CHAPTER 13

Heat Transfer and Mass Transport

Introduction

In this chapter, we present the first use in this text of the finite element method for solution of nonstructural problems. We first consider the heat-transfer problem, although many similar problems, such as seepage through porous media, torsion of shafts, and magnetostatics [3], can also be treated by the same form of equations (but with different physical characteristics) as that for heat transfer.

Familiarity with the heat-transfer problem makes possible determination of the temperature distribution within a body. We can then determine the amount of heat moving into or out of the body and the thermal stresses.

We begin with a derivation of the basic differential equation for heat conduction in one dimension and then extend this derivation to the two-dimensional case. We will then review the units used for the physical quantities involved in heat transfer.

In preceding chapters dealing with stress analysis, we used the principle of minimum potential energy to derive the element equations, where an assumed displacement function within each element was used as a starting point in the derivation. We will now use a similar procedure for the nonstructural heat-transfer problem. We define an assumed temperature function within each element. Instead of minimizing a potential energy functional, we minimize a similar functional to obtain the element equations. Matrices analogous to the stiffness and force matrices of the structural problem result.

We will consider one-, two-, and three-dimensional finite element formulations of the heat-transfer problem and provide illustrative examples of the determination of the temperature distribution along the length of a rod and within a two-dimensional body and show some three-dimensional heat transfer examples as well.

Next, we will consider the contribution of fluid mass transport. The one-dimensional mass-transport phenomenon is included in the basic heat-transfer differential equation. Because it is not readily apparent that a variational formulation is possible for this problem, we will apply Galerkin's residual method directly to the differential equation to obtain the finite element equations. (You should note that the mass transport stiffness matrix is asymmetric.) We will compare an analytical solution to the finite element solution for a heat exchanger design/analysis problem to show the excellent agreement.

Finally, we will present some computer program results for two-dimensional heat transfer.

13.1 Derivation of the Basic Differential Equation

One-Dimensional Heat Conduction (without Convection)

We now consider the derivation of the basic differential equation for the one-dimensional problem of heat conduction without convection. The purpose of this derivation is to present a physical insight into the heat-transfer phenomena, which must be understood so that the finite element formulation of the problem can be fully understood. (For additional information on heat transfer, consult texts such as References [1] and [2].) We begin with the control volume shown in Figure 13–1. By conservation of energy, we have

$$E_{\text{in}} + E_{\text{generated}} = \Delta U + E_{\text{out}} \tag{13.1.1}$$

or

$$q_x A\, dt + QA\, dx\, dt = \Delta U + q_{x+dx} A\, dt \tag{13.1.2}$$

where

E_{in} is the energy entering the control volume, in units of joules (J) or kW · h or Btu.

ΔU is the change in stored energy, in units of kW · h (kWh) or Btu.

q_x is the heat conducted (heat flux) into the control volume at surface edge x, in units of kW/m^2 or Btu/(h-ft^2).

q_{x+dx} is the heat conducted out of the control volume at the surface edge $x + dx$.

t is time, in h or s (in U.S. customary units) or s (in SI units).

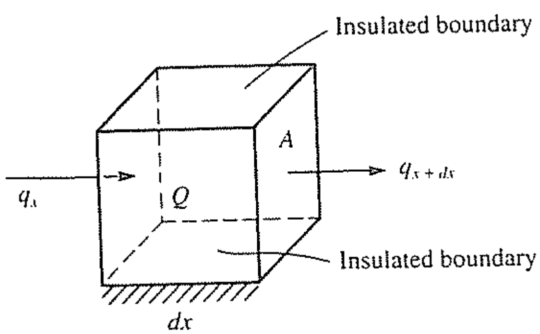


Figure 13–1 Control volume for one-dimensional heat conduction

Q is the internal heat source (heat generated per unit time per unit volume is positive), in kW/m^3 or $Btu/(h\text{-}ft^3)$ (a heat sink, heat drawn out of the volume, is negative).

A is the cross-sectional area perpendicular to heat flow q, in m^2 or ft^2.

By Fourier's law of heat conduction,

$$q_x = -K_{xx}\frac{dT}{dx} \tag{13.1.3}$$

where

K_{xx} is the thermal conductivity in the x direction, in $kW/(m \cdot {}^\circ C)$ or $Btu/(h\text{-}ft\text{-}{}^\circ F)$.

T is the temperature, in °C or °F.

dT/dx is the temperature gradient, in °C/m or °F/ft.

Equation (13.1.3) states that the heat flux in the x direction is proportional to the gradient of temperature in the x direction. The minus sign in Eq. (13.1.3) implies that, by convention, heat flow is positive in the direction opposite the direction of temperature increase. Equation (13.1.3) is analogous to the one-dimensional stress/strain law for the stress analysis problem—that is, to $\sigma_x = E(du/dx)$. Similarly,

$$q_{x+dx} = -K_{xx}\left.\frac{dT}{dx}\right|_{x+dx} \tag{13.1.4}$$

where the gradient in Eq. (13.1.4) is evaluated at $x + dx$. By Taylor series expansion, for any general function $f(x)$, we have

$$f_{x+dx} = f_x + \frac{df}{dx}dx + \frac{d^2f}{dx^2}\frac{dx^2}{2} + \cdots$$

Therefore, using a two-term Taylor series, Eq. (13.1.4) becomes

$$q_{x+dx} = -\left[K_{xx}\frac{dT}{dx} + \frac{d}{dx}\left(K_{xx}\frac{dT}{dx}\right)dx\right] \tag{13.1.5}$$

The change in stored energy can be expressed by

$$\begin{aligned}\Delta U &= \text{specific heat} \times \text{mass} \times \text{change in temperature}\\ &= c(\rho A\, dx)\, dT\end{aligned} \tag{13.1.6}$$

where c is the specific heat in $kW \cdot h/(kg \cdot {}^\circ C)$ or $Btu/(slug\text{-}{}^\circ F)$, and ρ is the mass density in kg/m^3 or $slug/ft^3$. On substituting Eqs. (13.1.3), (13.1.5), and (13.1.6) into Eq. (13.1.2), dividing Eq. (13.1.2) by $A\,dx\,dt$, and simplifying, we have the one-dimensional heat conduction equation as

$$\frac{\partial}{\partial x}\left(K_{xx}\frac{\partial T}{\partial x}\right) + Q = \rho c\frac{\partial T}{\partial t} \tag{13.1.7}$$

For steady state, any differentiation with respect to time is equal to zero, so Eq. (13.1.7) becomes

$$\frac{d}{dx}\left(K_{xx}\frac{dT}{dx}\right) + Q = 0 \tag{13.1.8}$$

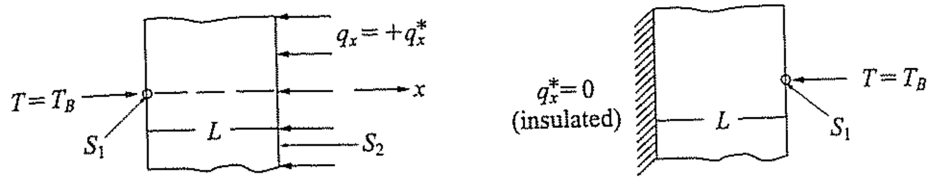


Figure 13–2 Examples of boundary conditions in one-dimensional heat conduction

For constant thermal conductivity and steady state, Eq. (13.1.7) becomes

$$K_{xx}\frac{d^2T}{dx^2} + Q = 0 \tag{13.1.9}$$

The boundary conditions are of the form

$$T = T_B \qquad \text{on } S_1 \tag{13.1.10}$$

where T_B represents a known boundary temperature and S_1 is a surface where the temperature is known, and

$$q_x^* = -K_{xx}\frac{dT}{dx} = \text{constant} \qquad \text{on } S_2 \tag{13.1.11}$$

where S_2 is a surface where the prescribed heat flux q_x^* or temperature gradient is known. On an insulated boundary, $q_x^* = 0$. These different boundary conditions are shown in Figure 13–2, where by sign convention, positive q_x^* occurs when heat is flowing into the body, and negative q_x^* when heat is flowing out of the body.

Two-Dimensional Heat Conduction (Without Convection)

Consider the two-dimensional heat conduction problem in Figure 13–3. In a manner similar to the one-dimensional case, for steady-state conditions, we can show that for material properties coinciding with the global x and y directions,

$$\frac{\partial}{\partial x}\left(K_{xx}\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_{yy}\frac{\partial T}{\partial y}\right) + Q = 0 \tag{13.1.12}$$

with boundary conditions

$$T = T_B \qquad \text{on } S_1 \tag{13.1.13}$$

$$q_n = q_n^* = K_{xx}\frac{\partial T}{\partial x}C_x + K_{yy}\frac{\partial T}{\partial y}C_y = \text{constant} \qquad \text{on } S_2 \tag{13.1.14}$$

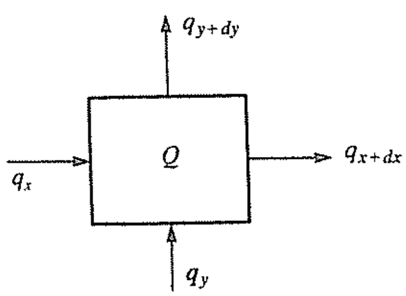


Figure 13–3 Control volume for two-dimensional heat conduction

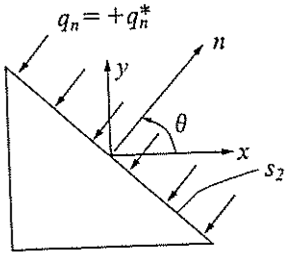


Figure 13–4 Unit vector normal to surface S_2

where C_x and C_y are the direction cosines of the unit vector n normal to the surface S_2 shown in Figure 13–4. Again, q_n^* is by sign convention, positive if heat is flowing into the edge of the body.

13.2 Heat Transfer with Convection

For a conducting solid in contact with a fluid, there will be a heat transfer taking place between the fluid and solid surface when a temperature difference occurs.

The fluid will be in motion either through external pumping action (**forced convection**) or through the buoyancy forces created within the fluid by the temperature differences within it (**natural** or **free convection**).

We will now consider the derivation of the basic differential equation for one-dimensional heat conduction with convection. Again we assume the temperature change is much greater in the x direction than in the y and z directions. Figure 13–5 shows the control volume used in the derivation. Again, by Eq. (13.1.1) for conservation of energy, we have

$$q_x A\,dt + QA\,dx\,dt = c(\rho A\,dx)\,dT + q_{x+dx}A\,dt + q_h P\,dx\,dt \tag{13.2.1}$$

In Eq. (13.2.1), all terms have the same meaning as in Section 13.1, except the heat flow by convective heat transfer is given by Newton's law of cooling

$$q_h = h(T - T_\infty) \tag{13.2.2}$$

where

h is the heat-transfer or convection coefficient, in kW/(m$^2\cdot$°C) or Btu/(h-ft^2-°F).

T is the temperature of the solid surface at the solid/fluid interface.

T_∞ is the temperature of the fluid (here the free-stream fluid temperature).

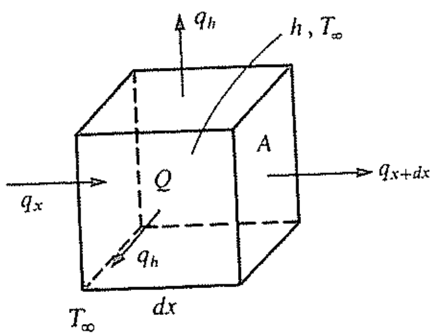


Figure 13–5 Control volume for one-dimensional heat conduction with convection

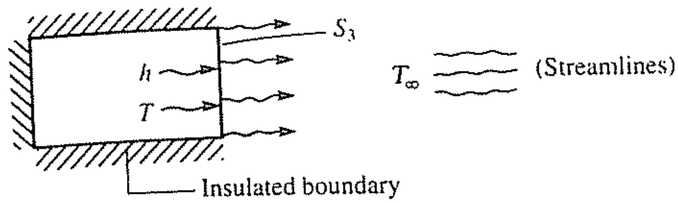


Figure 13–6 Model illustrating convective heat transfer (arrows on surface S_3 indicate heat transfer by convection)

P in Eq. (13.2.1) denotes the perimeter around the constant cross-sectional area A.

Again, using Eqs. (13.1.3)–(13.1.6) and (13.2.2) in Eq. (13.2.1), dividing by $A\,dx\,dt$, and simplifying, we obtain the equation for one-dimensional heat conduction with convection as

$$\frac{\partial}{\partial x}\left(K_{xx}\frac{\partial T}{\partial x}\right) + Q = \rho c\frac{\partial T}{\partial t} + \frac{hP}{A}(T - T_\infty) \tag{13.2.3}$$

with possible boundary conditions on (1) temperature, given by Eq. (13.1.10), and/or (2) temperature gradient, given by Eq. (13.1.11), and/or (3) loss of heat by convection from the ends of the one-dimensional body, as shown in Figure 13–6. Equating the heat flow in the solid wall to the heat flow in the fluid at the solid/fluid interface, we have

$$-K_{xx}\frac{dT}{dx} = h(T - T_\infty) \qquad \text{on } S_3 \tag{13.2.4}$$

as a boundary condition for the problem of heat conduction with convection.

13.3 Typical Units; Thermal Conductivities, *K*; and Heat-Transfer Coefficients, *h*

Table 13–1 lists some typical units used for the heat-transfer problem.

Table 13–2 lists some typical thermal conductivities of various solids and liquids. The thermal conductivity K, in Btu/(h-ft-°F) or W/(m · °C), measures the

Table 13–1 Typical units for heat transfer

Variable	SI	U.S. Customary
Thermal conductivity, K	kW/(m · °C)	Btu/(h-ft-°F)
Temperature, T	°C or K	°F or °R
Internal heat source, Q	kW/m^3	Btu/(h-ft^3)
Heat flux, q	kW/m^2	Btu/(h-ft^2)
Convection coefficient, h	kW/(m^2 · °C)	Btu/(h-ft^2-°F)
Energy, E	kW · h	Btu
Specific heat, c	(kW · h)/(kg · °C)	Btu/(slug-°F)
Mass density, ρ	kg/m^3	slug/ft^3

Table 13–2 Typical thermal conductivities of some solids and fluids

Material	K [Btu/(h-ft-°F)]	K [W/(m · °C)]
Solids		
Aluminum, 0 °C (32 °F)	117	202
Steel (1% carbon), 0 °C	20	35
Fiberglass, 20 °C (68 °F)	0.020	0.035
Concrete, 0 °C	0.468–0.81	0.81–1.40
Earth, coarse gravelly, 20 °C	0.300	0.520
Wood, oak, radial direction, 20 °C	0.098	0.17
Fluids		
Engine oil, 20 °C	0.084	0.145
Dry air, atmospheric pressure, 20 °C	0.014	0.0243

Table 13–3 Approximate values of convection heat-transfer coefficients (from Reference [1])

Mode	h [Btu/(h-ft²-°F)]	h [W/(m² · °C)]
Free convection, air	1–5	5–25
Forced convection, air	2–100	10–500
Forced convection, water	20–3,000	100–15,000
Boiling water	500–5,000	2,500–25,000
Condensation of water vapor	1,000–20,000	5,000–100,000

amount of heat energy (Btu or W · h) that will flow through a unit length (ft or m) of a given substance in a unit time (h) to raise the temperature one degree (°F or °C).

Table 13–3 lists approximate ranges of values of convection coefficients for various conditions of convection. The heat transfer coefficient h, in Btu/(h-ft^2-°F) or W/(m^2 · °C), measures the amount of heat energy (Btu or W · h) that will flow across a unit area (ft^2 or m^2) of a given substance in a unit time (h) to raise the temperature one degree (°F or °C).

Natural or **free convection** occurs when, for instance, a heated plate is exposed to ambient room air without an external source of motion. This movement of the air, experienced as a result of the density gradients near the plate, is called *natural* or *free convection*. **Forced convection** is experienced, for instance, in the case of a fan blowing air over a plate.

13.4 One-Dimensional Finite Element Formulation Using a Variational Method

The temperature distribution influences the amount of heat moving into or out of a body and also influences the stresses in a body. Thermal stresses occur in all bodies that experience a temperature gradient from some equilibrium state but are not free to expand in all directions. To evaluate thermal stresses, we need to know the temperature distribution in the body. The finite element method is a realistic method for predicting quantities such as temperature distribution and thermal stresses in a body. In this section, we formulate the one-dimensional heat-transfer equations using a variational method. Examples are included to illustrate the solution of this type of problem.

Step 1 Select Element Type

The basic element with nodes 1 and 2 is shown in Figure 13–7(a).

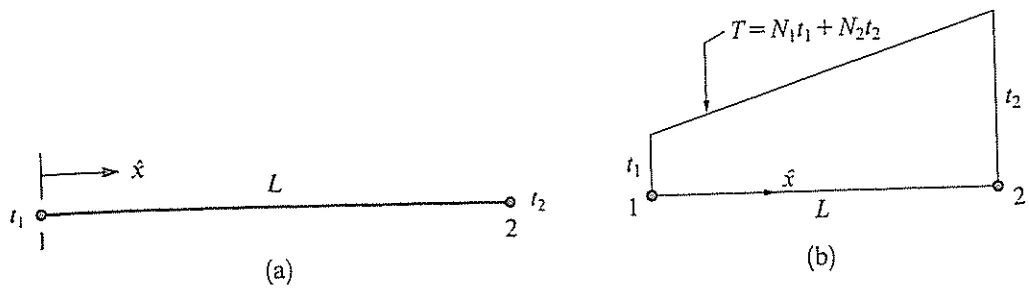


Figure 13–7 (a) Basic one-dimensional temperature element and (b) temperature variation along length of element

Step 2 Choose a Temperature Function

We choose the temperature function T [Figure 13–7(b)] within each element similar to the displacement function of Chapter 3, as

$$T(x) = N_1 t_1 + N_2 t_2 \tag{13.4.1}$$

where t_1 and t_2 are the nodal temperatures to be determined, and

$$N_1 = 1 - \frac{\hat{x}}{L} \qquad N_2 = \frac{\hat{x}}{L} \tag{13.4.2}$$

are again the same shape functions as used for the bar element. The $[N]$ matrix is then given by

$$[N] = \left[1 - \frac{\hat{x}}{L} \quad \frac{\hat{x}}{L}\right] \tag{13.4.3}$$

and the nodal temperature matrix is

$$\{t\} = \begin{Bmatrix} t_1 \\ t_2 \end{Bmatrix} \tag{13.4.4}$$

In matrix form, we express Eq. (13.4.1) as

$$\{T\} = [N]\{t\} \tag{13.4.5}$$

Step 3 Define the Temperature Gradient/Temperature and Heat Flux/Temperature Gradient Relationships

The temperature gradient matrix $\{g\}$, analogous to the strain matrix $\{\varepsilon\}$, is given by

$$\{g\} = \left\{\frac{dT}{d\hat{x}}\right\} = [B]\{t\} \tag{13.4.6}$$

where $[B]$ is obtained by substituting Eq. (13.4.1) for $T(\hat{x})$ into Eq. (13.4.6) and differentiating with respect to $\hat{x}$, that is,

$$[B] = \begin{bmatrix} \dfrac{dN_1}{d\hat{x}} & \dfrac{dN_2}{d\hat{x}} \end{bmatrix}$$

Using Eqs. (13.4.2) in the definition for $[B]$, we have

$$[B] = \begin{bmatrix} -\dfrac{1}{L} & \dfrac{1}{L} \end{bmatrix} \tag{13.4.7}$$

The heat flux/temperature gradient relationship is given by

$$q_x = -[D]\{g\} \tag{13.4.8}$$

where the material property matrix is now given by

$$[D] = [K_{xx}] \tag{13.4.9}$$

Step 4 Derive the Element Conduction Matrix and Equations

Equations (13.1.9)–(13.1.11) and (13.2.3) can be shown to be derivable (as shown, for instance, in References [4–6]) by the minimization of the following functional (analogous to the potential energy functional π_p):

$$\pi_h = U + \Omega_Q + \Omega_q + \Omega_h \tag{13.4.10}$$

where

$$U = \frac{1}{2}\iiint_V \left[K_{xx}\left(\frac{dT}{dx}\right)^2\right] dV$$

$$\Omega_Q = -\iiint_V QT\,dV \qquad \Omega_q = -\iint_{S_2} q^* T\,dS \qquad \Omega_h = \frac{1}{2}\iint_{S_3} h(T - T_\infty)^2\,dS \tag{13.4.11}$$

and where S_2 and S_3 are separate surface areas over which heat flow (flux) q^* (q^* is positive into the surface) and convection loss $h(T - T_\infty)$ are specified. We cannot specify q^* and h on the same surface because they cannot occur simultaneously on the same surface, as indicated by Eqs. (13.4.11).

Using Eqs. (13.4.5), (13.4.6), and (13.4.9) in Eq. (13.4.11) and then using Eq. (13.4.10), we can write π_h in matrix form as

$$\pi_h = \frac{1}{2}\iiint_V [\{g\}^T[D]\{g\}]\,dV - \iiint_V \{t\}^T[N]^T Q\,dV$$
$$- \iint_{S_2} \{t\}^T[N]^T q^*\,dS + \frac{1}{2}\iint_{S_3} h[(\{t\}^T[N]^T - T_\infty)^2]\,dS \tag{13.4.12}$$

On substituting Eq. (13.4.6) into Eq. (13.4.12) and using the fact that the nodal temperatures $\{t\}$ are independent of the general coordinates x and y and can therefore be taken outside the integrals, we have

$$
\begin{aligned}
\pi_h = {} & \frac{1}{2}\{t\}^T \iiint_V [B]^T[D][B]\,dV\{t\} - \{t\}^T \iiint_V [N]^T Q\,dV \\
& - \{t\}^T \iint_{S_2} [N]^T q^*\,dS + \frac{1}{2}\iint_{S_3} h[\{t\}^T[N]^T[N]\{t\} \\
& - (\{t\}^T[N]^T + [N]\{t\})T_\infty + T_\infty^2]\,dS
\end{aligned} \tag{13.4.13}
$$

In Eq. (13.4.13), the minimization is most easily accomplished by explicitly writing the surface integral S_3 with $\{t\}$ left inside the integral as shown. On minimizing Eq. (13.4.13) with respect to $\{t\}$, we obtain

$$
\begin{aligned}
\frac{\partial \pi_h}{\partial\{t\}} = {} & \iiint_V [B]^T[D][B]\,dV\{t\} - \iiint_V [N]^T Q\,dV \\
& - \iint_{S_2} [N]^T q^*\,dS + \iint_{S_3} h[N]^T[N]\,dS\{t\} \\
& - \iint_{S_3} [N]^T h T_\infty\,dS = 0
\end{aligned} \tag{13.4.14}
$$

where the last term hT_∞^2 in Eq. (13.4.13) is a constant that drops out while minimizing π_h. Simplifying Eq. (13.4.14), we obtain

$$
\left[\iiint_V [B]^T[D][B]\,dV + \iint_{S_3} h[N]^T[N]\,ds\right]\{t\} = \{f_Q\} + \{f_q\} + \{f_h\} \tag{13.4.15}
$$

where the force matrices have been defined by

$$
\begin{aligned}
& \{f_Q\} = \iiint_V [N]^T Q\,dV \qquad \{f_q\} = \iint_{S_2} [N]^T q^*\,dS \\
& \{f_h\} = \iint_{S_3} [N]^T h T_\infty\,dS
\end{aligned} \tag{13.4.16}
$$

In Eq. (13.4.16), the first term $\{f_Q\}$ (heat source positive, sink negative) is of the same form as the body-force term, and the second term $\{f_q\}$ (heat flux, positive into the surface) and third term $\{f_h\}$ (heat transfer or convection) are similar to surface tractions (distributed loading) in the stress analysis problem. You can observe this fact by comparing Eq. (13.4.16) with Eq. (6.2.46). Because we are formulating element equations of the form $f = \underline{k}t$, we have the element conduction matrix* for the heat-transfer problem given in Eq. (13.4.15) by

$$[k] = \iiint_V [B]^T[D][B]\,dV + \iint_{S_3} h[N]^T[N]\,dS \tag{13.4.17}$$

where the first and second integrals in Eq. (13.4.17) are the contributions of conduction and convection, respectively. Using Eq. (13.4.17) in Eq. (13.4.15), for each element, we have

$$\{f\} = [k]\{t\} \tag{13.4.18}$$

Using the first term of Eq. (13.4.17), along with Eqs. (13.4.7) and (13.4.9), the conduction part of the $[k]$ matrix for the one-dimensional element becomes

$$[k_c] = \iiint_V [B]^T[D][B]\,dV = \int_0^L \begin{Bmatrix} -\dfrac{1}{L} \\ \dfrac{1}{L} \end{Bmatrix} [K_{xx}] \begin{bmatrix} -\dfrac{1}{L} & \dfrac{1}{L} \end{bmatrix} A\,dx$$

$$= \frac{AK_{xx}}{L^2}\int_0^L \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} dx \tag{13.4.19}$$

or, finally,

$$[k_c] = \frac{AK_{xx}}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{13.4.20}$$

The convection part of the $[k]$ matrix becomes

$$[k_h] = \iint_{S_3} h[N]^T[N]\,dS = hP\int_0^L \begin{Bmatrix} 1 - \dfrac{\hat{x}}{L} \\ \dfrac{\hat{x}}{L} \end{Bmatrix} \begin{bmatrix} 1 - \dfrac{\hat{x}}{L} & \dfrac{\hat{x}}{L} \end{bmatrix} d\hat{x}$$

or, on integrating,

$$[k_h] = \frac{hPL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{13.4.21}$$

where

$$dS = P\,d\hat{x}$$

and P is the perimeter of the element (assumed to be constant). Therefore, adding Eqs. (13.4.20) and (13.4.21), we find that the $[k]$ matrix is

$$[k] = \frac{AK_{xx}}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{hPL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{13.4.22}$$

* The element conduction matrix is often called the *stiffness matrix* because *stiffness matrix* is becoming a generally accepted term used to describe the matrix of known coefficients multiplied by the unknown degrees of freedom, such as temperatures, displacements, and so on.

When h is zero on the boundary of an element, the second term on the right side of Eq. (13.4.22) (convection portion of $[k]$) is zero. This corresponds, for instance, to an insulated boundary.

The force matrix terms, on simplifying Eq. (13.4.16) and assuming Q, q^*, and product hT_∞ to be constant are

$$\{f_Q\} = \iiint_V [N]^T Q\,dV = QA\int_0^L \begin{Bmatrix} 1-\dfrac{\hat{x}}{L} \\ \dfrac{\hat{x}}{L} \end{Bmatrix} d\hat{x} = \frac{QAL}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \tag{13.4.23}$$

and
$$\{f_q\} = \iint_{S_2} q^*[N]^T\,dS = q^*P\int_0^L \begin{Bmatrix} 1-\dfrac{\hat{x}}{L} \\ \dfrac{\hat{x}}{L} \end{Bmatrix} d\hat{x} = \frac{q^*PL}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \tag{13.4.24}$$

and
$$\{f_h\} = \iint_{S_3} hT_\infty[N]^T\,dS = \frac{hT_\infty PL}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \tag{13.4.25}$$

Therefore, adding Eqs. (13.4.23)–(13.4.25), we obtain

$$\{f\} = \frac{QAL + q^*PL + hT_\infty PL}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \tag{13.4.26}$$

Equation (13.4.26) indicates that one-half of the assumed uniform heat source Q goes to each node, one-half of the prescribed uniform heat flux q^* (positive q^* enters the body) goes to each node, and one-half of the convection from the perimeter surface hT_∞ goes to each node of an element.

Finally, we must consider the convection from the free end of an element. For simplicity's sake, we will assume convection occurs only from the right end of the element, as shown in Figure 13–8. The additional convection term contribution to the stiffness matrix is given by

$$[k_h]_{\text{end}} = \iint_{S_{\text{end}}} h[N]^T[N]\,dS \tag{13.4.27}$$

Now $N_1 = 0$ and $N_2 = 1$ at the right end of the element. Substituting the N's into Eq. (13.4.27), we obtain

$$[k_h]_{\text{end}} = \iint_{S_{\text{end}}} h\begin{Bmatrix} 0 \\ 1 \end{Bmatrix}[0 \quad 1]\,dS = hA\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{13.4.28}$$

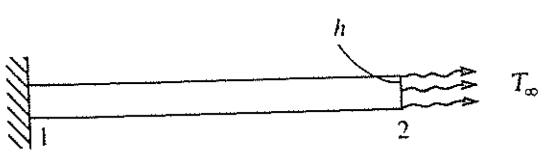


Figure 13–8 Convection force from the end of an element

The convection force from the free end of the element is obtained from the application of Eq. (13.4.25) with the shape functions now evaluated at the right end (where convection occurs) and with S_3 (the surface over which convection occurs) now equal to the cross-sectional area A of the rod. Hence,

$$\{f_h\}_{\text{end}} = hT_\infty A \begin{Bmatrix} N_1(\hat{x} = L) \\ N_2(\hat{x} = L) \end{Bmatrix} = hT_\infty A \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \tag{13.4.29}$$

represents the convective force from the right end of an element where $N_1(\hat{x} = L)$ represents N_1 *evaluated at* $\hat{x} = L$, and so on.

Step 5 Assemble the Element Equations to Obtain the Global Equations and Introduce Boundary Conditions

We obtain the global or total structure conduction matrix using the same procedure as for the structural problem (called the *direct stiffness method* as described in Section 2.4); that is,

$$[K] = \sum_{e=1}^{N} [k^{(e)}] \tag{13.4.30}$$

typically in units of kW/°C or Btu/(h-°F). The global force matrix is the sum of all element heat sources and is given by

$$\{F\} = \sum_{e=1}^{N} \{f^{(e)}\} \tag{13.4.31}$$

typically in units of kW or Btu/h. The global equations are then

$$\{F\} = [K]\{t\} \tag{13.4.32}$$

with the prescribed nodal temperature boundary conditions given by Eq. (13.1.13). Note that the boundary conditions on heat flux, Eq. (13.1.11), and convection, Eq. (13.2.4), are actually accounted for in the same manner as distributed loading was accounted for in the stress analysis problem; that is, they are included in the column of force matrices through a consistent approach (using the same shape functions used to derive $[k]$), as given by Eqs. (13.4.2).

The heat-transfer problem is now amenable to solution by the finite element method. The procedure used for solution is similar to that for the stress analysis problem. In Section 13.5, we will derive the specific equations used to solve the two-dimensional heat-transfer problem.

Step 6 Solve for the Nodal Temperatures

We now solve for the global nodal temperature, $\{t\}$, where the appropriate nodal temperature boundary conditions, Eq. (13.1.13), are specified.

Step 7 Solve for the Element Temperature Gradients and Heat Fluxes

Finally, we calculate the element temperature gradients from Eq. (13.4.6), and the heat fluxes, typically from Eq. (13.4.8).

To illustrate the use of the equations developed in this section, we will now solve some one-dimensional heat-transfer problems.

Example 13.1

Determine the temperature distribution along the length of the rod shown in Figure 13–9 with an insulated perimeter. The temperature at the left end is a constant 100 °F and the free-stream temperature is 10 °F. Let $h = 10$ Btu/(h-ft^2-°F) and $K_{xx} = 20$ Btu/(h-ft-°F). The value of h is typical for forced air convection and the value of K_{xx} is a typical conductivity for carbon steel (Tables 13–2 and 13–3).

The finite element discretization is shown in Figure 13–10. For simplicity's sake, we will use four elements, each 10 in. long. There will be convective heat loss only over the right end of the rod because we consider the left end to have a known temperature and the perimeter to be insulated. We calculate the stiffness matrices for each element as follows:

$$\frac{AK_{xx}}{L} = \frac{\pi(1\text{ in.})^2[20\text{ Btu/(h-ft-°F)}](1\text{ ft}^2)}{\left(\dfrac{10\text{ in.}}{12\text{ in./ft}}\right)(144\text{ in}^2)}$$

$$= 0.5236\text{ Btu/(h-°F)}$$

$$\frac{hPL}{6} = \frac{[10\text{ Btu/(h-ft}^2\text{-°F)}](2\pi)}{6}\left(\frac{1\text{ in.}}{12\text{ in./ft}}\right)\left(\frac{10\text{ in.}}{12\text{ in./ft}}\right) \tag{13.4.33}$$

$$= 0.7272\text{ Btu/(h-°F)}$$

$$hT_\infty PL = [10\text{ Btu/(h-ft}^2\text{-°F)}](10\text{ °F})(2\pi)\left(\frac{1\text{ in.}}{12\text{ in./ft}}\right)\left(\frac{10\text{ in.}}{12\text{ in./ft}}\right)$$

$$= 43.63\text{ Btu/h}$$

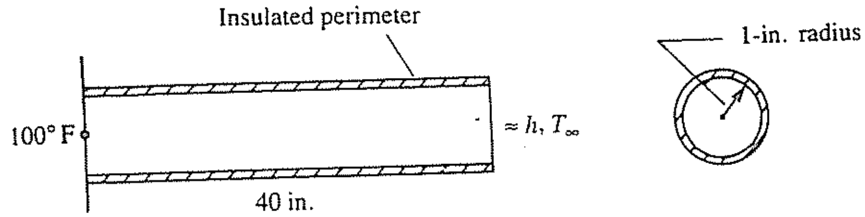


Figure 13–9 One-dimensional rod subjected to temperature variation

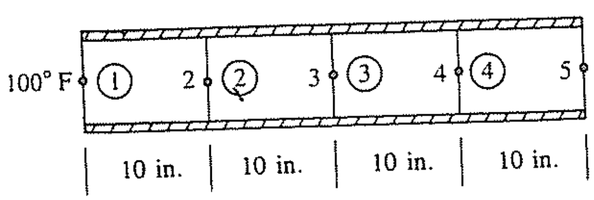


Figure 13–10 Finite element discretized rod

In general, from Eqs. (13.4.22) and (13.4.27), we have

$$[k] = \frac{AK_{xx}}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{hPL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + \iint_{S_{\text{end}}} h[N]^T[N]\,dS \tag{13.4.34}$$

Substituting Eqs. (13.4.33) into Eq. (13.4.34) for element 1, we have

$$[k^{(1)}] = 0.5236\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\ \text{Btu/(h-°F)} \tag{13.4.35}$$

where the second and third terms on the right side of Eq. (13.4.34) are zero because there are no convection terms associated with element 1. Similarly, for elements 2 and 3, we have

$$[k^{(2)}] = [k^{(3)}] = [k^{(1)}] \tag{13.4.36}$$

However, element 4 has an additional (convection) term owing to heat loss from the flat surface at its right end. Hence, using Eq. (13.4.28), we have

$$\begin{aligned}[k^{(4)}] &= [k^{(1)}] + hA\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\ &= 0.5236\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + [10\ \text{Btu/(h-ft}^2\text{-°F)}]\pi\left(\frac{1\ \text{in.}}{12\ \text{in./ft}}\right)^2\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0.5236 & -0.5236 \\ -0.5236 & 0.7418 \end{bmatrix}\ \text{Btu/(h-°F)}\end{aligned} \tag{13.4.37}$$

In general, we would use Eqs. (13.4.23)–(13.4.25), and (13.4.29) to obtain the element force matrices. However, in this example, $Q = 0$ (no heat source), $q^* = 0$ (no heat flux), and there is no convection except from the right end. Therefore,

$$\{f^{(1)}\} = \{f^{(2)}\} = \{f^{(3)}\} = 0 \tag{13.4.38}$$

and

$$\begin{aligned}\{f^{(4)}\} &= hT_\infty A\begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \\ &= [10\ \text{Btu/(h-ft}^2\text{-°F)}](10\,\text{°F})\pi\left(\frac{1\ \text{in.}}{12\ \text{in./ft}}\right)^2\begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \\ &= 2.182\begin{Bmatrix} 0 \\ 1 \end{Bmatrix}\ \text{Btu/h}\end{aligned} \tag{13.4.39}$$

The assembly of the element stiffness matrices [Eqs. (13.4.35)–(13.4.37)] and the element force matrices [Eqs. (13.4.38) and (13.4.39)], using the direct stiffness method, produces the following system of equations:

$$\begin{bmatrix} 0.5236 & -0.5236 & 0 & 0 & 0 \\ -0.5236 & 1.0472 & -0.5236 & 0 & 0 \\ 0 & -0.5236 & 1.0472 & -0.5236 & 0 \\ 0 & 0 & -0.5236 & 1.0472 & -0.5236 \\ 0 & 0 & 0 & -0.5236 & 0.7418 \end{bmatrix}\begin{Bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 0 \\ 0 \\ 0 \\ 2.182 \end{Bmatrix} \tag{13.4.40}$$

where F_1 corresponds to an unknown rate of heat flow at node 1 (analogous to an unknown support force in the stress analysis problem). We have a known nodal temperature boundary condition of $t_1 = 100\,°F$. This nonhomogeneous boundary condition must be treated in the same manner as was described for the stress analysis problem (see Section 2.5 and Appendix B.4). We modify the stiffness (conduction) matrix and force matrix as follows:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1.0472 & -0.5236 & 0 & 0 \\ 0 & -0.5236 & 1.0472 & -0.5236 & 0 \\ 0 & 0 & -0.5236 & 1.0472 & -0.5236 \\ 0 & 0 & 0 & -0.5236 & 0.7418 \end{bmatrix} \begin{Bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{Bmatrix} = \begin{Bmatrix} 100 \\ 52.36 \\ 0 \\ 0 \\ 2.182 \end{Bmatrix} \tag{13.4.41}$$

where the terms in the first row and column of the stiffness matrix corresponding to the known temperature condition, $t_1 = 100\,°F$, have been set equal to 0 except for the main diagonal, which has been set equal to 1, and the first row of the force matrix has been set equal to the known nodal temperature at node 1. Also, the term $(-0.5236) \times (100\,°F) = -52.36$ on the left side of the second equation of Eq. (13.4.40) has been transposed to the right side in the second row (as $+52.36$) of Eq. (13.4.41). The second through fifth equations of Eq. (13.4.41) corresponding to the rows of unknown nodal temperatures can now be solved (typically by Gaussian elimination). The resulting solution is given by

$$t_2 = 85.93\,°F \qquad t_3 = 71.87\,°F \qquad t_4 = 57.81\,°F \qquad t_5 = 43.75\,°F \tag{13.4.42}$$

For this elementary problem, the closed-form solution of the differential equation for conduction, Eq. (13.1.9), with the left-end boundary condition given by Eq. (13.1.10) and the right-end boundary condition given by Eq. (13.2.4) yields a linear temperature distribution through the length of the rod. The evaluation of this linear temperature function at 10-in. intervals (corresponding to the nodal points used in the finite element model) yields the same temperatures as obtained in this example by the finite element method. Because the temperature function was assumed to be linear in each finite element, this comparison is as expected. Note that F_1 could be determined by the first of Eqs. (13.4.40). ■

Example 13.2

To illustrate more fully the use of the equations developed in Section 13.4, we will now solve the heat-transfer problem shown in Figure 13–11. For the one-dimensional rod, determine the temperatures at 3-in. increments along the length of the rod and the rate of heat flow through element 1. Let $K_{xx} = 3$ Btu/(h-in.-°F), $h = 1.0$ Btu/(h-in^2-°F), and $T_\infty = 0\,°F$. The temperature at the left end of the rod is constant at 200 °F.

The finite element discretization is shown in Figure 13–12. Three elements are sufficient to enable us to determine temperatures at the four points along the rod, although more elements would yield answers more closely approximating the analytical solution obtained by solving the differential equation such as Eq. (13.2.3) with the partial derivative with respect to time equal to zero. There will be convective heat loss over the perimeter and the right end of the rod. The left end will not have convective heat loss. Using Eqs. (13.4.22) and (13.4.28), we calculate the stiffness matrices for the elements as follows:

$$\frac{AK_{xx}}{L} = \frac{(4\pi)(3)}{3} = 4\pi \text{ Btu/(h-°F)}$$
$$\frac{hPL}{6} = \frac{(1)(4\pi)(3)}{6} = 2\pi \text{ Btu/(h-°F)} \tag{13.4.43}$$
$$hA = (1)(4\pi) = 4\pi \text{ Btu/(h-°F)}$$

Substituting the results of Eqs. (13.4.43) into Eq. (13.4.22), we obtain the stiffness matrix for element 1 as

$$[k^{(1)}] = 4\pi\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + 2\pi\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$
$$= 4\pi\begin{bmatrix} 2 & -\frac{1}{2} \\ -\frac{1}{2} & 2 \end{bmatrix} \text{ Btu/(h-°F)} \tag{13.4.44}$$

Because there is no convection across the ends of element 1 (its left end has a known temperature and its right end is inside the whole rod and thus not exposed to fluid motion), the contribution to the stiffness matrix owing to convection from an end of the element, such as given by Eq. (13.4.28), is zero. Similarly,

$$[k^{(2)}] = [k^{(1)}] = 4\pi\begin{bmatrix} 2 & -\frac{1}{2} \\ -\frac{1}{2} & 2 \end{bmatrix} \text{ Btu/(h-°F)} \tag{13.4.45}$$

However, element 3 has an additional (convection) term owing to heat loss from the exposed surface at its right end. Therefore, Eq. (13.4.28) yields a contribution to the element 3 stiffness matrix, which is then given by

$$[k^{(3)}] = [k^{(1)}] + hA\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = 4\pi\begin{bmatrix} 2 & -\frac{1}{2} \\ -\frac{1}{2} & 2 \end{bmatrix} + 4\pi\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= 4\pi\begin{bmatrix} 2 & -\frac{1}{2} \\ -\frac{1}{2} & 3 \end{bmatrix} \text{ Btu/(h-°F)} \tag{13.4.46}$$

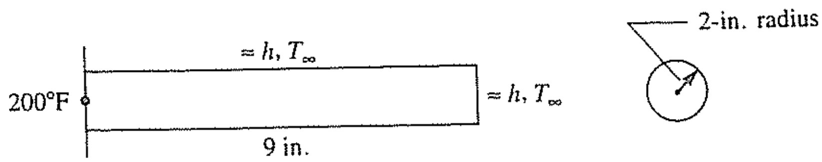


Figure 13–11 One-dimensional rod subjected to temperature variation

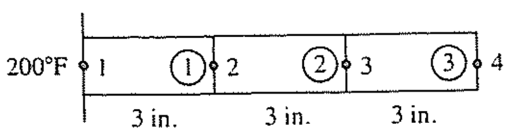


Figure 13–12 Finite element discretized rod of Figure 13–11

In general, we calculate the force matrices by using Eqs. (13.4.26) and (13.4.29). Because $Q = 0$, $q^* = 0$, and $T_\infty = 0\,^\circ\text{F}$, all force terms are equal to zero.

The assembly of the element matrices, Eqs. (13.4.44)–(13.4.46), using the direct stiffness method, produces the following system of equations:

$$4\pi \begin{bmatrix} 2 & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 4 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 4 & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 3 \end{bmatrix} \begin{Bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{13.4.47}$$

We have a known nodal temperature boundary condition of $t_1 = 200\,^\circ\text{F}$. As in Example 13.1, we modify the conduction matrix and force matrix as follows:

$$4\pi \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 4 & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 3 \end{bmatrix} \begin{Bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{Bmatrix} = \begin{Bmatrix} 800\pi \\ 400\pi \\ 0 \\ 0 \end{Bmatrix} \tag{13.4.48}$$

where the terms in the first row and column of the conduction matrix corresponding to the known temperature condition, $t_1 = 200\,^\circ\text{F}$, have been set equal to zero except for the main diagonal, which has been set to equal one, and the row of the force matrix has been set equal to the known nodal temperature at node 1. That is, the first row force is $(200)(4\pi) = 800\pi$, as we have left the 4π term as a multiplier of the elements inside the stiffness matrix. Also, the term $(-1/2)(200)(4\pi) = -400\pi$ on the left side of the second equation of Eq. (13.4.47) has been transposed to the right side in the second row (as $+400\pi$) of Eq. (13.4.48). The second through fourth equations of Eq. (13.4.48), corresponding to the rows of unknown nodal temperatures, can now be solved. The resulting solution is given by

$$t_2 = 25.4\,^\circ\text{F} \qquad t_3 = 3.24\,^\circ\text{F} \qquad t_4 = 0.54\,^\circ\text{F} \tag{13.4.49}$$

Next, we determine the heat flux for element 1 by using Eqs. (13.4.6) in (13.4.8) as

$$q^{(1)} = -K_{xx}[B]\{t\} \tag{13.4.50}$$

Using Eq. (13.4.7) in Eq. (13.4.50), we have

$$q^{(1)} = -K_{xx}\begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix}\begin{Bmatrix} t_1 \\ t_2 \end{Bmatrix} \tag{13.4.51}$$

Substituting the numerical values into Eq. (13.4.51), we obtain

$$q^{(1)} = -3\begin{bmatrix} -\frac{1}{3} & \frac{1}{3} \end{bmatrix}\begin{Bmatrix} 200 \\ 25.4 \end{Bmatrix}$$

or

$$q^{(1)} = 174.6 \text{ Btu/(h-in}^2) \tag{13.4.52}$$

We then determine the rate of heat flow $\bar{q}$ by multiplying Eq. (13.4.52) by the cross-sectional area over which q acts. Therefore,

$$\bar{q}^{(1)} = 174.6(4\pi) = 2194 \text{ Btu/h} \tag{13.4.53}$$

Here positive heat flow indicates heat flow from node 1 to node 2 (to the right). ■

Example 13.3

The plane wall shown in Figure 13–13 is 1 m thick. The left surface of the wall ($x = 0$) is maintained at a constant temperature of 200 °C, and the right surface ($x = L = 1$ m) is insulated. The thermal conductivity is $K_{xx} = 25$ W/(m · °C) and there is a uniform generation of heat inside the wall of $Q = 400$ W/m³. Determine the temperature distribution through the wall thickness.

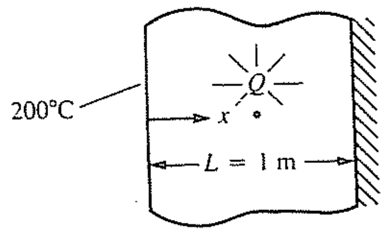


Figure 13–13 Conduction in a plane wall subjected to uniform heat generation

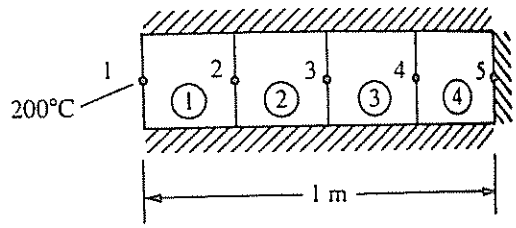


Figure 13–14 Discretized model of Figure 13–13

This problem is assumed to be approximated as a one-dimensional heat-transfer problem. The discretized model of the wall is shown in Figure 13–14. For simplicity, we use four equal-length elements all with unit cross-sectional area ($A = 1$ m²). The unit area represents a typical cross section of the wall. The perimeter of the wall model is then insulated to obtain the correct conditions.

Using Eqs. (13.4.22) and (13.4.28), we calculate the element stiffness matrices as follows:

$$\frac{AK_{xx}}{L} = \frac{(1\ \text{m}^2)[25\ \text{W}/(\text{m}\cdot{}^\circ\text{C})]}{0.25\ \text{m}} = 100\ \text{W}/{}^\circ\text{C}$$

For each identical element, we have

$$[k] = 100\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\ \text{W}/{}^\circ\text{C} \tag{13.4.54}$$

Because no convection occurs, h is equal to zero; therefore, there is no convection contribution to $\underline{k}$.

The element force matrices are given by Eq. (13.4.26). With $Q = 400$ W/m³, $q = 0$, and $h = 0$, Eq. (13.4.26) becomes

$$\{f\} = \frac{QAL}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \tag{13.4.55}$$

Evaluating Eq. (13.4.55) for a typical element, such as element 1, we obtain

$$\begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = \frac{(400\ \text{W/m}^3)(1\ \text{m}^2)(0.25\ \text{m})}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 50 \\ 50 \end{Bmatrix}\ \text{W} \tag{13.4.56}$$

The force matrices for all other elements are equal to Eq. (13.4.56).

The assemblage of the element matrices, Eqs. (13.4.54) and (13.4.56) and the other force matrices similar to Eq. (13.4.56), yields

$$100\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{Bmatrix} = \begin{Bmatrix} F_1 + 50 \\ 100 \\ 100 \\ 100 \\ 50 \end{Bmatrix} \tag{13.4.57}$$

Substituting the known temperature $t_1 = 200\,°\mathrm{C}$ into Eq. (13.4.57), dividing both sides of Eq. (13.4.57) by 100, and transposing known terms to the right side, we have

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{Bmatrix} = \begin{Bmatrix} 200\,°\mathrm{C} \\ 201 \\ 1 \\ 1 \\ 0.5 \end{Bmatrix} \tag{13.4.58}$$

The second through fifth equations of Eq. (13.4.58) can now be solved simultaneously to yield

$$t_2 = 203.5\,°\mathrm{C} \qquad t_3 = 206\,°\mathrm{C} \qquad t_4 = 207.5\,°\mathrm{C} \qquad t_5 = 208\,°\mathrm{C} \tag{13.4.59}$$

Using the first of Eqs. (13.4.57) yields the rate of heat flow out the left end:

$$F_1 = 100(t_1 - t_2) - 50$$

$$F_1 = 100(200 - 203.5) - 50$$

$$F_1 = -400\ \mathrm{W}$$

The closed-form solution of the differential equation for conduction, Eq. (13.1.9), with the left-end boundary condition given by Eq. (13.1.10) and the right-end boundary condition given by Eq. (13.1.11), and with $q_x^* = 0$, is shown in Reference [2] to yield a parabolic temperature distribution through the wall. Evaluating the expression for the temperature function given in Reference [2] for values of x corresponding to the node points of the finite element model, we obtain

$$t_2 = 203.5\,°\mathrm{C} \qquad t_3 = 206\,°\mathrm{C} \qquad t_4 = 207.5\,°\mathrm{C} \qquad t_5 = 208\,°\mathrm{C} \tag{13.4.60}$$

Figure 13–15 is a plot of the closed-form solution and the finite element solution for the temperature variation through the wall. The finite element nodal values and the closed-form values are equal, because the consistent equivalent force matrix has been used. (This was also discussed in Sections 3.10 and 3.11 for the axial bar subjected to distributed loading, and in Section 4.5 for the beam subjected to distributed loading.) However, recall that the finite element model predicts a linear temperature distribution within each element as indicated by the straight lines connecting the nodal temperature values in Figure 13–15.

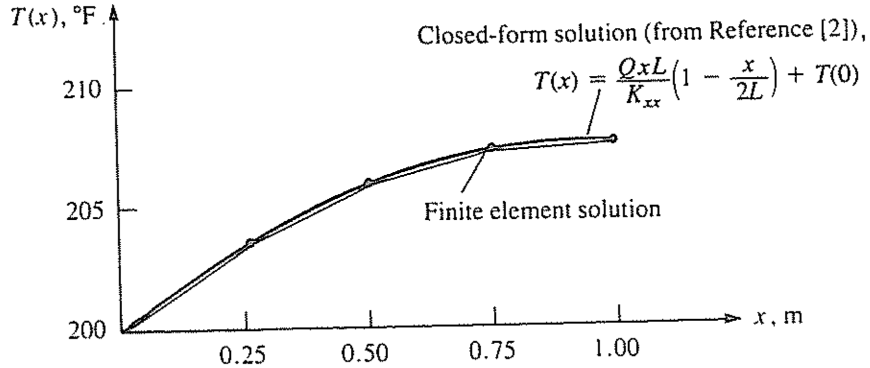


Figure 13–15 Comparison of the finite element and closed-form solutions for Example 13.3

Example 13.4

The fin shown in Figure 13–16 is insulated on the perimeter. The left end has a constant temperature of 100 °C. A positive heat flux of $q = 5000\ \mathrm{W/m^2}$ acts on the right end. Let $K_{xx} = 6\ \mathrm{W/(m\text{-}°C)}$ and cross-sectional area $A = 0.1\ \mathrm{m^2}$. Determine the temperatures at $L/4$, $L/2$, $3L/4$, and L, where $L = 0.4\ \mathrm{m}$.

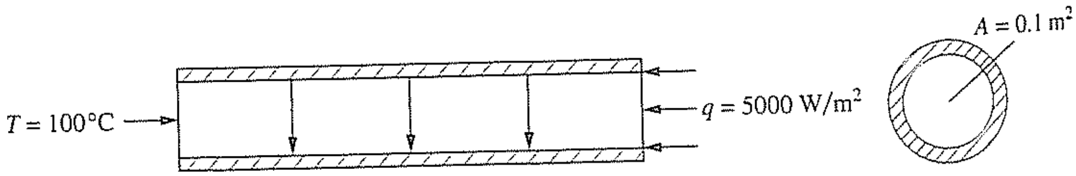


Figure 13–16 Insulated fin subjected to end heat flux

Using Eq. (13.4.22), with the second term set to zero as there is no heat transfer by convection from any surfaces due to the insulated perimeter and constant temperature on the left end and constant heat flux on the right end, we obtain

$$\underline{k}^{(1)} = \underline{k}^{(2)} = \underline{k}^{(3)} = \frac{AK_{xx}}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{(0.1\ \mathrm{m^2})(6\ \mathrm{W/(m\text{-}°C)})}{0.1\ \mathrm{m}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -6 \\ -6 & 6 \end{bmatrix}\mathrm{W/°C} \qquad (13.4.61)$$

$\underline{k}^{(4)} = \underline{k}^{(1)}$ also

$\underline{f}^{(1)} = \underline{f}^{(2)} = \underline{f}^{(3)} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$ as $Q = 0$ (no internal heat source) and $q^* = 0$ (no surface heat flux)

$$\underline{f}^{(4)} = qA\begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = (5000\ \mathrm{W/m^2})(0.1\ \mathrm{m^2})\begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 500 \end{Bmatrix}\mathrm{W} \qquad (13.4.62)$$

Assembling the global stiffness matrix from Eq. (13.4.61), and the global force matrix from Eq. (13.4.62), we obtain the global equations as

$$\begin{bmatrix} 6 & -6 & 0 & 0 & 0 \\ & 12 & -6 & 0 & 0 \\ & & 12 & -6 & 0 \\ & & & 12 & -6 \\ \text{Symmetry} & & & & 6 \end{bmatrix} \begin{Bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ 0 \\ 0 \\ 0 \\ 500 \end{Bmatrix} \tag{13.4.63}$$

Now applying the boundary condition on temperature, we have

$$t_1 = 100\,^\circ\text{C} \tag{13.4.64}$$

Substituting Eq. (13.4.64) for t_1 into Eq. (13.4.63), we then solve the second through fourth equations (associated with the unknown temperatures $t_2 - t_5$) simultaneously to obtain

$$t_2 = 183.33\,^\circ\text{C}, \quad t_3 = 266.67\,^\circ\text{C}, \quad t_4 = 350\,^\circ\text{C}, \quad t_5 = 433.33\,^\circ\text{C} \tag{13.4.65}$$

Substituting the nodal temperatures from Eq. (13.4.65) into the first of Eqs. (13.4.63), we obtain the nodal heat source at node 1 as

$$F_{1x} = 6(100\,^\circ\text{C} - 183.33\,^\circ\text{C}) = -500\,\text{W} \tag{13.4.66}$$

The nodal heat source given by Eq. (13.4.66) has a negative value, which means the heat is leaving the left end. This source is the same as the source coming into the fin at the right end given by $qA = (5000)(0.1) = 500$ W. ■

Finally, remember that the most important advantage of the finite element method is that it enables us to approximate, with high confidence, more complicated problems, such as those with more then one thermal conductivity, for which closed-form solutions are difficult (if not impossible) to obtain. The automation of the finite element method through general computer programs makes the method extremely powerful.

▲ 13.5 Two-Dimensional Finite Element Formulation

Because many bodies can be modeled as two-dimensional heat-transfer problems, we now develop the equations for an element appropriate for these problems. Examples using this element then follow.

Step 1 Select Element Type

The three-noded triangular element with nodal temperatures shown in Figure 13–17 is the basic element for solution of the two-dimensional heat-transfer problem.

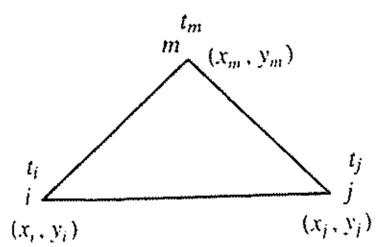


Figure 13–17 Basic triangular element with nodal temperatures

Step 2 Select a Temperature Function

The temperature function is given by

$$\{T\} = [N_i \quad N_j \quad N_m]\begin{Bmatrix} t_i \\ t_j \\ t_m \end{Bmatrix} \tag{13.5.1}$$

where t_i, t_j, and t_m are the nodal temperatures, and the shape functions are again given by Eqs. (6.2.18); that is,

$$N_i = \frac{1}{2A}(\alpha_i + \beta_i x + \gamma_i y) \tag{13.5.2}$$

with similar expressions for N_j and N_m. Here the α's, β's, and γ's are defined by Eqs. (6.2.10).

Unlike the CST element of Chapter 6 where there are two degrees of freedom per node (an x and a y displacement), in the heat transfer three-noded triangular element only a single scalar value (nodal temperature) is the primary unknown at each node, as shown by Eq. (13.5.1). This holds true for the three-dimensional elements as well, as shown in Section 13.7. Hence, the heat transfer problem is sometimes known as a scalar-valued boundary value problem.

Step 3 Define the Temperature Gradient/Temperature and Heat Flux/Temperature Gradient Relationships

We define the gradient matrix analogous to the strain matrix used in the stress analysis problem as

$$\{g\} = \begin{Bmatrix} \dfrac{\partial T}{\partial x} \\ \dfrac{\partial T}{\partial y} \end{Bmatrix} \tag{13.5.3}$$

Using Eq. (13.5.1) in Eq. (13.5.3), we have

$$\{g\} = \begin{bmatrix} \dfrac{\partial N_i}{\partial x} & \dfrac{\partial N_j}{\partial x} & \dfrac{\partial N_m}{\partial x} \\ \dfrac{\partial N_i}{\partial y} & \dfrac{\partial N_j}{\partial y} & \dfrac{\partial N_m}{\partial y} \end{bmatrix}\begin{Bmatrix} t_i \\ t_j \\ t_m \end{Bmatrix} \tag{13.5.4}$$

The gradient matrix $\{g\}$, written in compact matrix form analogously to the strain matrix $\{\varepsilon\}$ of the stress analysis problem, is given by

$$\{g\} = [B]\{t\} \tag{13.5.5}$$

where the $[B]$ matrix is obtained by substituting the three equations suggested by Eq. (13.5.2) in the rectangular matrix on the right side of Eq. (13.5.4) as

$$[B] = \frac{1}{2A}\begin{bmatrix} \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix} \tag{13.5.6}$$

The heat flux/temperature gradient relationship is now

$$\begin{Bmatrix} q_x \\ q_y \end{Bmatrix} = -[D]\{g\} \tag{13.5.7}$$

where the material property matrix is

$$[D] = \begin{bmatrix} K_{xx} & 0 \\ 0 & K_{yy} \end{bmatrix} \tag{13.5.8}$$

Step 4 Derive the Element Conduction Matrix and Equations

The element stiffness matrix from Eq. (13.4.17) is

$$[k] = \iiint_V [B]^T[D][B]\,dV + \iint_{S_3} h[N]^T[N]\,dS \tag{13.5.9}$$

where $$[k_c] = \iiint_V [B]^T[D][B]\,dV$$

$$= \iiint_V \frac{1}{4A^2} \begin{bmatrix} \beta_i & \gamma_i \\ \beta_j & \gamma_j \\ \beta_m & \gamma_m \end{bmatrix} \begin{bmatrix} K_{xx} & 0 \\ 0 & K_{yy} \end{bmatrix} \begin{bmatrix} \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix} dV \tag{13.5.10}$$

Assuming constant thickness in the element and noting that all terms of the integrand of Eq. (13.5.10) are constant, we have

$$[k_c] = \iiint_V [B]^T[D][B]\,dV = tA[B]^T[D][B] \tag{13.5.11}$$

Equation (13.5.11) is the true conduction portion of the total stiffness matrix Eq. (13.5.9). The second integral of Eq. (13.5.9) (the convection portion of the total stiffness matrix) is defined by

$$[k_h] = \iint_{S_3} h[N]^T[N]\,dS \tag{13.5.12}$$

We can explicitly multiply the matrices in Eq. (13.5.12) to obtain

$$[k_h] = h \iint_{S_3} \begin{bmatrix} N_iN_i & N_iN_j & N_iN_m \\ N_jN_i & N_jN_j & N_jN_m \\ N_mN_i & N_mN_j & N_mN_m \end{bmatrix} dS \tag{13.5.13}$$

To illustrate the use of Eq. (13.5.13), consider the side between nodes i and j of the triangular element to be subjected to convection (Figure 13–18). Then $N_m = 0$ along side i-j, and we obtain

$$[k_h] = \frac{hL_{i\text{-}j}t}{6} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{13.5.14}$$

where $L_{i\text{-}j}$ is the length of side i-j.

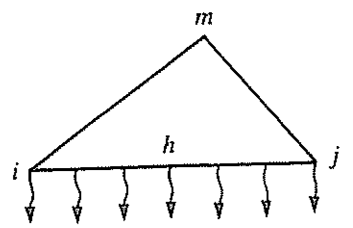


Figure 13–18 Heat loss by convection from side *i-j*

The evaluation of the force matrix integrals in Eq. (13.4.16) is as follows:

$$\{f_Q\} = \iiint_V Q[N]^T \, dV = Q \iiint_V [N]^T \, dV \tag{13.5.15}$$

for constant heat source Q. Thus it can be shown (left to your discretion) that this integral is equal to

$$\{f_Q\} = \frac{QV}{3} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \tag{13.5.16}$$

where $V = At$ is the volume of the element. Equation (13.5.16) indicates that heat is generated by the body in three equal parts to the nodes (like body forces in the elasticity problem). The second force matrix in Eq. (13.4.16) is

$$\{f_q\} = \iint_{S_2} q^* [N]^T \, dS = \iint_{S_2} q^* \begin{Bmatrix} N_i \\ N_j \\ N_m \end{Bmatrix} dS \tag{13.5.17}$$

This reduces to

$$\frac{q^* L_{i\text{-}j} t}{2} \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} \quad \text{on side } i\text{-}j \tag{13.5.18}$$

$$\frac{q^* L_{j\text{-}m} t}{2} \begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix} \quad \text{on side } j\text{-}m \tag{13.5.19}$$

$$\frac{q^* L_{m\text{-}i} t}{2} \begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix} \quad \text{on side } m\text{-}i \tag{13.5.20}$$

where $L_{i\text{-}j}$, $L_{j\text{-}m}$, and $L_{m\text{-}i}$ are the lengths of the sides of the element, and q^* is assumed constant over each edge. The integral $\iint_{S_3} hT_\infty [N]^T \, dS$ can be found in a manner similar to Eq. (13.5.17) by simply replacing q^* with hT_∞ in Eqs. (13.5.18)–(13.5.20).

Steps 5–7

Steps 5–7 are identical to those described in Section 13.4.

To illustrate the use of the equations presented in Section 13.5, we will now solve some two-dimensional heat-transfer problems.

Example 13.5

For the two-dimensional body shown in Figure 13–19, determine the temperature distribution. The temperature at the left side of the body is maintained at 100 °F. The edges on the top and bottom of the body are insulated. There is heat convection from the right side with convection coefficient $h = 20$ Btu/(h-ft^2-°F). The free-stream temperature is $T_\infty = 50$ °F. The coefficients of thermal conductivity are $K_{xx} = K_{yy} = 25$ Btu/(h-ft-°F). The dimensions are shown in the figure. Assume the thickness to be 1 ft.

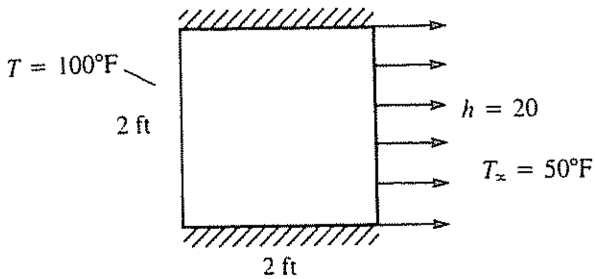


Figure 13–19 Two-dimensional body subjected to temperature variation and convection

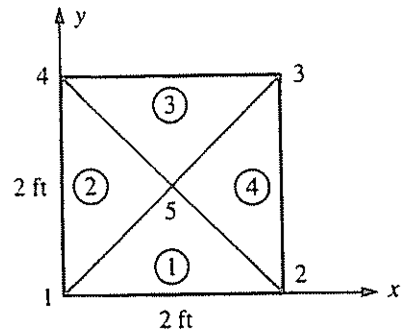


Figure 13–20 Discretized two-dimensional body of Figure 13–19

The finite element discretization is shown in Figure 13–20. We will use four triangular elements of equal size for simplicity of the longhand solution. There will be convective heat loss only over the right side of the body because the other faces are insulated. We now calculate the element stiffness matrices using Eq. (13.5.11) applied for all elements and using Eq. (13.5.14) applied for element 4 only, because convection is occurring only across one edge of element 4.

Element 1

The coordinates of the element 1 nodes are $x_1 = 0$, $y_1 = 0$, $x_2 = 2$, $y_2 = 0$, $x_5 = 1$, and $y_5 = 1$. Using these coordinates and Eqs. (7.2.10), we obtain

$$\begin{aligned} \beta_1 &= 0 - 1 = -1 \qquad \beta_2 = 1 - 0 = 1 \qquad \beta_5 = 0 - 0 = 0 \\ \gamma_1 &= 1 - 2 = -1 \qquad \gamma_2 = 0 - 1 = -1 \qquad \gamma_5 = 2 - 0 = 2 \end{aligned} \tag{13.5.21}$$

Using Eqs. (13.5.21) in Eq. (13.5.11), we have

$$[k_c^{(1)}] = \frac{1(1)}{2(2)} \begin{bmatrix} -1 & -1 \\ 1 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 25 & 0 \\ 0 & 25 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ -1 & -1 & 2 \end{bmatrix} \tag{13.5.22}$$

Simplifying Eq. (13.5.22), we obtain

$$[k_c^{(1)}] = \begin{array}{c} \begin{array}{ccc} 1 & 2 & 5 \end{array} \\ \begin{bmatrix} 12.5 & 0 & -12.5 \\ 0 & 12.5 & -12.5 \\ -12.5 & -12.5 & 25 \end{bmatrix} \end{array} \text{ Btu/(h-°F)} \tag{13.5.23}$$

where the numbers above the columns indicate the node numbers associated with the matrix.

Element 2

The coordinates of the element 2 nodes are $x_1 = 0$, $y_1 = 0$, $x_5 = 1$, $y_5 = 1$, $x_4 = 0$, and $y_4 = 2$. Using these coordinates, we obtain

$$\begin{array}{lll} \beta_1 = 1 - 2 = -1 & \beta_5 = 2 - 0 = 2 & \beta_4 = 0 - 1 = -1 \\ \gamma_1 = 0 - 1 = -1 & \gamma_5 = 0 - 0 = 0 & \gamma_4 = 1 - 0 = 1 \end{array} \tag{13.5.24}$$

Using Eqs. (13.5.24) in Eq. (13.5.11), we have

$$[k_c^{(2)}] = \frac{1}{4}\begin{bmatrix} -1 & -1 \\ 2 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 25 & 0 \\ 0 & 25 \end{bmatrix}\begin{bmatrix} -1 & 2 & -1 \\ -1 & 0 & 1 \end{bmatrix} \tag{13.5.25}$$

Simplifying Eq. (13.5.25), we obtain

$$[k_c^{(2)}] = \begin{array}{c} \begin{array}{ccc} 1 & 5 & 4 \end{array} \\ \begin{bmatrix} 12.5 & -12.5 & 0 \\ -12.5 & 25 & -12.5 \\ 0 & -12.5 & 12.5 \end{bmatrix} \end{array} \text{ Btu/(h-°F)} \tag{13.5.26}$$

Element 3

The coordinates of the element 3 nodes are $x_4 = 0$, $y_4 = 2$, $x_5 = 1$, $y_5 = 1$, $x_3 = 2$, and $y_3 = 2$. Using these coordinates, we obtain

$$\begin{array}{lll} \beta_4 = 1 - 2 = -1 & \beta_5 = 2 - 2 = 0 & \beta_3 = 2 - 1 = 1 \\ \gamma_4 = 2 - 1 = 1 & \gamma_5 = 0 - 2 = -2 & \gamma_3 = 1 - 0 = 1 \end{array} \tag{13.5.27}$$

Using Eqs. (13.5.27) in Eq. (13.5.11), we obtain

$$[k_c^{(3)}] = \begin{array}{c} \begin{array}{ccc} 4 & 5 & 3 \end{array} \\ \begin{bmatrix} 12.5 & -12.5 & 0 \\ -12.5 & 25 & -12.5 \\ 0 & -12.5 & 12.5 \end{bmatrix} \end{array} \text{ Btu/(h-°F)} \tag{13.5.28}$$

Element 4

The coordinates of the element 4 nodes are $x_2 = 2$, $y_2 = 0$, $x_3 = 2$, $y_3 = 2$, $x_5 = 1$, and $y_5 = 1$. Using these coordinates, we obtain

$$\begin{array}{lll} \beta_2 = 2 - 1 = 1 & \beta_3 = 1 - 0 = 1 & \beta_5 = 0 - 2 = -2 \\ \gamma_2 = 1 - 2 = -1 & \gamma_3 = 2 - 1 = 1 & \gamma_5 = 2 - 2 = 0 \end{array} \tag{13.5.29}$$

Using Eqs. (13.5.29) in Eq. (13.5.11), we obtain

$$[k_c^{(4)}] = \begin{array}{c} \begin{array}{ccc} 2 & 3 & 5 \end{array} \\ \begin{bmatrix} 12.5 & 0 & -12.5 \\ 0 & 12.5 & -12.5 \\ -12.5 & -12.5 & 25 \end{bmatrix} \end{array} \text{Btu/(h-°F)} \tag{13.5.30}$$

For element 4, we have a convection contribution to the total stiffness matrix because side 2–3 is exposed to the free-stream temperature. Using Eq. (13.5.14) with $i = 2$ and $j = 3$, we obtain

$$[k_h^{(4)}] = \frac{(20)(2)(1)}{6}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{13.5.31}$$

Simplifying Eq. (13.5.31) yields

$$[k_h^{(4)}] = \begin{array}{c} \begin{array}{ccc} 2 & 3 & 5 \end{array} \\ \begin{bmatrix} 13.3 & 6.67 & 0 \\ 6.67 & 13.3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \text{Btu/(h-°F)} \tag{13.5.32}$$

Adding Eqs. (13.5.30) and (13.5.32), we obtain the element 4 total stiffness matrix as

$$[k^{(4)}] = \begin{array}{c} \begin{array}{ccc} 2 & 3 & 5 \end{array} \\ \begin{bmatrix} 25.83 & 6.67 & -12.5 \\ 6.67 & 25.83 & -12.5 \\ -12.5 & -12.5 & 25 \end{bmatrix} \end{array} \text{Btu/(h-°F)} \tag{13.5.33}$$

Superimposing the stiffness matrices given by Eqs. (13.5.23), (13.5.26), (13.5.28), and (13.5.33), we obtain the total stiffness matrix for the body as

$$\underline{K} = \begin{bmatrix} 25 & 0 & 0 & 0 & -25 \\ 0 & 38.33 & 6.67 & 0 & -25 \\ 0 & 6.67 & 38.33 & 0 & -25 \\ 0 & 0 & 0 & 25 & -25 \\ -25 & -25 & -25 & -25 & 100 \end{bmatrix} \text{Btu/(h-°F)} \tag{13.5.34}$$

Next, we determine the element force matrices by using Eqs. (13.5.18)–(13.5.20) with q^* replaced by hT_∞. Because $Q = 0$, $q^* = 0$, and we have convective heat transfer only from side 2–3, element 4 is the only one that contributes nodal forces. Hence,

$$\{f^{(4)}\} = \begin{Bmatrix} f_2 \\ f_3 \\ f_5 \end{Bmatrix} = \frac{hT_\infty L_{2-3}t}{2}\begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} \tag{13.5.35}$$

Substituting the appropriate numerical values into Eq. (13.5.35) yields

$$\{f^{(4)}\} = \frac{(20)(50)(2)(1)}{2}\begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 1000 \\ 1000 \\ 0 \end{Bmatrix} \frac{\text{Btu}}{\text{h}} \tag{13.5.36}$$

Using Eqs. (13.5.34) and (13.5.36), we find that the total assembled system of equations is

$$\begin{bmatrix} 25 & 0 & 0 & 0 & -25 \\ 0 & 38.33 & 6.67 & 0 & -25 \\ 0 & 6.67 & 38.33 & 0 & -25 \\ 0 & 0 & 0 & 25 & -25 \\ -25 & -25 & -25 & -25 & 100 \end{bmatrix} \begin{Bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 1000 \\ 1000 \\ F_4 \\ 0 \end{Bmatrix} \tag{13.5.37}$$

We have known nodal temperature boundary conditions of $t_1 = 100\,^\circ\mathrm{F}$ and $t_4 = 100\,^\circ\mathrm{F}$. We again modify the stiffness and force matrices as follows:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 38.33 & 6.67 & 0 & -25 \\ 0 & 6.67 & 38.33 & 0 & -25 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & -25 & -25 & 0 & 100 \end{bmatrix} \begin{Bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{Bmatrix} = \begin{Bmatrix} 100 \\ 1000 \\ 1000 \\ 100 \\ 5000 \end{Bmatrix} \tag{13.5.38}$$

The terms in the first and fourth rows and columns corresponding to the known temperature conditions $t_1 = 100\,^\circ\mathrm{F}$ and $t_4 = 100\,^\circ\mathrm{F}$ have been set equal to zero except for the main diagonal, which has been set equal to one, and the first and fourth rows of the force matrix have been set equal to the known nodal temperatures. Also, the term $(-25)(100\,^\circ\mathrm{F}) + (-25) \times (100\,^\circ\mathrm{F}) = -5000$ on the left side of the fifth equation of Eq. (13.5.37) has been transposed to the right side in the fifth row (as $+5000$) of Eq. (13.5.38). The second, third and fifth equations of Eq. (13.5.38), corresponding to the rows of unknown nodal temperatures, can now be solved in the usual manner. The resulting solution is given by

$$t_2 = 69.33\,^\circ\mathrm{F} \qquad t_3 = 69.33\,^\circ\mathrm{F} \qquad t_5 = 84.62\,^\circ\mathrm{F} \tag{13.5.39}$$

■

Example 13.6

For the two-dimensional body shown in Figure 13–21, determine the temperature distribution. The temperature of the top side of the body is maintained at $100\,^\circ\mathrm{C}$. The body is insulated on the other edges. A uniform heat source of $Q = 1000\ \mathrm{W/m^3}$ acts over the whole plate, as shown in the figure. Assume a constant thickness of 1 m. Let $K_{xx} = K_{yy} = 25\ \mathrm{W/(m \cdot {}^\circ C)}$.

We need consider only the left half of the body, because we have a vertical plane of symmetry passing through the body 2 m from both the left and right edges. This vertical plane can be considered to be an insulated boundary. The finite element model is shown in Figure 13–22.

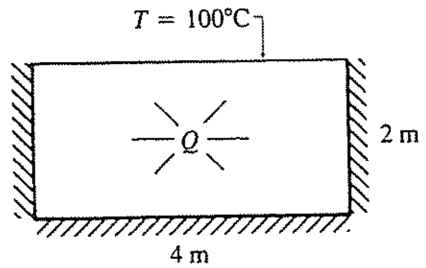


Figure 13–21 Two-dimensional body subjected to a heat source

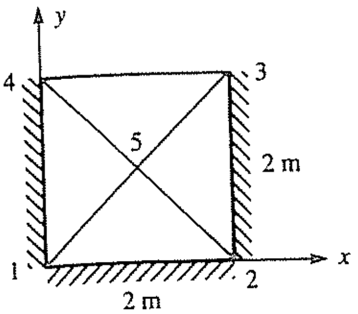


Figure 13–22 Discretized body of Figure 13–21

We will now calculate the element stiffness matrices. Because the magnitudes of the coordinates are the same as in Example 13.5, the element stiffness matrices are the same as Eqs. (13.5.23), (13.5.26), (13.5.28), and (13.5.30). Remember that there is no convection from any side of an element, so the convection contribution $[k_h]$ to the stiffness matrix is zero. Superimposing the element stiffness matrices, we obtain the total stiffness matrix as

$$\underline{K} = \begin{bmatrix} 25 & 0 & 0 & 0 & -25 \\ 0 & 25 & 0 & 0 & -25 \\ 0 & 0 & 25 & 0 & -25 \\ 0 & 0 & 0 & 25 & -25 \\ -25 & -25 & -25 & -25 & 100 \end{bmatrix} \text{W/°C} \tag{13.5.40}$$

Because the heat source Q is acting uniformly over each element, we use Eq. (13.5.16) to evaluate the nodal forces for each element as

$$\{f^{(e)}\} = \frac{QV}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \frac{1000(1\text{ m}^3)}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 333 \\ 333 \\ 333 \end{Bmatrix} \text{W} \tag{13.5.41}$$

We then use Eqs. (13.5.40) and (13.5.41) applied to each element, to assemble the total system of equations as

$$\begin{bmatrix} 25 & 0 & 0 & 0 & -25 \\ 0 & 25 & 0 & 0 & -25 \\ 0 & 0 & 25 & 0 & -25 \\ 0 & 0 & 0 & 25 & -25 \\ -25 & -25 & -25 & -25 & 100 \end{bmatrix} \begin{Bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{Bmatrix} = \begin{Bmatrix} 666 \\ 666 \\ 666 + F_3 \\ 666 + F_4 \\ 1333 \end{Bmatrix} \tag{13.5.42}$$

We have known nodal temperature boundary conditions of $t_3 = 100\,°\text{C}$ and $t_4 = 100\,°\text{C}$. In the usual manner, as was shown in Example 13.4, we modify the stiffness and force matrices of Eq. (13.5.42) to obtain

$$\begin{bmatrix} 25 & 0 & 0 & 0 & -25 \\ 0 & 25 & 0 & 0 & -25 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -25 & -25 & 0 & 0 & 100 \end{bmatrix} \begin{Bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{Bmatrix} = \begin{Bmatrix} 666 \\ 666 \\ 100 \\ 100 \\ 6333 \end{Bmatrix} \tag{13.5.43}$$

Equation (13.5.43) satisfies the boundary temperature conditions and is equivalent to Eq. (13.5.42); that is, the first, second, and fifth equations of Eq. (13.5.43) are the same as the first, second, and fifth equations of Eq. (13.5.42), and the third and fourth equations of Eq. (13.5.43) identically satisfy the boundary temperature conditions at nodes 3 and 4. The first, second, and fifth equations of Eq. (13.5.43) corresponding to the rows of unknown nodal temperatures, can now be solved simultaneously. The resulting solution is given by

$$t_1 = 180\,°\mathrm{C} \qquad t_2 = 180\,°\mathrm{C} \qquad t_5 = 153\,°\mathrm{C} \tag{13.5.44}$$

■

We then use the results from Eq. (13.5.44) in Eq. (13.5.42) to obtain the rates of heat flow at nodes 3 and 4 (that is, F_3 and F_4).

13.6 Line or Point Sources

A common practical heat-transfer problem is that of a source of heat generation present within a very small volume or area of some larger medium. When such heat sources exist within small volumes or areas, they may be idealized as **line** or **point sources.** Practical examples that can be modeled as line sources include hot-water pipes embedded within a medium such as concrete or earth, and conducting electrical wires embedded within a material.

A line or point source can be considered by simply including a node at the location of the source when the discretized finite element model is created. The value of the line source can then be added to the row of the global force matrix corresponding to the global degree of freedom assigned to the node. However, another procedure can be used to treat the line source when it is more convenient to leave the source within an element.

We now consider the line source of magnitude Q^*, with typical units of Btu/(h-ft), located at (x_o, y_o) within the two-dimensional element shown in Figure 13–23. The heat source Q is no longer constant over the element volume.

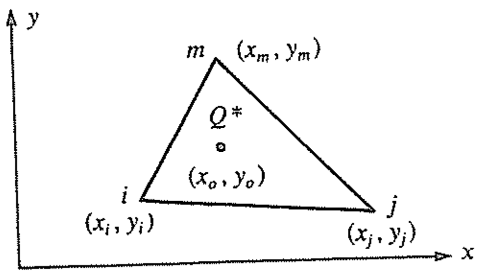


Figure 13–23 Line source located within a typical triangular element

Using Eq. (13.4.16), we can express the heat source matrix as

$$\{f_Q\} = \iiint_V \begin{Bmatrix} N_i \\ N_j \\ N_m \end{Bmatrix} \Bigg|_{x=x_o,\, y=y_o} \frac{Q^*}{A^*}\, dV \tag{13.6.1}$$

where A^* is the cross-sectional area over which Q^* acts, and the N's are evaluated at $x = x_o$ and $y = y_o$. Equation (13.6.1) can be rewritten as

$$\{f_Q\} = \iint_{A^*} \int_0^t \begin{Bmatrix} N_i \\ N_j \\ N_m \end{Bmatrix} \Bigg|_{x=x_o,\, y=y_o} \frac{Q^*}{A^*}\, dA\, dz \tag{13.6.2}$$

Because the N's are evaluated at $x = x_o$ and $y = y_o$, they are no longer functions of x and y. Thus, we can simplify Eq. (13.6.2) to

$$\{f_Q\} = \begin{Bmatrix} N_i \\ N_j \\ N_m \end{Bmatrix}\Bigg|_{x=x_o,\, y=y_o} Q^* t \text{ Btu/h} \tag{13.6.3}$$

From Eq. (13.6.3), we can see that the portion of the line source Q^* distributed to each node is based on the values of N_i, N_j, and N_m, which are evaluated using the coordinates (x_o, y_o) of the line source. Recalling that the sum of the N's at any point within an element is equal to one [that is, $N_i(x_o, y_o) + N_j(x_o, y_o) + N_m(x_o, y_o) = 1$], we see that no more than the total amount of Q^* is distributed and that

$$Q_i^* + Q_j^* + Q_m^* = Q^* \tag{13.6.4}$$

Example 13.7

A line source $Q^* = 65$ Btu/(h-in.) is located at coordinates $(5, 2)$ in the element shown in Figure 13–24. Determine the amount of Q^* allocated to each node. All nodal coordinates are in units of inches. Assume an element thickness of $t = 1$ in.

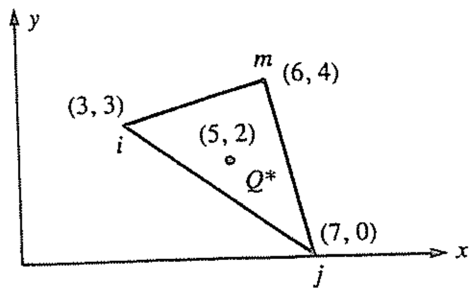


Figure 13–24 Line source located within a triangular element

We first evaluate the α's, β's, and γ's, defined by Eqs. (6.2.10), associated with each shape function as follows:

$$\alpha_i = x_j y_m - x_m y_j = 7(4) - 6(0) = 28$$
$$\alpha_j = x_m y_i - x_i y_m = 6(3) - 3(4) = 6$$
$$\alpha_m = x_i y_j - x_j y_i = 3(0) - 7(3) = -21$$
$$\beta_i = y_j - y_m = 0 - 4 = -4$$
$$\beta_j = y_m - y_i = 4 - 3 = 1 \tag{13.6.5}$$
$$\beta_m = y_i - y_j = 3 - 0 = 3$$
$$\gamma_i = x_m - x_j = 6 - 7 = -1$$
$$\gamma_j = x_i - x_m = 3 - 6 = -3$$
$$\gamma_m = x_j - x_i = 7 - 3 = 4$$

Also, $$2A = \begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_m & y_m \end{vmatrix} = \begin{vmatrix} 1 & 3 & 3 \\ 1 & 7 & 0 \\ 1 & 6 & 4 \end{vmatrix} = 13 \tag{13.6.6}$$

Substituting the results of Eqs. (13.6.5) and (13.6.6) into Eq. (13.5.2) yields

$$\begin{aligned} N_i &= \tfrac{1}{13}[28 - 4x - 1y] \\ N_j &= \tfrac{1}{13}[6 + x - 3y] \\ N_m &= \tfrac{1}{13}[-21 + 3x + 4y] \end{aligned} \tag{13.6.7}$$

Equations (13.6.7) for N_i, N_j, and N_m evaluated at $x = 5$ and $y = 2$ are

$$\begin{aligned} N_i &= \tfrac{1}{13}[28 - 4(5) - 1(2)] = \tfrac{6}{13} \\ N_j &= \tfrac{1}{13}[6 + 5 - 3(2)] = \tfrac{5}{13} \\ N_m &= \tfrac{1}{13}[-21 + 3(5) + 4(2)] = \tfrac{2}{13} \end{aligned} \tag{13.6.8}$$

Therefore, using Eq. (13.6.3), we obtain

$$\begin{Bmatrix} f_{Qi} \\ f_{Qj} \\ f_{Qm} \end{Bmatrix} = Q^* t \begin{Bmatrix} N_i \\ N_j \\ N_m \end{Bmatrix}_{\substack{x=x_O=5 \\ y=y_O=2}} = \frac{65(1)}{13} \begin{Bmatrix} 6 \\ 5 \\ 2 \end{Bmatrix} = \begin{Bmatrix} 30 \\ 25 \\ 10 \end{Bmatrix} \text{ Btu/h} \tag{13.6.9}$$

13.7 Three-Dimensional Heat Transfer Finite Element Formulation

When the heat transfer is in all three directions (indicated by q_x, q_y and q_z in Figure 13–25), then we must model the system using three-dimensional elements to account for the heat transfer. Examples of heat transfer that often is three-dimensional are shown in Figure 13–26. Here we see in Figure 13–26(a) and (b) an electronic component soldered to a printed wiring board [11]. The model includes a silicon chip, silver-eutectic die, alumina carrier, solder joints, copper pads, and the printed wiring board. The model actually consisted of 965 8-noded brick elements with 1395 nodes and 216 thermal elements and was modeled in Algor [10]. One-quarter of the

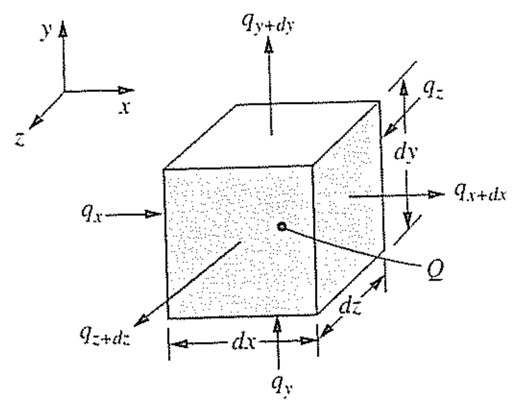
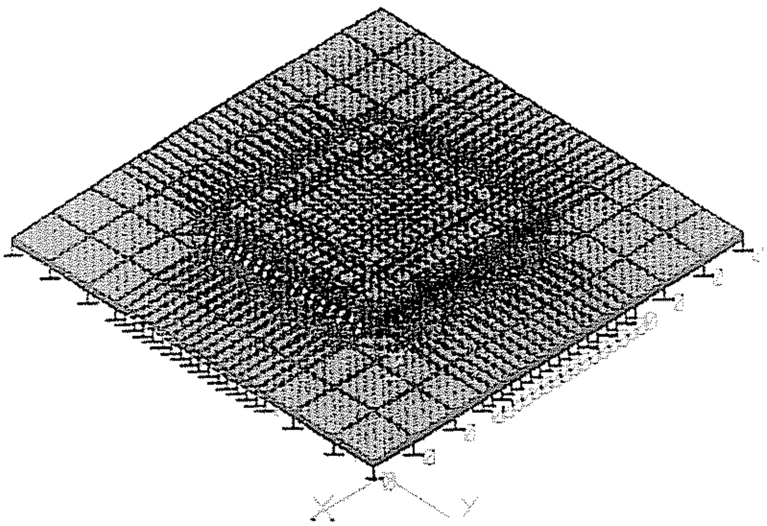


Figure 13–25 Three-dimensional heat transfer

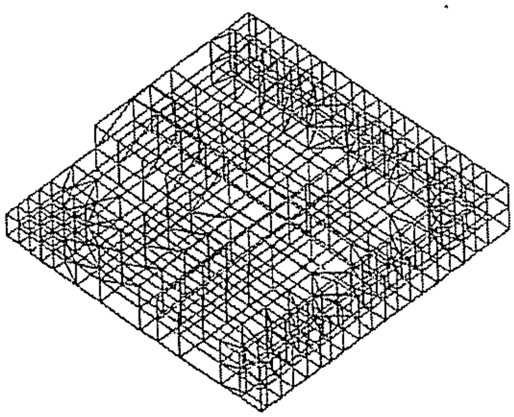
actual device was modeled. Figure 13–26(c) shows a heat sink used to cool a personal computer microprocessor chip (a two-dimensional model might possibly be used with good results as well). Finally, Figure 13–26(d) shows an engine block, which is an irregularly shaped three-dimensional body requiring a three-dimensional heat transfer analysis.

The elements often included in commercial computer programs to analyze three-dimensional heat transfer are the same as those used in Chapter 11 for three-dimensional stress analysis. These include the four-noded tetrahedral (Figure 11–2), the eight-noded hexahedral (brick) (Figure 11–4), and the twenty-noded hexahedral (Figure 11–5), the difference being that we now have only one degree of freedom at each node, namely a temperature. The temperature functions in the x, y, and z directions can now be expressed by expanding Eq. (13.5.2) to the third dimension or by using shape functions given by Eq. (11.2.10) for a four-noded tetrahedral element or by Eqs. (11.3.3) for the eight-noded brick or the Eqs. (11.3.11)–(11.3.14) for the twenty-noded brick. The typical eight-noded brick element is shown in Figure 13–27 with the nodal temperatures included.

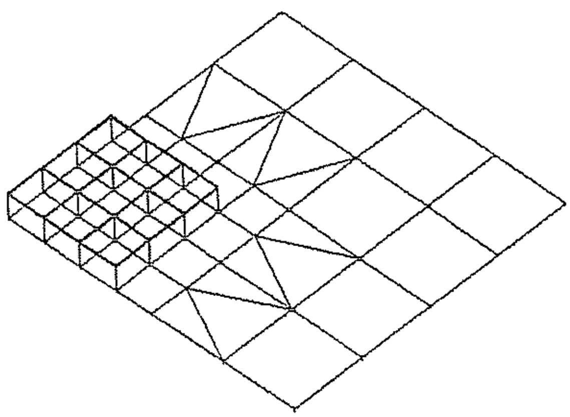


FEA model of 68-pin SMT component

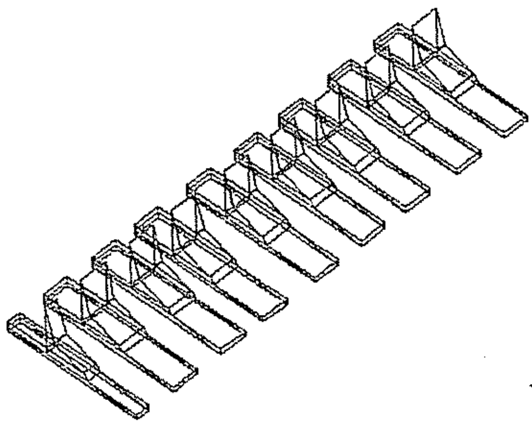
(a) Electronic component soldered to printed circuit board



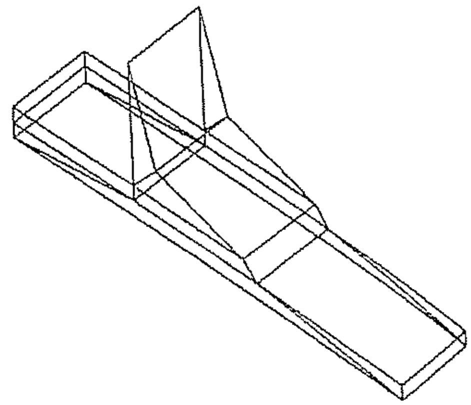
(b1) Carrier of the FEA model



(b2) Silicon chip (left side portion) and Au-Eutectic of FEA model

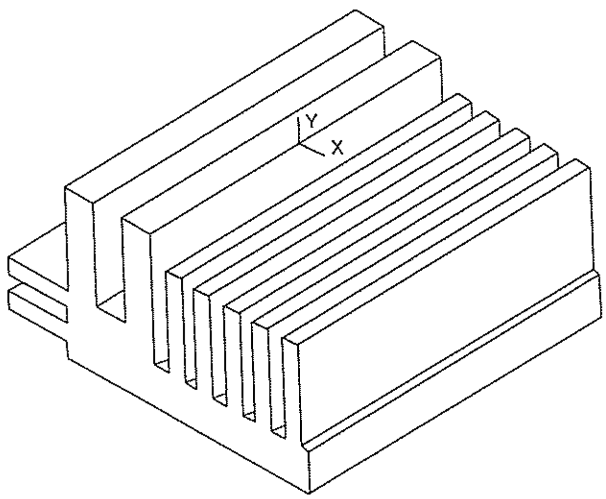


(b3) Solder joints and copper pads of FEA model

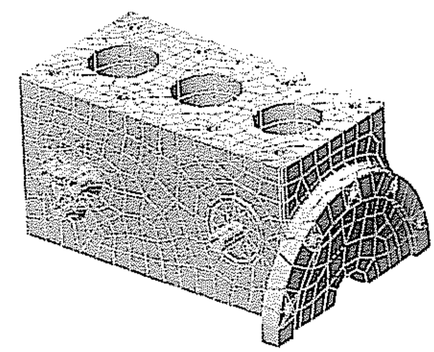


(b4) Close-up of solder and copper pad

(b) finite element model (quarter thermal model) showing the separate components



(c) Heat sink possibly used to cool a computer microchip



(d) Engine block

Figure 13–26 Examples of three-dimensional heat transfer

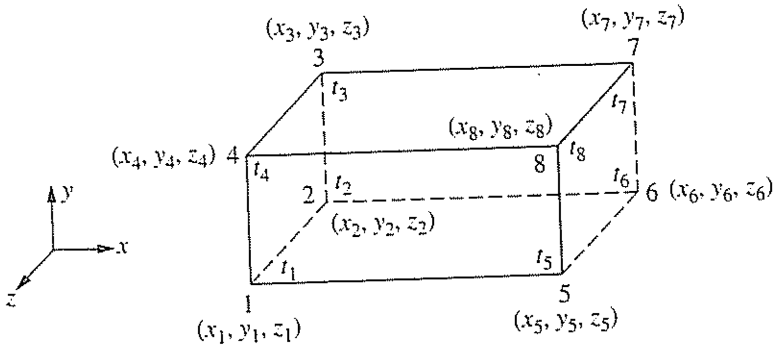


Figure 13–27 Eight-noded brick element showing nodal temperatures for heat transfer

13.8 One-Dimensional Heat Transfer with Mass Transport

We now consider the derivation of the basic differential equation for one-dimensional heat flow where the flow is due to conduction, convection, and **mass transport** (or **transfer**) of the fluid. The purpose of this derivation including mass transport is to show how Galerkin's residual method can be directly applied to a problem for which the variational method is not applicable. That is, the differential equation will have an odd-numbered derivative and hence does not have an associated functional of the form of Eq. (1.4.3).

The control volume used in the derivation is shown in Figure 13–28. Again, from Eq. (13.1.1) for conservation of energy, we obtain

$$q_x A\,dt + QA\,dx\,dt = c\rho A\,dx\,dT + q_{x+dx}A\,dt + q_h P\,dx\,dt + q_m\,dt \qquad (13.8.1)$$

All of the terms in Eq. (13.8.1) have the same meaning as in Sections 13.1 and 13.2, except the additional mass-transport term is given by [1]

$$q_m = \dot{m}cT \qquad (13.8.2)$$

where the additional variable $\dot{m}$ is the *mass flow rate* in typical units of kg/h or slug/h.

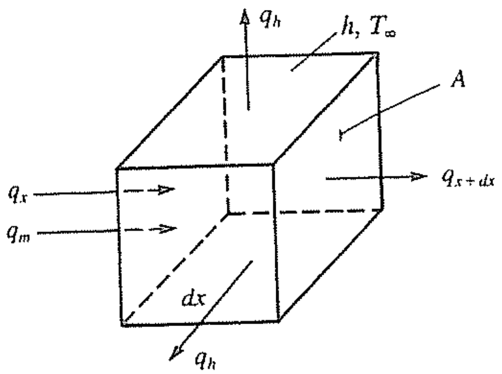


Figure 13–28 Control volume for one-dimensional heat conduction with convection and mass transport

Again, using Eqs. (13.1.3)–(13.1.6), (13.2.2), and (13.8.2) in Eq. (13.8.1) and differentiating with respect to x and t, we obtain

$$\frac{\partial}{\partial x}\left(K_{xx}\frac{\partial T}{\partial x}\right) + Q = \frac{\dot{m}c}{A}\frac{\partial T}{\partial x} + \frac{hP}{A}(T - T_\infty) + \rho c\frac{\partial T}{\partial t} \qquad (13.8.3)$$

Equation (13.8.3) is the basic one-dimensional differential equation for heat transfer with mass transport.

13.9 Finite Element Formulation of Heat Transfer with Mass Transport by Galerkin's Method

Having obtained the differential equation for heat transfer with mass transport, Eq. (13.8.3), we now derive the finite element equations by applying Galerkin's residual method, as outlined in Section 3.12, directly to the differential equation.

We assume here that $Q = 0$ and that we have steady-state conditions so that differentiation with respect to time is zero.

The residual R is now given by

$$R(T) = -\frac{d}{dx}\left(K_{xx}\frac{dT}{dx}\right) + \frac{\dot{m}c}{A}\frac{dT}{dx} + \frac{hP}{A}(T - T_\infty) \tag{13.9.1}$$

Applying Galerkin's criterion, Eq. (3.12.3), to Eq. (13.9.1), we have

$$\int_0^L \left[-\frac{d}{dx}\left(K_{xx}\frac{dT}{dx}\right) + \frac{\dot{m}c}{A}\frac{dT}{dx} + \frac{hP}{A}(T - T_\infty)\right] N_i\,dx = 0 \qquad (i = 1, 2) \tag{13.9.2}$$

where the shape functions are given by Eqs. (13.4.2). Applying integration by parts to the first term of Eq. (13.9.2), we obtain

$$\begin{aligned} u &= N_i & du &= \frac{dN_i}{dx}\,dx \\ dv &= -\frac{d}{dx}\left(K_{xx}\frac{dT}{dx}\right)dx & v &= -K_{xx}\frac{dT}{dx} \end{aligned} \tag{13.9.3}$$

Using Eqs. (13.9.3) in the general formula for integration by parts [see Eq. (3.12.6)], we obtain

$$\int_0^L \left[-\frac{d}{dx}\left(K_{xx}\frac{dT}{dx}\right)\right] N_i\,dx = -K_{xx}\frac{dT}{dx}N_i\bigg|_0^L + \int_0^L K_{xx}\frac{dT}{dx}\frac{dN_i}{dx}\,dx \tag{13.9.4}$$

Substituting Eq. (13.9.4) into Eq. (13.9.2), we obtain

$$\int_0^L \left(K_{xx}\frac{dT}{dx}\frac{dN_i}{dx}\right)dx + \int_0^L \left[\frac{\dot{m}c}{A}\frac{dT}{dx} + \frac{hP}{A}(T - T_\infty)\right] N_i\,dx = K_{xx}\frac{dT}{dx}N_i\bigg|_0^L \tag{13.9.5}$$

Using Eq. (13.4.2) in (13.4.1) for T, we obtain

$$\frac{dT}{dx} = -\frac{t_1}{L} + \frac{t_2}{L} \tag{13.9.6}$$

From Eq. (13.4.2), we obtain

$$\frac{dN_1}{dx} = -\frac{1}{L} \qquad \frac{dN_2}{dx} = \frac{1}{L} \tag{13.9.7}$$

By letting $N_i = N_1 = 1 - (x/L)$ and substituting Eqs. (13.9.6) and (13.9.7) into Eq. (13.9.5), along with Eq. (13.4.1) for T, we obtain the first finite element equation

$$\begin{aligned} &\int_0^L K_{xx}\left(-\frac{t_1}{L} + \frac{t_2}{L}\right)\left(-\frac{1}{L}\right)dx + \int_0^L \frac{\dot{m}c}{A}\left(-\frac{t_1}{L} + \frac{t_2}{L}\right)\left(1 - \frac{x}{L}\right)dx \\ &+ \int_0^L \frac{hP}{A}\left[\left(1 - \frac{x}{L}\right)t_1 + \left(\frac{x}{L}\right)t_2 - T_\infty\right]\left(\frac{1-x}{L}\right)dx = q_{x1}^* \end{aligned} \tag{13.9.8}$$

where the definition for q_x given by Eq. (13.1.3) has been used in Eq. (13.9.8). Equation (13.9.8) has a boundary condition q_{x1}^* at $x = 0$ only because $N_1 = 1$ at $x = 0$ and

$N_1 = 0$ at $x = L$. Integrating Eq. (13.9.8), we obtain

$$\left(\frac{K_{xx}A}{L} - \frac{\dot{m}c}{2} + \frac{hPL}{3}\right)t_1 + \left(-\frac{K_{xx}A}{L} + \frac{\dot{m}c}{2} + \frac{hPL}{6}\right)t_2 = q_{x1}^* + \frac{hPL}{2}T_\infty \quad (13.9.9)$$

where q_{x1}^* is defined to be q_x evaluated at node 1.

To obtain the second finite element equation, we let $N_i = N_2 = x/L$ in Eq. (13.9.5) and again use Eqs. (13.9.6), (13.9.7), and (13.4.1) in Eq. (13.9.5) to obtain

$$\left(-\frac{K_{xx}A}{L} - \frac{\dot{m}c}{2} + \frac{hPL}{6}\right)t_1 + \left(\frac{K_{xx}A}{L} + \frac{\dot{m}c}{2} + \frac{hPL}{3}\right)t_2 = q_{x2}^* + \frac{hPL}{2}T_\infty \quad (13.9.10)$$

where q_{x2}^* is defined to be q_x evaluated at node 2. Rewriting Eqs. (13.9.9) and (13.9.10) in matrix form yields

$$\left[\frac{K_{xx}A}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{\dot{m}c}{2}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} + \frac{hPL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right]\begin{Bmatrix} t_1 \\ t_2 \end{Bmatrix}$$

$$= \frac{hPLT_\infty}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} + \begin{Bmatrix} q_{x1}^* \\ q_{x2}^* \end{Bmatrix} \quad (13.9.11)$$

Applying the element equation $\{f\} = [k]\{t\}$ to Eq. (13.9.11), we see that the element stiffness (conduction) matrix is now composed of three parts:

$$[k] = [k_c] + [k_h] + [k_m] \quad (13.9.12)$$

where

$$[k_c] = \frac{K_{xx}A}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad [k_h] = \frac{hPL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad [k_m] = \frac{\dot{m}c}{2}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \quad (13.9.13)$$

and the element nodal force and unknown nodal temperature matrices are

$$\{f\} = \frac{hPLT_\infty}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} + \begin{Bmatrix} q_{x1}^* \\ q_{x2}^* \end{Bmatrix} \qquad \{t\} = \begin{Bmatrix} t_1 \\ t_2 \end{Bmatrix} \quad (13.9.14)$$

We observe from Eq. (13.9.13) that the mass transport stiffness matrix $[k_m]$ is asymmetric and, hence, $[k]$ is asymmetric. Also, if heat flux exists, it usually occurs across the free ends of a system. Therefore, q_{x1} and q_{x2} usually occur only at the free ends of a system modeled by this element. When the elements are assembled, the heat fluxes q_{x1} and q_{x2} are usually equal but opposite at the node common to two elements, unless there is an internal concentrated heat flux in the system. Furthermore, for insulated ends, the q_x^*'s also go to zero.

To illustrate the use of the finite element equations developed in this section for heat transfer with mass transport, we will now solve the following problem.

Example 13.8

Air is flowing at a rate of 4.72 lb/h inside a round tube with a diameter of 1 in. and length of 5 in., as shown in Figure 13–29. The initial temperature of the air entering the tube is 100 °F. The wall of the tube has a uniform constant temperature of 200 °F. The specific heat of the air is 0.24 Btu/(lb-°F), the convection coefficient

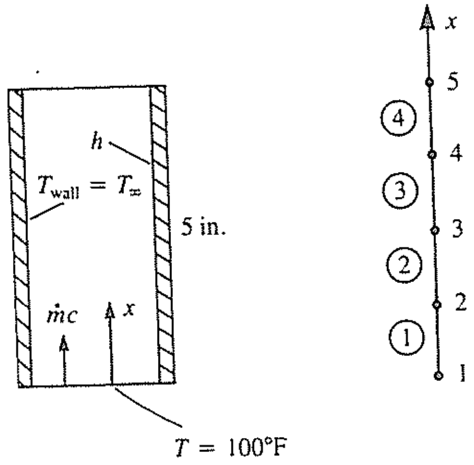


Figure 13–29 Air flowing through a tube, and the finite element model

between the air and the inner wall of the tube is 2.7 Btu/(h-ft^2-°F), and the thermal conductivity is 0.017 Btu/(h-ft-°F). Determine the temperature of the air along the length of the tube and the heat flow at the inlet and outlet of the tube. Here the flow rate and specific heat are given in force units (pounds) instead of mass units (slugs). This is not a problem because the units cancel in the $\dot{m}c$ product in the formulation of the equations.

We first determine the element stiffness and force matrices using Eqs. (13.9.13) and (13.9.14). To do this, we evaluate the following factors:

$$\frac{K_{xx}A}{L} = \frac{(0.017)\left[\dfrac{\pi(1)}{4(144)}\right]}{1.25/12} = 0.891 \times 10^{-3} \text{ Btu/(h-°F)}$$

$$\dot{m}c = (4.72)(0.24) = 1.133 \text{ Btu/(h-°F)} \tag{13.9.15}$$

$$\frac{hPL}{6} = \frac{(2.7)(0.262)(0.104)}{6} = 0.0123 \text{ Btu/(h-°F)}$$

$$hPLT_\infty = (2.7)(0.262)(0.104)(200) = 14.71 \text{ Btu/h}$$

We can see from Eqs. (13.9.15) that the conduction portion of the stiffness matrix is negligible. Therefore, we neglect this contribution to the total stiffness matrix and obtain

$$\underline{k}^{(1)} = \frac{1.133}{2}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} + 0.0123\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -0.542 & 0.579 \\ -0.554 & 0.591 \end{bmatrix} \tag{13.9.16}$$

Similarly, because all elements have the same properties,

$$\underline{k}^{(2)} = \underline{k}^{(3)} = \underline{k}^{(4)} = \underline{k}^{(1)} \tag{13.9.17}$$

Using Eqs. (13.9.14) and (13.9.15), we obtain the element force matrices as

$$\underline{f}^{(1)} = \underline{f}^{(2)} = \underline{f}^{(3)} = \underline{f}^{(4)} = \begin{Bmatrix} 7.35 \\ 7.35 \end{Bmatrix} \tag{13.9.18}$$

Assembling the global stiffness matrix using Eqs. (13.9.16) and (13.9.17) and the global force matrix using Eq. (13.9.18), we obtain the global equations as

$$\begin{bmatrix} -0.542 & 0.579 & 0 & 0 & 0 \\ -0.554 & 0.591-0.542 & 0.579 & 0 & 0 \\ 0 & -0.554 & 0.591-0.542 & 0.579 & 0 \\ 0 & 0 & -0.554 & 0.591-0.542 & 0.579 \\ 0 & 0 & 0 & -0.554 & 0.591 \end{bmatrix} \begin{Bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{Bmatrix}$$

$$= \begin{Bmatrix} F_1 + 7.35 \\ 14.7 \\ 14.7 \\ 14.7 \\ 7.35 \end{Bmatrix} \tag{13.9.19}$$

Applying the boundary condition $t_1 = 100\,°\text{F}$, we rewrite Eq. (13.9.19) as

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0.049 & 0.579 & 0 & 0 \\ 0 & -0.554 & 0.049 & 0.579 & 0 \\ 0 & 0 & -0.554 & 0.049 & 0.579 \\ 0 & 0 & 0 & -0.554 & 0.591 \end{bmatrix} \begin{Bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{Bmatrix} = \begin{Bmatrix} 100 \\ 14.7 + 55.4 \\ 14.7 \\ 14.7 \\ 7.35 \end{Bmatrix} \tag{13.9.20}$$

Solving the second through fifth equations of Eq. (13.9.20) for the unknown temperatures, we obtain

$$t_2 = 106.1\,°\text{F} \qquad t_3 = 112.1\,°\text{F} \qquad t_4 = 117.6\,°\text{F} \qquad t_5 = 122.6\,°\text{F} \tag{13.9.21}$$

Using Eq. (13.8.2), we obtain the heat flow into and out of the tube as

$$\begin{aligned} q_{\text{in}} &= \dot{m}ct_1 = (4.72)(0.24)(100) = 113.28 \text{ Btu/h} \\ q_{\text{out}} &= \dot{m}ct_5 = (4.72)(0.24)(122.6) = 138.9 \text{ Btu/h} \end{aligned} \tag{13.9.22}$$

where, again, the conduction contribution to q is negligible; that is, $-kA\Delta T$ is negligible. The analytical solution in Reference [7] yields

$$t_5 = 123.0\,°\text{F} \qquad q_{\text{out}} = 139.33 \text{ Btu/h} \tag{13.9.23}$$

The finite element solution is then seen to compare quite favorably with the analytical solution. ■

The element with the stiffness matrix given by Eq. (13.9.13) has been used in Reference [8] to analyze heat exchangers. Both double-pipe and shell-and-tube heat exchangers were modeled to predict the length of tube needed to perform the task of proper heat exchange between two counterflowing fluids. Excellent agreement was found between the finite element solution and the analytical solutions described in Reference [9].

Finally, remember that when the variational formulation of a problem is difficult to obtain but the differential equation describing the problem is available, a residual method such as Galerkin's method can be used to solve the problem.

13.10 Flowchart and Examples of a Heat-Transfer Program

Figure 13–30 is a flowchart of the finite element process used for the analysis of two-dimensional heat-transfer problems.

Figures 13–31 and 13–32 show examples of two-dimensional temperature distribution using the two-dimensional heat transfer element of this chapter (results obtained from Algor [10]). We assume that there is no heat transfer in the direction perpendicular to the plane.

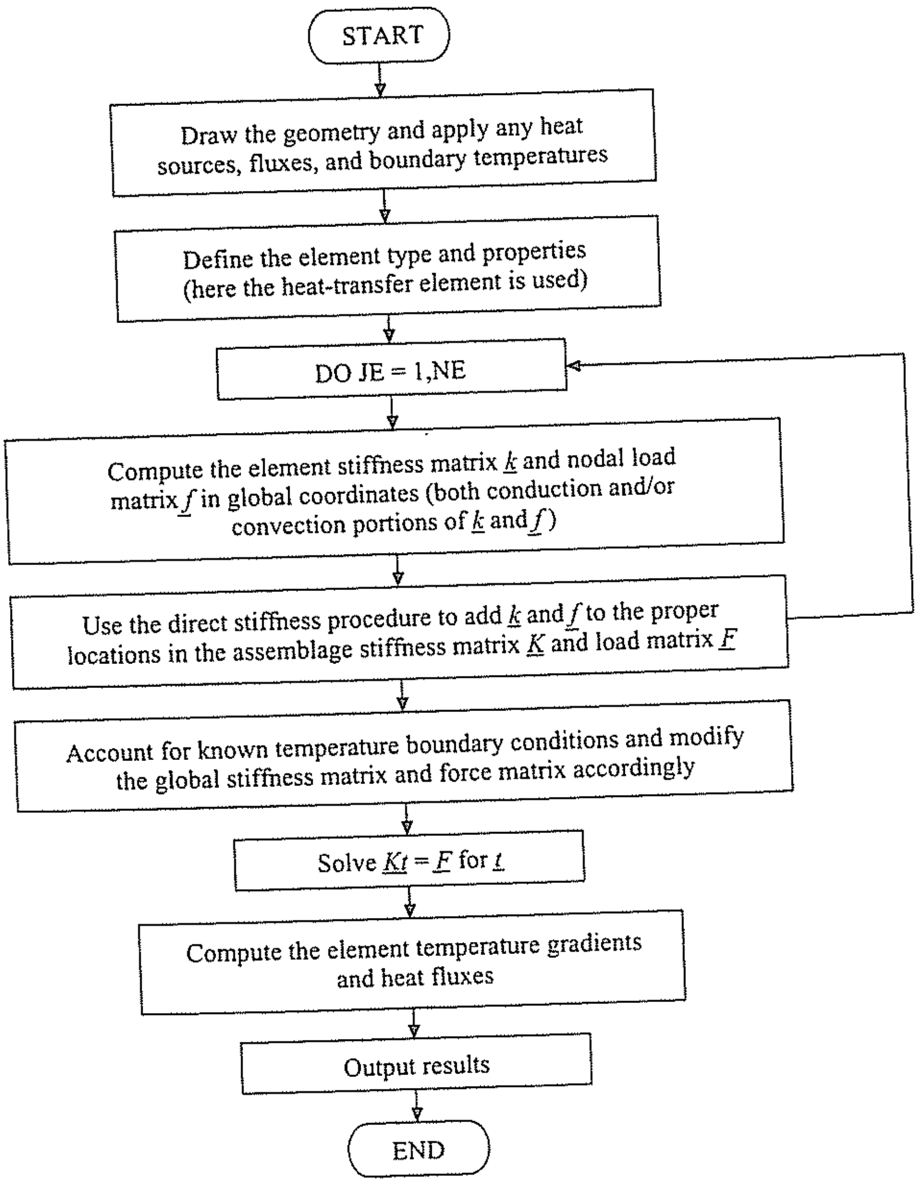
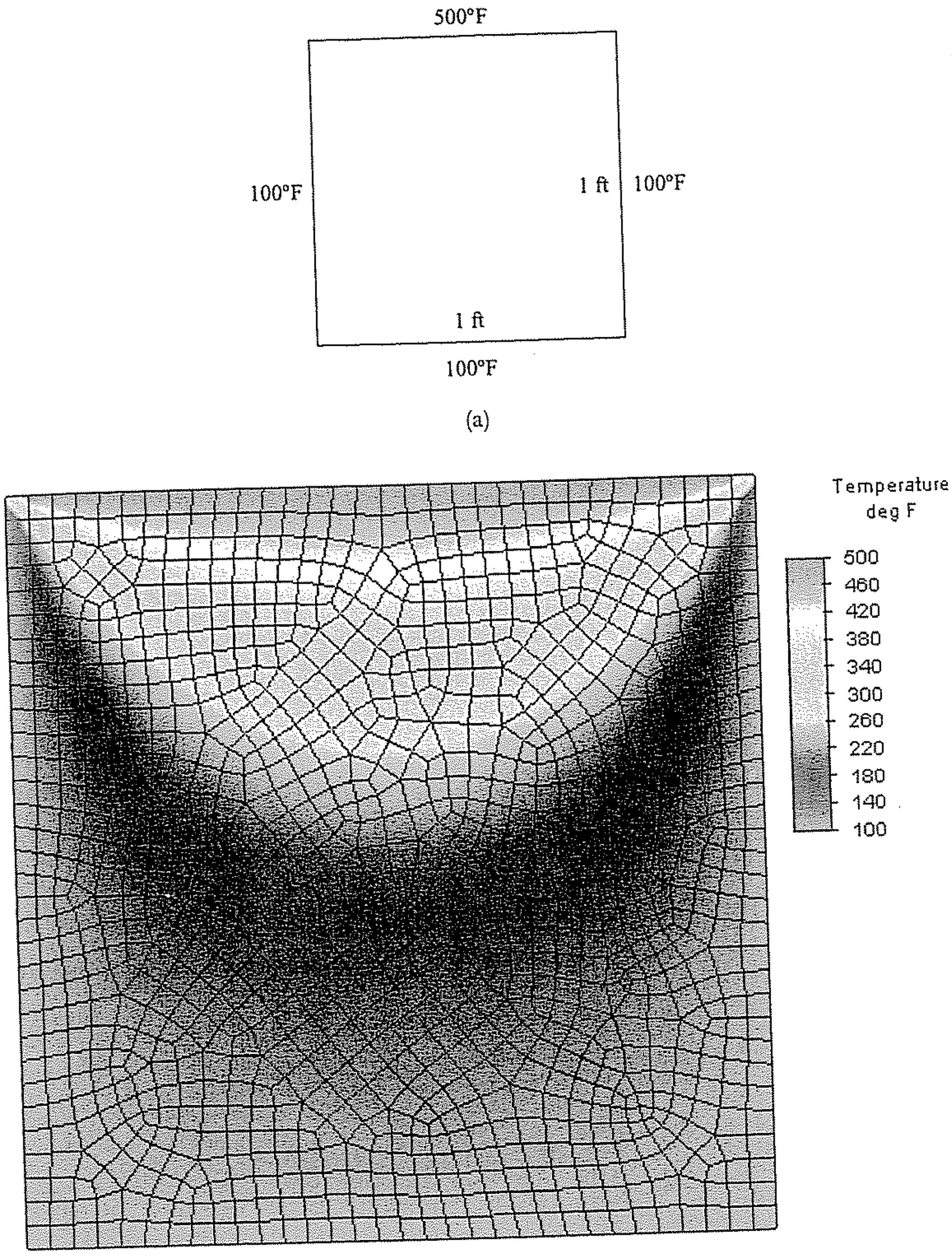


Figure 13–30 Flowchart of two-dimensional heat-transfer process



(b)

Figure 13–31 (a) Square plate subjected to temperature distribution and (b) finite element model with resulting temperature variation throughout the plate ((b) Courtesy of David Walgrave)

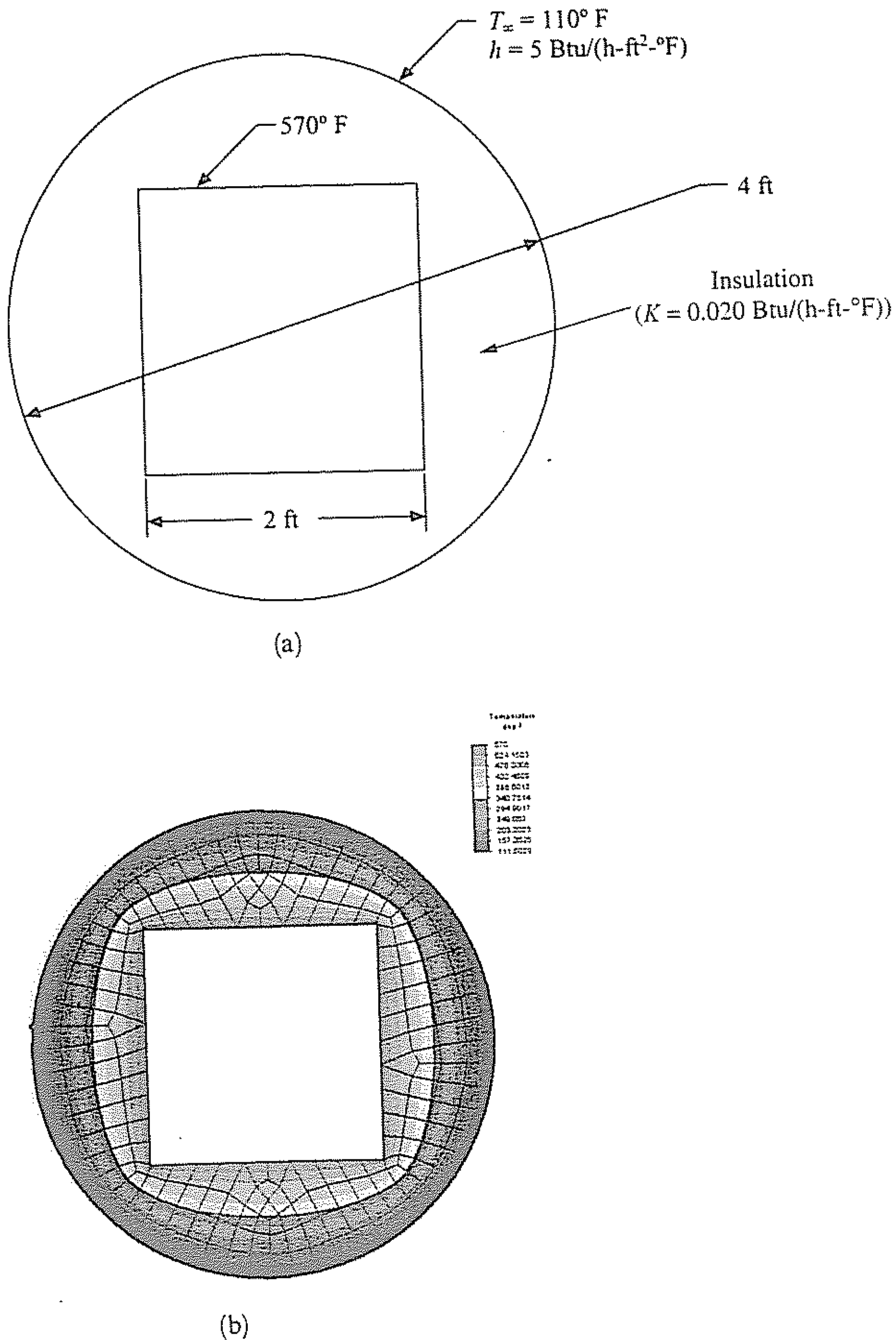


Figure 13–32 (a) Square duct wrapped by insulation and (b) the finite element model with resulting temperature variation through the insulation

Figure 13–31(a) shows a square plate subjected to boundary temperatures. Figure 13–31(b) shows the finite element model, along with the temperature distribution throughout the plate.

Figure 13–32(a) shows a square duct that carries hot gases such that its surface temperature is 570 °F. The duct is wrapped by a layer of circular fiberglass. The finite element model, along with the temperature distribution throughout the fiberglass is shown in Figure 13–32(b).

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Problems

13.1 For the one-dimensional composite bar shown in Figure P13–1, determine the interface temperatures. For element 1, let $K_{xx} = 200$ W/(m · °C); for element 2, let $K_{xx} = 100$ W/(m · °C); and for element 3, let $K_{xx} = 50$ W/(m · °C). Let $A = 0.1$ m^2. The left end has a constant temperature of 100 °C and the right end has a constant temperature of 300 °C.

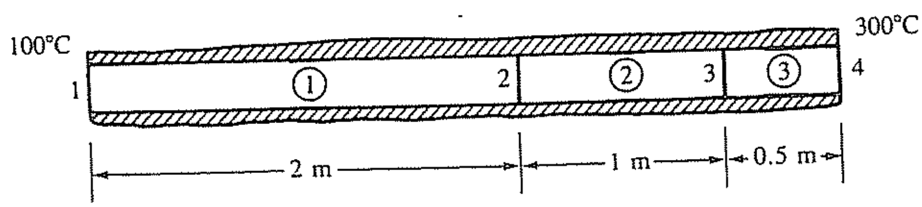

Figure P13–1

13.2 For the one-dimensional rod shown in Figure P13–2 (insulated except at the ends), determine the temperatures at $L/3$, $2L/3$, and L. Let $K_{xx} = 3$ Btu/(h.-in.-°F), $h = 1.0$ Btu/(h-in^2-°F), and $T_\infty = 0$ °F. The temperature at the left end is 200 °F.

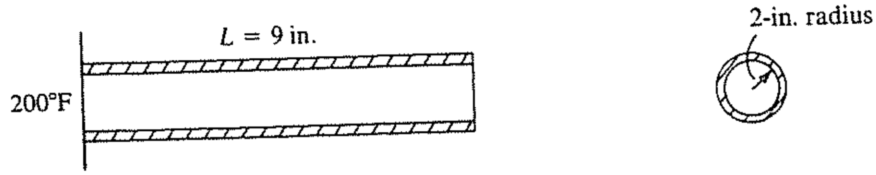

Figure P13–2

13.3 A rod with uniform cross-sectional area of 2 in^2 and thermal conductivity of 3 Btu/(h-in.-°F) has heat flow in the x direction only (Figure P13–3). The right end is insulated. The left end is maintained at 50 °F, and the system has the linearly distributed heat flux shown.

Use a two-element model and estimate the temperature at the node points and the heat flow at the left boundary.

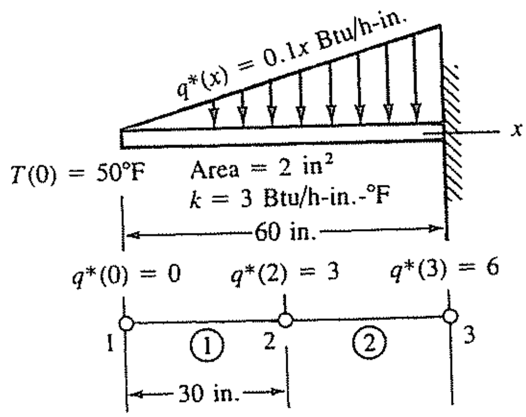

Figure P13–3

13.4 The rod of 1-in. radius shown in Figure P13–4 generates heat internally at the rate of uniform $Q = 10{,}000$ Btu/(h-ft^3) throughout the rod. The left edge and perimeter of the rod are insulated, and the right edge is exposed to an environment of $T_\infty = 100\,°$F. The convection heat-transfer coefficient between the wall and the environment is $h = 100$ Btu/(h-ft^2-°F). The thermal conductivity of the rod is $K_{xx} = 12$ Btu/(h-ft-°F). The length of the rod is 3 in. Calculate the temperature distribution in the rod. Use at least three elements in your finite element model.

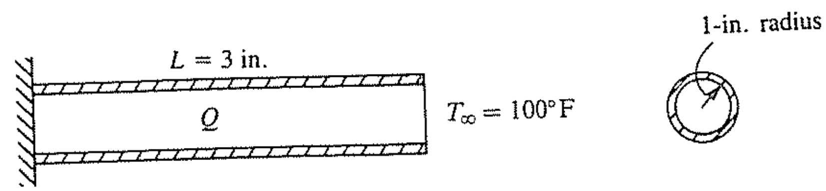

Figure P13–4

13.5 The fin shown in Figure P13–5 is insulated on the perimeter. The left end has a constant temperature of 100 °C. A positive heat flux of $q^* = 5000$ W/m^2 acts on the right end. Let $K_{xx} = 6$ W/(m · °C) and cross-sectional area $A = 0.1$ m^2. Determine the temperatures at $L/4$, $L/2$, $3L/4$, and L, where $L = 0.4$ m.

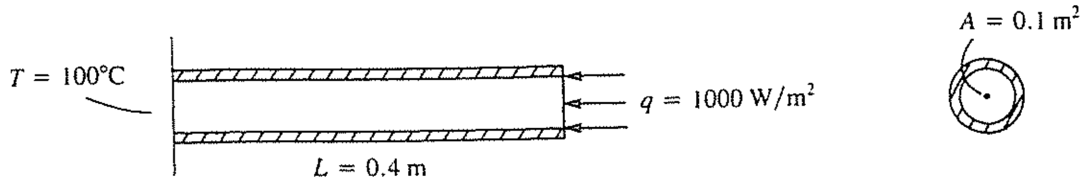

Figure P13–5

13.6 For the composite wall shown in Figure P13–6, determine the interface temperatures. What is the heat flux through the 8-cm portion? Use the finite element method. Use three elements with the nodes shown. 1 cm = 0.01 m.

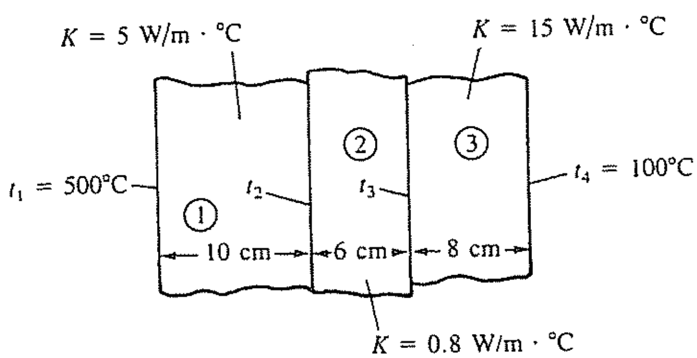

Figure P13–6

13.7 For the composite wall idealized by the one-dimensional model shown in Figure P13–7, determine the interface temperatures. For element 1, let $K_{xx} = 5$ W/(m · °C); for element 2, $K_{xx} = 10$ W/(m · °C); and for element 3, $K_{xx} = 15$ W/(m · °C). The left end has a constant temperature of 200 °C and the right end has a constant temperature of 600 °C.

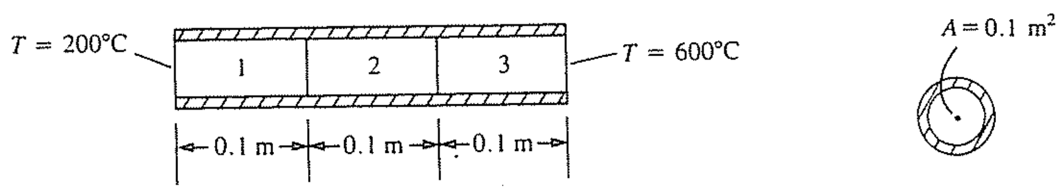

Figure P13–7

13.8 A double-pane glass window shown in Figure P13–8, consists of two 4 mm thick layers of glass with $k = 0.80$ W/m-°C separated by a 10 mm thick stagnant air space with $k = 0.025$ W/m-°C. Determine (a) the temperature at both surfaces of the inside layer of glass and the temperature at the outside surfaces of glass, and (b) the steady rate of heat transfer in Watts through the double pane. Assume the inside room temperature $T_{i\infty} = 20$ °C with $h_i = 10$ W/m²-°C and the outside temperature $T_{0\infty} = 0$ °C with $h_0 = 30$ W/m²-°C. Assume one-dimensional heat flow through the glass.

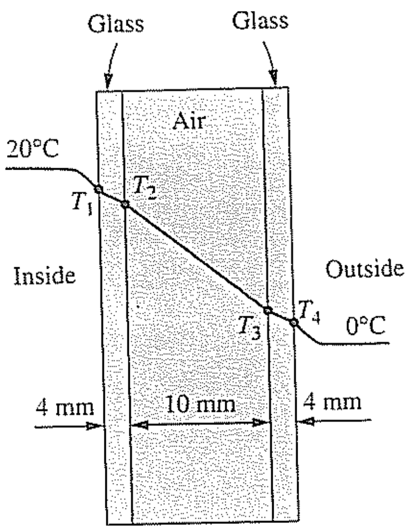

Figure P13–8

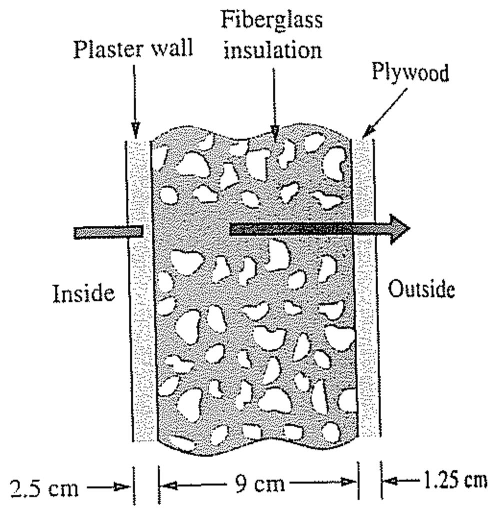

Figure P13–9

13.9 For the composite wall of a house, shown in Figure P13–9, determine the temperatures at the inner and outer surfaces and at the interfaces. The wall is composed of 2.5 cm thick plaster wall ($k = 0.20\ \text{W/m-°C}$) on the inside, a 9 cm thick layer of fiber glass insulation ($k = 0.038\ \text{W/m-°C}$), and a 1.25 cm plywood layer ($k = 0.12\ \text{W/m-°C}$) on the outside. Assume the inside room air is 20 °C with convection coefficient of $10\ \text{W/m}^2\text{-°C}$ and the outside air at −10 °C with convection coefficient of $20\ \text{W/m}^2\text{-°C}$. Also, determine the rate of heat transfer through the wall in Watts. Assume one-dimensional heat flow through the wall thickness.

13.10 Condensing steam is used to maintain a room at 20 °C. The steam flows through pipes that keep the pipe surface at 100 °C. To increase heat transfer from the pipes, stainless steel fins ($k = 15\ \text{W/m-°C}$), 20 cm long and 0.5 cm in diameter, are welded to the pipe surface as shown in Figure P13–10. A fan forces the room air over the pipe and fins, resulting in a heat transfer coefficient of $50\ \text{W/m}^2\text{-°C}$ at the base surface of the fin where it is welded to the pipe. However, the air flow distribution increases the heat transfer coefficient to $80\ \text{W/m}^2\text{-°C}$ at the fin tip. Assume the variation in heat transfer coefficient to then vary linearly from left end to right end of the fin surface. Determine the temperature distribution at L/4 locations along the fin. Also determine the rate of heat loss from each fin.

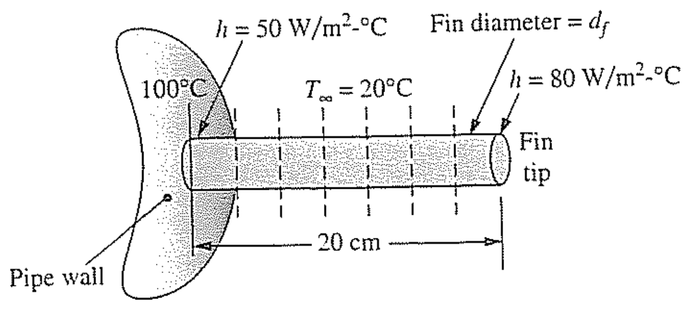

Figure P13–10

13.11 A tapered aluminum fin ($k = 200\ \mathrm{W/m\text{-}^\circ C}$), shown in Figure P13–11, has a circular cross section with base diameter of 1 cm and tip diameter of 0.5 cm. The base is maintained at 200 °C and looses heat by convection to the surroundings at $T_\infty = 10\ ^\circ\mathrm{C}$, $h = 150\ \mathrm{W/m^2\text{-}^\circ C}$. The tip of the fin is insulated. Assume one-dimensional heat flow and determine the temperatures at the quarter points along the fin. What is the rate of heat loss in Watts through each element? Use four elements with an average cross-sectional area for each element.

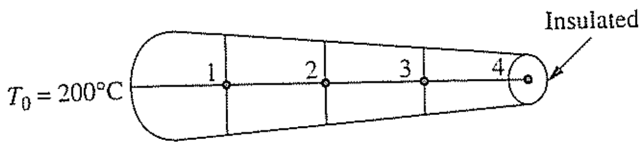

Figure P13–11

13.12 A wall is constructed of an outer layer of 0.5 inch thick plywood ($k = 0.80$ Btu/h-ft-°F), an inner core of 5 inch thick fiberglass insulation ($k = 0.020$ Btu/h-ft-°F), and an inner layer of 0.5 inch thick sheetrock ($k = 0.10$ Btu/h-ft-°F) (Figure P13–12). The inside temperature is 65 °F with $h = 1.5$ Btu/h-ft^2-°F, while the outside temperature is 0 °F with $h = 4$ Btu/h-ft^2-°F. Determine the temperature at the interfaces of the materials and the rate of heat flow in Btu/h through the wall.

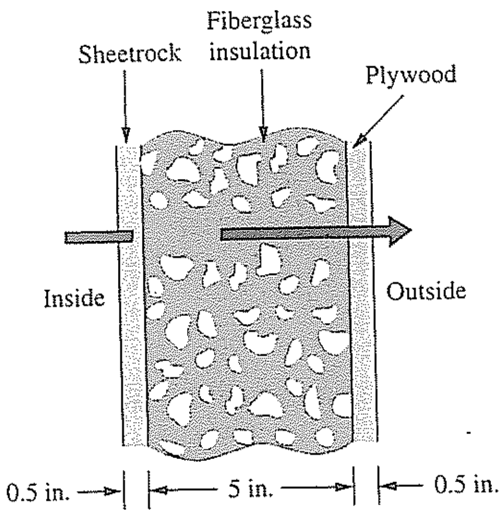

Figure P13–12

13.13 A large plate of stainless steel with thickness of 5 cm and thermal conductivity of $k = 15\ \mathrm{W/m\text{-}^\circ C}$ is subjected to an internal uniform heat generation throughout the plate at constant rate of $Q = 10 \times 10^6\ \mathrm{W/m^3}$. One side of the plate is maintained at 0 °C by ice water, and the other side is subjected to convection to an environment at $T_\infty = 35\ ^\circ\mathrm{C}$, with heat transfer coefficient $h = 40\ \mathrm{W/m^2\text{-}^\circ C}$, as shown in Figure P13–13. Use three elements in a finite element model to estimate the temperatures at each surface and in the middle of the plate's thickness. Assume a one-dimensional heat transfer through the plate.

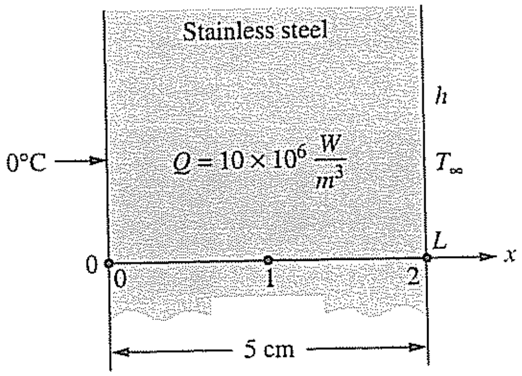

Figure P13–13

13.14 The base plate of an iron is 0.6 cm thick. The plate is subjected to 600 W of power (provided by resistance heaters inside the iron, as shown in Figure P13–14), over a base plate cross-sectional area of 150 cm^2, resulting in a uniform flux generated on the inside surface. The thermal conductivity of the metal base plate is $k = 20\,W/m\text{-}°C$. The outside temperature of the plate is 80 °C at steady state conditions. Assume one-dimensional heat transfer through the plate thickness. Using three elements, model the plate to determine the temperatures at the inner surface and interior one-third points.

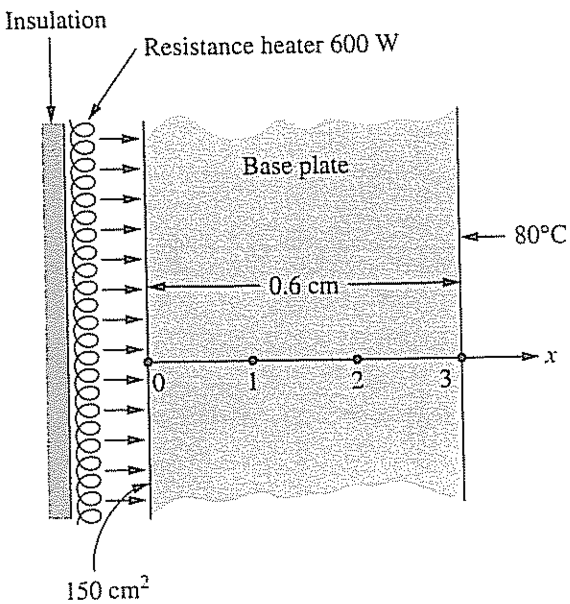

Figure P13–14

13.15 A hot surface is cooled by attaching fins (called pin fins) to it, as shown in Figure P13–15. The surface of the plate (left end of the pin) is 90 °C. The fins are 4 cm long and 0.25 cm in diameter. The fins are made of copper ($k = 400\,W/m\text{-}°C$). The temperature of the surrounding air is $T_\infty = 25\,°C$ with heat transfer coefficient on the surface (including the end surface) of $h = 30\,W/m^2\text{-}°C$. A model of the typical fin is also shown in Figure P13–15. Use four elements in your finite element model to determine the temperatures along the fin length.

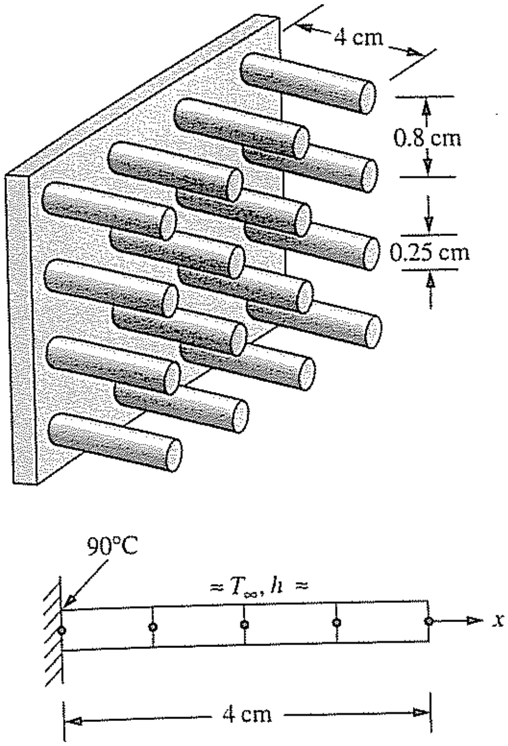

Figure P13–15

13.16 Use the direct method to derive the element equations for the one-dimensional steady-state conduction heat-transfer problem shown in Figure P13–16. The bar is insulated all around and has cross-sectional area A, length L, and thermal conductivity K_{xx}. Determine the relationship between nodal temperatures t_1 and t_2 (°F) and the thermal inputs F_1 and F_2 (in Btu). Use Fourier's law of heat conduction for this case.

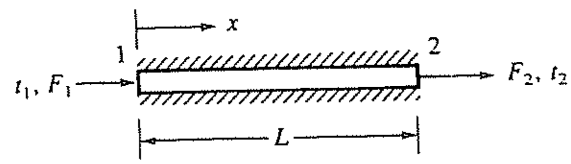

Figure P13–16

13.17 Express the stiffness matrix and the force matrix for convection from the left end of a bar, as shown in Figure P13–17. Let the cross-sectional area of the bar be A, the convection coefficient be h and the free stream temperature be T_∞.

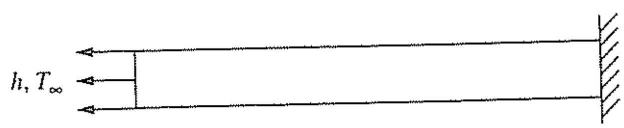

Figure P13–17

13.18 For the element shown in Figure P13–18, determine the $\underline{k}$ and $\underline{f}$ matrices. The conductivities are $K_{xx} = K_{yy} = 15$ Btu/(h-ft-°F) and the convection coefficient is $h = 20$ Btu/(h-ft^2-°F). Convection occurs across the *i-j* surface. The free-stream temperature is $T_\infty = 70\,°$F. The coordinates are expressed in units of feet. Let the line source be $Q^* = 150$ Btu/(h-ft) as located in the figure. Take the thickness of the element to be 1 ft.

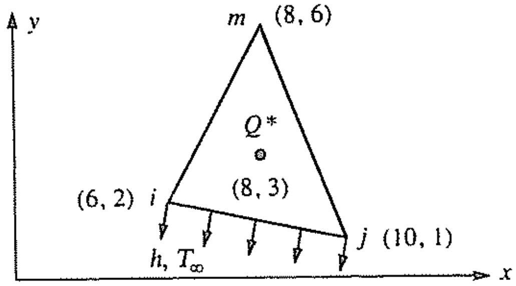

Figure P13–18

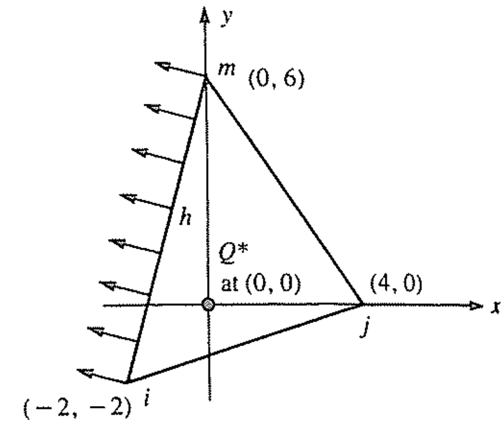

Figure P13–19

13.19 Calculate the $\underset{\sim}{k}$ and $\underset{\sim}{f}$ matrices for the element shown in Figure P13–19. The conductivities are $K_{xx} = K_{yy} = 15$ W/(m · °C) and the convection coefficient is $h = 20$ W/(m^2 · °C). Convection occurs across the i-m surface. The free-stream temperature is $T_\infty = 15\,^\circ$C. The coordinates are shown expressed in units of meters. Let the line source be $Q^* = 100$ W/m as located in the figure. Take the thickness of the element to be 1 m.

13.20 For the square two-dimensional body shown in Figure P13–20, determine the temperature distribution. Let $K_{xx} = K_{yy} = 25$ Btu/(h-ft-°F) and $h = 10$ Btu/(h-ft^2-°F). Convection occurs across side 4–5. The free-stream temperature is $T_\infty = 50\,^\circ$F. The temperatures at nodes 1 and 2 are 100 °F. The dimensions of the body are shown in the figure. Take the thickness of the body to be 1 ft.

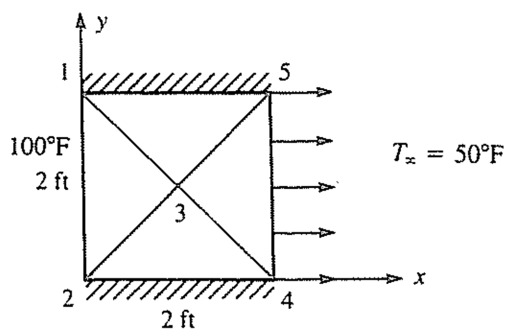

Figure P13–20

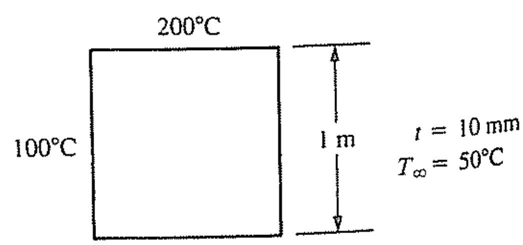

Figure P13–21

13.21 For the square plate shown in Figure P13–21, determine the temperature distribution. Let $K_{xx} = K_{yy} = 10$ W/(m · °C) and $h = 20$ W/(m^2 · °C). The temperature along the left side is maintained at 100 °C and that along the top side is maintained at 200 °C.

Use a computer program to calculate the temperature distribution in the following two-dimensional bodies.

13.22 For the body shown in Figure P13–22, determine the temperature distribution. Surface temperatures are shown in the figure. The body is insulated along the top and bottom edges, and $K_{xx} = K_{yy} = 1.0$ Btu/(h-in.-°F). No internal heat generation is present.

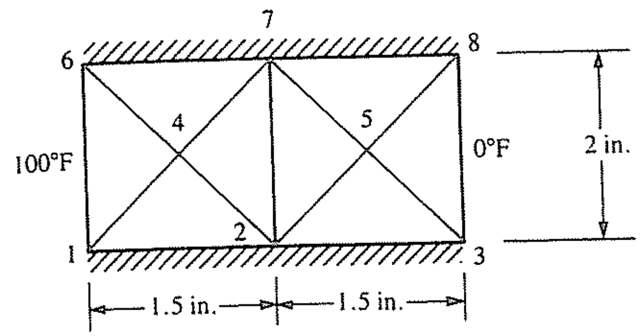

Figure P13–22

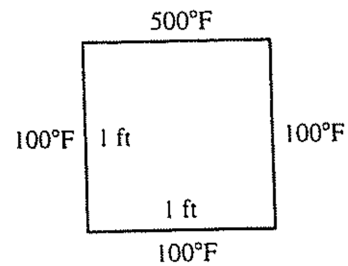

Figure P13–23

13.23 For the square two-dimensional body shown in Figure P13–23, determine the temperature distribution. Let $K_{xx} = K_{yy} = 10$ Btu/(h-ft-°F). The top surface is maintained at 500 °F and the other three sides are maintained at 100 °F. Also, plot the temperature contours on the body.

13.24 For the square two-dimensional body shown in Figure P13–24, determine the temperature distribution. Let $K_{xx} = K_{yy} = 10$ Btu/(h-ft-°F) and $h = 10$ Btu/(h-ft^2-°F). The top face is maintained at 500 °F, the left face is maintained at 100 °F, and the other two faces are exposed to an environmental (free-stream) temperature of 100 °F. Also, plot the temperature contours on the body.

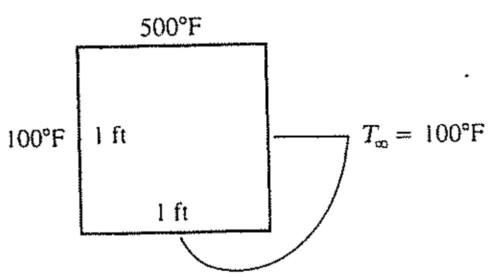

Figure P13–24

13.25 Hot water pipes are located on 2.0-ft centers in a concrete slab with $K_{xx} = K_{yy} = 0.80$ Btu/(h-ft-°F), as shown in Figure P13–25. If the outside surfaces of the concrete are at 85 °F and the water has an average temperature of 200 °F, determine the temperature distribution in the concrete slab. Plot the temperature contours through the concrete. Use symmetry in your finite element model.

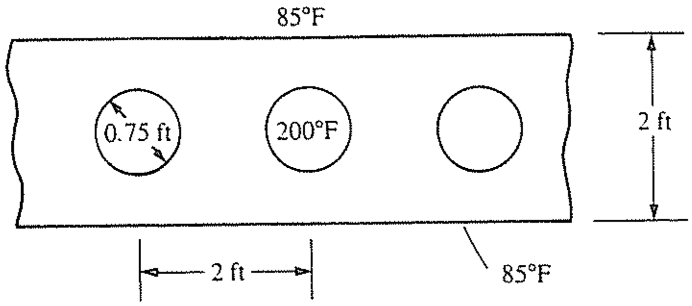

Figure P13–25

13.26 The cross section of a tall chimney shown in Figure P13–26 has an inside surface temperature of 330 °F and an exterior temperature of 130 °F. The thermal conductivity is $K = 0.5$ Btu/(h-ft-°F). Determine the temperature distribution within the chimney per unit length.

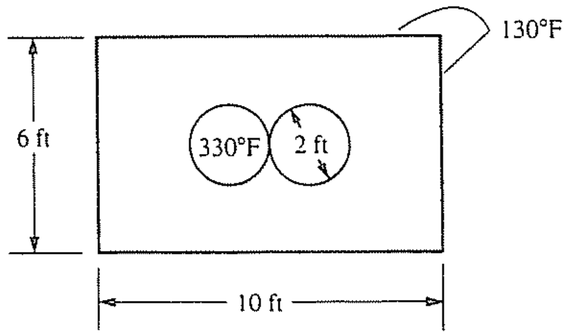

Figure P13–26

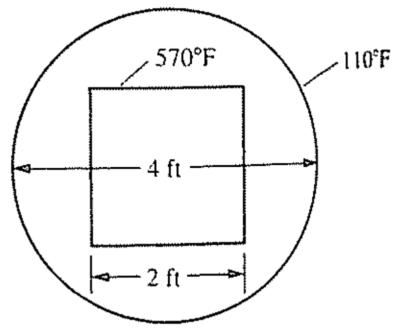

Figure P13–27

13.27 The square duct shown in Figure P13–27 carries hot gases such that its surface temperature is 570 °F. The duct is insulated by a layer of circular fiberglass that has a thermal conductivity of $K = 0.020$ Btu/(h-ft-°F). The outside surface temperature of the fiberglass is maintained at 110 °F. Determine the temperature distribution within the fiberglass.

13.28 The buried pipeline in Figure P13–28 transports oil with an average temperature of 60 °F. The pipe is located 15 ft below the surface of the earth. The thermal conductivity of the earth is 0.6 Btu/(h-ft-°F). The surface of the earth is 50 °F. Determine the temperature distribution in the earth.

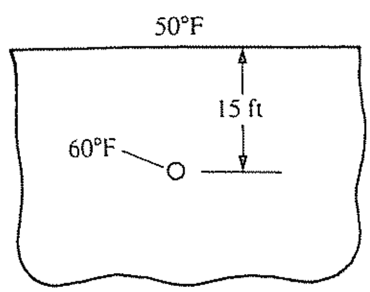

Figure P13–28

13.29 A 10-in.-thick concrete bridge deck is embedded with heating cables, as shown in Figure P13–29. If the lower surface is at 0 °F, the rate of heat generation (assumed to be the same in each cable) is 100 Btu/(h-in.) and the top surface of the concrete is at 35 °F. The thermal conductivity of the concrete is 0.500 Btu/(h-ft-°F). What is the temperature distribution in the slab? Use symmetry in your model.

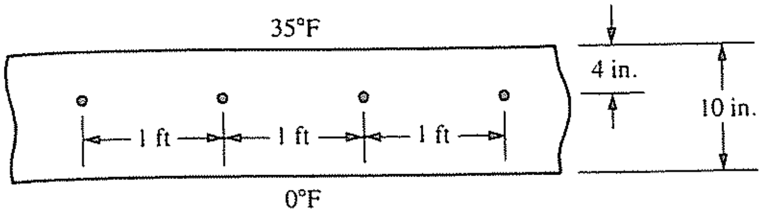

Figure P13–29

13.30 For the circular body with holes shown in Figure P13–30, determine the temperature distribution. The inside surfaces of the holes have temperatures of 150 °C. The outside of the circular body has a temperature of 30 °C. Let $K_{xx} = K_{yy} = 10$ W/(m · °C).

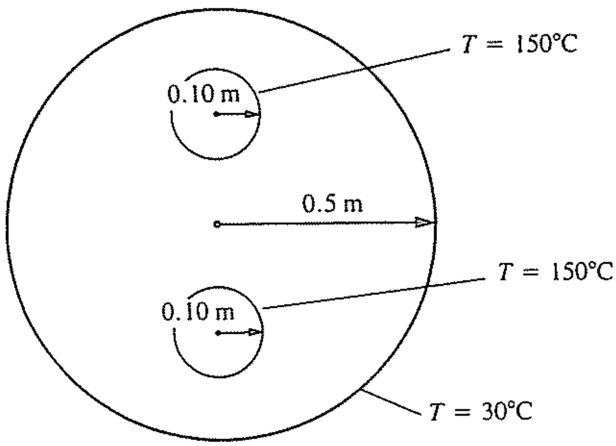

Figure P13–30

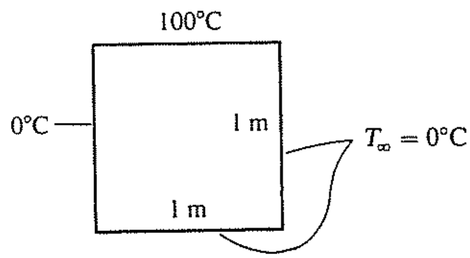

Figure P13–31

13.31 For the square two-dimensional body shown in Figure P13–31, determine the temperature distribution. Let $K_{xx} = K_{yy} = 10$ W/(m · °C) and $h = 10$ W/(m^2 · °C). The top face is maintained at 100 °C, the left face is maintained at 0 °C, and the other two faces are exposed to a free-stream temperature of 0 °C. Also, plot the temperature contours on the body.

13.32 A 200-mm-thick concrete bridge deck is embedded with heating cables as shown in Figure P13–32. If the lower surface is at −10 °C and the upper surface is at 5 °C, what is the temperature distribution in the slab? The heating cables are line sources generating heat of $Q^* = 50$ W/m. The thermal conductivity of the concrete is 1.2 W/(m · °C). Use symmetry in your model.

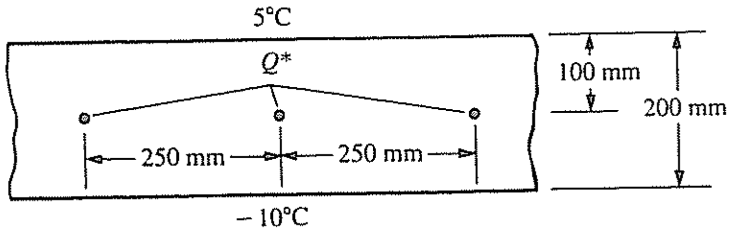

Figure P13–32

13.33 For the two-dimensional body shown in Figure P13–33, determine the temperature distribution. Let the left and right ends have constant temperatures of 200 °C and 100 °C, respectively. Let $K_{xx} = K_{yy} = 5$ W/(m · °C). The body is insulated along the top and bottom.

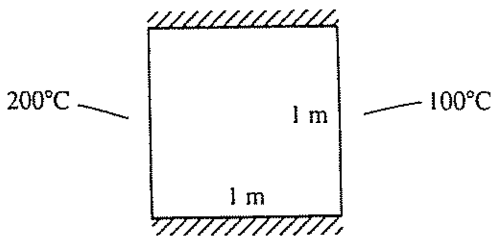

Figure P13–33

13.34 For the two-dimensional body shown in Figure P13–34, determine the temperature distribution. The top and bottom sides are insulated. The right side is subjected to heat transfer by convection. Let $K_{xx} = K_{yy} = 10$ W/(m · °C).

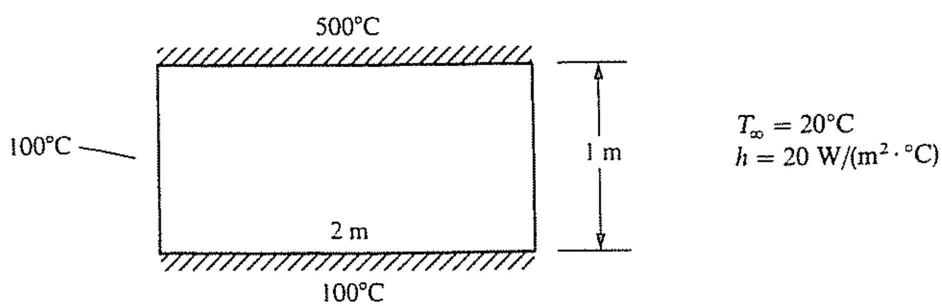

Figure P13–34

13.35 For the two-dimensional body shown in Figure P13–35, determine the temperature distribution. The left and right sides are insulated. The top surface is subjected to heat transfer by convection. The bottom and internal portion surfaces are maintained at 300 °C.

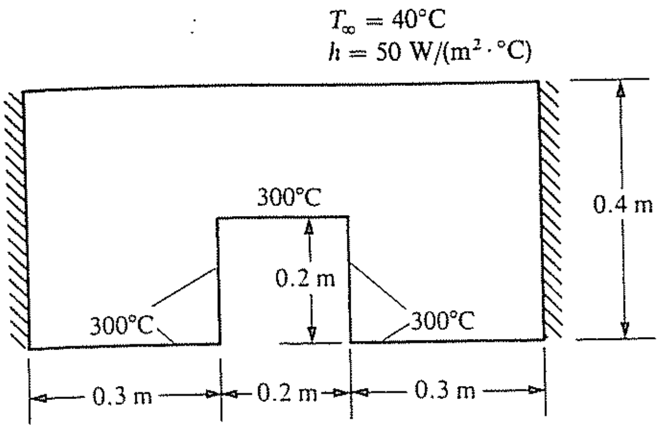

Figure P13–35

13.36 Determine the temperature distribution and rate of heat flow through the plain carbon steel ingot shown in Figure P13–36. Let $k = 60$ W/ m-K) for the steel. The top surface is held at 40°C, while the underside surface is held at 0 °C. Assume that no heat is lost from the sides.

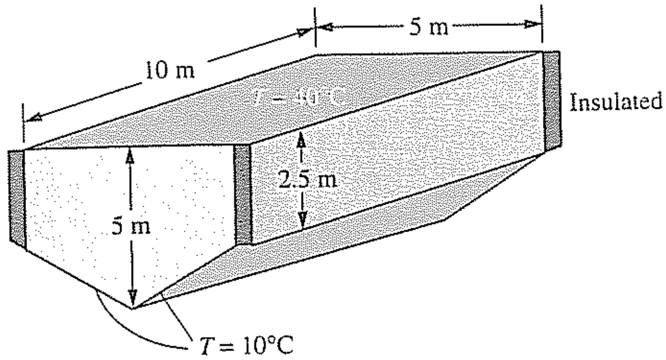

Figure P13–36

13.37 Determine the temperature distribution and rate of heat flow per foot length from a 5 cm outer diameter pipe at 180 °C placed eccentrically within a larger cylinder of insulation ($k = 0.058$ W/ m-°C) as shown in Figure P13–37. The diameter of the outside cylinder is 15 cm, and the surface temperature is 20 °C.

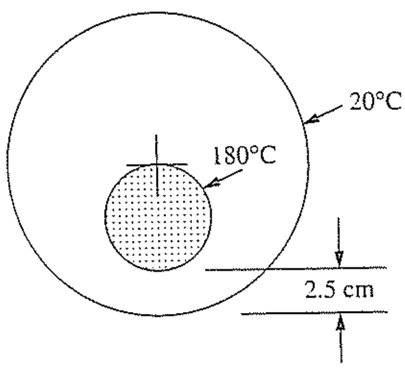

Figure P13–37

13.38 Determine the temperature distribution and rate of heat flow per foot length from the inner to the outer surface of the molded foam insulation ($k = 0.17$ Btu/h-ft-°F) shown in Figure P13–38.

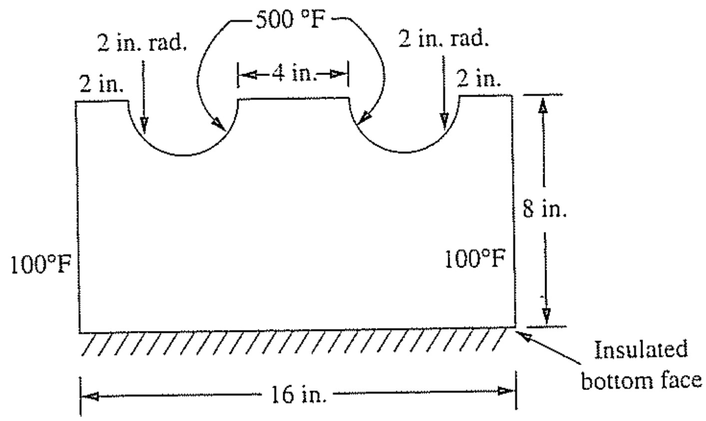

Figure P13–38

13.39 For the basement wall shown in Figure P13–39, determine the temperature distribution and the heat transfer through the wall and soil. The wall is constructed of concrete ($k = 1.0$ Btu/h-ft-°F). The soil has an average thermal conductivity of $k = 0.85$ Btu/h-ft-°F. The inside air is maintained at 70 °F with a convection coefficient $h = 2.0$ Btu/h-ft^2-°F. The outside air temperature is 10 °F with a heat transfer coefficient of $h = 6$ Btu/h-ft^2-°F. Assume a reasonable distance from the wall of five feet that the horizontal component of heat transfer becomes negligible. Make sure this assumption is correct.

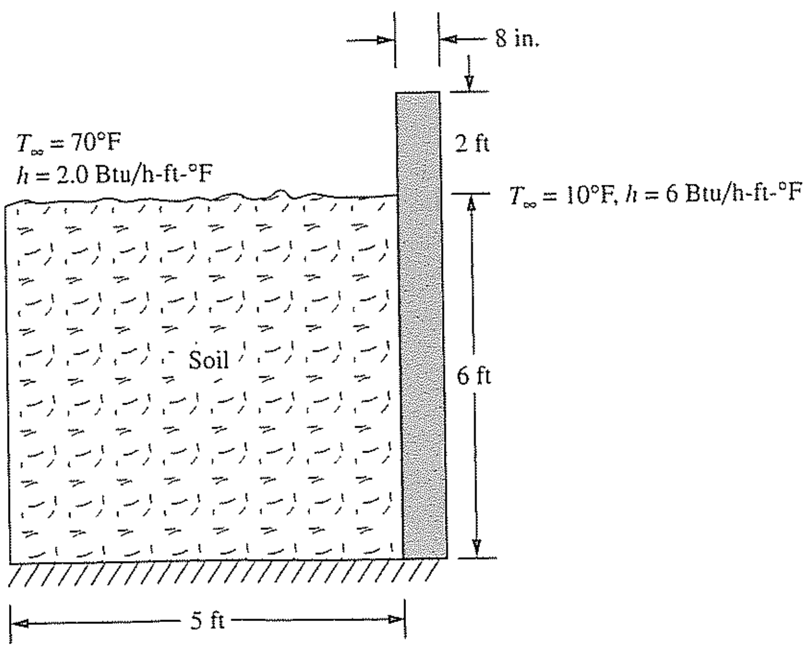

Figure P13–39

13.40 Now add a 6 in. thick concrete floor to the model of Figure P13–39 (as shown in Figure P13–40). Determine the temperature distribution and the heat transfer through the concrete and soil. Use the same properties as shown in P13–39.

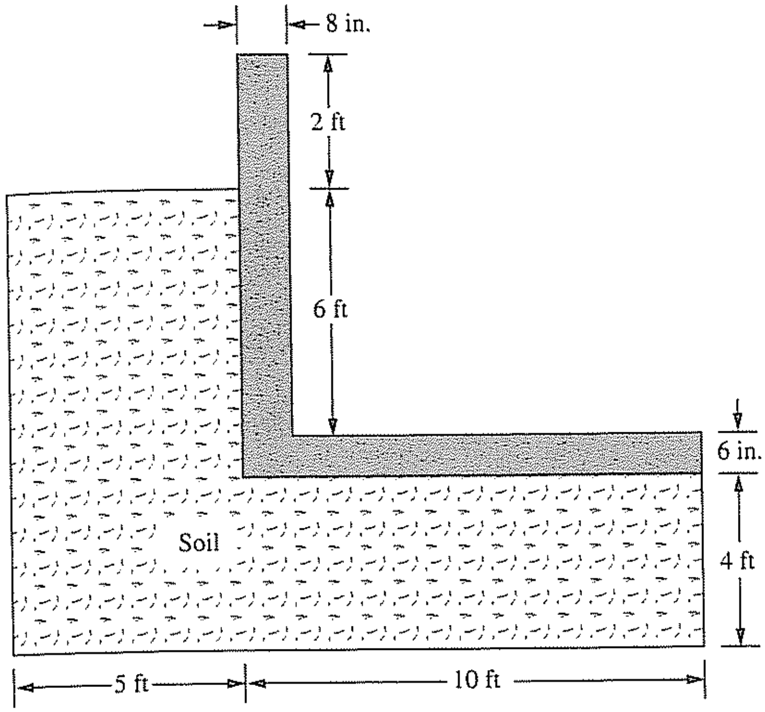

Figure P13–40

13.41 Aluminum fins ($k = 170$ W/m-K) with triangular profiles shown in Figure P13–41 are used to remove heat from a surface with a temperature of 160 °C. The temperature of the surrounding air is 25 °C. The natural convection coefficient is $h = 25$ W/m^2-K. Determine the temperature distribution throughout and the heat loss from a typical fin.

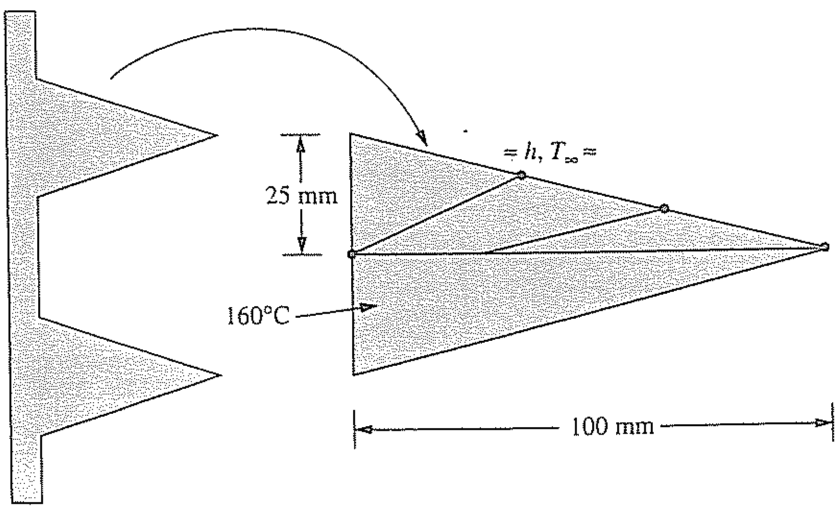

Figure P13–41

13.42 Air is flowing at a rate of 10 lb/h inside a round tube with diameter of 1.5 in. and length of 10 in., similar to Figure 13–29 on page 572. The initial temperature of the air entering the tube is 50 °F. The wall of the tube has a uniform constant temperature of 200 °F. The specific heat of the air is 0.24 Btu/(lb-°F), the convection coefficient between the air and the inner wall of the tube is 3.0 Btu/(h-ft^2-°F), and the thermal conductivity is 0.017 Btu/(h-ft-°F). Determine the temperature of the air along the length of the tube and the heat flow at the inlet and outlet of the tube.