Graph Theory HW3

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Problem 1. Consider a graph G where $V(G) = \{v_1, v_2, \dots, v_{2k}\}$, $E(G) = \{v_iv_j | 1 \le i < j \le k\} \cup \{v_iv_j | k+1 \le i < j \le 2k\} \cup \{v_iv_{i+k} | 1 \le i \le k\}$.
Let $A := \{v_1, v_2, \dots, v_k\}, B := \{v_{k+1}, v_{k+2}, \dots, v_{2k}\}$, and we can see that both the induced subgraph of A and the induced subgraph of B are complete graphs. – (1)
For any $S \subseteq V(G)$ with $|S| < k$, both the induced subgraph of $A \setminus S$ and the induced subgraph of B are connected by (1).
Also, since $C := \{v_iv_{i+k} | 1 \le i \le k\} \in E(G)$ are k edges that connect A, B, and no two of them shares the same vertex, after removing vertices in S, at least one of the edges in C is not removed in the induced subgraph of $V(G) \setminus S$.
$\therefore A, B$ are connected with each other.
$\therefore G - S$ is still a connected graph, by the definition, G is k-connected.
A, B are disjoint sets with size k.
Let $a_i := v_i, b_i := v_{2k+1-i}$.
All of the k vertex-disjoint A, B-paths have length 1, since $V(G) = A \cup B$ and each path has exactly one vertex in A and in B.
$\Rightarrow$ we cannot demand that there are vertex-disjoint paths from a_i to b_i for each $i \in [k]$ since the distance from $a_1 = v_1$ to $b_1 = v_{2k}$ is 2 not 1.

Problem 3. Recall from Corollary 4.7 in class:
A connected multigraph G has an Eulerian trail if and only if it has either 0 or 2 vertices of odd degree.
Since the number of odd-degree vertices is even, the above can also be written as:
A connected multigraph G has an Eulerian trail if and only if it has at most 2 vertices of odd degree. – (1)

(i) I think that this problem must request $e(G) \geq 1$, otherwise $\sigma(G) \leq -1$ is impossible. Therefore, I just assert that $e(G) \geq 1$.

Suppose that uv is an edge of G.

Consider a multigraph H.

$V(H) := V(G)$, $E(H) := E(G) \cup E(G) \setminus \{uv\}$, that is, double all edges in $E(G)$ except uv.

$\forall w \notin \{u, v\}, \deg_H(w) = 2\deg_G(w)$ is even.

H has at most 2 vertices of odd degree.

Also, H is connected since G is connected.

$\therefore$ by (1), H has an Eulerian trail p, and since $E(G) \subseteq E(H)$, p is a walk that covers every edge of G.

$\Rightarrow \sigma(G) \leq$ the length of $p = |E(H)| = 2e(G) - 1$.

(ii) If $\sigma(G) = e(G)$, the shortest walk that covers every edge of G passes through each edge for exactly one time, which is an Eulerian trail of G.

$\Rightarrow$ by (1), G has at most two vertices of odd degrees.

If G has at most two vertices of odd degrees, then by (1), G has an Eulerian trail.

Since by the definition of $\sigma(G)$, the walk passes through all edges.

$\therefore \sigma(G) \geq e(G)$.

Since G has an Eulerian trail, and it is a walk that passes through all edges.

$\therefore \sigma(G) \leq e(G)$.

$\therefore \sigma(G) = e(G)$.

(iii) No, such constant ℓ does not exist.

Consider G where $V(G) = \{v_1, v_2, \ldots, v_{4n+1}\}$ and $E(G) = \bigcup_{i=0}^{3}(\{v_{in+j}v_{in+j+1} | 1 \leq j \leq n-1\} \cup \{v_{in+1}v_{4n+1}\})$.

We can see that G has exactly 4 vertices of odd degree: $S := \{v_n, v_{2n}, v_{3n}, v_{4n}\}$.

Suppose that the shortest walk that covers every edge of G is p.

The multigraph H (where $V(G) = V(H)$ and the number of edge e in H is the number of its appearances in p) contains an Eulerian trail.

$\Rightarrow H - E(G)$ is a graph that only the 2 or 4 of the vertices in S have odd degree.

$\Rightarrow H - E(G)$ should contain a path from one of the vertex in S to another one in S.

$\Rightarrow |E(H - E(G))| \geq 2n$.

$\therefore \sigma(G) = |E(H)| \geq e(G) + 2n$.

$\Rightarrow \sigma(G) \not\leq e(G) + \ell$ since $2n$ is not a constant.

Problem 6. Consider a spanning tree T of G.

If $d_T(u, v) \leq 3$, then $d_G(u, v) \leq d_T(u, v) \leq 3$, and therefore $uv \in E(G^3)$.

Suppose that $|V(G)| = n$.

Let's prove that there exists a sequence of vertices $v_0(= v_n), v_1, v_2, \ldots, v_n$ where $V(G) = \{v_1, \ldots, v_n\}$ and $d_T(v_i, v_{i+1}) \leq 3,\ \forall 0 \leq i \leq n - 1$.

If such sequence exists, since "$d_T(u, v) \leq 3 \Rightarrow d_G(u, v) \leq d_T(u, v) \leq 3 \Rightarrow uv \in E(G^3)$", G^3 is Hamiltonian.

The exists of such sequence:

Choose an arbitrary vertex in T to be a root, and colors all vertices with even depth in black, all vertices with odd depth in white.

Since the depth of a neighbor u of a vertex v is exactly greater or less than that of v by 1, all adjacent neighbors have different colors.

Let's use the following algorithm to find the sequence.

Use DFS (depth-first search) to visit all the vertex.

We can treat DFS as a walk on the tree, and let p be the walk.

Claim: mark the first appearance of each black vertex in p, and the last appearance of each white vertex in p, and $v_i :=$ the $(i + 1)$-th (1-base) marked vertex in p for $0 \leq i \leq n - 1$, $v_n := v_0$. This sequence satisfies $d_T(v_i, v_{i+1}) \leq 3,\ \forall 0 \leq i \leq n - 1$. Furthermore, the depth of v_{n-1} is 1.

Proof: we can do induction on the number of vertices of the tree (I'll prove this claim holds for arbitrary tree).

For the tree with one vertex, it's trivial.

For the trees with more than one vertex, let's r be the root of the tree, and $s_1, s_2, \ldots, s_m$ be the children of r.

By the induction hypothesis, each of the m subtree of $s_1, s_2, \ldots, s_m$ has such sequence, and suppose that the sequence of the subtree of s_i is $v_{i0}, v_{i1}, \ldots, v_{in_i}$.
Since reversing the colors of all vertices (can) reverse the order of the sequence (by reversing the order of the children of all vertices being visited in DFS), the sequence is $r, v_{1(n_1-1)}, \ldots, v_{11}, v_{10}, v_{2(n_2-1)}, \ldots, \ldots, v_{m(n_m-1)}, \ldots, v_{m1}, v_{m0}, r$.
By the induction hypothesis, the depth of $v_{i(n_i-1)}$ is 2 and the depth of v_{i0} is 1, $d_T(r, v_{1(n-1)}) = 2, d_T(v_{m0}, r) = 1, d_T(v_{i0}, v_{i+1(n_{i+1}-1)}) = 3$ for all $1 \leq i \leq m-1$.
$\therefore$ the sequence satisfies condition in the claim.
$\therefore$ by induction, such sequence exists for all trees T.
$\therefore G^3$ is Hamiltonian.