Graph Theory HW4

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April 30, 2024

Problem 1.

(a) Let $X := \{a \in A : |N(a) \cap Y| < (d - \epsilon)|Y|\}$.
$e(X, Y) = \sum_{a \in X} |N(a) \cap Y| < \sum_{a \in X} (d - \epsilon)|Y| = (d - \epsilon)|X||Y|$.
$\Rightarrow d(X, Y) = \dfrac{e(X, Y)}{|X||Y|} < d - \epsilon = d(A, B) - \epsilon$.
If $X \geq \epsilon|A|$, then by the definition of ϵ-regular, $|d(X, Y) - d(A, B)| \leq \epsilon$, which contradicts to $d(X, Y) < d(A, B) - \epsilon$.
$\therefore |\{a \in A : |N(a) \cap Y| < (d - \epsilon)|Y|\}| = |X| < \epsilon|A|$.

(b) For any $C \subseteq X, D \subseteq Y$ with $|C| \geq \epsilon'|X|$, $|D| \geq \epsilon'|Y|$, there is $|C| \geq \epsilon'|X| \geq \dfrac{\epsilon}{\alpha}|X| \geq \epsilon|A|$, $|D| \geq \epsilon'|Y| \geq \dfrac{\epsilon}{\alpha}|Y| \geq \epsilon|B|$.
Since $\{A, B\}$ is an ϵ-regular pair, there is $|d(C, D) - d(A, B)| \leq \epsilon$.
Note that $|X| \geq \alpha|A| > \epsilon|A|, |Y| \geq \alpha|B| > \epsilon|B|$, there is $|d(X, Y) - d(A, B)| \leq \epsilon$.
$\Rightarrow |d(C, D) - d(X, Y)| \overset{\because \text{triangular inequality}}{\leq} |d(C, D) - d(A, B)| + |d(X, Y) - d(A, B)| \leq \epsilon + \epsilon \leq 2\epsilon \leq \epsilon'$.
$\therefore$ by the definition, $\{X, Y\}$ is an ϵ'-regular pair.

Problem 6.

(a) Let $f_i(x)$ denote the i-th lowest bit of x in its ternary expansion.
That is, if $x = \sum_{i=0}^{l-1} x_i 3^i$ where $x_i \in \{0, 1, 2\}$, then $f_i(x) := x_i$.
First, let's prove by contradiction to show that A is 3-AP-free.
Let $a - d, a, a + d$ be a 3-AP of A with $d \neq 0$.
Let i be the minimum j such that at least two of $f_j(a - d), f_j(a), f_j(a + d)$ are

different.

Such i exists because $d \neq 0$.

Let $b = \sum_{j=0}^{i-1} f_j(a)$.

$\Rightarrow a - d - b \equiv a - b \equiv a + d - b \equiv 0 \pmod{3^i}$.

$\Rightarrow (a - d - b) + (a - b) + (a + d - b) = 3(a - b) \equiv 0 \pmod{3^{i+1}}$.

$a-d-b \equiv 3^i f_i(a-d) \pmod{3^{i+1}}, a-b \equiv 3^i f_i(a) \pmod{3^{i+1}}, a+d-b \equiv 3^i f_i(a+d) \pmod{3^{i+1}}$.

$\Rightarrow 3^i(f_i(a - d) + f_i(a) + f_i(a + d)) \equiv 0 \pmod{3^{i+1}}$.

$\Rightarrow f_i(a - d) + f_i(a) + f_i(a + d) \equiv 0 \pmod 3$.

Since at least two of $f_i(a - d), f_i(a), f_i(a + d)$ are different by the definition of i, one of which is 0, and one of which is 1.

The remaining $\equiv 0 - 0 - 1 \equiv 2 \pmod 3$.

$\therefore$ the remaining is 2, which contradicts to that A does not contain any number with digit 2 in its ternary expansion.

$\therefore A$ is 3-AP-free.

Since every digit of A can be either 0 or 1, and there are l digits.

$\therefore |A| = 2^l = 2^{\log_3 n} = n^{\log_3 2}$.

(b) It is sufficient to show that:

(1) For every $x \notin A$, there are $y, z \in A \cap [0, x - 1]$ s.t. $\{x, y, z\}$ is a 3-AP.

(2) For every $x \in A$, there are no $y, z \in A \cap [0, x - 1]$ s.t. $\{x, y, z\}$ is a 3-AP.

(2) is because of that A is 3-AP-free, which is proved in (a).

Suppose $x \notin A$, and $S := \{i : f_i(x) = 2\}$. (The definition of f_i is in (a).)

By the definition of A, $S \neq \emptyset$.

Consider $d = \sum_{i \in S} 3^i$.

$x - d$ is the number changing all digit 2 of x into 1, and $x - 2d$ is the number changing all digit 2 of x into 0.

Also, since $d > 0$, there is $x - d, x - 2d \in [0, x - 1]$.

Hence $x - d, x - 2d \in A \cap [0, x - 1]$.

Take $y = x - d, z = x - 2d$, and we can see that $\{x, y, z\}$ is a 3-AP.

$\Rightarrow$ (1) is proved.

$\therefore A$ is the 3-AP-free set we get from the greedy algorithm.