Graph Theory HW4

許博翔 B10902085

Teammate: 黃芊禕 B10902029

November 7, 2023

Problem 2. Let $uv \in E(G),\ H := G - uv,\ N_H(w) := \{x | wx \in E(H)\}$.

Let $S := \{p | p$ is a Hamiltonian path in H with u being one of the endpoint $\}$, $p_i :=$ the i-th vertex on the path p ($p = p_1(= u)p_2p_3 \cdots p_n$), and $T_w := \{p | p \in S,\ p_n = w\}$.

Let $f : S \to 2^S$ (the power set of S), where $f(p) := \{p_1p_2 \cdots p_ip_np_{n-1} \cdots p_{i+1} | p_ip_n \in E(H),\ i \neq n-1\}$.

$\Rightarrow |f(p)| = |\{i | p_ip_n \in E(H),\ i \neq n-1\}| = \deg_H(p_n) - 1$.

Suppose that $q \in f(p)$, where $q = p_1p_2 \cdots p_ip_np_{n-1} \cdots p_{i+1}$.

$\Rightarrow p = q_1q_2 \cdots q_iq_nq_{n-1} \cdots q_{i+1}$.

$\because q_iq_n = p_ip_{i+1} \in E(H)$.

$\therefore p \in f(q)$ by the definition of f and Hamiltonian path.

$\therefore \forall p, q \in S,\ p \in f(q) \iff q \in f(p)$.

Consider a graph H' with vertex set S, and $pq \in E(H') \iff p \in f(q) (\iff q \in f(p))$.

$\deg_{H'}(p) = |f(p)| = \deg_H(p_n) - 1$.

The number of Hamiltonian cycles of G that contain $uv \equiv$ the number of Hamiltonian paths that starts at u and ends at v in $H \equiv \sum_{p_n=v} 1 \overset{\deg_H(v)\equiv\deg_G(v)-1\equiv 0 \pmod 2}{\equiv} \sum_{p_n=v} (\deg_H(v)-1) \overset{\deg_H(w)\equiv\deg_G(v)\equiv 1 \pmod 2 \text{ for } w\neq v}{\equiv} \sum_{w\in V}\sum_{p_n=w} (\deg_H(w)-1) \equiv \sum_{p\in S} (\deg_H(p_n)-1) \equiv \sum_{p\in S} \deg_{H'}(p) \equiv 0 \pmod 2$, which finishes this problem.

Problem 3. Let $m := \dfrac{n}{2}$.

Consider the graph $K_{m+1,m-1}$ (where $V(K_{m-1,m+1})$ has bipartition $A \cup B$, and

$|A| = m+1$), it does not have a perfect matching because $|N_S| < |S|$ for $S = A$ by Hall theorem.

$\therefore d(n) > \delta(K_{m+1,m-1}) = m-1$.

Claim: $d(n) = m$.

Proof: Let G be a graph with $\delta(G) \geq m$, and $S \subseteq V(G)$.

For $|S| > m$, $q(G \setminus S) \leq |V(G \setminus S)| \leq m-1 \leq |S|$.

For $|S| \leq m$:

The size of a connected component in $G \setminus S$ is at least $\delta(G \setminus S)+1$, because every vertex is connected to at least $\delta(G \setminus S)$ vertices.

$\therefore$ the number of connected components of $G\setminus S \leq \lfloor \dfrac{n-|S|}{\delta(G \setminus S)+1} \rfloor \overset{\delta(G\setminus S)\geq\delta(G)-|S|\geq m-|S|}{\leq} \lfloor \dfrac{2m-|S|}{m-|S|+1} \rfloor$.

Let $f(x) := \dfrac{2m-x}{(m-x+1)x}$ for $x \in (0, m+1)$ (so that $(m-x+1)x > 0$).

$f(x) \leq 1 \iff 2m - x \leq (m-x+1)x \iff x^2 - (m+2)x + 2m \leq 0 \iff (x-m)(x-2) \leq 0 \iff 2 \leq x \leq m$.—(1)

$$\Rightarrow \frac{q(G \setminus S)}{|S|} \leq \frac{\lfloor \frac{2m-|S|}{m-|S|+1} \rfloor}{|S|} \begin{cases} = \lfloor \dfrac{2m-1}{m} \rfloor = 1, \text{ if } x = 1 \\ \leq \dfrac{2m-|S|}{(m-|S|+1)|S|} \overset{(1)}{\leq} 1, \text{ if } x \geq 2 \end{cases}.$$

$\Rightarrow q(G \setminus S) \leq |S|$.

$\therefore$ by Tutte theorem, G has a perfect matching, which finishes the proof.

Problem 6. Consider G with bipartition $V(G) = A \cup B$.

$A = \{u_i | i \in [n]\}, B = \{v_i | i \in [n]\}$, where $u_i \in A, v_j \in B$ are adjacent $\iff$ the i-th column of L does not contain j.

$\because$ the size of a row is n, each element of $[n]$ appears exactly once in every row.

$\Rightarrow$ there are r js for all $j \in [n]$.

$\because$ no column contains two js.

$\Rightarrow \deg(v_j) = n-$ the number of columns that contain $j = n - r$.

$\because$ every column contains r different elements.

$\therefore \deg(u_i) = n - r \forall i \in [n]$.

$\therefore G$ is k-regular.

By corollary 5.10 in class, G has a perfect matching, and suppose that M is a perfect matching.

Let $f(j)$ denote the index of the other endpoint of the edge containing v_j in M (that is, $u_{f(j)}v_j \in M$).

Put j at the intersection of the $f(j)$-th column and the $(r+1)$-th row for all $j \in [n]$.
—(1)

By the definition of G, since $u_{f(j)}v_j \in M \subseteq E(G)$, j does not appears in the $f(i)$-th column of L.

Also, since M is a perfect matching, f is a bijection, which means the $(r+1)$-th row contains each element in $[n]$ exactly once.

$\therefore$ (1) is a valid way to extend L.