賽局論 HW6

許博翔

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Problem 1.

(a) Same as the reason in the powerpoint, when Alice holds A, she will always raise; when Bob holds A, he will always call; when Bob holds J, he will always fold.
Suppose that the probability that Alice raises when holding K, Q are p, q, respectively, and the probability that Bob calls when holding K, Q are r, s, respectively.
When holding Q, the expected value that Bob calls is -2, and that he folds is -1, since when he needs to decide whether calling or folding, Alice won't hold J.
$\therefore$ Bob will always fold when holding Q, $s = 0$.
When holding K, the expected value that Alice raises is $\frac{1}{3}(-2)+\frac{1}{3}(1)+\frac{1}{3}(1) = 0$, and that she checks is $\frac{1}{3}(-1) + \frac{1}{3}(1) + \frac{1}{3}(1) = \frac{1}{3}$.
$\therefore$ Alice will always checks when holding K, $p = 0$.
When holding Q, the expected value that Alice raises is $\frac{1}{3}(-2) + \frac{1}{3}(r(-2) + (1-r)(1)) + \frac{1}{3}(1) = -\frac{1}{3} + \frac{1}{3}(1 - 3r) = -r$, and that she checks is $\frac{1}{3}(-1) + \frac{1}{3}(-1) + \frac{1}{3}(1) = -\frac{1}{3}$.
Alice will indifferent between these two actions iff $-r = -\frac{1}{3} \iff r = \frac{1}{3}$.
When holding K, the expected value that Bob calls is $\frac{1}{1+q}(-2) + \frac{q}{1+q}(2)$, and that he folds is -1.
Bob will indifferent between these two actions iff $\frac{1}{1+q}(-2)+\frac{q}{1+q}(2) = -1 \iff \frac{q-1}{q+1} = -\frac{1}{2} \iff q = \frac{1}{3}$.
$\therefore$ there is a Nash equilibrium when Alice uses the mixed strategy $(1, 0, \frac{1}{3}, 0)$ and Bob uses the mixed strategy $(1, \frac{1}{3}, 0, 0)$.

(b) Let a_1, a_2, a_3, a_4 denote the probability that Alice raises when holding A, K, Q, J, respectively, and b_1, b_2, b_3, b_4 denote the probability that Bob calls when holding A, K, Q, J, respectively.

If there is a Nash equilibrium with $a_4 = 0$, then there are two cases.

Case 1: $a_1 < 1$.
When holding A, the expected value that Alice raises is $\frac{1}{3}(b_2(2)+(1-b_2)(1)) + \frac{1}{3}(b_3(2)+(1-b_3)(1)) + \frac{1}{3}(b_4(2)+(1-b_4)(1)) = \frac{1}{3}(b_2+b_3+b_4)+1 \geq 1 =$ the expected value that Alice checks, the equation holds $\iff b_2 = b_3 = b_4 = 0$.
In this case, since $\frac{1}{3}(b_2+b_3+b_4)+1 \leq 1$ must holds, there must be $b_2 = b_3 = b_4 = 0$.
When holding J, the expected value that Alice raises is $\frac{1}{3}(b_1(-2)+(1-b_1)(1)) + \frac{1}{3}(b_2(-2)+(1-b_2)(1)) + \frac{1}{3}(b_3(-2)+(1-b_3)(1)) \overset{\because b_2=b_3=0}{=} \frac{1}{3}(1-3b_1)+\frac{2}{3}$, and the expected value that Alice checks is -1.
$\because \frac{1}{3}(1-3b_1)+\frac{2}{3} \geq \frac{1}{3}(1-3\cdot 1)+\frac{2}{3} = 0 > -1$, $a_4 = 1$, which gets a contradiction.

Case 2: $a_1 = 1$.

When holding Q, the expected value that Bob calls is -2, and that he folds is -1, since when he needs to decide whether calling or folding, Alice won't hold J.

$\therefore$ Bob will always fold when holding Q, $b_3 = 0$.
When holding J, the expected value that Alice raises is $\frac{1}{3}(b_1(-2)+(1-b_1)(1)) + \frac{1}{3}(b_2(-2)+(1-b_2)(1)) + \frac{1}{3}(b_3(-2)+(1-b_3)(1)) \overset{\because b_3=0}{=} \frac{1}{3}(2-3b_1-3b_2)+\frac{1}{3}$, and the expected value that Alice checks is -1.
$\because \frac{1}{3}(2-3b_1-3b_2)+\frac{1}{3} = 1-b_1-b_2 \geq -1$, the equation holds $\iff b_1 = b_2 = 1$.
$\because a_4 = 0$, the equation must hold.

$\Rightarrow b_1 = b_2 = 1$.

When holding J, the expected value that Bob calls is -2, and that he folds is -1.

$\therefore$ Bob will always fold when holding J, $b_4 = 0$.
When holding Q, the expected value that Alice raises is $\frac{1}{3}(-2)+\frac{1}{3}(-2)+\frac{1}{3}(1) = -1$, and that she checks is $\frac{1}{3}(-1)+\frac{1}{3}(-1)+\frac{1}{3}(1) = -\frac{1}{3}$.

Since $-1 < -\frac{1}{3}$, Alice will always check when holding Q, $a_3 = 0$.
When holding K, the expected value that Bob calls is -2, that he folds is -1, since when he needs to decide whether calling or folding, Alice won't hold J or Q.

$\Rightarrow$ Bob will fold when holding K, contradicts to that $b_2 = 1$.

$\therefore$ there is no Nash equilibrium with $a_4 = 0$.

Problem 2 (21.11.13)**.** Let $B(v)$ denote the bidding function with respect to the value v.

The expected value a buyer gets is $P(B(v))(v - B(v)) + (1 - P(B(v)))(-B(v)) = P(B(v))v - B(v)$, where $P(p) := \mathbb{P}\{$ this buyer wins $\mid$ the bid $= p\}$.

$\because$ $B(v)$ is increasing and every player use it, $P(B(v)) = \mathbb{P}\{v > w$ for all the other players' values $w\} = v^{n-1}$.

$\Rightarrow P(p) = B^{-1}(p)^{n-1}$.

$\Rightarrow$ the expected value is $vC(B(v))^{n-1} - B(v)$.

Let $\beta = B(v)$, and C be the inverse of B.

Let $\frac{d(vC(B(v))^{n-1} - B(v))}{d\beta} = \frac{d((n-1)vC(\beta)^{n-2} - \beta)}{d\beta} = 0$.

$\Rightarrow (n-1)vC(\beta)^{n-2}C'(\beta) - 1 = 0$.

$\Rightarrow (n-1)v^{n-1}\frac{d(C(\beta))}{d\beta} = 1$.

$\Rightarrow (n-1)v^{n-1}\frac{dv}{dB(v)} = 1$.

$\Rightarrow (n-1)v^{n-1}dv = dB(v)$.

$\Rightarrow B(v) = \int (n-1)v^{n-1}dv = \frac{n-1}{n}v^n + C$.

$\because$ $B(0) = 0$, there is $B(v) = \frac{n-1}{n}v^n$.

Problem 3. Let $\frac{d(90 - q_i - q_j)q_i}{dq_i} = 0$.

$\Rightarrow -q_i + 90 - q_i - q_j = 0$.

$\Rightarrow q_i = \frac{90 - q_j}{2}$.

$\therefore q_1 = \frac{90 - q_2}{2}, q_2 = \frac{90 - q_1}{2}$.

$\Rightarrow q_1 = q_2 = 30$ is the Cournot equilibrium.

Problem 4. Let q_{21}, q_{22} denote the quantity of firm 2 when the unit cost is $10, 20$, respectively.

Similar to the above problem, $q_{21} = \dfrac{90 - q_1}{2}, q_{22} = \dfrac{80 - q_1}{2}$ at Cournot equilibrium.

Firm 1 wants to find $\max\limits_{q_1} \dfrac{1}{2}q_1(90 - q_1 - q_{21}) + \dfrac{1}{2}q_1(90 - q_1 - q_{22}) = \max\limits_{q_1} 90q_1 - q_1^2 - q_1(\dfrac{85 - q_1}{2}) = \max\limits_{q_1} -\dfrac{q_1^2}{2} + \dfrac{95}{2}q_1$.

Let $\dfrac{d(-\frac{q_1^2}{2} + \frac{95}{2}q_1)}{dq_1} = 0$.

$\Rightarrow -q_1 + \dfrac{95}{2} = 0$.

$\Rightarrow q_1 = \dfrac{95}{2}$.