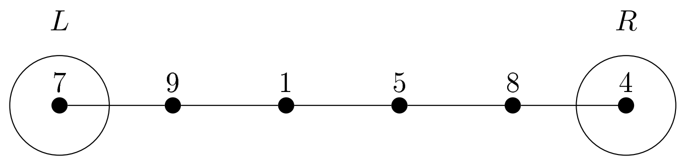
Contents

Reversing the correspondence is easy: given a tree T with two special vertices L and R, look at the unique path P of T connecting L and R. The vertices of P form the set M. Ordering the vertices of M gives us the first line of $f|_M$, the second line is given by the order of the vertices in P, from L to R.

Example (continued)**.**

$P =$



$$\begin{aligned} M &= \{1,4,5,7,8,9\}, \\ f|_M &= \begin{pmatrix} 1 & 4 & 5 & 7 & 8 & 9 \\ 7 & 9 & 1 & 5 & 8 & 4 \end{pmatrix}. \end{aligned}$$

The remaining values of f are then filled in accordance with the unique paths from the remaining vertices to P (directing these paths towards P). $\square$

3 Connectivity

While trees are connected graphs, their connectivity is very fragile — every edge is a cut-edge, so if any edge is removed, the graph is no longer connected. For many applications it is important to have graphs with more robust connectivity, and in this chapter we explore how this can be quantified.

3.1 Vertex connectivity

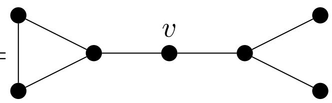
In this section we explore the question of how many vertices need to be removed to disconnect a graph.

Definition 3.1. A *vertex cut* in a connected graph $G = (V, E)$ is a set $S \subseteq V$ such that $G \backslash S := G[V \backslash S]$ has more than one connected component. A *cut vertex* is a vertex v such that $\{v\}$ is a cut.

Definition 3.2. G is called *k-connected* if $|V(G)| > k$ and if $G \backslash X$ is connected for every set $X \subseteq V$ with $|X| < k$. In other words, no two vertices of G are separated by fewer than k other vertices. Every (non-empty) graph is 0-connected and the 1-connected graphs are precisely the non-trivial connected graphs. The greatest integer k such that G is k-connected is the *connectivity* $\kappa(G)$ of G.

Example 3.3.

- $G = K_n$: $\kappa(G) = n - 1$

- $G = K_{m,n}$, $m \leq n$: $\kappa(G) = m$. Indeed, let G have bipartition $A \cup B$, with $|A| = m$ and $|B| = n$. Deleting A disconnects the graph. On the other hand, deleting $S \subset V$ with $|S| < m$ leaves both $A \backslash S$ and $B \backslash S$ non-empty and any $a \in A \setminus S$ is connected to any $b \in B \setminus S$. Hence $G \backslash S$ is connected.
- $G =$ 

: $\kappa(G) = 1$. Deleting v disconnects G, so v is a cut vertex.

Our first bound shows that in order to have high connectivity, you need all vertices to have large degree.

Proposition 3.4. *For every graph G, $\kappa(G) \leq \delta(G)$.*

Proof. If G is a complete graph then trivially $\kappa(G) = \delta(G) = |G| - 1$. Otherwise let $v \in G$ be a vertex of minimum degree $d(v) = \delta(G)$. Deleting $N(v)$ disconnects v from the rest of G. □

Remark 3.5. High minimum degree does not imply connectivity. Consider two disjoint copies of K_n.

However, the converse is *locally* true.

Theorem 3.6 (Mader 1972)**.** *Every graph of average degree at least $4k$ has a k-connected subgraph.*

Proof. For $k \in \{0, 1\}$ the assertion is trivial; we consider $k \geq 2$ and a graph $G = (V, E)$ with $|V| = n$ and $|E| = m$. For inductive reasons it will be easier to prove the stronger assertion that G has a k-connected subgraph whenever

(i) $n \geq 2k - 1$ and

(ii) $m \geq (2k - 3)(n - k + 1) + 1$.

(This assertion is indeed stronger, i.e. (i) and (ii) follow from our assumption of $\bar{d}(G) \geq 4k$: (i) holds since $n > \Delta(G) \geq 4k$, while (ii) follows from $m = \frac{1}{2}\bar{d}(G)n \geq 2kn$.)

We apply induction on n. If $n = 2k - 1$, then $k = \frac{1}{2}(n + 1)$, and hence

$$m \geq (n-2)\frac{n+1}{2} + 1 = \frac{1}{2}n(n-1)$$

by (ii). Thus $G = K_n \supseteq K_{k+1}$, proving our claim. We therefore assume that $n \geq 2k$. If v is a vertex with $d(v) \leq 2k - 3$, we can apply the induction hypothesis to $G \backslash v$ and are done. So we assume that $\delta(G) \geq 2k - 2$. If G is itself not k-connected, then there is a separating set $X \subseteq V$ with fewer than k vertices, such that $G \backslash X$ is disconnected. Let V_1 be one component of $G \backslash X$ and let V_2 be the union of the other components. Let $G_i = G[V_i \cup X]$, so that $G = G_1 \cup G_2$, and every edge of G is either in G_1 or G_2 (or both). Each vertex in each V_i has at least $\delta(G) \geq 2k - 2$ neighbours in G and thus also in G_i, so $|G_1|, |G_2| \geq 2k - 1$. Note that each $|G_i| < n$, so by the induction hypothesis, if no G_i has a k-connected subgraph then each

$$e(G_i) \leq (2k - 3)(|G_i| - k + 1).$$

Hence,

$$\begin{aligned} m &\leq e(G_1) + e(G_2) \\ &\leq (2k-3)(|G_1| + |G_2| - 2k + 2) \\ &\leq (2k-3)(n-k+1) \quad (\text{since } |G_1 \cap G_2| \leq k-1), \end{aligned}$$

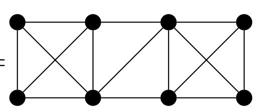
contradicting (ii). □

3.2 Edge connectivity

The previous section dealt with removing vertices from a graph. Here we consider disconnecting a graph by removing edges instead.

Definition 3.7. A *disconnecting set* of edges is a set $F \subseteq E(G)$ such that $G \setminus F$ has more than one component. Given $S, T \subset V(G)$, the notation $[S, T]$ specifies the set of edges having one endpoint in S and the other in T. An *edge cut* is an edge set of the form $[S, \overline{S}]$, where S is a non-empty proper subset of $V(G)$. A graph is *k-edge-connected* if every disconnecting set has at least k edges. The *edge-connectivity* of G, written $\kappa'(G)$, is the minimum size of a disconnecting set. A single edge that disconnects G is called a *bridge* (or a *cut-edge*).

Example 3.8.

- $G = K_n$: $\kappa'(G) = n - 1$.
- $G =$  : $\kappa'(G) = 3$, whereas $\kappa(G) = 2$.

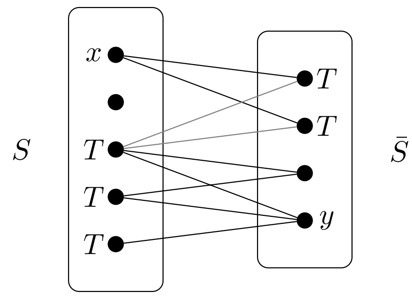
Remark 3.9. An edge cut is a disconnecting set but the reverse is not necessarily true. However, every minimal disconnecting set is a cut.

Theorem 3.10. $\kappa(G) \leq \kappa'(G) \leq \delta(G)$.

Proof. The edges incident to a vertex v of minimum degree form a disconnecting set; hence $\kappa'(G) \leq \delta(G)$. It remains to show $\kappa(G) \leq \kappa'(G)$. Suppose $|G| > 1$ and $[S, \overline{S}]$ is a minimum edge cut, having size $\kappa'(G)$.

If every vertex of S is adjacent to every vertex of $\overline{S}$, then $\kappa'(G) = |S||\overline{S}| = |S|(|G| - |S|)$. This expression is minimized at $|S| = 1$. By definition, $\kappa(G) \leq |G| - 1$, so the inequality holds.

Hence we may assume there exists $x \in S$, $y \in \overline{S}$ with x not adjacent to y. Let T be the vertex set consisting of all neighbours of x in $\overline{S}$ and all vertices of $S \backslash x$ that have neighbours in $\overline{S}$ (illustrated below). Deleting T destroys all the edges in the cut $[S, \overline{S}]$ (but does not delete x or y), so T is a separating set. Now, by the definition of T we can injectively associate at least one edge of $\left[S, \overline{S}\right]$ to each vertex in T, so $\kappa(G) \leq |T| \leq \left|[S, \overline{S}]\right| = \kappa'(G)$.



□

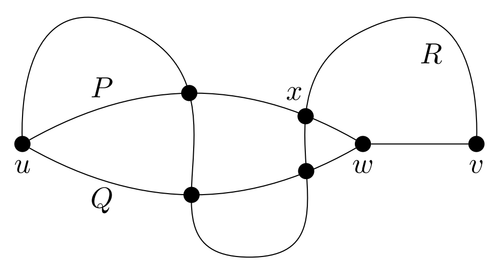
3.3 2-connected graphs

Definition 3.11. Two paths are *internally disjoint* if neither contains a non-endpoint vertex of the other. We denote the length of the shortest path from u to v (the *distance* from u to v) by $d(u, v)$.

Theorem 3.12 (Whitney 1932)**.** *A graph G having at least three vertices is* 2*-connected if and only if each pair $u, v \in V(G)$ is connected by a pair of internally disjoint u, v-paths in G.*

Proof. When G has internally disjoint u, v-paths, deletion of one vertex cannot separate u from v. Since this is given for every u, v, the condition is sufficient. For the converse, suppose that G is 2-connected. We prove by induction on $d(u, v)$ that G has two internally disjoint u, v paths. When $d(u, v) = 1$, the graph $G \backslash \{u, v\}$ is connected, since $\kappa'(G) \geq \kappa(G) = 2$. A u, v-path in $G \backslash \{u, v\}$ is internally disjoint in G from the u, v-path consisting of the edge $\{u, v\}$ itself.

For the induction step, we consider $d(u, v) = k > 1$ and assume that G has internally disjoint x, y-paths whenever $1 \leq d(x, y) < k$. Let w be the vertex before v on a shortest u, v-path. We have $d(u, w) = k - 1$, and hence by the induction hypothesis G has internally disjoint u, w-paths P and Q. Since $G \backslash w$ is connected, $G \backslash w$ contains a u, v-path R. If this path avoids P or Q, we are finished, but R may share internal vertices with both P and Q. Let x be the last vertex of R belonging to $P \cup Q$. Without loss of generality, we may assume $x \in P$. We combine the u, x-subpath of P with the x, v-subpath of R to obtain a u, v-path internally disjoint from $Q \cup \{(w, v)\}$.



□

Corollary 3.13. *G is* 2*-connected and $|G| \geq 3$ if and only if every two vertices in G lie on a common cycle.*

3.4 Menger's Theorem

Definition 3.14. Let $A, B \subseteq V$. An A-B path is a path with one endpoint in A, the other endpoint in B, and all interior vertices outside of $A \cup B$. Any vertex in $A \cap B$ is a trivial A-B path.

If $X \subseteq V$ (or $X \subseteq E$) is such that every A-B path in G contains a vertex (or an edge) from X, we say that X *separates* the sets A and B in G. This implies in particular that $A \cap B \subseteq X$.

Theorem 3.15 (Menger 1927)**.** *Let $G = (V, E)$ be a graph and let $S, T \subseteq V$. Then the maximum number of vertex-disjoint S-T paths is equal to the minimum size of an S-T separating vertex set.*

Proof. Obviously, the maximum number of disjoint paths does not exceed the minimum size of a separating set, because for any collection of disjoint paths, any separating set must contain a distinct vertex from each path. So we just need to prove there is an S-T separating set and a collection of disjoint S-T paths with the same size.

We use induction on $|E|$, the case $E = \varnothing$ being trivial. We first consider the case where S and T are disjoint.

Let k be the minimum size of an S-T separating vertex set. Choose $e = \{u, v\} \in E$. Let $G' = (V, E \setminus e)$. If each S-T separating vertex set in G' has size at least k, then inductively there exist k vertex-disjoint S-T paths in G', hence in G.

So we can assume that G' has an S-T separating vertex set C of size at most $k - 1$. Then $C \cup \{u\}$ and $C \cup \{v\}$ are S-T separating vertex sets of G of size k.

Since C is a separating set for G', no component of $G' \backslash C$ has elements from both S and T. Let V_S be the union of components with elements from S, and let V_T be the union of components with elements in T. If we were to add the edge (u, v) to $G' \backslash C$ then there would be a path from S to T (because C does not separate S and T in G). So, without loss of generality $u \in V_S$ and $v \in V_T$.

Now, each S-$(C \cup \{u\})$ separating vertex set B of G' has size at least k, as it is S-T separating in G. Indeed, each S-T path P in G intersects $C \cup \{u\}$. Let P' be the subpath of P that goes from S to the first time it touches $C \cup \{u\}$. If P' ends with a vertex in C, then $u \notin P'$ so P' is an S-$(C \cup \{u\})$ path in G'. If P' ends in u, then it is disjoint from C and so by the above it contains only vertices in V_S. So $v \notin P'$ and again P' is an S-$(C \cup \{u\})$ path in G'. In both cases we showed that P' is an S-$(C \cup \{u\})$ path in G' so P intersects B.

So by induction, G' contains k disjoint S-$(C \cup \{u\})$ paths. Similarly, G' contains k disjoint $(C \cup \{v\})$-T paths. Any path in the first collection intersects any path in the second collection only in C, since otherwise G' contains an S-T path avoiding C.

Hence, as $|C| = k - 1$, we can pairwise concatenate these paths to obtain $k - 1$ disjoint S-T paths. We can finally obtain a kth path by inserting the e between the path ending at u and the path starting at v.

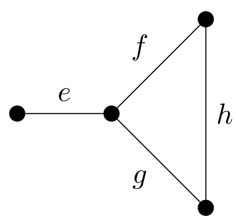
It remains to consider the general situation where S and T might not be disjoint. Let $X = S \cap T$ and apply the theorem with the disjoint sets $S' = S \backslash X$ and $T' = T \backslash X$, in the graph $G' = G \backslash X$. Let k' be the size of a minimum separating set in G'. We can obtain a $(k' + |X|)$-vertex S-T separating set in G by adding every vertex in X to an S'-T' separating set in G'. Similarly we can obtain a collection of $k' + |X|$ vertex-disjoint S-T paths by adding each vertex in X as a trivial path to a collection of vertex-disjoint S'-T' paths in G'. □

Corollary 3.16. *For $S \subseteq V$ and $v \in V \setminus S$, the minimum number of vertices distinct from v separating v from S in G is equal to the maximum number of paths forming an v-S fan in G. (that is, the maximum number of $\{v\}$-S paths which are disjoint except at v).*

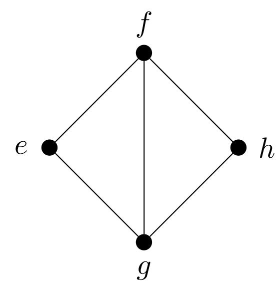
Proof. Apply Menger's Theorem with $T = N(v)$. Note that none of the resulting paths go through v; if one did, then it would contain two vertices of T, violating the definition of an S-T path. So we have a suitable number of vertex-disjoint S-T paths not including v, and we can append v to each path to give a v-S fan. □

Definition 3.17. The *line graph* of G, written $L(G)$, is the graph whose vertices are the edges of G, with $\{e, f\} \in E(L(G))$ when $e = \{u, v\}$ and $f = \{v, w\}$ in G (i.e. when e and f share a vertex).

Example 3.18.



G



$L(G)$

Note that a path in $L(G)$ corresponds to a sequence of distinct edges $e_0, \ldots, e_\ell$ in G such that every pair of consecutive edges is incident. This may not be a path in G, because the endpoints might not "line up": it might be that for some i the common endpoint of e_{i-1} and e_i is the same as the common endpoint of e_i and e_{i+1}. But we can always delete a few edges to obtain a path in G: if there exists i as described, we can simply delete the edge e_i, and repeatedly make such deletions until no conflicts remain.

Corollary 3.19. *Let u and v be two distinct vertices of G.*

1. *If $\{u, v\} \notin E$, then the minimum number of vertices different from u, v separating u from v in G is equal to the maximum number of internally vertex-disjoint u-v paths in G.*
2. *The minimum number of edges separating u from v in G is equal to the maximum number of edge-disjoint u-v paths in G.*

Proof. For (i), Apply Menger's Theorem with $S = N(u)$ and $T = N(v)$.

For (ii), Apply Menger's Theorem to the line graph of G, with S as the set of edges adjacent to u and T as the set of edges adjacent to v. □

Theorem 3.20 (Global Version of Menger's Theorem).

1. *A graph is k-connected if and only if it contains k internally vertex-disjoint paths between any two vertices.*
2. *A graph is k-edge-connected if and only if it contains k edge-disjoint paths between any two vertices.*

Proof. First we prove (i). if a graph G contains k internally disjoint paths between any two vertices, then $|G| > k$ and G cannot be separated by fewer than k vertices; thus, G is k-connected.

Conversely, suppose that G is k-connected (and, in particular, has more than k vertices) but contains vertices u, v not linked by k internally disjoint paths. By Corollary 3.19, u and v are adjacent; let $G' = G \backslash \{u, v\}$. Then G' contains at most $k-2$ internally disjoint u, v-paths. By Corollary 3.19, we can separate u and v in G' by a set X of at most $k-2$ vertices. As $|G| > k$, there is at least one further vertex $w \notin X \cup \{u, v\}$ in G. Now X separates w in G' from either u or v (say, from u). But then $X \cup \{v\}$ is a set of at most $k-1$ vertices separating w from u in G, contradicting the k-connectedness of G.

Then, (ii) follows straight from Corollary 3.19. □