機率與統計 HW3

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Definition 1. Define $\binom{n}{k} := 0$ when $k < 0$ or $k > n$.

Problem 1.

(a) $Y = \pi 1^2 \times \dfrac{X}{2\pi 1} = \dfrac{X}{2}$.

(b) $F_Y(y) = F_X(2y)$.

(c) $f_Y(y) = f_X(2y)$.

(d) $E[Y] = E[\dfrac{X}{2}] = \dfrac{1}{2}E[X]$.

Problem 2.

(a) $f_U(u) = \begin{cases} 0, \text{ if } u < -5 \\ \frac{1}{8}, \text{ if } -5 < u < -3 \\ 0, \text{ if } -3 < u < 3 \\ \frac{3}{8}, \text{ if } 3 < u < 5 \\ 0, \text{ if } u > 5 \end{cases}$.

$\mathrm{E}[U] = \int_{-\infty}^{\infty} u f_U(u) du = \int_{-5}^{-3} \frac{1}{8} u du + \int_{3}^{5} \frac{3}{8} u du = -1 + 3 = 2$.

$\mathrm{Var}[U] = \mathrm{E}[U^2] - (\mathrm{E}[U])^2 = \int_{-\infty}^{\infty} u^2 f_U(u) du - 4 = \int_{-5}^{-3} \frac{1}{8} u^2 du + \int_{3}^{5} \frac{3}{8} u^2 du - 4 = \frac{49}{3} - 4 = \frac{37}{3}$.

(b) $\mathrm{E}[2^U] = \int_{-\infty}^{\infty} 2^u f_U(u) du = \int_{-5}^{-3} \frac{1}{8} 2^u du + \int_{3}^{5} \frac{3}{8} 2^u du = \frac{1}{8\ln 2}(2^{-3} - 2^{-5}) + \frac{3}{8\ln 2}(2^5 - 2^3) = \frac{3}{8 \times 32 \ln 2} + \frac{9}{\ln 2} = \frac{2307}{256 \ln 2} \approx 13$.

Problem 3.

(a) The probability of each MM counted twice is p and they are independent.

$\Rightarrow R \sim \text{Bin}(20, p)$.

$\therefore P_R(r) = \binom{20}{r} p^r (1-p)^{20-r}$.

(b) $N = 20 + R$.

$\therefore P_N(n) = P_R(n-20) = \binom{20}{n-20} p^{n-20}(1-p)^{40-n}$.

Problem 4.

(a) $\because 1 - U \leq 1$, there is $X = -\ln(1-U) \geq 0$.

$\Rightarrow F_X(x) = 0$ for all $x \leq 0$.

$\Pr(X \leq x) = \Pr(-\ln(1-U) \leq x) = \Pr(\ln(1-U) \geq -x) = \Pr(1 - U \geq e^{-x}) = \Pr(U \leq 1 - e^{-x})$.

$$\therefore F_X(x) = \begin{cases} 1 - e^{-x}, \text{ if } x > 0 \\ 0, \text{ otherwise} \end{cases}.$$

(b) $\dfrac{dF_X(x)}{dx} = e^{-x}$.

$$\therefore f_X(x) = \begin{cases} e^{-x}, \text{ if } x > 0 \\ 0, \text{ otherwise} \end{cases}.$$

(c) $E[X] = \int_{-\infty}^{\infty} x f_X(x)dx = \int_0^{\infty} xe^{-x}dx = -xe^{-x}|_0^{\infty} + \int_0^{\infty} e^{-x}dx = 1$.

Problem 5.

(a) $P_N(n) = (1-p)^{n-1}p = 0.1 \times 0.9^{n-1}$

(b) $P_{N|B}(n) = 0$ for $n \leq 19$.

$P_{N|B}(n) =$ the probability that the $20, 21, \ldots, n-1$-th tests success but the n-th test fails, which is $(1-p)^{n-20}p = 0.1 \times 0.9^{n-20}$.

$$\therefore P_{N|B}(n) = \begin{cases} 0, \text{ if } n \leq 19 \\ 0.1 \times 0.9^{n-20}, \text{ otherwise} \end{cases}.$$

(c) $E[N|B] = \sum_{n=1}^{\infty} P_{N|B}(n) = \sum_{n=20}^{\infty} n0.1 \times 0.9^{n-20} = \sum_{n=1}^{\infty}(n+19)0.1 \times 0.9^{n-1} = 19\sum_{n=1}^{\infty} 0.1 \times 0.9^{n-1} + \sum_{n=1}^{\infty}\sum_{i=1}^{n} 0.1 \times 0.9^{n-1} = 19 \times \frac{0.1}{1-0.9} + \sum_{i=1}^{\infty}\sum_{n=i}^{\infty} 0.1 \times 0.9^{n-1} = 19 + \sum_{i=1}^{\infty} \frac{0.10.9^{i-1}}{1-0.9} = 19 + \sum_{i=1}^{\infty} 0.9^{i-1} = 19 + \frac{1}{1-0.9} = 19 + 10 = 29.$

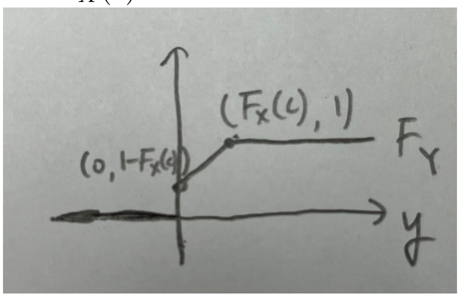
Problem 6. $g(x) = \Pr[x < X \leq c] = F_X(c) - F_X(x)$.

$\Rightarrow F_Y(y) = \Pr[Y \leq y] = \Pr[g(X) \leq y] = \Pr[F_X(c) - F_X(X) \leq y] = \Pr[F_X(X) \geq F_X(c) - y] = 1 - \Pr[F_X(X) \leq F_X(c) - y] \overset{\text{by the definition of CDF}}{=} 1 - F_X(c) + y$.

$\because g(x) \geq 0$ and $g(x) \leq \Pr[X \leq c] = F_X(c)$.

$$\therefore F_Y(y) = \begin{cases} 0, \text{ if } y < 0 \\ 1 - F_X(c) + y, \text{ if } 0 \leq y \leq F_X(c) \\ 1, \text{ if } y \geq F_X(c) \end{cases}.$$

Note that $F_Y(y)$ is not necessarily continuous since $\Pr[Y = 0] = \Pr[X \geq c] = 1 - F_X(c)$.



Problem 7.

(a) Since X is an exponential-1 random variable, there is $X \geq 0$ for all X.

Since if $X \leq 1$, then $Y = X \in [0,1]$, and if $X > 1$, then $Y = \frac{1}{X} \in (0,1)$.

$\therefore F_Y(y) = 0$ for $y \leq 0$, and $F_Y(y) = 1$ for $y > 1$.

For $y \in (0,1]$, $F_Y(y) = \Pr[Y \leq y] = \Pr[(X \leq 1 \wedge X \leq y) \vee (X > 1 \wedge \frac{1}{X} \leq y)] = \Pr[X \leq 1 \wedge X \leq y] + \Pr[X > 1 \wedge X \geq \frac{1}{y}] = F_X(y) + 1 - \Pr[X \leq \frac{1}{y}] = F_X(y) + 1 - F_X(\frac{1}{y})$.

$$\therefore F_Y(y) = \begin{cases} 1, \text{ if } y > 1 \\ 0, \text{ if } y \le 0 \\ F_X(y) + 1 - F_X(\frac{1}{y}), \text{ if } 0 < y \le 1 \end{cases}.$$

(b) $f_Y(y) = \dfrac{dF_Y(y)}{dy} = 0$ if $y > 1$ or $y < 0$.

Since X is an exponential-1 random variable, there is $f_X(x) = e^{-x}$.

$\Rightarrow \dfrac{d(F_X(y) + 1 - F_X(\frac{1}{y}))}{dy} = f_X(y) - f_X(\frac{1}{y})(-\frac{1}{y^2}) = f_X(y) + \dfrac{f_X(\frac{1}{y})}{y^2} = e^{-y} + \dfrac{e^{-\frac{1}{y}}}{y^2}$.

$$\therefore f_Y(y) = \begin{cases} 0, \text{ if } y < 0 \text{ or } y > 1 \\ e^{-y} + \dfrac{e^{-\frac{1}{y}}}{y^2}, \text{ if } 0 < y < 1 \end{cases}.$$

Problem 8. Let T be the random variable representing the time the tourist takes to arrive to the city.

$\mathrm{E}[T] = \frac{1}{3}(1 + \mathrm{E}[T]) + \frac{1}{3}(6 + \mathrm{E}[T]) + \frac{1}{3} \times 2.$

$\Rightarrow \frac{1}{3}\mathrm{E}[T] = \frac{1}{3} + 2 + \frac{2}{3} = 3.$

$\Rightarrow \mathrm{E}[T] = 9.$

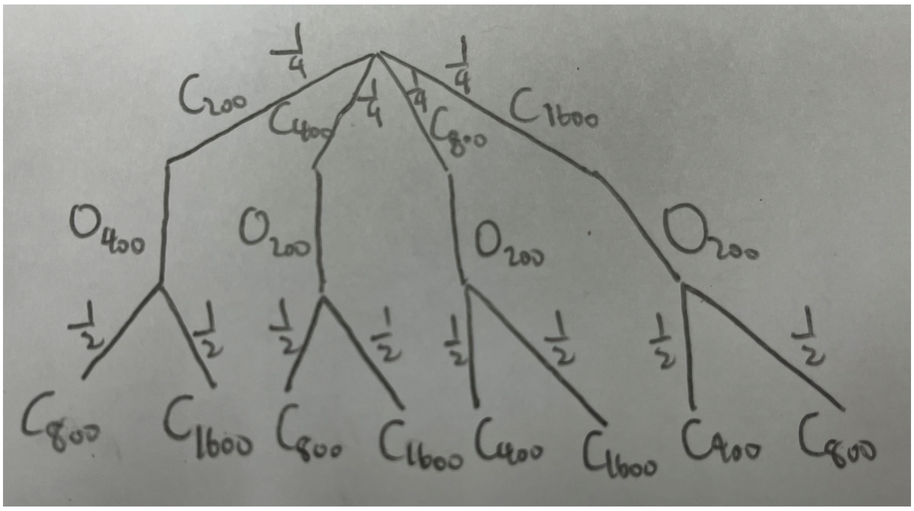
$\therefore$ the expected time is 9 hours.

Problem 9.

(a) $R \in \{200, 400, 800, 1600\}$.

$P_R(r) = \frac{1}{4},\ \forall r \in \{200, 400, 800, 1600\}$.

$\mathrm{E}[R] = \frac{1}{4}(200 + 400 + 800 + 1600) = 750.$

(b) 

$P_R(200) = 0, P_R(400) = 2 \times \frac{1}{8} = \frac{1}{4}, P_R(800) = P_R(1600) = 3 \times \frac{1}{8} = \frac{3}{8}$.
$\mathrm{E}[R] = 0 \times 200 + \frac{1}{4} \times 400 + \frac{3}{8}(800 + 1600) = 100 + 900 = 1000$.

Problem 10.

(a) The PDF of Erlang(4, 3) is $f(x) = \dfrac{3^4 x^3 e^{-3x}}{3!} = \dfrac{27x^3 e^{-3x}}{2}$.
$\Rightarrow$ 答對率 $= \displaystyle\int_0^3 g(t)f(t)dt = \int_0^3 \frac{4-(t-2)^2}{5} \times \frac{27t^3 e^{-3t}}{2} dt = \frac{1}{90} e^{-3t}(81t^5 - 189t^4 - 252t^3 - 252t^2 - 168t - 56)\Big|_0^3 = \frac{1}{45}(28 - 2629e^{-9}) \approx 0.615$.
$\therefore$ 考試分數期望值 $\approx 0.615 \times n$.

(b) 伍佰億聊天的時間 T 呈現 Exponential 機率分布，平均時間爲 60 分鐘
$\Rightarrow f_T(t) = \frac{1}{60} e^{-\frac{1}{60}t}, F_T(t) = 1 - e^{-\frac{1}{60}t}$.
$\Pr[T \geq 120 | T \geq 60] = \dfrac{\Pr[T \geq 120]}{\Pr[T \geq 60]} = \dfrac{e^{-2}}{e^{-1}} = \dfrac{1}{e}$.
所以之後已讀不回的機率 $= \frac{1}{e}$。

(c) $\Pr[T > 60 | T = 60] = 1 (\because \Pr[T = 60 | T = 60] = 0)$.
有好感的女生與伍佰億聊了 1 小時傳了 50 則訊息的機率 $= \dfrac{120^{50} e^{-120}}{50!}$，沒有好感的女生與伍佰億聊了 1 小時傳了 50 則訊息的機率 $= \dfrac{15^{50} e^{-15}}{50!}$。
所以其爲沒好感的女生且還持續聊天的機率爲 $\dfrac{0.2 \times \frac{15^{50} e^{-15}}{50!}}{0.2 \times \frac{15^{50} e^{-15}}{50!} + 0.8 \times \frac{120^{50} e^{-120}}{50!}}$
$= \dfrac{0.2 \times 15^{50} e^{-15}}{0.2 \times 15^{50} e^{-15} + 0.8 \times 120^{50} e^{-120}} \approx 0.411$.