Graph Theory HW2

許博翔 B10902085

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Problem 1.

(a) If two distinct edges have a common vertex and are colored in different colors, we call them a **coloring angle**.

In a non-monochromatic triangle, the multiset of the color of the edges is $\{R, R, B\}$ or $\{B, B, R\}$.

$\Rightarrow$ there is exactly one vertex that is incident to the edges of the same color in a non-monochromatic triangle.

$\Rightarrow$ there are exactly two vertices that are incident to edges of different colors in a non-monochromatic triangle.

$\Rightarrow$ there are exactly two coloring angles of a non-monochromatic triangle.

Also, for any two edges uv, uw that have a common vertex, there is a unique triangle uvw that uses both of the edges.

$\therefore$ every coloring angle is contained in exactly one non-monochromatic triangle.

$\therefore$ the number of non-monochromatic triangle $= \frac{1}{2} \times$ (the number of coloring angle).

Every coloring angle has a unique common vertex, so we can count the number of coloring angles by vertices.

The number of coloring angles having the common vertex v is the number of ways to select two edges incident to v with different colors, which is $r_v(n-1-r_v)$.

$\therefore$ the number of non-monochromatic triangles $= \frac{1}{2} \times$ (the number of coloring angle) $= \frac{1}{2}\sum_{i=1}^{n} r_i(n-1-r_i)$.

$\Rightarrow$ the number of monochromatic triangles $= \binom{n}{3} - \frac{1}{2}\sum_{i=1}^{n} r_i(n-1-r_i)$.

(b) By AM-GM inequality, $\sqrt{r_i(n-1-r_i)} \le \frac{n-1}{2}$.
$\Rightarrow \binom{n}{3} - \frac{1}{2}\sum_{i=1}^{n} r_i(n-1-r_i) \ge \binom{n}{3} - \frac{1}{2}\sum_{i=1}^{n}(\frac{n-1}{2})^2 = \binom{n}{3} - \frac{1}{8}n(n-1)^2 = \frac{1}{6}(n^3-3n^2+2n)-\frac{1}{8}(n^3-2n^2+n) = \frac{1}{24}n^3 - \frac{1}{4}n^2 + \frac{5}{24}n = (\frac{1}{24} - \frac{1}{4n} + \frac{5}{24n^2})n^3 = (\frac{1}{24} - o(1))n^3$.

Problem 2.

(a) For every coloring $c : 2^{[n]} \to [r]$, consider the following graph G and the corresponding edge-coloring d:
$G = \left([n], \binom{[n]}{2}\right)$, for all $uv \in E$, WLOG suppose that $u < v$, $d(uv) := c([v-1] \setminus [u-1])$.
By what we learned in class, $R(\underbrace{3, 3, \ldots, 3}_{r})$ is finite.
$\therefore$ for n large enough, for every edge-coloring of G, there exists a monochromatic triangle uvw.
Let n be large enough, and uvw be a monochromatic triangle, WLOG suppose that $u < v < w$.
$\Rightarrow d(uv) = d(vw) = d(wu)$.
$\Rightarrow c(\{u, u+1, \ldots, v-1\}) = c(\{v, v+1, \ldots, w-1\}) = c(\{u, u+1, \ldots, w-1\})$.
$\therefore X = \{u, u+1, \ldots, v-1\}, Y = \{v, v+1, \ldots, w-1\}$ satisfy that $X, Y, X \cup Y$ receive the same color.

(b) Suppose that the numbers in $\mathbb{N}$ are colored with r colors, and let $c : \mathbb{N} \to [r]$ be a coloring.
Consider the following graph G_n and the corresponding edge-coloring d:
$G_n := \left([n], \binom{[n]}{2}\right)$, for all $uv \in E$, $d(uv) := c(|u - v|)$.
By what we learned in class, $R(\underbrace{3, 3, \ldots, 3}_{r})$ is finite.
$\therefore \exists n$ s.t. for every edge-coloring of G_n, there exists a monochromatic triangle uvw.
WLOG suppose that $u < v < w$.
$\Rightarrow c(v-u) = c(w-v) = c(w-u)$.
Let $x = v-u, y = w-v, z = w-u$, and they are monochromatic satisfying

$x + y = z.$