Graph Theory HW4

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May 13, 2024

Problem 1.

(a) Suppose that $\mathcal{F}$ is maximal.

For any $S \in 2^{[n]}$:

$\because S \cap S^c = \emptyset$.

$\therefore$ at most one of S, S^c is in $\mathcal{F}$.

Suppose that $S^c \notin \mathcal{F}$.

Since $\mathcal{F}$ is maximal, this means that $\mathcal{F} \cup \{S^c\}$ is not intersecting.

$\Rightarrow \exists T \in \mathcal{F}$ s.t. $T \cap S^c = \emptyset$.

$\Rightarrow T \subseteq (S^c)^c = S$.

For any $U \in \mathcal{F}$, $T \cap U \neq \emptyset$ by the definition of intersecting.

$\Rightarrow T \cap U \subseteq S \cap U \neq \emptyset$.

$\therefore \mathcal{F} \cup \{S\}$ is intersecting.

Since $\mathcal{F}$ is maximal, $S \in \mathcal{F}$.

$\therefore$ exactly one of S, S^c is in $\mathcal{F}$.

Since $2^{[n]}$ can be partitioned into 2^{n-1} pairs of $\{S, S^c\}$.

$\therefore |\mathcal{F}| = 2^{n-1}$.

(b) Let $A := \{S \in 2^{[n]} : |S| = k\}$.

One can see that A can be partitioned into $\frac{1}{2}\binom{2k}{k}$ pairs of $\{S, S^c\}$.

For every pair of $\{S, S^c\}$, put exactly one of S, S^c into B.

Claim: No matter what B is, $F := \{S \in 2^{[n]} : |S| > k\} \cup B$ is intersecting and has size 2^{n-1}.

Proof:

$|F| = \binom{2k}{k+1} + \binom{2k}{k+2} + \cdots + \binom{2k}{2k} + \frac{1}{2}\binom{2k}{k}$
$= \frac{1}{2}\left(\binom{2k}{k+1} + \binom{2k}{k-1}\right) + \frac{1}{2}\left(\binom{2k}{k+2} + \binom{2k}{k-2}\right) + \cdots + \frac{1}{2}\left(\binom{2k}{2k} + \binom{2k}{0}\right) +$
$\frac{1}{2}\binom{2k}{k}$
$= \frac{1}{2}\left(\binom{2k}{0} + \binom{2k}{1} + \cdots + \binom{2k}{2k}\right)$
$= 2^{2k-1} = 2^{n-1}$.

For all $S, T \in F$, if $|S|+|T| > n$, then $|S \cap T| = |S|+|T|-|S \cup T| > n-|S \cup T| \geq n-n = 0$.

$\therefore S \cap T \neq \emptyset$.

If $|S|+|T| = n$, since $|S| \geq k, |T| \geq k$, the only possibility is that $|S| = |T| = k$, and hence $S, T \in B$.

Since $T \neq S^c$ by the definition of B, there is $T \not\subseteq S^c$.

$\Rightarrow S \cap T \neq \emptyset$.

$\therefore F$ is intersecting, and this finishes the proof of the claim.

Since there are $2^{\frac{1}{2}\binom{2k}{k}}$ choices of B, there are $2^{\frac{1}{2}\binom{2k}{k}}$ different F, and these F are distinct intersecting families of size 2^{n-1} in $2^{[n]}$.

Problem 2.

(i) Consider the bipartite graph G_k.

If $F \in V_k$, then $\deg(F)$ = the number of elements that can be added to $F = n-k$.

If $F \in V_{k+1}$, then $\deg(F)$ = the number of elements that can be removed from $F = k+1$.

$\forall S \subseteq V_k$, $|S|(n-k)$ = the number of edges between S and $N(S) \leq |N(S)|(k+1)$.

$\Rightarrow |S| \leq \frac{k+1}{n-k}|N(S)|$.

$\because k < \frac{n}{2} \Rightarrow 2k < n \Rightarrow 2k+1 \leq n \Rightarrow k+1 \leq n-k \Rightarrow \frac{k+1}{n-k} \leq 1$.

$\therefore |S| \leq \frac{k+1}{n-k}|N(S)| \leq |N(S)|$.

By Hall's theorem, there is a matching $M_k : V_k \hookrightarrow V_{k+1}$.

(ii) $k \geq \frac{n}{2}$.

$\Rightarrow n-1-k \leq n-1-\dfrac{n}{2} < \dfrac{n}{2}$.

By (i), there is a matching $M_{n-1-k} : V_{n-1-k} \hookrightarrow V_{n-k}$.

Let $F \in V_{k+1}$.

Construct $M_{k+1}(F) := M_{n-1-k}(V_{k+1}^c)^c$.

This is well-defined because $|V_{k+1}^c| = n-1-k$.

The image is V_k because $|M_{n-1-k}(V_{k+1}^c)^c| = n-(n-k) = k$.

There is an edge connecting F and $M_{k+1}(F)$ because $V_{k+1}^c \subset M_{n-1-k}(V_{k+1}^c) \Rightarrow M_{n-1-k}(V_{k+1}^c)^c \subset V_{k+1}$.

This is a matching because M_{n-1-k} and taking complement are both injective functions.

(iii) For every set F, suppose $|F| = k$, define $f(F) := \begin{cases} F, \text{ if } k = \lceil \frac{n}{2} \rceil \\ f(M_k(F)), \text{ otherwise} \end{cases}$.

Since if $k < \lceil \frac{n}{2} \rceil$, then $k \leq \lceil \frac{n}{2} \rceil - 1 \leq (\frac{n}{2} - \frac{1}{2}) < \frac{n}{2}$, and if $k > \lceil \frac{n}{2} \rceil$, then $k-1 \geq \lceil \frac{n}{2} \rceil \geq \frac{n}{2}$.

$\therefore M_k$ is well-defined for all $k \neq \lceil \frac{n}{2} \rceil$.

Also, $|M_k(F)| = k+1$ for $k < \lceil \frac{n}{2} \rceil$, and $|M_k(F)| = k-1$ for $k > \lceil \frac{n}{2} \rceil$.

$\therefore f$ is a well-defined function from $2^{[n]}$ to $V_{\lceil \frac{n}{2} \rceil}$.

For every $F \in V_{\lceil \frac{n}{2} \rceil}$, let $S_F := \{F' : f(F') = F\}$.

By the definition of f and M_k, S_F is a chain.

Since the preimage of f is $2^{[n]}$, $\{S_F : F \in V_{\lceil \frac{n}{2} \rceil}\}$ is a partition of $2^{[n]}$.

$\therefore \{S_F : F \in V_{\lceil \frac{n}{2} \rceil}\}$ is a partition of $\binom{n}{\lceil \frac{n}{2} \rceil}$ chains.

Since in an antichain, no two sets are in the same chain by the definition.

$\therefore$ there are at most $\binom{n}{\lceil \frac{n}{2} \rceil}$ elements in an antichain of $2^{[n]}$.

Problem 4. Let $A := \left\{ \sum_{i=1}^{n} \epsilon_i a_i,\ \epsilon_i \in \{-1, 1\} \right\}$, the set of all possible sums $\sum_{i=1}^{n} \epsilon_i a_i$.

Consider a surjective function $f : 2^{[n]} \to A$, where $f(S) := \sum_{i \in S} a_i - \sum_{i \in [n] \setminus S} a_i$.

Claim: If $S \subsetneq T$, then at most one of $f(S), f(T)$ is in I.

Proof:

$f(T) - f(S) = 2 \sum_{i \in T \setminus S} a_i \overset{\because a_i \geq 1}{\geq} 2|T \setminus S| \geq 2.$

Since I is an open interval of length at most 2, $f(S), f(T)$ can not both be in I.

This finishes the proof of the claim.

By the claim, $\{S \in 2^{[n]} : f(S) \in I\}$ is an antichain.

And since f is surjective, by Sperner's theorem, $|A \cap I| = |f(S) \in I : S \in 2^{[n]}| \leq |\{S \in 2^{[n]} : f(S) \in I\}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$.