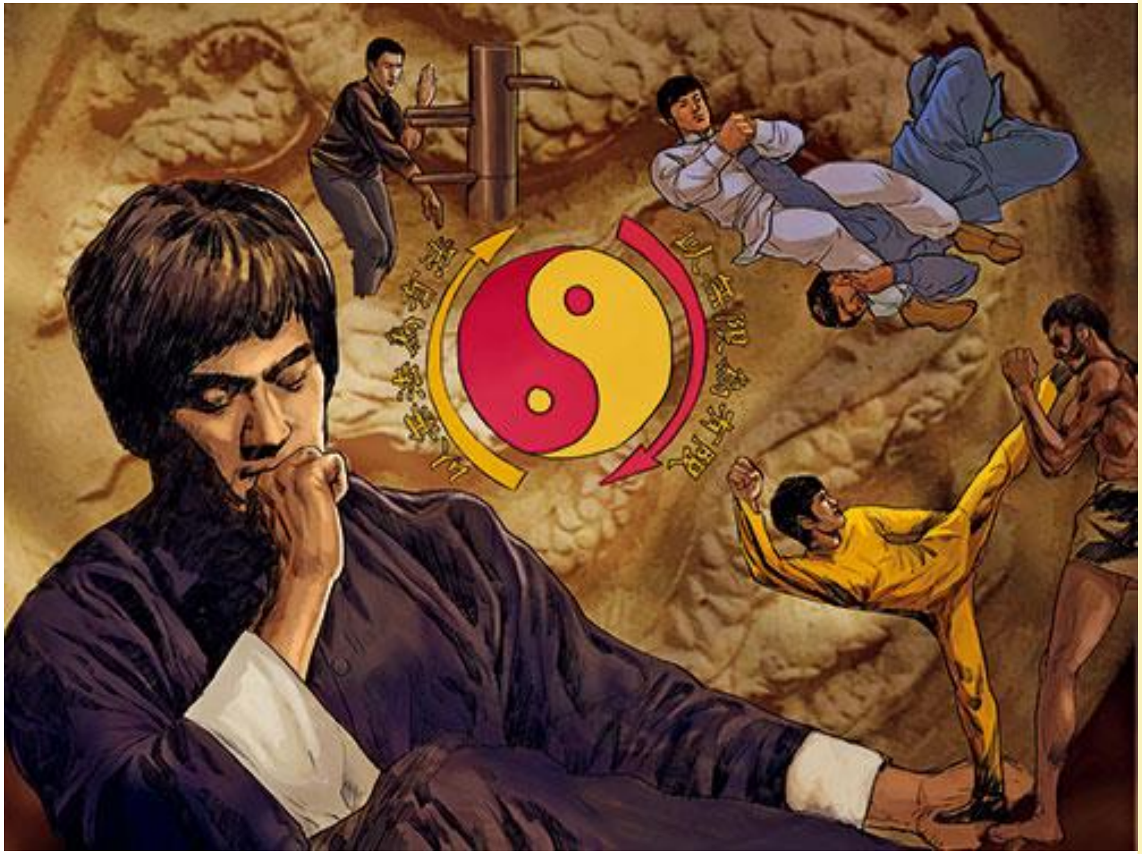
Lecture 25 Final Project



Good at
both Mathematical analysis and Computer programming

Final Class Examination will be held on Tuesday (May 9th , 8:00am-11:00am) at Hearst Mining Building 390

RR Week Office Hour Schedule

Prof. Li: Tu & Thu. 1:00pm – 3:00pm

GSIs:

Qingsong Tu: MWF 11:00 am – 12:30 pm

Jes Parker: Tu & Thus. 11:00am-12:30 pm

Ben Worsfold Butler: MWF 9:00 am – 10:30 am

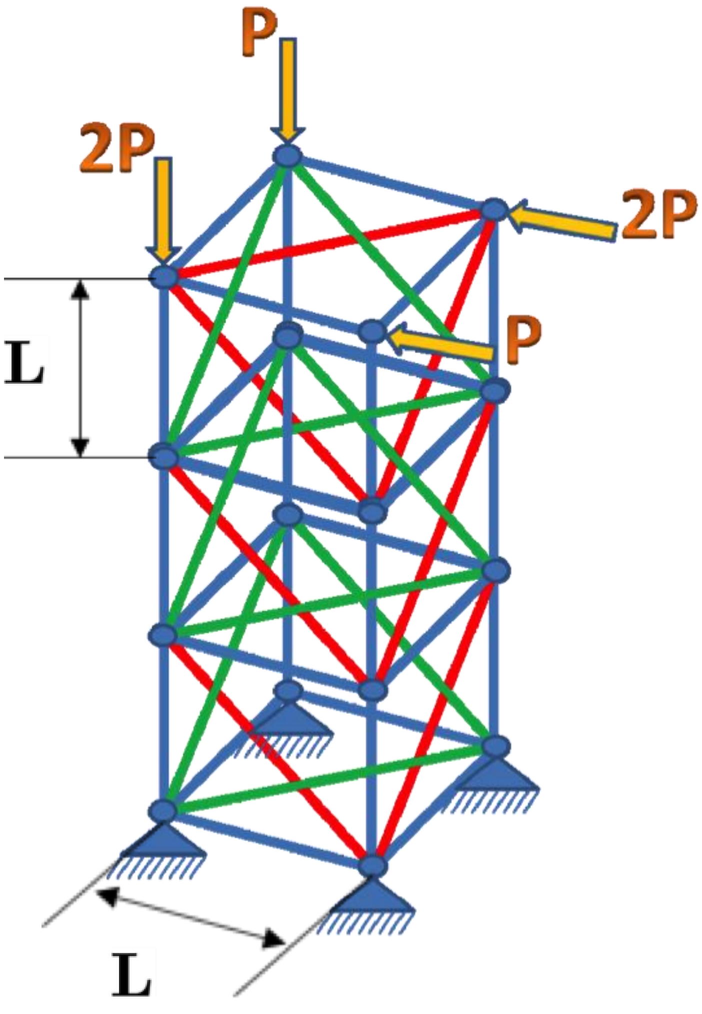
Final exam is 25% of overall grade.

It is a competition !

1	**A+**	**(95)**
3	**A**	**(90)**
4	**A-**	**(85)**
4	**B+**	**(80)**

It is a research, presentation, and oral examination.

Final Project I. Part I



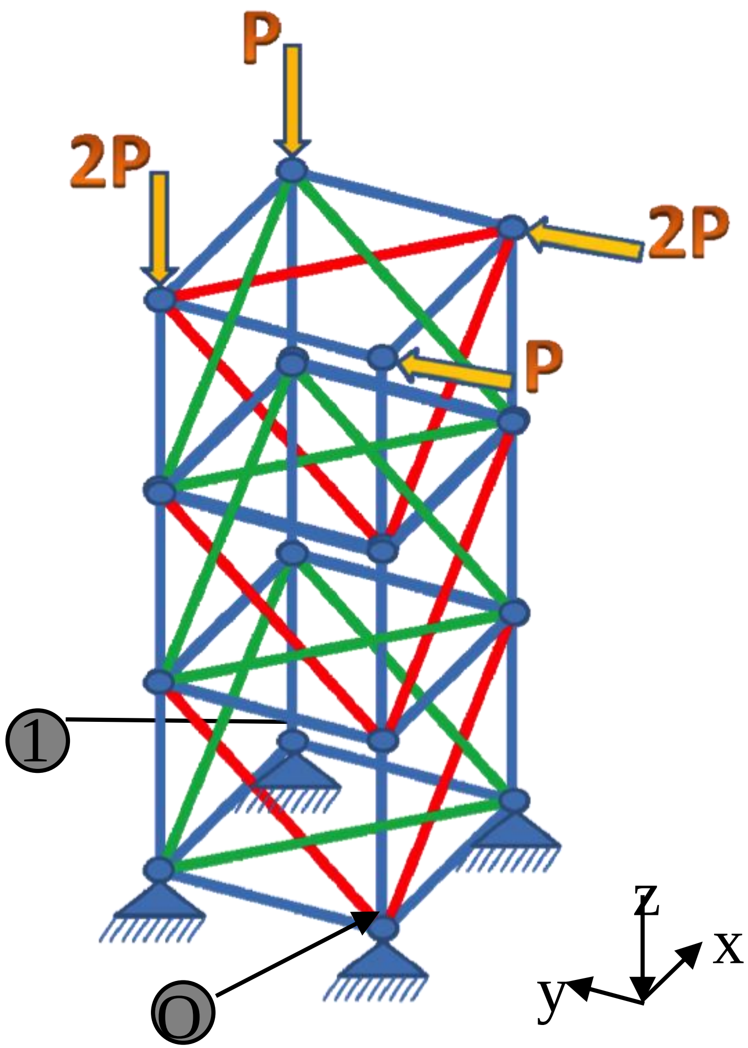
Use matrix analysis to find both the nodal displacements as well as the internal force for the 3D truss in the figure, with the following given values:

E = 400 MPa

L = 2.0 m

$A = 5 \times 10^{-4}$ m^2

P = 1000 N



There are 16 nodes.

- 1st node begins at the bottom back corner
- 16th node ends at the downward 2P force at top

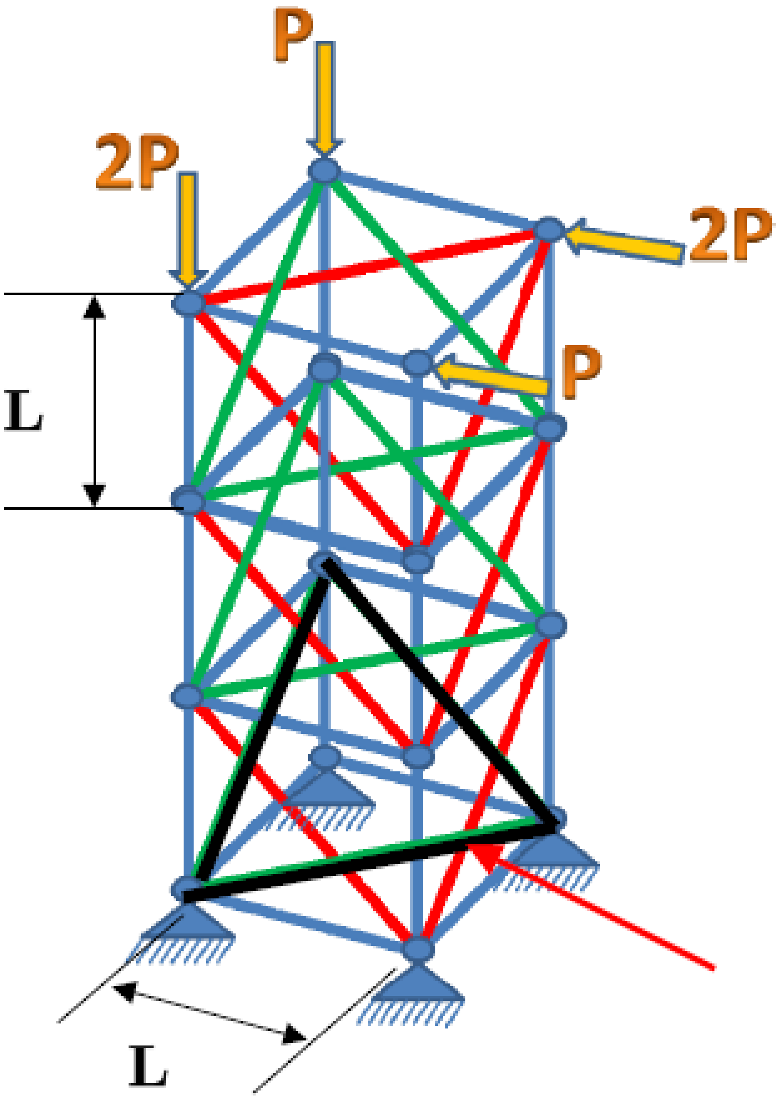
There are 44 members

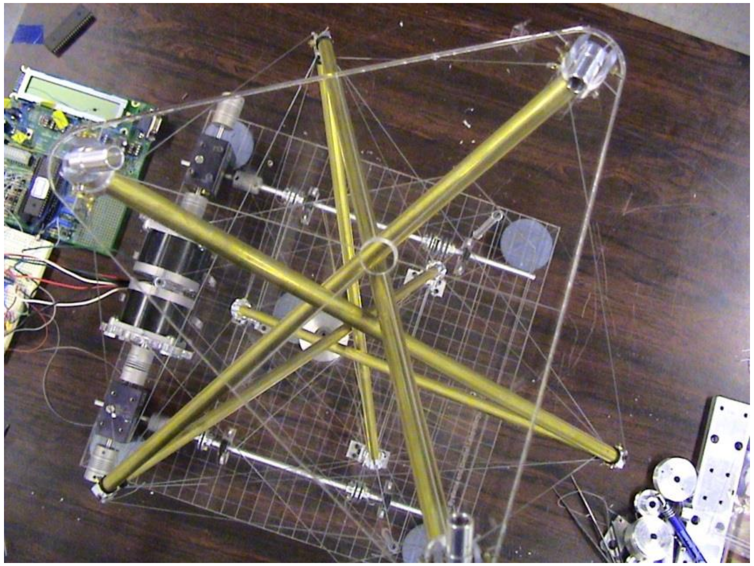
- 16 horizontal bars 1-16
- 12 vertical bars 17-28
- 9 green bars 29-37
- 7 red bars 38-44

Part II

- Replace lower green triangle (bar 29, 30, 31) with one rope with the same E and A

- Assume rope and nodes are frictionless, so the rope has a uniform tension throughout





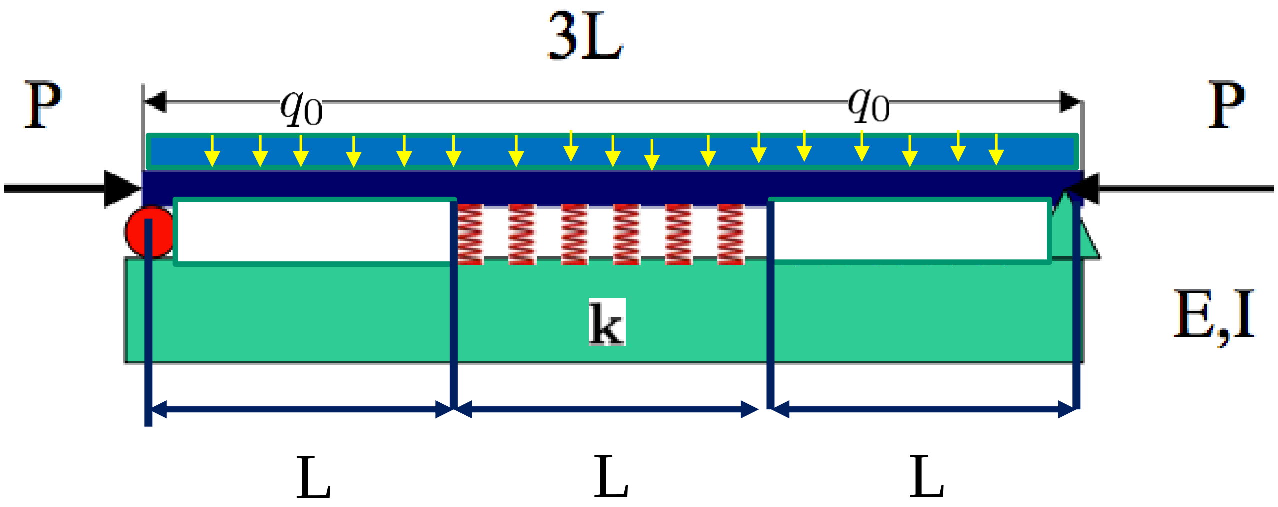
Pre-stressed tensegrity structure
as novel meta-materials

Tensegrity structures with active damping control

Final Project No. II

Buckling of a Beam on a Partial Winkler Foundation

Considering the following structure with $L = 2M$,
$EI = 100\ NM^2$, $k = 5000N/M^2$.
The beam is subjected a uniform load of $q_0 = 1000N/M$.
Use the beam element to find the critical load.

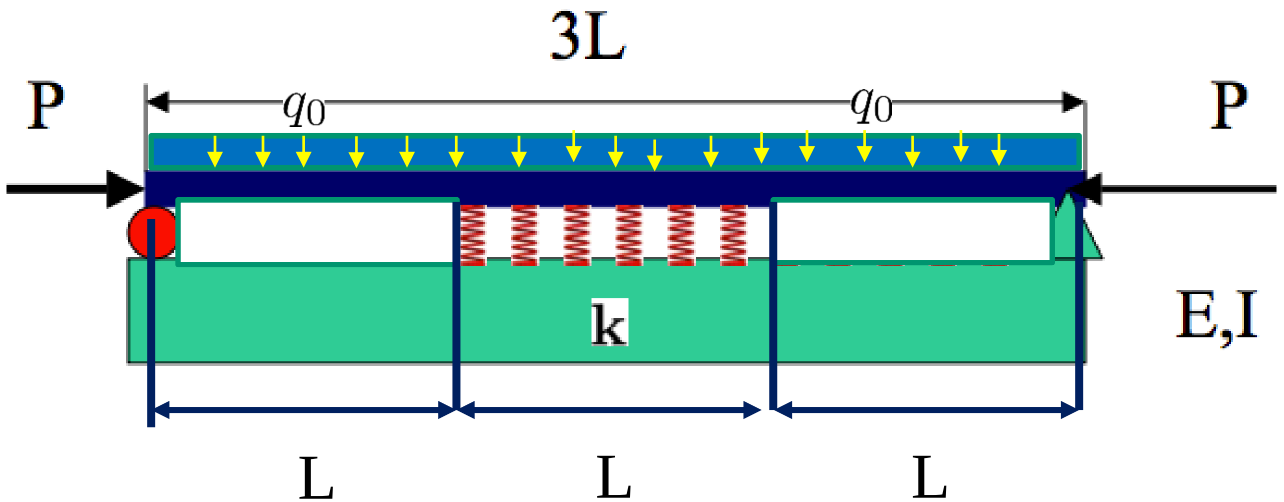


Final Project Topic No. II

Buckling of a Beam on a Partial Winkler Foundation

Extra Points

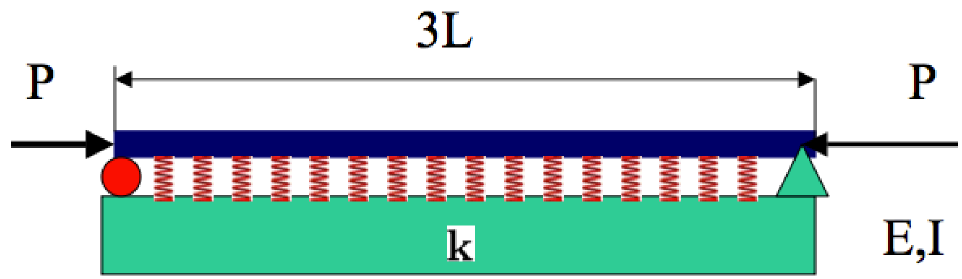
Find the critical laod when $q = 1000 \sin\left(\frac{\pi x}{3L}\right) N/M$**.**



(CE-130N-2013) Matlab Code Extract: solving for P_{crit}

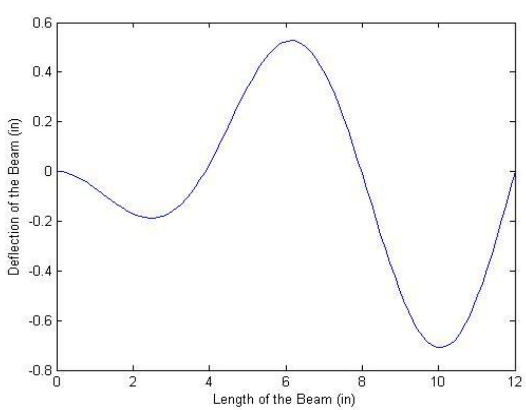
modified boundary conditions:

```
idf = [3:2*n-2 2*n];
idd = [1 2 2*n-1];
c(idd) = [0 0 0];
```



solving for critical buckling load:

```
[V,D] = eig(Kff,Gff);
Pcrit = D(1,1);
```



To solve the problem, k_{sp} is modified as follows:

$$k_{sp} = \int g_i \, k \, g_j \, dx$$

```
Ksp = k.* [13*dx/35       (11*dx^2)/210   9*dx/70       -13*dx^2/420;
           (11*dx^2)/210   dx^3/105        13*dx^2/420   -dx^3/140;
            9*dx/70       13*dx^2/420     13*dx/35      -11*dx^2/210;
           -13*dx^2/420   -dx^3/140       -11*dx^2/210    dx^3/105];
```

Pcrit = 8988N

Final Project

Mathematical model:

$$(\blacksquare)\nabla^2 w + (\blacktriangle) = 0\ .$$

- Anti-plane problem: $\blacksquare = Gt$ and $\blacktriangle = bt \ \rightarrow\ Gt\nabla^2 w + bt = 0.$
- Thin membrane: $\blacksquare = S$ and $\blacktriangle = p \ \rightarrow\ S\nabla^2 w + p = 0.$
- St. Venant Torsion: $\blacksquare = 1$ and $\blacktriangle = 0 \ \rightarrow\ \nabla^2 \psi = 0.$
- Prantl's membrane analogy: $\blacksquare = 1$ and $\blacktriangle = 2G\theta \ \rightarrow\ \nabla^2 \varphi + 2G\theta = 0.$
- Heat conduction problem: $\blacksquare = K$ and $\blacktriangle = Q \ \rightarrow\ K\nabla^2 T + Q = 0.$

1. Write a MATLAB function of the form

```
function [p,t,e]=pmesh(pv,hmax,nref)
```

which generates an unstructured triangular mesh of the polygon with vertices pv, with edge lengths approximately equal to $h_{\max}/2^{n_{\text{ref}}}$, using a simplified Delaunay refinement algorithm. The outputs are the node points p (N-by-2), the triangle indices t (T-by-3), and the indices of the boundary points e.

(a) The 2-column matrix pv contains the vertices x_i, y_i of the original polygon, with the last point equal to the first (a closed polygon).

(b) First, create node points along each polygon segment, such that all new segments have lengths $\leq h_{\max}$ (but as close to $h_{\max}$ as possible). Make sure not to duplicate any nodes.

(c) Triangulate the domain using the **delaunayn** command.

(d) Remove the triangles outside the domain (see the **inpolygon** command).

(e) Find the triangle with largest area A. If $A > h^2_{\max}/2$, add the circumcenter of the triangle to the list of node points.

(f) Retriangulate and remove outside triangles (steps (c)-(d)).

(g) Repeat steps (e)-(f) until no triangle area $A > h^2_{\max}/2$.

(h) Refine the mesh uniformly n_{ref} times. In each refinement, add the center of each mesh edge to the list of node points, and retriangulate. Again, make sure not to duplicate any nodes, see e.g. the command unique(p,'rows').

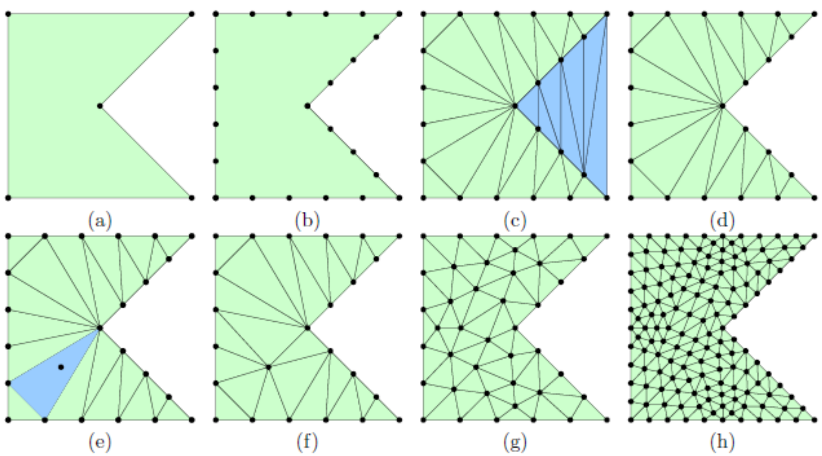
Finally, find the nodes e on the boundary using the **boundary_nodes** command. The following commands create the example in the figures. Also make sure that the function works with other inputs, that is, other polygons, $h_{\max}$, and n_{ref}.

```
pv=[0,0;1,0;.5,.5;1,1;0,1;0,0];
[p,t,e]=pmesh(pv,0.2,1);
tplot(p,t)
```

p is the array for nodal coordinates

t is connectivity array

e is the boundary node array

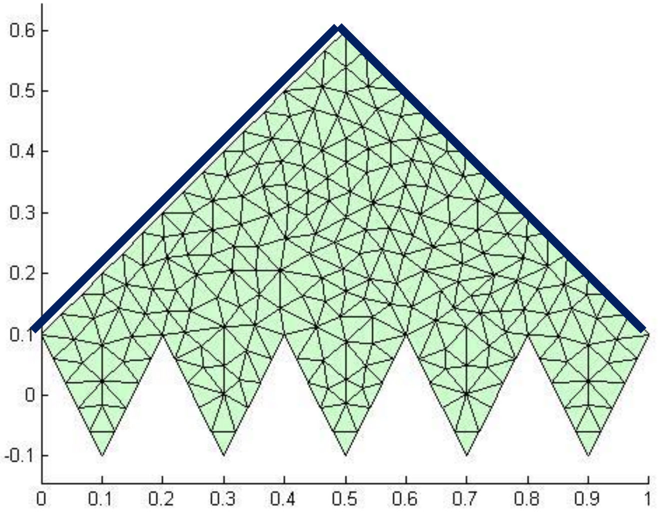


Name		Date modified	Type	Size
	Search boundary nodes			
boundary_nodes.m		11/9/2011 12:28 PM	MATLAB Code	1 KB
fempoi.m	FEM calculation	11/17/2011 3:46 PM	MATLAB Code	2 KB
fempoi_test.m	main program	4/24/2017 1:53 PM	MATLAB Code	1 KB
pmesh.m	generate mesh	11/17/2011 9:07 A...	MATLAB Code	4 KB
poiconv.m	Calculate error	11/17/2011 9:06 A...	MATLAB Code	1 KB
poiconv_test.m		11/16/2011 9:11 PM	MATLAB Code	1 KB
ps6.pdf		4/18/2016 3:39 PM	Adobe Acrobat D...	221 KB
Ref2-Chapter13.pdf		12/22/2011 5:44 PM	Adobe Acrobat D...	4,817 KB
tplot.m	plot results	11/15/2011 8:59 PM	MATLAB Code	1 KB

```
clear all; close all; clc;
 Square, Dirichlet left/bottom
pv=[0,0;1,0;1,1;0,1;0,0];
[p,t,e]=pmesh(pv,0.2,0);
e=e(p(e,1)==0 | p(e,2)==0);
u=fempoi(p,t,e);
figure; tplot(p,t,u)

% Circle, all Dirichlet
n=32; phi=2.0*pi*(0:n)'/n;
pv=[cos(phi),sin(phi)];
[p,t,e]=pmesh(pv,2*pi/n,0);
u=fempoi(p,t,e);
figure; tplot(p,t,u)

% Ellipse, all Dirichlet
a=2;
b=1;
n=32; phi=2.0*pi*(0:n)'/n;
pv=[a*cos(phi),b*sin(phi)];
[p,t,e]=pmesh(pv,2*pi/n,0);
u=fempoi(p,t,e);
figure; tplot(p,t,u)

% Triangle, all Dirichlet
a= sqrt(2)/2;
pv=[-a,0;a,0;0,1;-a,0];
[p,t,e]=pmesh(pv,0.2,0);
u=fempoi(p,t,e);
figure; tplot(p,t,u)
```

```
 %Complex polygon geometry, mixed Dirichlet/Neumann
x=(0:.1:1)';
y=.1*cos(10*pi*x);
pv=[x,y; .5,.6;0,0.1];
[p,t,e]=pmesh(pv,0.05,0);
e=e(p(e,2)>=.6-abs(p(e,1)-.5));
u=fempoi(p,t,e);
figure; tplot(p,t,u)
```

fempoi_test.m



Download Resource Files from Class website

Project III. Two-Dimensional Heat Transfer

(**100 Points**)**:** You are required to solve

$$\frac{\partial}{\partial x}\Big(K\frac{\partial T}{\partial x}\Big)+\frac{\partial}{\partial y}\Big(K\frac{\partial T}{\partial y}\Big)+Q=0\ \forall(x,y)\in\Omega;\ T=T_b\,\forall(x,y)\in\Gamma_T;\ \mathbf{n}\cdot\nabla T=q^*,\ \forall(x,y)\in\Gamma_q$$

where K is the thermal conductivity $(kW/(m\cdot C^o)$; Q is the heat source or sink (kW/m^3), and c is the specific heat $(kW\cdot h/kg\cdot C^o)$.

For steady state problem,

$$\frac{\partial}{\partial x}\Big(K\frac{\partial T}{\partial x}\Big)+\frac{\partial}{\partial y}\Big(K\frac{\partial T}{\partial y}\Big)+Q=0\quad\forall(x,y)\in\Omega$$

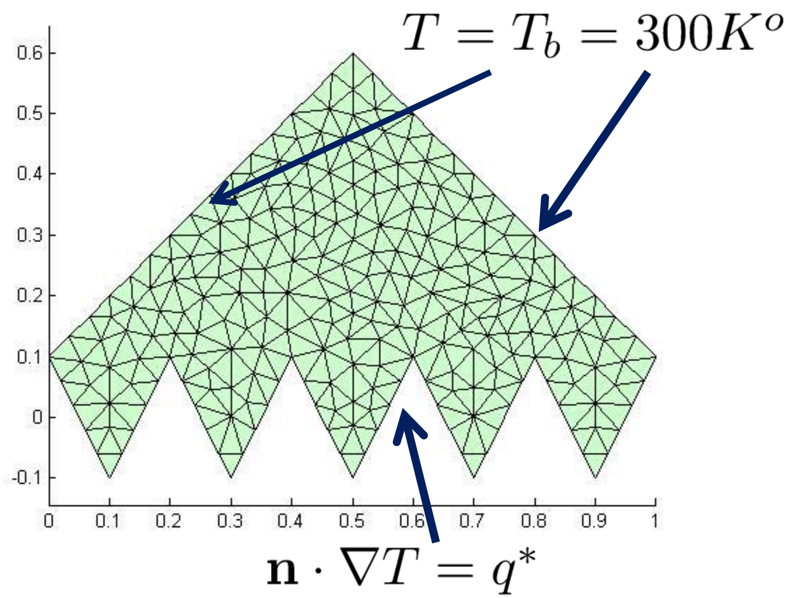
The potentail energy is

$$\Pi=\frac{Kt}{2}\int_{\Omega}\Big(\Big(\frac{\partial T}{\partial x}\Big)^2+\Big(\frac{\partial T}{\partial y}\Big)^2\Big)d\Omega-t\int_{\Omega}QTd\Omega-\int_{\Gamma_q}q^*TdS$$

The weak formulation is,

$$\delta\Pi = Kt\int_\Omega\Big(\frac{\partial T}{\partial x}\frac{\partial \delta T}{\partial x} + \frac{\partial T}{\partial y}\frac{\partial \delta T}{\partial y}\Big)d\Omega - t\int_\Omega Q\delta T d\Omega - \int_{\Gamma_q} q^*\delta T dS = 0$$

In the project, we consider Ω as a circular domain, $K = 1, Q = 1$, and $t = 1$; and $q^* = 0.2$.



Extra credits (**5 points**): solving the transient heat transfer problem,

$$\frac{\partial}{\partial x}\Big(K\frac{\partial T}{\partial x}\Big) + \frac{\partial}{\partial y}\Big(K\frac{\partial T}{\partial y}\Big) + Q = c\rho\frac{\partial T}{\partial t} \quad \forall(x,y)\in\Omega$$

$\rho = 100$ and $c = 10$.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Project IV FEM Calculation of Membrane

Use 2D triangle element to simulate membrane deformation **(100 Points)**.

Choose $S = 1$. Solve

$$S\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)+p(x,y) = 0, \quad \forall\,(x,y) \in \Omega, \quad \Omega = \left\{(x,y) \;\middle|\; \text{an elliptic membrane}\right\}$$

Boundary condition,

$$w(x,y) = 0, \quad \forall(x,y) \in \partial\Omega$$

Choose

$$p(x,y) = q_0, \quad |\mathbf{r}| \leq 1, \quad \text{where} \quad \mathbf{r} = x\mathbf{i} + y\mathbf{j}, \quad \text{and } q_0 = 1 \;.$$

Solve the problem using FEM.

You are required to solve

$$S\Big(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\Big) + p(x,y) = 0, \quad \forall (x,y) \in \Omega;$$

where S is the constant membrane tension.

The potentail energy is

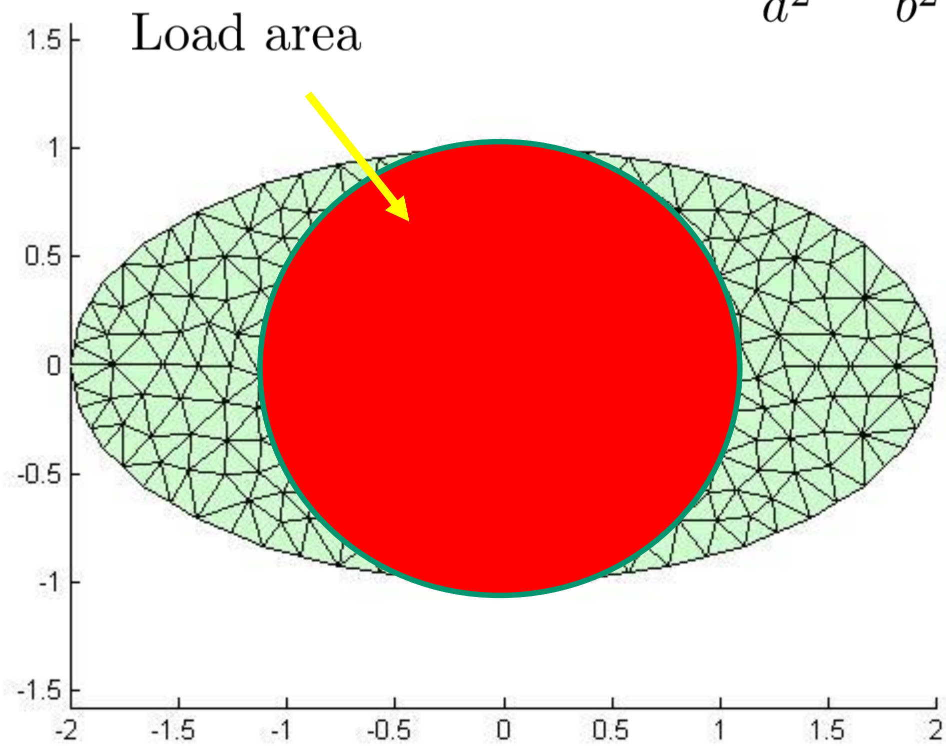
$$\Pi = \frac{S}{2}\int_\Omega \Big(\Big(\frac{\partial w}{\partial x}\Big)^2 + \Big(\frac{\partial w}{\partial y}\Big)^2\Big) d\Omega - \int_\Omega q w d\Omega$$

The virtual work principle is

$$S\int_\Omega \Big(\frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial \delta w}{\partial y}\Big) d\Omega = \int_\Omega q \delta w d\Omega$$

In the project, we consider Ω as an equilaterial triangle domain.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a = 2,\ b = 1$$



A Elliptic Membrane

Extra Poins **(5 Points)**:

Choose $S = 1$. Solve

$$S\Big(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\Big) + p(x,y) = \rho \frac{\partial^2 w}{\partial t^2} \quad \forall\ (x,y) \in \Omega,$$

where $\rho = 10$

$$\Omega = \Big\{(x,y)\ \Big|\ \text{is an ellipse}\Big\}$$

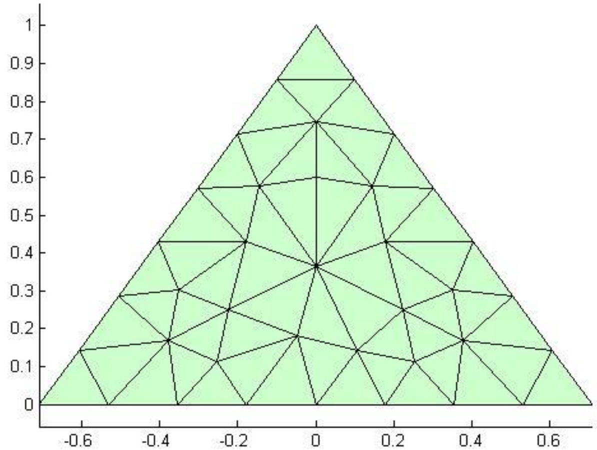
Boundary condition,

$$w(x,y) = 0, \quad \forall (x,y) \in \partial\Omega$$

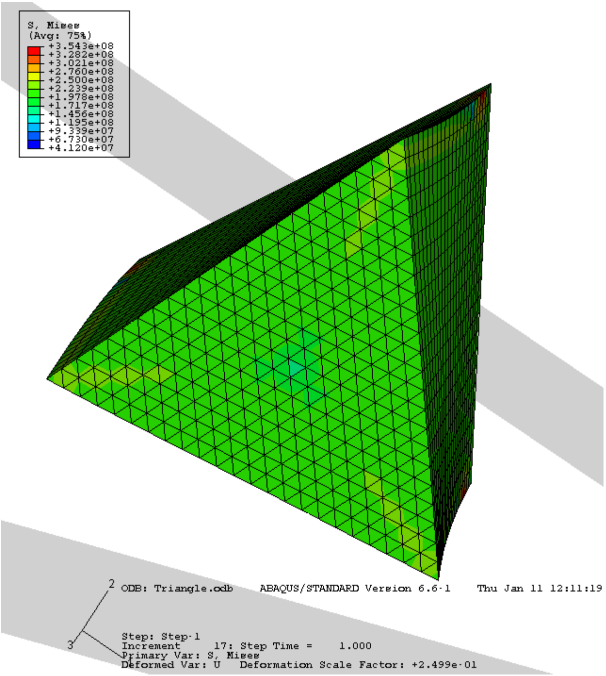
Solve the problem using FEM.

Project V.
FEM Calculation of Warping (Saint-Venant Torsion)

$(0, 1)$



$(-\sqrt{2}/2, 0)$ $(\sqrt{2}/2, 0)$



105 Points: Solving for $\psi, \sigma_{xz}, \sigma_{yz}$ and T; plot for ψ;

Summary of the St. Venant's Theory of Torsion

$$u = -\theta z y,\ v = \theta z x,\ w = \theta\psi$$

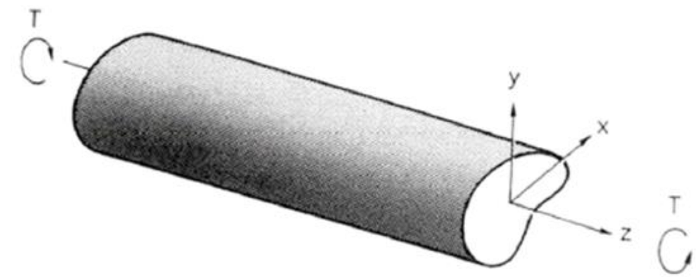
$$\sigma_{xz} = \theta G\Big(-y + \frac{\partial\psi}{\partial x}\Big),$$

$$\sigma_{yz} = \theta G\Big(x + \frac{\partial\psi}{\partial y}\Big)$$

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0,\ \forall(x,y)\in\Omega$$

$$\frac{\partial\psi}{\partial n} = (yn_x - xn_y),\ \forall(x,y)\in\partial\Omega$$

$$\theta = 0.2rad,\quad G = 50MP_a$$



$$\Pi = \frac{G\theta^2}{2}\int_\Omega\Big[\Big(\frac{\partial\psi}{\partial x} - y\Big)^2 + \Big(\frac{\partial\psi}{\partial y} + x\Big)^2\Big]d\Omega - T\theta$$

Integration by parts,

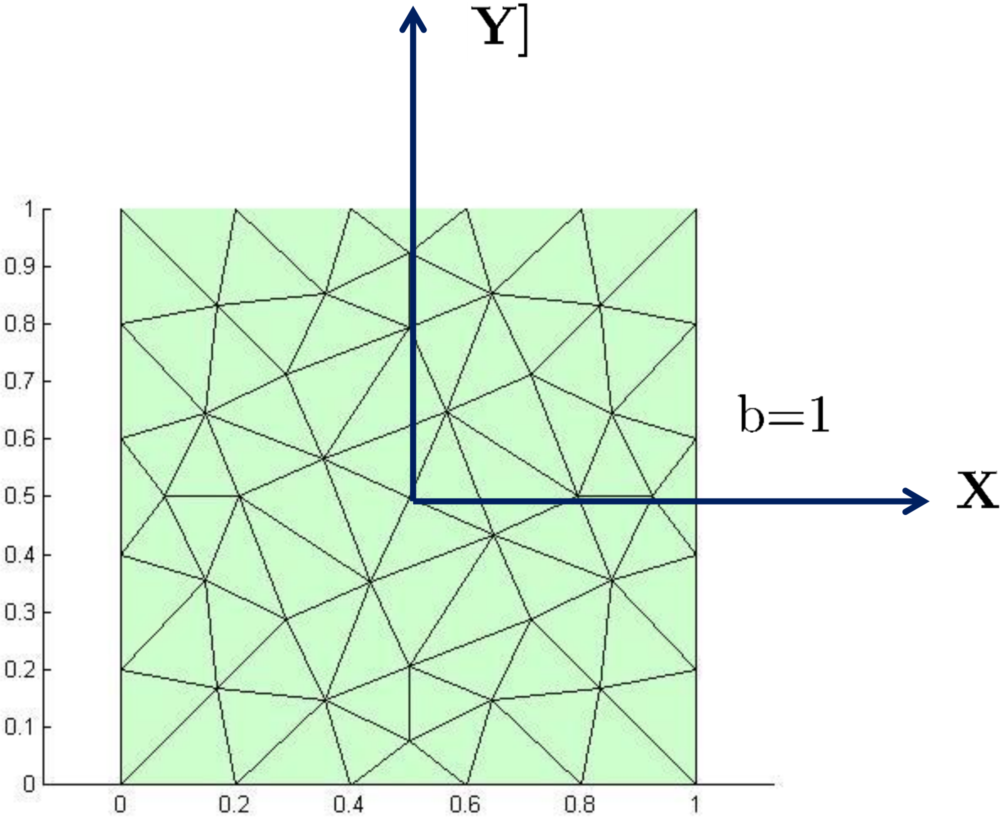
$$G\theta^2 \int_{\partial\Omega} \Big[\Big(\frac{\partial\psi}{\partial x} - y\Big)n_x + \Big(\frac{\partial\psi}{\partial y} + x\Big)n_y\Big]\delta\psi dS$$

$$-G\theta^2 \int_{\Omega} \Big(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\Big)\delta\psi d\Omega = 0 \ .$$

We recovered the governing equation and the boundary condition.

Given $\theta = 0.2rad$, we solve for ψ, T, and shear stress distributions.

Project VI Anti-plane Shear Problem



$a = 1$

Solve for $w, \gamma_{xz}, \gamma_{yz}, \sigma_{xz}$ and σ_{yz}. (**100 Points)**

Let $b(x,y) = b_0(1 - 4(x+y))$, the governing equation is

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + b(x,y) = 0 \quad \rightarrow$$

$$G\Big(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\Big) + b(x,y) = G\nabla^2 w + b(x,y) = 0 \ , \forall (x,y) \in \Omega$$

$$w = 0, \ \forall (x,y) \in \partial\Omega$$

where $G = 50MP_a$, t=1 M, and $b_0 = 10^5 N/M^3$.

$$\Pi = \frac{1}{2}Gt\int_A \Big[\Big(\frac{\partial w}{\partial x}\Big)^2 + \Big(\frac{\partial w}{\partial y}\Big)^2\Big] dA - \int_A (bt)w dA$$

Extra Pointe 5: Solve Dynamic Equations:

Let $b(x, y) = b_0(1 - 4(x + y)) \sin \omega t$, with $\omega = 2$.
The governing equation is

$$G\Big(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\Big) + b(x, y) = \rho \frac{\partial^2 w}{\partial t^2} \ , \forall (x, y) \in \Omega$$

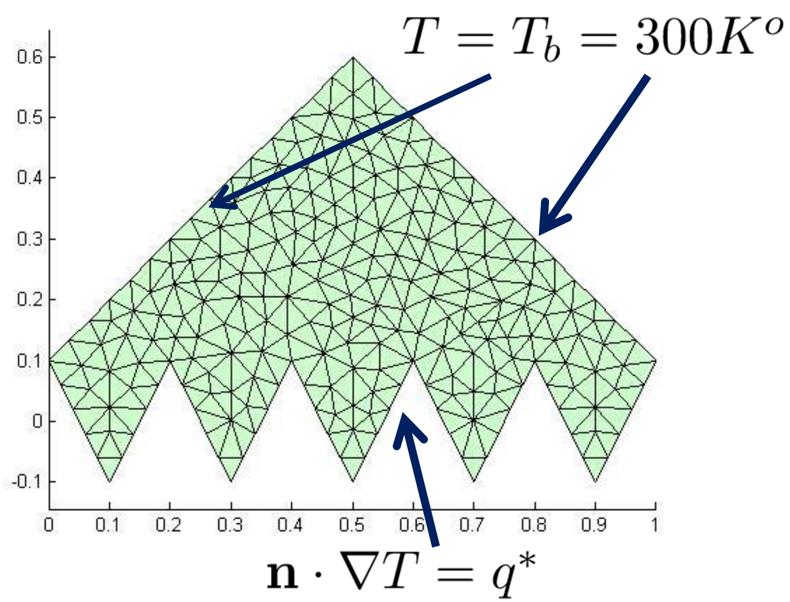
$$w = 0, \ \forall (x, y) \in \partial \Omega$$

where $G = 50 MP_a$ and $b_0 = 10^5 N/M^3$, with $\rho = 7000 kg/m^3$.
and proper initial conditions.

The Project III: Weak formulation is,

$$Kt \int_{\Omega} \Big(\frac{\partial T}{\partial x} \frac{\partial \delta T}{\partial x} + \frac{\partial T}{\partial y} \frac{\partial \delta T}{\partial y} \Big) d\Omega - t \int_{\Omega} Q \delta T d\Omega - \int_{\Gamma_q} q^* \delta T dS = 0$$

In the project, we consider Ω as a circular domain, $K = 1, Q = 1$, and $t = 1$; and $q^* = 0.2$.



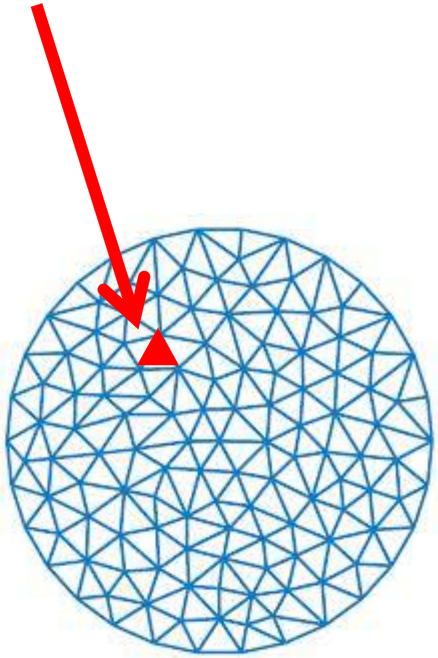
Finite Element Method

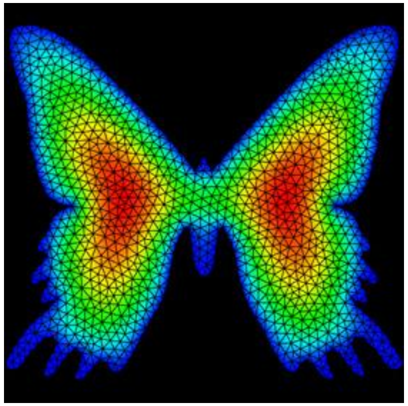
$$\delta\Pi = Kt \int_{\Omega} \nabla(\delta T) \cdot (\nabla T) d\Omega - \int_{\Omega} tQ\delta T d\Omega - \int_{\Gamma_q} \bar{q}\delta T dS = 0 \ .$$

$$\delta\Pi_e = Kt \int_{\Omega_e} \nabla(\delta T) \cdot (\nabla T) d\Omega - \int_{\Omega_e} tQ\delta T d\Omega - \int_{\Gamma_e^q} \bar{q}\delta T dS = 0 \ .$$

FEM discretization:

$$\Omega = \cup_{e=1}^{nelem} \Omega_e$$





Reproducing Property (Completness)

Let $T(x,y) = 1, x, y$, and

$$T^h(x,y) = N_1(x,y)T_1 + N_2(x,y)T_2 + N_3(x,y)T_3$$

Then

$$\begin{bmatrix} 1 \\ x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} N_1(x,y) \\ N_2(x,y) \\ N_3(x,y) \end{bmatrix}$$

We can solve it,

$$\begin{bmatrix} N_1(x,y) \\ N_2(x,y) \\ N_3(x,y) \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \end{bmatrix} .$$

$$\mathrm{Inv} \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

$$\text{However in Matlab code :} \quad \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} = \text{Inv} \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}$$

We then have

$$\begin{bmatrix} N_1(x,y) \\ N_2(x,y) \\ N_3(x,y) \end{bmatrix} = \begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \end{bmatrix} .$$

$$[\mathbf{g}] := [\nabla T] = \begin{bmatrix} \dfrac{\partial N_1}{\partial x} & \dfrac{\partial N_2}{\partial x} & \dfrac{\partial N_3}{\partial x} \\ \dfrac{\partial N_1}{\partial y} & \dfrac{\partial N_2}{\partial y} & \dfrac{\partial N_3}{\partial y} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} .$$

Hence

$$[B] = \begin{bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{bmatrix} [N_1, N_2, N_3] = \begin{bmatrix} c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \rightarrow [\mathbf{B}] = \begin{bmatrix} c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} .$$

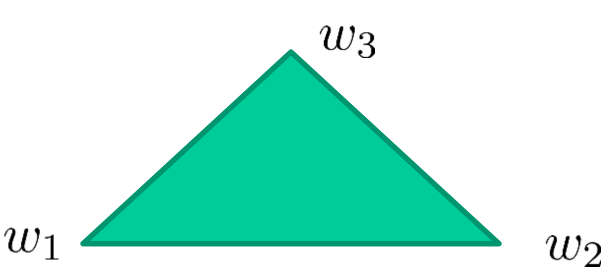
For the antiplane problems,

$$w^h(x,y) = N_1(x,y)w_1 + N_2(x,y)w_2 + N_3(x,y)w_3$$

$$[\gamma] := \begin{bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial w}{\partial x} \\[2ex] \dfrac{\partial w}{\partial y} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial N_1}{\partial x} & \dfrac{\partial N_2}{\partial x} & \dfrac{\partial N_3}{\partial x} \\[2ex] \dfrac{\partial N_1}{\partial y} & \dfrac{\partial N_2}{\partial y} & \dfrac{\partial N_3}{\partial y} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} .$$

Hence

$$[B] = \begin{bmatrix} \dfrac{\partial}{\partial x} \\[2ex] \dfrac{\partial}{\partial y} \end{bmatrix} [N_1, N_2, N_3] = \begin{bmatrix} c_{21} & c_{22} & c_{23} \\[2ex] c_{31} & c_{32} & c_{33} \end{bmatrix} \to [\mathbf{B}] = \begin{bmatrix} c_{21} & c_{22} & c_{23} \\[2ex] c_{31} & c_{32} & c_{33} \end{bmatrix} .$$



That is why this element is called the Constant Strain Triangle Element (CST).

$$\left(t\int_{\Omega_e}[B]^T[D][B]d\Omega\right)[T]-t\int_{\Omega_e}[N]Qd\Omega-\int_{\Gamma_e^q}[N]\bar{q}dS=0 \quad .$$

$$[K_e]=tK\int_{\Omega_e}\begin{bmatrix} c_{21} & c_{31}\\ c_{22} & c_{32}\\ c_{23} & c_{33}\end{bmatrix}\begin{bmatrix} c_{21} & c_{22} & c_{23}\\ c_{31} & c_{32} & c_{33}\end{bmatrix}d\Omega$$

$$=Kt\int_{\Omega_e}\begin{bmatrix} c_{21}^2+c_{31}^2 & c_{21}c_{22}+c_{31}c_{32} & c_{21}c_{23}+c_{31}c_{33}\\ c_{22}c_{21}+c_{32}c_{31} & c_{22}^2+c_{32}^2 & c_{22}c_{23}+c_{32}c_{33}\\ c_{23}c_{21}+c_{33}c_{31} & c_{23}c_{22}+c_{33}c_{32} & c_{23}^2+c_{33}^2\end{bmatrix}d\Omega$$

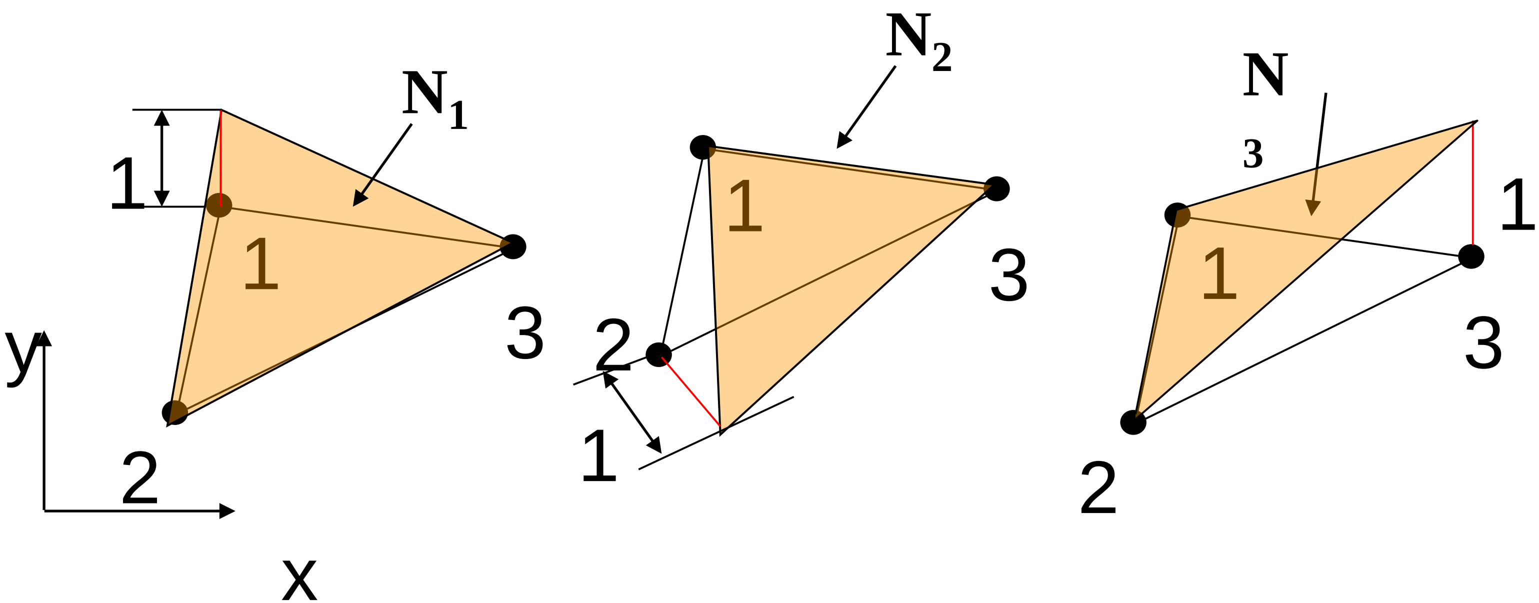
$$K_{ij}=C(2,i)C(2,j)+C(3,i)C(3,j),\quad i,j,=1,2,3$$

$$[f_Q]=t\int_{\Omega_e}[N]Qd\Omega,\quad \text{and}\quad [f_q]=\int_{\Gamma_e^q}\bar{q}[N]dS$$

For constant Q, we can calculate

$$[f_Q] = Qt \int_{\Omega_e} [N] d\Omega = \frac{tA_eQ}{3} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} .$$

$$\int_{\Omega_e} \begin{bmatrix} N_1(x,y) \\ N_2(x,y) \\ N_3(x,y) \end{bmatrix} d\Omega = \frac{A_e}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$



```
function u = fempoi(p,t,e)

% size of the system
N = size(p,1); T = size(t,1);

% matrix allocation
A = sparse(N,N); b= zeros(N,1);
Ak = zeros(3,3); bk = zeros(3,1);
for i = 1:T
   % get the node coordinates of triangle k
   V = [1, p(t(i,1),1), p(t(i,1),2);
      1, p(t(i,2),1), p(t(i,2),2);
      1, p(t(i,3),1), p(t(i,3),2)];
```

$$\mathrm{Inv}\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \; ?$$

$$= \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

fempoi.m

```
C = inv(V)

   area = 0.5*abs(det(V));
  for ik = 1:3
      for jk = 1:3
         Ak(ik,jk) = C(2,ik)*C(2,jk) + C(3,ik)*C(3,jk);
      end
      bk(ik) = area/3;
   end
% assembly, global stiffness and force vector
   A(t(i,:),t(i,:)) = A(t(i,:),t(i,:)) + Ak*area;
   b(t(i,:)) = b(t(i,:)) + bk;
end
```

$$[K_e] = tK\int_{\Omega_e} \begin{bmatrix} c_{21} & c_{31} \\ c_{22} & c_{32} \\ c_{23} & c_{33} \end{bmatrix} \begin{bmatrix} c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} d\Omega$$

$$K_{ij} = C(2,1)C(2,j) + C(3,i)C(3,j)$$

$$[f_Q] = \frac{t\Omega_e Q}{3} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} .$$

For constant q^*, we have

$$[f_q] = q^* \int_{\Gamma_e} [N] dS = q^* \int_{\Gamma_e} \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} dS$$

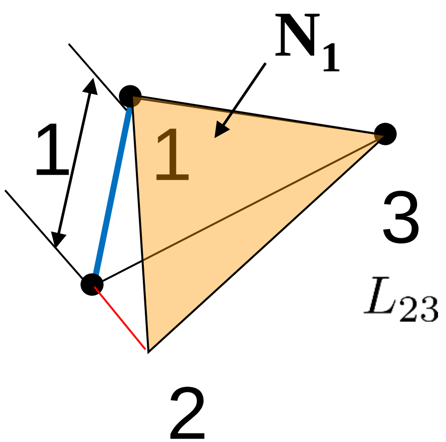
which can be reduced to

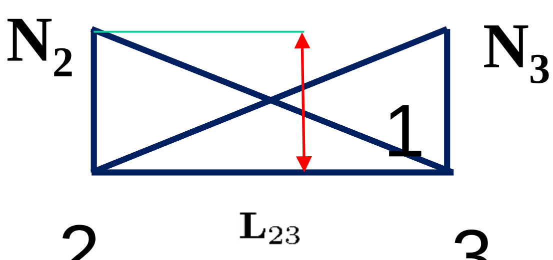
$$\frac{q^* L_{12}}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

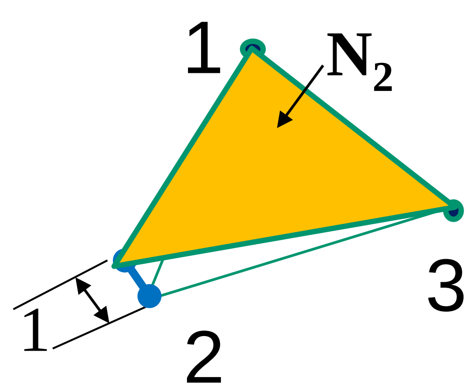
$$\frac{q^* L_{23}}{2} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

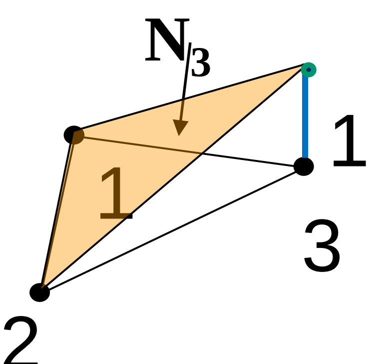
$$\frac{q^* L_{13}}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

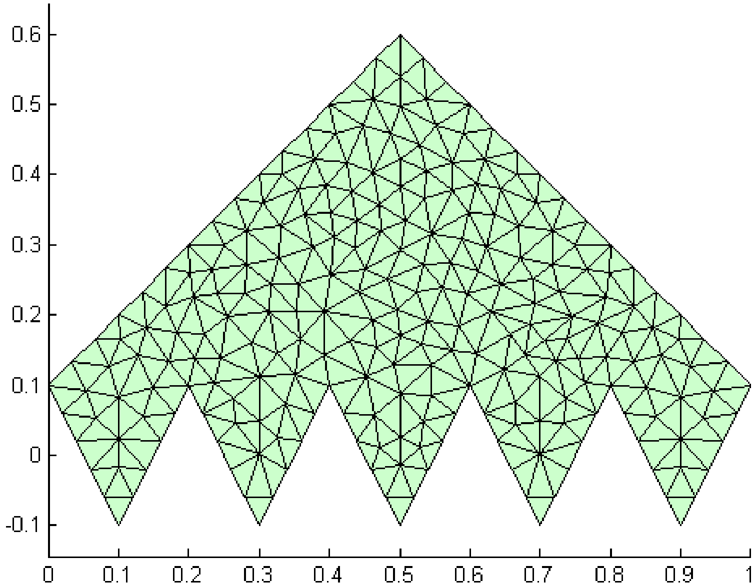
```
%% Loop over all line element
for i = 1:Nl
    ldx = [en(i),en(i+1)];    % The index of line element
    % The length of the current line element
    len = sqrt((p(ldx(1),1)-p(ldx(2),1))^2 + (p(ldx(1),2)-p(ldx(2),2))^2);
    % Assembly with q* = 0.2
    b(ldx) = b(ldx) + 0.2*len/2*[1;1];
end
```












```
% Complex polygon geometry,
% mixed Dirichlet/Neumann

x=(0:.1:1)';
y=.1*cos(10*pi*x);
pv=[x,y; .5,.6;0,0.1];
[p,t,e]=pmesh(pv,0.05,0);
e=e(p(e,2)>=.6-abs(p(e,1)-.5));
u=fempoi(p,t,e);
figure; tplot(p,t,u)
```

How to find the Neumann Boundary Node Set ?

You only need one line matlab sentence

```
%% Define boundary conditions
% index of Dirichlet boundary
ed=e(p(e,2)>=.6-abs(p(e,1)-.5));
% index of Neumann boundary
en = setdiff(e,ed);
en = [ed(1);en;ed(2)];
```

How to code:

$$[f_q] = q^* \int_{\Gamma_e} [N] dS = q^* \int_{\Gamma_e} \left[\begin{array}{c} N_1 \\ N_2 \\ N_3 \end{array} \right] dS$$

is your own business !

```
% assembly, global stiffness and force vector
    A(t(i,:),t(i,:)) = A(t(i,:),t(i,:)) + Ak*area;
    b(t(i,:)) = b(t(i,:)) + bk;
end

% for i = 1:length(e)
%    A(e(i),e(i)) = 1.0e30; b(e(i)) = 0;
% end

for i = 1:length(e)
    for j = 1:N
        A(e(i),j) = 0; A(j,e(i)) = 0;
    end
   A(e(i),e(i)) = 1; b(e(i)) = 0;
end



% solve the system;
u = A\b;

end
```

You cannot use this !

How to find idf and id
is your own business

$$\begin{bmatrix} \mathbf{K}_{ff} & \mathbf{K}_{fd} \\ \mathbf{K}_{df} & \mathbf{K}_{dd} \end{bmatrix} \begin{bmatrix} \mathbf{T}_f \\ \mathbf{T}_d \end{bmatrix} = \begin{bmatrix} \mathbf{F}_f \\ \mathbf{F}_d \end{bmatrix}$$

$$\boxed{\mathbf{T}_f = \mathbf{K}_{ff}^{-1}(\mathbf{F}_f - \mathbf{K}_{fd}\mathbf{T}_d)}$$

```
% Perform the solve
Aff =  A(idf,idf);
Afd =  A(idf,idd);
Adf =  A(idd,idf);
Add =  A(idd,idd);
Bf  =  b(idf);

u(idf) = Aff\(Bf-Afd*u(idd));
b(idd) = Adf*u(idf)+Add*u(idd)
```

Project 4:

Choose $S = 1$. Solve

$$S\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + p(x,y) = \rho \frac{\partial^2 w}{\partial t^2} \quad \forall\ (x,y) \in \Omega,$$

$$\Omega = \left\{ (x,y) \;\middle|\; \text{is an elliptic membrane} \right\}$$

Choose $\rho = 10$.

Boundary condition,

$$w(x,y) = 0, \quad \forall (x,y) \in \partial\Omega$$

Choose

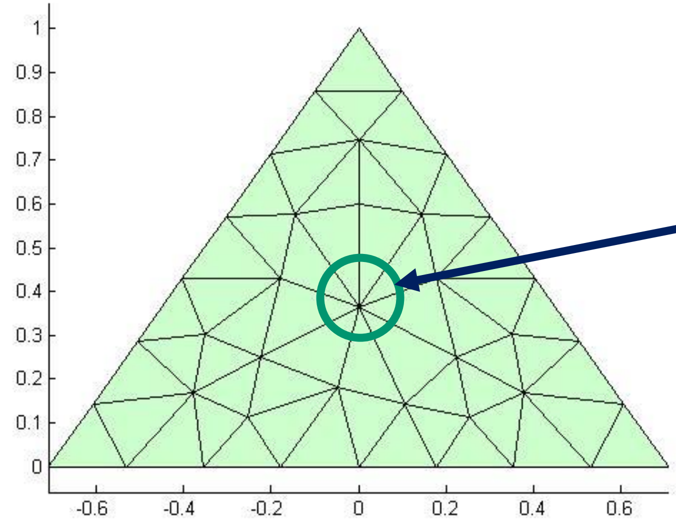
$$p(x,y) = q_0 \mathbf{j} \sin \omega t, \ \forall\ |\mathbf{r}| < 1 \text{ where } \ r = x\mathbf{i} + y\mathbf{j}, \ \omega = 10\ rad/s, \text{ and } q_0 = 1\ .$$

Project 5:

$$\delta\Pi = G\theta^2 \left\{ \int_\Omega \Big[\Big(\frac{\partial \psi}{\partial x} - y \Big) \frac{\partial \delta\psi}{\partial x} + \Big(\frac{\partial \psi}{\partial y} + x \Big) \frac{\partial \delta\psi}{\partial y} \Big] d\Omega \right\} = 0$$

Let $\mathbf{Q} = y\mathbf{i} - x\mathbf{j}$: $\quad \rightarrow \quad \int_\Omega \nabla\delta\psi \cdot \nabla\psi d\Omega - \int_\Omega \nabla\delta\psi \cdot \mathbf{Q} d\Omega = 0$.

$(0, 1)$



Fix it !

$(-\sqrt{2}/2, 0)$ $\quad$ $(\sqrt{2}/2, 0)$

Project 6: Solve for $w, \gamma_{xz}, \gamma_{yz}, \sigma_{xz}$ and σ_{yz}.

Let $b(x,y) = b_0(1 - 4(x+y))$, the governing equation is

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \cancel{\frac{\partial \sigma_{zz}}{\partial z}} + b(x,y) = 0 \quad \rightarrow$$

$$G\Big(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\Big) + b(x,y) = G\nabla^2 w + b(x,y) = 0 \ , \forall (x,y) \in \Omega$$

$$w = 0, \ \ \forall (x,y) \in \partial\Omega$$

where $G = 50MP_a$ and $b_0 = 10^5 N/M^3$.

$$\Pi = \frac{1}{2}Gt\int_A \Big[\Big(\frac{\partial w}{\partial x}\Big)^2 + \Big(\frac{\partial w}{\partial y}\Big)^2\Big]dA - \int_A (bt)w\,dA$$

$$\Pi = \frac{1}{2}Gt\int_A \Big[\Big(\frac{\partial w}{\partial x}\Big)^2 + \Big(\frac{\partial w}{\partial y}\Big)^2\Big]dA - \int_A (bt)w dA$$

How to calculate

$$t\int_{A_e} b(x,y)\delta w(x,y)dA \quad ?$$

$b(x,y) = b0(1-4(x+y))$.

This leads to

$$[f_Q] = t\int_{A_e} b(x,y)\begin{bmatrix} N_1(x,y) \\ N_2(x,y) \\ N_3(x,y) \end{bmatrix} dA$$

Table 10.1 *Exact integrals for a triangle*

m	n	$I = \int_A x^m y^n \, dx \, dy$
0	0	$\int dA = A = [\, x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\,]/2$
0	1	$\int y \, dA = A\bar{y} = A(y_1 + y_2 + y_3)/3$
1	0	$\int x \, dA = A\bar{x} = A(x_1 + x_2 + x_3)/3$
0	2	$\int y^2 \, dA = A(y_1^2 + y_2^2 + y_3^2 + 9\bar{y}^2)/12$
1	1	$\int xy \, dA = A(x_1 y_1 + x_2 y_2 + x_3 y_3 + 9\overline{x\,y}\,)/12$
2	0	$\int x^2 \, dA = A(x_1^2 + x_2^2 + x_3^2 + 9\bar{x}^2)/12$

Possible Questions ?

1. How to derive the weak form if you know the total potential energy ?
2. How to derive the element stiffness matrix ?
3. How to calculate element external force ?
4. How to assembe globale stiffness matrix ?

Project related problems:

5. What is Fourier Law ?
6. What is warping ?
7. What is anti-plane deformation ?
8. How to derive the potential energy for a membrane ?
9. How to derive the governing equation for a beam on the Winkler foundation ?
10. How to derive the A-matrix for a truss structure ?

11. How to take variational derivative ?

Example 7.2

Anti-Plane Shear

We illustrate here the proof that our multidimensional potential energy formulation is equivalent to the partial differential equations given above. We will do this for the anti-plane shear case. Consider a domain A with boundary ∂A, where w is stated to be zero on ∂A. In this case, $\mathcal{S} = \{w(x,y) \mid w = 0 \text{ for } (x,y) \in \partial A\}$ and $\mathcal{V} = \{\delta w(x,y) \mid \delta w = 0 \text{ for } (x,y) \in \partial A\}$. The minimization problem is

$$\min_{w \in \mathcal{S}} \Pi[w(x,y)] = \min_{w \in \mathcal{S}} \left[\int_A \frac{1}{2} G t \nabla w \cdot \nabla w \, dxdy - \int_A btw \, dxdy \right] .$$

12. What is essential (Dirichelt) boundary condition ? and what is natural (Neumann) boundary condition ?

13. Understand and can derive BC for both Saint-Venant torsion and Prandtle stress function problem ?

14. Understand and can derive BC for thermal conduction problem ?

15. How to construct kinematically admissible trial functions;

16. As a special Ritz method, what are the special advantages FEM has ?

17. What is the main difference between the Prandlt stress function method and the Saint-Venant torsion solution ?

18. What are the main differences between method of conservation energy and the minimum potential energy principle ?

19. What is the conservative force ?

20. Given a weak formulation of the assigned project, can you derive the strong form and the natural boundary conditions ?

21. Given a weak formulation of the assigned project, can you derive the FEM discrete algebraic equations ?

Wish you having a great presentation !