機率與統計 HW5

許博翔

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Problem 1.

(a) Recall that the CDF of Exponential(λ) is $1 - e^{-\lambda x}$, and its mean is $\frac{1}{\lambda}$.

False alarm probability $= \mathrm{P}[T > t_0 | H_0] = \mathrm{P}[T > t_0 | T \sim \text{Exponential}(\frac{1}{3})] = 1 - \mathrm{P}[T \leq t_0 | T \sim \text{Exponential}(\frac{1}{3})] = 1 - 1 + e^{-\frac{t_0}{3}} = e^{-\frac{t_0}{3}}$.

(b) Miss probability $= \mathrm{P}[T \leq t_0 | H_1] = \mathrm{P}[T \leq t_0 | T \sim \text{Exponential}(\frac{1}{\mu_O})] = 1 - e^{-\frac{t_0}{\mu_O}}$.

(c) We want $t \leq t_0 \iff f_{T|H_0}(t) \geq f_{T|H_1}(t)$.

Recall that the PDF of Exponential(λ) is $\lambda e^{-\lambda x}$.

$\therefore$ we want $t \leq t_{ML} \iff \frac{1}{\mu_O} e^{-\frac{t}{\mu_O}} \leq \frac{1}{3} e^{-\frac{t}{3}}$.

$\frac{1}{\mu_O} e^{-\frac{t}{\mu_O}} \leq \frac{1}{3} e^{-\frac{t}{3}} \iff \frac{3}{\mu_O} \leq e^{t(\frac{1}{\mu_O} - \frac{1}{3})} \iff \ln \frac{3}{\mu_O} \leq t(\frac{1}{\mu_O} - \frac{1}{3}) \overset{\because \mu_O \geq 3}{\iff} t \leq \frac{\ln \frac{3}{\mu_O}}{\frac{1}{\mu_O} - \frac{1}{3}}$.

$\therefore t_{ML} = \frac{\ln \frac{3}{\mu_O}}{\frac{1}{\mu_O} - \frac{1}{3}}$.

For $\mu_O = 6$, $t_{ML} = \frac{\ln \frac{1}{2}}{-\frac{1}{6}} = 6 \ln 2$.

For $\mu_O = 10$, $t_{ML} = \frac{\ln \frac{3}{10}}{-\frac{7}{30}} = \frac{30}{7} \ln \frac{10}{3}$.

(d) t_{MAP} is such that $t = t_{MAP}$ minimizes $f(t) := 0.8\mathrm{P}[T > t | H_0] + 0.2\mathrm{P}[T \leq t | H_1]$.

$\Rightarrow f'(t) = \frac{d(0.8 - 0.8\mathrm{P}[T \leq t | H_0] + 0.2\mathrm{P}[T \leq t | H_1])}{dt} = -0.8 f_{T|H_0}(t) + 0.2 f_{T|H_1}(t) = 0$.

$\Rightarrow f_{T|H_1}(t) = 4 f_{T|H_0}(t)$.

Since $g(t) := \frac{f_{T|H_1}(t)}{f_{T|H_0}(t)} = \frac{\mu_O}{3} e^{(\frac{1}{\mu_O} - \frac{1}{3})t}$ is a decreasing function of t ($\because \frac{1}{\mu_O} - \frac{1}{3} < 0$), there is $t_{MAP} = g^{-1}(\frac{1}{4}) > g^{-1}(1) = t_{ML}$.

(e) As in (d), $f_{T|H_1}(t) = 4f_{T|H_0}(t)$.

$\Rightarrow \dfrac{\mu_O}{3}e^{(\frac{1}{\mu_O}-\frac{1}{3})t_{MAP}} = \dfrac{1}{4}$.

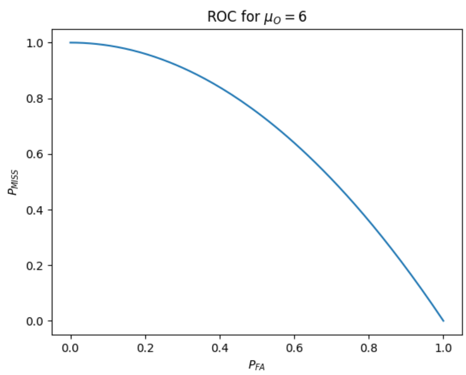
$\Rightarrow (\dfrac{1}{\mu_O} - \dfrac{1}{3})t_{MAP} = \ln\dfrac{3}{4\mu_O}$.

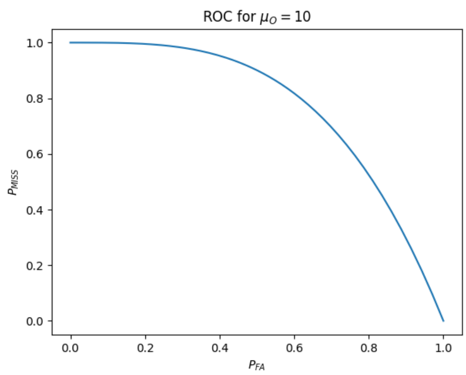
$\Rightarrow t_{MAP} = \dfrac{\ln\frac{3}{4\mu_O}}{\frac{1}{\mu_O}-\frac{1}{3}}$.

For $\mu_O = 6$, $t_{MAP} = \dfrac{\ln\frac{1}{8}}{-\frac{1}{6}} = 6\ln 8 = 18\ln 2$.

For $\mu_O = 10$, $t_{MAP} = \dfrac{\ln\frac{3}{40}}{-\frac{7}{30}} = \dfrac{30}{7}\ln\dfrac{40}{3}$.

(f)





I use the following code to draw the curve:

```
import numpy as np
import matplotlib.pyplot as plt

def draw(mu0, mu1):
        t=[i/100 for i in range(10000)]
        x=[np.exp(-mu0*i) for i in t]
        y=[1-np.exp(-mu1*i) for i in t]
        plt.plot(x, y)
        plt.xlabel('$P_{FA}$')
        plt.ylabel('$P_{MISS}$')
        plt.title('ROC for $\mu_O='+str(mu1)+'$')
        plt.savefig('mu0'+str(mu1)+'.png')
        plt.close()

```

```
draw(3, 6)
draw(3, 10)
```

Problem 2.

(a) Recall that the PDF of Gaussian$(0,1)$ is $\dfrac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$.

$x \le x_0 \iff x \in A_0 \iff L(x) = \dfrac{f_{X|H_0}(x)}{f_{X|H_1}(x)} \ge \gamma \iff \dfrac{\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}}{\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(x-v)^2}} \ge \gamma \iff e^{\frac{1}{2}((x-v)^2-x^2)} \ge \gamma \iff \dfrac{1}{2}(-2xv+v^2) \ge \ln\gamma \iff -xv \ge \ln\gamma - \dfrac{1}{2}v^2 \iff x \le -\dfrac{\ln\gamma}{v} + \dfrac{1}{2}v.$

$\therefore x_0 = -\dfrac{\ln\gamma}{v} + \dfrac{1}{2}v.$

(b) $\alpha = P_{FA} = \mathrm{P}[A_1|H_0] = \mathrm{P}[X \ge x_0|H_0] = 1 - \Phi(x_0).$

$\Rightarrow x_0 = \Phi^{-1}(1-\alpha).$

Problem 3.

(a) Recall that the mean and the variance of Exponential(λ) are $\dfrac{1}{\lambda}$ and $\dfrac{1}{\lambda^2}$, respectively.

$\Rightarrow$ the mean of 9 i.i.d. of Exponential(λ) is approximately Gaussian$(\dfrac{1}{\lambda}, \dfrac{1}{9\lambda^2})$ by the central limit theorem.

$\Rightarrow M_n(T)|H_0 \sim \text{Gaussian}(3,1), M_n(T)|H_1 \sim \text{Gaussian}(6,4).$

$\Rightarrow f_{M_n(T)|H_0}(t) = \dfrac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(t-3)^2}, f_{M_n(T)|H_1}(t) = \dfrac{1}{2\sqrt{2\pi}}e^{-\frac{1}{8}(t-6)^2}.$

Recall that t_{ML} is the solution of $f_{M_n(T)|H_0}(t) = f_{M_n(T)|H_1}(t)$.

$\Rightarrow \dfrac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(t-3)^2} = \dfrac{1}{2\sqrt{2\pi}}e^{-\frac{1}{8}(t-6)^2}.$

$\Rightarrow e^{\frac{1}{2}(t-3)^2-\frac{1}{8}(t-6)^2} = 2.$

$\Rightarrow \dfrac{1}{2}(t-3)^2 - \dfrac{1}{8}(t-6)^2 = \ln 2.$

$\Rightarrow \dfrac{3}{8}t^2 - \dfrac{3}{2}t = \ln 2.$

$\Rightarrow 3t^2 - 12t - 8\ln 2 = 0.$

Since $t_{ML} \ge 0$, there is $t_{ML} = \dfrac{6+\sqrt{36+24\ln 2}}{3} = 2 + 2\sqrt{1+\dfrac{2}{3}\ln 2} \approx 4.42.$

(b) The sum of n i.i.d. of Exponential(λ) is Erlang(n,λ).

$\therefore 9M_n(T)|H_0 \sim \text{Erlang}(9, \dfrac{1}{3}), 9M_n(T)|H_1 \sim \text{Erlang}(9, \dfrac{1}{6}).$

$\Rightarrow f_{M_n(T)|H_0}(t) = \dfrac{(9t)^8 e^{-\frac{1}{3}(9t)}}{3^9 8!}, f_{M_n(T)|H_1}(t) = \dfrac{(9t)^8 e^{-\frac{1}{6}(9t)}}{6^9 8!}.$

Recall that t_{ML} is the solution of $f_{M_n(T)|H_0}(t) = f_{M_n(T)|H_1}(t)$.

$\Rightarrow \dfrac{(9t)^8 e^{-\frac{1}{3}(9t)}}{3^9 8!} = \dfrac{(9t)^8 e^{-\frac{1}{6}(9t)}}{6^9 8!}.$

$\Rightarrow e^{(\frac{1}{3}-\frac{1}{6})(9t)} = 2^9.$

$\Rightarrow \dfrac{3}{2}t = 9\ln 2.$

$\Rightarrow t_{ML} = t = 6\ln 2 \approx 4.16.$

(c) Recall from problem 1 that t_{MAP} is the solution of $4f_{M_n(T)|H_0}(t) = f_{M_n(T)|H_1}(t)$.

$\Rightarrow 4\dfrac{(9t)^8 e^{-\frac{1}{3}(9t)}}{3^9 8!} = \dfrac{(9t)^8 e^{-\frac{1}{6}(9t)}}{6^9 8!}.$

$\Rightarrow e^{(\frac{1}{3}-\frac{1}{6})(9t)} = 2^{11}.$

$\Rightarrow \dfrac{3}{2}t = 11\ln 2.$

$\Rightarrow t_{MAP} = t = \dfrac{22}{3}\ln 2 \approx 5.08.$

(d) $P_{ERR} =$

Problem 4.

Problem 5.

Problem 6.

Problem 7.

Problem 8.

Problem 9.

Problem 10.