Graph Theory HW3

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Problem 1. Let's have an induction on r to prove the following claim:
Claim: For every k, there exists a least integer $n = n(k, r)$ such that whenever $[n]$ is r-coloured, there is a monochromatic k-AP $a_0, a_1, \dots, a_{k-1}$ whose common difference $d = a_1 - a_0$ is also the same colour.
For $r = 1$, $1, 2, \dots, k$ along with 1 are monochromatic, the claim holds.
Suppose for all $r < r'$, the claim holds.
For $r = r'$:
By Van der Waerden's Theorem: for all positive integers x and y, there exists a least integer $W = W(x, y)$ such that any y-coloring of $[W]$ contains a x-term monochromatic arithmetic progression.
Let $m = n(k, r-1)$, by Van der Waerden's Theorem, there exists $a > 0, d > 0$ such that $S = \{a + id | i \in \{0, 1, \dots, m(k-1)\}\}$ is monochromatic.
By the induction hypothesis, every $(r-1)$-coloring of $[m]$ contains a monochromatic k-AP along with its common difference, and so is every $(r-1)$-coloring of $\{di | i \in [m]\}$.
$\therefore$ either $\{di | i \in [m]\}$ contains r different colors, or a monochromatic k-AP along with its common difference. $n(k, r)$ exists for the latter case.
If $\{di | i \in [m]\}$ contains r different colors, let dj have the same color as S.
$\because a + dj(k-1) \leq a + dm(k-1)$.
$\therefore \{a + idj | i \in \{0, 1, \dots, k-1\}\} \subseteq S$.
$\Rightarrow \{a + idj | i \in \{0, 1, \dots, k-1\}\}$ along with dj have the same color.
$\Rightarrow n(k, r)$ exists.
$\therefore$ by induction, the claim holds for all k, r, and this finishes the proof of this problem.

Problem 2.

(a) For every coloring $c : 2^{[n]} \to [r]$, consider the following graph G and the corresponding edge-coloring d:
$G = \left([n], \binom{[n]}{2}\right)$, for all $uv \in E$, WLOG suppose that $u < v$, $d(uv) := c([v-1] \setminus [u-1])$.

By what we learned in class, $R(\underbrace{3, 3, \ldots, 3}_{r})$ is finite.

$\therefore$ for n large enough, for every edge-coloring of G, there exists a monochromatic triangle uvw.

Let n be large enough, and uvw be a monochromatic triangle, WLOG suppose that $u < v < w$.

$\Rightarrow d(uv) = d(vw) = d(wu)$.

$\Rightarrow c(\{u, u+1, \ldots, v-1\}) = c(\{v, v+1, \ldots, w-1\}) = c(\{u, u+1, \ldots, w-1\})$.

$\therefore X = \{u, u+1, \ldots, v-1\}, Y = \{v, v+1, \ldots, w-1\}$ satisfy that $X, Y, X \cup Y$ receive the same color.

(b) Suppose that the numbers in $\mathbb{N}$ are colored with r colors, and let $c : \mathbb{N} \to [r]$ be a coloring.

Consider the following graph G_n and the corresponding edge-coloring d:
$G_n := \left([n], \binom{[n]}{2}\right)$, for all $uv \in E$, $d(uv) := c(|u-v|)$.
By what we learned in class, $R(\underbrace{3, 3, \ldots, 3}_{r})$ is finite.

$\therefore \exists n$ s.t. for every edge-coloring of G_n, there exists a monochromatic triangle uvw.

WLOG suppose that $u < v < w$.

$\Rightarrow c(v-u) = c(w-v) = c(w-u)$.

Let $x = v-u, y = w-v, z = w-u$, and they are monochromatic satisfying $x + y = z$.