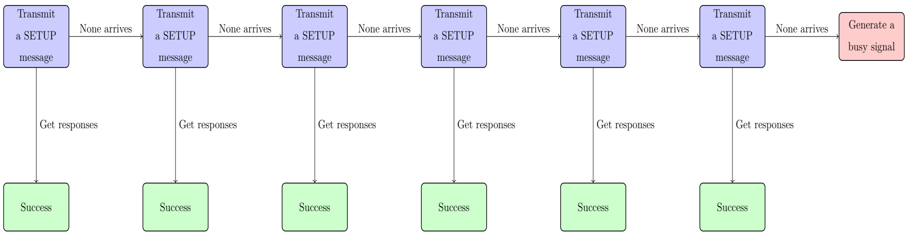
**Definition 1.** Define $\binom{n}{k} := 0$ when $k < 0$ or $k > n$.

**Problem 1.**

(a) 


(b) $$P_K(k) = \begin{cases} \underbrace{(1-p)^{k-1}}_{\text{The first } k-1 \text{ tries fail}} \underbrace{p}_{\text{The } k\text{-th try successes}}, & \text{if } 1 \le k \le 5 \\ \underbrace{(1-p)^5}_{\text{The first 5 tries fail}}, & \text{if } k = 6 \\ 0, \text{otherwise.} \end{cases}.$$

(c) It is the probability the all 6 tries fail, which is $(1-p)^6$.

(d) The probability that all $n$ tries fail $= (1-0.9)^n < 0.02$.

$\Rightarrow n = 2$ is sufficient since $0.1^2 = 0.01 < 0.02$.

**Problem 2.** Let $k \ge 1$.

The probability that rank $1, 2, \ldots, k-1$ are all received by men is $\frac{\binom{11-k}{5}}{\binom{10}{5}}$.

$\Rightarrow \Pr[X \ge k] = \frac{\binom{11-k}{5}}{\binom{10}{5}}$.

$\therefore p_X(x) = \Pr[X \ge x] - \Pr[X \ge x+1] = \frac{\binom{11-x}{5} - \binom{10-x}{5}}{\binom{10}{5}} = \frac{\binom{10-x}{4}}{\binom{10}{5}} = \frac{\binom{10-x}{4}}{252}$.

**Problem 3.** Let $d_k := p_{X_k}(0)$.

When randomly distributing $k$ hats of $k$ people, the probability that the specific $i$ people get their own hat is $\dfrac{(k-i)!}{k!}$.

$\Rightarrow$ by the inclusion-exclusion principle, $d_k = \displaystyle\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)!$.

$\Rightarrow p_{X_N}(x) = \displaystyle\binom{N}{x}d_{N-x} = \binom{N}{x}\sum_{i=0}^{N-x}(-1)^i\binom{N-x}{i}(N-x-i)! = \frac{N!}{x!(N-x)!}\sum_{i=0}^{N-x}(-1)^i\frac{(N-x)!}{i!} = \frac{N!}{x!}\sum_{i=0}^{N-x}\frac{(-1)^i}{i!}$.

**Problem 4.** The following is all possible results of the series, where W denotes win and L denotes lose.

WWW

(WWL/WLW/LWW)+(W/LW/LL)

(LLW/LWL/WLL)+(L/WL/WW)

LLL

And the probability of each result happens is its length.

(a) $P_N(3) = \dfrac{1}{4}, P_N(4) = \dfrac{6}{16} = \dfrac{3}{8}, P_N(5) = 1-\dfrac{1}{4}-\dfrac{3}{8} = \dfrac{3}{8}, P_N(n) = 0$ for $n \notin \{3,4,5\}$.

(b) $P_W(0) = \dfrac{1}{8}, P_W(1) = \dfrac{3}{16}, P_W(2) = \dfrac{6}{32} = \dfrac{3}{16}, P_W(3) = 1-\dfrac{1}{8}-\dfrac{3}{16}-\dfrac{3}{16} = \dfrac{1}{2}, P_W(w) = 0$ for $w \notin \{0,1,2,3\}$.

(c) Since Sixers and Celtics are equally likely to win any game, the number of L = the number of wins of Sixers = the number of wins of Celtics.

$\therefore P_L(l) = P_W(l),\ \forall l$.

**Problem 5.**

(a) $p_X(x) = \dbinom{n}{x}p^x(1-p)^{n-x}$.

$$\Rightarrow \frac{p_X(k+1)}{p_X(k)} = \frac{\frac{n!}{(k+1)!(n-k-1)!}p^k p(1-p)^{n-k-1}}{\frac{n!}{k!(n-k)!}p^k(1-p)^{n-k-1}(1-p)} = \frac{n-k}{k+1}\times\frac{p}{1-p}.$$

(b) For all $0 \le k_1 < k_2 \le n$, $\dfrac{n-k_1}{k_1+1} > \dfrac{n-k_2}{k_1+1} > \dfrac{n-k_2}{k_2+1}$.

$\Rightarrow \dfrac{n-k}{k+1}\times\dfrac{p}{1-p}$ is a strictly decreasing function of $k$ in $[0,n]$.

Solve the equation $\frac{n-k}{k+1} \times \frac{p}{1-p} = 1$, and we get $p(n-k) = (1-p)(k+1) \Rightarrow k = p(n+1) - 1$.

$$\therefore \begin{cases} \forall k < p(n+1)-1, \ \frac{p_X(k+1)}{p_X(k)} > 1 \Rightarrow p_X(k+1) > p_X(k) \\ \forall k > p(n+1)-1, \ \frac{p_X(k+1)}{p_X(k)} < 1 \Rightarrow p_X(k+1) < p_X(k) \\ \text{For } k = p(n+1)-1, \ \frac{p_X(k+1)}{p_X(k)} = 1 \Rightarrow p_X(k+1) = p_X(k) \end{cases}.$$

$\because k_{\max}$ is the largest integer that $k_{\max} > p(n+1)-1$ and $k_{\max} - 1 \leq p(n+1)-1$.

$\therefore p_X(k)$ has maximum at $k = k_{\max}$.

If $p(n+1) \in \mathbb{Z}, \ k_{\max} = p(n+1) - 1 \Rightarrow p_X(k_{\max}) = p_X(k_{\max} - 1)$.

$\therefore$ the maximum is also achieved at $k = k_{\max} - 1$.

**Problem 6.**

(a) Let $g(x) := \sum_{i=1}^{k} c_i f_i(x)$.

$\because g(x)$ is a valid PDF.

$\therefore \int_{-\infty}^{\infty} g(x)dx = 1$.

$\because f_i(x)$ are valid PDFs.

$\therefore \int_{-\infty}^{\infty} f_i(x)dx = 1$.

$$\Rightarrow \sum_{i=1}^{k} c_i = \sum_{i=1}^{k} c_i \int_{-\infty}^{\infty} f_i(x)dx = \int_{-\infty}^{\infty} \sum_{i=1}^{k} c_i f_i(x)dx = \int_{-\infty}^{\infty} g(x)dx = 1.$$

$\therefore$ "$g$ is a PDF $\Rightarrow \sum_{i=1}^{k} c_i = 1$".

Also, if $\sum_{i=1}^{k} c_i = 1$, then by the above, $\int_{-\infty}^{\infty} g(x)dx = 1$.

$\because f_i(x)$ are valid PDFs.

$\therefore f_i(x) \geq 0, \ \forall x$.

$\because$ the linear combination of continuous functions is also a continuous function, and $c_i$ are non-negative.

$\therefore g$ is continuous, and $g(x) \geq 0, \ \forall x$.

$\therefore g$ is a PDF $\iff \sum_{i=1}^{k} c_i = 1$.

(b) $\int_0^2 f_1(x)dx = 2c_1, \int_0^{\infty} f_2(x)dx = \int_0^{\infty} c_2 e^{-x}dx = c_2$.

$$1 = \int_{-\infty}^{\infty} f_X(x) = \frac{1}{2}\int_0^2 f_1(x)dx + \frac{1}{2}\int_0^{\infty} f_2(x)dx = c_1 + \frac{1}{2}c_2.$$

For $x = 3,\ f_X(x) = c_2e^{-3} \geq 0 \Rightarrow c_2 \geq 0.$

$f_X(x) = c_1 + c_2e^{-x} \geq 0$ for $x \in [0, 2]$.

$\Rightarrow c_1 \geq -c_2e^{-x} \geq -c_2e^{-2}.$

$$\therefore \text{ the conditions are } \begin{cases} c_1 + \frac{1}{2}c_2 = 1. \\ c_2 \geq 0. \\ c_1 \geq -c_2e^{-2}. \end{cases}.$$

**Problem 7.** The inter-arrival times are exponential random variables means that the arrival times follow a poisson distribution.

The rate is one passenger per minute.

$\therefore$ the probability that less than 7 passengers arrives in the first 10 minutes is $\displaystyle\sum_{i=0}^{6} \frac{10^i e^{-10}}{i!}$.

Use the following program to calculate the result and it is about 0.13.

```
from math import *
frac=[1]
for i in range(1, 100):
        frac.append(frac[i-1]*i)
ans=0
for i in range(7):
        ans+=10**i*exp(-10)/frac[i]
print(ans)
```

**Problem 8.** $\alpha = \dfrac{5}{n}$.

$\displaystyle\sum_{i=5}^{\infty} \frac{\alpha^i e^{-\alpha}}{i!} < 10\%.$

$\displaystyle\Rightarrow \sum_{i=0}^{4} \frac{(\frac{5}{n})^i e^{-\frac{5}{n}}}{i!} > 0.9.$

Write the following program and find the answer $n \geq 3$.

The probability that no orders are waiting $= \dfrac{(\frac{5}{3})^0 e^{-\frac{5}{3}}}{0!} \approx 0.19.$

```python
from math import *
frac=[1]
for i in range(1, 100):
        frac.append(frac[i-1]*i)
for n in range(1, 100):
        sm=0
        for i in range(5):
                sm+=(5/n)**i*exp(-5/n)/frac[i]
        if sm>0.9:
                print(n)
n=3
print(exp(-5/n))
```

**Problem 9.**

(a) $f_{X_n}(x) = \begin{cases} \dfrac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!}, \text{ if } x \geq 0, \\ \\ 0, \text{ otherwise.} \end{cases}$ .

$\because f_{X_n}(x) = 0$ for $x < 0$.

$\therefore F_{X_n}(x) \displaystyle\int_0^x \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!} dt.$

(b) Let $du = e^{-\lambda t}dt, v = \dfrac{\lambda^n t^{n-1}}{(n-1)!}$.

$\displaystyle\int vdu = uv - \int udv.$

$\Rightarrow F_{X_n}(x) = \displaystyle\int_0^x \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!} dt = -\frac{\lambda^{n-1} t^{n-1} e^{-\lambda t}}{(n-1)!}\bigg|_0^x - \int_0^x -\frac{\lambda^{n-1} t^{n-2} e^{-\lambda t}}{(n-2)!} dt = F_{X_{n-1}}(x) - \frac{\lambda^{n-1} x^{n-1} e^{-\lambda x}}{(n-1)!}.$

(c) $F_{X_n}(x) = F_{X_{n-1}}(x) - \dfrac{\lambda^{n-1} x^{n-1} e^{-\lambda x}}{(n-1)!} = F_{X_{n-2}}(x) - \dfrac{\lambda^{n-2} x^{n-2} e^{-\lambda x}}{(n-2)!} - \dfrac{\lambda^{n-1} x^{n-1} e^{-\lambda x}}{(n-1)!} = \cdots = F_1(x) - \displaystyle\sum_{k=1}^{n-1} \frac{\lambda^k x^k e^{-\lambda x}}{k!} = 1 - e^{-\lambda x} - \sum_{k=1}^{n-1} \frac{\lambda^k x^k e^{-\lambda x}}{k!} = 1 - \sum_{k=0}^{n-1} \frac{\lambda^k x^k e^{-\lambda x}}{k!}.$

Also, since $f_{X_n}(x) = 0$ for $x < 0$, we get that

$$F_{X_n}(x) = \begin{cases} 1 - \sum_{k=0}^{n-1} \frac{\lambda^k x^k e^{-\lambda x}}{k!}, \text{ if } x \geq 0 \\ 0, \text{ otherwise} \end{cases}.$$

**Problem 10.** $\Pr[R > 0] = \Pr[V > d] = \Pr[\max(X - k, 0) > d]$.

If $d < 0$, then $\Pr[\max(X - k, 0) > d] = 1$.

$\therefore d_0 \geq 0$.

$\Rightarrow \Pr[\max(X - k, 0) > d_0] = \Pr[X - k > d_0] = \frac{1}{2}$.

Since $X - k$ is a Guassian $(0, \sqrt{t})$, there is $d_0 = 0$.