Graph Theory HW2

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Problem 2.

(a) Suppose that $|V(G)| = n$.

Let's prove by contradiction, and suppose that for all $V' \subseteq V(G)$, the spanning subgraph H of G with vertices set V' satisfies $\delta(H) < \frac{1}{2}\bar{d}$. — (1)

Construct a sequence of graphs $\{G\}_{i=0}^{n-1}$, where $G_0 := G$, and $G_{i+1(0\le i\le n-2)} := G_i \setminus \{u_i\}$, where $\deg_{G_i}(u_i) < \frac{1}{2}\bar{d}$.

The construction is legal, since by (1), such u_i always exists in G_i.

$\Rightarrow |V(G_{n-1})| = |V(G_{n-2})| - 1 = |V(G_{n-3})| - 2 = \cdots = |V(G_0)| - (n-1) = 1.$

$|E(G_0)| = |E(G)| = \frac{1}{2}|V|\bar{d} = \frac{1}{2}n\bar{d}.$

Since $\deg_{G_i}(u_i) < \frac{1}{2}\bar{d}$, and all edges in $E(G_i) \setminus E(G_{i+1})$ are incident to u_i, there is $|E(G_{i+1})| = |E(G_i)| - |E(G_i) \setminus E(G_{i+1})| > |E(G_i)| - \frac{1}{2}\bar{d}$.

$\therefore |E(G_{n-1})| > |E(G_{n-2})| - \frac{1}{2}\bar{d} > |E(G_{n-3})| - d > \cdots > |E(G_0)| - \frac{n-1}{2}\bar{d} = \frac{1}{2}\bar{d}.$

Since each edge in G_{n-1} contributes two degrees to the only left vertex in G_{n-1}, the degree of that vertex $> \frac{1}{2}\bar{d} \times 2 > \frac{1}{2}\bar{d}$, which contradicts to (1).

$\therefore$ such subgraph should exist.

(b) I thought that the description missed a condition that G is a simple graph, so we assert G to be simple in the following.

Since $\bar{d}(G) \ge 2k - 2$, from the above, there exists $H \subseteq G$ s.t. $\delta(H) \ge k - 1$.

Since G is simple, H is also simple.

Let's have an induction on $m (1 \le m \le k)$ to prove that H contains every tree of order m — (2).

For $m = 1$, H contains a vertex, and therefore (2) holds.

Suppose for $m = m' < k$, (2) holds.

For $m = m' + 1$, for each tree T of order $m' + 1$, let v be a leaf of T, and $uv \in E(T)$. $T \setminus \{v\}$ is a tree of order m'.

By the induction hypothesis, H contains T' which is isomorphic to $T \setminus \{v\}$, where u' is isomorphic to u.

Since $\deg_H(u') \geq k - 1 > m' - 1$, there exists a neighbor v' of u' in H such that $v' \notin T'$.

$\Rightarrow H$ contains T since T is isomorphic to $T' \cup \{v'\} \cup \{u'v'\}$.

$\therefore$ by induction, H contains all trees of order k.

Problem 3. For each $e \in T_1$, suppose that $e = uv$.

Since T_2 is connected, there is a path $u_0(= u)u_1u_2 \cdots u_k(= v)$ from u to v.

Suppose that $U(, V) :=$ all vertices that are connected to $u(, v)$ on $T_1 - e$, respectively.

Since T_1 is connected, all vertices are connected to either u or v on $T_1 - e$.

$\therefore U, V$ is a partition of $[n]$.

Suppose that l is the minimum index such that $u_l \in V$.

Such l exists since $u_k = v \in V$.

We know that $u_0 = u \in U$.

$\Rightarrow l \geq 1$, and $u_{l-1} \in U$ by the definition of l.

$\Rightarrow$ take $f = u_{l-1}u_l$, and since $u_0(= u)u_1u_2 \cdots u_k(= v)$ is a path on T_2, $f \in T_2$.

Since $u_{l-1} \in U,\ u_l \in V$, f connects U and V.

$\Rightarrow T_1 - e + f$ is connected.

Since $|E(T_1 - e + f)| = n - 1$, $T_1 - e + f$ is a tree.

Problem 6. By the definition of k-connected, $\forall U_1 \subseteq V(G_1), U_2 \subseteq V(G_2)$ where $|U_1| < k, |U_2| < k$, $G_1 \setminus U_1, G_2 \setminus U_2$ are connected.

$\forall X \subseteq V(G_1) \cup V(G_2)$ with $|X| < k$, since $|X \cap V(G_1)| \leq |X| < k$ and $|X \cap V(G_2)| \leq |X| < k$, the vertices in $V(G_1) \setminus X$ are connected in $(G_1 \cup G_2) \setminus X$, and so are those in $V(G_2) \setminus X$.

Since $|V(G_1) \cap V(G_2)| \geq k$, $\exists u \in V(G_1) \cap V(G_2)$ s.t. $u \notin X$.

Since u is connected to all vertices in $V(G_1) \setminus X$, and connected to all vertices in

$V(G_2) \setminus X$, $(G_1 \cup G_2) \setminus X$ is connected.

$\therefore G_1 \cup G_2$ is k-connected by the definition.