機率與統計 HW4

許博翔

May 13, 2024

Lemma 0.1. Let $f(p,k) := \sum_{n=k}^{\infty} \binom{n}{k} p^n$.

Then $f(p,k) = \frac{1}{1-p}(\frac{p}{1-p})^k$.

Proof. Let's prove by induction on k.

For $k=0, f(p,0) = \sum_{n=0}^{\infty} \binom{n}{0} p^n = \frac{1}{1-p}$.

Suppose for $k=k'$, $f(p,k') = \frac{1}{1-p}(\frac{p}{1-p})^{k'}$.

For $k=k'+1$, $f(p,k'+1) = \sum_{n=k'+1}^{\infty} \binom{n}{k'+1} p^n = \sum_{n=k'+1}^{\infty} (\binom{n-1}{k'+1} + \binom{n-1}{k'}) p^n = \sum_{n=k'+1}^{\infty} (\binom{n}{k'+1} p^n + \binom{n-1}{k'} p^{n-1}) p = pf(p,k'+1) + \sum_{n=k'}^{\infty} \binom{n}{k'} p^{n-1} p = p(f(p,k'+1) + f(p,k'))$.

$\Rightarrow (1-p)f(p,k'+1) = pf(p,k')$.

$\Rightarrow f(p,k'+1) = \frac{p}{1-p} f(p,k') = \frac{1}{1-p}(\frac{p}{1-p})^{k'+1}$.

$\therefore$ by induction, **Lemma (0.1)** holds. ■

Problem 1. $\Pr(K=k \wedge X=x)$

= the probability that "the first x circuits are acceptable, the $x+1$-th is reject, and the $x+2$-th to n-th circuits contain exactly $k-1$ rejected circuits

$= p^x \times (1-p) \times \binom{n-x-1}{k-1} p^{n-x-1-k+1} (1-p)^{k-1}$

$= \binom{n-x-1}{k-1} p^{n-k} (1-p)^k$.

$$\therefore P_{K,X}(k,x) = \begin{cases} \binom{n-x-1}{k-1} p^{n-k}(1-p)^k, \text{ if } k+x \le n, k \ge 1, x \ge 0 \\ 0, \text{ otherwise} \end{cases}.$$

Problem 2. $P_N(n) = 0$ for $n < 0$.

For $n \geq 0$, $P_N(n) = \sum_{k=0}^{n} \frac{100^n e^{-100}}{(n+1)!} = (n+1) \times \frac{100^n e^{-100}}{(n+1)!} = \frac{100^n e^{-100}}{n!}$.

$$\therefore P_N(n) = \begin{cases} \frac{100^n e^{-100}}{n!}, \text{ if } n \in \mathbb{Z}^+ \cup \{0\} \\ 0, \text{ otherwise} \end{cases}.$$

$\because$ By the definition, if $k > n$, then $P_{N,K}(n,k) = 0$.

$\therefore P_K(k) = \sum_{n=0}^{\infty} P_{N,K}(n,k) = \sum_{n=k}^{\infty} P_{N,K}(n,k) = \sum_{n=k}^{\infty} \frac{100^n e^{-100}}{(n+1)!} = \sum_{n=k+1}^{\infty} \frac{100^{n-1} e^{-100}}{n!} = \frac{1}{100} \sum_{n=k+1}^{\infty} \frac{100^n e^{-100}}{n!} = \frac{1}{100} \sum_{n=k+1}^{\infty} P_N(n) = \frac{1}{100} \Pr(n > k)$.

Problem 3. The triangle is the same to Example 5.8.

The cases (a) (b) (c) (d) (e) are defined like those in Example 5.8.

(a): $F_{X,Y}(x,y) = 0$.

(e): $F_{X,Y}(x,y) = 1$.

(b): $F_{X,Y}(x,y) = \int_0^y \int_v^x 8uv\,du\,dv = \int_0^y 4v(x^2 - v^2)dv = (2v^2x^2 - v^4)\big|_0^y = 2x^2y^2 - y^4$.

(c): $F_{X,Y}(x,y) = \int_0^x \int_v^x 8uv\,du\,dv = \int_0^x 4v(x^2 - v^2)dv = (2v^2x^2 - v^4)\big|_0^x = 2x^4 - x^4 = x^4$.

(d): $F_{X,Y}(x,y) = \int_0^y \int_v^1 8uv\,du\,dv = \int_0^y 4v(1^2 - v^2)dv = (2v^2 - v^4)\big|_0^y = 2y^2 - y^4$.

$$\therefore F_{X,Y}(x,y) = \begin{cases} 0, \text{ if } x < 0 \text{ or } y < 0 \\ 2x^2y^2 - y^4, \text{ if } 0 \leq y \leq x \leq 1 \\ x^4, \text{ if } 0 \leq x < y, 0 \leq x \leq 1 \\ 2y^2 - y^4, \text{ if } 0 \leq y \leq 1, x > 1 \\ 1, \text{ if } x > 1, y > 1 \end{cases}.$$

Problem 4.

(a) If $x \geq 2$, then $F_X(x) = 1$.

If $x \leq 0$, then $F_X(x) = 0$.

If $0 \leq x \leq 2$, then $F_X(x) = \int_{-\infty}^x f_X(y)dy = \int_0^x \frac{y}{2}dy = \frac{x^2}{4}$.

$$\therefore F_X(x) = \begin{cases} 0, \text{ if } x \le 0 \\ \dfrac{x^2}{4}, \text{ if } 0 \le x \le 2 \\ 1, \text{ otherwise} \end{cases}.$$

(b) Since X_1, X_2 are independent, $\Pr[X_1 \le 1, X_2 \le 1] = \Pr[X_1 \le 1]\Pr[X_2 \le 1] = F_X(1)F_X(1) = (\frac{1}{4})^2 = \frac{1}{16}$.

(c) $F_W(1) = \Pr[\max(X_1, X_2) \le 1] = \Pr[X_1 \le 1, X_2 \le 1] = \frac{1}{16}$.

(d) $F_W(w) = \Pr[\max(X_1, X_2) \le w] = \Pr[X_1 \le w, X_2 \le w]$

$$\overset{\because X_1, X_2 \text{ are independent}}{=} \Pr[X_1 \le w]\Pr[X_2 \le w] = F_X(w)^2 = \begin{cases} 0, \text{ if } w \le 0 \\ \dfrac{x^4}{16}, \text{ if } 0 \le w \le 2 \\ 1, \text{ otherwise} \end{cases}.$$

Problem 5. First, $X \sim \text{Unif}[0, \frac{d}{2}], \Theta \sim \text{Unif}[0, \frac{\pi}{2}]$.

$$\Rightarrow f_X(x) = \begin{cases} \dfrac{2}{d}, \text{ if } 0 \le x \le \dfrac{d}{2} \\ 0, \text{ otherwise} \end{cases}, f_\Theta(\theta) = \begin{cases} \dfrac{2}{\pi}, \text{ if } 0 \le \theta \le \dfrac{\pi}{2} \\ 0, \text{ otherwise} \end{cases}.$$

The needle intersects one of the lines $\iff X \le \frac{l}{2}\sin\theta$.

$\therefore$ the probability that the needle will intersect one of the lines $= \Pr[X \le \frac{l}{2}\sin\theta]$.

Note that $l < d$, so the upperbound of X in the following integral is $\min(\frac{l}{2}\sin\theta, \frac{d}{2}) = \frac{l}{2}\sin\theta$.

And X, Θ are independent, so their joint pdf $f_{X,\Theta}(x, \theta) = f_X(x)f_\Theta(\theta)$.

$\Rightarrow \Pr[X \le \frac{l}{2}\sin\theta] = \int_0^{\frac{\pi}{2}} \int_0^{\frac{l}{2}\sin\theta} f_X(x)f_\Theta(\theta)dxd\theta = \int_0^{\frac{\pi}{2}} \frac{l}{2}\sin\theta\frac{2}{d}\frac{2}{\pi}d\theta = \frac{2l}{d\pi}$.

This experiment can get p: the approximated value of the probability that the needle will intersect one of the lines when the needle is dropped for sufficient large number of times.

And one can approximate $\pi \approx \frac{2l}{dp}$.

Problem 6. $r_{X,Y} = E[XY] = \int_{-1}^{1}\int_{-1}^{y} f_{X,Y}(x, y)xydxdy = \frac{1}{2}\int_{-1}^{1}\frac{y^2-1}{2}ydy = \frac{1}{4}(\frac{y^4}{4} - \frac{y^2}{2})\Big|_{-1}^{1} = 0$.

$E[e^{X+Y}] = \int_{-1}^{1}\int_{-1}^{y} f_{X,Y}(x,y)e^{x+y}dxdy = \frac{1}{2}\int_{-1}^{1}(e^{2y}-e^{y-1})dy = \left(\frac{1}{4}e^{2y} - \frac{1}{2}e^{y-1}\right)\Big|_{-1}^{1} =$
$\frac{1}{4}(e^2 - e^{-2}) - \frac{1}{2}(1 - e^{-2}) = \frac{1}{4}(e^2 - 2 + e^{-2}) = \left(\frac{e - \frac{1}{e}}{2}\right)^2.$

Problem 7. First, if $W < 1$, then both $\frac{X}{Y}, \frac{Y}{X}$ are less than 1.
Since $\Pr[X \le 0 \vee Y \le 0] = 0$ by the definition of $f_{X,Y}(x,y)$.
$\therefore \frac{X}{Y} < 1, \frac{Y}{X} < 1 \Rightarrow X < Y, Y < X$, which is impossible.
$\therefore$ there must be $W \ge 1$.
$F_W(w) = \Pr[W \le w] = \Pr[\max\left(\frac{X}{Y}, \frac{Y}{X}\right) \le w] = \Pr[\frac{X}{Y} \le w \wedge \frac{Y}{X} \le w] = \Pr[Y \ge \frac{X}{w} \wedge X \ge \frac{Y}{w}] = 1 - \Pr[X < \frac{Y}{w}] - \Pr[Y < \frac{X}{w}].$
Note that $\frac{Y}{w} \le \frac{a}{w} \le a, \frac{X}{w} \le \frac{a}{w} \le a.$
$\therefore 1 - \Pr[X < \frac{Y}{w}] - \Pr[Y < \frac{X}{w}] = 1 - \int_0^a \int_0^{\frac{y}{w}} f_{X,Y}(x,y)dxdy - \int_0^a \int_0^{\frac{x}{w}} f_{X,Y}dydx = 1 - 2\int_0^a \int_0^{\frac{y}{w}} \frac{1}{a^2}dxdy = 1 - \frac{2}{a^2}\int_0^a \frac{y}{w}dy = 1 - \frac{1}{a^2 w}a^2 = 1 - \frac{1}{w}.$
$\Rightarrow f_W(w) = F'_W(w) = \frac{1}{w^2}.$

Problem 8.

(a) Since at least one bus arrive, there is $n \ge 1$.

Since at most one bus arrives in a minute, there is $t \ge n$.

$\therefore$ the set is $\{(n,t) : t \ge n \ge 1, n, t \in \mathbb{Z}\}$.

(b) If $n > t$, then by (a), $P_{N,T}(n,t) = 0$.

If $n \le t$, it means that exactly $n-1$ buses passed through in the first $t-1$ minutes, and a bus passed through at the t-th minute, so the probability is $\binom{t-1}{n-1}p^n(1-p)^{t-n}$.
The probability that I didn't board the first $n-1$ buses but the n-th is $(1-q)^{n-1}q$.
$$\therefore P_{N,T}(n,t) = \begin{cases} \binom{t-1}{n-1}p^n(1-p)^{t-n}(1-q)^{n-1}q, \text{ if } t \ge n \ge 1, n,t \in \mathbb{Z} \\ 0, \text{ otherwise} \end{cases}.$$

(c) $P_N(n) =$ the probability that I didn't board the first $n-1$ buses but the n-th $= (1-q)^{n-1}q$.

$P_T(t) =$ the probability that in t minutes:

(1) the first $t-1$ minutes either the bus didn't come, or I didn't board the bus, which has probability $(1-pq)$.

(2) the t-th minute the bus came and I boarded the bus, which has probability pq.

$\therefore P_T(t) = (1-pq)^{t-1}pq$.

(d) $$P_{N|T}(n|t) = \frac{P_{N,T}(n,t)}{P_T(t)} = \begin{cases} \dfrac{\binom{t-1}{n-1}p^n(1-p)^{t-n}(1-q)^{n-1}q}{(1-pq)^{t-1}pq}, \text{ if } n \le t \\ 0, \text{ otherwise} \end{cases}.$$

$$P_{T|N}(t|n) = \frac{P_{N,T}(n,t)}{P_N(n)} = \begin{cases} \dfrac{\binom{t-1}{n-1}p^n(1-p)^{t-n}(1-q)^{n-1}q}{(1-q)^{n-1}q} = \dbinom{t-1}{n-1}p^n(1-p)^{t-n}, \text{ if } n \le t \\ 0, \text{ otherwise} \end{cases}.$$

Problem 9.

(a) $P_N(n) = 0$ for $n < 0$.

For $n \ge 0$, $P_N(n) = \displaystyle\sum_{k=0}^{n} \frac{100^n e^{-100}}{(n+1)!} = (n+1) \times \frac{100^n e^{-100}}{(n+1)!} = \frac{100^n e^{-100}}{n!}$.

$$\therefore P_N(n) = \begin{cases} \dfrac{100^n e^{-100}}{n!}, \text{ if } n \in \mathbb{Z}^+ \cup \{0\} \\ 0, \text{ otherwise} \end{cases}.$$

$$P_{K|N}(k|n) = \frac{P_{N,K}(n,k)}{P_N(n)} = \begin{cases} \dfrac{\frac{100^n e^{-100}}{n!}}{\frac{100^n e^{-100}}{(n+1)!}} = \dfrac{1}{n+1}, \text{ if } k = 0, 1, \ldots, n; n = 0, 1, \ldots \\ 0, \text{ otherwise} \end{cases}.$$

(b) $\mathrm{E}[K|N=n] = \displaystyle\sum_{k=0}^{n} kP_{K|N}(k|n) = \sum_{k=0}^{n} k \times \frac{1}{n+1} = \frac{n}{2}$.

(c) $\mathrm{E}[K|N] = \dfrac{N}{2}$.

$\mathrm{E}[K] = \mathrm{E}[\mathrm{E}[K|N]] = \mathrm{E}[\dfrac{N}{2}] = \displaystyle\sum_{n=0}^{\infty} \frac{n}{2}\frac{100^n e^{-100}}{n!} = \sum_{n=1}^{\infty} \frac{100^n e^{-100}}{(n-1)! \times 2} = 50\sum_{n=1}^{\infty} \frac{100^{n-1}e^{-100}}{(n-1)!} = 50\sum_{n=0}^{\infty} \frac{100^n e^{-100}}{n!} = 50\sum_{n=0}^{\infty} P_N(n) = 50$.

Problem 10. $J = J_6 + \frac{1}{2}(N_6+N_5+N_4+N_3+N_2+N_1+N_0) = 10^6 + \frac{1}{2}(N_6+\cdots+N_0)$.

$\Rightarrow \mathrm{E}[J] = 10^6 + \frac{1}{2}(\mathrm{E}[N_6] + \cdots + \mathrm{E}[N_0])$.

$\mathrm{E}[N_k|J_k] = J_k$ by the definition of N_k.

$\mathrm{E}[J_k|J_{k-1}, N_{k-1}] = J_{k-1} + N_{k-1}$.

$\therefore \mathrm{E}[N_k|N_{k+1}]\ \mathrm{E}[N_k] = \mathrm{E}[\mathrm{E}[N_k|N_{k+1}]]$.

Problem 11. 以下使用 **Lemma (0.1)** 的標號與結果來推導：

要剛好後退 3 公尺，要嘛是 3 個後退 1 公尺組成的（這個事件稱爲 A），要嘛是一個後退 3 公尺組成的（這個事件稱爲 B）。

令 X 爲剛好後退 3 公尺前，兩台吹風機都沒有正常運作的時間。

$\Rightarrow \Pr[A] = \sum_{x=0}^{\infty}(1-0.6)^x(1-0.4)^x 0.6 \times 0.4 = \sum_{x=0}^{\infty} 0.24^{x+1} = \frac{0.24}{1-0.24} = \frac{6}{19}$.

$\mathrm{E}[X|A]\Pr[A] = \sum_{x=0}^{\infty} 0.24^{x+1}x = 0.24 f(0.24, 1) = 0.24^2(1-0.24)^{-2} = (\frac{6}{19})^2$.

B 發生時，前 $x+2$ 單位時間有恰好兩單位時間有恰一個吹風機有正常運作，所以

$\Pr[B] = \sum_{x=0}^{\infty}\binom{x+2}{2}(1-0.6)^x(1-0.4)^x(0.6\times(1-0.4)+0.4\times(1-0.6))^3 = \sum_{x=0}^{\infty}\binom{x+2}{2}0.24^x 0.52^3 = \frac{0.52^3}{0.24^2}f(2, 0.24) = 0.52^3(1-0.24)^{-3} = 0.52^3 0.76^{-3}$.

$\mathrm{E}[X|B]\Pr[B] = \sum_{x=0}^{\infty}\binom{x+2}{2}0.24^x 0.52^3 x = \sum_{x=1}^{\infty}3\binom{x+2}{3}0.24^x 0.52^3 = \frac{3}{0.24^2}0.52^3 f(3, 0.24) = \frac{3}{0.24^2}0.52^3 \times 0.24^3(1-0.24)^{-4} = 3\times 0.24\times 0.52^3\times 0.76^{-4}$.

The answer $= \frac{\mathrm{E}[X+1|A]\Pr[A] + \mathrm{E}[X+3|B]\Pr[B]}{\Pr[A]+\Pr[B]}$

$= \frac{0.24^2\times 0.76^{-2} + 0.24\times 0.76^{-1} + 3\times 0.24\times 0.52^3\times 0.76^{-4} + 3\times 0.52^3\times 0.76^{-3}}{0.24\times 0.76^{-1} + 0.52^3\times 0.76^{-3}} = \frac{218925}{82897} \approx 2.641$.