機率與統計 HW3

許博翔

May 13, 2024

Problem 1. $\Pr(K = k \wedge X = x)$

= the probability that "the first x circuits are acceptable, the $x+1$-th is reject, and the $x+2$-th to n-th circuits contain exactly $k-1$ rejected circuits

$= p^x \times (1-p) \times \binom{n-x-1}{k-1} p^{n-x-1-k+1}(1-p)^{k-1}$

$= \binom{n-x-1}{k-1} p^{n-k}(1-p)^k$.

$$\therefore P_{K,X}(k,x) = \begin{cases} \binom{n-x-1}{k-1} p^{n-k}(1-p)^k, \text{ if } k+x \le n, k \ge 1, x \ge 0 \\ 0, \text{ otherwise} \end{cases}.$$

Problem 2. $P_N(n) = 0$ for $n < 0$.

For $n \ge 0$, $P_N(n) = \sum_{k=0}^{n} \frac{100^n e^{-100}}{(n+1)!} = (n+1) \times \frac{100^n e^{-100}}{(n+1)!} = \frac{100^n e^{-100}}{n!}$.

$$\therefore P_N(n) = \begin{cases} \frac{100^n e^{-100}}{n!}, \text{ if } n \in \mathbb{Z}^+ \cup \{0\} \\ 0, \text{ otherwise} \end{cases}.$$

$\because$ By the definition, if $k > n$, then $P_{N,K}(n,k) = 0$.

$\therefore P_K(k) = \sum_{n=0}^{\infty} P_{N,K}(n,k) = \sum_{n=k}^{\infty} P_{N,K}(n,k) = \sum_{n=k}^{\infty} \frac{100^n e^{-100}}{(n+1)!} = \sum_{n=k+1}^{\infty} \frac{100^{n-1} e^{-100}}{n!} = \frac{1}{100} \sum_{n=k+1}^{\infty} \frac{100^n e^{-100}}{n!} = \frac{1}{100} \sum_{n=k+1}^{\infty} P_N(n) = \frac{1}{100} \Pr(n > k)$.

Problem 3.

Problem 4.

(a) If $x \ge 2$, then $F_X(x) = 1$.

If $x \le 0$, then $F_X(x) = 0$.

If $0 \le x \le 2$, then $F_X(x) = \int_{-\infty}^{x} f_X(y)dy = \int_0^x \frac{y}{2}dy = \frac{x^2}{4}$.

$$\therefore F_X(x) = \begin{cases} 0, \text{ if } x \le 0 \\ \frac{x^2}{4}, \text{ if } 0 \le x \le 2 \\ 1, \text{ otherwise} \end{cases}.$$

(b) Since X_1, X_2 are independent, $\Pr[X_1 \le 1, X_2 \le 1] = \Pr[X_1 \le 1]\Pr[X_2 \le 1] = F_X(1)F_X(1) = (\frac{1}{4})^2 = \frac{1}{16}$.

(c) $F_W(1) = \Pr[\max(X_1, X_2) \le 1] = \Pr[X_1 \le 1, X_2 \le 1] = \frac{1}{16}$.

(d) $F_W(w) = \Pr[\max(X_1, X_2) \le w] = \Pr[X_1 \le w, X_2 \le w]$

$$\overset{\because X_1, X_2 \text{ are independent}}{=} \Pr[X_1 \le w]\Pr[X_2 \le w] = F_X(w)^2 = \begin{cases} 0, \text{ if } w \le 0 \\ \frac{x^4}{16}, \text{ if } 0 \le w \le 2 \\ 1, \text{ otherwise} \end{cases}.$$

Problem 5. First, $X \sim \text{Unif}[0, \frac{d}{2}], \Theta \sim \text{Unif}[0, \frac{\pi}{2}]$.

$$\Rightarrow f_X(x) = \begin{cases} \frac{2}{d}, \text{ if } 0 \le x \le \frac{d}{2} \\ 0, \text{ otherwise} \end{cases}, f_\Theta(\theta) = \begin{cases} \frac{2}{\pi}, \text{ if } 0 \le \theta \le \frac{\pi}{2} \\ 0, \text{ otherwise} \end{cases}.$$

The needle intersects one of the lines $\iff X \le \frac{l}{2}\sin\theta$.

$\therefore$ the probability that the needle will intersect one of the lines $= \Pr[X \le \frac{l}{2}\sin\theta]$.

Note that $l < d$, so the upperbound of X in the following integral is $\min(\frac{l}{2}\sin\theta, \frac{d}{2}) = \frac{l}{2}\sin\theta$.

And X, Θ are independent, so their joint pdf $f_{X,\Theta}(x, \theta) = f_X(x)f_\Theta(\theta)$.

$\Rightarrow \Pr[X \le \frac{l}{2}\sin\theta] = \int_0^{\frac{\pi}{2}} \int_0^{\frac{l}{2}\sin\theta} f_X(x)f_\Theta(\theta)dxd\theta = \int_0^{\frac{\pi}{2}} \frac{l}{2}\sin\theta\frac{2}{d}\frac{2}{\pi}d\theta = \frac{2l}{d\pi}$.

This experiment can get p: the approximated value of the probability that the needle will intersect one of the lines when the needle is dropped for sufficient large number of times.

And one can approximate $\pi \approx \frac{2l}{dp}$.

Problem 6. $r_{X,Y} = E[XY] = \int_{-1}^{1} \int_{-1}^{y} f_{X,Y}(x, y)xydxdy = \frac{1}{2}\int_{-1}^{1} \frac{y^2 - 1}{2}ydy =$

$\frac{1}{4}(\frac{y^4}{4}-\frac{y^2}{2})\Big|_{-1}^{1}=0.$

$E[e^{X+Y}]=\int_{-1}^{1}\int_{-1}^{y}f_{X,Y}(x,y)e^{x+y}dxdy=\frac{1}{2}\int_{-1}^{1}(e^{2y}-e^{y-1})dy=\left(\frac{1}{4}e^{2y}-\frac{1}{2}e^{y-1}\right)\Big|_{-1}^{1}=$ $\frac{1}{4}(e^2-e^{-2})-\frac{1}{2}(1-e^{-2})=\frac{1}{4}(e^2-2+e^{-2})=\left(\frac{e-\frac{1}{e}}{2}\right)^2.$

Problem 7. First, if $W<1$, then both $\frac{X}{Y},\frac{Y}{X}$ are less than 1.

Since $\Pr[X\le 0\vee Y\le 0]=0$ by the definition of $f_{X,Y}(x,y)$.

$\therefore \frac{X}{Y}<1,\frac{Y}{X}<1\Rightarrow X<Y,Y<X$, which is impossible.

$\therefore$ there must be $W\ge 1$.

$F_W(w)=\Pr[W\le w]=\Pr[\max\left(\frac{X}{Y},\frac{Y}{X}\right)\le w]=\Pr[\frac{X}{Y}\le w\wedge\frac{Y}{X}\le w]=\Pr[Y\ge \frac{X}{w}\wedge X\ge\frac{Y}{w}]=1-\Pr[X<\frac{Y}{w}]-\Pr[Y<\frac{X}{w}].$

Note that $\frac{Y}{w}\le\frac{a}{w}\le a,\frac{X}{w}\le\frac{a}{w}\le a.$

$\therefore 1-\Pr[X<\frac{Y}{w}]-\Pr[Y<\frac{X}{w}]=1-\int_0^a\int_0^{\frac{y}{w}}f_{X,Y}(x,y)dxdy-\int_0^a\int_0^{\frac{x}{w}}f_{X,Y}dydx=$ $1-2\int_0^a\int_0^{\frac{y}{w}}\frac{1}{a^2}dxdy=1-\frac{2}{a^2}\int_0^a\frac{y}{w}dy=1-\frac{1}{a^2w}a^2=1-\frac{1}{w}.$

$\Rightarrow f_W(w)=F_W'(w)=\frac{1}{w^2}.$

Problem 8.

(a) Since at least one bus arrive, there is $n\ge 1$.

Since at most one bus arrives in a minute, there is $t\ge n$.

$\therefore$ the set is $\{(n,t):t\ge n\ge 1,n,t\in\mathbb{Z}\}$.

(b) If $n>t$, then by (a), $P_{N,T}(n,t)=0$.

If $n\le t$, it means that exactly $n-1$ buses passed through in the first $t-1$ minutes, and a bus passed through at the t-th minute, so the probability is $\binom{t-1}{n-1}p^n(1-p)^{t-n}$.

The probability that I didn't board the first $n-1$ buses but the n-th is $(1-q)^{n-1}q$.

$$\therefore P_{N,T}(n,t)=\begin{cases}\binom{t-1}{n-1}p^n(1-p)^{t-n}(1-q)^{n-1}q, \text{ if } t\ge n\ge 1,n,t\in\mathbb{Z}\\ 0, \text{ otherwise}\end{cases}.$$

(c) $P_N(n)=$ the probability that I didn't board the first $n-1$ buses but the n-th

$= (1-q)^{n-1}q.$

$P_T(t) =$ the probability that in t minutes:

(1) the first $t-1$ minutes either the bus didn't come, or I didn't board the bus, which has probability $(1-pq)$.

(2) the t-th minute the bus came and I boarded the bus, which has probability pq.

$\therefore P_T(t) = (1-pq)^{t-1}pq.$

(d) $P_{N|T}(n|t) = \dfrac{P_{N,T}(n,t)}{P_T(t)} = \begin{cases} \dfrac{\binom{t-1}{n-1}p^n(1-p)^{t-n}(1-q)^{n-1}q}{(1-pq)^{t-1}pq}, \text{ if } n \le t \\ 0, \text{ otherwise} \end{cases}.$

$P_{T|N}(t|n) = \dfrac{P_{N,T}(n,t)}{P_N(n)} = \begin{cases} \dfrac{\binom{t-1}{n-1}p^n(1-p)^{t-n}(1-q)^{n-1}q}{(1-q)^{n-1}q} = \dbinom{t-1}{n-1}p^n(1-p)^{t-n}, \text{ if } n \le t \\ 0, \text{ otherwise} \end{cases}.$

Problem 9.

(a) $P_N(n) = 0$ for $n < 0$.

For $n \ge 0$, $P_N(n) = \displaystyle\sum_{k=0}^{n} \frac{100^n e^{-100}}{(n+1)!} = (n+1) \times \frac{100^n e^{-100}}{(n+1)!} = \frac{100^n e^{-100}}{n!}.$

$\therefore P_N(n) = \begin{cases} \dfrac{100^n e^{-100}}{n!}, \text{ if } n \in \mathbb{Z}^+ \cup \{0\} \\ 0, \text{ otherwise} \end{cases}.$

$P_{K|N}(k|n) = \dfrac{P_{N,K}(n,k)}{P_N(n)} = \begin{cases} \dfrac{\frac{100^n e^{-100}}{n!}}{\frac{100^n e^{-100}}{(n+1)!}} = \dfrac{1}{n+1}, \text{ if } k = 0, 1, \dots, n; n = 0, 1, \dots \\ 0, \text{ otherwise} \end{cases}.$

(b) $\mathrm{E}[K|N=n] = \displaystyle\sum_{k=0}^{n} kP_{K|N}(k|n) = \sum_{k=0}^{n} k \times \frac{1}{n+1} = \frac{n}{2}.$

(c) $\mathrm{E}[K|N] = \dfrac{N}{2}.$

$\mathrm{E}[K] = \mathrm{E}[\mathrm{E}[K|N]] = \mathrm{E}[\dfrac{N}{2}] = \displaystyle\sum_{n=0}^{\infty} \frac{n}{2}\frac{100^n e^{-100}}{n!} = \sum_{n=1}^{\infty} \frac{100^n e^{-100}}{(n-1)! \times 2} = 50\sum_{n=1}^{\infty} \frac{100^{n-1} e^{-100}}{(n-1)!} =$
$50\displaystyle\sum_{n=0}^{\infty} \frac{100^n e^{-100}}{n!} = 50\sum_{n=0}^{\infty} P_N(n) = 50.$

Problem 10.