Ch.21 Auctions and Biddings

November 16, 2023

Four Mechanisms

Bayes-Nash Equilibrium

Four Mechanisms

- English auction (ascending price auction), ex. 東港黑鮪魚拍賣
- First-price sealed bidding, ex. 政府工程招標
- Second-price sealed bidding, ex. E-Bay
- Dutch auction (descending price auction), ex. 內湖台北花市
- Which mechanism could maximize the seller's expected revenue?

Buyers' Value

- Common value: The value of the object is the same for every one.
  - Ex. the exploitation right of a new oil-field
  - The most optimistic oil company will bid the highest.
  - Winner's curse.
- Private value: Every buyer could have a different value.
  - Ex. A Picasso's painting
- We'll focus on an object with private value which is much easier to analyze.

Setting

- A, the seller, has a painting which is worth nothing to A.
- The value of the painting to B and C, two buyers, is i.i.d. U[0,1]. A buyer knows his own value, while the other two persons treat his value as a uniformly distributed random variable in [0,1].
- Every one is risk-neutral.

English Auction

- A raises the price continuously until one buyer drops out. The remaining buyer pays the last price and receives the painting. The one who quits pays nothing.
- Let v denote B's value.
- B has to determine the optimal timing to drop out.
- B's dominant strategy is to drop out when the price reaches v.
- B's expected payment = winning probability * E(the price triggering C to leave | B wins) $= v$ * $v/2 = v^2/2$.
- A similar reasoning applies to C.
- A's expected revenue $= 2 * Ev^2/2 = \int_0^1 v^2 dv = 1/3$.

Second-Price Sealed Bidding

- Let b and c be B's bid and C's bid.
- How should B choose b?
  - If $c \leq v$, it deserves to win and any bid no smaller than c, including v, will make B the winner.
  - If $c > v$, it is not worthwhile to win. Any bid smaller than c, including v, will make B to lose.
  - It's B's dominant strategy to set $b = v$.
- Similarly, C will bid at his true value.
- B's expected payment $= v * v/2 = v^2/2$.
- A's expected revenue $= 1/3$.

Dutch Auction

- A lowers the price continuously until one buyer stops A. That buyer pays the last price and receives the painting.
- At what price, b, will B stop A?
- $b = v$?

First-Price Sealed Bidding

- Let b be B's bid.
- If b is higher than C's bid, B pays b and receives the painting.
- $b < v$
- In both Dutch auction and the first-price sealed bidding, B has to decide b to beat C. If B wins, the payoff, $v - b$, is the same in two mechanisms. These two mechanisms are equivalent. So we only need to analyze one of them.

Symmetric BNE

- v is B's type. Let w be C's value, and hence C's type.
- It's a game with incomplete information, since a buyer does not know his opponent's preference.
- For each type, we have to figure out B's bid, i.e. we have to find a bidding function $B(v)$.
- Assume the bidding function B differentiable, $B' > 0$.
- Assume that both buyers have the same bidding function. So C will bid according to $B(w)$.
- What is the bidding function in equilibrium?

- Given C's bidding function $B(w)$, when B has a value v, what is B's best response β?
- B will solve the problem: $\max_\beta (v-\beta) * \Pr(\beta > B(w))$.
- Let C be the inverse bidding function, $C = B^{-1}$.
- $\Pr(\beta > B(w)) = \Pr(C(\beta) > w) = C(\beta)$
- B's problem is:

$$\max_\beta (v-\beta) * C(\beta).$$

- $$\max_{\beta}(v-\beta)*C(\beta).$$
- The FOC is:
$$-C(\beta)+(v-\beta)C'(\beta)=0$$
- In a symmetric equilibrium, $\beta = B(v)$ which implies:
$$-C(B(v))+(v-B(v))C'(B(v))=0$$
- Let $b = B(v)$, the above equation becomes:
$$-v+(v-b)\frac{dv}{db}=0, \text{ or } v=\frac{db}{dv}v+b.$$

From the last page,

$$v = \frac{db}{dv}v + b = \frac{d(bv)}{dv}$$

To integrate,

$$vB(v) = \frac{v^2}{2} + k.$$

When $v = 0$, $k = 0$, so

$$vB(v) = \frac{v^2}{2},\ B(v) = \frac{v}{2}.$$

B's expected payment is $v^2/2$, and A's expected revenue is still 1/3!

Revenue Equivalence

Theorem: So long as we assume v and w to be i.i.d., $v, w \geq 0$, A's expected revenue will be the same in four mechanisms.

Proof: Let $B(.)$ be the symmetric bidding function.
When C uses $B(w)$, if B bids at β, his expected payoff is:

$$vp(\beta) - f(\beta),$$

where $p(.)$ denotes B's winning probability, and $f(.)$ denote B's expected payment.
The F.O.C. is:

$$vp'(\beta) - f'(\beta) = 0.$$

- $$vp'(\beta) - f'(\beta) = 0.$$
- Let $P(v) = p(B(v))$ and $F(v) = f(B(v))$.
- $P'(v) = p'(B(v))B'(v)$, $F'(v) = f'(B(v))B'(v)$.
- To multiply the FOC by $B'(v)$, we have:

$$vP'(v) - F'(v) = 0.$$

- $$\begin{aligned} F(v) &= \int_0^v F'(u)du + F(0) \\ &= \int_0^v uP'(u)du \\ &= vP(v) - \int_0^v P(u)du \end{aligned}$$
- $P(v) = \Pr(v > w)$, because $B' > 0$.
- Hence $F(v) \perp$ mechanism.